دانشگاه مراغه

انجمن ریاضی ایران

بیست و هشتمین سمینار جبر ایران

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این کتاب الکترونیک بصورت رایگان در اختیار کلیه‌ی شرکت‌کنندگان در سمینار و متعاقباً سایر اشخاص حقیقی و حقوقی مرتبط قرار خواهد گرفت.

گردآوری، تدوین و صفحه‌آرایی: پروین نقی‌زاده

آدرس دبیرخانه

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پیام دبیر سمینار

بنام حضرت دوست که هر چه داریم از اوست. خداوند متعال را شاکریم که با برگزاری بیست و هشتمین سمینار جبر ایران، برگ زرین دیگری به سوابق درخشان گروه ریاضی دانشگاه مراغه افزوده شد. این گروه برگزاری بیستمین سمینار آنالیز ریاضی و کاربردهای آن (سال ۱۳۹۱)، ششمین سمینار آنالیز عددی (سال ۱۳۹۵) و نهمین سمینار هندسه و توپولوژی ایران (سال ۱۳۹۶) را در کارنامه فعالیت‌های علمی، اجتماعی ملی خود داشت که با برگزاری بیست و هشتمین سمینار جبر ایران، چهارمین تجربه خود را به ثبت رساند.

افزون بر این‌ها تعداد قابل توجه همایش‌های منطقه‌ای و کشوری نیز توسط گروه ریاضی برگزار شده است. جمله این فعالیت‌ها گویای وجود روحیه و فضای پویا و رو به رشد حاکم بر این گروه و این دانشگاه می‌باشند. هر تجربه از این قماش حاوی یافته‌ها و آموخته‌های جدید برای مجریان آن بوده است که در تجربیات آتی بکار گرفته می‌شود. بسیار شایسته است که از انجمن ریاضی ایران که باعث و بانی این سلسله سمینارهای ملی است تشکر و قدردانی کنیم و حامی استوار آن انجمن باشیم. همچنین وظیفه خود می‌دانیم از دانشگاه‌ها و مراکز برگزار کننده اولین تا بیست و هشتمین سمینار جبر ایران خاضعانه تشکر و سپاسگزاری کنیم و زحمات بی‌شائبه آن‌ها را ارج نهیم. نهایتاً، از ریاست محترم و معاونین محترم اداری و مالی، آموزشی و پژوهشی، دانشجویی و فرهنگی دانشگاه مراغه که حامی و پشتیبان اصلی سمینار بودند تقدیر و تشکر فراوان داشته باشیم.

از اساتید محترمی که به عنوان عضو کمیته علمی سمینار وقت گرانقدر خویش را صرف جلسات و همفکری‌های کمیته علمی نمودند خاضعانه تقدیر و تشکر می‌کنیم. از همکاران محترم دانشگاه مراغه و دانشجویان تحصیلات تکمیلی که در کمیته اجرائی سمینار فعالیت و همکاری نمودند و دبیرخانه سمینار را مساعدت و یاری کردند مجدانه قدردانی می‌کنم و توفیقات روزافزون برای همه این عزیزان آرزو دارم. در پایان، از نماینده محترم انجمن ریاضی ایران (جناب آقای دکتر یوسف زمانی) و دبیر علمی سمینار جناب آقای دکتر رحیم رحمتی اصغر و دبیر اجرایی سمینار جناب آقای دکتر فیروز پاشائی تشکر و قدردانی می‌نمایم.

همچنین، از همه یاری کنندگان سمینار بویژه، اعضای محترم کمیته راهبردی سپاسگزارم که هر یک از این بزرگواران نقش مهمی در برنامه‌ریزی و اجرای سمینار داشته‌اند. تشکر ویژه از سرکار خانم پروین نقی‌زاده و سرکار خانم ربابه عباسی به خاطر همکاری دراز مدت (حدود ۳ ماه) با دبیرخانه سمینار را وظیفه خود می‌دانم.

دکتر لیلا شهباز

پوستر سمینار

28th Iranian Algebra Seminar

بیست و هشتمین

سمینار جبر ایران

محورهای سمینار

جبر جامع
نظریه رسته
ساختارهای جبری مرتب
جبر جابجایی و جبر ناجابجایی
نظریه گروه‌ها
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هندسه جبری و نظریه جبری اعداد
جبر لی، جبر خطی و کاربردهای آن‌ها
نظریه کدهای جبری
آموزش ریاضی (با محوریت موضوعات جبری)

مهلت ثبت‌نام اولیه و ارسال اصل مقالات:
۱۵ اردیبهشت ۱۴۰۳
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۳۰ فروردین ۱۴۰۳
ثبت‌نام نهایی و پرداخت هزینه‌ها:
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۲۱-۲۰ تیر ماه ۱۴۰۳
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مراغه، میدان مادر، بلوار دکتر قنادی، دانشگاه مراغه، دانشکده علوم پایه، گروه ریاضی، دبیرخانه بیست و هشتمین سمینار جبر ایران

برگزارکنندگان و حامیان سمینار

برگزارکنندگان سمینار

حامیان سمینار

اهداف و محورهای سمینار

دانش جبر از نکته نظر تاریخی یکی از قدیمی‌ترین شاخه‌های ریاضیات است. علم جبر به عنوان شاخه‌ای بنیادی در ریاضیات، پس از نگارش کتاب "الجبر و المقابله" محمد بن موسی خوارزمی ریاضیدان و اخترشناس شهیر ایرانی قرن نهم میلادی انسجام مشهود و ملموس یافته است. بر اساس رده‌بندی صورت گرفته توسط انجمن ریاضی ایالات متحده، حدود ۲۲ درصد از ریاضیات قرن ۲۱ مبنای جبری دارند که این حقیقت گستره وسیع علم جبر و کاربردهای بسیار متنوع آن را نمایان می‌سازد. پژوهشگران شاخه جبر با طیف گسترده‌ای از ریاضیات سر و کار دارند و این امر ضرورت برگزاری کارگاه‌ها، سمینارها و حتی کنفرانس‌های متنوع در موضوع‌های جبری را دو چندان کرده است. سلسله سمینارهای جبر ایران فرصت بسیار مغتنمی است که فرصت تبادل نظرات و پژوهش‌های انجام شده بین پژوهشگران را فراهم می‌سازد و موجبات تشویق و ترغیب جوانان به این شاخه از ریاضیات می‌گردد.

محورهای این سمینار عبارتند از:

- جبر جامع
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- نظریه گروه‌ها
- جبر محاسباتی و ترکیبیات جبری
- هندسه جبری و نظریه جبری اعداد
- جبر لی، جبر خطی و کاربردهای آن‌ها
- نظریه کدهای جبری
- آموزش ریاضی (با محوریت موضوعات جبری)

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- دکتر رسول دانشفراز (معاون دانشجویی دانشگاه مراغه)
- دکتر عباس رضایی (معاون فرهنگی دانشگاه مراغه)
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- حجۀ الاسلام والمسلمین جناب آقای احمد زمانی (مسئول نهاد نمایندگی مقام معظم رهبری در دانشگاه مراغه)
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- دکتر رحیم رحمتی اصغر (دبیر علمی سمینار)
- دکتر لیلا شهباز (دبیر سمینار)

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- پروفسور مژگان محمودی (استاد دانشگاه شهید بهشتی)
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- دکتر ابوالفضل تاری‌زاده
- دکتر محمد شهریاری
- دکتر خدیجه کشوردوست
- دکتر مهدی جهانگیری

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- مهندس ربابه عباسی (کارمند دانشگاه مراغه)

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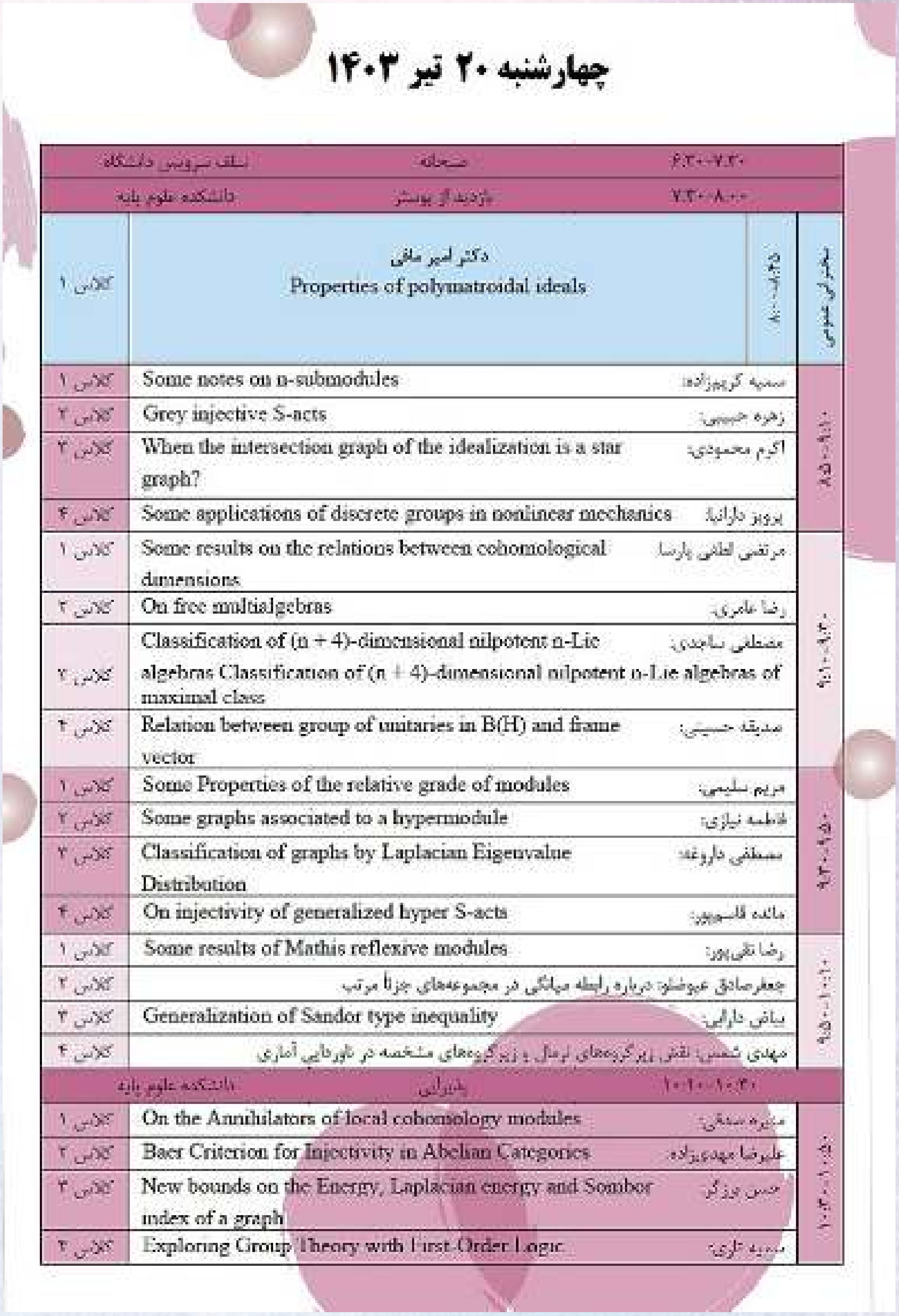

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تجلیل از دو استاد پیشکسوت در سمینار

در بیست و هشتمین سمینار جبر ایران از دو استاد برجسته علم جبر؛ پروفسور محمدمهدی ابراهیمی و پروفسور رحیم زارع نهندی تجلیل شد.

