Proceedings of The 8th Seminar on Functional Analysis and its Applications

Ayatollah Boroujerdi University

In the Name of God

The 8th Seminar on Functional Analysis and its Applications

6-7 November 2024, Ayatollah Boroujerdi University

Boroujerd, IRAN

Book of Abstracts


دانشگاه آیت الله العظمی بروجردی (ره)


دانشگاه آیت الله العظمی بروجردی (ره)

Organizers:

Moosa Gabeleh
Chair of the Scientific Committee

Abbas Zivari
Chair of the Organizing Committee

Scientific Committee:

- Dr. Ali Abkar (*Imam Khomeini International University*)
- Dr. Asadollah Aghajani (*Iran University of Science and Technology*)
- Dr. Massoud Amini (*Tarbiat Modares University*)
- Dr. Alireza Amini Harandi (*University of Isfahan*)
- Dr. Davood Alimohammadi (*Arak University*)
- Dr. Abasalt Bodaghi (*Islamic Azad University*)
- Dr. Mohammad Eslamian (*University of Science and Technology of Mazandaran*)
- Dr. Kamal Fallahi (*Payam Noor University*)
- Dr. Ali Farajzadeh (*Razi University*)
- Dr. Amir Ghasem Ghazanfari (*Lorestan University*)
- Dr. Mohammad Reza Haddadi (*Ayatollah Boroujerdi University*)
- Dr. Hoger Ghahramani (*University of Kurdistan*)
- Dr. Saeid Maghsoudi (*University of Zanjan*)
- Dr. Farshid Mirzaee (*Malayer University*)
- Dr. Sirous Moradi (*Lorestan University*)
- Dr. Iskandar Naraghi Rad (*Yasouj University*)
- Dr. Vahid Parvaneh (*Islamic Azad University*)
- Dr. Abdolrahman Razani (*Imam Khomeini International University*)
- Dr. Mohammad Sal Moslehian (*Ferdowsi University of Mashhad*)
- Dr. Mohammad Esmael Samei (*Buali Sina University*)
- Dr. Seyyed Mansour Vaezpour (*Amir Kabir University of Technology*)
- Dr. Mohammad Valaei Anvar (*Ayatollah Boroujerdi University*)

کمیته اجرایی

- دکتر عباس زیوری (دبیر اجرایی)
- دکتر پرویز گودرزی (ریاست دانشکده علوم پایه)
- دکتر لیلا گودرزی (معاون آموزشی دانشکده علوم پایه)
- دکتر آزاده نیک سرشت (مدیر گروه ریاضی)
- دکتر محمدرضا حدادی
- دکتر مرجان علی یاری
- دکتر محمد والایی انور
- دکتر منیژه گودرزی
- دکتر سید محمد حسینی
- دکتر وحید کشاورز (کارشناس LaTeX)
- آقای محمد امرایی (کارشناس آموزش دانشکده علوم پایه)
- خانم زهرا منصفی (دانشجوی مقطع کارشناسی ارشد)
- خانم مهدیه گودرزی (دانشجوی مقطع کارشناسی)
- خانم نرگس شرفی (دانشجوی مقطع کارشناسی)
- خانم مهسا کرزبر (دانشجوی مقطع کارشناسی)

Invited Talks

Prof. Ali Abkar
Imam Khomeini International University

Prof. Majid Fakhar
University of Isfahan

Prof. Ali Farajzadeh
Razi University

Prof. Alimohammad Nazari
Arak University

مقدمه

آنالیز تابعی به عنوان یکی از شاخه‌های مهم و بنیادی ریاضی، در سال‌های اخیر توجه اساتید و پژوهشگران را به خود جلب کرده است. بسیاری از اعضای هیات علمی و محققان کشور در این حوزه فعالیت‌های گسترده‌ای انجام می‌دهند و برگزاری سمینارها و کنفرانس‌های ملی و بین‌المللی به تقویت مباحث علمی و پژوهشی در این زمینه کمک شایانی می‌نماید.

سمینار آنالیز تابعی و کاربردهای آن، فرصتی مناسب برای تبادل نظر میان اساتید و پژوهشگران در زمینه‌های مختلف آنالیز تابعی و کاربردهای آن فراهم آورده است. اقدامات و تحقیقات در این عرصه نه تنها به غنای علمی کشور می‌افزاید، بلکه منجر به معرفی کشور عزیزمان ایران بعنوان یکی از مراکز علمی معتبر در سطح بین‌المللی خواهد شد.

این سمینار با محوریت آنالیز تابعی و کاربردهای آن تحت نظارت انجمن ریاضی ایران از سال 1365 آغاز به کار نمود و نخستین دوره آن در دانشگاه صنعتی شریف برگزار گردید و دوره‌های ششم و هفتم آن بترتیب در دانشگاه‌های اصفهان و بین‌المللی امام خمینی قزوین برگزار شد.

به حول و قوه الهی هشتمین سمینار آنالیز تابعی و کاربردهای آن در تاریخ شانزدهم و هفدهم آبان ماه 1403 به همت اعضای گروه ریاضی دانشگاه آیت‌الله بروجردی و با مشارکت پژوهشگران برجسته این حوزه برگزار گردید. به‌منظور ارتقای کیفی این سمینار، کمیته علمی متشکل از اساتید و محققین مؤثر در حوزه آنالیز تابعی تشکیل شد تا این رویداد بتواند اهداف علمی مورد نظر را به بهترین نحو محقق کند.

شایان ذکر است که تعداد 83 مقاله به دبیرخانه هشتمین سمینار آنالیز تابعی و کاربردهای آن ارسال گردید که پس از بررسی و داوری‌های لازم، تعداد 68 مقاله مورد پذیرش نهایی قرار گرفت.

در بخش سخنرانان مدعو نیز اساتید صاحب‌نام این حوزه از ریاضیات آقایان دکتر علی آبکار، دکتر مجید فخار، دکتر علی فرجزاده و دکتر علیمحمد نظری به ایراد سخنرانی و ارایه آخرین دستاوردهای علمی خود پرداختند.

در پایان، از سخنرانان و شرکت‌کنندگان سمینار به خاطر مشارکت ارزشمندشان سپاسگزاریم. همچنین از اعضای کمیته علمی، کمیته اجرایی و مسئولین محترم دانشگاه آیت‌الله بروجردی برای حمایت‌هایشان تقدیر می‌کنیم و امیدواریم که این سمینار توانسته باشد سهم کوچکی در پیشبرد اهداف و آرمان‌های متعالی علمی نظام مقدس جمهوری اسلامی داشته باشد.

و من الله توفیق

دکتر موسی گابله (دبیر علمی سمینار)

دکتر عباس زیوری (دبیر اجرایی سمینار)

CONTENTS

مجموعه مقاله های فارسی **282**

Invited Talks

COMPOSITION OPERATORS ACTING ON THE POLYDISK

ALI ABKAR*

Department of Pure Mathematics, Imam Khomeini International University, Qazvin 34149, Iran
abkar@sci.ikiu.ac.ir

Abstract. In this paper we first discuss composition and differentiation operators acting on the Hardy and Bergman spaces defined on the open unit disk in the complex plane. We then discuss the same operators on the polydisk in the n-dimensional complex space.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$, and let φ be an analytic self-map on $\mathbb{D}$. Let X be a Banach space of analytic functions on $\mathbb{D}$. The composition operator $C_\varphi : X \to X$ is defined by $C_\varphi(f) = f \circ \varphi$. This operator was first introduced by Erik Nordgren [2] where X was the Hardy Hilbert space H^2. Note that C_φ is a linear operator. Its properties like boundedness, compactness, Hilbert-Schmidtness, complete continuity, as well as its spectral properties depend on the symbol function φ, and the Banach space X.

Let us begin with the Hardy space H^2. An analytic function f belongs to H^2 if

$$\|f\|_{H^2}^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

2020 *Mathematics Subject Classification. Primary 47B33; Secondary 30H20*

Key words and phrases. Composition operator, differentiation operator, bounded operator, compact operator.

* Speaker.

It is easy to se that the vectors $\{z^n : n = 0, 1, 2, ...\}$ form an orthonormal basis for H^2 and that for each fixed $z \in \mathbb{D}$, the linear functional $\phi_z : H^2 \to \mathbb{C}$ given by $\phi_z(f) = f(z)$ is bounded. Hence, by Riesz representation theorem there exists a function $K_z(\cdot) \in H^2$ such that

$$f(z) = \phi_z(f) = \langle f, K_z \rangle, \quad f \in H^2.$$

The function K_z is called the reproducing kernel for the Hardy space. Indeed, we have

$$K_z(w) = \frac{1}{1 - \overline{z}w}, \quad z, w \in \mathbb{D}.$$

The Bergman space $A^2 = A^2(\mathbb{D})$ consists of analytic functions f in the unit disk for which

$$\|f\|_{A^2}^2 = \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty,$$

where

$$dA(z) = \pi^{-1} dx\, dy.$$

The inner product is defined by

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z)\overline{g(z)} dA(z).$$

It is easy to verify that the vectors

$$\{\sqrt{n+1} z^n : n = 0, 1, 2, ...\}$$

form an orthonormal bais for A^2, and

$$K_z(w) = \frac{1}{(1 - \overline{z}w)^2}$$

is the reproducing kernel for $A^2(\mathbb{D})$. Therefore, each $f \in A^2(\mathbb{D})$ can be written as

$$f(z) = \langle f, K_z \rangle = \int_{\mathbb{D}} f(w)\overline{K_z(w)} dA(w), \quad z \in \mathbb{D}.$$

In $\mathbb{C}^n$, the polydisk is defined by

$$\mathbb{D}^n = \{(z_1, ..., z_n) \in \mathbb{C}^n : |z_k| < 1, \, 1 \le k \le n\},$$

and the unit ball is defined by

$$\mathbb{B}_n = \{z = (z_1, ..., z_n) \in \mathbb{C}^n : |z|^2 = |z_1|^2 + \cdots + |z_n|^2 < 1\}.$$

Let $H(\mathbb{B}_n)$ denote the space of all holomorphic functions on the unit ball. The Bergman space on the unit ball $\mathbb{B}_n$ is defined as

$$A^2(\mathbb{B}_n) = H(\mathbb{B}_n) \cap L^2(\mathbb{B}_n, dv)$$

where $dv(z)$ is the normalized volume measure on $\mathbb{B}_n$.

Let $\mathbb{T}^n$ denote the distinguished boundary of $\mathbb{D}^n$, and let $\mathbb{S}_n$ denote the topological boundary of $\mathbb{B}_n$. Let $d\sigma$ denote the normalized Lebesgue measure on $\mathbb{T}^n$ or $\mathbb{S}_n$. The Hardy space $H^2(\mathbb{D}^n)$ consists of holomorphic functions on the polydisk such that

$$\sup_{0<r<1} \int_{\mathbb{T}^n} |f(r\zeta)|^2 \mathrm{d}\sigma(\zeta) < \infty.$$

Similarly, the Hardy space $H^2(\mathbb{B}_n)$ consists of holomorphic functions in the unit ball such that

$$\sup_{0<r<1} \int_{\mathbb{S}_n} |f(r\zeta)|^2 \mathrm{d}\sigma(\zeta) < \infty.$$

Recall that in one complex variable, composition operator is defined by $C_\varphi(f) = f \circ \varphi$ where φ is an analytic self-map of the unit disk, and f is a function in the Hardy or Bergman space. This notion can be generalized to several complex variables but we will not discuss this generalization here.

For an analytic map $\varphi : \mathbb{D} \to \mathbb{D}$, the operator C_φ is bounded on the Hardy and Bergman space [5]. Moreover, C_φ is compact on the Bergman space if and only if

$$\lim_{r\to 1^-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0.$$

2. Main Result

The composition-differentiation operator is defined by

$$D_\varphi(f) = f' \circ \varphi.$$

Note that D_φ is not bounded (on the Hardy and Bergman space) in general! D_φ is bounded on H^2 if and only if

$$\sup_{z\in\mathbb{D}} \frac{1-|z|}{(1-|\varphi(z)|)^3} < \infty,$$

and D_φ is compact on H^2 if and only if

$$\lim_{|z|\to 1^-} \frac{1-|z|}{(1-|\varphi(z)|)^3} = 0.$$

We aim to generalize the one-dimensional composition-differentiation operator $D_\varphi(f) = f' \circ \varphi$ to several variables.
For a holomorphic function f, we consider the radial derivative of f as

$$Rf(z) = \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}(z), \quad z = (z_1, ..., z_n).$$

If $f(z) = \sum_{k=0}^{\infty} f_k(z)$ is the homogenous expansion of f, then we have

$$Rf(z) = \sum_{k=1}^{\infty} k\, f_k(z).$$

We now introduce the composition-differentiation operator

$$E_\varphi(f) = ((Rf) \circ \varphi), \quad f \in H(\mathbb{B}_n),$$

and the higher order composition-differentiation operator

$$E^t_\varphi(f) = \big((R^t f) \circ \varphi\big), \quad t > 0,\ f \in H(\mathbb{B}_n),$$

where

$$R^t f(z) = \sum_{k=1}^{\infty} k^t f_k(z),$$

are fractional radial derivatives of f. The author has recently found conditions on φ to ensure that the operator E^t_φ is Hilbert-Schmidt [1]. Let $\Phi = (\varphi, \psi)$ where φ and ψ are two analytic self-maps of the unit disk. The composition operator $C_\Phi : A^2(\mathbb{D}^2) \to A^2(\mathbb{D})$ is defined by

$$C_\Phi(f)(z) = (f \circ \Phi)(z) = f(\varphi(z), \psi(z)), \quad z \in \mathbb{D}.$$

It is known that C_Φ is bounded; this was proved in [3] for the Hardy space, and in [4] for the Bergman space.

Theorem 2.1. *Let φ and ψ be analytic self-maps of the unit disk such that $\|\varphi\psi\|_\infty < 1$, and define $E_\Phi : A^2(\mathbb{D}^2) \to A^2(\mathbb{D})$ by*

$$\begin{aligned}(E_\Phi f)(z) &= (Rf \circ \Phi)(z) = (Rf)\big(\varphi(z), \psi(z)\big) \\ &= \varphi(z)\frac{\partial f}{\partial z_1}\big(\varphi(z), \psi(z)\big) + \psi(z)\frac{\partial f}{\partial z_2}\big(\varphi(z), \psi(z)\big).\end{aligned}$$

Then E_Φ is bounded. Moreover, if C_φ and C_ψ are compact, then E_Φ is compact.

References

[1] A. Abkar, *Hilbert-Schmidt composition-differentiation operators on the unit ball*, Anal. Math. Phys., **14** (2024).

[2] E. Nordgren, *Composition Operators*, Canadian Journal of Mathematics, **20** (1968), 442–449.

[3] K. J. Izuchi, Q. D. Nguyen and S. Ohno, *Composition operators induced by analytic maps to the polydisk*, Canadian J. Math., **64** (2012), 1329–1340.

[4] Z. Saeidiki and A. Abkar, *Composition operators on weighted Bergman spaces of polydisk*, Bull. Iran. Math. Soc., **49**, Artile 31, (2023).

[5] K. Zhu, *Operator Theory in Function Spaces*, AMS, 2007.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

ON FIXED POINT THEOREMS AND EQUILIBRIUM PROBLEMS

ALI FARAJZADEH

Department of mathematics, Razi University, Kermanshah, Iran
A.farajzadehali@razi.ac.ir, farajzadehali@gmail.com

Abstract. In this paper, the definition of pseudomonotone due to Brezis is extended to topological vector spaces. Then, three kinds of variational inequality problems, by investigating the relationship between their solution sets, in the setting of topological vector spaces are introduced and furthermore some existence theorems for being nonemptyness, convexity and compactness of their solution sets of them, under suitable assumptions are provided. Also Tychonoffs fixed point theorem will be extended from locally convex spaces to topolological vector spaces by using the notion of the quasi-pseudonorm. This class of spaces is between locally convex spaces and topological vector spaces. The method for proving the main theorem is based on the Fan-KKM Lemma.

1. Introduction and Preliminaries

In 1922, Banach [1] proved the following theorem, which is well known as "Banach's Fixed Point Theorem"

Theorem 1.1. *Let (E, d) be a complete metric space and $T : E \to E$ be a contractive mapping (that is, there exists $L \in [0, 1)$ such that*

$$d(Tx, Ty) \leq Ld(x, y),$$

for all $x, y \in E$). Then we have the following:

2020 *Mathematics Subject Classification. 46L05*
Key words and phrases. Quasi-norm, Equilibrium Problem, Radially continuous, Monotone.

(1) *T has a unique fixed point $x^* \in E$;*
(2) *Furthermore, for each $x_0 \in E$, the sequence $\{x_n\}$ defined by*

$$x_{n+1} = Tx_n,$$

for each $n \geq 0$ converges to the fixed point x^* of T, that is, $Tx^* = x^*$.

In 1912, the Brouwer [4] fixed point theorem appeared:

Theorem 1.2. *A continuous map from an compact convex subset of $\mathbb{R}^n$ to itself has a fixed point.*

In 1930, Schauder [10] extended Brouwer's fixed-point theorem from Euclidean spaces to Banach spaces.

Theorem 1.3. *Let C be a nonempty closed convex subset of a Banach space X. If $f : C \to C$ is continuous with a compact image, then f has a fixed point.*

To generalize the underlying spaces in fixed point theory, in 1934, Tychonoff [11] extended Schauder's fixed point theorem from Banach spaces to locally convex topological vector space.

Theorem 1.4. Tychonoff's fixed point theorem. *Let X be a Hausdorff locally convex topological vector space. For any nonempty compact convex set C in X, any continuous function $f : C \to C$ has a fixed point. Meanwhile, Schauder had the well-known conjecture.*

Schauder's Conjecture. Every continuous function, from a nonempty compact and convex set in a Hausdorff topological vector space into itself, has a fixed point.

In 1929, Knaster, Kuratowski, and Mazurkiewicz [7] (simply, KKM) obtained the following so-called KKM theorem:

Theorem 1.5. *Let A_i $(0 \leq i \leq n)$ be $n+1$ closed subsets of compact convex set $Co\{p_0, p_1, \ldots, p_n\}$ of $\mathbb{R}^n$. If the inclusion relation*

$$Co\{p_{i_0}, p_{i_1}, \ldots, p_{i_k}\} \subset A_{i_0} \cup A_{i_1} \cup \ldots \cup A_{i_k},$$

holds for all $p_{i_0}, p_{i_1}, \ldots, p_{i_k}$ $(0 \leq k \leq n, 0 \leq i_0 < i_1 < \ldots < i_k \leq n)$, then $\bigcap_{i=0}^{n} A_i \neq \emptyset$.

A milestone on the history of the KKM theory was erected by Ky Fan [5]. He extended the KKM theorem to infinite dimensional spaces and applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space.

Theorem 1.6. [5] *Let X be an arbitrary set in a topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of any finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in $\bigcup_{i=1}^{n} F(x_i)$.*

(ii) *$F(x)$ is compact for at least one $x \in X$. Then $\bigcap_{x\in X} F(x) \neq \emptyset$.*
This is usually known as the KKMF theorem.

2. Main Results

Theorem 2.1. *Let X be a subset of a Hausdorff topological space, Y be a nonempty set, and $F : Y \to 2^X$ be a set-valued mapping with nonempty intersectionally closed values and for at least one $y \in Y$, $F(y)$ is relatively compact. Assume that there exists a set-valued mapping $G : X \to 2^X$ such that family $\{G(x) : x \in X\}$ has finite intersection property and*

$$G(F(y)) \subseteq F(y), \ \forall y \in Y.$$

Then $\bigcap\limits_{y\in Y} F(y) \neq \emptyset$.

In 1994, Blum and Oettli [3] considered the following problem:

The Equilibrium Problem (EP) is the following: Find $x^* \in C$ such that $f(x^*, y) \geq 0$, for all $y \in C$.

In 1966, Hartman and Stampacchia [6] introduced the following variational inequality:

Lemma 2.2. *Let K be a compact convex subset in $\mathbb{R}^n$ and $f : K \to \mathbb{R}^n$ a continuous map. Then there exists $u_0 \in K$ such that*

$$\langle f(u_0), v - u_0\rangle \geq 0, \ \forall v \in K,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^n$.

Definition 2.3. Let $C \subseteq X$ be a nonempty set and $T : X \rightrightarrows X^*$, a set-valued mapping with $C \subseteq dom(T) = \{x \in X \ : \ T(x) \neq \emptyset\}$. An element $\bar{x} \in C$ is called a solution of

(a) weak variational inequality (WVI(T,C)), if for every $y \in C$, there exists $x^*(y) \in T(\bar{x})$ satisfying

$$\langle x^*(y), y - \bar{x}\rangle \ \geq 0.$$

(b) strong variational inequality (SVI(T,C)), if there exists $x^* \in T(\bar{x})$ satisfying

$$\langle x^*, y - \bar{x}\rangle \ \geq 0 \quad \text{for all} \quad y \in C.$$

Definition 2.4. Let X be a topological vector space and $T : X \rightrightarrows X^*$ a set-valued map. T is called B-pseudomonotone on a nonempty subset D of $dom(T)$ if, for every net $\{x_i\}$ in D such that $x_i \rightharpoonup x \in D$, and for every $x_i^* \in T(x_i)$, with $\limsup\limits_i \langle x_i^*, x_i - x\rangle \leq 0$, one has that for every $y \in D$, there exists $x^*(y) \in T(x)$ such that $\langle x^*(y), x - y\rangle \leq \ \liminf\limits_i \langle x_i^*, x_i - y\rangle$.

Definition 2.5. [9] A single-valued map $T : X \to X^*$ is said to be strictly monotone if for any $x \neq y$

$$\langle T(x) - T(y), x - y\rangle > 0.$$

Definition 2.6. [2] Let $T : X \rightrightarrows X^*$ and $D \subseteq dom(T)$. We say that T is radially continuous on D if its graph, that is $Gr(T) = \{(x, x^*) \in X \times X^* : x^* \in T(x)\}$, satisfies the condition: for all $x, y \in D$, $t_i \in [0,1]$, $t_i \to 0$, if

$$((1-t_i)x + t_i y, z_i^*) \in Gr(T), \quad \textit{and} \quad z_i^* \to z^*,$$

then $(x, z^*) \in Gr(T)$.

Proposition 2.7. *Let X be a Fréchet space with dual X^*, $C \subseteq int(dom(T))$, C closed and convex, and $T : X \rightrightarrows X^*$ be monotone, and radially continuous on C. Then T is B-pseudomonotone on C.*

Proposition 2.8. *Let X be a topological vector space with dual X^*, $C \subseteq dom(T)$, C closed and convex, and $T : X \rightrightarrows X^*$ be radially continuous, and locally bounded on C. If T is monotone on C, then T is B-pseudomonotone on C.*

Proposition 2.9. *Let X be a topological vector space with dual X^*, $C \subseteq dom(T)$, C closed and convex, and $T : X \rightrightarrows X^*$ be a lower semicontinuous mapping. If T is monotone on C, then T is B-pseudomonotone on C.*

Theorem 2.10. *Let C be a nonempty, closed and convex subset of a Hausdorff topological vector space E, and $T : C \to 2^{E^*}$ a set-valued mapping. Assume that the bifunction induced by T, that is $G_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x\rangle$, satisfies in the following assumptions:*

- (i) *for all $y \in C$ the set $\{x \in C : G_T(x, y) \geqslant 0\}$ is closed;*
- (ii) *there exist $D, K \subset C$, such that D is convex, compact and K is compact, and for every $x \in C \setminus K$, there exists $y \in D, G_T(x, y) < 0$.*

Then, the solution set of (EP) for G_T is nonempty and compact.

Theorem 2.11. *Let $T : X \rightrightarrows X^*$ and C be a nonempty, closed and convex subset of $dom(T)$. Suppose that:*

- (i) *$T(x)$ is compact for every $x \in C$;*
- (ii) *T is closed;*
- (iii) *there exist $D, K \subset C$, such that D is convex, compact and K is compact, and for every $x \in C \setminus K$, there exists $y \in D, \sup_{x^* \in T(x)} \langle x^*, y - x\rangle < 0$.*

Then the solution set of VI(T,C) is nonempty.

Theorem 2.12. *If the set-valued mapping $T : X \rightrightarrows X^*$ satisfies the hypotheses (i)-(iii) of Theorem A and the values of T are convex, then the solution set of SVI(T,C) is nonempty.*

By using Theorem 2.11, one can state another version of Theorem 2.12 as follows.

Theorem 2.13. *Let the set-valued mapping $T : X \rightrightarrows X^*$ be upper semicontinuous with compact values. If T satisfies the condition (iii) of Theorem A and the values of T are convex, then the solution set of SVI(T,C) is nonempty and compact.*

Theorem 2.14. *Let $T : X \rightrightarrows X^*$ be a closed set-valued mapping with compact values, $J : X \to X^*$ a continuous mapping, and C be a nonempty, closed and convex subset of dom(T). If there exist $D, K \subset C$, such that D is convex, compact and K is compact, and for every $x \in C \setminus K$, there exists $y \in D$, such that*

$$\sup_{x^* \in T(x)+\alpha J(x)} \langle x^*, y - x\rangle < 0, \qquad \textit{(coercivity)}$$

then, the solution set of $VI(T + \alpha J, C)$ is nonempty and compact subset of K. Moreover, if the values of T are convex, then all weak solutions of this problem are strong.

Theorem 2.15. *Let C_k, C be nonempty, closed and convex subsets of X, and $T_k, T : X \rightrightarrows X^*$ be set-valued mappings, and $J : X \to X^*$ be a monotone mapping such that*

(i) *$C_k \subseteq dom(T_k)$, and $C \cup (\bigcup_k C_k) \subseteq dom(T)$;*
(ii) *$\alpha_k \langle J(x), x_k\rangle \to 0$, for each $x \in \bigcap_k C_k$, $\{x_k\}$ unbounded sequence of $\bigcup_k C_k$ and $\alpha_k \to 0$;*
(iii) *if $\bigcup_k C_k$ is unbounded, the following coercivity holds: there exists $\tilde{x} \in \bigcap_k C_k$ such that, for each unbounded sequence $\{x_n\}$ of $\bigcup_k C_k$;*

$$\limsup_n G_T(x, \tilde{x}) < 0.$$

If x_k is a weak solution of $VI(T_k + \alpha_k J, C_k)$, for every $k \in \mathbb{N}$, then $\{x_k\}$ is bounded.

Theorem 2.16. *Suppose that $x_k \in SVI(T_k + \alpha_k J, C_k)$, $x_k \to x$, and T is closed and bounded on $\bigcup_k C_k$, $C \subset C_k$, for all $k \in \mathbb{N}$. If, J is continuous and for $z_k^* \in T_k(x_k)$, there exists $w_k^* \in T(x_k)$ such that $z_k^* - w_k^* \to 0$, then x is a strong solution of $VI(T, C)$.*

Proposition 2.17. *Let $T : X \rightrightarrows X^*$ be a set-valued B-pseudomonotone map with compact and convex values. Then the set of strong solutions of $VI(T, C)$ is closed.*

Proposition 2.18. *Let $T : X \rightrightarrows X^*$ be upper semicontinuous mapping with compact values. Then $S(T, C)$, the strong solutions of $VI(T, C)$, is closed.*

Theorem 2.19. *Let $T : X \rightrightarrows X^*$ be a set-valued map satisfying the conditions*

i. *T is B-pseudomonotone on $C \subseteq dom(T)$;*
ii. *$T(x)$ is compact and convex, for every $x \in C$.*

Then $S(T, C)$ is closed.

Theorem 2.20. *Let X be a compact subset of a Hausdorff topological space, Y be a nonempty set, and $f : X \times Y \to \mathbb{R}$, $g : X \times X \to \mathbb{R}$ be two bifunctions satisfying the following assumptions:*

(i) *The set-valued mapping* $F : Y \to 2^X$ *defined by*

$$F(y) = \{x \in X : f(x, y) \geq 0\},$$

is nonempty values and intersectionally closed.

(ii) *The family* $\{\{x \in X : g(x, z) \geq 0\}\}_{z \in X}$ *has finite intersection property.*

(iii) *For every* $x, z \in X$ *and* $y \in Y$ *the following implicaion holds*

$$g(x, z) \geq 0, f(z, y) \geq 0 \Rightarrow f(x, y) \geq 0.$$

Then, there exists $x_0 \in X$ *such that* $f(x_0, y) \geq 0$, *for all* $y \in Y$.

Jinlu [8] introduced a type of pseudonorms and quasipseudonorms on topological vector spaces, which is more general than seminorms. Then extended the Tychonoff's fixed point theorem to more general Hausdorff topological vector spaces. These spaces were named as pseudonorm adjoint topological vector spaces, which equipped with families of quasi-pseudonorms.

Definition 2.21. [8] Let X be a vector space. A mapping $p : X \to \mathbb{R}^+$ is called a pseudonorm on X if it satisfies the following conditions:

W_1. $p(x) \geq 0$, for all $x \in X$ and $p(0) = 0$; W_2. $p(-x) = p(x)$, for all $x \in X$; W_3. For any elements x_1, x_2 of X, and $0 \leq \alpha \leq 1$, one has

$$p(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha p(x_1) + (1 - \alpha)p(x_2).$$

It is clear to see that the condition W_3 implies that for any finite set of distinct elements $x_1, x_2, \ldots, x_n$ of X, and positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ satisfying $\sum_{i=1}^n \alpha_i = 1$, one has

$$p\left(\sum_{i=1}^n \alpha_i x_i\right) \leq \sum_{i=1}^n \alpha_i p(x_i).$$

Definition 2.22. [8] Let X be a vector space. A mapping $q : X \to \mathbb{R}^+$ is called a quasipseudonorm on X if there are a pseudonorm p on X and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

W_4. $q(x) \leq \varphi(p(x))$, for all $x \in X$;
W_5. $\varphi(0) = 0$.

Here, q is said to be adjoint with pseudonorm p and weighted function φ.

Definition 2.23. A topological vector space (X, τ) is called a pseudonorm adjoint topological vector space, if X is equipped with a family of τ-continuous quasipseudonorms $\{q_\lambda\}_{\lambda \in \Lambda}$ associated with a family of τ-continuous pseudonorms $\{p_\lambda\}_{\lambda \in \Lambda}$ and a family of weighted functions $\{\varphi_\lambda\}_{\lambda \in \Lambda}$.

Theorem 2.24. *Let* (X, τ) *be a Housdorrf and total pseudonorm adjoint topological vector space. Let* C *be a nonempty compact convex subset of* X. *Then every continuous mapping from* C *to itself has a fixed point.*

Definition 2.25. Let X be a vector space. A quasi-norm on X is a function $p : X \to \mathbb{R}$ satisfying

(i) $p(0) = 0$;
(ii) $p(x) \geq 0, \ \forall x \in X$;
(iii) $p(-x) = p(x), \ \forall x \in X$;
(iv) $p(x + y) \leq p(x) + p(y), \ \forall x, y \in X$;
(v) if $t_k, t \in \mathbb{F}$, $p(t_k - t) \to 0$ and $x_k, x \in X$, $p(x_k - x) \to 0$, then $p(t_k x_k - tk) \to 0$.

If the quasi-norm satisfies $p(x) = 0$ if and only if $x = 0$, then it is said to be total.

Remark 2.26. The topology of any topological vector space is generated by a family of quasi-norms.

Theorem 2.27. *Let X be a topological vector space, $\{p_\lambda\}_{\lambda \in \Lambda}$ be a total family of quasi-norms and quasi-convex. Let C be a nonempty compact convex subset of X. Then every continuous mapping from C to itself has a fixed point.*

References

[1] S. Banach, *Sur les operations dans les ensembles abstraits et leur applications aux equations integrales*, Fund. Math., **3** (1922), 133–181.
[2] M. Bianchi, G. Kassay and R. Pini, *Regularization of Brézis pseudomonotone variational inequalities*, Set-Valued and Variational Analysis, **29** (2021), 175–190.
[3] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student., **63** (1994), 123–145.
[4] LEJ. Brouwer, *Uber Abbildung von Mannigfaltigkeiten*, Math Ann., **7** (1) (1912), 9–115.
[5] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann., **142** (1961), 305–310.
[6] P. Hartman, G. Stampacchia, *On some non-linear elliptic differentialfunctional equations*, Acta Math., **115** (1966), 271–310.
[7] B. Knaster, K. Kuratowski and S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes fur n-Dimensionale Simplexe*, Fund. Math., **14** (1929), 132–137.
[8] J. Li, *An extension of Tychonoff's fixed point theorem to pseudonorm adjoint topological vector spaces*, Optimization, **70** (5-6) (2021), 1217–1229.
[9] A. S. Poznyak, *Advanced Mathematical Tools for Automatic Control Engineers*, Deterministic Techniques, 2009.
[10] J. Schauder, *Der Fixpunktsatz in Funktionalräumen* Studia Math., **2** (1930), 171–180.
[11] A. Tychonoff, *Ein Fixpunktsatz*, Mathematische Annalen., **111** (1935), 767–776.

LINEABILITY AND MAXIMAL LINEABILITY OF OPERATORS

MAJID FAKHAR*, MARYAM SOLEIMANI-MOURCHEHKHORTI

Department of Mathematics, University of Isfahan, Isfahan, Iran
School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran
fakhar@sci.ui.ac.ir; m-soleimani@ipm.ir

Abstract. In this article, we investigate lineability and maximal lineability of the set of norm attaining operators and non-absolutely p-summing operators between certain pairs of Banach spaces.

1. Introduction

In 1872 Weierstrass gave an example of a function that was continuous everywhere but differentiable nowhere.

$$f(x) = \sum_{k=0}^{\infty} a^k cos(b^k \pi x),$$

where $0 < a < 1$, b is any odd integer and $ab > 1 + \frac{3\pi}{2}$. After 1872 many other mathematicians also constructed similar functions. H. A. Schwarz (1873), M. G. Darboux (1874), U. Dini (1877), K. Hertz (1879), G. Peano (1890), D. Hilbert (1891), T. Takagi (1903), H. von Koch (1904), W. Sierpinski (1912), G. H. Hardy (1916), A. S. Besicovitch (1924), B. van der Waerden (1930), S. Mazurkiewicz (1931), S. Banach (1931), S. Saks (1932), and W.

2020 *Mathematics Subject Classification. Primary 46B87,; Secondary 15A03*
Key words and phrases. Lineable, Spaceable, Norm attaining operators, p-summing operators.
* Speaker.

Orlicz (1947). Banach (1931) showed that most continuous functions are nowhere differentiable. Specifically, the set of all continuous but nowhere differentiable functions on $\mathbb{R}$ is residual in $C(\mathbb{R})$, when endowed with the topology of uniform convergence in compacta.

Theorem 1.1. *(Gurariy [7]) The set of continuous nowhere differentiable functions on [0, 1] contains (except for the null function) an infinite dimensional vector subspace.*

Definition 1.2. Assume that X is a vector space (over the scalar field $\mathbb{R}$ or $\mathbb{C}$) and that μ is a cardinal number. Then a subset A of X is called

(1) lineable, if $A \cup \{0\}$ contains an infinite dimensional vector subspace,
(2) μ-lineable, if $A \cup \{0\}$ contains an μ-dimensional vector subspace (hence lineable means $\aleph_0$-lineable),
(3) maximal lineable, if A is $\dim(X)$-lineable.

Definition 1.3. If X is a topological vector space, then we say that A is

- spaceable, if $A \cup \{0\}$ contains some infinite dimensional closed vector subspace,
- maximal dense-lineable, if $A \cup \{0\}$ contains a dense vector subspace M of X with $\dim(M) = \dim(X)$.

2. Lineability of Norm Attaining Operators

Let E and F be two Banach spaces, We denote the space of linear and bounded operators from E to F (resp. finite rank and compact operators) by $\mathcal{L}(E, F)$ (resp. $\mathcal{F}(E, F)$ and $\mathcal{K}(E, F)$).
An operator $T \in \mathcal{L}(E, F)$ attain its norm at $x_0 \in B_E$ (the unit ball of E) if

$$\|T\| = \|T(x_0)\|.$$

Let us denote by $NA^{x_0}(E; F)$ the set of all operators from E to F that attain the norm at x_0.
T is said norm attaining, if it attain its norm at some $x \in B_E$. We write $NA(E, F)$ to denote the set of norm-attaining operators. If $F = \mathbb{R}$ *or* $\mathbb{C}$, then we denote $NA(E, F)$ by $NA(E)$.

Theorem 2.1. *(E. Bishop and R.R. Phelps [4]) $NA(E)$ is a dense subset in E^*.*

Theorem 2.2. *(James, 1963) If E is reflexive, then $NA(E) = E^*$ ($NA(E)$ is linear space).*

The following theorem shows why the lineability of $NA(E)$ is important.

Theorem 2.3. *(Petunin-Plichko [11]) Let E be a separable Banach space, W be closed subspace of E^*, separating and $W \subseteq NA(E)$ then W is an isometric predual of E.*

Example 2.4.

- $NA(c_0) = c_{00} \subseteq \ell_1$ (linear space, $NA(c_0)$ is lineable).

- $NA(\ell_1) = \{(\xi_n) \in \ell_\infty : \|\xi_n\|_\infty = \max_n |\xi_n|\}$ is not linear space. If choose $\xi = (-1, 0, 0, ...), \eta = (1, \frac{1}{2}, \frac{2}{3}, ..., \frac{n}{n+1}, ...)$, then $\xi, \eta \in NA(\ell_1)$ but $\xi + \eta \notin NA(\ell_1)$.

However, from Hahn-Banach Theorem, $c_0 \hookrightarrow NA(\ell_1)$ ($NA(\ell_1)$ is spaceable).

In 2001, Godefroy [6] raised the following problem.

Problem 2.5. Given an infinite dimensional Banach space E, is $NA(E)$ always lineable? Or, at least, can E always be equivalently renormed to make $NA(E)$ lineable?

Theorem 2.6. *(Acosta, Aizpuru, Aron, and García-Pacheco [1]) Let K be an infinite compact Hausdorff topological space. Then $NA(C(K))$ is lineable.*

Problem 2.7. (Bandyopadhyay and Godefroy [3]) Given an infinite-dimensional Banach space E, does $NA(E)$ contain at least a 2-dimensional vector subspace?

Theorem 2.8. *(M. Rmoutil, [10]) There is a Banach space E (= Reads space) for which $NA(E)$ does not contain 2-dimensional spaces.*

Theorem 2.9. *(M. Martín [8]) Given $n \in N$, there is a Banach space E_n such that $NA(E)$ contains n-dimensional spaces but not $(n+1)$-dimensional spaces.*

Theorem 2.10. *(D. Pellegrino and E. Texeira [9]) Let E and F be Banach spaces so that F contains an isometric copy of ℓ_q for some $1 \leq q < \infty$, and let $x_0 \in S_E$. Then $NA^{x_0}(E; F)$ is lineable in $\mathcal{L}(E; F)$. In particular*

- *$N(E, F)$ is lineable,*
- *$\mathcal{L}(E; F) \setminus N(E, F)$ is lineable if it is nonempty.*

Theorem 2.11.

- *If $x_0 \in S_E$, and F is any nonzero Banach space, then $(NA^{x_0}(E, F) \cap \mathcal{F}^1(E, F)) \cup \{0\}$ contains an isometric copy of F, so $NA^{x_0}(E, F) \cap \mathcal{F}^1(E, F)$ is spaceable.*
- *If E is not reflexive and F is any nonzero Banach space, then $\mathcal{F}^1(E, F) \setminus NA(E, F)$ is spaceable.*

Theorem 2.12. *If $NA(E)$ contains a (closed) subspace of dimension equal to μ, then $(NA(E, F) \cap \mathcal{F}^1(E, F)) \cup \{0\}$ also contains a (closed) subspace of dimension equal to μ.Therefore if $NA(E)$ is $\mu-$lineable, then $NA(E, F) \cap \mathcal{F}^1(E, F)$ is also $\mu-$lineable.*

Theorem 2.13. *If $x_0 \in S_E$ and $\{{x_i}^* : \ i \in \mathbb{N}\} \subset NA^{x_0}(E)$ is a linearly independent set in E^* and F has a $1-$unconditional basic sequence. Then*

- *$NA^{x_0}(E, F) \cap \mathcal{F}^i(E, F)$ is spaceable in $\mathcal{L}(E, F)$, for every $i \in \mathbb{N}$.*
- *$NA^{x_0}(E, F) \cap (\mathcal{N}(E, F) \setminus \mathcal{F}(E, F))$ is spaceable in $\mathcal{L}(E, F)$.*

3. Lineability of Non p-summing operators

Definition 3.1. Let $1 \leq p < \infty$ and $T : E \to F$ be a linear operator. We say that T is p-summing if there is a constant $c \geq 0$ such that for every $m \in \mathbb{N}$ and for every $x_1, ..., x_m \in E$ we have

$$(\sum_{i=1}^{m} \|Tx_i\|^p)^{\frac{1}{p}} \leq c \sup\{(\sum_{i=1}^{m} |\langle x^*, x_i\rangle|^p)^{\frac{1}{p}} : x^* \in B_{E^*}\}.$$

The space of p-summing operators from E to F is denoted by $\Pi_p(E, F)$. It is clear that $\Pi_p(E, F)$ is linear subspace of $\mathcal{L}(E, F)$.
A Banach space E is super-reflexive if and only if X admits an equivalent uniformly convex norm.

Theorem 3.2. *(Davis and Johnson[14]) If E is super-reflexive and F is arbitrary, then the set $\mathcal{K}(E, F) \setminus \Pi_p(E, F)$ is non-empty for all p.*

Puglisi and Seoane-Sepúlveda [16] raised the following problem.

Problem 3.3. If E is super-reflexive, F is any Banach space and $p \geq 1$. Is the set $\mathcal{L}(E, F) \setminus \Pi_p(E, F)$ lineable?

Theorem 3.4. *(Botelho, Diniz and Pellegrino [12]) Let $p \geq 1$ and E be superreflexive. If either E contains a complemented infinite-dimensional subspace with unconditional basis or F contains an infinite unconditional basic sequence then $\mathcal{K}(E, F) \setminus \Pi_p(E, F)$ is lineable (hence $\mathcal{L}(E, F) \setminus \Pi_p(E, F)$ is lineable).*

Problem 3.5. (Botelho, Diniz and Pellegrino, 2009) Under what circumstances is $\mathcal{L}(E, F) \setminus \Pi_p(E, F)$ μ-lineable for $\mu > \aleph_0$, when E is a super-reflexive Banach space?

Theorem 3.6. *(Botelho, Diniz, Pellegrino and Teixeira [13]) Let $p \geq 1$ and E be superreflexive. If either E contains a complemented infinite-dimensional subspace with unconditional basis or F contains an infinite unconditional basic sequence then $\mathcal{K}(E, F) \setminus \Pi_p(E, F)$ (hence $\mathcal{L}(E, F) \setminus \Pi_p(E, F)$) is $\mathfrak{c}$-lineable.*

Theorem 3.7. *(Kitson and Timoney [15]) If E is super-reflexive, for any Banach space F, $\mathcal{K}(E, F) \setminus \cup_{1 \leq p < \infty} \Pi_p(E, F)$ is spaceable in $\mathcal{K}(E, F)$.*

Hence, by using the fact that the dimension of every infinite dimensional Banach space is at least $\mathfrak{c}$, we conclude that if E is a super-reflexive Banach space the set $\mathcal{K}(E, F) \setminus \cup_{1 \leq p < \infty} \Pi_p(E, F)$ is $\mathfrak{c}$-lineable, for any Banach space F.

Theorem 3.8. *Let E be a super-reflexive Banach space and let one of the following conditions be satisfied.*

(a) *E is separable and $|F| = \mathfrak{c}$.*
(b) *F has a (Schauder) basis and $|E| = \mathfrak{c}$.*

Then $\mathcal{K}(E,F)\setminus \cup_{1\leq p<\infty} \mathit{\Pi}_p(E,F)$ is maximal lineable in $\mathcal{L}(E,F)$.

Theorem 3.9. *Assume that E is a super-reflexive Banach space, F is a Banach space satisfying that $\overline{\mathcal{F}(E,F)} = \mathcal{K}(E,F)$ and $|E| = |F| = \mathfrak{c}$. Then $|\mathcal{K}(E,F)| = \mathfrak{c}$, and specifically, $\mathcal{K}(E,F)\setminus \cup_{1\leq p<\infty} \mathit{\Pi}_p(E,F)$ is maximal lineable in $\mathcal{K}(E,F)$.*

References

1. M. D. Acosta, A. Aizpuru, R. M. Aron, F. J. García-Pacheco, *Functionals that do not attain their norm*, Bull. Belg. Math. Soc. Simon Stevin, **14** (2007), 407-418.
2. R.M. Aron, L. Bernal-González, D. Pellegrino and J.B. Seoane-Sepúlveda, *Lineability: The Search for Linearity in Mathematics*, Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2016.
3. Bandyopadhyay, G. Godefroy, *Linear structures in the set of norm-attaining functionals on a Banach space*, J. Convex Anal., **13** (2006), 489-497.
4. E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. **67** (1961), 97-98.
5. F.J. García-Pacheco and Puglisi, *Lineability of functionals and renormings*, Bull. Belg. Math. Soc. Simon Stevin, **25** (2018), 141-147.
6. G. Godefroy, *The Banach space c_0*, Extracta Math., **16** (2001), 1-25.
7. V. I. Gurariy, *Subspaces and bases in spaces of continuous functions*, Dokl. Akad. Nauk SSSR, (1966) (Russian)
8. M. Martín, *On proximinality of subspaces and the lineability of the set of norm-attaining functionals of Banach spaces*, J. Fuct. Anal., **278**(2020), 108353, 14p.
9. D. Pellegrino and E. V. Teixeira, *Norm optimization problem for linear operators in classical Banach spaces*, Bull. Braz. Math. Soc., **40** (2009) 417-431.
10. M. Rmoutil, *Norm-attaining functionals need not contain 2-dimensional subspaces*, J. Funct. Anal., **272** (2017), 918-928.
11. Yu. I. Petunin and A. N. Plichko, *Some properties of the set of functionals which attain their supremum on the unit sphere*, Ukrainiam Math. J., **26** (1974), 85-88.
12. G. Botelho, D. Diniz and D. Pellegrino, *Lineability of the set of bounded linear non-absolutely summing operators*, J. Math. Anal. Appl., **357** (1) (2009), 171-175.
13. G. Botelho, D. Diniz, D. Pellegrino and E.Teixeira, *A note on lineability*, arXiv:0905.2677v1 [math.FA] (2009)
14. W.J. Davis, J and W.B. Johnson, *Compact, non-nuclear operators*, Studia Math., **51** (1) (1974), 81-85.
15. D. Kitson and R.M. Timoney, *Operator ranges and spaceability*, J. Math. Anal. Appl., **378** (2) (2011), 680–686.
16. D. Puglisi and J.B. Seoane-Sepúlveda, *Bounded linear non-absolutely summing operators*, J. Math. Anal. Appl., **338** (1) (2008), 292–298.

ANALYSIS OF AN UNREDUCED ANTI-HESSENBERG MATRIX

ALIMOHAMMAD NAZARI* AND ATIYEH NEZAMI

Department of Mathematics, Arak University, Arak, Iran
a-nazari@araku.ac.ir; atiyeh.nezami@gmail.com

Abstract. This paper delves into the spectral properties and decomposition techniques of an Unreduced anti-Hessenberg matrix. Such matrices frequently emerge in graph theory and eigenvalue problems, where their specific structure allows for efficient computation of critical matrix properties. We provide a comprehensive analysis of the eigenvalues, determinant, inverse, and various decomposition methods, including LU and QR factorizations, as well as singular value decomposition (SVD). Our results demonstrate the unique behaviour of this matrix in different contexts and offer insights into its potential applications in numerical methods and theoretical studies. Special emphasis is given to the implications of the matrix's structure on its eigenvalue spectrum and stability. The findings contribute to a deeper understanding of this matrix's role in mathematical modelling and computational techniques, making it a valuable tool for researchers in applied mathematics and engineering disciplines.

1. Introduction

Matrices come in various forms and have numerous applications in engineering and different fields of science. Each matrix, based on its specific properties, can serve diverse purposes. Therefore, studying and analyzing

2020 *Mathematics Subject Classification. Primary 05C50; Secondary 05C76*
Key words and phrases. Unreduced anti-Hessenberg matrix, Eigenvalues, Determinant, Inverse.

* Speaker.

the characteristics of matrices is of great importance [1, 2, 3]. We begin by considering the following nonsingular matrix [1]:

$$A_n = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 & 1 & 1 \\ 1 & 1 & \cdots & \cdots & 1 & 1 & -1 \\ 1 & 1 & \cdots & \cdots & 1 & -2 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ 1 & -(n-1) & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

In this paper, we called matrix A_n an Unreduced anti-Hessenberg matrix. This matrix play a significant role in various fields of science and engineering due to their computational efficiency and unique properties [1]. This specific matrix structure often appears in graph theory, numerical linear algebra, and eigenvalue problems, where it serves as a fundamental tool for modelling and analysis. In this paper, we focus on matrix A_n. Understanding the spectral properties and decomposition techniques of such matrices is crucial for solving complex problems efficiently. We aim to explore the eigenvalues, determinant, inverse, LU and QR factorizations, and singular value decomposition of this matrix, providing insights into its mathematical behaviour and potential applications. This study not only enhances our understanding of the matrix's inherent characteristics but also contributes to the broader field of matrix analysis and its applications in computational methods. In the next section, we will discuss the aforementioned properties of this matrix. We will prove the first theorem and accept the remaining theorems without proof.

2. Main results

Theorem 2.1. *The determinant of matrix A_n is given by the following formula:*

$$\det(A_n) = (-1)^{\lfloor \frac{n}{2} \rfloor} n!.$$

Proof. We will prove this theorem by induction. If matrix A_n is of order $n = 2$, then its determinant is given by the following formula:

$$\det(A_2) = (-1)^{\lfloor \frac{1}{2} \rfloor} 2! = (-1)^1 2 = -2.$$

Assume that the determinant of matrix A_n of order $n = k-1$ is obtained from the following formula:

$$\det(A_{k-1}) = (-1)^{\lfloor \frac{k-1}{2} \rfloor} (k-1)!.$$

Now we will prove that the determinant of matrix A_n of order $n = k$ is given by the following formula:

$$\det(A_k) = (-1)^{\lfloor \frac{k}{2} \rfloor}(k)!.$$

Let k be an even number. For calculating the determinant of matrix A_k, we expand it along the last row. Thus, we have

$$\begin{aligned}
\det(A_k) &= -1 \times \det(A_{k-1}) + (-(k-1)) \times \det(A_{k-1}) \\
&= -(-1)^{\lfloor \frac{k-1}{2} \rfloor}(k-1)! - (k-1)(-1)^{\lfloor \frac{k-1}{2} \rfloor}(k-1)! \\
&= -(-1)^{\lfloor \frac{k-1}{2} \rfloor}(k-1)!(1 + k - 1) \\
&= (-1)^{\lfloor \frac{k}{2} \rfloor}(k)!.
\end{aligned}$$

Similarity, the theorem is proven for when k is an odd number. □

Theorem 2.2. *The inverse of matrix A_n is given by the following matrix:*

$$A_n^{-1} = \begin{bmatrix}
\frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} \\
\frac{1}{n(n-1)} & \frac{1}{n(n-1)} & \frac{1}{n(n-1)} & \cdots & \frac{1}{n(n-1)} & -\frac{1}{n} \\
\frac{1}{(n-1)(n-2)} & \frac{1}{(n-1)(n-2)} & \frac{1}{(n-1)(n-2)} & \cdots & -\frac{1}{n-1} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{(3)(4)} & \frac{1}{(3)(4)} & \frac{1}{(3)(4)} & \cdots & 0 & 0 \\
\frac{1}{(2)(3)} & \frac{1}{(2)(3)} & -\frac{1}{3} & \cdots & 0 & 0 \\
\frac{1}{2} & -\frac{1}{2} & 0 & \cdots & 0 & 0
\end{bmatrix}.$$

Theorem 2.3. *The eigenvalues of matrix A_n are calculated as follows:*

(1) *If n be even:*

$$\lambda_{i,n-(i-1)} = \pm\sqrt{i(n-i)+i}, \qquad i = 1, 2, \cdots, \frac{n}{2}.$$

(2) *If n be odd:*

$$\lambda_{i,n-i} = \pm\sqrt{i(n-i)+i}, \qquad i = 1, 2, \cdots, \frac{n-1}{2},$$

and

$$\lambda_n = \sqrt{\frac{n-1}{2}(n - \frac{n-1}{2}) + \frac{n-1}{2} + 1}.$$

Theorem 2.4. *The singular value of matrix A_n is follows the formula below:*

$$\sigma_1 = \sqrt{n},$$

$$\sigma_{i+2} = \frac{1}{n-i}\sqrt{(n-i)^3(n-i-1)}, \qquad i = 0, 1, \cdots, n-2.$$

Theorem 2.5. *The QR decomposition of matrix A_n is as follows:*

$Q = (A_n^{-1})^T R,$

$$R = \begin{bmatrix} \sqrt{n} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{n}\sqrt{n^3(n-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{n-1}\sqrt{(n-1)^3(n-2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3}\sqrt{3^3 \times 2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{2^3 \times 1} \end{bmatrix}.$$

Theorem 2.6. *The LU decomposition of matrix A_n is as follows:*

$$P = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & -n & -1 & -1 & \cdots & -1 & -1 \\ 0 & 0 & -(n-1) & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & -1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -2 \end{bmatrix}.$$

References

1. R. C. Diaz and A. I. Julio, *New results about the Inverse Eigenvalue Problem of a Graph*, arXiv preprint arXiv:2303.09739.
2. O. H. Hald, *Inverse eigenvalue problems for Jacobi matrices*, Linear Algebra Appl., **14** (1) (1976), 63–85.
3. K. H. Monfared and B. L. Shader, *Construction of matrices with a given graph and prescribed interlaced spectral data*, Linear Algebra Appl., **438** (11) (2013), 4348–4358.

Contributed Talks

ON THE STEFFENSEN-POPOVICIU MEASURE

NASER ABBASI*, ALI BARANI AND SAEED NAZARI PASARI

Department of Mathematics, Faculty of Science, Lorestan University, Khoramabad, Iran
abasi.n@lu.ac.ir; barani.a@lu.ac.ir; saeid.nazari@gmail.com

Abstract. Steffensen-Popovicio measure is investigated in plane. Several properties and results introduce in this setting. Suitable examples are also involved.

1. Introduction

The convex functions play important role in many mathematics topics, especialy in the optimization problems. The integral of a nonnegative convex functions, in terms of an arbitrary measure, may not be necessarily nonnegative, but their integral in terms of Steffensen-Popoviciu measures are nonnegative. Motivated by the above studies, we prove some inequalities by Steffensen-Popoviciu measures in dimension 1 and 2.

We recall the definition of Steffensen-Popoviciu measure from [5]:

Definition 1.1. A Steffensen-Popoviciu measure on interval $I = [a, b]$ is a real Borel measure μ on I such that

i) $\mu(I) = \int_a^b d\mu(x) > 0$,

ii) $\int_a^b f(x)d\mu(x) \geqslant 0$ for every nonnegative $f \in C$.

It is clear that every finite positive Borel measure is a Steffensen-Popoviciu measure.

2020 *Mathematics Subject Classification. Primary 26A51; Secondary 26D99*

Key words and phrases. Steffensen-Popovicio measure, co-ordinate convex function.

* Speaker.

The following result from T. Popoviciu [6] and A. M. Fink [3] gives a characterization of the Steffensen-Popoviciu measures. See also [4, p177], for details.

Lemma 1.2. *Let μ be a real Borel measure on interval $[a,b]$ such that $\mu([a,b]) > 0$. Then μ is a Steffensen-Popoviciu measure if and only if, it verifies the following conditions*

$$\int_a^t (t-x)d\mu(x) \geqslant 0 \quad \textit{and} \quad \int_t^b (x-t)d\mu(x) \geqslant 0$$

for every $t \in [a,b]$.

The measures

$$(x^2-1/6)^3 dx \quad on \quad [-1,1]$$

$$\left[(\frac{2x-a-b}{b-a})^2 + \lambda \right] dx \quad on \quad [a,b] \quad (\lambda \geqslant -\frac{1}{4})$$

$$\left[(\frac{2x-a-b}{b-a})^2 - \lambda(\frac{2x-a-b}{b-a}) \right] dx \quad on \quad [a,b] \quad (|\lambda| \leqslant \frac{2}{3})$$

are some examples of Steffensen-Popoviciu measures on intervals from [5].

2. Main results

In this section we introduce generalization of Steffensen-Popoviciu measure in case of two dimension and obtain some new inequalities by using these measures.

Remark 2.1. Let $I = [a,b]$ and $g, h : I \to \mathbb{R}$ are two Lebesgue integrable functions. Then

i) $g(x)dx$ is a Steffensen-Popoviciu mesure on I if and only if

$$\int_a^b g(x)dx > 0$$

and

$$\int_a^b f(x)g(x)dx \geqslant 0$$

for every nonnegative $f \in C$.

ii) Let $g \leqslant h$ and let $g(x)dx$ be a Steffensen-Popoviciu measure on interval I. Then $h(x)dx$ is also a Steffensen-Popoviciu measure on interval I.

iii) Let μ be a Steffensen-Popoviciu measure on interval I, $f \in C$, $i \in A(I)$ and $i \leqslant f$ on interval I, then

$$\int_a^b i(x)d\mu(x) \leqslant \int_a^b f(x)d\mu(x).$$

Next theorem is an extension of Theorem 4 in [5].

Theorem 2.2. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable function whose second derivative is an nonnegative continuous convex function. Then*

$$\frac{1}{b-a}\int_a^b f(x)dx \geqslant \frac{f(a)-f(b)}{2} + (1+\lambda)(\frac{b-a}{8})(f'(a)-f'(b))$$

where $\lambda \geqslant -\frac{1}{4}$.

Proof. Since f'' is nonnegative convex function and for every $\lambda \geqslant -\frac{1}{4}$, $[(\frac{2x-a-b}{b-a})^2 + \lambda]dx$ is a Steffensen-Popoviciu measure on interval $[a, b]$ then we obtain

$$0 \leqslant \int_a^b f''(x)[(\frac{2x-a-b}{b-a})^2 + \lambda]dx.$$

By integrating by parts the proof will be compelete. □

Fejér introduce the following celebrate inequality in [2]

Theorem 2.3. *let $f : [a, b] \to \mathbb{R}$ be a convex function and $w : [a, b] \to \mathbb{R}$ be a nonnegative integrable and symmetric function to $\frac{a+b}{2}$, i.e. $w(x) = w(a + b - x)$. Then*

$$f(\frac{a+b}{2}) \leqslant \frac{\int_a^b f(x)w(x)dx}{\int_a^b w(x)dx} \leqslant \frac{f(a)+f(b)}{2}. \tag{2.1}$$

The following inequality for convex functions can be obtained by replacing $w(x)$ by an appropriate Steffensen-Popoviciu measure in (2.1).

Theorem 2.4. *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then*

$$\frac{(b-a)^2}{3} f(\frac{a+b}{2}) \leqslant \frac{1}{b-a}\int_a^b f(x)(2x-a-b)^2 dx \leqslant \frac{(b-a)^2}{3}\left(\frac{f(a)+f(b)}{2}\right)$$

Proof. According to Remark 2.1, $\left(\frac{2x-a-b}{b-a}\right)^2 dx$ is a Steffensen-Popoviciu on $[a, b]$, then by replacing $w(x) := \left(\frac{2x-a-b}{b-a}\right)^2$ in (2.1), the proof will be complete. □

Now we introduce the notion of Steffensen-Popoviciu measure in two dimensions setting. In this section D will denote a nonempty compact convex subset of $\mathbb{R}$, and let $C(D)$ be the space of all two variable real-valued continuous functions on D and C be all of two variable real-valued continuous convex functions on D.

Definition 2.5. A Steffensen-Popoviciu measure on D is a real Borel measure μ on D such that

i) $\mu(D) = \int\int_D d\mu(x, y) > 0$,
ii) $\int\int_D f(x, y)d\mu(x, y) \geqslant 0$ for every nonnegative $f \in C$.

Now we recall convex on the co-ordinates function in [1].

Definition 2.6. The function $f : [a, b] \times [c, d] \to \mathbb{R}$ is called convex on the co-ordinates if the partial mappings $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ and $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ be convex for every $x \in [a, b]$ and $y \in [c, d]$ respectively.

It is clear that every two variable convex function on a compact convex area in $\mathbb{R}^2$, is convex on the co-ordinates function. See [1].

Definition 2.7. The function $w : [a, b] \times [c, d] \to \mathbb{R}$ is called symmetric on the co-ordinates if the partial mappings $w_x : [c, d] \to \mathbb{R}$, $w_x(v) = w(x, v)$ and $w_y : [a, b] \to \mathbb{R}$, $w_y(u) = w(u, y)$ be symmetric for every $x \in [a, b]$ about $\frac{c+d}{2}$ and $y \in [c, d]$ about $\frac{a+b}{2}$ respectively.

Now, by using the Theorem 1 in [7] and appropriate Steffensen-Popoviciu measure, we can get the following inequality for convex on the co-ordinates functions.

Corollary 2.8. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a convex on the co-ordinates integrable function. Then*

$$\begin{aligned} & \left(\frac{(b-a)(d-c)}{3}\right)^2 f(\frac{a+b}{2}, \frac{c+d}{2}) \\ \leqslant\ & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)(2x-a-b)^2(2y-c-d)^2 dydx \\ \leqslant\ & \left(\frac{(b-a)(d-c)}{3}\right)^2 \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

Proof. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a convex on the co-ordinates integrable function. By [7, Theorem 1] we have

$$\begin{aligned} f(\frac{a+b}{2}, \frac{c+d}{2}) &\leqslant \frac{\int_a^b \int_c^d f(x,y)w(x,y)dydx}{\int_a^b \int_c^d w(x,y)dydx} \\ &\leqslant \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

The measure $(\frac{(2x-a-b)(2y-c-d)}{(b-a)(d-c)})^2 dydx$ is a Steffensen-Popoviciu measure on $[a, b] \times [c, d]$. Now assume that

$$w(x,y) := \left(\frac{(2x-a-b)(2y-c-d)}{(b-a)(d-c)}\right)^2,$$

which is a nonnegative integrable and symmetric on the co-ordinates function on $f : [a, b] \times [c, d]$, then

$$f(\frac{a+b}{2}, \frac{c+d}{2})$$

$$\begin{aligned} &\leqslant \frac{\int_a^b \int_c^d f(x,y)(\frac{(2x-a-b)(2y-c-d)}{(b-a)(d-c)})^2 dydx}{\int_a^b \int_c^d (\frac{(2x-a-b)(2y-c-d)}{(b-a)(d-c)})^2 dydx} \\ &\leqslant \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

Therefore

$$\begin{aligned} &\left(\frac{(b-a)(d-c)}{3}\right)^2 f(\frac{a+b}{2}, \frac{c+d}{2}) \\ &\leqslant \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)(2x-a-b)^2(2y-c-d)^2 dydx \\ &\leqslant \left(\frac{(b-a)(d-c)}{3}\right)^2 \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

□

References

1. S. S. Dragomir, *On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane*, Taiwan. J Math., **4** (2001), 775–788.
2. L. Fejér, *Über die Fourierreihen*, Math. Naturwiss. Anz. Ungar. Akad. Wiss., **24** (1906), 369–390.
3. A. M. Fink, *A best possible Hadamard inequality*, Math. Inequal. Appl, **1** (2) (1998), 223-230.
4. C. P. Niculescu and L. E. Persson, *Convex Functions and their Applications. A Contemporary Approach*, CMS Books in Mathematics, **23** Springer-Verlag, Berlin-Heidelberg-New York, 2006.
5. C. P. Niculescu and M. M. Stanescu, *The Steffensen-Popoviciu measures in the context of quasiconvex functions*, J. Math. Inequal, **11** (2017), 469-483.
6. T. Popoviciu, *Notes sur les fonctions convexes d'ordre superieur IX*, Bull. Math. Soc. Roum. Sci., **43** (1941), 85-141.
7. K. L. Tseng, J. Pečariè and S. R. Hwang, *Inequalities of Hermite-Hadamard-Fejer type for convex functions and convex functions on the co-ordinates in a rectangle from the plane*, Taiwanese Journal of Mathematics, **1** (2008), 703-17.

ε-APPROXIMATIONS BY DOWNWARD SETS IN BANACH LATTICE SPACES

MAJID ABRISHAMI-MOGHADDAM*

Department of Mathematics, Birjand Branch, Islamic Azad University, Birjand, Iran.
m.abrishami.m@gmail.com

Abstract. In this paper we prove some results on characterization of ε-approximations of downward sets in Banach lattice spaces.

1. Introduction

Let X be a normed linear space, W a non-empty subset of X and $x \in X$. Define

$$d(x, W) = \inf_{w \in W} \|x - w\|.$$

We recall a point $w_0 \in W$ is called a ε-approximation to x from W if

$$\|x - w_0\| = d(x, W) + \varepsilon$$

and also

$$P_{W,\varepsilon}(x) := \{w \in W \ : \ \|x - w\| = d(x, W) + \varepsilon\}.$$

It is clear that $P_{W,\varepsilon}(x)$ is a closed and bounded subset of X. The concept of ε-approximation (or good approximation) has been studied by Singer [2]. In [1], Alizadeh et.al obtained results about the ε-simultaneous approximation of downward sets in vector lattice Banach spaces. In this paper according to the results obtained in [1], we prove some results on characterization of ε-approximations of downward sets in Banach lattice spaces.

2020 *Mathematics Subject Classification. Primary 41A65; Secondary 46B42.*
Key words and phrases. Banach lattice spaces, ε-approximation, Downward set.
* Speaker.

One advantage of cosidering the set $P_{W,\varepsilon}(x)$, instead of the set $P_W(x)$, is the set $P_{W,\varepsilon}(x)$ is always nonempty, for all $\varepsilon > 0$.

A real vector space X is defined as an ordered vector space if it has an order relation $\leq$ (i.e., $\leq$ is a reflexive, antisymmetric, and transitive binary relation on X). A vector lattice space (or a Riesz space) is an ordered vector space X with the additional property that for each pair of vectors $x, y \in X$, the $\sup\{x, y\}$ and the $\inf\{x, y\}$ both exist in X. As usual, $\sup\{x, y\}$ is denoted by $x \vee y$ and $\inf\{x, y\}$ by $x \wedge y$. Recall that a vector subspace S of a vector lattice space X is said to be a vector sublattice, whenever S is closed under the lattice operations of X, i.e., whenever for each pair $x, y \in S$ the vector $x \vee y$ and $x \wedge y$ (taken in X) belongs to S. A subset A of a vector lattice space is called solid whenever $|x| \leq |y|$ and $y \in A$ imply $x \in A$. A solid vector subspace of a vector lattice space is referred to as an ideal. An element $\mathbf{1} \in X$ is called a strong unit if for each $x \in X$ there exists $0 < \lambda \in \mathbb{R}$ such that $x \leq \lambda\mathbf{1}$. Then for each $x \in X$ there exists $0 < \lambda \in \mathbb{R}$ such that $|x| \leq \lambda\mathbf{1}$. Using $\mathbf{1}$ we can define a norm on X by

$$\|x\| = \inf\{\lambda > 0 \ : \ |x| \leq \lambda\mathbf{1}\}. \tag{1.1}$$

Recall that a norm $\|.\|$ on a vector lattice space is said to be a lattice norm whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. A vector lattice space equipped with a lattice norm is known as a normed vector lattice space. If a normed vector lattice space is also norm complete, then it is referred to as a Banach lattice. It is well known that X equipped with the norm (1.1) is a Banach lattice which is called a Banach lattice with strong unit $\mathbf{1}$.

The closed ball with center at x and radius r defined on Banach lattice X as follows:

$$B(x, r) = \{y \in X \ : \ \|y - x\| \leq r\} = \{y \in X \ : \ x - r\mathbf{1} \leq y \leq x + r\mathbf{1}\}.$$

A nonempty subset W of an ordered vector space X is called downward if

$$(w \in W \ , \ x \leq w) \Longrightarrow x \in W.$$

A simple example of a downward set is a set of the form $\{y \in X \ : \ y \leq g\}$, where $g \in X$. For another example, let $f : X \longrightarrow \mathbb{R}$ be an increasing function, then its lower level sets $S_c(f) = \{x \in X : f(x) \leq c\}$ for all $c \in \mathbb{R}$, are downward.

Let $\varphi : X \times X \longrightarrow \mathbb{R}$ be the function defined by

$$\varphi(x, y) := \{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq x + y\}, \tag{1.2}$$

for all $x, y \in X$. Since $\mathbf{1}$ is a strong unit, the set $\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq x + y\}$ is non-empty, closed and bounded from above by $\|x + y\|$.

The following theorems which are proved in reference [3], play a key role in the results obtained in this paper.

Theorem 1.1. [3] *Let W be a subset of X and φ be the coupling function defined by (1.2). Then the following assertions are equivalent:*

(1) *W is a downward set.*
(2) *For each $x \in X \backslash W$, we have*

$$\varphi(w, -x) < 0, \ \forall w \in W.$$

(2) *For each $x \in X \backslash W$, there exsist $l \in X$, such that*

$$\varphi(w, l) < 0 < \varphi(x, l), \ \forall w \in W.$$

Theorem 1.2. [3] *Let W be a closed donward subset of X , $x_0 \in X$. Let $y_0 \in W$ and $r_0 := \|x_0 - y_0\|$. Assume that φ is the function defined by (1.2). Then the following assettions are equivalent:*
(1) *$y_0 \in P_W(x)$.*
(2) *There exsits $l \in X$ such that*

$$\varphi(w, l) \leq 0 \leq \varphi(y, l), \tag{1.3}$$

for all $w \in W$ and $y \in B(x_0, r_0)$. Moreover, if (1.3) holds with $l = -y_0$, then $y_0 = \min P_W(x)$.

2. Main results

Let W be a closed subset of X and $x \in X \backslash W$. In addition, suppose that $w_0 \in intW \cap P_{W,\varepsilon}(x)$. Thus, there exists $\alpha > 0$ such that

$$V = \{y \in X \ : \ \|y - w_0\| < \alpha\} \subset W.$$

Lemma 2.1. *Let α be as above. Then, $\alpha \leq \varepsilon$.*

Proposition 2.2. *Let W be a closed subset of X and $x \in X \backslash W$. Then $P_{W,\varepsilon}(x) \subset V = \{w - \alpha \mathbf{1} \ : \ for\ some\ w \in bdW \text{ and } 0 \leq \alpha \leq \varepsilon\}$.*

Corollary 2.3. *Let W be a closed subset of X and $x \in X \backslash W$. Then $P_W(x) \subset bdW$.*

Proposition 2.4. *Let W be a closed donward subset of X and $x \in X \backslash W$. Then there exsits the least element $w_0 := \min P_{W,\varepsilon}(x)$.*

Proposition 2.5. *Let W be a closed donward subset of X , $x \in X \backslash W$, $w_0 \in P_{W,\varepsilon}(x)$ and φ be the function defined by (1.2). Then $\varphi(w, -w_0) \leq \varepsilon$, for all $w \in W$.*

Theorem 2.6. *Let W be a closed donward subset of X , $x \in X \backslash W$, $y_0 \in W$, $r_0 = \|x - y_0\|$ and φ be the function defined by (1.2). Then the following statements are equivalent:*
(1) *$y_0 \in P_{W,\varepsilon}(x)$.*
(2) *There exsits $l \in X$ such that*

$$\varphi(w, l) \leq \varepsilon \leq \varphi(y, l), \tag{2.1}$$

for all $w \in W$ and $y \in B(x, r_0)$. Moreover, if (2.1) holds with $l = -y_0$, then $y_0 = \min P_{W,\varepsilon}(x)$.

References

1. H. Alizadeh, Sh. Rezapour, S. M. Vaezpour, *ε-simultaneous approximations of downward sets*, Bulletin of the Iranian Mathematical Society, **39**(1) (2013), 193-203.
2. I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*,Springer-Verlag, New York, 1970.
3. H. Mohebi and A. M. Rubinov, *Best approximation by downward sets with applications*, Anal. Theory Appl. **22** (1) (2006), 20-40.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

SOME NOTES ON QUASI-MULTIPLIERS

MARJAN ADIB

Department of Mathematics, National University of Skills, (NUS) , Tehran, Iran
m.adib.pnu@gmail.com

ABSTRACT. In this paper, some properties and characterizations of quasi-multipliers on quasi-ideals of a Banach algebra has been taken into account. We apply our results to the quasi-ideals of the group algebra $M(G)$ and prove that $M(G)$ is isomorphic to a subspace of $QM_{gr}(L_1(G)^*)$ of all bilinear and separately continuous right generalized quasi-multiplier of $L_1(G)^*$.

1. INTRODUCTION

The notion of a quasi-multiplier is a generalization of the notion of a multiplier on a Banach algebra and was introduced by Akemann and Pedersen [6] for C^*-algebras. McKennon [7] extended the definition to a general complex Banach algebra A with a bounded approximate identity (b.a.i., for brevity) as follows. A bilinear mapping $m:\ A \times A \to A$ is a quasi-multiplier on A if

$$m(ab, cd) = a\, m(b, c)\, d \qquad (a, b, c, d \in A).$$

In this paper, we define the quasi-multipliers on quasi-ideals of a Banach algebra and extend the notion of strict topology on the space $QM_{gr}(A^*)$ of all generalized quasi-multipliers of A^*.

2020 *Mathematics Subject Classification. Primary 47B48; Secondary 46H25*
Key words and phrases. Quasi-multiplier, multiplier, Banach algebra, quasi-ideal.

2. Quasi-multipliers on quasi-ideals

Definition 2.1. A bilinear mapping $m : A^* \times A \to A^*$ is called a *right generalized quasi-multiplier of* A^* if

$$m(a \cdot \xi, b) = a \cdot m(\xi, b) \qquad \text{and} \qquad m(\xi, ba) = m(\xi, b) \cdot a \tag{2.1}$$

hold for arbitrary $\xi \in A^*$ and $a,\ b \in A$.

Let $QM_{gr}(A^*)$ be the set of all bilinear and separately continuous right generalized quasi-multipliers of A^*. It is obvious that $QM_{gr}(A^*)$ is a linear space. Moreover, it is a Banach space with respect to the norm

$$||m|| = \sup\{||m(\xi, a)||; \quad \xi \in A^*,\ a \in A,\ ||\xi|| \leq 1,\ ||a|| \leq 1\}.$$

Let A be a general Banach algebra. Then a map $T : A^* \to A^*$ is called a right generalized multiplier of A^* if

$$T(a \cdot \xi) = a \cdot T(\xi),$$

for all $\xi \in A^*, a \in A$. With $M_r(A^*)$ we denote the space of all bounded linear right generalized multipliers of A^*.

Definition 2.2. A bounded approximate identity $\{e_\lambda : \lambda \in I\}$ in a Banach algebra A is said to be ultra$^*-$approximate identity if, for all $m \in QM_{gr}(A^*)$ and $\xi \in A^*$, the net $\{m(\xi, e_\lambda) : \lambda \in I\}$ is Cauchy in A^*.

Theorem 2.3. *Let A be factorable with an ultra$^*-$approximate identity $\{e_\lambda\}$. Then the mapping $\rho :\ M_r(A^*) \to QM_{gr}(A^*)$, defined by*

$$\rho_T(\xi, a) = (T\xi) \cdot a \qquad (T \in M_r(A^*),\ \xi \in A^*,\ a \in A),$$

is a bijective with norm $\|\rho\| \leq 1$. If $\{e_\lambda\}$ is of norm one, then ρ is an isometry.

Let A be factorable with an ultra$^*-$approximate identity. We may therefore use the Theorem 2.3 to define a multiplication in $QM_{gr}(A^*)$ making it a Banach algebra. We outline the details as follows.

Let $m_1,\ m_2 \in QM_{gr}(A^*)$. By virtue of the above Theorem, there exist $T_1,\ T_2 \in M_{gr}(A^*)$ such that $m_1 = \rho_{T_1}$ and $m_2 = \rho_{T_2}$. Then

$$m_1 \circ_\rho m_2 = \rho_{T_1} \circ_\rho \rho_{T_2} := \rho_{T_2 T_1}$$

gives a well defined multiplication.

Note that $QM_{gr}(A^*)$ becomes an A-bimodule, as follows: For any $m \in QM_{gr}(A^*)$ and $a \in A$, we can define the products $a{*}m$ and $m{*}a$ as mappings from $A^* \times A$ into A^* given by

$$\begin{aligned} (a * m)(\xi, b) &= m(\xi \cdot a, b), \\ (m * a)(\xi, b) &= m(\xi, ab), \quad \xi \in A^*,\ b \in A. \end{aligned}$$

It is easy to see that $a * m,\ m * a \in QM_{gr}(A^*)$, so that $QM_{gr}(A^*)$ is an A-bimodule.

Definition 2.4. The strict topology β on $QM_{gr}(A^*)$ is defined by the seminorm

$$m \to \|m * a\| \qquad (a \in A,\ m \in QM_{gr}(A^*)).$$

Let A and B be two factorable Banach algebras with an ultra$^*-$approximate identity, and let $\varphi : B \to A$ be a homomorphism such that $\varphi^* : A^* \to B^*$ be onto. We define $\widetilde{\varphi} : QM_{gr}(A^*) \to QM_{gr}(B^*)$ by $[\widetilde{\varphi}(m)](\varphi^*(\xi), b) = \varphi^*(m(\xi, \varphi(b)))$ for each $\xi \in A^*$ and $b \in B$.

Theorem 2.5. *Let A, B, φ, φ^* and $\widetilde{\varphi}$ be as above and β_A and β_B denote the strict topology β on $QM_{gr}(A^*)$ and $QM_{gr}(B^*)$, respectively. If $\varphi : B \to A$ is continuous, then so is $\widetilde{\varphi} : (QM_{gr}(A^*), \beta_A) \to (QM_{gr}(B^*), \beta_B)$ a continuous homomorphism.*

Proof. Let $m, m' \in QM_{gr}(A^*)$. By Theorem 2.3, there exist $T, T' \in M_r(A^*)$ such that $m = \rho_T,\ m' = \rho_{T'}$. So $m \circ_\rho m' = \rho_{T'T}$ and

$$\begin{aligned}[\widetilde{\varphi}(m \circ_\rho m')](\varphi^*(\xi), b) &= \varphi^*((m \circ_\rho m')(\xi, \varphi(b))) \\ &= \varphi^*(\rho_{T'T}(\xi, \varphi(b))) = \varphi^*(T'T(\xi) \cdot \varphi(b)).\end{aligned}$$

On the other hand, $\widetilde{\varphi}(m) = \widetilde{\varphi}(\rho_T)$ and $\widetilde{\varphi}(m') = \widetilde{\varphi}(\rho_{T'})$ belong to $QM_{gr}(B^*)$. So there exist $S, S' \in M_{gr}(B^*)$ such that $\widetilde{\varphi}(\rho_T) = \rho_S$ and $\widetilde{\varphi}(\rho_{T'}) = \rho_{S'}$. This implies that $\varphi^*(\rho_T(\xi, \varphi(b))) = S(\varphi^*(\xi)) \cdot b$ and so $\varphi^*(T(\xi) \cdot \varphi(b)) = S(\varphi^*(\xi)) \cdot b$. Consequently,

$$[\widetilde{\varphi}(m) \circ_\rho \widetilde{\varphi}(m')](\varphi^*(\xi), b) = \rho_{S'S}(\varphi^*(\xi), b) = S'S(\varphi^*(\xi)) \cdot b = \varphi^*(T'T(\xi) \cdot \varphi(b))$$

hence $\widetilde{\varphi}$ is homomorphism.

Now, let us prove that $\widetilde{\varphi}$ is $(\beta_A, \beta_B)-$continuous. Let $\{m_\alpha\}$ be a net in $QM_{gr}(A^*)$ with $m_\alpha \to^{\beta_A} m \in QM_{gr}(A^*)$. Let $\xi \in B^*$, be arbitrary. There exist $\eta \in A^*$ with $\varphi^*(\eta) = \xi$. The continuity of φ^* implies that for each $b, c \in B$, we have

$$\begin{aligned}||(\widetilde{\varphi}(m_\alpha - m) * b)(\xi, c)|| &= ||\widetilde{\varphi}(m_\alpha - m)(\xi, bc)|| = ||\widetilde{\varphi}(m_\alpha - m)(\varphi^*(\eta), bc)|| \\ &= ||\varphi^*(m_\alpha - m)(\eta, \varphi(bc)))|| \to 0.\end{aligned}$$

That means $\widetilde{\varphi}(m_\alpha) \to^{\beta_B} \widetilde{\varphi}(m)$. □

Definition 2.6. A subalgebra A of an algebra B is called a quasi-ideal of B if $ABA \subset A$.

Definition 2.7. Let A be a quasi-ideal of the algebra B, then there is a linear mapping $\phi_B : B \to QM_{gr}(A^*)$ defined by, for each $b \in B$

$$\phi_B(b)(\xi, a) = \xi \cdot\ ba, \quad \textit{for all}\ \ \xi \in A^*,\ a \in A.$$

Note that for arbitrary $a' \in A$, we may define $\langle \xi \cdot\ ba, a' \rangle = \langle \xi, a'ba \rangle$.

Definition 2.8. The topology $\{\phi_B^{-1}(\vartheta) : \vartheta \in \beta_A\}$ on B, will be denoted by $u_{(B,A)}$. A net b_α in B is $u_{(B,A)}-$converges to some $b \in B$ precisely when $\lim_\alpha \|\xi \cdot b_\alpha a - \xi \cdot ba\| = 0$ for all $a \in A,\ \xi \in A^*$

Theorem 2.9. *Let A be factorable with an ultra$^*-$approximate identity $\{e_\alpha\}$. If A is a quasi-ideal of an algebra B, then the mapping ϕ_B is a $(u_{(B,A)}, \beta_A)$-continuous homomorphism of B to $QM_{gr}(A^*)$.*

Proof. Let $\{b_\alpha\}$ be a net in B with $b_\alpha \to^{u_{(B,A)}} b$. So by definition of $u_{(B,A)}-$topology there exist an index α_0 such that for all $a, c \in A$, $\xi \in A^*$ and all $\alpha \geq \alpha_0$, we have

$$\|(\phi_B(b_\alpha) * a)(\xi, c) - (\phi_B(b) * a)(\xi, c)\| = \|\xi \cdot b_\alpha ac - \xi \cdot bac\| \to 0.$$

That means $\phi_B(b_\alpha) \to^{\beta_A} \phi_B(b)$ and so ϕ_B is $(u_{(B,A)}, \beta_A)-$continuous.
Now, we show that ϕ_B is a homomorphism. Clearly, ϕ_B is additive. To show that ϕ_B is multiplicative, let $b_1, b_2 \in B$ be arbitrary. By Theorem 2.3, there exist $T, S \in M_{gl}(A^*)$ such that $\phi_B(b_1) = \rho_T$ and $\phi_B(b_2) = \rho_S$. So $\xi \cdot b_1 a = T(\xi) \cdot a$ and $\xi \cdot b_2 a = S(\xi) \cdot a$ for each $a \in A$, $\xi \in A^*$. Also by ([7] , p. 111), for each $a \in A$ and $b \in B$, $\lim_\alpha e_\alpha ba = ba$ and therefore $\lim_\alpha \|e_\alpha ba - ba\| = 0$. Now, since A is a quasi-ideal of B, it follows that

$$\begin{aligned}
[\phi_B(b_1) \circ_\rho \phi_B(b_2)](\xi, a) &= [\rho_T \circ_\rho \rho_S](\xi, a) = \rho_{(ST)}(\xi, a) = S(T(\xi)) \cdot a \\
&= T(\xi) \cdot b_2 a = \lim_\alpha T(\xi) \cdot e_\alpha b_2 a = \lim_\alpha \xi \cdot b_1 e_\alpha b_2 a
\end{aligned}$$

and hence

$$\begin{aligned}
\|[\phi_B(b_1) \circ_\rho \phi_B(b_2)](\xi, a) - \phi_B(b_1 b_2)(\xi, a)\| &= \lim_\alpha \|\xi \cdot b_1 e_\alpha b_2 a - \xi b_1 b_2 a\| \\
&\leq \lim_\alpha \|\xi b_1\| \|e_\alpha b_2 a - b_2 a\| = 0.
\end{aligned}$$

That means ϕ_B is a multiplicative. □

Corollary 2.10. *If the mapping ϕ_B is one to one. Then the algebra B may be regarded as a subset of $QM_{gr}(A^*)$.*

Example 2.11. Let G be a compact group and $A = L_1(G)$. Then the equation

$$(\phi_\mu(\xi, f) := (\xi * \mu) * f \qquad (\mu \in M(G), \xi \in L_\infty(G),\ f \in L_1(G)).$$

defines a linear isomorphism between $M(G)$ and a subspace of $QM_{gr}(A^*)$.

References

1. M. Adib, *Arens regularity of quasi-multipliers*, University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics. **80** (2018), 199-206.
2. M. Adib, *Some topology on the space of quasi-multipliers*, Bull. Iran. Math. Soc., **5** (2017), 1323-1332.
3. M. Adib and l.A. Khan, *Strict topologies on topological algebras of quasi-multipliers*, Bull. Iran. Math. Soc., **48** (2022), 435 - 446.
4. M. Adib, A. Riazi, and J. Bracic,*Quasi-multipliers of the dual of a Banach algebra*, Banach J. Math. Anal., **5** (2011), 6-14.
5. M. Adib, A. Riazi and L.A. Khan, *Quasi-multipliers on F-algebras*, Abstr. Appl. Anal, Article ID 235273 (2011), 1-30.
6. C.A. Akemann, and G.K. Pedersen, *Complications of semicontinuity in C^*-algebra theory*, Duke Math. J., **40** (1973), 785–795.
7. M. McKennon, *Quasi-multipliers*, Trans. Amer. Math. Soc., **233** (1977), 105–123.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

A GENERALIZATION OF QUASI-MULTIPLIERS

MARJAN ADIB

Department of Mathematics, National University of Skills, (NUS), Tehran, Iran
m.adib.pnu@gmail.com

Abstract. In this paper, we generalize the notion of quasi-multipliers. We discuss the strict and quasi-strict topologies on the algebra $QM_{gr}(A^*)$ of all bilinear and separately continuous generalized quasi-multipliers of A^* and study their various properties.

1. Introduction

The notion of a quasi-multiplier is a generalization of the notion of a multiplier on a Banach algebra and was introduced by Akemann and Pedersen [6] for C^*-algebras. McKennon [7] extended the definition to a general complex Banach algebra A with a bounded approximate identity (b.a.i., for brevity) as follows. A bilinear mapping $m: A \times A \to A$ is a quasi-multiplier on A if

$$m(ab, cd) = a\, m(b, c)\, d \qquad (a, b, c, d \in A).$$

In this paper, we extend the notion of quasi-multipliers and introduce several notions of strict and quasi-strict topologies on the space $QM_{gr}(A^*)$ of all generalized quasi-multipliers of A^*.

Definition 1.1. A bilinear mapping $m: A^* \times A \to A^*$ is called a *right generalized quasi-multiplier of* A^* if

$$m(a \cdot \xi, b) = a \cdot m(\xi, b) \qquad \text{and} \qquad m(\xi, ba) = m(\xi, b) \cdot a \qquad (1.1)$$

hold for arbitrary $\xi \in A^*$ and $a,\ b \in A$.

2020 *Mathematics Subject Classification. Primary 47B48; Secondary 46H25*
Key words and phrases. Quasi-multiplier, multiplier, Banach algebra, strict topology.

Let $QM_{gr}(A^*)$ be the set of all bilinear and separately continuous right generalized quasi-multipliers of A^*. It is obvious that $QM_{gr}(A^*)$ is a linear space. Moreover, it is a Banach space with respect to the norm

$$||m|| = \sup\{||m(\xi, a)||; \quad \xi \in A^*,\ a \in A,\ ||\xi|| \leq 1,\ ||a|| \leq 1\}.$$

Let A be a general Banach algebra. Then a map $T : A^* \to A^*$ is called a right generalized multiplier of A^* if

$$T(a \cdot \xi) = a \cdot T(\xi),$$

for all $\xi \in A^*, a \in A$. With $M_r(A^*)$ we denote the space of all bounded linear right generalized multipliers of A^*. It is obvious that for each $a \in A$, the right multiplication operator $R_a\xi = \xi \cdot a$ is a right generalized multiplier of A^*.

Definition 1.2. A bounded approximate identity $\{e_\lambda : \lambda \in I\}$ in a Banach algebra A is said to be ultra$^*-$approximate identity if, for all $m \in QM_{gr}(A^*)$ and $\xi \in A^*$, the net $\{m(\xi, e_\lambda) : \lambda \in I\}$ is Cauchy in A^*.

Theorem 1.3. *Let A be factorable with an ultra$^*-$approximate identity $\{e_\lambda\}$. Then the mapping $\rho :\ M_r(A^*) \to QM_{gr}(A^*)$, defined by*

$$\rho_T(\xi, a) = (T\xi) \cdot a \qquad (T \in M_r(A^*),\ \xi \in A^*,\ a \in A),$$

is a bijective with norm $\|\rho\| \leq 1$. If $\{e_\lambda\}$ is of norm one, then ρ is an isometry.

Let A be factorable with an ultra$^*-$approximate identity. We may therefore use the Theorem 1.3 to define a multiplication in $QM_{gr}(A^*)$ making it a Banach algebra. We outline the details as follows.

Let $m_1,\ m_2 \in QM_{gr}(A^*)$. By virtue of the above Theorem, there exist $T_1,\ T_2 \in M_{gr}(A^*)$ such that $m_1 = \rho_{T_1}$ and $m_2 = \rho_{T_2}$. Then

$$m_1 \circ_\rho m_2 = \rho_{T_1} \circ_\rho \rho_{T_2} := \rho_{T_2T_1}$$

gives a well defined multiplication.

2. Strict and quasi-strict topologies on $QM_{gr}(A^*)$

Note that $QM_{gr}(A^*)$ becomes an A-bimodule, as follows: For any $m \in QM_{gr}(A^*)$ and $a \in A$, we can define the products $a{*}m$ and $m{*}a$ as mappings from $A^* \times A$ into A given by

$$\begin{aligned}(a * m)(\xi, b) &= m(\xi \cdot a, b),\\ (m * a)(\xi, b) &= m(\xi, ab), \quad \xi \in A^*,\ b \in A.\end{aligned}$$

It is easy to see that $a * m,\ m * a \in QM_{gr}(A^*)$, so that $QM_{gr}(A^*)$ is an A-bimodule.

In the sequel, τ denotes the topology on $QM_{gr}(A^*)$ generated by the norm.

Definition 2.1. The strict topology β on $QM_{gr}(A^*)$ is defined by the seminorm

$$m \to \|m * a\| \qquad (a \in A,\ m \in QM_{gr}(A^*)).$$

Definition 2.2. The quasi-strict topology γ on $QM_{gr}(A^*)$ is defined by the seminorm

$$m \to \|m(\xi, a)\| \qquad (\xi \in A^*,\ a \in A,\ m \in QM_{gr}(A^*)).$$

Lemma 2.3. *Let A be factorable, then $\gamma \subseteq \beta \subseteq \tau$.*

Proposition 2.4. *Let A be factorable with an ultra$^*-$approximate identity. If A^* factors on the right then the mapping*

$$\phi_A : (A, \tau) \to (QM_{gr}(A^*), \beta)$$

by

$$(\phi_A(a))(\xi, b) = \xi \cdot ab$$

is a continuous monomorphism.

Proof. Let $\{a_\alpha\}$ be a net in A with $a_\alpha \to^\tau a \in A$. For each $\xi \in A^*$ and $b, c \in A$, we have

$$\begin{aligned}\|(\phi_A(a_\alpha) * b)(\xi, c) - (\phi_A(a) * b)(\xi, c)\| &= \|(\phi_A(a_\alpha))(\xi, bc) - (\phi_A(a))(\xi, bc)\| \\ &= \|\xi \cdot a_\alpha bc - \xi \cdot abc\| \to 0\end{aligned}$$

Hence $\phi_A(a_\alpha) \to^\beta \phi_A(a)$. Now, we show that ϕ_A is a multiplicative. Let $a_1, a_2 \in A$. By Theorem 1.3, there exist $T_1, T_2 \in M_{gr}(A^*)$ such that $\phi_A(a_1) = \rho_{T_1}$ and $\phi_A(a_2) = \rho_{T_2}$. Hence, for arbitrary $\xi \in A^*, b \in A$, we have

$$T_1(\xi) \cdot b = \xi \cdot a_1 b \qquad \text{and} \qquad T_2(\xi) \cdot b = \xi \cdot a_2 b.$$

It follows

$$\begin{aligned}(\phi_A(a_1) \circ_\rho \phi_A(a_2))(\xi, b) &= \rho_{T_2 T_1}(\xi, b) = T_2(T_1(\xi)) \cdot b = T_1\xi \cdot (a_2 b) \\ &= \xi \cdot a_1 a_2 b = \phi_A(a_1 a_2)(\xi, b).\end{aligned}$$

Assume that $\phi_A(a) = 0$ for $a \in A$. So for arbitrary $\xi \in A^*$ and $b \in A$, $\langle \xi, ab \rangle = 0$ which implies that $\langle \pi(a), b \cdot \xi \rangle = \langle b \cdot \xi, a \rangle = \langle \xi, ab \rangle = 0$. Since by the assumption, A^* factors on the right, we conclude that $a = 0$. Thus, ϕ_A is one to one. □

Proposition 2.5. *Let A be factorable with an ultra$^*-$approximate identity. If A^* factors on the right then $\phi_A(A)$ is $\beta-$closed.*

Proof. Let $m \in QM_{gr}(A^*)$ with $m \in \beta - cl(\phi_A(A))$. There exist a net $\{a_\alpha\}$ such that $\phi_A(a_\alpha) \to^\beta m$. It follows from the definition of $\beta-$topology that the net $\{a_\alpha\}$ is $\tau-$Cauchy. By completeness of A, there exist $a \in A$ such that $a_\alpha \to^\tau a$. By Proposition 2.4, ϕ_A is one to one and $(\tau, \beta)-$continuous, it is easy to see that $a_\alpha \to^\tau a$ iff $\phi_A(a_\alpha) \to^\beta \phi_A(a)$. Thus for each $\xi \in A^*$ and $y \in A$,

$$m(\xi, y) = \lim_\alpha \varphi_A(a_\alpha)(\xi, y) = \phi_A(a)(\xi, y),$$

this implies that $m = \phi_A(a)$ and hence $\phi_A(A)$ is $\beta-$closed.

□

Theorem 2.6. *Let A be factorable with an ultra$^*-$approximate identity. Then $(QM_{gr}(A^*), \gamma)$, $(QM_{gr}(A^*), \tau)$ and $(QM_{gr}(A^*), \beta)$ have the same bounded sets.*

Theorem 2.7. *Let A be factorable with an ultra$^*-$approximate identity.*
(i) Space $(QM_{gr}(A^*), \gamma)$ is complete.
(ii) If A has an approximate identity with norm one, then $(QM_{gr}(A^*), \beta)$ is complete.

Proof. (ii) Let $\{m_\alpha\}_{\alpha\in I}$ be a $\beta-$Cauchy net in $QM_{gr}(A^*)$. For each $a \in A$, the mapping $T_a^\alpha : A^* \to A^*$ which is given by $T_a^\alpha(\xi) = m_\alpha(\xi, a)$ defines elements in $M_{gr}(A^*)$. It is easy to show that $\rho_{T_a^\alpha} = m_\alpha * a$. By definition of $\beta-$topology, the net $\{\rho_{T_a^\alpha}\}_{\alpha\in I}$ is Cauchy in the norm of $QM_{gr}(A^*)$. By Theorem 1.3, ρ is isometry and therefore $\{T_a^\alpha\}$ is a Cauchy net in the norm of $M_{gr}(A^*)$. Now, since $M_{gr}(A^*)$ is complete, there exists $T_a \in M_{gr}(A^*)$ such that $||T_a^\alpha - T_a|| \to 0$. By virtue of the Lemma 2.3, the net $\{m_\alpha\}_{\alpha\in I}$ is $\gamma-$Cauchy and by (i), the space $QM_{gr}(A^*)$ is $\gamma-$complete. So there exist $m \in QM_{gr}(A^*)$ such that

$$\lim_\alpha m_\alpha(\xi, a) = m(\xi, a) \quad \text{for all} \quad \xi \in A^* \quad \text{and} \quad a \in A.$$

For each $b \in A$,

$$\begin{aligned}\rho_{T_a}(\xi, b) &= \lim_\alpha \rho_{T_a^\alpha}(\xi, b) = \lim_\alpha (m_\alpha * a)(\xi, b) = \lim_\alpha m_\alpha(\xi, ab)\\ &= m(\xi, ab) = m * a(\xi, b).\end{aligned}$$

Consequently

$$||m_\alpha * a - m * a|| = ||\rho_{T_a^\alpha} - \rho_{T_a}|| = ||T_a^\alpha - T_a|| \to 0,$$

which implies that m is the $\beta-$limit of the net $\{m_\alpha\}_{\alpha\in I}$, i.e., $QM_{gr}(A^*)$ is complete in $\beta-$topology. □

References

1. M. Adib, *Arens regularity of quasi-multipliers*, University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics. **80** (2018), 199-206.
2. M. Adib, *Some topology on the space of quasi-multipliers*, Bull. Iran. Math. Soc., **5** (2017), 1323-1332.
3. M. Adib and l.A. Khan, *Strict topologies on topological algebras of quasi-multipliers*, Bull. Iran. Math. Soc., **48** (2022), 435 - 446.
4. M. Adib, A. Riazi, and J. Bracic,*Quasi-multipliers of the dual of a Banach algebra*, Banach J. Math. Anal., **5** (2011), 6-14.
5. M. Adib, A. Riazi and L.A. Khan, *Quasi-multipliers on F-algebras*, Abstr. Appl. Anal, Article ID 235273 (2011), 1-30.
6. C.A. Akemann, and G.K. Pedersen, *Complications of semicontinuity in C^*-algebra theory*, Duke Math. J., **40** (1973), 785–795.
7. M. McKennon, *Quasi-multipliers*, Trans. Amer. Math. Soc., **233** (1977), 105–123.

