# 17th Iranian International Group Theory Conference

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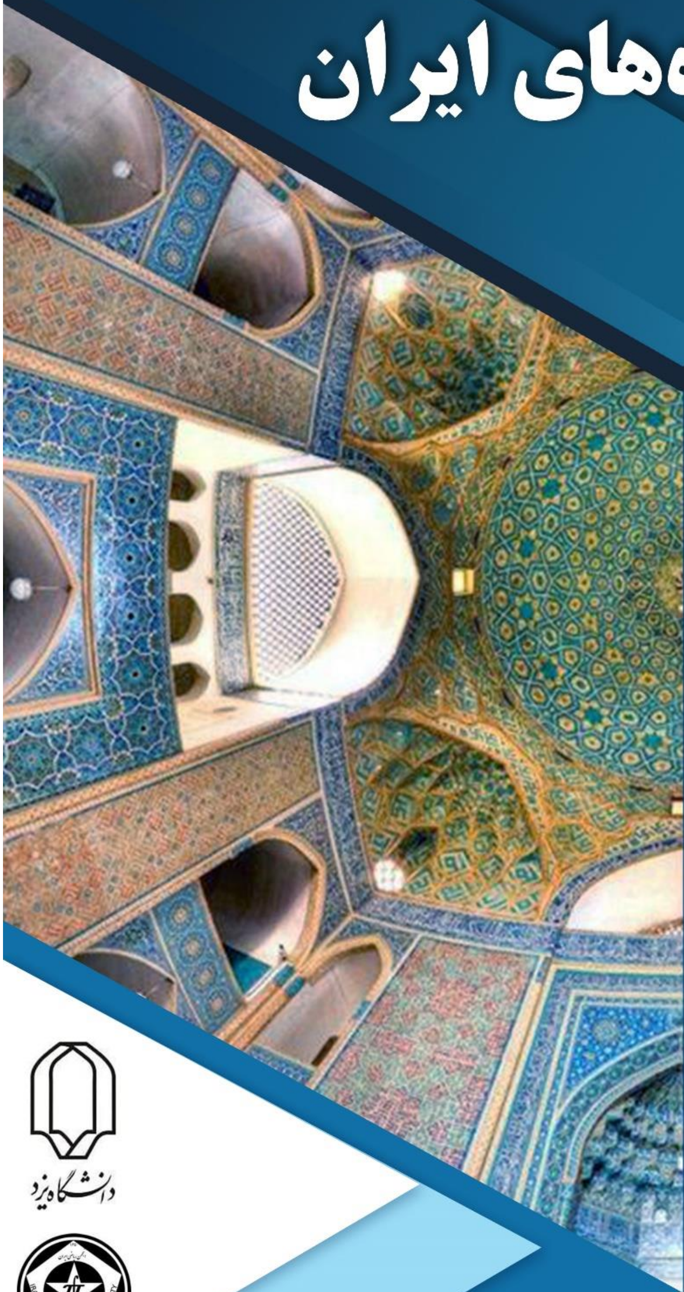

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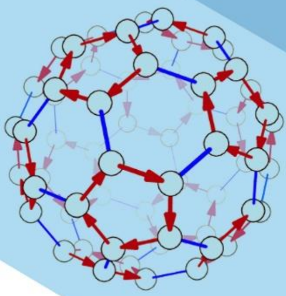

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# The 17th Iranian International Group Theory Conference (IGTC17)

# Abstract Booklet

**29-30 January 2025**
**Yazd University**
**Iran**

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# Message of Organizers

The 17th Iranian Group Theory Conference was held on the Department of Mathematical Science at Yazd University in Yazd, I. R. Iran during 29 and 30 January 2025. The Scientific Committee for the conference accepted 35 papers from more than 45 papers. The conference has seven invited speakers, three from I. R. Iran, one from Turkey, one from Oman and one from South Africa.

The purpose of the conference was to bring together experts and post graduate students in group theory to discuss about the new results in this area. The conference included seven 45 minutes lectures and thirty-three twenty-minutes contributed talks. The organizing committee of the conference warmly welcomes the participants, especially invited speakers and scientific committee to Yazd, and wishes a very enjoyable and fruitful stay. This conference coincides with the 60th anniversary of Professor Davvaz's birth, which we are happy about and celebrate. The organizers thanks President of Yazd University (Prof. Abbas Kalantari) and Research Deputy of Yazd University for making this conference possible through their financial support.

Prof. Saeid Alikhani

Conference Chairman

Prof. Mohammad Ali Iranmanesh

Scientific Committee Chairman

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# Invited Talks

# Subgroup perfect codes and regular sets in Cayley graphs

**Zeinab Akhlaghi**[1]

Amirkabir University of Technology, Tehran, Iran

**Abstract:** A subset $C$ of the vertex set of a graph $\Gamma$ is said to be $(a,b)$-regular if $C$ induces an $a$-regular subgraph and every vertex outside $C$ is adjacent to exactly $b$ vertices in $C$. In particular, if $C$ is an $(a,b)$-regular set of some Cayley graph on a finite group $G$, then $C$ is called an $(a,b)$-regular set of $G$ and a $(0,1)$-regular set is called a perfect code of $G$. In [Wang, Xia and Zhou, Regular sets in Cayley graphs, J. Algebr. Comb., 2023] it is proved that if $H$ is a normal subgroup of $G$, then $H$ is a perfect code of $G$ if and only if it is an $(a,b)$-regular set of $G$, for each $0 \leq a \leq |H|-1$ and $0 \leq b \leq |H|$ with $\gcd(2,|H|-1) \mid a$. We generalize this result and show that a subgroup $H$ of $G$ is a perfect code of $G$ if and only if it is an $(a,b)$-regular set of $G$, for each $0 \leq a \leq |H|-1$ and $0 \leq b \leq |H|$ such that $\gcd(2,|H|-1)$ divides $a$. Also, in [J. Zhang, Y. Zhu, A note on regular sets in Cayley graphs, Bull. Aust. Math. Soc., 2024] it is proved that if $H$ is a normal subgroup of $G$, then $H$ is an $(a,b)$-regular set of $G$, for each $0 \leq a \leq |H|-1$ and $0 \leq b \leq |H|$ such that $\gcd(2,|H|-1)$ divides $a$ and $b$ is even. We extend this result and we prove that the normality condition is not needed.

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# On flag-transitive automorphism groups of symmetric designs

**Seyed Hassan Alavi**[1]

Department of Mathematics, Faculty of Science, Bu-Ali Sina University, Hamedan, Iran

**Abstract:** This talk focuses on studying flag-transitive automorphism groups of symmetric designs. In particular, we present some reduction theorems and classification results on flag-transitive automorphism groups of symmetric designs with restricted parameter sets.

**Keywords** distinguishing number; group actions; subgroup.

**Mathematics Subject Classification (2010) : 05E15, 05C15, 20B25.**

A symmetric design $\mathcal{D}$ with parameter set $(v, k, \lambda)$ consists a point set of size $v$ and a block set of size $v$ such that each block is a $k$-subset of the point set and every pair of points is incident with $\lambda$ blocks. Symmetric designs admitting flag-transitive automorphism groups are of most interest. A classification of such designs with 2-transitive automorphism groups is obtained by Kantor [4]. Also he [5] significantly classified flag-transitive symmetric designs with $\lambda = 1$ (projective planes). Regueiro gave a complete classification of non-trivial symmetric designs with $\lambda = 2$ (biplanes) with flag-transitive automorphism groups apart from those admitting a 1-dimensional affine group, and Zhou and Dong studied non-trivial symmetric designs with $\lambda = 3$ (triplanes) and proved that if $\mathcal{D}$ is a non-trivial symmetric design with $\lambda = 3$ admitting a flag-transitive and point-primitive automorphism group $G$, then $\mathcal{D}$ has parameters $(11, 6, 3)$, $(15, 7, 3)$, $(45, 12, 3)$ or $G$ is a subgroup of $\mathrm{A\Gamma L}_1(q)$, where $q = p^n$ with $p \geq 5$ prime, see [6, 7] and therein references. In this talk, we present recent studies on symmetric designs with restricted parameter sets admitting flag-transitive automorphism groups [1, 2, 3]. The main results of this talk are derived from the main results in [1] and the joint papers [2, 3].

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# $\pi$-structure and $\pi$-characters

**Ali Iranmanesh**[1], **Kaveh Dastouri**

Department of Pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, Tehran, Iran

**Abstract:** Let $G$ be a $\pi$-seperable group. In Isaacs (2018), Isaacs used the $\mathrm{I}_\pi(G)$ characters for $\pi$-separable group to extend the idea of ($p$-)Brauer characters to the set $\pi$ in place of a set $p'$. If $G$ is a $\pi$-seperable group and $G^0$ is the set of $\pi$-elements of $G$, an Isaacs $\pi$-partial character $\varphi$ of $G$ is a complex class function defined on $G^0$ of the form $\chi^0$, where $\chi$ is a character of $G$ and $\chi^0$ is the restriction of $\chi$ to $G^0$. We say that an Isaacs $\pi$-partial character is irreducible if it cannot be written as the sum of two Isaacs $\pi$-partial characters and we denote $\mathrm{I}_\pi(G)$ to the set of all irreducible Isaacs $\pi$-partial characters of $G$.

In this talk, we present new results on the relationship between the degrees of irreducible Isaacs $\pi$-partial characters and the structure of finite groups. Specifically, we examine the influence of monomial and monolithic irreducible Isaacs $\pi$-partial character degrees on a group having a normal Hall $\pi'$-subgroup and a normal $p$-complement. Also, we will talk about the solvability of $\pi$-separable groups that every irreducible Isaacs $\pi$-partial character is monomial which we call $M_\pi$-group. The major part of the results of this talk are given from [1], [3] and [2].

**Keywords** $\pi$-partial character; normal Hall $\pi'$-subgroup; normal $p$-complement; monomial character; monolithic character.

**Mathematics Subject Classification (2010) : 20C20.**

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# The fixed-point-free action of groups

**İsmail Şuayip Güloğlu**[1]
Department of Computer Engineering, Doğuş University, Istanbul, Turkey

**Abstract:** This talk is a survey on the structure of finite groups G which admit a finite group $A$ acting via automorphisms and fixed-point-freely on $G$.

All the groups we are talking about are finite groups. We are going to present the interaction of two groups $G$ and $A$ where the second acts via automorphisms on the first and ask the question what sort of information can one deduce about the structure of $G$ if one knows something about $A$ and its action on $G$. Motivated by the 1959 theorem of Thompson which shows that a group having a fixed-point-free automorphism of prime order must be nilpotent we put the concrete question : How far is $G$ away from being nilpotent (and this we measure for a solvable group by the Fitting height of $G$ ), if A acts fixed point-freely on G? We distinguish between coprime and noncoprime actions and present the best known results about coprime case of this problem and introduce a new concept which could be a very useful group invariant to study the noncoprime case. In the last section we present some results from very recent articles of Dr.Y.Kızmaz which unfortunately remained unknown to most of the experts of finite groups and which introduce new concepts showing the importance of the study of fixed-point-free action for problems asking to bound the Fitting length of a solvable group.

**Keywords** Fixed-point-free action; Solvable Group, Nilpotent Group; Fitting height of a solvable group.

**Mathematics Subject Classification (2010) : 05E15, 05C15, 20B25.**

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# An overview on Baer's theorem and its extensions

**Saeed Kayvanfar**[1]

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**Abstract:** Baer's theorem is one of the cornerstone result in group theory, providing critical insights into the relationship between the finiteness of central factor group and commutator subgroup. Building upon Schur's foundational work, Baer's theorem connects the upper and lower central series, establishing constraints on group structure that have far-reaching implications. This paper provides a brief review of Baer's theorem, detailing its historical development, generalizations, and recent extensions. Some key results include exponents, bounds on central series, extensions to locally generalized radical groups, finite rank conditions and applications to automorphism-influenced properties are given. Particular attention is given to some of its applications in diverse areas of mathematics. Furthermore, the article explores open problems and potential research directions, highlighting the theorem's enduring significance and its role in shaping contemporary mathematical inquiry.

**Keywords** Baer's theorem; Schur-Baer class; Special rank; Locally generalized radical group; Hypercenter.

**Mathematics Subject Classification (2010) : 20F14, 20F19, 20F99.**

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# Three methods of finite simple groups genarations

**Thekiso Seretlo**

PAA Focus Group, School of Mathematical and Statistical Sciences, North-West University, Mahikeng Campus

**Abstract:** Let $G = PSL(3,7)$ be a finite simple group and $nX$ be a non-trivial conjugacy class of elements of order $n$ in $G$. In this paper we consider three methods of group generations. We first look at $(p,q,r)$-generations, where $p$, $q$, $r$ are primes with $p \leq q \leq r$. Then next we consider the conjugacy class ranks denoted rank$(G : nX)$. Last we look at nX-complementary generations. We approach this kind of generations using mainly the structure constant method. GAP [The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.12.2; 2022. (http://www.gap-system.org)] is used in our computations.

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# Prime subgroups and equational domains

**Mohammad Shahryari**[1]

Department of Mathematics, College of Science, Sultan Qaboos University, Muscat, Oman

**Abstract:** We introduce the concept of a prime subgroup in a given group. The corresponding quotient group will be an equational domain i.e., a group the union of every two algebraic set over which is an algebraic set. For the case of an existentially closed group $G$, we will prove that the prime subgroups of $G[x_1, x_2, \dots, x_n]$ are exactly the radicals of irreducible algebraic sets in $G^n$. The relation between prime subgroups and classes of maximal normal subgroups will be discussed.

**Keywords** algebraic geometry over groups; equational Noetherian groups; equational domains; prime subgroups; conjugately separable subgroups.

**Mathematics Subject Classification (2010) : 20F70.**

## 1 Introduction

We use the same notations as in [1] and [2]. Suppose $G$ is a group and $X = \{x_1, x_2, \dots, x_n\}$ is a set of variables. Let $\mathbb{F}[X]$ be the free group generated by $X$ and $G[X] = G * \mathbb{F}[X]$ be the free product of $G$ and $\mathbb{F}[X]$. Every element of $G[X]$ is a group word in variables $x_1, x_2, \dots, x_n$ and coefficients from $G$. If $w(x_1, \dots, x_n) \in G[X]$, then $w(x_1, \dots, x_n) \approx 1$ is called a group equation. For a given equation $w(x_1, \dots, x_n) \approx 1$, the set

$$\{(a_1, \dots, a_n) \in G^n : \ w(a_1, \dots, a_n) = 1\}$$

is the solution set of the given equation in $G$. A system of equations with coefficients from $G$ is any set of equations $S \approx 1$, where $S \subseteq G[X]$. The *algebraic set* corresponding to this system is the set of all common solutions of all equations in $S \approx 1$ in $G^n$. We denote this algebraic set by $V_G(S)$.

A topology can be defined on $G^n$ using the collection of algebraic sets as a sub-base of closed sets: every algebraic set, every finite union of algebraic sets, and every arbitrary intersection of unions of algebraic sets, is closed. This is called the Zariski topology on $G^n$. This topology is Noetherian if and only if for every $S \subseteq G[X]$, there exists a finite subset $S_0 \subseteq S$ such that $V_G(S) = V_G(S_0)$. In this case, we say that the group $G$ *equationally Noetherian*. If $G$ equationally Noetherian, then every

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algebraic set can be decomposed uniquely as a finite union of *irreducible* algebraic sets. The group $G$ is called an equational domain if and only if for any $n$, the union of every two algebraic sets in $G^n$ is again an algebraic set. In this case, every closed set in the Zariski topology is an algebraic set. There is another definition for the concept of domains in terms of *zero divisors*. An element $x \in G$ is called a zero divisor, if there exists a non-identity element $y \in G$ such that for every $g \in G$, we have $[x^g, y] = 1$. In [2], it is proved that a group $G$ is a domain if and only if $G$ does not contain any non-trivial zero divisor. This picture is more clear because now, one can see that a non-abelian finite group is a domain if and only if it is monolithic with a non-abelian socle i.e., it has a non-abelian unique minimal normal subgroup which is contained in all other non-trivial normal subgroups. In the case of infinite groups of course, there are many other groups which are domain but not monolithic. As an example, every non-abelian free group is a domain. This can be generalized to CSA groups (groups where all maximal abelian subgroups are malnormal). It is proved that every CSA group is a domain. Latter this result is generalized for a wider class of groups in [4] and [3]. This is the class of all $\mathrm{CSN}_k$ groups: groups where each maximal class $k$ nilpotent subgroup is malnormal. Now, it is a simple idea to call a normal subgroup $K$ in $G$ to be *prime* if and only if $G/K$ is a domain. Our aim is to present some examples of such subgroups as well as some of their properties.

## 2 Main results

A normal subgroup $K$ in $G$ is called *prime* if and only if $G/K$ is a domain. It is known that an algebraic set $Y \subseteq G^n$ is irreducible if and only if the radical $\mathrm{Rad}(Y)$ is a prime subgroup of $G[X]$. In the case when the group $G$ is existentially closed, these are the only prime subgroups of $G[X]$:

**Theorem 2.1.** *Let $G$ be an existentially closed equational Noetherian group. Then every prime subgroup of $G[X]$ has the form $\mathrm{Rad}(Y)$ for some irreducible algebraic subset $Y \subseteq G^n$.*

It is easy to see that every maximal normal subgroup $K \trianglelefteq G$ is a prime subgroup, if $G/K$ is not abelian. One can generalize this fact to the following theorem.

**Theorem 2.2.** *Let $G$ be a group with $\min - \mathrm{n}$. Let $k \geq 1$ and define*

$$T_k = \{A \trianglelefteq G : \ (\frac{G}{A})^{(k)} \neq 1\}.$$

*If $K$ is a maximal element of $T_k$ and $(G/K)^{(k)}$ is not abelian, then $K$ is a prime subgroup.*

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# Contributed Talks

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# The Lie 2-algebra associated to a 2-plectic manifold

**Masoud Aminizadeh**[1]

Department of Mathematics,Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran

**Abstract:** Lie n-algebras are examples of $L_\infty$-algebras: graded vector spaces equipped with a collection of skew-symmetric multi-brackets that satisfy a generalized Jacobi identity. In this paper for any 2-plectic manifold, we construct a Lie 2-algebra.

**Keywords** Multisymplectic manifold; Lie algebra.

**Mathematics Subject Classification (2010) : 53D05, 17B55, 70S05.**

## 1 Introduction

Multisymplectic structures in field theory play a role similar to that of symplectic structures in classical mechanics. In particular, they provide a natural geometric framework adapted to the variational character of the theory.

**Definition 1.1.** 2-plectic manifolds are smooth manifolds equipped with a closed, nondegenerate differential form of degree 3.

**Example 1.2.** If $G$ is a compact simple Lie group, there is a 3-form $\omega$ on $G$ that is invariant under both left and right translations, which unique up to rescaling. It is given by

$$\omega(v_1, v_2, v_3) =< v_1, [v_2, v_3] >$$

when $v_i$ are tangent vectors at identity of $G$, and $< ., . >$ is Killing form.

**Definition 1.3.** Let $p$ and $q$ be positive natural numbers. Further, let $S(n)$ be the symmmetric group on the numbers $1, ..., n$. A permutation $\sigma \in S(n)$ is a $(p, q)$-shuffle (p+q=n) iff $\sigma(i) < \sigma(i + 1)$ whenever $i \neq p$. The set of $(p, q)$-shuffles denoted by $Sh(p, q)$.

**Example 1.4.**

$$Sh(2, 1) = \{(1), (23), (123)\}.$$

$$Sh(1, 2) = \{(1), (12), (132)\}.$$

$$Sh(n, 0) = \{(1)\}, Sh(2, 2) = \{(1), (23), (243), (123), (1243), (13)(24)\}.$$

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## 2 Main results

In this section, for any 2-plectic manifold, we construct a Lie 2-algebra,

**Definition 2.1.** Let $M$ be a smooth manifold of dimension $n$, a smooth k-form (or simply a k-form) on $M$ is a smooth section of $\bigwedge^k T^*(M)$. The set of k-forms is denoted by $\Omega^k(M)$, and $\Omega(M) = \bigcup \Omega^k(M)$. Locally in a chart $U$ any smooth k-form can be represented as $\omega = \Sigma\omega_{i_1,...,i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}$, where $i_1 < ... < i_k$ is taking as increasing k-tuples in $\{1, ..., n\}$.

**Definition 2.2.** Let $M$ be a smooth manifold of dimension $n$, a derivation is a operator $d$ on $\Omega(M)$ satisfying:
1- $d$ is $\mathbb{R}$-linear.
2- If $\omega$ is a k-form, then $d\omega$ is a k+1-form.
3- $d(\omega \wedge \eta) = d\omega \wedge \eta + \omega \wedge d\eta$.
Similarly one can contract a differential k-form $\omega$ with a smooth vector field $X$, which gives a (k-1)-form $i_X\omega$ on $M$ defined as

$$i_X\omega(X_1, ..., X_{k-1}) = \omega(X, X_1, ..., X_{k-1}).$$

$i_X\omega$ is called the contraction of $\omega$ with respect to $X$.
Obviously the map $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is linear and satisfies $i_X \circ i_X = 0$.

**Definition 2.3.** Let $V$ be a graded vector space. Let $x_1, ..., x_n$ be elements of V and $\sigma \in S(n)$ a permutation. The Koszul sign $\epsilon(\sigma) = \epsilon(\sigma; x_1, ..., x_n)$ is defined by the equality

$$x_1 \wedge ... \wedge x_n = \epsilon(\sigma) = \epsilon(\sigma; x_1, ..., x_n)x_{\sigma(1)} \wedge ... \wedge x_{\sigma(n)}$$

which holds in the free graded commutative algebra generated by $V$.
Given $\sigma \in S(n)$, let $(-1)^\sigma$ denote the usual sign of a permutation. Note that $\epsilon(\sigma)$ does not include the sign $(-1)^\sigma$.

**Definition 2.4.** Let $(M, \omega)$ be a 2-plectic manifold. A 1-form $\alpha$ is Hamiltonian iff there exists a vector field $v_\alpha \in \chi(M)$ such that

$$d\alpha = -i_{v_\alpha}\omega.$$

We say $v_\alpha$ is Hamiltonian vecror field corresponding to $\alpha$. The set of Hamiltonian forms and the set of Hamiltonian vector fields on an 2-plectic manifold are both vector spaces and are denoted as $\Omega_{Ham}(M)$ and $\chi_{Ham}(M)$, respectively.

**Definition 2.5.** Given $\alpha, \beta \in \Omega_{Ham}(M)$, the bracket $\{\alpha, \beta\}$ is the form given by

$$\{\alpha, \beta\} = i_{v_\beta} i_{v_\alpha}\omega.$$

**Proposition 2.6.** *The bracket$\{.,.\}$ has the following properties:*
*(1) The bracket is skew-symmetric:*

$$\{\alpha, \beta\} = -\{\beta, \alpha\}.$$

*(2) The bracket of Hamiltonian forms is Hamiltonian:*

$$d\{\alpha, \beta\} = -i_{[v_\alpha, v_\alpha]}\omega.$$

**Proposition 2.7.** *The bracket $\{.,.\}$ satisfies the Jacobi identity up to an exact form:*

$$\{\alpha_1, \{\alpha_2, \alpha_3\}\} - \{\{\alpha_1, \alpha_2\}, \alpha_3\} - \{\alpha_2, \{\alpha_1, \alpha_3\}\} = -di_{(v_{\alpha_1} \bigwedge v_{\alpha_2} \bigwedge v_{\alpha_3})}\omega.$$

**Definition 2.8.** A Lie n-algebra is a graded vector space $L$ equipped with a collection

$$\{l_k : \bigotimes^k L \to L | 1 \leq k \leq n\}$$

of skew-symmetric linear maps such that the following identity holds for $1 \leq m \leq n$ :

$$\sum_{i+j=m+1} \sum_{\sigma \in Sh(i,m-i)} \epsilon(\sigma)(-1)^\sigma(-1)^{i(j-1)} L_j(L_i(v_{\sigma(1)}, ...v_{\sigma(i)}), v_{\sigma(i+1)}, ..., v_{\sigma(m)}) = 0.$$

**Theorem 2.9.** *(Main Theorem), Given a 2-plectic manifold $(M, \omega)$, there is a Lie 2-algebra associated to it.*

*Proof.* Let $L = L_0 \bigoplus L_1 \bigoplus L_2$ where $L_0 = \Omega_{Ham}(M)$, $L_1 = \chi(M)$ and $L_2 = C^\infty(M)$.
We define $l_1 : L \to L$ as $l_1(\alpha) = d\alpha$ for $\alpha \in C^\infty(M)$.
And $l_2 : L \bigotimes L \to L$ as $l_2(\alpha, \beta) = \{\alpha, \beta\}$ for $\alpha, \beta \in \Omega_{Ham}(M)$, and $l_i = 0$ for $i \geq 3$. □

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# Jacobson group of 2 by 2 matrix rings over finite local commutative rings

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**Abstract:** In this paper, for the finite local commutative rings $R$, we investigate complemetation property of Jacobson group of $M_2(R)$ $(J(M_2(R)) + \{I_2\})$. We state the role of surgective ring homomorphism in Jacobson groups and finally, introduce some properties of Jacobson group.

**Keywords** Jacobson group; matrix ring; finite ring.

**Mathematics Subject Classification (2010) : 16N20, 15B33, 16P10.**

## 1 Introduction

In the commutative ring $R$, the Jacobson radical of $R$ is defined as the intersection of all maximal ideals and for a non-commutative case, it is the annihilator ideal of all simple $R$-module. In general, $J(R)$ equals the intersection of all maximal right ideals of the ring. It is also true that $J(R)$ equals the intersection of all maximal left ideals, but it is not necessarily equal to the intersection of all maximal two-sided ideals [3, 4]. Let $\mathcal{J}(R) = 1 + J(R)$ denote the Jacobson group of $R$. We also denote the group of units of $R$ by $U(R)$.

Throuout this paper, assume that $p$ and $n$ are prime and natural numbers, resp. We also use the notation $M_n(R)$ and $UM_n(R)$ for the square and upper square matrix of order $n \times n$ over a ring $R$, resp.

One can see that the unitary group of any finite local ring and ring of order $p^n$ is semidirect product of Jacobson group and other subgroup [4]. We say that a normal subgroup $N$ in $G$ is complemented, if $G$ can be written as semidirect product of $N$ and an other subgroup of $G$. The writers in [1] show that $\mathcal{J}(R)$ is complemented in $U(R)$ when $R$ is a finite commutative ring or $R \cong M_2(\mathbb{Z}_{2^n})$ or $M_2(\mathbb{Z}_{3^n})$. Furthermore, [4] shows that $\mathcal{J}(R)$ is also complemented in $U(M_2(\mathbb{Z}_p))$.

Before entering to main section, we state a preliminary lemma as follows.

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**Lemma 1.1.** *Let $R$ be a finite local $p$-ring and $F_p$ is a finite field of characteristic $p$. Then*

*(i)* $J(M_2(\mathbb{Z}_{p^n})) \cong M_2(\langle p \rangle)$,

*(ii)* $\frac{U(M_2(R))}{\mathcal{J}(M_2(R))} \cong GL_2(F_p)$.

# 2 Main results

Where $R$ is a $p$-ring, $M_2(R)$ is also of order of $p^n$ and so, $|J(M_2((R))|$ is a power of $p$. Since $|J(M_2(R))| = |\mathcal{J}(M_2(R))|$, $\mathcal{J}(M_2(R))$ is a p-group. The relation $J(M_2(R)) = M_2(J(R))$ shows that the Jacobson group is trivial, if $R = \mathbb{Z}_p$. $J(R)$ in finite rings are nilpotent, because in all left Artinian rings, $J(R)$ is nilpotent, too. On the other hand, $p$-groups are nilpotent. The relation between length of ring and group series in $J(R)$ and $\mathcal{J}(R)$, respectively, can be an interesting subject to study. Furthermore, investing of properties of $1 + J(R)^n$, $n \in \mathbb{N}$, and lattice of subgroups of $\mathcal{J}(R)$ is another dependency of rings and groups that has received little study so far. The following easy lemma can obtain by definition of normal series.

**Lemma 2.1.** *Let $R$ be an Artinian unitary ring. Then the set $\{1 + J(R)^i\}_{i\in\mathbb{N}}$ forms a normal series for the Jacobson group $\mathcal{J}(R)$.*

In the next lemma, we investigate the role of surjective ring homomorphism in Jacobson group.

**Lemma 2.2.** *Assume that $R$ and $S$ are two finite commutative rings and $f : R \to S$ is an onto ring homomorphism. Then we have:*

*(i) By assumption $Ker(f) \subseteq J(R)$, if $\mathcal{J}(R)$ is complemented in $U(R)$, then $\mathcal{J}(S)$ is also complemented in $U(S)$,*

*(ii) If $\mathcal{J}(R)$ is complemented in $U(R)$, then $\mathcal{J}(\frac{R}{J(R)^2})$ is also complemented in $U(\frac{R}{J(R)^2})$,*

*(iii) By assumption $Ker(f) \subseteq J(R)$, if $\mathcal{J}(M_2(R))$ is complemented in $GL_2(R)$, then $\mathcal{J}(M_2(S))$ is also complemented in $GL_2(S)$.*

*Proof.* Items (ii) and (iii) are concluded from (i) and to prove (i), it is enough to consider that there exists a subgroup $H$ of $U(R)$ where $U(R) = H\mathcal{J}(R)$ in which $\mathcal{J}(R) = 1 + J(R)$, with $H \cap \mathcal{J}(R) = 1$. Then $U(S) = f(H)f(\mathcal{J}(R))$. Now, $f(\mathcal{J}(R))$ is a subset of $\mathcal{J}(S)$, but since $f$ is surjective and every unit is an image of a unit, we get $J(S) \subseteq f(J(R))$, so $f(\mathcal{J}(R)) = \mathcal{J}(S)$. To prove $f(H) \cap \mathcal{J}(S) = \{1\}$, choose $x \in f(H) \cap f(\mathcal{J}(R))$. Then there exist $y \in H$ and $z \in \mathcal{J}(R)$, so that $x = f(y) = f(z)$. It follows that $f(y^{-1}z) = 1$ and $1 - y^{-1}z \in Ker(f) \subseteq J(R)$. So, $y - z \in J(R)$ and $y \in H \cap \mathcal{J}(R) = \{1\}$. Thus $x = 1$.

□

In the next three theorems the role of complementation in Jacobson group appears.

**Theorem 2.3.** [1, 2] *Let $R$ be a finite local commutative $p$-ring, where $p \geq 5$. Then the Jacobson group of $M_2(R)$ is not complemented.*

**Theorem 2.4.** *Let $R$ be a finite local commutative $p$-ring. Then $\mathcal{J}(UM_2(R))$ is complemented in $UM_2(R)$, if and only if $p = 2$.*

**Theorem 2.5.** *[2] Let $R$ be a finite local commutative $p$-ring with the only maximal ideal $\mathcal{M}$, where $p \in \{2, 3\}$. Thus, the $\mathcal{J}(M_2(R))$ is complemented in $U(M_2(R))$ if and only if $\frac{R}{\mathcal{M}} \cong \mathbb{Z}_p$.*

Finite local commutative rings can not be written as a direct product of some rings. So, all Finite local commutative rings are $p$-rings. This conclusion is yielded from [5]. Since in each commutative ring intersection of all maximal ideals is Jacobson ideal, in the local commutative ring, Jaconson ideal ($J(R)$) is the only maximal ideal ($\mathcal{M}$). All elements of $\mathcal{M}$ are zero-divisors of $R$ and $R \setminus \mathcal{M}$ is the unitary group. If $R$ is a $p$-ring, its unitary group is not $p$ -group, but the Jacobson group is it. It is easy to see that the Jacobson group is $p$-sylow in the unitary abelian group of finite unitary commutative $p$-rings.

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# Non-extended commutative semigroups to the commutative rings

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**Abstract:** In this paper, we introduced some sets of commutative semigroups (by product) in which one can not be generalize them to commutative rings. To prove that, we use the zero-divisor graph of semigroups and rings.

**Keywords** semigroup; ring; commutative; zero-divisor graph.

**Mathematics Subject Classification (2010) : 18B40, 05C25, 05E40.**

## 1 Introduction

Throuout this paper, assume that $S$ and $R$ are commutative semigroup and ring, resp. Furtheremore, let $K_n$ be a complete graph by $n$ vertices and $P_n$ be a path graph of length $n$. We also denote a finite field of order $n$ by $\mathbb{F}_n$. But $F_n$ is Fan-shaped graph $P_n + K_1$ and different to $\mathbb{F}_n$.

The corona product of two graphs $G$ and $H$ is defined as the graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$ and joining the i-th vertex of $G$ to every vertex in the i-th copy of $H$. For a graph $X$, Sabidussi [6], has defined the $X$-join of a set of graphs $\{X_x\}_{x\in V(X)}$ as the graph $\Gamma$ obtained by replacing each vertex $x \in V(X)$ by the graph $X_x$ and inserting either all or none of the possible edges between vertices of $X_x$ and $X_y$ depending on whether or not $x$ and $y$ are joined by an edge in $X$. In fact, vertex and edge sets of $\Gamma$ are detailed as below:

$$V(\Gamma) = \{(x, y) \mid x \in V(X)\ ,\ y \in V(X_x)\},$$

$$E(\Gamma) = \{(x_1, y_1)(x_2, y_2) \mid x_1x_2 \in E(X) \text{ or else } (x_1 = x_2\ ,\ y_1y_2 \in E(X_x))\}.$$

The difference between semigroup and monid is that semigroups have zero and so, we can define the concept of zero-divisor graph over semigroups. Almost, according to the definition of zero-divisor graph of rings, zero-divisor graph of $S$ is a couple $(V, E)$ in which $V$ is non-zero zero-divisors of S

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and $E = \{xy \in S \mid x \neq y, (xy = 0 \text{ or } yx = 0)\}$. In the commutative case, $E$ can be written as $\{xy \in S \mid x \neq y, xy = 0\}$. Here, we show this graph by $\Gamma(S)$. Although, for the first time, zero-divisor graph of a ring introduced by [1], the zero-divisor graph of semigroups is developed by Demeyer and et.al. [2].

In this paper, we use product for the operation of semigroups. Extention of a semigroup to a ring performs by several ways. If $S = R$, we have to define an abelian group over $S$ with distributive law and if $S \subsetneq R$, then one can find an abelian group over underlying set $R$ in which restriction of $R$ to $S$ compliants to product table of $S$ with distributive law, too.

# 2 Main results

In this section, we introduced some commutative semigroups can not be extended to a commutative ring. To do this, we state a preliminary lemma about zero-divisor graphs.

**Lemma 2.1.** *If $R$ is an extention of $S$, then $\Gamma(S)$ is an induced subgraph of $\Gamma(R)$.*

**Theorem 2.2.** *The following 26 semigroups of order 7 ($S = \{0, x_1, x_2, x_3, x_4, x_5, x_6\}$) can not be the zero-divisors set of any commutative rings.*

*(1)* $x_1^2 = x_1x_2 = x_1x_4 = x_2x_4 = x_4^2 = x_3$ , $x_1x_3 = x_3x_4 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots = x_6x_6 = 0$ , $x_2x_3 = x_3^2 = x_1x_5 = x_3x_5 = x_5^2 = x_6$.

*(2)* $x_1^2 = x_1x_2 = x_1x_4 = x_2x_4 = x_4^2 = x_3$ , $x_1x_3 = x_3x_4 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots = x_6x_6 = x_3^2 = x_5^2 = 0$ , $x_2x_3 = x_1x_5 = x_3x_5 = x_6$.