پروفسور محمد مهدی ابراهیمی استاد تمام دانشگاه شهید بهشتی، پژوهشگر، مدرس، مؤلف و مترجم ایرانی در حوزهٔ ریاضیات می باشد. وی حدود ۳۰ عنوان کتاب تالیف و ترجمه دارند و مقالات متعددی در زمینهٔ جبر و ریاضی در کنفرانس‌ها و سمینارهای داخلی و دانشگاه‌های داخل و خارج کشور نوشته و ارائه داده‌اند.

پروفسور رحیم زارع نهندی، استاد تمام دانشگاه تهران، پژوهشگر، مدرس، مؤلف و مترجم، در زمینه‌های علمی و تحقیقاتی هندسه جبری با تألیفات مختلف می‌باشد. ایشان در سال ۱۹۸۲ میلادی (۱۳۶۱ شمسی) از دانشگاه مینه سوتای آمریکا در رشته هندسه جبری دریافت کردند. همکاری‌های دلسوزانه ایشان با سمینارها و کنفرانس‌های علمی کشور زبانزد جامعه ریاضی ایران است.

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    - Title of the Talk: How Degrees of Generators Control Resolutions
2. **Prof. Shahriar Shahriari**
    - William Polk Russell Professor of Mathematics and Statistics, Pomona College, Claremont, California
    - Title of the Talk: The Combinatorics of Vector Spaces Over Finite Fields
3. **Prof. Srikanth B. Iyengar**
    - Department of Mathematics, University of Utah, USA
    - Title of the Talk: Non-Existence of Ulrich Modules Over Cohen-Macaulay Local Rings
4. **Prof. Amir Mafi**
    - University of Kurdestan, Iran
    - Title of the Talk: Properties of Polymatroidal Ideals
5. **Prof. Amir Daneshgar**
    - Sharif University of Technology, Iran
    - Title of the Talk: Categories for the Working Computer Scientist
6. **Prof. Victoria Gould**
    - University of York, Canada
    - Title of the Talk: Coherency for Monoids and Purity for their Acts
7. **Prof. Alireza Abdollahi**
    - University of Isfahan, Iran
    - Title of the Talk: Applications of Representation Theory of Groups to Find Good Bbounds on Permuation Codes

Talk of Prof. S. Hamid Hassanzadeh

How Degrees of Generators Control Resolutions

Abstract: Syzygy is the most computational and informative word in commutative algebra. Sygygies determine the vector fields that leave a hypersurface invariant. Successive syzygies determine the free resolution of an ideal. Syzygies are full of geometry. In this talk, we show how knowing the degree of the generators of an ideal one can find bounds for the size of the syzygies and their degrees. We introduce the Boij-Soderberg theory on the shape of the cone of Betti tables and show how one may apply these results to study the space of vector fields.

This talk is based on a joint work with W. Alto and A. Simis

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Talk of Prof. Shahriar Shahriari

The Combinatorics of Vector Spaces Over Finite Fields

Abstract: Let V be a finite dimensional vector space over a finite field and consider the poset of subspaces of V ordered by inclusion. The combinatorial properties of this partially ordered set often resemble those of the Boolean lattices, the subsets of a finite set ordered by inclusion. Often the case of subsets is easier to handle, but, there are situations where a combinatorial question is easier to answer for subspaces. In this talk, we discuss two combinatorial problems in the poset of subspaces: Can you partition the set of subspaces of V into chains of almost uniform size? Given a small poset P , what is the largest number of subspaces of V that do not contain a copy of P?

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Talk of Prof. Srikanth B. Iyengar

Non-Existence of Ulrich Modules Over Cohen-Macaulay Local Rings

Abstract: Over a Cohen-Macaulay local ring, the minimal number of generators of a maximal Cohen-Macaulay module is bounded above by its multiplicity. In 1984 Ulrich asked whether there always exist modules for which equality holds; such modules are known nowadays as Ulrich modules. Over the years, Ulrich modules have been found over various classes of rings, but a few months ago Ma, Walker, Zhuang and I discovered two dimensional Cohen-Macaulay local rings that have no Ulrich modules. Some of these rings are Gorenstein normal domains; others are even complete intersection domains, though not normal. In my talk, I will give an overview of this topic, culminating in the recent work, which is available on the arxiv: https://arxiv.org/abs/2403.15566

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Talk of Prof. Amir Mafi

Properties of Polymatroidal Ideals

Abstract: Consider a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K and let I be apolymatroidal ideal of degree d. In this presentation, we delve into fundamental concept related to polymatroidal ideals within the realm of combinatorial combinatorial commutative algebra. Our focus includes properties of astab (associated prime stability) and dstab (depth stability) of polymatroidal ideals. Additionally, we explore Cohen-Macaulay and sequentially Cohen-Macaulay matroidal ideals.

In the end we give some examples such that the astab and dstab are unrelated and also we give some known problems and conjectures about this subject.

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Talk of Prof. Amir Daneshgar

Categories for the Working Computer Scientist

Abstract: The first part of this talk is dedicated to usefulness of category theory within and outside mathematics, reviewing some basic concepts and def- initions in the field. The main objective for the rest of the talk is to try to convince the audience that the statement “System Theory is a dual to Algebra” is true in a very broad and general sense. This point of view will definitely give rise to categorical presentations of many different system theoretic concepts within and outside mathematics in terms of Co-algebras and other Co-structures.

Based on some well-known and ongoing contributions, we already know that a large class of Dynamical Systems initiated in physics have already played the role of motivating examples for a search for generalized alge- braic/categorical structures. On the other hand, it is instructive to note that the concept of an algorithm may be described as a discrete dynamical system satisfying some finite presentability conditions. This shows that a large part of theoretical computer science such as Theory of Computation as the core of the field containing Type Theory and Theory of Formal Lan- guages and Automata as well as theory of Intelligent and Complex Systems and formal theory of Networks and Distributed Systems have categorical interpretations.

In the second part of this talk and based on well known contributions in the field, I will mention some categorical approaches and their conse- quences in system theory to serve as an introduction to the subject for the algebraists having some interest in Computer Science. The last couple of slides will be on “What are the main theoretical problems?” from this categorical point of view. I hope to be able to show that the structure of a Closed Monoidal Category and a need for some new Fixpoint-Limits play a central role.

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Talk of Prof. Victoria Gould

Coherency for Monoids and Purity for their Acts

Abstract: A monoid S may be represented via mappings of sets or, equivalently and more concretely, by S-acts. A right S-act is a set A together with a map $A \times S \to A$ where $(a, s) \mapsto as$, such that for all $a \in A$ and $s, t \in S$ we have $a1 = a$ and $(as)t = a(st)$. The monoid S itself and any right ideal of S, are right S-acts under the monoid multiplication. Cyclic right S-acts are of the form S/ρ where ρ is a right congruence. We say that S is right coherent if every finitely generated subact of every finitely presented right S-act is finitely presented and weakly right coherent if every finitely generated right ideal of S has a finite presentation.

Coherency properties are fascinating and somewhat elusive. They arise naturally from several directions, including model theory and ring theory. The corresponding notions of right coherency and weak right coherency for a ring R (where, of course, S-acts are replaced by R-modules) coincide, essentially because every cyclic R-module is determined by a right ideal. This is not the case for monoids and we must develop new strategies.

This talk will examine coherency notions for monoids from several angles, including the connection with other finitary conditions such as being right noetherian, axiomatisability of classes of algebraically closed right S-acts and so-called purity properties, which are related to injectivity of right S-acts. No prior knowledge will be assumed and technical details will be kept to a minimum.

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سخنرانی پروفسور علیرضا عبداللهی

کاربردهایی از نظریه نمایش گروه‌های متناهی برای پیدا کردن کران روی کدهای جایگشتی

چکیده: در این سخنرانی به برخی کاربردهای نظریه نمایش گروه‌های متناهی برای پیدا کردن کران‌هایی روی تعداد عناصر کدهای جایگشتی می پردازیم. یک کد جایگشتی یک زیرمجموعه غیر تهی از یک گروه متقارن با درجه‌ای مشخص است. فاصله‌ای که معمولاً برای کار کردن با این کدهای جایگشتی استفاده می شود، فاصله‌ای به نام کندال است. این فاصله به این صورت بین دو جایگشت σ و θ تعریف می شود که برابر با کمترین تعداد ترانهش‌هایی به صورت $(i, i+1)$ که برای نوشتن جایگشت $\sigma\theta^{-1}$ به صورت حاصلضربی از آنها نیاز است. با استفاده یک رابطه در جبر گروهی مختلط گروه متقارن و اعمال نمایش‌هایی خاص از گروه متقارن روی این رابطه و تبدیل آن به یک تساوی در ماتریس ها، شرایطی را بدست می آوریم که می توانیم از آنها برای بدست آوردن کران روی کدها استفاده کنیم، که در این سخنرانی به آنها می پردازیم.

این کار در گروه تحقیقاتی کاگ (کد، اسکیم، گروه) دانشگاه اصفهان انجام شده است و تیم تحقیقاتی مشتکل از اعضای هیات علمی: خانم دکتر مریم خاتمی، آقای دکتر باقریان، آقای دکتر فرزاد پرورش، آقای دکتر رضا سبحانی و اینجانب به همراه خانم دکتر فاطمه جعفری (پسا دکتری) انجام شده است. فعالیت‌های این گروه تحقیقاتی به این آدرس قابل مشاهده است: https://csg.ui.ac.ir

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A Note on Codegrees and Taketa's Inequality

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Abstract

Let G be a finite group and $\mathrm{cd}(G)$ will be the set of the degrees of the complex irreducible characters of G. Also let $\mathrm{cod}(G)$ be the set of codegrees of the irreducible characters of G. It has been conjectured by Isaacs and Seitz, that if G is solvable, then $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$, where $\mathrm{dl}(G)$ is the derived length of G. This inequality is named after Taketa. In this note, we show that $\mathrm{dl}(G) \leq |\mathrm{cod}(G)|$ in some cases and we conjecture that this inequality holds if G is a finite solvable group.

Keywords: Codegree, Character degree, Taketa's inequality

Mathematics Subject Classification [2010]: Primary: 20C20, Secondary: 20B15

1 Introduction and Preliminaries

Throughout this paper, G will be a finite group and $\mathrm{cd}(G)$ will be the set of the degrees of the complex irreducible characters of G. The number $\mathrm{cod}(\chi) = |G : \mathrm{Ker}(\chi)|/\chi(1)$ is called the codegree of χ. Also the set of codegrees of the irreducible characters of G is denoted by $\mathrm{cod}(G)$. This definition for codegrees first appeared in [10], where the authors use a graph-theoretic approach to compare the structure of a group with its set of codegrees. We follow [6] for group theoretic terminologies and notations not defined here.

Taketa showed that the derived length $\mathrm{dl}(G)$ of the group G is bounded by the number $|\mathrm{cd}(G)|$, i.e., $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$ (see [6, Corollary 5.13]). The Isaacs-Seitz conjecture asserts that the Taketa's inequality holds for solvable groups. This conjecture is also known as the Taketa problem or Taketa's inequality. There has been much work done regarding this conjecture in the last two decades. Moreover, T. Berger in [4] shows that this conjecture is true for groups of odd order. For more results we refer the reader to [2, 8, 13].