ORDER ISOMORPHISMS BETWEEN HOLDER-LIPSCHITZ ALGEBRAS WITH LIPSCHITZ INVOLUTION

DAVOOD ALIMOHAMMADI* AND ESMAT AHMADLOO

Department of Mathematics, Faculty of Science, Arak University, Arak 3848177584, Iran
d-alimohammadi@araku.ac.ir; e.ahmadloo.98@phd.araku.ac.ir

Abstract. In this paper, we study order-isomorphisms between real Holder-Lipschitz algebras. We first give some sufficent conditions that a map T between these algebras be an order isomorphism. We next characterize the structure of order isomorphisms T with certain properties between these algebras. Finally, we prove that every order isomorphism T between real Holder-Lipschitz algebras is an essential weighted composition operator.

1. Introduction

The symbol $\mathbb{K}$ denotes a field that can be $\mathbb{R}$ or $\mathbb{C}$. Let X be a nonempty set. We denote by $\mathcal{F}_{\mathbb{C}}(X)$ the set of all complex-valued functions on X. Then $\mathcal{F}_{\mathbb{C}}(X)$ is a complex linear space. Let $\mathcal{A}(X)$ be a linear subspace of $\mathcal{F}_{\mathbb{C}}(X)$ over $\mathbb{K}$. The pointwise order $\geq$ in $\mathcal{A}(X)$ is defined by $f \geq g$ if and only if f and g are real-valued functions in $\mathcal{A}(X)$ and $f(x) \geq g(x)$ for all $x \in X$. A function f in $\mathcal{A}(X)$ is called *positive* if $f(x) \geq 0$ for all $x \in X$.

Let X and Y be nonempty sets and let $\mathcal{A}(X)$ and $\mathcal{B}(Y)$ be linear spaces of $\mathcal{F}_{\mathbb{C}}(X)$ and $\mathcal{F}_{\mathbb{C}}(Y)$ over $\mathbb{K}$, respectively, with the pointwise order $\geq$. A map $T : \mathcal{A}(X) \to \mathcal{B}(Y)$ is called an *order preserving map* if $T(f) \geq T(g)$ for all $f, g \in \mathcal{A}(X)$ with $f \geq g$. An order isomorphism from $\mathcal{A}(X)$ to $\mathcal{B}(Y)$ is

2020 *Mathematics Subject Classification. Primary 46E05; Secondary 46J10, 47B65*
Key words and phrases. Lipschitz involution, Little Holder-Lipschitz algebra, Order isomorphism, Order preserving map.

* Speaker.

a bijective linear map $T : \mathcal{A}(X) \to \mathcal{B}(Y)$ such that T is an order preserving from $\mathcal{A}(X)$ to $\mathcal{B}(Y)$ and T^{-1} is an order preserving from $\mathcal{B}(Y)$ to $\mathcal{A}(X)$.

Let (X, d) be a metric space and let $\alpha \in (0, 1]$. Then the map $d^\alpha : X \times X \to \mathbb{R}$ defined by $d^\alpha(x, y) = (d(x, y))^\alpha$, $x, y \in X$, is a metric on X and the induced topology by d^α on X coincides with the induced topology by d on X. For a function $f : X \to \mathbb{K}$, the Lipschitz constant of f on (X, d^α) is denoted by $L_{(X,d^\alpha)}(f)$ and defined by

$$L_{(X,d^\alpha)}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d^\alpha(x, y)} : x, y \in X, x \neq y\right\}.$$

A function $f : X \to \mathbb{K}$ is called a $\mathbb{K}$-*valued Lipschitz function* on (X, d^α) (of order α on (X,d)) if $L_{(X,d^\alpha)}(f) < \infty$. We denote by $\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ the set of all $\mathbb{K}$-valued bounded Lipschitz functions on (X, d^α). Then $\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ is a commutative Banach algebra over $\mathbb{K}$ with the Lipschitz sum norm $\|\cdot\|_{\mathrm{Lip}(X,d^\alpha)}$ defined by

$$\|f\|_{\mathrm{Lip}(X,d^\alpha)} = \|f\|_X + L_{(X,d^\alpha)}(f), \qquad (f \in \mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)),$$

where $\|f\|_X = \sup\{|f(x)| : x \in X\}$. These algebras are called *big Holder-Lipschitz algebras of order* α on (X, d) over $\mathbb{K}$. We denote by $\mathrm{lip}_{\mathbb{K}}(X, d^\alpha)$ the set of all $f \in \mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ for which

$$\lim_{d(x,y)\to 0} \frac{|f(x) - f(y)|}{d^\alpha(x, y)} = 0.$$

Then $\mathrm{lip}_{\mathbb{K}}(X, d^\alpha)$ is a closed subalgebra of $\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ under the norm $\|\cdot\|_{\mathrm{Lip}(X,d^\alpha)}$. The algebra $\mathrm{lip}_{\mathbb{K}}(X, d^\alpha)$ is called *little Holder-Lipschitz algebra of order* α on (X, d) over $\mathbb{K}$. It is known that

$$\mathrm{Lip}(X, d^\alpha) \subseteq \mathrm{lip}(X, d^\beta) \subseteq \mathrm{Lip}(X, d^\beta)$$

whenever $0 < \beta < \alpha \leq 1$. The big Holder-Lipschitz algebra $\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ and little Holder-Lipschitz algebra $\mathrm{lip}_{\mathbb{K}}(X, d^\alpha)$ were first introduced in [5]. It is clear that $\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ and $\mathrm{lip}_{\mathbb{K}}(X, d^\alpha)$ are order linear spaces with the pointwise order $\geq$. It is known that $\mathrm{Lip}_{\mathbb{K}}(X, d)$ separates the points of X. We also write $\mathrm{Lip}(X, d^\alpha)$ and $\mathrm{lip}_(X, d^\alpha)$ instead of $\mathrm{Lip}_{\mathbb{C}}(X, d^\alpha)$ and $\mathrm{lip}_{\mathbb{C}}(X, d^\alpha)$, respectively.

Order isomorphisms between big Holder-Lipschitz algebras of order 1 studied in [2, 6, 7, 8] .In [3], Jiménez-Vargas and Villegas-Vallecillos determined the form of all order isomorphisms T between little Holder-Lipschitz algebras $\mathrm{lip}_{\mathbb{K}}(X, d^\alpha)$ and $\mathrm{lip}_{\mathbb{K}}(Y, \rho^\beta)$, using an approach depends on the analysis of the support map associated with T, where (X, d) and (Y, ρ) are compact metric spaces and $\alpha, \beta \in (0, 1)$.

For a topological space X, a self-map τ of X is called a *topological involution* on X if τ is continuous and $\tau(\tau(x)) = x$ for all $x \in X$. A subset E of X is called τ-*invariant* if $\tau(E)$ is a subset of E. Let X be a compact Hausdorff space and let τ be a topological involution on X. Then the map

$\tau^* : C(X) \to C(X)$ defined by $\tau^*(f) = \bar{f} \circ \tau$, $f \in C(X)$, is an algebra involution on $C(X)$ which is called the *algebra involution induced* by τ on $C(X)$ where $C(X)$ denotes the Banach algebra of all continuous complex-valued functions on X with the uniform norm $\|\cdot\|_X$. Define

$$C(X, \tau) = \{f \in C(X) : \tau^*(f) = f\}.$$

Then $C(X, \tau)$ is a self-adjoint uniformly closed real subalgebra of $C(X)$ that contains 1_X and separates the points of X. In addition, $C(X) = C(X, \tau) \oplus iC(X, \tau)$ and $i1_X \notin C(X, \tau)$. This algebra was first introduced in [4].

Let (X, d) be a metric space. A self-map τ of X is called a Lipschitz involution on (X, d) if τ is a Lipschitz mapping from (X, d) to (X, d) and $\tau(\tau(x)) = x$ for all $x \in X$.

Let (X, d) be a compact metric space, let τ be a Lipschitz involution on (X, d) and let τ^* be the algebra involution induced by τ on $C(X)$. It is shown [1, Lemma 2.4] $\tau^*(\mathrm{lip}(X, d^\alpha)) = \mathrm{lip}(X, d^\alpha)$ for $\alpha \in (0, 1)$. Define

$$\mathrm{lip}(X, d^\alpha, \tau) = \{f \in \mathrm{lip}(X, d^\alpha) : \tau^*(f) = f\}$$

for $\alpha \in (0, 1)$. It is shown [1, Theorem 2.7] that if $A = \mathrm{lip}(X, d^\alpha, \tau)$ and $B = \mathrm{lip}(X, d^\alpha)$ then A is a self-adjoint real subalgebra of $C(X, \tau)$ and B that separates the points of X, $1_X \in A$, $i1_X \notin A$, $B = A \oplus iA$, A is a real Banach algebra with the Lipschitz sum norm $\|\cdot\|_{\mathrm{Lip}(X,d^\alpha)}$ and $\mathrm{Re}\, f, i\mathrm{Im}\, f \in A$ for all $f \in A$. Note that $\mathrm{Lip}(X, d^\alpha, \tau) = \mathrm{Lip}_{\mathbb{R}}(X, d^\alpha)$ for $\alpha \in (0, 1]$ ($\mathrm{lip}(X, d^\alpha, \tau) = \mathrm{lip}_{\mathbb{R}}(X, d^\alpha)$ for $\alpha \in (0, 1)$) if and only if τ is the identity map on X. The algebras $\mathrm{lip}(X, d^\alpha, \tau)$ for $\alpha \in (0, 1)$ were first introduced in [1] and are called *real little Holder-Lipschitz algebras of order* α *with Lipschitz involution*.

2. Main results

In this section, we first give some sufficient conditions that a weighted composition operator between real little Holder-Lipschitz algebras with Lipschitz involution be an order real isomorphism.

Theorem 2.1. *Let (X, d) and (Y, ρ) be compact metric spaces and let τ and η be Lipschitz involutions on (X, d) and (Y, ρ), respectively. Suppose that $\alpha, \beta \in (0, 1)$, $\varphi : Y \to X$ is a Lipschitz homeomorphism from (Y, ρ^β) to (X, d^α) with $\varphi \circ \eta = \tau \circ \varphi$ on Y and a is a nonvanishing positive function in $\mathrm{lip}(Y, \rho^\beta, \eta)$. Let $T : \mathrm{lip}(X, d^\alpha, \tau) \to \mathrm{lip}(Y, \rho^\beta, \eta)$ be the map defined by*

$$T(f) = a \cdot (f \circ \varphi) \qquad (f \in \mathrm{lip}(X, d^\alpha, \tau)).$$

Then the following assertions hold:

(i) *T is well-defined, i. e., $a\cdot(f\circ\varphi) \in \mathrm{lip}(Y, \rho^\beta, \eta)$ for all $f \in \mathrm{lip}(X, d^\alpha, \tau)$.*
(ii) *T is an order real isomorphism from $\mathrm{lip}(X, d^\alpha, \tau)$ to $\mathrm{lip}(Y, \rho^\beta, \eta)$. In particular, $T^{-1}(h) = \frac{1}{a\circ\varphi^{-1}} \cdot (h \circ \varphi^{-1})$ for all $h \in \mathrm{lip}(X, d^\alpha, \tau)$.*
(iii) *$|T(f)| = T(|f|)$ for all $f \in \mathrm{lip}(X, d^\alpha, \tau)$.*
(iv) *$T(1_X) = a$ and so $T(1_X)$ is a nonvanishing positive function on Y.*

(v) $T^{-1}(1_Y) = \frac{1}{a\circ\varphi^{-1}}$ *and so* $T^{-1}(1_Y)$ *is a nonvanishing positive function on* X.

Let (X, d) and (Y, ρ) be compact metric spaces and let τ and η be Lipschitz involutions on (X, d) and (Y, ρ), respectively. Let $x_\tau = \{x, \tau(x)\}$ for $x \in X$ and let $y_\eta = \{y, \eta(y)\}$ for $y \in Y$. We denote by X_τ the set of all x_τ for which $x \in X$ and by Y_η the set of all y_η for which $y \in Y$. Let $\alpha, \beta \in (0, 1)$.

Now, we can obtain some properties of an order isomorphism T from $\mathrm{lip}(X, d^\alpha, \tau)$ to $\mathrm{lip}(Y, \rho^\beta, \eta)$ with $|T(i\mathrm{Im}\, f)| \leq T(|\mathrm{Im}\, f|)$ for all $f \in \mathrm{lip}(X, d^\alpha, \tau)$ and $|T^{-1}(i\mathrm{Im}\, h)| \leq T^{-1}(|\mathrm{Im}\, h|)$ for all $h \in \mathrm{lip}(Y, \rho^\beta, \eta)$, as follows.

Theorem 2.2. *Let* (X, d) *and* (Y, ρ) *be compact metric spaces, let* τ *and* η *be Lipschitz involutions on* (X, d) *and* (Y, ρ)*, respectively, and let* $\alpha, \beta \in (0, 1)$. *Suppose that* $T : \mathrm{lip}(X, d^\alpha, \tau) \to \mathrm{lip}(Y, \rho^\beta, \eta)$ *is an order isomorphism with* $|T(i\mathrm{Im}\, f)| \leq T(|\mathrm{Im}\, f|)$ *for all* $f \in \mathrm{lip}(X, d^\alpha, \tau)$ *and* $|T^{-1}(i\mathrm{Im}\, h)| \leq T^{-1}(|\mathrm{Im}\, h|)$ *for all* $h \in \mathrm{lip}(Y, \rho^\beta, \eta)$. *Then* T *is a homeomorphism from the real Banach space* $(\mathrm{lip}(X, d^\alpha, \tau), \|\cdot\|_{\mathrm{Lip}(X,d^\alpha)})$ *to the real Banach space* $(\mathrm{lip}(Y, \rho^\beta, \eta), \|\cdot\|_{\mathrm{Lip}(Y,\rho^\beta)})$, $T(1_X)$ *is a nonvanishing positive function on* Y, $T^{-1}(1_Y)$ *is a nonvanishing positive function on* X *and there exists a bijective map* $\Phi : Y_\eta \to X_\tau$ *such that*

(i) $T(f)(y) = T(1_X)(y)f(x)$ *for all* $y \in Y$, $x \in \Phi(y_\eta)$ *and* $f \in \mathrm{lip}(X, d^\alpha, \tau)$ *with* $f(x) \in \mathbb{R}$.

(ii) $T^{-1}(h)(x) = T^{-1}(1_Y)(x)h(y)$ *for all* $x \in X$, $y \in \Phi^{-1}(x_\tau)$ *and* $h \in \mathrm{lip}(Y, \rho^\beta, \eta)$ *with* $h(y) \in \mathbb{R}$.

(iii) $\Phi(y_\eta)$ *is a singleton whenever* $y \in Y$ *with* $\eta(y) = y$ *and* $\Phi^{-1}(x_\tau)$ *is a singleton whenever* $x \in X$ *with* $\tau(x) = x$.

Applying Theorem 2.2, we can characterize the structure of order preserving isomorphisms between $\mathrm{lip}_{\mathbb{R}}(X, d^\alpha)$- algebras.

Theorem 2.3. *Let* (X, d) *and* (Y, ρ) *be compact metric spaces and let* $\alpha, \beta \in (0, 1)$. *suppose that* $T : \mathrm{lip}_{\mathbb{R}}(X, d^\alpha) \to \mathrm{lip}_{\mathbb{R}}(Y, \rho^\beta)$ *is an order isomorphism. Then* $T(1_X)$ *is a nonvanishing positive function on* Y, $T^{-1}(1_Y)$ *is a nonvanishing positive function on* X *and there exists a Lipschitz homeomorphism* φ *from* (Y, ρ^β) *to* (X, d^α) *such that* $T(f) = T(1_X) \cdot (f \circ \varphi)$ *for all* $f \in \mathrm{lip}_{\mathbb{R}}(X, d^\alpha)$ *and* $T^{-1}(h) = T^{-1}(1_Y) \cdot (h \circ \varphi^{-1})$ *for all* $h \in \mathrm{lip}_{\mathbb{R}}(Y, \rho^\beta)$. *In particular,* $T(1_X) = \frac{1}{T^{-1}(1_Y)\circ\varphi}$ *and* $T^{-1}(1_Y) = \frac{1}{T(1_X)\circ\varphi^{-1}}$.

References

1. D. Alimohammadi and A. Ebadian, *Hedberg's theorem in real Lipschitz algebras*, Indian J. Pure Appl. Math., **32** (10) (2001), 1470-1493.
2. M. I. Garrido and J. A. Jaramillo, *Homomorphisms on function lattices*, Monatsh. Math. **141** (2004), 127-146.
3. A. Jiménez-Vargas and M. Villegas-Vallecillos, *Order isomorphisms of little Lipschitz algebras*, Houston J. Math. **34** (4) (2008), 1185-1196.

4. S. H. Kulkarni and B.V. Limaye, *Gleason parts of real function algebras*, Can. J. Math. **33** (1) (1981), 181-200.
5. D. R. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc. **111** (1964), 240-272.
6. N. Weaver, *Lattices of Lipschitz functions*, Pacific. J. Math. **164** (1994), 179-193.
7. N. Weaver, *Lipschitz Algebras*, World Scientific, Singapore, 1999.
8. N. Weaver, *Order completeness in Lipschitz algebras*, J. Funct. Anal. **130** (1995), 118-130.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

CONVEX ANALYSIS AND ITS APPLICATION IN MACHINE LEARNING

MARJAN ALIYARI*

Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran
m.aliyari@abru.ac.ir

Abstract. Convex analysis is a subset of mathematics that deals with the study of convex sets, functions, and their properties. It plays a significant role in optimization problems, especially within the field of machine learning. As machine learning algorithms are often designed to minimize or maximize an objective function while adhering to specific constraints, understanding the principles of convexity can lead to more effective algorithms and solutions.

1. Introduction

Convex analysis has emerged as a crucial area of study since the early 20th century, largely due to its applications in optimization problems, economics, and various applied sciences. The works of mathematicians such as Hermann Weyl and R. Tyrrell Rockafellar laid the foundational elements of convex analysis, establishing key theorems such as the Separation Theorem and duality principles1. This area of mathematics continues to evolve, integrating ideas from functional analysis and topology. Convex analysis is a branch of mathematics that studies the properties of convex sets and convex functions. It plays a crucial role in optimization, economics, and various fields of applied mathematics. Here are some key concepts.

2020 *Mathematics Subject Classification. Primary 46N10; Secondary 47N10*
Key words and phrases. Convex Analysis, Convex Optimization, Machine Learning, Supervised learning, Artificial Intelligence.
* Speaker.

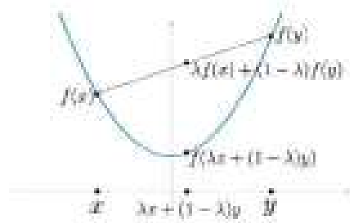


FIGURE 1. A geometric intuition behind convex functions.

1. Convex Sets: A set is defined as convex if for any two points within the set, the line segment joining these points also lies within the set. Mathematically,

$$\forall x, y \in S, \quad \lambda x + (1-\lambda)y \in S, \quad \forall \lambda \in [0,1].$$

2. Convex Functions: A function $f : R^n \to R$ is called convex if its domain is a convex set and it satisfies the following condition:

$$\forall x, y \in R^n, \lambda \in [0,1] : \quad f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

This intuitive geometric property implies that the function curve lies above the line segment connecting any two points on its graph(see fig1).

2. Main results

The intersection of any collection of convex sets is convex.The closure of a convex set is still convex. A convex combination of points within a convex set remains in that set. If a function is convex, it has a unique global minimum if it is continuous and defined over a convex set. Convex functions are continuous on the interior of their domains, and their level sets (i.e., sets where the function takes on a constant value) are also convex. In convex functions, any local minimum is also a global minimum. This characteristic is crucial for efficiency in finding optimal solutions. Also Convex functions admit subgradients, enabling the formulation of optimization algorithms that leverage this smoothness.Convex optimization problems are a crucial area in mathematical optimization that focus on minimizing convex functions over convex sets. These problems are well-structured in a way that allows for efficient algorithmic solutions, often guaranteeing global optimality under certain conditions.

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_j(x) = 0, \quad j = 1, \dots, p. \end{aligned}$$

Where:

- $x \in \mathbb{R}^n$ is the vector of variables,
- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is the convex objective function,

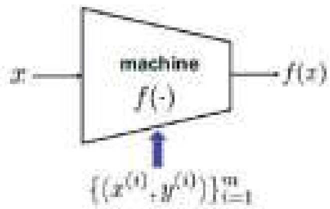


FIGURE 2. Supervised learning: A methodology for designing a computer system $f(.)$ with the help of a supervisor which offers input-output pair samples, called a training data set $\{(x^{(i)}, y^{(i)})\}_{i=1}^{m}$

- $f_i : \mathbb{R}^n \to \mathbb{R}$ represents the inequality constraints,
- $h_j : \mathbb{R}^n \to \mathbb{R}$ represents the equality constraints.

The least squares problem is a special case of convex optimization:

$$min_x \quad \|Ax - b\|_2^2, \tag{2.1}$$

where A is a matrix and b is a vector.

Also, a linear programming problem can be formulated as:

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b, \end{aligned} \tag{2.2}$$

where c is a vector of coefficients, and A and b define the inequality constraints.

Convex optimization is pivotal in various fields including economics, engineering, and machine learning. Understanding its principles and applications is essential for solving complex optimization problems effectively. In this section, we discuss its applications in machine learning [1].

3. Machine Learning

In this section, we will start investigating the field of machine learning and the role of optimization therein. What we are going to do are three folded. First of all, we will study what machine learning is and what the mission of the field is. We will then explore one very popular and traditional way to achieve the mission: **Supervised learning**. Lastly we will figure out how optimization techniques are related to supervised learning. Machine learning is about an algorithm which is defined to be a set of instructions that a computer system can execute. Formally speaking, machine learning is the study of algorithms with which one can train a computer system so that it can perform a specific task of interest. The input, usually denoted by x, indicates information employed to perform a task of interest. The output, usually denoted by y, denotes a task result (see Fig2)[2].

4. Importance of Convex Optimization in Machine Learning

Model Training: Many machine learning models, such as linear regression, logistic regression, and support vector machines (SVM), are formulated as convex optimization problems. For instance, SVM aims to maximize the margin between classes while minimizing a loss function, which is a convex function.
Loss Function Minimization: Convex optimization helps in finding the model parameters that minimize loss functions, which measure the difference between predicted and actual values. For example, in linear regression, the objective is to minimize the mean squared error.
Efficient Algorithms
Guaranteed Convergence: Convex optimization comes with convergence guarantees for various algorithms, notably gradient descent. If the objective function is convex and the learning rate is appropriately set, gradient descent will converge to the optimal solution 1. *Benefits in Practical Scenarios*
Robustness: Convex optimization problems tend to be less sensitive to noise and perturbations compared to non-convex problems, making them more reliable for real-world applications. This is crucial when dealing with large and noisy datasets common in machine learning .
Broad Applicability: Convex optimization techniques are applicable across numerous domains, including finance (portfolio optimization), engineering, economics, and healthcare, showcasing their versatility.
But convex optimization is limited to convex problems [3].

5. Conclusion

Convex analysis stands as a cornerstone in the world of optimization and machine learning. Its properties not only ensure the feasibility and efficiency of algorithms but also guarantee robust performance across diverse applications. Understanding convexity equips practitioners and researchers with the tools necessary to tackle complex optimization problems prevalent in today's data-driven landscape. In fact Convex analysis provides a robust framework for addressing optimization problems that arise in various machine learning applications.

References

1. D. Bertsekas, A. Nedic, A. Ozdaglar, *Convex analysis and optimization*, Athena Scientific, 2003 Mar 1.
2. S. Boyd, L.Vandenberghe, *Convex Optimization*, Cambridge University Press, 1. Cambridge, 2004.
3. J. Nocedal, S.J. Wright, *Numerical optimization*, Springer, 1999.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

.

STRONG CONVERGENCE THEOREM BY THE INERTIAL SHRINKING PROJECTION ALGORITHM FOR NONEXPANSIVE MAPPINGS IN HILBERT SPACES

SATTAR ALIZADEH*

Department of Mathematics, Marand Branch, Islamic Azad University, Marand, Iran
sattar.alizadeh10@gmail.com

Abstract. In this paper, we present a new modified inertial shrinking projection algorithm and prove a strong convergence theorem for a nonexpansive mapping in a Hilbert space. Our results improve some existing results in the literature.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle .,. \rangle$ and the induced norm $\|.\|$. Suppose that C is a nonempty closed and convex of H. A self mapping S of C is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad (x, y \in C).$$

We denote by $F(S)$ the set of fixed points of S.

It is well known that many of the most important nonlinear problems in mathematics can be reduced to finding the fixed points of a certain operator, and for many of these problems, contractive type conditions naturally arise. The methods for finding the fixed points of such mappings are therefore a

2020 *Mathematics Subject Classification.* 47H09, 47H10, 47J25
Key words and phrases. Inertial algorithm, nonexpansive mapping, shrinking projection, Strong convergence.
* Speaker.

fundamental topic in mathematics and therefore of interest to many mathematicians. Thus, Many algorithms have been introduced by researchers. These include the Mann iteration method [4] and the Ishikawa iteration method [3]. The underlying space must be satisfied in appropriate properties in order to achieve convergence in these methods. In addition, it is well known that we can prove only weak convergence of the sequences generated by the Mann iteration process, even in Hilbert spaces. Also, in Hilbert spaces, the Ishikawa iteration process for a Lipschitz pseudocontractive mapping is convergent, while the Mann process is not. Also, in Hilbert spaces, the Ishikawa iteration process is convergent for a Lipschitz pseudocontractive mapping. The Mann process is not.

Recently, in order to obtain weak or strong convergence, authors have used iteration methods in the framework of Hilbert spaces and Banach spaces, see [1, 5] and references therein. In addition, in order to achieve a strong convergence, many researchers have made extensive use of modified methods.

Let K be a closed convex subset of H and let P_K be metric (or nearest point) projection from H onto K (i.e., for $x \in H$, $P_K x$ is the only point in K such that $\|x - P_K x\| = inf\{\|x - z\| : z \in H\}$). Let $x \in H$ and $z \in K$, then $z = P_K x$ if and only if:

$$\langle x - z, y - z \rangle \leq 0, \tag{1.1}$$

for all $y \in K$

In 2008 Takahashi et al. [5] introduced an alternative projection method, called the shrinking projection method. It is as follows:
Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let S be a nonexpansive mapping of C into H such that $F(S) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1} x_0$, define a sequence u_n of C as follows:

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n) S u_n, \\ C_{n+1} = \{z \in C_n : \ \|(y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}} x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then, u_n converges strongly to $z_0 = P_{F(S)} x_0$.

In 2018 Dong et al. [2] proposed modified Mann inertial algorithm for a nonexpansive self-mapping S of a Hilbert space H with $F(S) \neq \emptyset$:

$$\begin{cases} x_0, x_1 \in H \textit{ chosen arbitrarily}, \\ w_n = x_n + \alpha_n (x_n - x_{n-1}), \\ v_n = (1 - \beta_n) w_n + \beta_n S w_n, \\ C_n = \{z \in H : \ \|y_n - z\| \leq \|w_n - z\|\}, \\ Q_n = \{z \in H : \ \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

for all $n \in \{0\} \cup \mathbb{N}$, where $\{\alpha_n\} \subset [\alpha_1, \alpha_2],\ \alpha_1 \in (-\infty, 0],\ \alpha_2 \in (\infty, 0],\ \{\beta_n\} \subset [\beta, 1],\ \beta \in (0, 1]$. Then $\{x_n\}$ is strongly convergent to $P_{F(S)}x_0$.

(1)

2. Main results

In this section, by modifying the inertial shrinking projection algorithm, we prove a strong convergence theorem for a nonexpansive mapping in a Hilbert space.

Theorem 2.1. *Let H be a real Hilbert space and $S : H \to H$ be a nonexpansive mapping such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be generated by the following algorithm:*

$$\begin{cases} x_0, x_1 \in H \text{ chosen arbitrarily}, \\ C_0 = H, \\ u_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = \beta_n z_n + (1 - \beta_n)Su_n, \\ z_n = \gamma_n u_n + (1 - \gamma_n)Su_n, \\ C_{n+1} = \{u \in C_n : \ \|y_n - u\| \leq \|u_n - u\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases}$$

for all $n \in \mathbb{N}$. Where $\alpha_n \in (0, 1)$ and $\{\beta_n\}, \{\gamma_n\} \in [0, 1]$ satisfy $\liminf\limits_{n\to\infty}(1 - \beta_n) > 0$ and $\lim\limits_{n\to\infty} \gamma_n = 0$. Then $\{x_n\}$ converges strongly to $P_{F(S)}x_0$.

Corollary 2.2. *Let H be a real Hilbert space and $S : H \to H$ be a nonexpansive mapping such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be generated by the following algorithm:*

$$\begin{cases} x_0, x_1 \in H \text{ chosen arbitrarily}, \\ C_0 = H, \\ u_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = \beta_n u_n + (1 - \beta_n)Su_n, \\ C_{n+1} = \{u \in C_n : \ \|y_n - u\| \leq \|u_n - u\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases}$$

for all $n \in \mathbb{N}$. Where $\alpha_n \in (0, 1)$ and $\{\beta_n\}, \in [0, 1]$ such that $\liminf\limits_{n\to\infty}(1 - \beta_n) > 0$. Then $\{x_n\}$ converges strongly to $P_{F(S)}x_0$.

Proof. Letting $\gamma_n = 1$ in Theorem 2.1, we get the desired result. □

References

1. S. Alizadeh and F. Moradlou, *A monotone hybrid algorithm for a family of generalized nonexpansive mappings in Banach spaces,* Int. J. Nonlinear Anal. Appl., **13** (2022) pp. 2347–2359.
2. Q. L. Dong, H. B. Yuan, Y. J. Cho and Th. M. Rassias, *Modified inertial Mann algorithm and inertial CQ-algorithm for nonexpansive mappings,* Optim. Lett., **12** (2018) 87–102.
3. S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., **40** (1974) 147–150.
4. W. R. Mann, *Mean value methods in iteration,* Proc. Amer. Math. Soc., **4** (1953) 506–510.
5. W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces,* J. Math. Anal. Appl., **341** (2008) 276–286.

STRONG CONVERGENCE THEOREMS BY A NEW HYBRID METHOD FOR FAMILIES OF NONLINEAR OPERATORS IN BANACH SPACES

SATTAR ALIZADEH*

Department of Mathematics, Marand Branch, Islamic Azad University, Marand, Iran
sattar.alizadeh10@gmail.com

Abstract. In this paper, by introducing a new hybrid method, we prove strong convergence theorems for families of generalized nonexpansive mappings related to this algorithm in Banach spaces. Using these theorems, we obtain some new results for these mappings.

1. Introduction

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers and real numbers. Also, let E be a real Banach space with the dual space E^* and let C be a nonempty closed convex subset of E.

A self-mapping T of C is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad (x, y \in C).$$

We denote by $F(T)$ the set of fixed points of T.

The theory of nonexpansive mappings is an important topic that can have a wide range of applications, see [2, 4] and references therein.

In recent years, many authors have proved weak or strong convergence theorems for a number of nonlinear mappings by means of various iteration

2020 *Mathematics Subject Classification.* 47H10; 47H09; 47H05.
Key words and phrases. Nonexpansive mapping, Fixed point, Hybrid method, Strong convergence theorem.

* Speaker.

methods in the framework of Hilbert spaces and Banach spaces, see [2, 1] and references therein.

Definition 1.1. Let E be a real Banach space with $\|.\|$ and dual space E^*. We denote the weak convergence and the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively and denote by J the normalized duality mapping from E into 2^{E^*} defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},$$

for all $x \in E$, where $\langle ., . \rangle$ denotes the generalized duality pairing between E and E^*.

Definition 1.2. Let C be a nonempty, closed convex subset of a smooth, strictly convex and reflexive Banach space E. We denote by ϕ the function $\phi : E \times E \to \mathbb{R}$ defined as follows:

$$\phi(x, y) = \|x\|^2 - \langle x, Jy \rangle + \|y\|^2, \ \ \forall x, y \in E.$$

Observe that if E is a Hilbert space then $\phi(x, y) = \|x - y\|^2$.

Definition 1.3. Let C be a nonempty, closed convex subset of a smooth, strictly convex and reflexive Banach space E. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, i.e., $\Pi_C x = x_0$, where x_0 is the solution to the minimization problem

$$\phi(x_0, x) = \inf_{y \in C} \phi(y, x).$$

Let T be a self-mapping of C. A point p in C is said to be an asymptotic fixed point of T [4], if there exists a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$. We denote by $\hat{F}(T)$ the set of all asymptotic fixed points of T. Further, $p \in C$ is called a strongly asymptotic fixed point of T if there exists $\{x_n\} \subset C$ such that $x_n \to p$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{\hat{F}}$ the set of strongly asymptotic fixed points of T. $p \in C$ is called a generalized asymptotic fixed point of T [5], if there exists $\{x_n\} \subset C$ such that $Jx_n \rightharpoonup Jp$ and $\|Jx_n - JTx_n\| \to 0$. We denote by $\check{F}(T)$ the set of generalized asymptotic fixed points of S.

Definition 1.4. [3] Let C be a nonempty, closed convex subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex. To study the equilibrium problem, for the bifunction $f : JC \times JC \to \mathbb{R}$, we assume that f satisfies the following conditions:

(A1) $f(x^*, x^*) = 0$ for all $x^* \in JC$;
(A2) f is monotone, i.e., $f(x^*, y^*) + f(y^*, x^*) \leq 0$ for all $x^*, y^* \in JC$;
(A3) for each $x^*, y^*, z^* \in JC$,

$$\lim_{t \downarrow 0} f(tz^* + (1 - t)x^*, y^*) \leq f(x^*, y^*);$$

(A4) for each $x^* \in JC$, $y^* \mapsto f(x^*, y^*)$ is convex and lower semicontinuous.

2. Main Result

In this section, by introducing a new hybrid method, we prove strong convergence theorems for families of generalized nonexpansive mappings.

Theorem 2.1. *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E such that JC is closed and convex. Let $f : JC \times JC \to \mathbb{R}$ be a bifunction satisfying $(A1)-(A4)$. Let $\{S_1, S_2, S_3\}$ and $\mathcal{F}$ be families of generalized nonexpansive self mappings of C such that $F(S_1) \cap F(S_2) \cap F(S_3) \cap F(\mathcal{F}) \neq \emptyset$, $EP(f) \cap F(\mathcal{F}) \neq \emptyset$ and $\hat{\hat{F}}(T) = F(T)$ for all $T \in \mathcal{F}$. Let $C_1 = C$ and $\{x_n\} \subset C$ be a sequence generated by $x_1 = x$*

$$\begin{cases} u_n \in C_n \text{ s.t. } f(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - x_n, Jy - Ju_n\rangle \geq 0, \ \ \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)(\gamma_n S u_n + \lambda_n T u_n + \mu_n U u_n), \\ z_n = \beta_n y_n + (1-\beta_n)(\gamma_n S_1 x_n + \lambda_n S_2 x_n + \mu_n S_3 x_n), \\ C_{n+1} = \{u \in C_n : \phi(z_n, u) \leq \phi(x_n, u)\}, \\ x_{n+1} = R_{C_{n+1}} x, \end{cases}$$

for all $n \in \mathbb{N}$, where $R_{C_{n+1}}$ is the sunny generalized nonexpansive retraction of E onto C_{n+1} and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\}$ and $\{\mu_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(i) $\liminf_{n\to\infty}(1-\alpha_n) > 0$; (ii) $\lim_{n\to\infty} \beta_n = 1$; (iii) $\gamma_n + \lambda_n + \mu_n = 1$;

(iv) $\liminf_{n\to\infty} \gamma_n\lambda_n > 0$, $\liminf_{n\to\infty} \gamma_n\mu_n > 0$ *and* $\liminf_{n\to\infty} \lambda_n\mu_n > 0$

and $r_n \subset (0,\infty)$ such that $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$ converges strongly to $w_0 = R_{F(\mathcal{F})\cap EP(f)}x$, where $R_{\mathcal{F}\cap EP(f)}$ is the sunny generalized nonexpansive retraction from E onto $F(\mathcal{F}) \cap EP(f)$.

Corollary 2.2. *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E such that JC is closed and convex. Let $f : JC \times JC \to \mathbb{R}$ be a bifunction satisfying $(A1)-(A4)$. Let $\{S_1, S_2, S_3\}$ and $\mathcal{F}$ be families of generalized nonexpansive self mappings of C such that $F(S_1) \cap F(S_2) \cap F(S_3) \cap F(\mathcal{F}) \neq \emptyset$, $EP(f) \cap F(\mathcal{F}) \neq \emptyset$ and $\hat{\hat{F}}(T) = F(T)$ for all $T \in \mathcal{F}$. Let $C_1 = C$ and $\{x_n\} \subset C$ be a sequence generated by $x_1 = x$*

$$\begin{cases} u_n \in C_n \text{ s.t. } f(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - x_n, Jy - Ju_n\rangle \geq 0, \ \ \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)(\gamma_n S u_n + \lambda_n T u_n + \mu_n U u_n), \\ C_{n+1} = \{u \in C_n : \phi(y_n, u) \leq \phi(x_n, u)\}, \\ x_{n+1} = R_{C_{n+1}} x, \end{cases}$$

for all $n \in \mathbb{N}$, where $R_{C_{n+1}}$ is the sunny generalized nonexpansive retraction of E onto C_{n+1} and $\{\alpha_n\}, \{\gamma_n\}, \{\lambda_n\}$ and $\{\mu_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(i) $\liminf_{n\to\infty}(1-\alpha_n) > 0$; (ii) $\gamma_n + \lambda_n + \mu_n = 1$;

(iii) $\liminf_{n\to\infty} \gamma_n\lambda_n > 0$, $\liminf_{n\to\infty} \gamma_n\mu_n > 0$, $\liminf_{n\to\infty} \lambda_n\mu_n > 0$

and $r_n \subset (0,\infty)$ *such that* $\liminf_{n\to\infty} r_n > 0$. *Then,* $\{x_n\}$ *converges strongly to* $w_0 = R_{F(\mathcal{F})\cap EP(f)}x$, *where* $R_{\mathcal{F}\cap EP(f)}$ *is the sunny generalized nonexpansive retraction from* E *onto* $F(\mathcal{F})\cap EP(f)$.

Remark 2.3. Since, in a Hilbert space, $\phi(x,y) = \|x-y\|^2$ for all $x,y \in H$ and J is the identity mapping and a nonexpansive mapping $T : C \to H$ with a fixed point is also a generalized nonexpansive mapping, so, we can obtain the following new result for three nonexpansive non- self mapping using Theorem 2.1 gives the desired result.

Theorem 2.4. *Let* C *be a nonempty closed convex subset of a Hilbert space* E. *Let* $f : C\times C \to \mathbb{R}$ *be a bifunction satisfying* $(A1)-(A4)$. *Let* $\{S_1, S_2, S_3\}$ *and* $\mathcal{F}$ *be families of nonexpansive self mappings of* C *such that* $F(S_1)\cap F(S_2)\cap F(S_3)\cap F(\mathcal{F}) \neq \emptyset$, $EP(f)\cap F(\mathcal{F}) \neq \emptyset$. *Let* $C_1 = C$ *and* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 = x$

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle u_n - x_n, y - u_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)(\gamma_n S x_n + \lambda_n T x_n + \mu_n U x_n), \\ z_n = \beta_n y_n + (1-\beta_n)(\gamma_n S x_n + \lambda_n T x_n + \mu_n U x_n), \\ C_n = \{u \in C : \|z_n - u\| \leq \|x_n - u\|\}, \\ Q_n = \{u \in C : \langle x_n - u, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}x, \end{cases}$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\}$ *and* $\{\mu_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\liminf_{n\to\infty}(1-\alpha_n) > 0$; *(ii)* $\lim_{n\to\infty} \beta_n = 1$; *(iii)* $\gamma_n + \lambda_n + \mu_n = 1$;

(iv) $\liminf_{n\to\infty} \gamma_n\lambda_n > 0$, $\liminf_{n\to\infty} \gamma_n\mu_n > 0$ *and* $\liminf_{n\to\infty} \lambda_n\mu_n > 0$

and $r_n \subset (0,\infty)$ *such that* $\liminf_{n\to\infty} r_n > 0$. *Then,* $\{x_n\}$ *converges strongly to* $w_0 = P_{\mathcal{F}\cap EP(f)}x$, *where* $P_{\mathcal{F}\cap EP(f)}$ *is the metric projection from* E *onto* $\mathcal{F}\cap EP(f)$.

References

1. S. Alizadeh and F. Moradlou, *A strong convergence theorem for equilibrium problems and generalized hybrid mappings,* Mediterr. J. Math. **13** (2016), 379–390.
2. S. Alizadeh and F. Moradlou, *A monotone hybrid algorithm for a family of generalized nonexpansive mappings in Banach spaces*, Int. J. Nonlinear Anal. Appl, **13**(2),(2022) pp. 2347–2359.
3. E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems, Mathematics Students,* **63** (1994), 123–145.
4. D. Butnariu, S. Reich and A. J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces,* J. Appl. Anal. **7** (2001) 151–174.
5. T. Ibaraki and W. Takahashi, *Generalized nonexpansive mappings and a proximal-type algorithm in Banach spaces.,* In: Nonlinear Analysis and Optimization I: Nonlinear Analysis, Contemp. Math. **513**, Amer. Math. Soc., Providence, RI, (2010), 169–180.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

THE INVESTIGATION OF SOLUTION OF NONHOMOGENEOUS NONLINEAR INTEGRAL EQUATIONS SYSTEMS BASED ON COMMON FIXED POINT OF α-ADMISSIBLE SELFMAPS

PARI AMIRI*

Department of Advanced Technologies, University of Mohaghegh Ardabili, Namin, Iran
p_amiri@uma.ac.ir

Abstract. This work discusses the existence of common fixed points for α-admissible self-maps in Perov-type metric spaces. A theorem is established based on these results, demonstrating the existence of solutions for nonhomogeneous nonlinear integral equation systems, supported by a relevant example that illustrates the applicability of the findings.

1. Introduction

Integral equations are essential for solving initial value problems in differential equations, which emerge from the mathematical modeling of various scientific issues, such as epidemic spread, heat conduction, semiconductor devices, energy storage in renewable and diesel generation systems, etc [4]. Numerous studies have examined solving integral equations, each has its advantages and disadvantages. What is clear is that it is essential to confirm the existence of solutions for the integral equations before applying the provided solution methods. One of the important tools in this context is the

2020 *Mathematics Subject Classification.* *47H10; 45G15*
Key words and phrases. Common fixed point, Perov-type metric space, α-admissible selfmap, Nonhomogeneous nonlinear integral equations system.
* Speaker.

fixed point theory. Extensive research has presented the existence of solutions of differential or integral equations systems using fixed point theory in various metric spaces (See [1, 2, 5], and references therein).
This work presents some results about the existence common fixed point of α-admissible selfmaps in Perov-type metric space. Then, a theorem is presented via obtained results demonstrating the existence of solutions for nonhomogeneous nonlinear integral equations systems, supported by an applicable example that highlights the results' applicability.

2. Main results

Let $\mathfrak{X}$ be an arbitrary nonempty set. Denote the usual component-wise partial ordered space by $(\mathbb{R}^{\mathrm{r}}, \preceq)$. Consider each element $\upsilon \in \mathbb{R}^{\mathrm{r}}$ as $\upsilon^{\mathrm{T}} = (\upsilon^1, \upsilon^2, \cdots, \upsilon^{\mathrm{r}})$. Then, for each $\upsilon, \omega \in \mathbb{R}^{\mathrm{r}}$, if $\upsilon^l \preceq \omega^l$ ($\upsilon^l \prec \omega^l$), for $l = 1, 2, \cdots, \mathrm{r}$, then we have $\upsilon \preceq \omega$ ($\upsilon \prec \omega$), respectively. Perov-type generalized metric is a mapping $\mathrm{m_p} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^{\mathrm{r}}_+$ which satisfies all the usual axioms [3]. Consider $\mathrm{M_{r,r}}(\mathbb{R}_+)$ and I as the set of all $\mathrm{r}\times\mathrm{r}$ matrices with positive elements and the identity $\mathrm{r}\times\mathrm{r}$ matrix, respectively. A matrix $\mathrm{Q} \in \mathrm{M_{r,r}}(\mathbb{R}_+)$ is convergent to zero when $\mathrm{Q}^l \to 0$. Let $\check{\mathrm{S}}, \check{\mathrm{T}} : \mathfrak{X} \to \mathfrak{X}$ be two selfmaps and $\alpha : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$. Then,

(i) $\check{\mathrm{S}}$ is called α-admissible if $\alpha(\upsilon, \omega) \geq 1$ implies $\alpha(\check{\mathrm{S}}\upsilon, \check{\mathrm{S}}\omega) \geq 1$;
(ii) The pair $(\check{\mathrm{S}}, \check{\mathrm{T}})$ is α-admissible if $\alpha(\upsilon, \omega) \geq 1$ implies $\alpha(\check{\mathrm{S}}\upsilon, \check{\mathrm{T}}\omega) \geq 1$.

Theorem 2.1. *Let* $(\mathfrak{X}, \mathrm{m_p})$ *be a complete Perov-type metric space,* Q *a matrix in* $\mathrm{M_{r,r}}(\mathbb{R}^+)$ *convergent to zero, and* $\check{\mathrm{S}}, \check{\mathrm{T}} : \mathfrak{X} \to \mathfrak{X}$ *a pair of continuous* α*-admissible selfmaps on* $\mathfrak{X}$*. Suppose that there exists* $\upsilon_\flat \in \mathfrak{X}$ *such that* $\alpha(\upsilon_\flat, \check{\mathrm{S}}\upsilon_\flat) \geq 1$*, and also for each* $\upsilon, \omega \in \mathfrak{X}$*, there exists* $\mathrm{L}_{\upsilon,\omega} \in \mathcal{A}_{\upsilon,\omega}$ *such that*

$$\alpha(\upsilon, \omega)\mathrm{m_p}(\check{\mathrm{S}}\upsilon, \check{\mathrm{T}}\omega) \preceq \mathrm{Q}\mathrm{L}_{\upsilon,\omega}$$

where

$$\begin{aligned}\mathcal{A}_{\upsilon,\omega} = \{&\mathrm{m_p}(\upsilon, \omega), \mathrm{m_p}(\upsilon, \check{\mathrm{S}}\upsilon), \mathrm{m_p}(\omega, \check{\mathrm{T}}\omega),\\ &(\mathrm{I} + \mathrm{Q})^{-1}\mathrm{m_p}(\upsilon, \check{\mathrm{T}}\omega), (\mathrm{I} + \mathrm{Q})^{-1}\mathrm{m_p}(\omega, \check{\mathrm{S}}\upsilon)\};\end{aligned}$$

Then $\check{\mathrm{S}}$ *and* $\check{\mathrm{T}}$ *have a common fixed point.*

Example 2.2. Let $\mathfrak{X} = [-2, 2]$, $\mathrm{Q} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix}$, and $\mathrm{m_p} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^2$ defined by

$$\mathrm{m}^{\mathrm{T}}_{\mathrm{p}}(\upsilon, \omega) = \big(|\upsilon - \omega|, \delta|\upsilon - \omega|\big), \quad (\forall \upsilon, \omega \in \mathfrak{X},\ \delta \in [0, \frac{4}{5}]).$$

Define $\check{S},\check{T}:\mathfrak{X}\to\mathfrak{X}$ and $\alpha(\upsilon,\omega):\mathfrak{X}\times\mathfrak{X}\to\mathbb{R}^+$ by

$$\check{S}\upsilon=\begin{cases}\upsilon+3 & \upsilon\in[-2,-1],\\ \upsilon-3 & \upsilon\in[1,2],\\ \frac{1}{3}\upsilon & \upsilon\in(-1,1).\end{cases},\qquad \check{T}\omega=\begin{cases}\frac{1}{3}\omega & \omega\in(-1,1),\\ \omega & else.\end{cases},$$

and

$$\alpha(\upsilon,\omega)=\alpha(\omega,\upsilon)=\begin{cases}\frac{4}{3} & \upsilon,\omega\in[-2,-1]\times[1,2],\\ 1 & \upsilon,\omega\in(-1,1),\\ \frac{1}{2} & else.\end{cases}$$

$\check{S}$ and $\check{T}$ satisfy the conditions of Theorem 2.1 and have the common fixed point $\upsilon^*=0$.

In the next, it is proved the existing of solution of a nonhomogeneous nonlinear integral equations system by utilizing the obtained results in this work. Let $t_1,t_2\in\mathbb{R}$, $t_1<t_2$, and $\mathfrak{X}=C([t_1,t_2],\mathbb{R})$, where $C([t_1,t_2],\mathbb{R})$ is the space of all continuous functions from $[t_1,t_2]$ into $\mathbb{R}$. Consider the following nonhomogeneous nonlinear integral equations system:

$$\begin{cases}\upsilon(t)=x(t)+\int_{t_1}^{t_2}\mathcal{K}(t,\xi,\upsilon(\xi))d\xi\\ \omega(t)=y(t)+\int_{t_1}^{t_2}\mathcal{G}(t,\xi,\omega(\xi))d\xi,\end{cases}\quad(\forall t,\xi\in[t_1,t_2]),$$

where $\mathcal{K},\mathcal{G}:[t_1,t_2]^2\times\mathbb{R}\to\mathbb{R}$ and $\upsilon,\omega,x,y\in\mathfrak{X}$. Let $m_p:\mathfrak{X}^2\to\mathbb{R}^2_+$ is defined by:

$$m_p^T(\upsilon,\omega)=(\|\upsilon-\omega\|_\infty,\|\upsilon-\omega\|_1),\quad(\forall\upsilon,\omega\in\mathfrak{X}),\tag{2.1}$$

in which

$$\|\upsilon-\omega\|_\infty=\sup_{t\in[t_1,t_2]}|\upsilon(t)-\omega(t)|,\quad \|\upsilon-\omega\|_1=\int_{t_1}^{t_2}|\upsilon(t)-\omega(t)|dt.$$

It is obvious that $(\mathfrak{X},m_p(\upsilon,\omega))$ is a complete Perov-type metric space. Define the selfmaps $\check{S},\check{T}:\mathfrak{X}\to\mathfrak{X}$ as follows:

$$\begin{aligned}\check{S}\upsilon(t)&=x(t)+\int_{t_1}^{t_2}\mathcal{K}(t,\xi,\upsilon(\xi))d\xi,\\ \check{T}\omega(t)&=y(t)+\int_{t_1}^{t_2}\mathcal{G}(t,\xi,\omega(\xi))d\xi,\end{aligned}\quad(\forall t,\xi\in[t_1,t_2]),\tag{2.2}$$

where $\mathcal{K}$ and $\mathcal{G}$ are bounded continuous functions.

Theorem 2.3. *Let* $\mathfrak{X}=C([t_1,t_2],\mathbb{R})$, $Q=\begin{pmatrix}q_{11} & q_{12}\\ q_{21} & q_{22}\end{pmatrix}\in M_{r,r}(\mathbb{R}_+)$ *convergent to zero, the metric function* $m_p:\mathfrak{X}^2\to\mathbb{R}^2_+$ *and selfmaps* $\check{S},\check{T}:\mathfrak{X}\to\mathfrak{X}$ *are defined by* (2.1) *and* (2.2)*,respectively, and* $\alpha(\upsilon,\omega)=\alpha(\upsilon(t),\omega(t))$*, for all* $\upsilon,\omega\in\mathfrak{X}$ *and* $t\in[t_1,t_2]$*, which may has different constant values on different subintervals of* $[t_1,t_2]$*. Consider for each* $t,\xi\in[t_1,t_2]$,

$$\big|\mathcal{K}(t,\xi,\upsilon(\xi))-\mathcal{G}(t,\xi,\omega(\xi))\big|\le\frac{1}{t_2-t_1}\Big(\frac{\gamma|\upsilon(t)-\omega(t)|}{\alpha(\upsilon,\omega)}-|x(t)-y(t)|\Big),\tag{2.3}$$

where $\gamma=\min\{q_{11},q_{22}\}$*. Then* $\check{S}$ *and* $\check{T}$ *have a common fixed point in* $\mathfrak{X}$*.*

Example 2.4. Consider the bellow integral equations system:

$$\begin{cases} \upsilon(\mathrm{t}) = \mathrm{x}(\mathrm{t}) + \int_0^{\frac{\pi}{4}} \mathcal{K}(\mathrm{t}, \xi, \upsilon(\xi))d\xi, \\ \omega(\mathrm{t}) = \mathrm{y}(\mathrm{t}) + \int_0^{\frac{\pi}{4}} \mathcal{G}(\mathrm{t}, \xi, \omega(\xi))d\xi, \end{cases} \tag{2.4}$$

inwhich for all $\mathrm{t}, \xi \in [0, \frac{\pi}{4}]$,

$$\mathcal{K}(\mathrm{t}, \xi, \upsilon(\xi)) = \sin^2(\mathrm{t})\big(\frac{3}{2} - \upsilon(\xi)\big), \quad \mathcal{G}(\mathrm{t}, \xi, \omega(\xi)) = \mathrm{t}\big(\frac{3}{\pi}\frac{\omega(\xi)}{\xi} - \sin^2(\xi)\big),$$

$$\mathrm{x}(\mathrm{t}) = \big(\frac{\pi}{4}\sin(\mathrm{t})\big)^2, \quad \mathrm{y}(\mathrm{t}) = \frac{\pi}{8}\mathrm{t}.$$

It can be easily show that the equations solutions in (2.4) are

$$\upsilon(\mathrm{t}) = \frac{\pi}{2}\sin^2(\mathrm{t}), \quad \omega(\mathrm{t}) = \mathrm{t},$$

$\upsilon, \omega, \mathrm{x}, \mathrm{y} \in C([0, \frac{\pi}{4}], \mathbb{R})$, and $\mathcal{K}$ and $\mathcal{G}$ are bounded continuous functions on $(0, \frac{\pi}{4}]$. Figure 1 shows the common solutions $(\mathrm{t} = 0, \frac{\pi}{4})$ of $\upsilon(\mathrm{t})$ and $\omega(\mathrm{t})$ for the system (2.4). Let

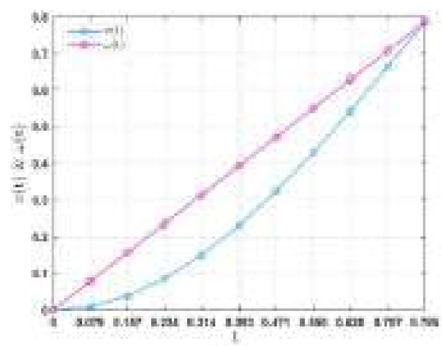


FIGURE 1. Graphical show of common solutions for the system (2.4).

$$\alpha(\upsilon(\mathrm{t}), \omega(\mathrm{t})) = \begin{cases} \sqrt{1.1} & \mathrm{t} \in [\frac{7\pi}{32}, \frac{\pi}{4}], \\ \frac{3}{4} & else. \end{cases}$$

for all $\upsilon, \omega \in \mathfrak{X}$, and $Q = \begin{pmatrix} \frac{9}{10} & \frac{1}{10} \\ 0 & \frac{9}{10} \end{pmatrix}$. Clearly, $Q \in M_{2,2}(\mathbb{R}_+)$ is convergent to zero and it can be easily check that inequality (2.3) holds for $\mathrm{t}, \xi \in [0, \frac{\pi}{4}]$.

REFERENCES

1. P. Amiri, H. Afshari, *Common fixed point results for multi-valued mappings in complex-valued double controlled metric spaces and their applications to the existence of solution of fractional integral inclusion systems*, Chaos, Solitons & Fractals, **154** (2022), 111622.

2. P. Amiri, ME. Samei, *Existence of urysohn and atangana–baleanu fractional integral inclusion systems solutions via common fixed point of multi-valued operators*, Chaos, Solitons & Fractals, **165** (2022), 112822.
3. AI. Perov, *On the Cauchy problem for a system of ordinary differential equations*, Pviblizhen. Met. Reshen. Differ. Uvavn, **2** (1964), 115–134.
4. M. Rahman, *Integral equations and their applications*, WIT press, (2007).
5. A. Thabet, RP. Agarwal, E. Karapınar, and PS. Kumari, *Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space*, Symmetry, **11** (2019), 686.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

THE SOLUTIONS' EXISTENCE OF HOMOGENEOUS NONLINEAR INTEGRAL EQUATIONS SYSTEMS VIA FIXED POIN THEORY

PARI AMIRI*

Department of Advanced Technologies, University of Mohaghegh Ardabili, Namin, Iran
p_amiri@uma.ac.ir

Abstract. This work aims to prove the existence of the solution for homogeneous nonlinear integral equation systems based on an approach of common fixed points for α-admissible selfmaps within the double controlled Perov-type metric spaces. An applicable example highlights the practical relevance of the findings.

1. Introduction

Integral equations have significant applications in medicine, science, and engineering. Numerous studies have examined solving linear/nonlinear and homogeneous/nonhomogeneous integral equations so far [3]. Verifying the existence of solutions for the integral equations is crucial before applying the proposed solution methods. One of the key tools in this context is fixed point theory. Research has shown the varied approaches of the fixed point theory for proving the existence of solutions for different integral systems across various metric spaces (See [1, 2], and references therein).
This research introduces the double controlled Perov-type metric space and

2020 *Mathematics Subject Classification.* *47H10; 45G15*

Key words and phrases. Common fixed point, double controlled Perov-type metric space, α-admissible selfmap, Homogeneous nonlinear integral equations system.

* Speaker.

discusses the existence of common fixed points for α-admissible selfmaps within this space. A theorem is subsequently presented that shows the existence of solutions for homogeneous nonlinear integral equation systems based on this research outputs. Finally, an example is provided to illustrate the practical applicability of the results.

2. Main results

Let $\mathfrak{X}$ be an arbitrary nonempty set and $(\mathbb{R}^{\mathrm{r}}, \preceq)$ the usual component-wise partial ordered space. Consider each element $\xi \in \mathbb{R}^{\mathrm{r}}$ as $\xi^{\mathrm{T}} = (\xi^1, \xi^2, \cdots, \xi^{\mathrm{r}})$. For $\xi, \eta \in \mathbb{R}^{\mathrm{r}}$, if $\xi^\kappa \preceq \eta^\kappa$ ($\xi^\kappa \prec \omega^\kappa$), for $\kappa = 1, \cdots, \mathrm{r}$, then we call $\xi \preceq \eta$ ($\xi \prec \eta$).

Definition 2.1. Consider two functions $\sigma_1, \sigma_2 : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}_+$ such that $\sigma_{\mathrm{j}}(\xi, \eta) \geq 1$, for each $\xi, \eta \in \mathfrak{X}$ and $\mathrm{j} = 1, 2$. The mapping $\Delta_{\mathrm{p}} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^{\mathrm{r}}_+$ is called double controlled Perov-type metric if for all $\xi, \eta, \zeta \in \mathfrak{X}$

(i) $\Delta_{\mathrm{p}}(\xi, \eta) \succeq 0$, $\big(\Delta_{\mathrm{p}}(\xi, \eta) = 0$ iff $\xi = \eta\big)$;
(ii) $\Delta_{\mathrm{p}}(\xi, \eta) = \Delta_{\mathrm{p}}(\eta, \xi)$;
(iii) $\Delta_{\mathrm{p}}(\xi, \eta) \preceq \sigma_1(\xi, \zeta)\Delta_{\mathrm{p}}(\xi, \zeta) + \sigma_2(\zeta, \eta)\Delta_{\mathrm{p}}(\zeta, \eta)$.

(Perov-type metric is a mapping $\mathrm{d} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^{\mathrm{r}}_+$ which satisfies all the usual axioms [4]). The pair $(\mathfrak{X}, \Delta_{\mathrm{p}})$ is called a double controlled Perov-type metric space. The consepts of Cauchy and convergent sequence in a Δ_{p}-metric space and completeness of space are defined similar to those in a metric space. Clearly, Δ_{p}-metric space is the Perov-type b-metric space, if $\sigma_{\mathrm{j}}(\xi, \eta) = \mathrm{b} \geq 1$, for all $\xi, \eta \in \mathfrak{X}$ and $\mathrm{j} = 1, 2$.

Denote $\mathrm{M}_{\mathrm{r,r}}(\mathbb{R}_+)$ and I as the set of $\mathrm{r} \times \mathrm{r}$ matrices with positive elements and identity $\mathrm{r} \times \mathrm{r}$ matrix, respectively. A matrix $\mathrm{Q} \in \mathrm{M}_{\mathrm{r,r}}(\mathbb{R}_+)$ is convergent to zero if $\mathrm{Q}^\kappa \to 0$. Let mappings $\Phi, \Psi : \mathfrak{X} \to \mathfrak{X}$ and $\alpha : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$. Then,
(i) Φ is called α-admissible if $\alpha(\xi, \eta) \geq 1$ implies $\alpha(\Phi\xi, \Phi\eta) \geq 1$;
(ii) (Φ, Ψ) is α-admissible if $\alpha(\xi, \eta) \geq 1$ implies $\alpha(\Phi\xi, \Psi\eta) \geq 1 \& \alpha(\Psi\eta, \Phi\xi) \geq 1$.

Theorem 2.2. *Let $(\mathfrak{X}, \Delta_{\mathrm{p}})$ be a complete double controlled Perov-type metric space, $\mathrm{Q} \in \mathrm{M}_{\mathrm{r,r}}(\mathbb{R}^+)$ convergent to zero, and (Φ, Ψ) a pair of continuous α-admissible selfmaps on $\mathfrak{X}$. Assume that*
(i) For any arbitrary sequences $\{\xi_{\mathrm{j}}\}_{\mathrm{j} \geq 0}$ and $\{\eta_{\mathrm{j}}\}_{\mathrm{j} \geq 0}$ in $\mathfrak{X}$,

$$\sup_{\ell \geq 0} \Big(\lim_{\mathrm{j} \to \infty} \sigma_1(\xi_{\mathrm{j}+1}, \xi_{\mathrm{j}+2}) \sigma_2(\xi_{\mathrm{j}+1}, \eta_\ell) \frac{\mathrm{R_i}(\mathrm{Q}^{\mathrm{j}+1}) \cdot \Delta_{\mathrm{p}}(\xi_\circ, \xi_1)}{\mathrm{R_i}(\mathrm{Q}^{\mathrm{j}}) \cdot \Delta_{\mathrm{p}}(\xi_\circ, \xi_1)} \Big) < 1,$$

where $\mathrm{R_i}(\mathrm{Q}^{\mathrm{j}})$ is the i-th row of the matrix Q^{j}, for $\mathrm{i} = 1, 2, \cdots, \mathrm{r}$;
(ii) $\forall \xi, \eta \in \mathfrak{X}$, there exists ${\mathrm{L}^*}_{\xi,\eta} \in {\mathcal{A}^*}_{\xi,\eta}$; $\alpha(\xi, \eta)\Delta_{\mathrm{p}}(\Phi\xi, \Psi\eta) \preceq \mathrm{Q}{\mathrm{L}^*}_{\xi,\eta}$, where

$${\mathcal{A}^*}_{\xi,\eta} = \Big\{ \Delta_{\mathrm{p}}(\xi, \eta), \Delta_{\mathrm{p}}(\xi, \Phi\xi), \Delta_{\mathrm{p}}(\eta, \Psi\eta), (\mathrm{I} + \sigma_2(\Phi\xi, \Psi\eta)\mathrm{Q})^{-1} \Delta_{\mathrm{p}}(\xi, \Psi\eta), (\mathrm{I} + \sigma_1(\Phi\xi, \Psi\eta)\mathrm{Q})^{-1} \Delta_{\mathrm{p}}(\eta, \Phi\xi) \Big\};$$

(iii) There exists $\xi_\circ \in \mathfrak{X}$ *such that* $\alpha(\xi_\circ, \Phi\xi_\circ) \geq 1$.
Then Φ *and* Ψ *have a common fixed point.*

Example 2.3. Let $\mathfrak{X} = [-\frac{3}{2}, \frac{3}{2}]$, $\Delta_p^T(\xi,\eta) = \big(|\xi-\eta|^{\frac{1}{2}}, |\xi-\eta|\big)$, for each $\xi, \eta \in \mathfrak{X}$, and $Q = \begin{pmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{1}{3} & \frac{1}{10} \end{pmatrix}$. Define functions $\sigma_1, \sigma_2 : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ by $\sigma_1(\xi,\eta) = \sqrt{2}$ and

$$\sigma_2(\xi,\eta) = \sigma_2(\eta,\xi) = \begin{cases} |\xi-\eta|^{\frac{1}{2}} & (\xi,\eta) \in [-\frac{3}{2}, -1) \times (1, \frac{3}{2}], \\ \sqrt{2} & else, \end{cases}$$

The condition (i) satisfies for any $\{\xi_\ell\}, \{\eta_\ell\} \subseteq \mathfrak{X}$, and Δ_p is complete double controlled Perov-type metric. Define $\Phi, \Psi : \mathfrak{X} \to \mathfrak{X}$ & $\alpha(\xi,\eta) : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ by

$$\Phi\xi = \begin{cases} \xi + \frac{5}{2} & \xi \in [-\frac{3}{2}, -1], \\ \xi - \frac{5}{2} & \xi \in [1, \frac{3}{2}], \\ -\xi & \xi \in (-1, 1). \end{cases}, \qquad \Psi\eta = \begin{cases} -(\eta + \frac{5}{2}) & \eta \in [-\frac{3}{2}, -1], \\ -(\eta - \frac{5}{2}) & \eta \in [1, \frac{3}{2}], \\ -\eta & \eta \in (-1, 1). \end{cases},$$

$$\alpha(\xi,\eta) = \alpha(\eta,\xi) = \begin{cases} \frac{9}{8} & \xi, \eta \in [-\frac{3}{2}, -1] \times [1, \frac{3}{2}], \\ 1 & \xi, \eta \in (-1, 1),\ \xi = \eta, \\ 0 & else. \end{cases}$$

Φ and Ψ satisfy all rules of Theorem 2.2 and $\xi^* = 0$ is common fixed point.

Let $J = [t_1, t_2] \subseteq \mathbb{R}$ and $\mathfrak{X} = L^2(J)$. Consider the bellow system:

$$\begin{cases} \xi(t) = \int_{t_1}^{t_2} \mathcal{K}(t, s, \xi(s)) ds, \\ \eta(t) = \int_{t_1}^{t_2} \mathcal{G}(t, s, \eta(s)) ds, \end{cases} \quad (\forall t, s \in J),$$

where $\xi, \eta \in \mathfrak{X}$ and $\mathcal{K}, \mathcal{G} : J^2 \times \mathbb{R} \to \mathbb{R}$. Define $\Delta_p : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}_+^2$ by:

$$\Delta_p^T(\xi,\eta) = (\sqrt{\|\xi-\eta\|_1}, \|\xi-\eta\|_2), \quad (\forall \xi, \eta \in \mathfrak{X}), \tag{2.1}$$

where $\|\xi-\eta\|_1 = \int_{t_1}^{t_2} |\xi(t) - \eta(t)| dt$ and $\|\xi-\eta\|_2 = \big(\int_{t_1}^{t_2} |\xi(t) - \eta(t)|^2 dt\big)^{\frac{1}{2}}$. $\Delta_p(\xi,\eta)$ is complete double controlled Perov-type metric with $\sigma_j(\xi,\eta) = \sqrt{2}$.

Theorem 2.4. *Let* $\mathfrak{X} = L^2(J)$, $\Delta_p : \mathfrak{X}^2 \to \mathbb{R}_+^2$ *is the metric defined by* (2.1), $\alpha(\xi,\eta) = \alpha(\xi(t), \eta(t))$, *for* $\xi, \eta \in \mathfrak{X}$ *and* $t \in J$. *Let* $Q = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_{r,r}(\mathbb{R}_+)$ *be convergent to zero such that*

$$\lim_{j\to\infty} \frac{R_i(Q^{j+1}) \cdot \Delta_p(\xi_\circ, \xi_1)}{R_i(Q^j) \cdot \Delta_p(\xi_\circ, \xi_1)} < \frac{1}{2}, \quad (\forall \xi_\circ, \xi_1 \in \mathfrak{X},\ i = 1, 2).$$

Define the selfmaps $\Phi, \Psi : \mathfrak{X} \to \mathfrak{X}$ *by* $\begin{cases} \Phi\xi(t) = \int_{t_1}^{t_2} \mathcal{K}(t, s, \xi(s)) ds, \\ \Psi\eta(t) = \int_{t_1}^{t_2} \mathcal{G}(t, s, \eta(s)) ds, \end{cases}$ *for* $t \in J$, *inwhich* $\mathcal{K}$ *and* $\mathcal{G}$ *are bounded continuous. Let* $\delta = \min\{a_{11}^2, \alpha(\xi,\eta) a_{22}\}$ *and*

$$\big|\mathcal{K}(t, s, \xi(s)) - \mathcal{G}(t, s, \eta(s))\big| \leq \frac{\delta|\xi(t) - \eta(t)|}{\alpha^2(\xi,\eta)(t_2 - t_1)}, \quad (\forall t, s \in J). \tag{2.2}$$

Then Φ and Ψ have a common fixed point in $\mathfrak{X}$.

Example 2.5. Consider the following integral equations system:

$$\begin{cases} \xi(\mathrm{t}) = \int_{\frac{1}{4}}^{\frac{1}{2}} \mathcal{K}(\mathrm{t}, \mathrm{s}, \xi(\mathrm{s}))ds, \\ \eta(\mathrm{t}) = \int_{\frac{1}{4}}^{\frac{1}{2}} \mathcal{G}(\mathrm{t}, \mathrm{s}, \eta(\mathrm{s}))ds, \end{cases} \tag{2.3}$$

where

$$\mathcal{K}(\mathrm{t}, \mathrm{s}, \xi(\mathrm{s})) = \frac{2^{2(\mathrm{t}-1)} \ln 2}{(\sqrt{2}-1)\xi(\mathrm{s})}, \quad \mathcal{G}(\mathrm{t}, \mathrm{s}, \eta(\mathrm{s})) = \frac{\mathrm{t} \ln 2}{\sqrt{2}-1} 4^{1-\eta(\mathrm{s})},$$

for all $\mathrm{t}, \mathrm{s} \in [\frac{1}{4}, \frac{1}{2}]$. It can be verify that the solutions of equations in the system (2.3) are $\xi(\mathrm{t}) = 4^{\mathrm{t}-1}$ and $\eta(\mathrm{t}) = \mathrm{t}$. Undoubtly, $\xi, \eta \in \mathrm{L}^2([\frac{1}{4}, \frac{1}{2}])$, and $\mathcal{K}, \mathcal{G}$ are bounded continuous on $[\frac{1}{4}, \frac{1}{2}]$. Figure 1 shows that $\mathrm{t} = \frac{1}{2}$ is the common solution of $\xi(\mathrm{t})$ and $\eta(\mathrm{t})$ for the system (2.3). Assume that

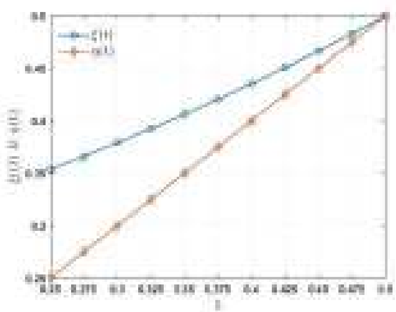


FIGURE 1. Graph of common solution of $\xi(\mathrm{t})$ and $\eta(\mathrm{t})$ for the system (2.3).

$$\alpha(\xi, \eta) = \alpha(\xi(\mathrm{t}), \eta(\mathrm{t})) = \begin{cases} \sqrt{1.1} & \mathrm{t} \in [\frac{15}{32}, \frac{1}{2}], \\ 0.5 & else \end{cases}$$

for all $\xi, \eta \in \mathfrak{X}$, $\mathrm{Q} = \begin{pmatrix} \sqrt{0.97} & 0 \\ 0.1 & 0.93 \end{pmatrix}$, and $\Phi, \Psi : \mathrm{L}^2([\frac{1}{4}, \frac{1}{2}]) \to \mathrm{L}^2([\frac{1}{4}, \frac{1}{2}])$ are defined by:

$$\Phi\xi(\mathrm{t}) = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{4^{\mathrm{t}-\mathrm{s}} \ln 2}{\sqrt{2}-1} ds, \quad \Psi\eta(\mathrm{t}) = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\mathrm{t} 4^{1-\mathrm{s}} \ln 2}{\sqrt{2}-1} ds, \quad (\forall \mathrm{t}, \mathrm{s} \in [\frac{1}{4}, \frac{1}{2}]).$$

One can easily verify that (2.2) holds for $\mathrm{t}, \mathrm{s} \in [\frac{1}{4}, \frac{1}{2}]$ and $\mathrm{t} = \frac{1}{2}$ is a solution of the integral equations system (2.3).

References

1. P. Amiri, H. Afshari, *Common fixed point results for multi-valued mappings in complex-valued double controlled metric spaces and their applications to the existence of solution of fractional integral inclusion systems*, Chaos, Solitons & Fractals, **154** (2022), 111622.
2. P. Amiri, ME. Samei, *Existence of urysohn and atangana–baleanu fractional integral inclusion systems solutions via common fixed point of multi-valued operators*, Chaos, Solitons & Fractals, **165** (2022), 112822.
3. M. Rahman, *Integral equations and their applications*, WIT press, 2007.
4. AI. Perov, *On the Cauchy problem for a system of ordinary differential equations*, Pviblizhen. Met. Reshen. Differ. Uvavn, **2** (1964), 115–134.

WEIGHTED SPACES OF HOLOMORPHIC FUNCTIONS ON TWO SUBSETS OF COMPLEX PLANE

MOHAMMAD ALI ARDALANI*

Department of Mathematics, University of Kurdistan, Sanandaj, Iran
m.ardalani@uok.ac.ir

Abstract. In this paper, we present isomorphic classification of weighted on two new subsets of complex plane. Also we obtain a necessary and sufficient condition for boundedness of composition operators on these spaces.

1. Introduction

For a holomorphic function $f : O \longrightarrow \mathbb{C}$ we define the weighted sup-norm

$$\|f\|_v = \sup_{z \in O} \mid f(z) \mid v(z)$$

and weighted spaces

$$H_v(O) = \{f : O \longrightarrow \mathbb{C} : \|f\|_v < \infty\}$$

and

$$H_{v_0}(O) = \{f \in H_v(O) :\mid f(z) \mid v(z) \text{ vanishes at infinity}\}.$$

$\mid f \mid v$ vanishes at infinity if for any $\varepsilon > 0$ there is a compact subset $K \subset O$ such that $\mid f(z) \mid v(z) < \varepsilon$ for all $z \in O \setminus K$.
Firstly consider the following set.

$$\mathbb{C}_{\mathbb{I}} := \{\omega \in \mathbb{C} : \omega \neq 0,\ -\pi < \arg\ \omega < \pi\} = \mathbb{C} \setminus (-\infty, 0].$$

2020 *Mathematics Subject Classification.* Primary 46E15; Secondary 46B03
Key words and phrases. Weighted spaces, holomorphic functions, Isomorphic classification, Composition operators.

* Speaker.

By a standard weight υ on $\mathbb{C}_{\mathbb{I}}$ we mean a continuous function $\upsilon : \mathbb{C}_{\mathbb{I}} \to (0, \infty)$ such that

$$\upsilon(s) < \upsilon(r) \ \text{ if } \ 0 < s < r \ \text{ and } \ \lim_{r \to 0^+} \upsilon(r) = 0. \tag{1.1}$$

Also we are deal with

$$\mathbb{D}_{\mathbb{I}} := \mathbb{D} \cap \mathbb{C}_{\mathbb{I}} = \mathbb{D} \setminus [-1, 0].$$

where $\mathbb{D} := \{\omega \in \mathbb{C} : \mid \omega \mid < 1\}$ is the unit disc.
Here by a standard weight υ on $\mathbb{D}_{\mathbb{I}}$ we mean a continuous function $\upsilon : \mathbb{C}_{\mathbb{I}} \to (0, \infty)$ which satisfies similarly relation (1.1) on $\mathbb{D}_{\mathbb{I}}$.

Definition 1.1. (a) A standard weight $\upsilon : \mathbb{C}_{\mathbb{I}} \to (0, \infty)$ satisfies $(*)_{\mathbb{C}_{\mathbb{I}}}$ if

$$\sup_{k \in \mathbb{Z}} \frac{\upsilon(2^{2k})}{\upsilon(2^{2(k-1)})} < \infty$$

and satisies $(**)_{\mathbb{C}_{\mathbb{I}}}$ if

$$\inf_{n \in \mathbb{N}} \sup_{k \in \mathbb{Z}} \frac{\upsilon(2^{2(k-1)})}{\upsilon(2^{2(k-1+n)})} < 1.$$

(b) A standard weight $\upsilon : \mathbb{D}_{\mathbb{I}} \longrightarrow (0, \infty)$ satisfies $(*)_{\mathbb{D}_{\mathbb{I}}}$ if

$$\sup_{k \in \mathbb{Z}} \frac{\upsilon(\frac{2^k}{2^k+2})}{\upsilon(\frac{2^{k-1}}{2^{k-1}+2})} < \infty$$

and satisfies $(**)_{\mathbb{D}_{\mathbb{I}}}$ if

$$\inf_{n \in \mathbb{N}} \sup_{k \in \mathbb{Z}} \frac{\upsilon(\frac{2^k}{2^k+2})}{\upsilon(\frac{2^{k-1+n}}{2^{k-1+n}+2})} < 1.$$

Example 1.2. (a) For any ω in $\mathbb{C}_{\mathbb{I}}$ we define $\upsilon_1(\omega) = (Im(i\omega^{\frac{1}{2}}))^2$ and $\upsilon_2(\omega) = \min(\upsilon_1(\omega), 1)$. υ_1 satisfies both $(*)_{\mathbb{C}_{\mathbb{I}}}$ and $(**)_{\mathbb{C}_{\mathbb{I}}}$ while υ_2 satisfies only $(*)_{\mathbb{C}_{\mathbb{I}}}$.
(b) For any ω in $\mathbb{D}_{\mathbb{I}}$ we define $\upsilon_3(\omega) = \mid \frac{2\omega}{1-\omega} \mid$ and $\upsilon_4(\omega) = \min(\upsilon_3(\omega), 1)$. υ_3 satisfies both $(*)_{\mathbb{D}_{\mathbb{I}}}$ and $(**)_{\mathbb{D}_{\mathbb{I}}}$ while υ_4 satisfies only $(*)_{\mathbb{D}_{\mathbb{I}}}$.

By constructing two conformal maps from $\mathbb{C}_{\mathbb{I}}$ and $\mathbb{D}_{\mathbb{I}}$ onto the upper half-plane respectively and using results of [2] and [3] we characterize isomorphic classification of $H_\upsilon(\mathbb{C}_{\mathbb{I}}), H_{\upsilon_0}(\mathbb{C}_{\mathbb{I}}), H_\upsilon(\mathbb{D}_{\mathbb{I}})$ and $H_{\upsilon_0}(\mathbb{D}_{\mathbb{I}})$. Besides we use results of [1] for obtaing a necessary and sufficient condition for boundedness of composition operators.

2. Main results

As our main results on the isomorphic classifications of weighted spaces $H_\upsilon(\mathbb{C}_{\mathbb{I}}), H_{\upsilon_0}(\mathbb{C}_{\mathbb{I}}), H_\upsilon(\mathbb{D}_{\mathbb{I}})$ and $H_{\upsilon_0}(\mathbb{D}_{\mathbb{I}})$ we obtain

Theorem 2.1. *Let υ be a standard weight on $\mathbb{C}_{\mathbb{I}}$ satisfying $(*)_{\mathbb{C}_{\mathbb{I}}}$. Then*

(i) *$H_\upsilon(\mathbb{C}_\mathbb{I})$ is isomorphic to ℓ_∞ if and only if υ satisfies $(**)_{\mathbb{C}_\mathbb{I}}$.*

(ii) *$H_{\upsilon_0}(\mathbb{C}_\mathbb{I})$ is isomorphic to c_0 if and only if υ satisfies $(**)_{\mathbb{C}_\mathbb{I}}$.*

(iii) *$H_\upsilon(\mathbb{C}_\mathbb{I})$ is isomorphic to $H_\infty(\mathbb{D})$ if and only if υ does not satisfy $(**)_{\mathbb{C}_\mathbb{I}}$. Where $H_\infty(\mathbb{D})$ is the space of all bounded analytic functions on the unit disc $\mathbb{D}$.*

Similarly for $H_\upsilon(\mathbb{D}_\mathbb{I})$ and $H_{\upsilon_0}(\mathbb{D}_\mathbb{I})$ we have

Theorem 2.2. *Let υ be a standard weight on $\mathbb{D}_\mathbb{I}$ satisfying $(*)_{\mathbb{D}_\mathbb{I}}$. Then*

(i) *$H_\upsilon(\mathbb{D}_\mathbb{I})$ is isomorphic to ℓ_∞ if and only if υ satisfies $(**)_{\mathbb{D}_\mathbb{I}}$.*

(ii) *$H_{\upsilon_0}(\mathbb{D}_\mathbb{I})$ is isomorphic to c_0 if and only if υ satisfies $(**)_{\mathbb{D}_\mathbb{I}}$.*

(iii) *$H_\upsilon(\mathbb{D}_\mathbb{I})$ is isomorphic to $H_\infty(\mathbb{D})$ if and only if υ does not satisfy $(**)_{\mathbb{D}_\mathbb{I}}$.*

For boundedness of composition opeartors on $H_\upsilon(\mathbb{C}_\mathbb{I})$ and $H_\upsilon(\mathbb{D}_\mathbb{I})$ we present two following theorems:

Theorem 2.3. *Let υ_1 and υ_2 be standard weights on $\mathbb{C}_\mathbb{I}$ such that υ_1 satisifies $(*)_{\mathbb{C}_\mathbb{I}}$. Also suppose $\varphi : \mathbb{C}_\mathbb{I} \longrightarrow \mathbb{C}_\mathbb{I}$ is a nonconstant holomorphic map. Then the composition operator $C_\varphi : H_{\upsilon_1}(\mathbb{C}_\mathbb{I}) \longrightarrow H_{\upsilon_2}(\mathbb{C}_\mathbb{I})$ (defined by $C_\varphi(f) = f \circ \varphi$) is a bounded operator if and only if*

$$\sup_{\omega \in \mathbb{C}_\mathbb{I}} \frac{\upsilon_2(\omega)}{\upsilon_1(\varphi(\omega))} < \infty.$$

Theorem 2.4. *Let υ_1 and υ_2 be standard weights on $\mathbb{D}_\mathbb{I}$ such that υ_1 satisifies $(*)_{\mathbb{D}_\mathbb{I}}$. Also suppose $\varphi : \mathbb{D}_\mathbb{I} \longrightarrow \mathbb{D}_\mathbb{I}$ is a nonconstant holomorphic map. Then the composition operator $C_\varphi : H_{\upsilon_1}(\mathbb{D}_\mathbb{I}) \longrightarrow H_{\upsilon_2}(\mathbb{D}_\mathbb{I})$ is a bounded operator if and only if*

$$\sup_{\omega \in \mathbb{D}_\mathbb{I}} \frac{\upsilon_2(\omega)}{\upsilon_1(\varphi(\omega))} < \infty.$$

References

1. M. A. Ardalani and W. Lusky, *Bounded operators on weighted spaces of holomorphic functions on the upper halfplane*, Studia Math. **209** (2012), 225-234.
2. M. A. Ardalani and W. Lusky, *Weighted spaces of holomorphic functions on the upper halfplane*, Math. Scand. **111** (2012), 244-260.
3. A. Harutyunyan and W. Lusky, *A remark on the isomorphic classification of weighted spaces of holomorphic functions on the upper halfplane*, Ann. Univ. Sci. Budapest Sect. Comp. **39** (2013), 125-135.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

CONCERNING ABOUT A TOPOLOGY OF PARTIAL METRIC SPACES

MEHDI ASADI*

Department of Mathematics, Zanjan Branch, Islamic Azad University, Zanjan, Iran
masadi.azu@gmail.com; masadi@iau.ir

Abstract. In this talk, we speak about a introducing a weaker than its topology which is made by Matthews in partial metric spaces(PMS). Some illustrated examples are included. Also, we also, focus in the definition of partial metric space to improve it.

1. Introduction

After introducing partially metric spaces by Matthews in [3] many papers are written specially in fixed point theory which all of them turn on $p(a,a)$ is not zero. In this paper we make a weaker than its topology and we remove the condition $p(x,x) \leq p(x,y)$ in the following main definition of partial metric. For more detail refer to [1, 2].

Definition 1.1 ([3])**.** Let X be a nonempty set and $p : X \times X \to \mathbb{R}^+$ be a self mapping of X such that for all $x, y, z \in X$ the followings are satisfied:

p1 $x = y \iff p(x,x) = p(x,y) = p(y,y)$,
p2 $p(x,x) \leq p(x,y)$,
p3 $p(x,y) = p(y,x)$,
p4 $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

2020 *Mathematics Subject Classification. Primary 47H10; Secondary 54H25*
Key words and phrases. Topology of partially metric; Partially metric space; Fixed point.

* Speaker.

Then p is called partial metric on X and the pair (X, p) is called partial metric space (in short PMS).

Note p on X generates a T_0 topology τ_p on X, whose base is a family of open p-balls
$$\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$$
where
$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\},$$
for all $x \in X$ and $\varepsilon > 0$.

2. Main results

Now we introduce new definition of partially metric.

Definition 2.1. Let X be a nonempty set and $p : X \times X \to \mathbb{R}^+$ be a self mapping of X such that for all $x, y, z \in X$ the followings are satisfied:

$p1$ $p(x, x) = p(x, y) = p(y, y) \iff x = y$,
$p3$ $p(x, y) = p(y, x)$,
$p4$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then p is called partial metric on X and the pair (X, p) is called partial metric space.

Put
$$d(x, y) := p(x, y) - \min\{p(x, x), p(y, y)\} + k|p(x, x) - p(y, y)|. \tag{2.1}$$

Proposition 2.2 ([1]). *d is a metric on X.*

If we define weak topology τ_d by the balls
$$B^k_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\},$$
for every $k \in (0, 1)$. Then τ_d is T_0.

Theorem 2.3 ([1]). *Balls $B^k_d(x, \varepsilon)$ for every $x \in X$ and $\varepsilon > 0$ makes a base for topology τ_d.*

Corollary 2.4 ([1]). *Let (X, p) be a complete partially metric space. T a self mapping on X and*
$$p(Tx, Ty) - \min\{p(Tx, Tx), p(Ty, Ty)\} \leq l(p(x, y) - \min\{p(x, x), p(y, y)\}),$$
for some $l \in [0, 1)$ and for every $x, y \in X$. Then T has a unique fixed point on X.

Acknowledgement

The author express his deep gratitude to the referees for their valuable comments and suggestions.

References

1. Mehdi Asadi, *Some notes on topology of partially metric spaces*, Filomat, **38**(5) (2024), 1745–1750.
2. Mehdi Asadi, Farshid Khojasteh *New topology on M-metric space*, Appear in Journal of New researches in Mathematics, 2023.
3. S. G. Matthews, *Partial metric topology*, Proceedings of the 8th summer conference on topology and its applications, Annals of the New York Academy of Sciences, **728** (1994), 183-197.

J-CLASS IS PRESERVED UNDER ISOMORPHISM

MEYSAM ASADIPOUR* AND MOHAMMAD REZA AZIMI

[1] *Department of Mathematics, College of Sciences, Yasouj University, Yasouj, Iran.; Asadipour.mey@gmail.com; Asadipour@yu.ac.ir*

[2] *Department of Mathematics, Faculty of Sciences, University of Maragheh, 55181-83111, Golshahr, Maragheh, Iran. mhr.azimi@maragheh.ac.ir*

ABSTRACT. Let X be an infinite-dimensional complex Banach space. For an operator $T \in L(X)$ and a vector $x \in X$, let $J_T(x)$ denote the set of all points $y \in X$ such that $x_n \longrightarrow x$ and $T^{k_n}x_n \longrightarrow y$ for some sequences $\{x_n\} \subset X$, $\{k_n\} \subset \mathbb{N}$ and $k_{n+1} > k_n$. If $J_T(x) = X,\ x \neq 0$, then T is called a J-class operator. In this paper, we show that J-class is preserved under isomorphism, Furthermore, to get more familiar with J-class operators we will give a nontrivial example of J-class operator on $\ell^2(\mathbb{N})$.

1. INTRODUCTION

Assume that X is a complex Banach space and T is a linear continuous operator on X. If for every nonempty open subsets U, V of X, there exists a positive integer n so that $T^n(U) \cap V$ is nonempty, then T is called transitive. If the underlying space is considered as a separable Banach space, then transitivity is equivalent to hypercyclicity. To clarify the notion of hypercyclicity, note that if for an $x \in X$, $orb(T, x) = \{T^n x;\ n = 0, 1, 2, \cdots\}$ is a dense subset of X, then x is called a hypercyclic vector for T and in

2020 *Mathematics Subject Classification.* *47A16; 37B99*
Key words and phrases. Orbit, Hypercyclicity, J-class.
* Speaker.

this case T is called a hypercyclic operator. It is well known that X is an infinite-dimensional separable Banach space if and only if there exists a hypercyclic operator on it, [5]. Therefore every non-separable Banach space is ignored in the hypercyclicity. Thus the authors in [3] proposed $J(x)$ for an operator T and a vector x in Banach space X as following:

$$J_T(x) = \{z \in X : \text{there exist a sequence } \{z_n\} \subset X \text{ and a strictly increasing sequence of positive integers } \{m_n\}, \text{ such that } z_n \to x \text{ and } T^{m_n} z_n \to z\}.$$

Note that, an operator T is called a J-class operator if there exists a non-zero $x \in X$ such that $J_T(x) = X$ and every hypercyclic operator is an example of the J-class operators. It worth to mention that, there exist many J-class operators on the non-separable infinite Banach spaces $\ell^\infty(\mathbb{N})$, [3]. Also note that, an equivalent definition for the set $J_T(x)$ through the open sets was introduced, [2]. To be precise;

$$J_T(x) = \{z \in X : \text{for every pair of neighborhoods } U,\ V \text{ of } x,\ z \text{ respectively, and every } N \in \mathbb{N}, \text{ there exists an integer } n > N \text{ such that } T^n U \cap V \neq \phi\}.$$

More information can be seen in [1], [4] and [5].

In the following, we will show that J-class is preserved under isomorphism. Furthermore, to get more familiar with J-class operators we will give a non-trivial example of J-class operator on $\ell^2(\mathbb{N})$.

2. Main results

It is well known that, if a hypercyclic operator T is isomorphic to $\widetilde{T}$, then $\widetilde{T}$ is also hypercyclic; that is, hypercyclicity is preserved under isomorphism. Hence It is natural to raise the following question;
Question: Is the J-class preserved under isomorphism?
The answer to this question can be seen in the following theorem.

Theorem 2.1. *Let T be a J-class operator on X and also let T be isomorphic to $\widetilde{T}$ on Y. Then $\widetilde{T}$ is a J-class operator.*

Proof. Assume that $J(x) = X, x \neq 0$ and $S : Y \to X$ is a linear and continuous isomorphism and $\widetilde{T} := S^{-1}TS$. We want to show that $S^{-1}x$ is a J-class for $\widetilde{T}$ or equivalently $J_{\widetilde{T}}(S^{-1}x) = Y$. Thus for an arbitrary $y \in Y$ there exist some sequences $\{x_n\} \subset X$ and a strictly increasing sequence of positive integers $\{m_n\}$ such that

$$x_n \to x \quad \text{and} \quad T^{m_n} x_n \to Sy.$$

Hence

$$S^{-1}x_n \to S^{-1}x \quad \text{and} \quad \widetilde{T}^{m_n} S^{-1} x_n = S^{-1} T^{m_n} x_n \to y$$

Therefore the proof is completed. □

To get more familiar with J-class operators we give a nontrivial example of J-class operator on $\ell^2(\mathbb{N})$ in the following.

Example 2.2. Consider weighted backward shift operator T on $\ell^2(\mathbb{N})$ given by:

$$T(x^1, x^2, \cdots) = (2x^2, \frac{3}{2}x^3, \frac{4}{3}x^4, \cdots).$$

Also let Y be the set of finite sequences with entries $z \in \mathbb{C}$ that $Re(z) \in \mathbb{Q}$, $Im(z) \in \mathbb{Q}$. Since Y is dense in $\ell^2(\mathbb{N})$, so there are strictly increasing sequence $\{2k\}_k$, sequence $\{x_k\} \subset Y$ that

$$x_k = (x^1, 0, x^3, 0, \cdots, x^{2k-1}, 0, 0, \cdots),$$

$x_k \longrightarrow 0$ as $k \longrightarrow \infty$ and $T^{2k}x_k = 0$. Now, for the random member

$$y = (y^1, 0, y^3, 0, \cdots, y^{2m+1}, 0, 0, \cdots) \in Y$$

and $k \geq 1$, we set;

$$w_{2k}(y) = (\underbrace{0, \cdots, 0}_{2k-times}, \frac{y^1}{2k+1}, 0, \frac{3y^3}{2k+3}, 0, \cdots, \frac{(2m+1)y^{2m+1}}{2(k+m)+1}, 0, 0, \cdots).$$

Clearly, for every $k \in \mathbb{N} \cup \{0\}$, $w_{2k}(y)$ belongs to Y and the sequence $\{w_{2k}(y)\}$ is a sequence in $\ell^2(\mathbb{N})$. Since

$$||w_{2k}(y)||^2 = \sum_{j=1}^{2m} |\frac{j}{2k+j}y^j|^2 \leq \frac{4m^2}{(2k+1)^2}||y||^2,$$

so $w_{2k}(y) \longrightarrow 0$, as $k \longrightarrow \infty$. Note that for $n \geq 1$:

$$T^n(x^1, x^2, x^3, \cdots) = \left((n+1)x^{n+1}, \frac{1}{2}(n+2)x^{n+2}, \frac{1}{3}(n+3)x^{n+3}, \cdots\right),$$

thus

$$T^{2k}w_{2k}(y) = \Bigg((2k+1)\frac{1}{2k+1}y^1, 0, (\frac{2k+3}{3})(\frac{3}{2k+3})y^3, 0, \cdots, (\frac{2(k+m)+1}{(2m+1)})(\frac{2m+1}{2(k+m)+1})y^{2m+1}, 0, 0, \cdots\Bigg) = y.$$

Hence all conditions of the J-class Criterion in [1] holds and $J_T(0) = \ell^2(\mathbb{N})$ and consequently [3, Proposition5.13] implies that $J_T(x) = \ell^2(\mathbb{N})$ for some nonzero $x \in \ell^2(\mathbb{N})$.

References

1. M. Asadipour and B. Yousefi, *On some properties of J-class operators*, Commun. Korean Math. Soc., **34** (1) (2019), 145–154.
2. M. R. Azimi, *J-class sequences of linear operators*, Complex Anal. Oper. Theory, **12** (2018), 293–303.
3. G. Costakis and A. Manoussos, *J-class operators and hypercyclicity*, J. Operator Theory, **67** (2012), 101–119.

4. G. Costakis and A. Manoussos, *J-class Weighted Shifts on the Space of Bounded Sequences of Complex Numbers*, Integral Equations and Operator Theory, **62** (2008), 149–158.
5. K.G. Grosse-Erdmann and A. Peris, *Linear Chaos*, Springer, 2011.

EXPANSIVE WEIGHTED GENERALIZED SHIFT OPERATORS ON ℓ^p SPACES

FATEMAH AYATOLLAH ZADEH SHIRAZI*

Faculty of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran
f.a.z.shirazi@ut.ac.ir

Abstract. The main aim of the following text is to classify all expansive weighted generalized shift operators on ℓ^p spaces.