*(3)* $x_1^2 = x_1x_2 = x_1x_4 = x_2x_4 = x_4^2 = x_3$ , $x_1x_3 = x_3x_4 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots = x_6x_6 = x_5^2 = 0$ , $x_2x_3 = x_3^2 = x_1x_5 = x_3x_5 = x_6$.

*(4)* $x_1^2 = x_1x_2 = x_2^2 = x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_4^2 = x_1x_5 = x_3x_5 = x_5^2 = x_6$ , $x_1x_3 = x_3x_4 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots x_6x_6 = 0$.

*(5)* $x_1^2 = x_1x_2 = x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_4^2 = x_1x_5 = x_3x_5 = x_5^2 = x_6$ , $x_1x_3 = x_3x_4 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots x_6x_6 = x_2^2 = 0$.

*(6)* $x_1^2 = x_1x_2 = x_2^2 = x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_4^2 = x_1x_5 = x_3x_5 = x_6$ , $x_1x_3 = x_3x_4 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots x_6x_6 = x_5^2 = 0$.

*(7)* $x_1x_2 = x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_4^2 = x_1x_5 = x_3x_5 = x_5^2 = x_6$ , $x_1^2 = x_1x_3 = x_2^2 = x_3x_4 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots x_6x_6 = 0$.

*(8)* $x_1^2 = x_1x_2 = x_2^2 = x_2x_3 = x_1x_4 = x_2x_4 = x_4^2 = x_1x_5 = x_3x_5 = x_6$ , $x_1x_3 = x_3x_4 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots x_6x_6 = x_3^2 = x_5^2 = 0$.

*(9)* $x_1^2 = x_1x_2 = x_2x_3 = x_1x_4 = x_2x_4 = x_3^2 = x_4^2 = x_1x_5 = x_3x_5 = x_6$ , $x_1x_3 = x_2^2 = x_3x_4 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots x_6x_6 = x_5^2 = 0$.

(10) $x_1^2 = x_1x_2 = x_2x_3 = x_1x_4 = x_2x_4 = x_4^2 = x_1x_5 = x_3x_5 = x_5^2 = x_6$ , $x_1x_3 = x_2^2 = x_3x_4 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots x_6x_6 = x_3^2 = 0$.

(11) $x_1x_2 = x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_4^2 = x_1x_5 = x_3x_5 = x_6$ , $x_1^2 = x_1x_3 = x_2^2 = x_3x_4 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots x_6x_6 = x_5^2 = 0$.

(12) $x_1^2 = x_1x_2 = x_2x_3 = x_1x_4 = x_2x_4 = x_4^2 = x_1x_5 = x_3x_5 = x_6$ , $x_1x_3 = x_2^2 = x_3x_4 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots x_6x_6 = x_3^2 = x_5^2 = 0$.

(13) $x_1x_2 = x_2x_3 = x_1x_4 = x_2x_4 = x_4^2 = x_1x_5 = x_3x_5 = x_6$ , $x_1^2 = x_1x_3 = x_2^2 = x_3x_4 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots x_6x_6 = x_3^2 = x_5^2 = 0$.

(14) $x_1^2 = x_1x_2 = x_2^2 = x_4$ , $x_1x_3 = x_3x_4 = x_4^2 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots = x_6x_6 = 0$ , $x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_1x_5 = x_3x_5 = x_5^2 = x_6$.

(15) $x_1^2 = x_1x_2 = x_2^2 = x_4$ , $x_1x_3 = x_3x_4 = x_4^2 = x_2x_5 = x_3^2 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots = x_6x_6 = x_5^2 = 0$ , $x_2x_3 = x_1x_4 = x_2x_4 = x_1x_5 = x_3x_5 = x_6$.

(16) $x_1^2 = x_1x_2 = x_2^2 = x_4$ , $x_1x_3 = x_3x_4 = x_4^2 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots = x_6x_6 = x_5^2 = 0$ , $x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_1x_5 = x_3x_5 = x_6$.

(17) $x_1^2 = x_1x_2 = x_2^2 = x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_1x_5 = x_3x_5 = x_5^2 = x_6$ , $x_1x_3 = x_3x_4 = x_4^2 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots = x_6x_6 = 0$.

(18) $x_1^2 = x_1x_2 = x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_1x_5 = x_3x_5 = x_5^2 = x_6$ , $x_1x_3 = x_2^2 = x_3x_4 = x_4^2 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots = x_6x_6 = 0$.

(19) $x_1^2 = x_1x_2 = x_2^2 = x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_1x_5 = x_3x_5 = x_6$ , $x_1x_3 = x_3x_4 = x_4^2 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots = x_6x_6 = x_5^2 = 0$.

(20) $x_1x_2 = x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_1x_5 = x_3x_5 = x_5^2 = x_6$ , $x_1^2 = x_1x_3 = x_2^2 = x_3x_4 = x_4^2 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots = x_6x_6 = 0$.

(21) $x_1^2 = x_1x_2 = x_2^2 = x_2x_3 = x_1x_4 = x_2x_4 = x_1x_5 = x_3x_5 = x_6$ , $x_1x_3 = x_3x_4 = x_4^2 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots = x_6x_6 = x_3^2 = x_5^2 = 0$.

(22) $x_1^2 = x_1x_2 = x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_1x_5 = x_3x_5 = x_6$ , $x_1x_3 = x_3x_4 = x_4^2 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots = x_6x_6 = x_2^2 = x_5^2 = 0$.

(23) $x_1^2 = x_1x_2 = x_2x_3 = x_1x_4 = x_2x_4 = x_1x_5 = x_3x_5 = x_5^2 = x_6$ , $x_1x_3 = x_3x_4 = x_4^2 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots = x_6x_6 = x_2^2 = x_3^2 = 0$.

(24) $x_1x_2 = x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_1x_5 = x_3x_5 = x_6$ , $x_1^2 = x_1x_3 = x_3x_4 = x_4^2 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots = x_6x_6 = x_2^2 = x_5^2 = 0$.

(25) $x_1^2 = x_1x_2 = x_2x_3 = x_1x_4 = x_2x_4 = x_1x_5 = x_3x_5 = x_6$ , $x_1x_3 = x_3x_4 = x_4^2 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots = x_6x_6 = x_2^2 = x_3^2 = x_5^2 = 0$.

*(26)* $x_1x_2 = x_2x_3 = x_1x_4 = x_2x_4 = x_1x_5 = x_3x_5 = x_6$ , $x_1^2 = x_1x_3 = x_3x_4 = x_4^2 = x_2x_5 = x_4x_5 = x_1x_6 = x_2x_6 = \cdots = x_6x_6 = x_2^2 = x_3^2 = x_5^2 = 0$.

*Proof.* According to the [8], zero-divisor graph of these 26 semigroups are Fan-shaped graph $(P_5+K_1)$. All zero-divisor graph of commutative rings with 6 vertices are in form bipartite graphs $K_{2,4}, K_{3,3}$, complete graph $K_6$ and corona product of $K_3$ in $K_1$ [4, 5]. These graphs conclude the result. □

**Theorem 2.3.** *The* 26 *semigroups mentioned in Theorem 2.2, can not be extended to the following* 100 *rings with* $|R| \geq 7$ *(These commutative rings are all commutative rings with zero-divisor graphs of order* $6 \leq |\Gamma(R)| \leq 14$*).*

$$\mathbb{Z}_8 \quad \frac{\mathbb{Z}_2[x]}{\langle x^3\rangle} \quad \frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2\rangle} \quad \frac{\mathbb{Z}_2[x,y]}{\langle x,y\rangle^2} \quad \frac{\mathbb{Z}_4[x]}{\langle 2,x\rangle^2} \quad \frac{\mathbb{F}_4[x]}{\langle x^2\rangle} \quad \frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$$

$$\mathbb{Z}_2\times\mathbb{F}_4 \quad \mathbb{Z}_3\times\mathbb{Z}_3 \quad \mathbb{Z}_{25} \quad \frac{\mathbb{Z}_5[x]}{\langle x^2\rangle} \quad \mathbb{Z}_2\times\mathbb{Z}_5 \quad \mathbb{Z}_3\times\mathbb{F}_4 \quad \mathbb{Z}_2\times\mathbb{Z}_4 \quad \mathbb{Z}_2\times\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$$

$$\mathbb{Z}_3\times\mathbb{Z}_5 \quad \mathbb{F}_4\times\mathbb{F}_4 \quad \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2 \quad \mathbb{Z}_{49} \quad \frac{\mathbb{Z}_7[x]}{\langle x^2\rangle} \quad \mathbb{Z}_2\times\mathbb{Z}_7 \quad \mathbb{F}_4\times\mathbb{Z}_5 \quad \mathbb{Z}_3\times\mathbb{Z}_4$$

$$\mathbb{Z}_3\times\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \quad \mathbb{Z}_{16} \quad \frac{\mathbb{Z}_2[x]}{\langle x^4\rangle} \quad \frac{\mathbb{Z}_4[x]}{\langle x^2+2\rangle} \quad \frac{\mathbb{Z}_4[x]}{\langle x^2+3x\rangle} \quad \frac{\mathbb{Z}_4[x]}{\langle x^3-2, 2x^2, 2x\rangle}$$

$$\frac{\mathbb{Z}_2[x,y]}{\langle x^3, xy, y^2\rangle} \quad \frac{\mathbb{Z}_8[x]}{\langle 2x, x^2\rangle} \quad \frac{\mathbb{Z}_4[x]}{\langle x^3, 2x^2, 2x\rangle} \quad \frac{\mathbb{Z}_4[x]}{\langle x^2+2x\rangle} \quad \frac{\mathbb{Z}_8[x]}{\langle 2x, x^2+4\rangle} \quad \frac{\mathbb{Z}_2[x,y]}{\langle x^2, y^2-xy\rangle}$$

$$\frac{\mathbb{Z}_4[x,y]}{\langle x^2, y^2-xy, xy-2, 2x, 2y\rangle} \quad \frac{\mathbb{Z}_4[x,y]}{\langle x^2, y^2, xy-2, 2x, 2y\rangle} \quad \frac{\mathbb{Z}_2[x,y]}{\langle x^2, y^2\rangle} \quad \frac{\mathbb{Z}_4[x]}{\langle x^2\rangle} \quad \frac{\mathbb{Z}_4[x]}{\langle x^3-x^2-2, 2x^2, 2x\rangle}$$

$$\frac{\mathbb{Z}_2[x,y,z]}{\langle x,y,z\rangle^2} \quad \frac{\mathbb{Z}_4[x,y]}{\langle x^2, y^2, xy, 2x, 2y\rangle} \quad \frac{\mathbb{F}_8[x]}{\langle x^2\rangle} \quad \frac{\mathbb{Z}_4[x]}{\langle x^3+x+1\rangle} \quad \mathbb{Z}_2\times\mathbb{F}_8 \quad \mathbb{Z}_3\times\mathbb{Z}_7 \quad \mathbb{Z}_5\times\mathbb{Z}_5$$

$$\mathbb{Z}_{27} \quad \frac{\mathbb{Z}_9[x]}{\langle 3x, x^2-3\rangle} \quad \frac{\mathbb{Z}_9[x]}{\langle 3x, x^2-6\rangle} \quad \frac{\mathbb{Z}_3[x]}{\langle x^3\rangle} \quad \frac{\mathbb{Z}_3[x,y]}{\langle x,y\rangle^2} \quad \frac{\mathbb{Z}_9[x]}{\langle 3,x\rangle} \quad \frac{\mathbb{F}_9[x]}{\langle x^2\rangle} \quad \frac{\mathbb{Z}_9[x]}{\langle x^2+1\rangle}$$

$$\mathbb{Z}_2\times\mathbb{F}_9 \quad \mathbb{Z}_3\times\mathbb{F}_8 \quad \mathbb{F}_4\times\mathbb{Z}_7 \quad \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3 \quad \mathbb{Z}_4\times\mathbb{F}_4 \quad \frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}\times\mathbb{F}_4$$

$$\mathbb{Z}_3\times\mathbb{F}_9 \quad \mathbb{F}_4\times\mathbb{F}_8 \quad \mathbb{Z}_5\times\mathbb{Z}_7 \quad \mathbb{Z}_{121} \quad \frac{\mathbb{Z}_{11}[x]}{\langle x^2\rangle} \quad \mathbb{Z}_2\times\mathbb{Z}_{11} \quad \mathbb{F}_4\times\mathbb{F}_9 \quad \mathbb{Z}_5\times\mathbb{F}_8 \quad \mathbb{Z}_2\times\mathbb{Z}_9$$

$$\mathbb{Z}_2\times\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle} \quad \mathbb{Z}_5\times\mathbb{Z}_4 \quad \mathbb{Z}_5\times\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \quad \mathbb{Z}_2\times\mathbb{Z}_8 \quad \mathbb{Z}_2\times\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle} \quad \mathbb{Z}_2\times\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2\rangle} \quad \mathbb{Z}_2\times\frac{\mathbb{Z}_2[x,y]}{\langle x,y\rangle^2}$$

$$\mathbb{Z}_2\times\frac{\mathbb{Z}_4[x]}{\langle 2,x\rangle^2} \quad \mathbb{Z}_4\times\mathbb{Z}_4 \quad \mathbb{Z}_4\times\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \quad \frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}\times\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \quad \mathbb{Z}_3\times\mathbb{Z}_{11} \quad \mathbb{Z}_5\times\mathbb{Z}_9 \quad \mathbb{Z}_7\times\mathbb{Z}_7$$

$$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{F}_4 \quad \mathbb{Z}_{169} \quad \frac{\mathbb{Z}_{13}[x]}{\langle x^2\rangle} \quad \mathbb{Z}_2\times\mathbb{Z}_{13} \quad \mathbb{F}_4\times\mathbb{Z}_{11} \quad \mathbb{Z}_7\times\mathbb{F}_8 \quad \mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3 \quad \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$$

$$\mathbb{Z}_2\times\mathbb{Z}_2\times\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \quad \mathbb{Z}_3\times\mathbb{Z}_{13} \quad \mathbb{Z}_5\times\mathbb{Z}_{11} \quad \mathbb{Z}_7\times\mathbb{F}_9 \quad \mathbb{F}_8\times\mathbb{F}_8 \quad \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2 \quad \mathbb{Z}_3\times\mathbb{Z}_9$$

$$\mathbb{Z}_3\times\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$$

*Proof.* Zero-divisor graph of the 26 semigroups is Fan-shaped graph $F_5$ and this graph is not an induced subgraph of $\Gamma(R)$ for the above 100 rings ([4] and [5]). □

**Theorem 2.4.** *Extention of the following* 10 *commutative semigroups of order* 7 *is not* 100 *commutative rings mentioned in theorem 2.3.*

*(1)* $x_1^2 = x_1x_2 = x_2^2 = x_1x_3 = x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_3x_4 = x_1x_5 = x_2x_6 = x_3x_6 = 0$ , $x_4^2 = x_4x_5 = x_4x_6 = x_5x_6 = x_4$ , $x_2x_5 = x_3x_5 = x_2$ , $x_5^2 = x_5$ , $x_1x_6 = x_1$ , $x_6^2 = x_6$.

*(2)* $x_1x_2 = x_2^2 = x_1x_3 = x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_3x_4 = x_1x_5 = x_2x_6 = x_3x_6 = 0$ , $x_4^2 = x_4x_5 = x_4x_6 = x_5x_6 = x_4$ , $x_2x_5 = x_3x_5 = x_2$ , $x_5^2 = x_5$ , $x_1^2 = x_1x_6 = x_1$ , $x_6^2 = x_6$.

*(3)* $x_1^2 = x_1x_2 = x_1x_3 = x_2x_3 = x_1x_4 = x_2x_4 = x_3x_4 = x_1x_5 = x_2x_6 = x_3x_6 = 0$ , $x_3^2 = x_3x_5 = x_3$ , $x_4^2 = x_4x_5 = x_4x_6 = x_5x_6 = x_4$ , $x_2^2 = x_2x_5 = x_2$ , $x_5^2 = x_5$ , $x_1x_6 = x_1$ , $x_6^2 = x_6$.

*(4)* $x_1x_2 = x_1x_3 = x_2x_3 = x_1x_4 = x_2x_4 = x_3x_4 = x_1x_5 = x_2x_6 = x_3x_6 = 0$ , $x_3^2 = x_3x_5 = x_3$ , $x_4^2 = x_4x_5 = x_4x_6 = x_5x_6 = x_4$ , $x_2^2 = x_2x_5 = x_2$ , $x_5^2 = x_5$ , $x_1^2 = x_1x_6 = x_1$ , $x_6^2 = x_6$.

*(5)* $x_1^2 = x_1x_2 = x_2^2 = x_1x_3 = x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_3x_4 = x_1x_5 = x_2x_6 = x_3x_6 = 0$ , $x_4^2 = x_4x_5 = x_4x_6 = x_5x_6 = x_4$ , $x_2x_5 = x_2$ , $x_3x_5 = x_3$ , $x_5^2 = x_5$ , $x_1x_6 = x_1$ , $x_6^2 = x_6$.

*(6)* $x_1x_2 = x_2^2 = x_1x_3 = x_2x_3 = x_3^2 = x_1x_4 = x_2x_4 = x_3x_4 = x_1x_5 = x_2x_6 = x_3x_6 = 0$ , $x_4^2 = x_4x_5 = x_4x_6 = x_5x_6 = x_4$ , $x_2x_5 = x_2$ , $x_3x_5 = x_3$ , $x_5^2 = x_5$ , $x_1^2 = x_1x_6 = x_1$ , $x_6^2 = x_6$.

*(7)* $x_1^2 = x_1x_2 = x_2^2 = x_1x_3 = x_2x_3 = x_1x_4 = x_2x_4 = x_3x_4 = x_1x_5 = x_2x_6 = x_3x_6 = 0$ , $x_4^2 = x_4x_5 = x_4x_6 = x_5x_6 = x_4$ , $x_2x_5 = x_3^2 = x_2$ , $x_3x_5 = x_3$ , $x_5^2 = x_5$ , $x_1x_6 = x_1$ , $x_6^2 = x_6$.

*(8)* $x_1x_2 = x_2^2 = x_1x_3 = x_2x_3 = x_1x_4 = x_2x_4 = x_3x_4 = x_1x_5 = x_2x_6 = x_3x_6 = 0$ , $x_4^2 = x_4x_5 = x_4x_6 = x_5x_6 = x_4$ , $x_2x_5 = x_3^2 = x_2$ , $x_3x_5 = x_3$ , $x_5^2 = x_5$ , $x_1^2 = x_1x_6 = x_1$ , $x_6^2 = x_6$.

*(9)* $x_1^2 = x_1x_2 = x_2^2 = x_1x_3 = x_2x_3 = x_1x_4 = x_2x_4 = x_3x_4 = x_1x_5 = x_2x_6 = x_3x_6 = 0$ , $x_4^2 = x_4x_5 = x_4x_6 = x_5x_6 = x_4$ , $x_2x_5 = x_2$ , $x_3x_5 = x_3^2 = x_3$ , $x_5^2 = x_5$ , $x_1x_6 = x_1$ , $x_6^2 = x_6$.

*(10)* $x_1x_2 = x_2^2 = x_1x_3 = x_2x_3 = x_1x_4 = x_2x_4 = x_3x_4 = x_1x_5 = x_2x_6 = x_3x_6 = 0$ , $x_4^2 = x_4x_5 = x_4x_6 = x_5x_6 = x_4$ , $x_2x_5 = x_2$ , $x_3x_5 = x_3^2 = x_3$ , $x_5^2 = x_5$ , $x_1^2 = x_1x_6 = x_1$ , $x_6^2 = x_6$.

*Proof.* Zero-divisor graph of these semigroups are isomorphic to an $X$-join of some graphs in which $X$ is a bull graph, the corresponding graph to two pendant vertices of $X$ is $K_1$ and the only vertex of degree 2 in the bull graph corresponds $K_1$, too. Replacement of the other vertices of $X$ happened by $K_1$ and $K_2$. This described graph is not an induced subgraph of zero-divisor graph of the commutative rings. Thus, These semigroups are non-extended to the rings mentioned in theorem 2.3. □

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# The Bogomolov multiplier of groups of order $p^7$ and exponent $p$

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**Abstract:** In this paper, we conduct an in-depth investigation into the structure of the Bogomolov multiplier for groups of order $p^7$ ($p > 2$) and exponent $p$.

**Keywords** Commutativity-Preserving exterior product; $\tilde{B_0}$-pairing; Curly exterior square; Bogomolov multiplier.

**Mathematics Subject Classification (2010) : 13A50, 14E08, 14M20, 20D15.**

## 1 Introduction

In 1988 Bogomolov in [2] showed that the unramified cohomology group $H^2_{nr}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ is canonically isomorphic to

$$B_0(G) = \bigcap_{A \leq G} \ker\{res^G_A : H^2(G, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^2(A, \mathbb{Q}/\mathbb{Z})\},$$

where $res^G_A$ is the usual cohomological restriction maps and $A$ is an abelian subgroup of $G$. Therefore, the group $B_0(G)$ is a group-theorical invariant presented as an obstraction to Noether's problem and in [3] called it the *Bogomolov Multiplier* of $G$.

In the following, we introduce farther properties of $\tau(G)$ and $[G, G^\varphi]$ that will be useful.

**Lemma 1.1.** *[1, Lemmas 9, 10, 11] Let $G$ be a group. The following statements, for all $x, y, z, v, w \in G$ and all $n, m \in \mathbb{N}$, hold:*

(i) $[x, yz] = [x, z][x, y][x, y, z]$ and $[xy, z] = [x, z][x, z, y][y, z]$.

(ii) If $G$ is nilpotent of class $c$, then $\tau(G)$ is nilpotent of class at most $c + 1$.

(iii) If $G$ is nilpotent of class $\leq 2$, then $[G, G^\varphi]$ is abelian.

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(iv) $[x, y^\varphi] = [x^\varphi, y]$.

(v) $[x, y, z^\varphi] = [x, y^\varphi, z] = [x^\varphi, y, z] = [x^\varphi, y^\varphi, z] = [x^\varphi, y, z^\varphi] = [x, y^\varphi, z^\varphi]$.

(vi) $[[x, y^\varphi], [v, w^\varphi]] = [[x, y], [v, w]^\varphi]$.

(vii) $[x^n, y^\varphi] = [x, y^\varphi]^n = [x, (y^\varphi)^n]$, where $[x, y] = 1$.

(viii) If $[G, G]$ is nilpotent of class $c$, then $[G, G^\varphi]$ is nilpotent of class $c$ or $c + 1$.

(ix) If $x$ and $y$ are commuting elements of $G$ of orders $m$ and $n$, respectively, then the order of $[x, y^\varphi]$ divides $\gcd(m, n)$.

In the process of the proofs in the third section, we need the following lemma to expanding the commutators.

**Lemma 1.2.** *Let $G$ be a nilpotent group of class at most 6. Then*

$$\begin{aligned}[x^n, y] =&[x, y]^n [x, y, x]^{\binom{n}{2}} [x, y, x, x]^{\binom{n}{3}} [x, y, x, x, x]^{\binom{n}{4}} [x, y, x, [x, y]]^{a(n)}\\ &[x, y, x, x, x, x]^{\binom{n}{5}} [x, y, x, [x, y], x]^{\binom{n}{3}+2\binom{n}{4}} [x, y, x, x, [x, y]]^{\binom{n}{3}+\binom{n}{4}},\end{aligned}$$

*for all $x, y \in \tau(G)$ and every positive integer $n$, where $a(n) = n(n-1)(2n-1)/6$.*

**Definition 1.3.** [?, Definition 16] A presentation $\{x_1, ..., x_n | R\}$ is a polycyclic presentation, if there is a sequence $S = (s_1, ..., s_n)$ with $s_i \in \mathbb{N} \cup \{\infty\}$ and integers $a_{i,k}, b_{i,j,k}, c_{i,j,k}$ such that $R$ consists of the following relations

$$x_i^{s_i} = x_{i+1}^{a_{i,i+1}} ... x_n^{a_{i,n}} \quad \text{for} \quad 1 \leq i \leq n \ \text{ with } \ s_i < \infty,$$

$$x_j^{-1} x_i x_j = x_{i+1}^{b_{i,j,j+1}} ... x_n^{b_{i,j,n}} \quad \text{for} \quad 1 \leq j < i \leq n,$$

$$x_j x_i x_j^{-1} = x_{j+1}^{c_{i,j,j+1}} ... x_n^{c_{i,j,n}} \quad \text{for} \quad 1 \leq j < i \leq n.$$

If $G$ is defined by such a polycyclic presentation, then $G$ is called a *PC* group. In addition to every *PC* group can be defined by a polycyclic presentation.

**Proposition 1.4.** *[1, Proposition 20] Let $G$ be a finite group with a polycyclic generating sequence $x_1, \ldots, x_n$, then the group $[G, G^\varphi]$ is generated by*

$$\{[x_i, x_j^\varphi] \mid i, j = 1, \ldots, n, i > j\}.$$

# 2 Groups of order $p^7$ and exponent $p$ $(p > 5)$ with trivial Bogomolov multiplier

In this section, we show how to prove the triviallity of the Bogomolov multiplier of groups of order $p^7$ and exponent $p$ $(p > 5)$.

**Proposition 2.1.** *The group* $G_{20} :< a, b, c, d, e, f, g \mid [b, a] = c, [c, a] = d, [c, b] = e, [d, a] = f >$ *has trivial Bogomolov multiplier.*

*Proof.* By using Proposition 1.4, $[G_{20}, G_{20}^{\varphi}]$ is generated by $[b, a^{\varphi}]$, $[c, a^{\varphi}]$, $[c, b^{\varphi}]$, $[d, a^{\varphi}]$ modulo $\mathcal{M}_0{}^*(G_{20})$. Using Lemma 1.1 (vi), we have

$$[[b, a^{\varphi}], [c, a^{\varphi}]] = [[b, a], [c, a]^{\varphi}] = [c, d^{\varphi}] \in \mathcal{M}_0{}^*(G_{20}),$$

and

$$[[b, a^{\varphi}], [c, b^{\varphi}]] = [[b, a], [c, b]^{\varphi}] = [c, e^{\varphi}] \in \mathcal{M}_0{}^*(G_{20}).$$

Similarly,

$$[[b, a^{\varphi}], [d, a^{\varphi}]], [[c, a^{\varphi}], [c, b^{\varphi}]], [[c, a^{\varphi}], [d, a^{\varphi}]], [[c, b^{\varphi}], [d, a^{\varphi}]] \in \mathcal{M}_0{}^*(G_{20}).$$

Thus any two elements of the generating set of $[G_{20}, G_{20}^{\varphi}]$, are commuting modulo $\mathcal{M}_0{}^*(G_{20})$, and each element of $[G_{20}, G_{20}^{\varphi}]$ can be expressed as

$$[b, a^{\varphi}]^{\alpha_1}[c, a^{\varphi}]^{\alpha_2}[c, b^{\varphi}]^{\alpha_3}[d, a^{\varphi}]^{\alpha_4}\tilde{w},$$

where $\tilde{w} \in \mathcal{M}_0{}^*(G_{20})$, and $1 \leq i \leq 4$, $\alpha_i \in \mathbb{Z}$.
Let $w = [b, a^{\varphi}]^{\alpha_1}[c, a^{\varphi}]^{\alpha_2}[c, b^{\varphi}]^{\alpha_3}[d, a^{\varphi}]^{\alpha_4}\tilde{w} \in \mathcal{M}^*(G_{20})$, then
$1 = \kappa^*(w) = c^{\alpha_1}d^{\alpha_2}e^{\alpha_3}f^{\alpha_4}$. Since $c, d, e, f$ are in the polycyclic generating sequence and $exp(G_{20}) = p$, we have $c^{\alpha_1} = d^{\alpha_2} = e^{\alpha_3} = f^{\alpha_4} = 1$ and $p$ divides $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4$, respectively. Now using Lemmas 1.1 and 1.2, we have

$$1 = [c^p, b^{\varphi}] = [c, b^{\varphi}]^p[c, b^{\varphi}, c]^{\binom{p}{2}}[c, b^{\varphi}, c, c]^{\binom{p}{3}}.$$

Since

$$[c, b^{\varphi}, c]^{\binom{p}{2}} = [c, b, c^{\varphi}]^{\binom{p}{2}} = [e, c^{\varphi}]^{\binom{p}{2}} = [e^{\binom{p}{2}}, c^{\varphi}] = 1$$

and

$$[c, b^{\varphi}, c, c]^{\binom{p}{3}} = [c, b, c, c^{\varphi}]^{\binom{p}{3}} = [e, c, c^{\varphi}]^{\binom{p}{3}} = [1, c^{\varphi}]^{\binom{p}{3}} = 1,$$

$[c, b^{\varphi}]^p = 1$. Similarly, $[b, a^{\varphi}]^p = [c, a^{\varphi}]^p = [d, a^{\varphi}]^p = 1$. Thus $w = \tilde{w}$. Hence $\mathcal{M}^*(G_{20}) \subseteq \mathcal{M}_0{}^*(G_{20})$ and $\tilde{B}_0(G_{20}) = 0$. □

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# 4-valent vertex-transitive graphs of order $6p$ with no regular subgroup of automorphisms

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**Abstract:** In this paper, we class all 4-valent vertex-transitive graphs of order $6p$, $p$ a prime, that their automorphism group have no regular subgroup using permutation group theory techniques.

**Keywords** vertex-transitive graph; regular subgroup; solvable radical.

**Mathematics Subject Classification (2010) : 20B20, 05C25.**

## 1 Introduction

Let $G$ be a group and $H$ be a subgroup of $G$. Let $D$ be an inversed-closed union of some double cosets of $H$ in $G$. Then the *coset graph* $\text{Cos}(G, H, D)$ of $G$ with respect to $H$ and $D$ is a graph with vertex set $[G : H]$, the set of right cosets of $H$ in $G$, and two vertices $Hx, Hy$ for $x, y \in G$ are adjacent whenever $xy^{-1} \in D$. Note that the valency of $\text{Cos}(G, H, D)$ is $|D|/|H|$ and it is connected if and only if $G = \langle D\rangle$. Indeed the action of $G$ on $V(\text{Cos}(G, H, D))$ by right multiplication induces a vertex-transitive automorphism group, which is arc-transitive if and only if $D$ is a single double coset. Moreover $\text{Cos}(G, H, D) \cong \text{Cos}(G, H^\beta, D^\beta)$ for every $\beta \in \text{Aut}(G)$. Conversely, let $X$ be a graph and let $A$ be a vertex-transitive subgroup of $\text{Aut}(X)$. Then the graph $X$ is isomorphic to a coset graph $\text{Cos}(A, H, D)$, where $H = A_u$ is the stabilizer of $u \in V(X)$ in $A$ and $D$ consists of all elements of $A$ which map $u$ to one of its neighbors (see for example [4]). In the case that $H = 1$, the resulting coset graph is called a Cayley graph. Equivalently, a graph $\Gamma$ is a Cayley graph over a group $G$ if $\text{Aut}(\Gamma)$ admits a regular subgroup isomorphic to $G$.

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Although Cayley graphs are vertex-transitive, there are vertex-transitive graphs which are not Cayley graphs and the smallest one is the well-known Petersen graph. Such a graph will be called a *vertex-transitive non-Cayley graph*, or a $\mathcal{VNC}$-graph for short.

In [3], Marušič asked for a determination of the set $\mathcal{NC}$ of non-Cayley numbers, that is, those positive numbers $n$ for which there exists a $\mathcal{VNC}$-graph of order $n$. Marušič investigated the problem when $n$ is a prime power, and then to settle this question, a lot of $\mathcal{VNC}$-graphs were constructed in the literature. In [1], Feng considered of determining the smallest valency for $\mathcal{VNC}$-graphs of a given order and it was solved for the graphs of odd prime power order. In this paper, we solve the problem for 4-valent vertex-transitive graphs of order $6p$.

## 2 Main results

A finite simple group $G$ is said to be a $K_3$-group if its order has exactly three distinct prime divisors. By [2, Pages 12-14], $G$ is isomorphic to one of the following groups:

$$A_5, A_6, \mathrm{PSL}_2(7), \mathrm{PSL}_2(8), \mathrm{PSL}_2(17), \mathrm{PSL}_3(3), \mathrm{U}_3(3), \mathrm{U}_4(2). \quad (1)$$

The *socle* of a group $G$ is the subgroup generated by the set of all minimal normal subgroups of $G$, it is denoted by $\mathrm{soc}(G)$. Also a group $G$ is said to be *almost simple* if $T \leq G \leq \mathrm{Aut}(T)$, where $T$ is a non-abelian simple group. It is well known that $G$ is an almost simple group if and only if $\mathrm{soc}(G) = T$ for some non-abelian simple group $T$.

Now we prove the following lemma.

**Lemma 2.1.** *Let $G$ be an almost simple group and $\mathrm{soc}(G) = A$, where $A$ is a non-abelian simple $K_3$-group. Then*

(*i*) *If $A \cong A_5$ then $G \cong A_5$ or $S_5$.*

(*ii*) *If $A \cong A_6$ then $G \cong A_6$ or $S_6$ or $S_6 \rtimes \mathbb{Z}_2$ or $\mathrm{M}_{10}$ or $P\Sigma L_2(9)$.*

(*iii*) *If $A \cong \mathrm{PSL}_2(p)$, where $p \in \{7, 17\}$, then $G \cong \mathrm{PSL}_2(p)$ or $\mathrm{PGL}_2(p)$.*

(*iv*) *If $A \cong \mathrm{PSL}_2(8)$ then $G \cong \mathrm{PSL}_2(8)$ or $G/\mathrm{PSL}_2(8) \cong \mathbb{Z}_3$.*

(*v*) *If $A \cong \mathrm{PSL}_3(3)$ then $G \cong \mathrm{PSL}_3(3)$ or $G/\mathrm{PSL}_3(3) \cong \mathbb{Z}_2$.*

(*vi*) *If $A \cong \mathrm{U}_3(3)$ then $G \cong \mathrm{U}_3(3)$ or $G/\mathrm{U}_3(3) \cong \mathbb{Z}_2$.*

(*vii*) *If $A \cong \mathrm{U}_4(2)$ then $G \cong \mathrm{U}_4(2)$ or $G/\mathrm{U}_4(2) \cong \mathbb{Z}_2$.*

*Proof.* Since $G$ is an almost simple group and $\mathrm{soc}(G) = A$, it implies that $A \leq G \leq \mathrm{Aut}(A)$. If $A \cong A_5$, then $A_5 \leq G \leq \mathrm{Aut}(A_5)$. Now since $|\mathrm{Out}(A_5)| = 2$, it follows that $G \cong A_5$ or $S_5$ and (*i*) holds. Also if $A$ is isomorphic to one of $\mathrm{PSL}_2(7)$, $\mathrm{PSL}_2(17)$, $\mathrm{PSL}_3(3)$, $\mathrm{U}_3(3)$ or $\mathrm{U}_4(2)$ then $|\mathrm{Out}(A)| = 2$ and the assertions in $(iii), (v), (vi)$ and $(vii)$ hold. If $A \cong A_6$ then $|\mathrm{Out}(A_6)| = 4$ and $\mathrm{Aut}(A_6) \cong S_6 \rtimes \mathbb{Z}_2$. Thus $G \cong A_6$ or $S_6$ or $S_6 \rtimes \mathbb{Z}_2$ or $\mathrm{M}_{10}$ or $P\Sigma L_2(9)$ and (*ii*) holds. Finally if $A \cong \mathrm{PSL}_2(8)$ then $|\mathrm{Out}(\mathrm{PSL}_2(8))| = 3$. Hence $G \cong \mathrm{PSL}_2(8)$ or $G/\mathrm{PSL}_2(8) \cong \mathbb{Z}_3$ and (*iv*) holds. □

**Proposition 2.2.** *Let $X$ be a graph, $A = \mathrm{Aut}(X)$ and $N$ be a normal subgroup of $A$. Also let $\Omega$ be the set of orbits of $N$ on $V(X)$ and $X_N$ be a Cayley graph on a subgroup of $A/\mathcal{K}$, say $T/\mathcal{K}$, where $\mathcal{K}$ is the kernel of the action of $N$ on $\Omega$ and $|T| = |V(X)|$. Then $X$ is a Cayley graph on $T$.*

**Theorem 2.3.** *Let $X$ be a connected tetravalent vertex-transitive graph of order $6p$, where $p$ is a prime. Then $X$ is a $\mathcal{VNC}$-graph if and only if $X \cong X_{3,p,t}$, where $1 \leq t \leq p-1$ and $t^2 \equiv -1 \pmod p$ or $X$ is one of the nine certain graphs.*

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# A kind of algebraic hypergraph on modules

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**Abstract:** We are interested in studying a hypergraph on a given $S$-module $B$, where the vertices of all nontrivial submodules of $B$ and a subset $E$ of the set of nontrivial submodules of $B$ forms a hyperedge in case the intersection of each two distinct elements of $E$ is a $\delta$-small submodule of $B$ and $E$ is maximal with respect to this property.