By [1, Lemma 3.1, Theorem 3.4 and Theorem 3.5], we know that if $|\mathrm{cod}(G)| \leq 3$, then $\mathrm{dl}(G) \leq |\mathrm{cod}(G)|$. In this paper we show that if G is a nilpotent group then under some conditions, $\mathrm{dl}(G) \leq |\mathrm{cod}(G)|$. Moreover, we show that if G has square free order then $\mathrm{dl}(G) \leq |\mathrm{cod}(G)|$. Also we show that if $\mathrm{cod}(G)$ consists of power of p then $\mathrm{dl}(G) < |\mathrm{cod}(G)|$. Thus we may propose the the following conjecture, inspired by Taketa's inequality:

Conjecture: Let G be a finite solvable group. Then $\mathrm{dl}(G) \leq |\mathrm{cod}(G)|$.

Also we need the following results for our proofs.

Proposition 1.1. [3, Theorem 141.2] *Let G be a nonabelian p-group all of whose nonabelian maximal subgroups have a cyclic center. Then one of the following holds:*

---

[*] Speaker.

(a) *G is minimal nonabelian.*

(b) *G is a 2-group of maximal class.*

(c) *$G = LZ(G)$, where $L \cong D_8, Q_8$ or M_{p^n}, $n \geq 3$ (if $p = 2$ then $n \geq 4$), and either $Z(G)$ is cyclic with $Z(G) > Z(L)$ or $G = L \times \langle t \rangle$, where t is an element of order p.*

(d) *G has exactly one abelian maximal subgroup A of rank≥ 2, $Z(G)$ is cyclic, and G possesses exactly one normal abelian subgroup of type (p, p)(contained in A).*

Proposition 1.2. [1, Lemma 3.1] *Let G be a group. Then $|\mathrm{cod}(G)| = 2$ if and only if G is an elementary abelian group*

2 Main Results

In this section, we investigate Taketa inequality for some groups. Also we prove Taketa inequality for p-groups with the following assumption.

Assumption A. All maximal subgroup of each p-group have a faithful character.

We prove the following lemma which is crucial for some of our results.

Lemma 2.1. *Let G be a p-group and $|G| = p^n$, where p is prime and $n \geq 1$. Then by considering assumption A we have* $\mathrm{dl}(G) \leq |\mathrm{cod}(G)|$.

Proof. If G is abelian group then clearly the assertion holds. Thus we may suppose that G is not abelian group. Also if $|\mathrm{cod}(G)| = 2$ then by Proposition 1.2, G is an elementary abelian group and so $\mathrm{dl}(G) = 1 \leq |\mathrm{cod}(G)|$. Therefore in the following we assume that $|\mathrm{cod}(G)| \geq 3$. Let H be a maximal subgroup of index p in G. Thus H is normal in G and by our assumption $Z(H)$ is cyclic. If H is an abelian group then $\mathrm{dl}(G) \leq \mathrm{dl}(G/H) + \mathrm{dl}(H) = 2$ and so $\mathrm{dl}(G) \leq |\mathrm{cod}(G)|$. Thus we may suppose that H is nonabelian group. So G is one of the group in Proposition 1.1. For Case (a) we see that $G^{'}$ and $G/G^{'}$ are abelian. Hence $\mathrm{dl}(G) \leq \mathrm{dl}(G/G^{'}) + \mathrm{dl}(G^{'}) = 2$ and the assertion holds. For Case (b), by [5, P. 194, Theorem 4.5], G is isomorphic to dihedral, generalized quaternion or semidihedral group. Also by [5, P.191, Theorem 4.3], $\mathrm{dl}(G) = 2$ and hence $\mathrm{dl}(G) \leq \mathrm{cod}(G)$ as wanted. For Case (c), we know that M_{p^n} has cyclic subgroup of index p. Therefore it is easy to see that $\mathrm{dl}(M_{p^n}) \leq 2$. Also clearly $\mathrm{dl}(Q_8) = \mathrm{dl}(D_8) = 2$. Now if $G = L \times \langle t \rangle$, where t is an element of order p then $\mathrm{dl}(G) \leq \mathrm{dl}(G/L) + \mathrm{dl}(L) \leq 3$. Therefore $\mathrm{dl}(G) \leq \mathrm{cod}(G)$. Hence we may assume that $G = LZ(G)$, where $Z(G)$ is cyclic with $Z(G) > Z(L)$. We have $G/Z(G) \cong L/L \cap Z(G)$. If $L \cong Q_8$ or D_8 then $\mathrm{dl}(G) \leq \mathrm{dl}(G/Z(G)) + \mathrm{dl}(Z(G)) \leq 2$ and the assertion holds. Also if $L \cong M_{p^n}$ then since the center of M_{p^n} has order p^{n-2} we have $\mathrm{dl}(M_{p^n}/Z(M_{p^n})) = 1$. Therefore $\mathrm{dl}(L/L \cap Z(G)/Z(L)/L \cap Z(G)) = 1$ and so $\mathrm{dl}(L/L \cap Z(G)) \leq 2$. Now $\mathrm{dl}(G) \leq \mathrm{dl}(G/Z(G)) + \mathrm{dl}(Z(G)) = \mathrm{dl}(L/L \cap Z(G)) + \mathrm{dl}(Z(G)) \leq 3$, as wanted. Thus the assertion holds. Finally for Case (d), G has a unique abelian maximal subgroup A. Also by our assumption, we see that A has a cyclic center and by the proof of Proposition 1.1, G is a 2-group of maximal class. Now by Case (b) the proof is complete.

□

By the above lemma we show that $\mathrm{dl}(G) \leq |\mathrm{cod}(G)|$, where G is a nilpotent group.

Theorem 2.2. *Let G be a nilpotent group. Then by considering assumption A,* $\mathrm{dl}(G) \leq |\mathrm{cod}(G)|$.

Proof. If G is abelian then $\mathrm{dl}(G) = 1 \leq |\mathrm{cod}(G)|$. Thus we may suppose that G is not abelian. Since G is nilpotent, it follows that G is the direct product of its Sylow p-subgroups. Let $G = P_1 \times P_2 \times \cdots \times P_n$, where $P_i \in Syl_{p_i}(G)$ and $(|P_i|, |P_j|) = 1$ for each $1 \leq i, j \leq n$. By [12, Corollary 2.3] we know that $\mathrm{cod}(G) = \mathrm{cod}(P_1) \times \cdots \times \mathrm{cod}(P_n)$. Also we know that $\mathrm{dl}(G) \leq \max\{\mathrm{dl}(P_i) \mid 1 \leq i \leq n\}$. Thus $\mathrm{dl}(G) \leq \mathrm{dl}(P_j)$ for some Sylow P_j, where $1 \leq j \leq n$. Now by Lemma 2.1, $\mathrm{dl}(G) \leq \mathrm{dl}(P_i) \leq |\mathrm{cod}(P_j)| \leq |\mathrm{cod}(G)|$. □

Theorem 2.3. *Let G be a finite group and p be a fixed prime. If* $\mathrm{cod}(G)$ *consists of power of p then by considering assumption A,* $\mathrm{dl}(G) < |\mathrm{cod}(G)|$.

We recall that a positive integer n is square-free if and only if in the prime factorization of n, no prime factor occurs with an exponent larger than one.

Theorem 2.4. *Let G be a square-free order. Then* $\mathrm{dl}(G) \leq |\mathrm{cod}(G)|$.

Proof. Since G is square-free order, it implies that all Sylow subgroups of G are cyclic. Thus $G^{'}$ is cyclic group (see [7, Theorem 5.16]). Now by $\mathrm{dl}(G) \leq \mathrm{dl}(G/G^{'}) + \mathrm{dl}(G^{'}) = 2 \leq |\mathrm{cod}(G)|$. □

The proofs of the following results are the same as the proofs of Proposition 2.7, Theorem 2.8 and Corollary 2.9 of [13].

Lemma 2.5. *Suppose that C is a class of finite solvable groups which is closed with respect to taking quotient. Suppose there exists a group in C, say G, such that* $\mathrm{dl}(G) \nleq |\mathrm{cod}(G)|$ *and also G has the smallest possible order. Then G has a unique minimal normal subgroup, say N, such that* $\mathrm{cod}(G) = \mathrm{cod}(G/N)$.

Proof. If G is abelian group then $G \cong \mathbb{Z}_p$ and so $\mathrm{dl}(G) \leq |\mathrm{cod}(G)|$, a contradiction. Thus G is nonabelian solvable group and G has minimal normal subgroup. Let M and N be two minimal normal subgroups of G. Thus $M \cap N = 1$ and so G is isomorphic to a subgroup of $G/M \times G/N$. By our assumption we see that $\mathrm{dl}(G) \leq \max\{\mathrm{dl}(G/M), \mathrm{dl}(G/N)\} \leq \max\{|\mathrm{cod}(G/M)|, |\mathrm{cod}(G/N)|\} \leq |\mathrm{cod}(G)|$, a contradiction. Thus G has a unique minimal normal subgroup, say N. Since G is solvable, it follows that N is abelian. Thus $\mathrm{dl}(G) \leq \mathrm{dl}(N) + \mathrm{dl}(G/N) = 1 + \mathrm{dl}(G/N) \leq 1 + |\mathrm{cod}(G/N)| \leq 1 + |\mathrm{cod}(G)| \leq \mathrm{dl}(G)$. Thus $|\mathrm{cod}(G/N)| = |\mathrm{cod}(G)|$ and so $\mathrm{cod}(G) = \mathrm{cod}(G/N)$, as wanted. □

Theorem 2.6. *Suppose that G is a solvable group and for all $\chi, \psi \in \mathrm{Irr}(G)$, $\ker(\chi) = \ker(\psi)$ if $\chi(1) = \psi(1) > 1$. Then* $\mathrm{dl}(G) \leq |\mathrm{cod}(G)|$.

Proof. Suppose that G is a minimal counter example to this theorem. By Lemma 2.5, G has a unique minimal normal subgroup M and $\mathrm{cod}(G/M) = \mathrm{cod}(G)$. If G is abelian then $\mathrm{dl}(G) \leq |\mathrm{cod}(G)|$, as wanted. Thus $G^{'} \neq 1$ and so $M \subseteq G^{'}$. Therefore $\mathrm{cod}(G \mid M) \subseteq \mathrm{cod}(G \mid G^{'}) \subseteq \mathrm{cod}(G) - 1$. Let $k \in \mathrm{cod}(G \mid M)$ such that $k \neq 1$. Thus there exists an irreducible character χ of G such that $\chi(1) = k$ and $M \nsubseteq \ker(\chi)$. On the other hand, $k \in \mathrm{cod}(G) = \mathrm{cod}(G/M)$ and so there exists an irreducible character ψ of G such that $\psi(1) = k$ and $M \subseteq \ker(\psi)$. By hypothesis $\ker(\chi) = \ker(\psi)$, a contradiction. □

By using the proof of the Theorem 2.4 we have the following result.

Corollary 2.7. *Suppose that G is a solvable group in which distinct nonlinear irreducible characters have distinct* $\mathrm{cod}(G)$*. Then* $\mathrm{dl}(G) \leq |\mathrm{cod}(G)|$.

Finally we guess that if G is a finite p-group. Then witout considering Assumption A, Lemma 2.1 holds. So we may propose the following open question.

Problem 1. Suppose that G is a finite p-group. Could we say $\mathrm{dl}(G) \leq |\mathrm{cod}(G)|$?

3 Conclusion

In this note, we show that $\mathrm{dl}(G) \leq |\mathrm{cod}(G)|$ in some cases and we conjecture that this inequality holds if G is a finite solvable group.

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Trace and Linked Ideals

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Abstract

In this talk we consider two various kinds of ideals: trace ideals and linked ideals and study their connections. We show that some linked ideals are trace but the converse does not hold any more, although, in some special cases it does.