1. Introduction

If Γ is a nonempty set and $\varphi : \Gamma \to \Gamma$ is arbitrary, for arbitrary set X, let's call $\sigma_\varphi : X^\Gamma \to X^\Gamma$ with $\sigma_\varphi((x_i)_{i\in\Gamma}) = (x_{\varphi(i)})_{i\in\Gamma}$ (for $(x_i)_{i\in\Gamma} \in X^\Gamma$), a *generalized shift*, which has been introduced for the first time in [2], as a generalization of one–sided shift and two–sided shift. Now suppose X is a vector space over field F and choose *weight vector* $\mathfrak{a} = (\mathfrak{a}_i)_{i\in\Gamma} \in F^\Gamma$, then one may consider *weghted generalized shift* $\sigma_{\varphi,\mathfrak{a}} : X^\Gamma \to X^\Gamma$ *with* $\sigma_{\varphi,\mathfrak{a}}((x_i)_{i\in\Gamma}) = (\mathfrak{a}_i x_{\varphi(i)})_{i\in\Gamma}$ *(for* $(x_i)_{i\in\Gamma} \in X^\Gamma$*), weighted generalized shifts are a common generalization of weighted shifts and generalized shifts, they are known as weighted composition operators too. Let's refer the interested reader to* [1].
Consider nonzero cardinal number τ, $\mathbb{C}^\tau$ *is a vector space over* $\mathbb{C}$ *in a natural way. For* $p \in [1, +\infty)$ *let* $\ell^p(\tau) = \{x \in \mathbb{C}^\tau : ||x||_p < +\infty\}$ *and equipp* $\ell^p(\tau)$ *with norm* $||\ ||_p$.
For metric space (Z, D) *we call a homeomorphism* $f : Z \to Z$ *expansive*

2020 *Mathematics Subject Classification. Primary 46C99*
Key words and phrases. Expansive, Weighted composition operator, Weighted generalized shift.
* Speaker.

if there exists $\varepsilon > 0$ *such that* $\sup\{D(f^n(x), f^n(y)) : n \in \mathbb{Z}\} > \varepsilon$ *for each distinct* $x, y \in Z$.
Expansive generalized shifts on Cantor sets have been studied in [3]. *In this text we study expasivity of "suitable" weighted generalized shifts. .*

2. Main results

In this section suppose τ *is a nonzero cardinal number,* $\varphi : \tau \to \tau$ *is arbitrary,* $p \in [1, +\infty)$ *and* $\mathfrak{w} = (\mathfrak{w}_\alpha)_{\alpha\in\tau} \in \mathbb{C}^\tau$,

Remark 2.1. For $p \in [1, +\infty)$, the following statements are equivalent [1]:

- $\sigma_{\varphi,\mathfrak{w}}(\ell^p(\tau)) \subseteq \ell^p(\tau)$,
- $\sigma_{\varphi,\mathfrak{w}}(\ell^p(\tau)) \subseteq \ell^p(\tau)$ and $\sigma_{\varphi,\mathfrak{w}} \restriction_{\ell^p(\tau)}: \ell^p(\tau) \to \ell^p(\tau)$ is continuous,
- $\sup\{||(\mathfrak{w}_\alpha)_{\alpha\in\varphi^{-1}(\beta)}||_p : \beta \in \varphi(\tau)\} < +\infty$.

For $\alpha \in \tau$ *suppose* $\pi_\alpha : \ell^p(\tau) : \ell^p(\tau) \to \mathbb{C}$ *is the projection map on the* α*'s coordinate. And* $\mathsf{e}_\alpha = (\delta_{\beta,\alpha})_{\beta\in\tau}$ *where* $\delta_{\alpha,\alpha} = 1$ *and* $\delta_{\beta,\alpha} = 0$ *for* $\beta \neq \alpha$.

Lemma 2.2. *If* $\sigma_{\varphi,\mathfrak{w}}(\ell^p(\tau)) \subseteq \ell^p(\tau)$, *then the following statents are equivalent:*

(1) $\sigma_{\varphi,\mathfrak{w}} \restriction_{\ell^p(\tau)}: \ell^p(\tau) \to \ell^p(\tau)$ *is a homeomorphism,*
(2) $\varphi : \tau \to \tau$ *is bijective and*

$$0 < \inf\{|\mathfrak{w}_\alpha| : \alpha \in \tau\} \leq \sup\{|\mathfrak{w}_\alpha| : \alpha \in \tau\} < +\infty\,.$$

Proof. (1) $\Rightarrow$ (2): Suppose $\sigma_{\varphi,\mathfrak{w}} \restriction_{\ell^p(\tau)}: \ell^p(\tau) \to \ell^p(\tau)$ is a homeomorphism, then for each $\theta \in \tau$ we have (since $\sigma_{\varphi,\mathfrak{w}} \restriction_{\ell^p(\tau)}: \ell^p(\tau) \to \ell^p(\tau)$ is surjective):

$$\begin{aligned}\mathbb{C} &= \pi_\theta(\ell^p(\tau)) = \pi_\theta(\sigma_{\varphi,\mathfrak{w}}(\ell^p(\tau))) \\ &= \{\pi_\theta((\mathfrak{w}_\alpha x_{\varphi(\alpha)})_{\alpha\in\Gamma}) : (x_\alpha)_{\alpha\in\Gamma} \in \ell^p(\tau)\} \\ &= \{\mathfrak{w}_\theta x_{\varphi(\theta)} : (x_\alpha)_{\alpha\in\Gamma} \in \ell^p(\tau)\} = \{\mathfrak{w}_\theta z : z \in \pi_{\varphi(\theta)}(\ell^p(\tau))\} = \mathfrak{w}_\theta\mathbb{C}\end{aligned}$$

so $\mathfrak{w}_\theta \neq 0$. If φ is not one–to–one then there exist distinct $\lambda, \mu \in \tau$ such that $\varphi(\mu) = \varphi(\lambda) =: \psi$ agin using surjectivity of $\sigma_{\varphi,\mathfrak{w}} \restriction_{\ell^p(\tau)}: \ell^p(\tau) \to \ell^p(\tau)$ we have:

$$\begin{aligned}\mathbb{C}\times\mathbb{C} &= \{(\pi_\mu(x), \pi_\lambda(x)) : x \in \ell^p(\tau)\} \\ &= \{(\pi_\mu(x), \pi_\lambda(x)) : x \in \sigma_{\varphi,\mathfrak{w}}(\ell^p(\tau))\} \\ &= \{(\pi_\mu(\sigma_{\varphi,\mathfrak{w}}(y)), \pi_\lambda(\sigma_{\varphi,\mathfrak{w}}(y))) : y \in \ell^p(\tau)\} \\ &= \{(\mathfrak{w}_\mu y_{\varphi(\mu)}, \mathfrak{w}_\lambda y_{\varphi(\lambda)}) : (y_\alpha)_{\alpha\in\Gamma} \in \ell^p(\tau)\} \\ &= \{(\mathfrak{w}_\mu y_\psi, \mathfrak{w}_\lambda y_\psi) : (y_\alpha)_{\alpha\in\Gamma} \in \ell^p(\tau)\} = (\mathfrak{w}_\mu, \mathfrak{w}_\lambda)\mathbb{C}\end{aligned}$$

which is a contradiction, hence $\varphi : \tau \to \tau$ is one–to–one. Since $\sigma_{\varphi,\mathfrak{w}} \restriction_{\ell^p(\tau)}$ is one–to–one we have $(0)_{\alpha\in\tau} = \sigma_{\varphi,\mathfrak{w}}((0)_{\alpha\in\tau}) \neq \sigma_{\varphi,\mathfrak{w}}(\mathsf{e}_\theta) = \sigma_{\varphi,\mathfrak{w}}((\delta_{\alpha,\theta})_{\alpha\in\tau}) = (\mathfrak{w}_\alpha\delta_{\varphi(\alpha),\theta})_{\alpha\in\tau} = \sum\limits_{\alpha\in\varphi^{-1}(\theta)} \mathfrak{w}_\alpha$, thus $\varphi^{-1}(\theta) \neq \varnothing$ and $\varphi : \tau \to \tau$ is onto.

Thus $\varphi : \tau \to \tau$ is bijective. By Remark 2.1, $\sup\{|\mathfrak{w}_\alpha| : \alpha \in \tau\} < +\infty$. One can verify easily $\sigma_{\varphi,\mathfrak{w}} \restriction_{\ell^p(\tau)}^{-1} = \sigma_{\varphi^{-1},\mathfrak{v}} \restriction_{\ell^p(\tau)}$ where $\mathfrak{v} = (\frac{1}{\mathfrak{w}_\alpha})_{\alpha\in\tau}$, agin by

Remark 2.1, $\sup\{|\frac{1}{\mathfrak{w}_\alpha}| : \alpha \in \tau\} < +\infty$ hence $\inf\{|\mathfrak{w}_\alpha| : \alpha \in \tau\} > 0$.
(2) $\Rightarrow$ (1): Use Remark 2.1. □

Theorem 2.3. *Suppose $\sigma_{\varphi,\mathfrak{w}} \restriction_{\ell^p(\tau)}: \ell^p(\tau) \to \ell^p(\tau)$ is a homeomorphism, then the following statements are equivalent:*

1. *$\sigma_{\varphi,\mathfrak{w}} \restriction_{\ell^p(\tau)}: \ell^p(\tau) \to \ell^p(\tau)$ is expansive,*
2. *there exists $\varepsilon > 0$ such that for each nonzero vector $x \in \ell^p(\tau)$, $\sup\{||\sigma^n_{\varphi,\mathfrak{w}}(x)||_p : n \in \mathbb{Z}\} > \varepsilon$,*
3. *for all nonzero vector $x \in \ell^p(\tau)$, $\sup\{||\sigma^n_{\varphi,\mathfrak{w}}(x)||_p : n \in \mathbb{Z}\} = +\infty$,*
4. *$T := \sup(\{|\mathfrak{w}_{\varphi^{-1}(\theta)} \cdots \mathfrak{w}_{\varphi^{-n}(\theta)}| : n \geq 1\} \cup \{|1/(\mathfrak{w}_{\varphi(\theta)} \cdots \mathfrak{w}_{\varphi^n(\theta)})| : n \geq 1\}) = +\infty$, for each $\theta \in \tau$.*

Proof. (1) $\Rightarrow$ (2): It is evident by the definition of expansivity.
(2) $\Rightarrow$ (3): Choose $\varepsilon > 0$ such that for each nonzero vector $x \in \ell^p(\tau)$, $\sup\{||\sigma^n_{\varphi,\mathfrak{w}}(x)||_p : n \in \mathbb{Z}\} > \varepsilon$, thus for all nonzero vector $x \in \ell^p(\tau)$ and $s > 0$ we have $\varepsilon < \sup\{||\sigma^n_{\varphi,\mathfrak{w}}(sx)||_p : n \in \mathbb{Z}\} = s\sup\{||\sigma^n_{\varphi,\mathfrak{w}}(x)||_p : n \in \mathbb{Z}\}$. Therefore $\varepsilon/s < \sup\{||\sigma^n_{\varphi,\mathfrak{w}}(x)||_p : n \in \mathbb{Z}\}$, so $\sup\{||\sigma^n_{\varphi,\mathfrak{w}}(x)||_p : n \in \mathbb{Z}\} = +\infty$.
(3) $\Rightarrow$ (4): Suppose (3) is valid. For each $\theta \in \tau$ and $r \in (0,1)$ there exists $m \in \mathbb{Z}(\setminus\{0\})$ such that $1 < ||\sigma^m_{\varphi,\mathfrak{w}}(r\mathsf{e}_\theta)||_p$. We have the following cases:
Case I: $m \geq 1$. In this case

$$\begin{aligned} 1 &< ||\sigma^m_{\varphi,\mathfrak{w}}(r\mathsf{e}_\theta)||_p = ||(r\mathfrak{w}_\alpha\mathfrak{w}_{\varphi(\alpha)} \cdots \mathfrak{w}_{\varphi^{m-1}(\alpha)}\delta_{\varphi^m(\alpha),\theta})_{\alpha\in\tau}||_p \\ &= r|\mathfrak{w}_{\varphi^{-m}(\theta)}\mathfrak{w}_{\varphi^{-m+1}(\theta)} \cdots \mathfrak{w}_{\varphi^{-1}(\theta)}| \leq rT\,. \end{aligned}$$

Case II: $m = -k \leq -1$. In this case for $\mathfrak{v} = (\mathfrak{v}_\alpha)_{\alpha\in\tau} = (1/\mathfrak{w}_\alpha)_{\alpha\in\tau}$ wehave (note that $\sigma_{\varphi,\mathfrak{w}} \restriction^{-1}_{\ell^p(\tau)} = \sigma_{\varphi^{-1},\mathfrak{v}} \restriction_{\ell^p(\tau)}$)

$$\begin{aligned} 1 &< ||\sigma^{-k}_{\varphi,\mathfrak{w}}(r\mathsf{e}_\theta)||_p = ||\sigma^k_{\varphi^{-1},\mathfrak{v}}(r\mathsf{e}_\theta)||_p \\ &= ||(r\mathfrak{v}_\alpha\mathfrak{v}_{\varphi^{-1}(\alpha)} \cdots \mathfrak{v}_{\varphi^{-k+1}(\alpha)}\delta_{\varphi^{-k}(\alpha),\theta})_{\alpha\in\tau}||_p \\ &= ||(r/(\mathfrak{w}_\alpha\mathfrak{w}_{\varphi^{-1}(\alpha)} \cdots \mathfrak{w}_{\varphi^{-k+1}(\alpha)})\delta_{\varphi^{-k}(\alpha),\theta})_{\alpha\in\tau}||_p \\ &= r|1/(\mathfrak{w}_{\varphi^k(\theta)}\mathfrak{w}_{\varphi^{k-1}(\theta)} \cdots \mathfrak{w}_{\varphi(\theta)})| \leq rT\,. \end{aligned}$$

Using the above two cases $1/r < T$ for all $r \in (0,1)$, thus $T = +\infty$.
(4) $\Rightarrow$ (3): Suppose (4) is valid and $x = (x_\alpha)_{\alpha\in\tau} \in \ell^p(\tau)$ is a nonzero vector. Choose $\theta \in \tau$ such that $x_\theta \neq 0$. We have the following cases:
Case A. $\sup\{|\mathfrak{w}_{\varphi^{-1}(\theta)} \cdots \mathfrak{w}_{\varphi^{-n}(\theta)}| : n \geq 1\} = +\infty$. For $M > 0$ there exists $n \geq 1$ such that $|x_\theta\mathfrak{w}_{\varphi^{-1}(\theta)} \cdots \mathfrak{w}_{\varphi^{-n}(\theta)}| > M$. Using a similar method described in Proof of "(3) $\Rightarrow$ (4)" (Case I) we have:

$$\begin{aligned} M &> |x_\theta\mathfrak{w}_{\varphi^{-1}(\theta)} \cdots \mathfrak{w}_{\varphi^{-n}(\theta)}| = ||\sigma^n_{\varphi,\mathfrak{w}}(x_\theta\mathsf{e}_\theta)||_p \\ &= ||(\mathfrak{w}_\alpha\mathfrak{w}_{\varphi(\alpha)} \cdots \mathfrak{w}_{\varphi^{n-1}(\alpha)}x_\theta\delta_{\varphi^n(\alpha),\theta})_{\alpha\in\tau}||_p \\ &\geq ||(\mathfrak{w}_\alpha\mathfrak{w}_{\varphi(\alpha)} \cdots \mathfrak{w}_{\varphi^{n-1}(\alpha)}x_{\varphi^n(\alpha)})_{\alpha\in\tau}||_p = ||\sigma^n_{\varphi,\mathfrak{w}}(x)||_p \end{aligned}$$

Therefore in this case $\sup\{||\sigma^n_{\varphi,\mathfrak{w}}(x)||_p : n \geq 1\} = +\infty$.
Case B: $\sup\{|1/(\mathfrak{w}_{\varphi(\theta)} \cdots \mathfrak{w}_{\varphi^n(\theta)})| : n \geq 1\} = +\infty$. For $M > 0$ there exists

$n \geq 1$ such that $|x_\theta/(\mathfrak{w}_{\varphi(\theta)} \cdots \mathfrak{w}_{\varphi^n(\theta)})| > M$. Using a similar method described in Proof of (Case A), $M > ||\sigma_{\varphi,\mathfrak{w}}^{-n}(x)||_p$. Thus in this case $\sup\{||\sigma_{\varphi,\mathfrak{w}}^k(x)||_p : n \leq -1\} = +\infty$.
Use Case (A) and (B) to obtain (3).
(3) $\Rightarrow$ (2): It is evident.
(2) $\Rightarrow$ (1): Use the fact that $\sigma_{\varphi,\mathfrak{w}} \restriction_{\ell^p(\tau)}: \ell^p(\tau) \to \ell^p(\tau)$ is linear. $\square$

Acknowledgement

The author wish to dedicate this paper to our lady, Fatimah-Zahra (as).

References

1. F. Ayatollah Zadeh Shirazi, F. Ebrahimifar, R. Rezavand, *Weighted generalized shift operators on ℓ^p spaces*, Rendiconti del Circolo Matematico di Palermo, **71** (1) (2022), 1-12.
2. F. Ayatollah Zadeh Shirazi, N. Karami Kabir, F. Heydari Ardi, *A note on shift theory*, Mathematica Pannonica, Proceedings of ITES-2007, **19** (2) (2008), 187-195.
3. F. Ayatollah Zadeh Shirazi, H. Zabeti, *E-chaotic generalized shift dynamical systems*, Punjab University Journal of Mathematics (Lahore), **47** (1) (2015), 135-140.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

THE EXTENDED LIMIT SET OF A SEQUENCE OF LINEAR OPERATORS

MOHAMMAD REZA AZIMI*

Department of Mathematics, Faculty of Sciences, University of Maragheh, P. O. Box: 55181-83111, Golshahr, Maragheh, Iran
mhr.azimi@maragheh.ac.ir

Abstract. The aim of this work is to investigate some basic properties of the extended limit sets of a sequence of bounded linear operators $\{T_n\}_{n=1}^{\infty}$. Afterwards, we characterize the hypercyclicity of $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{B}(X)$ based upon these sets.

1. Introduction

Let X be a separable Banach space over the field of complex numbers $\mathbb{C}$. By X^* we denote the dual space of X. Throughout this paper the interior of a set $A \subseteq X$ is denoted by A°.

A sequence of operators $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{B}(X)$ is called ***hypercyclic*** if there exists a vector $x \in X$ such that the set $\{T_n x : n \in \mathbb{N}\}$ is dense in X. In such a case, x is called a hypercyclic vector for the sequence of operators $\{T_n\}_{n=1}^{\infty}$. A vector $x \in X$ is called ***cyclic*** for $\{T_n\}_{n=1}^{\infty}$ whenever $\text{span}\{T_n x : n \in \mathbb{N}\}$ is dense in X. We say that an operator $T : X \to X$ is hypercyclic if the sequence of its iterates $\{T^n\}_{n=1}^{\infty}$ is hypercyclic.

A sequence of operators $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{B}(X)$ is called ***topologically transitive*** if for any pair U, V of nonempty open subsets of X, there exits $n \in \mathbb{N}$

2020 *Mathematics Subject Classification. Primary 47A16; Secondary 37B99*
Key words and phrases. Sequences of operators, Hypercyclic operators, J-class operators.

* Speaker.

such that

$$T_n(U) \cap V \neq \emptyset.$$

For more detail see [2, 4, 6]

Definition 1.1. Let $T \in \mathcal{B}(X)$ and let $x \in X$. The ***extended limit set*** of x, denoted by $J_T(x)$ consists of those vectors y in X for which there exist a strictly increasing sequence of positive integers $\{k_n\}$ and a sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $T^{k_n}x_n \to y$. An operator T is called J-class operator whenever $J_T(x) = X$ for some nonzero vector $x \in X$.([3])

Definition 1.2. Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{B}(X)$ and $x \in X$. The *limit set* of x under $\{T_n\}_{n=1}^{\infty}$, denoted by $L_{\{T_n\}}(x)$ consists of those vectors $y \in X$ for which there exists a strictly increasing sequence of positive integers $\{k_n\}$ such that $T_{k_n}x \to y$.([1])

In this work, we study the dynamics of a sequence of linear operators by using the extended limit sets. It turns out that the extended limit sets are strongly related to the hypercyclicity and being J-class of a sequence of linear operators.([1])

2. Main results

Lemma 2.1. *Let $x_n \to x$ and $y_n \to y$ for some $x, y \in X$. If $y_k \in J_{\{T_n\}}(x_k)$ for every $k = 1, 2, 3, ...$, then $y \in J_{\{T_n\}}(x)$.*

Lemma 2.2. *Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{B}(X)$ such that $T_n \to T$ uniformly on X for some $T \in \mathcal{B}(X)$. If $T_n y \in J_{\{T_n\}}(T_n x)$ for every $n = 1, 2, 3, ...$ and for arbitrary $x, y \in X$, then $Ty \in J_{\{T_n\}}(Tx)$.*

Lemma 2.3. *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mutually commuting operators in $\mathcal{B}(X)$. For all $x \in X$, the sets $J_{\{T_n\}}(x)$ and $L_{\{T_n\}}(x)$ are closed. Moreover, for any $S \in \mathcal{B}(X)$ commuting with the sequence $\{T_n\}_{n=1}^{\infty}$ we have $SJ_{\{T_n\}}(x) \subseteq J_{\{T_n\}}(Sx)$ and $SL_{\{T_n\}}(x) \subseteq L_{\{T_n\}}(Sx)$.*

Corollary 2.4. *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of invertible and mutually commuting operators in $\mathcal{B}(X)$. Then, for every $l \in \mathbb{N}$ and $x \in X$, $T_l^{-1}J_{\{T_n\}}(T_l x) = J_{\{T_n\}}(x)$.*

Proof. Choose $l \in \mathbb{N}$ and $x \in X$ arbitrarily. By Lemma 2.3, we have $J_{\{T_n\}}(x) \subseteq T_l^{-1}J_{\{T_n\}}(T_l x)$. For the reverse inclusion, pick an arbitrary vector $y \in T_l^{-1}J_{\{T_n\}}(T_l x)$. According to the definition of the extended limit set $J_{\{T_n\}}(T_l x)$, there are a strictly increasing sequence of positive integers $\{k_i\}$ and a sequence $\{x_i\}$ in X such that $x_i \to T_l x$ and $T_{k_i}x_i \to T_l y$ as $i \to \infty$. Hence $T_l^{-1}x_i \to x$ and $T_{k_i}T_l^{-1}x_i = T_l^{-1}T_{k_i}x_i \to y$ as $i \to \infty$. That is, $y \in J_{\{T_n\}}(x)$ and the proof is completed. □

Proposition 2.5. *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of invertible operators in $B(X)$. Then $y \in J_{\{T_n\}}(x)$ if and only if $x \in J^{-1}_{\{T_n\}}(y)$ for every $x, y \in X$.*

Theorem 2.6. *For $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{B}(X)$ assume that $\sup_n \|T_n\| < \infty$. Then $J_{\{T_n\}}(x) = L_{\{T_n\}}(x)$ for every $x \in X$.*

Proof. Note that the inclusion $L_{\{T_n\}}(x) \subseteq J_{\{T_n\}}(x)$ holds trivially. Now suppose $y \in J_{\{T_n\}}(x)$. Then there exist a strictly increasing sequence of positive integers $\{k_n\}$ and a sequence $\{x_n\}$ in X such that $x_n \to x$ and $T_{k_n} x_n \to y$. Put $M = \sup_n \|T_n\|$. Thus we have

$$\begin{aligned} \|T_{k_n}x - y\| &\leq \|T_{k_n}x - T_{k_n}x_n\| + \|T_{k_n}x_n - y\| \\ &\leq M\|x_n - x\| + \|T_{k_n}x_n - y\| \to 0 \end{aligned}$$

as $n \to \infty$. Therefore, $y \in L_{\{T_n\}}(x)$ and the the proof is complete. □

Remark 2.7. By the Banach-Steinhaus theorem, the assumption $\sup_n \|T_n\| < \infty$ is equivalent to saying that for every $x \in X$, each orbit $\{T_n x : n \geq 1\}$ is bounded. Hence, the original assumption of the above theorem can be replaced by the latter.

Theorem 2.8. *Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{B}(X)$. Then the set $H = \{x \in X : J_{\{T_n\}}(x) = X\}$ is closed and connected. Moreover, if $\{T_n\}$ is a mutually commuting sequence of surjective operators then $T_m H \subseteq H$ for any $m \in \mathbb{N}$.*

Proposition 2.9. *Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{B}(X)$ and let the set $H_\sigma = \bigcup_{n=1}^{\infty}\{x \in X : J_{T_n}(x) = X\}$. Then $\bigcup_{n=1}^{\infty} T_n H_\sigma \subseteq H_\sigma$. Moreover, if $\{T_n\}_{n=1}^{\infty}$is a mutually commuting sequence, then $\bigcup_{i=1}^{\infty} T_i J_{\{T_n\}}(x) \subseteq J_{\{T_n\}}(Sx)$ for some $S \in \mathcal{B}(X)$ which commutes with the sequence $\{T_n\}_{n=1}^{\infty}$.*

Theorem 2.10. *Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{B}(X)$ be a mutually commuting sequence of surjective operators. The following are equivalent.*

(i) *The sequence $\{T_n\}$ is hypercyclic;*
(ii) *The equality $J_{\{T_n\}}(x) = X$ holds for every $x \in X$;*
(iii) *The set $H = \{x \in X : J_{\{T_n\}}(x) = X\}$ is dense in X;*
(iv) *The set $H = \{x \in X : J_{\{T_n\}}(x) = X\}$ has nonempty interior.*

Proof. To prove that (i) implies (ii), let $x, y \in X$ be arbitrary. Remind that the set of all hypercyclic vectors for $\{T_n\}$ is a G_δ and dense set in X, see [**?**] for more details. Hence there exists a sequence of hypercyclic vectors $\{x_n\}$ such that $x_n \to x$. However in view of the hypercyclicity definition, being the set $\{T_n x : n \in \mathbb{N}\}$ dense in X, one may find a strictly increasing sequence of positive integers $\{k_n\}$ such that $T_{k_n} x_n \to y$. This means that $y \in J_{\{T_n\}}(x)$ and so $J_{\{T_n\}}(x) = X$.
The implication $(ii) \Rightarrow (iii)$ is trivial. Moreover, applying Lemma 2.1 makes sure that (iii) implies (ii).
Now we show that (iv) implies (ii). Fix $x \in H^\circ$ (the interior of H) and let $y \in X$ be an arbitrary. Since $y \in X = J_{\{T_n\}}(x)$ there exist a strictly increasing sequence of positive integers $\{k_n\}$ and a sequence $\{x_n\}$ in X such that $x_n \to x$ and $T_{k_n} x_n \to y$. Without loss of generality we may assume that $x_n \in H$ for every $n \in \mathbb{N}$. Indeed, this can be done in sake of the our assumption that $x \in H^\circ$. By Proposition 2.8 we may now claim that

$T_{k_n}x_n \in H$ for every $n \in \mathbb{N}$. But H is closed and $T_{k_n}x_n \to y$ therefore, $y \in H$.

To prove the implication $(ii) \Rightarrow (i)$, let $\{B_j\}$ be a countable open basis for the relative topology of X. By Baire category theorem and the hypercyclic vectors description ([?]) it is sufficient to show that the set $\bigcup_{n=1}^{\infty} T_n^{-1}(B_j)$ is dense in X for every j. In fact, the set of all hypercyclic vectors for $\{T_n\}$ equals

$$\bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} T_n^{-1}(B_j).$$

Now consider the equivalent definition of $J_{\{T_n\}}(x)$ described in Remark **??**. Let $x \in X$ and let a neighborhood U of x and B_j be given. Since $J_{\{T_n\}}(x) = X$, there exists $y \in U$ and $n \in \mathbb{N}$ (arbitrarily large enough) such that $T_n y \in B_j$ or, equivalently, $y \in T_n^{-1}(B_j)$. This fact completes the proof, since U was chosen arbitrarily. □

Example 2.11. A historical example of hypercyclic operator is due to S. Rolewicz ([5]) who proved that for any $\lambda \in \mathbb{C}$, the multiple of the ***backward shift*** $\lambda B : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by $\lambda B(x_1, x_2, x_3, ...) = (\lambda x_2, \lambda x_3, \lambda x_4, ...)$ is hypercyclic whenever $|\lambda| > 1$.

Consider the sequence $T_n : \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ defined by $T_n = (2I \oplus 2B)^n$ where B is the backward shift on $\ell^2(\mathbb{N})$ and I is the identity map. Let $x \in \ell^2(\mathbb{N})$ be a nonzero hypercyclic vector for $2B$. Note that the operator $2I \oplus 2B$ is not hypercyclic itself but is J-class. Now we prove that $J_{\{T_n\}}(0 \oplus x) = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$. For if, let $z \oplus z' \in \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ be an arbitrary vector. Then by the hypercyclicity of $2B$, there exists a strictly increasing sequence of positive integers $\{k_n\}$ such that $(2B)^{k_n}x \to z'$. Define $x_n := \frac{z}{2^{k_n}} \oplus x$. Then $x_n \to 0 \oplus x$ and $T_{k_n}x_n \to z \oplus z'$. From this we deduce that $J_{\{T_n\}}(0 \oplus x) = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$.

References

1. M.R. Azimi, *J-class sequences of linear operators,* Complex Anal. Oper. Theory, **12** (1) (2018), 293–303.
2. F. Bayart, ´E. Matheron, *Dynamics of linear operators, 179 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2009.*
3. *G. Costakis, A. Manoussos, J-class operators and hypercyclicity, J. Operator Theory,* **67** *(1) (2012), 101-119.*
4. *K.-G. Grosse-Erdmann, A.P. Manguillot, Linear chaos, Universitext, Springer, London, 2011.*
5. *S. Rolewicz, On orbits of elements, Studia Math.,* **32** *(1969), 17-22.*
6. *H. N. Salas, Hypercyclic weighted shifts, Trans. Amer. Math. Soc.,* **347** *(3) (1995), 993-1004.*

INTEGRAL JENSEN INEQUALITY FOR STRONGLY PREINVEX FUNCTIONS

ALI BARANI* AND NASER ABBASI

Department of Mathematics, Lorestan University, Khoramabad, Iran
barani.a@lu.ac.ir; abasi.n@lu.ac.ir

Abstract. In this paper a version of Jensen's integral type inequality for strongly preinvex functions is given.

1. Introduction

Convexity plays a crucial role in many fields of mathematics. In recent years, considerable efforts have been done to extend the concept of convexity and its applications. Jensen's inequality play an important role in convex analysis and probability theory, etc see [5]. Many inequalities can be obtained from it. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, is said to be strongly convex with modules $\lambda > 0$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \lambda t(1-t)(x-y)^2,$$

for every $x, y \in I$ and $t \in [0,1]$. Strongly convex functions were introduced by Polyak [7] which have useful properties and applications in generalized convex optimization theory. The integral type of Jensen's inequality in setting of strongly convex functions introduced in [4] as follows

$$f(m) \leq \frac{1}{d-c}\int_c^d f(g(x))dx - \frac{\lambda}{d-c}\int_c^d (g(x)-m)^2 dx, \tag{1.1}$$

2020 *Mathematics Subject Classification.* Primary 26D15; Secondary 26D07
Key words and phrases. Strongly preinvex functions, Jensen's type inequality.
* Speaker.

where $m := \frac{1}{d-c}\int_c^d g(x)dx$ and g is an integrable function on $[c,d]$ with $a \leq g(x) \leq b$ and f is a strongly convex with modules $\lambda > 0$ on $[a,b]$. In recent years, many papers which dealing with refinements of Jensen's inequality for important generalized convex functions have been appeared in the literature, see [1, 3, 4, 5, 6] and references therein.

An important generalization of the class of convex functions is that the class of preinvex functions was introduced in [9, 10] by Weir and Mond and Weir and Jeyakumar and then applied it to the establishment of the sufficient optimality conditions and duality in nonlinear programming. There have been some works in the literature which are investigated by preinvex functions (e.g. see [2, 8, 9] and references therein). First we recall some notions from invexity analysis which will be used in sequel. A set $S \subseteq \mathbb{R}$ is said to be invex with respect to the map $\eta : S \times S \to \mathbb{R}$, if

$$y + t\eta(x,y) \in S. \tag{1.2}$$

for every $x, y \in S$ and $t \in [0,1]$. It is obvious that every convex set is invex with respect to $\eta(x,y) = x - y$, but there exist invex sets which are not convex. Recall that for $x, y \in S$ the $\eta-$path P_{xy} is a subset of S defined by

$$P_{xy} := \{x + t\eta(y,x)|\ 0 \leq t \leq 1\}.$$

Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$. Then, the function $f : S \to \mathbb{R}$ is said to be preinvex with respect to η, if we have

$$f(y + t\eta(x,y)) \leq tf(x) + (1-t)f(y). \tag{1.3}$$

for every $x, y \in S$ and $t \in [0,1]$. Every convex function is preinvex with respect to the map $\eta(x,y) = x - y$ but the converse does not holds. Recall that the mapping $\eta : S \times S \to \mathbb{R}$ is said to be satisfies the conditions C if

$$\begin{aligned}\eta(y, y + t\eta(x,y)) &= -t\eta(x,y),\\ \eta(x, y + t\eta(x,y)) &= (1-t)\eta(x,y).\end{aligned}$$

for every $x, y \in S$ and $t \in [0,1]$. From conditions C we have

$$\eta(y + t_2\eta(x,y), y + t_1\eta(x,y)) = (t_2 - t_1)\eta(x,y). \tag{1.4}$$

Integral Jensen's inequality for preinvex functions defined on invex subsets of real line is introduced in [6] as follows;

Theorem 1.1. *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$ and η satisfies conditions C. Suppose that $f : S \to \mathbb{R}$ is a preinvex function. Assume that $g : J \to \mathbb{R}$ is an integrable function for some interval $J \subseteq \mathbb{R}$. Let $c, d \in J$, $c < d$ be such that for every $x \in [c,d]$, $g(x) \in P_{ab}($ or $P_{ba})$ for $a := g(c), b := g(d)($ or $b := g(c), a := g(d))$. Then the following inequality holds*

$$f\left(\frac{1}{d-c}\int_c^d g(x)dx\right) \le \frac{1}{\eta(b,a)}\int_{P_{ab}} fog(x)dx\Big(\ or\ \frac{1}{\eta(a,b)}\int_{P_{ba}} fog(x)dx\Big), \tag{1.5}$$

provided that fog is integrable, where $\int_{P_{ab}}$ is denoted for integral over P_{ab}.

Now we recall the notion of strongly preinvex functions.

Definition 1.2. Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$. Then, the function $f : S \to \mathbb{R}$ is said to be strongly preinvex respect to η, with modulus $c > 0$ if

$$f(y + t\eta(x, y)) \le tf(x) + (1 - t)f(y) - ct(1 - t)(\eta(x, y))^2. \tag{1.6}$$

for every $x, y \in S$ and $t \in [0, 1]$.

Note that every strongly preinvex function respect to an η is preinvex with respect to η, but the converse is not true in general. Moreover while $\eta(x, y) := x - y$ then strongly preinvexity reduce to strongly convexity of functions. The main purpose of this paper is investigate some generalized versions of integral Jensen's inequality in setting of real valued strongly preinvex functions.

2. Main results

Our goal in this section is to establish an analogue of the classical Jensen inequality in setting of strongly invexity of functions. We start with the next result which is an improvement of proposition 1.1.2 in [1], is useful in our investigations.

Theorem 2.1. *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$. Suppose that f is a real valued function on S. Then,*
(i) If $f : S \to \mathbb{R}$ is a strongly preinvex function with modulus $c > 0$ and η satisfies condition C then, the restriction of f to any $\eta-$path in S is a strongly convex function with modulus c.
(ii) If for every $x, y \in S$, $f(x + \eta(y, x)) \le f(y)$ and the restriction of f to any $\eta-$path in S is a strongly convex function with modulus $c > 0$, then f is a strongly preinvex function with modulus c on S.

By combining Theorem 2.1 and Proposition 1.1.2 in [1] we obtain the following result.

Proposition 2.2. *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$. Suppose that η satisfies condition C. Then,*
(i) If $f : S \to \mathbb{R}$ is a strongly preinvex function with modulus $c > 0$ then the restriction of the function $g : S \to \mathbb{R}$ defined by

$$g(u) := f(u) - cu^2,$$

to each $\eta-$path in S is a convex function.
(ii) If for every $x, y \in S$, $f(x+\eta(y,x)) \leq f(y)$ and the restriction of the function $g : S \to \mathbb{R}$ defined by $g(u) := f(u) - cu^2$, to each $\eta-$path in S is a convex function then, f is a strongly preinvex function with modulus c.

Finally, we are in a position to establish the next theorem which is an improvement of Jensen's integral type inequality in strongly preinvex functions setting.

Theorem 2.3. *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$ and η satisfies conditions C. Suppose that $f : S \to \mathbb{R}$ is a strongly preinvex respect to η, with modulus $\lambda > 0$. Assume that $g : J \to \mathbb{R}$ is an integrable function for some interval $J \subseteq \mathbb{R}$. Let $c, d \in J$, $c < d$ be such that for every $x \in [c,d]$, $g(x) \in P_{ab}($ or $P_{ba})$ for $a := g(c), b := g(d)($ or $b := g(c), a := g(d))$. Then the following inequality holds*

$$\begin{aligned} &f(m) - \lambda m^2 \\ &\leq \frac{1}{\eta(b,a)} \int_{P_{ab}} fog(x)dx - \frac{\lambda}{d-c}\int_c^d g^2(x)dx \\ &\Big(or \frac{1}{\eta(a,b)} \int_{P_{ba}} fog(x)dx - \frac{\lambda}{d-c}\int_c^d g^2(x)dx \Big), \end{aligned} \tag{2.1}$$

provided that fog is integrable, where $\int_{P_{ab}}$ is denoted for integral over P_{ab}, where $m := \frac{1}{d-c}\int_c^d g(x)dx$.

References

1. J.B. Hiriart-Urruty, C. Lemaréchal, *Fundamentals of Convex Analysis.* Springer, Berlin 2001.
2. S.R. Mohan and S. K. Neogy, *On invex sets and preinvex functions*, J. Math. Anal. Appl. **189** (1995), 901-908.
3. A. Matković and J.E. Pečarić, *A variant of Jensen's inequality for convex functions of several variables*, J. Math. Inequ. Appl. **1** (2007), 45-51.
4. N. Merentes and K. Nikodem, *Remarks on strongly convex functions*, Aequationes Math. **80** (2010), 193-199.
5. C.P. Niculescu, C.I. Spridon, *New Jensen-type inequalities*, J. Math. Anal. Appl. **401** (2013), 343-348.
6. S. N. Pasari, A. Barani and N. Abbasi, *Generalized integral Jensen inequality*, J. I. Appl., **(2024)** (25) (2024).
7. B.T. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions.* Soviet Math. Dokl. **7** (1966), 72-75.
8. Z. Peng, Y. Liu and X.J. Long, *Remarks on new properties of preinvex functions*, Modern Applied Science, **3** (2009), 11-16.
9. T. Weir, and B. Mond, *Preinvex Functions in multiple Objective Optimization*, J. Math. Anal. Appl., **136** (1998), 29-38.
10. T. Weir and V. Jeyakumar, *A class of nonconvex functions and mathematical programming.* Bull. Austral. Math. Soc. **38** (1988), 177-189.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

FUZZY NORM AND FUZZY INNER PRODUCT SPACES

BAYAZ DARABY* AND ASGHAR RAHIMI

Department of Mathematics, University of Maragheh, Maragheh, Iran.
bdaraby@maragheh.ac.ir; rahimi@maragheh.ac.ir

Abstract. In this paper, we show that every classical inner product on a linear space induces the fuzzy inner product and it consequences fuzzy norm at the point of view of Bag and Samanta. We prove the Cauchy-Schwarz inequality on fuzzy Hilbert spaces.

1. Introduction

The idea of fuzzy norms on a linear space first introduced by Katsaras [7] in 1984. R. Biswas [3] and A. M. El-Abye and H. M. El-Hamouly [6] tried to give a meaningful definition of fuzzy inner product space and associated fuzzy norm function with those definition are restricted to the real linear space only. P. Mazumder and S. K. Samanta introduced the definition of fuzzy inner product space in point of veiw of Bag and Samanta fuzzy norm[1]. Recently, B. Daraby and et al. [4] studied some properties of fuzzy Hilbert spaces and they showed that all results in classical Hilbert spaces are immediate consequences of the corresponding results for Felbin-fuzzy Hilbert spaces. Also by an example, they showed that the spectrum of the category of Felbin- fuzzy Hilbert spaces is broader than the category of classical Hilbert spaces [5].

2020 *Mathematics Subject Classification.* 46S40, 54A40.
Key words and phrases. Fuzzy norm, Fuzzy inner product space, Fuzzy Hilbert space, Fuzzy frame.

* Speaker.

2. Some preliminaries

Definition 2.1. [1]. Let U be a linear space over the field F. A fuzzy subset N of $U \times \mathbb{R}$ is called a fuzzy norm on U if for all $x, u \in U$ and $c \in F$, the following conditions are satisfied:

(N1) $\forall t \in \mathbb{R}$ with $t \leq 0$, $N(x,t) = 0$;
(N2) $(\forall t \in \mathbb{R}, t > 0, N(x,t) = 1)$ iff $x = \underline{0}$;
(N3) $\forall t \in \mathbb{R}, t > 0, N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $\forall s, t \in \mathbb{R}, x, u \in U, N(x+u, s+t) \geq \min\{N(x,s), N(u,t)\}$;
(N5) $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

The pair (U, N) will be referred to as a fuzzy normed linear space.

Theorem 2.2. [1] *Let (U, N) be a fuzzy normed linear space. Assume further that,*

(N6) $\forall t > 0, N(x,t) > 0 \Rightarrow x = \underline{0}$.

Define $\|x\|_\alpha = \bigwedge\{t > 0 : N(x,t) \geq \alpha\}$, $\alpha \in (0,1)$. Then $\{\|.\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on U and they are called α- norms on U corresponding to the fuzzy norm N on U.

Definition 2.3. [8] Let U be a linear space over the field $\mathbb{C}$ of complex numbers. Let $\mu : U \times U \times \mathbb{C} \longrightarrow I = [0,1]$ be a mapping such that the following hold:

(FIP1) for $s, t \in \mathbb{C}, \mu(x+y, z, |t|+|s|) \geq \min\{\mu(x,z,|t|), \mu(y,z,|s|)\}$;
(FIP2) for $s, t \in \mathbb{C}, \mu(x,y,|st|) \geq \min\{\mu(x,x,|s|^2), \mu(y,y,|t|^2)\}$;
(FIP3) for $t \in \mathbb{C}, \mu(x,y,t) = \mu(y,x,\bar{t})$;
(FIP4) $\mu(\alpha x, y, t) = \mu(x, y, \frac{t}{|\alpha|}), \alpha (\neq 0) \in \mathbb{C}, t \in \mathbb{C}$;
(FIP5) $\mu(x,x,t) = 0, \forall t \in \mathbb{C}\backslash\mathbb{R}^+$;
(FIP6) $(\mu(x,x,t) = 1, \forall t > 0)$ iff $x = \underline{0}$;
(FIP7) $\mu(x,x,.) : \mathbb{R} \longrightarrow I$ is a monotonic non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} \mu(\alpha x, x, t) = 1$.

We call μ fuzzy inner product function on U and (U, μ) fuzzy inner product space (FIP space).

Theorem 2.4. [8] *Let U be a linear space over $\mathbb{C}$. Let μ be a FIP on U. Then*

$$N(x,t) = \begin{cases} \mu(x,x,t^2) & \text{if } t \in \mathbb{R}, t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

is a fuzzy norm on U. Now if μ satisfies the following conditions:

(FIP8) $\left(\mu(x,x,t^2) > 0, \forall t > 0\right) \Rightarrow x = \underline{0}$ *and*

(FIP9) *for all $x, y \in U$ and $p, q \in \mathbb{R}$,*

$$\mu(x+y, x+y, 2q^2) \bigwedge \mu(x-y, x-y, 2p^2) \geq \mu(x,x,p^2) \bigwedge \mu(y,y,q^2),$$

then $\|x\|_\alpha = \bigwedge\{t>0 : N(x,t)\geq\alpha\}, \alpha\in(0,1)$ *is an ordinary norm satisfying parallelogram law. By using polarization identity, we can get ordinary inner product, called the* $\langle .,.\rangle_\alpha$*-inner product, as follows:*

$$\langle x,y\rangle_\alpha = \frac{1}{4}\left(\|x+y\|_\alpha^2 - \|x-y\|_\alpha^2\right) + \frac{1}{4}i\left(\|x+iy\|_\alpha^2 - \|x-iy\|_\alpha^2\right), \forall\alpha\in(0,1).$$

3. Main Results

Proposition 3.1. *Let* (U,μ) *be a FIP space satisfying* $(FIP8)$ *and* $(FIP9)$. *A fuzzy inner product space* (U,μ) *with its corresponding norm* N *satisfies the Schwartz inequality*

$$|\langle x,y\rangle_\alpha| \leq \|x\|_\alpha\|y\|_\alpha \quad \forall\alpha\in(0,1].$$

Proof. At the first, we show that for all $\alpha\in(0,1)$, $\langle x,x\rangle_\alpha = \|x\|_\alpha^2$.
According to the definition of α-fuzzy inner product by supposing $x=y$ we have:

$$\begin{aligned}
\langle x,x\rangle_\alpha &= \frac{1}{4}(\|x+x\|_\alpha^2 - \|x-x\|_\alpha^2) + \frac{i}{4}(\|x+ix\|_\alpha^2 - \|x-ix\|_\alpha^2)\\
&= \frac{1}{4}(4\|x\|_\alpha^2 - 0) + \frac{i}{4}x(\|1+i\|_\alpha^2 - \|1-i\|_\alpha^2)\\
&= \|x\|_\alpha^2.
\end{aligned}$$

Therefore $\langle x,x\rangle_\alpha = \|x\|_\alpha^2$. Let $x,y\in U$ be arbitrary, in the special case where $y=0$, the corollary is trivially true. Now assume that $y\neq 0$. Considering $\lambda\in\mathbb{C}$ and by $\lambda = \frac{\langle x,y\rangle_\alpha}{\|y\|_\alpha^2}$ for all $\alpha\in(0,1)$, we have:

$$\begin{aligned}
0 &\leq \|x-\lambda y\|_\alpha^2\\
&= \langle x,x\rangle_\alpha - \langle\lambda y,x\rangle_\alpha - \langle x,\lambda y\rangle_\alpha + \langle\lambda y,\lambda y\rangle_\alpha\\
&= \langle x,x\rangle_\alpha - \lambda\langle y,x\rangle_\alpha - \overline{\lambda}\langle x,y\rangle_\alpha + \lambda\overline{\lambda}\langle y,y\rangle_\alpha\\
&= \|x\|_\alpha^2 - \lambda\overline{\langle x,y\rangle_\alpha} - \overline{\lambda}\langle x,y\rangle_\alpha + \lambda\overline{\lambda}\|y\|_\alpha^2\\
&= \|x\|_\alpha^2 - \frac{|\langle x,y\rangle_\alpha|^2}{\|y\|_\alpha^2} - \frac{|\langle x,y\rangle_\alpha|^2}{\|y\|_\alpha^2} + \frac{|\langle x,y\rangle_\alpha|^2}{\|y\|_\alpha^2}\\
&= \|x\|_\alpha^2 - \frac{|\langle x,y\rangle_\alpha|^2}{\|y\|_\alpha^2}.
\end{aligned}$$

Therefore $0\leq \|x\|_\alpha^2 - \dfrac{|\langle x,y\rangle_\alpha|^2}{\|y\|_\alpha^2}$, It follows that $|\langle x,y\rangle_\alpha| \leq \|x\|_\alpha\|y\|_\alpha$. □

Example 3.2. Let $(U,\langle .,.\rangle)$ be a real inner product space. Define a function $\mu : U\times U\times\mathbb{C}\to[0,1]$ by

$$\mu(x,y,t) = \begin{cases} \dfrac{|t|}{|t|+\|x\|\|y\|} & \text{if } t>\|x\|\|y\|,\\ 0 & \text{if } t\leq\|x\|\|y\|,\\ 0 & \text{if } t\in\mathbb{C}\setminus\mathbb{R}^+. \end{cases}$$

Hence, we conclude that every classic inner product induces the fuzzy inner product.

Definition 3.3. Let (U,μ) and (V,μ) be two fuzzy Hilbert spaces satisfying $(FIP8)$ and $(FIP9)$ where μ is the same fuzzy inner product. Let T be a fuzzy bounded linear operator from U to V. If there exists an operator T^* from V to U such that for all $\alpha \in (0,1)$

$$\langle Tx, y\rangle_\alpha = \langle x, T^*y\rangle_\alpha, \quad \forall x \in U, y \in V,$$

then the operator T^* is called fuzzy adjoint of T.

Theorem 3.4. *If $T : (U, N_1) \longrightarrow (V, N_2)$ is a strongly fuzzy bounded operator, where (U, N_1) and (V, N_2) are fuzzy normed linear spaces that N_1 and N_2 induce from fuzzy inner products on U and V respectively, then there exists $T^* : (V, N_2) \longrightarrow (U, N_1)$ such that for all $x \in U, y \in V$ and for all $\alpha \in (0,1)$*

$$\langle x, T(y)\rangle_\alpha = \langle T^*(x), y\rangle_\alpha . \tag{3.1}$$

Theorem 3.5. *Let (U,μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$ and $\alpha \in (0,1)$. Let T be a fuzzy operator on (U,μ). Then T^* is also a fuzzy linear operator on (U,μ) and following properties hold:*

i) $(T^*)^* = T$;
ii) $(T_1 + T_2)^* = T_1^* + T_2^*$;
iii) $(\lambda T)^* = \bar{\lambda} T^*, \quad \forall \lambda \in \mathbb{C}$;
iv) $(ST)^* = T^*S^*$.

Corollary 3.6. *Let (U,μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$ and $\alpha \in (0,1)$. Let T be a fuzzy operator on (U,μ). Then*

$$\|T^*T\|_\alpha = \|TT^*\|_\alpha = \|T\|_\alpha^2.$$

References

1. T. Bag and S. K. Samanta, *Finite Dimensional Fuzzy Normed Linear Spaces*, J. Fuzzy Math., **11** (3) (2003), 687-705.
2. B. T. Bilalov and F. A. Guliyeva, *The Weakly Basicity of System in the Intuitionistic Fuzzy Metric Space*, J. Fuzzy Math., **9** (1) (2014), 33-45.
3. R. Biswas, *Fuzzy Inner Product Spaces and Fuzzy Norm Functions*, Inform. Sci., **53** (2) (1991), 185-190.
4. B. Daraby, Z. Solimani and A. Rahimi, *Some Properties of Fuzzy Hilbert Spaces*, Complex Anal. Oper. Theory., **11** (1) (2017), 119-138.
5. B. Daraby, Z. Solimani and A. Rahimi, *A Note on Fuzzy Hilbert Spaces*, J. Intell. Fuzzy Syst., **31** (1) (2016), 313-319.
6. A. M. El-Abyad and H. M. El-Hamouly, *Fuzzy Inner Product Spaces*, Fuzzy Sets Syst., **44** (2) (1991), 309-326.
7. A. K. Katsaras, *Fuzzy Topological Vector Spaces II*, Fuzzy Sets and Systems, **12** (2) (1984), 143-154.
8. P. Mazumdar, S. K. Samanta, *On Fuzzy Inner Product Spaces*, Fuzzy Math., **16** (1) (2008), 377-392.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

ON EXISTENCE OF SOLUTION OF ELLIPTIC EQUATIONS VIA MINIMIZATION

ZAHRA DONYARI *AND MOHSEN ZIVARI-REZAPOUR

Department of Mathematics, Faculty of Mathematical Sciences & Computer, Shahid Chamran University of Ahvaz, Ahvaz, Iran. z-donyari@stu.scu.ac.ir; mzivari@scu.ac.ir

Abstract. In this paper, we will prove the existence of a non trivial solution for a nonhomogeneous nonlinear equation via minimization on the Nehari manifold.

1. Introduction

1.1. **A radial problem in $\mathbb{R}^N$.** A first case of lack of compactness takes place when one works in unbounded open sets. For simplicity we treat the case where $\Omega = \mathbb{R}^N$, with $N \geq 3$, by studying the following problem. Consider the p-Laplacian problem

$$\begin{cases} -\Delta_p u = |u|^{r-2}u + \lambda |u|^{q-2}\, u \quad in\, \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{1.1}$$

and let $1 < p < N$ and $p < q < r < p^* = [1, \frac{Np}{N-p}]$, $\lambda \in \mathbb{R}$, q, $r \in (2, 2^*)$. A weak solution of (1.1) is a function $u \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \, \nabla v \, dx = \int_{\mathbb{R}^N} |u|^{r-2} uv \, dx + \lambda \int_{\mathbb{R}^N} |u|^{q-2} uv \, dx \quad \forall v \in H^1(\mathbb{R}^N).$$

2020 *Mathematics Subject Classification. Primary: 35j25*

Key words and phrases. Nehari manifold, nonhomogeneous, equations, energy functional, minimization.

* Speaker.

A problem like (1.1) is not compact. A natural attempt to overcome this problem is to guess that translational invariance is the only reason for the failure of compactness, and to try work in a space of functions where translations are not allowed. This is possible in this case because the problem is also invariant under rotations, so that one can try to work in spaces of radial functions. We define $D_r = \{u \in D^{1,2}(\mathbb{R}^N) \,|\, u \textit{ is radial } \}$, $H_r = \{u \in H^1(\mathbb{R}^N) \,|\, u \textit{ is radial } \}$.

Lemma 1.1. *Let $p \in (2, 2^*)$. Then the embedding of H_r into $L^p(\mathbb{R}^N)$ is compact.*

Proof. See Lemma 3.1.4 [5]. □

The energy functional and the Nehari manifold define for this problem and prove the existence of a nontrival solution via minimization on the nehari manifold. We recall that the functional associated to it is $I(u) = \frac{1}{p}\int\limits_\Omega |\nabla u|^p dx - \frac{1}{r}\int\limits_\Omega |u|^r\, dx - \frac{\lambda}{q}\int\limits_\Omega |u|^q dx$, defined and differentiable on $W_0^{1,p}(\Omega)$. Recall that I is unbounded below, one again we restrict I to a suitable set in order to get red of this problem. Now we use the set $\mathcal{N} = \{u \in W_0^{1,p}(\Omega) \,:\, u \neq 0\,,\, I'(u)u = 0\} = \{u \in W_0^{1,p}(\Omega) \,:\, u \neq 0\,,\, \|u\|^p_{W_0^{1,p}(\Omega)} = \int\limits_\Omega |u|^r dx + \lambda \int\limits_\Omega |u|^q dx\}$. This set is called the Nehari manifold. Thus, $I(u) = (\frac{1}{p} - \frac{1}{q})\|u\|^p + (\frac{1}{q} - \frac{1}{r})\,|u|_r^r\,$.

1.2. **Main results.**

Lemma 1.2. *The Nehari manifold is not empty.*

Proof. For every not identically zero $u \in W_0^{1,p}(\Omega)$, one sees immediately that $tu \in \mathcal{N}$ for some $t > 0$. Indeed, $tu \in \mathcal{N}$ is equivalent to

$$tu \in \mathcal{N} \Leftrightarrow \|tu\|^p_{W_0^{1,p}(\Omega)} = \int\limits_\Omega |tu|^r dx + \lambda \int\limits_\Omega |tu|^q dx,$$

$$t^p\, \|u\|^p_{W_0^{1,p}(\Omega)} = \int\limits_\Omega t^r |u|^r dx + \lambda \int\limits_\Omega t^q |u|^q dx,$$

$$\|u\|^p_{W_0^{1,p}(\Omega)} = \int\limits_\Omega t^{r-p} |u|^r dx + \lambda \int\limits_\Omega t^{q-p} |u|^q dx.$$

By the Mean Value Theorem we have $\exists t > 0 \;\; s.t \;\, f(t) = 0$, let $f(t) = At^{r-p} + Bt^{q-p} - C$ and $\lim\limits_{t\to 0^+} f(t) = -C$, $\lim\limits_{t\to+\infty} f(t) = \lim\limits_{t\to+\infty} t^{r-p} = +\infty$. Thus, $\exists t > 0 \;\; s.t \;\, f(t) = 0$. Thus Nehari manifold is not empty. □

It is obvious, we have $m = \inf\limits_{u\in N} I(u) > 0$.

Lemma 1.3. *The level m is attained by a nonnegative function, namely there exists $u \in \mathcal{N}$, $u(x) \geq 0$ a.e, such that $I(u) = m$.*

Proof. We first show that we can take a minimizing sequence for m in $\mathcal{N} \cap H_r$. To this aim, let $\{v_k\}_k \subseteq \mathcal{N}$ be a minimizing sequence. As usual we can assume $v_k \geq 0$. Let $w_k = v_k^* \in H_r$ be the nonnegative radial functional given by Theorem 3.1.5 [5]. We have $\|w_k\|^p = \int\limits_{\mathbb{R}^N} |\nabla v_k^*|^p \, dx \leq \int\limits_{\mathbb{R}^N} |\nabla v_k|^p \, dx = \int\limits_{\mathbb{R}^N} |v_k|^r \, dx + \lambda \int\limits_{\mathbb{R}^N} |v_k|^q \, dx = \int\limits_{\mathbb{R}^N} |v_k^*|^r \, dx + \lambda \int\limits_{\mathbb{R}^N} |v_k^*|^q \, dx = \int\limits_{\mathbb{R}^N} |w_k|^r \, dx + \lambda \int\limits_{\mathbb{R}^N} |w_k|^q \, dx$. Hence if we set $\gamma(t) = I'(tw_k)tw_k = t^p\|w_k\|^p - t^r \, |w_k|_r^r - \lambda \, |w_k|_q^q$. By lemma 1.2, there exists $t_t \in (0,1]$ such that $\gamma(t_k) = 0$, that is, $t_k w_k \in \mathcal{N}$. We obtain $m \leq I(t_k w_k) = t_k^p(\frac{1}{p} - \frac{1}{q})\|w_k\|^p + t_k^r(\frac{1}{q} - \frac{1}{r}) \, |w_k|_r^r \leq (\frac{1}{p} - \frac{1}{q})\|w_k\|^p + (\frac{1}{q} - \frac{1}{r}) \, |w_k|_r^r \leq (\frac{1}{p} - \frac{1}{q})\|v_k\|^p + (\frac{1}{q} - \frac{1}{r}) \, |v_k|_r^r = I(v_k)$. This implies that $\{t_k w_k\}_k$ is minimizing sequence for m and $t_k w_k \in H_r$, as we had claimed. In the sequel we set $u_k = t_k w_k$. Of course, $u_k \geq 0$, and we can assume that, up to subsequence $u_k \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. By Lemma 1.1 we obtain $u_k \to u$ *in* $L^p(\mathbb{R}^N)$ *and in* $L^r(\mathbb{R}^N)$, and, again up to subsequences, $u_k(x) \to u(x)$ almost every where, so that $u(x) \geq 0$ a.e and $u \in H_r$. We now prove that weak limit u belongs $\mathcal{N}$ and $I(u) = m$. Then we have by weak lower semicontinuity,

$$
\begin{aligned}
I(u) &= \frac{1}{p}\|u\|^p - \frac{1}{r}\int\limits_{\Omega} |u|^r \, dx - \frac{\lambda}{q}\int\limits_{\Omega} |u|^q dx \\
&\leq \liminf_{k\to\infty}(\frac{1}{p}\|u_k\|^p - \frac{1}{r}\int\limits_{\Omega} |u_k|^r \, dx - \frac{\lambda}{q}\int\limits_{\Omega} |u_k|^q dx) \\
&= \liminf_{k\to\infty} \; I(u_k) = m \qquad (1.2)
\end{aligned}
$$

$$
\|u\|^p \leq \liminf_{k\to\infty} \; \|u_k\|^p = \liminf_{k\to\infty}(\int\limits_{\Omega} |u_k|^r dx + \lambda \int\limits_{\Omega} |u_k|^q dx).
$$

Since $u_k \in \mathcal{N}$, we have $\|u\|^p = \int\limits_{\Omega} |u_k|^r dx + \lambda \int\limits_{\Omega} |u_k|^q dx$. By (1.2) it cannot be $\|u_k\| \to 0$, and therefore $\int\limits_{\Omega} |u_k|^r dx + \lambda \int\limits_{\Omega} |u_k|^q dx$ cannot tend to zero, thus, by strong convergence, $\int\limits_{\Omega} |u_k|^r dx + \lambda \int\limits_{\Omega} |u_k|^q dx \neq 0$, which shows that $u \neq 0$. passing to the limit we obtain

$$
\|u\|^p \leq \int\limits_{\Omega} |u_k|^r dx + \lambda \int\limits_{\Omega} |u_k|^q dx. \qquad (1.3)
$$

If

$$
\|u\|^p = \int\limits_{\Omega} |u_k|^r dx + \lambda \int\limits_{\Omega} |u_k|^q dx, \qquad (1.4)
$$

then $u \in \mathcal{N}$ and shows that u is the required minimizer. Since (1.4) holds, we only have to treat the case where $\|u\|^p < \int\limits_{\Omega} |u_k|^r dx + \lambda \int\limits_{\Omega} |u_k|^q dx$. We now show that if this happens, we reach a contradiction. Indeed, take $t > 0$ such that $tu \in \mathcal{N}$, namely. Since we have by lemma 1.2, we deduce that $0 < t < 1$. But $tu \in \mathcal{N}$, so that

$$\begin{aligned} 0 < m \leq I(tu) &= (\frac{1}{p} - \frac{1}{q})\|tu\|^p + (\frac{1}{q} - \frac{1}{r})\,|tu|_r^r \\ &= t^p(\frac{1}{p} - \frac{1}{q})\|u\|^p + t^r(\frac{1}{q} - \frac{1}{r})\,|u|_r^r \\ &< (\frac{1}{p} - \frac{1}{q})\|u\|^p + (\frac{1}{q} - \frac{1}{r})\,|u|_r^r \\ &\leq \liminf_k\,(\frac{1}{p} - \frac{1}{q})\|u_k\|^p + \liminf_k\,(\frac{1}{q} - \frac{1}{r})\,|u_k|_r^r \\ &= \liminf_k((\frac{1}{p} - \frac{1}{q})\|u_k\|^p + (\frac{1}{q} - \frac{1}{r})\,|u_k|_r^r) = \liminf_k\, I(u_k) = m. \end{aligned}$$

This contradiction proves that $\|u\|^p = \int\limits_\Omega |u_k|^r dx + \lambda \int\limits_\Omega |u_k|^q dx$, therefore $u \in \mathcal{N}$. By the weak lower semicontinuity of the norm it is straightforward to deduce that $I(u) \leq \liminf\limits_{k\to\infty}\ I(u_k) = m$, and the lemma is proved. □

References

1. M.Willem, *Minimax Theorems (Progress in Nonlinear Differential Equations and Their Applications)*, 24, Birkhauser, Boston, 1996.
2. R. Adams, *Sobolev spaces*, Academic Press, New York, 1975.
3. K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.
4. L.C.Evans,*Partial Differential Equations*, Graduate Studies in Mathematics, 19, American Mathematical Society, Providence, 1998.
5. M. Badiale and E. Serra, *Semilinear Elliptic Equations for Beginners*, Existence Results via the Variational Approach, Springer, London, 2011.
6. Ch. Chu, sh. Le and H.Suo, *Existence of a positve radial solution for semilinear elliptic problem involving variable exponent*, Electronic Research Archive., USA, **31** (5) (2023), 2472-2482.

A NEW STRONG CONVERGENCE THEOREM FOR SOLVING A FIXED POINT PROBLEM USING THE BREGMAN DISTANCE

MOSTAFA GHADAMPOUR*

Department of Mathematics, Payame Noor University P. O. Box 19395-4697, Tehran, Iran
ghadampour.m@pnu.ac.ir

Abstract. In this paper, using the Bregman distance, we introduce a new projection-type algorithm for finding an element of the set of fixed points of a resolvent operator in a reflexive Banach space.

1. Introduction

Let $f : X \to (-\infty, \infty]$ be a Gâteaux differentiable function. The bifunction $D_f : dom f \times int\, dom f \to [0, +\infty]$ defined by

$$D_f(y, x) := f(y) - f(x) - \langle y - x, \nabla f(x)\rangle, \tag{1.1}$$

is called the Bregman distance with respect to f.

For all $z \in X$, we have

$$D_f\Big(z, \nabla f^*\big(\sum_{i=1}^{N} t_i \nabla f(x_i)\big)\Big) \leq \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{1.2}$$

where $\{x_i\} \subset X$ and $\{t_i\} \subset (0, 1)$ with $\Sigma_{i=1}^{N} t_i = 1$.

2020 *Mathematics Subject Classification.* 47H09,47H10

Key words and phrases. Variational inequality, Fixed point problem, Fréchet differentiable.

* Speaker.

The equilibrium problem for a bifunction $g : C \times C \to \mathbb{R}$ satisfying the condition $g(x, x) = 0$ for every $x \in C$ is stated as follows

$$\text{Find } y^* \in C \text{ such that } g(x, y^*) \leq 0, \tag{1.3}$$

for all $x \in C$. The set of solutions of (1.3) is denoted by $EP(g)$.

Throughout this paper, the conditions B1-B4 are the same as [3]. The resolvent of φ is the operator $Res_\varphi^f : X \to 2^C$ defined by [5].

$$Res_\varphi^f x = \{z \in C : \varphi(z, y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0, \ \ \forall y \in C\}.$$

If $f : X \to (-\infty, \infty]$ be a Gâteaux differentiable and strongly coercive function and φ satisfies conditions B1-B4. Then $dom\ Res_\varphi^f = X$ (see [5, Lemma 1]).

2. Main results

Theorem 2.1. *Let C be a nonempty closed convex subset of a real reflexive Banach space X, and let $f : X \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying B1-B4 and $F(Res_\varphi^f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$ and*

$$\begin{cases} v_n = \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f(Res_\varphi^f x_n)), \\ u_n \in C \ \ \text{such that} \\ \varphi(u_n, y) + \langle \nabla f(u_n) - \nabla f(v_n), y - u_n\rangle \geq 0, \quad \forall \ y \in C, \\ C_{n+1} = \{v \in C_n : D_f(v, u_n) \leq D_f(v, x_n)\}, \\ x_{n+1} = \overleftarrow{Proj}^f_{C_{n+1}} x. \end{cases} \tag{2.1}$$

where $\{\alpha_n\} \subset (0, 1)$.

Then, the sequence $\{x_n\}$ converges strongly to $\overleftarrow{Proj}^f_{F(Res_\varphi^f)} u$.

Proof. We first prove that $F(Res_\varphi^f)$ and C_n both are closed and convex subset of C for all $n \geq 0$. In fact, it follows from [5, Lemma 2] that $EP(\varphi)$ is closed and convex. Furthermore, it is obvious that $C_0 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \geq 1$. Since the inequality $D_f(v, u_n) \leq D_f(v, x_n)$ is equivalent to

$$\langle \nabla f(x_n), v - x_n\rangle - \langle \nabla f(u_n), v - u_n\rangle \leq f(u_n) - f(x_n).$$

Therefore, we have

$$C_{n+1} = \{v \in C_n : \langle \nabla f(x_n), v - x_n\rangle - \langle \nabla f(u_n), v - u_n\rangle \leq f(u_n) - f(x_n)\}.$$

Therefore C_{n+1} is closed and convex.

Now, we prove that $F(Res_\varphi^f) \subset C_n$ for all $n \geq 0$. Indeed, it is obvious that $F(Res_\varphi^f) \subset C_0 = C$. Assume that $F(Res_\varphi^f) \subset C_n$. From (1.2), [5, Lemma 2] and fact that $u_n = Res^f_\varphi(v_n)$, we have

$$\begin{aligned} D(u,u_n) = D_f(u, Res^f_\varphi(v_n)) \leq D_f(u,v_n) \\ =& D_f(u, \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Res^f_\varphi x_n)) \\ \leq& \alpha_n D_f(u,x_n) + (1-\alpha_n) D_f(u, Res^f_\varphi x_n) \\ \leq& \alpha_n D_f(u,x_n) + (1-\alpha_n) D_f(u,x_n) \leq D_f(u,x_n). \end{aligned} \tag{2.2}$$

Hence, $u \in C_{n+1}$, i.e., $F(Res^f_\varphi) \subset C_n$, for all $n \in \mathbb{N} \cup \{0\}$. Therefore, $\{x_n\}$ is well defined.

It follows from [3, Lemma 2.6] that

$$D_f(u,x_n) = D_f(u, \overleftarrow{Proj}^f_{C_n} x) \leq D_f(u,x) - D_f(\overleftarrow{Proj}^f_{C_n} x, x) \leq D_f(u,x),$$

for all $n \geq 0$ and $u \in F(Res^f_\varphi)$. This implies that $D_f(u,x_n)$ is bounded. From [3, Lemma 2.5], we conclude that the sequence $\{x_n\}$ is bounded. Since $f : X \to \mathbb{R}$ is uniformly Fr´echet differentiable and bounded on bounded subsets of X, it follows from [4, Proposition 2.1, p. 474] that ∇f is uniformly continuous and bounded on bounded subsets of X. Hence, $\{\nabla f(x_n)\}$ is bounded. Since $x_{n+1} \in C_{n+1} \subset C_n$, then $D_f(x_n,x) \leq D_f(x_{n+1},x)$ for all $n \geq 0$. Therefore, $D_f(x_n,x)$ is nondecreasing. So, the limit of $D_f(x_n,x)$ exists. Since

$$\begin{aligned} D_f(x_{n+1},x_n) &= D_f(x_{n+1}, \overleftarrow{Proj}^f_{C_n} x) \leq D_f(x_{n+1},x) - D_f(\overleftarrow{Proj}^f_{C_n} x, x) \\ &= D_f(x_{n+1},x) - D_f(x_n,x), \end{aligned}$$

for all $n \geq 0$. we have $\lim_{n\to\infty} D_f(x_{n+1},x_n) = 0$. Now from $x_{n+1} \in C_{n+1}$, we have $D_f(x_{n+1},u_n) \leq D_f(x_{n+1},x_n)$ for all $n \in \mathbb{N} \cup \{0\}$. Hence, $\lim_{n\to\infty} D_f(x_{n+1},u_n) = 0$. By [2, Lemma 2.1.2] and boundedness of $\{x_n\}$, we obtain $\lim_{n\to\infty} \|x_{n+1}-x_n\| = \lim_{n\to\infty} \|x_{n+1}-u_n\| = 0$. Therefore,

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{2.3}$$

Hence,

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(u_n)\| = 0. \tag{2.4}$$

It follows from [1, Theorem 1.8] that

$$\lim_{n\to\infty} |f(x_n) - f(u_n)| = 0. \tag{2.5}$$

From the definition of the Bregman distance, we obtain that

$$\begin{aligned} D_f(v,x_n) - D_f(v,u_n) =& f(u_n) - f(x_n) + \langle \nabla f(u_n) - \nabla f(x_n), v - u_n \rangle \\ &+ \langle \nabla f(x_n), x_n - u_n \rangle, \end{aligned} \tag{2.6}$$

for all $v \in F(Res^f_\varphi)$. From (2.3)-(2.6), we obtain that

$$\lim_{n\to\infty} \big(D_f(v,x_n) - D_f(v,u_n) \big) = 0. \tag{2.7}$$

Now, by $u_n = Res_{\varphi}^{f} v_n$ and [5, Lemma 2], for all $v \in F(Res_{\varphi}^{f})$ we have that

$$\begin{aligned} D_f(u_n, v_n) &= D_f(Res_{\varphi}^{f} v_n, v_n) \leq D_f(v, v_n) - D_f(v, Res_{\varphi}^{f} v_n) \\ &\leq D_f(v, x_n) - D_f(v, Res_{\varphi}^{f} v_n) = D_f(v, x_n) - D_f(v, v_n). \end{aligned} \tag{2.8}$$

It follows from (2.7) that $D_f(u_n, v_n) = 0$. Therefore, from [2, Lemma 2.1.2], we conclude that $\lim_{n\to\infty} \|u_n - v_n\| = 0$. It follows from [4, Proposition 2.1, p. 474] that

$$\lim_{n\to\infty} \|\nabla f(u_n) - \nabla f(v_n)\| = 0. \tag{2.9}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $x_{n_m} \rightharpoonup x^*$. Then from (2.3), we conclude that $u_n \rightharpoonup x^*$. Since $u_n = Res_{\varphi}^{f} v_n$, then we have

$$\varphi(u_n, y) + \langle \nabla f(u_n) - \nabla f(v_n), y - u_n \rangle \geq 0, \quad \forall\, y \in C.$$

Now, substituting n_m instead of n in the above, we conclude that

$$\varphi(u_{n_m}, y) + \langle \nabla f(u_{n_m}) - \nabla f(v_{n_m}), y - u_{n_m} \rangle \geq 0, \quad \forall\, y \in C.$$

By (B2), we obtain

$$\langle \nabla f(u_{n_m}) - \nabla f(v_{n_m}), y - u_{n_m} \rangle \geq -\varphi(u_{n_m}, y) \geq \varphi(y, u_{n_m}), \tag{2.10}$$

for all $y \in C$. Now, we know from (B4) that $\varphi(x, .)$ is convex and lower semicontinuous. Hence, letting $m \to \infty$, we conclude from (2.9) and (2.10) that $\varphi(y, q) \leq 0$ for all $y \in C$. Then from (1.3), $q \in EP(\varphi)$. So by the condition (iii) of [5, Lemma 2], we have that $q \in F(Res_{\varphi}^{f})$.

□

References

[1] A. Ambrosetti and G. Prodi, *A Primer of Nonlinear Analysis*, Cambridge University Press, Cambridge 1993.

[2] D. Butnariu and A. N. Iusem, *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*, Kluwer Academic Publishers, Dordrecht 2000.

[3] M. Ghadampour., E. Soori., R.P. Agarwal Oand D. O'Regan, *A Strong Convergence Theorem for Solving an Equilibrium Problem and a Fixed Point Problem Using the Bregman Distance*, J Optim Theory Appl, **195** (2022), 854-877.

[4] S. Reich and S. Sabach, *A strong convergence theorem for proximal type- algorithm in reflexive Banach spaces*, J. Nonlinear Convex Anal, **10** (2009), 471-485.

[5] S. Reich and S. Sabach, *Two strong convergence theorems for Bregman strongly non-expansive operators in reflexive Banach spaces*, Nonlinear Analysis: Theory,Methods & Applications, **73** (2010), 122-135.

NEW INEQUALITIES INVOLVING KANTOROVICH CONSTANT FOR SECTOR MATRICES

AMIR GHASEM GHAZANFARI*AND SOMAYEH MALEKINEJAD

Department of Mathematics Lorestan University, Khoramabad, Iran
ghazanfari.a@lu.ac.ir
Department of Mathematics Payame Noor University, P.O. Box 19395-3697 Tehran, Iran.
maleki60313@pnu.ac.ir; maleki60313@gmail.com

Abstract. The main goal of this paper is to discuss the famous inequalities from positive definite matrices to sector matrices in a more general setting. For instance, we show that if f is an operator convex function on $(0,\infty)$ and Φ is a unital positive linear map on $\mathcal{B}(\mathcal{H})$, then, for every $A \in S_\alpha$

$$\cos^2(\alpha)\mathcal{R}\Phi(f(A)) \leq K(h,2)\mathcal{R}f(\Phi(A)) \leq K(h,2)\sec^2(\alpha)\mathcal{R}\Phi(f(A)),$$

where $h = \frac{M}{n}$ whenever $0 < mI \leq \mathcal{R}A \leq MI$.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of bounded linear operators acting on $\mathcal{H}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is positive definite (resp. positive semi-definite) if $\langle Ax, x\rangle > 0$ (resp. $\langle Ax, x\rangle \geq 0$) holds for all non-zero $x \in \mathcal{H}$. If A is positive semi-definite, we denote $A \geq 0$. Let $\mathcal{PS}, \mathcal{P} \subseteq \mathcal{B}(\mathcal{H})$ be the sets of all positive semi-definite operators and positive definite operators, respectively.

2020 *Mathematics Subject Classification.* *47A63, 47A64, 47B65*
Key words and phrases. Kantorovich constant, Monotone operator, Positive linear map.
* Speaker.

An operator $A \in \mathcal{B}(\mathcal{H})$ is called accretive if $\mathcal{R}A \geq 0$ in its Cartesian (or Toeplitz) decomposition $A = \mathcal{R}A + i\mathcal{I}A$, where $\mathcal{R}A = \frac{A+A^*}{2}$, $\mathcal{I}A = \frac{A-A^*}{2}$.

Let $\mathbb{M}_n$ denote the set of $n \times n$ complex matrices. The numerical rang of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

When talking about accretive matrices, we need to introduce sectorial matrices. A matrix $A \in \mathbb{M}_n$ is said to be sectorial if $W(A) \subset \mathcal{S}_\alpha$ for some $0 \leq \alpha < \frac{\pi}{2}$, where $\mathcal{S}_\alpha$ denote the sector regions in the complex plane as follows:

$$\mathcal{S}_\alpha = \{z \in \mathbb{C} : \mathcal{R}z \geq 0, |\mathcal{I}z| \leq (\mathcal{R}z)\tan(\alpha)\}.$$

The study of sector matrices, we refer the reader to [2, 5, 3]. A linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is called positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. If $\Phi(I) = I$, where I denoted the identity operator, then we say that Φ is unital. We know that if Φ is a positive linear map, then $\Phi(\mathcal{R}A) = \mathcal{R}\Phi(A)$ and $\Phi(\mathcal{I}A) = \mathcal{I}\Phi(A)$. This shows that $W(\Phi(A)) \subset \mathcal{S}_\alpha$ if $W(A) \subset \mathcal{S}_\alpha$. In particular, if $A \in \mathcal{S}_\alpha$, then so is $\Phi(A)$.

2. Main result

Let A and B be two positive definite operators on a Hilbert space $\mathcal{H}$ and Φ be a unital positive linear map on $\mathcal{B}(\mathcal{H})$. Ando [1, Theorem 3] showed the following property of a positive linear map in connection with the operator geometric mean.

$$\Phi(A \sharp B) \leq \Phi(A) \sharp \Phi(B). \tag{2.1}$$

Inequality (2.1) is extended to an operator mean σ in Kubo-Ando theory as follows:

$$\Phi(A\sigma B) \leq \Phi(A)\sigma\Phi(B),$$

In particular for the weighted geometric mean, we have

$$\Phi(A \sharp_\nu B) \leq \Phi(A) \sharp_\nu \Phi(B), \tag{2.2}$$

where ν is a real number in $(0, 1]$. A complementary inequality to inequality (2.2), is the following important inequality[4]:
Let $0 < m_1 I \leq A \leq M_1 I$ and $0 < m_2 I \leq B \leq M_2 I$ and $0 < \nu \leq 1$, then

$$K(h, \nu)\Phi(A) \sharp_\nu \Phi(B) \leq \Phi(A \sharp_\nu B), \tag{2.3}$$

where $h = \frac{M_1 M_2}{m_1 m_2}$ and $K(h, \nu)$ is the generalization Kantorovich constant defined by

$$K(h, \nu) = \frac{h^\nu - h}{(\nu - 1)(h - 1)} \left(\frac{(\nu - 1)(h^\nu - 1)}{\nu(h^\nu - h)} \right)^\nu.$$

The special case, $K(h, 2) = K(h, -1) = \frac{(1+h)^2}{4h}$ is called the Kantorovich constant. The generalization Kantorovich constant $K(h, \nu)$ has the following properties:

$$K(h,\nu)=K(h,1-\nu)$$
$$0<K(h,\nu)\leq 1 \text{ for } 0<\nu\leq 1,$$

and $K(h,\nu)$ is decreasing for $\nu\leq\frac{1}{2}$ and increasing for $\nu>\frac{1}{2}$ therefore for all $\nu\in\mathbb{R}$, $K(h)=K(h,\frac{1}{2})=\frac{2h^{\frac{1}{4}}}{1+h^{\frac{1}{2}}}\leq K(h,\nu)$.

Now, we state another complementary inequality to (2.2) with respect to $R(A,B)$ as follows:

Proposition 2.1. *Let A and B be two positive definite operators in $\mathcal{P}$, then*

$$K\left(R(A,B)^2,\nu\right)\Phi(A)\sharp_\nu\Phi(B)\leq\Phi(A\sharp_\nu B). \tag{2.4}$$

The inequality (2.4) with $\nu=\frac{1}{2}$, becomes the following inequality,

$$\frac{2R(A,B)^{\frac{1}{2}}}{1+R(A,B)}\Phi(A)\sharp\Phi(B)\leq\Phi(A\sharp B). \tag{2.5}$$

Next, we recall the Kadison's Schwarz inequalities

$$\Phi(A^2)\geq\Phi(A)^2,\ \ \Phi(A^{-1})\geq\Phi(A)^{-1} \tag{2.6}$$

and two complementary inequalities to them, whenever $0<m\leq A\leq M$:

$$\frac{(m+M)^2}{4mM}\Phi(A)^2\geq\Phi(A^2),\ \ \frac{(m+M)^2}{4mM}\Phi(A)^{-1}\geq\Phi(A^{-1}). \tag{2.7}$$

The following inequality unifies Kadison's Schwarz inequalities into a single form.

$$\Phi(BA^{-1}B)\geq\Phi(B)\Phi(A)^{-1}\Phi(B). \tag{2.8}$$

If $0<m\leq A,B\leq M$, then the following inequality is a complementary inequality to (2.8)

$$\frac{(m+M)^2}{4mM}\Phi(B)\Phi(A)^{-1}\Phi(B)\geq\Phi(BA^{-1}B). \tag{2.9}$$

The similar to the proof of Proposition 2.1, we give another complementary inequality to (2.8) with respect to $R(A,B)$ as follows:

$$\frac{(1+R(A,B)^2)^2}{4R(A,B)}\Phi(B)\Phi(A)^{-1}\Phi(B)\geq\Phi(BA^{-1}B). \tag{2.10}$$

To compare the inequalities (2.3) with (2.5) and (2.9) with (2.10), the following examples show that neither Kantorovich constants in (2.3) and (2.9) nor Kantorovich constants in (2.5) and (2.10) are uniformly better than the other.

Example 2.2. Let $A=\begin{bmatrix}2&0\\0&\frac{1}{3}\end{bmatrix}$ and $B=\begin{bmatrix}4&0\\0&\frac{1}{2}\end{bmatrix}$. Clearly, $m_1I=\frac{1}{3}I\leq A\leq 2I=M_1I$ and $m_2I=\frac{1}{2}I\leq B\leq 4I=M_2I$. Then $A^{-1}B=\begin{bmatrix}2&0\\0&\frac{3}{2}\end{bmatrix}$ and

$B^{-1}A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$, therefore $R(A,B)^2 = 4 \leq h = \frac{M_1M_2}{m_1m_2} = 48$. Consequently, $K(R(A,B)^2, \frac{1}{2}) \geq K(h, \frac{1}{2})$ and $K(R(A,B)^2, 2) \leq K(h, 2)$.

Example 2.3. Let $C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Clearly, $m_1I = \frac{3-\sqrt{5}}{2}I \leq C \leq \frac{3+\sqrt{5}}{2}I = M_1I$ and $m_2I = I \leq D \leq 2I = M_2I$. Then $C^{-1}D = \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}$ and $D^{-1}C = \begin{bmatrix} 2 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, therefore $R(C,D)^2 = \left(\frac{5+\sqrt{17}}{2}\right)^2 \geq h = \frac{M_1M_2}{m_1m_2} = \frac{(3+\sqrt{5})^2}{2}$.
Consequently, $K(R(C,D)^2, \frac{1}{2}) \leq K(h, \frac{1}{2})$ and $K(R(C,D)^2, 2) \geq K(h, 2)$.

Theorem 2.4. *If f is an operator convex function on $(0,\infty)$, Φ is a unital positive linear map on $\mathcal{B}(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})$ with $0 < mI \leq A \leq MI$, then*

$$K(h,2)f(\Phi(A)) \geq \Phi(f(A)) \geq f(\Phi(A)), \tag{2.11}$$

where $h = \frac{M}{m}$.

Corollary 2.5. *If f is an operator convex function on $(0,\infty)$ and Φ is a unital positive linear map on $\mathcal{B}(\mathcal{H})$. Then, for every $A \in S_\alpha$*

$$\cos^2(\alpha)\mathcal{R}\Phi(f(A)) \leq K(h,2)\mathcal{R}f(\Phi(A)) \leq K(h,2)\sec^2(\alpha)\mathcal{R}\Phi(f(A)),$$

where $h = \frac{M}{n}$ whenever $0 < mI \leq \mathcal{R}A \leq MI$.

Corollary 2.6. *Let $A, B \in \mathcal{S}_\alpha$. Then*

$$K\left(R(\mathcal{R}A, \mathcal{R}B)^2, \nu\right)\mathcal{R}\left(\Phi(A)\sharp_\nu\Phi(B)\right) \leq \sec^2(\alpha)\left(\Phi(A\sharp_\nu B)\right), \tag{2.12}$$

where $0 < \nu \leq 1$.

References

1. T. Ando, *Concavity of certain maps on positive definite matrices and applications to Hadamard prodcts*, Linear Algebra Appl. **26** (1979), 203-241.
2. Y. Bedrani, F. Kittaneh and M. Sababeh, *From positive to accretive matrices*, Positivity, https://doi.org/10.1007/s11117-021-00831-8(2021).
3. A. G. Ghazanfari and S. Malekinejad, *Operator monotone functions on accretive matrices,* Positivity **27** (5) (2023), 59.
4. J. Mićić, J. Pečarić and Y. Seo, *Complementary inequalities to inequalities of Jensen and Ando based on the Mond-Pečarić method*, Linear Algebra Appl., **318** (2000), 87-107.
5. M. Raissouli, M.S. Moslehian and S.furuichi, *Relative entropy and Tsallis entropy of two accretive operators*, C. R. Acad. Sci. Paris Ser I, **355** (2017) 687-693.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

UNITARILY INVARIANT NORMS OF OPERATOR MONOTONE FUNCTIONS ON ACCRETIVE MATRICES

AMIR GHASEM GHAZANFARI* AND SOMAYEH MALEKINEJAD

Department of Mathematics Lorestan University, Khoramabad, Iran
ghazanfari.a@lu.ac.ir
Department of Mathematics Payame Noor University, P.O. Box 19395-3697 Tehran, Iran.
maleki60313@pnu.ac.ir; maleki60313@gmail.com

Abstract. In this study, we demonstrate that for an operator monotone function $f:(0,\infty)\to\mathbb{R}$ and an accretive matrix $A\in\mathbb{M}_n$, the expression f(A) can be written as:

$$f(A)=a+bA+\int_0^\infty \lambda A(\lambda+A)^{-1}d\mu(\lambda),$$

where $b\geq 0$, $a\in\mathbb{R}$, and μ is a positive measure on the closed positive half-line $[0,\infty)$ satisfying the condition $\int \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty$.

We also show that if $f:(0,\infty)\to(0,\infty)$ be an operator monotone function and $A\in\mathcal{S}_\alpha$, then for any unitarily invariant norm $|||.|||$,

$$|||f(\mathcal{R}A)|||\leq|||\mathcal{R}f(A)|||\leq\sec^2(\alpha)|||f(\mathcal{R}A)|||.$$

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of bounded linear operators acting on $\mathcal{H}$. An operator $A\in\mathcal{B}(\mathcal{H})$ is positive definite (resp. positive semi-definite) if $\langle Ax,x\rangle>0$ (resp. $\langle Ax,x\rangle\geq 0$) holds for all non-zero $x\in\mathcal{H}$. If A is positive semi-definite, we denote $A\geq 0$.

2020 *Mathematics Subject Classification.* 47A63, 47A64, 47B65
Key words and phrases. Accretive matrix, Monotone operator, Positive linear map.
* Speaker.

Let $\mathcal{PS}, \mathcal{P} \subseteq \mathcal{B}(\mathcal{H})$ be the sets of all positive semi-definite operators and positive definite operators, respectively.

An operator $A \in \mathcal{B}(\mathcal{H})$ is called accretive if $\mathcal{R}A \geq 0$ in its Cartesian (or Toeplitz) decomposition $A = \mathcal{R}A + i\mathcal{I}A$, where $\mathcal{R}A = \frac{A+A^*}{2}$, $\mathcal{I}A = \frac{A-A^*}{2}$.

Let $\mathbb{M}_n$ denote the set of $n \times n$ complex matrices. The numerical rang of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

When talking about accretive matrices, we need to introduce sectorial matrices. A matrix $A \in \mathbb{M}_n$ is said to be sectorial if $W(A) \subset \mathcal{S}_\alpha$ for some $0 \leq \alpha < \frac{\pi}{2}$, where $\mathcal{S}_\alpha$ denote the sector regions in the complex plane as follows:

$$\mathcal{S}_\alpha = \{z \in \mathbb{C} : \mathcal{R}z \geq 0, |\mathcal{I}z| \leq (\mathcal{R}z)\tan(\alpha)\}.$$

The study of sector matrices, we refer the reader to [1, 5, 4]. A linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is called positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. If $\Phi(I) = I$, where I denoted the identity operator, then we say that Φ is unital. We know that if Φ is a positive linear map, then $\Phi(\mathcal{R}A) = \mathcal{R}\Phi(A)$ and $\Phi(\mathcal{I}A) = \mathcal{I}\Phi(A)$. This shows that $W(\Phi(A)) \subset \mathcal{S}_\alpha$ if $W(A) \subset \mathcal{S}_\alpha$. In particular, if $A \in \mathcal{S}_\alpha$, then so is $\Phi(A)$.

The operator norm $\|A\|$ of $A \in \mathbb{M}_n$ is defined by

$$\|A\| = \sup\{\langle Ax, y\rangle : x.y \in \mathbb{C}^n, \|x\| = \|y\| = 1\}.$$

Recall that a norm $|||\cdot|||$ on $A \in \mathbb{M}_n$ is unitarily invariant if $|||UAV||| = |||A|||$ for any $A \in \mathbb{M}_n$ and for all unitary matrices $U, V \in \mathbb{M}_n$. The study of norm inequalities involving operator monotone functions has always been of interest, we refer the reader to [1, 2, 3]. Recently, Bedrani et al. [1] showed that when $f : (0, \infty) \to (0, \infty)$ be an matrix monotone function with $f(1) = 1$ and $A \in \mathcal{S}_\alpha$,

$$f(\mathcal{R}(A)) \leq \mathcal{R}(f(A)) \leq \sec^2(\alpha) f(\mathcal{R}(A)). \tag{1.1}$$

We extend relation (1.1) from the setting of positive matrix monotone functions to real matrix monotone functions.

2. Unitarily invariant norms

Let A be an accretive operator on $\mathcal{B}(\mathcal{H})$, then

$$\mathcal{R}(Sp(A)) \subseteq \mathcal{R}(\overline{W(A)}) \subseteq \overline{\mathcal{R}(W(A))} = \overline{W(\mathcal{R}(A))}.$$

It is trivial if $\lambda, \mu \geq 0$ and $\upsilon \in \mathbb{R}$, then $\frac{\lambda}{\sqrt{(\lambda+\mu)^2+\upsilon^2}} \leq 1$. Therefore, if A be an accretive operator on $\mathcal{B}(\mathcal{H})$, then

$$\begin{aligned}\|\lambda(\lambda + A)^{-1}\| &= \sup\left\{\left|\frac{\lambda}{\lambda+\omega}\right| : \omega \in Sp(A)\right\} \\ &= \sup\left\{\frac{\lambda}{\sqrt{(\lambda+\mu)^2+\upsilon^2}} : \omega = \mu + i\upsilon \in Sp(A)\right\} \leq 1.\end{aligned}$$

Theorem 2.1. *Let $f : (0,\infty) \to \mathbb{R}$ be an operator monotone function and $A \in \mathbb{M}_n$ be accretive matrix. Then*

$$f(A) = a + bA + \int_0^\infty \lambda A(\lambda + A)^{-1} d\mu(\lambda), \tag{2.1}$$

where $b \geq 0$, $a \in \mathbb{R}$ and μ is a positive measure on the closed positive half-line $[0,\infty)$ with$\int \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty$.

Proof. It is known that every operator monotone function f on $(0,\infty)$ has a special integral representation as follows [2, (V.53)]:

$$f(t) = a + bt + \int_0^\infty \frac{\lambda t}{\lambda + t} d\mu(\lambda), \tag{2.2}$$

where a, b are real numbers, $b \geq 0$, and μ is positive measure with $\int \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty$. This representation shows that f has the following analytic continuation on complex plane except on $(-\infty, 0]$,

$$f(z) = a + bz + \int_0^\infty \frac{\lambda z}{\lambda + z} d\mu(\lambda). \tag{2.3}$$

Since $\mathcal{I}\left(\frac{z}{\lambda+z}\right) = \frac{\lambda^2 \mathcal{I} z}{|\lambda+z|^2}$, the function defined above in (2.3) maps the upper half-plane into itself.

$$\lambda \leq \lambda + \mathcal{R}z \leq |\lambda + z| \Rightarrow \left|\frac{(\lambda+1)z}{\lambda+z}\right| \leq \frac{(\lambda+1)(|\mathcal{R}z| + |\mathcal{I}z|)}{\lambda + \mathcal{R}z} \leq M < \infty.$$

Therefore,

$$\int_0^\infty \frac{\lambda z}{\lambda + z} d\mu(\lambda) = \int_0^\infty \frac{(\lambda+1)z}{\lambda+z} \frac{\lambda}{\lambda+1} d\mu(\lambda) < \infty.$$

This implies that if D is a diagonal matrice, then

$$f(D) = a + bD + \int_0^\infty \lambda D(\lambda + D)^{-1} d\mu(\lambda).$$

If A is diagonalizable, then $A = V^{-1}DV$ and

$$f(A) = f(V^{-1}DV) = V^{-1}f(D)V = a + bA + \int_0^\infty \lambda A(\lambda + A)^{-1} d\mu(\lambda).$$

Now, for a general accretive matrix A, let (D_n) be a sequence of digonalizable matrices converging to A (by density of diagonalizable matrices), in the operator norm. From $D_n \to A$, we get $\langle Ax, x\rangle = \lim_n \langle D_n x, x\rangle$ and $\mathcal{R}\langle Ax, x\rangle = \lim_n \mathcal{R}\langle D_n x, x\rangle$. Therefore without losing generality, we can assume that the sequence (D_n) is accretive diagonalizable.

Since $\|(\lambda(\lambda + A)^{-1})\| \leq 1$, the following inequalities hold

$$\begin{aligned}
&\|\lambda D_n(\lambda + D_n)^{-1} - \lambda A(\lambda + A)^{-1}\| \\
&= \|\lambda(\lambda + D_n)^{-1}\left[D_n(\lambda + A) - (\lambda + D_n)A\right](\lambda + A)^{-1}\| \\
&= \|\lambda(\lambda + D_n)^{-1}(D_n - A)\lambda(\lambda + A)^{-1}\| \leq \|\lambda(\lambda + D_n)^{-1}\|\|D_n - A\|\|\lambda(\lambda + A)^{-1}\| \\
&\leq \|D_n - A\|.
\end{aligned}$$

This implies that

$$\int_0^\infty \lambda D_n(\lambda + D_n)^{-1} d\mu(\lambda) \to \int_0^\infty \lambda A(\lambda + A)^{-1} d\mu(\lambda).$$

Consequently $f(D_n) \to f(A)$. Therefore if $A \in \mathbb{M}_n$ is accretive, then

$$f(A) = a + bA + \int_0^\infty \lambda A(\lambda + A)^{-1} d\mu(\lambda).$$

□

Proposition 2.2. *Let $f : (0, \infty) \to \mathbb{R}$ be an operator monotone function and $A \in \mathbb{M}_n$ be accretive matrix, then for any unital positive linear map Φ on $\mathbb{M}_n$, the following inequalities hold*

$$\mathcal{R}f(\Phi(A)) \geq f(\mathcal{R}\Phi(A)) = f(\Phi(\mathcal{R}A)) \geq \Phi(f(\mathcal{R}A)), \tag{2.4}$$

and

$$\mathcal{R}\Phi\left(f(A)\right) \leq \sec^2(\alpha)\ \mathcal{R}f\left(\Phi(A)\right).$$

Corollary 2.3. *Let $f : (0, \infty) \to (0, \infty)$ be an operator monotone function and $A \in \mathcal{S}_\alpha$. Then for any unitarily invariant norm $|||.|||$,*

$$|||f(\mathcal{R}A)||| \leq |||\mathcal{R}f(A)||| \leq \sec^2(\alpha)|||f(\mathcal{R}A)|||.$$

References

1. Y. Bedrani, F. Kittaneh and M. Sababeh, From positive to accretive matrices, Positivity, https://doi.org/10.1007/s11117-021-00831-8(2021).
2. R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.
3. F. Hiai and X. Zhan, *Inequalities involving unitarily invariant norms and operator monotone functions*. Linear Algebra Appl., **341** (2002), 151-169.
4. A. G. Ghazanfari and S. Malekinejad,*Operator monotone functions on accretive matrices,* Positivity, **27**(5) (2023), 59.
5. M. Raissouli, M.S. Moslehian and S.furuichi, *Relative entropy and Tsallis entropy of two accretive operators*, C. R. Acad. Sci. Paris Ser I, **355** (2017), 687-693.

OPERATOR QUASI-CONVEXITY OF BERNSTEIN FUNCTIONS

A. G. GHAZANFARI*AND F. LOTFI

Department of Mathematics, Lorestan University, P. O. Box 465, Khoramabad, Iran.
ghazanfari.a@lu.ac.ir; lotfi.fa@fs.lu.ac.ir

Abstract. Let $\mathcal{M}_n$ denote the algebra of all $n \times n$ complex matrices. Let A, B be positive definite matrices in $\mathcal{M}_n$ and $|||\cdot|||$ be a unitarily invariant norm on $\mathcal{M}_n$. We establish some estimates of the right hand side of some Hermite-Hadamard type inequalities in which Bernstein functions are involved. Furthermore, if f is a Bernstein function on $(0, \infty)$, we obtain a bound for $|||f(A) - f(B)|||$ in terms of $|||A - B|||$, as follows

$$|||f(A) - f(B)||| \leq \max\{\|f'(A)\|, \|f'(B)\|\} \, |||A - B||| \, .$$

for all positive definite matrices A, B in $\mathcal{M}_n$.