**Keywords** $\delta$-small submodule; hypergraph; hyperedge; $\delta$-small intersection hypergraph.

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## 1 Introduction

Studying algebraic structures such as modules and rings via graphs and hypergraphs can provide insights and visual representations of the relationships between the elements and operations of the structures.

As given in [4, 5], a hypergraph $\mathcal{H} = (V; E)$ on a finite set of vertices (or nodes) $V = \{v_1, \ldots, v_n\}$ is defined as a family of hyperedges $E = \{e_j \mid 1 \leq j \leq m\}$ where each hyperedge is a non-empty subset of $V$ and such that $\cup_{j=1}^{m} e_j = V$. It means that in a hypergraph, a hyperedge links one or more vertices.

According to the basic graph theory, any complete subgraph of a given graph forms a clique. Cliques appear in many areas of graph theory. In particular, they are intrinsic in the study of perfect graphs and play an important role in the extremal graph theory ([10]). They are also well-studied in computer science. In fact, it is known that the problem of computing the clique number is NP-hard (despite the fact that it can be computed in polynomial time for planner graphs and also for chordal graphs). Besides their prominent graph theoretical aspect, they have also real-world applications, for instance, in modeling clusters in graph-based data mining ([3]).

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Back to our main goal, we may interested in intersection and co-intersection graphs on modules studied in [1] and [2]. Also small intersection graph on a module defined and studied by authors in [6].

Finding cliques of a graph can be viewed as an important issue in graph theory while the solution of this problem may help researchers in network sciences to determine the structure of networks. Maybe first efforts on computing cliques of a graph via hypergraphs were done in [7] and [8]. In [7], the authors introduced co-intersection hypergraph on a given module and as a tool used them to find maximal cliques in intersection graph of [1]. Let $B$ be a right $S$-module. Then $\mathcal{CIH}_S(B)$ is a hypergraph where the vertices are all nontrivial submodules of $B$ and a set $U$ containing some vertices forms a hyperedge provided the intersection of each two elements of $U$ is nonzero and $U$ is maximal with respect to this property. Dually, in [8], intersection hypergraph of a module $B$ namely $\mathcal{IH}_S(B)$ is defined to be a hypergraph such that vertices are all nontrivial submodules of $B$ and a set $E$ of some vertices is a hyperedge in case the intersection of each two elements of $E$ is zero submodule and $E$ is maximal with respect to this condition.

Let $B$ be a right $S$-module. Consider a graph assigned to $B$ denoted by $\mathcal{DSI}_S(B)$ as follows: $V(\mathcal{DSI}_S(B)) = \{N < B \mid N \neq 0\}$ and two distinct vertices $N$ and $K$ are adjacent provided their intersection is $\delta$-small in $B$. This definition generalizes the co-intersection graph defined in [2]. So it is natural and can be of interest to generalize this definition to hypergraphs. By the way, $\mathcal{D}_S(B)$ can be the interested hypergraph where the vertices are all nontrivial submodules of $B$ and a subset $E_i$ of the set of all nontrivial submodules of $B$ with $\mid E_i \mid \geq 2$, is a hyperedge of $\mathcal{D}_S(B)$ provided the intersection of each two elements of $E_i$ is $\delta$-small in $B$ and $E_i$ is maximal with respect to this property. By the definition, we may conclude each hyperedge in $\mathcal{D}_S(B)$ introduces a maximal clique in $\mathcal{DSI}_S(B)$ and in turn each maximal clique in $\mathcal{DSI}_S(B)$ gives us a hyperedge in $\mathcal{D}_S(B)$. So, the number of hyperedges of $\mathcal{D}_S(B)$ is equal to the number of maximal cliques of $\mathcal{DSI}_S(B)$.

Following [12], a submodule $K$ of $B$ is $\delta$-small in $B$ denoted by $K \ll_\delta B$, if $B = K + L$ and $B/L$ is singular then $B = L$. Every small submodule of an $S$-module is $\delta$-small in that module.

The sum of all simple submodules of a module $B$ is said to be the Socle of $B$, denoted by $Soc(B)$. If $B$ has no essential submodules, then $Soc(B) = B$. In this case, we say $B$ is semisimple. The radical of $B$, denoted by $Rad(B)$ is the sum of all small submodules of $B$, that is equivalent to the intersection of all maximal submodules of $B$ (see [11]). Analogues to $Rad(B)$, we have the notion $\delta(B)$ which is defined to the sum of all $\delta$-small submodules of $B$.

Section 1 is devoted to present some preliminaries about graphs and hypergraphs and their importance. Some motivations to create such work are presented in Section 1. In Section 2, we characterize modules $B$ such that $\mathcal{D}_S(B)$ has only one hyperedge containing all vertices. Also, we provide equivalent conditions based on the module $B$ such that $\mathcal{D}_S(B)$ is disconnected. We show that the diameter of a connected small intersection hypergraph $\mathcal{D}_S(B)$ is at most three. The girth of $\mathcal{D}_S(B)$ is also computed. We provide some examples of $\mathcal{D}_S(B)$ via their corresponding figures. In the last Section, we state the conclusions of the current work and further plans are presented. The readers can be referred to [11] for extra information about modules and rings. Also [9], may contain any unexplained concept related to graphs and hypergraphs.

## 2 Defining a hypergraph on modules via singularity

Applying singularity, we are able to define and investigate a new hypergraph on a given $S$-module $B$.

**Definition 2.1.** Let $B$ be an $S$-module. Applying the concept of $\delta$-small submodules of a module, we define a hypergraph $\mathcal{D}_S(B)$, where the vertices are all nontrivial submodules of $B$ and a set $U_i$ (such that $\mid U_i \mid \geq 2$) of nontrivial submodules of $B$ forms a hyperedge provided the intersection of each two distinct elements of $U_i$ is $\delta$-small in $B$ and $U_i$ is maximal with respect to this property.

Throughout this manuscript, we consider modules with at least two nontrivial submodules.

Recall that a module $B$ is $\delta$-hollow in case every proper submodule of $B$, is $\delta$-small in $B$. A hollow module is $\delta$-hollow.

*Remark* 2.2. (1) If $B$ is an $S$-module with exactly two nontrivial submodules $N$ and $L$, then $(N \cap L) + N = N$ and $(N \cap L) + L = L$. So that $N \cap L$ is a small and hence a $\delta$-small submodule of $B$. This means that $\mathcal{D}_S(B)$ has a hyperedge with two nontrivial submodules $N$ and $L$.

(2) For a $\delta$-hollow module $B$, the hypergraph $\mathcal{D}_S(B)$ has just one hyperedge containing all vertices. Note that each nontrivial submodule of $B$ is $\delta$-small in $B$.

(3) Any nontrivial $\delta$-small submodule of $B$ is included in any hyperedge of $\mathcal{D}_S(B)$.

(4) For a module $B$ with $\delta(B) = 0$, the hypergraph $\mathcal{D}_S(B)$ is exactly as $\mathcal{IH}_S(B)$.

The following determines the structure of modules $B$ such that their hypergraph $\mathcal{D}_S(B)$ are null.

**Proposition 2.3.** *The hypergraph $\mathcal{D}_S(B)$ has no hyperedges if and only if $B$ is uniform and $\delta(B) = 0$.*

Note that $\delta(\mathbb{Z}) = 0$ and $\mathbb{Z}$ as an $\mathbb{Z}$-module is uniform, overall imply $\mathcal{D}_{\mathbb{Z}}(\mathbb{Z})$ is null.

Modules $B$ for which $\mathcal{D}_S(B)$ has just one hyperedge containing all vertices, will be discussed below.

**Theorem 2.4.** *If $B$ is an $S$-module. Then the following are equivalent:*

(1) *$\mathcal{D}_S(B)$ has just exactly one hyperedge containing all vertices;*

(2) *For each two distinct nontrivial submodules $N$ and $L$ of $B$, we have $N \cap L \ll_\delta B$;*

(3) *Each nontrivial submodule of $B$ is either $\delta$-small or maximal in $B$;*

Observe that any nontrivial submodule of the $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is maximal. In fact the intersection of each two nontrivial submodules of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is zero.

**Example 2.5.** Consider the semisimple $\mathbb{Z}$-module $B = \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Then $C_1 = \{(0,0),(0,1),(0,2)\}$, $C_2 = \{(0,0),(1,0),(2,0)\}$, $C_3 = \{(0,0),(1,1),(2,2)\}$ and $C_4 = \{(0,0),(1,2),(2,1)\}$ are all nontrivial submodules of $B$. It is clear that every nontrivial submodule of $B$ is maximal in $B$. Hence, $V = \{C_1, C_2, C_3, C_4\}$,
$E = \{\{C_1, C_2, C_3, C_4\}\})$ and the hypergraph $\mathcal{D}_{\mathbb{Z}}(B)$ has the form (Theorem 2.3):

In the following theorem, we present some equivalent conditions for $\mathcal{D}_S(B)$ to be disconnected. Compare Theorem 2.5 with Proposition 1.1.

**Theorem 2.6.** *The following statements are equivalent for a module $B$:*

(1) *The hypergraph $\mathcal{D}_S(B)$ is disconnected;*

(2) $\delta(B) = 0$ *and $B$ has a non-trivial essential submodule;*

(3) *The hypergraph $\mathcal{D}_S(B)$ has an isolated vertex.*

**Corollary 2.7.** *Let $B$ be an $S$-module. Then the hypergraph $\mathcal{D}_S(B)$ is connected if and only if $\delta(B) \neq 0$ or $B$ is semisimple.*

**Example 2.8.** Suppose that $B = \mathbb{Z}_{p^2q}$ as an $\mathbb{Z}$-module where $p, q$ are distinct prime numbers and $p < q$. The list of all nontrivial submodules of $B$ is:

$T_1 =< pq >$, $T_2 =< p^2 >$, $T_3 =< q >$ and $T_4 =< p >$.

By the way, $\mathcal{D}_S(B)$ has two hyperedges $L_1 = \{T_1, T_2, T_3\}$ and $L_2 = \{T_1, T_3, T_4\}$.

Now, we are in a position to find the diameter of $\mathcal{D}_S(B)$ when it is a connected hypergraph. By definition, if $\mathcal{D}_S(B)$ is not connected, then diam$(\mathcal{D}_S(B) = \infty$.

**Proposition 2.9.** *Let $B$ be an $S$-module. If $\mathcal{D}_S(B)$ is a connected hypergraph, then diam$(\mathcal{D}_S(B)) \leq 3$.*

**Example 2.10.** Let $B = \mathbb{Z}_{p^3q}$ as an $\mathbb{Z}$-module. All nontrivial submodules are $W_1 =< p^2q >$, $W_2 =< p^3 >$, $W_3 =< pq >$, $W_4 =< p^2 >$, $W_5 =< q >$ and $W_6 =< p >$. Then $\mathcal{D}_S(B)$ has three hyperedges $E_1 = \{W_1, W_2, W_3, W_5\}$, $E_2 = \{W_1, W_3, W_5, W_6\}$ and $E_3 = \{W_1, W_3, W_4, W_5\}$. The hypergraph $\mathcal{D}_S(B)$ has the following figure:

Note here that *diam*$(\mathcal{D}_{\mathbb{Z}}(\mathbb{Z}_{p^3q}) = 2$.

Recall that by $l_S(B)$, we mean the length of the $S$-module $B$. In other words, we say $B$ has length $n \in \mathbb{N}$, provided $n$ is the length of the largest chain of submodules of $B$. If no such largest chain exists, then $l_S(B) = \infty$.

**Lemma 2.11.** *Let $B$ be an $S$-module. Assume that $\Delta(\mathcal{D}_S(B)) < \infty$ and $\delta(\mathcal{D}_S(B)) \geq 1$. Then $l_S(B) \leq \Delta(\mathcal{D}_S(B)) + 1$ and every nontrivial submodule of $B$ has finitely many submodules.*

The following theorem gives the girth of $\mathcal{D}_S(B)$.

**Theorem 2.12.** *Let $B$ be a module with $\delta(B) = 0$. If $\mathcal{D}_S(B)$ contains a cycle, then* gr$(\mathcal{D}_S(B) \leq 4$. *Further if $\mathcal{D}_S(B)$ is connected, then $gr(\mathcal{D}_S(B)) = 3$.*

**Example 2.13.** Consider the $\mathbb{Z}$-module $B = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ where the nontrivial submodules are:
$A_1 = \{(0,0), (0,1), (0,2), (0,3)\}$, $A_2 = \{(0,0), (1,1), (0,2), (1,3)\}$,
$A_3 = \{(0,0), (1,0)\}$, $A_4 = \{(0,0), (1,2)\}$, $A_5 = \{(0,0), (0,2), (1,0), (1,2)\}$ and $A_6 = \{(0,0), (0,2)\}$.
All hyperedges are $E_1 = \{A_1, A_2, A_3, A_4, A_6\}$ and $E_2 = \{A_1, A_2, A_5, A_6\}$. Here is the figure of $\mathcal{D}_{\mathbb{Z}}(B)$:

Following the figure, we can write $gr(\mathcal{D}_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_4)) = 3$.

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# A note on commuting automorphisms of some finite $p$-groups

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**Abstract:** Let $G$ be a group. If the set $\mathcal{A}(G) = \{\alpha \in Aut(G) : x\alpha(x) = \alpha(x)x \quad for\ all\ x \in G\}$ forms a subgroup of $Aut(G)$, then $G$ is called $\mathcal{A}$-group. In this paper, we prove that if $G$ is a finite $p$-group such that $|G/G'Z(G)| \leq p^2$, then $G$ is an $\mathcal{A}$-group. Also, we give some consequences of our main results. Finally, we give an application of main results.

**Keywords** commuting automorphism; $p$-group; $\mathcal{A}$-group.

**Mathematics Subject Classification (2010) : 20F28, 20D45, 20D15.**

## 1 Introduction

Let $G$ be a group. By $Aut(G)$ we denote the group of all automorphisms of $G$. An automorphism $\alpha$ of $G$ is called a commuting automorphism if $g\alpha(g) = \alpha(g)g$ for all $g \in G$. The set of all commuting automorphisms of the group $G$ is denoted by $\mathcal{A}(G)$.

Throughout, $p$ denotes a fixed prime and $d(G)$ denotes the minimum number of generators of $G$. The nilpotency class of $G$ is denoted by $c(G)$. If $|G| = p^n$ and $c(G) = c$, then the coclass of $G$ is $cc(G) = n - c$. Groups of coclass 1 and coclass 2 are called $p$-groups of maximal class and almost maximal class, respectively. By $Z_i(G)$, we denote the i-th terms of the upper central series of $G$. The i-th terms of the lower central series of $G$ for $i \geq 2$ are denoted by $\gamma_i(G)$ and $\gamma_2(G)$ is denoted also by $G'$. Let $\alpha \in Aut(G)$ then $C_G(\alpha) = \{g \in G : \alpha(g) = g\}$. Finally, $\mathbb{Z}_m^n$ is a direct product of $n$ copies of $\mathbb{Z}_m$.

In 2002, Deaconescu, Silberberg and Walls, showed that even though $\mathcal{A}(G)$ has a number of the properties of a group, but it is not necessarily a subgroup of $Aut(G)$ (see [4]).

If the set $\mathcal{A}(G)$ is a subgroup of $Aut(G)$, then $G$ is called $\mathcal{A}$-group ( for brevity $G \in \mathcal{A}$).

Fouladi and Orfi showed that, if $G$ is a finite AC-group, a finite $p$-group of maximal class, or a finite metacyclic $p$-group, then $G$ is an $\mathcal{A}$-group (see [5]). We note that a group $G$ is called an AC-group if the centralizer of every non-central element of G is abelian.

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In 2015 Rai proved that a finite $p$-group $G$ of almost maximal class, for an odd prime $p$, is an $\mathcal{A}$-group (see [7]).

For finite 2-groups of almost maximal class, the situation is more complicated. In [2], together with Vosooghpour, we proved that a 2-group of almost maximal class is an $\mathcal{A}$-group. Then, we proved the minimum coclass of a non-$\mathcal{A}$, $p$-group is equal to 3. We also discussed the direct product of certain $\mathcal{A}$-groups and subsequently, we gave some applications of our results.

In this paper, we prove that if $G$ is a finite $p$-group such that $|G/G'Z(G)| \leq p^2$ then $G$ is an $\mathcal{A}$-group. Also, we give some consequences of our main results. Finally, we give an application of main results.

## 2 Main results

First, we collect some results on commuting automorphisms, required in the proof of main results.

**Lemma 2.1.** ([8, Lemma 2.2]) *Let $G$ be a group of nilpotency class 2. If $d(G/Z(G)) = 2$, then $G$ is an $\mathcal{A}$-group.*

**Lemma 2.2.** ([3, Theorem 1.1]) *Let $G$ be a group generated by a set $X$ of elements. If $\gamma_i(G) = \langle Y, \gamma_{i+1}(G)\rangle$, then $\gamma_{i+1}(G)$ is generated by $\gamma_{i+2}(G)$ together with all commutator $[x, y]$, where $x, y$ run through $X$, $Y$ respectively. This is true for $i = 1, 2, ...,$ provided that $\gamma_1(G)$ is interpreted to mean $G$.*

**Lemma 2.3.** ([3, Theorem 1.4]) *If $G$ is a group, $x \in G$ and $y \in \gamma_r(G)$, then for any integer $n$,*

$$[x^n, y] \overset{\gamma_{r+2}(G)}{\equiv} [x, y^n] \overset{\gamma_{r+2}(G)}{\equiv} [x, y]^n$$

**Lemma 2.4.** ([4, Lemma 2.2]) *Let $G$ be a group. If $\alpha \in \mathcal{A}(G)$ and $x \in G$, then $x^{-1}\alpha(x) \in C_G(G')$.*

**Lemma 2.5.** ([4, Lemma 2.1]) *If $\alpha \in \mathcal{A}(G)$ and $x, y \in G$, then $[\alpha(x), y] = [x, \alpha(y)]$.*

**Lemma 2.6.** ([4, Lemma 2.6]) *If $\alpha \in \mathcal{A}(G)$, then*

(i) $[G', \alpha] \leq Z(G)$.

(ii) $\gamma_3(G) \leq C_G(\alpha)$.

**Lemma 2.7.** ([4, Lemma 2.4]) *Let $G$ be a group and $\alpha, \beta \in \mathcal{A}(G)$. Then $\alpha\beta \in \mathcal{A}(G)$ if and only if $[\alpha(x), \beta(x)] = 1$ for all $x \in G$.*

Now we turn to the proof our main theorem.

**Theorem 2.8.** *Let $G$ be a finite $p$-group such that $|G/G'Z(G)| \leq p^2$. Then $G$ is an $\mathcal{A}$-group.*

*Proof.* It is clear that if $G$ is abelian, then $G$ is an $\mathcal{A}$-group. Also, if $c(G) = 2$, then $G$ is an $\mathcal{A}$-group by Lemma 1.2. Suppose that $c(G) \geq 3$ and $G/G'Z(G) = \langle xG'Z(G)\rangle \times \langle yG'Z(G)\rangle \cong \mathbb{Z}_p^2$. So, $G = \langle x, y, Z(G)\rangle$ and every element of $G$ can be expressed as $x^i y^j gz$ where $0 \leq i, j < p$, $g \in G'$ and $z \in Z(G)$. By Lemma 1.1, $G' = \langle [x, y], \gamma_3(G)\rangle$ and $\gamma_3(G) = \langle [x, y, x], [x, y, y], \gamma_4(G)\rangle$. Now by Lemma 2.3, exponent $\gamma_3(G)/\gamma_4(G)$ is equal to $p$ and $|G'/\gamma_3(G)| = p$. Let $\alpha \in \mathcal{A}(G)$ and $\alpha(x) = x^i y^j gz$ where $0 \leq i, j < p$, $g \in G'$ and $z \in Z(G)$. Then $1 = [\alpha(x), x] = [y^j, x]^g[g, x]$, so $[y, x]^j \in \gamma_3(G)$. Therefore $p|j$ but $0 \leq j < p$, so $j = 0$. By a similar argument we obtain $\alpha(y) = y^{j'}g'z'$ where $0 \leq j' < p$, $g' \in G'$ and $z' \in Z(G)$. We claim that $i = j'$. By Lemma 2.5, we have $[\alpha(x), y] = [x, \alpha(y)]$. Therefore by a simple calculation we have $[x, y]^i \overset{\gamma_3(G)}{\equiv} [x, y]^{j'}$. Since $|G'/\gamma_3(G)| = p$ and $0 \leq i, j' < p$, we have $i = j'$. Next, we prove that $i = 1 = j'$.

First, we show that $Z(G) \cap G' \leq \gamma_3(G)$. Let $a \in Z(G) \cap G'$, since $a \in G'$, we have $a = [x, y]^l t$ where $0 \leq l < p$ and $t \in \gamma_3(G)$. Since $a \in Z(G)$, we have $[[x, y]^l t, x] = 1$, so $[x, y, x]^l \in \gamma_4(G)$ by Lemma 2.3. By a similar argument, since $[[x, y]^l t, y] = 1$, we obtain $[x, y, y]^l \in \gamma_4(G)$. Hence $l = 0$, since otherwise $(l, p) = 1$. Therefore $[x, y, y], [x, y, x] \in \gamma_4(G)$ and $\gamma_3(G) = \langle [x, y, x], [x, y, y], \gamma_4(G)\rangle = \gamma_4(G)$ which is a contradiction.

Now by Lemma 2.6, $[x, y]^{-1}[x^i gz, y^i g'z'] \in Z(G) \cap G' \leq \gamma_3(G)$. So, $[x, y]^{i^2-1} \in \gamma_3(G)$. Recalling that $|G'/\gamma_3(G)| = p$, therefore $p|i^2 - 1$. Now we have two cases:

(a) If $p = 2$, then it is trivial that $i = 1$.

(b) If $p$ is an odd prime, then $i = 1$ or $i = p-1$. Let $i = p-1$. Therefore $x^{-1}\alpha(x) = x^{p-2}gz \in C_G(G')$ by Lemma 2.4, so $[x^{p-2}gz, [x, y]] = 1$. Therefore $[x, y, x]^2 \in \gamma_4(G)$. Since $p$ is an odd prime, $[x, y, x] \in \gamma_4(G)$. By a similar argument, since $y^{-1}\alpha(y) \in C_G(G')$, we have $[x, y, y] \in \gamma_4(G)$. Therefore $\gamma_3(G) = \langle [x, y, x], [x, y, y], \gamma_4(G)\rangle = \gamma_4(G)$, a contradiction.

So $\alpha(x) = xgz$ and $\alpha(y) = yg'z'$ where $g, g' \in G' \cap C_G(G') = Z(G')$ by Lemma 2.4. Now by Lemma 2.6 and a simple calculation, we obtain $h^{-1}\alpha(h) \in Z(G')Z(G)$ for all $h \in G$ and all $\alpha \in \mathcal{A}(G)$. Let $\alpha, \beta \in \mathcal{A}(G)$ and $h \in G$ such that $\alpha(h) = huc$ and $\beta(h) = hvc'$ where $u, v \in Z(G')$ and $c, c' \in Z(G)$. Then

$$[\alpha(h), \beta(h)] = [huc, hvc'] = [h, v]^u[u, v][u, h]^v = 1,$$

because $1 = [\alpha(h), h] = [u, h]$, $1 = [\beta(h), h] = [v, h]$ and $u, v \in Z(G')$. So by Lemma 2.7, $G$ is an $\mathcal{A}$-group. □

Now, we give some immediate consequences of our main theorem.

Now, it is clear that Theorem 3.4 [5], follows from Theorem 2.8, so we may deduce the following result.

**Corollary 2.9.** *If $G$ is a $p$-group of maximal class, then $G$ is an $\mathcal{A}$-group.*

Next, we need the following result.

**Lemma 2.10.** ([2, Lemma 3.11]) *Let $G$ be a finite group and $G = H \times K$, where groups $H$ and $K$ have no common direct factor. Then $G$ is an $\mathcal{A}$-group if and only if $H, K$ are $\mathcal{A}$-groups.*

Now, we apply our main theorem to a larger class of groups.

**Corollary 2.11.** *Let $G$ be a finite nilpotent group. If all of it's sylow subgroups satisfy in the conditions of Theorem 2.8, then $G$ is an $\mathcal{A}$-group.*

**Definition 2.12.** Let $m, n$ be integers and $3 \leq m \leq n$. The set of all groups $G$ of order $p^n$ and $c(G) = m-1$, in which $|\gamma_i(G) : \gamma_{i+1}(G)| = p$ $(i = 2, ...., m-1)$ is denoted by $CF(m, n, p)$.

In [3] Blackburn studied this class of groups.

**Theorem 2.13.** ([3, Theorem 2.4]) *If $G \in CF(m, n, p)$, then $Z_i(G) \cap G' = \gamma_{m-i}(G)$ for $0 \leq i \leq m-2$.*

Now, by Theorem 2.13 and Theorem 2.8, we have the following result.

**Corollary 2.14.** *If $G \in CF(m, n, p)$ and $|Z(G)| \geq p^{n-m+1}$, then $G$ is an $\mathcal{A}$-group.*

**Example 2.15.** Let $H$ be a $p$-group of maximal class and $K$ be a cyclic group of order $p$. If $G = H \times K$, then $G/G'Z(G) \cong \mathbb{Z}_p^2$, so by Theorem 2.8, $G$ is an $\mathcal{A}$-group.

## An application of our results

As an application of our results we provide a short proof, to verifying commuting automorphisms $p$-groups of order $p^n$ where $p$ is a prime and $n \leq 5$ (see [7] and [8]).

We will use the classification of group of order $p^5$ by James in the proof. We note the James classified these groups in 10 isoclinism families. The families are denoted by $\phi_k$ for $k = 1, 2, ..., 10$ (see [6]).

To prove our aim, we use the following results.

**Theorem 2.16.** ([8, Theorem 1.5]) *For a given prime $p$, the minimal number of generators of a non-$\mathcal{A}$ $p$-group of order $p^5$ and of nilpotency calss 2 is equal to 4.*

**Theorem 2.17.** ([7, Theorem 1.5]) *Let $G$ be a group of order $p^5$ for a prime $p$. Then $G$ is a non-$\mathcal{A}$ group if and only if $G$ is an extra-special $p$-group for an odd prime or $G$ is an extra-special 2-group of plus type, i.e, the central product of two dihedral groups of order 8.*

**Theorem 2.18.** ([8, Theorem 1.2]) *There exists a non-$\mathcal{A}$ $p$-group $G$ of order $p^n$ for all $n \geq 5$.*

**Theorem 2.19.** ([1, Theorem 1.5]) *Let $G$ be a $p$-group of nilpotency class $c$ where $p$ is an odd prime. If $G' \cap Z(G) = \gamma_c(G)$ and $|Z_2(G)/Z(G)| \leq p^2$, then $G$ is an $\mathcal{A}$-group. In particular if $G$ is non-abelian, then $p||Inn(G) \cap \mathcal{A}(G)|$.*

Now, let $G$ be a $p$-group of order $p^n$ where $p$ is a prime and $n \leq 5$.

If $G$ is an abelian group, then $\mathcal{A}(G) = Aut(G)$ and $G$ is an $\mathcal{A}$-group. So, let $G$ be a non-abelian group of order $p^n$, $n > 2$.

(i) If $n = 3$, then $G$ is a $p$-group of maximal class and by Corollary 2.9, $G \in \mathcal{A}$.

(ii) If $n = 4$, then we have two cases:

(ii.1) If $c(G) = 2$, then $G/Z(G) \cong \mathbb{Z}_p^2$, so by Theorem 2.8, $G \in \mathcal{A}$.

(ii.2) If $c(G) = 3$, then $G$ is a $p$-group of maximal class and by Corollary 2.9, $G \in \mathcal{A}$.

(iii) If $n = 5$, then we have two cases:

(iii.1) If $p$ is an odd prime, then we have the following cases:

(iii.1.1) In the families $\phi_2$, $\phi_3$ and $\phi_6$, $G/G'Z(G) \cong \mathbb{Z}_p^2$, so by Theorem 2.8, $G \in \mathcal{A}$.

(iii.1.2) In the families $\phi_4$, $c(G) = 2$ and $d(G) = 3$, so by Theorem 2.16, $G \in \mathcal{A}$.

(iii.1.3) The family $\phi_5$, consists of extra special $p$-groups, so by Theorem 2.18, $G \notin \mathcal{A}$.

(iii.1.4) In the families $\phi_7$ and $\phi_8$, we have $|Z_2(G)/Z(G)| \leq p^2$, $|Z(G)| = p$ and $G' \cap Z(G) = \gamma_3(G)$, so by Theorem 2.19, $G \in \mathcal{A}$.

(iii.1.5) The families $\phi_9$ and $\phi_{10}$ consists of $p$-groups of maximal class and by Corollary 2.9, $G \in \mathcal{A}$.

(iii.2) Theorem 2.17, deals with the case $p = 2$.

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# Abelian groups with epimorphisms between their subgroups

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**Abstract:** In this note, we will examine abelian groups which for any both subgroups of them, there is an epimorphism from one to another. In particular, we study this concept for subgroups of $\mathbb{Q}$, finitely generated and quasi-injective abelian groups.

**Keywords** epimorphism; subgroup; quasi-injective.

**Mathematics Subject Classification (2010) : 20K15, 20K27, 20K30.**

## 1 Introduction

Let $P$ be the set of prime numbers and $(\alpha_p)_{p\in P}$ is a sequence such that for any $p \in P$, $\alpha_p$ is a non-negative integer or $\infty$. Two such sequences $(\alpha_p)_{p\in P}$ and $(\beta_p)_{p\in P}$ are *equivalent* if $\alpha_p = \beta_p$ for all but a finite number $p$ and $\alpha_p = \beta_p$ if either $\alpha_p = \infty$ or $\beta_p = \infty$. It is not hard to see that this relation is an equivalence relation. Now let $A$ be a subgroup of $\mathbb{Q}$ (the group of rational numbers) and $a \in A$. We consider the sequence $(\alpha_p)_{p\in P}$ as follows:

$\alpha_p = n \geq 0$ with $a \in p^n A - p^{n+1}A$ and if no such n exists, then we set $\alpha_p = \infty$. Then it is assigned the equivalence class $[(\alpha_p)_{p\in P}]$ to group $A$. It is shown that the assigned sequence is independent of the choice of $a$ in $A$. Also it is well-known that there is a one-to-one correspondence between the set of subgroups of $\mathbb{Q}$ and the set of equivalence classes $[(\alpha_p)_{p\in P}]$.

Let $A$ be an abelian group. We say that $A$ satisfies $(*)$-*condition* if for any two subgroups of $A$, there is an epimorphism from one to another. In this note, using the assigned sequence, we show that a subgroup of $\mathbb{Q}$ satisfies $(*)$-*condition* if and only if it is isomorphic to $\mathbb{Z}$. Then we determine which finitely generated abelian groups satisfy $(*)$-*condition*. Finally, the injective (quasi-injective) abelian groups satisfied $(*)$-*condition* are classified. See [1, 2]

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## 2 Main results

**Example 2.1.** It is easy to see that the assigned sequences to groups $\mathbb{Z}$ and $\mathbb{Q}$ are $(0, 0, 0, \ldots)$ and $(\infty, \infty, \infty, \ldots)$, respectively. Also the abelian group generated by $\{1/2, 1/3, 1/5, \ldots\}$ is denoted by sequence $(1, 1, 1, \ldots)$.

**Example 2.2.** The assigned sequences to group $< 1/2^m, 1/3^n \mid m, n \in \mathbb{N} >$, generated by $\{1/2^m, 1/3^n \mid m, n \in \mathbb{N}\}$, is $(\infty, \infty, 0, 0, 0, \ldots)$.

**Theorem 2.3.** *A subgroup of $\mathbb{Q}$ satisfies $(*)$-condition if and only if it is isomorphic to $\mathbb{Z}$.*

**Theorem 2.4.** *If a finite abelian group $A$ satisfies $(*)$-condition, then $|A| = p^n$ where $p$ is a prime number and $n$ is a non-negative integer.*

The converse of above theorem is not true in general. For example $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ dose not satisfy $(*)$-*condition.*

**Theorem 2.5.** *A finitely generated abelian group $A$ satisfies $(*)$-condition if and only if $A$ is isomorphic to either $\underbrace{\mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p}_{n-times}$, $\underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n-times}$, $\underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n-times} \oplus \mathbb{Z}_{p^m}$ or $\mathbb{Z}_{p^n}$, for some prime number $p$ and integers $n$, $m$.*

Recall that an abelian group $A$ is said to be *quasi-injective* if for any subgroup $B$ of $A$, any $f \in Hom_{\mathbb{Z}}(B, A)$ cab be extended to an endomorphism of $A$.

**Theorem 2.6.** *A nonzero quasi-injective abelian group $A$ satisfies $(*)$-condition if and only if $A$ is isomorphic to either $\mathbb{Z}_{p^n}$, $\mathbb{Z}_{p^\infty}$ or $\mathbb{Z}_p^{(\gamma)}$, where $p$ is a prime number, $n \geq 2$ and $\gamma$ is a set.*

**Corollary 2.7.** *A nonzero injective abelian group $A$ satisfies $(*)$-condition if and only if $A$ is isomorphic to $\mathbb{Z}_{p^\infty}$, for some prime number $p$.*

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# Groups with prime power Isaacs $\pi$-partial character codegrees

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**Abstract:** Let $\pi$ be a set of prime numbers, $p \notin \pi$ be a prime and $G$ be a $\pi$-solvable group. We prove in this note that if $\mathrm{cod}(\varphi)$ is $p$-power for every nonlinear, monomial and monolithic $\varphi \in \mathrm{I}_\pi(G)$, then $G$ has a normal Hall $\pi'$-subgroup.