Keywords: Trace ideals, Linked ideals, Reflexive modules.

Mathematics Subject Classification [2010]: Primary: 13C13,13C40, 13E05.

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1 Introduction and Preliminaries

Let R be a non-trivial commutative Noetherian ring and M and N be two R-modules. The trace of M in N is defined to be

$$Tr_M(N) := \sum_{\alpha \in Hom_R(M,N)} Im(\alpha).$$

The R- module X is said to be a trace module if it there are R- modules M and N such that $X = Tr_M(N)$. Also, the ideal $\mathfrak{a}$ is called a trace ideal if it is the trace of an R- module in R, i.e. there exists an R-module M such that $\mathfrak{a} = Tr_M(R)$. Trace ideals is a main object of this paper.

Trace modules have long been useful technical tools in commutative algebra. For instance, there are some characterizations of good classes of rings, such as semisimple, Gorenstein and regular rings in terms of trace ideals. For example, it is proved that R is semisimple if and only if every R-module is trace in its injective envelope. Also, the trace of a module implies some properties of the module. For instance M is projective if and only if its trace ideal is idempotent and recently, H. Lindo discussed the role of trace ideal of a module in studying the center of its endomorphism ring, see [6]

Although, the theory of trace modules is a useful technical tool in commutative algebra, but recently it has attracted new attentions as an interesting object in its own right, see [5], [6], [2] and [3].

The other main object of this paper is *linked ideals*. Linkage theory is an important topic in commutative algebra and algebraic geometry. It refers to Halphen (1870) and M. Noether (1882) who worked to classify space curves. Following Peskin and Szpiro, two proper ideals $\mathfrak{a}$ and $\mathfrak{b}$ in a Cohen-Macaulay local ring R is said to be linked if there is a regular sequence $\underline{x}$ in their intersection such that $\mathfrak{a} = (\underline{x}) :_R \mathfrak{b}$ and $\mathfrak{b} = (\underline{x}) :_R \mathfrak{a}$.

In [4], the authors introduced the concept of linkage of ideals over a module and studied some of its basic properties. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two non-zero ideals of R and M denotes a non-zero finitely generated R-module. Assume that $\mathfrak{a}M \neq M \neq \mathfrak{b}M$ and let $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ be an ideal generating by an M-regular sequence. Then the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be linked by I over M, denoted by

$\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$, if $\mathfrak{b}M = IM :_M \mathfrak{a}$ and $\mathfrak{a}M = IM :_M \mathfrak{b}$. This concept is the classical concept of linkage of ideals introduced by Peskin and Szpiro, where $M = R$.

In this paper we consider trace and linked ideals and study their connections. More precisely we show that the ideals which are linked by zero ideal are trace and the converse holds in some cases.

2 Main Results

We keep the notations and settings in the introduction throughout the paper.

Lemma 2.1 ([5, Lemma 2.5]). *Let M be a cyclic R-module. Then $Tr_M(R) = Ann_R(Ann_R(M))$.*

Proposition 2.2. *Assume that $\mathfrak{a}$ is an ideal which is linked by zero ideal on R. Then $\mathfrak{a}$ is a trace ideal.*

Proof. Let $x_1, \ldots, x_n \in R$ such that $\mathfrak{a} = (x_1, \ldots, x_n)$. Then, by the above lemma,

$$\begin{aligned}
Tr_{\mathfrak{a}}(R) &= \sum_{\alpha \in Hom_R(\mathfrak{a},R)} \alpha(\mathfrak{a}) \\
&= \sum_{\alpha \in Hom_R(\mathfrak{a},R)} \sum_{i=1}^{i=n} \alpha(x_iR) \\
&\subseteq \sum_{i=1}^{n} \sum_{\alpha \in Hom_R(x_iR,R)} \\
&= \sum_{i=1}^{n} Tr_{x_iR}(R) \\
&= \sum_{i=1}^{n} Ann_R(Ann_R(x_iR)) \\
&\subseteq Ann_R(Ann_R(\mathfrak{a})).
\end{aligned}$$

By the assumption there exists ideal $\mathfrak{b}$ such that $\mathfrak{a} = Ann_R(\mathfrak{b})$ and $\mathfrak{b} = Ann_R(\mathfrak{a})$, in other words $\mathfrak{a} = Ann_R(Ann_R(\mathfrak{a}))$. So, by the above equation

$$\mathfrak{a} \subseteq Tr_{\mathfrak{a}}(R) \subseteq Ann_R(Ann_R(\mathfrak{a})) = \mathfrak{a}.$$

This implies taht $\mathfrak{a} = Tr_{\mathfrak{a}}(R)$ is a trace ideal. ☐

The converse of the above statement does not hold any more, as we show in the next example.

Lemma 2.3 ([7, Corollary 2.8]). *A radical ideal $\mathfrak{a}$ is linked if and only if $\mathfrak{a} = \bigcap_{\mathfrak{p} \in \Lambda} \mathfrak{p}$ for some R-regular sequence $\mathfrak{x}$ and some $\Lambda \subseteq Ass_R(\frac{R}{\mathfrak{x}})$.*

Example 2.4. Let $R := \frac{\mathbb{C}[X,Y]}{(X^2-Y^2)}$ and $\mathfrak{a} := (x, y)$. Then by [2, Example 6.10], $\mathfrak{a}$ is a trace ideal, while the above lemma shows it is not linked by zero ideal on R.

We recall that an R-module M is said to be reflexive if the natural homomorphism $\alpha_M : M \to Hom_R(Hom_R(M, R), R)$, where $\alpha_M(m)(\Phi) = \Phi(m)$, is an isomorphism. In the next proposition we show, in spite of the above example, some trace ideals are linked.

Lemma 2.5 ([6, Proposition 2.8]). *Let M be a finitely generated reflexive R-module. Then $Ann_R(Tr_M(R)) = Ann_R(M)$.*

Proposition 2.6. *Assume that M is a reflexive cyclic R-module. Then*

(i) $Tr_M(R)$ is linked by zero ideal on R.

(ii) *For any finitely generated projective R-module P of positive rank $Tr_{M\otimes_R P}(R)$ is linked by zero ideal on R.*

Proof. (i) Since M is cyclic, by 2.1, $Tr_M(R) = Ann_R(Ann_R(M))$ and by 2.5 $Tr_M(R) = Ann_R(Ann_R(Tr_M(R)))$. This proves the claim.

(ii) By [1, Corollary 3.3], $Tr_{M\otimes_R P}(R) = Tr_M(R)$ and the result follows by part (i).

□

3 Conclusion

Trace and linked ideals are two various kind of ideals and they have many applications in commutative algebra and it is worth to find their relations. In this paper we show the ideals which are linked by zero ideal are trace and the converse holds in some cases.

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Rate of Powers of Maximal Ideals

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Abstract

Let R be a standard graded k-algebra with maximal graded ideal $\mathfrak{m}$. The rate of a graded R-module M denoted by $\mathrm{rate}_R(M)$, is a measure of the growth of the shifts in the minimal graded free resolution of M. The notion of the rate for an algebra first introduced by Backelin. The Backelin rate of the algebra R denoted by $\mathrm{Rate}(R)$ and it coincides with the rate of $\mathfrak{m}(1)$, the graded maximal ideal shifted by 1.

We study the rate of powers of the maximal ideal of R. If R is not artinian, we show that $\mathrm{rate}(m^s(s)) \leq \mathrm{Rate}(R) \leq 2\,\mathrm{rate}(m^s(s))$ for $s \gg 0$. We can consider $m^s(s)$ as a graded module over the Veronese subring $R^{(c)}$. It is proved that $\mathrm{rate}_{R^{(c)}}\,\mathfrak{m}^s(s) \leq \lceil \mathrm{Rate}(R)/c \rceil$ for all $c \geq 1$.

Keywords: Regularity, Rate, Minimal free resolutions, Koszul algebras

Mathematics Subject Classification [2010]: 13D02 (primary), 16W50 , 13D07 (secondary), 16W70, 16S37.

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1 Introduction and Preliminaries

Throughout k denotes a field and $R = \oplus_{i\geq 0} R_i$ is a commutative standard graded algebra k. We denote by $\mathfrak{m}$ the homogeneous maximal ideal of R. Let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a finitely generated graded R-module.

There are several invariants that we associate to M. For each $i \geq 0$, we set

$$t_i^R(M) := \max\left\{j : \mathrm{Tor}_i^R(M,\mathsf{k})_j\right\}$$

provided that $\mathrm{Tor}_i^R(M,\mathsf{k}) \neq 0$, otherwise we set $t_i^R(M) = -\infty$. Indeed $t_i^R(M)$ is the maximum degree of a minimal generators of the i-th syzygy of M.

The Castelnuovo-Mumford regularity of M is defined by:

$$\mathrm{reg}_R(M) := \sup\left\{t_i^R(M) - i : i \geq 0\right\},$$

The regularity can be infinite. For example if $R = \mathsf{k}[x]/(x^3)$, then $\mathrm{reg}_R(\mathsf{k}) = +\infty$.

The regularity plays an important role in the study of homological properties of M. Avramove and peeva in [4] proved that $\mathrm{reg}_R(\mathsf{k})$ is zero or infinite. The algebra R is called Koszul if $\mathrm{reg}_R(\mathsf{k}) = 0$.

Another important invariant is the rate of graded modules. The Backelin rate of the k-algebra R is defined as

$$\mathrm{Rate}(R) := \sup\left\{t_i^R(\mathsf{k}) - 1/i - 1 : i \geq 2\right\}.$$

The notion of rate for a k-algebra R is introduced by Backelin ([5]) to study the Koszul property of R. He used complex arguments about a latice of ideals derived from a presentation of R as a quotient of a free noncommutative algebra to prove that the Veronese subring $R^{(c)}$ is a Koszul algebra for all sufficiently large c.

The notion of rate is generalized in [1] for graded modules. The rate of a finitely generated graded R-module M is defined by

$$\mathrm{rate}_R(M) := \sup\left\{t_i^R(M)/i : i \geq 1\right\}.$$

This invariant is always finite (see [1]). The Backelin rate of the algebra R is denoted by $\mathrm{Rate}(R)$ and is equal to $\mathrm{rate}^R(\mathfrak{m}(1))$, the rate of the unique homogenous maximal ideal of R which is shifted by 1.

We always have $\mathrm{Rate}(R) \geq 1$ and the equality holds if and only if R is Koszul. Indeed the rate of a graded algebra R is an invariant that measures how far R is from being Koszul.

The goal of this paper is to study the rate of powers of the maximal ideal $\mathfrak{m}$. More precisely we investigate upper bounds for $\mathfrak{m}^s(s)$ as a graded module (which is generated in degree zero) over the subring $R^{(c)}$ and over R itself.

2 Main Results

We recall that throughout R is the standard graded algebra over a filed k with graded maximal ideal of R.

In this section we study the rate of powers of the maximal ideal $\mathfrak{m}$. The first step in this direction is the following lemma which provide upper bound for the maximal degree of generators of

Lemma 2.1. *Let R be a standard graded k-algebra with homogeneous maximal ideal $\mathfrak{m}$. Then*

$$t_i^R(\mathfrak{m}^s(s)) \leq t_i^R(\mathfrak{m}(1)) \tag{1}$$

for all integers $s > 0$ and $i \geq 0$.