We also show that if $A, B, X \in \mathcal{M}_n$ such that A and B are positive definite and f is a Bernstein function on $(0, \infty)$. Then

$$|||f(A)X - Xf(B)||| \leq \max\left\{\|f'(A)\|, \|f'(B)\|\right\} |||AX - XB||| \, .$$

These results extend and unify some known results for operator monotone functions; a subclass of Bernstein functions.

1. Introduction and Preliminary

Theory of functions has played a major role in advancements of mathematical inequalities among real numbers and Hilbert space operators. Among

2020 *Mathematics Subject Classification. 26D10, 47A63, 26D15.*
Key words and phrases. Operator monotone functions, unitarily invariant norms, Bernstein functions.
* Speaker.

the most useful functions are those called convex functions. Recall that a function $f : J \to \mathbb{R}$ is said to be convex on the interval J if

$$f((1-\lambda)x+\lambda y) \leq (1-\lambda)f(x)+\lambda f(y),$$

for all $x, y \in J$ and $\lambda \in [0,1]$. We notice that a convex function satisfies the weaker inequality

$$f((1-\lambda)x+\lambda y) \leq \max\{f(x), f(y)\}; x, y \in J, 0 \leq \lambda \leq 1.$$

Functions satisfying the latter inequality are usuall called quasi-convex functions. Clearly any monotone function (increasing or decreasing) is quasi-convex.

Differential calculus has its own impact on the study of convex and monotone functions. Recall that a twice differentiable function is convex if and only if $f'' \geq 0$, while it is increasing if and only if $f' \geq 0$.

Extending these notions, completely monotone and Bernstein functions were defined analogously. In the sequel, the notation C^∞ refers to the class of infinitely differentiable functions on the open interval $(0,\infty)$. Recall that a completely monotone function is a function $f : (0,\infty) \to \mathbb{R}$ satisfying $(-1)^k f^{(k)} \geq 0$ for all $k \in \mathbb{N}$. A function whose negative is completely monotone is called a Bernstein function. Thus, a Bernstein function satisfies $(-1)^{k-1} f^{(k)} \geq 0, k \in \mathbb{N}$.

We notice that a completely monotone function is decreasing and convex while a Bernstein function is increasing and concave.

When A is Hermitian, $\sigma(A) \subset \mathbb{R}$ and $A = UD(\lambda_i)U^*$ for some unitary matrix U and a diagonal matrix $D(\lambda_i)$; whose diagonal entries are the eigenvalues λ_i of A. Now if $f : J \to \mathbb{R}$ is a real valued function defined on an interval J that contains $\sigma(A)$, for some Hermitian matrix A, we define $f(A) = UD(f(\lambda_i))U^*$.

A function $f : J \to \mathbb{R}$ is said to be operator monotone if $f(A) \leq f(B)$ whenever $A, B \in \mathcal{M}_n$ are Hermitian such that $A \leq B$. Thus, operator monotone functions preserve the order among Hermitian matrices. On the other hand, f is said to be operator convex (operator concave) if

$$f\left((1-\lambda)A+\lambda B\right) \leq (\geq)(1-\lambda)f(A)+\lambda f(B),$$

for all Hermitian matrices A, B with spectra in J and all $0 \leq \lambda \leq 1$. For some fundamental results on operator convex (operator concave) and operator monotone functions, see [2, 4].

It is well known that an operator monotone function on $(0,\infty)$ is necessarily a Bernstein function, [2, Problem V.5.6., page 148].

The following monotonicity property, which follows immediately from the definition, will be useful in our analysis:
If $X \in \mathcal{M}_n$ is Hermitian with spectrum $\sigma(X)$, and f, g are continuous real valued functions on an interval containing $\sigma(X)$, then

$$f(t) \geq g(t),\ t \in \sigma(X) \Rightarrow\ f(X) \geq g(X).$$

For more details about this property, the reader is referred to [4].

2. Quasi-convexity and operator monotone functions

We begin our main results by presenting certain properties of the derivatives of a Bernstein function. In the sequel, the following integral representation of Bernstein functions will be crucial.

Proposition 2.1. [5, Theorem 3.2] *Let $f : (0,\infty) \to \mathbb{R}$ be a Bernstein function. Then positive numbers α, β exist such that*

$$f(x) = \alpha + \beta x + \int_0^\infty \left(1 - e^{-tx}\right) d\mu(t), \tag{2.1}$$

for some positive measure μ on $(0,\infty)$ satisfying

$$\int_0^\infty \min\{1, t\} d\mu(t) < \infty.$$

The first result is about operator quasi-convexity of the derivative of a Bernstein function.

Theorem 2.2. *Let f be a Bernstein function on $(0,\infty)$. Then the norm of $f^{(n)}$ is quasi-convex on M_n^{++}, for $n = 0, 1, 2, \cdots$. That is*

$$\|f^{(n)}((1-\nu)A + \nu B)\| \le \max\{\|f^{(n)}(A)\|, \|f^{(n)}(B)\|\}, \tag{2.2}$$

for all positive definite matrices A, B and $0 \le \nu \le 1$.

In [2, Theorem X.3.4], it is shown that an operator monotone function f on $(0,\infty)$ satisfies

$$|||Df(A)||| \le \|f'(A)\|, A \in \mathcal{M}_n^{++},$$

for any unitarily invariant norm $|||\cdot|||$ on $\mathcal{M}_n$.

The following result extends this inequality to the class of Bernstein functions; a wider class than that of operator monotone functions.

Theorem 2.3. *Let f be a Bernstein function on $(0,\infty)$ and $|||\cdot|||$ be a unitarily invariant norm on $\mathcal{M}_n$. If $A \in \mathcal{M}_n^{++}$, then*

$$|||Df(A)||| \le \|f'(A)\|.$$

Since the operator norm is a unitarily invariant norm, Theorem 2.3 implies the following.

Corollary 2.4. *Let $f : (0,\infty) \to \mathbb{R}$ be a Bernstein function. Then $f \in \mathcal{D}^{(1)}$.*

We emphasize that this corollary extends [2, Exercise X.3.5].

3. Applications

As an important application of the results in this paper, we find bounds for $\|f(B) - f(A)\|$ in terms of $\|B - A\|$, which is one of the central problems in perturbation theory.

Theorem 3.1. *If f is a Bernstein function on $(0,\infty)$. Then, for every unitarily invariant norm $|||.|||$ and every positive definite operators A, B*

$$|||f(B)-f(A)||| \leq \max\{\|f'(A)\|, \|f'(B)\|\}\,|||B-A|||\,. \tag{3.1}$$

If f is an operator monotone function (therefore a Bernstein function) on $(0,\infty)$ and if $A, B \in \mathcal{M}_n^{++}$ are such that $A \geq a1_H$ and $B \geq a1_H$ for the positive number a, then by [2, Theorem X.3.8], the following inequality holds

$$|||f(B)-f(A)||| \leq f'(a)\,|||B-A||| \tag{3.2}$$

for every unitarily invariant norm $|||\cdot|||$ on $\mathcal{M}_n$.

Remark 3.2. Let $A, B \in \mathcal{M}_n^{++}$ and $X \in \mathcal{M}_n$. If f is a completely monotone function on $(0,\infty)$, Aujla in [1, Theorem 2.1] showed that

$$|||f(A)X - Xf(B)||| \leq |f'(m(A,B))|\,|||AX-XB|||\,,$$

for every unitarily invariant norm $|||\cdot|||$ on $\mathcal{M}_n$, where $m(A,B)$ is the smallest among the eigenvalues of A and B.

By the same method used in the proof of [1, Theorem 2.1], we can prove that if $A, B \in \mathcal{M}_n^{++}$, $X \in \mathcal{M}_n$ and f is a Bernstein function on $(0,\infty)$, then

$$|||f(A)X - Xf(B)||| \leq \max\left\{\|f'(A)\|, \|f'(B)\|\right\}|||AX-XB|||\,. \tag{3.3}$$

Since f' is a decreasing function, we have

$$\begin{aligned} f'(m(A,B)) &= f'(\min\{a_0,b_0\}) = \max\{f'(a_0), f'(b_0)\} \\ &= \max\left\{\|f'(A)\|, \|f'(B)\|\right\}. \end{aligned}$$

Consequently, (3.3) is a generalization of [3, Theorem 1].

References

1. J. S. Aujla, *Some norm inequalities for completely monotone functions-II*, Linear Algebra Appl., **359** (2003), 59-65.
2. R. Bhatia, *Matrix analysis*, Springer-Verlag, New York, 1997.
3. A.G. Ghazanfari, *Refined Heinz operator inequalities and norm inequalities*, Oper. Matrices, **15** (1) (2021), 239-252.
4. J. E. Pečarić, T. Furuta, J. Mićić Hot and Y. Seo, *Mond-Pečarić Method in operator inequalities*, Element, Zagreb, 2005.
5. R. L. Schilling, Z. Song and R. Vondraček, *Bernstein Functions Theory and Applications*,Studies in Mathematics 37, De Gruyter, 2010.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

A NEW VERSION OF EKELAND'S VARIATIONAL PRINCIPLE AND EQUILIBRIUM PROBLEMS

M. GHOBADI* AND A.P. FARAJZADEH

Department of Mathematics, Razi University, Iran/67149-Kermanshah
ghobadimahmood83@gmail.com; farajzadehali@gmail.com

Abstract. In this paper, a new version of Ekeland's variational principle by using the concept of τ-distance for bounded from below functions which are not necessarily lower semicontinuous is provided. This new version will be applied to establish an existence theorem for a solution of the quasi-equilibrium problem in the complete metric spaces. Also, we deal with equilibrium problems without convexity assumptions either for the domain or for involved bifunctions. Our approach is based on the concepts of cyclically monotone and cyclically antimonotone for bifunctions.

1. Introduction

The first appearance of equilibrium problems as we understand them now is due to Muu and Oettli. A quasi-equilibrium problem is an equilibrium problem where the feasible set in moving depending on the considered point. This kind of problems encompasses many relevant problems like quasi-optimization problems. Ekeland's variational principle was first expressed by Ekeland, which was inspired by the Bishop-Phelps Theorem provides one of the most powerful tools in modern nonlinear analysis. Convexity and generalized convexity and closure of the set C, monotonity, cyclically

2020 *Mathematics Subject Classification. Primary 47B35; Secondary 30H05*
Key words and phrases. Ekeland's variational principle, Transfer lower continuous, transfer weakly lower continuous, τ-distance.
* Speaker.

monotonity and cyclically antimonotonity, together with some continuity for the function h, were the most used conditions in dealing with the existence of the solutions of equilibrium problems [1, 2]. In order to avoid any assumption of convexity both for the domain C and for the bifunction f, some authors proposed a different approach in which the existence of solutions for equilibrium problem is obtained assuming the inequality $f(x,y) \geq h(y) - h(x)$.

Definition 1.1. [3] Given a nonempty subset C of a topological space X, a function $h : C \to \mathbb{R}$ is said to be:

- **lower semicontinuous** (upper semicontinuous) if for each $x \in C$ and each $\lambda \in \mathbb{R}$ such that $h(x) > \lambda$, there exists a neighborhood V_x of x such that $h(y) > \lambda$, for all $y \in V_x \cap C$ (if $-h$ is lower semicontinuous) or equivalently for any net $\{x_\alpha\} \subset C$, $x_\alpha \to x$ implies $h(x) \leq \liminf_\alpha h(x_\alpha)$ $(h(x) \geq \limsup_\alpha h(x_\alpha))$.
- **transfer lower continuous** (transfer upper continuous) if for each $x, y \in C$ such that $h(x) > h(y)$, there exists $w \in C$ and a neighborhood V_x of x such that $h(z) > h(w)$, for all $z \in V_x \cap C$ (if $-h$ is transfer lower continuous).
- **transfer weakly lower continuous** (transfer weakly upper continuous) if for each $x, y \in C$ such that $h(x) > h(y)$, there exists $w \in C$ and a neighborhood V_x of x such that $h(z) \geq h(w)$, for all $z \in V_x \cap C$ (if $-h$ is transfer weakly lower continuous).

Theorem 1.2. [3] *Let C be a compact and nonempty subset of a topological space X and $h : C \to \mathbb{R}$ be a function. Then, the set* $\arg\min_C h$ *(set* $\arg\max_C h$*) is nonempty and compact if and only if h is transfer lower continuous (transfer upper continuous).*

Theorem 1.3. [3] *Let C be a compact and nonempty subset of a topological space X and $h : C \to \mathbb{R}$ be a function. Then, the set* $\arg\min_C h$ *(set* $\arg\max_C h$*) is nonempty if and only if h is transfer weakly lower continuous (transfer weakly upper continuous).*

Given a nonempty subset C of a topological space X, it is a basic fact from real analysis that every function $h : C \to \mathbb{R}$ (not necessarily lower semicontinuous) admits a lower semicontinuous regularization $\overline{h} : C \to \mathbb{R} \cup \{-\infty\}$ which is defined by $\overline{h}(x) = \liminf_{y \to x} h(y) = \sup_U \inf_{y \in U \cap C} h(y)$, where U runs all neighborhoods of x. It is easy to verify that $\mathrm{Epi}(\overline{h}) := \overline{\mathrm{Epi}(h)}$, the closure in $C \times \mathbb{R}$.

Definition 1.4. [1] For a given nonempty set C, a bifunction $f : C \times C \to \mathbb{R}$ is said to be **cyclically antimonotone** (cyclically monotone) on C if for all $n \in \mathbb{N}$ and all $x_0, x_1, ..., x_n \in C$, with $x_{n+1} = x_0$, $\sum_{i=0}^{n} f(x_i, x_{i+1}) \geq 0$ $(\sum_{i=0}^{n} f(x_i, x_{i+1}) \leq 0)$ or equivalently there exists a function $h : C \to \mathbb{R}$ such that $f(x,y) \geq h(y) - h(x)$ $(f(x,y) \leq h(y) - h(x))$.

Definition 1.5. Let (X,d) be a metric space. Then a function $p : X \times X \to \mathbb{R}$ is called a τ-distance on X if
(1) $p(x,z) \leq p(x,y) + p(y,z) \quad \forall x, y, z \in X$;
moreover, there exists a function $\eta : X \times \mathbb{R}^+ \to \mathbb{R}^+$ which is concave and continuous in its second argument and satisfying the following conditions:
(2) $\eta(x,0) = 0$ and $\eta(x,t) \geq t \quad \forall x \in X,\, t \in \mathbb{R}^+$;
(3) $\lim_n x_n = x$ and $\limsup_n\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply

$$p(w,x) \leq \liminf_n p(w,x_n) \quad \forall w \in X;$$

(4) $\limsup_n\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$;
(5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

Note that if d is a meter then it is τ-distance, by taking $p = d$ and $\eta(x,t) = t$.

2. Equilibrium and Quasi-equilibrium Problems

Let C be a nonempty subset of a topological space X and $f : C \times C \to \mathbb{R}$, be a given bifunction. We denote by $\mathrm{EP}(f,C)$ ($\mathrm{MEP}(f,C)$) the solution set of the equilibrium problem (Minty equilibrium problem)

$$\text{Find } x \in C \text{ such that } f(x,y) \geq 0 \ (f(y,x) \leq 0),\ \forall\, y \in C.$$

Remark 2.1. If $f(x,y) \leq h(y) - h(x)$ for some function $h : C \to \mathbb{R}$ (f is cyclically monotone), then

$$\mathrm{EP}(f, C) \subset \arg\min_C h \subset \mathrm{MEP}(f, C). \tag{2.1}$$

Given a subset C of a complete metric space (X,d), a bifunction $f : C \times C \to \mathbb{R}$ and a set-valued mapping $K : C \rightrightarrows C$, we denote by $\mathrm{MQEP}(f,K)$ the solution set of the so-called Minty quasi-equilibrium problem

$$\text{Find } x \in C \text{ such that } x \in K(x) \ \wedge \ f(y,x) \leq 0,\ \forall y \in K(x).$$

Remark 2.2. Given f and K as the above, If $f(x,y) \leq h(y) - h(x)$ for some function $h : C \to \mathbb{R}$ (in other words, f is cyclically monotone), then

$$\arg\min_C h \cap \mathrm{Fix}(K) \subset MQEP(f,K). \tag{2.2}$$

3. Main results

The next result provides an existence result for MQEP and it extends Theorem 4.2 [1].

Theorem 3.1. *Let C be a nonempty closed subset of a complete metric space (X,d), let $K : C \rightrightarrows C$ be a set-valued mapping, and $f : C \times C \to \mathbb{R}$ be a bifunction. Assume that there exists a τ-distance p on X. Assume that the following conditions hold.*
(i) $\mathrm{Fix}(K)$ *is compact and nonempty;*

(ii) there exists a bounded from below function $h : C \to \mathbb{R}$ such that $\arg\min_C h = \arg\min_C \overline{h}$*, in other words, h is transfer lower continuous ([1], Proposition 2.3) and $f(x,y) \leq h(y) - h(x),\ \forall x,y \in C$. Suppose that for each $\varepsilon > 0$ and each $x_0 \in X$ with $p(x_0,x_0) = 0$ the following implication holds: for all $x \in C$, $\overline{h}(x) + \varepsilon p(x,x_0) \leq h(x_0) \Longrightarrow \exists y \in K(x), h(y) + \varepsilon p(x,y) \leq \overline{h}(x)$. Then, the set MQEP$(f,K)$ is nonempty.*

Theorem 3.2. *Let C be a compact and nonempty subset of a topological space X and $f : C \times C \to \mathbb{R}$ be a bifunction. If there exists a transfer weakly lower continuous function $h : C \to \mathbb{R}$ with*

$$f(x,y) \geq h(y) - h(x),\ \forall x,y \in C.$$

then, the set EP(f,C) is nonempty.

Corollary 3.3. *Let C be a compact and nonempty subset of a topological space X and $f : C \times C \to \mathbb{R}$ be a bifunction. If there exists a transfer weakly lower continuous function $h : C \to \mathbb{R}$ with*

$$f(x,y) \leq h(y) - h(x),\ \forall x,y \in C.$$

then, the solution set of MEP(f,C) is nonempty.

Now, we provide a link between the set of solutions of the equilibrium problem and of the Minty equilibrium problem.

(A) Let C be a nonempty subset of a topological space X and $f : C \times C \to \mathbb{R}$ be a bifunction. for all $x,y \in C$ and U a neighborhood of x, there exist $z \in U$ and $t \in (0,1]$ such that $tf(z,y) + (1-t)f(z,x) \geq 0$.

Theorem 3.4. [2] *Let C be a topological space and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying condition (A). Assume that, for each $y \in C$, the set $F(y) = \{x \in C : f(x,y) < 0\}$ is open. Then, every solution of the Minty equilibrium problem is a solution of the equilibrium problem.*

Theorem 3.5. *Let C be a compact and nonempty subset of a topological space X and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying condition (A). Assume that*
(i) there exists a transfer weakly lower continuous function $h : C \to \mathbb{R}$ with $f(x,y) \leq h(y) - h(x),\ \forall x,y \in C$;
(ii) f is upper semicontinuous with respect to its first argument.
Then, the equilibrium problem has a solution.

References

1. J. Cotrina, M. Thera and J. Zuniga, *An Existence Result for Quasi-equilibrium problems via Ekeland's Variational Principle*, J. Optim Theory Appl, **187** (2020), 336–355.
2. M. Giuli, *Cyclically monotone equilibrium problems and Ekeland's principle*, Decis. Econ. Finance, **40** (2017), 231–242.
3. G. Tian and J. Zhou, *Transfer continuities, generalizations of the Weierstrass and maximum theorems*, J. Math. Econ, **24** (1995), 281-303.

NEW GENERALIZATION OF ISHIKAWA ALGORITHM FOR FINDING THE FIXED POINT

M. R. HADDADI*

Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran
haddadi@abru.ac.ir

Abstract. In this paper, In the following we give an iterative algorithm for fixed point problems in the Banach space. We show that the iterative algorithm of this paper is under weaker conditions and stronger results with respect to Ishikawa algorithm.

1. Introduction

Ishikawa's algorithm supports an iterative approach to problem-solving. This iterative process is particularly important for fixed point problems, as they may involve multiple feedback loops or recursive relationships that require repeated analysis and refinement of the problem structure. Ishikawa's algorithm focuses on identifying the root causes of the fixed point problem, rather than just addressing the symptoms. Hence, in this paper, we give a new generalization of Ishikawa algorithm for finding the fixed point.

Let A be a nonempty subset of a Banach spaces X. We recall that a self-mapping $T : A \to A$ is said to be *nonexpansive* provided that $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in A$. It was announced that if X is a uniformly convex Banach space and A is a bounded, closed and convex subset of X, then the nonexpansive mapping T has a fixed point.

2020 *Mathematics Subject Classification. Primary 90C48; Secondary 47H09*

Key words and phrases. Fixed point; Uniformly convex Banach space; Iterative sequence.

* Speaker.

Lemma 1.1. *([3]) Let $\{a_n\} \subset [0, \infty)$, $\{b_n\} \subset [0, \infty)$ and $\{c_n\} \subset [0, 1)$ be sequences of real numbers such that $a_{n+1} \leq (1 - c_n)a_n + b_n$ $\forall n \in \mathbb{N}$ and*

$$\sum_{n=1}^{\infty} c_n = \infty, \sum_{n=1}^{\infty} b_n < \infty.$$

Then, $\lim_{n\to\infty} a_n = 0$.

Theorem 1.2. [4] *If E is a convex compact subset of a Hilbert space H, T is a Lipschitzian pseudo-contractive map from E into itself and x, is any point in E, then the sequence $\{x_n\}$ converges strongly to a fixed point of T, where x_n is defined iteratively for each n $\mathbb{N}$ by*

$$y_n = \alpha_n T x_n + (1 - \alpha_n)x_n,$$
$$x_{n+1} = \beta_n T y_n + (1 - \beta_n)y_n, \ n \in \mathbb{N}, x_0 \in X,$$

where $0 \leq \alpha_n \leq \beta_n \leq 1$ for all n, $\lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

2. Main Results

In the following we give an iterative algorithm for fixed point problems in the Banach space. We show that the iterative algorithm of this paper is under weaker conditions and stronger results with respect to Ishikawa algorithm.

We denote by $\mathcal{A}$ the family of functions $\Theta : X \to (0, \infty)$ so that:
(i) If $\Theta(x) = \Theta(y)$, then $x = y$;
(iii) for each sequence $\{\rho_n\} \subseteq X$, $\lim\limits_{n\to\infty} \Theta(\rho_n) = \Theta(\rho)$ iff $\lim\limits_{n\to\infty} \rho_n = \rho$;

Example 2.1. Let X be a norm space. It is clear that $f(t) = e^{\|t\|}$ is an element of $\mathcal{A}$. Other examples are $f(t) = e^{-\|t\|}$, $f(t) = \cosh \|t\|$, $f(t) = \frac{2\cosh \|t\|}{1+\cosh \|t\|}$, $f(t) = 1 + \ln(1 + \|t\|)$, $f(t) = \frac{2+2\ln(1+\|t\|)}{2+\ln(1+\|t\|)}$ and $f(t) = e^{\|t\| e^{\|t\|}}$.

A function $\natural|.\natural : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as follows

$$\natural a \natural = \begin{cases} a & \text{if } a \geq 1, \\ \dfrac{1}{a} & \text{if } a < 1. \end{cases}$$

Definition 2.2. Let X be a vector space and $F : X \to X$ such that

$$\natural \frac{\Theta(Fx)}{\Theta(Fy)} \natural \leq \natural (\frac{\Theta(x)}{\Theta(y)}) \natural^{\alpha}, \ x, y \in X$$

where $\alpha \in (0, 1)$. Then we say F is a Θ-contraction mapping.

Theorem 2.3. *Let X be a Banach space. and let $F : X \to X$ be a Θ-contraction. Then F has a unique fixed point $u \in X$. Furthermore, for any $x \in X$ we have*

$$\lim_{n\to\infty} F^n(x) = u$$

with

$$\natural \Theta(F^n x)\Theta(u)^{-1} \natural \leq \natural \Theta(x)(\Theta(Fx))^{-1} \natural^{\frac{\alpha^n}{1-\alpha}}.$$

Theorem 2.4. *Let $\mathcal{X}$ be a Banach space. A mapping $F : \mathcal{X} \to \mathcal{X}$ be an Θ-contraction mapping, $0 < \alpha \leq 1$ and $x_0 \in \mathcal{X}$. We define a sequence $\{x_n\} \subseteq \mathcal{X}$ by*

$$\Theta(y_n) = \Theta(Fx_n)^{\alpha_n}\Theta(x_n)^{1-\alpha_n},$$
$$\Theta(x_{n+1}) = \Theta(Fy_n)^{\beta_n}\Theta(y_n)^{1-\beta_n},$$

where $n \in \mathbb{N}$, $0 \leq \alpha_n \leq \beta_n \leq 1$, $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}$ is convergent strongly to $p \in \mathcal{X}$ such that $Fp = p$.

The Theorem 2.4 is under weaker conditions and stronger results with respect to Theorem 1.2. Because (i) It does not need Hilbert space and (ii) the condition $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$ replaced by $\sum_{n=1}^{\infty} \alpha_n = \infty$. On the other hand, we show in the next section that the algorithm of Theorem 2.4 has more convergence rate with respect to the Ishikawa algorithm

Now, in the following we will consider and compare two algorithms in the following by an example. Algorithm 1 is the Ishikawa algorithm defined by

$$y_n = \alpha_n Fx_n + (1 - \alpha_n)x_n,$$
$$x_{n+1} = \beta_n Fy_n + (1 - \beta_n)y_n,$$

algorithm 2 is the algorithm of Theorem 2.4 defined by

$$\Theta(y_n) = \Theta(Fx_n)^{\alpha_n}\Theta(x_n)^{1-\alpha_n},$$
$$\Theta(x_{n+1}) = \Theta(Fy_n)^{\beta_n}\Theta(y_n)^{1-\beta_n},$$

where $n \in \mathbb{N}$, $x_0 \in A$, $0 \leq \alpha_n \leq \beta_n \leq 1$, $\lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

References

1. M. R. Haddadi, *Further generalization of the cyclic contraction for the best proximity point problem*, I. J. of Nonlinear Anal. and Appl., **14** (9), 379-284.
2. N. Hussain, V. Parvaneh, B. Samet and C. Vetro, *Some fixed point theorems for generalized contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2015** **185** (2015).
3. H. K. Xu, *An iterative approach to quadratic optimization*, J. Optim. Theory Appl., **116** (2003) 659-678.
4. S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., **44** (1974).

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

EMPLOYING HYBRID FIXED POINT THEOREM OF DHAGE TO EXISTENCE OF SOLUTION FOR A COUPLE SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

ALIREZA HATAMI*

Department of Mathematics, Faculty of Science, Bu-Ali Sina University, Hamedan, Iran
a.hatami@sci.basu.ac.ir

Abstract. In this research, we study the existence results for a couple system of boundary value problem with fractional order. The hybrid fixed point theorem of Dhage for the sum of three operators are used for proving the main results. An example is provided to illustrate the theory.

1. Introduction

Fractional differential equations (FDEs) have been got proper attention from many researchers because of their usability to model complex phenomena of real world process [1, 5]. Due to the large numbers of applications of FDEs in sciences and engineering, plenty of research papers have been written in this area.

2020 *Mathematics Subject Classification. Primary 47B35; Secondary 30H05*
Key words and phrases. Banach algebra, Fractional differential equations, hybrid fixed point theorem.
* Speaker.

This paper deals, with the existence and uniqueness of solutions for boundary-value problem (BVP) of FDE

$$\begin{cases} {}^{\mathrm{C}}\mathcal{D}^{\alpha}\left[\frac{{}^{\mathrm{C}}\mathcal{D}^{\beta}x(t)-f(t,\mathcal{I}^{\nu}x(t))}{\Psi(t,\mathcal{I}^{\nu-1}x(t))}\right]=g\left(t,\mathcal{I}^{\nu}x(t)\right), \\ {}^{\mathrm{C}}\mathcal{D}^{\delta}\left[\frac{{}^{\mathrm{C}}\mathcal{D}^{\gamma}y(t)-h(t,\mathcal{I}^{\eta}y(t))}{\Phi(t,\mathcal{I}^{\eta-1}x(t))}\right]=z\left(t,\mathcal{I}^{\eta}y(t)\right), \\ x(0)=0,\ \left[\frac{{}^{\mathrm{C}}\mathcal{D}^{\beta}x(t)-f(t,\mathcal{I}^{\nu}x(t))}{\Psi(t,\mathcal{I}^{\nu-1}x(t))}\right]_{t=0}=0, \\ y(0)=0,\ \left[\frac{{}^{\mathrm{C}}\mathcal{D}^{\gamma}y(t)-h(t,\mathcal{I}^{\eta}y(t))}{\Phi(t,\mathcal{I}^{\eta-1}x(t))}\right]_{t=0}=0, \end{cases} \tag{1.1}$$

for $t\in \mathrm{J}:=[0,1]$, Where ${}^{\mathrm{C}}\mathcal{D}^{\alpha}, {}^{\mathrm{C}}\mathcal{D}^{\beta}, {}^{\mathrm{C}}\mathcal{D}^{\delta}$ and ${}^{\mathrm{C}}\mathcal{D}^{\gamma}$ denote the Caputo fractional derivatives of orders α,β,δ and γ, respectively, $\alpha,\beta\in(1,2],\delta,\gamma\in(0,1]$, $\mathcal{I}^{\nu},\mathcal{I}^{\eta}$ denote the Riemann-Liouville fractional integrals of orders ν,η, respectively, $\nu\in(1,2]$ and $\eta\in(2,3]$, $f\big(0,\mathcal{I}^{\nu}x(0)\big)=0, h\big(0,\mathcal{I}^{\eta}x(0)\big)=0, \Phi,\Psi\in C\big(\mathrm{J}\times\mathbb{R},\mathbb{R}\setminus\{0\}\big)$ and $f,g,h,z\in C\left(\mathrm{J}\times\mathbb{R}\right)$. We place the BVP of coupled in banach space $C\big(\mathrm{J},\mathbb{R}\big)$.

2. Main results

In this section, we recall some definitions.

Definition 2.1 ([3]). The Riemann-Liouville fractional integral of order $\alpha>0$ for a continuous function $f:(0,\infty)\to\mathbb{R}$ is given by

$$\mathcal{I}^{\alpha}x(t)=\tfrac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-\eta)^{\alpha-1}x(\eta)\,\mathrm{d}\eta, \tag{2.1}$$

Definition 2.2 ([3]). T1he Caputo fractional derivative of order $\alpha>0$ for a continuous function $f:(0,\infty)\to\mathbb{R}$ defined by

$${}^{\mathrm{C}}\mathcal{D}^{\alpha}x(t)=\tfrac{1}{\Gamma(n-\alpha)}\int_0^{\tau}\tfrac{x(\eta)^{(n)}}{(\tau-\eta)^{\alpha-n+1}}\mathrm{d}\eta, \tag{2.2}$$

Lemma 2.3 ([3]). *Let $\alpha,\beta\geq 0, x\in C[a,b]$ then for all $t\in[a,b]$, we have* (a) $\mathcal{I}^{\alpha}\mathcal{I}^{\beta}x(t)=\mathcal{I}^{\alpha+\beta}x(t)$; (b) ${}^{C}\mathcal{D}^{\alpha}\mathcal{I}^{\alpha}x(t)=x(t)$; (c) ${}^{C}\mathcal{D}^{\alpha}\mathcal{I}^{\beta}x(t)=\mathcal{I}^{\beta-\alpha}x(t)$.

Lemma 2.4 ([4]). *The following result holds for fractional differential equations: $\mathcal{I}^{\nu}{}^{C}\mathcal{D}\varsigma(\omega)=\varsigma(\omega)+\lambda_0+\lambda_1\omega+\lambda_2\omega^2+\cdots+\lambda_{n-1}\omega^{n-1}$, for $\lambda_i\in\mathbb{R}, i=1,2,\cdots,n-1$.*

Lemma 2.5 ([4]). *The general solution of the fractional differential equation ${}^{C}\mathcal{D}x(t)=0$ is $x(t)=\lambda_0+\lambda_1 t+\lambda_2 t^2+\cdots+\lambda_{n-1}t^{n-1}$, where $\lambda_i\in\mathbb{R}, i=1,2,\cdots,n-1$.*

Lemma 2.6. *The unique solution of the hybrid fractional integro-differential problem 1.1 is given by*

$$x(t) = \int_0^t \frac{(t-\omega)^{\beta-1}}{\Gamma(\beta)} \Psi\left(\omega, \mathcal{I}^{\nu-1}x(\omega)\right) \left(\int_0^\tau \frac{g(s,\mathcal{I}^\nu x(s))}{\Gamma(\alpha)}(\tau-s)^{\alpha-1}\mathrm{d}s\right)\mathrm{d}\omega + \int_0^t \frac{f(\omega,\mathcal{I}^\nu x(\omega))}{\Gamma(\beta)}(t-\omega)^{\beta-1}\mathrm{d}\omega$$

and

$$y(t) = \int_0^t \frac{(t-\omega)^{\gamma-1}}{\Gamma(\gamma)} \Phi\left(\omega, \mathcal{I}^{\eta-1}x(\omega)\right) \left(\int_0^\tau \frac{z(s,\mathcal{I}^\eta y(s))}{\Gamma(\delta)}(\tau-s)^{\delta-1}\,\mathrm{d}s\right)\mathrm{d}\omega + \int_0^t \frac{h(\omega,\mathcal{I}^\eta y(\omega))}{\Gamma(\gamma)}(t-\omega)^{\gamma-1}\mathrm{d}\omega$$

The following hybrid fixed point theorem of Dhage for three operators in a Banach algebra, due to Dhage, will be used to prove the existence result for 1.1.

Theorem 2.7 (hybrid fixed point theorem [2])**.** *Let S be a nonempty, closed convex and bounded subset of a Banach algebra E and let $\mathcal{A},\mathcal{C} : E \to E$ and $\mathcal{B} : S \to E$ be three operators satisfying:*

(a) *$\mathcal{A},\mathcal{C}$ are Lipschitzian with Lipschitz constants L_A and L_C, respectively,*
(b) *$\mathcal{B}$ is compact and continuous,*
(c) *$x = \mathcal{A}x\mathcal{B}y + \mathcal{C}x \Rightarrow x \in S, \forall y \in S$, and*
(d) *$L_\mathcal{A}M_\mathcal{B} + L_\mathcal{C} < 1$, where $M_\mathcal{B} = \|\mathcal{B}(S)\| = \sup\{\|\mathcal{B}x\| : x \in S\}$.*

Then the operator equation $x = \mathcal{A}x\mathcal{B}y + \mathcal{C}x$ has a solution in S.

We need the following assumptions in the sequel.

(H_1) The functions $f, g, h, z : \mathrm{J} \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist two positive functions ϕ, ψ, Ξ, ζ with bound $\|\phi\|, \|\psi\|, \|\Xi\|$ and $\|\zeta\|$, respectively, s.t

$$\left|f\big(t,x(t)\big) - f\big(t,\bar{x}(t)\big)\right| \le \phi(t)|x(t)-\bar{x}(t)|,$$
$$\left|g\big(t,x(t)\big) - g\big(t,\bar{x}(t)\big)\right| \le \psi(t)|x(t)-\bar{x}(t)|,$$
$$\left|h\big(t,x(t)\big) - h\big(t,\bar{x}(t)\big)\right| \le \Xi(t)|x(t)-\bar{x}(t)|,$$

and $\left|z\big(t,x(t)\big) - z\big(t,\bar{x}(t)\big)\right| \le \zeta(t)|x(t)-\bar{x}(t)|$, for $t \in \mathrm{J}$ and $x, y \in \mathbb{R}$.

(H_2) We have

$$\|\mathcal{A}x - \mathcal{A}\bar{x}\| \le \left(\frac{\|\psi\|}{\Gamma(\nu+1)\Gamma(\alpha+1)} + \frac{\|\zeta\|}{\Gamma(\eta+1)\Gamma(\delta+1)}\right)\|x-\bar{x}\|,$$
$$\|\mathcal{C}x - \mathcal{C}\bar{x}\| \le \left(\frac{\|\phi\|}{\Gamma(\nu+1)\Gamma(\gamma+1)} + \frac{\|\Xi\|}{\Gamma(\eta+1)\Gamma(\delta+1)}\right)\|x-\bar{x}\|,$$

(H_3) There exist a function $p \in C\big(\mathrm{J},\mathbb{R}^+\big)$ and a continuous nondecreasing function $\Lambda : [0,\infty) \to (0,\infty)$ s.t $|\Psi\big(t,x(t)\big)| \le p(t)\Lambda(|x|)$ and $|\Phi\big(t,x(t)\big)| \le p(t)\Lambda(|x|)$.

Theorem 2.8 (Existence)**.** *Assume that hypotheses* (H_1)*-*(H_3) *holds. Then the* BVP 1.1 *has a solution defined on* J.

Example 2.9. Consider the following BVP

$$\begin{cases} {}^{\mathrm{C}}\mathcal{D}^{3/2}\left[\frac{{}^{\mathrm{C}}\mathcal{D}^{3/2}x(t)-f\left(t,\mathcal{I}^{3/2}x(t)\right)}{\Psi\left(t,\mathcal{I}^{1/2}x(t)\right)}\right]=\frac{e^{-t}\sin|\mathcal{I}^{3/2}x(t)|}{\sqrt{81}+215}, \\ {}^{\mathrm{C}}\mathcal{D}^{1/2}\left[\frac{{}^{\mathrm{C}}\mathcal{D}^{1/2}y(t)-h\left(t,\mathcal{I}^{5/2}y(t)\right)}{\Phi\left(t,\mathcal{I}^{3/2}x(t)\right)}\right]=\frac{e^{-t}\cos|\mathcal{I}^{1/2}x(t)|}{80}, \\ x(0)=0,\ \left[\frac{{}^{\mathrm{C}}\mathcal{D}^{3/2}x(t)-f\left(t,\mathcal{I}^{3/2}x(t)\right)}{\Psi\left(t,\mathcal{I}^{1/2}x(t)\right)}\right]_{t=0}=0, \\ y(0)=0,\ \left[\frac{{}^{\mathrm{C}}\mathcal{D}^{1/2}y(t)-h\left(t,\mathcal{I}^{5/2}y(t)\right)}{\Phi\left(t,\mathcal{I}^{3/2}x(t)\right)}\right]_{t=0}=0, \end{cases} \tag{2.3}$$

Clearly, $\alpha=\beta=\frac{3}{2}\in(1,2]$, $\delta=\gamma=\frac{1}{2}\in(0,1]$, $\nu=\frac{3}{2}\in(1,2]$ and $\eta=\frac{5}{2}\in(2,3]$. We take $g\big(t,\mathcal{I}^{\nu}x(t)\big)=\frac{e^{-t}\cos|\mathcal{I}^{\frac{3}{2}}x(t)|}{\sqrt{81}+215}$, $\Phi\big(t,\mathcal{I}^{3/2}x(t)\big)=\frac{t}{60}\sin|\mathcal{I}^{\frac{3}{2}}x(t)|$, $f\big(t,\mathcal{I}^{\nu}x(t)\big)=\frac{e^{-5t}\sin|\mathcal{I}^{\frac{3}{2}}x(t)|}{10}$, $h\big(t,\mathcal{I}^{\eta}y(t)\big)=\frac{e^{-5t}\sin|\mathcal{I}^{\frac{1}{2}}y(t)|}{50}$, $z\big(t,\mathcal{I}^{\nu-1}x(t)\big)=\frac{e^{-t}\cos|\mathcal{I}^{\frac{3}{2}}x(t)|}{80}$ and $\Psi\big(t,\mathcal{I}^{1/2}x(t)\big)=\frac{t}{100}\sin|\mathcal{I}^{\frac{1}{2}}x(t)|$.

References

1. B Ahmad, K. S. Ntouyas and J. Tariboon, *On hybrid Caputo fractional integro-differential inclusions with nonlocal conditions*, Boundary Value Problems, **9** (6) (2016), 4235–4246.
2. B. C. Dhage, *A Fixed Point Theorem in Banach Algebras Involving Three Operators with Applications*, Kyungpook mathematical journal, **44** (1) (2004), 145–155.
3. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 2006, 204., Elsevier Science B.V. Amsterdam.
4. L. Podlubny, *Fractional differential equations*, Academic Press, 1999, New York.
5. S.Sitho, K. S. Ntouyas and J. Tariboon, *Existence results for hybrid fractional integro-differential equations,* Boundary Value Problems, **113** (2015).

SOME RESULTS ON E-FRAMES IN HILBERT SPACES

HASSAN HEDAYATIRAD* AND TAYEBE LAL SHATERI

Department of Mathematics, Hakim Sabzevari University, Sabzevar, Iran
hasan.hedayatirad@hsu.ac.ir; t.shateri@hsu.ac.ir

Abstract. For a separable Hilbert space $\mathcal{H}$, an E-frame is a sequence that its E-transform is a frame for $\mathcal{H}$ where E is an invertible infinite matrix mapping on the Hilbert space $\bigoplus_{n=1}^{\infty} \mathcal{H}$. This is a generalization of the notion of frames for $\mathcal{H}$. In this paper, we have stated some results about this concept. Specially we show that an ordinary frame can be an E-frame under certain conditions.

1. Introduction

The concept of frames for Hilbert spaces was first introduced by Duffin and Schaeffer [4]. It reintroduced by Daubechies, Grossman, and Meyer and attracts mathematicians from then on [3]. Frames have very important properties which makes them very useful in several area of mathematics, physics, and engineering. Many generalizations of frames have been published so far. In one of this generalizations, Talebi and Dehghan [5] have introduced the concept of E-frames for a separable Hilbert space and study some properties of them.

In this paper, we give some results on E-frames. Throughout this paper, we assume that $\mathcal{H}$ is a separable Hilbert space. A countable family $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such

2020 *Mathematics Subject Classification. Primary 47B35; Secondary 30H05*
Key words and phrases. Frame, E-frame, Hilbert space, Matrix mapping.
* Speaker.

that

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} \left|\left\langle f, f_k \right\rangle\right|^2 \leq B\|f\|^2 \qquad (f \in \mathcal{H}). \tag{1.1}$$

A and B are called the frame bounds.

For the sequence $(\mathcal{H}_n)_{n=1}^{\infty}$ of separable Hilbert spaces, suppose that

$$\bigoplus_{n=1}^{\infty} \mathcal{H}_n = \left\{ \{f_n\}_{n=1}^{\infty} | f_n \in \mathcal{H}_n, \sum_{n=1}^{\infty} \|f_n\|^2 < \infty \right\}.$$

We can define a well defined inner product $\langle ., . \rangle$ on $\bigoplus_{n=1}^{\infty} \mathcal{H}_n$ defined by

$$\left\langle \{f_n\}_{n=1}^{\infty}, \{g_n\}_{n=1}^{\infty} \right\rangle = \sum_{n=1}^{\infty} \left\langle f_n, g_n \right\rangle.$$

It is well known that $\bigoplus_{n=1}^{\infty} \mathcal{H}_n$ is a Hilbert space with respect to this inner product which is called the Hilbert space direct sum of $(\mathcal{H}_n)_{n=1}^{\infty}$ [2]

Let $\mathcal{X}$ and $\mathcal{Y}$ be two sequence spaces and $E = (E_{n,k})_{n,k \geq 1}$ be an infinite matrix of real or complex numbers. We say that E defines a matrix mapping from $\mathcal{X}$ into $\mathcal{Y}$, if for every sequence $x = \{x_n\}_{n=1}^{\infty}$ in $\mathcal{X}$, the sequnce $Ex = \{(Ex)_n\}_{n=1}^{\infty}$ is in $\mathcal{Y}$, where

$$(Ex)_n = \sum_{k=1}^{\infty} E_{n,k} x_k, \quad n = 1, 2, \ldots.$$

Using the matrix mapping concept, the authors in [5], have introduced a new notion of frames which is called E-frame. Let E be an invertible infinite matrix mapping on $\bigoplus_{n=1}^{\infty} \mathcal{H}$. Then for each $\{f_k\}_{k=1}^{\infty} \in \bigoplus_{n=1}^{\infty} \mathcal{H}$,

$$E\{f_k\}_{k=1}^{\infty} = \left\{ \sum_{k=1}^{\infty} E_{n,k} f_k \right\}_{n=1}^{\infty}.$$

Definition 1.1. [5] The sequence $\{f_k\}_{k=1}^{\infty}$ is called an E-frame for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} \left|\left\langle f, \left(E\{f_k\}\right)_n \right\rangle\right|^2 \leq B\|f\|^2 \qquad (f \in \mathcal{H}). \tag{1.2}$$

If just the right inequality in (1.2) holds, then $\{f_k\}_{k=1}^{\infty}$ is called an E-Bessel sequence for $\mathcal{H}$ with E-Bessel bound B. One can easily check that for given constant $B > 0$, the sequence $\{f_k\}_{k=1}^{\infty}$ is an E-Bessel sequence if and only if the operator T_E defined by

$$T_E : \ell^2(\mathbb{N}) :\longrightarrow \mathcal{H}\, , T\{c_k\}_{k=1}^{\infty} = \sum_{n=1}^{\infty} c_k \left(E\{f_k\}_{k=1}^{\infty}\right)_n$$

is a bounded operator from $\ell^2(\mathbb{N})$ in to $\mathcal{H}$ with $\|T_E\| \leq \sqrt{B}$. We call T_E the pre E-frame operator. Its adjoint, the analysis operator, is given by

$$T_E^* : \mathcal{H} \longrightarrow \ell^2(\mathbb{N})\, , T_E^* f = \left\{ \left\langle f, \left(E\{f_k\}_{k=1}^{\infty}\right)_n \right\rangle \right\}_{n=1}^{\infty}. \tag{1.3}$$

Composing T_E and T_E^*, the E-frame operator

$$S_E : \mathcal{H} \longrightarrow \mathcal{H}\,, S_E f = \sum_{n=1}^{\infty} \langle f, \big(E\{f_k\}_{k=1}^{\infty}\big)_n \rangle \big(E\{f_k\}_{k=1}^{\infty}\big)_n$$

is obtained. S_E is bounded, invertible, self-adjoint and positive [5]. This leads us to the following reconstruction formulas for all $f \in \mathcal{H}$

$$f = \sum_{n=1}^{\infty} \langle f, \big(E\{S_E^{-1} f_k\}_{k=1}^{\infty}\big)_n \rangle \big(E\{f_k\}_{k=1}^{\infty}\big)_n \tag{1.4}$$

$$f = \sum_{n=1}^{\infty} \langle f, \big(E\{f_k\}_{k=1}^{\infty}\big)_n \rangle \big(E\{S_E^{-1} f_k\}_{k=1}^{\infty}\big)_n,$$

2. Main results

Theorem 2.1. *Let $\{f_k\}_{k=1}^{\infty}$ be an E-frame for $\mathcal{H}$ with bounds A and B. Assume that $D : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded invertible operator on $\mathcal{H}$. Then $\{Df_k\}_{k=1}^{\infty}$ is an E-frame with bounds $A\|D^{-1}\|^{-2}$ and $B\|D\|^2$.*

Lemma 2.2. *If $\{f_k\}_{k=1}^{\infty}$ belongs to $\bigoplus_{n=1}^{\infty} \mathcal{H}$, then it is a Bessel sequence.*

Remark 2.3. Suppose that $E = (E_{n,k})_{n,k\geq 1}$ is an infinite complex matrix which defines an operator on $\ell^2(\mathbb{N})$ equipped with the condition that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |E_{n,k}|^2 < \infty. \tag{2.1}$$

First note that for any sequence $\{c_k\}_{k=1}^{\infty}$ in $\ell^2(\mathbb{N})$,

$$\|E\{c_k\}_{k=1}^{\infty}\|_{\ell^2}^2 = \sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} E_{n,k} c_k \right|^2 \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |E_{n,k}|^2 \sum_{k=1}^{\infty} |c_k|^2 .$$

Hence 2.1 implies that E is bounded. This makes that $\{E_{k,n}\}_{k=1}^{\infty}$ and $\{E_{n,k}\}_{k=1}^{\infty}$ belongs to $\ell^2(\mathbb{N})$ for all $n \in \mathbb{N}$.
Now we define a matrix mapping $\tilde{E}$ on $\bigoplus_{n=1}^{\infty} \mathcal{H}$ using E as follow

$$\tilde{E} : \bigoplus_{n=1}^{\infty} \mathcal{H} \longrightarrow \bigoplus_{n=1}^{\infty} \mathcal{H} \quad ; \quad \tilde{E}\{f_k\}_{k=1}^{\infty} = \left\{ \sum_{k=1}^{\infty} E_{n,k} f_k \right\}_{n=1}^{\infty} . \tag{2.2}$$

The above discussion together with Lemma 2.2 and [1, Corollary 3.2.5] makes $\tilde{E}$ well-defined. Moreover for any sequence $\{f_k\}_{k=1}^{\infty}$ in $\bigoplus_{n=1}^{\infty} \mathcal{H}$ we see by assumption that

$$\sum_{n=1}^{\infty} \left\| \sum_{k=1}^{\infty} E_{n,k} f_k \right\|^2 \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |E_{n,k}|^2 \sum_{k=1}^{\infty} \|f_k\|^2 < \infty.$$

Thus $\tilde{E}\{f_k\}_{k=1}^{\infty}$ belongs to $\bigoplus_{n=1}^{\infty} \mathcal{H}$ and our aim is achieved. Therefore any infinite complex matrix which defines an operator on $\ell^2(\mathbb{N})$ and satisfies 2.1 induces a matrix mapping on $\bigoplus_{n=1}^{\infty} \mathcal{H}$.

Motivated by Remark 2.3, in the next result, we prove a classical frame can be an E-frame.

Theorem 2.4. *Suppose that $E = (E_{n,k})_{n,k\geq 1}$ is an infinite invertible complex matrix which defines an operator on $\ell^2(\mathbb{N})$ and satisfies 2.1. Suppose that $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with bounds A and B. Then $\{f_k\}_{k=1}^{\infty}$ is an $\tilde{E}$-frame with bound $\|E\|^2 B$ and CA, for some $C > 0$.*

Theorem 2.5. *Let $\{f_k\}_{k=1}^{\infty}$ be a Bessel sequence on $\mathcal{H}$. Suppose that $E = (E_{n,k})_{n,k\geq 1}$ is an infinite complex matrix satisfying the conditions of theorem 2.4. Then $\{f_k\}_{k=1}^{\infty}$ is an $\tilde{E}$-Bessel sequence if and only if $\overline{E}(T^* f) \in \ell^2(\mathbb{N})$, where T^* is the analysis operator of $\{f_k\}_{k=1}^{\infty}$.*

References

1. O. Christensen, *An introduction to frames and Riesz bases, Birkhauser, Boston, 2016.*
2. *J. B. Conway, A Course in Functional Analysis, 2nd Edition, Springer-Verlag, 1990.*
3. *I. Daubechies, A. Grassman and Y. Meyer, Painless nonothogonal expanisions, J. Math. Phys., **27** (1986), 1271–1283.*
4. *R.J. Duffin and A.C. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc., **72** (1952), 341–366.*
5. *G. Talebi, M. A. Dehghan, On E-frames in separable Hilbert spaces, Banach J. Math. Anal. 9(3) (2015), 43–74.*

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

EXISTENCE FIXED POINT THEOREMS FOR MAPPINGS IN CONTROLLED METRIC TYPE SPACES

PARASTOO HEIATIAN NAEINI*

Department of Mmathematics, Payame Noor University , P. O. Box 19395-4697, Tehran, Iran
parastoo.heiatiannaeini@pnu.ac.ir

Abstract. In this article we want to take advantage of the notion of CMTS to present new contractive conditions and making use of our new contractions to formulate new results related to FP of a mapping that satisfies a set of conditions.

1. Introduction

The fixed point (FP) theory technique is widely used by scientists to prove the existence of solutions to problems in science involving integral equations or diferential equations. So, the appeal of fixed point theory to a large number of scientists is understandable. After Banach launched and proved Banach's contraction theorem, many mathematicians extended this well-known theorem into more general forms either by enhancing Banach's contraction into more general forms or by extending metric space (MS) into new ones, such as cone MS, G-MS, partial MS and so on. One of the important generalizations of MS is the idea of b-MS introduced by Baktain [1] and Abdeljawad et al. [9] used the idea of partial b-MS to enhance some known FP results. Shatanawi et al. made use of ordered relations to present

2020 *Mathematics Subject Classification. Primary 37C25, Secondary47H10*
Key words and phrases. fixed point, controlled metric space of type (γ, β), extended b-metric space).
* Speaker.

a new type of Banach's contraction theorem. Rasham et al. established a generalization of Banach's contraction theorem on fuzzy metric spaces. Also, Gupta at el. initiated several fixed point results in the setting of fuzzy metric spaces. Gamal et al. took the advantage of weakly compatible maps to present new fixed point findings via various contractions in multiplicative metric spaces and to examine some applications. Meanwhile, other authors introduced diferent types of contraction conditions, and to examine some applications in their obtained results, see for example. In the last few years, Kamran et al. [2] presented a good idea to extend the concept of b-MS in a clever way based on a control function with domain [1; +1) and named their concept “extended bmetric spaces (EbMS)”. Recently, Mlaiki et al. [3] extended the idea of b-MS to a new idea, which they named “controlled metric type space (CMTS)” by inserting a control function $\theta(u,l)$ in the triangular inequality of the definition of the metric space in a luminous way. Also, Mlaiki et al. [3] provided an example showing that the concept of a CMTS is not an EbMS. For more results in extended b-metric spaces and controlled metric spaces.

From now on, F stands for a non-empty set.

Definition 1.1. [1] For $b \geq 1$, the function $v : F \times F \to [0,\infty)$ is called a b-metric if $\forall v, l, s \in F$, we have

(1) $v(v,l) = 0 \Longleftrightarrow l = v$,
(2) $v(v,l) = v(l,v)$,
(3) $v(v,l) \leq b[v(v,s) + v(s,l)]$.

The pair (F, v) is called a b-MS.

The above concept has been generalized by two diferent ways. The first way was given by Kamran et al. [2] as follows:

Definition 1.2. [2] Consider the function $\theta : F \times F \to [1,\infty)$, and the function $v : F \times F \to [0,\infty)$ is called an extended b-metric if $\forall v, l, s \in F$, we have

(1) $v(v,l) = 0 \Longleftrightarrow l = v$,
(2) $v(v,l) = v(l,v)$,
(3) $v(v,l) \leq \theta(v,l)[v(v,s) + v(s,l)]$.

The pair (F, v) is referred to as an EbMS.

For some examples on EbMS, see [2]. The second way for generalizing the b-MS was given by Mlaiki et al. [3] as follows:

Definition 1.3. [3] Consider the function $\theta : F \times F \to [1,\infty)$, and the function $v : F \times F \to [0,\infty)$ is called a controlled metric type if $\forall v, l, s \in F$, we have

(1) $v(v,l) = 0 \Longleftrightarrow l = v$,
(2) $v(v,l) = v(l,v)$,
(3) $v(v,l) \leq \theta(v,s)v(v,s) + \theta(s,l)v(s,l)$.

The pair (F, v) is called a CMTS.

In the present work, we will establish and prove some fixed point theorems for mappings that satisfy a set of conditions in controlled metric type spaces introduced by Mlaiki et al. Our technique in constructing our new contraction conditions is to insert the control function $\theta(u, l)$ that appears on the right hand side of the triangular inequality of the definition of the controlled metric spaces in the right hand side of our proposed contraction conditions.

2. Main results

From now on, CCMTS is a complete controlled metric type space, and CbMS is a complete b-metric space with constant b.

Theorem 2.1. *On CCMTS (F, v), assume there exists $r \in (0, 1]$ such that $Q : F \to F$ satisfies*

$$v(Ql, Qv) \leq r\theta(l, v)v(l, v),$$

for all $\forall v, l \in F$. Assume $\lim_{t\to\infty} sup\theta(l_{i+1}, l_m)\theta(l_{i+1}, l_{i+2})$ exists and less than $\frac{1}{r}$, where $l_i = Q^i l_0$ for $l_0 \in F$. Also, suppose that for any $\forall v, l \in F$, we have $\lim_{i\to+\infty} sup\theta(v, Q^i l)$ and $\lim_{i\to+\infty} sup\theta(Q^i l, v)$ exist and are finite. Then, Q has a FP in F.

Theorem 2.2. *On CCMTS (F, v), assume there exists $r \in (0, 1]$ such that $Q : F \to F$ satisfies*

$$v(Ql, Qv) \leq r\theta(l, v)v(l, v).$$

for all for all $\forall l, v \in Q$. Suppose that for any $m \in N$, $\lim_{i\to\infty} sup\theta(l_i, l_m)\theta(l_i, l_{i+1})$ exists and is less than $\frac{1}{r}$, where $l_i = Q^i l_0$ for $l_0 \in F$. If θ is continuous in its variables, then Q has a FP in F.

The uniqueness of the FP in Theorem 2.1 or 2.2 can be obtained if an appropriate condition is added.

Theorem 2.3. *On CCMTS(F, v), assume there exists $r \in (0, 1]$ such that $Q : F \to F$ satisfies*

$$v(Ql, Qv) \leq r\theta(l, v)v(l, v)$$

for all $l, v \in F$. Assume that $\lim_{i\to\infty} sup\theta(l_i, l_m)\theta(l_i, l_{i+1})$ exists and is less than $\frac{1}{r}$, where $l_i = Q^i l_0$ for $l_0 \in Q$. Moreover, assume that for any $l, s_0 \in M$, $\lim_{i\to\infty} sup\theta(l, Q^i l_0)$ exists and is finite, or θ is continuous. If $l, s \in F$, we have $\lim_{i\to\infty} sup\theta(Q^i v, Q^i l)$ exists and is less than $\frac{1}{r}$, then T has only one FP in Q.

The following known result can be obtained immediately from our Theorem 2.3 by simply defining θ to be the constant function b.

Corollary 2.4. *On CbMS (F, v), assume there exists $r \in (0, 1]$ with $b^2 r < 1$ such that $Q : F \to F$ satisfies $v(Ql, Qv) \leq rbv(l, v)$, for all $l, v \in F$. Then, Q has only one FP in F.*

Theorem 2.5. *On $CCMTS(F, v)$, assume there exist $r, a \in [0, 1]$ (both are not 0) and $h \in [0, 1)$ such that $Q : F \to F$ satisfies*

$$v(Ql, Qv) \leq r\theta(l, v)v(l, v) + a\theta(l, Ql)v(l, Ql) + hv(v, Qv),$$

for all $l, v \in F$. Also, suppose that for any $m \in N$,

$$\lim_{j\to\infty} sup\theta(l_{j+1}, l_m)\theta(l_{j+1}, l_{i+2}) < \frac{1-h}{r+a},$$

where $l_i = Q^i l_0$ for $l_0 \in F$. Moreover, assume that for any $v \in F$, we have $\lim sup_{i\to+\infty}\theta(v, l_i)$ exists and is finite, and $\lim sup_{i\to+\infty}\theta(l_i, v)$ exists, is less than $\frac{1}{h}$ and is finite. Then, T has a FP in F.

In our next result, we assume that θ is continuous in its variables.

Theorem 2.6. *On CCMTS (F, v), assume there exist $r, a \in [0, 1]$ (both are not 0) and $h \in [0, 1)$ such that $Q : F \to F$ satisfies*

$$v(Ql, Qv) \leq r\theta(l, v)v(l, v) + a\theta(l, Ql)v(l, Ql) + hv(v, Qv),$$

for all $l, v \in F$. Also, suppose that for any $m \in N$,

$$\lim_{j\to\infty} sup\theta(l_{j+1}, l_m)\theta(l_{j+1}, l_{i+2}) < \frac{1-h}{r+a},$$

where $l_i = Q^i l_0$ for $l_0 \in F$. Also, suppose for $v \in Q$, we have $\theta(v, Qv) < \frac{1}{h}$. If θ is continuous in its variables, then Q has a FP in F.

The uniqueness of FP can be achieved in Theorem 2.5 or 2.6 if a suitable condition is added.

Theorem 2.7. *On CCMTS(F, v), assume there exist $r \in (0, 1]$, $a \in [0, 1]$ and $h \in [0, 1)$ such that $Q : F \to F$ satisfies*

$$v(Ql, Qv) \leq r\theta(l, v)v(l, v) + a\theta(l, Ql)v(l, Ql) + hv(v, Qv).$$

for all $l, v \in F$. To the addition of all conditions in Theorem 2.4 or 2.5, suppose Q satisfies $\lim_{t\to\infty} sup\theta(Q^t l, Q^t v) < \frac{1}{r}$, for all $l, v \in F$. Then, Q has only one FP in F.

References

1. I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal., **30** (1989), 26–37.
2. T. Kamran, M. Samreen and Q. U. L. Ain, *A generalization of b-metric space and some fixed point theorems*, Mathematics, **5** (2017), 1–7.
3. N. Mlaiki, H. Aydi, N. Souayah and T. Abdeljawad, *Controlled metric type spaces and the related contraction principle*, Mathematics, **6** (2018), 1–7.

NOTION OF ORTHOGONAL SETS AND EXTENSION OF BANACH FIXED POINT THEOREM

PARASTOO HEIATIAN NAEINI*

Department of Mmathematics, Payame Noor University , P. O. Box 19395-4697, Tehran, Iran.
parastoo.heiatiannaeini@pnu.ac.ir

Abstract. In this paper, we introduce the notion of the orthogonal sets and then we give an extension of Banach fixed point theorem. Finally, we study the existence and uniqueness of solution for a first-order ordinary diferential equation.

1. Introduction

The purpose of this paper is to introduce the notion of orthogonality of sets which contains the notion of orthogonality in normed spaces (see [1] and [2]). By using this concept, we discuss an analogue of [3] in orthogonal sets. The main result of [3] is the following theorem:

Theorem 1.1. *Let X be a partially ordered set such that every pair $x, y \in X$ has a lower bound and an upper bound. Furthermore, let d be a metric on X such that (X, d) is a complete metric space. If F is a continuous, monotone mapping from X into X such that*

- *there exists $k \in (0, 1)$ with $d(F(x), F(y)) \leq kd(x, y), \ \forall x \geq y$,*
- *there exists $x_0 \leq F(x_0)$ or $x_0 \geq F(x_0)$.*

2020 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. Orthogonal set, fixed point, diferential equation, Picard operator.

* Speaker.

Then F is a Picard operator (briefly, PO), that is, F has a unique fixed point x^* and $\lim_{n\to\infty} F^n(x) = x^*$ *for each $x \in X$.*

In this paper, we introduce the notion of the orthogonal sets and give a real generalization of Banach' fixed point theorem. As an application, we find the existence of solution for a first-order ordinary differential equation. Banach' fixed point theorem and other fixed point theorems do not work to prove this problem.

We start our work with the following definition, which can be consider the main definition of our paper.

Definition 1.2. Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be an binary relation. If $\perp$ satisfies the following condition:

$$\exists x_0 : (\forall y, y \perp x_0) \textit{ or } (\forall y; x_0 \perp y),$$

then it is called an orthogonal set (briefly O-set). We denote this O-set by $(X, \perp)$.

2. Main results

In this section, we prove the main theorem of the present paper. To this end, we need the following definitions:

Definition 2.1. Let $(X, \perp)$ be O-set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called an orthogonal sequence (briefly, O-sequence) if

$$(\forall n, x_n \perp x_{n+1}) \textit{ or } (\forall n, \forall x_{n+1} \perp x_n).$$

Definition 2.2. Let $(X, \perp, d)$ be an orthogonal metric space ($(X, \perp)$ is an O-set and (X, d) is a metric space). Then $f : X \to X$ is said to be orthogonally continuous (or $\perp$-continuous) in $a \in X$ if, for each O-sequence $\{a_n\}_{n\in N}$ in X with $a_n \to a$, we have $f(a_n) \to f(a)$. Also, f is said to be $\perp$-continuous on X if f is $\perp$-continuous in each $a \in X$.

We can not prove the following problem about the continuity of functions on inner product spaces:

Let X be an inner product space with the inner product $\langle ., . \rangle$. Define $x \perp y$ if $\langle x, y \rangle = 0$. Let $f : X \to X$ be $\perp$-continuous on X. Is f continuous on X? We cannot prove the above problem for $X = \mathbb{R}^n$ and its inner product.

Definition 2.3. Let $(X, \perp, d)$ be an orthogonal set with the metric d. Then X is said to be orthogonally complete (briefly, O-complete) if every Cauchy O-sequence is convergent.

It is easy to see that every complete metric space is O-complete and the converse is not true. In the next example, X is O-complete and it is not complete.

Definition 2.4. Let $(X, \perp, d)$ be an orthogonal metric space and $0 < \lambda < 1$. A mapping $f : X \to X$ is called an orthogonal contraction (briefly, $\perp$-contraction) with Lipschitz constant λ if, for all $x, y \in X$ with $x \perp y$,

$$d(fx, fy) \leq \lambda d(x, y).$$

Definition 2.5. Let $(X, \perp)$ be an O-set. A mapping $f : X \to X$ is said to be $\perp$-preserving if $f(x) \perp f(y)$ if $x \perp y$. Also, $f : X \to X$ is said to be weakly $\perp$-preserving if $f(x) \perp f(y)$ or $f(y) \perp f(x)$ if $x \perp y$.

Now, we are ready to prove the main theorem of this paper which can be consider as a real extension of Banach contraction principle.

Theorem 2.6. *Let $(X, \perp, d)$ be an O-complete metric space (not necessarily com- plete metric space) and $0 < \lambda < 1$. Let $f : X \to X$ be $\perp$-continuous, $\perp$-contraction with Lipschitz constant λ and $\perp$-preserving. Then f has a unique fixed point $x^* \in X$. Also, f is a Picard operator, that is,* $\lim f^n(x) = x^*$ *for all $x \in X$.*

Now, we show that our theorem is a real extension of Banach's contraction principle.

Corollary 2.7. *(Banach's contraction principle) Let (X, d) be a complete metric space and $f : X \to X$ be a mapping such that, for some $\lambda \in (0, 1]$,*

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for all $x, y \in X$. Then f has a unique fixed point in X.

In following, our purpose here is to apply Theorem 2.6 to prove the existence of a solution for the following diferential equation:

$$\begin{cases} u^{'}(t) = f(t, u(t)), a.e. t \in I = [0, T], \\ u(t) = a, a \geq 1, \end{cases} \tag{2.1}$$

where $f : I \times R \to R$ is an integrable function satisfying the following conditions:

(c1) $f(s, x) \geq 0$ for all $x \geq 0$ and $s \in I$,

(c2) there exists $\alpha \in L^1(I)$ such that

$$|f(s, x) - f(s, y)| \leq \alpha(s)|x - y|$$

for all $t \in I$ and $x, y \geq 0$ with $xy \geq (x \vee y)$, where $x \vee y = x$ *or* y.

Note that $f : I \times \mathbb{R} \to \mathbb{R}$ is not necessarily Lipschitz from the given condition (c2). For example, the function

$$f(s, x) = \begin{matrix} sx, & x \leq \frac{1}{2}, \\ 0, & x > \frac{1}{2} \end{matrix}$$

satisfies the conditions (c1) and (c2) while f is not continuous and monotone. Also, for $s \neq 0$,

$$|f(s, \frac{1}{2}) - f(s, \frac{2}{3})| = s\frac{1}{2} > s\frac{1}{6} = s|\frac{1}{2} - \frac{2}{3}|.$$

Theorem 2.8. *Under these conditions, for all $T > 0$, the differential equation (2.1) has a unique positive solution.*

References

1. J. Diestel, *Geometry of Banach Spaces*, Springer-Verlag, Berlin-Heidelberg-New York, 1975.
2. C.R. Diminnie, *A new orthogonality relation for normed linear spaces*, Math. Nachr., **114** (1983), 197-203.
3. A.C.M. Ran and M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), 1435-1443.

SOME PROPERTIES OF CONTROLLED G-FUSION FRAMES IN HILBERT C*-MODULES

SEDIGHEH HOSSEINI*

Department of Mathematics, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran
rsana7238@gmail.com;

Abstract. In this paper, the concept of controlled *-g-fusion frames is introduced in Hilbert C*-modules. The equivalent condition for controlled *-g-fusion frame is established using the operator theoretic approach. Some characterizations of controlled *-g-fusion frames are found out. Moreover, the relationship between *-g-fusion frames and controlled *-g-fusion frames are established.

1. Introduction

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [3] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [2] by Daubechies et al., frame theory began to be widely used, particularly in the more specialized context of wavelet frame and Gabor frame [4]. Frames have been used in signal processing, image processing, data compression, and sampling theory. The controlled frame was introduced by Balazs et al [1]. with the aim to improve the efficiency of the iterative algorithms constructed for inverting the frame operator. Controlled frames in Hilbert C*-modules was introduced by Rashidi-Kouchi and Rahimi [5].

2020 *Mathematics Subject Classification. Primary 42C15; Secondary 46A32*
Key words and phrases. Controlled *-g-Fusion Frames, Hilbert C*-modules.
* Speaker.

2. Main results

2.1. **Basic definitions and Preliminaries.** Let A be a unital C^* -algebra, let J be countable index set. Throughout this paper H and L are countably generated Hilbert $A-$ modules and H_j, $j \in J$ is a sequence of submodules of L. For each $j \in J$, $End_A^*(H, H_j)$ is the collection of all adjointable $A-$linear maps from H to H_j , and $End_A^*(H, H)$ is denoted by $End_A^*(H)$. Also let $GL^+(H)$ be the set of all positive bounded linear invertible operators on H with bounded inverse. Throughout this paper, we assume that A is a unital C^*-algebra and H is a Hilbert A-module.

Firstly we give the definition of $*$-G-fusion frame in Hilbert C^*-modules.

Definition 2.1. Let $\{W_j\}_{j\in J}$be a sequence of closed submodules orthogonally complemented of H, $\{v_j\}_{j\in J}$ be a family of weights in $\mathcal{A}$, ie., each v_j is positive invertible element from the center of $\mathcal{A}$ and $\Lambda_j \in End_A^*(H, H_j)$ for each $j \in J$. We say that $\Lambda = \{W_j, \Lambda_j, v_j\}_{j\in J}$ is a *-g-fusion frame for H if there exist two strictly nonzero elements A and B in $\mathcal{A}$ such that

$$A\langle f, f\rangle A^* \leq \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \leq B\langle f, f\rangle B^* \tag{2.1}$$

for all $f \in H$

The constants A and B are called the lower and upper bounds of the *-g-fusion frame, respectively. If $A = B$ thenΛ is called tight *-g-fusion frame and if $A = B = 1$ then we say Λ is a Parseval *-g-fusion frame. The operator $S : H \to H$ defined by $Sf = \sum_{j\in J} v_j^2 P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} f$ for all $f \in H$. Is called *-g-fusion frame operator.

Now we define the notion of $(C, \acute{C})$ -controlled *-g-fusion frame in Hilbert C*-modules.

Definition 2.2. Let $(C, \acute{C}) \in GL^+(H)$, $\{W_j\}_{j\in J}$ be a sequence of closed sub modules orthogonally complemented of H, $\{v_j\}_{j\in J}$ be a family of weights in $\mathcal{A}$, ie., each v_j is positive invertible element from the center of $\mathcal{A}$ and $\Lambda_j \in End_A^*(H, H_j)$ for each $j \in J$. We say that $\Lambda_{C\acute{C}} = \{W_j, \Lambda_j, v_j\}_{j\in J}$ is a $(C, \acute{C})$ controlled *-g-fusion frame for H if there exist two strictly nonzero elements A and B in $\mathcal{A}$ such that

$$A\langle f, f\rangle A^* \leq \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} C f, \Lambda_j P_{W_j} \acute{C} f\rangle \leq B\langle f, f\rangle B^* \tag{2.2}$$

for all $f \in H$ The constants A and B are called the lower and upper bounds of the $(C, \acute{C})$ controlled *-g-fusion frame, respectively. When $A = B$, the sequence $\Lambda_{C\acute{C}} = \Lambda = \{W_j, \Lambda_j, v_j\}_{j\in J}$ is called $(C, \acute{C})$ controlled tight *-g-fusion frame, and when $A = B = 1$, it is called a $(C, \acute{C})$ controlled Parseval *-g-fusion frame . If only upper inequality of the above inequality hold, then $\Lambda_{C\acute{C}}$ is called an $(C, \acute{C})$ controlled *-g-fusion bessel sequence for H.

Suppose that $\Lambda_{C\acute{C}}$ be a $(C, \acute{C})$ controlled *-g-fusion bessel sequence for H. The bounded linear operator $T_{(C,\acute{C})} : l^2(H_{j_{j\in J}}) \to H$ define by

$$T_{(C,\acute{C})}(\{f_j\}_{j\in J}) = \sum_{j\in J} v_j (C,\acute{C})^{\frac{1}{2}} P_{W_j}\Lambda_j^* f_j \tag{2.3}$$

for all $\{f_j\}_{j\in J} \in l_2(\{H_j\}_{j\in J})$

is called the synthesis operator for the $(C,\acute{C})$-controlled *-g-fusion frame $\Lambda_{C\acute{C}}$. The adjoint operator $T^*_{(C,\acute{C})} : H \to l_2(\{H_j\}_{j\in J})$ given by

$$T^*_{(C,\acute{C})}(g) = \{v_j \Lambda_j P_{W_j}(C,\acute{C})^{\frac{1}{2}} g\} \tag{2.4}$$

is called the analysis operator for the $(C,\acute{C})$-controlled *-g-fusion frame $\Lambda_{C\acute{C}}$. When $C$and $\acute{C}$ commute with each other, and commute with the operator $P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}$, for each $j \in J$, then the $(C,\acute{C})$-controlled *-g-fusion frame operator $S_{(C,\acute{C})} : H \to H$ is defined

$$S_{(C,\acute{C})} = T_{(C,\acute{C})} T^*_{(C,\acute{C})} f = \sum_{j\in J} v_j^2 \acute{C} P_{W_j}\Lambda_j^*\Lambda_j P_{W_j} C f \tag{2.5}$$

for all $f \in H$. And we have

$$\langle S_{(C,\acute{C})} f, f\rangle = \sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} C f, \Lambda_j P_{W_j}\acute{C} f\rangle \tag{2.6}$$

for all $f \in H$.

From now we assume that C and $\acute{C}$ commute with each other, and commute with the operator $P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}$, for each $j \in J$

Lemma 2.3. *Let $\Lambda_{C\acute{C}}$ be a $(C,\acute{C})$-controlled *-g-fusion frame for H. Then the $(C,\acute{C})$-controlled *-g-fusion frame operator $S_{(C,\acute{C})}$ is positive, self-adjoint and invertible.*

We estabilish an equivalent definition of $(C,\acute{C})$-controlled *-g-fusion frame operator.

Theorem 2.4. *Let $\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} C f, \Lambda_j P_{W_j}\acute{C} f\rangle$ converge in norm $\mathcal{A}$, $\Lambda_{C\acute{C}} = \{W_j, \Lambda_j, v_j\}_{j\in J}$ is a $(C,\acute{C})$ controlled *-g-fusion frame for H, if and only if*

$$\|A^{-1}\|^2\langle f,f\rangle \leq \|\sum_{j\in J} v_j^2 \langle \Lambda_j P_{W_j} C f, \Lambda_j P_{W_j}\acute{C} f\rangle\| \leq \|B\|^2\langle f,f\rangle \tag{2.7}$$

for all $f \in H$ and strictly nonzero elements $A, B \in \mathcal{A}$.

When $C = \acute{C}$ we say that the sequence $\{W_j, \Lambda_j, v_j\}_{j\in J}$ is a C^2-controlled *-g-fusion frame for H.

Theorem 2.5. *Let $C \in Gl^+(H)$ The $\{W_j, \Lambda_j, v_j\}_{j\in J}$ is a *-g-fusion frame if and only if The sequence $\Lambda_{C,C} = \{W_j, \Lambda_j, v_j\}_{j\in J}$ is a C^2- controlled *-g-fusion frame for H.*

Proposition 2.6. *Let* $\{W_j, \Lambda_j, v_j\}_{j\in J}$ *is a *-g-fusion frame for* H *with frame operator* S *and let* $C, \acute{C} \in GL^+(H)$. *Then* $\{W_j, \Lambda_j, v_j\}_{j\in J}$ *is a* $(C, \acute{C})$-*controlled *-g-fusion frame for* H.

Theorem 2.7. *Let* $C, \acute{C} \in GL^+(H)$, *and* $C, \acute{C}$ *commute with each other and commute with* $P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}$, *for each* $j \in J$. *Let* $\Lambda C\acute{C} = \{W_j, \Lambda_j, v_j\}_{j\in J}$ *is a* $(C, \acute{C})$-*controlled *-g-fusion bessel sequence for* H *with bound* B, *then the operator* $T_{(C,\acute{C})} : l_2(\{H_j\}_{j\in J}) \to H$ *given by*

$$T_{(C,\acute{C})}(\{g_j\}_{j\in J}) = \sum_{j\in J} v_j (C\acute{C})^{\frac{1}{2}} P_{W_j}\Lambda_j^* g_j$$

for all $\{g_j\}_{j\in J} \in l_2(\{H_j\}_{j\in J})$ *is well defined and bounded operator with,* $\|T_{(C,\acute{C})}\| \le \|B\|$.

Proposition 2.8. *Let* $\{W_j, \Lambda_j, v_j\}_{j\in J}$ *be a* $(C, \acute{C})$-*Controlled *-g-fusion frame with bounds* A *and* B *with operator frame* $S_{(C,\acute{C})}$. *Let* $\theta \in End_A^*(H)$ *be injective and has a closed range. Suppose that* θ *commute with* $C, \acute{C}$ *and* P_{W_j} *for all* $j \in J$. *Then* $\{W_j, \Lambda_j\theta, v_j\}_{j\in J}$ *is a* $(C, \acute{C})$-*Controlled *-g-fusion frame for* H.

Lemma 2.9. *Let* $\{W_j, \Lambda_j, v_j\}_{j\in J}$ *be a* $(C, \acute{C})$-*Controlled *-g-fusion frame with bounds* A *and* B. *Let* $\theta \in End_A^*(L, H)$ *be injective and has a closed range. Suppose that* θ *commute with* $\Lambda_j P_{W_j}\acute{C}$ *and* $\Lambda_j P_{W_j} C$ *for all* $j \in J$. *Then* $\{W_j, \theta\Lambda_j, v_j\}_{j\in J}$ *is a* $(C, \acute{C})$-*Controlled *-g-fusion frame for* H.

Theorem 2.10. *Let* $(H, A, \langle .,. \rangle_A)$ *and* $(H, B, \langle .,. \rangle_B)$ *be two Hilbert C*-modules and let* $\varphi : A \to B$ *be a *-homomorphisme and* θ *be a map on* H *such that* $\langle \theta f, \theta g\rangle_B = \varphi(\langle f, g\rangle_A)$ *for all* $f, g \in H$. *Suppose that* $\Lambda C\acute{C} = \{W_j, \Lambda_j, v_j\}_{j\in J}$ *is a* $(C, \acute{C})$-*Controlled *-g-fusion frame for* $(H, A, \langle .,. \rangle_A)$ *with frame operator* S_A *and lower and upper bounds* A *and* B *respectively. If* θ *is surjective such that* $\theta\Lambda_j P_{W_j} = \Lambda_j P_{W_j}\theta$ *for each* $j \in J$ *and* $\theta C = C\theta$ *and* $\theta\acute{C} = \acute{C}\theta$

Then $\{W_j, \Lambda_j, v_j\}_{j\in J}$ *is a* $(C, \acute{C})$-*Controlled *-g-fusion frame for* $(H, B, \langle .,. \rangle_B)$ *with frame operator* S_B *and lower and upper bounds* A *and* B *respectively and* $\langle S_B\theta f, \theta g\rangle_B = \varphi(\langle S_A f, g\rangle_A)$.

References

1. P. Balazs, J. P. Antoine and A. Grybos, *Weighted and controlled frames*, Int. J. Wavelets Multiresolut. Inf. Process., **8**(1) (2010) 109-132.
2. I. Daubechies, A. Grassman and Y. Meyer, *Painless nonothogonal expanisions*, J. Math. Phys., **27** (1986), 1271-1283.
3. R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic fourier series*, Proc. Amer. Math. soc., **72** (1952), 341-366.
4. D. Gabor, *Theory of communications*, Journal of Electrical Engineering. **93** (1946), 429–457.
5. M. Rashidi-Kouchi and A. Rahimi, *on controlled frames in Hilbert C*-modules*, International journal of wavelets. **15** (4) (2017), article 1750038.

GENERALIZED MULTIPLIERS FOR CONTROLLED BESSEL SEQUENCES

HESSAM HOSSEINNEZHAD*

Department of Basic Sciences, Shahid Sattari Aeronautical University of Science and Technology, Tehran, Iran.
hosseinnezhad@ssau.ac.ir; hosseinnezhad_h@yahoo.com

Abstract. This note includes a general version of Bessel multipliers for controlled Bessel sequences in Hilbert spaces. In fact, by combining analysis, an operator on ℓ^2, and synthesis, we reach so called generalized Bessel multipliers. Because of their importance for applications, we investigate some properties of them. Moreover, some necessary or sufficient conditions for the invertibility of such operators are presented.

1. Introduction

In [3], Schatten provided a detailed study of ideals of compact operators using their singular decomposition. He investigated the operators of the form $\sum_{i\in\mathbb{I}} \lambda_i g_i \otimes f_i$, where $\{f_i\}_{i\in\mathbb{I}}$ and $\{g_i\}_{i\in\mathbb{I}}$ are orthonormal families. In [1], the orthonormal families were replaced with Bessel and frame sequences to define Bessel and frame multipliers. These are operators that combine analysis, a multiplication with a fixed sequence (called the symbol) and synthesis. Multipliers have important applications for signal processing and acoustics. Recently, the concept of multipliers has been extended and introduced for continuous frames, fusion frames, p-Bessel sequences, generalized frames, controlled frames, Banach frames, Hilbert C^*-modules and etc. In

2020 *Mathematics Subject Classification. Primary 42C15; Secondary 46C05*
Key words and phrases. Controlled Bessel sequence, Controlled frame, Controlled Riesz basis, Generalized multiplier.
* Speaker.

this paper, by replacing the fixed multiplicative operator by a bounded and linear operator U on $\ell^2(\mathbb{I})$ and give a generalization of the controlled Bessel multipliers. Moreover, some new results are presented.

2. Notation and preliminaries

In this section, we collect the basic notation and some preliminary results. Throughout the paper, $\mathcal{H}$ is a separable Hilbert space and $\mathbb{I}$ is an at most countable index set. Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded and linear operators on $\mathcal{H}$. For $U \in \mathcal{B}(\mathcal{H})$,the notations U^* is the adjoint operator of U. We define $\mathcal{GL}(\mathcal{H})$ as the set of all bounded and linear operators on $\mathcal{H}$ with a bounded inverse. Given $0 < p < \infty$, we define the Schatten p-class of $\mathcal{H}$, denoted $\mathcal{S}_p(\mathcal{H})$, as the space of all compact operators U on $\mathcal{H}$ for which singular value sequence $\{\lambda_i\}_{i\in\mathbb{I}}$ belongs to $\ell^2(\mathbb{I})$. It is proved that $\mathcal{S}_p(\mathcal{H})$ is a two sided $*$-ideal of $\mathcal{B}(\mathcal{H})$.

Definition 2.1. Let $C, C' \in \mathcal{GL}(\mathcal{H})$ and $F = \{f_i\}_{i\in\mathbb{I}}$ be a sequence in $\mathcal{H}$. Then, F is called a (C, C')-controlled frame if there exist two constants $0 < A \leq B < \infty$ such that for every $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} \langle f, Cf_i\rangle\langle C'f_i, f\rangle \leq B\|f\|^2. \tag{2.1}$$

If only the right inequality in (2.1) holds, then we call F a (C, C')-controlled Bessel sequence. We call the (C, C)-controlled Bessel sequence and (C, C)-controlled frame, C^2-controlled Bessel sequence and C^2-controlled frame, respectively.

Let $F = \{f_i\}_{i\in\mathbb{I}}$ be a C^2-controlled Bessel sequence in $\mathcal{H}$. Then, the analysis operator $T_{CF} : \mathcal{H} \to \ell^2(\mathbb{I})$ is defined as

$$T_{CF}f := \{\langle f, Cf_i\rangle\}, \ (f \in \mathcal{H}).$$

Clearly, the operator T_{CF} is bounded and linear and it's adjoint, which is called the synthesis operator, is defined as

$$T^*_{CF} : \ell^2(\mathbb{I}) \to \mathcal{H}, \ T^*_{CF}\{a_i\}_{i\in\mathbb{I}} := \sum_{i\in\mathbb{I}} a_i Cf_i, \ (\{a_i\}_{i\in\mathbb{I}} \in \ell^2(\mathbb{I})).$$

Definition 2.2. Let $C, C' \in \mathcal{GL}(\mathcal{H})$ and $F = \{f_i\}_{i\in\mathbb{I}}$ be a sequence in $\mathcal{H}$. Then, F is called a (C, C')-controlled Riesz basis if F is complete and there exist $0 < A \leq B < \infty$ such that for every $\{a_i\}_{i\in\mathbb{I}} \in \ell^2(\mathbb{I})$,

$$A\sum_{i\in\mathbb{I}} |a_i|^2 \leq \|\sum_{i\in\mathbb{I}} a_i Cf_i\|^2 \leq B\sum_{i\in\mathbb{I}} |a_i|^2.$$

Now, we remember the concept of multipliers for controlled Bessel sequences, which is defined, of course by one contoroller operator, in [2].

Definition 2.3. Let $m \in \ell^\infty(\mathbb{I})$ and $F = \{f_i\}_{i\in\mathbb{I}}$ and $G = \{g_i\}_{i\in\mathbb{I}}$ be two C^2 and C'^2 controlled Bessel sequence in $\mathcal{H}$, respectively. Then, the operator $M_{m,(C'G),(CF)} : \mathcal{H} \to \mathcal{H}$ defined as

$$M_{m,(C'G),(CF)}f := \sum_{i\in\mathbb{I}} m_i\langle f, Cf_i\rangle C'g_i, \ (f \in \mathcal{H}),$$

is called the (C, C')-controlled Bessel multiplier operator with symbol m. It is clear that

$$M_{m,(C'G),(CF)} = T^*_{C'G}\mathfrak{M}_m T_{CF},$$

where $\mathfrak{M}_m : \ell^2(\mathbb{I}) \to \ell^2(\mathbb{I})$ is the mapping given by

$$\mathfrak{M}_m\{a_i\} := m_i a_i, \ (\{a_i\}_{i\in\mathbb{I}} \in \ell^2(\mathbb{I})).$$

3. Main Results

In this section, we introduce the concept of generalized controlled Bessel multipliers and investigate some properties of them.

Definition 3.1. Let $U \in \mathcal{B}(\ell^2(\mathbb{I}))$ and $F = \{f_i\}_{i\in\mathbb{I}}$ and $G = \{g_i\}_{i\in\mathbb{I}}$ be two C^2 and C'^2 controlled Bessel sequences in $\mathcal{H}$, respectively. Then, the operator $M_{(C'G)U(CF)} : \mathcal{H} \to \mathcal{H}$ defined as $M_{(C'G)U(CF)} = T^*_{C'G}UT_{CF}$ is called the generalized (C, C')-controlled Bessel multiplier.

In the first proposition, we summarize some properties of $M_{(C'G)U(CF)}$.

Proposition 3.2. *Let $F = \{f_i\}_{i\in\mathbb{I}}$ and $G = \{g_i\}_{i\in\mathbb{I}}$ be two C^2 and C'^2 controlled Bessel sequences in $\mathcal{H}$ with upper bounds B and B', respectively, and $M_{(C'G)U(CF)}$ be the Bessel multipliers associated with F and G.*

- *$M_{(C'G)U(CF)}$ is a bounded and linear operator with $\|M_{(C'G)U(CF)}\| \leq \sqrt{BB'}\|U\|$.*
- *If $U \in \mathcal{S}_p(\ell^2(\mathbb{I}))$, then $M_{(C'G)U(CF)} \in \mathcal{S}_p(\mathcal{H})$. The converse is true only if F and G are two (C, C')-controlled Riesz bases.*

The following proposition provides some conditions under which the underlying controlled Bessel sequences of a generalized (C, C')- controlled Bessel multiplier become controlled frames.

Proposition 3.3. *Assume that $F = \{f_i\}_{i\in\mathbb{I}}$ and $G = \{g_i\}_{i\in\mathbb{I}}$ are two C^2 and C'^2 controlled Bessel sequences in $\mathcal{H}$. Moreover, suppose that there exists $A > 0$ such that for every $f \in \mathcal{H}$, $A\|f\|^2 \leq \left|\langle M_{(C'G)U(CF)}f, f\rangle\right|$. Then, F and G are C^2 and C'^2 controlled frames in $\mathcal{H}$, respectively.*

The next result gives a necessary and sufficient condition for invertibility of the generalized multiplier.

Proposition 3.4. *Let $\{f_i\}_{i\in\mathbb{I}}$ and $G = \{g_i\}_{i\in\mathbb{I}}$ be two C^2 and C'^2 controlled Riesz bases in $\mathcal{H}$. Then the multiplier $M_{(C'G)U(CF)}$ is invertible if and only if U is invertible.*

Proposition 3.5. *Suppose that* $\{f_i\}_{i\in\mathbb{I}}$ *and* $G=\{g_i\}_{i\in\mathbb{I}}$ *are two* C^2 *and* C'^2 *controlled Bessel sequences in* $\mathcal{H}$*. If* $M_{(C'G)U(CF)}$ *is invertible, then* F *and* G *are* C^2 *and* C'^2 *controlled frames in* $\mathcal{H}$*, respectively.*

References

1. P. Balazs, *Basic definition and properties of Bessel multipliers*, J. Math. Anal. Appl. **325** (2) (2007), 581–588.
2. M. Rashidi-Kouchi, A. Rahimi, and F. A. Shah, *Duals and multipliers of controlled frames in Hilbert spaces*, International Journal of Wavelets, Multiresolution and Information Processing, **16** (06) (2018), 1850057.
3. R. Schatten, *Norm Ideals of Completely Continuous Operators*, Springer, Berlin, 1960.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

WEAKLY HYPERCYCLICITY OF COMPOSITION OPERATORS

J. IZADI AND KHADIJE AMIRI*

Department of Mathematics, Payame Noor University (PNU), Tehran, Iran
j.izadi@pnu.ac.ir; khadijeamiri1877@gmail.com

Abstract. In this paper, we investigate conditions which a weakly hypercyclic composition operator acting on the space of holomorphic functions.

1. Introduction

Let X be a Banach space of analytic functions on the open unit disk U of the plane and φ be a self map of the open unit disk. The composition operator $C_\varphi : X \to X$ is defined by $C_\varphi(f)(z) = f(\varphi(z))$ for every $f \in X$ and $z \in U$.
Let E be a topological vector space, then a continuous linear mapping $T : X \to X$ is called weakly hypercyclic provided there exist some $x \in E$ such that the set

$$Orb(T, x) = \{\lambda T^n x \; : \; n \geq 0, \; \lambda \in \mathbb{C}\}$$

is weakly dense in E. The first example of hypercyclic operator in the space of entire functions, by Birkhoff [1] in 1929. He showed the hypercyclicity of the translation operator, while Maclane [4] proved in 1952 the hypercyclicity of the differentiation operator. Hypercyclicity on Banach spaces started in 1960 by S. Rolewicz [5] who show λB is hypercyclic whenever B is unilateral

2020 *Mathematics Subject Classification.* Primary 47B33; Secondary 47B38
Key words and phrases. weighted operator, Hypercyclic operator, weakly Hypercyclicity.
* Speaker.

backward shift(on l^p and c_0) and $|\lambda| > 1$. Weakly hypercyclic acting on some function spaces has been invested in [2], [3]. We see that although the weakly hypercyclicity of composition operator follows by a corollary from [7] but our proof is direct, simple and independent. Our attention in the present paper is the weakly hypercyclic properties for a composition operator acting on a Banach space of analytic functions on the open unit disk.

2. Main results

In the following, we consider the consider the composition operator C_φ acting on a Banach space of analytic functions and we will investigate the weakly hypercyclicity of C_φ.
The Banach space X under consideration satisfy the following axioms:
Axiom 1. X is a subspace of the space of all analytic functions on the open unit disk U.
Axiom 2. $z \in X \subset X$ and $1 \in X$.
Axiom 3. For each $\lambda \in U$, the linear functional of evaluation at λ, e_λ, is bounded on X.

Theorem 2.1. *Let $X \subset C\left(\bar{U}\right)$ and $e_\lambda \in X^*$ for all $\lambda \in \bar{U}$. If φ has boundery fixed point, then C_φ is not weakly hypercyclic.*

Proof. Let w be a boundery fixed point of φ. since the functional $e_w \in X^*$, for all $f \in X$ we get

$$\langle f, C_\varphi^* e_w \rangle = \langle C_\varphi f, e_w \rangle = f \circ \varphi(w) = \langle f, e_w \rangle.$$

Thus $C_\varphi^* e_w = e_w$ and so 1 is an eigenvalue of C_φ^*. But it is well known that the point spectrum of the adjoint of a weakly hypercyclic operator is empty, hence C_φ^* can not be weakly hypercyclic. □

Corollary 2.2. *Under the conditions of Theorem 2.1, C_φ^* is not hypercyclic.*

Corollary 2.3. *Under the conditions of Theorem 2.1, λC_φ^* can not be weakly hypercyclic for all $\lambda \in \mathbb{C}$.*

Proof. In the proof of Theorem 2.1, substitute C_φ^* by λC_φ^*. □

Theorem 2.4. *Let $X \subset C(\bar{U})$ and $e_\lambda \in X^*$ for all $\lambda \in \bar{U}$. If φ has boundary fixed point, then λC_φ is not weakly hypercyclic for all $\lambda \in \mathbb{C}$.*

Now we want to investigate Banach spaces satisfying the conditions of the above theorems. for this let $\{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers with $\beta(0) = 1$ and $1 < p < \infty$. Consider the space of sequences $f = \{\hat{f}(n)\}_{n=-\infty}^{\infty}$ such that

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

The notation$f(z)=\sum_{n=-\infty}^{\infty}\hat{f}(n)z^n$ shall be used whether or not the series converges for any value of z. these are called formal Laurent series and $L^p(\beta)$ denotes the space of such formal Laurent series. Note that when n ranges on $\mathbb{N}\bigcup\{0\}$, they are called formal power series and are denoted by $H^P(\beta)$. These are reflexive Banach spaces with the norm $\|\cdot\|_\beta$. The following lemma has been proved in [6].

Lemma 2.5. *For $1<p<\infty$, the dual of dual of $H^p(\beta)$ is $H^q(\beta^{\frac{p}{q}})$ where $\frac{1}{p}+\frac{1}{q}=1$ and $\beta^{\frac{p}{q}}=\{(\beta(n))^{\frac{p}{q}}\}_n$.*

Proposition 2.6. *Let $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Then $H^p(\beta)^*=H^q(\beta^{-1})$, where $\beta^{-1}=\{\beta^{-1}(n)\}_{n=0}^{\infty}$.*

Proof. Define $L:H^q(\beta^{\frac{p}{q}})\to H^q(\beta^{-1})$ by $L(f)=F$ where

$$f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$$

and

$$F(z)=\sum_{n=0}^{\infty}\hat{f}(n)\beta^p(n)z^n.$$

Then

$$\|F\|^q_{H^q(\beta^{-1})}=\sum_{n=0}^{\infty}\frac{|\hat{f}(n)|^q\beta^{pq}(n)}{\beta^q(n)}=\sum_{n=0}^{\infty}|\hat{f}(n)|^q\beta^p(n)=\|f\|^q_{H^p(\beta^{\frac{p}{q}})}$$

Thus L is an isometry. It is also surjectve. because if

$$F(z)=\sum_{n=0}^{\infty}\hat{F}(n)z^n\in H^q(\beta^{-1}),$$

then $L(f)=F$, where

$$f(z)=\sum_{n=0}^{\infty}(\frac{\hat{F}(n)}{\beta^p(n)})z^n.$$

Hence $H^p(\beta^{\frac{p}{q}})$ and $H^q(\beta^{-1})$ are norm isomorphic. Since

$$H^p(\beta)^*=H^q(\beta^{-1}),$$

the proof is complete. ☐

Example 2.7. It is well known that a complex number λ is a bounded point Evaluation on $H^P(\beta)$ iff $\{\frac{\lambda^n}{\beta(n)}\}_n$ where $\frac{1}{p}+\frac{1}{q}=1$ ([29,30]).If $\lim_n\frac{\beta_{n+1}}{\beta_n}\to 1$, then each function in $H^P(\beta)$ is analytic on the open unit disk U, so clearly the Banach space $H^p(\beta)$ satisfies the axioms 1, 2, 3. We know that in the case $p=2$, the classical Hardy space, Bergman and Dirichlet spaces are weighted Hardy spaces with $\beta(n)=1$ and $\beta(n)=(n+1)^{-\frac{1}{2}}$ and $\beta(n)=(n+1)^{\frac{1}{2}}$ respectively. So in the special case, Hardy, Bergman and Dirichlet spaces are examples satisfying the axioms 1, 2, 3. Furthermore, note that under the condition $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^p} < \infty$, we have

$$e_\lambda \in H^p(\beta)^* = H^q(\beta^{-1})$$

for all λ in the closed unit disk. Indeed, in this case , the spaces $H^p(\beta)$ are small and consist of functions that are continuous on the unit circle. A necessary condition for the operator C_φ to be hypercyclic on $H^p(\beta)$ is that φ is univalent and this condition guarantees boundedness of the composition operator C_φ on some small weighted Hardy spaces [8]. By above discussions together with the earlier results, we can conclude the following theorem.