**Keywords** $\pi$-partial character; codegree ; Hall subgroup; $\pi$-solvable.
**Mathematics Subject Classification (2010) : 20C15, 20C20.**

## 1 Introduction

Throughout this paper, $G$ will be a finite group. We will use $\mathrm{Irr}(G)$ and $\mathrm{IBr}_p(G)$ to denote the set of the ordinary irreducible characters and irreducible ($p-$) Brauer characters of $G$. With the similar way, for a set of prime numbers $\pi$, we write $\mathrm{I}_\pi(G)$ to denote the sets of irreducible Issacs $\pi$-partial characters of $G$ for a $\pi$-separable group $G$. Studying the structure of the group $G$ through the properties of its irreducible characters is a significant topic in group theory. Many works have been conducted in this regard. For example, the well-known theorems by Itô and Thompson establish the relationship between the existence of a normal Sylow $p$-subgroup or a normal $p$-complement and the divisibility of irreducible character degrees by $p$, both in ordinary and modular cases. Additionally, there are papers that establish relationships between the numerical properties of Isaacs' $\pi$-partial character degrees and the structure of the group.

Recall that a character is called monomial if it can be induced from a linear character of some subgroup. A character $\chi$ is referred to as monolithic if $G/\ker(\chi)$ contains a unique minimal normal subgroup. For an irreducible character $\chi$, the codegree of $\chi$ is defined as $\mathrm{cod}(\chi) = |G : \ker(\chi)|/\chi(1)$. This terminology was introduced by Qian, Wang, and Wei in [6]. X. Chen and M. Lewis, in [1], proved that if $G$ is solvable and $\mathrm{cod}(\chi)$ is a $p$-power for every nonlinear, monomial, and monolithic

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$\chi \in \mathrm{Irr}(G)$, or for every nonlinear, monomial, and monolithic $\chi \in \mathrm{IBr}_p(G)$, then $G$ has a normal Sylow $p$-subgroup, where $p$ is a prime. In this paper, we present a similar result for Isaacs $\pi$-partial characters and generalize their theorem. Specifically, we prove that if $\mathrm{cod}(\varphi)$ is a $p$-power ($p \notin \pi$) for every nonlinear, monomial, and monolithic $\varphi \in \mathrm{I}_\pi(G)$, then $G$ has a normal Hall $\pi'$-subgroup. Note that by setting $\pi = p'$, their result is recovered as a special case.

## 2 Isaacs $\pi$-partial characters

Let $G$ be $\pi$-separable group for some set $\pi$ of primes. The element $g$ of $G$ is said $\pi$-element if every prime divisor of the order of $g$ lies in $\pi$. In other words, the order of $g$ is $\pi$-number. We denote $G^0$ the set of $\pi$-elements of $G$. Also, we say that a conjugacy class is $\pi$-class if all of its members are $\pi$-elements.

Let $\chi$ be a complex character. We denote $\chi^0$ the restriction of $\chi$ to $G^0$. A complex-valued function $\varphi$ on $G^0$ is said to be an Isaacs $\pi$-partial character of $G$ if $\varphi = \chi^0$ for some character $\chi$ of $G$. In other words, we could say Isaacs $\pi$-partial character of $G$ is a class function of $G^0$ which lifts to a complex character (not necessarily unique) of $G$. An Isaacs $\pi$-parial character of $G$ is irreducible if it cannot be written as a sum of two Isaacs $\pi$-partial characters and the set of all irreducible Isaacs $\pi$-partial characters of $G$ is denoted by $\mathrm{I}_\pi(G)$. We say $\varphi(1)$ is degree of Isaacs $\pi$-partial character $\varphi$ and the Isaacs $\pi$-partial character $\varphi$ is linear if $\varphi(1) = 1$.

Let $\varphi = \chi^0$ be an Isaacs $\pi$-partial character. We define $\ker\varphi = \ker\chi$ and by [3, Crollary 4.22] $\bigcap_{\varphi \in \mathrm{I}_\pi(G)} \ker\varphi = O_{\pi'}(G)$. We can also induce Isaacs $\pi$-partial characters by using the usual formula for induced character, but applied only for $\pi$-elements of $G$. We refer the readers to [5] for Brauer character theory and [3] for $\pi$-partial characters.

**Theorem 2.1** (**[3, Theorem 5.11]**). *Suppose that $\varphi \in \mathrm{I}_\pi(G)$, where $N \trianglelefteq G$ and $G$ is $\pi$-separable, and let $T$ be the stabilizer of $\theta$ in $G$. Then induction defines a bijection $\mathrm{I}_\pi(T|\theta) \to \mathrm{I}_\pi(G|\theta)$. Also, if $\alpha \in \mathrm{I}_\pi(T|\theta)$ and $\beta = \alpha^G$, then $\alpha$ is the unique irreducible constituent of $\beta_T$ that lies in $I_\pi(T|\theta)$.*

## 3 Main results

In this section, we state some of new results.

**Lemma 3.1.** *Let $G = HN$ be a $\pi$-separable group where $H$ is a subgroup of $G$ and $N$ is a normal subgroup of $G$ with $H \cap N = 1$. Then each linear Isaacs $\pi$-partial character of $N$ which is invariant under $G$ extends to an Isaacs $\pi$-partial character of $G$.*

**Theorem 3.2.** *Let $G$ be a $\pi$-solvable group and $\mathrm{cod}(\varphi)$ is $p$-power for every nonlinear, monomial and monolithic $\varphi \in \mathrm{I}_\pi(G)$ such that $p \notin \pi$. Then $G$ has a normal Hall $\pi'$-subgroup.*

*Proof.* Suppose that $\mathrm{cod}(\varphi)$ are $p$-powers for all nonlinear, monomial and monolithic irreducible Issacs $\pi$-partial characters. We work by induction on $|G|$. First suppose that $G$ has more than two minimal normal subgroups. Let $M_1, \ldots, M_s$ are different minimal normal subgroups of $G$. By induction, the quotient groups $G/M_i$ for $1 \leq i \leq s$ has a normal Hall $\pi'$-subgroup and since $G$ is embedded in $\prod_{i=1}^{s} G/M_i$. Hence, $G$ also has a normal Hall $\pi'$-subgroup. Thus, we may assume that $G$ has a unique minimal normal subgroup, say $M$. We know that $M$ is $\pi'$-group or elementary abelian $q$-group for some $q \in \pi$. If $M$ is $\pi'$-group then by induction $G/M$ has a normal Hall $\pi'$-subgroup and does $G$. Thus, we may assume that $M$ is an elementary abelian $q$-group for some $q \in \pi$ . Hence, $M$ is $\pi$-number and $\mathrm{Irr}(M) = \mathrm{I}_\pi(\mathrm{M})$.

Let $K$ be a Hall $\pi'$-subgroup of $G$. By Frattini argument we have $G = MKN_G(K) = MN_G(K)$. By unique minimality of $M$, we have either $M \cap N_G(K) = 1$ or $M \cap N_G(K) = M$.

We know that $G = MN_G(K)$, then we have $T = M.(N_G(K) \cap T)$. Thus, $M$ is complemented in $T$. By Lemma 3.1, there exsits some $\mu \in \mathrm{I}_\pi(T)$ such that $\mu_M = \lambda$. By Theorem 2.1, $\varphi = \mu^G \in \mathrm{I}_\pi(G)$ is monomial and we have $\varphi(1) = |G : T|$. Since $M$ is unique minimal normal subgroup of $G$, then either $\ker(\varphi) = 1$ or $M \subseteq \ker(\varphi)$. But $M \subseteq \ker(\varphi)$ is impossible, since if $M \subseteq \ker(\varphi)$ then $M \subseteq \ker(\mu)$ and so $\lambda$ is principal character of $M$. We conclude that $\ker(\varphi) = 1$ and hence, $\varphi$ is monolithic. Also, $\varphi$ cannot be linear, as $\varphi(1) = |G : T| = 1$ implies that $\lambda$ is invariant under $G$, and hence under $K$. This leads to a contradiction, as $|K| = |C_K(\lambda)| < \sqrt{|K|}$.

Therefore, by the hypothesis, $\mathrm{cod}(\varphi) = \frac{|G|}{\varphi(1)}$ is $p$-power then $|G|_\pi = \varphi(1)_\pi$. Also

$$\varphi(1)^2_{\pi'} = |G : T|^2_{\pi'} > |G|_{\pi'} = |K|.$$

Thus, we have

$$\varphi(1)^2_{\pi'} |G|_\pi > |G|_{\pi'} |G|_\pi = |G| \geq |G|_{\pi'}.$$

We have $\varphi(1)^2 < |G|$ and so $(|G|_\pi \varphi(1)_{\pi'})^2 < |G|$. Hence, $|G|_{\pi'} > \varphi(1)^2_{\pi'} |G|_\pi$, which is a contradiction. □

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# Some results on the soft HG-hypergroupoids

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**Abstract:** Soft sets are a very suitable solution for uncertainty problems. One of the hyperstructures, is hypergroup-hypergroupoid. In this paper, is defined soft hypergroup-hypergroupoids. This the concept, is defined by using the concepts of soft hypergroup and soft hypergroupoid. In addition, we present the category of soft hypergroup-hypergroupoids.

**Keywords** Groupoid, Soft set, Soft groupoid, Soft hyperoupoid, Soft HG-hypergroupid.

**Mathematics Subject Classification (2010) : 20N20, 18E45.**

## 1 Introduction

Today, we see that many issues of different sciences, including technical and engineering, chemistry, physics, medicine, and even economics, social and environmental sciences, and many other sciences, are not defined with definite and completely clear data. In other words, because we are not faced with completely specific data, we cannot solve these problems with classical and usual mathematical methods. To solve uncertainty problems, various methods have been stated and studied. Fuzzy set theory[7] and rough set theory[5], are among the famous theories that can be mentioned in this passage. For further reading, refer to [6, 3, 2]. These methods, with all the positive things they have, also have some problems. To improve the previous methods and solve problems in the study of uncertainty problems, in 1999, the concept of soft sets was presented by Molodtsov [4].

Let $U$ be an initial universe set and $E$ be a set of parameters. $\mathcal{P}(U)$ denotes the power set of $U$ and $A \subseteq E$.

**Definition 1.1.** A pair $(F, A)$ is called a soft set over $U$, where $F$ is a mapping given by $F : A \longrightarrow \mathcal{P}(U)$. In fact, a soft set over $U$ is a parameterized family of subsets of the universe $U$. For $\alpha \in A$, $F(\alpha)$ may be considered as the set of $\alpha$-approximate elements of the soft set $(F, A)$.

A pair $(F, A)$ is called a soft set over $U$, where $F$ is a mapping given by $F : A \longrightarrow \mathcal{P}(U)$. In fact,

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a soft set over $U$ is a parameterized family of subsets of the universe $U$. For $\alpha \in A$, $F(\alpha)$ may be considered as the set of $\alpha$-approximate elements of the soft set $(F, A)$.

**Definition 1.2.** For two soft sets $(\mathcal{F}, R)$ and $(\mathcal{G}, T)$ over $U$, we say that $(\mathcal{F}, R)$ is a soft subset of $(\mathcal{G}, T)$, denoted by $(\mathcal{F}, R) \subseteq (\mathcal{G}, T)$, if the following conditions hold:

(i) $R \subseteq T$,

(ii) for all $\kappa \in R, \mathcal{F}(\kappa)$ and $\mathcal{G}(\kappa)$, are identical approximations.

Two soft sets $(\mathcal{F}, R)$ and $(\mathcal{G}, T)$ over $U$ are called soft equal, if $(\mathcal{F}, R) \subseteq (\mathcal{G}, T)$ and $(\mathcal{G}, T) \subseteq (\mathcal{F}, R)$.

**Definition 1.3.** Let $(\mathcal{F}, R)$ be a soft set. The set $Supp(\mathcal{F}, R) = \{x \in R | \mathcal{F}(x) \neq \phi\}$ is called the support of the soft set $(\mathcal{F}, R)$. A soft set $(\mathcal{F}, R)$ is non-null, if $Supp(\mathcal{F}, R) \neq \phi$.

A polygroup is a multi-valued system $\mathcal{M} =< P, \circ, e, {}^{-1} >$, with $e \in P$, ${}^{-1} : P \longrightarrow P$, $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$, where the following axioms hold, for all $r, s, t \in P$:

(i) $(r \circ s) \circ t = r \circ (s \circ t)$

(ii) $e \circ r = r \circ e = r$

(iii) $r \in s \circ t$ implies $s \in r \circ t^{-1}$ and $t \in s^{-1} \circ r$.

$\mathcal{P}^*(P)$ is the set of all the non-empty subsets of $P$, and also if $x \in P$ and $R$, $T$ are non-empty subsets of $P$, then we have $R \circ T = \bigcup\limits_{\substack{b \in T \\ a \in R}} a \circ b$, $x \circ T = \{x\} \circ T$ and $R \circ x = R \circ \{x\}$.

The following, are the facts that are clearly concluded from the principles of the polygroups: $e \in r \circ r^{-1} \cap r^{-1} \circ r$, $e^{-1} = e$ and $(r^{-1})^{-1} = r$.

**Example 1.4.** If we consider the set $P$ as $P = \{e, r, s\}$, then $P =< P, \circ, e, {}^{-1} >$ along with polyaction which have shown in the Table 2.1. is a polygroup.

| $\circ$ | $e$ | $r$ | $s$ |
|---|---|---|---|
| $e$ | $e$ | $r$ | s |
| $r$ | $r$ | $\{e, s\}$ | $\{r, s\}$ |
| $s$ | $s$ | $\{r, s\}$ | $\{e, r\}$ |

Table 2.1.

**Definition 1.5.** A crossed hypermodule $\chi = (C, P, \partial, \kappa)$ is consists of hypergroups $< C, *, e, {}^{-1} >$ and $< P, \circ, e, {}^{-1} >$ together with a strong homomorphism $\partial : C \longrightarrow P$ and a (left) action $\kappa : P \times C \longrightarrow \mathcal{P}^*(C)$ on $C$, satisfying the following conditions:

(i) $\partial({}^{p}c) = p \circ \partial(c) \circ p^{-1}$, for all $c \in C$ and $p \in P$,

(ii) ${}^{\partial(c)}c' = c * c' * c^{-1}$, for all $c, c' \in C$.

**Example 1.6.** (i) In every hyperygroup, the set containing only the identity member is always a subhypergroup, and this subhypergroup is normal in the hypergroup. Therefore, we have crossed hypermodule $(1, P) = (1, P, c_1, id \mid_{c_1})$.

(ii) Every hypergroup $P$ contains the whole hypergroup $P$ as a normal subhypergroup. So, we always have crossed hypermodule $(P, P) = (P, P, c, id_P)$.

(iii) Consider the following hypergroup morphisms of an abelian hypergroup $P$, written multiplicatively,

$$l : 1 \to Aut(P) \qquad i \to id_P \qquad k : P \to 1 \qquad p \to 1$$

So, we have a crossed hypermodule $(P, 1) = (P, 1, l, k)$.

## 2 Category of soft crossed Hg-hypergroupoids

In this section, we introduce soft crossed Hg-hypergroupoids.

**Definition 2.1.** A hypergroup-hypergroupoid (or HG-hypergroupoid), is a hypergroup object in the category of hypergroupiods.

**Definition 2.2.** If $H$ is a hg-hypergroupoid and $\mathcal{P}(H)$ is all subhg-hypergroupoids of $H$, and $F$ is mapping $F : A \longrightarrow \mathcal{P}(H)$, where $A$ is a set of parameters, such that the set $F(\alpha)$is a subhg-hypergroupoid of $H$, for all $\alpha \in A$, then the pair $(F, A)$ is called a soft hg-hypergroupoid on $H$, and it is represented by the $(H, F, A)$.

*Remark* 2.3. A soft hg-hypergroupoid on $H$ hg-hypergroupoid can be defined as a parametrized family of subhg-hypergroupoid of hg-hypergroupoid $H$.

*Remark* 2.4. Every hg-groupoid has a hypergroupoid structure. Hence every soft hg-hypergroupoid, has a soft hypergroupoid structure.

**Theorem 2.5.** *The soft hypergroup $(F, A)$, on the abelian hypergroup $H$, is a soft hg-hypergroupoid.*

**Definition 2.6.** If $(H, F, A)$ and $(H', F', A')$ are two soft hg-hypergroupoids on their hg-hypergroupoids $H$ and $H'$, respectively, then $(f, g) : (H, F, A) \longrightarrow (H', F', A')$ is called hg-hypergroupoid homomorphism, when $(f, g)$ is a soft homomorphism.

*Remark* 2.7. A new category is obtained by taking the objects as soft hg-hypergroupoids and their morphisms as soft hg-hypergroupoid homomorphisms between them. We call this category, the category of soft hg-hypergroupoids and we show that with SHG-HGC.

**Example 2.8.** Let $(H, F, A)$ be a soft hg-hypergroupoid and $(H', F', A')$ be a soft subhypergroupoid of $(H, F, A)$. If $Ob(F'(\alpha)) \leq Ob(F(\alpha))$ and $Mor(F'(\alpha)) \leq Mor(F(\alpha))$, for all $\alpha \in A'$, then $(H', F', A')$ is called a soft subhg-hypergroupoid $(H, F, A)$.

**Definition 2.9.** Let $(H', F', A')$ be a soft hg-hypergroupoid of soft hg-hypergroupoid $(H, F, A)$, such that, $Ob(F'(\alpha)) \trianglerighteq Ob(F(\alpha))$ and $Mor(F'(\alpha)) \trianglerighteq Mor(F(\alpha))$ for all $\alpha \in A'$. Then $(H', F', A')$ is called a normal soft subhg-hypergroupoid of $(H, F, A)$.

In 1976, Brown and Spenccer proved that the category of crossed modules and the category of group-groupoids are equivalent [1]. We prove that soft crossed hypermodules and soft hg-hypergroupoids, are equivalent categories.

**Theorem 2.10.** *A soft hg-hypergroupoid, can be obtained from each soft crossed hypermodule.*

**Theorem 2.11.** *A soft crossed hypermodule, can be obtained from each soft hg-hypergroupoid.*

**Theorem 2.12.** *The category of soft hg-hypergroupoids is equivalent to the category of soft crossed hypermodules.*

# 3 Conclusion

In this paper, after recalling the concepts of soft sets and soft groups, We defined and checked soft crossed Hg-hypergroupoids studied their properties.

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# Some results on the covering of groups

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**Abstract:** Let $G$ be a group. An irredundant $n-$covering of a group is a finite set of $n$ proper subgroups of $G$ such that $G = \cup_{i=1}^{n} H_i$ but $G \neq \cup_{i \neq j} H_i$ for any $1 \leq j \leq n$. Let $\Sigma(G) = \{n \in \mathbb{N}$ :There is an irredundant $n-$covering of $G\}$. Then $\sigma(G) = min\ \Sigma(G)$ is called the covering number of $G$. It is well known that for any group $G$ we have $2 \notin \Sigma(G)$ We prove that for any $n \geq 3$ there is a group $G$ such that $n \in \Sigma(G)$. Also we prove that a covering of the Galois group of an extension $E/F$ gives a sum decomposition of $E$. Also we have a decomposition of group rings $RG = RH_1 + \cdots + RH_n$.

**Keywords** Covering number, Sum of subfields, Galois group.

**Mathematics Subject Classification (2010) : 20D60, 12F10.**

## 1 Introduction

The study of a mathematical structure by decomposing it to substructures is a main theme in Mathematics. Also different decompositions in different contexts could be transformed to each other by a functor. For example a partitition of a set $X$ gives a direct sum decomposition of the free module on X. Conversly a direct sum decomposition of a free module $M$ gives a partition of a basis of $M$. Another decomposition comes from the isomorphism $S_R(M \bigoplus N) \cong S_R(M) \bigotimes S_R(N)$ for any two free modules over a commutative ring $R$ where $S_R(M)$ is the symmetric algebra of $M$ over $R$. An interesting problem in the decomosition is that if all components have a property $P$ then the whole object has property $P$ or not. For example, In [1] it is proved that if a ring $R$ is sum of two nilpotent subrings $R_1$ and $R_2$ then $R$ is nilpotent. Another example in module theory states that a direct sum $\bigoplus M_i$ of modules is projective if and only if each $M_i$ is a projective module. In group theory the representation of a groups as union of finitely many proper subgroups is studied widely. An irredundant $n-$covering of a group $G$ is a finite set of $n$ proper subgroups of $G$ such that $G = \cup_{i=1}^{n} H_i$ but $G \neq \cup_{i \neq j} H_i$ for any $1 \leq j \leq n$. Let $\Sigma(G) = \{n \in \mathbb{N}$ :There is an irredundant $n-$covering of $G\}$. Then $\sigma(G) = min\ \Sigma(G)$ is called the covering number of $G$. It is well known that for any group $G$ we have $2 \notin \Sigma(G)$ We

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prove that for any $n \geq 3$ there is a group $G$ such that $n \in \Sigma(G)$. Also we prove that a covering of the Galois group of an extension $E/F$ gives a sum decomposition of $E$. It is $RG = RH_1 + \cdots + RH_n$.If $G = \cup_{i=1}^{n} H_i$ where $H_i$'s are proper subgroups of $G$ and $n$ is minimum with this property then we call $n$ the covering number of $G$ and denote it by $\sigma(G)$. If there is no such decomposition we set $\sigma(G) = \infty$. It is clear that $\sigma(G) \leq \sigma(G/N)$ for any normal subgroup $N$ of $G$. The group $G$ is said to be $n$-elementary if for every non trivial normal subgroup $N$ of $G$ we have $n = \sigma(G) < \sigma(G/N)$. By a theorem of B. H. Neumann [4] if $G = \cup_{i=1}^{n} H_i$ then the subgroups of infinite index may be omitted. So it is assumed that $[G : \cap_{i=1}^{n} H_i]$ is finite. So there is a normal subgroup $N$ which contained in $\cap_{i=1}^{n} H_i$ and $G/N$ is finite. So $\sigma(G) = \sigma(G/N)$. So every $n$-elementary group is finite. It is well known that no group is union of two proper subgroups. Also it is proved by Scorza [5] that a group $G$ is the union of three proper subgroups $A, B, C$ if and only if $[G : A] = [G : B] = [G : C] = 2$ and $[G : A \cap B \cap C] = 4$. Also this is equivalent to that $G$ has $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient group. Also in [6] it is proved that there is no group $G$ such that $\sigma(G) = 7$. In contrast we prove that for any $n \geq 3$ there is a group $G$ such that $n \in \Sigma(G)$. We also show that an irredundant covering of a group $G$ gives an unshortenable sum decomposition of the group ring $RG$. Further a covering of some Galois groups $Gal(E/F)$ gives an unshortenable sum decomposition of the field $E$.

## 2 Main results

In this section, we state our main results.

**Theorem 2.1.** *Let $3 \leq n \in \mathbb{N}$. Then there is a group $G$ such that $n \in \Sigma(G)$.*

*Proof.* We prove that if $n \in \Sigma(H)$ then $n+2 \in \Sigma(H \times \mathbb{Z}_2 \times \mathbb{Z}_2)$. Let $H = \cup_{i=1}^{n} H_i$. Then $H \times \mathbb{Z}_2 \times \mathbb{Z}_2 = H \times \mathbb{Z}_2 \times \{0\} \bigcup H \times \{0\} \times \mathbb{Z}_2 \bigcup \bigcup_{i=1}^{n} H_i \times \langle (1,1) \rangle$. □

Also the following gives another version of Theorem 2.1 which is given in [2]:

**Theorem 2.2.** *If $n \geq 2$ then $n + 1 \in \Sigma(\mathbb{Z}_2^n)$.*

*Proof.* Set $H_i = \{(a_1, \cdots .a_n) \in \mathbb{Z}_2^n : a_i = 0\}$ for $1 \leq i \leq n$ and $H_{n+1} = \langle (1, \cdots, 1) \rangle$. Then $\mathbb{Z}_2^n = \bigcup_{i=1}^{n+1} H_i$ is an irredundant $n + 1-$covering. □

*Remark* 2.3. The above covering of $\mathbb{Z}_2^n$ is an example of vector space covering. Assume $F_{q^n}$ denotes a field with $q^n$ elements. If $d|n$ then $F_{q^d}$ is a subfield of $F_{q^n}$. Let $m = \frac{q^n-1}{q^d-1}$ and $F_{q^n}^* = \bigcup_{i=1}^{m} a_i F_{q^d}^*$ be the union of cosets of $F_{q^d}^*$ in $F_{q^n}^*$. Then $F_{q^n} = \bigcup_{i=1}^{m} a_i F_{q^d}$ is an irredundant $m-$covering of $F_{q^n}$. This implies $m \in \Sigma(F_p^{kn})$ wher $q = p^k$.

The proof of the following theorem is easy.

**Theorem 2.4.** *If $H = \cup_{i=1}^{n} H_i$ is an irredundant covering of the (semi)group $H$ then $RH = RH_1 + \cdots + RH_n$ is an unshortenable sum decomposition of $RH$.*

**Theorem 2.5.** *Let $E/F$ be a Galois extension of degree four such that $Gal(E/F) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $Char(F) \neq 2$. Then $E = E_1 + E_2 + E_3$ where $E_i$'s are three proper intermediate subfields of $E/F$.*

*Proof.* Since $G = Gal(E/F)$ has only three subgroup of index two, so $E/F$ has only three intermediate subfields $E_1, E_2, E_3$ of degree two. Since $E_i/F$ is separable so $E_i = F[a_i]$ where $a_i^2 = b_i \in F^*$. Also $E = F(a_1, a_2)$. So $\{1, a_1, a_2, a_1a_2\}$ is a basis for $E/F$. Thus $a_1a_2 \notin F[a_1] + F[a_2]$. Since $(a_1a_2)^2 = b_1b_2 \in F^*$ so $[F(a_1a_2) : F] = 2$. Hence $E_3 = F[a_1a_2]$.This implies $E_1 + E_2 + E_3 = E$. □

**Example 2.6.** Let $E = \mathbb{Q}[\sqrt{2}, \sqrt{3}]$. Then $[K : \mathbb{Q}] = 4$ and $Gal(E/Q) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Also $E = \mathbb{Q}[\sqrt{2}] + \mathbb{Q}[\sqrt{3}] + \mathbb{Q}[\sqrt{6}]$.

**Example 2.7.** Let $F = \mathbb{Q}[\omega]$ where $\omega$ is a 3−primitive root of unity and $E = F[\sqrt[3]{2}, \sqrt[3]{3}]$. Then $[K : F] = 9$ and $Gal(E/Q) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\sigma(\mathbb{Z}_3 \times \mathbb{Z}_3) = 4$. Also $E = F[\sqrt[3]{2}] + F[\sqrt[3]{3}] + F[\sqrt[3]{6}] + F[\sqrt[3]{12}]$.

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# Automorphism groups of finite topological spaces

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**Abstract:** In this note we give some results about the automorphism groups of some finite topological spaces and some finite topologies. We compute the automorphism groups of spaces $X$ with $|X| \leq 4$. Also we compute the automorphism groups of some spaces $X$ when its topology is finite.

**Keywords** Finite topological space, Automorphism group, Finite topology.

**Mathematics Subject Classification (2010) : 37F20, 20B25.**

## 1 Introduction

An Alexandroff space is a topological space such that any intersection of open subsets is open. Theses spaces are studied in a paper by P.S. Alexandroff [1]. Finite topological spaces are Alexandroff spaces. In the above mentioned paper, he showed that there is a natural equivalence between the category of finite ($T_0$) topological spaces and finite preordered sets(posets)(By a preorder we mean a reflexive and transitive relation). In this correspondence continuous functions correspond to order preserving maps. The following are three main themes in the study of finite topological spaces:

(i) Computing the finite topologies on a set $X$ and Counting the number of topologies on a finite set: For example it is shown that there are 33 inequivalent topology on a set with four elements. Also there are 139 inequivalent topology on a set with five elements. The following table is taken from wikipedia.

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**Number of topologies on a set with $n$ points**

| $n$ | Distinct topologies | Distinct $T_0$ topologies | Inequivalent topologies | Inequivalent $T_0$ topologies |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 |
| 2 | 4 | 3 | 3 | 2 |
| 3 | 29 | 19 | 9 | 5 |
| 4 | 355 | 219 | 33 | 16 |
| 5 | 6942 | 4231 | 139 | 63 |
| 6 | 209527 | 130023 | 718 | 318 |
| 7 | 9535241 | 6129859 | 4535 | 2045 |
| 8 | 642779354 | 431723379 | 35979 | 16999 |
| 9 | 63260289423 | 44511042511 | 363083 | 183231 |
| 10 | 8977053873043 | 6611065248783 | 4717687 | 2567284 |
| OEIS | A000798 | A001035 | A001930 | A000112 |

(ii) Algebraic topology of finite topological spaces: The results of [4] and [6] provide a correspondence up to weak homotopy equivalence between finite topological spaces and finite simplicial complexes, proving that these two classes of spaces exhibit precisely the same homotopy groups. See also [3].

(iii) Computing the associated groups to a finite topological spaces: For a topological space $X$ we denote the group of homeomorphism on $X$ by $Aut(X)$. It is clear that if $|X| = n$ then $Aut(X) \leq S_n$. In [2] it is proved that for any finite group $G$ of order $n$ with $r$ generators there is a finite topological space $X$ with $n(r+2)$ points such that $Aut(X) \cong G$. This bound is not necessarily optimal. For example, If $X$ is a discrete space with $n$ elements then $Aut(X) = S_n$. Let $G$ be a group. We say $G \in T(n)$ if there is a topological space $X$ with $n$ elements such that $Aut(X) \cong G$. In this note we give some results about $T(n)$.

## 2 Main results

In this section, we state our main results. The following tables compute $Aut(X)$ when $|X| \leq 4$.

Inequivalent Topologies on two point set X={a,b}

1. {∅, {a,b}} (the trivial topology) $Aut(X) \cong S_2$
2. {∅, {a}, {a,b}} $Aut(X) \cong S_1$
3. {∅, {a}, {b}, {a,b}} (the discrete topology) $Aut(X) \cong S_2$

Inequivalent topologies on
three point set X={a,b,c}

There are 29 distinct topologies on *X* but only 9 inequivalent topologies:

| | | |
|---|---|---|
| 1. | {∅, {a,b,c}} | $Aut(X) \cong S_3$ |
| 2. | {∅, {c}, {a,b,c}} | $Aut(X) \cong S_2$ |
| 3. | {∅, {a,b}, {a,b,c}} | $Aut(X) \cong S_2$ |
| 4. | {∅, {c}, {a,b}, {a,b,c}} | $Aut(X) \cong S_2$ |
| 5. | {∅, {c}, {b,c}, {a,b,c}} ($T_0$) | $Aut(X) \cong S_1$ |
| 6. | {∅, {c}, {a,c}, {b,c}, {a,b,c}} ($T_0$) | $Aut(X) \cong S_2$ |
| 7. | {∅, {a}, {b}, {a,b}, {a,b,c}} ($T_0$) | $Aut(X) \cong S_2$ |
| 8. | {∅, {b}, {c}, {a,b}, {b,c}, {a,b,c}} ($T_0$) | $Aut(X) \cong S_1$ |
| 9. | {∅, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}} ($T_0$) | $Aut(X) \cong S_3$ |

Inequivalent topologies on four element set X={a,b,c,d}

There are 355 distinct topologies on *X* but only 33 inequivalent topologies:

| | | |
|---|---|---|
| 1. | {∅, {a, b, c, d}} | $Aut(X) \cong S_4$ |
| 2. | {∅, {a, b, c}, {a, b, c, d}} | $Aut(X) \cong S_3$ |
| 3. | {∅, {a}, {a, b, c, d}} | $Aut(X) \cong S_3$ |
| 4. | {∅, {a}, {a, b, c}, {a, b, c, d}} | $Aut(X) \cong S_2$ |
| 5. | {∅, {a, b}, {a, b, c, d}} | $Aut(X) \cong S_2 \times S_2$ |
| 6. | {∅, {a, b}, {a, b, c}, {a, b, c, d}} | $Aut(X) \cong S_2$ |
| 7. | {∅, {a}, {a, b}, {a, b, c, d}} | $Aut(X) \cong S_2$ |
| 8. | {∅, {a}, {b}, {a, b}, {a, b, c, d}} | $Aut(X) \cong S_2 \times S_2$ |
| 9. | {∅, {a, b, c}, {d}, {a, b, c, d}} | $Aut(X) \cong S_3$ |
| 10. | {∅, {a}, {a, b, c}, {a, d}, {a, b, c, d}} | $Aut(X) \cong S_2$ |
| 11. | {∅, {a}, {a, b, c}, {d}, {a, d}, {a, b, c, d}} | $Aut(X) \cong S_2$ |
| 12. | {∅, {a}, {b, c}, {a, b, c}, {a, d}, {a, b, c, d}} | $Aut(X) \cong S_2$ |
| 13. | {∅, {a, b}, {a, b, c}, {a, b, d}, {a, b, c, d}} | $Aut(X) \cong S_2 \times S_2$ |
| 14. | {∅, {a, b}, {c}, {a, b, c}, {a, b, c, d}} | $Aut(X) \cong S_2$ |
| 15. | {∅, {a, b}, {c}, {a, b, c}, {a, b, d}, {a, b, c, d}} | $Aut(X) \cong S_2$ |
| 16. | {∅, {a, b}, {c}, {a, b, c}, {d}, {a, b, d}, {c, d}, {a, b, c, d}} | $Aut(X) \cong S_2 \times S_2$ |
| 17. | {∅, {b, c}, {a, d}, {a, b, c, d}} | $Aut(X) \cong D_4$ |
| 18. | {∅, {a}, {a, b}, {a, b, c}, {a, b, d}, {a, b, c, d}} ($T_0$) | $Aut(X) \cong S_2$ |
| 19. | {∅, {a}, {a, b}, {a, c}, {a, b, c}, {a, b, c, d}} ($T_0$) | $Aut(X) \cong S_2$ |
| 20. | {∅, {a}, {b}, {a, b}, {a, c}, {a, b, c}, {a, b, c, d}} ($T_0$) | $Aut(X) \cong S_1$ |
| 21. | {∅, {a}, {a, b}, {a, b, c}, {a, b, c, d}} ($T_0$) | $Aut(X) \cong S_1$ |
| 22. | {∅, {a}, {b}, {a, b}, {a, b, c}, {a, b, c, d}} ($T_0$) | $Aut(X) \cong S_2$ |
| 23. | {∅, {a}, {a, b}, {c}, {a, c}, {a, b, c}, {a, b, d}, {a, b, c, d}} ($T_0$) | $Aut(X) \cong S_1$ |
| 24. | {∅, {a}, {a, b}, {a, c}, {a, b, c}, {a, b, d}, {a, b, c, d}} ($T_0$) | $Aut(X) \cong S_1$ |
| 25. | {∅, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}, {a, b, c, d}} ($T_0$) | $Aut(X) \cong S_2 \times S_2$ |
| 26. | {∅, {a}, {b}, {a, b}, {a, c}, {a, b, c}, {a, b, d}, {a, b, c, d}} ($T_0$) | $Aut(X) \cong S_1$ |
| 27. | {∅, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, d}, {a, b, d}, {a, b, c, d}} ($T_0$) | $Aut(X) \cong S_1$ |
| 28. | {∅, {a}, {a, b}, {a, c}, {a, b, c}, {a, d}, {a, b, d}, {a, c, d}, {a, b, c, d}} ($T_0$) | $Aut(X) \cong S_3$ |
| 29. | {∅, {a}, {b}, {a, b}, {a, c}, {a, b, c}, {a, d}, {a, b, d}, {a, c, d}, {a, b, c, d}} ($T_0$) | $Aut(X) \cong S_2$ |
| 30. | {∅, {a}, {b}, {a, b}, {c}, {a, c}, {b, c}, {a, b, c}, {a, b, d}, {a, b, c, d}} ($T_0$) | $Aut(X) \cong S_2$ |
| 31. | {∅, {a}, {b}, {a, b}, {c}, {a, c}, {b, c}, {a, b, c}, {a, d}, {a, b, d}, {a, c, d}, {a, b, c, d}} ($T_0$) | $Aut(X) \cong S_1$ |
| 32. | {∅, {a}, {b}, {a, b}, {c}, {a, c}, {b, c}, {a, b, c}, {a, b, c, d}} ($T_0$) | $Aut(X) \cong S_3$ |
| 33. | {∅, {a}, {b}, {a, b}, {c}, {a, c}, {b, c}, {a, b, c}, {d}, {a, d}, {b, d}, {a, b, d}, {c, d}, {a, c, d}, {b, c, d}, {a, b, c, d}} ($T_0$) | $Aut(X) \cong S_4$ |

**Theorem 2.1.** *Let $(X, T)$ be a topological space which has an invariant clopen subset $U$ i.e for any $f \in Aut(X)$, $f(U) = U$. Then $\Phi : Aut(X) \longrightarrow Aut(U) \times Aut(X \backslash U), \Phi(f) = (f_{|U}, f_{|X\backslash U})$ is an isomorphism .*

*Proof.* Note that a subset $W$ of $X$ is open if and only if $W \cap U$ and $W \backslash U$ are open. □

**Theorem 2.2.** *If $Aut(X)$ contains a cycle of length three then $S_3 \leq Aut(X)$.*

*Proof.* Let $X = \{1, 2, 3, \cdots\}$ and $f = (1 \quad 2 \quad 3)$ be a cycle of length three in $Aut(X)$. Set $H = \langle f \rangle$. Every element $g$ of $S_3$ acts on $X$ which fixes $X \backslash \{1, 2, 3\}$. Let $U$ be an open set in $X$. Since $H$ is transitive on subsets of the same size in $\{1, 2, 3\}$ so there is an element $h \in H$ such that $g(U \cap \{1, 2, 3\}) = h(U \cap \{1, 2, 3\})$. Also $h(X \backslash \{1, 2, 3\}) = g(X \backslash \{1, 2, 3\}) = X \backslash \{1, 2, 3\}$ since $g$ and $h$ act trivially on $X \backslash \{1, 2, 3\}$. This implies $g(U) = h(U)$. So $g(U)$ is an open subset of $X$. Since $g^6 = id_X$, so $g^{-1} = g^5$ is also open. So $g$ is a homeomorphism and $S_3 \leq Aut(X)$. □

The following corollary is obtained in [5] with a different proof.