Proof. We prove by induction on s. The case $s = 1$ is obvious. Let $s \geq 2$. Consider the exact sequence

$$0 \hookrightarrow \mathfrak{m}^s \to \mathfrak{m}^{s-1} \to \mathfrak{m}^{s-1}/\mathfrak{m}^s \to 0$$

By applying $\mathrm{Tor}(,\mathsf{k})$, we get the exact sequence

$$\mathrm{Tor}_{i+1}\left(\mathfrak{m}^{s-1}/\mathfrak{m}^s, \mathsf{k}\right)_j \to \mathrm{Tor}_i\left(\mathfrak{m}^s, \mathsf{k}\right)_j \to \mathrm{Tor}_i\left(\mathfrak{m}^{s-1}, \mathsf{k}\right)_j,$$

for all $i \geq 0$. This yields the inequality

$$t_i^R(\mathfrak{m}^s(s)) \leq \max\left\{t_{i+1}^R\left(\mathfrak{m}^{s-1}/\mathfrak{m}^s\right), t_i^R\left(\mathfrak{m}^{s-1}\right)\right\}. \tag{2}$$

Notice that $\mathfrak{m}^{s-1}/\mathfrak{m}^s \simeq \mathsf{k}(-s+1)^n$ for some integer n and that $t_{i+1}^R(\mathsf{k}) = t_i^R(\mathfrak{m})$. Now using 2 and inductive hypothesis we conclude the assertion. ☐

If R is artinian, then there exists positive integer s such that $\mathfrak{m}^{s+1} = 0$ and $\mathfrak{m}^s \neq 0$. Then $\mathfrak{m}^s$ is isomorphic to $\mathsf{k}(-s)^r$ for some integer r. It follows then $\mathrm{Rate}(R) = \mathrm{rate}(m^s(s))$.

In the following result is about the asymptotic behavior of powers of maximal ideals when R is not artinian.

Theorem 2.2. *Let R be a non-artinian standard graded k-algebra with homogeneous maximal ideal $\mathfrak{m}$. Then there exists $s_0 > 0$ such that*

$$\mathrm{rate}(m^s(s)) \leq \mathrm{Rate}(R) \leq 2\,\mathrm{rate}(m^s(s)) \tag{3}$$

for all $s > s_0$.

The left inequality in 3 is obtained by applying Lemma 2.1. For the right inequality in 3 we used the fact that there exists $s_0 > 0$ such that the natural map $\mathrm{Tor}_i(\mathfrak{m}^s, \mathsf{k}) \to \mathrm{Tor}_i(\mathfrak{m}^{s-1}, \mathsf{k})$ is zero for all i and all $s > s_0$ (see for example the proof of [3, Theorem 6.3.6]).

Definition 2.3 (Veronese submodules)**.** Let c and d be integers such that $c > 0$ and $0 \leq d \leq c-1$. Let M a finitely generated graded R-module.

(1) Define $R^{(c)} := \bigoplus_{i\in\mathbb{Z}} R_{ic}$, then $R^{(c)}$ is a standard graded K-algebra and is a subring of R. We refer to $R^{(c)}$, with this grading, as the c-th Veronese subring of R. Then M can be consider as a finitely generated graded $R^{(c)}$-module via $R^{(c)} \hookrightarrow R$.

(2) We define $M^{(c,d)} := \bigoplus_{i\in\mathbb{Z}} M_{ic+d}$, an $R^{(c)}$-submodule of M. This called the (c, d)-th Veronese submodule of M. In the case $d = 0$, we denote $M^{(c,0)}$ by $M^{(c)}$. Notice that M as a graded $R^{(c)}$-module decomposes in to the direct sum $M = \bigoplus_{d=0}^{c} M^{(c,d)}$.

It is easy to see that $(-)^{(c,d)}$ is an exact functor from the category of graded R-modules to the category of graded $R^{(c)}$-modules.

Let x be a real number, then we denote $\lceil x\rceil$ to be the smallest integer larger than the x.

Theorem 2.4. *Let R be a standard graded K-algebra and M be a finitely generated positively graded R-module. Then*

$$\mathrm{rate}_{R^{(c)}}(\mathfrak{m}^s(s)) \leq \lceil \mathrm{Rate}(R)/c\rceil$$

for all integers $s, c \geq 1$. Here $\lceil x\rceil$ denotes the smallest integer larger than the real number x.

In particular

$$\mathrm{Rate}(R^{(c)}) \leq \lceil \mathrm{Rate}(R)/c\rceil$$

for all integers $c \geq 1$.

Corollary 2.5. *Let be in the situation of the above theorem. Then the following hold.*

(1) *for all $c \geq \mathrm{Rate}(R)$ and $s \geq 1$,*

$$\mathrm{reg}_{R^{(c)}}(\mathfrak{m}^s(s)) = 0$$

(2) *for all $c \geq \mathrm{Rate}(R)$*

$$\mathrm{reg}_{R^{(c)}}(R) = 0$$

In particular $\mathrm{reg}_{R^{(c)}}(R^{(c,d)}) = 0$ for all $c \geq \mathrm{Rate}(R)$ and $0 \leq d \leq c-1$

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Cofiniteness of Top Local Cohomology Modules with Respect to the Class of Modules in Dimension Less than a Fixed Integer

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Abstract

Let R be a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of R, M a finitely generated R-module, and n a non-negative integer. In this paper, we prove that the top local cohomology module $\mathrm{H}_{\mathfrak{a}}^{\dim_R(M)-n}(M)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module and the set $\{\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^{\dim_R(M)-n}(M)) : \dim(R/\mathfrak{p}) \geq n\}$ is finite. As a consequence, we observe that if R is a semi-local ring, then $\mathrm{Supp}_R(\mathrm{H}_{\mathfrak{a}}^{\dim_R(M)-1}(M))$ is a finite set.

Keywords: Associated prime ideals, Cofinite modules, Local cohomology modules

Mathematics Subject Classification [2010]: 13D07, 13D45

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1 Introduction and Preliminaries

Throughout, let R denote a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of R, M a finite (i.e., finitely generated) R-module, X an arbitrary R-module which is not necessarily finite, and n a non-negative integer. For basic results, notations, and terminology not given in this paper, readers are referred to [2,3].

Hartshorne, in [5], defined an $\mathfrak{a}$-torsion R-module X to be $\mathfrak{a}$-*cofinite* if $\mathrm{Ext}_R^i(R/\mathfrak{a}, X)$ is a finite R-module for all i and asked the following question:

Question 1.1. (see [5, First Question]) Under what hypotheses, is $\mathrm{H}_{\mathfrak{a}}^i(M)$ an $\mathfrak{a}$-cofinite R-module for all i?

We also have the following question as a related question to Question 1.1.

Question 1.2. (see [6, Problem 4]) Under what hypotheses, is $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^i(M))$ a finite set for all i?

Even though $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^i(M))$ is not a finite set (and so, $\mathrm{H}_{\mathfrak{a}}^i(M)$ is not an $\mathfrak{a}$-cofinite R-module) in general (see [9, Section 4] and [7, Section 1]), $\mathrm{H}_{\mathfrak{a}}^i(M)$ is an $\mathfrak{a}$-cofinite R-module (and hence, $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^i(M))$ is a finite set) in the case that $i = \dim_R(M)$. In fact, the top local cohomology module $\mathrm{H}_{\mathfrak{a}}^{\dim_R(M)}(M)$ is an $\mathfrak{a}$-cofinite Artinian R-module (see [8, Proposition 5.1]).

Recall that X is said to be an $\mathrm{FD}_{<n}$ (or *in dimension* $< n$) R-module if there exists a finite submodule X' of X such that $\dim_R(X/X') < n$. We say that X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-*cofinite* R-module if X is an $\mathfrak{a}$-torsion R-module and $\mathrm{Ext}_R^i(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all i. Note that X is a finite R-module if and only if X is an $\mathrm{FD}_{<0}$ R-module, and so X is an $\mathfrak{a}$-cofinite R-module if and only if X is an $(\mathrm{FD}_{<0}, \mathfrak{a})$-cofinite R-module. Thus, it is natural to raise the following questions as generalizations of Questions 1.1 and 1.2. In this paper, for a subset A of $\mathrm{Spec}(R)$, the set $\{\mathfrak{p} \in A : \dim(R/\mathfrak{p}) \geq n\}$ is denoted by $A_{\geq n}$.

Question 1.3. Under what hypotheses, is $\mathrm{H}^i_{\mathfrak{a}}(M)$ an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all i?

Question 1.4. Under what hypotheses, is $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))_{\geq n}$ a finite set for all i?

In this paper, we study Questions 1.3 and 1.4. As a generalization of [8, Proposition 5.1], we prove that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module (and hence, $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))_{\geq n}$ is a finite set) in the case that $i = \dim_R(M) - n$. In fact, the top local cohomology module $\mathrm{H}^{\dim_R(M)-n}_{\mathfrak{a}}(M)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite $\mathrm{D}_{<n+1}$ R-module. Here, the class of all R-modules X with $\dim_R(X) < n$ is denoted by $\mathrm{D}_{<n}$. As a consequence, we show that $\mathrm{Supp}_R(\mathrm{H}^{\dim_R(M)-1}_{\mathfrak{a}}(M))$ is a finite set whenever R is a semi-local ring. We also provide new proofs for previous results about cofiniteness of local cohomology modules based on the main result of this paper.

2 Main Results

The following lemma is needed in the proof of the main result of this paper.

Lemma 2.1. *Suppose that X is an arbitrary R-module with finite Krull dimension d. Then $\mathrm{H}^{d-i}_{\mathfrak{a}}(X)$ is a $\mathrm{D}_{<i+1}$ R-module for all $i \leq d$. In particular, $\mathrm{H}^{d-i}_{\mathfrak{a}}(X)$ is an $\mathrm{FD}_{<i+1}$ R-module for all $i \leq d$.*

Proof. Assume by way of contradiction that $\mathrm{H}^{d-i}_{\mathfrak{a}}(X)$ is not a $\mathrm{D}_{<i+1}$ R-module for a non-negative integer $i \leq d$. Then $\dim_R(\mathrm{H}^{d-i}_{\mathfrak{a}}(X)) > i$ and so there exists a prime ideal $\mathfrak{p}$ of $\mathrm{Supp}_R(\mathrm{H}^{d-i}_{\mathfrak{a}}(X))$ such that $\dim(R/\mathfrak{p}) > i$. Therefore $\mathrm{H}^{d-i}_{\mathfrak{a}R_{\mathfrak{p}}}(X_{\mathfrak{p}}) \neq 0$ and $d - i > \dim_{R_{\mathfrak{p}}}(X_{\mathfrak{p}})$. This contradicts [2, Theorem 6.1.2]. □

For an arbitrary R-module X, we denote the largest integer i in which $\mathrm{H}^i_{\mathfrak{a}}(X)$ is not zero, is not in $\mathrm{D}_{<n}$, and is not in $\mathrm{FD}_{<n}$ by $\mathrm{cd}(\mathfrak{a}, X)$, $\mathrm{cd}_{\mathrm{D}_{<n}}(\mathfrak{a}, X)$, and $\mathrm{cd}_{\mathrm{FD}_{<n}}(\mathfrak{a}, X)$, respectively. It is well known that $\mathrm{cd}(\mathfrak{a}, X) \leq \dim_R(X)$ and so $\mathrm{cd}_{\mathrm{D}_{<0}}(\mathfrak{a}, X) \leq \dim_R(X)$ and $\mathrm{cd}_{\mathrm{FD}_{<0}}(\mathfrak{a}, X) \leq \dim_R(X)$. The following corollary generalizes this result.