Theorem 2.8. *Let $1 < p < \infty$, φ be a univalent self map of U and $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^p} < \infty$. Then λC_φ can not be weakly hypercyclic on $H^p(\beta)$ for all $\lambda \in \mathbb{C}$.*

References

1. G. D. Birkhoff, *Demonstrtion d'un theoreme elementaire surles functions entieres*, C. R. Acad. Sci. Paris, **189** (1929), 473-475.
2. K. C. Chan and R. Sanders, *A weakly hypercyclic operator that is not norm hypercyclic*, Journal of Operator Theory, (2004), 39-59.
3. Z. Kamali,*Weakly Hypercyclic Composition Operators on some Hilbert Spaces of Analytic Functions*, J. Math. Extention, **7** (4) (2013), 95-99.
4. G. R. MacLane, *Sequences of derivatives and normal families*, Journal d'Analyse Mathématique **2** (1) (1952), 72-87.
5. S. Rolewics, *on orbits of elements*, Studia Math, **32** (1969), 17-22.
6. B. Yousefi, *on the space $l^p(\beta)$, Rendiconti Del Circolo Matematico di Palermo*, **49** (2000), 115-120.
7. B. Yousefi and J. Izadi, *Weighted composition operators and supercyclicity criterion*, International Journal of Mathematics and Mathematical Sciences, Volume 2011, Article ID 514370, 5 pages.
8. N. Zorboska, *Cyclic composition operators on smooth weighted Hardy spaces*, The Rocky Mountain journal of mathematics (1999), 725-740.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

REMARKS ON APPROXIMATE IDEAL AMENABILITY OF BANACH ALGEBRAS

SEYEDEH SOMAYEH JAFARI*

Department of mathematics, Payame Noor University, Tehran, Iran
ss.jafari@pnu.ac.ir

Abstract. Let A be a Banach algebra, and let I be a closed rtwo-sided ideal of A. In this note, we present some conditions which guarantee that approximate I-weak amenability of A. In particular, as an povital result, we show that weak*-approximate 3-I-weak amenability of A follows that approximate I-weak amenability.

1. Introduction

Let A be a Banach algebra, and let X be a Banach A-bimodule. Then the dual space X^* is a Banach A-bimodule by the following actions

$$\langle a \cdot f, \xi\rangle = \langle f, \xi \cdot a\rangle, \quad \langle f \cdot a, \xi\rangle = \langle f, a \cdot \xi\rangle, \quad (a \in A, \xi \in X, f \in X^*)$$

A *derivation* $D : A \longrightarrow X$ is a bounded linear map that satisfies

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in A).$$

Moreover, a derivation D is called *inner* if there exists an element $\xi \in X$ such that

$$D(a) = a \cdot \xi - \xi \cdot a \quad (a \in A).$$

2020 *Mathematics Subject Classification. Primary 47B35; Secondary 30H05*
2020 *Mathematics Subject Classification. Primary 46H05; Secondary 46H20*
Key words and phrases. Amenable, Banach algebra, Derivation.
* Speaker.

In addition, a derivation D is *approximately inner* if D is the strong limit of a net of inner derivations; that is, there exists a net $(\xi_\alpha) \subseteq X$ such that

$$D(a) = \lim_\alpha a \cdot \xi_\alpha - \xi_\alpha \cdot a \quad (a \in A).$$

Note that (ξ_α) is not necessarily bounded. The same strategy as Lemma 1 in [2] shows that D is approximately inner if and only if it is weakly approximately inner; that is, D is the weak limit of a net of inner derivations. Banach algebra A is called *amenable* if $H^1(A, X^*) = \{0\}$ for all Banach A-bimodule X, where $H^1(A, X^*)$ is defined as [4]. This means that each continuous derivation from A into dual of every A-bimodule is inner.

A Banach algebra A is called *weakly amenable* (resp. *approximately weakly amenable*) if every continuous derivation $D : A \to A^*$ is inner (resp. approximately inner). Let I be a closed two-sided ideal of a Banach algebra A. Then A is called *approximately I-weakly amenable* (resp. *I-weakly amenable*) if every continuous derivation $D : A \longrightarrow I^*$ is approximately inner (resp. inner). Moreover, A is called *approximately ideally amenable* (resp. *ideally amenable*) if A is approximately I-weakly amenable (resp. I-weakly amenable) for every closed two-sided ideal I of A; see [1] and [5] for more details and examples.

For a Banach algebra A and a closed two-sided ideal I of A, it is known 3-I-weak amenability of A follows that A is I-weakly amenable. So, A is ideally amenable if $H^1(A, I^{***}) = 0$ for each closed two-sided ideal I of A. We next give this statement holds in the approximate case with fewer conditions.

Definition 1.1. Let A be a Banach algebra and I be a closed two-sided ideal of A. Then A is called *weak**-approximately 3-I-weakly amenable if for each continuous derivation $D : A \longrightarrow I^{***}$, there is a net $(\psi_\alpha) \subseteq I^{***}$ such that $a \cdot \psi_\alpha - \psi_\alpha \cdot a \xrightarrow{w^*} D(a)$ for all $a \in A$.

We now state the following result as a main tool in this note.

Theorem 1.2. *Let A be a Banach algebra and let I be a closed two-sided ideal of A. Suppose A is weak*-approximately 3-I-weakly amenable. Then A is approximate I-weakly amenable.*

As some immediate consequences of Theorem 1.2, we have the following corollary.

Corollary 1.3. *Let A be a Banach algebra.*

(a) *If A is weak$^*-$approximately $3 - I-$weakly amenable for every two-sided closed ideal I of A, then it is approximately ideally amenable.*

(b) *If every continuous derivation $D : A \to A^{***}$ is approximately inner in w^*-topology, then A is approximately weakly amenable.*

Note that if A is a reflexive Banach algebra, then the converse of the above results is true.

Theorem 1.4. *Let A be a Banach algebra and for every closed ideal J of A such that $J = \overline{AJ \cup JA}$, $A^\sharp$ is weak$^*-$approximately $3-J-$weakly amenable. Then A is approximately ideally amenable.*

Theorem 1.5. *Suppose that A be an (approximately) ideally amenable Banach algebra and let J be a non-closed ideal of A. Let*

(a) J is a Banach algebra under a norm $||\cdot||_J$ such that for $a \in A$, $b \in J$ we have

$$||ab||_J \leq ||a||\ ||b||_J,\ \ ||ba||_J \leq ||a||\ ||b||_J,$$

(b) J has an approximate identity which also an approximate identity for A.

Then J is (approximately) ideally amenable.

We end the work with the following theorem.

Theorem 1.6. *Let I be a closed two-sided ideal in A and let A be a ideal of A^{**} with bounded approximate identity. If A^{**} is weak$^*-$approximately $3-I-$weakly amenable, then A is approximately $I-$weakly amenable.*

References

1. M. Eshaghi-Gordji and T. Yazdanpanah , *derivations into duals of ideals of Banach algebras*, Proc. Ind. Acad. Sci. (Math. Sci.) **114(4)** (2004), 399-403.
2. F. Gourdeau, *Amenability of Lipschitz algebras*, *Math. Proc. Camb. Phil. Soc.* **112** (1992), 581-588.
3. F. Ghahramani and R. J. Loy, *Generalized notions of amenability*, *J. Funct. Anal.* **208** (2004), 229-260.
4. B. E. Johnson, *Cohomology in Banach algebras*, *Mem. Amer. Math. Soc.* **127** (1972).
5. O. T. Mewomo, *On approximate ideal amenability in Banach algebras*, An. Stiinf. Univ. Al. I. Cuza Iasi. Mat (N.S.). Tomul Lv I, **1** (2010), 199-208.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

ON SOME ESTIMATES OF NUMERICAL RADIUS INEQUALITIES FOR 2×2 OPERATOR MATRICES

HOSNA JAFARMANESH*

Department Mathematics, South Tehran Branch, Islamic Azad University, Tehran, Iran.
hosna.jafarmanesh@yahoo.com

Abstract. In this paper, we establish several numerical radius inequalities for 2×2 operator matrices, which improve on existing ones. Applications of our inequalities are also provided.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, we denote by $|A|$ the absolute value operator of A, that is, $|A| = (A^*A)^{\frac{1}{2}}$, where A^* is the adjoint operator of A. The numerical radius and the operator norm of A are defined respectively by

$$\omega(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle| \quad \textit{and} \ \|A\| = \sup_{\|x\|=1} \|Ax\|.$$

It is well known that the numerical radius, i.e., $\omega(.)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\|.\|$. In fact, for any $A \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|.$$

2020 *Mathematics Subject Classification.* Primary 47A63; Secondary 47A64
Key words and phrases. Numerical radius, operator matrix, inequality.
* Speaker.

A general 2×2 operator matrix in $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is an operator of the form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A, B, C, D \in \mathcal{B}(\mathcal{H})$. The operator $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ is called the diagonal part of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, and $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ is the off-diagonal part.
In [3], it has been shown that if $T = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, then

$$\omega^r(T) \leq \frac{1}{2} \max \left\{ \| |A|^r + |A^*|^r \|, \| |D|^r + |D^*|^r \| \right\} \tag{1.1}$$

for $r \geq 1$. Bhunia et. al. [1] proved that if $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, then

$$\omega^2(S) \leq \frac{1}{2} \max \left\{ \| |B^*|^2 + |C|^2 \|, \| |B|^2 + |C^*|^2 \| \right\}. \tag{1.2}$$

In this paper, we derive some upper bounds for the numerical radius of diagonal parts and off-diagonal parts of 2×2 operator matrices. In particular, our results refine the inequalities (1.1) and (1.2).

2. Main results

We first recall that if $h : [0, \infty) \to [0, \infty)$, then h is called doubly convex if h is convex in the usual sense and

$$h(a^{1-t} b^t) \leq h^{1-t}(a) h^t(b), \quad a, b \geq 0, \ 0 \leq t \leq 1.$$

To prove our numerical radius inequalities , we need the following lemma.

Lemma 2.1. [4] *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$, Then*

(a) $\omega(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}) = \max(\omega(A), \omega(D))$.

(b) $\omega(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}) = \frac{1}{2}\|A\|$.

Theorem 2.2. *Let $T = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ and f, g be two non-negative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ $(t \geq 0)$ and let $h : [0, \infty) \to [0, \infty)$ be an increasing doubly convex function. Then, two partial isometry U and V exist such that*

$$h(\omega(T)) \leq \frac{1}{2} \max \left\{ h\left(\omega\left(f(|A|) U g(|A^*|)\right)\right), h\left(\omega(f(|D|) V g(|D^*|))\right) \right\}$$
$$+ \frac{1}{4} \max \left\{ \| h\left(g^2(|A^*|)\right) + h\left(f^2(|A|)\right) \|, \| h\left(g^2(|D^*|)\right) + h\left(f^2(|D|)\right) \| \right\}.$$

In particular, for all $r \geq 1$ and $0 \leq t \leq 1$,

$$\omega^r(T) \le \frac{1}{2}\max\left\{\omega^r\left(|A|^t U |A^*|^{1-t}\right), \omega^r\left(|D|^t V |D^*|^{1-t}\right)\right\} + \frac{1}{4}\max\left\{\left\||A^*|^{2r(1-t)} + |A|^{2rt}\right\|, \left\||D^*|^{2r(1-t)} + |D|^{2rt}\right\|\right\}.$$

Remark 2.3. By letting $t = \frac{1}{2}$ in the above inequality, it is easy to sea that the above inequality refines inequality (1.1).

Theorem 2.4. *Let* $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ *and* f, g *be two non-negative continuous functions on* $[0, \infty)$ *such that* $f(t)g(t) = t$ $(t \ge 0)$. *Then*

$$\begin{aligned}\omega(S) \le \frac{\sqrt{2}}{4}\max\Big\{&\|f(|C|)g^2(|C|)f(|C|) + g(|B^*|)f^2(|B^*|)g(|B^*|)\|^{\frac{1}{2}},\\ &\|g(|C^*|)f^2(|C^*|)g(|C^*|) + f(|B|)g^2(|B|)f(|B|)\|^{\frac{1}{2}}\Big\}\\ &+ \frac{1}{4}\max\left\{\left\|g^2(|B^*|) + f^2(|C|)\right\|, \left\|g^2(|C^*|) + f^2(|B|)\right\|\right\}.\end{aligned}$$

In particular, for all $0 \le t \le 1$,

$$\omega(S) \le \frac{\sqrt{2}}{4}\max\left\{\left\||C|^2 + |B^*|^2\right\|^{\frac{1}{2}}, \left\||C^*|^2 + |B|^2\right\|^{\frac{1}{2}}\right\} + \frac{1}{4}\max\left\{\left\||B^*|^{2(1-t)} + |C|^{2t}\right\|, \left\||C^*|^{2(1-t)} + |B|^{2t}\right\|\right\}.$$

Remark 2.5. By letting $t = \frac{1}{2}$ in the above inequality, it is easy to sea that the above inequality refines inequality (1.2).

Corollary 2.6. *Let* $A, B \in \mathcal{B}(\mathcal{H}$. *Then*

$$\omega^{\frac{1}{2}}(B^*A) \le \frac{\sqrt{2}}{4}\max\{\||A|^2 + |B|^2\|^{\frac{1}{2}}, \||A^*|^2 + |B^*|^2\|^{\frac{1}{2}}\} + \frac{1}{4}\max\{\||B| + |A|\|, \||A^*| + |B^*|\|\}.$$

In particular, if A *and* B *are normal, then*

$$\omega(B^*A) \le \left(\frac{\sqrt{2}}{4}\left\||A|^2 + |B|^2\right\|^{\frac{1}{2}} + \frac{1}{4}\left\||B| + |A|\right\|\right)^2,$$

which is clearly smaller than the upper bound given in [2], *namely*

$$\omega(B^*A) \le \frac{1}{2}\left\||A|^2 + |B|^2\right\|.$$

References

1. P. Bhunia, S. Bag and K. Paul , *Bounds for zeros of a polynomials using numerical radius of Hilbert space operators*, Annals of Functional Analysis, **12** (21) (2021).
2. S. S. Dragomir, *Power inequalities for the numerical radius of product of two operators in Hilbert spaces*, Sarajevo J. Math., **5** (18) (2) (2009) 269-278.
3. M, Hajmohamadi, R. Lashkaripour and M. Bakherad, *Some generalizations of numerical radius on off-diagonal part of* 2×2 *operator matrices*, J. Math. Inequal., **12** (2) (2018), 447-457.
4. F. Kittaneh, *A numerical radius inequality and an estimate for the numerical radius of the frobenius companion matrix*, Studia Math., **158** (2003), 11-17.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

BEST PROXIMITY POINT OF CYCLIC G-φ-WEAK CONTRACTIVE MAPPINGS IN GRAPHICAL METRIC SPACES

SOMAYE JALALI[1]* AND KAMAL FALLAHI[2]

1 *Department of Mathematics, Payame Noor University, Tehran, Iran*
somi_63@yahoo.com
2 *Department of Mathematics, Razi University, Kermanshah, Iran*
k.fallahi@razi.ac.ir; fallahi1361@gmail.com

Abstract. The main purpose of this paper is to show the existence of best proximity points for a famous class of contractive mappings, named cyclic G-φ-weak contractive mappings, in graphical metric spaces. As an application, some consequences are also considered for the special graphs, which can demonstrate the importance of the main theorems.

1. Introduction

In 2006, Eldred and Veeremani [3] showed the existence of best proximity point of contractive mappings on uniformly convex Banach spaces. Suppose that $\mathcal{A}, \mathcal{B} \neq \emptyset$ are two subsets of a metric space $(\mathcal{X}, d)$, $d(\mathcal{A}, \mathcal{B}) = \inf\{d(a, b) : a \in \mathcal{A}, b \in \mathcal{B}\}$ and $\mathcal{F} : \mathcal{A} \to \mathcal{B}$. The best proximity point of $\mathcal{F}$ is each point $a \in \mathcal{A}$, where $d(a, \mathcal{F}a) = d(\mathcal{A}, \mathcal{B})$. In 2009, Suzuki et al. [8] proved the existence of such point of cyclic contractive mappings in a metric space by virtue of an UC (stands for unconditionally Cauchy) property. Note that a mapping $\mathcal{F} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is called cyclic, introduced by Kirk et al. [7], whenever $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{B}$ and $\mathcal{F}(\mathcal{B}) \subseteq \mathcal{A}$. Also, the pair $(\mathcal{A}, \mathcal{B})$

2020 *Mathematics Subject Classification. Primary 41A52, 47H10; Secondary 05C40*
Key words and phrases. Cyclic G-φ-weak contractive mappings; Graphical metric spaces; Orbitally G-continuous mapping; UC property; Best proximity point.
* Speaker.

is said to have UC property if for $\{a_n\}$ and $\{a'_n\}$ in $\mathcal{A}$ and a $\{b_n\}$ in $\mathcal{B}$, $\lim_{n\to\infty} d(a_n, b_n) = \lim_{n\to\infty} d(a'_n, b_n) = d(\mathcal{A}, \mathcal{B})$ implies $\lim_{n\to\infty} d(a, a'_n) = 0$. In 2011-2012, Abkar and Gabeleh obtained the partially ordered version of these results. On the other hand, in 2008, Jachymski [6] introduced a graphical metric space, a metric space $(\mathcal{X}, d)$ endowed by a graph $G = (V, E)$ in which V is vertex set and E is edge set containing all loops, and extended several concepts and fixed point theorems. Regarding all concepts mentioned above, we prove the existence of best proximity points for cyclic G-φ-weak contractive mappings in graphical metric spaces. For this, we need some notations and definitions. Suppose that $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is a mapping.

- $\mathcal{C}_{\mathcal{F}} = \{a \in \mathcal{X} : (\mathcal{F}^m a, \mathcal{F}^n a) \in E(\tilde{G}) \text{ for } m, n = 0, 1, \cdots\}$.
- Take Ψ the class of all continuous and nondecreasing functions $\varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ in which φ is positive on $(0, +\infty)$ and $\varphi(0) = 0$

Definition 1.1. ([6]) A mapping $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is orbitally G-continuous on $\mathcal{X}$ when $\mathcal{F}^{b_n} a \to b$ implies $\mathcal{F}(\mathcal{F}^{b_n} a) \to \mathcal{F} b$ for each $a, b \in \mathcal{X}$ and positive sequence $\{b_n\}$ so that $(\mathcal{F}^{b_n} a, \mathcal{F}^{b_n+1} a) \in E(G)$ for every $n \in \mathbb{N}$.

Definition 1.2. ([6]) G is named a C-graph on $\mathcal{X}$ if $a \in \mathcal{X}$ and $\{a_n\}$ is a sequence in $\mathcal{X}$ so that $a_n \to a$ and $(a_{n+1}, a_n) \in E(G)$ for all $n \in \mathbb{N}$, then there is $\{a_{2n_i}\}$ of $\{a_n\}$ so that $(a_{2n_i}, a) \in E(G)$ for each $i \in \mathbb{N}$.

2. Main results

Definition 2.1. Let $(\mathcal{X}, d)$ be a graphical metric space. A mapping $\mathcal{F} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is said to be a cyclic φ-contractive mapping when

$$d(\mathcal{F}a, \mathcal{F}^2 b) \leq d(a, \mathcal{F}b) - \varphi\big(d(a, \mathcal{F}b)\big) + \varphi\big(d(\mathcal{A}, \mathcal{B})\big) \tag{2.1}$$

for any $(a, b) \in \mathcal{A} \times \mathcal{A}$ with $(a, b) \in E(G)$, where $\varphi \in \Psi$.

The following is the first main result of this paper.

Theorem 2.2. *Assume that $\mathcal{A}, \mathcal{B} \neq \emptyset$ are two closed subsets of a graphical metric space $(\mathcal{X}, d)$, $\mathcal{F} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic G-φ-weak contractive mapping on $\mathcal{A}$ and $\mathcal{F}^2$ keeps the edges of G on $\mathcal{A}$, i.e. $(a, b) \in E(G)$ implies $(\mathcal{F}^2 a, \mathcal{F}^2 b) \in E(G)$ for each $a, b \in \mathcal{A}$. Moreover, suppose that $C_{\mathcal{F}}|_{\mathcal{A}} \neq \emptyset$ and $a_{n+1} = \mathcal{F} a_n$. If G is C-graph on $\mathcal{A}$ and $\{a_{2n}\}$ has a convergent subsequence in $\mathcal{A}$, then $\mathcal{F}$ possesses a best proximity point in $\mathcal{A}$.*

Proof. Using the partially ordered version of the proof of this theorem mentioned by Abkar and Gabeleh [1, 2] and replacing the hypothesises of graphs, we can prove the existence of a best proximity point by a well-known way of computations in graphical best proximity point theory. □

The following corollaries for the special choices of graphs can be immediately obtained from 2.2. First, take $G = G_0$ where G_0 is a complete graph, i.e. G_0 is a graph in which $V(G_0) = \mathcal{X}$ and $E(G_0) = \mathcal{X} \times \mathcal{X}$.

Corollary 2.3. *Assume that $\mathcal{A}, \mathcal{B} \neq \emptyset$ are two closed subsets of a graphical metric space $(\mathcal{X}, d)$ and $\mathcal{F} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic G_0-φ-weak contractive mapping on $\mathcal{A}$. Moreover, suppose that $a_{n+1} = \mathcal{F}a_n$. If G_0 is C-graph on $\mathcal{A}$ and $\{a_{2n}\}$ has a convergent subsequence in $\mathcal{A}$, then $\mathcal{F}$ possesses a best proximity point in $\mathcal{A}$.*

Second, suppose that $(\mathcal{X}, \leqslant)$ is a partially ordered set. Now, take G_1 a graph on $\mathcal{X}$ in which $V(G_1) = \mathcal{X}$ and $E(G_1) = \{(a, b) \in \mathcal{X} \times \mathcal{X} : a \leqslant b\}$.

Corollary 2.4. [2, Theorem 3.4, Abkar and Gabeleh (2012)] *Assume that $(\mathcal{X}, \leqslant)$ is a partially ordered set, $\mathcal{A}, \mathcal{B} \neq \emptyset$ are two closed subsets of a graphical metric space $(\mathcal{X}, d)$, $\mathcal{F} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic G_1-φ-weak contractive mapping and $\mathcal{F}^2$ is nondecreasing on $\mathcal{A}$. Moreover, suppose that $x_0 \in \mathcal{A}$ with $x_0 \leqslant \mathcal{F}^2 x_0$ exists and $a_{n+1} = \mathcal{F}a_n$. If G_1 is C-graph on $\mathcal{A}$ and $\{a_{2n}\}$ has a convergent subsequence in $\mathcal{A}$, then $\mathcal{F}$ has a best proximity point in $\mathcal{A}$.*

Next, take G_2 a graph on $\mathcal{X}$ in which $V(G_2) = \mathcal{X}$ and $E(G_2) = \{(a, b) \in \mathcal{X} \times \mathcal{X} : a \leqslant b \text{ or } b \leqslant a\}$, i.e. a and b are comparable members of $\mathcal{X}$.

Corollary 2.5. *Assume that $(\mathcal{X}, \leqslant)$ is a partially ordered set, $\mathcal{A}, \mathcal{B} \neq \emptyset$ are two closed subsets of a graphical metric space $(\mathcal{X}, d)$ and $\mathcal{F} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic G_2-φ-weak contractive mapping. Also, suppose that if a and b are comparable for $a, b \in \mathcal{A}$, then $\mathcal{F}^2 a$ and $\mathcal{F}^2 b$ are comparable. Moreover, assume that $x_0 \in \mathcal{A}$ exists such that x_0 and $\mathcal{F}^2 x_0$ are comparable and $a_{n+1} = \mathcal{F}a_n$. If G_2 is C-graph on $\mathcal{A}$ and $\{a_{2n}\}$ has a convergent subsequence in $\mathcal{A}$, then $\mathcal{F}$ possesses a best proximity point in $\mathcal{A}$.*

Ultimately, assume that $\varepsilon > 0$ is fixed and recall that $a, b \in \mathcal{X}$ are said to be ε-close whenever $d(a, b) < \varepsilon$. Define the ε-graph G_ε by $V(G_\varepsilon) = \mathcal{X}$ and $E(G_\varepsilon) = \{(a, b) \in \mathcal{X} \times \mathcal{X} : d(a, b) < \varepsilon\}$.

Corollary 2.6. *Assume that $\mathcal{A}, \mathcal{B} \neq \emptyset$ are two closed subsets of a graphical metric space $(\mathcal{X}, d)$ and $\mathcal{F} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic G_ε-φ-weak contractive mapping. Also, suppose that if a and b are ε-close for $a, b \in \mathcal{A}$, then $\mathcal{F}^2 a$ and $\mathcal{F}^2 b$ are ε-close. Moreover, assume that $x_0 \in \mathcal{A}$ exists such that x_0 and $\mathcal{F}^2 x_0$ are ε-close and $a_{n+1} = \mathcal{F}a_n$. Setting just $\varphi(f) = (1 - t)f$ with $t \in (0, 1)$ and $f \geq 0$ in Theorem 2.2, we have next theorem.*

Theorem 2.7. *Assume that $\mathcal{A}, \mathcal{B} \neq \emptyset$ are two closed subsets of a graphical metric space $(\mathcal{X}, d)$ and $\mathcal{F} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic G-weak contractive mapping on $\mathcal{A}$, i.e. $d(\mathcal{F}a, \mathcal{F}^2 b) \leq td(a, \mathcal{F}b) + (1-t)d(\mathcal{A}, \mathcal{B})$ for every $(a, b) \in \mathcal{A} \times \mathcal{A}$ with $(a, b) \in E(G)$ in which $t \in (0, 1)$. Also, suppose that $\mathcal{F}^2$ keeps the edges of G on $\mathcal{A}$, $C_{\mathcal{F}}|_{\mathcal{A}} \neq \emptyset$ and $a_{n+1} = \mathcal{F}a_n$. If G is C-graph on $\mathcal{A}$ and $\{a_{2n}\}$ has a convergent subsequence in $\mathcal{A}$, then $\mathcal{F}$ possesses a best proximity point in $\mathcal{A}$.*

Corollary 2.8. [1, Theorem 4.1, Abkar and Gabeleh (2011)] *Assume that $(\mathcal{X}, \leqslant)$ is a partially ordered set, $\mathcal{A}, \mathcal{B} \neq \emptyset$ are two closed subsets of a graphical metric space $(\mathcal{X}, d)$, $\mathcal{F} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic G_1-weak contractive mapping. Also, suppose that $\mathcal{F}^2$ is nondecreasing on $\mathcal{A}$, $x_0 \in \mathcal{A}$ with*

$x_0 \leqslant \mathcal{F}^2 x_0$ exists and $a_{n+1} = \mathcal{F} a_n$. If G_1 is C-graph on $\mathcal{A}$ and $\{a_{2n}\}$ has a convergent subsequence in $\mathcal{A}$, then $\mathcal{F}$ possesses a best proximity point in $\mathcal{A}$.

The following is the second main result of this paper.

Theorem 2.9. *Assume that $\mathcal{A}, \mathcal{B} \neq \emptyset$ are subsets of a graphical metric space $(\mathcal{X}, d)$, $\mathcal{A}$ is complete, both $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{A})$ have the UC property, $\mathcal{F} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic G-φ-weak contractive mapping on both $\mathcal{A}$ (and $\mathcal{B}$) in which $\mathcal{F}$ and $\mathcal{F}^2$ keep the edges of G on $\mathcal{A}$. If either $\mathcal{F}$ is orbitally G-continuous on $\mathcal{A}$ or G is a C-graph on $\mathcal{A}$, then $\mathcal{F}$ possesses a best proximity point $\mathcal{A}$ whenever $a_0 \in \mathcal{A}$ with $a_0 \in C_{\mathcal{F}}$ exists.*

Proof. Again, using the partially ordered version of the proof of this theorem mentioned by Abkar and Gabeleh [1, 2] and replacing the hypothesises of graphs, we can prove the existence of a best proximity point by a well-known way of computations in graphical best proximity point theory. □

Remark 2.10. Note that we can obtain some consequences from Theorem 2.9 regarding special graphs similar to discussion mentioned for the first main result. For example, if we take $G = G_1$ in Theorem 2.9, we can obtain Theorem 3.5 of Abkar and Gabeleh [2]. Again, if we put $\varphi(f) = (1-t)f$, then we obtain Theorem 4.3 of Abkar and Gabeleh [1].

It is not difficult to construct a few examples to demonstrate the validity of the main results and also to develop some applications to demonstrate their effectiveness. For example, one can refer to the examples of Abkar and Gabeleh's papers [1, 2] or Fallahi et al's works [4, 5]. Moreover, we can obtain integral version of contractions mentioned above as some new applications that left to the reader.

References

1. A. Abkar and M. Gabeleh, *Best proximity points for cyclic mappings in ordered metric spaces*, J. Optim Theory Appl., **151** (2011), 418-424.
2. A. Abkar and M. Gabeleh, *Generalized cyclic contractions in partially ordered metric spaces*, Optim Lett., **6** (2012), 1819-1830.
3. A. A. Eldred and P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl., **323** (2006), 1001-1006.
4. K. Fallahi and S. Jalali, *Graphical cyclic $\mathcal{J}$-integral Banach type mappings and the existence of their best proximity points*, J. Linear. Topol. Algebra., **13** (1) (2024), 19-28.
5. K. Fallahi, G. Soleimani Rad and A. Fulga, *Best proximity points for $(\varphi - \psi)$-weak contractions and some applications*, Filomat, **37** (6) (2023), 1835-1842.
6. J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136** (2008), 1359-1373.
7. W. A. Kirk, P. S. Srinivasan and P. Veeramani, *Fixed points for mappings satisfying cyclic contractive conditions*, Fixed Point Theory, **4** (1) (2003), 79-86.
8. T. Suzuki, M. Kikkawa and C. Vetro, *The existence of best proximity points in metric spaces with the property UC*, Nonlinear Anal., **71** (2009), 2918-2926.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

CLASSIFICATION OF JUST-INFINITE C^*-ALGEBRAS

HASAN KARIMI*

Department of Mathematics, Farhangian University, Tehran, Iran
hassankarimi7799@gmail.com

Abstract. In this paper, we examine the just-infinite dimensional C^*-algebras. More precisely, we describe a classification of just-infinite C^*-algebras in terms of their primitive ideal space along with some results.

1. Introduction

A C^*-algebra $\mathcal{A}$ is said to be primitive if it admits a faithful irreducible representation on some Hilbert space. It is said to be prime if, whenever I and J are closed two-sided ideals in $\mathcal{A}$ such that $I \cap J = 0$, then either $I = 0$, or $J = 0$. It is easy to see that every primitive C^*-algebra is prime, and it is a non-trivial result that the converse holds for all separable C^*-algebras; cf. [1, Proposition 4.3.6]. However, there are (complicated) examples of non-separable C^*-algebras that are prime, but not primitive, see [2]. A closed two-sided ideal I in a C^*-algebra $\mathcal{A}$ is said to be primitive if $I \neq \mathcal{A}$ and I is the kernel of an irreducible representation of $\mathcal{A}$ on some Hilbert space. The primitive ideal space, $\mathrm{Prim}(\mathcal{A})$, is the set of all primitive ideals in $\mathcal{A}$. A closed two-sided ideal I of $\mathcal{A}$ is primitive if and only if the quotient $\mathcal{A}/I$ is a primitive C^*-algebra. In particular, $0 \in \mathrm{Prim}(\mathcal{A})$ if and only if $\mathcal{A}$ is primitive.

2020 *Mathematics Subject Classification. Primary 46L05; Secondary 46L35*
Key words and phrases. C^*-algebra, Just-infinite, Residually finite dimensional, Primitive ideal space.
* Speaker.

A C^*-algebra $\mathcal{A}$ is said to be residually finite dimensional (RFD), if it admits a separating family of finite dimensional representations. The finite dimensional representations can be taken to be irreducible and pairwise (unitarily) inequivalent.

Assume that $\{\pi_i\}_{i\in\mathbb{I}}$ is a family of irreducible and pairwise inequivalent finite dimensional representations of a C^*-algebra $\mathcal{A}$. Let k_i be the dimension of the representation π_i, and identify the image of π_i with $M_{k_i}(\mathbb{C})$. We then get a *-homomorphism

$$\Psi_{\mathbb{I}} = \bigoplus_{i\in\mathbb{I}} \pi_i : \mathcal{A} \to \prod_{i\in\mathbb{I}} M_{k_i}(\mathbb{C})$$

Note that $\Psi_{\mathbb{I}}$ is injective if and only if $\bigcap_{i\in\mathbb{I}} Ker(\pi_i) = \{0\}$, which again happens if and only if $\{Ker(\pi_i) : i \in \mathbb{I}\}$ is a dense subset of $\mathrm{Prim}(\mathcal{A})$.

The purpose of this note is to investigate just-infinite dimensional C^*-algebras, defined to be infinite dimensional C^*-algebras for which all proper quotients by closed two-sided ideals are finite dimensional.

2. Main results

The following lemma holds:

Lemma 2.1. *A C^*-algebra $\mathcal{A}$ is RFD if and only if* $\mathrm{Prim}(\mathcal{A})$ *contains a dense subset $\mathcal{P}$ such that $\mathcal{A}/I$ is a full matrix algebra, for each $I \in \mathcal{P}$.*

Definition 2.2. A unital C^*-algebra $\mathcal{A}$ is said to be strictly residually finite dimensional (strictly RFD) if there exists an infinite family $\{\pi_i : \mathcal{A} \to M_{k_i}(\mathbb{C})\}_{i\in\mathbb{I}}$ of irreducible, pairwise inequivalent, finite dimensional representations of $\mathcal{A}$ such that the map

$$\Psi_{\mathbb{J}} = \bigoplus_{i\in\mathbb{J}} \pi_i : \mathcal{A} \to \prod_{i\in\mathbb{J}} M_{k_i}(\mathbb{C})$$

is surjective, for each infinite subset $\mathbb{J}$ of $\mathbb{I}$.

Proposition 2.3. *A unital separable C^*-algebra $\mathcal{A}$ is strictly* RFD *if and only if there exists an infinite subset $\mathcal{P}$ of* $\mathrm{Prim}(\mathcal{A})$ *such that each of its infinite subsets is dense in* $\mathrm{Prim}(\mathcal{A})$*, and such that $\mathcal{A}/I$ is finite dimensional, for each $I \in \mathcal{P}$.*

Definition 2.4. A C^*-algebra $\mathcal{A}$ is said to be just-infinite if it is infinite dimensional, and for each non-zero closed two-sided ideal I in $\mathcal{A}$, the quotient $\mathcal{A}/I$ is finite dimensional.

Lemma 2.5. *Every just-infinite C^*-algebra is prime.*

Remark 2.6. Note that no commutative C^*-algebra is just-infinite, since no commutative C^*-algebra other than $\mathbb{C}$ is prime. This also shows that no commutative C^*-algebra has a just-infinite quotient.

The primitive ideal space of a just-infinite C^*-algebra turn out to be homeomorphic to one of the T_0-spaces in the following class:

Example 2.7. For each $n \in \{0, 1, 2, ..., \infty\}$, consider the T_0-space Y_n defined to be the disjoint union $Y_n = \{0\} \cup Z_n$, where Z_n is a set with n elements, if n is finite, and Z_n has countably infinitely many elements, if $n = \infty$. Equip Y_n with the topology for which the closed subsets of Y_n are precisely the following sets: $\emptyset, Y_n$, and all finite subsets of Z_n. In the following take Z_n to be $\{1, 2, ..., n\}$, if $1 \leq n < \infty$, and Z_∞ to be $\mathbb{N}$.

Lemma 2.8. *Let $\mathcal{A}$ be a separable C^*-algebra. The following hold:*

(i) Prim*($\mathcal{A}$) is homeomorphic to Y_n, for some $n \in \{0, 1, 2, \cdots, \infty\}$ if and only if the following three conditions hold:*
 (a) *$\mathcal{A}$ is primitive,*
 (b) *$\mathcal{A}/I$ is simple, for each non-zero primitive ideal I in $\mathcal{A}$,*
 (c) *If* Prim*($\mathcal{A}$) is infinite, then $\bigcap_{I \in \mathcal{P}} I = 0$, for each infinite subset $\mathcal{P}$ of* Prim*($\mathcal{A}$).*

(ii) *If* Prim*($\mathcal{A}$) is infinite and $\mathcal{A}$ satisfies* (b) *and* (c)*, then it automatically satisfies* (a)*. If $\mathcal{A} = I$ is finite dimensional, for each non-zero $I \in \mathrm{Prim}(\mathcal{A})$, then condition* (b) *holds.*

(iii) *If $\mathcal{A}$ is just-infinite, then $\mathrm{Prim}(\mathcal{A}) = Y_n$, $n \in \{0, 1, 2, \cdots, \infty\}$.*

The just-infinite C^*-algebras are classified as follows(see [3]):

Theorem 2.9. *Let $\mathcal{A}$ be a separable C^*-algebra. Then $\mathcal{A}$ is just-infinite if and only if* Prim*($\mathcal{A}$) is homeomorphic to Y_n, for some $n \in \{0, 1, 2, ..., \infty\}$, and each non-faithful irreducible representation of $\mathcal{A}$ is finite dimensional. (If $n = 0$, we must also require that $\mathcal{A}$ is infinite dimensional; this is automatic when $n \geq 1$.) Moreover:*

(α) Prim*($\mathcal{A}$)$= Y_0$ if and only if $\mathcal{A}$ is simple. Every infinite dimensional simple C^*-algebra is just-infinite.*

(β) Prim*($\mathcal{A}$) $= Y_n$, for some integer $n \geq 1$, and $\mathcal{A}$ is just-infinite, if and only if $\mathcal{A}$ contains a simple non-zero essential infinite dimensional ideal I_0 such that $\mathcal{A}/I_0$ is finite dimensional. In this case, n is equal to the number of simple summands of $\mathcal{A}/I_0$.*

(γ) *The following conditions are equivalent:*
 (i) *$\mathcal{A}$ is just-infinite and* Prim*($\mathcal{A}$)$= Y_\infty$,*
 (ii) *$\mathcal{A}$ is just-infinite and RFD,*
 (iii) Prim*($\mathcal{A}$) is an infinite set, all of its infinite subsets are dense, and $\mathcal{A}/I$ is finite dimensional, for each non-zero $I \in \mathrm{Prim}(\mathcal{A})$,*
 (iv) Prim*($\mathcal{A}$) is an infinite set, the direct sum representation $\bigoplus_{i \in T} \pi_i$ is faithful for each infinite family $\{\pi_i\}_{i \in T}$ of pairwise inequivalent irreducible representations of A, and each non-faithful irreducible representation of A is finite dimensional.*

The following results follows immediately from Proposition 2.3 and Theorem 2.9:

Corollary 2.10. *Each separable* RFD *just-infinite C^*-algebra is strictly RFD.*

Note that not all strictly *RFD* C^*-algebras are just-infinite, see [3, Section 4.3].

Corollary 2.11. *The primitive ideal space of a separable just-infinite* C^**-algebra is countable. Moreover, any* RFD *just-infinite separable* C^**-algebra has countably infinitely many equivalence classes of finite dimensional irreducible representations.*

References

1. G. K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.
2. N. Weaver, *A prime C^*-algebra that is not primitive*, J. Funct. Anal., **203** (2) (2003), 356–361.
3. R. I. Grigorchuk, M. Musat and M. Rørdam, *Just-infinite C^*-algebras*, Comment. Math. Helv., **93** (1) (2018), 157–201.

A NOVEL INTELLIGENT COMPUTATIONAL TECHNIQUE FOR THERMAL ANALYSIS OF LONGITUDINAL FIN VIA CHEBYSHEV POLYNOMIALS WITH INTERIOR POINT ALGORITHM

MAHDI KESHTKAR* AND ELYAS SHIVANIAN

Department of Mathematics, Buein Zahra Technical University, Buein Zahra, Qazvin, Iran
Department of Applied Mathematics, Imam Khomeini International University, Qazvin, 34149-16818, Iran
keshtkarmahdi@gmail.com, shivanian@sci.ikiu.ac.ir

Abstract. In this work, heat transfer in a longitudinal rectangular fin with temperature-dependent thermal properties and internal heat generation is studied and more accurate results obtained in respect of the previous investigations. The advanced heat transfer models have been used to study the effects of thermo-geometric parameters, coefficient of heat transfer and thermal conductivity parameters on the temperature distribution, heat transfer and thermal performance of the longitudinal rectangular fin. It is applied a novel intelligent computational approach for searching the solution. In order to achieve this aim, the governing equation is transformed into an equivalent problem whose boundary conditions are such that they are convenient to apply reformed version of Chebyshev polynomials of the first kind. These Chebyshev polynomials based functions construct approximate series solution with unknown weights. The mathematical formulation of optimization problem consists of an unsupervised error which is minimized by tuning weights via interior point method. The trial approximate solution is validated by imposing tolerance constrained into optimization problem.

2010 *Mathematics Subject Classification. Primary 47A55; Secondary 39B52, 34K20.*
Key words and phrases. polynomial of the first kind; Interior point method; Heat transfer analysis.

* Speaker.

1. Introduction

Fins, at a very high frequency of occurrence, are encountered in many engineering applications to enhance heat transfer. Many researchers have assigned their works to the heat transfer treatment of the fins [1]. Consider the following energy equation

$$(1 + \beta\theta)\,\theta'' + \beta\left(\theta'\right)^2 - M^2\theta^{m+1} + M^2 Q\gamma\theta + M^2 Q = 0, \quad 0 < x < 1, \tag{1.1}$$

subject to the boundary conditions

$$\theta'(0) = 0, \quad \text{at the tip}, \tag{1.2}$$

$$\theta(1) = 1, \quad \text{at the base}. \tag{1.3}$$

The problem (1.1)-(1.3) has been exactly solved for some special cases. It has been very recently studied also numerically by applying Galerkin's method of weighted residual.

There are many techniques well-described in the literature which can be applied to solve the problem (1.1)-(1.3), some of them are semi-analytical methods such as Adomian decomposition method [2], homotopy perturbation method, differential transformation method and homotopy analysis method [3], and some of them are numerical ones such as finite difference method and spectral collocation method. While the main deficiency of the semi-analytical methods is being convergence hardly, the numerical methods in general works perfect, the convergence is still problem though.

In this article, the main aim is to propose a new intelligent computational approach to obtain solution for the non-linear second-order boundary value problem (1.1)-(1.3). First, we transform the governing equation into an equivalent problem whose boundary conditions are $[-1, 1]$. In this way, they are convenient to apply reformed version of Chebyshev polynomials of the first kind. Then we optimize Chebyshev polynomials of the first kind to construct approximate series solution with unknown weights. Furthermore, it is set up an optimization problem based on unsupervised error as objective function subject to a tolerance as constraint. This optimization problem is minimized by tuning weights via interior point method. Therefore, the current method is mostly convergence in according to both numerical and semi-analytical methods, hence, as a result, the method is more efficiency.

2. High order derivatives of Basis functions

Chebyshev polynomials are very useful as orthogonal polynomials on the interval $[-1, 1]$ of the real line. These polynomials have very good properties in the approximation of functions so that appear frequently in several fields of mathematics, physics and engineering.

2.1. Basic Properties of Chebyshev Polynomials.

The Chebyshev polynomials of the first kind, known as $T_n(x) = \cos(n \arccos x)$, can be obtained by means of Rodrigue's formula

$$T_n(x) = \frac{\Gamma(\frac{1}{2})}{(-2)^n\Gamma(n+\frac{1}{2})}\sqrt{1-x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(1-x^2\right)^{n-\frac{1}{2}}, \quad n = 0, 1, 2, \cdots. \tag{2.1}$$

The Chebyshev polynomials of the first kind can be developed by means of the generating function too, as follows:

$$\frac{1-tx}{1-2tx+t^2} = \sum_{n=0}^{+\infty} T_n(x)t^n. \tag{2.2}$$

The first two Chebyshev polynomials $T_n(x) = 1$ and $T_n(x) = x$ are known from (2.1), all other polynomials $T_n(x), n \geq 2$ can be obtained by means of the recurrence formula

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{2.3}$$

Theorem 2.1. ***(Slevinsky-Safouhi)*** *Let $G(x)$ be a function kth differentiable and with the term $\left(\frac{d}{x\,dx}\right)^k G(x)$ welldefined. The term $\frac{d^kG}{dx^k}$ can be expressed by:*

$$\frac{d^kG}{dx^k} = \sum_{i=\lfloor\frac{k+1}{2}\rfloor}^{k} \hat{A}_k^i x^{2i-k}\left(\frac{d}{x\,dx}\right)^k G(x), \tag{2.4}$$

3. Proposed Method

By the change of variable $x \mapsto \frac{1}{2}x + \frac{1}{2}$, the boundary value problem (1.1)-(1.3) can be rewritten as

$$4\frac{d^2\theta}{dx^2} + 4\beta\theta\frac{d^2\theta}{dx^2} + 4\beta(\frac{d\theta}{dx})^2 - M^2\theta^{m+1} + M^2Q(1+\gamma\theta) = 0, \tag{3.1}$$

$$\theta(1) = 1, \theta'(-1) = 0. \tag{3.2}$$

Define $\hat{T}_n,\ n \geq 1$ as

$$\hat{T}_n(x) = T_n(x) - (-1)^{n+1}n^2x + (-1)^{n+1}n^2 - 1, \quad n \geq 1, \tag{3.3}$$

Definition 3.1. Define a approximate series solution of order M as

$$\Theta_M(x) = \sum_{n=1}^{M} \alpha_n\hat{T}_n(x), \tag{3.4}$$

and consider the number of N regularly distributed nodal points in interval $[-1, 1]$, namely $x_i, i = 1, 2, ..., N$, then we define the unsupervised errors as the sum of mean squared errors:

$$\epsilon(N, \alpha) = \frac{1}{N}\sum_{i=1}^{N}\left\{4\sum_{n=1}^{M}\alpha_n\hat{T}_n''(x_i) + 4\beta(\sum_{n=1}^{M}\alpha_n\hat{T}_n(x_i)+1)\sum_{n=1}^{M}\alpha_n\hat{T}_n''(x_i) + 4\beta(\sum_{n=1}^{M}\alpha_n\hat{T}_n'(x_i))^2 - M^2(\sum_{n=1}^{M}\alpha_n\hat{T}_n(x_i)+1)^{m+1} + M^2Q\gamma(\sum_{n=1}^{M}\alpha_n\hat{T}_n(x_i)+1) + M^2Q\right\}^2, \tag{3.5}$$

Now, define the following optimization problems

$$\begin{aligned}&\min_{\alpha} \epsilon(N,\alpha)\\&\text{subject to} \quad \epsilon(N,\alpha)-\varepsilon \leq 0,\end{aligned} \tag{3.6}$$

where ε is a given tolerance. In our approach, the interior point method (IPM) is used for tuning of weights of the approximate series solution (3.4).

References

1. H. Chen, J. Ma and H. Liu, *Least square spectral collocation method for nonlinear heat transfer in moving porous plate with convective and radiative boundary conditions*, International Journal of Thermal Sciences **132** (2018), 335-343.
2. A.M. Wazwaz, *Adomian decomposition method for a reliable treatment of the Bratu-type equations*, Applied Mathematics and Computation **166** (3) (2005), 652-663.
3. S. Abbasbandy, E. Magyari and E Shivanian, *The homotopy analysis method for multiple solutions of nonlinear boundary value problems*, Communications in Nonlinear Science and Numerical Simulation **14** (9) (2009), 3530-3536.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

SOME FIXED POINT THEOREM ON B-METRIC SPACE VIA SIMULATION FUNCTION

PARVANEH LO′LO′* AND MARYAM SHAMS

Department of Mathematics, Behbahan Khatam Alanbia University of Technology, Behbahan, Iran
Department of Mathematics, University of Sharekord, Sharekord, Iran
lolo@bkatu.ac.ir; maryam.shams@sku.ac.ir

Abstract. In this work, we investigate the existence and uniqueness of a common fixed point of almost contractions via simulation functions in b-metric spaces. Moreover, some results and examples are given to support the availability of the obtained results.

1. Introduction

In the paper of Czerwik [4], the notion of b-metric space has been introduced and some fixed point theorems for single-valued and multi-valued mappings in b-metric spaces proved. Successively, this notion has been reintroduced by Khamsi and Hussain [5], with the name of metric-type space. Many researchers have been keeping their interest in finding the existence of fixed points of single-valued self-maps and set-valued mappings in b-metric space, we refer [2, 7].

Definition 1.1. [4] Let X be a non-empty set. A function $d_b : X \times X \to \mathbb{R}$ is said to be a b-metric if the following conditions are satisfied:

i) $d_b(x, y) \geq 0$ for all $x, y \in X$ and $d_b(x, y) = 0$ if and only if $x = y$,

2020 *Mathematics Subject Classification. Primary 47H10; Secondary 47H25*
Key words and phrases. complete b-metric space, common fixed point, simulation function.
* Speaker.

ii) $d_b(x,y) = d_b(y,x)$ for all $x, y \in X$,
iii) there exists $\rho \geq 0$ such that $d_b(x,t) \leq \rho[d_b(x,y) + d_b(y,t)]$ for all $x, y, t \in X$.

In this case, the pair (X, d_b, ρ) is called a b-metric space. Clearly, for $\rho = 1$, b-metric space is a metric space.

Definition 1.2. Let (X, d_b, ρ) be a b-metric space and $\{x_n\} \subseteq X$ a sequence

i) $\{x_n\} \subseteq X$ converges to a point $x \in X$ if $\lim_{n\to\infty} d_b(x_n, x) = 0$,
ii) $\{x_n\} \subseteq X$ is Cauchy if, for each $\varepsilon > 0$, there is some $n(\varepsilon) \in \mathbb{N}$ such that $d_b(x_n, x_m) < \varepsilon$ for all $m, n \geq n(\varepsilon)$,
iii) (X, d_b, ρ) is said to be complete if every Cauchy sequence is convergent in X.

In [7], lo′lo′ et al. extended the fixed point theorems for four mappings in b-metric space.

Theorem 1.3. [7] *Let (X, d_b, ρ) be a complete b-metric space. Let $h, k, M, N : X \to X$ be given mapping satisfying for each $x, y \in X$,*

$$d_b(hx, ky) \leq$$

$$\delta \max\{d(Mx, Ny), d(hx, Mx), d(Ny, ky), \frac{1}{2\rho}[d(Mx, ky) + d(hx, Ny)]\},$$

where δ is a non-negative number such that $\delta < \frac{1}{\rho}$. We suppose the following hypotheses:

i) *$\{h, M\}$ and $\{k, N\}$ commuting,*
ii) *h, k, M and N are continuous,*
iii) *$hX \subseteq Nx$ and $kX \subseteq MX$,*

Then h, k, M and N have a unique common fixed point.

In the following, we explain the concept of simulation function, which plays an essential role in proving the fixed point theorems of this work.

Definition 1.4. Let $\zeta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ be a function. Therefore ζ is named a simulation function if it satisfies the following conditions:

ζ_1) $\zeta(\alpha, \beta) < \beta - \alpha$, for all $\alpha, \beta > 0$
ζ_3) if $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, +\infty)$ such that $\lim_{n\to+\infty} \alpha_n = \lim_{n\to+\infty} \beta_n > 0$ then $\limsup_{n\to+\infty} \zeta(\alpha_n, \beta_n) < 0$.

In the beginning, the simulation function was defined by khojasteh et al. [6] as a mapping $\zeta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ satisfying $\zeta(0, 0) = 0$ besides the conditions (ζ_1) and (ζ_2) of Definition 1.4. In this work, a impproved definition of Argoubi et al. [1] is used.

Example 1.5. Let $\zeta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ be the function defined by

$$\zeta(\alpha, \beta) = \begin{cases} 1, & if (\alpha, \beta) = (0, 0), \\ \lambda\beta - \alpha, & otherwise, \end{cases}$$

where $\lambda \in]0,1[$. Then ζ is a simulation function with $\zeta(0,0) > 0$.

The following theorem is the main result of [6].

Theorem 1.6. *Let (X,d) be a complete metric space and $h : X \to X$ be a $\mathcal{Z}$-contraction with respect to a certain simulation function ζ, that is,*

$$\zeta(d(hx,hy), d(x,y)) \geq 0, \textit{for all } x,y \in X.$$

Then h has a unique fixed point.

More research on fixed point results via the simulation function can be found in [8].
In [9], Olgun et al. proved the following theorem in complete metric space.

Theorem 1.7. [9] *Let (X,d) be a complete metric space and $h : X \to X$ be a mapping. Suppose that there exists a simulation function ζ such that*

$$\zeta(d(hx,hy), L(x,y)) \geq 0 \quad \textit{for all } x,y \in X,$$

where

$$L(x,y) = \max\{d(x,y), d(x,hx), d(y,hy), \frac{1}{2}[d(x,hy) + d(y,hx)]\}.$$

Then h has a unique fixed point.

Recently, Babu et al.[3] extended the above theorem for two functions in b-metric space.

Theorem 1.8. [3] *Let (X, d_b, ρ) be a complete b-metric space and $h, k : X \to X$ be a mappings. Suppose that there exists a simulation function ζ such that*

$$\zeta(\rho^4 d_b(hx,ky), L(x,y)) \geq 0 \quad \textit{for all } x,y \in X,$$

where

$$L(x,y) = \max\{d_b(x,y), d_b(x,hx), d_b(y,ky), \frac{1}{2\rho}[d_b(x,ky) + d_b(y,hx)]\}.$$

If h or k is continuous, then h and k have a unique common fixed point.

In this work, by generalizing the ideas in [3] and [7] , we defin generalized contractions and prove the existence and uniqueness of the common fixed point for four non-self-mappings in b-metric space.

2. Main results

Theorem 2.1. *Let (X, d_b, ρ) be a complete b-metric space and $h, k, M, N : X \to X$ be given selfmappings. If there exists simulation function ζ such that*

$$\zeta(\rho^4 d_b(hx,ky), L(x,y)) \geq 0 \ \textit{forall } x,y \in X \tag{2.1}$$

where

$$L(x,y) =$$

$$\max\{d_b(Mx,Ny), d_b(hx,Mx), d_b(Ny,ky), \frac{1}{2\rho}[d_b(Mx,ky) + d_b(hx,Ny)]\}$$

We suppose the following hypotheses:

i) *M and N commute with h and k, respectively,*
ii) *$hX \subseteq Nx$ and $kX \subseteq MX$,*
iii) *h, k, M and N are continuous,*

Then h, k, M and N have unique common fixed point.

Remark 2.2. By setting $M = N = I$ in Theorem2.1, the main result of [3], Theorem 1.8, is obtained.

Theorem 2.3. *Let (X, d_b, ρ) be a complete b-metric space and $h, k, M, N : X \to X$ be given selfmappings. If there exists simulation function ζ and a nonnegative number $s < \frac{1}{\rho}$ such that*

$$\zeta(d_b(hx, ky), sL(x, y)) \geq 0 \ forall \ x, y \in X \tag{2.2}$$

where

$$L(x, y) = \max\{d_b(Mx, Ny), d_b(hx, Mx), d_b(Ny, ky), \frac{1}{2\rho}[d_b(Mx, ky) + d_b(hx, Ny)]\}$$

We suppose the following hypotheses:

i) *M and N commute with h and k, respectively,*
ii) *$hX \subseteq Nx$ and $kX \subseteq MX$,*
iii) *h, k, M and N are continuous,*

Then h, k, M and N hav unique common fixed point.

Remark 2.4. Theorem 2.1 of [7] can be easily concluded using Theorem 2.3 and property ζ_1 of simulation function ζ.

References

1. H. Argoubi, B. Samet and C. vetro, *Nonlinear contractions involving simulation function in metric space with a partial order,* J. Nonlinear Sci. Appl., **8** (2015), 1082-1094
2. H. Aydi, M.F. Bota, E. Karapinar and S. Mitroviȁc, *A fixed point theorem for set-valued quasicontractions in b-metric spaces*, Fixed Point Theory Appl., **2012** (88) (2012).
3. G. V. R. Babu, T. M. Dula and P. S. Kumar, *A Common fixed point theorem in b-metric spaces via simulation function*, Journal of Fixed Point Theory and Applications, 12(2018), 2018:12.
4. S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostrava, **1** (1993), 5-11.
5. M.A. Khamsi, N. Hussain, *KKM mappings in metric type spaces*, Nonlinear Anal., **73** (2010), 3123-3129.
6. F. Khojasteh, S. Shukla and S. Radenović,*A new approach to the study of fixed point theorems via simulation functions.* Filomat 29. 1189-1194(2015)
7. P. Lo′lo′, S. M. Vaezpour and J. Esmaily, *Common best proximity points theorem for four mappings in metric-type spaces*, Fixed Point Theory Appl. 2015, 2015, 47
8. P. Lo′lo′, M. Shams and M. De la Sen,*Existence of a solution for a nonlinear integral equation by nonlinear contractions involving simulation function in partially order metric space*, Nonlinear Anal. Model. Control, **28** (3) (2023), 578–596.
9. M. Olgun, O. Bicer and T. Alyildiz, *A new aspect to Picard operators with simulation functions*, Turk. J. Math., **40** (2016), 832-837.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

SOLUTIONS OF VARIATIONAL INEQUALITIES IN TOPOLOGICAL VECTOR SPACE

M. MANSOURIRAD*AND A.P. FARAJZADEH

Department of Mathematics, Razi University, Iran/67149-Kermanshah
maryammansourirad@gmail.com; farajzadehali@gmail.com

Abstract. In this paper, we first extend the definition of pseudomonotone Brezis in topological vector spaces. Then, we introduce three kinds of variationational inequality problems in the setting of topological vector spaces by investigating The relationship between their solution sets. By using the KKM theory some existence theorems for a solution of the introduced variational inequality problems are provided. The convexity and compactness of the solution sets of the variational inequality problems under suitable assumptions are given. Finally, it is shown that all weak solutions of this problem are strong by using the B-pseudomonotonicity and Sion lemma.

1. Introduction

In order to investigate the existence of solutions of $VI(T, C)$, we will focus on a notion of monotonicity property for the operator T that was introduced by Brézis for single-valued operators in 1968 (see [4]), called in the sequel B-pseudomonotonicity.
Let X be a real, topological vector space, with dual X^*, and write $\langle x, x^* \rangle$ in place of $x^*(x)$ for $x \in X$ and $x^* \in X^*$. We should be topologized in such a

2020 *Mathematics Subject Classification.* Primary 47B35; Secondary 30H05
Key words and phrases. variational inequality, B-pseudomonotone, coercivity condition.
* Speaker.

way that the canonical bilinear form $\langle .,. \rangle$ is continuous on $X \times X^*$ (we can consider, X^* with weak*-topology).

Definition 1.1. Let $C \subseteq X$ be a nonempty set. Given a set-valued map $T : X \rightrightarrows X^*$, with $C \subseteq dom(T) = \{x \in X \ : \ T(x) \neq \emptyset\}$. An element $\bar{x} \in C$ is a solution of

(a) variational inequality (VI(T,C)), if for all $y \in C$,

$$\sup_{x^* \in T(\bar{x})} \langle x^*, y - \bar{x} \rangle \ \geq 0.$$

(b) weak variational inequality (WVI(T,C)), if for every $y \in C$, there exists $x^*(y) \in T(\bar{x})$ satisfying $\langle x^*(y), y - \bar{x} \rangle \ \geq 0$.

(c) strong variational inequality (SVI(T,C)), if there exists $x^* \in T(\bar{x})$ satisfying $\langle x^*, y - \bar{x} \rangle \ \geq 0$ for all $y \in C$.

Remark 1.2. If T has weak*-compact values for all $x \in C$, then the continuouse functional $f : X^* \to \mathbb{R}$ with mapping $f(x^*) = \langle x^*, y - \bar{x} \rangle = (\widehat{y - \bar{x}})(x^*)$ with $VI(T, C)$ is equivalent to finding $\bar{x} \in C$, such that, for every $y \in C$, there exists $x^*(y) \in T(\bar{x})$ satisfying $\langle x^*(y), y - \bar{x} \rangle \geq 0$. In this case, $\bar{x}$ is said to be a weak solution of $VI(T, C)$. If $x^*(y)$ can be chosen independently of $y \in C$, then $\bar{x}$ is said to be a strong solution of $VI(T, C)$. Note that, by the Sion minimax theorem, every weak solution is a strong solution in case T is weakly*-compact and convex valued on C.

Definition 1.3. [1] Let X and Y be a topological vector space and B be the unite ball of Y, we shall say that a set-valued map T from X to Y is upper semi-continuous at $x_0 \in X$ if, for all$\epsilon > 0$, there exists a neighbourhood $N(x_0)$ of x_0 such that $T(x) \subset T(x_0) + \epsilon B$ for all $x \in N(x_0)$. it is upper semi-continuous if it is upper semi-continuous at all points $x_0 \in X$.

The following definition extendes of defintion of B-pseudomonotone given in [3].

Definition 1.4. We say that $T : X \rightrightarrows X^*$ is B-pseudomonotone on a nonempty subset D of $dom(T)$ if, for every net $\{x_i\}$ in D such that $x_i \rightharpoonup x \in D$, and for every $x_i^* \in T(x_i)$, with $\limsup_i \langle x_i^*, x_i - x \rangle \ \leq 0$, one has that for every $y \in D$, there exists $x^*(y) \in T(x)$ such that $\langle x^*(y), x - y \rangle \leq \liminf_i \langle x_i^*, x_n - y \rangle$.

It is obvious that the solution set of VI(T,C) associated to (T, C) equals to finding the set of all elements:

$$\text{(EP)} \quad x \in C \text{ such that } G_T(x, y) \geq 0 \text{ for all } y \in C,$$

where $G_T : C \times C \to \mathbb{R} \cup \{+\infty\}$ given by

$$G_T(x, y) = \sup_{x^* \in T(x)} < x^*, y - x >,$$

It is clear that the solution set of (EP) for G_T equals to $\bigcap_{y\in C} F(y)$, where $F : C \to 2^C$ with $F(y) = \{x \in C : G_T(x,y) \geq 0\}$. In the other words,

$$(\bar{x} \in \cap_{y\in C} F(y) \Leftrightarrow G_T(\bar{x},y) \geq 0 \quad \forall y \in C).$$

The following result plays a crucial role in proving existence theorems for (EP).

Theorem 1.5. [5] *In a Hausdorff topological vector space, let Y be a convex set and $\emptyset \neq X \subset Y$. For each $x \in X$, let $F(x)$ be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1, x_2, ..., x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^n F(x_i)$ (that is KKM property). If there is a non-empty subset X_0 of X such that the intersection $\bigcap_{x\in x_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y, then $\bigcap_{x\in X} F(x) \neq \emptyset$.*

2. Main results

The following result provides sufficient conditions which under them the solution set of (EP) for G_T is nonempty and compact. Further, it has an important role in next theorem.

Theorem 2.1. *Let C be a nonempty, closed and convex subset of a Hausdorff topological vector space E, and $F : C \to 2^C$ with mapping $F(y) = \{x \in C : G_T(x,y) \geq 0\}$, be satisfying the following assumptions:*

(i) *for all $y \in C$ the set $\{x \in C : G_T(x,y) \geqslant 0\}$ is closed;*
(ii) *for all $x \in C$ the set $\{y \in C : G_T(x,y) < 0\}$ is convex;*
(iii) *there exist $D, K \subset C$, such that D is convex, compact and K is compact, and for every $x \in C \setminus D$, there exists $y \in K, G_T(x,y) < 0$.*

Then, there exists $\bar{x} \in C$ such that $G_T(\bar{x},y) \geqslant 0$ for all $y \in C$.

Now, we are ready to present our main theorem which is a generalization of reflexive Banach spaces to Hausdorff topological spaces (Theorem 2 in [3]) by omitting the B-pseudomonotonicity and a weaker coercivity condition. Moreover, our assumptions guarantee the solution set of VI(T,C) is compact and convex.

Theorem 2.2. *Let $T : X \rightrightarrows X^*$ and C be a nonempty, closed and convex subset of dom(T). Suppose that:*

(i) *$T(x)$ is weak*-compact for every $x \in C$;*
(ii) *T is closed;*
(iii) *there exist $D, K \subset C$, such that D is convex, compact and K is compact, and for every $x \in C \setminus D$, there exists $y \in K, \sup_{x^*\in T(x)} \langle x^*, y - x\rangle < 0$.*

Then, the solutions of $VI(T,C)$ is nonempty, convex and compact.

Remark 2.3. It is straightforward, by using the B-pseudomonotonicity and Sion Lemma, to show that all weak solutions of this problem are strong.

Now, we going to state different version of Theorem 4 in [2] from reflexive Banach spaces to Hausdorff topological vector spaces by reducing the use of Hausdorff distance properties, moreover, our assumptions are guaranteed to hold for weak solutions.

Theorem 2.4. *Let C_k, C be nonempty, closed and convex subsets of X, and $T_k, T : X \rightrightarrows X^*$ be set-valued mappings, and $J : X \to X^*$ is a monotone mapping, such that*

(i) *$C_k \subseteq dom(T_k)$, and $C \cup \{\bigcup_k C_k\} \subseteq dom(T)$;*
(ii) *$\alpha_k \langle J(x), x_k \rangle \to 0$, for each $x \in \bigcap_k C_k$, and $\{x_k\}$ unbounded sequence of $\bigcup_k C_k$, $\alpha_k \to 0$;*
(iii) *if $\bigcup_k C_k$ is unbounded, the following coercivity holds: there exists $\tilde{x} \in \bigcap_k C_k$ such that, for each unbounded sequence $\{x_k\}$ of $\bigcup_k C_k$;*

$$\limsup_k G_T(x_k, \tilde{x}) < 0.$$

If x_k is a weak solution of $VI(T_k + \alpha_k J, C_k)$, for every $k \in \mathbb{N}$, then, $\{x_k\}$ is bounded.

Example 2.5. An example that has weak solution, is not strong solution. by considering $X = \mathbb{R}$ and $C = [-1, 1]$, with mapping $T(x) = \mathbb{R} - \{0\}$.

Now, we provided conditions leading to the weak convergence of a sequence of strong approximate solutions of $VI(T_k + \alpha_k J, C_k)$ to a weak (strong) solution of the original variational inequality $VI(T, C)$.

Proposition 2.6. *Let $T : X \rightrightarrows X^*$ be a set-valued B-pseudomonotone mapping with compact and convex values. Then the set of strong solutions of $VI(T, C)$ is weakly closed.*

Theorem 2.7. *Let $T : X \rightrightarrows X^*$ be upper semi-continiuos mapping with compact values. Then $S(T, C)$ is closed.*

Theorem 2.8. *Let $T : X \rightrightarrows X^*$ be an operator satisfying the conditions*

i. *T is B-pseudomonotone on $C \subseteq dom(T)$;*
ii. *$T(x)$ is weakly compact and convex, for every $x \in C$.*

Then $S(T, C)$ is weakly closed.

References

1. Aubin, Jean-Pierre. Optima and equilibria: an introduction to nonlinear analysis, Springer Science , Business Media, 140 2002.
2. M. Bianchi, G. Kassay, and R. Pini, *Regularization of Brezis pseudomonotone variational inequalities*, Set-Valued and Variational Analysis, **29** (2021), 175-190.
3. M. Bianchi, I. Konnov, and R. Pini, *On a threshold descent method for quasi-equilibria*, Optimization Letters, (2023), 1-15.
4. Brezis, Haïm,Equations et inéquations non linéaires dans les espaces vectoriels en dualité, Annales de l'institut Fourier, **18**(1) (1968), 115-175.
5. Ky Fan, *Some properties of convex sets related to fixed point theorems*, Mathematische Annalen, **266**(4) (1984), 519-537.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

(T,α)-DUALS OF GENERALIZED FRAMES IN HILBERT SPACES

M. MIRZAEE AZANDARYANI AND M. POURGHOLAMHOSSEIN*

Department of Mathematics, University of Qom, Qom, Iran
m.mirzaee@qom.ac.ir; purgol@yahoo.com

Abstract. In this paper, the concept of (T,α)-duals for g-frames is considered and studied, where T is an invertible operator on a separable Hilbert space and α is an integer. Mainly, we focus on direct sums and tensor products of (T,α)-duals.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space and let I be a finite or countable index set. A family $\mathcal{F}=\{f_i\}_{i\in I}\subseteq\mathcal{H}$ is a *frame* for $\mathcal{H}$, if there exist two positive numbers A and B such that

$$A\|f\|^2\leq\sum_{i\in I}|\langle f,f_i\rangle|^2\leq B\|f\|^2,$$

for each $f\in\mathcal{H}$. A and B are the *lower* and *upper* frame bounds, respectively, (see [5, 3]).

One of the most important generalizations of frames is g-frame introduced in [9].

For each $i\in I$, let $\mathcal{H}_i$ be a Hilbert space. In this paper, $L(\mathcal{H},\mathcal{H}_i)$ is the set of all bounded operators from $\mathcal{H}$ into $\mathcal{H}_i$ and $L(\mathcal{H},\mathcal{H})$ is denoted by $L(\mathcal{H})$.

2020 *Mathematics Subject Classification.* 42C15

Key words and phrases. Frame, Generalized frame, (T,α)-dual, tensor product, direct sum.

* Speaker.

Definition 1.1. We call $\Lambda = \{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ a *generalized frame* or a *g-frame* for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ if there exist two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2,$$

for each $f \in \mathcal{H}$. If only the second inequality is required, we call it a *g-Bessel sequence* with upper bound B.

Note that

$$\oplus_{i\in I}\mathcal{H}_i := \left\{\{f_i\}_{i\in I} | f_i \in \mathcal{H}_i, \|\{f_i\}_{i\in I}\|_2{}^2 = \sum_{i\in I} \|f_i\|^2 < \infty\right\}$$

with pointwise operations and the inner product defined by

$$\langle\{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle := \sum_{i\in I}\langle f_i, g_i\rangle$$

is a Hilbert space. If $\mathcal{H}_i = \mathcal{H}$ for each $i \in I$, we denote $\oplus_{i\in I}\mathcal{H}_i$ by $\ell^2(I, \mathcal{H})$.

A g-Bessel sequence $\Gamma := \{\Gamma_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called an *alternate g-dual* or a *g-dual* for a g-Bessel sequence Λ if

$$f = \sum_{i\in I} \Gamma_i^*\Lambda_i f = \sum_{i\in I} \Lambda_i^*\Gamma_i f,$$

for each $f \in \mathcal{H}$.

For a g-Bessel sequence $\Lambda := \{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ the *synthesis operator* is $T_\Lambda : \oplus_{i\in I}\mathcal{H}_i \longrightarrow \mathcal{H}$, $T_\Lambda(\{f_i\}_{i\in I}) := \sum_{i\in I} \Lambda_i^* f_i$ and its adjoint operator which is $T_\Lambda^*(f) = \{\Lambda_i f\}_{i\in I}$ is called the *analysis operator* of Λ. The operator S_Λ is defined by $S_\Lambda := T_\Lambda T_\Lambda^*$. If Λ is a g-frame, then S_Λ is invertible.

Direct sums and tensor products of g-frames have been studied recently (see [1, 6, 7] and the references therein). In the present note, we obtain some results for the tensor product and direct sum of (T, α)-duals for g-frames, mostly, the obtained results for duals in [6, 7] and for α-duals in [8] are generalized to (T, α)-duals.

2. Main results

First, we recall the next definition from [4].

Definition 2.1. Let $\{f_i\}_{i\in I}$ be a frame in a Hilbert space $\mathcal{H}$. Then, a frame $\{g_i\}_{i\in I}$ is called a generalized dual for $\{f_i\}_{i\in I}$ if there is an invertible operator on $\mathcal{H}$ like T such that

$$f = \sum_{i\in I}\langle Tf, g_i\rangle f_i, \quad \forall f \in \mathcal{H}.$$

We recall the following definition from [2].

Definition 2.2. Let $\alpha \in \mathbb{Z}$ and let $\Lambda = \{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame. A g-frame $\Gamma = \{\Gamma_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called an α-dual of $\{\Lambda_i\}_{i\in I}$ if $\sum_{i\in I} \Lambda_i^* \Gamma_i f = S_\Lambda^\alpha f$, for each $f \in \mathcal{H}$.

Now, we state the definition of (T, α)-duals of a g-frame.

Definition 2.3. Assume that T is an invertible operator on a Hilbert space $\mathcal{H}$. Let $\alpha \in \mathbb{Z}$ and let $\Lambda = \{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame. A g-frame $\Gamma = \{\Gamma_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a (T, α)-dual of $\{\Lambda_i\}_{i\in I}$ if $\sum_{i\in I} \Lambda_i^* \Gamma_i T f = S_\Lambda^\alpha f$, for each $f \in \mathcal{H}$.

In this paper I, J and I_k, for each $1 \leq k \leq n$, are finite or countable index sets. $\mathcal{H}$, $\mathcal{H}_j$, $\mathcal{H}_k$, $\mathcal{H}_{kj}$, $\mathcal{H}_{i(k)}$ and $\mathcal{H}_{i(k)j}$ are separable Hilbert spaces for each $j \in J$, $k \in \{1, \dots, n\}$ and $i(k) \in I_k$. $\Phi_j := \{\Lambda_{ij} \in L(\mathcal{H}_j, \mathcal{H}_{ij}) : i \in I\}$, $\Psi_j := \{\Gamma_{ij} \in L(\mathcal{H}_j, \mathcal{H}_{ij}) : i \in I\}$, $\Phi^{(k)} := \{\Lambda_{i(k)} \in L(\mathcal{H}_k, \mathcal{H}_{i(k)})\}_{i(k)\in I_k}$, $\Psi^{(k)} := \{\Gamma_{i(k)} \in L(\mathcal{H}_k, \mathcal{H}_{i(k)})\}_{i(k)\in I_k}$, $\otimes_{k=1}^n \Phi^{(k)}$ is

$$\{\Lambda_{i(1)} \otimes \ldots \otimes \Lambda_{i(n)} \in L(\otimes_{k=1}^n \mathcal{H}_k, \mathcal{H}_{i(1)} \otimes \ldots \otimes \mathcal{H}_{i(n)})\}_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)},$$

and $\Phi_j^{(k)} := \{\Lambda_{i(k)j} \in Ł(\mathcal{H}_{kj}, \mathcal{H}_{i(k)j})\}_{i(k)\in I_k}$.

Recall that if $\mathcal{H}_k$ is a Hilbert space for each $1 \leq k \leq n$, then the (Hilbert) tensor product $\otimes_{k=1}^n \mathcal{H}_k := \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$ is a Hilbert space. The inner product for simple tensors is defined by $\langle \otimes_{k=1}^n f_k, \otimes_{k=1}^n g_k \rangle := \Pi_{k=1}^n \langle f_k, g_k \rangle$, where $f_k, g_k \in \mathcal{H}_k$. If U_k is a bounded linear operator on $\mathcal{H}_k$, then the tensor product $\otimes_{k=1}^n U_k$ is a bounded linear operator on $\otimes_{k=1}^n \mathcal{H}_k$. Also $(\otimes_{k=1}^n U_k)^* = \otimes_{k=1}^n U_k^*$ and $\| \otimes_{k=1}^n U_k \| = \Pi_{k=1}^n \|U_k\|$.

Tensor products have important applications, for example tensor products are useful in the approximation of multi-variate functions of combinations of univariate ones.

Now, the next result for (T, α)-duals of g-frames are obtained.

Theorem 2.4. *Let T_k be an invertible operator on a Hilbert space $\mathcal{H}_k$, for each $k \in \{1, \dots, n\}$. Suppose that $\Phi^{(k)}$'s and $\Psi^{(k)}$'s are g-frames. If $\Psi^{(k)}$ is a (T_k, α)-dual of $\Phi^{(k)}$, for each $k \in \{1, \dots, n\}$, then $\otimes_{k=1}^n \Psi^{(k)}$ is a $(\otimes_{k=1}^n T_k, \alpha)$-dual of $\otimes_{k=1}^n \Phi^{(k)}$.*

The next result, obtained in [8], is a special case of the above theorem.

Corollary 2.5. *Suppose that $\Phi^{(k)}$'s and $\Psi^{(k)}$'s are g-frames. If $\Psi^{(k)}$ is an α-dual of $\Phi^{(k)}$, for each $k \in \{1, \dots, n\}$, then $\otimes_{k=1}^n \Psi^{(k)}$ is an α-dual of $\otimes_{k=1}^n \Phi^{(k)}$.*

Let $\Phi_j = \{\Lambda_{ij} \in L(\mathcal{H}_j, \mathcal{H}_{ij}) : i \in I\}$ be a g-Bessel sequence for $\mathcal{H}_j$, $j \in J$, with upper bound B_j such that $B := sup\{B_j : j \in J\} < \infty$. Then $\{\Phi_j\}_{j\in J}$ is called a *B-Bounded family of g-Bessel sequences* or shortly *B-BFGBS*.

Let $\Phi_j = \{\Lambda_{ij} \in L(\mathcal{H}_j, \mathcal{H}_{ij}) : i \in I\}$ be an (A_j, B_j) g-frame for $\mathcal{H}_j$, $j \in J$, such that $A := inf\{A_j : j \in J\} > 0$ and $B := sup\{B_j : j \in J\} < \infty$. Then we say that $\{\Phi_j\}_{j\in J}$ is an (A, B)-*bounded family of g-frames* or shortly (A, B)-*BFGF*.

Theorem 2.6. *Let $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ be BFGF. Assume that T_j is an invertible operator on $\mathcal{H}_j$ with $\sup_{j\in J}\|T_j\| < \infty$ and $\inf_{j\in J}\|T_j^{-1}\|^{-1} > 0$. If Ψ_j is a (T_j, α)-dual of Φ_j, for each $j \in J$, then $\oplus_{j\in J}\Psi_j := \{\oplus_{j\in J}\Gamma_{ij} : i \in I\}$ is a $(\oplus_{j\in J}T_j, \alpha)$-dual for $\oplus_{j\in J}\Phi_j := \{\oplus_{j\in J}\Lambda_{ij} : i \in I\}$.*

The following corollary, obtained in [8], is a special case of the above theorem.

Corollary 2.7. *Let $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ be BFGF. If Ψ_j is an α-dual of Φ_j, for each $j \in J$, then $\oplus_{j\in J}\Psi_j := \{\oplus_{j\in J}\Gamma_{ij} : i \in I\}$ is an α-dual for $\oplus_{j\in J}\Phi_j := \{\oplus_{j\in J}\Lambda_{ij} : i \in I\}$.*

Corollary 2.8. *Assume that $T_j^{(k)}$ is an invertible operator on $\mathcal{H}_{kj}$, for each $1 \le k \le n$ and for each $j \in J$ such that $\sup_{j\in J}\|T_j^{(k)}\| < \infty$ and $\inf_{j\in J}\|{T_j^{(k)}}^{-1}\|^{-1} > 0$. Let $\{\Phi_j^{(k)}\}_{j\in J}$ and $\{\Psi_j^{(k)}\}_{j\in J}$ be BFGF, for each $1 \le k \le n$ and let $\Phi_j^{(k)}$ be a $(T_j^{(k)}, \alpha)$-dual of $\Psi_j^{(k)}$, for each $j \in J$ and $k \in \{1, \ldots, n\}$. Then $\otimes_{k=1}^n(\oplus_{j\in J}\Phi_j^{(k)})$ is an $(\otimes_{k=1}^n(\oplus_{j\in J}T_j^{(k)}), \alpha)$-dual of $\otimes_{k=1}^n(\oplus_{j\in J}\Psi_j^{(k)})$.*

By considering $T_j^{(k)} = Id_{\mathcal{H}_{jk}}$, we get the next result which is Corollary 2.10 in [8].

Corollary 2.9. *Let $\{\Phi_j^{(k)}\}_{j\in J}$ and $\{\Psi_j^{(k)}\}_{j\in J}$ be BFGF, for each $1 \le k \le n$ and let $\Phi_j^{(k)}$ be an α-dual of $\Psi_j^{(k)}$, for each $j \in J$ and $k \in \{1, \ldots, n\}$. Then $\otimes_{k=1}^n(\oplus_{j\in J}\Phi_j^{(k)})$ is an α-dual of $\otimes_{k=1}^n(\oplus_{j\in J}\Psi_j^{(k)})$.*

References

[1] A. Abdollahi and E. Rahimi, *Generalized frames on super Hilbert spaces*, Bull. Malays. Math. Sci. Soc, **35** (2012), 807–818.

[2] M. R. Abdollahpour and A. Najati, *G-frames and Hilbert-Schmidt operators*, Bull. Iranian Math. Soc, **4** (2011), 141–155.

[3] I. Daubechies, A. Grossmann and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys, **27** (1986), 1271–1283.

[4] M. A. Dehghan and M. A. Hasankhani Fard, *G-dual frames in Hilbert spaces*, Politehn. Univ. Bucharest Sci. Bull. Ser A, **75** (2013), 129–140.

[5] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc, **72** (1952), 341–366.

[6] A. Khosravi and M. Mirzaee Azandaryani, *Fusion frames and g-frames in tensor product and direct sum of Hilbert spaces*, Appl. Anal. Discrete Math, **6** (2012), 287–303.

[7] A. Khosravi and M. Mirzaee Azandaryani, *G-frames and direct sums*, Bull. Malays. Math. Sci. Soc, **36** (2013), 313–323.

[8] M. Mirzaee Azandaryani and M. Pourgholamhossein, *Duality and α-duality of g-frames and fusion frames*, Mathematical Analysis & Convex Optimization, **4** (2023), 1–6.

[9] W. Sun, *G-frames and g-Riesz bases*, J. Math. Anal. Appl, **322** (2006), 437–452.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

SOME INEQUALITIES FOR DISTRIBUTION OF A MATRIX MEAN ON ANOTHER

HAMIDEH MOHAMMADZADEHKAN*

Department of Mathematics Payame Noor University, P.O. Box 19395-3697 Tehran, Iran.
h_mohamadzadeh@pnu.ac.ir; mohammadzadeh83@gmail.com

Abstract. Let $\mathcal{S}_\alpha(\ 0 \leq \alpha < \frac{\pi}{2})$ denote the set of all complex matrices, σ_1, σ_2 be matrix means and $A, B, C \in \mathcal{S}_\alpha$. We are investigating some inequalities, specifically if

$$\mathcal{R}\left((A\sigma_1 B)\sigma_2(A\sigma_1 C)\right) \leq \cos^4(\alpha) \ \mathcal{R}\left(A\sigma_1(B\sigma_2 C)\right);$$

then

$$\mathcal{R}\left((A\sigma_1^* B)\sigma_2^* \ (A\sigma_1^* C)\right) \geq \cos^4(\alpha) \ \mathcal{R}\left(A\sigma_1^*(B\sigma_2^* C)\right).$$

We also prove some results for the norm, determinant, and largest singular value of the above adjointation inequality mentioned above.

1. Introduction

A matrix $A \in \mathbb{M}_n$ is said to be accretive if in its Cartesian or Toeplitz decomposition, $A = \mathcal{R}A + i\mathcal{I}A$, $\mathcal{R}A > 0$, where $\mathcal{R}A = \frac{A+A^*}{2}$, $\mathcal{I}A = \frac{A-A^*}{2i}$; and A is called accretive–dissipative if $\mathcal{R}A$ and $\mathcal{I}A$ are both positive definite.

The numerical range of $A \in \mathbb{M}_n$ is defined as

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, \ \|x\| = 1\}.$$

The sector region $\mathcal{S}_\alpha$ is defined as follows:

$$\mathcal{S}_\alpha = \{z \in \mathbb{C} : \mathcal{R}z > 0, \quad |\mathcal{I}z| \leq (\mathcal{R}z)\tan(\alpha)\}.$$

2020 *Mathematics Subject Classification.* 47A63, 47A64, 15B485

Key words and phrases. Inequality mean, Matrix mean, Sector matrix.

* Speaker.

If $W(A) \subset \mathcal{S}_\alpha$ for some $0 \le \alpha < \frac{\pi}{2}$, A is called a sectorial matrix and is denoted by $A \in \mathcal{S}_\alpha$.

A matrix mean σ is defined by an matrix monotone function $f : (0, \infty) \to (0, \infty)$ with $f(1) = 1$ as

$$A\sigma B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

for positive invertible matrices A and B. The function f that satisfies the conditions is called the "representing function" of σ. The matrix mean with representing function $f(t^{-1})^{-1}$ is called the adjoint of σ and denoted by σ^*. In other words

$$A\sigma^* B = (A^{-1}\sigma B^{-1})^{-1} \qquad \textit{for invertible } A \textit{ and } B.$$

Lemma 1.1. *([1]), Let $A, B \in \mathcal{S}_\alpha$. Then $A\sigma B \in \mathcal{S}_\alpha$ and*

$$\mathcal{R}A\sigma\mathcal{R}B \le \mathcal{R}(A\sigma B) \le \sec^2(\alpha)\ (\mathcal{R}A\sigma\mathcal{R}B).$$

Lemma 1.2. *([1]), Let $f : (0, \infty) \to (0, \infty)$ is an operator monotone function with $f(1) = 1$ and $A, B \in \mathcal{S}_\alpha$ for some $0 \le \alpha < \frac{\pi}{2}$. Then*

$$f(\mathcal{R}A) \le \mathcal{R}(f(A)) \le \sec^2(\alpha)\ f(\mathcal{R}A)$$

Lemma 1.3. *([5]), Let $\alpha \ge 1$. If $f : (0, \infty) \to (0, \infty)$ is operator monotone function, then $f(\alpha t) \le \alpha f(t)$.*

Lemma 1.4. *([3]), Let σ_1 and σ_2 be nonzero connections. Then the following statements are equivalent:*

(i) $A\sigma_1(B\sigma_2 C) \le (A\sigma_1 B)\sigma_2(A\sigma_1 C)$ *for all* $A, B, C \ge 0$.
(ii) $A\sigma_1^*(B\sigma_2^* C) \ge (A\sigma_1^* B)\sigma_2^*(A\sigma_1^* C)$ *for all* $A, B, C \ge 0$.

2. Main results

In Lemma 1.4, Chansangiam established equivalent inequalities for positive matrices. Now we will prove the following theorem for sector matrices.

Theorem 2.1. *Let σ_1 and σ_2 be nonzero connections and $A, B, C \in \mathcal{S}_\alpha$. If*

$$\mathcal{R}\left((A\sigma_1 B)\sigma_2(A\sigma_1 C)\right) \le \cos^4(\alpha)\ \mathcal{R}\left(A\sigma_1(B\sigma_2 C)\right),$$

then

$$\mathcal{R}\left((A\sigma_1^* B)\sigma_2^*\ (A\sigma_1^* C)\right) \ge \cos^4(\alpha)\ \mathcal{R}\left(A\sigma_1^*(B\sigma_2^* C)\right).$$

Theorem 2.1 is extension of (2) $\Rightarrow$ (1) of [3, Theorem 3].

As a special case, if A is accretive-dissipative, then $e^{-i\frac{\pi}{4}} A \in \mathcal{S}_{\frac{\pi}{4}}$.

Corollary 2.2. *Let σ_1 and σ_2 be nonzero connections and A, B and C be accretive-dissipative. If*

$$4\mathcal{R}\left((A\sigma_1 B)\sigma_2(A\sigma_1 C)\right) \le \mathcal{R}\left(A\sigma_1(B\sigma_2 C)\right),$$

then

$$4\mathcal{R}\left((A\sigma_1^* B)\sigma_2^*\ (A\sigma_1^* C)\right) \ge \mathcal{R}\left(A\sigma_1^*(B\sigma_2^* C)\right).$$

Proposition 2.3. *Let σ_1 and σ_2 be nonzero connections and $A, B, C \in \mathcal{S}_\alpha$. If $f : (0, \infty) \to (0, \infty)$ is an operator monotone function with $f(1) = 1$, such that*

$$\mathcal{R}\left((A\sigma_1 B)\sigma_2(A\sigma_1 C)\right) \leq \cos^4(\alpha)\ \mathcal{R}\left(A\sigma_1(B\sigma_2 C)\right),$$

then

$$\mathcal{R}\left(f((A\sigma_1^* B)\sigma_2^*\ (A\sigma_1^* C))\right) \geq \cos^6(\alpha)\ \mathcal{R}\left(f(A\sigma_1^*(B\sigma_2^* C))\right).$$

A norm $\|.\|$ defined on $\mathbb{M}_n$ is said to be unitarily invariant if $\|UAV\| = \|A\|$ for any $A \in \mathbb{M}_n$ and for all unitary matrices $U, V \in \mathbb{M}_n$.

Next, by applying the following lemma, we get the norm version of Theorem 2.1.

Lemma 2.4. *([2]), Let $A \in \mathcal{S}_\alpha$. Then*

$$\|\mathcal{R}A\| \leq \|A\| \leq sec(\alpha)\ \|\mathcal{R}A\|.$$

for any unitarily invariant norm $\|.\|$ on $\mathbb{M}_n$.

Theorem 2.5. *Let σ_1 and σ_2 be nonzero connections and $A, B, C \in \mathcal{S}_\alpha$. If*

$$\mathcal{R}\left((A\sigma_1 B)\sigma_2(A\sigma_1 C)\right) \leq \cos^4(\alpha)\ \mathcal{R}\left(A\sigma_1(B\sigma_2 C)\right),$$

then

$$\|(A\sigma_1^* B)\sigma_2^*\ (A\sigma_1^* C)\| \geq \cos^5(\alpha)\|A\sigma_1^*(B\sigma_2^* C)\|.$$

The determinant version of Theorem 2.1 is obtained through the following lemma.

Lemma 2.6. *([4]), If $A \in \mathcal{S}_\alpha$, then*

$$det(\mathcal{R}A) \leq |det(A)| \leq sec^n(\alpha)\ det(\mathcal{R}A).$$

Theorem 2.7. *Let σ_1 and σ_2 be nonzero connections and $A, B, C \in \mathcal{S}_\alpha$. If*

$$\mathcal{R}\left((A\sigma_1 B)\sigma_2(A\sigma_1 C)\right) \leq \cos^4(\alpha)\ \mathcal{R}\left(A\sigma_1(B\sigma_2 C)\right),$$

then

$$|det\left((A\sigma_1^* B)\sigma_2^*\ (A\sigma_1^* C)\right)| \geq \cos^{5n}(\alpha)|det\left(A\sigma_1^*(B\sigma_2^* C)\right)|.$$

Using the following lemma, Theorem 2.9 can be derived.

Lemma 2.8. *([5]), Let $A \in \mathcal{S}_\alpha$. Then*

$$\lambda_j(\mathcal{R}A) \leq s_j(A) \leq sec^2(\alpha)\ \lambda_j(\mathcal{R}A), \qquad j = 1, \ldots, n,$$

where s_j and λ_j denote the jth largest singular value and eigenvalue of a matrix.

Theorem 2.9. *Let σ_1 and σ_2 be nonzero connections and $A, B, C \in \mathcal{S}_\alpha$. If*

$$\mathcal{R}\left((A\sigma_1 B)\sigma_2(A\sigma_1 C)\right) \leq \cos^4(\alpha)\ \mathcal{R}\left(A\sigma_1(B\sigma_2 C)\right),$$

then

$$s_j\left((A\sigma_1^* B)\sigma_2^*\ (A\sigma_1^* C)\right) \geq \cos^6(\alpha)s_j\left(A\sigma_1^*(B\sigma_2^* C)\right).$$

It is well known that the numerical radius $\omega(A)$ of $A \in \mathbb{M}_n$ is defined by

$$\omega(A) = \sup\{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}.$$

When $A \in \mathcal{S}_0$, we have $\omega(A) = \|A\|$, and therefore $\omega(\mathcal{R}A) = \|\mathcal{R}A\|$. Bedrani et al. [1] showed that if $A \in \mathcal{S}_\alpha$, then

$$\omega(\mathcal{R}A) \leq \omega(A) \leq \sec(\alpha)\ \omega(\mathcal{R}A). \tag{2.1}$$

By inequality (2.1) and Theorem 2.1, we obtain the following theorem.

Theorem 2.10. *Let σ_1 and σ_2 be nonzero connections and $A, B, C \in \mathcal{S}_\alpha$. If*

$$\mathcal{R}\left((A\sigma_1 B)\sigma_2(A\sigma_1 C)\right) \leq \cos^4(\alpha)\ \mathcal{R}\left(A\sigma_1(B\sigma_2 C)\right),$$

then

$$\omega\left((A\sigma_1^* B)\sigma_2^*\ (A\sigma_1^* C)\right) \geq \cos^5(\alpha)\omega\left(A\sigma_1^*(B\sigma_2^* C)\right).$$

References

1. Y. Bedrani, F. Kittaneh, and M. Sababeh, *From positive to accretive matrices*, Positivity, **25**(4) (2021), 1601-1629.
2. R. Bhatia, *Matrix analysis* , Springer, New York (1997).
3. P. Chansangiam, *Adjointations of operator inequalities and characterizations of operator monotonicity via operator means*, Commun. Math. Appl., **7**(2) (2016), 93.
4. R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press., Cambridge, 2013.
5. H. Jafarmanesh, M. Khosravi and A. Sheikhhosseini, *Some operator inequalities involving operator monotone functions*, Bull. Sci. Math., **166** (2021), 102938.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

SOME FIXED POINT THEOREMS FOR GENERALIZED HYBRID MAPPINGS IN THE FRAMEWORK OF CONVEX METRIC SPACES

MOHAMMAD MOOSAEI*

Department of Mathematics, Faculty of Science, Bu-Ali Sina University, Iran
m.moosaei@basu.ac.ir

ABSTRACT. In this manuscript, we first introduce $(\alpha, \beta)-$generalized hybrid mappings in the framework of convex metric spaces, next we show that if S a self-mapping of a nonempty closed convex subset D of a uniformly convex complete metric space (M, ρ, W), then S has at least a fixed point if and only if there is some z in D such that the sequence $\{S^n(z)\}_{n=1}^{\infty}$ is bounded, moreover; if D is bounded, then the fixed points set of S is nonempty, closed and convex.

1. Introduction and Preliminaries

In 1970, Takahashi [8] introduced the notion of convexity in metric spaces and proved that normed spaces and convex subsets are convex metric spaces. He also presented some examples of convex metric spaces which are not embedded in any normed/Banach spaces. Let D be a nonempty subset of a Hilbert space $\mathbb{H}$, and let S be a self-mapping of D. The set of all fixed points of S is denoted by $F(S) := \{x \in D : S(x) = x\}$. The mapping S is called *nonexpansive* if $\|S(x) - S(y)\|d \leq \|x - y\|$ for all $x, y \in D$. The mapping S is said to be *firmly nonexpansive* if $\left\|S(x) - S(y)\right\|^2 \leq \left\langle x - y, S(x) - S(y)\right\rangle$ for

2010 *Mathematics Subject Classification. Primary 47H09; Secondary 47H10*

Key words and phrases. Asymptotic center, Convex metric space, Fixed point, Uniformly convex metric space .

* Speaker.

all $x, y \in D$; see [3]. In 2008, Kohsaka and Takahashi [5] introduced a new class of nonlinear operators which contains the class of firmly nonexpansive mappings in a Hilbert space. They were called *nonspreading* mappings, as follows.

$$2\|S(x) - S(y)\|^2 \leq \|x - S(y)\|^2 + \|y - S(x)\|^2$$

for all $x, y \in D$. In 2010, Takahashi [9] introduced the class of hybrid mappings in Hilbert spaces which is defined as follows. The mapping S is said to be *hybrid* if

$$3\|S(x) - S(y)\|^2 \leq \|x - y\|^2 + \|x - S(y)\|^2 + \|y - S(x)\|^2$$

for all $x, y \in D$. In 2010, Aoyoma *et al.* [2] given the class of λ-hybrid mappings in a Hilbert space. This class of mappings contains the classes of nonexpansive mappings, nonspreading mappings and hybrid mappings as follows. The mapping S is said to be λ-*hybrid* if there exists an $\lambda \in \mathbb{R}$ such that

$$(1 + \lambda)\|S(x) - S(y)\|^2 - \lambda\|x - S(y)\|^2 \leq (1 - \lambda)\|x - y\|^2 + \lambda\|y - S(x)\|^2$$

for all $x, y \in D$. We observe that 0-hybrid, 1-hybrid and $\frac{1}{2}$-hybrid are nonexpansive, nonspreading and hybrid, respectively. In 2010, Kocurek *et al.* [4] given a more extensive class of nonlinear operators than the class of λ-hybrid mappings as follows. The mapping S is called an (α, β)-generalized hybrid if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|S(x) - S(y)\|^2 + (1 - \alpha)\lambda\|x - S(y)\|^2 \leq \beta\|S(x) - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in D$. It is obvious that an $(1 + \lambda, \lambda)$-generalized hybrid mapping is an λ-hybrid mapping. One can easy to show that the class of (α, β)-generalized hybrid mappings includes the classes of nonexpansive mappings, nonspreading mappings, hybrid mappings and λ-hybrid mappings.