**Corollary 2.3.** *If* $Aut(X) \cong \mathbb{Z}_3$ *then* $|X| \geq 6$.

The following example is constructed by methods of [2].

**Example 2.4.** Let $B = \{\{1\}, \{4\}, \{7\}, \{1, 2\}, \{4, 5\}, \{7, 8\}, \{1, 2, 3, 4\}, \{4, 5, 6, 7\}, \{1, 7, 8, 9\}\}$ be a basis for a topology on $X = \{1, 2, \cdots, 9\}$. Then $Aut(X) = \langle(1\ 4\ 7)(2\ 5\ 8)(3\ 6\ 9)\rangle \cong \mathbb{Z}_3$..

**Theorem 2.5.** *Let $G$ and $H$ be two groups:*

*(i) If $H \in T(n)$ then $H \in T(n + 1)$.*

*(ii) If $G \in T(m)$ and $H \in T(n)$ then $G \times H \in T(m + n)$.*

*Proof.* (i) Assume $H \cong Aut(X, T)$ where $X = \{1, \cdots n\}$. Set $Y = \{1, \cdots, n+1\}$ and $T_Y = T \cup \{Y\}$. Then it is easily seen that $Aut(Y, T_Y) \cong Aut(X, T) \cong H$. So $H \in T(n + 1)$.

(ii) Assume $G \cong Aut(X, T_X)$ and $H \cong Aut(Y, T_Y)$ where $|X| = m, |Y| = n$ and $X \cap Y = \emptyset$.. Set $T = T_X \cup \{X \cup A : A \in T_Y\}$. Then $T$ is a topology on $X \cup Y$ and $Aut(X \cup Y, T) \cong G \times H$. Hence $G \times H \in T(m + n)$.

□

**Theorem 2.6.** *Let $\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = X$. Set $T = \{A_0, A_1, \cdots A_n\}$ and $m_i = |A_i \backslash A_{i-1}|$ for each $1 \leq i \leq n$. Then $T$ is a topology and $Aut(X) \cong \Pi_{i=1}^n S_{m_i}$*

**Theorem 2.7.** *Let $(X, T)$ be a topological space.*

*(i) If $|T| = 2$ then $T = \{\emptyset, X\}$ and $Aut(X) \cong S_X$.*

*(ii) If $|T| = 3$ then $T = \{\emptyset, A, X\}$ where $\emptyset \subsetneq A \subsetneq X$. Then $Aut(X) \cong S_A \times S_{X\backslash A}$*

*(iii) If $|T| = 4$ then $T = \{\emptyset, A, B, X\}$ where $\emptyset \subsetneq A \subsetneq B \subsetneq X$ or $T = \{\emptyset, A, X\backslash A, X\}$. In the first case $Aut(X) \cong S_A \times S_{B\backslash A} \times S_{X\backslash B}$. In the second case if $|A| = |X\backslash A|$ then $Aut(X) \cong S_A \times S_{X\backslash A} \times \mathbb{Z}_2$ else $Aut(X) \cong S_A \times S_{X\backslash A}$.*

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# The average character degree and $p$-solvability in a finite group

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**Abstract:** Assume that $G$ is a finite group, $N$ is a non-trivial normal subgroup of $G$ and $p$ is an odd prime. Let $\mathrm{Irr}_p(G) = \{\chi \in \mathrm{Irr}(G) : \chi(1) = 1 \text{ or } p \mid \chi(1)\}$ and $\mathrm{Irr}_p(G|N) = \{\chi \in \mathrm{Irr}_p(G) : N \not\leq \ker\chi\}$. The average character degree of irreducible characters of $\mathrm{Irr}_p(G)$ and the average character degree of irreducible characters of $\mathrm{Irr}_p(G|N)$ are denoted by $\mathrm{acd}_p(G)$ and $\mathrm{acd}_p(G|N)$, respectively. In this talk, we see that if $\mathrm{Irr}_p(G|N) \neq \emptyset$ and $\mathrm{acd}_p(G|N) < \mathrm{acd}_p(\mathrm{PSL}_2(p))$, then $G$ is $p$-solvable and $O^{p'}(G)$ is solvable. We find examples that make this bound best possible. Moreover, we see that if $\mathrm{Irr}_p(G|N) = \emptyset$, then $N$ is $p$-solvable and and for every $P \in \mathrm{Syl}_p(G)$, $P \cap N$ and $PN/N$ are abelian.

**Keywords** $p$-solvable group, average character degree.

**Mathematics Subject Classification (2010) : 20C15, 20D05.**

## 1 Introduction

In this paper, $G$ is a finite group and $p$ is a prime divisor of $|G|$. Let $\mathrm{Irr}(G)$ denote the set of (complex) irreducible characters of $G$. For a normal subgroup $N$ of $G$ and $\theta \in \mathrm{Irr}(N)$, let $\mathrm{Irr}(G|N) = \{\chi \in \mathrm{Irr}(G) : N \not\leq \ker\chi\}$ and $\mathrm{Irr}(\theta^G)$ denote the set of the irreducible constituents of the induced character $\theta^G$. The average character degree of $G$ is denoted by $\mathrm{acd}(G)$ (see [5, 7]) and it is defined as follows:

$$\mathrm{acd}(G) = \frac{\Sigma_{\chi\in\mathrm{Irr}(G)}\chi(1)}{|\mathrm{Irr}(G)|}.$$

By $\mathrm{acd}(G|N)$, we mean the average character degree of the irreducible characters in $\mathrm{Irr}(G|N)$ (see [3]). In recent years, the examination of the relationship between the structure of finite groups and the average character degrees of these groups has attracted the attention of many researchers. In 2013, I. M. Isaacs, M. Loukaki and A. Moretó proved that if $\mathrm{acd}(G) \leq 3$, then $G$ is solvable. Also, let $\mathrm{acd}(G) \leq 3/2$, then $G$ is supersolvable and if $\mathrm{acd}(G) \leq 4/3$, then $G$ is nilpotent. In 2014, A.

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Moretó and H.N. Nguyen improved this bound in [5], demonstrating that if $\mathrm{acd}(G) < 16/5$, then $G$ is solvable. If $\mathrm{acd}_{even}(G) < 18/5$, then $G$ is solvable, where $\mathrm{acd}_{even}(G)$ denotes the average of the degrees of the irreducible characters of $G$ of even degree, by [5]. N. Ahanjideh in [1], it has been shown that if $\mathrm{acd}(G|N) < \max\{\mathrm{acd}(\mathrm{PSL}_2(p), 16/5\}$, then $G$ is $p$-solvable.

We write

$$\begin{aligned} \mathrm{Irr}_p(G) &= \{\chi \in \mathrm{Irr}(G) : \chi(1) = 1 \text{ or } p \mid \chi(1)\} \\ \mathrm{Irr}_p(G|N) &= \mathrm{Irr}_p(G) \cap \mathrm{Irr}(G|N). \end{aligned}$$

Let $\mathrm{acd}_p(G)$ and $\mathrm{acd}_p(G|N)$ be the average degree of irreducible characters belonging to $\mathrm{Irr}_p(G)$ and $\mathrm{Irr}_p(G|N)$, respectively. H.N. Nguyen and P.H. Tiep in [6] have shown that if either $p \geq 5$ and $\mathrm{acd}_p(G) < \mathrm{acd}_p(\mathrm{PSL}_2(p))$ or $p \in \{2, 3\}$ and $\mathrm{acd}_p(G) < \mathrm{acd}_p(\mathrm{PSL}_2(5))$, then $G$ is $p$-solvable and $O^{p'}(G)$ is solvable, where $O^{p'}(G)$ is the minimal normal subgroup of $G$ whose quotient is a $p'$-group. Z. Akhlaghi in [2] has proved that if $N$ is a non-trivial normal subgroup of $G$ with $\mathrm{Irr}_2(G|N) \neq \emptyset$ and $\mathrm{acd}_2(G|N) < 5/2$, then $G$ is solvable.

In [4], we continue the previous investigation and we show that considering the appropriate bound for $\mathrm{acd}_p(G|N)$ instead of $\mathrm{acd}_p(G)$ can leads us to the $p$-solvability of $G$.

Let $f(p) = \mathrm{acd}_p(\mathrm{PSL}_2(p))$ if $p \geq 5$ and otherwise, let $f(p) = \mathrm{acd}_p(\mathrm{PSL}_2(5))$. So,

$$f(p) = \begin{cases} (p+1)/2, & \text{if } p \geq 5 \\ 7/3, & \text{if } p = 3 \\ 5/2, & \text{if } p = 2 \end{cases}.$$

In this talk, we see that fallowing results which come frame [4]:

**Theorem 1.1.** *Let $1 \neq N \trianglelefteq G$ and $p$ be an odd prime divisor of $|G|$. If $G/N$ is not $p$-solvable, then* $\mathrm{acd}_p(\lambda^G) \geq f(p)$ *for every $\lambda \in \mathrm{Irr}(N)$ with* $\mathrm{Irr}_p(\lambda^G) \neq \emptyset$.

**Theorem 1.2.** *Let $p$ be an odd prime and $1 \neq N \trianglelefteq G$ with* $\mathrm{acd}_p(G|N) < f(p)$. *Then either*

(i) *$G$ is $p$-solvable and $O^{p'}(G)$ is solvable;*

(ii) *or* $\mathrm{Irr}_p(G|N) = \emptyset$, *$N$ is $p$-solvable and for every $P \in \mathrm{Syl}_p(G)$, $P \cap N$ and $PN/N$ are abelian.*

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# A different look at the Paul Erdose's problem

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**Abstract:** The class $\mathbb{E}_2(\infty)$ of groups is the class of groups in which the second center has finite index with some properties.

**Keywords** Erdös problem; FC-group; infinite group.

**Mathematics Subject Classification (2010) : 20F99, 20F19, 20E26.**

## 1 Introduction

Paul Erdös posed the following question [5]:
Suppose that every infinite set of elements of a group $G$ contains a pair of elements which commute. Does there exist an upper bound for the order of (finite) subsets of $G$ consisting of pairwise non-commuting elements?
The affirmative answer to this question was given in [5] by B. H. Neumann who proved that an infinite group $G$ is center-by-finite if and only if every infinite subset of $G$ contains two distinct commuting elements.
Since this first paper, problems of a similar nature have been the object of several articles (for example [1]-[5]).

Let $G$ be a group and $\chi$ a class of groups. We say that $G$ satisfies the condition $(\chi, \infty)$ if every infinite subset of $G$ contains a pair of elements which generate a subgroup in the class $\chi$. Also we say that $G$ satisfies condition $\mathbb{E}_2(\infty)$ (or $G$ is in $\mathbb{E}_2(\infty)$ ) if every infinite set of elements of $G$ contains two elements $x, y$ such that

$$x \neq y, \qquad |\langle x, y\rangle'| \leq 2.$$

We denote by $\chi_0$ the class of groups $G$ which has a subgroup $A \leqslant Z_2(G)$ with finite index such that $A^2 \leqslant Z(G)$.

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# 2 Main results

In this section, we state some of new results in order to give an answer to Erdös's question.

**Proposition 2.1.** *Let $G$ be an infinite group.the following statements are equivalent:*

*(i)* *$G$ is an $\mathbb{E}_2(\infty)$-group;*

*(ii)* *Every infinite subset $X$ of $G$ has two distinct elements $x, y$ such that* $[x^2, y] = 1 = [x, y^2] = [x, y]^2 = 1$*;*

*(iii)* *Every infinite subset $X$ of $G$ has two distinct elements $x, y$ such that* $[x, y]^2 = 1 = [x, y, y] = [y, x, x]$.

**Theorem 2.2.** *If $G \in \chi_0$ then $G \in \mathbb{E}_2(\infty)$.*

**Theorem 2.3.** *Let $G$ be a group. If $G^2$ and $\frac{G}{Z(G)}$ are $FC$-groups, then $G \in \chi_0$ or $G \notin \mathbb{E}_2(\infty)$.*

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# On fully simple semihypergroups

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**Abstract:** In this paper, we consider the fundamental relations $\beta$ and $\gamma$ in simple and 0-simple semihypergroups. In particular, we introduce all simple and 0-simple semihypergroups having order 3 where $\beta$ is not transitive, a part of isomorphism. Moreover, we show that the least order for which there exist a strongly simple semihypergroup where $\beta$ is not transitive is 4. Also, we prove that $\gamma$ is transitive in all simple semihypeergroups and determine necessary and sufficient conditions for a 0-simple semihypergroup to have $\gamma$ transitive. Finally, we study the class of fully simple semihypergroups and we completely characterize these semihypergroups in terms of a small set of simple semihypergroups of size three.

**Keywords** Fully simple Semihypergroup; Fundamental Relation; Hypergroup; Simple semihypergroup; 0-simple semihypergroup.

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# Zeros of some irreducible characters in extenstions of Suzuki simple groups

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**Abstract:** Let $S$ be a Suzuki simple group ${}^2B_2(q)$, where $q = 2^{2k+1} \geq 8$ and let $G$ be an almost simple group with socle $S$. In this paper, we determine the number of conjugacy classes of $G$ that are zeros of irreducible characters of prime power degree of $G$.

**Keywords** Suzuki simple group, Conjugacy class, vanishing element, prime power degree.

**Mathematics Subject Classification (2010) : 20D05, 20E45, 20D15.**

## 1 Introduction

Let $G$ be a finite group. A classical theorem of Burnside states that any non-linear irreducible character of $G$ always vanishes on some element of $G$. In [**?**, Theorm A], Malle, Navarro, and Olsson proved that every irreducible character of $p$-power degree of $G$ vanishes on some $p$-element of $G$. Motivated by the result, we define the set $\mathrm{Van}_{ppd}(G)$ as follows:

$$\mathrm{Van}_{ppd}(G) = \{x \in G | \chi(x) = 0 \quad \textit{for some } \chi \in \mathrm{Irr}(G) \text{ of prime power degree}\}$$

where $\mathrm{Irr}(G)$ is the set of all irreducible characters of $G$.

On the other hand, we know that the projective special linear groups $\mathrm{PSL}_2(\mathrm{q})$ and the Suzuki groups ${}^2B_2(q)$ represent two families of non-abelian simple groups with a small number of conjugacy class sizes, distinct element orders, and character degrees. Consider $S$ be a Suzuki simple group ${}^2B_2(q)$, where $q = 2^{2k+1} \geq 8$, and $S \leq G \leq \mathrm{Aut}(S)$. In [1], the author determines the set of character degrees of $G$. In this paper, we find the cardinality of the set of conjugacy classes contained in $\mathrm{Van}_{ppd}(G)$. Additionally, we identify the set of element orders and the number of conjugacy classes of $G$ whenever $2k + 1$ is prime.

**Notation**. We denote the cardinality of the set of conjugacy classes contained in a subset $A$ by $k_G(A)$ and the set of orders of the elements of $G$ by $\omega(G)$.

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## 2 Main results

The results of [3] tells us that if $S$ is a Suzuki simple group ${}^2B_2(q)$ where $q = 2^{2k+1} \geq 8$, then

$$\omega(S) = \{2, 4, \text{all factors of } q-1,\ q-r+1,\ \text{and}\ q+r+1 \text{ where } r = \sqrt{2q}\}$$

and $S$ has $q+3$ distinct conjugacy classes.

**Theorem 2.1.** *Let $S$ be a Suzuki simple group ${}^2B_2(q)$, where $q = 2^{2k+1} \geq 8$ and let $S \leq G \leq \mathrm{Aut}(S)$. Then*

$$k_G(\mathrm{Van}_{ppd}(G)) = 3(2k+1)$$

**Theorem 2.2.** *Let $S$ be a Suzuki simple group ${}^2B_2(q)$, where $q = 2^{2k+1} \geq 8$ and let $S \leq G \leq \mathrm{Aut}(S)$. If $2k+1$ is prime, then the following situation occurs:*

*(1)* $k_G(G) = k_G(S) + k_G(G-S) = [5 + ((q-2)/(2k+1))] + 10k$.

*(2)* $\omega(G) = \omega(S) \cup \{(2k+1), 2(2k+1), 4(2k+1), 5(2k+1)\}$.

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# Derived length and sum of element orderes in groups

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**Abstract:** Given a finite group $G$, write $\psi(G)$ to denote the sum of the orders of the elements of $G$. In this paper, among other things, we show that the derived length of a solvable group $G$ is $\leq \log_{2p-1} \psi(G)$, where $p$ is the smallest prime divisor of $|G|$.

**Keywords** Element orders, solvable group, simple group.

**Mathematics Subject Classification (2010) : 20G40, 20E28.**

## 1 Introduction and results

For a nonempty $X$ of a finite group $G$, define the function

$$\psi(G) = \sum_{x \in X} o(x)$$

where $o(x)$ is the order of $x \in G$. Amiri et al. [1], proved that the maximum value of $\psi$ on groups of the same order occurs at the cyclic group. In fact, they showed that for each positive integer $n$, the cyclic group of order $n$ is uniquely determined up to isomorphism by its order and the sum of the orders of its elements (see also Proposition 2.5, below). For more details, see [2], [4] and [5].

Here we give some sufficient condition on $\psi(G)$ of a group $G$ to be nilpotent or solvable.

**Theorem 1.1.** *Let $G$ be finite group. Then we have*
*1). if $\psi(G) \leq 6p$ where $p$ is the smallest prime divisor of $|G|$, then $G$ is nilpotent.*
*2). if $\psi(G) \leq 210$, then $G$ is solvable and $\psi(A_5) = 211$.*

Finding a upper bound for the solvability length of a solvable group is an important problem in the theory of groups, for example see [9]. Here, we show that the solvability length of a nontrivial solvable group is bounded by a function of $\psi(G)$.

**Theorem 1.2.** *Suppose that $G$ is a solvable group of derived length $d$ and $p$ is the smallest prime divisor of $|G|$. Then $d \leq \log_{2p-1} \psi(G)$.*

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## 2 Some properties of the function $\psi$

**Lemma 2.1.** *Let $G$ be a cyclic group of order $n = \prod_{i=1}^{k} p_i^{m_i}$, where $p_i$ are distinct primes. Then*

$$\psi(G) = \prod_{i=1}^{k} \frac{(p_i^{2m_i+1}) + 1}{p_i + 1}.$$

*Proof.* Since the function $\psi$ is multiplicative, that is, $\psi(H \times K) = \psi(H) \times \psi(K)$ if and only if $(|H|, |K|) = 1$, and clearly $\psi(C_{p^t}) = (p^{2t+1} + 1)/(p + 1)$ where $C_{p^t}$ is a cyclic group of order $p^t$, the result follows. □

**Lemma 2.2.** *Let $G$ be a group and $p$ be the smallest prime divisor of $|G|$. Then*
*1). $p|G| - p + 1 \leq \psi(G)$.*
*2). if $G$ is noncyclic, then $\psi(G) \leq (p + |G|^2 - |G|)/p$.*

*Proof.* 1). Since every nonidentity element of $G$ has order at least $p$, $1 + p(|G| - 1) \leq \psi(G)$ and so $p|G| - p + 1 \leq \psi(G)$.
2). As $G$ is noncyclic, every nonidentity element of $G$ has order at most $|G|/p$. Therefore $\psi(G) \leq 1 + (|G| - 1)|G|/p$ and so $\psi(G) \leq (p + |G|^2 - |G|)/p$. □

The following result may be compared to main results of [1] and [3].

**Proposition 2.3.** *Let $G$ be a noncyclic nilpotent group of order $n = \prod_{i=1}^{k} p_i^{m_i}$, where $p_i$ are distinct primes and $P_1, \cdots, P_t$ are noncyclic Sylow $p_i$-subgroups of $G$, respectively $(1 \leq i \leq t)$. Then*

$$\psi(G) < \prod_{i=1}^{t} (\frac{p_i + 1}{p_i^2}) \psi(C_n).$$

*Proof.* As the function $\psi$ is multiplicative and $G$ is a noncyclic nilpotent group, for proof we may assume that $G$ is a noncyclic group of order $p^m$. Now it is easy to verify that

$$(p + |G|^2 - |G|)/p < \frac{p+1}{p^2}(p^{2m+1} + 1)/(p + 1) = \frac{p+1}{p^2}\psi(C_{p^m}),$$

and so the result follows from Case (2) of Lemma 2.2 and Lemma 2.1.

□

**Corollary 2.4.** *Let $G$ be a noncyclic nilpotent group of odd order and $|G| = n = \prod_{i=1}^{k} {p_i}^{m_i}$, where $p_i$ are distinct primes. Then*

$$\psi(G) < \psi(C_n)/2^t,$$

*where $t$ is the number of noncyclic Sylow $p_i$-subgroups of $G$.*

## 3 Proofs

First, we prove Theorems 1.1

**Proof of Theorem 1.1.**

(1) As $\psi(G) \leq 6p$, by Lemma 2.2, we get $|G| \leq (6p + p - 1)/p < 7$ and the result follows (note that $\psi(S_3) = 13$).

(2) Suppose, on the contrary, that there exists a non-solvable group $G$ such that $\psi(G) \leq 210$. Therefore, there exists $H$ normal in $K$, with $K$ contained in $G$ and $K/H$ nonabelian simple. Then

$$|K/H| < \psi(K/H) \leq \psi(K) \leq \psi(G) \leq 210.$$

The only simple groups with order $< 210$ are $L_2(5)$ and $L_2(7)$, with orders 60 and 168, and the values of $\psi$ for these two groups are 211 and 715, so in neither case do we have $\psi(K/H) \leq 210$. That completes the proof.

Note that it is easy to see that these bounds cannot be improved (in fact, $S_3$ is only non-nilpotent group with $\psi(G) = 13$ and $A_5$ is only non-solvable group with $\psi(G) = 211$).

Subsequently, we prove Theorem 1.2.

**Lemma 3.1.** *Let $G$ be a group and $N$ a nontrivial normal subgroup of $G$. Then*

$$\psi(G) \geq \psi(G/N) + \psi(N) - 1.$$

*Proof.* Clearly $\psi(G) = \psi(G \backslash N) + \psi(N)$. Now as $\psi(G/N) = 1 + \sum_{a \notin N} o(aN)$ and $o(aN) | o(a)$ for every element $a \in G \backslash N$, one can obtain that $\psi(G/N) \geq 1 + \psi(G \backslash N)$. Thus $\psi(G) \geq \psi(G/N) + \psi(N) - 1$. □

**Corollary 3.2.** *Let $G$ be a group. Then for every nontrivial normal subgroup $N$ of $G$, $\psi(G) \neq \psi(G/N)$ and $\psi(G) \neq \psi(N)$.*

**Proof of Theorem 1.2.** Let $p$ be the smallest prime divisor of $|G|$. We argue by induction on solvability length. According to Lemma 2.2, we have $p \leq |G| \leq (\psi(G) + p - 1)/p$ and so $2p - 1 \leq p^2 - p + 1 \leq \psi(G)$. Therefore we may assume that $d \geq 2$. If $G$ is a metabelian group ($d = 2$), then $p^2 + p \leq |G|$. Therefore, according to Lemma 2.2, $p^2 + p \leq (\psi(G) + p - 1)/p$ and so

$$(2p - 1)^2 \leq p^3 + p^2 - p + 1 \leq \psi(G).$$

Thus for $d = 2$, the result holds. So assume that $d \geq 3$ and the result holds for $d - 1$. Now $G/G^{d-1}$ has solvability length $d - 1$. Thus, by the induction hypothesis, we have $d - 1 \leq log_{2q-1}\psi(G/G^{d-1})$, where $q$ is the smallest prime divisor of $|G/G^{d-1}|$. It follows, by Lemma 3.1, that

$$d - 1 \leq \log_{2q-1}(\psi(G) - \psi(G^{d-1}) + 1) < \log_{2q-1}(\psi(G)),$$

and so $d \leq \log_{2p-1} \psi(G)$ (note that $\psi(G^{d-1}) \geq 3$ and $p \leq q$), as wanted.

*Remark* 3.3. If $G$ is a nilpotent group of class $c$ and $p$ is the smallest prime divisor of $|G|$, then by an argument similar to the one in the proof of Theorem 1.2 (by replacing $G/Z(G)$ by the factor group $G/G^{d-1}$, where $Z(G)$ is the center of $G$), we can obtain that

$$c \leq log_{2p-1}\psi(G),$$

and so $d \leq [log_2(log_{2p-1}\psi(G))] + 1$.

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# Independence polynomial of zero divisor graphs of commutative rings and some Cayley graphs

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**Abstract:** Let $i_k$ be the number of independent sets of size $k$ of graph $G$. The independence polynomial of $G$ is $I(G,x) = \sum_{k=0}^{\alpha(G)} i_k x^k$. We study the independence polynomials of zero-divisor graphs of the ring $\mathbb{Z}_n$ where $n \in \{2p, p^2, pq\}$, and $p, q, r$ are primes with $p > q > r > 2$. We also study the independence polynomial of some Cayley graphs.

**Keywords** independence polynommial, Cayley graph, zero divisor.
**Mathematics Subject Classification (2010) : 05C31.**

## 1 Introduction

Let $G = (V, E)$ be an undirected and simple graph. A subset $S \subseteq V$ is independent set if $xy \notin E$ for every $x, y \in S$. The maximum cardinality of independent set is called the independence number of $G$ and is denoted by $\alpha(G)$. The independence polynomial of $G$ is $I(G,x) = \sum_{k=0}^{\alpha(G)} i_k x^k$, where $i_k$ is the number of the independent sets of size $k$ of $G$ ([1]).

Let $R$ be a commutative ring with identity and $Z(R)$ be the set of its zero-divisors. The zero-divisor graph of $R$, denoted by $\Gamma(R)$, is the simple graph with vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices $x$ and $y$ are adjacent if $xy = 0$. That is,

$$V(\Gamma(R)) = Z^*(R) = Z(R) \setminus \{0\}$$
$$E(\Gamma(R)) = \{\{x, y\} \ : \ xy = 0 \ \wedge \ x, y \in V(\Gamma(G))\}$$

An integer $d$ is called a proper divisor of $n$ if $1 < d < n$ and $d|n$. Let $d_1, \ldots, d_k$ be the distinct proper divisors of $n$. For $1 \leq i \leq k$, consider the following sets:

$$V_{d_i} = \{x \in \mathbb{Z}_n \ : \ gcd(x, n) = d_i\}$$

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The sets $V_{d_1}, \ldots, V_{d_k}$ are pairwise disjoint and we can partition the vertex set of $\Gamma(\mathbb{Z}_n)$ as

$$V(\Gamma(\mathbb{Z}_n)) = \bigcup_{i=1}^{k} V_{d_i}.$$

Let $G$ be a finite group and let $S \subseteq G$ be a subset. The corresponding Cayley graph $\Gamma(G) = Cay(G, S)$ has vertex set $G$. Two vertices $g, h \in G$ are joined by an edge if and only if there exists $s \in S$ such that $g = sh$. If $H$ is a graph such that there exists a group $G$ and generating set $S \subseteq G$ with $H \cong \Gamma(G)$, then $H$ is said to be Cayley. Any Cayley graph is a $|S|$-regular graph. If $S$ generates $G$, then $\Gamma(G) = Cay(G, S)$ is connected. We use [2, 3] to obtain more results in this paper.

## 2 Main results

In this section, we study the independence polynomial of zero-divisor graphs of rings $\mathbb{Z}_n$, where $n \in \{2p, p^2, pq, p^2q, pqr, p^\alpha\}$ for distinct prime numbers $p$, $q$ and $r$. We also study the independence polynomial of some Cayley graphs.

**Theorem 2.1.** *If $p$ is a prime number, then*

$$I(\Gamma(\mathbb{Z}_{p^2}); x) = 1 + (p-1)x$$

**Theorem 2.2.** *If $p$ is a prime number, then*

$$I(\Gamma(\mathbb{Z}_{2p}); x) = x + \sum_{k=0}^{p-2} \binom{p-2}{k} x^k.$$

**Theorem 2.3.** *If $p > q > 2$ are prime numbers, then,*

$$I(\mathbb{Z}_{pq}; x) = (1+x)^{p-1} + (1+x)^{q-1} - 1$$

**Theorem 2.4.** *Let $G$ be a group and $H \neq G$ be subgroup of $G$ such that $[G : H] = t$. If $S = G \setminus H$, then*

*(i)* $\alpha(Cay(G, S)) = |H|$.

*(ii)* $I(Cay(G, S), x) = t(1+x)^{|H|} - 1$.

**Theorem 2.5.** *Let $\mathbb{Z}_{2^n}$ and $S = \{k < n : gcd(k, n) = 1\}$. Then*

$$I(Cay(\mathbb{Z}_{2^n} : S), x) = 2(1+x)^{2^{n-1}} - 1.$$

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# The application of Möbius transformations in constructing functional representations of finite groups

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**Abstract:** In this article, a connection between finite groups of low rank and Möbius transformations is proposed, and through this connection, functional representations for a set of finite groups are obtained. Among the well-known finite groups is the set of complex $n$-th roots of unity, and algebraic methods for deriving these roots in lower orders are presented here.

**Keywords** Möbius transformation; functional representation.

**Mathematics Subject Classification (2010) : 05E15, 05C15, 20B25.**

## 1 Introduction

We aim to use Möbius transformations of the form $f(z) = \frac{az+b}{cz+d}$, where $a$, $b$, $c$, and $d$ are real or complex coefficients, to construct representations in the form of a finite set of functions. By considering specific correspondences within the set of functions, we derive functional representations of certain finite groups.

For example, the set of functions

$$\{f_0(z) = z,\ f_1(z) = 1 - z,\ f_2(z) = \frac{1}{z}, f_3(z) = \frac{z}{1-z},\ f_4(z) = \frac{1-z}{z},\ f_5(z) = \frac{1}{1-z}\}, \tag{1.1}$$

where $f_i : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$ are defined for $i = 0, \ldots, 5$, forms a group under the operation of function composition. This group is isomorphic to the symmetric group $S_3$ of order 6 [2]. Here, we have:

$$f_1^2 = f_2^2 = f_3^2 = f_0 = \mathrm{Id}, \tag{1.2}$$

and

$$f_4^3 = f_5^3 = f_0 = \mathrm{Id}. \tag{1.3}$$

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It is known that the set of Möbius transformations of the form $f(z) = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$ (or $ad - bc = 1$), corresponds to the set of matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with the condition $ad - bc \neq 0$ (or $ad - bc = 1$) under the one-to-one mapping:

$$\xi : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{az+b}{cz+d}. \tag{1.4}$$

This mapping is a homomorphism [3]. We note that both sides of this correspondence are groups: Möbius transformations form a group under the operation of function composition, and $2 \times 2$ matrices form a group under matrix multiplication.

On the other hand, for any arbitrary $2 \times 2$ matrix

$$A_{2\times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we have:

$$\xi(-A) = \xi\left(\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}\right) = \frac{-az-b}{-cz-d} = \frac{az+b}{cz+d} = \xi(A). \tag{1.5}$$

Thus, the matrices $A$ and $-A$ are both mapped to the same element of the Möbius transformations [1]. Consequently, the group of Möbius transformations with the condition $ad - bc = 1$ is isomorphic to the group $\frac{\mathrm{SL}(2,\mathbb{R})}{\{\pm I\}}$, where $I$ is the $2 \times 2$ identity matrix, and SL is the special linear group of $2 \times 2$ matrices with real entries, defined as:

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, ad - bc = 1 \right\}. \tag{1.6}$$

## 2 Main results

In this section, using the homomorphism presented in Eq. (1.4), we aim to obtain functional representations of certain finite groups and analyze their geometric and algebraic properties. In this regard, we present an interesting algebraic method for solving the equation $z^n = 1$ for small values of $n$, which ultimately leads to the usual algebraic approaches.

The foundation of this work lies in using the Cayley-Hamilton theorem for matrices, specifically for matrices in the group $\mathrm{SL}(2,\mathbb{R})$. The characteristic equation of a $2 \times 2$ matrix, whose roots are the eigenvalues of the matrix, is given by:

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0. \tag{2.1}$$

If $\det(A) = 1$, the characteristic equation becomes:

$$\lambda^2 - (a+d)\lambda + 1 = 0.$$

For a general matrix equation $\lambda^2 = m\lambda + n$, we have $-n = ad - bc$ and $m = a + d$, and higher powers of $\lambda$ are derived as follows:

$$\lambda^3 = m\lambda^2 + n\lambda = m(m\lambda + n) + n\lambda, \tag{2.2}$$

$$\lambda^4 = (m^3 + 2mn)\lambda + m^2 n + n^2, \tag{2.3}$$

and

$$\lambda^5 = (m^4 + 3m^2 n + n^2)\lambda + n(m^3 + 2mn). \tag{2.4}$$

For example, if $\lambda^3 = 1$, the goal is to find the real solutions of the equation:

$$m^2 + n = 0\ , \quad mn = 1 \implies n = -m^2 \implies -m^3 = 1 \implies m = -1.$$

Thus, $a + d = -1$ and $ad - bc = 1$, leading to:

$$a(-1 - a) - bc = 1 \implies a^2 + a + bc + 1 = 0.$$

Assuming $bc = -3bc = -3$, we have:

$$a^2 + a - 2 = 0 \implies (a - 1)(a + 2) = 0,$$

resulting in:

$$\begin{cases} a = 1 \\ or \\ a = -2 \end{cases} . \tag{2.5}$$

Thus, one of the two matrices

$$\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix},$$

and consequently one of the transformations

$$f(x) = \frac{x + 3}{-x - 2} \quad \text{or} \quad f(x) = \frac{-2x + 3}{-x + 1}$$

will result.