Corollary 2.2. *Suppose that n is a non-negative integer and that X is an arbitrary R-module. Then $\mathrm{cd}_{\mathrm{D}_{<n}}(\mathfrak{a}, X) \leq \dim_R(X) - n$. In particular, $\mathrm{cd}_{\mathrm{FD}_{<n}}(\mathfrak{a}, X) \leq \dim_R(X) - n$.*

By Lemma 2.1 or Corollary 2.2, $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i > \dim_R(X) - n$. Let X be an $\mathrm{FD}_{<n}$ R-module. The following proposition shows that $\mathrm{H}^{\dim_R(X)-n}_{\mathfrak{a}}(X)$ may not be an $\mathrm{FD}_{<n}$ R-module. Even though $\mathrm{H}^{\dim_R(X)-n}_{\mathfrak{a}}(X)$ may not be an $\mathrm{FD}_{<n}$ R-module, we prove in the main result of this paper that it is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module.

Proposition 2.3. *Suppose that n is a non-negative integer, X is an $\mathrm{FD}_{<n}$ R-module with finite Krull dimension $d > n$, and $\mathfrak{p}$ is a prime ideal of R with $\dim(R/\mathfrak{p}) = n$ and $\dim_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) = d - n$. Then $\mathrm{H}^{d-n}_{\mathfrak{p}}(X)$ is not an $\mathrm{FD}_{<n}$ R-module (and so $\mathrm{cd}_{\mathrm{FD}_{<n}}(\mathfrak{p}, X) = \dim_R(X) - n$).*

Proof. Assume by way of contradiction that $\mathrm{H}^{d-n}_{\mathfrak{p}}(X)$ is an $\mathrm{FD}_{<n}$ R-module. Then $X_{\mathfrak{p}}$ and $\mathrm{H}^{d-n}_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}})$ are finite $R_{\mathfrak{p}}$-modules which contradicts [2, Corollary 7.3.3]. □

Melkersson, in [8, Proposition 5.1], showed that the top local cohomology module $\mathrm{H}^{\dim_R(M)}_{\mathfrak{a}}(M)$ is an $\mathfrak{a}$-cofinite R-module. In the main result of this paper, we generalize Melkersson's result [8, Proposition 5.1] and prove that, for a non-negative integer n, the top local cohomology module $\mathrm{H}^{\dim_R(M)-n}_{\mathfrak{a}}(M)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module.

Theorem 2.4. *Suppose that n is a non-negative integer and that X is an $\mathrm{FD}_{<n}$ R-module with finite Krull dimension d. Then $\mathrm{H}^{d-n}_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module. In particular, $\mathrm{H}^{d-i}_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<i}, \mathfrak{a})$-cofinite R-module for all $i \leq d$ whenever X is a finite R-module.*

Proof. We can, and do, assume that X is a finite R-module. We argue by induction on d. Since X is a finite R-module, there is nothing to prove in the case that $d = 0$. Suppose that $d > 0$ and that $d - 1$ is settled. We can, and do, assume that $\Gamma_{\mathfrak{a}}(X) = 0$ from [2, Lemma 2.1.2 and Corollary

2.1.7(iii)]. Thus $\mathfrak{a}$ contains an element a which is a non-zerodivisor on X by [2, Lemma 2.1.1(ii)]. The short exact sequence

$$0 \longrightarrow X \xrightarrow{a} X \longrightarrow X/aX \longrightarrow 0$$

induces an exact sequence

$$\mathrm{H}^{(d-1)-n}_{\mathfrak{a}}(X/aX) \longrightarrow \mathrm{H}^{d-n}_{\mathfrak{a}}(X) \xrightarrow{a} \mathrm{H}^{d-n}_{\mathfrak{a}}(X)$$

of local cohomology modules. Since $\dim_R(X/aX) \leq d-1$, $\mathrm{H}^{(d-1)-n}_{\mathfrak{a}}(X/aX)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite $\mathrm{D}_{<n+1}$ R-module by the inductive hypothesis and Lemma 2.1. Hence $(0 :_{\mathrm{H}^{d-n}_{\mathfrak{a}}(X)} a)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite $\mathrm{D}_{<n+1}$ R-module from the latter exact sequence. Therefore $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{d-n}_{\mathfrak{a}}(X)) \cong \mathrm{Hom}_R(R/\mathfrak{a}, (0 :_{\mathrm{H}^{d-n}_{\mathfrak{a}}(X)} a))$ is an $\mathrm{FD}_{<n}$ R-module. Thus $\mathrm{H}^{d-n}_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module by Lemma 2.1 and [11, Lemma 2.1]. □

Corollary 2.5. *Suppose that n is a non-negative integer and X is an $\mathrm{FD}_{<n}$ R-module with finite Krull dimension d. Then the set $\mathrm{Ass}_R(\mathrm{H}^{d-n}_{\mathfrak{a}}(X))_{\geq n}$ is finite. In particular, $\mathrm{Ass}_R(\mathrm{H}^{d-i}_{\mathfrak{a}}(X))_{\geq i}$ is finite for all $i \leq d$ whenever X is a finite R-module.*

Proof. This follows from Theorem 2.4 and [3, Exercise 1.2.28]. □

Concerning Question 1.1, in [8, Theorem 7.10], Melkersson proved that $\mathrm{H}^{i}_{\mathfrak{a}}(M)$ is an $\mathfrak{a}$-cofinite R-module for all i if $\dim(R) \leq 2$. As an improvement of [8, Theorem 7.10], by [4, Corollary 5.2], $\mathrm{H}^{i}_{\mathfrak{a}}(M)$ is an $\mathfrak{a}$-cofinite R-module for all i whenever $\dim_R(M) \leq 2$. With respect to Question 1.3, the author and Papari-Zarei, in [12, Corollary 3.2], proved that $\mathrm{H}^{i}_{\mathfrak{a}}(M)$ an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all i if $\dim(R) \leq n+2$ which is a generalization of Melkersson's result [8, Theorem 7.10]. As a generalization of [4, Corollary 5.2] and an improvement of [12, Corollary 3.2], the authors in [10, Theorem 2.1] showed that $\mathrm{H}^{i}_{\mathfrak{a}}(M)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all i whenever $\dim_R(M) \leq n+2$. In the next corollary, we present a new proof of this result based on Theorem 2.4.

Corollary 2.6. *Suppose that n is a non-negative integer and M is a finite R-module such that $\dim_R(M) \leq n+2$. Then $\mathrm{H}^{i}_{\mathfrak{a}}(M)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all i.*

Proof. From Theorem 2.4, for all $i \geq 2$, $\mathrm{H}^{i}_{\mathfrak{a}}(M)$ is an $(\mathrm{FD}_{<\dim_R(M)-i}, \mathfrak{a})$-cofinite R-module and so an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module. Since M is a finite R-module, $\mathrm{H}^{0}_{\mathfrak{a}}(M)$ is also an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module. Thus $\mathrm{H}^{1}_{\mathfrak{a}}(M)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module by [1, Theorem 4.2]. □

Corollary 2.7. (see [8, Proposition 5.1]) *Suppose that M is a finite R-module with finite Krull dimension d. Then $\mathrm{H}^{d}_{\mathfrak{a}}(M)$ is an $\mathfrak{a}$-cofinite Artinian R-module.*

Proof. By putting $i = 0$ in Lemma 2.1 and Theorem 2.4, $\mathrm{H}^{d}_{\mathfrak{a}}(M)$ is an $\mathfrak{a}$-cofinite $\mathrm{D}_{<1}$ R-module. Therefore $\mathrm{H}^{d}_{\mathfrak{a}}(M)$ is an $\mathfrak{a}$-cofinite Artinian R-module. □

Corollary 2.8. *Suppose that R is a semi-local ring and X is an $\mathrm{FD}_{<1}$ R-module with finite Krull dimension d. Then $\mathrm{Supp}_R(\mathrm{H}^{d-1}_{\mathfrak{a}}(X))$ is a finite set.*

Proof. By taking $i = 1$ in Lemma 2.1 and Theorem 2.4, $\mathrm{H}^{d-1}_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<1}, \mathfrak{a})$-cofinite $\mathrm{D}_{<2}$ R-module. Therefore $\mathrm{Supp}_R(\mathrm{H}^{d-1}_{\mathfrak{a}}(X))$ is a finite set. □

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Cousin Functors Commute with Direct Limits

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Abstract

Let R be a commutative Noetherian ring with non-zero identity and $\mathcal{F}$ a filtration of $\operatorname{Spec}(R)$. We show that the Cousin functor with respect to $\mathcal{F}$, $C_R(\mathcal{F}, -) : \mathcal{C}_{\mathcal{F}}(R) \longrightarrow \operatorname{Comp}(R)$, where $\mathcal{C}_{\mathcal{F}}(R)$ is the category of R-modules which are admitted by $\mathcal{F}$ and $\operatorname{Comp}(R)$ is the category of complexes of R-modules, commutes with the formation of direct limits. We observe that an R-module X is balanced big Cohen-Macaulay if $(R, \mathfrak{m})$ is a local ring, $\mathfrak{m}X \neq X$, and every finitely generated submodule of X is a big Cohen-Macaulay R-module with respect to some system of parameters for R.

Keywords: Cousin complexes, Cousin functors, Direct limits

Mathematics Subject Classification [2010]: 13D02

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1 Introduction

Throughout R will denote a commutative Noetherian ring with non-zero identity. For basic results, notations, and terminology not given in this paper, readers are referred to [2, 3, 6].

The notion of Cousin complex was introduced in [4] and it has a commutative algebra analogue given by Sharp in [7]. In [10], Sharp generalized this concept to the Cousin complex for an R-module X with respect to a filtration $\mathcal{F}$ of $\operatorname{Spec}(R)$ and denoted this complex by $\mathrm{C}_R(\mathcal{F}, X)$. He approved it as a powerful tool by characterizing Gorenstein rings, Cohen-Macaulay modules, local cohomology modules, and balanced big Cohen-Macaulay modules in terms of Cousin complexes (see [7, Theorem 5.4], [8, Theorem 2.4], [9, Theorem], and [10, Corollary 3.7]).

Let $\mathcal{C}_{\mathcal{F}}(R)$ be the category of R-modules which are admitted by $\mathcal{F}$ and let $\operatorname{Comp}(R)$ be the category of complexes of R-modules. In [1], the authors introduced the Cousin functor with respect to $\mathcal{F}$, $C_R(\mathcal{F}, -) : \mathcal{C}_{\mathcal{F}}(R) \longrightarrow \operatorname{Comp}(R)$. They used this functor to construct Cousin spectral sequences with respect to $\mathcal{F}$ and study the extension functors of Cousin cohomologies (i.e., the cohomology modules of Cousin complexes). By using this functor, they also found some equivalent conditions for vanishing of Cousin cohomologies and gave some results for modules with finite (i.e., finitely generated) Cousin cohomologies.

In this paper, we study Cousin functors and show that they commute with direct limits. As a consequence, we observe that, in the case that R is local with maximal ideal $\mathfrak{m}$, an R-module X is balanced big Cohen-Macaulay if $\mathfrak{m}X \neq X$ and every finite submodule of X is a big Cohen-Macaulay R-module with respect to some system of parameters for R.

2 Main Results

A filtration of $\operatorname{Spec}(R)$ is a descending sequence $\mathcal{F} = (F_i)_{i\geq 0}$ of subsets of $\operatorname{Spec}(R)$ with the property that, for all $i \geq 0$, each member of $F_i \setminus F_{i+1}$ is a minimal member of F_i with respect to inclusion. We say that $\mathcal{F}$ admits an R-module X if $\operatorname{Supp}_R(X) \subseteq F_0$.