The purpose of this manuscript is to study the existence theorems for (α, β)- generalized hybrid mappings in uniformly convex complete metric spaces which are more general than Hilbert spaces.

Before going to the main results, let us recall some notations, definitions and results which will be need in the sequel.

Throughout this paper, we denote by $\mathbb{R}$ and $\mathbb{N}$ the set of all real numbers and the set of all positive integers, respectively'. Let D be a nonempty subset of a metric (M, ρ), and S be a mapping from D into itself. Let $m \in M, r > 0$. The closed ball centred at m with radius r is denoted by $B[m; r]$ and is as follows,

$$B[m; r] := \{x \in M : \rho(x, m) \leq r\}.$$

The distance m from D and the set of all fixed points of S are denoted by $dist(m, D)$ and $F(D)$, respectively and are as follows,

$$dist(m, D) := \{\rho(x, m) : x \in M\} \text{ and } F(S) := \{x \in D : S(x) = x\}.$$

The mapping S is called *nonexpansive*if $\rho(S(x),S(y)) \leq \rho(x,y)$ for all $x,y \in D$. The map S is called *quasi-nonexpansive* if $F(S)$ is nonempty and $\rho(S(x),p) \leq \rho(x,p)$ for all $x \in D, p \in F(S)$. S is said to be an (α,β)-*generalized hybrid* mapping if

$$\alpha\rho^2(S(x),S(y)) + (1-\alpha)\rho^2(x,S(y)) \leq \beta\rho^2(S(x),y) + (1-\beta)\rho^2(x,y)$$

for all $x,y \in D$, where $\alpha,\beta \in \mathbb{R}$ and $\rho^2(x,y) := (\rho(x,y))^2$.

Let D be a nonempty subset of a metric space (M,ρ) and $\{x_n\}$ be a bounded sequence in M. Define the function $r_a(.,\{x_n\}) : M \to \mathbb{R}^+$ by

$$r_a(x,\{x_n\}) = \limsup_{n\to\infty} \rho(x_n,x) \text{ for every } x \in M.$$

The infimmum of $r_a(.,\{x_n\})$ over D is called the asymptotic radius of $\{x_n\}$ with respect to D and denoted by $r_a(D,\{x_n\})$. A point u in D is called an asymptotic center of the sequence $\{x_n\}$ with respect to D if

$$r_a(u,\{x_n\}) = r_a(D,\{x_n\}).$$

The set of all asymptotic centers of $\{x_n\}$ with respect to D is denoted by $Z_a(D,\{x_n\})$.

The concept of convexity in a metric space was introduced by Takahashi [8] and is as follows.

Definition 1.1. Let (M,ρ) be a metric space and let $I = [0,1]$. A mapping $W : M \times M \times I \to M$ is said to be a convex structure on M, if

$$\rho(z,W(x,y,\lambda)) \leqslant \lambda\rho(z,x) + (1-\lambda)\rho(z,y)$$

for all $x,y,z \in X$ and $\lambda \in I$. The metric space (M,ρ) together with a convex structure W is called a convex metric space and denoted by (M,ρ,W). A nonempty subset D of M is said to be convex if $W(x,y,\lambda) \in D$ for all $x,y \in D$ and $\lambda \in I$.

Let M be a convex metric space. Takahashi [8] proved that the open balls and the closed balls are convex subsets of M. If $\{D_\alpha\}_{\alpha\in J}$ is a family of convex subsets of M, then $\bigcap_{\alpha\in J} D_\alpha$ is a convex subset of M (see [8, 1] for more details). All normed spaces and their convex subsets are convex metric spaces, but there are some examples of convex metric spaces which are not embedded in any normed space (see [8]).

Definition 1.2. Let (M,ρ,W) be a convex metric space. A real-valued function f on a nonempty convex D of M is called convex if it satisfies

$$f(W(x,y,\lambda)) \leqslant \lambda f(x) + (1-\lambda)f(y)$$

for each $x,y \in D$ and $\lambda \in I$.

Definition 1.3. (see [7]) A convex metric space (M,ρ,W) is said to have property (C) if every decreasing sequence of nonempty bounded closed convex subsets of M has nonempty intersection.

Definition 1.4. (see [6]) A convex metric space (M, ρ, W) is called strictly convex if

$$\rho(a, W(x, y, \frac{1}{2})) < r$$

for all $x, y, a \in M$ with $\rho(x, a) = \rho(y, a) = r$, $x \neq y$ and $r > 0$.

Definition 1.5. (see [7]) A convex metric space (M, ρ, W) is said to be uniformly convex if for any $r > 0$, $\epsilon \in (0, 2]$, there exists $\alpha = \alpha(\epsilon) \in (0, 1)$ such that for all $x, y, z \in X$ with $\rho(z, x) \leqslant r$, $\rho(z, y) \leqslant r$ and $\rho(x, y) \geqslant r\epsilon$, we have

$$\rho(z, W(x, y, \frac{1}{2})) < r(1 - \alpha).$$

It is obvious that uniform convexity implies strict convexity.

The following example can be found in [7].

Example 1.6. (i) Uniformly convex Banach spaces are uniformly convex complete metric spaces.
(ii) Let $\mathbb{H}$ be a Hilbert space and M be a nonempty closed subset of the unit sphere of H such that if $x, y \in M, \lambda \in I$, then

$(\lambda x + (1 - \lambda)y)/\|\lambda x + (1 - \lambda)y\| \in M$ and $diam(M) \leqslant \sqrt{2}/2$, where $diam(M)$ is the diameter of M.

Put $\rho(x, y) = \cos^{-1}(< x, y >)$ for each $x, y \in M$, where $<, >$ is the inner product in $\mathbb{H}$. The mapping W defined by

$$W(x, y, \lambda) = \frac{\lambda x + (1 - \lambda)y}{\|\lambda x + (1 - \lambda)y\|}$$

for all $(x, y, \lambda) \in M \times M \times I$, is a structure on M, and (M, ρ, W) is a uniformly convex complete metric space .

We will need the following lemma and theorem.

Lemma 1.7. *(see [8]) Let M be a convex metric space. Then for all $x, y \in M$, and $\lambda \in I$, the following statements hold:*
(i) $\rho(x, y) = \rho(x, W(x, y, \lambda)) + \rho(y, W(x, y, \lambda))$.
(ii) $\rho(x, W(x, y, \lambda)) = (1 - \lambda)\rho(x, y)$ and $\rho(y, W(x, y, \lambda)) = \lambda\rho(x, y)$.

Theorem 1.8. *(see [7]) Let M be a uniformly convex complete metric space. Then M has property (C).*

2. Main results

In this section, we suppose that D is a nonempty subset of a convex metric space (M, ρ, W) and S is a self-mapping on D.

The proof of the following lemma is clear.

Lemma 2.1. *If the mapping S is $(\alpha, \beta)-$generelaized hybrid with $F(S) \neq \emptyset$, then S is quasi-nonexpansive.*

Lemma 2.2. *Suppose that the set D is closed and convex and M is strictly convex. If the mapping S is $(\alpha, \beta)-$generelaized hybrid, then the fixed point set of S is also closed and convex.*

Proposition 2.3. *Let D be a closed convex subset of M and S be an $(\alpha, \beta)-$generelaized hybrid with $\alpha \in \mathbb{R} \sim (0,1), \beta \in I$. Then $Z_a(D, \{x_n\})$ is closed, convex. Moreover; it is S-invariant if the sequence $\{S^n(x)\}$ is bounded for some $x \in D$.*

The following theorems are main results of this section.

Theorem 2.4. *Let D be a closed convex subset of a uniformly convex complete metric space (M, ρ, W) and S be an $(\alpha, \beta)-$generelaized hybrid with $\alpha \in \mathbb{R} \sim (0,1), \beta \in I$. Then the set of fixed points of S, namely $F(S)$, is nonempty if only if there exists $z \in D$ such that the sequence $\{S^n(z)\}$ is bounded. Moreover; it is closed and convex.*

The following theorem is a direct consequence of the above theorem.

Theorem 2.5. *Let D be a closed convex bonded subset of a uniformly convex complete metric space (M, ρ, W) and S be an $(\alpha, \beta)-$generelaized hybrid with $\alpha \in \mathbb{R} \sim (0,1), \beta \in I$. Then $F(S)$ is nonempty, closed and convex.*

References

1. R. P. Agarwal, D. O'Regan and D. R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer, New York, 2003.
2. K. Aoyama, S. Iemoto, F. Kohsaka and W. Takahashi, *Fixed point and ergodictheorems for λ-hybrid mappings in Hilbert spaces*, J. Nonlinear convex Anal, **11** (2010), 335-343.
3. F. E. Browder, *Convergence theorems for sequences of nonliner operators in Banach spaces*, Math. Z, **100** (1967), 201-225.
4. P. Kocourek, W. Takahashi and J.-C. Yao, *Fixed point theorems and weak convergencetheorems for generalized hybrid mappings in Hilbert spaces*, Taiwanes J. Math, **14** (2010), 2497-2511.
5. F. Kohsaka and W. Takahashi, *Fixed point theorems for a class ofnonlinear mappingsrelated to maximal monotone operators in Banach spaces*, Arch. Math. (Basel), **91** (2008), 166-177.
6. M. Moosaei, *The structure of fixed points set of a type of generalized nonexpansive mappings in convex metric spaces*, Thai. J. Math, **19** (2) (2021), 315-324.
7. T. Shimizu and W. Takahashi, *Fixed points of multivalued mappings in certain convex metric spaces*, Topol. Methods Nonlinear Anal, **8** (1996), 197-203.
8. T. Takahashi, *A convexity in metric spaces and nonexpansive mappings I*, Kodai Math. Sem. Rep, **22** (1970), 142-149.
9. W. Takahashi, *Fixed point theorems for new nonlinear mappingsin a Hilbert space*, J. Nonlinear convex Anal, **11** (2010), 79-88.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

STRONG CONVERGENCE OF EXTENDED PROXIMAL POINT ALGORITHMS

SIROUS MORADI* AND BEHZAD DJAFARI ROUHANI

Department of Mathematics, Faculty of Sciences, Lorestan University, Khorramabad, 6815144316, Iran
moradi.s@lu.ac.ir, sirousmoradi@gmail.com
Department of Mathematical Sciences, University of Texas at El Paso, El Paso, Texas 79968, USA
behzad@math.utep.edu

Abstract. In this paper we consider an extended proximal point algorithm with errors for amaximal monotone operator in a real Hilbert space, previously studied by Boikanyo and Morosanu, and Kanzow and Shehu. Also, we give a necessary and sufficient condition for the common zero set of finite operators to be nonempty, and by showing that in this case, this iterative sequence converges strongly to the metric projection of some point onto the set of common zeros of operators.

1. Introduction and Preliminaries

In 1970, Rockafellar [1] proved that the subdifferential of a proper, convex and lower semicontinuous function is a maximal monotone operator. The proximal point algorithm (PPA) was introduced by Martinet [2] for minimizing a convex function on $\mathbb{R}^n$. One of the most effective iterative methods for approximating the zeroes of a maximal monotone operator is the proximal point algorithm (PPA) which was initiated by Martinet [2] and subsequently

2020 *Mathematics Subject Classification.* *47J25, 47H05, 47H09, 90C29, 90C90.*
Key words and phrases. Maximal monotone operator, Proximal point algorithm, Resolvent operator.
* Speaker.

improved by Rockafellar [3]. By assuming $A^{-1}(0) \neq \emptyset$, in 2010, Boikanyo and Morosanu [4] studied the convergence of a PPA with errors for a maximal monotone operator A. In this paper, we significantly improve their results by giving a necessary and sufficient condition for the zero set of A to be nonempty, and by showing that in this case, this PPA converges strongly to the metric projection of u onto the zero set of A, without assuming the boundedness of the error sequence.
Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. An operator $T : D(T) \subseteq H \rightrightarrows H$ is said to be monotone if its graph $G(T)$ is a monotone subset of $H \times H$. It is clear that if T is monotone, then its inverse defined by $T^{-1} := \{(y,x) : (x,y) \in G(T)\}$ is also a monotone operator. We say that T is maximal monotone if T is monotone and the graph of T is not properly contained in the graph of any other monotone operator. It is known that in Hilbert space, this is equivalent to the range of the operator $(I+T)$ being all of H, where I is the identity operator on H, i.e. $R(I+T) = H$. Also it is clear that T is maximal monotone if and only if T^{-1} is maximal monotone.
For a maximal monotone operator T, and for every $t > 0$, the operator $J_t : H \longrightarrow H$ defined by $J_t^T(x) := (I+tT)^{-1}(x)$ is well defined, single valued and nonexpansive on H. It is called the resolvent of T.
Consider the following set valued problem:

$$find\, x \in D(T)\, such\, that\, 0 \in T(x). \tag{1.1}$$

One of the most effective iterative methods for solving problem (1.1) is the proximal point algorithm (abbreviated PPA). The PPA generates an iterative sequence (x_n) as follows:

$$x_{n+1} = J_{\gamma_n}^T(x_n + e_n), \tag{1.2}$$

for all $n \geq 0$, where $x_0 \in H$ is a given starting point, $(\gamma_n) \subset (0,+\infty)$ and (e_n) is a sequence of computational errors.

Recently, Boikanyo and Morosanu [4] considered the sequence generated by the following algorithm:

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J_{\beta_n}^T(x_n) + e_n,\ n \geq 0 \tag{1.3}$$

where $x_0, u \in H$, $\alpha_n \in (0,1)$, $\beta_n \in (0,+\infty)$, for all $n \geq 0$. They showed that if $T^{-1}(0) \neq \emptyset$ and $\alpha_n \to 0$, $\beta_n \to +\infty$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and either $\sum_{n=0}^{\infty} \|e_n\| < \infty$ or $\frac{\|e_n\|}{\alpha_n} \to 0$, then (x_n) converges strongly to an element of $T^{-1}(0)$ which is nearest to u.
In this paper, we first introduce our regularization method which provides also an affirmative answer to an open question raised by Boikanyo and Morosanu [4, p. 640], concerning the design of a PPA where $\lim_{n\to+\infty} \|e_n\| = 0$ and the sequence (c_n) is bounded. Also our method will provide us with a second affirmative answer to the open question of Boikanyo and Morosanu.

2. Main results

In the following theorem, we give a necessary and sufficient condition for the zero set of T to be nonempty, in which case we show the strong convergence of the sequence generated by (2.1).

Theorem 2.1. *Let $T : D(T) \subseteq H \rightrightarrows H$ be a maximal monotone operator. For any fixed $x_0, u \in H$, let the sequence (x_n) be generated by*

$$x_{n+1} = J^T_{c_n}((1-t_n)x_n + t_n u + e_n), \tag{2.1}$$

for all $n \geq 0$, where $t_n \in (0,1)$, $c_n \in (0,+\infty)$ and $e_n \in H$ for all $n \geq 0$. Then the following statements hold:
(i) If

$$\limsup_{m\to\infty} \sum_{k=1}^{m} (1-t_k)(1-t_{k+1})...(1-t_m) < \infty, \tag{2.2}$$

$\lim_{n\to+\infty} c_n = +\infty$ and $(e_n) \subset H$ is bounded, then $T^{-1}(0) \neq \emptyset$ if and only if $\liminf_{n\to\infty}(\|x_{n+1}\| + \|x_n\|) < \infty$ if and only if (x_n) is bounded.
(ii) If $F := T^{-1}(0) \neq \emptyset$, then for every sequence $(t_n)_{n=1}^\infty \subset (0,1)$, $(c_n)_{n=1}^\infty \subset (0,\infty)$ and $(e_n) \subset H$ where (2.2) holds, $\lim_{n\to+\infty} c_n = +\infty$ and $\lim_{n\to+\infty} \frac{\|e_n\|}{t_n} = 0$, we have that the sequence (x_n) converges strongly to $P_F u$.

Remark 2.2. If $(t_n)_{n=1}^\infty \subset (0,1)$ and $\liminf_{n\to+\infty} t_n \geq \alpha$ for some $\alpha \in (0,1)$, then we clearly have:

$$\limsup_{m\to\infty} \sum_{k=1}^{m} (1-t_k)(1-t_{k+1})...(1-t_m) < \infty. \tag{2.3}$$

However, from (2.3) we cannot conclude that $\liminf_{n\to+\infty} t_n \geq \alpha$ for some $\alpha \in (0,1)$. For example, if $t_n = \frac{1}{ln(n+2)}$ then the condition (2.3) holds, but $\lim_{n\to\infty} t_n = 0$.
Therefore, the following corollary is a direct consequence of Theorem 2.1.

Corollary 2.3. *Let $T : D(T) \subseteq H \rightrightarrows H$ be a maximal monotone operator. For any fixed $x_0, u \in H$, let the sequence (x_n) be generated by*

$$x_{n+1} = J^T_{c_n}((1-t_n)x_n + t_n u + e_n), \tag{2.4}$$

for all $n \geq 0$, where $t_n \in (0,1)$, $c_n \in (0,+\infty)$ and $e_n \in H$ for all $n \geq 0$. Then the following statements hold.
(i) If $\liminf_{n\to+\infty} t_n \geq \alpha$ for some $\alpha \in (0,1)$, $\lim_{n\to+\infty} c_n = +\infty$ and $(e_n) \subset H$ is bounded, then $T^{-1}(0) \neq \emptyset$ if and only if $\liminf_{n\to\infty}(\|x_{n+1}\| + \|x_n\|) < \infty$ if and only if (x_n) is bounded.
(ii) If $F := T^{-1}(0) \neq \emptyset$ then for every sequence $(t_n)_{n=1}^\infty \subset (0,1)$, $(c_n)_{n=1}^\infty \subset (0,\infty)$ and $(e_n) \subset H$ where $\liminf_{n\to+\infty} t_n \geq \alpha$ for some $\alpha > 0$, $\lim_{n\to+\infty} c_n = +\infty$, and $\lim_{n\to+\infty} \|e_n\| = 0$, we have that the sequence (x_n) generated by (2.4) converges strongly to $P_F u$.

Alike of above corollary, we can prove a similar theorem for the scheme (1.3) considered by Boikanyo and Morosanu [4].

The following examples show that without additional assumptions, we cannot replace the condition $\lim_{n\to+\infty} c_n = +\infty$ with the boundedness condition for (c_n). In the first example $T^{-1}(0) = \emptyset$, and in the second one $T^{-1}(0) \neq \emptyset$.

Example 2.4. Let $T : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $Tx = 1$. Obviously T is a maximal monotone operator. By taking $c_n = 1$, $e_n = 0$, $t_n = \frac{1}{2}$ for all $n \geq 0$, $x_0 = 0$ and $u = 0$ we have $x_{n+1} = \frac{1}{2}x_n - 1$ (the sequence (x_n) is generated by (2.1)). Then (x_n) is a decreasing sequence, and obviously $\lim_{n\to+\infty} x_n = -2$, but $T^{-1}(0) = \emptyset$.

Example 2.5. Let $T : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $Tx = x + 1$. Obviously T is a maximal monotone operator. By taking $c_n = 1$, $e_n = 0$, $t_n = \frac{1}{2}$ for all $n \geq 0$, $x_0 = 1$ and $u = 0$ we have $x_{n+1} = \frac{1}{4}x_n - \frac{1}{2}$ (the sequence (x_n) is generated by (2.1)). Then (x_n) is a decreasing sequence, and obviously $\lim_{n\to+\infty} x_n = \frac{-2}{3}$, but $T^{-1}(0) = \{-1\}$.

In the following theorem, we give another necessary and sufficient condition for the zero set of T to be nonempty, and show the strong convergence of the corresponding PPA.

Theorem 2.6. *Let $T : D(T) \subseteq H \rightrightarrows H$ be a maximal monotone operator. For any fixed $x_0, y_0, z_0, u \in H$, let the sequences (x_n), (y_n) and (z_n) be generated by*

$$x_{n+1} = J^T_{c_n}((1-t_n)x_n + t_n u + e_n), \tag{2.5}$$

$$y_{n+1} = J^T_{c_n}((1-t_n)y_n), \tag{2.6}$$

$$z_{n+1} = J^T_{\gamma}((1-t_n)z_n), \tag{2.7}$$

for all $n \geq 0$, where $t_n \in (0,1)$, $c_n \in (\gamma, +\infty)$ for some $\gamma \in (0, +\infty)$, and $e_n \in H$ for all $n \geq 0$. Suppose that

$$\limsup_{m\to\infty} \sum_{k=1}^{m} (1-t_k)(1-t_{k+1})...(1-t_m) < \infty. \tag{2.8}$$

Then the following statements hold:
(i) If (e_n) is bounded, then (x_n) is bounded if and only if (y_n) is bounded. Also if (y_n) is bounded then (z_n) is bounded too.
(ii) If (e_n) is bounded, $\lim_{n\longrightarrow\infty} t_n = 0$ and

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < +\infty \tag{2.9}$$

then $F = T^{-1}(0) \neq \emptyset$ if and only if (x_n) is bounded.
(iii) If $\lim_{n\to\infty} t_n = 0$, $\lim_{n\to\infty} \frac{\|e_n\|}{t_n} = 0$, the inequality (2.9) holds and $F = T^{-1}(0) \neq \emptyset$, then $s-\lim_{n\longrightarrow\infty} x_n = P_F(0)$ and $s-\lim_{n\longrightarrow\infty} y_n = P_F(0)$.

Remark 2.7. The above theorem provides another affirmative answer to the open question raised by Boikanyo and Morosanu [4, p. 640].

3. Applications

We can apply the main results to find a minimizer of a function f. Let $f : H \longrightarrow (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. Then the subdifferential $\partial f : H \rightrightarrows H$ defined as follows:

$$\partial f(z) := \{\zeta \in H : f(y) - f(z) \geq (\zeta, y - z), \forall y \in H\}. \tag{3.1}$$

We know that ∂f as a proper, convex and l.s.c. is maximal monotone. Also

$$z \in argminf := \{x \in H : f(x) \leq f(y), \forall y \in H\} \Longleftrightarrow 0 \in \partial f(z). \tag{3.2}$$

Therefore the proximal point algorithm for $\partial f(z)$ provides a scheme for approximating a minimizer of f. Following we present one of these applications.

Theorem 3.1. *Let H be a real Hilbert space and $f : H \longrightarrow (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. For any $x_0, u \in H$, let the sequence (x_n) be generated by (2.1) for $T = \partial f$, where $u \in H$, $(t_n)_{n=1}^{\infty} \subset (0, 1)$ and $(c_n)_{n=1}^{\infty} \subset (0, \infty)$ such that (2.2) holds and $c_n \to +\infty$ as $n \to +\infty$. Suppose that $(e_n)_{n=1}^{\infty} \subset H$ is a sequence with $\lim_{n\to+\infty} \frac{\|e_n\|}{t_n} = 0$. If (x_n) is bounded, then $argminf \neq \emptyset$ and (x_n) converges strongly to $P_F u$, the metric projection of u onto $F := \partial f^{-1}(0) = argminf$.*

4. Future Direction

As a future direction for research, since numerous other PPA have been developed and their convergence studied by many authors, it might be interesting to investigate the possibility of implementing the ideas and methods developed in this paper to these other PPA, as well as extending these methods to more general spaces, such as Banach and metric spaces.

References

1. O. A. Boikanyo and G. Morosanu, *A proximal point algorithm converging strongly for general errors*, Optim. Lett., **4** (2010), 635–641.
2. B. Martinet, *Règularisation dinéquations variationnelles par approximations successives* Rev. Française Informat. Recherche Opérationnelle **4** (1970), 154–158.
3. R. T. Rockafellar,*On the maximal monotonicity of subdifferential mappings* Pac. J. Math., **33** (1970), 209–216.
4. R. T. Rockafellar, *Monotone operators and the proximal point algorithm* SIAM J. Control Optim., **14** (1976), 877–898.
5. H. K. Xu, *A regularization method for the proximal point algorithm*, J. Global Optim., **36** (2006), 115–125.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

REGULARIZED TERNARY DYNAMICAL SYSTEMS ON HILBERT MODULES

MAYSAM MOSADEQ*

Department of Mathematics, Larestan Branch, Islamic Azad University, Larestan, Iran
maysammosaddegh@yahoo.com

Abstract. In this talk, we introduce the concept of the implemented Hilbert modules. We closely examine the concepts of regularized ternary dynamical systems of Hilbert modules and show that ternary derivations are appeared as the infinitesimal generator of regularized ternary dynamical systems. Let X and Y be full Hilbert modules over C^*-algebras A and B, respectively. We bring forward a one to one correspondence between the groups $TU(X)$ and $TU(Y)$ to show that each regularized ternary dynamical system on X implements a regularized ternary dynamical system on Y.

1. Introduction

Let X be a Banach space. A one parameter family $\{\alpha_t\}_{t\in\mathbb{R}}$ of bounded linear operators on X is called a *regularized one parameter group* if there exists an injective bounded linear operator c on X such that $\alpha_0 = c$ and $c\alpha_{t+s} = \alpha_t\alpha_s$ for every $t, s \in \mathbb{R}$. For convenience, we say that such a regularized one parameter group is a c-one parameter group. A c-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ is called strongly continuous if $\lim_{t\to 0}\alpha_t(x) = c(x)$, for each $x \in X$. The infinitesimal generator δ of a c-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ is

2020 *Mathematics Subject Classification. Primary 46L08; Secondary 47D03, 46L57*
Key words and phrases. C^*-Dynamical systems, Hilbert C^*-module, ternary derivation, one parameter group.

* Speaker.

a mapping $\delta : D(\delta) \subseteq X \to X$ such that $\delta(x) = c^{-1} \lim_{t \to 0} \dfrac{\alpha_t(x) - c(x)}{t}$ where $D(\delta) = \{x \in X : \lim_{t \to 0} \dfrac{\alpha_t(x) - c(x)}{t}$ exists in the range of $c\}$.

The notion of regularized semigroups was introduced by Davies and Pang in 1987 [3]. Trivially, if c is the identity operator on X, then a regularized one parameter group is nothing than a one parameter group in the usual sense (see [8, p. 8]).

One parameter groups of bounded linear operators and their extensions are of more considerable magnitude because of their applications in the theory of dynamical systems. The classical C^*-dynamical systems are expressed by means of strongly continuous one parameter groups of $*$-automorphisms on C^*-algebras. On the other hand, the infinitesimal generator of a C^*-dynamical system is a closed densely defined $*$-derivation.

As an extension of C^*-algebras it can be pointed to Hilbert C^*-modules. A (left) Hilbert C^*-module over a C^*-algebra A is an algebraic left A-module X equipped with an A-valued inner product $\langle .,. \rangle$ which is A-linear in the first and conjugate linear in the second variable such that X is Banach space with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. The Hilbert A-module X is called *full* if $A_X := span\{\langle x, x\rangle : x, y \in X\}$ is dense in A. Note that A_X is an ideal in A, called the *range ideal* of X. We denote by $\langle X, X\rangle$ the closure of A_X and call it the *support* of X. Therefore, X is a full Hilbert A-module if $\langle X, X\rangle$ is equal to A. The referred to [4] for more details on Hilbert C^*-modules.

Recently, some generalized notions of derivations have been investigated in the the setting of Hilbert C^*-modules. For instance it can be pointed to "ternary derivation" .

A *ternary derivation* is a densely defined linear map δ of X into X such that $\delta(\langle x, y\rangle z) = \langle x, y\rangle \delta(z) + (\langle \delta(x), y\rangle + \langle x, \delta(y)\rangle)z$ for each $x, y, z \in D(\delta)$ where $D(\delta)$ is a ternary subalgebra of X in the sense that $D(\delta)$ is invariant under the ternary product $(x, y, z) \to \langle x, y\rangle z$ (i.e. $\langle x, y\rangle z \in D(\delta)$ for every $x, y, z \in D(\delta)$). As an example, let δ be an adjointable operator with the adjoint $-\delta$. Then, δ is a ternary derivation.

In each case of generalization of derivations, a noted point drawing the attention of analysts is trying to represent a suitable dynamical system whose infinitesimal generator is exactly the desired extended derivation as well as being an extension of a C^*-dynamical system. Some approaches to preparing new dynamical systems and their applications have been explained in [1, 5, 6, 7] and references therein.

In the remainder of this section, we introduce some important classes of operators between Hilbert modules. Let Let X and Y be Hilbert modules over C^*-algebras A and B, respectively.

Let $\varphi : A \to B$ be an injective morphism of C^*-algebras. A surjective map $T : X \to Y$ is called *φ-unitary* if T is a *φ-morphism* in the sense that $\langle T(x), T(y)\rangle = \varphi(\langle x, y\rangle)$ for all $x, y \in X$.

It is notable that each φ-unitary map T is necessarily a linear operator, a φ-module map and an isometry. Moreover, if N is a full Hilbert B-module, then φ is surjective and so it is an isomorphism (see [2, Remark 2.9]).

Following [1], we call a bijective linear map $T : X \to Y$ a *ternary isomorphism* if $T(\langle x, y\rangle z) = \langle T(x), T(y)\rangle T(z)$ for all $x, y, z \in M$. As an example of a ternary isomorphism, let $T : M \to N$ be a φ-unitary operator. Hence, T is a bijection and $T(\langle x, y\rangle z) = \varphi(\langle x, y\rangle)T(z) = \langle T(x), T(y)\langle T(z)$ for all $x, y, z \in M$. We denote by $TU(X)$ the group of all bounded ternary automorphisms of X onto X.

In the present paper, we introduce the concept of the implemented Hilbert modules. We closely examine the concepts of regularized ternary dynamical systems of Hilbert modules and show that ternary derivations are appeared as the infinitesimal generator of regularized ternary dynamical systems. Let X and Y be full Hilbert modules over C^*-algebras A and B, respectively. We bring forward a one to one correspondence between the groups $TU(X)$ and $TU(Y)$ to show that each regularized ternary dynamical system on X implements a regularized ternary dynamical system on Y.

2. Main results

In the following Theorem, we apply an isomorphism between C^*-algebras A and B, and a bijection between two different Hilbert A-modules to construct a Hilbert B-module.

Theorem 2.1. *Suppose that X and Y are Hilbert A-module, $\varphi : A \to B$ is a linear isomorphism of C^*-algebras and $T : X \to Y$ is a bijective linear operator. Define the module action $b.y := T(\varphi^{-1}(b).T^{-1}(y))$ on Y. Then, Y equipped with the inner product $\langle z, w\rangle_B := \varphi(\langle T^{-1}(z), T^{-1}(w)\rangle_A)$ can also be regarded as a Hilbert B-module. Moreover, if X is a full A-module, then Y is a full Hilbert B-module.*

We call the above alternative Hilbert B-module Y *the Hilbert B-module implemented by* (φ, T) or briefly *the implemented Hilbert B-module* and denote it by $Y^{(\varphi,T)}$. Trivially, the map T in the implemented Hilbert B-module $Y^{(\varphi,T)}$ is a φ-unitary. Consequently, T is a ternary automorphism.

The next result establishes the converse of Theorem 2.1.

Theorem 2.2. *Let $(X$, be a full Finsler A-module, let Y be a Finsler B-module, and let $T : X \to Y$ be a bijective linear operator. If there exists a map $\varphi : A \to B$ such that $a.x = T^{-1}(\varphi(a).T(x))$ and $\varphi(\langle x, y\rangle_A) = \langle T(x), T(y)\rangle_B$ $(a \in A,\ x, y \in X)$, then φ is a $*$-isomorphism of C^*-algebras if and only if Y is full.*

the following theorem shows that each ternary derivation of a Hilbert A-module X defines a ternary derivation of a Hilbert B-module Y by a φ-unitary $T : X \to Y$.

Theorem 2.3. *Let $\varphi : A \to B$ be an injective morphism of C^*-algebras and let T be a φ-unitary from a Hilbert A-module X to Hilbert B-module Y. Suppose that $\delta_1 : D(\delta_1) \subseteq X \to X$ is a ternary derivation. Then, the mapping $\delta_2 : T(D(\delta_1)) \subseteq Y \to Y$ defined by $\delta_2(T(x)) := T(\delta_1(x))$ is a ternary derivation.*

The next result is an immediate consequence of the preceding theorem.

Corollary 2.4. *Let X be a Hilbert A-module and let $Y^{(\varphi,T)}$ be the implemented Hilbert B-module. Then, each ternary derivation of X defines a ternary derivation of $Y^{(\varphi,T)}$.*

The following theorem establishes a one to one correspondence between terenary automorphisms groups of possibly different full Hilbert modules.

Theorem 2.5. *Let $\varphi : A \to B$ be an isomorphism of C^*-algebras and T be a φ-unitary from a full Hilbert A-module X to a full Hilbert B-module Y. Then, the map $\Phi : TU(X) \to TU(Y)$ defined by $\Phi(S) = TST^{-1}$ is a group isomorphism.*

Definition 2.6. Let c be a ternary automorphisms on a full Finsler A-module X. A *regularized ternary dynamical system* is a mapping $t \to \alpha_t$ from the additive group $\mathbb{R}$ into the group $TU(X)$ of ternary automorphisms on X for which $\alpha_0 = c$ and $c\alpha_{t+s} = \alpha_t\alpha_s$ for every $t, s \in \mathbb{R}$, and $\lim_{t\to 0} \alpha_t(x) = c(x)$ for each $x \in X$. We define the *infinitesimal generator* δ of a regularized ternary dynamical system $\{\alpha_t\}_{t\in\mathbb{R}}$ as a mapping $\delta : D(\delta) \subseteq X \to X$ such that $\delta(x) = c^{-1} \lim_{t\to 0} \frac{\alpha_t(x) - c(x)}{t}$ where $D(\delta) = \{x \in X : \lim_{t\to 0} \frac{\alpha_t(x) - c(x)}{t} exists\}$.

The following theorem shows that ternary derivations of Hilbert modules are appeared as the infinitesimal generator of regularized ternary dynamical systems.

Theorem 2.7. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a regularized ternary dynamical system on a full Hilbert A-module X and δ_1 be its generator. Then, $D(\delta_1)$ is a dense ternary subalgebra of X and δ_1 is a ternary derivation.*

Now, we are ready to present the following main result.

Theorem 2.8. *Let X and Y be full Hilbert modules over C^*-algebras A and B, respectively. Suppose that $\varphi : A \to B$ is an injective morphism and $T : X \to Y$ is a surjective φ-morphism. If $\{\alpha_t\}_{t\in\mathbb{R}}$ is a regularized ternary dynamical system on X with the infinitesimal generator δ_1, then $\{\alpha_t\}_{t\in\mathbb{R}}$ implements a regularized ternary dynamical system $\{T\beta_tT^{-1}\}_{t\in\mathbb{R}}$ on Y. Moreover, the map δ_2 (introduced in Theorem 2.3) is its infinitesimal generator.*

References

1. Gh. Abbaspour and M. Skeide, *Generators of dynamical systems on Hilbert modules*, Commun. Stoch. Anal., **1** (2007), 193-207.
2. D. Bakic and B. Guljas, *On a class of module maps of Hilbert C^*-modules*, Math. Commun., **7** (2002), 177-192.
3. E. B. Davies and M. M. H. Pang, *The Cauchy problem and a generalization of the Hille-Yosida theorem*, Proc. Lond. Math. Soc., **55** (1987), 181-208.
4. E. C. Lance, *Hilbert C^*-Modules*, Lecture Note Series 210, Cambridge Univ. Press, Cambridge, 1995.
5. M. Mosadeq, *σ-C^*-dynamics of $\mathcal{K}(H)$*, J. Math. Ext., **16** (2022), 1-22.
6. M. Mosadeq, M. Hassani and A. Niknam, *(σ, γ)-generalized dynamics on modules*, J. Dyn. Syst. Geom. Theor., **9** (2011), 171-184.
7. M. Mosadeq, M. Hassani and A. Niknam, *Approximately quasi inner generalized dynamics on modules*, J. Sci. Islam. Repub. Iran., **23** (2012), 245-250.
8. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

INNERNES OF UNITARIES ON HILBERT MODULES AND APPLICTIONS TO GENERALIZED DYNAMICS

MAYSAM MOSADEQ*

Department of Mathematics, Larestan Branch, Islamic Azad University, Larestan, Iran
maysammosaddegh@yahoo.com

Abstract. In this talk, we introduce the concept of inner unitary operators on Hilbert C^*-modules to demonstrate the definition of approximately inner generalized dynamical system. As an application, we provide the existence of a class of non-approximately inner generalized dynamical systems.

1. Introduction

A Hilbert C^*-module over a C^*-algebra A is an algebraic left A-module M equipped with an A-valued inner product $\langle .,. \rangle$, which is A-linear in the first and conjugate linear in the second variable for which M is a Banach space with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. The Hilbert module M is called full if the closed linear span $\langle M, M\rangle$ of all elements of the form $\langle x, y\rangle$ $(x, y \in M)$ is equal to A. The referred to [3] for more details on Hilbert C^*-modules.

The classical C^*-dynamical systems are expressed by means of strongly continuous one parameter groups of linear $*$-automorphisms on C^*-algebras. If $\{\varphi_t\}_{t\in\mathbb{R}}$ is a C^*-dynamical system on a C^*-algebra A, then there exists

2020 *Mathematics Subject Classification.* Primary 47D03; Secondary 22D25,46L57,16W20

Key words and phrases. C^*-Dynamical systems, generalized derivation, Hilbert C^*-module, unitary operator.

* Speaker.

a closed densely defined linear operator d given by $d(a) = \lim_{t\to 0} \frac{\varphi_t(a) - a}{t}$, where $D(d) = \{a \in A : \lim_{t\to 0} \frac{\varphi_t(a) - a}{t} exists\}$. The mentioned operator d is called the infinitesimal generator of $\{\varphi_t\}_{t\in\mathbb{R}}$. A subset D of $D(d)$ is called a core for d if for each $a \in D(d)$, there exists a sequence $\{a_n\}$ in D such that $a_n \to a$ and $d(a_n) \to d(a)$.

A linear functional ω on a unital C^*-algebra A is called a state if $\|\omega\| = 1$ and ω is positive in the sense that $\omega(aa^*) \geq 0$, for each $a \in A$. Suppose that d is the infinitesimal generator of a C^*-dynamical system $\{\varphi_t\}_{t\in\mathbb{R}}$ on a unital C^*-algebra A. A state ω on A is said to be a ground state for $\{\varphi_t\}_{t\in\mathbb{R}}$ if $-i\omega\,(a^*.d(a)) \geq 0$ for all a in a core D of d. A C^*-dynamical system $\{\varphi_t\}_{t\in\mathbb{R}}$ on A is called approximately inner if there exists a sequence $\{h_n\}$ of self-adjoint elements of A such that $\lim_{n\to\infty} \varphi_{e^{ith_n}}(a) = \varphi_t(a)$, where $\varphi_{e^{ith_n}}(a) := e^{ith_n} a e^{-ith_n}$ $(a \in A)$. Powers and Sakai proved in [7] that if $\{\varphi_t\}_{t\in\mathbb{R}}$ is an approximately inner C^*-dynamical system on a unital C^*-algebra A, then there exists a ground state ω for $\{\varphi_t\}_{t\in\mathbb{R}}$.

Let d be the infinitesimal generator of a C^*-dynamical system $\{\varphi_t\}_{t\in\mathbb{R}}$ on a C^*-algebra A. It is known that d is a $*$-derivation of A. Therefore, the theory of C^*-dynamical systems concerns the theory of derivations. Recently, various generalized notions of derivations have been investigated in the context of Banach algebras and Banach modules. One of them is “generalized derivation” which is defined as follows.

A linear mapping δ from a dense subspace $D(\delta)$ of a full Hilbert A-module M into M is called a generalized derivation if there exists a mapping d from a dense subalgebra $D(d)$ of A into A for which $D(\delta)$ is an algebraic left $D(d)$-module, and $\delta(a.x) = a.\delta(x) + d(a).x$ for each $a \in D(d)$, and $x \in D(\delta)$.

The method has been used in [1] shows that d is a derivation. For convenience, we say that such a generalized derivation δ is a d-derivation.

In each case of generalization of derivations, a noted point drawing the attention of analysts is trying to represent a suitable dynamical system whose infinitesimal generator is exactly the desired extended derivation as well as being an extension of a C^*-dynamical system. Some approaches to preparing new dynamical systems and their applications have been explained in [1, 4, 5, 6] and references therein.

In the remainder of this section, we introduce an important class of operators between Hilbert modules.

Let M and N be Hilbert modules over C^*-algebras A and B, respectively. Suppose that $\phi : A \to B$ is a linear $*$-homomorphism. A map $T : M \to N$ is called ϕ-morphism if $\langle T(x), T(y)\rangle = \phi(\langle x, y\rangle)$ for all $x, y \in M$. It is easy to see that if T is a ϕ-morphism, the T is necessarily a linear operator and a ϕ-module map in the sense that $T(ax) = \phi(a)T(x)$ $(a \in A,\, x \in M)$.

A surjective map $T : M \to N$ is called unitary if there is an injective linear $*$-homomorphism $\phi : A \to B$ such that T is a ϕ-morphism.

Let $\phi : A \to B$ be an injective homomorphism of C^*-algebras and let $T : M \to N$ be a ϕ-morphism. It is known from [2, Remark 2.9] that T is an isometry. Thus, each unitary operator of Hilbert modules is an isometry. Moreover, if N is a full Hilbert B-module, then ϕ is surjective and so it is an isomorphism of C^*-algebras. We denote by $U(M)$ the group of all unitary operators from full Hilbert module M onto M.

In 2005, Abbaspour, Moslehian and Niknam [1] applied unitary operators to provide an extension of C^*-dynamical systems on Hilbert modules as follows.

Let M be a full Hilbert A-module. A generalized dynamical system is a group homomorphism $t \to \alpha_t$ from the additive group $\mathbb{R}$ into the group $U(M)$ of unitary operators on M for which $\lim_{t\to 0} \alpha_t(x) = x$ for each $x \in M$. We define the infinitesimal generator δ of a generalized dynamical system $\{\alpha_t\}_{t\in\mathbb{R}}$ as a mapping $\delta : D(\delta) \subseteq M \to M$ such that $\delta(x) = \lim_{t\to 0} \frac{\alpha_t(x) - x}{t}$, where $D(\delta) = \{x \in M : \lim_{t\to 0} \frac{\alpha_t(x) - x}{t} exists\}$.

It has been proved that in [1] that if $\{\alpha_t\}_{t\in\mathbb{R}}$ is a generalized dynamical system on M with the generator δ, then there is a (unique) C^*-dynamical system $\{\varphi_t\}_{t\in\mathbb{R}}$, which we call it *the associated C^*-dynamical system*, on A and a (unique) derivation d of A such that α_t is a surjective φ_t-morphism ($t \in \mathbb{R}$), and d turns δ into a d-derivation.

2. Main results

We begin this section with the definition of "inner unitary operator" as follows.

Definition 2.1. Let A be a C^*-algebra and M be a Hilbert A-module. A linear operator $\alpha : M \to M$ is called *inner unitary* if there exists a unitary element $u \in A$ such that α is a surjective φ_u-morphism, where $\varphi_u(a) = uau^*$ ($a \in A$).

Example 2.2. Let A be a C^*-algebra and take $M := A$. Then, M is a Hilbert A-module via $< a, b >= ab^*$. Suppose that θ is a linear $*$-automorphism on A and v is an arbitrary unitary element in A. Define $\alpha : A \to A$ by $\alpha(x) := \theta(v)x$. Then, by putting $u := \theta(v)$, it follows that u is a unitary element of A and α is an inner unitary operator.

Inspiring some ideas of [6], we present the concept of approximately innerness for generalized dynamical systems.

Definition 2.3. A generalized dynamical system $\{\alpha_t\}_{t\in\mathbb{R}}$ on M is said to be *approximately inner* if one can approximate it by a sequence $\{\alpha_{n,t}\}_{t\in\mathbb{R}}$ of uniformly continuous one parameter group of inner unitaries on M in the sense that there exist a sequence $\{h_n\}$ of self-adjoint elements of A

and a sequence $\{\alpha_{n,t}\}_{t\in\mathbb{R}}$ of uniformly continuous one parameter group of surjective $\varphi_{e^{ith_n}}$-morphisms such that $\lim_{n\to\infty} \alpha_{n,t}(x) = \alpha_t(x)$ for all $x \in M$.

The following theorem provides a conditin under which the C^*-dynamics associated with a generalized dynamical system has a ground state.

Theorem 2.4. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be an approximately inner generalized dynamical system on M. Then, one can correspond a unique approximately inner C^*-dynamical system $\{\varphi_t\}_{t\in\mathbb{R}}$ on A which turns α_t into a φ_t-morphism ($t \in \mathbb{R}$). Consequently, $\{\varphi_t\}_{t\in\mathbb{R}}$ has a ground state.*

In the rest of the paper, we are going to construct a generalized dynamical system which is not approximately inner. For this aim, let M_i be a Hilbert module over C^*-algebra A_i $(i = 1, 2)$. Put, $M := M_1 \times M_2$, and $A := A_1 \times A_2$. Then, M is a Hilbert A-module. Suppose that $\{\alpha^i_t\}_{t\in\mathbb{R}}$ is a generalized dynamical system on M_i with the infinitesimal generator δ_i $(i = 1, 2)$, and define the map $\alpha : \mathbb{R} \to U(M)$ by $\alpha_t := \alpha^1_t \oplus \alpha^2_t$, where $\alpha^1_t \oplus \alpha^2_t(x, y) = \left(\alpha^1_t(x), \alpha^2_t(y)\right)$.

Theorem 2.5. *$\{\alpha_t\}_{t\in\mathbb{R}}$ is a generalized dynamical system on M and the mapping $\delta : D(\delta_1) \times D(\delta_2) \subseteq M \to M$ defined by $\delta := \delta_1 \oplus \delta_2$ is its infinitesimal generator.*

Theorem 2.6. *$\{\alpha_t\}_{t\in\mathbb{R}}$ is approximately inner if and only if so is $\{\alpha^i_t\}_{t\in\mathbb{R}}$ ($i = 1, 2$).*

Now, we are ready to present the following main result.

Theorem 2.7. *Let $\{\alpha^1_t\}_{t\in\mathbb{R}}$ be a generalized dynamical system on M_1 such that its associated C^*-dynamical system on A_1 has no ground state. Then, for each generalized dynamical system $\{\alpha^2_t\}_{t\in\mathbb{R}}$ on M_2, $\{\alpha_t\}_{t\in\mathbb{R}}$ can not be approximately inner.*

References

1. Gh. Abbaspour, M. S. Moslehian and A. Niknam, *Dynamical systems on Hilbert C^*-modules*, Bull. Iranian Math. Soc., **31** (2005), 25-35.
2. D. Bakic and B. Guljas, *On a class of module maps of Hilbert C^*-modules*, Math. Commun., **7** (2002), 177-192.
3. E. C. Lance, *Hilbert C^*-Modules*, Lecture Note Series 210, Cambridge Univ. Press, Cambridge, 1995.
4. M. Mosadeq, *σ-C^*-dynamics of $\mathcal{K}(H)$*, J. Math. Ext., **16** (2022), 1-22.
5. M. Mosadeq, *Dynamial systems implemented by isomorphic groups of unitaries*, Sahand Commun. Math. Anal. Theor., **20** (2023), 51-68.
6. M. Mosadeq, M. Hassani and A. Niknam, *Approximately quasi inner generalized dynamics on modules*, J. Sci. Islam. Repub. Iran., **23** (2012), 245-250.
7. R. T. Powers and S. Sakai, *Existence of ground states and KMS states for approximately inner Dynamics*, Commun. Math. Phys., **39** (1975), 273-288.

GRAMSCHMIDT REORTHONORMALIZATION PROCEDURE

N.NAJAFI* AND M.GHASEMI KAMALVAND

Department of Mathematics, Lorestan University, Khorramabad, Iran
najafi.ns@fs.lu.ac.ir; ghasemi.m@lu.ac.ir

Abstract. In this article, we will find a new method for obtaining Gram–Schmidt orthonormalization algorithm. In fact, we first obtain the QR factorization according to sum of rank one matrices, and then we conclude in the Gram–Schmidt orthonormalization procedure.

1. Introduction

We have always used the row-by-column multiplication method to calculate the product of two matrices. That means if A and B are two matrices and $A \times B$ is defined, then:

$$A \times B = \begin{bmatrix} a_1^t \\ a_2^t \\ \vdots \\ a_m^t \end{bmatrix}_{m\times n} \times \begin{bmatrix} b_1 & b_2 & \dots & b_q \end{bmatrix}_{n\times q} = \begin{bmatrix} a_1^t b_1 & a_1^t b_2 & \dots & a_1^t b_q \\ a_2^t b_1 & a_2^t b_2 & \dots & a_2^t b_q \\ \vdots & \vdots & \ddots & \vdots \\ a_m^t b_1 & a_m^t b_2 & \dots & a_m^t b_q \end{bmatrix}_{m\times q}$$

which a_i^t is the i-th row of matrix A and b_j is the j-th column of matrix B. However, it is possible to see the multiplication of $A \times B$ in another way. If

---

2020 *Mathematics Subject Classification.* Primary 65F25; Secondary 65F25

Key words and phrases. QR factorization, *Gram–Schmidtorthonormalization.*

* Speaker.

$A = \begin{bmatrix} a_1 & a_2 & ... & a_n \end{bmatrix}$, $B = \begin{bmatrix} b_1^t \\ b_2^t \\ \vdots \\ b_n^t \end{bmatrix}$ then

$$A \times B = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}_{m\times n} \times \begin{bmatrix} b_1^t \\ b_2^t \\ \vdots \\ b_n^t \end{bmatrix}_{n\times q} = a_1 b_1^t + a_2 b_2^t + \cdots + a_n b_n^t$$

which a_i is the i-th column of matrix A, and b_j^t is the j-th row of matrix B and $a_i b_i^t$ is a $m \times n$ rank one matrix. In fact, by doing this, we express $A \times B$ as the sum of n rank one matrices. It is called column-row algorithm. We now use this idea and write the non-singular matrix A as the sum of n rank one matrices in the form of $a_i b_i^t$. We find a a_i, b_i^t such that a_is be orthonormal and b_is be the rows of an upper triangular matrix. This task yields QR factorization. For more information refer to [1], [2] and [3].

2. QR Factorization:

We can reach to QR factorization of matrix A with this algorithm: assume a_i is the i-th column of $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$;

(1) $r_{11} = \|a_1\|$, $q_1 = \dfrac{a_1}{r_{11}}$, $r_1^t = q_1^t A$
put $A = q_1 r_1^t + \begin{bmatrix} 0 & A^{(2)} \end{bmatrix}$.
Here, q_1 is perpendicular to all columns of $A^{(2)}$:
$\begin{bmatrix} q_1^t 0 & q_1^t A^{(2)} \end{bmatrix} = q_1^t A - r_1^t = 0 \Rightarrow q_1^t a_i^{(2)} = 0.$ $1 \leq$ i $\leq n-1$.
In which $a_i^{(2)}$ corresponds to the i-th column of $A^{(2)}$.

(2) put $q_2 = \dfrac{{a_1}^{(2)}}{\|a_1^{(2)}\|}$, $r_2^t = q_2^t \begin{bmatrix} 0 & A^{(2)} \end{bmatrix}$.
we have $A = q_1 r_1^t + q_2 r_2^t + \begin{bmatrix} 0 & 0 & A^{(3)} \end{bmatrix}$ or $\begin{bmatrix} 0 & A^{(2)} \end{bmatrix} = q_2 r_2^t + \begin{bmatrix} 0 & 0 & A^{(3)} \end{bmatrix}$.
All columns in $A^{(3)}$ are perpendicular to q_1, q_2.

(3) By continuing this algorithm, we will have:

$$A = q_1 r_1^t + q_2 r_2^t + \dots q_n r_n^t = \begin{bmatrix} q_1 & q_2 & \dots & q_n \end{bmatrix} \begin{bmatrix} r_1^t \\ r_2^t \\ \vdots \\ r_n^t \end{bmatrix} = QR.$$

R is an upper triangular matrix $r_k^t = q_k^t \begin{bmatrix} 0 & 0 & \dots & 0 & A^{(k)} \end{bmatrix}$.
$A^{(k)}$ is a $n \times (n-(k-1))$ matrix, so R is an upper triangular matrix. Due to the way of construction of q_is, $q_1, q_2, \dots, q_n$ are unitary and pairwise perpendicular, then Q is an orthogonal matrix.

The above algorithm can be rewritten as follows:
Algorithm 1:
$input : A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$, $q_1 = \dfrac{a_1}{\|a_1\|}$.

$C = A;$
$for\ l = 1\ to\ n$
$for\ k = l\ to\ n$
$r_l^t = q_l^t \begin{bmatrix} 0 & 0 & \dots & 0 & C\begin{bmatrix} 1:n & k:n \end{bmatrix} \end{bmatrix}$
end
$B = \sum_{i=1}^{l} q_i r_i^t$
$C = A - B$
$q_{l+1} = \dfrac{C_{l+1}}{\|C_{l+1}\|}$
end

$$output : Q = \begin{bmatrix} q_1 & q_2 & \dots & q_n \end{bmatrix} , R = \begin{bmatrix} r_1^t \\ r_2^t \\ \vdots \\ r_n^t \end{bmatrix}$$

According to $q_{l+1} = \dfrac{C_{l+1}}{\|C_{l+1}\|}$, it is seen that only column $l+1$ of matrix A is needed. If put $w = C_{l+1}$, it means $w = a_{l+1} - b_{l+1}(b_{l+1}$ is the $(l+1)-th$ column of $B)$ and because $q_{l+1} = \dfrac{w}{\|w\|}$ then
$\|w\|q_{l+1} = a_{l+1} - b_{l+1} \Rightarrow \ \|w\|q_j^t q_{l+1} = q_j^t a_{l+1} - q_j^t b_{l+1}$, $1 \le j \le l$
$\Rightarrow q_j^t a_{l+1} = q_j^t[\sum_{i=1}^{l} q_i r_i^t]_{l+1} = [\sum_{i=1}^{l} q_j^t q_i r_i^t]_{l+1}$
$\Rightarrow q_j^t a_{l+1} = [\ q_j^t q_j\ r_j^t]_{l+1} \ \Rightarrow r_{j,l+1} = q_j^t a_{l+1},\ \ 1 \le j \le l$
Moreover
$\|w\|q_{l+1} = a_{l+1} - b_{l+1} \ \Rightarrow \|w\|q_{l+1}^t q_{l+1} = q_{l+1}^t a_{l+1} - q_{l+1}^t\ [\sum_{i=1}^{l} q_i r_i^t]_{l+1} \Rightarrow$
$\|w\| = q_{l+1}^t q_{l+1} - [\sum_{i=1}^{l} q_{l+1}^t q_i r_i^t]_{l+1} \Rightarrow r_{l+1,l+1} = \|w\|$.
Therefore, it has been shown that Algorithm 1 could be transformed as follows:
Algorithm 2:
$input : A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix};$

$q_1 = \dfrac{a_1}{\|a_1\|}$, $r_{11} = \|a_{11}\|$

$for\ l = 1\ to\ n$
$for\ k = 1\ to\ l$
$r_{k,l+1} = q_k^t a_{l+1}$
end
$b = \sum_{j=1}^{l} r_{j,l+1} q_j$
$w = a_{l+1} - b$
$r_{l+1,l+1} = \|w\|$
$q_{l+1} = \dfrac{w}{\|r_{l+1,l+1}\|}$

end
output : $R = (r_{ij})$, $Q = \begin{bmatrix} q_1 & q_2 & \dots & q_n \end{bmatrix}$.
Algorithm 2 is same the well-known algorithm of *GramSchmidt* applied to the columns of matrix A.

References

1. R. A. Horn and C. R. Johnson, *Matrix Analysis, Second Edition*, Cambridge university press (2013).
2. Y. Saad, *Iterative methods for sparse linear systems, second edition, copyright*, J. Analysis, (2003) by the society for industrial and applied mathematics.
3. E. Strang, *Multiplying and Factoring Matrices*, The American Mathematical Monthly, **125** (3) (2018), 223-230.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

A REFINEMENT OF THE LIU AND WANG INEQUALITY

LEILA NASIRI[1]* AND MAHMOOD SHAKOORI[2]

Department of Mathematics, Faculty of science, Farhangian University, Tehran, Iran.
leilanasiri468@gmail.com
Department of Mathematics and computer science, Faculty of science, Lorestan University, Khorramabad, Iran.
mahmoodshakoori@gmail.com

Abstract. Inspired by the recent work by Furuichi et. al., we give several inequalities for sector matrices which refine the obtained results by Liu et. al and Lin.

1. Introduction

Let $\mathbb{M}_n$ be the set of all $n \times n$ complex matrices. For $A \in \mathbb{M}_n$, the conjugate transpose of A is denoted by A^*. We say $A \in \mathbb{M}_n$ is accretive, if its real part which defined by $\Re A = \frac{A+A^*}{2}$ is positive definite. For two Hemitian matrices $A, B \in \mathbb{M}_n$, we use $A \geq B$ to mean $A - B$ is positive semidefinite. For $0 \leq \alpha < \frac{\pi}{2}$, S_α denote the sector regions in the complex plane as $\mathcal{S}_\alpha = \{z \in \mathbb{C} : \Re z > 0, |\Im z| \leq (\Re z) \tan \alpha\}$. Recall that the numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^* Ax : x \in \mathbb{C}^n, x^* x = 1\}.$$

2020 *Mathematics Subject Classification. Primary 15A45; Secondary 15A60*
Key words and phrases. Sector matrices, The AM-GM-HM inequality.
* Speaker.

A matrix $A \in \mathbb{M}_n$ whose numerical range is contained in a sector region S_α is called a sector matrix. For $A, B \in \mathbb{M}_n$, we have the following inequality:

$$\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1} \leq A\sharp B \leq \frac{A+B}{2}, \tag{1.1}$$

which is well khnow as the AM-GM-HM inequality. Liu et. al [2] and Lin [4] extended the inequalities (1.1) to sector matrices as follows:

$$\cos^4 \alpha \Re\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1} \leq \Re(A\sharp B) \leq \sec^2 \alpha \Re\left(\frac{A+B}{2}\right). \tag{1.2}$$

Our main topic in this paper is to refine the inequalities (1.2).

2. Main results

The authors [1, Eq.(2.6)] showed that if $0 \leq \alpha \leq 1$ and $0 < x \leq 1$, then $\psi_\alpha(x)x^\alpha \leq (1-\alpha)+\alpha x$, where $\psi_\alpha(x) = 1+\frac{2^\alpha \alpha(1-\alpha)(x-1)^2}{(x+1)^{\alpha+1}}$. When $\alpha = \frac{1}{2}$, we get

$$\psi_{\frac{1}{2}}(x)x^{\frac{1}{2}} \leq \frac{x+1}{2}, \tag{2.1}$$

where $\psi_{\frac{1}{2}}(x) = 1+\frac{\sqrt{2}(x-1)^2}{4(x+1)^{\frac{3}{2}}}$.

Lemma 2.1. *Let $A, B \in \mathbb{M}_n$ with $0 < sA \leq B \leq tA$ for positive real numbers $0 < s \leq t$. Then*

$$\min\left\{\psi_{\frac{1}{2}}(s), \psi_{\frac{1}{2}}(t)\right\} A\sharp B \leq \frac{A+B}{2}. \tag{2.2}$$

Proof. If we apply (2.1) for strictly positive matrix X, then

$$\min_{s\leq x\leq t} \psi_{\frac{1}{2}}(x)X^{\frac{1}{2}} \leq \frac{X+1}{2}. \tag{2.3}$$

Note that $s \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq t$. By setting the operator $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (2.3) instead of X, we have

$$\min_{s\leq x\leq t} \psi_{\frac{1}{2}}(x)\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}} \leq \frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I}{2}, \tag{2.4}$$

If we multiply the both sides of (2.4) by $A^{\frac{1}{2}}$, we get (2.5). □

Remark 2.2. Let $A, B \in \mathbb{M}_n$ and $0 < m \leq A, B \leq M$. We can show that $0 < \frac{m}{M}A \leq B \leq \frac{M}{m}A$. By considering $s = \frac{m}{M}$ and $t = \frac{M}{m}$ in Lemma 2.1, we have

$$\psi_{\frac{1}{2}}(h)A\sharp B \leq \frac{A+B}{2}, \tag{2.5}$$

where $h = \frac{m}{M}$.

Remark 2.3. Since $\psi_{\frac{1}{2}}(h) \geq 1$, (2.5) is considerable refinement of (1.1).

To prove our main result, we need to recall the following Lemmas:

Lemma 2.4. ([4, 3]) *Let $A \in S_\alpha$. Then we have $\Re(A^{-1}) \le \Re^{-1}(A) \le \sec^2(\alpha)\Re(A^{-1})$. The first inequality holds for an accretive matrix $A \in \mathbb{M}_n$.*

Lemma 2.5. ([5]) *If $A, B \in \mathbb{M}_n$ be accretive and $0 < \lambda < 1$. Then*

$$\Re A \sharp_\lambda \Re B \le \Re(A \sharp_\lambda B).$$

Theorem 2.6. *Let $A, B \in S_\alpha$ with $0 < m \le \Re A, \Re B \le M$. Then*

$$\cos^4 \alpha \psi_{\frac{1}{2}}(h) \Re\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1} \le \Re(A\sharp B) \le \sec^2 \alpha \psi_{\frac{1}{2}}^{-1}(h) \Re\left(\frac{A+B}{2}\right). \tag{2.6}$$

Proof. When $\lambda = \frac{1}{2}$, [6, Lemma 5] follows $\Re(A\sharp B)\& \le \sec^2 \alpha \Re(A)\sharp\Re(B)$. On the other hand, applying (2.5) for $\Re(A)$ and $\Re(B)$, we get

$$\Re(A)\sharp\Re(B) \le \psi_{\frac{1}{2}}^{-1}(h)\Re\left(\frac{A+B}{2}\right),$$

where $h = \frac{m}{M}$. From two inequalities above, we obtain the second inequality of (2.6) as claimed. If we first apply Lemma 2.5 for special case $\lambda = \frac{1}{2}$ and for A^{-1} and B^{-1} replacement A and B and then use the second inequality of (2.6), we obtain

$$\Re(A^{-1})\sharp\Re(B^{-1}) \le \Re(A^{-1}\sharp B^{-1}) \le \sec^2 \alpha \psi_{\frac{1}{2}}^{-1}(h)\Re\left(\frac{A^{-1}+B^{-1}}{2}\right),$$

so

$$\Re(A^{-1})\sharp\Re(B^{-1}) \le \sec^2 \alpha \psi_{\frac{1}{2}}^{-1}(h)\Re\left(\frac{A^{-1}+B^{-1}}{2}\right).$$

By the operator monotone decreasing of the reverse, we get

$$(\Re(A^{-1})\sharp\Re(B^{-1}))^{-1} \ge \cos^2 \alpha \psi_{\frac{1}{2}}(h)\Re^{-1}\left(\frac{A^{-1}+B^{-1}}{2}\right),$$

or

$$\sec^2 \alpha \psi_{\frac{1}{2}}^{-1}(h)(\Re(A^{-1})\sharp\Re(B^{-1}))^{-1} \ge \Re^{-1}\left(\frac{A^{-1}+B^{-1}}{2}\right). \tag{2.7}$$

Compute

$$\begin{aligned}
\Re\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1} &\le \Re^{-1}\left(\frac{A^{-1}+B^{-1}}{2}\right) \text{(by Lemma 2.4)} \\
&\le \sec^2 \alpha \psi_{\frac{1}{2}}^{-1}(h)(\Re(A^{-1})\sharp\Re(B^{-1}))^{-1} \text{(by (2.7))} \\
&= \sec^2 \alpha \psi_{\frac{1}{2}}^{-1}(h)(\Re^{-1}(A^{-1})\sharp\Re^{-1}(B^{-1})) \\
&\le \sec^4 \alpha \psi_{\frac{1}{2}}^{-1}(h)(\Re A\sharp\Re B) \text{ (by Lemma 2.4)} \\
&\le \sec^4 \alpha \psi_{\frac{1}{2}}^{-1}(h)\Re(A\sharp B) \text{ (by Lemma 2.5)}.
\end{aligned} \tag{2.8}$$

□

Remark 2.7. From $\psi_{\frac{1}{2}}^{-1}(h) \leq 1(\psi_{\frac{1}{2}}(h) \geq 1)$, it is apparent that the inequalities in (2.6) are better than ones in (1.2).

Note that $\|\cdot\|_u$ denote any unitarily invariant norm on $\mathbb{M}_n$, which satisfies $\|X\|_u = \|UXV\|_u$ for any unitary matrices U, V and any $X \in \mathbb{M}_n$.

Lemma 2.8. **([7])** *Let $A \in S_\alpha$. Then*

$$\|\Re(A)\|_u \leq \|A\|_u \leq \sec\alpha \|\Re(A)\|_u.$$

The following Theorem is an application of (2.6) for norms.

Theorem 2.9. *Consider A, B as defined in Theorem 2.6. Then*

$$\cos^5\alpha\psi_{\frac{1}{2}}(h)\left\|\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right\|_u \leq \|A\sharp B\|_u \leq \sec^3\alpha\psi_{\frac{1}{2}}^{-1}(h)\left\|\frac{A+B}{2}\right\|_u. \tag{2.9}$$

Proof. From Lemma 2.8 and the left-side of the inequality (2.6), we can see that

$$\begin{aligned}\left\|\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right\|_u \leq \sec\alpha\left\|\Re\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right\|_u &\leq \sec^5\alpha\psi_{\frac{1}{2}}^{-1}(h)\|\Re(A\sharp B)\|_u \\ &\leq \sec^5\alpha\psi_{\frac{1}{2}}^{-1}(h)\|A\sharp B\|_u.\end{aligned}$$

From Lemma 2.8 with the right-side of the inequality (2.6), the second inequality of (2.9) is similar. □

Remark 2.10. Since $\psi_{\frac{1}{2}}(h) \geq 1$, the inequality (2.9) refine [2, Eq. (3.6)] and [4, Eq. (14)].

References

1. S. Furuichi and H. Moradi, *Some refinements of classical inequalities*, Rocky Mountain J. Math. **48** (7)(2018), 2289-2309.
2. J. Liu and Q. Wang, *More inequalities for sector matrices,* Bull. Iran. Math. Soc., **44**(2018), 1059-1066.
3. M. Lin, *Extension of a result of Hanynsworth and Hartfiel*, Arch. Math., **1** (2015), 93–100.
4. M. Lin, *Some inequalities for sector matrices*, Oper. Matrices, **10** (4)(2016), 915-921.
5. M. Raissouli, M. S. Moslehian and S. Furuichi, *Relative entropy and tsallis entropy of two accretive operators*, C. R. Acad. Sci. Paris Ser. I, **355**(2017), 687-693.
6. F. Tan and H. Chen, *Inequalities for sector matrices and positive linear maps*, Electronic Journal of Linear Algebra, **35**(2019), 418-423.
7. F. Zhang, *A matrix decomposition and its applications*, Linear Multilinear Algebra, **63**(2015), 2033–2042.

NON LINEAR MAPS PRESERVING THE WEYL DOMAIN IN THE SPECTRUM

ROHOLLAH PARVINIANZADEH*

Department of Mathematics, University of Yasouj, Yasouj, Iran
r.parvinian@yu.ac.ir

Abstract. Let X and Y be infinite-dimensional complex Banach spaces, and $B(X)$ (resp. $B(Y)$) be the algebra of all bounded linear operators on X (resp. on Y). For an $T \in B(X)$, let $\sigma_w(T)$ denotes the Weyl domain in the spectrum. We characterize surjective maps φ_1 and φ_2 from $B(X)$ onto $B(Y)$ satisfy

$$\sigma_w(\varphi_1(T)\varphi_2(S)) = \sigma_w(TS)$$

for every $T, S \in B(X)$.

1. Introduction

In recent years, there has been considerable interest in studying nonlinear preserver problems. These problems involve maps between algebras that leave invariant certain properties or subsets or relations without assuming any algebraic condition like linearity or additivity or multiplicativity. The first result of this kind is due to Kowalski and Słodkowski [4] and dates back to 1980. One of the most famous problems in this direction is Kaplansky's problem [3] asking whether every surjective unital invertibility preserving

---

2020 *Mathematics Subject Classification.* Primary 47B49; Secondary 47A10, 47B48
Key words and phrases. Fredholm domain, semi-Fredholm domain, Wey domain, preserver map.
* Speaker.

linear map between two semisimple Banach algebras is a Jordan homomorphism. Throughout this paper, X is an infinite-dimensional complex Banach spaces, and the dual space of X is denoted by X^*. For an operator $T \in B(X)$, the adjoint, the null space and the range of T are denoted by T^*, $ker(T)$ and $R(T)$, respectively. Recall that an operator $T \in B(X)$ is called upper (resp. lower) semi-Fredholm if $R(T)$ is closed and $ker(T)$ (resp. $\frac{X}{R(T)}$) is finite dimensional. The operator T is called semi-Fredholm if it is either upper or lower semi-Fredholm and T is Fredholm if it is both upper and lower semi-Fredholm. If T is a semi-Fredholm operator, then the index, $ind(T)$, is defined by

$$ind(T) = dim(kerT) - codim(R(T)).$$

Obviously, any Fredholm operator has a finite index. An operator $T \in B(X)$ is called a Weyl operator, if it is Fredholm of index zero. The Weyl domain in the spectrum of $T \in B(X)$ is defined by

$$\sigma_w(T) = \{\lambda \in \sigma(T) \ : \ T - \lambda I \ is \ a \ Weyl \ operator\}.$$

Several authors have studied the linear maps which preserve Fredholm or semi-Fredholm domain (see [1, 5, 6]). Shi and Ji [6] combined the spectrum with semi-Fredholm or Fredholm domain and described additive maps on $B(X)$ preserving the intersection of semi-Fredholm or Fredholm domain with the spectrum. In [5] Hajighasemi and Hejazian characterised the form of maps preserving the semi-Fredholm domain, the Fredholm domain and the Weyl domain in the spectrum.

In this paper, we investigate the form of all maps φ_1 and φ_2 from $B(X)$ onto $B(Y)$ such that, for every T and S in $B(X)$, the Weyl domain in spectrum of TS and $\varphi_1(T)\varphi_2(S)$ are the same.

For a vector $x \in X$ and $f \in X^*$, let $x \otimes f$ stands for the operator of rank at most one defined by $(x \otimes f)y = f(y)x$ for every $y \in X$. Note that, $(x \otimes f)^* = f \otimes \hat{x}$, where $\hat{x}$ denotes the canonical image of x in X^{**}. We denote $F_1(X)$ the set of all rank-one operators on X and $N_1(X)$ be the set of nilpotent operators in $F_1(X)$. It is clear that $x \otimes f \in N_1(X)$ if and only if $f(x) = 0$. Note that every rank one operator in $B(X)$ can be written in this form, and that every finite rank operator $A \in B(X)$ can be written as a finite sum of rank one operators; i.e., $A = \sum_{i=1}^{n} x_i \otimes f_i$ for some $x_i \in X$ and $f_i \in X^*$, $i = 1, 2, ..., n$. We denote by $F(X)$ the set of all finite rank operators in $B(X)$.

The first lemma is an elementary observation that gives the spectral domains of any rank one operator.

Lemma 1.1. *(See [5, Remark 1.1].) Let $x \in X$ and $f \in X^*$. We have*

$$\sigma_w(x \otimes f) := \begin{cases} \emptyset & \text{if } f(x) = 0, \\ \{f(x)\} & \text{if } f(x) \neq 0. \end{cases}$$

The following result characterizes in term of the spectral domains when two operators are the same.

Lemma 1.2. *(See [5, Lemma 2.1].) Let $A, B \in B(X)$. The following statements are equivalent.*
(1) $A = B$.
(2) $\sigma_w(AR) = \sigma_w(BR)$ *for all* $R \in B(X)$.
(2) $\sigma_w(AR) = \sigma_w(BR)$ *for all* $R \in F_1(X) \backslash N_1(X)$.

The third lemma gives a spectral characterization of rank one operators in term of the spectral domains $\sigma_w(.)$.

Lemma 1.3. *([5, Lemma 2.2].) For a nonzero operator $R \in B(X)$, the following statements are equivalent.*
(a) R has rank one.
(b) $\sigma_w(RT)$ contains at most one element for all $A \in B(X)$.
(c) $\sigma_w(RT)$ contains at most one element for every rank two operator $A \in B(X)$.

2. Main results

The following lemma is a useful observation needed to establish the linearity of surjective maps preserving $\sigma_w(.)$.

Lemma 2.1. *Let $A, B \in B(X)$. For two vectors $x, y \in X$ and a linear functional $f \in X^*$, the following statements hold.*
(1) $\sigma_w((x + y) \otimes f) = \sigma_w(x \otimes f) + \sigma_w(y \otimes f)$
(2) $\sigma_w((A + B)R) = \sigma_w(AR) + \sigma_w(BR)$ *for all* $R \in F_1(X)$.

Proof. The proof is easy with Lemma 1.1.

□

The followig theorem is the main result of this paper.

Theorem 2.2. *Let φ_1 and φ_2 be maps from $B(X)$ onto $B(Y)$ which satisfy*

$$\sigma_w(\varphi_1(A)\varphi_2(B)) = \sigma_w(AB), \ (A, B \in B(X)).$$

Then there exist two linear bijective mappings $P : X \to Y$ and $Q : Y \to X$ such that $\varphi_1(A) = PAQ$ and $\varphi_2(A) = Q^{-1}AP^{-1}$ for all $A \in B(X)$.

Proof. The proof of it will be completed after checking several claims.

Claim 1. φ_1 is injective and $\varphi_1(0) = 0$.

Claim 2. φ_1 preserves rank one operators in both directions.

Claim 3. φ_1 is linear.

Claim 4. φ_1 and φ_1 have the desired forms.

□

References

[1] Y. Bouramdane and M. Ech-Chérif El Kettani, *Maps preserving some spectral domains of skew products of operators*, Linear Multilinear Algebra, **70** (21) (2022), 6978–6988.
[2] M. Elhodaibi and S. Saber, *Maps preserving some spectral domains of Jordan product of operators*, Acta Scientiarum Mathematicarum, **89** (2023), 621–634.
[3] I. Kaplansky, *Algebraic and Analytic Aspects of Operator Algebras*, Amer. Math. Soc. Providence, 1970.
[4] S. Kowalski, Z. Słodkowski, *A characterization of multiplicative linear functionals in Banach algebras*, Studia Math., **67** (3) (1980) 215–223.
[5] S. Hajighasemi and S. Hejazian, *Maps preserving some spectral domains of operator products*, Linear Multilinear Algebra, **70** (12)(2022), 2367–2381.
[6] Shi W, Ji G. *Additive maps preserving the semi-Fredholm domain in spectrum*, Ann Funct Anal., **7**(2) (2016), 254—260.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

BASIS LIKE SYSTEMS IN QUATERNIONIC HILBERT SPACES

A.RAHIMI* AND B. DARABY

Department of Mathematics, Faculty of Basic Sciences, University of Maragheh, Maragheh, Iran
rahimi@maragheh.ac.ir; bdaraby@maragheh.ac.ir

Abstract. Frames and the similar systems in Hilbert spaces which help us to represent any signal or vectors in terms of elementary elements plying very important role in functional analysis and its applications. Recently, quaternions and quaternionic Hilbert spaces have attracted the attention of many mathematicians and signal engineers, and every day new applications of them occur in the mathematical community. In this article, we discuss and reproducing generative systems such as frames, continuous frames, and fusion.

1. Quaternionic Hilbert spaces

Quaternions can be considered as the combination of a real scalar and a 3D vector that has real coefficients. This vector forms the imaginary part of the quaternion. Birkhoff and von Neumann in their celebrated seminal work on Quantum Logic in 1936 [1], Quantum Mechanics may alternatively be formulated on a Hilbert space where the ground field of complex numbers is replaced for the division algebra of quaternions. Nowadays, the picture is more clear on the one hand and more strict on the other hand, after the efforts started in 1964 by Piron [4] and concluded in 1995 by Sol'er [6].

2020 *Mathematics Subject Classification. Primary 42C15; Secondary 42A38*
Key words and phrases. quaternionic Hilbert spaces, frames, Bessel sequences, fusion frames.

* Speaker.

Irish mathematician William Rowan Hamilton introduced quaternions in 1843. He aimed to extend the idea of complex numbers, which represent points in plane (with addition and multiplication properties), to points in space. While points in space are represented as coordinates of triplets, adding and subtracting them was known, but multiplying and dividing them posed a challenge unlike complex numbers. The great breakthrough in quaternions finally came, when Hamilton carved the formula for the quaternions given by

$$i^2 = j^2 = k^2 = ijk = -1.$$

Quaternions are used in pure as well as applied mathematics, especially in calculations involving three-dimensional rotations, as seen in computer graphics, crystallographic texture analysis, robotics and physics. Rotations represented using quaternions are more concise, effective, and numerically stable for computation as compared to representations using matrices. In modern terms, quaternions are a four dimensional non-commutative extension of the complex numbers over the set of real numbers. It was the first non-commutative four-dimensional associative normed division algebra over the real numbers, and therefore a ring, being a skew-field.

Quaternions are a four dimensional non-commutative extension of the complex numbers over $\mathbb{R}$ and are generally denoted by $\mathbb{H}$. The set of quaternions $\mathbb{H}$ contain elements of the form

$$q = q_0 + iq_1 + jq_2 + kq_3, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}.$$

These elements, called quaternions, form a four-dimensional vector space over $\mathbb{R}$ under component-wise addition and component-wise scalar multiplication with basis set $\{1, i, j, k\}$. Moreover, the product of any two quaternions (also called Hamilton product) is determined by the multiplication of the basis elements and the distributive law. Within the set $\mathbb{H}, 0$ represents the null element, while 1 represents the identity element with respect to multiplication. This enables $\mathbb{H}$ to form a non-commutative ring with unity, thereby forming a skew field.

For any $q = q_0 + iq_1 + jq_2 + kq_3 \in \mathbb{H}$. q_0 is called the real (or scalar part) and $iq_1 + jq_2 + kq_3$ is called the imaginary (or vector part) of q. The conjugate of q is given by $\overline{q} := q_0 - iq_1 - jq_2 - kq_3$. The norm of q is defined as $|q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$

For any non-zero quaternion $q = q_0 + iq_1 + jq_2 + kq_3 \in \mathbb{H}$. , there exists a unique inverse q^{-1} given by $q^{-1} = \frac{\overline{q}}{|q|^2}$

Right quaternionic vector spaces are basically right modules over the ring $\mathbb{H}$. The formal definition of a right quaternionic vector space is as follows:

Definition 1.1. [5] A right quaternionic vector space $\mathcal{H}_{\mathbb{H}}$ is a linear vector space with the right multiplication of scalars which satisfies the following properties:

(i) $(h_1 + h_2)q = h_1q + h_2q, \quad h_1, h_2 \in \mathcal{H}_{\mathbb{H}}, q \in \mathbb{H}.$
(ii) $h(p + q) = hp + hq, \quad h \in \mathcal{H}_{\mathbb{H}}, p, q \in \mathbb{H}.$

(iii) $h(pq) = (hp)q, \quad h \in \mathcal{H}_{\mathbb{H}}, p, q \in \mathbb{H}$.

Definition 1.2. [5] A right quaternionic pre-Hilbert space (or right quaternionic inner product space) $\mathcal{H}_{\mathbb{H}}$is a right quaternionic vector space endowed with the inner product $\langle .|. \rangle : \mathcal{H}_{\mathbb{H}}\mathcal{H}_{\mathbb{H}}$ which satisfies the following properties:
(i) $\langle h|h_1 + h_2 \rangle = \langle h|h_1 \rangle + \langle |h_2 \rangle, \quad h, h_1, h_2 \in \mathcal{H}_{\mathbb{H}}$.
(ii) $\langle h|h \rangle > 0, \quad h \neq 0$.
(iii) $\overline{\langle h_1|h_2 \rangle} = \langle h_2|h_1 \rangle, \quad h_1, h_2 \in \mathcal{H}_{\mathbb{H}}$.
(iv) $\langle h_1|h_2 q \rangle = \langle h_1|h_2 \rangle q, \quad h_1, h_2 \in \mathcal{H}_{\mathbb{H}}, q \in \mathbb{H}$.

A right quaternionic Hilbert space is a right quaternionic pre-Hilbert space which is complete with respect to the norm induced by the above defined inner product. Define the quaternionic norm $\|.\| : \mathcal{H}_{\mathbb{H}} \to \mathbb{H}^+$ on $\mathcal{H}_{\mathbb{H}}$ by $\|h\| = \sqrt{\langle h|h \rangle}$.

2. Frames in complex and quaternionic Hilbert spaces

Traditionally, most of the mathematical and analytical topics have been studied on complex or real Hilbert spaces. In the last few decades, many concepts of functional analysis and operator theory on quaternion Hilbert spaces have been investigated and studied, and by using the ideas in complex or real Hilbert has tried to generalize those concepts in quaternion Hilbert spaces, and interestingly, almost similar proofs have been presented for most of the relevant. theorems. In this category, we intend to study and examine the concept of frames and continuous frames in these spaces.

Frames for Hilbert spaces have been first introduced by Duffin and Scheaffer in the study of some problems in nonharmonic Fourier series in 1952, [3]. A discrete frame is a countable family of elements in a separable Hilbert space which allows for a stable, not necessarily unique, decomposition of an arbitrary element into an expansion of the frame elements.

The many applications of frames in Hilbert spaces have led to the introducing of new generalizations of the theory of frames, such as fusion frames [2], continuous frames, g frames, woven frames, K-frames, semi-frames and etc.

Recall that for a Hilbert space $\mathcal{H}$ and a countable index set I, a family of vectors $\{f_i\}_{i \in I} \subseteq \mathcal{H}$ is called a discrete frame for $\mathcal{H}$, if there exist constants $0 < A \leq B < +\infty$ such that

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H},$$

the constants A and B are called frame bounds. The frame $\{f_i\}_{i \in I}$ is called tight if $A = B$ and Parseval if $A = B = 1$. The frame decomposition is the most important frame result. It shows for the frame $\{f_i\}_{i \in I}$, every element in $\mathcal{H}$ has a representation as an infinite linear combination of the frame elements; i.e., there exist coefficients $\{c_i(f)\}_{i \in I}$ such that $f = \sum_{i \in I} c_i(f) f_i$, where $f \in \mathcal{H}$ is arbitrary.

The notion of frames has been extended to quaternionic Hilbert spaces by S. K. Sharma and S. Goel [5].

Definition 2.1. Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in I}$ be a sequence in $V_R(\mathfrak{Q})$. Then $\{u_i\}_{i\in H}$ is said to be a frame for $V_R(\mathfrak{Q})$ if there exist two finite constants with $0 < A \leq B < \infty$ such that

$$A\|u\|^2 \leq \sum_{i\in I} |\langle u|u_i\rangle|^2 \leq B\|u\|^2, \quad u \in V_R(\mathfrak{Q}).$$

Definition 2.2. Let $\{W_i^R\}_{i\in I}$ be a sequence of right closed subspaces of a right separable quaternionic Hilbert space $V_R(\mathfrak{Q})$ and $\{w_i\}_{i\in I}$ be a family of weights i.e. $w_i > 0, i \in I$. Then $\{(W_i^R, w_i)\}_{i\in I}$ is called a fusion frame (frame of subspaces), if there exist constants $0 < A \leq B < 8$called fusion frame bounds such that

$$A||u||^2 \leq \sum_{i\in I} w_i^2 ||\pi_{W_i^R}(u)||^2 \leq B||u||^2, \forall u \in V_R(\mathfrak{Q})$$

Inspired by the above ideas, the concept of continuous frames for quaternionic Hilbert spaces can be presented as follows.

Definition 2.3. Let (Ω, μ) be a measure space with positive measure μ and $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space. A weakly-measurable mapping $F : \Omega \to V_R(\mathfrak{Q})$ is called a continuous frame for $V_R(\mathfrak{Q})$ with respect to (Ω, μ) if there exist constants $0 < A \leq B < \infty$ such that

$$A\|u\|^2 \leq \int_\Omega |\langle u|F(\omega)\rangle|^2 d\mu(\omega) \leq B\|u\|^2, \quad u \in V_R(\mathfrak{Q}).$$

The constants A and B are called continuous frame bounds. This mapping F is called tight continuous frame if $A = B$ and if $A = B = 1$ it called a Parseval continuous frame. The mapping is called Bessel if the second inequality holds. In this case, B is called Bessel constant.

References

1. G. Birkhoff and J. von Neumann, *The logic of quantum mechanics*, Ann. of Math., **37** (4) (1936), 823-843.
2. P.G. Casazza, G. Kutyniok, *Frames of subspaces* Contemp. Math., **345**, Amer. Math. Soc., Providence, RI, 2004, 87-113.
3. R. J. Duffin, A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., **472** (1952), 341-366.
4. C. Piron, *Axiomatique quantique*, Helv. Phys. Acta, **37** (1964), 439-468.
5. S. K. Sharma and S. Goel, *Frames in quaternionic Hilbert spaces*, J. Math. Phys.Anal. Geom., **15** (3) (2019), 395–411.
6. M. P. Solèr, *Characterization of Hilbert spaces by orthomodular spaces*, Comm. Algebra, **23** (1) (1995), 219-243.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

ON SECOND TEOPLITZ DETERMINANTS FOR THE SUBCLASS OF ANALYTIC FUNCTIONS

HORMOZ RAHMATAN*

Department of Mathematics, Payame Noor University (PNU), Tehran, Iran
h.rahmatan@gmail.com

Abstract. We consider functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic in the unit disk and satisfy in the inequality $Re\, \frac{f(z)}{zf'(z)} > \alpha$, $0 \leq \alpha < 1$. Also, upper bounds for the second symmetric Teoplitz determinants for the classes of almost starlike functions of order α are given.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.
Toeplitz determinants are closely related to Hankel determinants [3]. Toeplitz matrices have constant entries along the diagonal. Toeplitz matrices have some applications in pure and applied mathematics[6].

2020 *Mathematics Subject Classification. Primary 30C45*
Key words and phrases. Analytic functions, univalent functions, almost starlike functions, second symmetric Teoplitz determinants.
* Speaker.

Thomas and Halim in [5] introduced the symmetric determinant $T_q(n)$ for analytic functions f of the form (1.1) defined by,

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_n & \dots & a_{n+q-2} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n+q-1} & a_{n+q-2} & \dots & a_n \end{vmatrix} \qquad q \in \mathbb{N} \setminus 1, \quad n \in \mathbb{N}.$$

The study of exact upper bound of $|T_q(n)|$ for different subclasses of analytic functions has attracted some authors. The Toeplitz determinant $T_q(n)$ for class $\mathcal{S}$ of univalent functions was studied and improved by Ali et al[1] and [4].

Let Ω be the family functions $\omega(z)$ in the unit disc $\mathbb{U}$ satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in \mathbb{U}$. Note that $f(z) \prec g(z)$ if there is a function $\omega(z) \in \Omega$ such that $f(z) = g(\omega(z))$. [2].

In this paper, we find sharp upper bounds of $|T_2(1)|$ and $|T_2(2)|$ for the almost starlike functions.

Suppose that $\mathcal{P}$ denote the class of analytic functions p of the type

$$p(z) = 1 + \sum_{n=2}^{\infty} c_n z^n, \tag{1.2}$$

To prove our main results, we need following lemmas and theorems.

Lemma 1.1. *If $p \in \mathcal{P}$ and of the form (1.2), then for $n \in \mathbb{N} = \{1, 2, \dots\}$, the following sharp inequality hold*

$$|c_n| \leq 2. \tag{1.3}$$

Lemma 1.2. *If $p \in \mathcal{P}$ and of the form (1.2), then*

$$2c_2 = c_1^2 + (4 - c_1^2)x,$$

for some x, with $|x| \leq 1$.

2. Main results

In the section by using of subordination method sharp upper bounds of initial coefficients, $|T_2(1)|$ and $|T_2(2)|$ for the almost starlike functions are obtained.

The class consisting of almost starlike functions order of α,$(0 \leq \alpha < 1)$ denoted by $\check{\mathcal{S}}^*(\alpha)$.

Especially the class $\check{\mathcal{S}}^*(0) = \check{\mathcal{S}}^*$.

Definition 2.1. A function $f(z)$ given by 1.1 on $\mathbb{U}$ is said to be in the class $\check{\mathcal{S}}^*(\alpha)$, if the following condition is satisfied:

$$Re\, \frac{f(z)}{zf'(z)} > \alpha,$$

where $z \in \mathbb{U}$, $\quad 0 \leq \alpha < 1$.

2.1. **Estimation of coefficient bounds for the function class $\check{\mathcal{S}}^*(\alpha)$.**

Theorem 2.2. *If the function $f(z)$ in $\check{\mathcal{S}}^*(\alpha)$ is given by (1.1), then*

$$(i) : |a_2| \leq 2k,$$
$$(ii) : |a_3| \leq \frac{1}{2}k(2+8k),$$

where $k = |\alpha - 1|$ and $0 \leq \alpha < 1$.

Proof. Let $f(z) \in \check{\mathcal{S}}^*(\alpha)$, by using of subordination method and lemma 1.1, proof of the theorem 2.2 is completed. □

Corollary 2.3. *If the function $f(z)$ in $\check{\mathcal{S}}^*(0)$ is given by (1.1), then*

$$(i) : |a_2| \leq 2,$$
$$(ii) : |a_3| \leq 5$$

2.2. **Upper bound of $|T_2(1)|$ for the class $\check{\mathcal{S}}^*(\alpha)$.**

Theorem 2.4. *Let $f(z) \in \check{\mathcal{S}}^*(\alpha)$ be given by 1.1. Then*

$$|T_2(1)| \leq 1 + 4k^2,$$

where $k = |\alpha - 1|$ and $0 \leq \alpha < 1$.

Proof. From defnition of subordination, using triangle inequality and by applying lemma 1.1, this completes the proof of the theorem. □

Corollary 2.5. *If the function $f(z) \in \check{\mathcal{S}}^*(0)$ is given by (1.1), then*

$$|T_2(1)| \leq 5,$$

Theorem 2.6. *Let $f(z) \in \check{\mathcal{S}}^*(\alpha)$ be given by 1.1. Then*

$$|T_2(2)| \leq k^2(4 + 48k + 2),$$

where $k = |\alpha - 1|$ and $0 \leq \alpha < 1$.

Proof. From defnition of subordination and by applying lemmas 1.1 and 1.2, this completes the proof of the theorem. □

Corollary 2.7. *If the function $f(z) \in \check{\mathcal{S}}^*(0)$ is given by (1.1), then*

$$|T_2(1)| \leq 54,$$

3. Conclusion

In this paper, we have introduced a new class $\check{\mathcal{S}}^*(\alpha)$, where $(0 \leq \alpha < 1)$, and have worked on the estimation of coefficient and upper bound of second toeplitz determinant. readers and interested scholars can extend our work further by working on higher order toeplitz determinant.

References

1. M. F. Ali, D. K. Thomas and A. Vasudevarao, *Toeplitz determinants whose elements are the coefficients of analytic and univalent functions*, Bull. Aust. Math. Soc, (2020), 1–12.
2. P. L. Duren, *Univalent functions*, Grundlehern der Mathematischen Wissenschaften, 259, Springer, New York, (1983).
3. Sh. Najafzade, H. Rahmatan and H. Haji, *Application of subordination for estimating the Hankel determinant for subclass of univalent functions*, Creat. Math. Inform, **30** (2021), 69–74.
4. H. Rahmatan, *Bounds of the fifth Toeplitz determinant for the classes of functions with bounded turnings*, Int. J. Nonlinear Anal. Appl, **14** (7) (2023), 99–106.
5. D. K. Thomas and A. A. Halim, *Toeplitz matrices whose elements are the coefficients of starlike and close-to-convex functions*, Bull. Malays. Math. sci. Soc., **40** (4) (2017),1781–1790.
6. K. Ye and L. H.Lim, *Every Matrix is product of Toeplitz matrices*, Found. Comput., **16** (2016), 577–598.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

A RELIABLE NUMERICAL TECHNIQUE FOR SOLVING A CLASS OF FRACTIONAL OPTIMAL CONTROL PROBLEMS WITH INTEGRAL EQUATION CONDITION

MOHAMMADREZA ROSTAMI*

Faculty of Science, Mahallat Institute of Higher Education, Mahallat, Iran.
Rostami.Mohamadreza@gmail.com

Abstract. In this paper, we investigate a class of optimal control problems via fractional integro-differential equations. Also, we apply the Ritz method for solving this problem. By this method, the given problem is reduced to the problem for solving a system of algebraic equations using shifted Legendre polynomials basis functions. An approximate solution for this problem is obtained by solving the system. Finally by a example will be showing the accurately and applicability for this method.

1. Introduction

The fractional calculus defines integrals and derivatives of non-integer order. Let $\alpha > 0$ be a real number and $n \in \mathbb{N}$ be such that $n-1 < \alpha < n$. Here we follow [3]. The left and right Caputo fractional derivative of order α are defined by

$$ {}_a^C D_x^{\alpha} y(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} y^{(n)}(t)\, dt, \tag{1.1}$$

2020 *Mathematics Subject Classification.* 49K05, 49M30, 33F05
Key words and phrases. Optimal control problem, Integro-differential equations, Polynomial basis functions, Ritz method.
* Speaker.

and

$$ {}_{x}^{C}D_{b}^{\alpha}y\left(x\right)=\frac{\left(-1\right)^{n}}{\Gamma\left(n-\alpha\right)}\int_{x}^{b}\left(t-x\right)^{n-\alpha-1}y^{\left(n\right)}\left(t\right)dt, \tag{1.2} $$

respectively. Here Γ is the well-known Gamma function.

In this paper, we consider the following problem

$$ Min\ \ J=\int_{0}^{1}\mathcal{J}\left(t,x\left(t\right),u\left(t\right)\right)dt, \tag{1.3} $$

subject to dynamic control system and and the initial boundary condition

$$ \begin{cases} {}_{0}^{C}D_{t}^{\alpha}\,x\left(t\right)=f\left(t,x\left(t\right),u\left(t\right)\right)+\int_{0}^{t}g\left(t,s,x\left(s\right),u\left(s\right)\right)ds, & 0<t\leq 1,\\ x^{(i)}\left(0\right)=x_{i}, \quad i=0,1,\ldots,n-1, \end{cases} \tag{1.4} $$

where ${}_{0}^{C}D_{t}^{\alpha}$ denotes the fractional derivative of Caputo type of order α, $n-1<\alpha\leq n$.

2. Preliminaries

In this section we consider shifted Legendre polynomials and we study the approximation of functions with the use of these polynomials.

Let $p_m(x)$ be the shifted Legendre polynomials of order m which are defined on the interval $[0,1]$ and can be determined with the following recurrence formula

$$ p_{0}\left(x\right)=1, \qquad\qquad p_{1}\left(x\right)=2x-1, $$

$$ p_{m+1}\left(x\right)=\frac{2m+1}{m+1}\left(2x-1\right)p_{m}\left(x\right)-\frac{m}{m+1}p_{m-1}\left(x\right), \qquad m=1,2,\ldots \tag{2.1} $$

The orthogonality of shifted Legendre polynomials is expressed by the relation

$$ \int_{0}^{1}p_{i}\left(x\right)p_{j}\left(x\right)dx=\begin{cases}\frac{1}{2i+1} & i=j\\ 0 & i\neq j\end{cases} \tag{2.2} $$

We consider $f(x)$ be an arbitrary continues function on $[0,1]$, then we can express it in the shifted Legendre polynomials form,

$$ f\left(x\right)\simeq P_{N}\left(x\right)=\sum_{j=0}^{N}c_{j}p_{j}\left(x\right), \tag{2.3} $$

where the coefficients c_j, for shifted Legendre polynomials are obtained from following relation

$$ c_{j}=\left(2j+1\right)\int_{0}^{1}p_{j}\left(x\right)f\left(x\right)dx, \qquad j=0,1,\ldots,N. \tag{2.4} $$

Lemma 2.1. *Let $p(x)$ be a polynomial which satisfies the following conditions*

$$ p^{(k)}(0)=y_{k}, \qquad 0\leq k\leq n. \tag{2.5} $$

then $p(x)$ has the following form

$$p\left(x\right)=\sum_{j=0}^{N}c_{j}x^{n+1}p_{j}\left(x\right)+H\left(x\right), \tag{2.6}$$

where, $N\in Z^{+}$, $c_j\in\mathbb{R}$, $p_j\left(x\right)$ shifted Legendre polynomials and $H(x)$ is the interpolating polynomial which satisfies the above conditions.

3. Solving the problem

In this section, we apply the presented method for solving the optimal control problems of fractional integro-differential equations. We refer the interested reader to [1] for more information. There are many direct techniques for solving the various optimization problems such as Ritz method and Epsilon-Ritz method. For more study we can see the report [5]. Now we consider the following problem (1.3) subject to dynamic control system and and the initial boundary condition (1.4).