If $\lambda^4 = 1$, then:

$$m^3 + 2mn = 0, \quad n(m^2 + n) = 1. \tag{2.6}$$

Assuming $n = -1$, we find:

$$-1(m^2 - 1) = 1 \implies m^2 - 1 = -1 \implies m = 0, \tag{2.7}$$

which also satisfies the equation $m^3 + 2mn = 0$. Thus, the trace of our matrix must be zero, meaning:

$$m = a + d = 0 \quad \text{and} \quad ad - bd = 1.$$

Given $ad - bc = 1$, we have:

$$ad - bc = 1 \Longrightarrow -a^2 - bc - 1 = 0 \Longrightarrow a^2 + bc + 1 = 0. \tag{2.8}$$

With suitable choices for $a$, $b$, and $c$, different matrices can be obtained. For example, choosing $a = 1$, $b = 2$, and $c = -1$, we have:

$$A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \Longrightarrow \det(A) = 1, \quad \text{trace}(A) = 0. \tag{2.9}$$

Additionally:

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I \Longrightarrow A^4 = I. \tag{2.10}$$

For the corresponding function $f(x) = \frac{x+2}{-x-1}$, it is easily seen that:

$$(f \circ f)(x) = x. \tag{2.11}$$

If in the equation $m^3 + 2mn = 0$, the value of mm is nonzero, then assuming $n = -1$, we have:

$$m^2 = 2, \qquad \text{hence} \qquad m = \pm\sqrt{2}.$$

However, this solution does not satisfy the second equation. Nevertheless, it is a solution to the equation $\lambda^4 = -1$, because:

$$\lambda^4 + 1 = 0 \Longrightarrow \lambda^4 + 2\lambda^2 - 2\lambda^2 + 1 = 0 \Longrightarrow (\lambda^2 + 1)^2 = 2\lambda^2 \Longrightarrow \lambda^2 = \pm\sqrt{2}\lambda - 1.$$

For $\lambda^4 = -1$:

$$m^4 + 3nm^2 + n^2 = 0 \Longrightarrow m^2 = \frac{-3n \pm \sqrt{5n^2}}{2}. \tag{2.12}$$

If $n = -1$, then:

$$m = \frac{1 \pm \sqrt{5}}{2} \qquad \text{or} \qquad m = \frac{-1 \pm \sqrt{5}}{2}. \tag{2.13}$$

The question arises: Do these solutions satisfy the equation?
For $a + d = \frac{1+\sqrt{5}}{2}$, we have:

$$n(m^3 + 2mn) = 1 \Longrightarrow m^3 - 2m = -1. \tag{2.14}$$

# 3 Conclusion

This paper presented a framework to derive functional representations of some finite groups using Möbius transformations. Since these functional representations are generally derived from complex functions, and the structure of fractional linear transformations in the complex numbers has a specific homographic structure in the real case, this method can be analyzed from both the perspectives of complex and real analysis, as well as the geometric forms in both cases.

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# Targeted effective topological complexity of symmetric motion

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**Abstract:** In this lecture, using group action on topological spaces, symmetrical and similar movement of robots has been mathematically modelled. Then, by using this modelling, ways to ease the planning of robot movements have been presented; Including that you can use similar programs for similar and repeated movements. Also, by limiting the movement target points for the robot, using the group action, these programs can be reduced to a smaller number. It is mentioned that the modelling of robot movement using the action of groups in order to stabilize the movement of the robot as much as possible is beneficial for optimizing the programming costs as well as launching and maintaining the robot.

**Keywords** distinguishing number; group actions; subgroup.

**Mathematics Subject Classification (2010) : 05E15, 05C15, 20B25.**

## 1 Introduction and preliminaries

From the year 2000 [1], studies were conducted for robot motion planning using algebraic topology tools by Farber, which was an attempt to optimize robot motion. He has taken a valuable step towards solving this problem by stating the problem of motion planning in terms of mappings between topological spaces. By defining a concept called topological complexity, in addition to providing motion planning in many spaces, Farber also provided ways to minimize the number of programs and make the robot motion more stable.

Following Farber's studies, many mathematicians published and published articles to optimize these calculations . Also, in order to be more compatible in different robot movement environments, studies were also conducted . Among these researches, the type of topological complexity is of interest in this work, which deals with finding similar paths between points. Then, it provides a method to use the same programs to plan the robot's motion in these same paths and reduce the need to define new motion planning. The existence of such similar movements is recognized and corresponded

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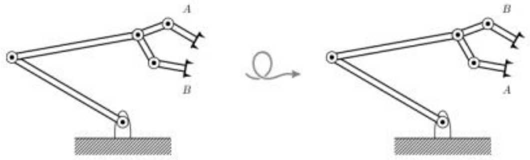
in mathematics with a relationship called symmetry. As a simple example, consider a robot with reciprocating motions. One can model this motion as a movement $s$ and its reverse $-s$ whose scales chosen from group $\mathbb{Z}_2$.




Let $k \geq 1$ be an integer, $G$ a topological group and $X$ be a $G$-space. Assume

$$P_k(X) = \{(1, ..., \gamma_k) \in (PX)^k | G_{\gamma_i(1)} = G_{\gamma_{i+1}(0)} for 1 \leq i \leq k-1\}.$$

Note that $P_1(X) = PX$ and $P_2(X) = PX \times_{X/G} PX$. Define the map $\pi_k : P_k(X) \to X \times X$ by the rule $\pi k(\gamma_1, ..., \gamma_k) = \gamma_1(0), \gamma_k(1)$ for $(\gamma_1, ..., \gamma_k) \in P_k(X)$. This map was defined in [2] as a fibration.

**Definition 1.1** ([2]). (i) A $(G, k)$-motion planner on an open subset $U \subseteq X \times X$ is a section of $\pi_k$ over $U$, i.e. a map $s : U \to P_k(X)$ such that $\pi_k \circ s = i_U$.

(ii) Denote by $TC^{G,k}(X)$ the least integer $k$ greater or equal 1 such that there exists an open cover of $X \times X$ by $k$ sets which admit $(G, k)$-motion planners.

It is clear that $TC^{G,1}(X) = TC(X)$. Note that we do not require $(G, k)$- motion planners to be equivariant, hence $TC^{G,2}(X) \leq TC^G(X)$.

## 2 Main results

In this section we define targeted effective topological complexity to study symmetric motion of robots which go towards a specific goal. The motion space of the robot is denoted by $X$ and the set considered as the target of the movement is denoted by $Y$. The first idea of this notation was defined and studied in [3].

**Definition 2.1.** Let $k \geq 1$ be an integer, $G$ a topological group. Given a $G$-space $X$ and subspace $Y \subseteq X$, write

$$P_k^{X \times Y}(X) = \{(\gamma_1, \gamma_2, ..., \gamma_k) \in (PX)^k : G\gamma_i(1) = G\gamma_{i+1}(0), \gamma_k(1) \in Y for 1 \leq i \leq k-1\}.$$

Define the map $\pi_k^{'} : P_k^{X \times Y}(X) \longrightarrow X \times Y$ by

$$\pi_k^{'}(\gamma_1, \gamma_2, ..., \gamma_k) = (\gamma_1(0), \gamma_k(1)) for (\gamma_1, \gamma_2, ..., \gamma_k) \in P_k^{X \times Y}(X).$$

This map is a fibration because it is pullback of $\pi_k$ by the inclusion map $X \times Y \hookrightarrow X \times X$. Then, $TC^{G,k}(X, Y)$ is the Schwarz genus of $\pi_k^{'}$. in other word, $TC^{G,k}(X, Y) = genus(\pi_k^{'})$

**Proposition 2.2.** *For* $Y \subseteq X$, $TC^{G,k}(X, Y) \leq TC^{G,k}(X)$.

It is clear that $TC^{G,1}(X, Y) = TC(X, Y)$.

**Lemma 2.3.** *The following inequalities hold for any $k \geq 1$ and any subgroup $H \subseteq G$:*

*(1)* $TC^{G,k}(X,Y) \leq TC^{H,k}(X,Y)$,

*(2)* $TC^{G,k}(X,Y) \leq TC^{G,k}(X,Y)$.

**Theorem 2.4.** *Let $Z \subseteq X, Y$. If there exists a G-map $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $fog \simeq id_Y$, then $TC^{G,k}(X,Z) \leq TC^{G,k}(Y,Z)$. In particular, if $X$ and $Y$ are G-homotopy equivalent, then $TC^{G,k}(X,Z) = TC^{G,k}(Y,Z)$.*

Now we prove a result from relative topological complexity concept.

**Theorem 2.5.** *Let $X$ be a topological space and $Z_1, Z_2 \subseteq X$ be subspaces of $X$. If there exist continuous maps $f : Z_1 \longrightarrow Z_2$ and $g : Z_2 \longrightarrow Z_1$ such that $fog \simeq id_{Z_2}$, then $TC(X,Z_2) \leq TC(X,Z_1)$. in particular, if $X$ and $Y$ are homotopy equivalent, then $TC(X,Z_1) = TC(X,Z_2)$.*

**Definition 2.6.** Let $k_0 \geq 1$ be the minimal number such that $TC^{G,i}(X,Y) = TC^{G,i+1}(X,Y)$ for any $i \geq k_0$. we write

$$TC^{G,\infty}(X,Y) = TC^{G,k_0}(X,Y)$$

and it called the *relative effective topological complexity* of $X$.

Note that $TC^{G,\infty}(X,Y)$ there exists because $TC^{G,i}(X,Y)_{i=0}^{\infty}$ is a decreasing sequence that is bounded from below by 1.

**Theorem 2.7.** *If a G-space $X$ is G-contractible, then $TC^{G,\infty}(X,Y) = 1$.*

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# Enumeration $EL$-Hypergroups of Order 3

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**Abstract:** The $EL$-hyperstructures are formulated from (quasi) partially ordered (semi)groups through the application of the "Ends Lemma". In this paper, we enumerate $EL$-hypergroups of order 3, derived from ordered semigroups of order 3. Our approach is straightforward, leading to the identification of 27 distinct $EL$-hypergroups of order 3.

**Keywords** Ends lemma, $EL$-hypergroups, quasi order relation.

**Mathematics Subject Classification (2010) : 16Y99, 20N20.**

## 1 Introduction

The relationship between hyperstructures and ordering has been extensively studied by mathematicians. A specific class of hyperstructures determined by binary relations is known as $EL$-hyperstructures. This concept was first introduced by Chvalina [2] in his investigation of quasi-ordered sets and hypergroups. Further studies on ordered semigroups and ordered groups in connection with $EL$-hyperstructures were conducted by Novak [?, ?, ?].

Ghazavi et al. generalized $EL$-hyperstructures [4]. However, fewer works have focused on enumerating these structures. Nordo employing a program written in PASCAL, calculated that there are 3,999 non-isomorphic hypergroups of the same order [10]. Ghazavi and Mirvakili [5, ?] classified $EL$-(semi)hypergroups with 2 elements.

In this paper, we classify $EL$-hypergroups of order 3 using a straightforward approach that employs two procedures in MATLAB. Initially, we identify all ordered semigroups of order 3. Subsequently, we enumerate the $EL$-hypergroups of order 3, identifying all such hypergroups up to isomorphism and up to equivalent.

A hypergroupoid is a pair $(H, \circ)$ where $H$ is a nonempty set and $\circ : H \times H \longrightarrow P^*(H)$ is a binary hyperoperation. ($P^*(H)$ is the system of all nonempty subsets of $H$). A semihypergroup is an

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associative hypergroupoid, i.e. hypergroupoid satisfying the equality $a \circ (b \circ c) = (a \circ b) \circ c$ for every triad $a, b, c \in H$. If moreover the semihypergroup $H$ satisfies $a \circ H = H = H \circ a$, for all $a \in H$, it is called a *hypergroup*. This condition is known as *reproduction axiom*.

By a quasi (partially) ordered (semi)group, we mean a triple $(S, \cdot, R)$, where $(S, \cdot)$ is a (semi)group and $R$ is a quasi (partially) order relation on $S$ such that for all $x, y, z \in S$ with the property $xRy$ there holds $(x \cdot z)R(y \cdot z)$ and $(z \cdot x)R(z \cdot y)$. This property is known as monotone condition. Moreover, the notation $[x)_R$ used below stands for the set $\{s \in S; xRs\}$ and also $[A)_R = \bigcup_{x \in A} [x)_R$. Similarly, $(x]_R = \{s \in S; sRx\}$ and $(A]_R = \bigcup_{x \in A} (x]_R$. The *EL*-hyperstructures or *Ends lemma* based hyperstructures are hyperstructures constructed from a quasi (partially) ordered (semi)groups using "Ends lemma". This concept was first introduced by Chvalina in 1995 [2]. In particular, Chvalina proved that:

**Lemma 1.1.** *([2], Theorem 1.3) Let $(S, \cdot, R)$ be a partially ordered semigroup. Binary hyperoperation $\circ : S \times S \longrightarrow P^*(S)$ defined by $a \circ b = [a \cdot b)_R = \{x \in S, a \cdot bRx\}$ is associative. The semihypergroup $(S, \circ)$ is commutative if and only if the semigroup $(S, \cdot)$ is commutative.*

**Theorem 1.2.** *([2], Theorem 1.4) Let $(S, \cdot, R)$ be a partially ordered semigroup. The following conditions are equivalent:*
*I) For any pair $(a, b) \in S^2$ there exists a pair $(c, c_1) \in S^2$ such that $(b \cdot c)Ra$ and $(c_1 \cdot b)Ra$.*
*II) The associated semihypergroup $(S, \circ)$ is a hypergroup.*

## 2 Main results

Let $(S, \circ, \leq)$ and $(S', \circ', \leq')$ be two semihypergroups of the same order. A bijection $\sigma : S \to S'$ is an order preserving isomorphism if $\sigma(a \circ b) = \sigma(a) \circ' \sigma(b)$ and $a \leq b$ implies that $\sigma(a) \leq' \sigma(b)$

**Theorem 2.1.** *Let $(S, \cdot, \leq)$ and $(S', \cdot', \leq')$ be order preserving isomorph semigroups then $(S, \circ)$ and $(S', \circ')$ are isomorph semihypergroups.*

**Lemma 2.2.** *All non-empty quasi order relation on the set $H = \{1, 2, 3\}$ are 28 and name them $R_1, R_2, \cdots, R_{28}$.*

Chotchaisthit in [1] used elementary ideas and showed that there are only 24 non-isomorphic semigroups of order 3. Set $S_i = \{1, 2, 3\}$. Although, they are known by their standard names, we name them as $(S_1, \cdot_1), \cdots, (S_{24}, \cdot_{24})$.

There are 24*28=672 triple $(S_i, \cdot_i, R_j)$ for $1 \leq i \leq 24$ and $1 \leq j \leq 28$. But we look after the ones which are quasi ordered semigroups (i.e. which one has the monotone condition). Using the given procedure in appendix A, in MATLAB software, we can see that among all 672 triple $(S_i, \cdot_i, R_j)$ for $1 \leq i \leq 24$ and $1 \leq j \leq 28$ there are 359 ones which satisfies monotone condition.

**Theorem 2.3.** *There are only 27 non-isomorphic EL-hypergroup of order 3 obtain from ordered semigroup of order 3.*

**Theorem 2.4.** *The group $\mathbb{Z}_3$ is an $EL$-hypergroup of order 3 which can be constructed by only one way in $EL$-construction.*

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# Beyond bi-gyrogroups: Introducing bi-gyrosemigroups

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**Abstract:** In this paper, we introduce an algebraic structure termed bi-gyrosemigroups, which extends the concepts of (semi)groups, gyro(semi)groups, and bi-gyrogroups. We explore various examples and properties of this structure.

**Keywords** Semigroup, groupoid, bi-gyrosemigroup, bi-gyrogroup.

**Mathematics Subject Classification (2010) : 20N02; 18B40, 20M75, 20N05, 15A30.**

## 1 Introduction

The concept of gyrogroups, rooted in the algebra of Möbius transformations of the complex unit disc, serves as a natural extension of groups and vector spaces. Gyrogroups can be classified into gyrocommutative and non-gyrocommutative types. Certain gyrocommutative gyrogroups permit scalar multiplication, leading to gyrovector spaces. These structures retain the essence of their classical counterparts while establishing a valuable connection between non-associative algebra and hyperbolic geometry. Einstein's velocity addition law reveals a profound structure characterized as a gyrocommutative gyrogroup and gyrovector space, surpassing 150 years of ongoing investigations since Möbius' initial findings [9]. Ungar's significant contribution to the field of mathematics lies in his work on the generalization of gyrogroups to bi-gyrogroups, [7] and [8]. In particular, he focused on generalized Lorentz transformation groups denoted as $\Gamma = SO(m, n)$, where $m$ and $n$ are natural numbers. These transformation groups possess a distinctive bi-decomposition structure denoted as $\Gamma = H_L B H_R$, where $B$ represents a subset of $\Gamma$, while $H_L$ and $H_R$ denote subgroups within $\Gamma$. This bi-decomposition structure induces a group-like organization for $B$, which is referred to as a bi-gyrogroup. A semigroup is a fundamental construct comprising a set with an associative binary operation, extending groups with applications in automata theory, dynamical systems, algebraic geometry, functional analysis, and probability theory [2, 3, 4]. Asharfi et al. [1, 5] constructed and enumerated gyrogroup of order

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les than 32. Gyrosemigroups introduced by Mirvakili et al. [6] and bi-gyrogroups studied by Suksumran and Ungar [7] extend this knowledge. This study introduces bi-gyrosemigroups, generalizing gyrosemi(groups), (semi)groups, and bi-gyrogroups, exploring various examples and properties of this structure.

## 2 Main results

In this section, we start with the definition of bi-gyrogroups:

**Definition 2.1.** [7] (Bi-gyrogroup). A groupoid $(B, \oplus)$ is a bi-gyrogroup if its binary operation satisfies the following axioms.

(1) There is an element $0 \in B$ such that $0 \oplus a = a \oplus 0 = a$ for all $a \in B$.

(2) For each $a \in B$, there is an element $b \in B$ such that $b \oplus a = 0$.

(3) Each pair of $a$ and $b$ in $B$ corresponds to a left automorphism $lgyr[a, b]$ and a right automorphism $rgyr[a, b]$ in $Aut(B, \oplus)$ such that for all $c \in B$,

$$(a \oplus b) \oplus lgyr[a, b]c = rgyr[b, c]a \oplus (b \oplus c)$$

(4) For all $a, b \in B$,

 (a) $rgyr[a, b] = rgyr[a \oplus b, lgyr[a, b]b]$

 (b) $lgyr[a, b] = lgyr[a \oplus b, rgyr[a, b]b]$

(5) For all $a \in B$, $lgyr[a, 0]$ and $rgyr[a, 0]$ are the identity automorphism of $B$.

We explore and analyze the notion of bi-gyrosemigroup. A groupoid which satisfies the third and forth axioms of Definition 2.1 is called a *bi-gyrosemigroup*.

Next, the bi-gyrosemigroups of order $n$ with non-trivial left and right gyrators.

**Theorem 2.2.** *If $(S = \{1, \ldots, n\}, \oplus)$ be a groupoid with the following Cayley's table and $rgyr[a, b] = F$ for every $a, b \in S$, where $F : S \to S$ is a permutation of order $n$, then $(S, \oplus)$ is bi-gyrosemigroup for every $lgyr[a, b] \in Aut(S, \oplus)$.*

| $\cdot$ | 1 | 2 | $\cdots$ | $n$ |
|---|---|---|---|---|
| 1 | $F(1)$ | $F(1)$ | $\cdots$ | $F(1)$ |
| 2 | $F(2)$ | $F(2)$ | $\cdots$ | $F(2)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $n$ | $F(n)$ | $F(n)$ | $\cdots$ | $F(n)$ |

*Table 1: Bi-gyrosemigroup of order $n$ with non-trivial left and right gyrators*

In the following, two spacial cases of Theorem 2.2 are presented.

**Example 2.3.** Let $S$ be a non-empty set. We define $\oplus : S \times S \to S$ by $a \oplus b = a$, for all $S$. Let $lgyr[a, b]$ be a automorphism of $S$ and $rgyr[a, b]$ the identity automorphism for every $a, b \in S$. Then $rgyr[a, b] = rgyr[a \oplus b, lgyr[a, b]b]$ , $lgyr[a, b] = lgyr[a \oplus b, rgyr[a, b]b]$ and $(a \oplus b) \oplus lgyr[a, b]c = rgyr[b, c]a \oplus (b \oplus c)$, for every $a, b, c \in S$.

**Theorem 2.4.** *If $(G, \cdot)$ is an Abelian group, then $(G, \oplus)$ is a bi-gyrosemigroup, when for every $a, b \in G$, $a \oplus b = a^{-1} \cdot b$, $lgyr[a, b]c = c$ and $rgyr[a, b]c = c^{-1}$.*

**Theorem 2.5.** *Let $(S, \oplus)$ is a groupoid with non-identity automorphism, then there exists a left and right gyrators, $lgyr$ and $rgyr$, such that $(S, \oplus, lgyr, rgyr)$ is not a bi-gyrosemigroup.*

A gyrosemigroup is a bi-gyrosemigroup where $lgyr[a, b]$ is an identity. Gyrosemigroups introduced by Mirvakili and et. al. [6] and they characterize gyrosemigroups of order two up to gyroisomorphism.

**Theorem 2.6.** *If $(S, \oplus, gyr)$ is a gyrosemigroup then it is a bi-gyrosemigroup.*

*Proof.* Set $lgyr[a, b] = id$ and $rgyr[a, b] = gyr[a, b]$ for every $a, b \in S$. It is not difficult to see that $(S, \oplus, lgyr, rgyr)$ is a bi-gyrosemigroup. □

**Example 2.7.** The converse of Theorem 2.6 is not true. Let $S = \{0, 1\}$ by the following Cayley's Table:

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 0 | 0 |

| $lgyr$ | 0 | 1 |
|---|---|---|
| 0 | $A$ | $T$ |
| 1 | $T$ | $A$ |

| $rgyr$ | 0 | 1 |
|---|---|---|
| 0 | $T$ | $T$ |
| 1 | $T$ | $T$ |

where $A$ is identity automorphism and $T$ is permutation $(01)$. This means that $T(0) = 1$ and $T(1) = 0$.

We have $(S, \oplus, lgyr, rgyr)$ is a bi-gyrosemigroup but $(S, \oplus, rgyr)$ is not a gyrosemigroup.

**Theorem 2.8.** *There are exactly 22 non-bi-gyroisomorphic bi-gyrosemigroups of order 2.*

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# On meta and para-$\mathfrak{Nil}$-Hamiltonian groups

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**Abstract:** Let $\mathfrak{Nil}$ be the class of nilpotent groups. This article explores the finiteness of meta and para-$\mathfrak{Nil}$-Hamiltonian groups or their derived subgroups when these groups contain a soluble subgroup of finite index or a non-nilpotent (or insoluble) subgroup of finite order respectively.

**Keywords** Meta-$\mathfrak{Nil}$-Hamiltonian; Para-$\mathfrak{Nil}$-Hamiltonian.

**Mathematics Subject Classification (2010) : 20F19, 20F22.**

## 1 Introduction

Let $\mathfrak{Nil}$ be a class of nilpotent groups. *The group $G$ is said to be meta-$\mathfrak{Nil}$-Hamiltonian if all its non-nilpotent subgroups is normal.* Also, *we say that $G$ is para-$\mathfrak{Nil}$-Hamiltonian if $G$ is a non-nilpotent group and every non-normal subgroup of $G$ is either nilpotent or minimal non-nilpotent.* Also, *$G$ is called biminimal non-$\mathfrak{Nil}$ group if it is neither a nilpotent nor a minimal non-nilpotent group, but each proper subgroup of $G$ either is nilpotent or is a minimal non-nilpotent group.* Para-$\mathfrak{Nil}$-Hamiltonian groups are a natural extension of biminimal non-nilpotent groups.

If $\mathfrak{A}$ is the class of abelian groups, then the class of meta-$\mathfrak{A}$-Hamiltonian is called metahamiltonian and the class of para-$\mathfrak{A}$-Hamiltonian groups is called parahamiltonian. A significant amount of research has been conducted on metahamiltonian groups, which can be referred to [2, 4, 5, 6]. Parahamiltonian groups were first introduced by Atlihan and de Giovanni[1]. They proved that the commutator subgroup of a locally graded parahamiltonian group is finite [1, Lemma 5]. Recall that a group $G$ is known as locally graded if every non-trivial finitely generated subgroup of $G$ contains a proper subgroup of finite index.

In the article [3], it is shown that the para-$\mathfrak{Nil}$-Hamiltonian group $G$, if contains a minimal non-nilpotent subgroup whose normal closure is locally graded, then $G'$ is finite and if $G$ is insoluble, so $G$ is finite [3, Theorem 4.6, Corollary 4.7]. Furthermore, it is shown that any finite insoluble para-$\mathfrak{Nil}$-Hamiltonian group is isomorphic either to $A_5$ or $\mathrm{SL}(2,5)$ ([3, Theorem 3.6]).

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The structure of insoluble meta-$\mathfrak{Nil}$-Hamiltonian groups has been studied in both perfect and non-perfect cases ([3, Propositions 5.8, 5.9, 5.10]). It has also been demonstrated that every locally graded meta-$\mathfrak{Nil}$-Hamiltonian group is solvable [3, Lemma 5.11].

## 2 Main results

Let $G$ be non-nilpotent meta-$\mathfrak{Nil}$-Hamiltonian group and $H$ be a non-normal subgroup of $G$. Then $H$ must be nilpotent. Therefore $G$ is a para-$\mathfrak{Nil}$-Hamiltonian group such that all of whose non-normal subgroups are nilpotent. So we must focus on normal subgroup of $G$, in particular normal subgroup of finite index.

**Lemma 2.1.** *Let $G$ be a finitely generated soluble group with torsion-free abelian subgroup of finite index. Then $G$ is polycyclic and* $\mathrm{Fit}(G)$ *is nilpotent of finite index.*

**Theorem 2.2.** *(i) Let $G$ be a finitely generated non-nilpotent meta-$\mathfrak{Nil}$-Hamiltonian group. If $G$ is a soluble-by-finite, then $G'$ is finite.*

*(ii) Let $G$ be a locally graded meta-$\mathfrak{Nil}$-Hamiltonian group. Then either $G$ is locally nilpotent or $G'$ is finite.*

**Theorem 2.3.** *Let $G$ be a non-nilpotent meta-$\mathfrak{Nil}$-Hamiltonian group. If $G$ is polycyclic-by-finite, then $G'$ is finite, $G$ is finite-by-polycyclic and* $\mathrm{Fit}(G)$ *is nilpotent-by-finite.*

**Theorem 2.4.** *Let $G$ be a finitely generated non-nilpotent meta-$\mathfrak{Nil}$-Hamiltonian group. If $G$ has a torsion-free soluble subgroup of finite index, then:*

*(i) $G'$ is finite and $G$ is soluble;*

*(ii) $G$ is center-by-finite;*

*(iii) $G$ is polycyclic and* $\mathrm{Fit}(G)$ *is nilpotent of finite index;*

*(iv) there are only finitely many maximal subgroups $M$ such that $G = MG'$, obviously $M$ is nilpotent.*

Any locally graded meta-$\mathfrak{Nil}$-Hamiltonian group is soluble by [3, Lemma 5.11], so every finite meta-$\mathfrak{Nil}$-Hamiltonian group is soluble. In the following remark, we examine the insoluble meta-$\mathfrak{Nil}$-Hamiltonian groups.

*Remark* 2.5. Let $G$ be an insoluble meta-$\mathfrak{Nil}$-Hamiltonian group. Then $G$ is infinite.

(a) Assume that $G$ is perfect, then $G$ is minimal non-nilpotent.

(a-i) If $G$ has a maximal subgroup, then $G$ is finitely generated and $G/\Phi(G)$ is non-abelian simple group by [3, propositions 5.8, 5,9]. In additional, by Theorem 2.2 (i), $G$ does not contain any soluble subgroup of finite index, so any maximal subgroup of $G$ is non-normal of infinite index and every finite normal subgroup of $G$ is contained in $\Phi(G)$ and is central by $N/C$-Theorem.

(a-ii) Otherwise $G$ is Fitting $p$-group for some prime $p$ by [7, Theorem 3.3 (i) and (ii)].

(b) Assume that $G$ is not perfect, then $G'' = \gamma_3(G)$ is perfect insoluble and is satisfied (a).

**Theorem 2.6.** *Let $G$ be a finitely generated insoluble meta-$\mathfrak{Nil}$-Hamiltonian. If $G$ is periodic, then:*

*(i) $G'' = \gamma_3(G)$ is perfect minimal non-nilpotent of finite index;*

*(ii) the Frattini factor group of $\gamma_3(G)$ is non-abelian simple group;*

*(iii) $\gamma_3(G)$ is intersection of any subgroup of finite index;*

*(iv) $\gamma_3(G)$ does not any subgroup of finite index.*

Assume that $G$ is a para-$\mathfrak{Nil}$-Hamiltonian group and $H \ntrianglelefteq G$. If $|H|$ is finite then $H$ is soluble, for $H$ is nilpotent or minimal-non-nilpotent. So, all finite insoluble subgroups of $G$ are normal.

**Theorem 2.7.** *Let $G$ be a para-$\mathfrak{Nil}$-Hamiltonian group. If $G$ has an insoluble subgroup of finite order, then $G$ is finite and so $G \cong A_5$ or $\mathrm{SL}(2,5)$.*

**Theorem 2.8.** *Let $G$ be a finitely generated para-$\mathfrak{Nil}$-Hamiltonian group. If $G$ contains a nilpotent subgroup of finite index. Then for some $n$, $\gamma_n(G)$ is finite. In additional, if $G$ is insoluble, then $G$ is finite.*

**Theorem 2.9.** *Let $G$ be a insoluble para-$\mathfrak{Nil}$-Hamiltonian group. Assume that $X \leqslant G$ is a non-nilpotent subgroup of finite order.*

*(i) If $X^G$ the normal closer of $X$ is finite, as well as, $|X^G : X|$ is finite, then $G$ is finite.*

*(ii) If $X \trianglelefteq G$ or $X \ntrianglelefteq G$ but $X \trianglelefteq X^G$, then $G$ is finite.*

*(iii) Assume that $X^G$ is of infinite order, then $\gamma_3(G) = \gamma_\infty(G)$ is finitely generated perfect insoluble group that its Fitting factor is non-abelian simple. In additional, if $\gamma_3(G)$ is minimal non-nilpotent, then $\mathrm{Fit}(\gamma_3(G)) = \Phi(\gamma_3(G))$.*

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# On decomposability of torsion-free abelian groups of rank two

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**Abstract:** This talk is mainly about the decomposability of additive groups of rank two torsion-free rings without zero-divisors. In fact, we show that such additive groups are not necessarily decomposed as a direct sum of two isomorphic groups of rank one.

**Keywords** ring; p-adic; torsion-free.

**Mathematics Subject Classification (2010) : 20K15.**

## 1 Introduction

In the past decades, considerable progress has been made in the construction of rings with given additive group and the related problem of characterizing the additive groups of rings satisfying various conditions. The results for rings over torsion-free abelian groups are limited. In [2], the authors study the rings over torsion-free abelian groups of rank two which yields to a characterization of a group $A$ in terms of groups of rank one. Then, they find a necessary and sufficient condition that there exist a non-commutative ring over $A$ and determine all such rings. Moreover, they prove that a ring $R$ is a ring without zero-divisors over $A$ exactly if it is isomorphic to a subring of a quadratic field extension of the rational numbers. Finally, they conjecture that if $R$ is a rank two torsion-free ring without zero devisors, then $R^+$ decomposes as a direct sum $U \oplus V$, where $U \cong V$ and $V$ is not a nil group. Aghdam and the author study this claim in [1]. In this talk, we present their study which shows that this belief is misplaced.

## 2 Main results

We recall some notions related to the subject under investigation. Given an abelian group $A$, we call $R$ a ring over $A$ if the additive group $R^+ = A$. A subgroup $G$ of $A$ is called $p-$pure exactly if $p^n G = G \cap p^n A$ for all $n \in \mathbb{Z}$; it is called pure if it is $p-$pure for all prime numbers $p$. A system $\{a_1, a_2, \ldots, a_k\}$

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of non-zero elements of a group $A$ is called linearly independent if $n_1a_1+n_2a_2+\ldots+n_ka_k=0$ implies that $n_1a_1=n_2a_2=\ldots=n_ka_k=0$, where $n_i\in\mathbb{Z}$. A system of elements is dependent if it is not independent. By the rank of a group $A$ is meant the cardinal number of a maximal independent system containing only elements of infinite and prime power orders. Let $x$ and $y$ be independent elements of a torsion-free group $A$ of rank two. Each element of $A$ has the unique representation $ux+vy$, where $u$ and $v$ are rational numbers. Let,

$$U_0=\{u_0\in\mathbb{Q}:u_0x\in G\},\quad U=\{u\in\mathbb{Q}:ux+vy\in G\quad\text{for some}\quad v\in\mathbb{Q}\},$$

$$V_0=\{v_0\in\mathbb{Q}:v_0y\in G\},\quad V=\{v\in\mathbb{Q}:ux+vy\in G\quad\text{for some}\quad u\in\mathbb{Q}\}.$$

Then $U_0$ and $V_0$ are subgroups of $U$ and $V$ respectively. The groups, $U,U_0,V$ and $V_0$ are called the groups of rank one belonging to the independent set $\{x,y\}$. Given a prime $p$, the largest integer $k$ such that $p^k$ divides $x$ in the torsion-free group $A$ ($x\in A$) is called the $p-$height , $h_p(x)$, of $x$; if no such maximal integer $k$ exists, then we set $h_p^A(x)=\infty$. Now let $p_1,p_2,\cdots,p_n,\cdots$ be an increasing sequence of all primes. Then, the sequence $\chi_A(x)=(h_{p_1}^A(x),h_{p_2}^A(x),\cdots,h_{p_n}^A(x),\cdots)$, is said to be the height-sequence of $x$. We omit the subscript $A$ if no ambiguity arises. Two height-sequences $\chi=(k_1,k_2,\cdots,k_n,\cdots)$ and $\mu=(l_1,l_2,\cdots,l_n,\cdots)$, will be considered equivalent if $\sum_n|k_n-l_n|$ is finite. An equivalence class of height-sequences is called a type. A torsion-free group $A$ in which all non-zero elements are of the same type is called homogeneous.

Now, we give a brief review about the structure of the ring of $p$-adic integers. Let $p$ be a prime and $\mathbb{Q}_p$ the ring of rational numbers whose denominators are prime to $p$. The non-zero ideals of $\mathbb{Q}_p$ are principal ideals generated by $p^k$ with $k=0,1,\ldots$. If the ideals $(p^k)$ are considered as a fundamental system of neighborhoods of 0, then $\mathbb{Q}_p$ becomes a topological ring, and we may form the completion $\mathbb{Q}_p^*$ of $\mathbb{Q}_p$ in this topology. $\mathbb{Q}_p^*$ is again a ring whose ideals are $(p^k)$ with $k=0,1,\ldots$, and which is complete in the topology defined by its ideals. The elements of $\mathbb{Q}_p^*$ may be represented as $\pi=s_0+s_1p+s_2p^2+\ldots+s_np^n+\ldots$ with $s_n=0,1,\ldots,p-1$. The arising ring $\mathbb{Q}_p^*$ is a commutative domain called the ring of $p-$adic integers.Its cardinality is the power of the continuum. For the additive group of $\mathbb{Q}_p^*$ we shall use the symbol $J_p$. The following result is of utmost importance about the group of $p-$adic integers.