Suppose that $\mathcal{F} = (F_i)_{i\geq 0}$ is a filtration of $\operatorname{Spec}(R)$ which admits an R-module X. The Cousin complex $C_R(\mathcal{F}, X)$ for X with respect to $\mathcal{F}$ is of the form

$$0 \xrightarrow{d_X^{-2}} X \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^{0}} \cdots \xrightarrow{d_X^{i-3}} X^{i-2} \xrightarrow{d_X^{i-2}} X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^{i}} \cdots$$

where, for all $i \geq 0$, $X^i = \bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} \left(\operatorname{Coker} d_X^{i-2}\right)_{\mathfrak{p}}$ and $d_X^{i-1}(x) = \left\{\left(x + \operatorname{Im} d_X^{i-2}\right)/1\right\}_{\mathfrak{p}\in F_i\setminus F_{i+1}}$ for every element x of X^{i-1}; and satisfies

(P1) $\operatorname{Supp}_R(X^i) \subseteq \operatorname{Supp}_R(X) \cap F_i$,

(P2) $\operatorname{Supp}_R(\operatorname{Coker} d_X^{i-2}) \subseteq \operatorname{Supp}_R(X) \cap F_i$,

(P3) $\operatorname{Supp}_R(\mathrm{H}^{i-1}(C_R(\mathcal{F}, X))) \subseteq \operatorname{Supp}_R(X) \cap F_{i+1}$, and

(P4) the natural R-homomorphism $\xi_{X^i} : X^i \longrightarrow \bigoplus_{\mathfrak{p}\in F_i\setminus F_{i+1}} (X^i)_{\mathfrak{p}}$, where $\xi_{X^i}(x) = \{x/1\}_{\mathfrak{p}\in F_i\setminus F_{i+1}}$ for every element x of X^i, is an isomorphism

(see [7, Proposition 2.2 and Corollary 2.3], [10, Definitions 1.1, Definition 1.3, and Proposition 1.4], and [5, Proposition 1.1 and Lemma 1.2]). We adopt the convention that $X^{-1} = X$.

Bamdad and the author proved the following lemma and used it to introduce the Cousin functor with respect to $\mathcal{F}$, $C_R(\mathcal{F}, -) : \mathcal{C}_{\mathcal{F}}(R) \longrightarrow \operatorname{Comp}(R)$, which is R-linear and covariant (see [1, Theorem 2.2]).

Lemma 2.1. (see [1, Lemma 2.1]) *Let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of $\operatorname{Spec}(R)$ which admits R-modules X and Y. Then for every R-homomorphism $f : X \longrightarrow Y$, there exists a unique morphism of complexes $C_R(\mathcal{F}, f) = (f^i)_{i\geq -2} : C_R(\mathcal{F}, X) \longrightarrow C_R(\mathcal{F}, Y)$ such that $f^{-1} = f$.*

In the following theorem, we prove that the Cousin functor with respect to $\mathcal{F}$ preserves direct limits.

Theorem 2.2. *Let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of $\operatorname{Spec}(R)$. Then the Cousin functor with respect to $\mathcal{F}$, $C_R(\mathcal{F}, -) : \mathcal{C}_{\mathcal{F}}(R) \longrightarrow \operatorname{Comp}(R)$, commutes with the formation of direct limits.*

Proof. Assume that $(\Lambda, \preceq)$ is a (non-empty) directed partially ordered set and that $(\{X_\alpha\}_{\alpha\in\Lambda}, \{\phi_\beta^\alpha : X_\alpha \longrightarrow X_\beta\}_{\alpha\preceq\beta})$ is a direct system in $\mathcal{C}_{\mathcal{F}}(R)$ with direct limit $(\varinjlim X_\alpha, \{\phi_\alpha : X_\alpha \longrightarrow \varinjlim X_\alpha\}_{\alpha\in\Lambda})$. $(\{C_R(\mathcal{F}, X_\alpha)\}_{\alpha\in\Lambda}, \{C_R(\mathcal{F}, \phi_\beta^\alpha) : C_R(\mathcal{F}, X_\alpha) \longrightarrow C_R(\mathcal{F}, X_\beta)\}_{\alpha\preceq\beta})$ is a direct system in $\operatorname{Comp}(R)$ because $C_R(\mathcal{F}, -)$ is a covariant functor. Set $(\varinjlim C_R(\mathcal{F}, X_\alpha), \{\psi_\alpha\}_{\alpha\in\Lambda})$ be the direct limit of the later direct system where

$$\varinjlim C_R(\mathcal{F}, X_\alpha) = 0 \xrightarrow{\overrightarrow{d^{-2}}} \varinjlim X_\alpha \xrightarrow{\overrightarrow{d^{-1}}} \varinjlim (X_\alpha)^0 \xrightarrow{\overrightarrow{d^{0}}} \cdots \xrightarrow{\overrightarrow{d^{i-1}}} \varinjlim (X_\alpha)^i \xrightarrow{\overrightarrow{d^{i}}} \cdots .$$

Since $(\varinjlim X_\alpha, \{\phi_\alpha : X_\alpha \longrightarrow \varinjlim X_\alpha\}_{\alpha\in\Lambda})$ is the direct limit of $(\{X_\alpha\}_{\alpha\in\Lambda}, \{\phi_\beta^\alpha : X_\alpha \longrightarrow X_\beta\}_{\alpha\preceq\beta})$, $C_R(\mathcal{F}, \varinjlim X_\alpha)$ is an object in $\operatorname{Comp}(R)$, $C_R(\mathcal{F}, \phi_\alpha) : C_R(\mathcal{F}, X_\alpha) \longrightarrow C_R(\mathcal{F}, \varinjlim X_\alpha)$ is a morphism in $\operatorname{Comp}(R)$ for all $\alpha \in \Lambda$, and $C_R(\mathcal{F}, \phi_\beta) C_R(\mathcal{F}, \phi_\beta^\alpha) = C_R(\mathcal{F}, \phi_\alpha)$ for all $\alpha \preceq \beta$. Thus there exists a unique morphism

$$f = (f^i)_{i\geq -2} : \varinjlim C_R(\mathcal{F}, X_\alpha) \longrightarrow C_R(\mathcal{F}, \varinjlim X_\alpha)$$

such that $f\psi_\alpha = C_R(\mathcal{F}, \phi_\alpha)$ for all $\alpha \in \Lambda$. Therefore we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r, "\overrightarrow{d^{-2}}"] & \varinjlim X_\alpha \arrow[r, "\overrightarrow{d^{-1}}"] \arrow[d, "f^{-1}"] & \varinjlim (X_\alpha)^0 \arrow[r, "\overrightarrow{d^{0}}"] \arrow[d, "f^{0}"] & \cdots \arrow[r, "\overrightarrow{d^{i-1}}"] & \varinjlim (X_\alpha)^i \arrow[r, "\overrightarrow{d^{i}}"] \arrow[d, "f^{i}"] & \cdots \\
0 \arrow[r, "d^{-2}_{\varinjlim X_\alpha}"] & \varinjlim X_\alpha \arrow[r, "d^{-1}_{\varinjlim X_\alpha}"] & (\varinjlim X_\alpha)^0 \arrow[r, "d^{0}_{\varinjlim X_\alpha}"] & \cdots \arrow[r, "d^{i-1}_{\varinjlim X_\alpha}"] & (\varinjlim X_\alpha)^i \arrow[r, "d^{i}_{\varinjlim X_\alpha}"] & \cdots .
\end{tikzcd}$$

By using induction on $i \geq -1$, we prove that f^i is an isomorphism. The case $i = -1$ is clear. Suppose that $i \geq 0$ and that f^j is an isomorphism for all $-1 \leq j \leq i-1$. Let $\mathfrak{p} \in F_i \setminus F_{i+1}$. By (P1) and (P3),

$$\begin{tikzcd}
(\varinjlim (X_\alpha)^{i-2})_{\mathfrak{p}} \arrow[r, "(\overrightarrow{d^{i-2}})_{\mathfrak{p}}"] \arrow[d, "(f^{i-2})_{\mathfrak{p}}"] & (\varinjlim (X_\alpha)^{i-1})_{\mathfrak{p}} \arrow[r, "(\overrightarrow{d^{i-1}})_{\mathfrak{p}}"] \arrow[d, "(f^{i-1})_{\mathfrak{p}}"] & (\varinjlim (X_\alpha)^{i})_{\mathfrak{p}} \arrow[r, "(\overrightarrow{d^{i}})_{\mathfrak{p}}"] \arrow[d, "(f^{i})_{\mathfrak{p}}"] & 0 \\
((\varinjlim X_\alpha)^{i-2})_{\mathfrak{p}} \arrow[r, "(d^{i-2}_{\varinjlim X_\alpha})_{\mathfrak{p}}"] & ((\varinjlim X_\alpha)^{i-1})_{\mathfrak{p}} \arrow[r, "(d^{i-1}_{\varinjlim X_\alpha})_{\mathfrak{p}}"] & ((\varinjlim X_\alpha)^{i})_{\mathfrak{p}} \arrow[r, "(d^{i}_{\varinjlim X_\alpha})_{\mathfrak{p}}"] & 0
\end{tikzcd}$$

is a commutative diagram with exact rows. Hence $(f^i)_{\mathfrak{p}}$ is an isomorphism from the Five Lemma [6, Proposition 2.72]. Thus

$$\bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (f^i)_{\mathfrak{p}} : \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (\varinjlim (X_\alpha)^i)_{\mathfrak{p}} \longrightarrow \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} ((\varinjlim X_\alpha)^i)_{\mathfrak{p}}$$

is an isomorphism. On the other hand, since $\xi_{(X_\alpha)^i} : (X_\alpha)^i \longrightarrow \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} ((X_\alpha)^i)_{\mathfrak{p}}$ is an isomorphism for all $\alpha \in \Lambda$ by (P4), $\overrightarrow{\xi_i} : \varinjlim (X_\alpha)^i \longrightarrow \varinjlim (\bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} ((X_\alpha)^i)_{\mathfrak{p}})$ is an isomorphism. Let $\nu : \varinjlim (\bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} ((X_\alpha)^i)_{\mathfrak{p}}) \longrightarrow \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} \varinjlim ((X_\alpha)^i)_{\mathfrak{p}}$ and $\mu : \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} \varinjlim ((X_\alpha)^i)_{\mathfrak{p}} \longrightarrow \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (\varinjlim (X_\alpha)^i)_{\mathfrak{p}}$ be the natural isomorphisms. Thus

$$\xi^{-1}_{(\varinjlim X_\alpha)^i} \left(\bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (f^i)_{\mathfrak{p}} \right) \mu \nu \overrightarrow{\xi_i} : \varinjlim (X_\alpha)^i \longrightarrow (\varinjlim X_\alpha)^i$$

is an isomorphism. Since $\mu \nu \overrightarrow{\xi_i} = \xi_{\varinjlim (X_\alpha)^i}$ and $\xi_{(\varinjlim X_\alpha)^i} f^i = (\bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (f^i)_{\mathfrak{p}}) \xi_{\varinjlim (X_\alpha)^i}$, f^i is an isomorphism as we desired. □

Corollary 2.3. *Let $\mathcal{F} = (F_i)_{i \geq 0}$ be a filtration of* Spec(R)*. Then the Cousin functor with respect to $\mathcal{F}$, $C_R(\mathcal{F}, -) : \mathcal{C}_{\mathcal{F}}(R) \longrightarrow$* Comp$(R)$*, commutes with direct sums.*