By using the hermit interpolation method, we obtain polynomial $H(t)$, then,

$$x_{N}\left(t\right)=\sum_{l=0}^{N}c_{l}t^{n}p_{l}\left(t\right)+H\left(t\right), \tag{3.1}$$

and

$$u_{M}\left(t\right)=\sum_{j=0}^{M}\hat{c}_{l}p_{l}\left(t\right). \tag{3.2}$$

Obviously $x_N\left(t\right)$, satisfy in the initial boundary conditions (1.4). Substituting (3.1) and (3.2) in (1.3) and (1.4), we achieve

$$J[c_0,c_1,\ldots,c_N,\hat{c}_0,\hat{c}_1,\ldots,\hat{c}_M]=\int_0^1\mathcal{J}\left(t,x_N\left(t\right),u_M\left(t\right)\right)dt$$

and

$$\mathcal{G}\left(t\right)={}_{0}^{C}D_{t}^{\alpha}\,x_N\left(t\right)-f\left(t,x_N\left(t\right),u_M\left(t\right)\right)-\int_0^t g\left(t,s,x_N\left(s\right),u_M\left(s\right)\right)ds=0.$$

Now, we collocate function $\mathcal{G}\left(t\right)$ at points s_i, $1\le i\le k$, which are the roots of the shifted Chebyshev polynomials of degree k on $[0,1]$. We have

$$\mathcal{G}_i\left(s_i\right)={}_{0}^{C}D_{t}^{\alpha}\,x_N\left(s_i\right)-f\left(s_i,x_N\left(s_i\right),u_M\left(s_i\right)\right)-\int_0^{s_i} g\left(s_i,s,x_N\left(s\right),u_M\left(s\right)\right)ds=0,$$

and

$$\mathcal{F}_{N,M}^{k}=J+\lambda\mathcal{G},$$

which $\lambda=\left(\lambda_1,\lambda_2,\cdots,\lambda_k\right)$ and $\mathcal{G}=\left(\mathcal{G}_1,\mathcal{G}_2,\cdots,\mathcal{G}_k\right)^T$.

By calculating the $\mathcal{F}_{N,M}^{k}$, we obtain an algebraic function of unknowns c_l for $1\le l\le N$ and $\hat{c}_j$ for $1\le j\le M$ and λ_i for $1\le i\le k$. According to differential calculus, there exists the following system of equations, as necessary conditions of optimization for the functions, which should be satisfied by optimizing the unknown coefficients

$$\begin{cases} \frac{\partial F}{\partial c_l} = 0, & 1 \leq l \leq N, \\ \frac{\partial F}{\partial \hat{c}_j} = 0, & 1 \leq j \leq M, \\ \frac{\partial F}{\partial \lambda_i} = 0, & 1 \leq i \leq k. \end{cases}$$

Then we achieve the following algebraic system of equations and by solving the above system, with Newton iterative method we can determine optimizing values of the coefficients.

4. Applications and numerical results

In this section, we apply the method presented in this paper to solve the one test examples. The table of absolute error shows that this method is useful for solving optimal control problems of fractional integro-differential equations.

Example 4.1. Consider the optimal control of minimizing

$$J = \int_0^1 \left[\left(x(t) - e^{t^2} \right)^2 + \left(u(t) - (2t+1) \right)^2 \right] dt,$$

subject to the following non-linear Volterra integral equation:

$$\begin{cases} {}_0^C D_t^{\alpha} x(t) = u(t) - x(t) + t(2t+1) \int_0^t e^{s(t-s)} x(s)\, ds, \\ x(0) = 1, \end{cases}$$

where $\alpha \in (0, 1]$ and $t \in (0, 1]$.

The analytical solution for $\alpha = 1$ is given by $x(t) = e^{t^2}$, $u(t) = 2t+1$ and $J = 0$. The numerical values of the cost function J with $\alpha = 1$ and different values of N and M and $k = 5$ are reported in Table 1 and are compared with the solutions derived by another methods.

Methods	J
method in [2] with $(k = 2, M = 6)$	$3.2956\ e{-9}$
method in [4] with $(k = 1, M = 6)$	$1.1559\ e{-8}$
Present method with $(N = 4, M = 1)$	$2.5315\ e{-7}$
Present method with $(N = 6, M = 1)$	$3.8947\ e{-10}$
Present method with $(N = 8, M = 1)$	$5.9660\ e{-15}$

References

1. I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, Prentice-Hall, 1963.
2. R. Khanduzi, A. Ebrahimzadeh and M. R. Peyghami, *A modified teaching-learning-based optimization for optimal control of Volterra integral systems*, Soft Comput. **22** (17) (2018), 5889-5899.
3. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, In: North-Holland Mathematics Studies, 204. Elsevier, Amsterdam, 2006.

4. S. Sabermahani, Y. Ordokhani, K. Rabiei and M. Razzaghi, *Solution of optimal control problems governed by volterra integral and fractional integro-differential equations*, J. Vib. Control, (2022).
5. K. Sayevand and M.R. Rostami, *Fractional optimal control problems: Optimality conditions and numerical solution*, IMA J. Math. Control I.

EXISTENCE AND CONVERGENCE OF BEST PROXIMITY POINTS FOR SEMI-CYCLIC φ-CONTRACTIONS IN METRIC SPACES

A. SAFARI-HAFSHEJANI*

Department of Pure Mathematics, Payame Noor University (PNU), P. O. Box: 19395-3697, Tehran, Iran
email: asafari@pnu.ac.ir

Abstract. In this paper, in the setting of metric spaces, we introduce the notion of semi-cyclic φ-contraction as generalization of cyclic φ-contractions. Then we study the existence, convergence and uniqueness of the best proximity points for such mappings by using the UC property. Also, we have furnished iterative algorithms to determine the best proximity points. The presented results extend and improve some recent results in the literature.

1. Introduction and Preliminaries

Let metric space (X, d) and non-empty subsets A and B of it. A self mapping $T : A \cup B \to A \cup B$ is said to be *cyclic* provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. A point $z \in A \cup B$ is called a *best proximity point* for T if $d(z, Tz) = d(A, B)$, where $d(A, B) = \inf\{d(a, b) : \ a \in A, \ b \in B\}$. If $d(A, B) = 0$, z is called a *fixed point* of T.

In 2006, Anthony Eldred and Veeremani [2] introduced cyclic contraction mappings on uniformly convex Banach spaces and studied the existence of

2020 *Mathematics Subject Classification. Primary 47H10; Secondary 54H25*
Key words and phrases. Best proximity point, semi-cyclic φ-contraction, UC property.
* Speaker.

the best proximity point of them. The cyclic self mapping T on $A \cup B$, is said to be a *cyclic contraction* if

$$d(Tx, Ty) \leq \lambda d(x, y) + (1 - \lambda)d(A, B)),$$

for some $\lambda \in (0, 1)$ and every $x \in A$ and $y \in B$.

Theorem 1.1. [2] *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space. Suppose self mapping T on $A \cup B$, be a cyclic contraction map, then there is a unique best proximity point z in A. Further, if $z_0 \in A$ and $z_{n+1} := Tz_n$ for each $n \geq 0$, then z_{2n} converges to this best proximity point.*

Since then, the problem of the existence of a best proximity point for cyclic mappings, has been studied by many authors; see for instance [3–6] and references therein.

In 2009, *cyclic φ-contraction mappings* as a generalization of cyclic-contractions, were introduced and studied by Al-Thagafi and Shahzad [1] on uniformly convex Banach spaces. The cyclic self mapping T on $A \cup B$, is said to be *cyclic φ-contraction* if $\varphi : [0, +\infty) \to [0, +\infty)$ is a strictly increasing map and

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)),$$

for all $x \in A$ and $y \in B$. The authors presented the following existence theorem.

Theorem 1.2. [1] *Let A and B be nonempty convex subsets of a uniformly convex Banach space X such that A is closed. Let self mapping T on $A \cup B$, be a cyclic φ-contraction map. Then T has a unique best proximity point x in A which is the only fixed point of T^2 in A and the sequence $\{T^{2n}z_0\}$ converges to z for every starting point $z_0 \in A$.*

In this paper, in the setting of metric spaces, we introduce a new class of contractions on $A \cup B$. Then we study the existence, convergence and uniqueness of the best proximity points for such mappings with the UC property. Also, we have furnished iterative algorithms to determine the best proximity points. Our results, while generalizing some recent results in [1].

2. Main Results

Let I be the identity map. We introduce a new class of contractions to establish our main results.

Definition 2.1. Let A and B be nonempty subsets of the metric space (X, d). Let $T_1 : A \to B$ and $T_2 : B \to A$, (T_1, T_2) is said to be a semi-cyclic φ-contraction if there exists a strictly increasing map $\varphi : [0, +\infty) \to [0, +\infty)$ such that

(a) $d(T_1x, T_2T_1x') \leq (I - \varphi)(d(x, T_1x')) + \varphi(d(A, B))$ for all $x, x' \in A$;

(b) $d(T_2y, T_1T_2y') \leq d(y, T_2y')$ for all $y, y' \in B$.

The following useful result will be needed later.

Lemma 2.2. *Let A and B be nonempty subsets of the metric space (X, d). Let $T_1 : A \to B$ and $T_2 : B \to A$, (T_1, T_2) be a semi-cyclic φ-contraction. For $z_0 \in A$ define $z_{2n+1} := T_1z_{2n}$ and $z_{2n+2} := T_2z_{2n+1}$ for each $n \geq 0$. Then*

$$\lim_{n\to\infty} d(z_n, z_{n+1}) = d(A, B).$$

Theorem 3 of [1] and Proposition 3.1 of [2] are special cases of Lemma 2.2.

Lemma 2.3. *Let A and B be nonempty subsets of the metric space (X, d). Let $T_1 : A \to B$ and $T_2 : B \to A$, (T_1, T_2) be a semi-cyclic φ-contraction. For $z_0 \in A$ define $z_{2n+1} := T_1z_{2n}$ and $z_{2n+2} := T_2z_{2n+1}$ for each $n \geq 0$. If $\{z_{2n}\}$ has a convergence subsequence in A, then there exists $z \in A$ such that $d(z, T_1z) = d(A, B)$. Further, if (A, B) has the UC property, then every best proximity point of T_1 in A, is a fixed point of T_2T_1 in A.*

Theorem 4 of [1] and Proposition 3.2 of [2] are special cases of Lemma 2.3.

Lemma 2.4. *Let A and B be nonempty subsets of the metric space (X, d) such that (A, B) has the UC property. Let $T_1 : A \to B$ and $T_2 : B \to A$, (T_1, T_2) be a semi-cyclic φ-contraction. For $z_0 \in A$ define $z_{2n+1} := T_1z_{2n}$ and $z_{2n+2} := T_2z_{2n+1}$ for each $n \geq 0$. Then $d(z_{2n}, z_{2n+2}) \to 0$ as $n \to \infty$.*

Theorem 5 of [1] is a special case of the next lemma.

Lemma 2.5. *Let A and B be nonempty subsets of the metric space (X, d) such that (A, B) and (B, A) have the UC property. Let $T_1 : A \to B$ and $T_2 : B \to A$, (T_1, T_2) be a semi-cyclic φ-contraction. For $z_0 \in A$ define $z_{2n+1} := T_1z_{2n}$ and $z_{2n+2} := T_2z_{2n+1}$ for each $n \geq 0$. Then for every $\epsilon > 0$ there exists a positive integer N_0 such that for all $m > n \geq N_0$*

$$d(z_{2m}, z_{2n+1}) < d(A, B) + \epsilon.$$

Lemma 2.6. *Let A and B be nonempty subsets of the metric space (X, d) such that (A, B) and (B, A) have the UC property. Let $T_1 : A \to B$ and $T_2 : B \to A$, (T_1, T_2) be a semi-cyclic φ-contraction. For $z_0 \in A$ define $z_{2n+1} := T_1z_{2n}$ and $z_{2n+2} := T_2z_{2n+1}$ for each $n \geq 0$. Then $\{z_{2n}\}$ is a Cauchy sequences.*

Next theorem is our main result in this article which is an extension of Theorem 1.2.

Theorem 2.7. *Let A and B be nonempty subsets of the metric space (X, d) such that (A, B) and (B, A) have the UC property and A is complete. Let $T_1 : A \to B$ and $T_2 : B \to A$, (T_1, T_2) be a semi-cyclic φ-contraction. For $z_0 \in A$ define $z_{2n+1} := T_1z_{2n}$ and $z_{2n+2} := T_2z_{2n+1}$ for each $n \geq 0$. Then*

(i) *T_1 has a unique best proximity point z in A;*
(ii) *z is a unique fixed point of T_2T_1 in A;*
(iii) *the sequence $\{z_{2n}\}$ converges to z.;*
(iv) *T_1z is a unique best proximity point T_2 in B and $\{z_{2n+1}\}$ converges to T_1z.*

The next theorem is a generalization of Theorem 6 of [1].

Theorem 2.8. *Let A and B be nonempty subsets of the metric space (X, d) such that A is complete. Let $T_1 : A \to B$ and $T_2 : B \to A$, (T_1, T_2) be a semi-cyclic φ-contraction. If $d(A, B) = 0$, Then there exists a unique fixed point $z \in A$ such that $\{z_n\}$ converges to z for every $z_0 \in A$, where $z_{2n+1} := T_1z_{2n}$ and $z_{2n+2} := T_2z_{2n+1}$ for each $n \geq 0$.*

References

1. M.A. Al-Thagafi and N. Shahzad, *Convergence and existence results for best proximity points*, Nonlinear Anal., Theory Methods Appl., **70** (10) (2009), 3665-3671.
2. A. Anthony Eldred and P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl., **323** (2006), 1001-1006.
3. R. Espínola and A. Fernández-León, *On best proximity points in metric and Banach space*, Canad. J. math, **63** (3) (2011), 533-550.
4. M. Gabeleh, G. Sankara Raju Kosuru, *Some remarks on convergence of best proximity points and Semi-cyclic contractions*, Rend. Circ. Mat. Palermo, II. (2022), 14 pages.
5. A. Safari-Hafshejani, *The existence of best proximity points for generalized cyclic quasi-contractions in metric spaces with the UC and ultrametric properties*, Fixed Point Theory, **23** (2) (2022), 507-518.
6. T. Suzuki, M. Kikawa and C. Vetro, *The existence of best proximity points in metric spaces with the property UC*, Nonlinear Anal., **71** (2009), 2918-2926.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

EEXISTENCE AND CONVERGENCE OF FIXED POINTS OF NONCYCLIC GERAGHTY CONTRACTIONS

A. SAFARI-HAFSHEJANI*

Department of Pure Mathematics, Payame Noor University (PNU), P. O. Box: 19395-3697, Tehran, Iran
email: asafari@pnu.ac.ir

Abstract. In this article, in the setting of metric spaces we introduce the notion of noncyclic Geraghty contraction maps. We give new conditions for existence, convergence and uniqueness of fixed points for such mappings and iterative algorithms to determine of fixed points of them. Presented results extend and improve some recent results in the literature.

1. Introduction and Preliminaries

Let Φ denote the set of all functions $\beta : [0, +\infty) \to [0, 1)$ satisfy

$$\beta(t_n) \to 1 \quad \text{implies} \quad t_n \to 0, \text{ as } n \to \infty.$$

In 1973, Geraghty presented an extension of the Banach contraction principle as below.

Theorem 1.1. [2] *Let (X, d) be a complete metric space and $\Gamma : X \to X$ be a self-mapping such that there exists $\beta \in \Phi$ for which*

$$d(\Gamma x, \Gamma y) \leq \beta\big(d(x, y)\big) d(x, y), \quad \forall x, y \in X.$$

Then Γ has a unique fixed point.

2020 *Mathematics Subject Classification. Primary 47H10; Secondary 54H25*
Key words and phrases. Fixed point, Noncyclic Geraghty contraction, UC property.

* Speaker.

Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of a metric space (X,d). A self mapping $\Gamma:\mathcal{A}\cup\mathcal{B}\to\mathcal{A}\cup\mathcal{B}$ is called noncyclic provided that $\Gamma(\mathcal{A})\subseteq\mathcal{A}$ and $\Gamma(\mathcal{B})\subseteq\mathcal{B}$. To generalize the obtained fixed point result's Geraghty to noncyclic maps, we use the property UC was introduced by Suzuki et al. [5] and proved the existence of the fixed points for noncyclic contraction type mappings in metric spaces.

In the sequel, we recall some definitions and facts will be used hereafter

Definition 1.2. ([5]) Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of a metric space (X,d). Then $(\mathcal{A},\mathcal{B})$ is said to satisfies property UC, if for all sequences $\{x_n\},\{x'_n\}\subseteq\mathcal{A}$ in F and $\{y_n\}\subseteq\mathcal{B}$, we have

$$\left.\begin{array}{l}\lim_{n\to\infty} d(x_n,y_n)=\operatorname{dist}(\mathcal{A},\mathcal{B}),\\ \lim_{n\to\infty} d(x'_n,y_n)=\operatorname{dist}(\mathcal{A},\mathcal{B}),\end{array}\right\} \Rightarrow \lim_{n\to\infty} d(x_n,x'_n)=0;$$

It was announced in [1] that every nonempty, closed and convex pair in a uniformly convex Banach space X has the property UC. It is obvious that if $\mathcal{A}$ and $\mathcal{B}$ are nonempty subsets of a metric space (X,d) such that $\operatorname{dist}(\mathcal{A},\mathcal{B})=0$, then $(\mathcal{A},\mathcal{B})$ satisfies the property UC.

Lemma 1.3. [5, Lemma 2] *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of a metric space (X,d). Assume that $(\mathcal{A},\mathcal{B})$ has the property* UC. *Let $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathcal{A}$ and $\mathcal{B}$ respectively, such that one of the following equalities holds*

$$\lim_{m\to\infty}\sup_{n\geq m} d(x_m,y_n)=\operatorname{dist}(\mathcal{A},\mathcal{B}) \ \textit{or} \ \lim_{n\to\infty}\sup_{m\geq n} d(x_m,y_n)=\operatorname{dist}(\mathcal{A},\mathcal{B}).$$

Then $\{x_n\}$ is a Cauchy sequence.

2. Main Results

To establish our main results of this section, we introduce the following class of noncyclic contraction type mappings.

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of a metric space (X,d). A mapping $\Gamma:\mathcal{A}\cup\mathcal{B}\to\mathcal{A}\cup\mathcal{B}$ is said to be a *noncyclic Geraghty contraction* whenever

(i) $\Gamma(\mathcal{A})\subseteq\mathcal{A}$ and $\Gamma(\mathcal{B})\subseteq\mathcal{B}$;
(ii) $\exists\ \beta\in\Phi\ ;\ d^*(\Gamma x,\Gamma y)\leq\beta\big(d^*(x,y)\big)d^*(x,y),\quad \forall(x,y)\in\mathcal{A}\times\mathcal{B}$,

where $d^*(x,y)=d(x,y)-\operatorname{dist}(\mathcal{A},\mathcal{B})$ for all $(x,y)\in\mathcal{A}\times\mathcal{B}$.

Lemma 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of a metric space (X,d) and let $\Gamma:\mathcal{A}\cup\mathcal{B}\to\mathcal{A}\cup\mathcal{B}$ be a cyclic Geraghty contraction map* [3]. *Then T^2 is a noncyclic Geraghty contraction map.*

The following useful lemmas are efficient to prove the main result of this section.

Lemma 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of a metric space (X, d) and let $\Gamma : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a cyclic Geraghty quasi-contraction. For $x_0 \in \mathcal{A}$, define $x_{n+1} := \Gamma x_n$ and for $y_0 \in \mathcal{B}$, define $y_{n+1} := \Gamma y_n$ for each $n \geq 0$.*

$$\lim_{n\to\infty} d(x_n, y_n) = \operatorname{dist}(\mathcal{A}, \mathcal{B}).$$

Lemma 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of a metric space (X, d) such that $(\mathcal{A}, \mathcal{B})$ has the property* UC*. Let $\Gamma : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a noncyclic Geraghty quasi-contraction. For $x_0 \in \mathcal{A}$, define $x_{n+1} := \Gamma x_n$ and for $y_0 \in \mathcal{B}$, define $y_{n+1} := \Gamma y_n$ for each $n \geq 0$. Then $d(x_n, x_{n+1}) \to 0$ and $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$.*

Lemma 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of the metric space (X, d) such that $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{A})$ have the property* UC*. Let $\Gamma : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a noncyclic Geraghty quasi-contraction. For $x_0 \in \mathcal{A}$, define $x_{n+1} := \Gamma x_n$ and for $y_0 \in \mathcal{B}$, define $y_{n+1} := \Gamma y_n$ for each $n \geq 0$. Then for every $\epsilon > 0$ there exists a positive integer N_0 such that for all $m > n \geq N_0$*

$$d(x_m, y_n) < \operatorname{dist}(\mathcal{A}, \mathcal{B}) + \epsilon.$$

Now we are ready to state our main results in this section.

Theorem 2.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of the metric space (X, d) such that $d(A, B) = 0$. Let $\Gamma : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a noncyclic Geraghty quasi-contraction. For $x_0 \in A$ define $x_{n+1} := \Gamma x_n$ for each $n \geq 0$. Then Γ has a unique fixed point $x \in A \cap B$ and $x_n \to x$ as $n \to \infty$.*

Theorem 2.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of a metric space (X, d) such that $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{A})$ have the property* UC *and $\mathcal{A}$ is complete. Let $\Gamma : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a noncyclic Geraghty quasi-contraction. For $x_0 \in \mathcal{A}$, define $x_{n+1} := \Gamma x_n$ for each $n \geq 0$. Then there exists a unique fixed point $x \in \mathcal{A}$ such that $x_n \to x$.*

Corollary 2.8. [4] *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of a metric space (X, d) such that $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{A})$ have the property* UC *and $\mathcal{A}$ is complete. Let $\Gamma : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a noncyclic mapping such that*

$$d(Tx, Ty) \leq cd(x, y) + (1 - c)d(A, B),$$

for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$ where $c \in [0, 1)$ is constant. For $x_0 \in \mathcal{A}$, define $x_{n+1} := \Gamma x_n$ for each $n \geq 0$. Then there exists a unique fixed point $x \in \mathcal{A}$ such that $x_n \to x$.

Corollary 2.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of a metric space (X, d) such that $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{A})$ have the property* UC *and $\mathcal{A}$ is complete. Let $\Gamma : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a cyclic Geraghty quasi-contraction. For $x_0 \in \mathcal{A}$, define $x_{n+1} := \Gamma x_n$ for each $n \geq 0$. Then there exists a unique best proximity point of Γ in $\mathcal{A}$ that is a fixed point of Γ^2 and $x_{2n} \to x$.*

References

1. A. Anthony Eldred, P. Veeramani,*Existence and convergence of best proximity points*, J. Math. Anal. Appl., **323** (2006), 1001-1006.
2. M. A. Geraghty, *On contractive mappings*, Proc. Am. Math. Soc., **40** (2) (1973), 604-608.
3. V. Parvaneh, M.R. Haddadi, H. Aydi, *On best proximity point results for some type of mappings*, J. Function Spaces, **2020**, Article ID: 6298138, 6 pages.
4. A. Safari-Hafshejani, A. Amini-Harandi, M. Fakhar, *Best proximity points and fixed points results for noncyclic and cyclic Fisher quasi-contractions*, Numer. Funct. Anal. Optim. **40** (2019), 603-619.
5. T. Suzuki, M. Kikawa, C. Vetro, *The existence of best proximity points in metric spaces with the property* UC, Nonlinear Anal., **71** (7) (2009), 2918-2926.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

CONFORMABLE STURM–LIOUVILLE PROBLEMS WITH EIGEN–PARAMETER DEPENDENT AND TRANSMISSION CONDITIONS

MOHAMMAD SHAHRIARI*

Department of Mathematics, Faculty of Science, University of Maragheh, P.O. Box 55136-553, Maragheh, Iran
shahriari@maragheh.ac.ir

Abstract. In this paper, we study conformable Sturm-Liouville operators of order α $(0 < \alpha \leq 1)$, with eigen-parameter dependent and a jump conditions. Also, we examine the asymptotic form of solutions, eigenvalues and eigenfunctions of the problem. Finally, we provide some examples to better understand the problem.

1. Introduction

In 2014, Khalil et al. in [3] define a new well-behaved (local) simple fractional derivative called the conformable fractional derivative(CFD) depending just on the basic limit definition of the derivative. Unlike other definitions of fractional derivative such as Riemann-Liouville, Caputo, etc., this definition enables us to prove many properties similar to derivatives of integer order. For more information about the CF derivative, refer to [1, 2]. However, the CF derivative has its drawbacks. Its derivative has some disadvantages and some unusual properties, e.g. The zeroth derivative of a function does not return the function.

In this part, we will present some necessary definitions and properties related to conformable fractional calculus theory which can be found in [1, 3].

2020 *Mathematics Subject Classification. Primary 34B20, 34B24; Secondary 47A10, 26A33*
Key words and phrases. Sturm–Liouville problem, conformable derivative, eigen-parameter dependent conditions.

* Speaker.

Definition 1.1. For the function $h : [0,\infty) \to \mathbb{R}$ and $\alpha \in (0,1]$, the CF derivative is defined by:

$$D^\alpha h(\tau) = \lim_{\delta\to 0} \frac{h(\tau+\delta\tau^{1-\alpha}) - h(\tau)}{\delta}, \qquad D^\alpha h(0) = \lim_{\tau\tau\to 0^+} D^\alpha h(\tau). \tag{1.1}$$

for all $\tau > 0$. If h is differentiable, then

$$D^\alpha f(\tau) = \tau^{1-\alpha} h'(\tau). \tag{1.2}$$

Definition 1.2. For the function $h : [0,\infty) \to \mathbb{R}$ and $\alpha \in (0,1]$, the CF integral is defined by:

$$J_\alpha h(\tau) = \int_0^\tau h(t)\mathrm{d}_\alpha t = \int_0^\tau t^{\alpha-1} h(t)\mathrm{d}t. \tag{1.3}$$

So that the above integral is the Riemann improper integral.

2. Main problem and spectral properties

Let us consider the following boundary value problem

$$\ell_\alpha y := -D^\alpha D^\alpha y + qy = \lambda y \tag{2.1}$$

$$L_1(y) := \lambda y(0) - hD^\alpha y(0) = 0, \quad L_2(y) := \lambda y(\pi) + HD^\alpha y(\pi) = 0 \tag{2.2}$$

with the jump conditions

$$\begin{aligned} \mathcal{D}_1(y) &:= y(p+0) - ay(p-0) = 0,\\ \mathcal{D}_2(y) &:= D^\alpha y(p+0) - bD^\alpha y(p-0) - cy(p-0) = 0, \end{aligned} \tag{2.3}$$

where $q(x)$ is real-valued function in $L^1_\alpha[0,\pi]$. We also assume that h, H, a, b, c and p are real numbers and $ab = 1$. For the reader's convenience, we use the notation $L_\alpha = L_\alpha(q(x); h; H; p)$, for the problem (2.1)–(2.3). To obtain a self-adjoint operators, we define the weighted inner products as follows

$$\langle F, G\rangle_{\mathcal{H}} := \int_0^\pi f\overline{g}\mathrm{d}_\alpha x + \frac{1}{h} f_1\overline{g_1} + \frac{1}{H} f_2\overline{g_2}, \quad F = \begin{pmatrix} f(x)\\ f_1\\ f_2\end{pmatrix}, G = \begin{pmatrix} g(x)\\ g_1\\ g_2\end{pmatrix}. \tag{2.4}$$

In order to obtain a new eigenvalue problem using space $\mathcal{H} := L^2_\alpha(0,\pi) \oplus \mathbb{C}^2$ with the operator

$$A_\alpha : \mathcal{H} \to \mathcal{H}.$$

Next we introduce

$$R_1(y) := y(0),\ R_1'(y) := hD^\alpha y(0), \quad R_2(y) := y(\pi),\ R_2'(y) := HD^\alpha y(\pi).$$

In this space we construct with domain

$$\mathrm{dom}\,(A_\alpha) = \left\{ F = \begin{pmatrix} f(x)\\ f_1\\ f_2\end{pmatrix} \middle| \begin{array}{c} f, D^\alpha f \in AC\big([0,p)\cup(p,\pi]\big),\ \ell f \in L(0,\pi)\\ f_1 = R_1(f),\ f_2 = R_2(f),\ \mathcal{D}_1(f) = \mathcal{D}_2(f) = 0, \end{array} \right\}$$

by

$$A_\alpha F = \begin{pmatrix} \ell f\\ R_1'(f)\\ -R_2'(f)\end{pmatrix} \quad \text{with } F = \begin{pmatrix} f(x)\\ R_1(f)\\ R_2(f)\end{pmatrix} \in \mathrm{dom}\,(A_\alpha).$$

By construction the eigenvalue problem of the foolowing form

$$A_\alpha Y = \lambda Y, \qquad Y := \begin{pmatrix} y(x) \\ R_1(y) \\ R_2(y) \end{pmatrix} \in \operatorname{dom}(A_\alpha).$$

It is easy to see that this problem is equivalent to the eigenvalue problem (2.1)–(2.3).

Lemma 2.1. *If $0 < \alpha \le 1$, then the operator A_α is self-adjoint on $L^2_\alpha((0,\pi);w) \oplus \mathbb{C}^2$.*

Especially, the eigenvalues of A_α and consequently L_α are real and simple. Assume that $u(x,\lambda)$ and $v(x,\lambda)$ are solutions of (2.1) with

$$u(0,\lambda) = h, \quad D^\alpha u(0,\lambda) = \lambda, \quad v(\pi,\lambda) = -H, \quad D^\alpha v(\pi,\lambda) = \lambda, \tag{2.5}$$

and the transmission conditions (2.3), respectively. Note that the equation (2.1) with initial conditions (2.5) has a unique solution. These solutions are the entire function in $\lambda \in \mathbb{C}$ for $x \in [0,d) \cup (d,\pi]$. Similarly, we can write

$$\Delta(\lambda) := W_\alpha(u(\lambda), v(\lambda)) = L_1(v(\lambda)) = -w(\pi) L_2(u(\lambda)), \tag{2.6}$$

whereW_α is the modified conformable Wronskian. The function $\Delta(\lambda)$ is called the characteristic function. All roots λ_n of $\Delta(\lambda)$ are the eigenvalues of L_α.

3. Asymptotic formulas for eigendata

In this section, we study the asymptotic forms of solutions and eigenvalues. For these aims, we prove some lemmas and theorems as follows.

Theorem 3.1. *Let $\lambda = \rho^2$ and $\tau := |\mathrm{Im}\rho|$. The asymptotic forms of solutions and the characteristic function for PDCFSLP (2.1)– (2.3) as $|\lambda| \to \infty$, are established as follows:*

$$u(x,\lambda) = \begin{cases} \rho^2 \cos\left(\frac{\rho}{\alpha} x^\alpha\right) + \rho\left[q_1(x) \sin\left(\frac{\rho}{\alpha} x^\alpha\right) + \frac{1}{2}\int_0^x q(t) \sin\left(\frac{\rho}{\alpha}\left(x^\alpha - 2t^\alpha\right)\right) \mathrm{d}_\alpha t\right] \\ \qquad + O\left(\exp\left(\frac{\tau}{\alpha} x^\alpha\right)\right), \qquad 0 \le x < p, \\ \rho^2\left[b_1 \cos\left(\frac{\rho}{\alpha} x^\alpha\right) + b_2 \cos\left(\frac{\rho}{\alpha}(2p^\alpha - x^\alpha)\right)\right] + \rho\left[f_1(x) \sin\left(\frac{\rho}{\alpha} x^\alpha\right)\right. \\ \qquad \left. + f_2(x) \sin\left(\frac{\rho}{\alpha}(2p^\alpha - x^\alpha)\right)\right] + O\left(\exp\left(\frac{\tau}{\alpha} x^\alpha\right)\right), \qquad p \le x < \pi, \end{cases} \tag{3.1}$$

$$D^\alpha_x u(x,\lambda) = \begin{cases} -\rho^3 \sin\left(\frac{\rho}{\alpha} x^\alpha\right) + \rho^2\left[q_1(x) \cos\left(\frac{\rho}{\alpha} x^\alpha\right) + \frac{1}{2}\int_0^x q(t) \cos\left(\frac{\rho}{\alpha}\left(x^\alpha - 2t^\alpha\right)\right) \mathrm{d}_\alpha t\right] \\ \qquad + O\left(\frac{1}{\rho} \exp\left(\frac{\tau}{\alpha} x^\alpha\right)\right), \qquad 0 \le x < p, \\ \rho^3\left[-b_1 \sin\left(\frac{\rho}{\alpha} x^\alpha\right) + b_2 \sin\left(\frac{\rho}{\alpha}(2d^\alpha - x^\alpha)\right)\right] + \rho^2\left[f_1(x) \cos\left(\frac{\rho}{\alpha} x^\alpha\right)\right. \\ \qquad \left. - f_2(x) \cos\left(\frac{\rho}{\alpha}(2p^\alpha - x^\alpha)\right)\right] + O\left(\rho \exp\left(\frac{\tau}{\alpha} x^\alpha\right)\right), \qquad d \le x < \pi, \end{cases} \tag{3.2}$$

where

$$q_1(x) = \frac{1}{2}\int_0^x q(t) \mathrm{d}_\alpha t - h, \;\; f_1(x) = b_1\left(\frac{1}{2}\int_0^x q(t) \mathrm{d}_\alpha t - h\right) + \frac{c}{2},$$

$$f_2(x) = b_2\left(-\frac{1}{2}\int_0^x q(t) \mathrm{d}_\alpha t + \int_0^p q(t) \mathrm{d}_\alpha t - h\right) - \frac{c}{2}, \;\; b_1 = \frac{a+b}{2}, \;\; b_2 = \frac{a-b}{2}.$$

The characteristic function satisfies

$$\Delta(\lambda) = w(\pi)\left[\rho^4\left(b_1\cos\left(\frac{\rho}{\alpha}\pi^\alpha\right)+b_2\cos\left(\frac{\rho}{\alpha}(2p^\alpha-\pi^\alpha)\right)\right)+\rho^3\left[-w_1\sin\left(\frac{\rho}{\alpha}\pi^\alpha\right)\right.\right.$$
$$\left.\left.+w_2\sin\left(\frac{\rho}{\alpha}(2d^\alpha-\pi^\alpha)\right)\right]+o\left(\exp\left(\frac{\tau}{\alpha}\pi^\alpha\right)\right)\right]. \tag{3.3}$$

where

$$w_1 = b_1\left(H-h+\frac{1}{2}\int_0^\pi q(t)\mathrm{d}_\alpha t\right)+\frac{c}{2},$$
$$w_2 = b_2\left(H-h-\frac{1}{2}\int_0^\pi q(t)\mathrm{d}_\alpha t+\int_0^p q(t)\mathrm{d}_\alpha t\right)-\frac{c}{2}. \tag{3.4}$$

One can see that from Valiron's theorem [4] and the above calculations, we obtain:

Theorem 3.2. *The eigenvalues $\lambda_n=\rho_n^2$ of the (2.1)–(2.3) satisfy*

$$\rho_n = \alpha\pi^{1-\alpha}n+O(1) \qquad \text{as } n\to\infty. \tag{3.5}$$

Example 3.3. Consider the following problem (2.1)–(2.3) with $q(x)=0$, $h=H=1$ and $a=1$ with one jump point $p=\frac{\pi}{4}$. The eigenvalues are presented in Table 1. We use the **fzero** function in **MATLAB R2015a** to compute the zeros $\rho_{n,\alpha}$ of the function $\Delta(\lambda)$.

TABLE 1. Eigenvalues and asymptotic results for Example 3.3.

	$\rho_{n,\alpha}$				$\zeta_{n,\alpha}$			
n	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=0.99$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=0.99$
2	1.2747	1.3266	1.3520	1.3501	0.646	0.657	0.670	0.674
3	2.2651	2.2556	2.2608	2.2801	0.765	0.745	0.747	0.759
4	3.5339	3.5366	3.4232	3.3073	0.895	0.876	0.848	0.826
10	9.2301	9.6384	9.3520	9.6268	0.935	0.955	0.926	0.961
20	19.3728	19.7019	19.5224	19.4549	0.973	0.976	0.967	0.971
30	29.2630	29.4561	29.8097	29.5722	0.985	0.983	0.984	0.988
40	38.9354	39.7493	40.0483	39.5571	0.986	0.984	0.988	0.988

We compared the eigenvalues with first term of asymptotic form (3.5) as $\zeta_{n,\alpha}=\frac{\rho_n}{n\alpha\pi^{1-\alpha}}$. The eigenvalue and ratio ζ_n are presented in Tables 1. According to asymptotic form (3.5), the values of $\zeta_{n,\alpha}$ must be tend to one, that hold for results of $\zeta_{n,\alpha}$ in Tables 1.

REFERENCES

1. T. Abdeljawad, *On conformable fractional calculus, J Comput Appl Math.,* ***279*** *2015, 57–66.*
2. *A. Atangana, D. Baleanu, and A. Alsaedi, New properties of conformable derivative, Open Math.,* ***13*** *2015, 889–898.*
3. *R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, A new definition of fractional derivative, J Comput Appl Math.,* ***264*** *2014, 65–70.*
4. *H. Mortazaasl and A. Jodayree Akbarfam, Trace formula and inverse nodal problem for a conformable fractional Sturm-Liouville problem, Inverse Probl. Sci. Eng.,* ***28(4)*** *2020, 524-555.*

The 8[th] Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

INVERSE CONFORMABLE STURM–LIOUVILLE PROBLEMS BY THREE SPECTRA WITH JUMP CONDITIONS

MOHAMMAD SHAHRIARI*

Department of Mathematics, Faculty of Science, University of Maragheh, P.O. Box 55136-553, Maragheh, Iran
shahriari@maragheh.ac.ir

Abstract. In this manuscript, we consider the fractional conformable Sturm–Liouville problem (CFSLP) with finite numbers of transmission conditions at an interior point in $[0, \pi]$. Also, we study the uniqueness theorem for inverse second order of fractional differential operators by applying three spectra with a finite number of discontinuities at interior points. For this aim, we investigate the CFSLP in three intervals $[0, \pi]$, $[0, p]$, and $[p, \pi]$ such that $p \in (0, \pi)$ is an interior point.

1. Introduction

Sturm–Liouville problem is one of the most important problems in mathematics, physics and engineering. This problem arises in the modeling of many systems in vibration theory, quantum mechanics, hydrodynamics, and etc. [8].

There are two types of CFSLP: direct and inverse problems. In direct problems, the eigenvalues, eigenfunctions, and other properties are estimated from the known coefficients [5]. The inverse spectral problem can

2020 *Mathematics Subject Classification. Primary 34B20, 34B24; Secondary 47A10, 26A33*

Key words and phrases. Conformable Sturm–Liouville problem, internal discontinuities, three spectra.

* Speaker.

be considered as three aspects: existence, uniqueness, and reconstruction of coefficients with special property of eigenvalues and eigenfunctions, (see [6, 7] and the references therein). The main purpose of this manuscript is to study the inverse CFSLP by using three spectra.

In 2014, Khalil et al. in [4] define a new well-behaved (local) simple fractional derivative called the conformable fractional derivative(CFD) depending just on the basic limit definition of the derivative. Unlike other definitions of fractional derivative such as Riemann-Liouville, Caputo, etc., this definition enables us to prove many properties similar to derivatives of integer order. For more information about the CF derivative, refer to [1, 2]. However, the CF derivative has its drawbacks. Its derivative has some disadvantages and some unusual properties, e.g. The zeroth derivative of a function does not return the function.

In this part, we will present some necessary definitions and properties related to conformable fractional calculus theory which can be found in [1, 4].

Definition 1.1. For the function $h : [0,\infty) \to \mathbb{R}$ and $\alpha \in (0,1]$, the CF derivative is defined by:

$$D^\alpha h(\tau) = \lim_{\delta\to 0} \frac{h(\tau+\delta\tau^{1-\alpha}) - h(\tau)}{\delta}, \qquad D^\alpha h(0) = \lim_{\tau\tau\to 0^+} D^\alpha h(\tau). \quad (1.1)$$

for all $\tau > 0$. If h is differentiable, then

$$D^\alpha f(\tau) = \tau^{1-\alpha} h'(\tau). \quad (1.2)$$

Definition 1.2. For the function $h : [0,\infty) \to \mathbb{R}$ and $\alpha \in (0,1]$, the CF integral is defined by:

$$J_\alpha h(\tau) = \int_0^\tau h(t)\mathrm{d}_\alpha t = \int_0^\tau t^{\alpha-1} h(t)\mathrm{d}t. \quad (1.3)$$

So that the above integral is the Riemann improper integral.

Let us consider the following three CFSLPs

$$\ell_0 y := -D^\alpha D^\alpha y + qy = \lambda y \quad (1.4)$$

with

$$\mathrm{B}_1(y) := D^\alpha y(0) + h\, y(0) = 0, \quad (1.5)$$

$$\mathrm{B}_2(y) := D^\alpha y(\pi) + H\, y(\pi) = 0, \quad (1.6)$$

subject to the following jump conditions

$$\begin{aligned} U_k(y) &:= y(p_k+0) - b_k y(p_k-0) = 0, \\ V_k(y) &:= D^\alpha y(p_k+0) - c_k D^\alpha y(p_k-0) - d_k y(p_k-0) = 0, \end{aligned} \quad (1.7)$$

$$\ell_1 y := -D^\alpha D^\alpha y + q_1 y = \lambda y \quad (1.8)$$

with

$$\mathrm{B}_1(y) = 0, \quad \mathrm{B}_3(y) := D^\alpha y(p) + H_1\, y(p) = 0, \tag{1.9}$$

subject to the following jump conditions

$$U_k(y) = 0, \quad V_k(y) = 0, \text{ for } k = 1, 2, \ldots, p-1, \tag{1.10}$$

and

$$\ell_2 y := -D^\alpha D^\alpha y + q_2 y = \lambda y \tag{1.11}$$

with

$$\mathrm{B}_4(y) := D^\alpha y(p) + H_2\, y(p) = 0, \ \mathrm{B}_2(y) = 0, \tag{1.12}$$

subject to the following jump conditions

$$U_k(y) = 0, \quad V_k(y) = 0, \text{ for } k = p+1, 2, \ldots, m-1, \tag{1.13}$$

where D^α is the CF derivative of order $0 < \alpha \leq 1$, $q(x) \in L^1_\alpha[0,\pi]$, $q_1 = q|_{[0,p)}$, and $q_2 = q|_{(p,\pi]}$ are real valued functions. Also, h, H, H_1, H_2, b_k, c_k, d_k, and p_k, $k = 1, 2, \ldots, m-1$ (with $m \geq 2$) are real numbers. The parameter λ is the spectral parameter. In this paper, we suppose that $c_k b_k > 0$, $p_0 = 0 < p_1 < p_2 < \ldots < p_{m-1} < p_m = \pi$.
In this section, we assume that $p = p_s$ for $1 \leq s \leq m-1$. As well as, we use the notations $L_0 = L(q(x); h; H; p_k)$, $L_1 = L(q_1(x); h; H_1; p_k)$, $L_2 = L(q_2(x); H_2; H; p_k)$ for the problems (1.4)–(1.13). Using the jump conditions (1.7) in the transmission point $p = p_s$, $(1 \leq s \leq m-1)$, we must have $d_s = 0$ and

$$H_2 = \frac{c_s}{b_s} H_1, \text{ for } H_1 \in (0, \infty). \tag{1.14}$$

2. **Uniqueness result**

In this section, we study the inverse CFSLP of the reconstruction of a boundary value problem L_0 from its spectral characteristics. For this purpose, we consider three boundary value problems L_i, $(i = 0, 1, 2)$, from three spectra $\{\lambda_n, \mu_n, \nu_n\}_{n \geq 0}$. To prove the uniqueness theorem, we use an adaptation of this technique, firstly it was discussed by F. Gesztesy and B. Simon in [3].

Theorem 2.1. *If $\lambda_n = \tilde{\lambda}_n$, $\mu_n = \tilde{\mu}_n$, and $\nu_n = \tilde{\nu}_n$ for $n \geq 0$, and $r(x) = \tilde{r}(x)$, $h = \tilde{h}$, and $H = \tilde{H}$, and if $\{\mu_n\}_{n=1}^{+\infty}$ and $\{\nu_n\}_{n=1}^{+\infty}$ are pairwise disjoint, then $L = \tilde{L}$.*

Assuming $b_s = c_s = 1$ in Eq. (1.14) we have $H_1 = H_2$. From this assumptions, the main result (Theorem 2.1) can be extended to the case $p \in (p_s - 1, p_{s+1})$.

Corollary 2.2. *Let $\lambda_n = \tilde{\lambda}_n$, $\mu_n = \tilde{\mu}_n$, and $\nu_n = \tilde{\nu}_n$ for $n \geq 0$, and $r(x) = \tilde{r}(x)$, $h = \tilde{h}$, $H = \tilde{H}$, $b_s = 1$, and $c_s = 1$, and if $\{\mu_n\}_{n=1}^{+\infty}$ and $\{\nu_n\}_{n=1}^{+\infty}$ are pairwise disjoint, then $L = \tilde{L}$.*

Let $b_i = c_i = 1$, $d_i = 0$ for $i = 1, 2, \ldots, m-1$ in Eqs. (1.7), then our CFSLP changes to the continuous case equation.

Corollary 2.3. *If $\lambda_n = \tilde{\lambda}_n$, $\mu_n = \tilde{\mu}_n$, and $\nu_n = \tilde{\nu}_n$ for $n \geq 0$, $h = \tilde{h}$, $H = \tilde{H}$, and if $\{\mu_n\}_{n=1}^{+\infty}$ and $\{\nu_n\}_{n=1}^{+\infty}$ are pairwise disjoint, then $L = \tilde{L}$.*

References

1. T. Abdeljawad, *On conformable fractional calculus, J Comput Appl Math.,* ***279*** *2015, 57–66.*
2. *A. Atangana, D. Baleanu, and A. Alsaedi, New properties of conformable derivative, Open Math.,* ***13*** *2015, 889–898.*
3. *F. Gesztesy and B. Simon, On the determination of a potential from three spectra, Amer. Math. Soc. Trans.,* **189** *(1999), 85–92.*
4. *R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, A new definition of fractional derivative, J Comput Appl Math.,* ***264*** *2014, 65–70.*
5. *A.S. Ozkan and İ. Adalar, Inverse problems for a conformable fractional Sturm–Liouville operator, J. Inverse Ill-Posed Probl.,* ***28(6)*** *775–782, 2020.*
6. *M. Shahriari, A.J. Akbarfam and G. Teschl, Uniqueness for inverse Sturm–Liouville problems with a finite number of transmission conditions, J. Math. Anal. Appl.,* ***395*** *(2012), 19–29.*
7. *M. Shahriari and H. Mirzaei, Inverse Sturm-Liouville problem with conformable derivative and transmission conditions, Hacet. J. Math. Stat.,* **52(3)** *(2023), 753–767.*
8. *G. Teschl, Mathematical Methods in Quantum Mechanics; With Applications to Schrödinger Operators, Graduate Studies in Mathematics, Amer. Math. Soc., Rhode Island, 2009.*

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

EXISTENCE OF POSITIVE SOLUTIONS FOR AN ECOLOGICAL MODEL INVOLVING NONLOCAL OPERATOR

S. SHAKERI*

Department of Mathematics, Ayatollah Amoli Branch ,Islamic Azad University , Amol , Iran
Saleh.Shakeri@iau.ac.ir

Abstract. In this article, Using the method of sub-super solutions and comparison principle, we study the existence of positive solutions for an ecological model. This model describes the steady states of a logistic growth model and the dynamics of the fish population with grazing, natural predation and constant yield harvesting.

1. Introduction

In this paper, we are interested in the existence of positive solutions for the p-Kirchhoff-type problems

$$\begin{cases} -M(\int_\Omega |\nabla u|^2 dx)\Delta u = \dfrac{au - bu^{\gamma-1} - c}{u^\alpha}, & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases} \tag{1.1}$$

where $M : [0, \infty) \to (0, \infty)$ is a continuous and increasing function, $c \geq 0$, $a, b > 0$, Ω is a bounded domain with smooth boundary, $\Delta u = \operatorname{div}(\nabla u)$ is the Laplacian operator , $1 < \gamma$ and $\alpha \in (0, 1)$.
Here u is the population density and $\frac{au - bu^{\gamma-1} - c}{u^\alpha}$ represents logistics growth.

2020 *Mathematics Subject Classification. Primary 47B35; Secondary 30H05*
Key words and phrases. sub-super solutions, comparison principle, sign-changing weights.
* Speaker.

This model describes grazing of a fixed number of grazers on a logistically growing species (see [11]). The herbivore density is assumed to be a constant which is a valid assumption for managed grazing systems and the rate of grazing is given by $\frac{c}{u^{\alpha}}$. At high levels of vegetation density this term saturates to c as the grazing population is a constant. This model has also been applied to describe the dynamics of fish populations ([12]). In the case of the fish population the term $\frac{c}{u^{\alpha}}$ corresponds to natural predation.

Problem (1.1) is called nonlocal because of the term $-M(\int_{\Omega}|\nabla u|^r dx)$ which implies that the first two equations in (1.1) are no longer pointwise equalities.it is well known that the Kirchhoff type systems has a mechanical and biological motivation[3, 13]. In this article, we are motivated by the ideas introduced in [7, 9, 11] and properties of Kirchhoff-type operators in [4, 5], we study problem (1.1) in semipositone case (i.e., $\lim_{s\to 0^+} f(s) = -\infty$; $f(s) = \frac{as-bs^{\gamma-1}-c}{s^{\alpha}}$; see [6]).
Using sub-supersolution techniques, we prove the existence of a positive solution for the problem.
To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -\Delta\phi = \lambda\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Let ϕ be the eigenfunction corresponding to the first eigenvalue λ_1 of (1.3) such that $\phi(x) > 0$ in Ω and $\|\phi\|_{\infty} = 1$. It can be shown that $\frac{\partial\phi}{\partial n} < 0$ on $\partial\Omega$. Here n is the outward normal. Let $m, \delta > 0$ and $\mu > 0$ be such that:

$$\mu \le \phi \le 1, x \in \Omega - \overline{\Omega_{\delta}}, \tag{1.3}$$

$$|\nabla\phi|^2 \ge m, x \in \overline{\Omega_{\delta}}, \tag{1.4}$$

with $\overline{\Omega_{\delta}} := \{x \in \Omega | d(x, \partial\Omega) \le \delta\}$. This is possible since $|\nabla\phi|^2 \ne 0$ on $\partial\Omega$ while $\phi = 0$ on $\partial\Omega$. We will also consider the unique solution $e \in W_0^{1,2}(\Omega)$ of the boundary value problem

$$\begin{cases} -\Delta e = 1, & x \in \Omega, \\ e = 0, & x \in \partial\Omega. \end{cases}$$

to discuss our existence result, it is known that $e > 0$ in Ω and $\frac{\partial e}{\partial n} < 0$ on $\partial\Omega$.

2. Existence results

In this section, we shall establish our existence result via the method of sub-supersolution. A function w is said to be a subsolution of (1.1), if it is in $W_0^{1,2}(\Omega)$ such that

$$-M\left(\int_\Omega |\nabla\psi|^2 dx\right)\int_\Omega \nabla\psi\cdot\nabla w dx \leq \int_\Omega \left[\frac{a\psi - b\psi^{\gamma-1} - c}{\psi^\alpha}\right] w dx,$$

and z is said supersolution of (1.1), if it is in $W_0^{1,2}(\Omega)$ such that

$$-M\left(\int_\Omega |\nabla z|^2 dx\right)\int_\Omega \nabla z\cdot\nabla w dx \leq \int_\Omega \left[\frac{az - bz^{\gamma-1} - c}{z^\alpha}\right] w dx,$$

for all $w \in W = \{w \in C_0^\infty(\Omega) | w \geq 0,\ x \in \Omega\}$. Then the following result holds:
Then the following result holds:

Lemma 2.1. *([1, 2, 8]) Suppose there exist sub and supersolutions ψ and z respectively of (1.1) such that $\psi \leq z$. Then (1.1) has a solution u such that $\psi \leq u \leq z$.*

Now we state our main result.

Theorem 2.2. *Let there exist constants $M_0 > 0$ and $M_\infty \geq 0$ such that $M_0 \leq M(t) \leq M_\infty$ for all $t \in [0,\infty)$. Given $a, b > 0$, $1 < \gamma$, and $\alpha \in (0,1)$, there exists a constant $c_1 = c_1(a, b, \alpha, \gamma, \Omega) > 0$ such that for $c < c_1$, (1.1) has a positive solution.*

Remark 2.3. In the nonsingular case ($\alpha = 0$), positive solutions exist only when $a > \lambda_1$ (the principle eigenvalue) ([9, 10]). But in the singular case, we establish the existence of a positive solution for any $a > 0$.

Acknowledgments. This research work has been supported by a research grant from Islamic Azad University-Ayatollah Amoli Branch.

References

1. G.A. Afrouzi, N.T. Chung and S. Shakeri, *Existence of positive solutions for Kirchhoff type equations*, Electron. J. Diff. Eqs., **180** (2013), 1-8.
2. S. Cui, *Existence and nonexistence of positive solution for singular semilinear elliptic boundary value problems*, Nonlinear Anal., (2000), 149-176.
3. C. O. Alves and F. J. S. A. Corrêa,*On existence of solutions for a class of problem involving a nonlinear operator*, Commun. Appl. Nonlinear Anal. **8** (2001), 43-56.
4. G. Dai, *Three solutions for a nonlocal Dirichlet boundary value problems involving the $p(x)$-Laplacian*, Appl. Anal. **92** (1) (2013), 191-210.
5. G. Dai and R. Ma, *Solutions for a $p(x)$-Kirchhoff type equation with Neumann boundary data*, Nonlinear Anal. Real World Appl., **12** (2011), 2666-2680.
6. J. Goddard ll, E.K. Lee and L. Sankarand R. Shivaji, *Existence results for classses of infinite semipositone problems*, Boundary Value Problems, **2013** (97) (2013), 1-9.

7. E.K. Lee, R. Shivaji and J. Ye, *Positive solutions for infinite semipositone problems with falling zeros*, Nonlinear Anal., **72** (2010), 4475-4479.
8. E.K. Lee, R. Shivaji and J. Ye, *Classes of infinite semipositone systems*, Proc. R. Soc. Edinb., **139A** (2009), 853-865.
9. S. Oruganti, J. Shi and R. Shivaji, *Diffusive logistic equation with constant yield harvesting, I: steady states*, Trans. Am. Math. Soc., **345** (9) (2002), 3601-3619
10. S. Oruganti, J. Shi and R. Shivaji, *Logistic equation with the p-Laplacian and constant yeild harvesting*, Abstr. Appl. Anal., **9** (2004), 723-727.
11. S. H. Rasouli,*Existence of solutions for singular* (p, q)*-Kirchhoff type systems with multiple parameters,Electronic Journal of Differential Equations*, **69**(2016), 1-8.
12. J.H. Steele and E.W. Henderson, *Modelling long term fluctuations in fish stocks*, Science, **224** (1984), 985-987.
13. J.D.Murray, Mathematical biology II, *spatial models and biomedical applications, Interdisciplinary Applied Mathematics* . **18** (2003), (springer).

EXISTENCE OF ϕ-BEST PROXIMITY POINT FOR CONTRACTIVE MAPPINGS

MARYAM SHAMS AND PARVANEH LO′LO′*

Department of Mathematics, University of Sharekord, Sharekord, Iran
Department of Mathematics, Behbahan Khatam Alanbia University of Technology, Behbahan, Iran
maryam.shams@sku.ac.ir; lolo@bkatu.ac.ir

Abstract. In this article, we first introduce the (F, ϕ, ψ)-proximal contraction mappings and then present theorems on the existence and uniqueness of the ϕ-best proximity point of these mappings. Furthermore, sufficient conditions to ensure the existence of a unique solution for a variational inequality problem are also discussed.

1. Introduction

Let A and B be nonempty subsets of the metric space (X, d). Also, suppose that $f : A \to B$ is non-self mapping. If $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ and $d(a, fa) = d(A, B)$, then a is called the best proximity point. When a mapping does not have a fixed point, studying the best proximity point theory is a suitable way to obtain optimal approximate solutions. Therefore, optimization theory is developed with the theory of the best proximity points. If the mapping under study is self-mapping, the best proximity point is the fixed point. Therefore, the best proximity point theorems also act as a natural extension of fixed point theorems. Interesting best proximity point theorems with different control functions and various

2020 *Mathematics Subject Classification. Primary 47B35; Secondary 30H05*
Key words and phrases. ϕ-best proximity, (F, ϕ, ψ)-proximal contraction, inner product.
* Speaker.

spaces can be seen in [3]. The existence and convergence of the best point of proximity has been investigated by al-Taghfi and Shahzad in [1]. In this article, we first introduce the (F, ϕ, ψ)-proximal contraction mappings and then present theorems on the existence and uniqueness of the ϕ-best proximity point of these mappings.

2. Preliminaries

Let X be a nonempty set and $\phi : X \to [0, \infty]$, then we denote the center of the function ϕ by $Z_\phi = \{x \in X : \phi(x) = 0\}$. Let A and B be two nonempty subsets of a metric space (X, d); the following notions will be used all over the article.

$$A_0 = \{a \in A : d(a, b) = d(A, B), \text{for some } b \in B\},$$
$$B_0 = \{b \in B : d(a, b) = d(A, B), \text{for some } a \in A\}.$$
$$B_{est}(T) = \{x \in A : d(x, Tx) = d(A, B)\}.$$

Definition 2.1. [4] An element $x^* \in A$ is said to be a ϕ-best proximity point of the non-self mapping $T : A \to B$ if $x^* \in B_{est}(T) \bigcap Z_\phi$.

Let (X, d) be a metric space. We denote the set of functions F that satisfying in the following conditions by $\mathcal{F}$.

F1) $\max\{a, b\} \leq F(a, b, c)$, $a, b, c \in [0, \infty)$.
F2) $F(0, 0, 0) = 0$.
F3) F is continuous.

Here are some simple examples for member functions of $\mathcal{F}$:

1) $F(a, b, c) = a + b + c$, $a, b, c \in [0, \infty)$.
2) $F(a, b, c) = \max\{a + b\} + c$, $a, b, c \in [0, \infty)$

Also we denote the set of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfy in the following conditions with Ψ

ψ1) ψ be non-descending,
ψ2) $\Sigma_{n=1}^{\infty} \psi^n(t) < \infty$, $t > 0$.

In 2017, Isik and his colleagues [4] provided the following definition and proved some theorems on these mappings.

Definition 2.2. Let A and B be two nonempty subsets of a metric space (X, d), $\phi : A \to [0, \infty)$ be a given function and $F \in \mathcal{F}$. Then, $T : A \to B$ is (F, ϕ)-proximal contraction, if there exists $k \in (0, 1)$ such that

$$\begin{cases} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{cases} \text{implies } F(d(u, v), \phi(u), \phi(v)) \leq kF(d(x, y), \phi(x), \phi(y)).$$

for all $u, v, x, y \in A$

3. Main results

In this section, we present the concept of (F, ϕ, ψ)-proximal contraction mappings and investigate the existence and uniqueness of the ϕ-best proximity point for these mappings.

Definition 3.1. Let A and B be two nonempty subsets of a metric space (X,d). Consider $\phi : A \to [0,\infty)$ and $F \in \mathcal{F}$. Then, $T : A \to B$ is (F,ϕ,ψ)-proximal contraction, if there exists $\psi \in \Psi$ such that

$$\begin{cases} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{cases} \text{implies } F(d(u,v),\phi(u),\phi(v)) \leq \psi(F(d(x,y),\phi(x),\phi(y))).$$

for all $u,v,x,y \in A$

Theorem 3.2. *Let A and B be two nonempty subsets of a metric space (X,d) and non-self mapping $T : A \to B$ satisfies in the following conditions*

1) *A_0 is nonempty and closed,*
2) *$T(A_0) \subseteq B_0$.*
3) *$\phi : A \to [0,\infty)$ is lower semi-continuous;*
4) *T is a (F,ϕ,ψ)-proximal contraction*

Then T has a unique ϕ-best proximity point $x^* \in A$. Moreover for each $x \in X$, we have $\lim_{n\to\infty} T^n x = x^*$.

Corollary 3.3. *Let A and B be two nonempty subsets of a metric space (X,d). If the following assumptions hold, then there exists a unique $x^* \in A$ such that $B_{est}(T) \cap Z_\phi = \{x^*\}$. Moreover, for each $x \in X$, we have $\lim_{n\to\infty} T^n x = x^*$.*

1) *A_0 is nonempty and complete with respect to the topology induced by d;*
2) *$T(A_0) \subseteq B_0$.*
3) *$\phi : A \to [0,\infty)$ is lower semi-continuous;*
4) *$T : A \to B$ is an (F,ϕ)-proximal contraction.*

The following is an example of a theorem.

Example 3.4. Let $X = \{0,1,2,3,...\}$,$A = \{0,1,3,5,...\}$ and $B = \{0,2,4,6,...\}$. We consider the meter d as follows:

$$d(x,y) = \begin{cases} x+y & x \neq y, \\ 0 & x = y. \end{cases}$$

Let $F : [0,\infty) \to [0,\infty)$, $\phi : A \to [0,\infty)$ and $\psi : [0,\infty) \to [0,\infty)$ be defined by $F(a,b,c) = a+b+c$, $\phi(x) = x$ and $\psi(x) = (1/2)x$. So, it is clear $\phi \in \mathcal{F}$ and $Z_\phi = \{0\}$. Define $T : A \to B$ by

$$T(x) = \begin{cases} x-1 & x \in \{3,5,7,...\}, \\ 0 & x = 0,1. \end{cases}$$

Then here $A_0 = \{0\}$, $B_0 = \{0\}$ and $d(A,B) = 0$. Let us consider $d(u,Tx) = d(A,B) = 0$ and $d(v,Ty) = d(A,B) = 0$, so $u = v = Tx = Ty = 0$ therefore $x,y \in \{0,1\}$. By distinguishing four cases, we have T is a (F,ϕ,ψ)-proximal contraction. So, by an application of Theorem ,we conclude $B_{est}(T) \cap Z_\phi = \{0\}$.

We mention that by H we denote a real Hilbert space, with inner product $< .,. >$ and induced norm $\| . \|$. Let K be a nonempty closed and convex subset of H and $S : H \to H$ be an operator. We consider the following variational inequality problem: Find $u \in K$ such that $< Su, v - u > \geqslant 0$ for all $v \in K$. We use the theoretical results concerning the metric projection operator $P_K : H \to K$ see [5, 6]; and recall that for each $u \in H$, there exists a unique nearest point $P_K u \in K$ satisfying the inequality $\| u - P_K u \| \leqslant \| u - v \|$, for all $v \in K$.

Lemma 3.5. *Let $S : H \to H$. Then, $u \in K$ is a solution of $< Su, v - u > \geqslant 0$, for all $v \in K$, if and only if $u = P_K(u - \lambda Su)$, with $\lambda > 0$.*

Theorem 3.6. *Let $\phi : K \to [0, \infty)$ is lower semi-continuous and $P_K(I_K - \lambda S) : K \to K$, with $\lambda > 0$, is a (F, ϕ, ψ)-proximal contraction; then the above problem admits a unique solution $u^* \in K$ such that $Fix(P_K(I_K - \lambda S)) \bigcap Z_\phi = \{u^*\}$. Moreover, for each $u_0 \in K$, there exists a sequence $\{u_n\} \subseteq K$ such that $u_{n+1} = P_K(u_n - \lambda Su_n)$ for every $n \in N \cup \{0\}$ and $\lim_{n\to\infty} u_n = u^*$.*

Proof. We consider the operator $T : K \to K$ defined by $Tx = P_K(x - \lambda Sx)$ for all $x \in K$. By the previous Lemma, $u^* \in K$ is a solution of $< Su^*, v - u^* > \geqslant 0$ for all $v \in K$ if and only if $u^* = Tu^*$. Now, T satisfies all the hypotheses of Theorem 3.4 by setting $A = B = K$. It now follows from Theorem 3.4 that the fixed point problem $u = Tu$ admits a unique solution $u^* \in K$, which means that $Fix(T)$ is a singleton. □

Acknowledgement

This research was supported by University of Shahrekord

References

1. MA. Al-Thagafi and N. Shahzad, *Convergence and existence results for best proximity points*, Nonlinear Analysis: Theory, Methods & Applications, **70** (10) (2009), 3665–3671.
2. M. A. Alghamdi, N. Shahzad and F. Vetro, *Best proximity points for some classes of proximal contractions*, in Abstract and Applied Analysis, Hindawi, **2013**, (2013).
3. P. Magadevan, S. Karpagam and E. Karapınar, *Existence of fixed point and best proximity point of p-cyclic orbital ϕ-contraction map*, Nonlinear Analysis: Modelling and Control, **27** (1) (2022), 91–101.
4. H. Isik, M. S. Sezen and G.Vetro *ϕ-best proximity point theorems and applications to variational inequality problems*, J. Fixed Point Theory Appl., **19** (2017), 3177-3189.
5. F. Deutsch, *Best approximation in Inner* , J. product spaces, Springer-Verlag, New York 2001.
6. K. Goebel and S. Reich *Uniform convexity, hyperbolic geometry and nonexpansive mappings*, Marcel Dekker, New York 1984.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

NUMERICAL RADIUS OF CONDITIONAL TYPE OPERATORS

M. SOHRABI*AND A. REZAEE

Department of mathematics, Lorestan University, khorramabad, Iran
Department of mathematics, Payame Noor university, Tehran, Iran
Sohrabi.mo@lu.ac.ir; a.rezaee70@pnu.ac.ir

Abstract. We state some basic results on Moore-Penrose inverse of conditional type operators on $L^2(\Sigma)$. For instance, we show, polar decomposition of $T^\dagger$. In addition, we determine a lower and upper bounds for the numerical radius of T and $T^\dagger$.

1. Introduction

Let (X, Σ, μ) be a complete σ-finite measure space. For any complete σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$ the Hilbert space $L^2(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is abbreviated to $L^2(\mathcal{A})$ where $\mu_{|\mathcal{A}}$ is the restriction of μ to $\mathcal{A}$. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$ and $L^0_+(\Sigma) = \{f \in L^0(\Sigma) : f \geq 0\}$. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. All sets and functions statements are to be interpreted as being valid almost everywhere with respect to μ. For each nonnegative $f \in L^0(\Sigma)$ or $f \in L^2(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique $\mathcal{A}$-measurable function $E^{\mathcal{A}}(f)$ such that

$$\int_A f d\mu = \int_A E^{\mathcal{A}}(f) d\mu,$$

2020 *Mathematics Subject Classification.* Primary 47B35; Secondary 30H05
Key words and phrases. Conditional operator, Moore-Penrose inverse, Numerical radius.
* Speaker.

where A is any $\mathcal{A}$-measurable set for which $\int_A f d\mu$ exists. Now associated with every complete σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$, the mapping $E^{\mathcal{A}} : L^2(\Sigma) \to L^2(\mathcal{A})$ uniquely defined by the assignment $f \mapsto E^{\mathcal{A}}(f)$, is called the conditional expectation operator with respect to $\mathcal{A}$. Put $E = E^{\mathcal{A}}$. The mapping E is a linear orthogonal projection. Note that $\mathcal{D}(E)$, the domain of E, contains $L^2(\Sigma) \cup \{f \in L^0(\Sigma) : f \geq 0\}$. For more details on the properties of E see [5, 7, 8].

Given a complex separable Hilbert space H, let $B(H)$ denotes the linear space of all bounded linear operators on H. $\mathcal{N}(T)$ and $\mathcal{R}(T)$ denote the null-space and range of an operator T, respectively. Recall that for $T \in B(H)$, there is a unique factorization $T = U|T|$, where $\mathcal{N}(T) = \mathcal{N}(U) = \mathcal{N}(|T|)$, U is a partial isometry; i.e. $UU^*U = U$ and $|T| = (T^*T)^{1/2}$ is a positive operator. This factorization is called the polar decomposition of T. It is a classical fact that the polar decomposition of T^* is $U^*|T^*|$. Associated with $T \in B(H)$ there is a useful related operator $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$, called the Aluthge transform of T. For important properties of Aluthge transform see [3, 4].

Let $CR(H)$ be the set of all bounded linear operators on H with closed range. For $T \in CR(H)$, the Moore-Penrose inverse of T, denoted by $T^\dagger$, is the unique operator $T^\dagger \in CR(H)$ that satisfies following:

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T. \tag{1.1}$$

We recall that $T^\dagger$ exists if and only if $T \in CR(H)$. The Moore-Penrose inverse is designed as a measure for the invertibility of an operator. If $T = U|T|$ is invertible, then $T^{-1} = T^\dagger$, U is unitary and so $|T|$ is invertible. For other important properties of $T^\dagger$ see [1, 2].
The numerical range $W(T)$ of an operator $T \in B(H)$ is defined by $W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$. Also, $\omega(T) = \sup\{|\lambda| : \lambda \in W(T)\}$ denote the numerical radius of T.

2. Main results

Lemma 2.1. *Let $\omega \in L^0(\Sigma)$, $0 \leq v \in L^0(A)$ and let $A := M_{v\bar{\omega}}EM_\omega \in B(L^2(\Sigma))$. Then for each $p \in (0, \infty)$ and $f \in L^2(\Sigma)$,*

$$A^p(f) = v^p\bar{\omega}E(|\omega|^2)^{p-1}E(\omega f).$$

Proof. [6]. □

Lemma 2.2. *Let $T = M_wEM_u$ be a weighted conditional operator on $L^2(\Sigma)$. Then the following assertions hold.*

(a) $T \in B(L^2(\Sigma))$ if and only if $E(|w|^2)E(|u|^2) \in L^\infty(\mathcal{A})$, and in this case $\|T\| = \|E(|w|^2)E(|u|^2)\|_\infty^{1/2}$.

(b) Let $T \in B(L^2(\Sigma))$*,* $0 \leq u \in L^0(\Sigma)$ *and* $\upsilon = u(E(|w|^2))^{\frac{1}{2}}$*. If* $E(\upsilon) \geq \delta$ *on* $\sigma(\upsilon)$*, then* T *has closed range.*

(c) Let $U|T|$ *be the polar decomposition of* T*. Then*

$$|T|(f) = \left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}} \chi_S \bar{u} E(uf);$$

$$U(f) = \left(\frac{\chi_{S \cap G}}{E(|w|^2)E(|u|^2)}\right)^{\frac{1}{2}} wE(uf),$$

where where $S = \sigma(E(u))$*,* $G = \sigma(E(w))$ *and* $f \in L^2(\Sigma)$*.*

(d) The Aluthge transformation of T *is*

$$\widetilde{T}(f) = \frac{\chi_S E(uw)}{E(|u|^2)} \bar{u} E(uf), \qquad f \in L^2(\Sigma).$$

Proof. [6]. □

From now on, we assume that $u, w \in L^0_+(\Sigma)$, $T = M_w E M_u \in B(L^2(\Sigma))$ and $K := S \cap G$, where $G = \sigma(E(w))$ and $S = \sigma(E(u))$.

Proposition 2.3. $T \in CR(L^2(\Sigma))$*. Then* $T^\dagger = M_{\frac{\chi_K}{E(u^2)E(w^2)}} T^*$*.*

Proof. It is easy to check that T satisfy all equations in (1.1). □

Proposition 2.4. *Let* $T \in CR(L^2(\Sigma))$ *and let* $U_\dagger |T^\dagger|$ *be the polar decomposition of* $T^\dagger$*. Then*

$$|T^\dagger|(f) = (\frac{\chi_K}{E(u^2)(E(w^2))^3})^{\frac{1}{2}} wE(wf);$$

$$U_\dagger(f) = (\frac{\chi_K}{E(u^2)E(w^2)})^{\frac{1}{2}} uE(wf).$$

Proof. [6]. □

Now, we determine a lower and upper estimates for the numerical radius of $T^\dagger$.

Proposition 2.5. *Let* $T, \widetilde{T} \in CR(L^2(\Sigma))$*. Then*

$$\|E(u)E(w)\|_\infty \leq \omega(T) \leq \|\sqrt{E(u^2)E(w^2)}\|_\infty;$$

$$\frac{1}{\mu(B)} \int_B \frac{E(u)E(w)}{E(u^2)E(w^2)} d\mu \leq \omega(T^\dagger) \leq \int_K \frac{d\mu}{\sqrt{E(u^2)E(w^2)}},$$

where B *is the largest* A*-measurable set contained in* K *with* $\mu(B) < \infty$*.*

Proof. [6]. □

The following is an example of this propositon.

Example 2.6. Let $X = [-\frac{1}{2}, \frac{1}{2}]$, $d\mu = dx$, Σ be the Lebesgue sets, and let $\mathcal{A} \subseteq \Sigma$ be the σ-algebra generated by the symmetric sets about the origin. Then for each $f \in \mathcal{D}(E)$, $E(f)(x) = \frac{f(x)+f(-x)}{2}$. Put $u(x) = 2x + 5$, $w(x) = \cos x$ and $T = M_w E M_u$. Then $K = B = X$, $E(u) = 5$, $E(w) = \cos x$, $E(u^2) = 4x^2 + 25$ and $E(w^2) = \cos^2(x)$. Note that

$$u\sqrt{E(w^2)} = (2x + 5)(\cos x) \geq 3.9;$$

$$E(u)\frac{E(uw)}{\sqrt{E(u^2)}} = \frac{125\cos x}{\sqrt{4x^2 + 25}} \geq \frac{125\cos\frac{1}{2}}{\sqrt{26}} \geq 24.5\ .$$

So by Lemma 2.2, $T, \widetilde{T} \in CR(L^2(\Sigma))$. Also, it is easy to check that

$$\int_{[-\frac{1}{2},\frac{1}{2}]} \frac{E(u)E(w)}{E(u^2)E(w^2)} d\mu = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{5\cos x dx}{(4x^2 + 25)(\cos^2(x))} = 0.2060;$$

$$\int_{[-\frac{1}{2},\frac{1}{2}]} \frac{d\mu}{\sqrt{E(u^2)E(w^2)}} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{(x^2 + 4)(x^2 + 9)}} = 0.2074;$$

$$\|T\| = \|\sqrt{(4x^2 + 25)(\cos^2(x))}\|_\infty = 5;$$

$$\|T^\dagger\| = \|\frac{1}{\sqrt{E(u^2)E(w^2)}}\|_\infty = 0.2235;$$

$$\|\widetilde{T}\| = \|E(uw)\|_\infty = 5.$$

Thus, $\|\widetilde{T}\| = \|T\| = \omega(T)$ and by Proposition 2.5 we have

$$0.2060 \leq \omega(T^\dagger) \leq 0.2074 \leq \|T^\dagger\| \leq \frac{1}{2}\omega(T).$$

of the text but before the references.

References

1. A. Ben-Israel and T. N. E. Greville, *Generalized inverses: theory and applications*, Second Ed., Springer, 2003.
2. D. S. Djordjević and N. C. Dinčić, *Reverse order law for the Moore–Penrose inverse*, J. Math. Anal. Appl., **361** (2010), 252-261.
3. T. Furuta, *Invitation to linear operators*, Taylor & Francis, Ltd., London, 2001.
4. I. B. Jung, E. Ko and C. Pearcy, *Aluthge transforms of operators*, Integral Equations Operator Theory, **37** (2000), 437-448.
5. J. Herron, *Weighted conditional expectation operators*, Oper. Matrices **5** (2011), 107-118.
6. M.R. Jabbarzadeh and M. S. Chegeni, *Moore-Penrose inverse of conditional type operators*, Oper. Matrices, **11** (2017), 289-299.
7. A. Lambert, *L^p- multipliers and nested sigma-algebras*, Oper. Theory Adv. Appl., **104** (1998), 147-153.
8. M. M. Rao, *Conditional measure and applications*, Marcel Dekker, New York, 1993.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

FIXED POINT APPROXIMATIONS FOR MAPPINGS IN GEODESIC SPACES

HOSSEIN SOLEIMANI*

Department of Mathematics, Malayer Branch, Islamic Azad University, Malayer, Iran
hsoleimani@iau.ac.ir

Abstract. In this paper, we study fixed point approximations for mappings in geodesic spaces, and we prove some stability results in fixed point theory for contraction mappings in geodesic spaces. we also consider some extention theorems for these spaces.

Let(X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subseteq R$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t_0)) = |t - t_0|$ for all $t, t_0 \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic is denoted $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.
A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\triangle(x1, x2, x3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{y}_j) = d(x_i, y_j)$ for $i, j \in \{1, 2, 3\}$.

2020 *Mathematics Subject Classification. Primary 54H25; Secondary 47H09*
Key words and phrases. $CAT(0)$ space; Geodesic space; Fixed point.
* Speaker.

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.
Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$

Definition 0.1. Let X be a $CAT(0)$ space and $A \subseteq X$. We say X is convex, if for all $x, y \in A$ we have $[x, y] \subseteq X$.

Definition 0.2. let X be a $CAT(0)$ space and $A \subseteq X$. A is called geodesically bounded if A does not contain a geodesic ray.

we give some basic properties of metric segments in $CAT(0)$ spaces.

Remark 0.3. ([6]) Let X be a $CAT(0)$ space and let $x, y \in X$ such that $x \neq y$ and $s, t \in [0, 1]$. Then $(1-t)x \bigoplus ty = (1-s)x \bigoplus sy$ if and only if $s = t$.

Lemma 0.4. *([6]) Let X be a $CAT(0)$ space and let $x, y \in X$ such that $x \neq y$. then*

1. $[x, y] = \{(1-t)x \oplus ty | t \in [0, 1]\}$.
2. $d(x, z) + d(z, y) = d(x, y)$ *if and only if* $z \in [x, y]$.
3. *The mapping* $f : [0, 1] \longrightarrow [x, y], f(t) = (1-t)x \oplus ty$ *is continuous and bijective.*

Lemma 0.5. *([6]) Let X be a $CAT(0)$ space. then*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Lemma 0.6. *([1]) Let (X, d) be a $CAT(0)$ space, $p, q, x, y \in X$ and $t \in [0, 1]$. Then*

$$d((1-t)p \oplus tq, (1-t)x \oplus ty) \leq \max\{d(p, x), d(q, y)\}.$$

Fixed point theorems in $CAT(0)$ spaces have been developed in several recent papers including [2, 3, 4, 5]. The existence of fixed points for nonexpansive mappings in a complete $CAT(0)$ space was proved by Kirk ([2]) as follows:

Theorem 0.7. *Suppose K is a nonempty, bounded, closed and convex subset of complete $CAT(0)$space and suppose $f : K \longrightarrow K$ is nonexpansive. Then f has a fixed point.*

Theorem 0.8. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. Suppose $f : K \longrightarrow X$ is nonexpansive mapping for which*

$$\inf\{d(x, f(x)) : x \in K\} = 0$$

Then f has a fixed point in K.

1. Main results

Let $K \subseteq X$ be a nonempty, compact and convex subset of a complete $CAT(0)$ space (X,d). Consider

$$\mathcal{A} := \{A : A \text{ is a contraction selfmap on } K \text{ with } Fix(A) \neq \emptyset\},$$

with

$$\rho(A,B) = \sup\{d(Ax,Bx) : x \in K\}$$

for every $A, B \in \mathcal{A}$. Then $(\mathcal{A}, \rho)$ is a complete metric space.

Definition 1.1. Let $K \subseteq X$ be a nonempty, compact and convex subset of a complete $CAT(0)$ space (X,d). A map $A : K \to K$ is said to have the stable fixed point property if there exist $x \in Fix(A)$ such that

$$\forall \varepsilon > 0 \exists \delta > 0 \ni (B \in \mathcal{A}, \rho(A,B) \leq \delta) \Longrightarrow$$
$$\exists z \in K \ni (Bz = z, d(z,x) \leq \varepsilon).$$

Theorem 1.2. *Let $K \subseteq X$ be a nonempty, bounded, closed and convex subset of a complete $CAT(0)$ space (X,d), $x \in X$, and $A \in \mathcal{A}$. Soppose $\lambda : X \to [0,1]$ be a continuous map. Define B on K by $Bz = \lambda(z)x \oplus (1 - \lambda(z))Az$. Then B is contraction and $Bz \in K$ for all $z \in K$.*

Theorem 1.3. *Let $A \in \mathcal{A}$ and $\varepsilon > 0$. then there exists $\delta > 0$ such that for each $B \in \mathcal{A}$ satisfying $\rho(A,B) \leq \delta$ and each $x \in K$ satisfying $Bx = x$, there exists $y \in Fix(A)$ such that $d(x,y) \leq \varepsilon$.*

Example 1.4. Put $X := \mathbb{R}$, $K := [0,1]$ and $Ax = x$ for all $x \in K$ so $Fix(A) = K$. and let $A_n x = (1 - \frac{1}{n})x$ and $B_n x = \min\{x + \frac{1}{n}, 1\}$ for all n. Therefore $A_n, B_n \to A$ and $Fix(A_n) = \{0\}$ and $Fix(B_n) = [1 - \frac{1}{n}, 1]$.

This example shows that in general the answer to our question is negative. Nevertheless, we show in this paper that for a typical $A \in \mathcal{A}$ the answer is positive.

Theorem 1.5. *Let (X,d) be a geodesically bounded normal complete $CAT(0)$ space, $A \in \mathcal{A}$, $\varepsilon > 0$ and $x \in Fix(A)$. Then there exist $B \in \mathcal{A}$ and $\delta > 0$ such that $\rho(A,B) \leq \varepsilon$ and $Bz = x$ for each $z \in K$ satisfying $d(z,x) \leq \delta$.*

Theorem 1.6. *Let (X,d) be a geodesically bounded normal complete $CAT(0)$ space, $A \in \mathcal{A}$, $\varepsilon > 0$ and $x \in Fix(A)$. Let $B \in \mathcal{A}$ and $\delta > 0$ be as guaranteed by theorem (1.5). Then for each $C \in \mathcal{A}$ which $\rho(C,B) \leq \delta$, there exists $y \in K$ such that $Cy = y$ and $d(y,x) \leq \rho(B,C)$.*

Theorem 1.7. *Let (X,d) be a geodesically bounded normal complete $CAT(0)$ space. Then there exists a subset $\mathcal{F}$ of $\mathcal{A}$ which is a countable intersection of open subsets of $(\mathcal{A},\rho)$ so that for each $A \in \mathcal{F}$, A have the stable fixed point property.*

References

1. A. Amini-Harandi and A.P. Farajzadeh, *Best approximation, coincidence and fixed point theorems for set-valued maps in* $\mathbb{R}$*-trees*, Nonlinear Analysis, **71** (2009) 1649-1653.
2. W. A. Kirk, *Fixed point theorems in* $CAT(0)$ *spaces and* $\mathbb{R}$*-trees* Fixed Point Theory Appl., (2004), 309-316.
3. W. A. Kirk *An abstract fixed point theorem for nonexpansive mappings*, Proc. Amer. Math. Soc., **82** (1981), 640-642.
4. W. A. Kirk, *Geodesic geometry and fixed point theorey II, in: International Conference on Fixed Point Theory and Applications,* Yokohama Publ., Yokohama, 2004, 113-142.
5. S. Dhompongsa and B. Panyanak, *On* $\triangle-$*convergence theorems in* $CAT(0)$*spaces* Computers and Mathematics with Applications **56** (2008), 2572-2579.
6. S. Reich and A. J. Zaslavski, *A Stability Result in Fixed Point Theory* Fixed Point Theory, **6** (1) (2005), 113-118.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

ON GENERALIZATIONS OF SYMMETRIC BI-DERIVATIONS ON GROUP ALGEBRAS

EBRAHIM TAMIMI* AND HAMID REZA SANAEIFAR

Department of Mathematics, Velayat University, Iranshahr, Iran
e.tamimi@velayat.ac.ir; hamidsanaeifar57@gmail.com

Abstract. In this paper, we investigate generalizations of symmetric bi-derivations on $L_0^\infty(G)^*$. For $k \in \mathbb{N}$, we prove that if $B : L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ is a symmetric bi-derivation such that $[B(m,m), m^k] \in Z(L_0^\infty(G)^*)$ for all $m \in L_0^\infty(G)^*$, then $B = 0$. Also, we characterize symmetric generalized biderivations on group algebras.

1. Introduction

Let G denote a locally compact abelian group with a fixed left Haar measure λ. We know that $L^1(G)$ and $L^\infty(G)$ are Banach algebras. Consider that $L^\infty(G)$ is the continuous dual of $L^1(G)$. We denote by $L_0^\infty(G)$ the subspace of $L^\infty(G)$ consisting of all functions $g \in L^\infty(G)$ that vanish at infinity. For every $n \in L_0^\infty(G)^*$ and $g \in L_0^\infty(G)$ we define the functional $ng \in L_0^\infty(G)^*$ by

$$\langle ng, \varphi \rangle = \langle n, g\varphi \rangle$$

in which $\langle \varphi, \psi \rangle = \langle g, \varphi * \psi \rangle$ and

$$\varphi * \psi(x) = \int_G \phi(y)\psi(y^{-1}x)d\lambda(y) \tag{1.1}$$

2020 *Mathematics Subject Classification.* Primary 43A15; Secondary 47B47

Key words and phrases. Bi-derivation, Locally compact abelian group, κ-centralizing mapping, κ-skew centralizing mapping.

* Speaker.

for all $\varphi, \psi \in L_1(G)$ and $x \in G$. We equippe $L_0^\infty(G)^*$ to the first Arens product "·" defined by the formula $\langle m.n, g\rangle = \langle m, ng\rangle$ for all $m, n \in L_0^\infty(G)^*$ and $g \in L_0^\infty(G)^*$. Then $L_0^\infty(G)^*$ is a Banach algebra with the mentioned product. For more information of $L_0^\infty(G)^*$ see [4]. The notion of symmetric bi-derivations is investigated in [1, 8]. Let $\mathcal{A}$ be an algebra and $B(.,.) : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be a symmetric bi-linear mapping; that is, $B(x, y) = B(y, x)$, $B(\alpha x, y) = \alpha B(x, y)$ and $B(x + y, z) = B(x, z) + B(y, z)$ for all $x, y, z \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. The mapping $f : \mathcal{A} \to \mathcal{A}$ defined by $f(x) = B(x, x)$ is called the trace of B. We say that B is called a symmetric bi-derivation if we have

$$B(xy, z) = B(x, z)y + xB(y, z) \tag{1.2}$$

for all $x, y, z \in \mathcal{A}$. Also, B is called a symmetric generalized bi-derivation if there exists a symmetric bi-derivation $\tilde{\mathrm{B}}$ of $\mathcal{A}$ such that

$$B(xy, z) = xB(y, z) + \overline{B}(x, z)y \tag{1.3}$$

for all $x, y, z \in \mathcal{A}$. A symmetric generalized bi-derivation B associated with a symmetric bi-derivation $\tilde{\mathrm{B}}$ is denoted by $B_{\overline{B}}$. For $\kappa \in \mathbb{N}$, a linear mapping $T : \mathcal{A} \to \mathcal{A}$ is called κ-(skew) centralizing if

$$[T(x), x^\kappa] \in Z(\mathcal{A}) \qquad (T(x) \circ x^\kappa \in Z(\mathcal{A})) \tag{1.4}$$

for all $x \in \mathcal{A}$, in a special case, if for every $x \in \mathcal{A}$

$$[T(x), x^\kappa] = 0 \qquad (T(x) \circ x^\kappa = 0));$$

then T is called κ-(skew) commuting, where $Z(\mathcal{A})$ is the center of $\mathcal{A}$, $[x, y] = xy - yx$ and $x \circ y := x.y + y.x$ for all $x, y \in \mathcal{A}$. If, $\kappa = 1$, T is called (skew) centralizing and (skew) commuting, respectively. Symmetric bi-derivations on rings have been introduced and studied by Maksa [5]. Vukman [9] proved that if $B : R \times R \to R$ is a symmetric bi-derivation such that for every $x \in R$

$$[[f(x), x], x] \in Z(R);$$

then $B = 0$, where R is a noncommutative prime ring of characteristic not two and three. He conjectured that if there exists $\kappa \in \mathbb{N}$ such that for every $x \in R$ we have $f_\kappa(x) \in Z(R)$; then $B = 0$, where

$$f_{i+1}(x) = [f_i(x), x]$$

for $i > 1$ and $f_1(x) = f(x)$. In [3], Deng gave an affrmative answer to the Vukman's conjecture. For related results on symmetric bi-derivations on Banach algebras see [7]; see also [2] for study of generalized bi-derivations.

The mapping $B(.,.) : L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ defined by $B(m, n) = r.m.n$ is a nonzero bi-derivation. These facts lead us to investigate symmetric bi-derivations on $L_0^\infty(G)^*$. In this paper, we first study symmetric bi-derivations on $L_0^\infty(G)^*$ and prove that they map $L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ into the radical of $L_0^\infty(G)^*$. We also show that if $B : L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ is a symmetric bi-derivation and f is κ-centralizing for some $\kappa \in \mathbb{N}$, then B is zero map. In the case that, B is a symmetric generalized bi-derivation, we prove that there exists $\theta \in L_0^\infty(G)^*$ such that $B(m,n) = m.n.\theta$ for all $m.n \in L_0^\infty(G)^*$.

2. Main results

In the sequel, we use the symbols D, for symmetric bi-derivations. The following result is an analogue of [1, Proposition 2.1.] for bi-derivations.

Lemma 2.1. *Let $D_1, D_2 : L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ be symmetric bi-derivations. Then $D_1 + D_2$ maps $L_0^\infty(G)^* \times L_0^\infty(G)^*$ into the radical of $L_0^\infty(G)^*$.*

Proof. For every $m \in L_0^\infty(G)^*$ we define the mapping $\Delta_m : L_0^\infty(G)^* \to L_0^\infty(G)^*$ by

$$\Delta_m(n) = D(m,n). \tag{2.1}$$

For every $m \in L_0^\infty(G)^*$, Δ_m is a derivation on $L_0^\infty(G)^*$ and hence Δ_m maps $L_0^\infty(G)^*$ into its radical for all $m \in L_0^\infty(G)^*$; see [6]. Since $D_i(L_0^\infty(G)^* \times L_0^\infty(G)^*) = \bigcup_m \Delta_m(L_0^\infty(G)^*)$, $i = 1, 2$, then D_i maps $L_0^\infty(G)^* \times L_0^\infty(G)^*$ into the radical of $L_0^\infty(G)^*$. □

Theorem 2.2. *Let $D_1, D_2 : L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ be symmetric bi-derivations, and let f and g be the trace of D_1 and D_2, respectively. Then the following are equivalent.*

(a) *there exists $\kappa \in \mathbb{N}$ such that $f(t^\kappa) = g(t^\kappa) = 0$ for all $t \in L_0^\infty(G)^*$;*
(b) *there exists $\kappa \in \mathbb{N}$ such that $f + g$ is κ-commuting;*
(c) *there exists $\kappa \in \mathbb{N}$ such that $f + g$ is κ-centralizing;*
(d) *there exists $\kappa \in \mathbb{N}$ such that $f + g$ is κ-skew commuting;*
(e) *there exists $\kappa \in \mathbb{N}$ such that $f + g$ is κ-skew centralizing;*
(f) *$D_1 + D_2 = 0$.*

Proof. Let $\kappa \in \mathbb{N}$ and $t \in L_0^\infty(G)^*$. We obtain $(f+g)(t^\kappa) = D(t^\kappa, t^\kappa) + D(t^\kappa, t^\kappa) = f(t)(t^{2\kappa-2}) + g(t)(t^{2\kappa-2})$. Also, we obtain

$$f(t).t^\kappa = \langle f(t), t^\kappa \rangle, \qquad g(t).t^\kappa = \langle g(t), t^\kappa \rangle \tag{2.2}$$

□

Corollary 2.3. *Let $D_1, D_2 : L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ be symmetric bi-derivations, and let f and g be the trace of D_1 and D_2, respectively. Then the following assertions are equivalent.*

(a) *$f + g$ is (skew) centralizing;*
(b) *there exists $\kappa \in \mathbb{N}$ such that $f + g$ is κ-(skew) centralizing;*
(c) *for every $\kappa \in \mathbb{N}$, $f + g$ is κ-(skew) centralizing;*
(d) *$D_1 + D_2 = 0$.*

Corollary 2.4. *Let Let* $D_1 : L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ *and* $D_2 : L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ *be symmetric bi-derivations,* f *and* g *be the trace of* D_1 *and* D_2*, respectively. Then the following assertions are equivalent.*

(a) $f + g$ *is commuting;*
(b) $f + g$ *is centralizing;*
(c) $f + g$ *is skew commuting;*
(d) $f + g$ *is skew centralizing;*
(e) $D_1 + D_2 = 0$.

Acknowledgement

The authors would like to thank the referee of the paper for invaluable comments.

References

1. M. H. Ahmadi Gandomani and M. J. Mehdipour, *Symmetric Bi-derivations and their Generalizations on Group Algebras*, Filomat, **35** (4) (2021) 1233–1240.
2. M. Bresar, *On generalized bi-derivations and related maps*, J. Algebra, **122** (1995), 764–786.
3. Q. Deng, *On a conjecture of Vukman*, Internat. J. Math. Math. Sci. **20** (2) (1997), 263–266.
4. A. T. Lau and J. Pym, *Concerning the second dual of the group algebra of a locally compact group*, J. London Math. Soc. 41 (1990) 445–460.
5. G. Maksa, *On the trace of symmetric bi-derivations*, C. R. Math. Rep. Acad. Sci. Canada, **9** (1987), 303—307.
6. M. J. Mehdipour and Z. Saeedi, *Derivations on group algebras of a locally compact abelian group*, Monatsh. Math., **180** (3) (2016) 595-–605.
7. C. Park, *Biderivations and bihomomorphisms in Banach algebras*, Filomat **33** (8) (2019) 2317-–2328.
8. E. Tamimi and A. Ghaffari, *A Survey on* χ*-module Connes amenability of semigroup algebras*, JAS. 29 (2023), Accepted to Online Publish. https://doi.org/10.21203/rs.3.rs-2676837/v1
9. J. Vukman, *Two results concerning symmetric bi-derivations on prime rings*, Aequations Math., **40** (1990), 181–189.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

CHARACTERIZATION OF (APPROXIMATE) DIAGONALS OF LAU PRODUCT BANACH ALGEBRAS

MOHAMMAD VALAEI ANVAR

Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran
m.valaei@abru.ac.ir

Abstract. In this paper we compare the (approximate) diagonals of Lau product Banach algebras $\mathcal{A} \times_\theta \mathcal{B}$ and these notions of two Banach algebras $\mathcal{A}$ and $\mathcal{B}$, where θ is a homomorphism from $\mathcal{B}$ to $\mathcal{A}$.

1. Introduction

Definition 1.1. [1] Let $\mathcal{A}$ be a Banach algebra.

(i) An element $\mathbf{m} \in \mathcal{A}\hat{\otimes}\mathcal{A}$ is called a *diagonal* for $\mathcal{A}$, if

$$a \cdot \mathbf{m} = \mathbf{m} \cdot a, \quad a\Delta\mathbf{m} = a, \qquad (a \in \mathcal{A}).$$

(ii) A bounded net $\{\mathbf{m}_\alpha\}_\alpha$ in $\mathcal{A}\hat{\otimes}\mathcal{A}$ is called an *approximate diagonal* for $\mathcal{A}$ if

$$a\Delta_{\mathcal{A}}\mathbf{m}_\alpha \to a \textit{ and } a \cdot \mathbf{m}_\alpha - \mathbf{m}_\alpha \cdot a \to 0, \quad (a \in \mathcal{A}).$$

Let $\mathcal{A}$, $\mathcal{B}$ be Banach algebras and $\theta \in \sigma(\mathcal{B})$, where $\sigma(\mathcal{B})$ is the set of all nonzero characters on $\mathcal{B}$. Then the direct product $\mathcal{A} \times \mathcal{B}$ equipped with the algebra multiplication

$$(a, b) \times_\theta (c, d) = (ac + \theta(d)a + \theta(b)c, bd), \quad (a, c \in \mathcal{A}, b, d \in \mathcal{B}),$$

and with the l^1-norm $\|(a, b)\| = \|a\| + \|b\|$ is a Banach algebra which is called the *Lau product* of $\mathcal{A}$ and $\mathcal{B}$ and is denoted by $\mathcal{A} \times_\theta \mathcal{B}$.

2020 *Mathematics Subject Classification. Primary 47B35; Secondary 30H05*

Key words and phrases. Banach algebras, approximate diagonal.

Bhatt and Dabhi [2] replaced the character by a homomorphism in Lau product and presented a new type of the mentioned earlier product. Let $\theta : \mathcal{B} \longrightarrow \mathcal{A}$ be an algebra homomorphism with $\|\theta\| \leqslant 1$. Then, $\mathcal{A} \times \mathcal{B}$ equipped with the algebra multiplication

$$(a,b) \times_\theta (c,d) = (ac + a\theta(d) + \theta(b)c, bd), \quad (a, c \in \mathcal{A}, b, d \in \mathcal{B}),$$

and with the l^1-norm turns into a Banach algebra which is denoted by $\mathcal{A}\times_\theta\mathcal{B}$.

2. DIAGONALS OF LAU PRODUCTS

Proposition 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, such that one the following is satisfied.*

(i) *$\theta \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ with $\|\theta\| \leqslant 1$;*
(ii) *$\theta \in \sigma(\mathcal{B})$.*

Then, $\mathcal{A} \times_\theta \mathcal{B}$ has a diagonal, whenever $\mathcal{A}$ and $\mathcal{B}$ have diagonals.