**Theorem 2.1.** *$p-$pure subgroups of the $p-$adic integers are indecomposable.*

*Proof.* See [3, Theorem 88.1]. □

The following result is the starting point of this study about the additive group of torsion-free domains of rank two.

**Theorem 2.2.** *If there exists a ring $R$ without zero-divisors over the torsion-free group $A$ of rank two, then $A$ contains independent elements $x$ and $y$ such that the groups of rank one belonging to $x$ and $y$ satisfy:*

*(i) $U\cong V$ and $U_0\cong V_0$.*

*(ii) none of the groups $U, U_0, V$ and $V_0$ are of nil type.*

*Proof.* See [2, Theorem 5.]. □

The authors in [2] believe that there is a stronger theorem than Theorem 2.2, namely that if $R$ is a ring without zero divisors over $A$, then $A$ decomposes as a direct sum $U \oplus V$, where $U \cong V$ and $V$ is not of nil type. Now, we show that this claim is not true in general. In this context, we give a result at first.

**Theorem 2.3.** *Let $A$ be a finite rank group supporting a ring without divisors of zero, then $A$ is homogeneous.*

*Proof.* See [1, Theorem 5.]. □

**Theorem 2.4.** *Let $\pi$ be a rational element in $\mathbb{Q}_p^*$ such that $\pi^2$ is not rational. Let $A = \langle 1, \pi \rangle^*$ be the pure subgroup of $\mathbb{Q}_p^*$ generated by $\{1, \pi\}$. Then, $A$ is an indecomposable group of rank two.*

*Proof.* We observe that the choice of $\pi$ implies that $A$ is a rank two subring of the ring of $p$-adic integers. On the other hand, by Theorem 2.1, $A$ is indecomposable. □

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# Some well-covered graphs defined by additive groups of finite rings

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**Abstract:** We study when the Cayley graph of the additive group of a finite ring $R$ and a related graph constructed from the additive group of $R$, called the unit graph of $R$, are equal. Then we use this to characterize all finite rings which have a well-covered unit graph.

**Keywords** unitary Cayley graph, well-covered graph, finite ring, finite Abelian group.

**Mathematics Subject Classification (2010) : 05C25, 05C69, 15B33, 16P10, 05E40.**

## 1 Introduction

Throughout this paper, all groups and rings are assumed to be finite. Groups are also assumed to be Abelian unless explicitly stated otherwise and rings are associative and with a nonzero identity. Also $R$ always denotes a ring and $G$ is a group. Additionally, all graphs considered here are simple and undirected.

Many interesting properties of rings can be stated in terms of the additive group or the multiplicative semigroup of the of the ring and hence can be studied from the view point of group theory. For example, factorization properties of elements of $R$ are defined just using the multiplicative semigroup $(R, \cdot)$ and these properties are also studied extensively from a semigroup view point (see, for example, [4], and the references therein). Another example is to use the rich theory of Cayley graphs of groups (see, for example, the survey paper [3]) to study properties of rings. Recall that if $G$ is an Abelian group and $S \subseteq G$ with $S = -S$, then the (undirected) *Cayley graph* of $G$ with respect to $S$, denoted by $\mathrm{Cay}(G, S)$ is the graph with elements of $G$ as its vertices where $x, y \in G$ are adjacent if and only if $x - y \in S$. Now, one can consider Cayley graphs of $(R, +)$ with respect to certain subsets of $R$ which have ring theoretic importance, such as the set of unit elements $\mathrm{U}(R)$ of $R$, or the zero-divisors $\mathrm{Zd}(R)$ of $R$, see for example [1, 6].

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We study the graph $\mathrm{Cay}(R, \mathrm{U}(R))$ which is called the unitary Cayley graph of $R$ and a related graph which we denote by $\mathrm{UG}(R)$ and which is again defined using the additive group of $R$. The latter graph was first introduced by Grimaldi in [2] for $\mathbb{Z}_n$, the group of integers module $n$, with $S$ as the set of elements of $\mathbb{Z}_n$ which are coprime to $n$, with a definition similar to $\mathrm{Cay}(\mathbb{Z}_n, S,)$ with just one difference: in the definition of the adjacency of the vertices, he used $+$ instead of $-$. More generally, the graph $\mathrm{UG}(R)$, called the *unit graph* of $R$, is the (undirected) graph with vertex set $R$, where two distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in \mathrm{U}(R)$. Note that the only difference of $\mathrm{Cay}(R, \mathrm{U}(R))$ and $\mathrm{UG}(R)$, is the $+$ or $-$ in the definition of the adjacency of the vertices.

Here, we will characterize when $\mathrm{Cay}(R, \mathrm{U}(R))$ and $\mathrm{UG}(R)$ are equal. Moreover, we use this to characterize all rings $R$, such that $\mathrm{UG}(R)$ is well-covered. Recall that a graph is called *well-covered*, if all its maximal independent sets are of the same size. Well-covered graphs have been extensively studied in graph theory from various perspectives, including algorithmic, algebraic, and structural characterizations (see, for example, [5]).

## 2 The main results

In our first main theorem, we present a necessary and sufficient condition under which unit graphs and unitary Cayley graphs coincide. Here $J(R)$ denotes the Jacobson radical of $R$, that is, the intersection of all maximal left (or equivalently, right) ideals of $R$.

**Theorem 2.1.** *If $R$ is a finite ring, then* $\mathrm{Cay}(R, \mathrm{U}(R)) = \mathrm{UG}(R)$ *if and only if* $\mathrm{char}\,(R/J(R)) = 2$.

Now, we aim to characterize finite rings for which the unit graph is well-covered. We start with the following lemma.

**Lemma 2.2.** *Suppose $R$ is a finite ring with $2 \notin \mathrm{U}(R)$. Then, $\mathrm{UG}(R)$ is a well-covered graph if and only if so is $\mathrm{UG}((R/J(R)))$.*

It is noteworthy that this correspondence does not hold for unit graphs in general. For instance, $\{1, 2\}$ forms a maximal independent set of $\mathrm{UG}(\mathbb{Z}_3)$, but its inverse image in $\mathrm{UG}(\mathbb{Z}_9)$, namely $\{1, 2, 4, 5, 7, 8\}$, is not even an independent set. Indeed, we show that $\mathrm{UG}(R)$ is not well-covered whenever $2 \in \mathrm{U}(R)$. First, we deal with rings of matrices. Recall that a diagonal matrix whose entries are $+1$ or $-1$ is called a *signature matrix*.

**Lemma 2.3.** *Let $F$ be a finite field with $\mathrm{char}\,(F) \neq 2$ and let $n$ be a positive integer. Then, $\S_n = \{\mathrm{diag}(a_1, \ldots, a_n) \mid a_i = \pm 1\}$ is a maximal independent set of $\mathrm{UG}(M_n(F))$.*

In the above Lemma, we constructed a maximal independent set of $\mathrm{UG}(M_n(F))$, whose size is a power of two. On the other hand, it is not hard to see that the family of matrices in $M_n(F)$ whose first row is zero forms a maximal independent set of size $|F|^{n^2-n}$, which is an odd number. Consequently, we obtain the following result.

**Corollary 2.4.** *For any finite field $F$ and positive integer $n$, if $\mathrm{char}\,(F) \neq 2$, then $\mathrm{UG}(M_n(F))$ is not well-covered.*

From this, we can derive our second main theorem.

**Theorem 2.5.** *If $R$ is a finite ring and $2 \in \mathrm{U}(R)$, then $\mathrm{UG}(R)$ is not well-covered.*

Using Theorems 2.1 and 2.5 and Lemma 2.2, it is not hard to deduce the following Corollary.

**Corollary 2.6.** *Let $R$ be a finite ring. Then, the unit graph of $R$ is well-covered if and only if the unitary Cayley graph of $R$ is well-covered and* $\mathrm{char}\,(R/J(R)) = 2$.

**Corollary 2.7.** *Let $R$ be a finite ring. Then, the unit graph $\mathrm{UG}(R)$ is well-covered if and only if either*

*(i) $R/J(R)$ is isomorphic to one of $F$, $F \times F$, or $M_2(F)$ for some finite field $F$ with characteristic $2$;*

*(ii) or $R/J(R)$ is isomorphic to $\mathbb{Z}_2^k$ for some $k \in \mathbb{N}$.*

In addition to examples provided in [6, Example 3.6], we present several non-commutative rings for which the unit graph is well-covered. Recall that by letting $F$ be a finite field and $G$ a finite group, a *group algebra* $F[G]$, is defined as the ring of all (formal) linear combinations of elements of $G$ with coefficients in $F$ with addition and multiplication defined as

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g) g$$
$$(\sum_{g\in G} a_g g) \cdot (\sum_{g\in G} b_g g) = \sum_{g\in G} (\sum_{x\in G} a_x b_{x^{-1}g}) g,$$

where $a_g, b_g \in F$ for each $g \in G$.

**Example 2.8.** Let $F$ be a finite field with characteristic 2 and $G$ a finite non-abelian 2-group. Consider the group algebra $R = F[G]$, which is a non-commutative ring. It is known that $R$ is a local ring and $R/J(R) \simeq F$. Moreover, by considering $R' = R \times R$, we observe that $R'/J(R') \simeq R/J(R) \times R/J(R) \simeq F \times F$. Furthermore, if we assume that $S = (\mathbb{Z}_2[G])^k$ for some $k \in \mathbb{N}$, then $S/J(S) \simeq \mathbb{Z}_2^k$. Also, by letting $T = M_2(R)$, we infer that $T/J(T) \simeq M_2(R/J(R)) \simeq M_2(F)$. Therefore, for all the rings $R$, $R'$, $S$ and $T$, the unit graph is well-covered. The key step in this example is to find a finite non-commutative local ring with characteristic two, so any other ring with this property would provide a new family of examples.

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# The 2-nilpotent multiplier of $n$-Lie algebras

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**Abstract:** In this paper, we remind the concept of $c$-nilpotent multiplier and the formula for calculating the number of basic commutators in $n$-Lie algebras. Then we give the structure of 2-nilpotent multiplier of the direct sum of two $n$-Lie algebras. Next we calculate the dimension of 2-nilpotent multiplier of every abelian $n$-Lie algebras and Heisenberg $n$-Lie algebras $H(n, m)$, and then give a bound for a dimension of 2-nilpotent multiplier of any nilpotent $n$-Lie algebras of class 2. Finaly, we obtain a bound for the dimension of $\gamma_3(L)$, where $L$ is an $n$-Lie algebras with $\dim L/Z_2(L) = d$.

**Keywords** $n$-Lie algebra, Basic commutators, Free $n$-Lie algebras, 2-nilpotent multiplier, Heisenberg $n$-Lie algebras.

**Mathematics Subject Classification (2010) : 17B30; Secondary 17B05, 17B60.**

## 1 Introduction

Lie polynomials apeared at the end of 19$^{th}$ century and the begining of the 20$^{th}$ century in the work of Campbell, Baker and Hausdorff on exponential maping in a Lie group, which has lead to the so called Campbell-Baker-Hausdorff formula. Around 1930, E. Witt introduced the Lie algebra of Lie polynomials and showed that the Lie algebra of Lie polynomials is actally the free Lie algebra, and that its enveloping algebra is the associative algebra of noncommutative polynomials. He proved what is now called the Poincare-Birkhoff-Witt theorem, and showed how he free Lie algebra is related lower central series of the free group. About at the same time P. Hall and Magnus, with their commuutator calculus, opened the way to bases of the free Lie algebra, by M. Hall.

The consept of basic commutators is defined in groups and Lie algebras and there is also a way to construct and identify them. Moreover, a formula for calculating their number is obtained.

In 1962, A.I. Shirshov [6] gave a method that generalizes Hall's method for choosing a basis in a free Lie algebra.

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Basic commutators are of particular importance in calculating the dimensions of different spaces and is therefore highly regarded. P. Niroomand and M. Parvizi in [4], obtained more results about 2-nilpotent multiplier $\mathcal{M}^{(2)}(L)$ of a finite dimensional nilpotent Lie algebra $L$ and by using the Witt formula.

In 1985, Filippov [2] introduced the concept of *n-Lie algebras*, as an $n$-ary multilinear and skew-symmetric operation $[x_1, \ldots, x_n]$, which satisfies the following generalized Jacobi identity

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n].$$

Clearly, such an algebra becomes an ordinary Lie algebra when $n = 2$.

Let $L$ be an $n$-Lie algebra over a field $\mathbb{F}$ with a free presentation

$$0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0,$$

where $F$ is a free $n$-Lie algebra. Then the $c$-nilpotent multiplier $\mathcal{M}^{(c)}(L)$ of $L$ is defined as

$$\mathcal{M}^{(c)}(L) := \frac{R \cap F^{c+1}}{\gamma_{c+1}[R, F, \ldots, F]}.$$

So far, several studies have been done in case $n = 2$, that is, for Lie algebras. For more information, you can refer to [1, 4]

In [5], the authors introduced the concept of free $n$-Lie algebras and then defined the concept of basic commutators of weight $w$ in $d$-dimensional $n$-Lie algebras and also proved some of its properties and found a formula to calculate the number of them. In the sequal, we review it.

**Theorem 1.1** ([5]). *Let the set $X = \{x_i | x_{i+1} > x_i;\ i = 1, 2, \ldots, d\}$ be an ordered set and a basis for the free $n$-Lie algebra $F$ and $w$ be an positive integer. Then the number of basic commutators of weight $w$ is*

$$l_d^n(w) = \sum_{j=1}^{\alpha_0} \beta_{j^*} \left( \sum_{i=2}^{w-1} \alpha_i \binom{\binom{d}{n-1}}{w-i} \right), \tag{1.1}$$

*where $\alpha_0 = \binom{d-1}{n-1}$, $\alpha_i$, $(2 \leq i \leq w-1)$ is the coefficient of the $(i-2)$-th term in binomial expansion $(a+b)^{w-3}$, (i.e. $\alpha_i = \binom{w-3}{i-2}$) and if $\binom{k-1}{n-1} + 1 \leq j \leq \binom{k}{n-1}$, (for $k = n-1, n, n+1, n+2, \ldots, d-1$), then $j^* = \binom{k-1}{n-1} + 1$ and $\beta_{j^*} = (d - n - j^* + 2)$.*

**Corollary 1.2** ([5]). *If $n = d$, then*

*(i)* $l_n^n(1) = n$.

*(ii)* $l_n^n(2) = \binom{n}{n} = 1$.

*(iii)* $l_n^n(3) = \binom{n}{n-1} = n$.

*(iv)* $l_n^n(4) = \binom{n}{2} + n$.

**Theorem 1.3** ([5]). *Let $F$ be a free $n$-Lie algebra and $F^i$ be the $i$-th term of the lower central series of $F$, for each $i \in \mathbb{N}$. Then $\frac{F^i}{F^{i+c}}$ is abelian of dimension $\sum_{j=0}^{c-1} l_d^n(i+j)$ where $c = 1, 2, \ldots$.*

## 2 2-multipliers of $n$-Lie algebras

The next theorem plays an essential role in proving the main results of this paper. In this theorem, we obtain the 2-nilpotent multiplier of the direct sum of two $n$-Lie algebras.

**Theorem 2.1.** *Let $L$ and $M$ be two finite dimensional $n$-Lie algebras. Then*

$$\mathcal{M}^{(2)}(L \oplus M) \cong \mathcal{M}^{(2)}(L) \oplus \mathcal{M}^{(2)}(M) \oplus ((L^{ab} \otimes^n_{\text{mod}} L^{ab}) \otimes^n_{\text{mod}} M^{ab}) \oplus ((M^{ab} \otimes^n_{\text{mod}} M^{ab}) \otimes^n_{\text{mod}} L^{ab}).$$

Note that since every one dimensional $n$-Lie algebra $L$ is abelian and hence isomorphic to $A(1)$, so by the definition of $c$-nilpotent multiplier of $n$-Lie algebras, $\mathcal{M}^{(2)}(L) = 0$.

In the sequal, we state a result that is also proved in Lie algebras.

**Theorem 2.2.** *Let $L$ be an abelian $n$-Lie algebra with the finite dimensional $d$. Then* $\dim \mathcal{M}^{(c)}(L) = l^n_d(c+1)$*. In particular,* $\dim \mathcal{M}(L) = l^n_d(2) = \frac{1}{2}d(d-1)$*.*

Eshrati and et.al. in [1] prove that every finite dimensional nilpotent $n$-Lie algebra can be decomposed into the direct sum of a Heisenberg $n$-Lie algebra and an abelian $n$-Lie algebra. So in order to get the 2-nilpotent multiplier of every $n$-Lie algebra, we must first calculate the 2-multiplier of each Heisenberg $n$-Lie algebra. In the following two theorems, we identify its 2-nilpotent multiplier.

**Theorem 2.3.** *Let $H(n,1)$ be a Heisenberg $n$-Lie algebra of dimension $n+1$. Then*

$$\mathcal{M}^{(2)}(H(n,1)) \cong A\left(\frac{n^2+3n}{2}\right).$$

We know that a Heisenberg $n$-Lie algebra $H(n,m)$ is capable if and only if $m = 1$. Thus the following theorem can be proved for non-capable Heisenberg $n$-Lie algebra.

**Theorem 2.4.** *Let $H(n,m)$ be a Heisenberg $n$-Lie algebra of dimension $mn+1$, with $m \geq 2$. Then*

$$\mathcal{M}^{(2)}(H(n,m)) \cong A(l^n_{mn}(3)).$$

In the following theorem, we determine the 2-nilpotent multiplier of $n$-Lie algebras with one-dimensional derived subalgebra.

**Theorem 2.5.** *Let $L$ ba an $n$-Lie algebra of dimension $d$ and with* $\dim L^2 = m$*. Then*

$$\mathcal{M}^{(2)}(L) = \begin{cases} A\left(\frac{n^2+3n}{2} + l^n_{d-n-1}(3) + \sum_{i=1}^{n-1} n^{ni}(d-n-1)^{n-i} + (d-n-1)^{ni}n^{n-i}\right), & m = 1, \\ A\left(l^n_{mn}(3) + l^n_{d-mn-1}(3) + \sum_{i=1}^{n-1} (mn)^{ni}(d-mn-1)^{n-i} + (d-mn-1)^{ni}(mn)^{n-i}\right), & m \geq 2. \end{cases}$$

The next theorem is the main result of this section.

**Theorem 2.6.** *Let $L$ be a nilpotent $n$-Lie algebra of dimension $d$, with* $\dim L^2 = k \geq 1$*. Then*

$$\dim \mathcal{M}^{(2)}(L) \leq \frac{n^2+3n}{2} + l^n_{d-n-k}(3) + \sum_{i=1}^{n-1} n^{ni}(d-n-k)^{n-i} + (d-n-k)^{ni}n^{n-i} + (d-k)^{2n-2} - k + 1.$$

The following theorem is similar to the Lemma 4.2 of [3]. Moneyhun prove that if $\dim L/Z(L) = d$, then $\dim L^2 = \dim \gamma_2(L) \leq \frac{1}{2}d(d-1) = l_d(2)$.

**Theorem 2.7.** *Let $L$ be an $n$-Lie algebra such that $\dim L/Z_2(L) = d$. Then the dimension of $\gamma_3(L)$ is at most $l_d^n(3)$.*

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# The dimension of Schur multiplier and nonabelian tensor square of some nilpotent $n$-Lie algebras

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**Abstract:** Let $L$ be an $n$-Lie algebra over a field $\mathbb{F}$. In this paper, we state some bounds for the dimension of the nonabelian tensor square $L \otimes L$ and the Schur multiplier $\mathcal{M}(L)$ of a given $n$-Lie algebra $L$ as well as the dimension of $L \wedge L$, the non-abelian exterior (wedge) product of $L$.

**Keywords** nonabelian tensor square; Schur multiplier; nilpotent $n$-Lie algebras.

**Mathematics Subject Classification (2010) : 19C09, 17B99.**

Lie polynomials apeared at the end of 19$^{th}$ century and the begining of the 20$^{th}$ centuryy in the work of Campbell, Baker and Hausdorff on exponential maping in a Lie group, which has lead to the so called Campbell-Baker-Hausdorff formula. Around 1930, E. Witt introduced the Lie algebra of Lie polynomials and showed that the Lie algebra of Lie polynomials is actally the free Lie algebra, and that its enveloping algebra is the associative algebra of noncommutative polynomials. He proved what is now called the Poincare-Birkhoff-Witt theorem, and showed how he free Lie algebra is related lower central series of the free group. About at the same time P. Hall and Magnus, with their commuutator calculus, opened the way to bases of the free Lie algebra, by M. Hall. For more details about a historical review of free Lie algebras, we refer the reader to the reference [5] and the references therein.

The consept of basic commutators is defined in groups and Lie algebras and there is also a way to construct and identify them. Moreover, a formula for calculating their number is obtained.

In 1962, A.I. Shirshov [7] gives a method that generalizes Hall's method for choosing a basis in a free Lie algebra.

Basic commutators is of particular importance in calculating the dimensions of different spaces and is therefore highly regarded. P. Niroomand and M. Parvizi in [4], investigate to obtain some more results about 2-nilpotent multiplier $\mathcal{M}^{(2)}(L)$ of a finite dimensional nilpotent Lie algebra $L$ and by using the Witt formula, calculate its dimension.

In 1985, Filippov [2] introduced the concept of *$n$-Lie algebras*, as an $n$-ary multilinear and skew-

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symmetric operation $[x_1, \ldots, x_n]$, which satisfies the following generalized Jacobi identity

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n].$$

Clearly, such an algebra becomes a ordinary Lie algebra when $n = 2$.

Let $L$ be an $n$-Lie algebra over a field $\mathbb{F}$ with a free presentation

$$0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0,$$

where $F$ is a free $n$-Lie algebra. Then the $c$-nilpotent multiplier $\mathcal{M}^{(c)}(L)$ of $L$ is defined as

$$\mathcal{M}^{(c)}(L) := \frac{R \cap F^{c+1}}{\gamma_{c+1}[R, F, \ldots, F]}.$$

So far, several studies have been done in case $n = 2$, that is, for Lie algebras. For more information, you can refer to [?, 1, 4]

In [6], the authors introduced the concept of free $n$-Lie algebras and then defined the concept of basic commutators of weight $w$ in $d$-dimensional $n$-Lie algebras and also proved some its properties and the formula to calculate the number of them. In the sequal, we review it.

**Theorem 0.8** ([6]). *Let the set $X = \{x_i | x_{i+1} > x_i;\ i = 1, 2, \ldots, d\}$ be an ordered set and a basis for the free $n$-Lie algebra $F$ and $w$ be an positive integer number. Then the number of basic commutators of weight $w$ is*

$$l_d^n(w) = \sum_{j=1}^{\alpha_0} \beta_{j^*} \left( \sum_{i=2}^{w-1} \alpha_i \binom{\binom{d}{n-1}}{w-i} \right), \tag{0.1}$$

*where $\alpha_0 = \binom{d-1}{n-1}$, $\alpha_i$, $(2 \le i \le w-1)$ is the coefficient of the $(i-2)$-th sentence in Newton's binomial expansion $(a+b)^{w-3}$, (i.e. $\alpha_i = \binom{w-3}{i-2}$) and if $\binom{k-1}{n-1} + 1 \le j \le \binom{k}{n-1}$, (for $k = n-1, n, n+1, n+2, \ldots, d-1$), then $j^* = \binom{k-1}{n-1} + 1$ and $\beta_{j^*} = (d - n - j^* + 2)$.*

**Corollary 0.9** ([6]). *If $n = d$, then*

*(i)* $l_n^n(1) = n$.

*(ii)* $l_n^n(2) = \binom{n}{n} = 1$.

*(iii)* $l_n^n(3) = \binom{n}{n-1} = n$.

*(iv)* $l_n^n(4) = \binom{n}{2} + n$.

**Theorem 0.10** ([6]). *Let $F$ be a free $n$-Lie algebra and $F^i$ be the $i$-th term of the lower central series of $F$, for each $i \in \mathbb{N}$. Then $\frac{F^i}{F^{i+c}}$ is the abelian of dimension $\sum_{j=0}^{c-1} l_d^n(i+j)$ where $c = 1, 2, \ldots$.*

# 1 2-multipliers of $n$-Lie algebras

The next theorem plays an essential role in proving the main results of this paper. In this theorem, we get the 2-nilpotent multiplier of the direct sum of two $n$-Lie algebras.

**Theorem 1.1.** *Let $L$ and $M$ be two finite dimensional $n$-Lie algebras. Then*

$$\mathcal{M}^{(2)}(L\oplus M)\cong \mathcal{M}^{(2)}(L)\oplus \mathcal{M}^{(2)}(M)\oplus ((L^{ab}\otimes^n_{\text{mod}} L^{ab})\otimes^n_{\text{mod}} M^{ab})\oplus ((M^{ab}\otimes^n_{\text{mod}} M^{ab})\otimes^n_{\text{mod}} L^{ab}).$$

Note that since every one dimensional $n$-Lie algebra $L$ is abelian and hence isomorphic to $A(1)$, so by the definition of $c$-nilpotent multiplier of $n$-Lie algebras, $\mathcal{M}^{(2)}(L)=0$.

In the sequal, we state a result that is also proved in Lie algebras.

**Theorem 1.2.** *Let $L$ be an abelian $n$-Lie algebra with the finite dimensional $d$. Then* $\dim \mathcal{M}^{(c)}(L)= l^n_d(c+1)$. *In particular,* $\dim \mathcal{M}(L)=l^n_d(2)=\frac{1}{2}d(d-1)$.

Eshrati and et.al. in [1] prove that every finite dimensional nilpotent $n$-Lie algebra can be decomposed into the direct sum of one Heisenberg $n$-Lie algebra and one abelian $n$-Lie algebra. So in order to get the 2-nilpotent multiplier of each $n$-Lie algebra, we must first calculate the 2-multiplier of each Heisenberg $n$-Lie algebra. In the following two theorems, we identify its 2-nilpotent multiplier.

**Theorem 1.3.** *Let $H(n,1)$ be a Heisenberg $n$-Lie algebra of dimension $n+1$. Then*

$$\mathcal{M}^{(2)}(H(n,1))\cong A\left(\frac{n^2+3n}{2}\right).$$

We know that a Heisenberg $n$-Lie algebra $H(n,m)$ is capable if and only if $m=1$, and if $m\geq 2$, then $H(n,m)$ is not capable. Thus the following theorem can be proved for non-capable Heisenberg $n$-Lie algebra.

**Theorem 1.4.** *Let $H(n,m)$ be a Heisenberg $n$-Lie algebra of dimension $mn+1$, with $m\geq 2$. Then*

$$\mathcal{M}^{(2)}(H(n,m))\cong A(l^n_{mn}(3)).$$

In the following theorem, we determine the 2-nilpotent multiplier of $n$-Lie algebras with one-dimensional derived subalgebra.

**Theorem 1.5.** *Let $L$ ba an $n$-Lie algebra of dimension $d$ and with* $\dim L^2=m$. *Then*

$$\mathcal{M}^{(2)}(L)=\begin{cases} A\left(\dfrac{n^2+3n}{2}+l^n_{d-n-1}(3)+\displaystyle\sum_{i=1}^{n-1} n^{ni}(d-n-1)^{n-i}+(d-n-1)^{ni}n^{n-i}\right) & m=1\\ A\left(l^n_{mn}(3)+l^n_{d-mn-1}(3)+\displaystyle\sum_{i=1}^{n-1}(mn)^{ni}(d-mn-1)^{n-i}+(d-mn-1)^{ni}(mn)^{n-i}\right) & m\geq 2\end{cases}$$

The next theorem is the main result of this section.

**Theorem 1.6.** *Let $L$ be a nilpotent $n$-Lie algebra of dimension $d$, with* $\dim L^2=k\geq 1$. *Then*

$$\dim \mathcal{M}^{(2)}(L)\leq \frac{n^2+3n}{2}+l^n_{d-n-k}(3)+\sum_{i=1}^{n-1} n^{ni}(d-n-k)^{n-i}+(d-n-k)^{ni}n^{n-i}+(d-k)^{2n-2}-k+1.$$

The following theorem is similar to the Lemma 4.2 of [3]. Moneyhun prove that if $\dim L/Z(L)=d$, then $\dim L^2=\dim \gamma_2(L)\leq \frac{1}{2}d(d-1)=l_d(2)$.

**Theorem 1.7.** *Let $L$ be an $n$-Lie algebra such that* $\dim L/Z_2(L)=d$. *Then the dimension of $\gamma_3(L)$ is at most $l^n_d(3)$.*

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# Some bounds for dimension the Schur multiplier of nilpotent $n$-Lie algebras

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**Abstract:** We give a bound on the dimension of the Schur multiplier of a finite dimensional nilpotent $n$-Lie algebra and the structure of all nilpotent $n$-Lie algebras that attain this bound. Also, we obtain a new bound for the Schur multiplier of a $d$-dimensional $c$-step nilpotent $n$-Lie algebr with the derived subalgebra of dimention $d-n$ which sharpens the earlier known bounds.

**Keywords** nilpotent $n$-Lie algebra, Schur multiplier, capable $n$-Lie algebra, maximal class.

**Mathematics Subject Classification (2010) : 17B05, 17B30, 17B99.**

## 1 Introduction

In 1985, Filippov [5] introduced the concept of $n$-Lie algebras, as $n$-Lie algebra $L$ is a vector space $L$ with an $n$-arymultiplier and skew-symmetric operation $[x_1, \dots, x_n]$ which satisfies the generalized Jacobi identity

$$[[x_1, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^{n} [x_1, \dots, x_{i-1}, [x_i, y_2, \dots, y_n], x_{i+1}, \dots, x_n].$$

Clearly, such an algebra becomes an ordinary Lie algebra when $n = 2$. Let $L$ be a finite dimensional $n$-Lie algebra. By $L^i$ and $Z_i(L)$ we denote the $i$th term in the lower and upper central series of $L$, respectively. Let $L$ be an $n$-Lie algebra and $L \cong F/R$ for a free $n$-Lie algebra $F$. It is well-known the Schur multiplier $\mathcal{M}(L)$ of $L$ is isomorphic to $(R \cap F^2)/[R, F, \dots, F]$. For a $d$-dimenional $c$-step nilpotent $n$-Lie algebra, with $dimL^2 = m$, Darabi and second author in [2] showed that $\dim \mathcal{M}(L) \leq \binom{d}{n}$. Improving this bound, Eshrati, et.al. [4] prived that $\dim \mathcal{M}(L) \leq \binom{d}{n} - m$. Later this bound was improved in [**?**, Theorem 3.10]. They established the following bound

$$\dim \mathcal{M}(L) \leq \binom{d-m+1}{n} + (m-2)\binom{d-m}{n-1} + n - m, \tag{1.1}$$

and also, when $m = 1$ the bound is attained if and only if $L \cong H(n,1) \oplus F(d-n-1)$, where $H(n,t)$ and $F(s)$ are used to denote the Heisenberg and abelian $n$-Lie algebras of dimension $nt+1$ and $s$,

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respectively. Moreover, Akbarossadat and second author succecded to characterize the structure of all nonabelian $n$-Lie algebra $L$ that attain the bound (1.1) in [1].

## 2 Preliminaries results

**Theorem 2.1** ([4]). *Suppose that $L$ be a $d$-dimensional $n$-Lie algebra and that $K$ is an ideal of $L$. Then the following inequalities hold:*

*(1)* $\dim \mathcal{M}(L) + \dim(K \cap L^2) \leq \dim \mathcal{M}\left(\frac{L}{K}\right) + \dim \mathcal{M}(K) + a\binom{b}{n-1}$, *where* $a = \dim K$ *and* $b = \dim(L/K)/(L/K)^2$.

*(2)* $\dim \mathcal{M}(L) + \dim(K \cap L^2) \leq \binom{d}{n}$.

**Proposition 2.2** ([3]). *Suppose that $L$ is an $n$-Lie algebra and that for integer $2 \leq i \leq c$ ($j = (n-1)(i-1)+1$), the map* $\Psi_i : \underbrace{L \times L \times \cdots \times L}_{j-times} \longrightarrow \underbrace{L^{ab} \otimes \cdots \otimes L^{ab}}_{(n-1)-times} \otimes \frac{L^i}{L^{i+1}}$, *is defined as follows:*

$$\underbrace{(x_{1,1}, x_{1,2}, \ldots, x_{1,n}, x_{2,2}, \ldots, x_{2,n}, \ldots, x_{i-2,2}, \ldots, x_{i-2,n}, x_{i-1,2}, \ldots, x_{i-1,n}, x_{i,2}, \ldots, x_{i,n})}_{-j} \longmapsto$$

$$\begin{aligned}
&\overline{\Big[\Big[\cdots[[[x_{1,1}, x_{1,2}, \ldots, x_{1,n}], x_{2,2}, \ldots, x_{2,n}], \ldots], x_{i-2,2}, \ldots, x_{i-2,n}\Big], x_{i,2}, \ldots, x_{i,n}\Big]} \otimes \overline{x_{i-1,2}} \otimes \cdots \otimes \overline{x_{i-1,n}} \cdots \\
+ \quad &\overline{\Big[[\cdots[[[x_{1,1}, x_{1,2}, \ldots, x_{1,n}], x_{2,2}, \ldots, x_{2,n}], \ldots], x_{i-2,2}, \ldots, x_{i-2,n}\Big]} \otimes \overline{x_{i-1,2}} \otimes \cdots \otimes \overline{\Big[x_{i-1,n-1}, x_{i,2}, \ldots, x_{i,n}\Big]} \otimes \overline{x_{i-1,n}} \\
+ \quad &\cdots \\
+ \quad &\overline{x_{1,1}} \otimes \overline{x_{1,2}} \otimes \cdots \otimes \overline{\Big[x_{1,n-1}, x_{2,2}, x_{2,3}, \cdots x_{2,n-1}, [x_{2,n}, x_{3,2}, \ldots, x_{3,n-1}, [\ldots, x_{i-1,n-1}, [x_{i-1,n}, x_{i-2,2}, \ldots, x_{i-2,n-1},}\\
&\quad \overline{[x_{i-2,n}, x_{i-1,2}, \ldots, x_{i-1,n-1}, [x_{i-1,n}, x_{i,2}, \ldots, x_{i,n}]]]]]\Big]} \otimes \overline{x_{1,n}} \\
- \quad &\overline{\Big[[\cdots[[[x_{1,1}, x_{1,2}, \ldots, x_{1,n}], x_{2,2}, \ldots, x_{2,n}], \ldots], x_{i-2,2}, \ldots, x_{i-2,n}], x_{i-1,2}, \ldots, x_{i-1,n}\Big]} \otimes \overline{x_{i,2}} \otimes \cdots \otimes \overline{x_{i,n}},
\end{aligned}$$

*for every $x_{1,2}, \ldots, x_{1,n}, x_{2,2}, \ldots, x_{2,n}, \ldots, x_{i-1,2}, \ldots, x_{i-1,n}, x_{i,2}, \ldots, x_{i,n} \in L$. Then* $\mathrm{Im}\Psi_i \subseteq \ker \lambda_i$ *for $2 \leq i \leq c$.*

In the following theorem, we find a bound that improving the bound of (1.1), also, we classify all nilpotent $n$-Lie algebras, that attain the bound.