Proof. Take the bounded nets $(a_n)_n$, $(c_n)_n$ in $\mathcal{A}$ and $(b_n)_n$, $(d_n)_n$ in $\mathcal{B}$ such that $\mathbf{m}_\mathcal{A} = \sum_{n=1}^\infty a_n \otimes c_n$ and $\mathbf{m}_\mathcal{B} = \sum_{n=1}^\infty b_n \otimes d_n$ are diagonals for $\mathcal{A}$ and $\mathcal{B}$, respectively. If $\theta \in$Hom$(\mathcal{A}, \mathcal{B})$ that $\|\theta\| \leqslant 1$, set

$$\mathbf{m}_{\mathcal{A}\times_\theta\mathcal{B}} := \sum_{n=1}^\infty \big((a_n - \theta(b_n), b_n)\big) \otimes \big((c_n - \theta(d_n), d_n)\big),$$

and in the case that $\theta \in \sigma(\mathcal{B})$, put

$$\mathbf{m}_{\mathcal{A}\times_\theta\mathcal{B}} := \sum_{n=1}^\infty \big((a_n - \theta(b_n)\Delta_\mathcal{A}\mathbf{m}_\mathcal{A}, b_n)\big) \otimes \big((c_n - \theta(d_n)\Delta_\mathcal{A}\mathbf{m}_\mathcal{A}, d_n)\big).$$

It is easily verified that $\mathbf{m}_{\mathcal{A}\times_\theta\mathcal{B}}$ is a diagonal for $\mathcal{A} \times_\theta \mathcal{B}$. □

The next proposition may be considered as a converse of Proposition 2.1.

Proposition 2.2. *Let $\mathcal{A} \times_\theta \mathcal{B}$ has a diagonal. Under the following conditions, $\mathcal{A}$ and $\mathcal{B}$ have diagonals.*

(1) *$\theta \in Hom(\mathcal{A}, \mathcal{B})$ with $\|\theta\| \leqslant 1$;*
(2) *$\mathcal{A}$ is unital and $\theta \in \sigma(\mathcal{B})$.*

Proof. Let $\mathbf{m}_{\mathcal{A}\times_\theta\mathcal{B}} = \sum_{n=1}^\infty (a_n, b_n) \otimes (c_n, d_n)$ be a diagonal for $\mathcal{A} \times_\theta \mathcal{B}$. If $\theta \in$Hom$(\mathcal{A}, \mathcal{B})$ that $\|\theta\| \leqslant 1$, define

$$\mathbf{m}_\mathcal{A} := \sum_{n=1}^\infty \big(a_n + \theta(b_n)\big) \otimes \big(c_n + \theta(d_n)\big) \text{ and } \mathbf{m}_\mathcal{B} = \sum_{n=1}^\infty b_n \otimes d_n,$$

and moreover in the case where $\theta \in \sigma(\mathcal{B})$ and $e_\mathcal{A}$ is an identity for $\mathcal{A}$, consider

$$\mathbf{m}_\mathcal{A} := \sum_{n=1}^\infty \big(a_n + \theta(b_n)e_\mathcal{A}\big) \otimes \big(c_n + \theta(d_n)e_\mathcal{A}\big), \quad \mathbf{m}_\mathcal{B} = \sum_{n=1}^\infty b_n \otimes d_n.$$

Clearly, $\mathbf{m}_\mathcal{A}$ and $\mathbf{m}_\mathcal{B}$ are diagonals for $\mathcal{A}$ and $\mathcal{B}$, respectively. □

3. Approximate diagonals of Lau products

In the following example we present an approximate diagonal for the Cartesian product $\mathcal{A}$ and $\mathcal{B}$ according to the approximate diagonals for these Banach algebras.

Example 3.1. *Let $(\boldsymbol{n}_\alpha)_\alpha$ and $(\boldsymbol{k}_\beta)_\beta$ be the approximate diagonals for $\mathcal{A}$ and $\mathcal{B}$, respectively. Then, they may be considered as follows: $\boldsymbol{n}_\alpha = \sum_{n=1}^\infty a_n^{(\alpha)} \otimes c_n^{(\alpha)}$ with $\sum_{n=1}^\infty ||a_n^{(\alpha)}||\ ||c_n^{(\alpha)}|| < \infty$ and $\boldsymbol{k}_\beta = \sum_{n=1}^\infty b_n^{(\beta)} \otimes d_n^{(\beta)}$ with $\sum_{n=1}^\infty ||b_n^{(\beta)}||\ ||d_n^{(\beta)}|| < \infty$. Suppose that $\theta : \mathcal{B} \longrightarrow \mathcal{A}$ be the zero map. Then, $(\boldsymbol{m}_{\alpha,\beta})_{\alpha,\beta}$ is an approximate diagonal for $\mathcal{A} \times_0 \mathcal{B} = \mathcal{A} \times \mathcal{B}$, that*

$$\boldsymbol{m}_{\alpha,\beta} = \sum_{n=1}^{\infty} \big(a_n^{(\alpha)}, b_n^{(\beta)}\big) \otimes \big(c_n^{(\alpha)}, d_n^{(\beta)}\big).$$

Conversely, let $(\boldsymbol{m}_\alpha)_\alpha$ be an approximate diagonal for the cartesian product Banach algebras $\mathcal{A} \times \mathcal{B}$, that

$$\boldsymbol{m}_\alpha = \sum_{n=1}^{\infty} \big(a_n^{(\alpha)}, b_n^{(\alpha)}\big) \otimes \big(c_n^{(\alpha)}, d_n^{(\alpha)}\big).$$

Then, $(\boldsymbol{n}_\alpha)_\alpha$ and $(\boldsymbol{k}_\alpha)_\alpha$ are approximate diagonals for $\mathcal{A}$ and $\mathcal{B}$, that

$$\boldsymbol{n}_\alpha = \sum_{n=1}^{\infty} a_n^{(\alpha)} \otimes c_n^{(\alpha)}, \quad \boldsymbol{k}_\alpha = \sum_{n=1}^{\infty} b_n^{(\alpha)} \otimes d_n^{(\alpha)}.$$

Theorem 3.2. *If $\mathcal{A} \times_\theta \mathcal{B}$ has an approximate diagonal and one the following is satisfied, then $\mathcal{A}$ and $\mathcal{B}$ have approximate diagonals.*

(i) *$\theta \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ with $\|\theta\| \leqslant 1$;*
(ii) *$\mathcal{A}$ is unital and $\theta \in \sigma(\mathcal{B})$.*

Proof. Let the bounded net $(\mathbf{m}_\alpha)_\alpha$ be an approximate diagonal for $\mathcal{A} \times_\theta \mathcal{B}$ such that $\mathbf{m}_\alpha = \sum_{n=1}^\infty (a_n^{(\alpha)}, b_n^{(\alpha)}) \otimes (c_n^{(\alpha)}, d_n^{(\alpha)})$.

Suppose that (i) holds. Therefore, the bounded net $\left(\sum_{n=1}^\infty b_n^{(\alpha)} \otimes d_n^{(\alpha)}\right)_\alpha$ is an approximate diagonal for $\mathcal{B}$. Also, the bounded net $\left(\sum_{n=1}^\infty (a_n^{(\alpha)} + \theta(b_n^{(\alpha)})) \otimes (c_n^{(\alpha)} + \theta(d_n^{(\alpha)}))\right)_\alpha$ is an approximate diagonal of $\mathcal{A}$.

Assume that (ii) is given. Let $e_{\mathcal{A}}$ be the identity of $\mathcal{A}$. Then, the nets $\big(\mathbf{n}_\alpha\big)_\alpha$ and $\big(\mathbf{k}_\alpha\big)_\alpha$ are approximate diagonals for $\mathcal{A}$ and $\mathcal{B}$, respectively, where

$$\mathbf{n}_\alpha = \sum_{n=1}^{\infty} (a_n^{(\alpha)} + \theta(b_n^{(\alpha)})e_{\mathcal{A}}) \otimes (c_n^{(\alpha)} + \theta(d_n^{(\alpha)})e_{\mathcal{A}}) \text{ and } \mathbf{k}_\alpha = \sum_{n=1}^{\infty} b_n^{(\alpha)} \otimes d_n^{(\alpha)}.$$

□

For more details in proof see [3]. The next result is the converse of Theorem 3.2.

Theorem 3.3. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ have approximate diagonals. Under one of the following assertions, $\mathcal{A} \times_\theta \mathcal{B}$ has an approximate diagonal.*

(i) $\theta \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ *with* $\|\theta\| \leqslant 1$*;*
(ii) $\mathcal{A}$ *is unital and* $\theta \in \sigma(\mathcal{B})$*.*

Proof. Let $(\mathbf{n}_\alpha)_\alpha$ be an approximate diagonal for $\mathcal{A}$ and $(\mathbf{k}_\beta)_\beta$ be an approximate diagonal for $\mathcal{B}$. Then, $\mathbf{n}_\alpha = \sum_{n=1}^{\infty} a_n^{(\alpha)} \otimes c_n^{(\alpha)}$ where $\sum_{n=1}^{\infty} ||a_n^{(\alpha)}|| \, ||c_n^{(\alpha)}|| < \infty$ and $\mathbf{k}_\beta = \sum_{n=1}^{\infty} b_n^{(\beta)} \otimes d_n^{(\beta)}$ with $\sum_{n=1}^{\infty} ||b_n^{(\beta)}|| \, ||d_n^{(\beta)}|| < \infty$.

If (i) is valid, Consider

$$\mathbf{m}_{\alpha,\beta} = \sum_{n=1}^{\infty} (a_n^{(\alpha)} - \theta(b_n^{(\beta)}), b_n^{(\beta)}) \otimes (c_n^{(\alpha)} - \theta(d_n^{(\beta)}), d_n^{(\beta)}).$$

Clearly, $(\mathbf{m}_{\alpha,\beta})_{\alpha,\beta}$ is an approximate diagonal of $\mathcal{A} \times_\theta \mathcal{B}$. For (ii), we consider

$$\mathbf{m}_{\alpha,\beta} = \sum_{n=1}^{\infty} \big(a_n^{(\alpha)} - \theta(b_n^{(\beta)}) e_{\mathcal{A}}, b_n^{(\beta)}\big) \otimes \big(c_n^{(\alpha)} - \theta(d_n^{(\beta)}) e_{\mathcal{A}}, d_n^{(\beta)}\big).$$

Similar to the above, one can show that the bounded net $(\mathbf{m}_{\alpha,\beta})_{\alpha,\beta}$ is an approximate diagonal for $\mathcal{A} \times_\theta \mathcal{B}$. □

Example 3.4. *Let* $\mathcal{A} := \mathbb{M}_n$ *be the Banach algebra of complex* $n \times n$ *matrices of dimension* n^2*. Then* $\mathcal{A}$ *is unital, with the identity* $e := \sum_{i=1}^{n} e_{ii}$*. Define* $\boldsymbol{M} = \frac{1}{n} \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji}$*. Then,* $\boldsymbol{M}$ *is the unique element of* $\mathcal{A} \hat{\otimes} \mathcal{A}$ *that is a diagonal for both* $\mathcal{A}$ *and* $\mathcal{A}^{op}$*. Let* $\theta \in$*Hom*$(\mathcal{A}, \mathcal{A})$ *with* $\|\theta\| \leqslant 1$*. Then*

$$\frac{1}{n} \sum_{i,j=1}^{n} \big(e_{ij} - \theta(e_{ij}), e_{ij}\big) \otimes \big(e_{ji} - \theta(e_{ji}), e_{ji}\big)$$

is only common diagonal for $\mathcal{A} \times_\theta \mathcal{A}^{op}$ *and* $\mathcal{A}^{op} \times_\theta \mathcal{A}$*. If* $\theta \in \sigma(\mathcal{A})$*, then*

$$\frac{1}{n} \sum_{i,j=1}^{n} \big(e_{ij} - \theta(e_{ij}) e, e_{ij}\big) \otimes \big(e_{ji} - \theta(e_{ji}) e, e_{ji}\big)$$

is only common diagonal for $\mathcal{A} \times_\theta \mathcal{A}^{op}$ *and* $\mathcal{A}^{op} \times_\theta \mathcal{A}$*.*

References

1. H. G. Dales, *Banach algebras and automatic continuity*, LMS Monographs 24, Clarenden Press, Oxford, 2000.
2. S. J. Bhatt and P. A. Dahbi, *Arens regularity and amenability of Lau product of Banach algebras defined by a Banach algebra morphism*, Bull. Aust. Math. Soc, **87** (2013), 195-206.
3. A. Bodaghi and M. Valaei, *diagonals of Lau product Banach algebras*, J. of Algebra and its Applications, doi.org/10.1142/S0219498824500439.

The 8th Seminar on Functional Analysis and its Applications
6-7 November 2024
Ayatollah Boroujerdi University
Boroujerd, Iran

A COMPARATIVE STUDY OF COMBINATION RULES IN EVIDENCE THEORY FOR MULTI-SOURCE INFORMATION FUSION

FARNAZ YAQUB AZARI* AND ILDAR SADEQI

Department of Mathematics, Faculty of Basic Sciences and Engineering, Sahand University of Technology, Tabriz, Iran
fyaqubazari@gmail.com; esadeqi@sut.ac.ir

Abstract. Evidence theory, also known as Dempster-Shafer theory, is a powerful framework for reasoning under uncertainty with multiple sources of evidence. This paper presents a comprehensive comparative study of different combination rules within the evidence theory framework. We focus on evaluating the performance of common rules, including Dempster's rule, Yager's rule, Dubois and Prade's rule, and the Smets' rule of combination. We analyze their theoretical properties, computational aspects, and effectiveness in handling conflicting evidence. The paper concludes with a discussion of the advantages and limitations of each rule, offering practical recommendations for choosing the most suitable combination rule based on the characteristics of the evidence sources and the specific application context.

1. Introduction

In many real-world applications, decisions must be made based on information from multiple, often conflicting, sources. Evidence theory provides a robust framework for representing and combining uncertain information from multiple sources, taking into account both belief and uncertainty. This

2020 *Mathematics Subject Classification.* Primary 47B35; Secondary 30H05
Key words and phrases. Evidence theory, Combination Rules, Dempster's rule.
* Speaker.

paper delves into the core of evidence theory: combination rules. We aim to provide a comprehensive comparison of the most commonly used rules, facilitating the informed selection of the appropriate rule for diverse applications.

2. Preliminaries

Definition 2.1. [1] If $\mathbf{\Theta} = \{\theta_1, \theta_2, \cdots \theta_r\}$ is a finite complete set of r mutually exclusive elements, it is called a Frame of Discernment (FOD). Set $\mathbf{\Theta}$ is a FOD, and its power set constitutes a set of propositions. If a function $m : 2^{\mathbf{\Theta}} \to [0,1]$ satisfies the following formula:

$$\begin{cases} m(\varnothing) = 0 \\ \sum_S m(S) = 1 \end{cases}$$

the mass function m is a Basic Probability Assignment (BPA). In this definition, $m(S)$ is the basic probability number of proposition S, and indicates the belief assigned to S.

Definition 2.2. [3] Let m be a BPA on FOD $\mathbf{\Theta}$, if $Bel : 2^{\mathbf{\Theta}} \to [0,1]$ statisfies; $Bel(S) = \sum_{B \subseteq S} m(B), S \in 2^{\mathbf{\Theta}}$ then, $Bel(S)$ is called the belief measure of proposition S.
$Pl(S)$ is the plausibility function that is defined as follows:

$$Pl(S) = 1 - Bel(\bar{S}) = \sum_{B \cap S \neq \varnothing} m(B)$$

$Pl(S)$ is the degree to which you do not disagree with proposition S.

Definition 2.3. [2] Suppose m_1 and m_2 are two BPAs for $\mathbf{\Theta}$. Then Dempster's Rule defined as:
$(m_1 \bigoplus m_2)(a) = K^{-1} \sum b_1, b_2 \in 2^{\mathbf{\Theta}} b_1 \cap b_2 = a m_1(b_1) m_2(b_2)$, where K is a normalization constant given by $K = 1 - \sum b_1, b_2 \in 2^{\mathbf{\Theta}} b_1 \cap b_2 = a m_1(b_1) m_2(b_2)$.

Combination rules in evidence theory combine evidence from multiple sources, represented by their respective BPAs, to produce a new BPA that reflects the combined knowledge. The choice of combination rule significantly impacts the results, and understanding their strengths and weaknesses is crucial for effective information fusion.

3. Main results

In this section, the comparison of composition rules in evidence theory is discussed in terms of properties Local Computation Availability (LCA), Neutral Element Commitment(NEM), Possibility Reservation(PR), Convergence toward Certainty (CTC), Invariance of Iterated Indifference Evidence ($IIIE$) and Weak Specialisation (WS) [2]. The combined rules are taken from articles [5], [1], [4], [3], [6].

TABLE 1. Comparison of combination rules

Definition	LCA	NEM	PR	CTC	$IIIE$	WS
Dempster	yes	yes	no	yes	yes	yes
Yager	yes	yes	no	no	no	no
Dubois-Prade	no	yes	yes	yes	yes	yes
Smets	yes	yes	no	no	no	no
Murphy	no	no	yes	yes	yes	yes
Ma, Jiang	yes	yes	no	yes	yes	yes
Zhang	yes	yes	yes	no	yes	yes
Mixing	yes	yes	yes	yes	no	yes
Inagaki	no	yes	-	-	-	-
Disjunctive Consensus Pooling	no	yes	yes	-	-	-

Due to the fact that in various articles, Dempster's Rule, Yager's Rule, Dubois-Prade's Rule and Smets' Rule have been used abundantly, we evaluate the performance of the combination rules using a set of benchmark scenarios, including:

(1) Conflict handling:** How well does the rule handle conflicting evidence from different sources?
(2) Computational complexity:** How computationally expensive is the rule to implement?
(3) Sensitivity to noise:** How robust is the rule to noisy or inaccurate evidence?
(4) Interpretability:** How easily can the results of the combination rule be understood and interpreted?

The results of the comparison of the combined rules based on the above questions showed that:
Dempster's Rule, combines evidence by normalizing the joint probability of focal elements and are well-defined. Also, it is prone to counterintuitive results with high conflict and potentially biased towards conflicting evidence. Therefore, Dempster's rule is mathematically elegant and widely used, but its performance degrades with high conflict.
Yager's Rule Combines evidence based on the conjunctive combination of focal elements. It's advantages is handles conflict gracefully, avoids counterintuitive results. Also, it's limitations is less mathematically rigorous than Dempster's rule, potentially overly conservative. Hense, Yager's rule is more robust to conflict but might be overly conservative.
Dubois - Prade's Rule combines evidence based on the intersection of focal elements, taking into account uncertainty. It's advantages are handles conflict without normalization and provides a more nuanced representation of uncertainty. It's limitations are can be computationally intensive, might not be as widely adopted as other rules. So, Dubois-Prade's rule offers a nuanced approach to conflict but can be computationally intensive.
Smets' Rule of Combination (also known as the Transferable Belief Model)

combines evidence by distributing conflicting mass to the set of all possible hypotheses. It's advantages are handles conflict without normalization, allows for belief functions to express ignorance. Their limitations are less intuitive than other rules, requires a different interpretation of belief functions.So, Smets' rule is well-suited for applications where ignorance and conflict are expected and a different interpretation of belief functions is acceptable.

4. Conclusion

This comparative study provides a comprehensive overview of the most commonly used combination rules in evidence theory. By understanding their strengths and limitations, researchers and practitioners can choose the appropriate rule for their specific application. This will facilitate the effective fusion of multi-source information for decision making in various domains.

References

1. D. Dubois and H. Prade, *Properties of measures of information in evidence and possibility theories*, Fuzzy Sets Syst, **24**, (1987), 161–182.
2. W. Ma, X. Luo and Y. Jiang, *A flexible rule for evidential combination in Dempster-Shafer theory of evidence*, Int. J. Intell. Syst, **25**, (2023), 529–559.
3. M. Qin, T. Yongchuan and W. Junhao, *An improved total uncertainty measure in the evidence theory and its application in decision making,*Entropy, **22**, 4 (2020), 487.
4. Y. Yang and D. Han, *A New Distance-Based Total Uncertainty Measure in the Theory of Belief Functions*, Know. Based Syst., **94**, (2016), 114–123.
5. K. Yuan, F. Xiao, L. Fei, B. Kang, and Y. Deng, *Conflict management based on belief function entropy in sensor fusion*, SpringerPlus,**5** (2016), 639-656.
6. Y. Zhang, F. huang, X. Deng and W. Jiang. *A New Total Uncertainty Measure from A Perspective of Maximum Entropy Requirement* , Entropy, **23** (2021), 1-20.

مجموعه مقاله های فارسی

بررسی مزایای رویکرد غیر خطی در ماشین بردار پشتیبان با استفاده از ترفند هسته

محسن اسماعیل بیگی [1] * و زهرا کریمی [1]

[1] گروه ریاضی، دانشکده علوم ریاضی و آمار، دانشگاه ملایر
m.esmaeilbeigi@malayeru.ac.ir

چکیده. در این مقاله، رویکرد غیر خطی در ماشین بردار پشتیبان با بهره‌گیری از ترفند هسته مورد بررسی قرار می‌گیرد. نتایج مطالعات محققان در زمینه یادگیری ماشین نشان می‌دهد که استفاده از ترفند هسته باعث بهبود دقت و کارایی مدل‌ها در مواجهه با داده‌هایی که از توزیع غیر خطی برخوردارند می‌گردد. آشنایی با جنبه‌های ریاضی این رویکرد به درک بهتر از چگونگی نگاشت داده‌ها از فضای ورودی به فضای ویژگی و تاثیر آن بر روی سطوح تصمیم کمک می‌کند. این تحقیق با ارائه برخی مثال‌ها و تحلیل‌های نظری، نقش کلیدی رویکرد غیر خطی در بهبود عملکرد ماشین بردار پشتیبان را به خوبی نمایان می‌سازد.

۱. پیش‌گفتار

یکی از قدرتمندترین الگوریتم‌ها در زمینه یادگیری ماشین، ماشین بردار پشتیبان است که به دلیل دقت بالا و قابلیت تعمیم مناسب، مورد توجه بسیاری از پژوهشگران قرار گرفته است. ماشین بردار پشتیبان در ابتدا به عنوان یک الگوریتم خطی معرفی شد که قادر بود داده‌ها را به بهترین نحو ممکن با یک ابرصفحه طبقه بندی نماید. اما در بسیاری از مسائل دنیای واقعی، داده‌ها به شکلی پیچیده و غیر خطی توزیع شده‌اند که مدل‌های خطی قادر به تفکیک آنها نیستند. برای غلبه بر این محدودیت، ترفند هسته معرفی شد که امکان تبدیل فضای داده‌های اولیه به یک فضای ویژگی با ابعاد بالاتر را فراهم می‌کند. این تبدیل باعث می‌شود که مسائل غیر خطی در فضای اولیه، در فضای ویژگی جدید به صورت خطی قابل تفکیک باشند. در این مقاله، به بررسی مزایای رویکرد غیر خطی در ماشین بردار پشتیبان (SVM) با استفاده از ترفند هسته می‌پردازیم. هدف اصلی این تحقیق، ارائه تحلیل‌های ریاضی دقیق و مقایسه عملکرد ماشین بردار پشتیبان خطی و غیر خطی در مسائل مختلف طبقه بندی است.

2020 Mathematics Subject Classification. 41A65; 68W25

واژگان کلیدی. ماشین بردار پشتیبان، ترفند هسته، هسته گاوسی، هسته چندجمله‌ای، فضای ویژگی .

* سخنران

۲. پیشینه‌ی تحقیق

ماشین بردار پشتیبان که توسط Vapnik در سال ۱۹۹۵ معرفی شد، یکی از محبوب‌ترین الگوریتم‌های یادگیری ماشین برای مسائل طبقه‌بندی است [۱]. SVM به‌دنبال یافتن یک ابرصفحه است که داده‌ها را با حداکثر حاشیه ممکن از هم جدا کند . یکی از محدودیت‌های SVM خطی این است که نمی‌تواند داده‌های غیر خطی را به‌خوبی طبقه‌بندی کند. برای رفع این محدودیت، ترفند هسته معرفی شد. در این رویکرد از هسته‌ها به‌عنوان تابعی برای نگاشت داده‌ها به فضای ویژگی با ابعاد بالاتر استفاده می‌شود. انواع مختلف هسته‌ها عبارتند از:

هسته خطی (Linear Kernel): ساده‌ترین نوع هسته که فضای ویژگی‌ها را تغییر نمی‌دهد. این هسته برای داده‌هایی که به‌طور خطی جداشدنی هستند، مناسب است.

هسته چندجمله‌ای (Polynomial Kernel): نگاشت غیر خطی داده‌ها به فضای ویژگی با ساختارچندجمله‌ای رادر پی دارد. این هسته مناسب برای داده‌هایی با توزیع پیچیده‌تر است.

هسته تابع پایه شعاعی (RBF Kernel): این هسته داده‌ها را به‌طور شعاعی به فضای ویژگی با ابعاد بالاتر نگاشت می‌کند. این هسته برای داده‌هایی با توزیع پیچیده و غیرخطی بسیار مناسب است.

هر نوع هسته دارای مزایا و معایب خاص خود است. برای مثال، هسته خطی ساده است و محاسبات کمتری نیاز دارد، اما در مواجهه با داده‌های غیرخطی کارایی پایینی دارد. هسته RBF بسیار انعطاف پذیر است و می‌تواند داده‌های پیچیده را به‌خوبی طبقه‌بندی کند، اما نیاز به تنظیم دقیق دارد. مطالعات متعددی نشان داده‌اند که استفاده از هسته‌های غیرخطی می‌تواند بهبود قابل توجهی در عملکرد SVM داشته باشد [۲].

۳. ماشین بردار پشتیبان مبتنی بر هسته‌ها (ترفند هسته)

در مواجهه با داده‌هایی که طبقه‌بندی آن ها با استفاده از ابرصفحه‌ها امکان‌پذیر باشد، استفاده از ماشین بردار پشتیبان خطی معمول می‌باشد. این در حالی است که در بسیاری از داده‌های موجود با توجه به ماهیت آن‌ها، استفاده از ساختار ماشین بردار پشتیبان خطی برای طبقه‌بندی آن‌ها از کارایی لازم برخوردار نیست. در چنین مواقعی استفاده از ساختار ماشین بردار پشتیبان مبتنی بر هسته‌ها به دلیل استفاده از فضای ویژگی داده‌ها مدنظر می‌باشد. در حقیقت زمانی که با استفاده از ابرصفحه‌ها در فضای ورودی داده‌ها قادر به تفکیک خطی آنها نباشیم، این اطمینان وجود دارد که با استفاده از یک نگاشت ویژگی مناسب، Φ، داده‌ها به فضای ویژگی جدید انتقال یابد به گونه‌ای که در این فضا امکان تفکیک خطی با استفاده از ابرصفحه‌ها امکان‌پذیر باشد [۳]. لازم به ذکر است که داده X_i در فضای ورودی، با ویژگی آن یعنی Φ_{X_i} در فضای ویژگی جایگزین می‌گردد. بنابراین ابرصفحه جدا کننده داده‌ها به صورت رابطه ۱.۳ تعیین می‌گردد:

$$\Phi_X^T \boldsymbol{w} + b = 0 \tag{۱.۳}$$

و مساله دوگان برای ساختار ماشین بردار پشتیبان به فرم رابطه ۲.۳ تبدیل می‌شود:

$$\begin{cases} \max \left(\sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j y_i y_j \Phi_{x_i}^T \Phi_{x_j} \right) \\ s.t. \quad \sum_{i=1}^{N} \alpha_i y_i = 0 \\ 0 \leq a_i \leq c \end{cases} \tag{۲.۳}$$

لازم به ذکر است، دستیابی به فضای ویژگی داده‌ها در چهارچوب فضای هیلبرت هسته باز تولید امکان‌پذیر است. به عبارت دیگر، نگاشت ویژگی معرفی شده به صورت $\Phi : \Omega \to H_K(\Omega)$ در نظر گرفته می‌شود، به طوری که تحت نگاشت

$$X \longmapsto \Phi_X = K(.,X), \tag{۳.۳}$$

مجموعه داده‌ها از فضای Ω به فضای ویژگی $H_K(\Omega)$ انتقال می‌یابد. جایی‌که $H_K(\Omega)$ فضای هیلبرت هسته بازتولید نظیر به هسته K می‌باشد. با توجه به خصوصیات فضای هیلبرت هسته بازتولید، ضرب داخلی در فضای ویژگی به دست آمده، $H_K(\Omega)$ ، با استفاده از هسته K به سادگی امکان‌پذیر است، یعنی:

$$K(X,Z) = \langle \Phi_X, \Phi_Z \rangle = \Phi_X^T \Phi_Z \tag{۴.۳}$$

حال مساله دوگان بهینه‌سازی درجه دو در ساختار ماشین بردار پشتیبان مبتنی بر فضای ویژگی به صورت رابطه زیر خواهد بود:

$$\begin{cases} \max \left(\sum_{i=1}^N \alpha_i - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j y_i y_j K(X_i, X_j) \right) \\ s.t. \quad \sum_{i=1}^N \alpha_i y_i = 0 \\ 0 \le \alpha_i \le C \end{cases} \tag{۵.۳}$$

پس تفکیک کننده در این مسئله به صورت زیر خواهد بود:

$$y = sign\Big(\sum_{i \in Su} \alpha_i y_i K(X_i, X) + b \Big),$$

که Su مجموعه دربردارنده بردارهای پشتیبان است و مقدار b به صورت زیر قابل محاسبه است:

$$b = \frac{1}{|Su|} \sum_{i \in Su} [y_i - \sum_{j \in Su} \alpha_j y_j K(X_j, X_i)].$$

مثال ۱.۳. فرض کنید که $\Omega \subseteq \mathbb{R}^2$ و $K : \Omega \times \Omega \to \mathbb{R}$ یک هسته چندجمله ای مرتبه دو با ضابطه زیر باشد:

$$K(X,Z) = (X^T Z)^2; \qquad \forall X, Z \in \Omega.$$

می‌توان نشان داد که نگاشتی غیرخطی مانند Φ وجود دارد به طوری که فضای داده‌های ورودی یعنی Ω با بعد دو را به یک فضای ویژگی از داده‌ها با بعد سه می‌نگارد که در ادامه این نگاشت معرفی می‌گردد. با توجه به آن که هسته به کار رفته یک هسته درجه دو می‌باشد خواهیم داشت:

$$\begin{aligned} K(X,Z) = (X^T Z)^2 &= \left(\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)^2 = (x_1 z_1 + x_2 z_2)^2 \\ &= \begin{pmatrix} x_1^2 & \sqrt{2} x_1 x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} z_1^2 \\ \sqrt{2} z_1 z_2 \\ z_2^2 \end{pmatrix} = \begin{pmatrix} x_1^2 \\ \sqrt{2} x_1 x_2 \\ x_2^2 \end{pmatrix}^T \begin{pmatrix} z_1^2 \\ \sqrt{2} z_1 z_2 \\ z_2^2 \end{pmatrix} \\ &= \Phi^T(x_1, x_2) \Phi(z_1, z_2) = \Phi^T(X) \Phi(Z) = \langle \Phi(X), \Phi(Z) \rangle. \end{aligned}$$

بنابراین

$$K(X,Z)=\langle\Phi(X),\Phi(Z)\rangle,$$

که در آن

$$\forall X=(x_1,x_2)\in\Omega\subseteq\mathbb{R}^2;\qquad \Phi(X)=\begin{pmatrix}x_1^2\\ \sqrt{2}x_1x_2\\ x_2^2\end{pmatrix}$$

توجه داریم که $\Phi(X)\in\mathbb{R}^3$ می‌باشد و نگاشتی است که داده‌های ورودی از بعد دو را به فضای ویژگی از بعد سه منتقل می‌کند. در حقیقت ترفند هسته باعث می‌گردد بی نیاز از محاسبه ضرب داخلی و بدون نیاز به دسترسی مستقیم به نگاشت غیرخطی، تنها با استفاده از هسته به کار رفته قادر باشیم اجزا و عناصر ماتریس کوواریانس در روش ماشین بردار پشتیبان را به دست آوریم. در ادامه چهارچوب انتقال از فضای داده به فضای ویژگی در حالت کلی بیان می‌گردد.

به کمک مفاهیم فضای هیلبرت هسته بازتولید امکان بیان هر هسته معین مثبت به کمک بسط مرکر فراهم می‌باشد، یعنی:

$$K(X,Z)=\sum_{n=1}^{\infty}\lambda_n\varphi_n(X)\varphi_n(Z)=\langle\Phi_X,\Phi_Z\rangle_{l_2},\tag{۶.۳}$$

که

$$\Phi_X=(\sqrt{\lambda_1}\varphi_1(X),\sqrt{\lambda_2}\varphi_2(X),\cdots).$$

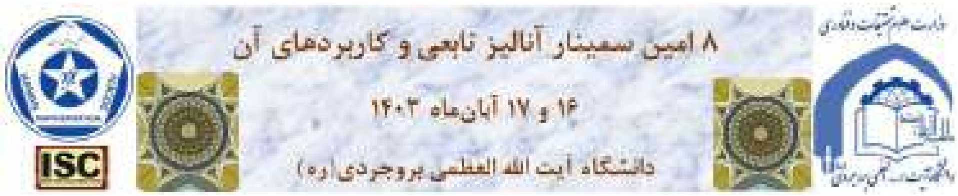
در حقیقت به کمک بسط فوق قادر خواهیم بود که هسته $K(X,Z)$ را در قالب یک ضرب داخلی بین ویژگی Φ_X و ویژگی Φ_Z بیان نماییم. در این حالت فضای ویژگی ممکن است نامتناهی البعد باشد، اما جای نگرانی نیست چرا که به کمک ترفند هسته این ضرب داخلی را می‌توان بدون نیاز به دسترسی مستقیم به نگاشت Φ، تنها به کمک هسته مورد استفاده انجام داد.

۴. نتیجه‌گیری و پیشنهادات

در این پژوهش، به بررسی مزایا و کاربردهای ترفند هسته در ماشین بردار پشتیبان پرداخته شد. به کمک ترفند هسته قادر به مدل‌سازی روابط پیچیده و غیر خطی بین داده‌ها خواهیم بود که این امر در مواجهه با داده های دنیای واقعی از اهمیت بسزایی برخوردار است. بر اساس مطالعه اخیر، توسعه و آزمایش هسته‌های جدید با توجه به ویژگی‌ها و نیازهای خاص مسائل مختلف می‌تواند زمینه‌ساز بهبود عملکرد ماشین بردار پشتیبان شود.

مراجع

1. V. N. Vapnik, *The Nature of Statistical Learning Theory*, Springer, New York 1995.
2. C. C. Chang, C. J. Lin, *LIBSVM: A library for support vector machines*, ACM Transactions on Intelligent Systems and Technology, **2** (3) (2011), 1-27.
3. M. Esmaeilbeigi, A. Daneshkhah, O. Chatrabgoun, *Some Computational Considerations for Kernel-Based Support Vector Machine*, Digital Twin Technologies and Smart Cities. Internet of Things. Springer, Cham. 2020.



وجود جواب‌های تناوبی دسته‌ای از معادلات دیفرانسیل مرتبه‌ی دوم غیرخطی به‌کمک قضیه نقطه ثابت باناخ و کاربردهای آن

مرتضی بیات [1] *

[1] گروه ریاضی، واحد زنجان، دانشگاه آزاد اسلامی، زنجان، ایران
baayyaatt@gmail.com

چکیده. در این مقاله وجود جواب تناوبی دسته‌ای از معادلات دیفرانسیل مرتبه‌ی دوم غیرخطی را بررسی می‌کنیم. بدین منظور، ابتدا معادله انتگرالی نظیر معادله دیفرانسیل مرتبه‌ی دوم غیرخطی را با ساختن تابع گرین مناسب بدست می‌آوریم و سپس با استفاده از قضیه نقطه ثابت باناخ، وجود جواب تناوبی معادله دیفرانسیل حاصل را نشان می‌دهیم. در ادامه به‌عنوان یک کاربرد، مدل ریاضی سیستم سیلندر، پیستون و سوپاپ خودرو را به‌صورت یک معادله دیفرانسیل مرتبه‌ی دوم غیرخطی با ضرایب متناوب ارائه می‌دهیم. در پایان، یک روش عددی برای تقریب جواب‌های تناوبی معادله دیفرانسیل پیشنهاد می‌دهیم.

۱. پیش‌گفتار

از زمان نیوتن در سال ۱۶۸۶، قوانین طبیعت با معادلات دیفرانسیل توصیف شده‌اند. معادلات دیفرانسیل غیرخطی مرتبه دوم ابزاری ضروری برای مدل‌سازی مسائل فیزیکی هستند که در زمینه‌های مختلف مانند زیست‌شناسی، شیمی، فیزیک، مکانیک، الکترونیک، مهندسی، اقتصاد، انرژی اتمی و نظریه ارتباطات ایجاد می‌شوند. معادلات دیفرانسیل غیرخطی برای مدل‌سازی نوسانگرهای میرا و متحرک، پیش‌بینی وقوع زلزله، مدل‌سازی مغز، مدل‌سازی قلب و تجزیه و تحلیل تصادف استفاده می‌شوند.
در مقاله [۵]، مهری و همدانی وجود جواب تناوبی برای دو دسته معادله دیفرانسیل زیر را بدست آورده‌اند:

$$x'' + f(t, x) = 0, \qquad x'' + f(t, x, x') = 0.$$

2020 Mathematics Subject Classification. 34C25; 34A34; 47H10; 46A50
واژگان کلیدی. معادله دیفرانسیل مرتبه‌ی دوم غیرخطی، جواب تناوبی، قضیه نقطه ثابت باناخ، تابع گرین، معادله انتگرالی، نگاشت انقباضی.
* سخنران

همچنین مهری وجود جواب تناوبی معادله زیر را به‌کمک قضیه نقطه ثابت لری-شاودر داده است:

$$x'' + \big(k + h(x)\big)x' + f(t,x) = p(t).$$

اسماعیل‌زاده و همکاران در مقاله [۱]، وجود جواب تناوبی برای معادله خودگردان زیر به‌کمک قضیه نقطه ثابت شاودر داده‌اند:

$$x'' + p(x)x' + q(x) = 0.$$

مارتینز در مقاله [۴]، وجود جواب تناوبی زیر را با استفاده از نظریه درجه توپولوژیکی داده است:

$$u'' + f(u)u' + g(t,u) = e(t).$$

اخیرا در مقاله [۳]، گیدونی یک شرط لازم و کافی برای وجود جواب منحصربفرد T- تناوبی برای معادله زیر داده شده است:

$$x'' + f(t,x) + p(t,x,x') = 0,$$

که در آن f یک تابع فوق خطی نسبت به x که به‌طور یکنواخت نسبت به‌زمان رشد می‌کند و نیز تابع p کراندار است.
همچنین در مقاله [۲]، ایزی و همکاران وجود جواب تناوبی معادله تعمیم یافته دافینگ زیر را بدست آورده‌اند:

$$x'' + cx' + ax + bx^2 + 2x^3 = p(t),$$

که در آن a، b و c ثابت‌های حقیقی و $p : [0, 2\pi] \to \mathbb{R}^n$ تابع پیوسته است و این معادله در شرایط اولیه تناوبی زیر صدق می‌کند:

$$x(0) = x(2\pi), \qquad x'(0) = x'(2\pi).$$

در این مقاله، مدل ریاضی سیستم سیلندر، پیستون و سوپاپ خودرو را به‌یک معادله دیفرانسیل غیرخطی مرتبه‌ی دوم تبدیل می‌شود که تقریب بهتری از مدل خطی کلاسیک آن می‌دهد. هدف اصلی این مقاله اثبات وجود جواب تناوبی معادله بدست آمده با استفاده از قضیه نقطه ثابت باناخ است. در پایان روشی برای تقریب عددی جواب تناوبی معادله داده خواهد شد.
اینک معادله غیرخطی زیر را در نظر می‌گیریم:

$$ax'' + f(t)x' + g(t,x) = h(t), \qquad (۱.۱)$$

که a یک عدد حقیقی ناصفر و g, f و h توابع پیوسته و متناوب نسبت به t با دوره‌ی تناوب ω می‌باشند. در ضمن مشتق‌ها x' و x'' نسبت به t گرفته می‌شوند. حال وجود جواب تناوبی این معادله را بر اساس قضیه نقطه ثابت باناخ (برای نگاشت انقباضی) تحت شرایط زیر اثبات می‌کنیم:

$$x(0) = x(\omega), \qquad x'(0) = x'(\omega). \qquad (۲.۱)$$

یک حالت خاص معادله‌ی (۱.۱)، معادله حرکت آونگ نامیرا زیر است:

$$x'' + \frac{c}{m}x' + \frac{g}{a}\sin(x) = 0,$$

که در آن x زاویه‌ی انحراف یک آونگ نامیرا به‌طول a و m جرم گلوله‌ی آن است.
یک مدل واقعی از (۱.۱)، در مطالعه مدل ریاضی سیلندر، پیستون و سوپاپ خودرو می‌باشد. مدل ریاضی ارتعاشی سوپاپ را می‌توان مطابق قانون دوم نیوتن بب اساس معادله دیفرانسیل حرکت حاکم به‌صورت زیر نوشت:

$$mx'' + c(t)x' + k(t,x)x = p(t), \tag{۳.۱}$$

معادله (۳.۱) یک حالت خاص معادله (۱.۱) با در نظر گرفتن $g(x,t) = k(t,x)x$ می‌باشد.
چنانچه ω دوره‌ی تناوب حرکت ارتعاشی معادله (۳.۱) با شرط (۲.۱) باشد، نشان خواهیم داد که این معادله دارای جواب ω- تناوبی است.
در قضیه زیر یک شرط کافی برای وجود جواب تناوبی معادله (۱.۱) با شرط (۲.۱) داده می‌شود.

قضیه ۱.۱. فرض کنیم f، g و h توابع پیوسته و متناوب نسبت به t با دوره‌ی تناوب ω باشند. معادله دیفرانسیل مرتبه‌ی دوم غیرخطی (۱.۱) با شرط (۲.۱) حداقل یک جواب ω- تناوبی دارد، هرگاه دو شرط زیر برقرار باشد:

(الف). ضریب ناصفر a در نامساوی زیر صدق کند:

$$|a| < 1 - \frac{M\omega}{2}, \tag{۴.۱}$$

که در آن $M = \|f\|_\infty$.

(ب). برای $K > 0$ که در شرط لیپشتس

$$\big|g(t,x(t)) - g(t,y(t))\big| \leq K\|x-y\|_\infty, \qquad (\text{برای هر } t\in[0,\omega]) \tag{۵.۱}$$

صدق می کند، نامساوی زیر برقرار باشد:

$$K < \frac{8}{\omega^2}\Big(1 - |a| - \frac{M\omega}{2}\Big). \tag{۶.۱}$$

در اثبات وجود و یگانگی جواب تناوبی در قضیه بالا از قضیه نقطه ثابت باناخ با ساختن یک عملگر انقباضی مناسب وجود نقطه ثابت (نظیر وجود جواب تناوبی) استفاده می‌شود. و نیز به کمک نامساوی‌های ساخته شده روی فضای باناخ حاصل، کران مناسبی برای تضمین وجود جواب تناوبی داده می‌شود.

مثال ۲.۱. طبق قضیه ۱.۱ معادله دیفرانسیل زیر دارای جواب $\frac{\pi}{4}$- تناوبی است:

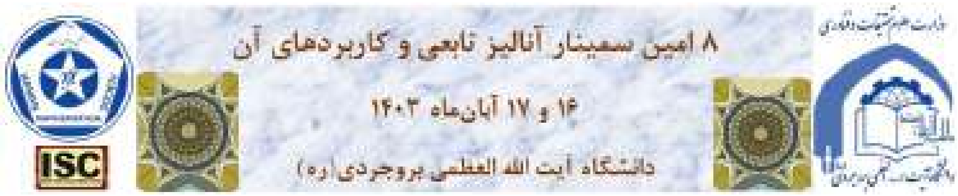
$$0.6x'' + \sin^2(8t)x' + 0.008x^2 = \cos(8t),$$

برای $a = 0.6$، $M = 1$، $\omega = \frac{\pi}{4}$ و $K = 0.09$، با محاسبات کامپیوتری نتایج زیر بدست می‌آید:

$$x(0) = -0.30611 \qquad x'(0) = 0.02736,$$
$$x(\frac{\pi}{4}) = -0.30611 \qquad x'(\frac{\pi}{4}) = 0.02736.$$

مراجع

1. E. Esmailzadeh, B. Mehri and G. Nakhaie-Jazar, *Periodic solution of a second order, autonomous, nonlinear system*, Nonlinear Dynamics, **10** (1996), 307–316.
2. E.O. Eze, U.E. Obasi, G. Ezugorie and E. E. Hannah, *Existence, uniqueness, boundedness and stability of periodic solutions of a certain second-order nonlinear differential equation with damping and resonance effects*, Biometrical Letters, **60** (2023), 109–124.
3. P. Gidoni, *Existence of a periodic solution for superlinear second order ODEs*, Journal of Differential Equations, **345** (2023), 401–417.
4. R.F. Martins, *Existence of periodic solutions for second-order differential equations with singularities and the strong force condition*, Journal of Mathematical Analysis and Applications, **317** (2006), 1–13.
5. B. Mehri and G.G. Hamedani, *On the Existence of Periodic Solutions of Nonlinear Second Order Differential Equations*, SIAM Journal on Applied Mathematics, **29** (1975), 72–76.



نگاشت های جمعی حافظ شعاع طیفی بین جبرهای باناخ

روح الله پروینیان زاده *

گروه ریاضی، دانشکده علوم پایه، دانشگاه یاسوج
r.parvinian@yu.ac.ir

چکیده. فرض کنید A یک C^* −جبر از رتبه حقیقی صفر و B یک جبر باناخ باشند. در این مقاله، اگر $T : A \to B$ یک نگاشت جمعی حافظ شعاع طیفی باشد، همریختی جردن بودن T را مورد برسی قرار می دهیم.

۱. پیش‌گفتار

مسائل نگهدارنده‌ی خطی یک مسأله قدیمی است. اولین بار فربینیوس [۷] در سال ۱۸۹۷ این مسائل را مطرح کردند، ایشان نگاشت های خطی حافظ دترمینان روی جبر ماتریس ها را مورد مطالعه قرار دادند. یکی از مهم ترین مسأله ها، در مسائل نگهدارنده‌ی خطی، مسأله کاپلانسکی [۷] است، کاپلانسکی در سال ۱۹۷۰ این سوال را مطرح کرد که آیا هر نگاشت خطی یکدار پوشا حافظ طیف بین جبر های باناخ نیم ساده، یکریختی جردن است؟ مساله کاپلانسکی در حالتی که حداقل یکی از جبرها جابجایی باشد توسط گلیسون، کاهان و زلاسکو[۴] حل شد. در حالت غیر جابجایی بهترین نتایج توسط سرور و اوپتیت بدست آمد. جعفریان و سرور[۵] نشان دادند که نگاشتهای خطی دو سویی یکدار حافظ معکوسپذیری روی جبر عملگر های خطی کراندار روی یک فضای باناخ، حافظ عملگرهای رتبه یک می‌باشد و با استفاده از حافظ رتبه یک بودن، مساله کاپلانسکی را حل کردند. اوپتیت[۲] مساله کاپلانسکی را برای جبرهای فون نیومان ثابت کرد، او ثابت کرد که نگاشتهای خطی دو سویی یکدار حافظ عناصر خودتوان می باشد و با استفاده از این مطلب یک جواب مثبت برای مساله کاپلانسکی بدست اورد. در مسأله کاپلانسکی مارتین متیو[۸] فرض حافظ طیف را به حافظ شعاع طیفی کاهش داد و این سوال را

2020 Mathematics Subject Classification. Primary 46L05, 47A10 Secondary 47B48, 46H10

واژگان کلیدی. نگاشت های جمعی، حافظ شعاع طیفی، جبرهای باناخ.

* سخنران

مطرح کرد که آیا هر نگاشت خطی یکدار پوشا و حافظ شعاع طیفی بینC^*- جبرها، یکریختی جردن است؟ این حدس در حالت کلی اثبات نشده است و همچنان یک مسأله باز است. هدف از انجام این مقاله، مطالهی نگاشت های حافظ شعاع طیفی و مسألهی کاپلانسکی میباشد و حدس متیو را با فرضیات اضافی مورد بررسی قرار میدهیم.

۲. دستآوردهای پژوهش

در این مقاله فرض میکنیم A و B جبرهای باناخ یکدار باشند و عناصر یکه هر دو جبر A و B را با e نشان می دهیم.

تعریف ۱.۲. فرض کنید A یک جبر باناخ باشد و $a \in A$. طیف a را با $\sigma(a)$ نشان می دهیم و برابر است با مجموعه همه $\lambda \in C$ که $a - \lambda e$ در A معکوسپذیر نباشد. شعاع طیفی a را با $r(a)$ نشان می دهیم و به صورت زیر تعریف می شود.

$$r(a) = sup\{|\lambda| : \lambda \in A\}.$$

تعریف ۲.۲. نگاشت $T : A \to B$ را حافظ طیف مینامیم، هرگاه برای هر $a \in A$،

$$\sigma(Ta) = \sigma(a). \tag{۱.۲}$$

نگاشت $T : A \to B$ را حافظ شعاع طیفی مینامیم، هرگاه برای هر $a \in A$،

$$r(Ta) = r(a). \tag{۲.۲}$$

تعریف ۳.۲. نگاشت $T : A \to B$ را همریختی جردن گوییم، هرگاه برای هر a متعلق به A،

$$T(a^2) = T(a)^2. \tag{۳.۲}$$

نگاشت T از جبر باناخ A به توی جبر باناخ B را حافظ معکوسپذیر گوییم، هرگاه $a \in A^{-1}$، آنگاه $T(a) \in B^{-1}$، که A^{-1} و B^{-1} به ترتیب مجموعه عناصر معکوسپذیر A و B می باشند.

قضیه زیر توسط زیمنک ثابت شد و به ویژگی های زیمنک معروف است،(مرجع [۱] را ببینید).

قضیه ۴.۲. اگر A جبر باناخ یکدار باشد، آنگاه شرایط زیر معادل هستند.
(۱) a در رادیکال جیکبسون A قرار دارد
(۲) برای هر عنصر $x \in A$، $\sigma(a + x) = \sigma(x)$
(۳) برای هر عنصر شبه پوچ توان $x \in A$، $r(a + x) = 0$.

از لم های زیر در اثبات قضایای اصلی استفاده می شود.

لم ۵.۲. فرض کنید A و B جبرهای باناخ باشند و $T : A \to B$ یک نگاشت پوشا و حافظ شعاع طیفی باشد، آنگاه $T(rad(A)) = rad(B)$ که $rad(A)$ رادیکال جیکبسون A می باشد.

لم ۶.۲. فرض کنید A، B جبر باناخ و $T: A \to B$ یک نگاشت یکدار پوشا و حافظ شعاع طیفی باشد. اگر A نیم ساده باشد. آنگاه T پیوسته است.

لم ۷.۲. فرض کنید $T: A \to B$ یک نگاشت پوشا و حافظ شعاع طیفی باشد، اگر A جبر باناخ یکدار، نیم‌ساده و جابجایی باشد، آنگاه B جابجایی هست.

تعریف ۸.۲. C^* - جبر A از رتبه حقیقی صفر است اگر مجموعه عناصر خودالحاق معکوسپذیر A در عناصر خودالحاق A چگال باشند. این معادل است با اینکه هر عضو خودالحاق A حد ترکیبات خطی از تصاویر متعامد A است.

با استفاده از پیوستگی T و اینکه به راحتی ثابت می شود T حافظ عناصر خودالحاق می باشد، قضیه زیر را داریم.

قضیه ۹.۲. فرض کنید A یک C^* -جبر از رتبه حقیقی صفر و B یک جبر باناخ یکدار باشد. اگر $T: A \to B$ یک نگاشت پوشای جمعی حافظ طیف باشد. آنگاه $T(a^2) = (Ta)^2$ برای هر عنصر خودالحاق a.

در قضیه زیر تحدید T روی عناصر خودالحاق A را به یک همریختی جردن روی A توسیع می دهیم.

قضیه ۱۰.۲. فرض کنید A یک C^* -جبر از رتبه حقیقی صفر و B یک جبر باناخ یکدار باشد. اگر $T: A \to B$ یک نگاشت پوشای جمعی حافظ طیف باشد. آنگاه همریختی جردن $S: A \to B$ وجود دارد بطوریکه $T(a) = S(a)$ برای هر عنصر خودالحاق a.

قضیه زیر توسط برشر [۳] ثابت شد.

قضیه ۱۱.۲. فرض کنید X یک فضای هیلبرت و $\varphi: B(X) \to B(X)$ یک نگاشت خطی پوشا و حافظ شعاع طیفی باشد. آنگاه اسکالر مختلط c با $|c| = 1$ و یا عملگر خطی دوسویی $T: X \to X$ وجود دارد بطوریکه $\varphi(A) = cTAT^{-1}$ برای هر $A \in B(H)$؛ یا عملگر خطی دوسویی $T: X^* \to X$ وجود دارد بطوریکه $\varphi(A) = cTA^*T^{-1}$ برای هر $A \in B(H)$.

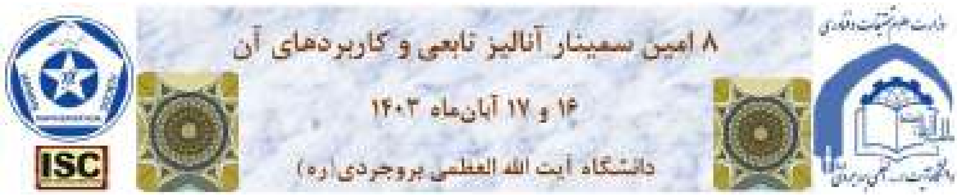
قضیه زیر یکی از نتایج اصلی مقاله می باشد و اثبات آن با استفاده از قضایای بالا می باشد.

قضیه ۱۲.۲. فرض کنید H یک فضای هیلبرت و $\varphi: B(H) \to B(H)$ یک نگاشت پوشای جمعی و حافظ شعاع طیفی باشد. آنگاه اسکالر مختلط c با $|c| = 1$ و عملگر خطی دوسویی $T: H \to H$ وجود دارد بطوریکه $\varphi(A) = cTAT^{-1}$ یا $\varphi(A) = cTA^*T^{-1}$ برای هر عنصر خودالحاق $A \in B(H)$.

مراجع

1. B. Aupetit, *A primeron spectral theory*, Springer-Verlage, New York.
2. B. Aupetit, *Spectrum-preserving linear mappings between Banach algebras or Jordan-Banach algebras*, J. London Math. Soc.**62** (2000), 917–924.
3. M. Bresar, *Linear maps preserving the spectral radius*, J. Funct. Anal., **142** (2000), 360–368.

4. A. Gleason, J. Kahane and W. Zelazko, *A characterization of maximal ideals in commutative Banach algebras*, Studia Math., **29** (1968), 339–343.
5. A. A Jafarian, A. R Surour, *Spectrum perserving linear maps*, J. Funct. Anal., **66**(1986), 255–261.
6. G. Frobenius, *Ueber die Darstellung der endlichen Gruppen durch lineare Substitutionen*, Berl Ber. Appl., **203**(1897) 994–1015.
7. I. Kaplansky, *Algebraic and analytic aspects of operator algebras*, CBMS regional conference seriesin Mathematics, Amer. Math. Soc, providence, 1970.
8. M. Mathieu, *Towards a non-selfadjoint version of Kadison's theorem*, Ann. Math. Inf., **32** (2005), 87–94.



بررسی برخی شباهت ها و تفاوت های ابردوری و محدب-دوری

حمیده پورانصاری [1] و میثم اسدی پور [2]*

[1] گروه ریاضی، دانشکده علوم پایه، دانشگاه یاسوج
h pouransary@gmail.com
[2] گروه ریاضی، دانشکده علوم پایه، دانشگاه یاسوج
Asadipour.mey@gmail.com

چکیده. دراین مقاله برخی تفاوت‌ها و شباهت‌های بین عملگرهای ابردوری و محدب-دوری معرفی خواهند شد. این تفاوت‌ها در دو زمینه بعدفضای زمینه و همچنین توان های دلخواه یک عملگر و ازطرفی شباهت براساس فشرده نبودن این دو نوع عملگر، مورد بررسی قرار خواهند گرفت.

۱. پیش‌گفتار

فرض کنیم X یک فضای باناخ روی میدان اعداد مختلط $\mathbb{C}$ و B یک زیرمجموعه ناتهی از X است. اگر $T \in B(X)$ آنگاه مدار B تحت عملگر T به صورت

$$orb(T,B) := \{T^n x; x \in B, n = 0,1,2,...\}$$

است. در این صورت با توجه به موارد زیر، عملگر T ابردوری یا فرادوری نامیده می‌شود.
۱) اگر به ازای بردار $x \in X$ زیرمجموعه B به صورت مضارب اسکالر x یعنی $B = \mathbb{C}.x$ باشد و همچنین $orb(T,B)$ در X چگال باشد، آنگاه بردار x یک بردار فرادوری برای T و عملگر T یک عملگر فرادوری نامیده می‌شود. مجموعه بردارهای فرادوری عملگر T را با نماد $SC(T)$ نمایش می‌دهیم.
۲) اگر به ازای $x \in X$ زیرمجموعه B به صورت مجموعه تک عضوی $\{x\}$ و $orb(T,B)$ در X چگال باشد، آنگاه بردار x یک بردار ابردوری برای T و عملگر T یک عملگر ابردوری نامیده می‌شود. $HC(T)$ معرف مجموعه بردارهای ابردوری عملگر T است.

2020 Mathematics Subject Classification. 47A16; 47B37

واژگان کلیدی. عملگر ابردوری، عملگر محدب-دوری، غلاف محدب.

* سخنران

به راحتی قابل مشاهده است که اگر عملگر $T \in B(X)$ ابردوری باشد آنگاه فرادوری نیز است اما لزوماً عکس این مطلب صادق نیست، [۳]. همانگونه که مشاهده می‌گردد وجود عملگر ابردوری روی فضای باناخ تفکیک‌ناپذیر ممکن نیست و از طرفی وجود یک عملگر ابردوری روی هرفضای باناخ تفکیک‌پذیر ثابت شده است، [۱]. علاوه بر دو نوع عملگرهای بالا، نوع دیگری از عملگر به نام عملگر محدب-دوری وجود دارد که به صورت زیر تعریف می‌شود.

تعریف: فرض کنیم برای عملگر $T \in B(X)$ بردار $x \in X$ وجود دارد بطوریکه غلاف محدب $orb(T, X)$ یعنی مجموعه $\{p(T)x \ : \ p$ چندجمله‌ای محدب است$\}$ در X چگال است. در این صورت x را بردار محدب-دوری برای T نامیم و نماد $CC(T)$ معرف مجموعه بردارهای محدب-دوری عملگر T است. لازم به ذکر است که در تعریف بالا منظور از چندجمله‌ای محدب p یک چندجمله‌ای به صورت زیر است

$$p(z) = \sum_{i=0}^{n} a_i z^i, \quad a_i \geq 0, \quad \sum_{i=0}^{n} a_i = 1.$$

باتوجه به مفهوم چگال بودن این مطلب واضح است که هر عملگر ابردوری را می‌توان به عنوان مثالی از یک عملگر محدب-دوری در نظر گرفت ولی سوالی که دراین مقاله پاسخ به آن حائز اهمیت است این است که:

تفاوت‌ها و شباهت‌های بین مفهوم ابردوری بودن و محدب-دوری بودن چیست؟

لازم به ذکرست دلیل اهمیت پاسخ به این سئوال این است که بر اساس بررسی شباهت‌ها، شناخت بیشتری از این دو مفهوم حاصل می‌گردد و از طرفی بررسی تفاوت های این دو مفهوم، نشان می دهد که این دو مفهوم در حالت کلی معادل یکدیگر نیستند. بنابراین در بخش آتی برخی شباهت‌ها و تفاوت‌های ابردوری و محدب-دوری را ارائه خواهیم داد.

۲. برخی تفاوت‌ها و شباهت‌های بین ابردوری و محدب-دوری

همانگونه که در بخش پیش‌گفتار توضیح داده شد، دراین بخش برخی تفاوت‌ها و شباهت‌های بین ابردوری و محدب-دوری معرفی خواهند شد. این تفاوت‌ها و شباهت‌های در سه زمینه بعدفضای زمینه، فشرده بودن عملگر و توان های عملگر، مورد بررسی قرار خواهند گرفت. ابتدا یک تفاوت براساس بعد فضای زمینه را معرفی میکنیم.

۱.۲. بعدفضای زمینه: برخلاف عملگر ابردوری که فقط بر فضای باناخ با بعد نامتناهی وجود دارد، در مقاله [۴] عملگرهای محدب-دوری روی فضای $\mathbb{R}^n$ و $\mathbb{C}^n$ مورد بررسی قرار گرفته‌اند. برای تاکید بر ارتباط بین بعد فضای زمینه و مفهوم ابردوری بودن قضیه زیر که اثبات آن در [۳] قابل مشاهده است را ارائه می‌دهیم.

قضیه ۱.۲. عملگر ابردوری T روی فضای باناخ X وجود دارد اگر و فقط اگر X با بعد نامتناهی باشد.

۲.۲. **فشرده بودن عملگر:** وجه اشتراکی که می‌توان برای دومفهوم مورد نظر ذکر کرد این است که هیچ عملگر فشرده ابردوری و همچنین هیچ عملگر فشرده محدب-دوری روی فضای باناخ با بعد نامتناهی وجود ندارد و این شباهت بر اساس نتیجه ۷.۳ از [۴] و گزاره ۱۱.۵ از [۳] به شرح ذیل است.

قضیه ۲.۲. هیچ عملگر فشرده محدب-دوری یا فشرده ابردوری روی فضای باناخ X با بعد نامتناهی وجود ندارد.

۳.۲. **توان های یک عملگر ابردوری و محدب-دوری:** در ادامه یک تفاوت اساسی را میان دو مفهوم ابردوری بودن و محدب دوری بودن ارائه می‌دهیم.
در [۳] می توان مشاهده کرد که بردار $x \in X$ یک بردار ابردوری برای عملگر $T \in B(X)$ است اگر و فقط اگر برای هر توان T^p از T نیز یک بردار ابردوری باشد.

قضیه ۳.۲. فرض کنیم $T \in B(X)$ در این صورت به ازای هر $p \in \mathbb{N}$،

$$HC(T) = HC(T^p).$$

بنابراین اکنون دوسؤال مطرح است:

سؤال۱: اگر $T \in B(X)$ محدب دوری باشد، آیا میتوان نتیجه گرفت که به ازای هر عدد $p \in \mathbb{N}$، T^p نیز یک محدب دوری است؟

توجه داریم که اگر عملگر T ابردوری باشد، آنگاه بنابر قضیه قبل داریم که هر دو مجموعه $CC(T)$ و $CC(T^p)$ همواره ناتهی هستند، بنابراین دراین حالت خاص، پاسخ سؤال فوق مثبت است اما در حالت عمومی، مثالهایی از عملگرهای محدب-دوری T وجود دارند که برخی از توانهای آنها، فاقد هرگونه بردار محدب-دوری می‌باشند.

مثال ۴.۲. فرض کنیم $T := r_0\alpha I_{\mathbb{C}} \oplus \mu B \in B(\mathbb{C} \bigoplus \ell_p)$ $1 \leq p < \infty$ که B عملگر شیفت به عقب روی فضای باناخ $\ell^p(\mathbb{N})$ در نظر گرفته شده است. در این صورت بر اساس قضیه ۳ از [۵] نتیجه می‌شود که

$$CC(T) \neq \emptyset, \quad CC(T^3) = \emptyset.$$

براساس سؤال قبل و توضیحات و مثال قبل مشخص شد که ممکن است عملگر T و هر توان آن محدب-دوری باشد ولی در حالت عمومی این مطلب صادق نیست. در نتیجه ارائه سؤال زیر طبیعی است.
سؤال۲: فرض کنیم به ازای یک $p_0 \in \mathbb{N}$ هر دو عملگر T^{p_0} و T محدب-دوری باشد. در این صورت آیا میتوان نتیجه گرفت که $CC(T) = CC(T^{p_0})$ ؟

برای پاسخ به این سؤال، قضیه زیر را ارائه می دهیم.

قضیه ۵.۲. فرض کنیم B عملگر شیفت به عقب روی $\ell^2(\mathbb{N})$ در این‌صورت عملگر $T = 2B$ و T^2 دو عملگر محدب-دوری هستند که $CC(T) \neq CC(T^2)$.

برهان. فرض کنیم $T := 2B$ و عملگر D روی $\ell^2(\mathbb{N})$ به‌صورت زیر تعریف شده‌است:

$$D(x_0, x_1, x_2, ...) = (x_0, x_0, x_1, x_1, x_2, x_2, ...).$$

براساس گزاره $3 \cdot 3$ از [۲] بردار $x \in CC(T)$ وجود دارد به‌طوری‌که

$$\mathrm{Orb}(T^2, x) \subseteq D(\ell^2(\mathbb{N}))$$

و در نتیجه

$$\overline{\mathrm{Co}(\mathrm{Orb}(T^2x))} \subseteq \overline{\mathrm{Span}\{\mathrm{Orb}(T^2x)\}} \subseteq D(\ell^2(\mathbb{N})) \neq \ell^2(\mathbb{N}).$$

توجه دایم که $D(\ell^2(\mathbb{N}))$ یک زیرفضای بسته نابدیهی از $\ell^2(\mathbb{N})$ است و بنابراین

$$x \notin CC(T^2)$$

و برهان قضیه کامل است. □

مراجع

1. S. I. Ansari, *Existence of hypercyclic operators on topological vector spaces*, J. Funct. Anal., **148** (1997), 384–390.
2. T. Bermudez, A. Bonilla and N. S. Feldman, *On convex-cyclic operators*, J. Math. Anal. Appl. **434** (2016), 1166–1181.
3. K.G. Grosse-Erdmann and A. Peris, *Linear Chaos*, Springer, 2011.
4. H. Rezaei, *On the convex hull generated by orbit of operators*, Lin. Al. Appl., **438** (2013), 4190–4203.
5. F. Saavedra and M. P. Rosa, *Powers of convex-cyclic operators*, Abstract and Applied Analysis, **2014** (2014), Article ID 631894, 3 pages.

نیم‌گروه‌های تک توان مطلقاً بسته

شراره تهمتن

شراره تهمتن، گروه ریاضی، واحد بروجرد، دانشگاه آزاد اسلامی، بروجرد، ایران
Taham_sh@yahoo.com

چکیده. فرض کنید e خانواده‌ای از نیم‌گروه‌های توپولوژیک T_1 شامل همه نیم‌گروه‌های توپولوژیک صفر بعدی هاسدورف باشد. نیم‌گروه توپولوژیک X ، e بسته است اگر X در هر $Y \in e$ که $X \subseteq Y$ ، بسته باشد. X ، e بسته یک به یک است، اگر برای هر همریختی یک به یک $h\colon X \to Y$ که $Y \in e$ ، تصویر $h[X]$ در Y بسته باشد. نیم‌گروه X تک توان است اگر شامل یک عنصر خودتوان یکتا باشد. دراین مقاله ثابت می‌شود نیم‌گروه جابه‌جایی و تک توان X ، e بسته است اگر و فقط اگر کراندار، غیر تکین و گروه-متناهی باشد. همچنین ثابت می‌شود که برای هر نیم‌گروه تک توان e بسته یک به یک مانند X ، مرکز $Z(X)$ نیز e بسته یک به یک است.

۱. پیش‌گفتار و نتایج اصلی

در موارد متعددی ویژگی‌های یک ساختار توپولوژیک را می‌توان از بسته بودن آن در یک ساختار توپولوژیکی، به عنوان زیر ساختار تشخیص داد. به عنوان مثال فضای متریک X کامل است اگر و فقط اگر X در هر فضای متریکِ شامل آن، به عنوان یک زیرفضا، بسته باشد . نیم گروه توپولوژیک، یک نیم‌گروه و فضای توپولوژیک X است که در آن عمل دوتایی و شرکت‌پذیر $X \times X \to X$, $(x.y) \to xy$ پیوسته است. در خصوص نیم‌گروه‌های توپولوژیک نیز می‌توان خواصی را از طریق بسته بودن آن‌ها در نیم‌گروه‌های توپولوژیک محیطی‌شان تشخیص داد.

تعریف ۱.۱. فرض کنید e خانواده‌ای از نیم‌گروه‌های توپولوژیکی و X یک نیم گروه توپولوژیک باشد.

- X ، e بسته نامیده می‌شود هرگاه برای هر $Y \in e$ که $X \subseteq Y$ ، X در Y بسته باشد.
- X ، e بسته یک به یک نامیده می‌شود هرگاه برای هر $Y \in e$ و هر همریختی یک به یک و پیوسته $h\colon X \to Y$ ، تصویر $h[X]$ در Y بسته باشد.
- X ، e بسته مطلق نامیده می‌شود هرگاه برای هر $Y \in e$ و هر همریختی پیوسته $h\colon X \to Y$ ، تصویر $h[X]$ در Y بسته باشد.

به وضوح برای هر نیم گروه توپولوژیکی X داریم:

$$X\ ,\ e \text{ بسته مطلق است } \Rightarrow X\ ,\ e \text{ بسته یک به یک است } \Rightarrow X\ ,\ e \text{ بسته است}$$

هر نیم‌گروه توپولوژیک که دارای یکی از شرایط بسته بودن فوق باشد، مطلقاً بسته نامیده می‌شود. در این مقاله e را دسته‌ای از نیم گروه‌های توپولوژیکی در نظر می‌گیریم که هر زیر مجموعه متناهی از آن‌ها یک مجموعه بسته باشد (این ویژگی اصل جداسازی T_1 گفته می‌شود). احکامی که اثبات می‌شوند در ادامه مقاله‌های [۱] و [۴] و [۵] می‌باشد.

تعریف ۲.۱. اگر X یک نیم گروه باشد:

- X تکین نامیده می‌شود هرگاه یک زیر مجموعه نامتناهی $A \subseteq X$ وجود داشته باشد طوری که AA تک عضوی باشد.
- X زنجیره ـ متناهی نامیده می‌شود هرگاه هر زیر مجموعه نامتناهی X ، شامل عناصر $x.y$ باشد طوری که $xy \notin \{x.y\}$.
- X متناوب نامیده می‌شود هرگاه برای هر $x \in X$ عدد طبیعی n وجود داشته باشد طوری که x^n عنصری خودتوان باشد.
- X کراندار نامیده می‌شود هرگاه عدد طبیعی n موجود باشد طوری که برای هر $x \in X$ ، x^n عنصری خودتوان باشد.
- X گروه ـ متناهی نامیده می‌شود هرگاه هر زیرگروه X متناهی باشد.
- X گروه ـ کراندار نامیده می‌شود هرگاه هر زیرگروه X کراندار باشد.

مشخصات زیر برای نیم گروه‌های جابه‌جایی e بسته در [۱] ثابت شده است.

قضیه ۳.۱. یک نیم گروهِ جابجایی، e بسته است اگر و فقط اگر زنجیره ـ متناهی، متناوب، غیر تکین و گروه ـ کراندار باشد.
در مقاله حاضر برای نیم گروه‌های تک توان، مشخصات فوق به صورت زیر ساده‌سازی می‌شود.

قضیه ۴.۱. یک نیم گروه تک توان X ، e بسته است، اگر و فقط اگر کراندار و غیر تکین باشد.

نتیجه اصولی دیگر این مقاله، مشخصه زیر از نیم‌گروه‌های تک توانِ e بسته یک به یک است.

قضیه ۵.۱. نیم گروه جابه جایی و تک توان X ، e بسته یک به یک است، اگر و فقط اگر X کراندار، غیرتکین و گروه ـ متناهی باشد.
برای نیم گروه X مرکز آن به صورت $Z(X) = \{z \in X\colon X\colon \forall x \in X(zx = xz)\}$ تعریف می‌شود. در منابع [۱] و [۳] ویژگی‌های زیر از مرکز یک نیم گروه e بسته (یک به یک) به اثبات رسیده است.

قضیه ۶.۱. مرکز هر نیم گروه e بسته (یک به یک)، زنجیره متناهی، متناوب، غیر تکین (گروه ـ متناهی) است.

نتیجه ۷.۱. مرکز نیم گروه e بسته یک به یکِ X ، e بسته یک به یک است.

نتیجه ۸.۱. مرکز نیم گروه تک توان و e بسته مطلقِ X ، متناهی و بنابراین e بسته مطلق است.

۲. تعاریف و قراردادها

برای هر عضو a از نیم‌گروه X قرار می‌دهیم: $H_a = \{x \in X : (xX^1 = aX^1) \wedge (X^1x = X^1a)\}$ که $X^1 = X \cup \{1\}$ و ۱ عنصری بی اثر فرض شده است به عبارت دیگر برای هر $x \in X^1$ داریم $x1 = x = 1x$. طبق منبع [۲] اگر $e \in X$ عنصری خود توان باشد، H_e شامل زیرگروه ماکسیمال از X و عنصر e می‌باشد.
برای یک زیر مجموعه A از X و عدد طبیعی n قرار می‌دهیم:

$$\sqrt[n]{A} = \{x \in X : x^n \in A\} \quad . \quad \sqrt[\mathbb{N}]{A} = \bigcup_{n \in \mathbb{N}} \sqrt[n]{A}$$

لم زیر در [۱] ثابت شده است.
لم ۱.۲. برای هر عنصر خود توان $e \in X$ ، $(\sqrt[\mathbb{N}]{H_e}H_e) \cup (H_e\sqrt[\mathbb{N}]{H_e}) \subseteq H_e$.

۳. اثبات قضیه (۴.۱)

قضیه (۴.۱) از لم‌های زیر نتیجه می‌شود.
لم ۱.۳. فرض کنید X نیم گروهی جابجایی و متناوب با عنصر خودتوان و منحصر به فرد e و زیرگروه ماکسیمال بدیهی H_e باشد. اگر X کراندار نباشد آنگاه یک زیر مجموعه نامتناهی $A \subseteq X$ وجود دارد طوری که $AA = \{A\}$.

برهان. فرض کنید برای هر زیر مجموعه نامتناهی $A \subseteq X$ داشته باشیم $AA \neq \{e\}$ قرار می‌دهیم $W = \mathbb{N} \cup \{0\}$.
دنباله $(x_k)_{k \in W}$ در X و دنباله اعداد $(n_k)_{k \in W}$ را طوری می‌یابیم که برای هر $k \in W$ شرایط زیر را داشته باشند.

$$(i)\ x_k^{n_k} \notin \{e\} \cup \{x_i^{n_i} : i < k\} \quad , \quad (ii)\ x_k^{2n_k} = e \quad , \quad (iii)\ Max_{i<k}|x_i^{n_i}| < n_k$$

برای مجموعه نامتناهی $A = \{x_k^{n_k} : k \in W\}$ داریم $AA = \{e\}$ و این مطلب با فرض در تناقض است. □

لم ۲.۳. فرض کنید X یک نیم گروه جابه جایی با عنصر خودتوان و یکتای e و نیم گروه ماکسیمال و کراندار H_e باشد. اگر X کراندار نباشد آنگاه زیر مجموعه نامتناهی $A \subseteq X$ وجود دارد که $AA = \{e\}$.

برهان. $p \in \mathbb{N}$ وجود دارد که برای هر $x \in H_e$ ، $x^p = e$. مجموعه $P = \{x^p : x \in X\}$ یک زیرنیم گروه از X است و کراندار نمی‌باشد. همچنین $P \cap H_e = \{e\}$. به دلیل آنکه زیرگروه ماکسیمال P بدیهی است طبق لم (۱.۳) مجموعه نامتناهی $A \subseteq P \subseteq X$ وجود دارد که $AA = \{e\}$. □

لم ۳.۳. برای یک نیم گروه جابه جایی و تک توان X ، شرایط زیر معادلند.
e, X (1) بسته است. X (2) ، متناوب، غیرتکین و گروه ـ کراندار است. X (3) ، کراندار و غیر تکین است.

برهان. طبق قضیه (۳.۱) احکام ۱ و ۲ معادلند. طبق لم (۲.۳) نتیجه $3 \to 2$ حاصل می‌شود و $2 \to 3$ نیز واضح است. □

۴. اثبات قضیه(۵.۱)

قضیه۵.۱ را می‌توان از لم‌های زیر نتیجه گرفت.

لم ۱.۴. اگر نیم گروه تک توان X ، e بسته یک به یک باشد، آنگاه $Z(X)$ کراندار، غیر تکین و گروه ـمتناهی است.

برهان. طبق قضیه۶.۱ نیم گروه $Z(X)$ متناوب، غیر تکین و گروه ـ متناهی است. اگر $Z(X)$ تهی باشد آنگاه کراندار محسوب می‌شود و اگر تهی نباشد زیرنیم گروهِ نیم گروه تک توان X می‌باشد و طبق لم ۳.۳ کراندار است. □

لم ۲.۴. هر زیر نیم گروه جابه جاییِ تک توان، گروه ـمتناهی، غیر تکین و کراندار X از یک نیم گروه توپولوژیکیِ T_1 مانند Y در $\bar{Y}$ بسته و گسسته است.

برهان. به جای Y بستار Y را قرار می‌دهیم. پس فرض می‌کنیم که X در Y متراکم یا چگال است. □

برای هر $x \in X$ و هر $y \in Y$ همسایگی $U \subseteq Y$ از y وجود دارد طوری که مجموعه $x(U \cap X)$ متناهی است.

برهان. فرض کنید چنین نباشد. عناصر $a \in X = \cup_{i\in\mathbb{N}} \sqrt[i]{H_e}$ و $y \in Y$ را طوری در نظر می‌گیریم که برای هر همسایگی $U \subseteq Y$ از y مجموعه $a(U \cap X)$ نامتناهی باشد. فرض کنید k بزرگترین عددی باشد که برای هر $x \in \sqrt[k]{H_e}$ همسایگی $U \subseteq Y$ از y موجود باشد به گونه‌ای که مجموعه $x(U \cap X)$ متناهی باشد. $b \in \sqrt[k+1]{H_e}$ وجود دارد طوری که برای هر همسایگی $V \subseteq Y$ از y مجموعه $b(V \cap X)$ متناهی است. داریم $b^2 \in \sqrt[k]{H_e}$. همسایگی $U \subseteq Y$ از y را طوری در نظر می‌گیریم که مجموعه b^2U در X تک عضوی باشد. برای هر $u \in U \cap X$ داریم $(b^2u)^k \in H_e$ پس $b^2y = b^2u \in \sqrt[k]{H_e}$. مجدداً همسایگی $V \subseteq U$ وجود دارد به طوری ک مجموعه $(b^2y)Y$ در X تک عضوی است. پس برای مجموعه نامتناهی $A = b(V \cap X)$ داریم $AA \subseteq (b^2y)V$ که با فرض غیر تکین بودن X در تناقض است. □

برای هر $x \in X$ و هر $y \in Y$ همسایگی $V \subseteq Y$ از y وجود دارد طوری که مجموعه xV^n در X تک عضوی است.

برهان. طبق ادعای ۱ و مشابه اثبات آن این ادعا ثابت می‌شود. □

برای هر $n \in \mathbb{N}$ و $x \in X$ و $y \in Y$ همسایگی $V \subseteq Y$ از y وجود دارد طوری که مجموعه xV^n در X تک عضوی است.

برهان. حالت $n = 1$ همان ادعای ۲ می‌باشد. فرض کنید برای $n \in \mathbb{N}$ و هر $x \in X$ و $y \in Y$ همسایگی $U \subseteq Y$ از y یافت شود طوری که مجموعه xV^n در X تک عضوی به صورت $\{a\}$ باشد. طبق ادعای ۲ همسایگی $V \subseteq Y$ از y وجود دارد طوری که مجموعه aV در X تک عضوی است. پس مجموعه $V^{n+1} \subseteq aV$ نیز در X تک عضوی است. □

برای هر $k \in \mathbb{N}$ زیر فضای $\sqrt[k]{H_e}$ از Y گسسته است.

برهان. فرض کنید چنین نباشد. k را کوچکترین عددی می‌گیریم که $\sqrt[k]{H_e}$ گسسته نباشد. الزاماً $k > 1$. y را یک نقطه غیر ایزوله از $\sqrt[k]{H_e}$ در نظر می‌گیریم. همسایگی $V_0 \subseteq Y$ از y وجود دارد طوری که $V_0V_0 \cap \sqrt[k-1]{H_e}$.
طبق ادعای ۲ می‌توانیم فرض کنیم $V_0y = \{y^2\}$. با استقرا دنباله $(x_n)_{n\in W}$ از نقاط $\sqrt[k]{H_e}$ و دنباله کاهشی $(V_n)_{n\in W}$ از مجموعه‌های باز در Y را طوری می‌یابیم که برای هر $n\ inW$ شرایط زیر را داشته باشند.

$(i)\ x_n \in V_n \cap \sqrt[k]{H_e} \backslash \{x_i\}_{i<n}$. $(ii)\ y \in V_{n+1} \subseteq V_n$. $x_nV_{n+1} = \{y^2\}$

حال برای مجموعه نامتناهی $A = \{x_n\}_{n\in W}$ داریم $AA = \{y^2\}$ که متناقض با غیر تکین بودن X است. □

برای هر $k \in \mathbb{N}$ زیر فضای $\sqrt[k]{H_e}$ در Y بسته است.

برهان. فرض کنید چنین نباشد. k را کوچکترین عددی می‌گیریم که $\sqrt[k]{H_e}$ در Y بسته نباشد. الزاماً $k > 1$. برای هر نقطه ثابت $y \in \overline{\sqrt[k]{H_e}} \backslash \sqrt[k]{H_e}$ داریم $y^2 \in \overline{\{x^2 \colon x \in \sqrt[k]{H_e}} \subseteq \overline{\sqrt[k-1]{H_e}\}} = \sqrt[k-1]{H_e} \subseteq \sqrt[k]{H_e}$ طبق لم۳.۳ $\sqrt[k]{H_e}$ گسسته است پس همسایگی $V \subseteq Y$ از y وجود دارد به گونه‌ای که $UU \cap \sqrt[k]{H_e}$. به دلیل آن که $y \in \overline{\sqrt[k]{H_e}} \backslash \sqrt[k]{H_e}$ مجموعه $A = U \cap \sqrt[k]{H_e}$ نامتناهی است و $AA \subseteq UU \cap \sqrt[k]{H_e}$ که متناقض با غیر تکین بودن X است. □

مراجع

[1] T. Banakh and S. Bardyla, *Characterizing categorically closed commutative semigroups*, J. Algebra, **591** (2021), 84-110

[2] J. Howie, *Fundamentals of Semigroup Theory*, Clarendon Press: Oxford, UK, 1995.

[3] T. Banakh and S. Bardyla, *Subgroups of categorically closed semigroups*, arXiv (2022), arXiv: 2209.08013.

[4] T. Banakh, *Categorically closed topological groups*, Axioms, (2017), 6-32.

[5] T. Banakh and S. Bardyla, *Characterizing chain-finite and chain- compact*, topologicalsemilattices, Semigroup Forum, **98** (2019), 234-250.

۸ امین سمینار آنالیز تابعی و کاربردهای آن
۱۶ و ۱۷ آبان‌ماه ۱۴۰۳
دانشگاه آیت الله العظمی بروجردی (ره)

یافتن بهترین نقاط تقریبی به کمک دنباله‌های تکراری

لیلا رستگاری [1] * و محمود پورغلامحسین [2]

[1] دانشجوی دکترای ریاضی، گروه ریاضی، دانشکده علوم پایه، دانشگاه قم
rastegari969696@gmail.com
[2] گروه ریاضی، دانشکده علوم پایه، دانشگاه قم
purgol@yahoo.com

چکیده. در این مقاله، از الگوریتم مَن مربوط به نگاشت‌های غیر خود ریخت غیر انبساطي برای حل مسائل مینیمم‌سازی سرتاسری شامل بهترین نقاط تقریبي استفاده می‌کنیم سپس به ارائه همگرايي قوی و ضعیف الگوریتم پیشنهادی تحت شرایطي مناسب روی فضاهای هیلبرت می‌پردازیم. در پایان با چند مثال عددی به بررسي در راستای مشاهده چگونگي رفتار همگرايي الگوریتم پیشنهادی خواهیم پرداخت.

۱. پیشگفتار

ابتدا درفضاهای هیلبرت برای نگاشت‌های غیرخودریخت غیرانبساطی دو الگوریتم ترکیبی ارائه می‌شود که بوسیله آن‌ها، دنباله تقریبی جهت یافتن بهترین نقاط تقریب که به بهترین نقطه تقریب همگرا هستند به‌دست می‌آید در پایان الگوریتم عمومی مَن رامعرفی خواهیم کرد [۱، ۲، ۳، ۴، ۵].

تعریف ۱.۱. فرض کنید (X, d) یک فضای متریک باشد و $A, B \subseteq X$ مجموعه‌های بسته باشند. مجموعه‌های A و B را داری ویژگی (P)گویند هرگاه به ازای $x_1, x_2 \in A_0$ و $y_1, y_2 \in B_0$ چنانچه

$$d(x_1, y_1) = d(x_2, y_2) = d(A, B),$$

آنگاه نتیجه بگیریم: $d(x_1, x_2) = d(y_1, y_2)$.

تعریف ۲.۱. فرض کنید (X, d) یک فضای متریک باشد و $A, B \subseteq X$ مجموعه‌های ناتهی بسته باشند. زوج (A, B) را دارای خاصیت (UC) گوییم هرگاه برای هر دو دنباله $\{x_n\}$ و

واژگان کلیدی. الگوریتم کلی مَن؛ مینیمم‌سازی سرتاسری؛ بهترین نقطه تقریبی؛ نگاشت غیر انبساطی..

* سخنران

$\{z_n\}$ در A و دنباله $\{y_n\}$ در B به طوری که

$$\lim_{n\to\infty} d(x_n, y_n) = d(A, B),$$
$$\lim_{n\to\infty} d(z_n, y_n) = d(A, B),$$

آنگاه نتیجه بگیریم:

$$\lim_{n\to\infty} d(x_n, z_n) = 0.$$

لم ۳.۱. [۶] فرض کنید E یک فضای باناخ محدب یکنواخت و A و B زیرمجموعه‌های ناتهی و بسته از آن باشند به طوری که A محدب باشد. همچنین فرض کنید $T : A \to B$ یک غیرخودنگاشت، غیر انبساطی باشد. آنگاه T دارای ویژگی تقریبی است.

اکنون درفضاهای هیلبرت برای نگاشت‌های غیرخودریخت غیرانبساطی دو الگوریتم ترکیبی معرفی می‌کنیم سپس الگوریتم مَن را بیان خواهیم کرد:

الگوریتم ترکیبی ۱.

گام اول: $x_0 \in A_0$ را به طور دلخواه اختیار می‌کنیم.

گام دوم: برای هر $n \in \mathbb{N} \cup \{0\}$ قرار دهید:

$$y_n = \alpha_n x_n + (1 - \alpha_n) P_A(T(x_n)),$$

که در آن وجود دارد $a \in [0, 1)$ به طوری که $\alpha_n \in [0, a]$.

گام سوم: قرار دهید:

$$C_n = \{z \in A_0 : \|y_n - z\| \leq \|x_n - z\|\}.$$

گام چهارم: قرار دهید:

$$Q_n = \{z \in A_0 : \langle x_n - z, x_n - x_0 \rangle \leq 0\}.$$

گام پنجم: قرار دهید:

$$x_{n+1} = P_{C_n \cap Q_n}(x_0).$$

الگوریتم ترکیبی ۲.

گام اول: $x_0 \in A_0$ را به طور دلخواه اختیار می‌کنیم.

گام دوم: برای هر $n \in \mathbb{N} \cup \{0\}$ داریم:

$$y_n = \alpha_n P_B(x_n) + (1 - \alpha_n) T(x_n),$$

که در آن $\alpha_n \in [0, 1]$ و $\lim_{n\to\infty} \alpha_n = 0$.

گام سوم:

$$C_n = \{z \in A_0 : \|y_n - z\| \leq \|x_n - z\| + d(A, B)\}.$$

گام چهارم:

$$Q_n = \{z \in A_0 : \langle x_n - z, x_n - x_0 \rangle \leq 0\}.$$

گام پنجم:

$$x_{n+1} = P_{C_n \cap Q_n}(x_0).$$

الگوریتم مَن.

فرض کنید H یک فضای هیلبرت و A و B زیرمجموعه‌های ناتهی محدب و بسته از آن باشند. همچنین فرض کنید $T : A \to B$ یک نگاشت باشد به طوری که $T(A_0) \subseteq B_0$. بهترین نقاط تقریبی برای نگاشت T روی A را می‌توان با الگوریتم عمومی مَن که دارای دو گام به صورت زیر است، بدست آورد.

گام اول: برای $n = 1$ نقطه $x_1 \in A_0$ را دلخواه در نظر می‌گیریم.

گام دوم: قرار دهید:

$$x_{n+1} = P_A(\alpha_n P_B(x_n) + (1 - \alpha_n)Tx_n),$$

که در آن $\{\alpha_n\}$ دنباله‌ای در بازه $(0, 1)$ است.

قضیه ۴.۱. [۵] فرض کنید H یک فضای هیلبرت و A و B زیرمجموعه‌های ناتهی محدب و بسته از آن باشند. همچنین فرض کنید $T : A \to B$ یک نگاشت باشد به طوری که $T(A_0) \subseteq B_0$. فرض کنید $\{x_n\}$ دنباله الگوریتم ۱ باشد به طوری که $P(A^T) \neq \emptyset$. آنگاه برای هر $z \in P(A^T)$ حد $\lim_{n\to\infty} \|x_n - z\|$ وجود دارد.

۲. نتایج عددی

دراین بخش برای درک بهتر، یک مثال عددی ارائه می‌کنیم.

مثال ۱.۲. فضای هیلبرت $\mathbb{R}^2$ با نرم اقلیدسی را درنظر می‌گیریم. زیرمجموعه‌های A و B از آن را به صورت زیر تعریف می‌کنیم:

$$A = \{(x, y) \in \mathbb{R}^2 : \quad x \leq -2,\ 1 \leq y \leq 4\},$$
$$B = \{(x, y) \in \mathbb{R}^2 : \quad x \geq 2\}.$$

همچنین نگاشت $T : A \to B$ با ضابطه $T(x, y) := (-x, 2 - \cos(y - 1))$ را در نظر می‌گیریم. در ابتدا مجموعه‌های A_0 و B_0 را به دست می‌آوریم.

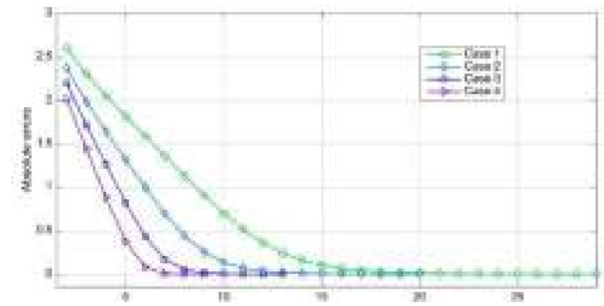
$$A_0 = \{(x, y) \in A : \quad x = -2\},$$
$$B_0 = \{(x, y) \in B : \quad x = 2,\ 1 \leq y \leq 4\}.$$

داریم:

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

واضح است که T غیر انبساطی است. همچنین $T(A_0) \subseteq B_0$.
بنابراین $T(x, y) = (2, z)$ به طوری که $1 \leq z \leq 4$، و این یعنی $T(x, y) \in B_0$.
علاوه بر این $P(A^T) = \{z\}$، به طوری که $z = (-2, 1)$.
حال قصد داریم این بهترین نقطه تقریبی را طبق الگوریتم مَن با نقطه شروع $x_1 = (-2, 4)$ به صورت تقریبی بیابیم. در واقع می‌خواهیم به صورت عددی همگرایی دنباله را مشاهده کنیم.

برای این کار دنباله α_n را در ۴ حالت مختلف بررسی کرده و نتایج عددی را در جداول چهارگانه زیر بررسی می‌کنیم. همچنین خطای همگرایی را به صورت $\|x_n - z\| < 10^{-4}$ در نظر گرفته‌ایم.



شکل ۱: خطا مطلق ۴ حالت

شکل ۱ خطای همگرایی رانشان می‌دهد. در این نمودار خطای مطلق چهارحالت در رنگ‌های مختلف دیده می‌شود.

جدول ۱: نتیجه عددی حالت ۱

تکرار	نقطه تکراری	$\|x_n - z\|$	خطای مطلق
2	$(-2, 3.5960)$	4.0681	2.5960
3	$(-2, 3.2995)$	4.0499	2.2995
4	$(-2, 3.0461)$	4.0431	2.0461
...	...	...	...
27	$(-2, 1.0002)$	4.0000	0.0002
28	$(-2, 1.0001)$	4.0000	0.0001
29	$(-2, 1.0001)$	4.0000	$8.3040e-05$

در جدول ۱ مقدار $\alpha_n = \frac{3}{5}$ است و دنباله به‌دست آمده با قرار دادن $x_1 = (-2, 4)$ در الگوریتم مَن، در جمله $n = 29$ به نقطه تقریب $x = (-2, 1)$ همگرا شده است.

در جدول ۲ مقدار $\alpha_n = \frac{n+1}{2(n+2)}$ است و دنباله به‌دست آمده با قرار دادن $x_1 = (-2, 4)$ در الگوریتم مَن، در جمله $n = 20$ به نقطه تقریب $x = (-2, 1)$ همگرا شده است. در این جدول مقدار α_n درمقایسه با حالت قبل کوچکتر شده است و سرعت همگرایی بیشتر است.

در جدول ۳ مقدار مقدار $\alpha_n = \frac{n+2}{4(n+3)}$ است و دنباله به‌دست آمده با قرار دادن $x_1 = (-2, 4)$ در الگوریتم مَن، در جمله $n = 13$ به نقطه تقریب $x = (-2, 1)$ همگرا شده است. در این جدول مقدار α_n در مقایسه باحالت قبل کوچکتر شده است و سرعت همگرایی بیشتر است.

جدول ۲: نتیجه عددی حالت ۲

خطای مطلق	$\|x_n - z\|$	نقطه تکراری	تکرار
2.3687	4.0529	$(-2, 3.3687)$	2
1.9771	4.0421	$(-2, 2.9771)$	3
1.6376	4.0405	$(-2, 2.6376)$	4
...	...	...	...
0.0004	4.0000	$(-2, 1.0004)$	18
0.0002	4.0000	$(-2, 1.0002)$	19
$8.1220e-05$	4.0000	$(-2, 1.0001)$	20

جدول ۳: نتیجه عددی حالت ۳

خطای مطلق	$\|x_n - z\|$	نقطه تکراری	تکرار
2.1920	4.0462	$(-2, 3.1920)$	2
1.7091	4.0406	$(-2, 2.7091)$	3
1.2603	4.0398	$(-2, 2.2603)$	4
...	...	...	...
0.0007	4.0000	$(-2, 1.0007)$	11
0.0002	4.0000	$(-2, 1.0002)$	12
$3.6071e-05$	4.0000	$(-2, 1.0000)$	13

جدول ۴: نتیجه عددی حالت ۴

خطای مطلق	$\|x_n - z\|$	نقطه تکراری	تکرار
2.0110	4.0425	$(-2, 3.0110)$	2
1.4387	4.0405	$(-2, 2.4387)$	3
0.8808	4.0333	$(-2, 1.8808)$	4
0.0763	4.0116	$(-2, 1.3749)$	5
1.6376	4.0007	$(-2, 1.0763)$	6
0.0046	4.0000	$(-2, 1.0046)$	7
0.0001	4.0000	$(-2, 1.0001)$	8
$2.6658e-06$	4.0000	$(-2, 1.0000)$	9

در جدول ۴ مقدار $\alpha_n = \frac{n+3}{40(n+4)}$ است و دنباله به‌دست آمده با قرار دادن $x_1 = (-2, 4)$ در الگوریتم مَن، در جمله $n = 9$ به نقطه تقریب $x = (-2, 1)$ همگرا شده است. در این جدول مقدار α_n در مقایسه باحالت قبل کوچکتر شده است و سرعت همگرایی بیشتر است.

۳. نتیجه

با توجه به جداول در می‌یابیم که هر چقدر α_n به صفر نزدیکتر باشد، سرعت همگرایی دنباله بیشتر است. مثلاً در حالت چهارم بعد از تکرار نهم به جواب $(-2, 1.0000)$ رسیده‌ایم که خیلی به جواب اصلی نزدیک است و در حالی که در حالت سوم، در تکرار سیزدهم به همین جواب رسیده‌ایم.

مراجع

1. A. Abkar and M. Gabeleh, *Best proximity points of non-self mappings*, TOP, **21** (2) (2013), 287-295.
2. M. Gabeleh, *Best proximity point theorems via proximal non-self mappings*, J. Optim. Theory Appl., **164** (164) (2015), 565–576.
3. M.Gabeleh and N. Shahzad, *Best proximity points, cyclic Kannan maps and geodesic metric spaces*, J. Fixed Point Theory Appl., **18** (2016), 167–188.
4. M.R. Haddadi, *Best proximity point iteration for nonexpansive mapping in Banach spaces*, J. Nonlinear Sci. Appl., **7** (2014), 126–130.
5. G.K. Jacob, M. Postolache, M. Marudai and V. Raja, *Norm convergence iterations for best proximity points of non-self non-expansive mappings*, Politeh. Univ. Buchar. Sci. Bull. Ser. A Appl. Math. Phys., **79** (2017), 49–56.
6. R. Suparatulatorn, W. Cholamjiak and S. Suantai, *Existence and Convergence Theorems for Global Minimization of Best Proximity Points in Hilbert Spaces*, Acta Applicandae Mathematicae., **165** (2020), 81–90.

مجموعه مقاله های

هشتمین سمینار آنالیز تابعی و کاربردهای آن

۱۶ و ۱۷ آبان ۱۴۰۳

دانشگاه آیت ا... العظمی بروجردی(ره)