**Theorem 2.3** ([3]). *Let $L$ be a $c$-step nilpotent $n$-Lie algebra with $\dim L = d$ and $\dim L^2 = m \geq 1$. Then*

$$\dim \mathcal{M}(L) \leq \binom{d-m+1}{n} + (m-2)\binom{d-m}{n-1} + n - m \quad + \binom{d-m}{n-1} - n - \sum_{i=2}^{a}(d-m-(n+(i-2)(n-1))), \tag{2.1}$$

*in which* $a = \min\left\{c, \left[\frac{d-m-n}{n-1}\right] + 2\right\}$.

**Theorem 2.4** ([3]). *Let $L$ be a $d$-dimensional $n$-Lie algebra with nilpotency class two such that* $\dim L^2 = m$. *Then the dimension of $\mathcal{M}(L)$ attains the bound of (2.1) if and only if $L$ is an $n$-Lie algebra isomorphic with one of the following $n$-Lie algebras: $L_{5,8}$, $L_{6,26}$, $H(2,1) \oplus F(d-3)$, $H(n,1)$.*

**Proposition 2.5** ([3]). *Let $L$ be a $d$-dimensional $n$-Lie algebra with nilpotency class $c \geq 3$ such that the dimension of $\mathcal{M}(L)$ attains the bound of (2.1). Suppose that $I$ is a one-dimensional ideal contained in $Z(L) \cap L^2$. Then the dimension of $\mathcal{M}(L/I)$ attains the bound of (2.1).*

**Theorem 2.6** ([3]). *There is no $d$-dimensional 3-step nilpotent $n$-Lie algebra , say $L$ such that the dimension of $\mathcal{M}(L)$ attains the bound of (2.1).*

In what follows, we prove a theorem that states that every nonabelian nilpotent $n$-Lie algebra that attains the bound of the Schur multiplier dimension is of nilpotency class 2.

**Theorem 2.7** ([3]). *There is no $d$-dimensional $n$-Lie algebra with nilpotency class $c \geq 3$, say $L$ such that the dimension of $\mathcal{M}(L)$ attains the bound of (2.1). In other words, if $L$ is a $d$-dimensional $n$-Lie algebra with nilpotency class $c \geq 3$ with $\dim L^2 = m$, then*

$$\dim \mathcal{M}(L) \leq \binom{d-m+1}{n} + (m-2)\binom{d-m}{n-1} + n - m + \binom{d-m}{n-1} - n - \sum_{i=2}^{a}(d-m-(n+(i-2)(n-1))) - 1.$$

**Corollary 2.8** ([3]). *Nonabelian nilpotent $n$-Lie algebras that their Schur multiplier dimension attains the bound (2.1) read as follows: $L_{5,8}$, $L_{6,26}$, $H(1) \oplus F(d-3)$, $H(n,1)$.*

## 3 The Schur multiplier of a nilpotent $n$-Lie algebra with derived subalgebra of maximum dimension

In the present section, we first attempt to introduce a bound of the dimension of the Schur multiplier which sharpen the later known bounds for some classes of nilpotent $n$-Lie algebras.

**Theorem 3.1** ([3]). *Suppose that $L$ is an $n$-Lie algebra with nilpotency class $c$ and finite dimension $d$ that $\dim L^2 = d - n$. Then*

$$\dim \mathcal{M}(L) \leq (d-n)(n-1) + 1 - \left[\frac{c-1}{n}\right]. \tag{3.1}$$

The following corollary is easily obtained from the previous theorem and introduces a bound for the Schur multiplier dimension of nilpotent $n$-Lie algebras of maximal class, which are a special case of $n$-Lie algebras discussed in the previous theorem.

**Corollary 3.2** ([3]). *Let $L$ be a nilpotent $n$-Lie algebras of maximal class and finite dimension $d$. Then*

$$\dim \mathcal{M}(L) \leq \begin{cases} (d-n)(n-1) + 1 - \dfrac{d-2-(n-2)}{n}; & d \overset{n}{\equiv} 0 \\ (d-n)(n-1) + 1 - \dfrac{d-2-(n-1)}{n}; & d \overset{n}{\equiv} 1 \\ (d-n)(n-1) + 1 - \dfrac{d-2-(i-2)}{n}; & d \overset{n}{\equiv} i; \qquad 2 \leq i \leq n-1. \end{cases}$$

By (1) and Theorems 2.1, and 3.1, we have the following three bounds for the dimension of Schur multiplier of $d$-dimensional nilpotent $n$-Lie algebra $L$ with $\dim L^2 = m$.

$$\dim \mathcal{M}(L) \leq \binom{d-m+1}{n} + (m-2)\binom{d-m}{n-1} + n - m \tag{3.2}$$

$$\dim \mathcal{M}(L) \leq (d-n)(n-1); \qquad d-n=m,\ d \geq n+2 \tag{3.3}$$

$$\dim \mathcal{M}(L) \leq (d-n)(n-1) + 1 - \left[\frac{c-1}{n}\right]; \qquad d-n=m \tag{3.4}$$

Now, the following corollary compares the three bounds.

**Corollary 3.3** ([3]). *Suppose that $L$ is a $d$-dimensional $n$-Lie algebra with nilpotency class $c$ and $\dim L^2 = m$.*

*(i) For all $c \geq n+1$, (3.4) is bounded better than (3.2). Also for $2 \leq c \leq n$, these two bounds are equal.*

*(ii)* *For* $c \geq 2n+1$, (3.4) *is bounded better than* (3.3). *Also, for* $3 \leq c \leq 2n$, *these two bounds are equal.*

**Corollary 3.4** ([3]). *Suppose that* $L$ *is a* $d$*-dimensional* $n$*-Lie algebra with nilpotency class* $c$ *and* $\dim L^2 = d - n$. *Then* $\dim \mathcal{M}(L) \leq (d-n)r + 1$, *where* $r = cc(L)$.

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# Galois groups and Newton polygons

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**Abstract:** This work explores the application of Newton polygons to decompose polynomials into irreducible factors over $p$-adic fields, a key tool for computing Galois groups of Taylor polynomials associated with transcendental functions such as $\log x$ and $\exp x$. We demonstrate that the Galois groups of the splitting fields of the Taylor polynomials of $1 + \log(1 - x)$ are either the full symmetric group $S_n$ or the alternating group $A_n$, depending on the discriminant and residue of $n$ modulo 4. Similar methods extend to other elementary functions like Cos$x$.

**Keywords** Discriminants, Galois theory, Newton polygon, $p$-adic theory.

**Mathematics Subject Classification (2010) : Primary: 11R32, 11F85, 11R29.**

## 1 Introduction

In the early 20th century, Hilbert established the existence of irreducible polynomials over $\mathbb{Q}$ with Galois group $S_n$ for every degree $n$, though his proof was non-constructive. Later, Schur [4] provided an explicit construction, showing that the Galois groups of the exponential Taylor polynomials

$$f_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

are $A_n$ if $4 \mid n$ and $S_n$ otherwise. Coleman [2] later reinterpreted this result using $p$-adic Newton polygons.

Newton's polygon, introduced in 1676 to solve implicit algebraic equations via Puiseux series, became pivotal in local field theory after Hensel's 1892 introduction of $p$-adic numbers. For a polynomial $P(x) = \sum a_i x^i$ over a valued field $L$, its Newton polygon is the lower convex hull of the points $(i, \nu_L(a_i))$. This structure encodes factorization patterns over local fields like $\mathbb{Q}_p$.

We apply Coleman's framework to compute Galois groups of Taylor polynomials of $1 + \log(1 - x)$,

$$f_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n},$$

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proving their irreducibility and determining their Galois groups as $S_n$ or $A_n$. The discriminant criterion shows $S_n$ arises for $n \equiv 0, 2, 3 \pmod 4$ or prime $n \equiv 1 \pmod 4$. Analogous results hold for Cos$x$ [3].

## 2 Newton Polygons over $\mathbb{Q}_p$

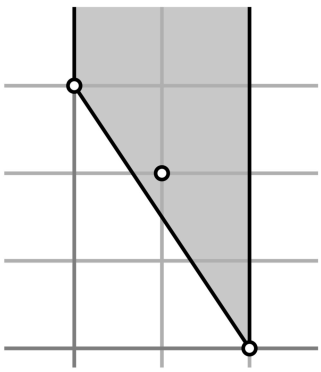
Let $\mathbb{Q}_p$ be the $p$-adic field with valuation $\nu_p$. For $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Q}_p[x]$, its **Newton polygon** is the lower convex hull of $\{(i, \nu_p(a_i))\}_{i=0}^{n}$.

**Theorem 2.1** (Cassels [1])**.** *If $f(x)$ has vertices $(x_0, y_0), \ldots, (x_\ell, y_\ell)$ on its Newton polygon, then $f(x)$ factors over $\mathbb{Q}_p$ as $f(x) = f_1(x) \cdots f_\ell(x)$, where $\deg(f_i) = x_i - x_{i-1}$. All roots of $f_i(x)$ in $\overline{\mathbb{Q}}_p$ have valuation $-\frac{y_i - y_{i-1}}{x_i - x_{i-1}}$.*

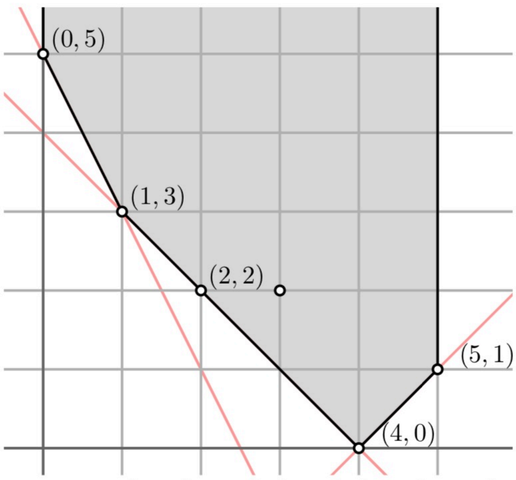
**Corollary 2.2.** *If $f_i(x)$ is irreducible and $d \mid \deg(f_i)$, then $d$ divides the order of the Galois group of $f(x)$ over $\mathbb{Q}$.*

*Proof.* Let $\alpha$ be a root of $f_i(x)$. Then $[\mathbb{Q}_p(\alpha) : \mathbb{Q}_p] = \deg(f_i)$, implying $d$ divides $|\mathrm{Gal}(f)|$. The embedding $\mathrm{Gal}(f/\mathbb{Q}_p) \hookrightarrow \mathrm{Gal}(f/\mathbb{Q})$ completes the proof. $\square$

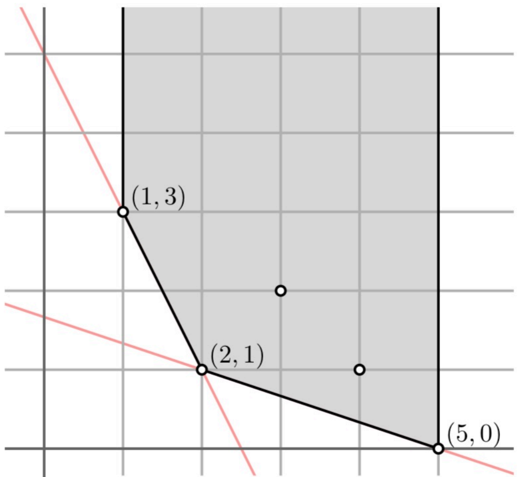
Some examples of Newton polygons can be seen in the following figures:



$p^3 + p^2 x + x^2$




$P(x) = p^5 + p^3 x + p^2 x^2 + p^2 x^3 - x^4 + p x^5$




$P(x) = p^3 x + p x^2 + p^2 x^3 - p x^4 + x^5$

## 3 Galois groups of logarithmic Taylor polynomials

For $f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k}$, we proceed as follows:

(i) **Irreducibility:** To show the irreducibility of $f_n(x) \in \mathbb{Q}[x]$, we first choose a prime number $p$ strictly between $n/2$ and $n$ using Chebyshev theorem, and consider the polynomial $f_n$ over $\mathbb{Q}_p$. Then the Newton polygon of $f_n(x)$ has only three vertices $(0,0), (p,-1), (n,0)$ and therefore using the above Theorem, $f_n$ factors in $\mathbb{Q}_p[x]$ as

$$f_n(x) = g_n(x)h_n(x),$$

in which $g_n$ is a polynomial of degree $p$ with roots of valuations $1/p$. Hence the polynomial $g_n(x) \in \mathbb{Q}_p[x]$ is an irreducible factor of $f_n(x) \in \mathbb{Q}_p[x]$. Repeating the same argument for another prime $\ell \neq p$ between $n/2$ and $n$, the polynomial $f_n(x) \in \mathbb{Q}_\ell[x]$ has an irreducible factor of degree $\ell$. Note that for $n \geq 12$ there are at least two primes between $n$ and $n/2$. Now assuming that the degree $n$ polynomial $f_n(x) \in \mathbb{Q}[x]$ is reducible, it must have two irreducible factors of degree $p > n/2$ and $\ell > n/2$ which is a contradiction.

(ii) **Transitivity and $p$-cycles:** Irreducibility implies $G_n \leq S_n$ is transitive. By Corollary 2.2, $G_n$ contains a $p$-cycle. Jordan's theorem then forces $G_n \supseteq A_n$.

(iii) **Discriminant criterion:** The discriminant $\Delta(f_n)$ is a square only if $n \equiv 1 \pmod 4$. Thus:

- For $n \not\equiv 1 \pmod 4$, $G_n = S_n$.
- For prime $n \equiv 1 \pmod 4$, $G_n = S_n$ (as $f_n$ cannot be contained in $A_n$).

## 4 Conclusion

The Newton polygon method robustly computes Galois groups for Taylor polynomials of transcendental functions. For $1 + \log(1-x)$, $G_n = S_n$ except possibly when $n \equiv 1 \pmod 4$ and composite. Similar techniques apply to Cos$x$ [3], underscoring the versatility of $p$-adic tools in Galois theory.

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# Polynomials, number fields and Galois groups

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**Abstract:** Computing the Galois groups of polynomials over rational numbers plays an important role in many problems in number theory. Although by a probabilistic argument the Galois group of a random irreducible polynomial with integer coefficients is the full symmetric group on $n$ roots, there is no algorithm to compute such Galois groups in general. In this note, after a short review of some classical results, we study the Galois groups of some important examples and of certain families of trinomials and quadrinomials, and provide some applications.

**Keywords** Galois theory, Discriminant, Ramification.

**Mathematics Subject Classification (2010) : Primary: 11R32, 11R29; Secondary: 11S15.**

## 1 Introduction

Let $f(x) \in \mathbb{Z}[x]$ be a monic and irreducible polynomial. The Galois group of $f(x)$ over the rational numbers $\mathbb{Q}$, denoted by $G_f$, is the Galois group of the splitting field of $f(x)$, i.e. the field gerenerated by all roots of $f(x)$, over $\mathbb{Q}$. One can think about an elelement of $G_f$ as a permutation of roots of $f(x)$ in $S_n$. It is well-known that probabilistically the Galois group of a random irreducible polynomial of degree $n$ is $S_n$. In this regards, let 's mention van der Waerden conjecture which is recently proved by Bhargava [1]: the upper bound for the count of polynomials $x^n + a_1x^{n-1} + \cdots + a_n \in \mathbb{Z}[x]$ with $max\{|a_1|, \cdots, |a_n|\} \leq H$ whose Galois groups are not $S_n$ is $O(H^{n-1})$.

However, it is an open question that any finite group is the Galois group of some polynomial over $\mathbb{Q}$. In 1897 Hilbet showed the existence of a polynomial of degree $n$ whose Galois group is $S_n$, and in 1930 Schur explicitly constructed such extensions. For polynomials with small degrees some classical results, e.g. Dedekind theorem, compute the corresponding Galois group. What about arbitrary degrees? For arbitrary degree polynomials, people started to compute the discriminant $D_f$ and then the inertia groups of primes dividing $D_f$ to obtain the Galois group $G_f$. By this method certain

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families of trinomials $f(x) = x^n + ax^s + b$, and of quadrinomials $f(x) = x^n + ax^{n-1} + bx^{n-2} + c$ have been computed. It is also proven that all sporadic simple groups with exception Mathieu group $M_{23}$ are the Galois groups of some polynomials over $\mathbb{Q}$. For example the monster group is obtained using the so-called rigidity method for the inverse Galois problems. One can consult [12] for more information.

A historical application of computing such Galois groups is as follows: the roots of the polynomial $f(x)$ are expressed in terms of radicals if and only if the Galois group $G_f$ is a solvable group.

## 2 classical results

We recall the followng general propositions in Galois theory for any irreducible and monic polynomial $f(x) \in \mathbb{Q}[x]$ of degree $n$ with Galois group $G_f$:

- $n|\ |G_f|$.
- $G_f$ is a transitive subgroup of $S_n$.
- if $n = p$ is prime, then $G_f$ contains a $p$-cycle.
- If $G_n$ contains a $p$-cycle for $n/2 < p < n-2$, then $A_n \subseteq G_f$ for alternating group $A_n$.
- $G_f \subseteq A_n$ if and only if the rational number $disc(f)$ is square.
- (Dedekind Theorem:) for any prime $p$ not dividing $disc(f)$, let

  $$f(x) \equiv f_1(x) \cdots f_k(x) \pmod{p}$$

  be the decomposition of $f(x)$ mod $p$. Then $G_f$ has a permutation of the type $(deg(f_1), \cdots, deg(f_k))$.

**Example 2.1.** The monic polynomial $f(x) = x^3 - cx - 1 \in \mathbb{Z}[x]$ is irreducible, and its Galois group $G_f$ is $A_3$ for $c = 3$, and $S_3$ otherwise (cf. [5]).

**Example 2.2.** If we use Dedekind theorem for monic and irreducible polynomial $f(x) \in \mathbb{Z}[x]$ of prime degree $p$, where $f(x) \in \mathbb{F}_\ell[x]$ has all but two roots in $\mathbb{F}_\ell$, then we can conclude that $G_f$ is the full symmetric group $S_\ell$.

**Example 2.3.** Since any transition and any $p$-cycle generates $S_p$ for $p$ prime, the Galois group over $\mathbb{Q}$ of any polynomial of prime degree $p$ which has only two non-real roots, is the full symmetric group $S_p$.

**Example 2.4.** In [9] it is shown that any permutation $(a \quad b)(c \quad d)$ and any $p$-cycle generates $S_p$ for prime $p > 7$. Hence, the Galois group over $\mathbb{Q}$ of any polynomial of prime degree $p$ which has only four non-real roots, is the full symmetric group $S_p$.

**Example 2.5.** The Galois group of $x^7 - 154x + 99$ over $\mathbb{Q}$ is the group $PSL_3(2)$ (see for example [6]).

**Example 2.6.** It is easily shown that for any prime $p$, the Galois group of binomial $x^p - a$ where $a \in \mathbb{Q}^n$, i.e. $G_f = Gal(\mathbb{Q}(e^{2i\pi/p}, \sqrt[n]{a})/\mathbb{Q})$ is the semi-direct product of $(\mathbb{Z}/p\mathbb{Z})^\times \simeq \mathbb{Z}/(p-1)\mathbb{Z}$ and the cyclic group $\mathbb{Z}/p\mathbb{Z}$.

To see the computations of various polynomials of degrees between 3 and 7 one can see [11].

# 3 certain familes of trinomials and quadrinomials

In recent decades other tools such as the inertia groups as a subgroups of the decomposition groups, elliptic curves, $p$-adic analysis, étale fundamental groups, ... are used to compute the Galois groups of polynomials. We first review some results on trinomials.

Using the classification of finite simple groups by Feit drew up the list of groups susceptible to be the Galois group of irreducible prime degree trinomials over the rational numbers.

Let $f(x) = x^n + ax^s + b \in \mathbb{Z}[x]$ be an irreducible polynomial of degree $n$ where $s$ is prime and $gcd(nb, as(n-s)) = 1$. Then $G_f$ is either $S_n$ or $A_n$ (cf. [4] and [10]).

Let $f(x) = x^p + ax^s + a \in \mathbb{Z}[x]$ be an irreducible polynomial of prime degree $p$. In the case $p$ divides $a$ only once, then $G_f$ is either the full symmetric group $S_p$ or the affine group $Aff(\mathbb{F}_p)$. In the case $s = 1$ and $p \not| a$, then $G = S_p$ if the discriminant $disc(f)$ is not square, and otherwise $G_f$ is the alternating group $A_p$ or the projective special linear group $PSL_2(2^e)$. The latter case is possible if $p$ is a Fermat prime. In the case $s = p - 1$ and either $a \neq \pm 1$ or $p \not\equiv 2 \pmod 3$, the Galois group $G_f$ is the full symmentric group $S_n$ (cf. [2], [4], [3], [8]).

Now let's concentrate on Galois groups of some (infinite) familes of qudrinomials. For any irreducible polynomial of the form

$$f(x) = x^n + ax^{n-1} + bx^{n-2} \pm 1 \in \mathbb{Z}[x]$$

with $a^2 = 4b$ and $gcd(n, a) = 1$, the Galois group $G_f$ over $\mathbb{Q}$ is the full symmetric group $S_n$. Let $\ell$ be a prime number, and assume that $gcd(2a(\ell-2), \ell c) = 1$ and $(-1)^{\ell-2}4(\frac{a}{2})^\ell(\ell-2)^{\ell-2} + \ell^\ell c$ is not a perfect square. Then, for any irreducible polynomial of the form $f(x) = x^\ell + ax^{\ell-1} + bx^{\ell-2} + c \in \mathbb{Z}[x]$ with $a^2 = 4b$, the Galois group $G_f$ of the splitting field of $f(x)$ over $\mathbb{Q}$ is also the full symmetric group $S_\ell$ (cf. [7]). In [7], it is shown the existence of an infinite families of quadrinomials satisfying these hypotheses. Here the discriminant is computed as follows:

$$disc(f) = (-1)^{\frac{n(n-1)}{2}} c^{n-2}\Big((-1)^{n-2}4(\frac{a}{2})^n(n-2)^{n-2} + n^n c\Big). \tag{3.1}$$

The Galois group of any irreducible quadrinomial of the form

$$x^\ell + ax^{\ell-1} + bx^{\ell-2} + c \in \mathbb{Z}[x]$$

with $\ell \equiv 1 \pmod 4$ an odd prime is again the full symmetric group $S_\ell$ if the coefficients $a$, $b$ and $c$ are all odd integers and $a + b + c \equiv 1 \pmod 4$ (see [7]). Therefore, all these families of polynomials are not solvable by radicals.

It is worth mentioning that in [7] polynomials of degree $n$ with arbitrary terms are constructed under some mild hypotheses whose Galois groups are $S_n$.

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# On weakly $n$-slender groups

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**Abstract:** In this talk, we generalize the class of n-slender groups to weakly n-slender groups, which is closed under direct products and inverse limits. Also, we show that for a space $X$ with first countability at $p$, if the fundamental group of $X$ at $p$ is (weakly) n-slender, then $X$ is semilocally simply connected (homotopically Hausdorff) at $p$.

**Keywords** slender groups; n-slender groups; fundamental groups.
**Mathematics Subject Classification (2010) : 57M07, 55Q52.**

## 1 Introduction

Let $n \in \mathbb{N}$. The following notation is needed in what follows. The element $e_n \in \prod_{\mathbb{N}} \mathbb{Z}$ is defined by $e_n(k) = 1$ if $k = n$ and $e_n(k) = 0$ for all $k \neq n$. We recall that an abelian group $A$ is slender if for every homomorphism $h : \prod_{\mathbb{N}} \mathbb{Z} \to A$, then $\{n \in \mathbb{N} | h(e_n) \neq 0\}$ is a finite set. Every free abelian group is slender, however, divisible groups (e.g $\mathbb{Q}$, $J_p$ the p-adic integer group) are not slender. The subgroups and the direct sum of slender groups are slender [2, Section 94]. However, the direct product of slender groups is not slender in general. For example, the free abelian group $\mathbb{Z}$ is slender and $\prod_{\mathbb{N}} \mathbb{Z}$ is not slender.

Recall from [1] that a group $G$ is called noncommutative slender (n-slender) group, if for every homomorphism $h : \times_{\mathbb{N}}\mathbb{Z} \to G$, $h(\times_{\mathbb{N}\setminus\{1,2,\cdots,n\}}\mathbb{Z}) = 0$ for some $n \in \mathbb{N}$. An abelian group A is n-slender, if and only if A is slender; therefore, free abelian groups are n-slender. Furthermore, the subgroups, the free product (e.g. free groups), and the weak direct product of n-slender groups are n-slender [1]. But the direct product of n-slender groups is not n-slender in general. For example, the free abelian group $\mathbb{Z}$ is n-slender and $\prod_{\mathbb{N}} \mathbb{Z}$ is not n-slender, since the abelian group $\prod_{\mathbb{N}} \mathbb{Z}$ is not slender. Also the complete product of n-slender groups is not n-slender in general. For example, the free abelian group $\mathbb{Z}$ is n-slender, but $\times_{\mathbb{N}}\mathbb{Z}$ is not n-slender.

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## 2 Weakly slender

We generalize the concept of slender group, named weakly slender group, as follows.

**Definition 2.1.** We call an abelian group $A$ weakly slender if for every homomorphism $h : \prod_{\mathbb{N}} \mathbb{Z} \to A$ such that, for every $n \in \mathbb{N}$, $h(e_n) = a$ for some $a \in A$ we have $h = 0$.

It is easy to see that if $A$ is weakly slender and $B \leqslant A$, then $B$ is weakly slender. In the following proposition, we show that weakly slenderness is closed under extension.

**Proposition 2.2.** *Let $B$ be a subgroup of an abelian group $A$ such that $B$ and $\frac{A}{B}$ are weakly slender; then $A$ is weakly slender.*

**Theorem 2.3.** *(i) Every slender group is weakly slender.*
*(ii) The direct product $\prod_{\alpha \in I} A_\alpha$ of abelian groups is weakly slender if and only if $A_\alpha$ is weakly slender for every $\alpha \in I$ .*

**Example 2.4.** The free abelian group $\mathbb{Z}$ is slender, so the above theorem implies that $\prod_{\mathbb{N}} \mathbb{Z}$ is weakly slender but it is not slender.

**Example 2.5.** Assume that $\{c_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers. Let $A = \{(x_n)_{n\in\mathbb{N}} \in \prod_{\mathbb{N}} \mathbb{Z} | \sum\limits_{n=1}^{\infty} c_n x_n < \infty\}$. Indeed $A$ is a subgroup of $\prod_{\mathbb{N}} \mathbb{Z}$, which implies that $A$ is weakly slender since $\prod_{\mathbb{N}} \mathbb{Z}$ is weakly slender. For instance, $B = \{(x_n)_{n\in\mathbb{N}} \in \prod_{\mathbb{N}} \mathbb{Z} | \sum\limits_{n=1}^{\infty} \frac{x_n}{n^2} < \infty\}$ is a weakly slender group.

The fact that the inverse limit of groups is a subgroup of the product of groups implies the following corollary.

**Corollary 2.6.** *The class of weakly slender groups is closed under inverse limit.*

**Example 2.7.** Let $n \in \mathbb{N}$. Suppose that $G_n = \bigoplus\limits_{i=1}^{n} \mathbb{Z}$. Since $G_n$ is a free abelian group, it is a slender group. Therefore $G_n$ is weakly slender which implies that the inverse limit of the sequence $\{G_n\}_{n\in\mathbb{N}}$ is a weakly slender group.

## 3 Weakly noncommutative slender

We generalize the concept of noncommutative slender group which is a noncommutative version of weakly slender abelian group as follows.

**Definition 3.1.** We call a group $G$ is weakly noncommutative slender (weakly n-slender) if every homomorphism $h : \pi_1(HA) \to G$ is trivial, where $HA$ is the Harmonic Archipelago space.

It is easy to see that if $H$ is a weakly n-slender group and $H \trianglelefteq G$ with $\frac{G}{H}$ is weakly n-slender, then $G$ is weakly n-slender.

**Theorem 3.2.** *Every n-slender group is weakly n-slender.*

By Example 2.4, $\prod_{\mathbb{N}} \mathbb{Z}$ is an abelian weakly slender group, which is not slender group. Since $\prod_{\mathbb{N}} \mathbb{Z}$ is abelian but it is not slender, so it is not n-slender, too. On the other hand, by the next theorem, $\prod_{\mathbb{N}} \mathbb{Z}$ is weakly n-slender. Since the fundamental group of $\prod_{\mathbb{N}} S^1$ is isomorphic to $\prod_{\mathbb{N}} \mathbb{Z}$, the fundamental group of $\prod_{\mathbb{N}} S^1$ is a weakly n-slender group, which is not n-slender. In the Example 3.5 we show that there is a nonabelian weakly n-slender group, which is not n-slender.

**Theorem 3.3.** *Let $A$ be an abelian group. Then $A$ is weakly n-slender if and only if it is weakly slender.*

By projection homomorphisms we conclude the following theorem, which shows that the class of weakly n-slender groups is closed under the direct product.

**Theorem 3.4.** *Let $\{G_\alpha\}_{\alpha \in I}$ be a family of groups. Then for every $\alpha \in I$, $G_\alpha$ is weakly n-slender if and only if $\prod_{\alpha \in I} G_\alpha$ is weakly n-slender.*

Since the inverse limit of groups is a subgroup of the product of groups, so the class of weakly n-slender groups is closed under inverse limit.

**Example 3.5.** It is well known that the fundamental group of the Hawaiian Earring is a subgroup of the inverse limit of free groups; therefore $\pi_1(HE)$ is weakly n-slender, since every free group is n-slender. In other hand $\times_{\mathbb{N}} \mathbb{Z} \cong \pi_1(HE)$ and $\times_{\mathbb{N}} \mathbb{Z}$ is not n-slender.

**Example 3.6.** Let $n \in \mathbb{N}$. Suppose that $G_n = \mathop{*}\limits_{i=1}^{n} \mathbb{Z}$. Since $G_n$ is a free group, it is an n-slender group. Therefore $G_n$ is weakly n-slender which implies that the inverse limit of the sequence $\{G_n\}_{n \in \mathbb{N}}$ is a weakly n-slender group. Also, if $X = \prod_{n \in \mathbb{N}} \bigvee\limits_{i=1}^{n} S^1$, then the fundamental group of $X$ is weakly n-slender, since it is isomorphic to $\prod_{\mathbb{N}} G_n$.

**Theorem 3.7.** *Let $X$ be a topological space, and let $x_0$ be a point in $X$ such that $X$ is first countable at $x_0$. Then*
*(i) if $\pi_1(X, x_0)$ is n-slender, then $X$ is semilocally simply connected at $x_0$.*
*(ii) if $\pi_1(X, x_0)$ is weakly n-slender, then $X$ is homotopically Hausdorff at $x_0$.*

Note that the converse of (i) and (ii) in the above theorem is not true in general. For example, let $G$ be a group which is not n-slender. The Eilenberg–MacLane (CW) complex $K(G, 1)$ is a first countable, semilocally simply connected space with the fundamental group $G$. Also for a nonweakly n-slender group $H$, the Eilenberg–MacLane (CW) complex $K(H, 1)$ is a first countable, homotopically Hausdorff space with the fundamental group $H$.

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# Some results on the edge covering of Cayley graphs

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**Abstract:** Let $G$ be a simple graph of order $n$ and size $m$. An edge covering of a graph is a set of edges such that every vertex of the graph is incident to at least one edge of the set. The edge cover polynomial of $G$ is the polynomial $E(G,x) = \sum_{i=\rho(G)}^{m} e(G,i)x^i$, where $e(G,i)$ is the number of edge covering of $G$ of size $i$ and $\rho(G)$ is the edge cover number which is a minimum edge covering number of $G$. In this paper, we study the edge cover number and the edge cover polynomial of some Cayley graphs.

**Keywords** edge cover polynomial; number of edge cover polynomial; Cayley graphs.
**Mathematics Subject Classification (2010) : 05E15, 05C15, 20B25.**

## 1 Introduction

Let $G$ be a finite group and let $S \subseteq G$ be a subset. The corresponding Cayley graph $\Gamma(G) = Cay(G,S)$ has vertex set equal to $G$. Two vertices $g, h \in G$ are joined by a directed edge from $g$ to $h$ if and only if there exists $s \in S$ such that $g = sh$. Each edge is labeled to denote that it corresponds to $s \in S$. If $H$ is a graph such that there exists a group $G$ and generating set $S \subseteq G$ with $H \cong \Gamma(G)$, then $H$ is said to be Cayley. Any Cayley graph is a $|S|$-regular graph. If $S$ generates $G$, then $\Gamma(G) = Cay(G,S)$ is connected. Also Cayley graph is a simple and vertex transitive.

Let $\Gamma$ be a simple graph of order $n$ and size $m$. An edge covering of a graph is a set of edges such that every vertex of the graph is incident to at least one edge of the set. The edge cover polynomial of $\Gamma$ is the polynomial $E(\Gamma,x) = \sum_{i=\rho(\Gamma)}^{m} e(\Gamma,i)x^i$, where $e(\Gamma,i)$ is the number of edge covering of $\Gamma$ of size $i$ and $\rho(\Gamma)$ is a minimum edge covering number of $\Gamma$ (see [1, 2]).

A matching in a graph $\Gamma$ is a set of edges such that no two edges share a common vertex. A maximum matching is a matching that contains the largest possible number of edges and the matching number of a graph $\Gamma$ is the size of a maximum matching of $\Gamma$ that denoted by $\alpha'(\Gamma)$.

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## 2 Main results

In this section, we state some results on the edge cover number and the edge cover polynomial of some Cayley graphs. We use the references [3, 4].

**Theorem 2.1.** *For a Cayley graph $\Gamma(G) = Cay(G, S)$ of order $n$,*

$$\lceil \frac{n}{2} \rceil \leq \rho(\Gamma(G)) \leq n - \lfloor \frac{|S|}{2} \rfloor.$$

**Theorem 2.2.** *If the Cayley graph $\Gamma(G) = Cay(G, S)$ has order $n$ and $|S| = n - 1$, then $\rho(\Gamma(G)) = \lceil \frac{n}{2} \rceil$.*

**Theorem 2.3.** *Let $G$ be a group and $H \neq G$ be subgroup of $G$ such that $[G : H] = t$. If $S = G \setminus H$, then $\Gamma(G) = Cay(G, S)$ is a complete $t$-partite graph $K_{n_1,n_2,...,n_t}$ and $\rho(\Gamma(G)) = max\{n_1, n_2, ..., n_t\}$.*

**Theorem 2.4.** *Let $\Gamma(G) = Cay(G, S)$ be a Cayley graph of order $n$. If $H$ is a graph such that $E(H, x) = E(\Gamma(G), x)$, then $H$ is a $|S|$-regular graph of order $n$.*

We end this note by the following theorem:

**Theorem 2.5.** *If $E(\Gamma(G), x) = \sum_{i=\lceil \frac{n}{2} \rceil}^{m} e(\Gamma(G), i)x^i$ is the edge cover polynomial of $\Gamma = Cay(G, S)$, then*

*(i) For every $i$, $e(\Gamma(G), i) \geq \binom{m}{i} - n\binom{m-|S|}{i}$.*

*(ii) For every $i$, $i \geq m - 2|S| + 2$, $e(\Gamma(G), i) = \binom{m}{i} - n\binom{m-|S|}{i}$.*

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