Corollary 2.4. *Let $\mathcal{F} = (F_i)_{i \geq 0}$ be a filtration of* Spec(R) *which admits an R-module X. Then $C_R(\mathcal{F}, X) \cong \varinjlim_{\alpha \in \Lambda} C_R(\mathcal{F}, X_\alpha)$, where X_α is a finite submodule of X for all $\alpha \in \Lambda$.*

Corollary 2.5. *Let $\mathcal{F} = (F_i)_{i \geq 0}$ be a filtration of* Spec(R) *which admits an R-module X. Then, for all $i \geq -1$, $\mathrm{H}^i(C_R(\mathcal{F}, X)) \cong \varinjlim_{\alpha \in \Lambda} \mathrm{H}^i(C_R(\mathcal{F}, X_\alpha))$, where X_α is a finite submodule of X for all $\alpha \in \Lambda$.*

Corollary 2.6. *Suppose that $\mathcal{F} = (F_i)_{i \geq 0}$ is a filtration of* Spec(R) *which admits an R-module X. Assume also that $X \cong \varinjlim_{\alpha \in \Lambda} X_\alpha$ where, for all $\alpha \in \Lambda$, X_α is a finite submodule of X such that $C_R(\mathcal{F}, X_\alpha)$ is exact. Then $C_R(\mathcal{F}, X)$ is exact. In particular, $C_R(\mathcal{F}, X)$ is exact if $C_R(\mathcal{F}, Y)$ is exact for every finite submodule Y of X.*

Let R be a local ring, X an arbitrary R-module, and $r_1, \ldots, r_n$ a system of parameters for R. Recall that, X is said to be big Cohen-Macaulay with respect to $r_1, \ldots, r_n$ if $r_1, \ldots, r_n$ is a regular sequence on X. Also, X is said to be balanced big Cohen-Macaulay if X is big Cohen-Macaulay with respect to every system of parameters for R. It is well known that if X is a finite big Cohen-Macaulay with respect to some system of parameters for R, then X is a balanced big Cohen-Macaulay R-module.

Corollary 2.7. *Suppose that R is a local ring with maximal ideal $\mathfrak{m}$ and that X is an arbitrary R-module such that $\mathfrak{m}X \neq X$. Assume also that $X \cong \varinjlim_{\alpha \in \Lambda} X_\alpha$ where, for all $\alpha \in \Lambda$, X_α is a finite submodule of X and also is a big Cohen-Macaulay R-module with respect to some system of parameters for R. Then X is balanced big Cohen-Macaulay. In particular, X is balanced big Cohen-Macaulay if every finite submodule of X is a big Cohen-Macaulay R-module with respect to some system of parameters for R.*

Proof. By assumptions, for all $\alpha \in \Lambda$, X_α is a balanced big Cohen-Macaulay R-module. Thus, from [10, Corollary 3.7] (or [1, Corollary 4.6]), for all $\alpha \in \Lambda$, the Cousin complex for X_α with respect to the dimension filtration (i.e., $\mathcal{D}(R) = (D_i(R))_{i \geq 0}$ where $D_i(R) = \{\mathfrak{p} \in \operatorname{Spec}(R) : \dim(R) - \dim_R(R/\mathfrak{p}) \geq i\}$ for all $i \geq 0$) is exact. Hence the Cousin complex for X with respect to the dimension filtration is exact by Corollary 2.6. Therefore, again from [10, Corollary 3.7] (or [1, Corollary 4.6]), X is a balanced big Cohen-Macaulay R-module. □

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درباره هم‌متناهی بودن مدول‌های کوهمولوژی موضعی تعمیم‌یافته

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چکیده. فرض کنیم R حلقه نوتری، جابجایی و یکدار و I ایده‌آلی از R باشد به‌طوری که $q(I,R)\leq ۱$. فرض کنیم M و N، R-مدول‌های متناهی‌مولد باشند به‌طوری که برای هر $i\geqslant ۲$، $H^i_I(M,N)$ آرتینی است. دراین‌صورت نشان می‌دهیم برای هر $j\geqslant ۰$، $H^j_I(M,N)$، I-هم‌متناهی است.

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۱. مقدمه

در سراسر این مقاله R یک حلقه جابجایی یکدار و نوتری، I ایده‌آلی از R و M، N، R-مدول‌های متناهی‌مولد هستند.

مدول‌های کوهمولوژی موضعی توسط گروتندیک معرفی شد [۳]. فرض کنیم M یک R-مدول باشد. برای هر $i\geq ۰$، i-امین مدول کوهمولوژی موضعی M نسبت به ایده‌آل I به صورت زیر تعریف می‌شود:

$$H^i_I(M)\cong \varinjlim_{n\geq ۱}\operatorname{Ext}^i_R(R/I^n,N).$$

هرزوگ در [۵] مدول‌های کوهمولوژی موضعی تعمیم‌یافته را معرفی کرد. فرض کنیم I ایده‌آلی از R باشد. در این صورت برای هر $i\geq ۰$، فانکتور $H^i_I(-,-)$ برای هر جفت از R-مدول‌های M و N به صورت زیر تعریف می‌شود:

$$H^i_I(M,N)\cong \varinjlim_{n\geq ۱}\operatorname{Ext}^i_R(M/I^nM,N).$$

بدیهی است که $H^i_I(R,N)$ همان کوهمولوژی موضعی $H^i_I(N)$ است.

می‌دانیم که اگر $(R,\mathfrak{m},k)$ حلقه موضعی و نوتری باشد، آنگاه برای هر R-مدول متناهی‌مولد M و هر $i\in\mathbb{N}_۰$، R-مدول $H^i_{\mathfrak{m}}(M)$ آرتینی است. بنابراین R-مدول $\operatorname{Hom}_R(k,H^i_{\mathfrak{m}}(M))$ متناهی‌مولد است. با توجه به این واقعیت، گروتندیک حدس زیر را مطرح کرد:

برای هر ایده‌آل I از حلقه نوتری R و هر R-مدول متناهی‌مولد M، R-مدول $\operatorname{Hom}_R(R/I,H^i_I(M))$ برای هر $i\in\mathbb{N}_۰$، متناهی‌مولد است. هارتشورن در [۴] با بیان مثال نقض، حدس گروتندیک را رد کرد و نشان داد که این حدس حتی در حالت منظم بودن حلقه نیز صحیح نمی‌باشد. هارتشورن مدول‌های هم‌متناهی را به صورت زیر تعریف کرد:

2020 Mathematics Subject Classification. Primary: 13D45, 14B15; Secondary: 13E05.

واژگان کلیدی. حلقه نوتری، کوهمولوژی موضعی، مدول کوهمولوژی موضعی تعمیم‌یافته، مدول هم‌متناهی.

R-مدول M را I-هم‌متناهی گویند هرگاه $\operatorname{Supp}(M) \subseteq \operatorname{V}(I)$ و R-مدول $\operatorname{Ext}^i_R(R/I, M)$ برای هر $i \geq 0$، متناهی‌مولد باشد.

برای هر ایده‌آل I از R، $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq I\}$ را با $\operatorname{V}(I)$ نشان می‌دهیم.

او همچنین پرسش زیر را مطرح کرد:

به‌ازای کدام حلقه R و کدام ایده‌آل I از R، R-مدول‌های $H^i_I(M)$ به‌ازای‌هر i و هر R-مدول متناهی‌مولد M، I-هم‌متناهی هستند؟

به‌عنوان تعمیمی از حدس هارتشورن می‌توان این سوال را مطرح کرد که تحت چه شرایطی مدول‌های کوهمولوژی موضعی تعمیم‌یافته $H^i_I(M, N)$، I-هم‌متناهی هستند؟

همان‌طور که بیان شد یکی از مسائل مهم در جبر جابجایی، هم‌متناهی بودن مدول‌های کوهمولوژی موضعی و مدول‌های کوهمولوژی موضعی تعمیم‌یافته است و در رابطه با این موضوع هم پژوهش‌های زیادی انجام شده است. در این مقاله هم سعی شده است که چند نکته جدید دیگر را در مورد هم‌متناهی بودن مدول‌های کوهمولوژی موضعی تعمیم‌یافته بیان کنیم. ابتدا به چند تعریف مقدماتی در رابطه با موضوع اشاره می‌کنیم.

تعریف ۱.۱. اگر I ایده‌آلی از حلقه R و M یک R-مدول باشد، آنگاه بعد کوهمولوژیک M نسبت به I به صورت زیر تعریف می‌شود:

$$\operatorname{cd}(I, M) = \sup\{i \in \mathbb{N}_0 : H^i_I(M) \neq 0\}.$$

تعریف ۲.۱. نماد $q(I, R)$ توسط هارتشورن به‌صورت زیر تعریف شده است:

$$q(I, R) = \sup\{i \in \mathbb{N}_0 : H^i_I(R) \text{ آرتینی نیست}\},$$

اگر این مجموعه کوچکترین کران بالا داشته باشد، درغیراین‌صورت $-\infty$ تعریف می‌کنیم.

در این مقاله نشان می‌دهیم اگر R حلقه نوتری و جابجایی یکدار و I ایده‌آلی از R باشد با شرط $q(I, R) \leq 1$ و M و N، R-مدول‌های متناهی‌مولد باشند به‌طوری‌که برای هر $i \geq 0$، $H^i_I(M, N)$ آرتینی است، آنگاه برای هر $j \geq 0$، $H^j_I(M, N)$، I-هم‌متناهی می‌باشد.

برای مفاهیم و اصطلاحاتی که در این مقاله استفاده شده است، خواننده می‌تواند به مرجع [۱] مراجعه کند.

۲. نتایج اصلی

تعریف ۱.۲. فرض کنیم I ایده‌آلی از R باشد. بعد آرتینی R-مدول‌های M و N نسبت به ایده‌آل I را به‌صورت زیر تعریف می‌کنیم:

$$q_I(M, N) = \sup\{i \in \mathbb{N}_0 : H^i_I(M, N) \text{ آرتینی نیست}\},$$

اگر این مجموعه کوچکترین کران بالا داشته باشد، درغیراین‌صورت $-\infty$ تعریف می‌کنیم.

تعریف ۲.۲. فرض کنیم R یک حلقه نوتری و I ایده‌آلی از R باشد. فرض کنیم M و N دو R-مدول باشند. دراین‌صورت تعریف می‌کنیم:

$$\tilde{q}_I(M, N) = \sup\{i \in \mathbb{N}_0 : H^i_I(M, N) \text{ آرتینی و } I\text{-هم‌متناهی نیست}\}.$$

قضیه ۳.۲ ([۶]). فرض کنیم R حلقه نوتری و I ایده‌آلی از R باشد به‌طوری‌که $q(I, R) \leq 1$. دراین‌صورت برای R-مدول‌های متناهی‌مولد M و N و برای هر $i \geq 0$، $H^i_I(M, N)$، I-هم‌متناهی است.

قضیه زیر اولین نتیجه این مقاله است.

قضیه ۴.۲. فرض کنیم R حلقه نوتری و I ایده‌آلی از R باشد به‌طوری که $q(I,R) \leq ۱$. دراین‌صورت برای ایده‌آل‌های متناهی‌مولد M و N داریم $q_I(M,N) = \tilde{q}_I(M,N)$.

برهان. واضح است که $q_I(M,N) \leq \tilde{q}_I(M,N)$. کافی است نشان دهیم $\tilde{q}_I(M,N) \leq q_I(M,N)$.

فرض کنیم $q_I(M,N) = t$. دراین‌صورت طبق تعریف برای هر $i > t$، $H_I^i(M,N)$ آرتینی است. از طرفی طبق قضیه ۳.۲، برای هر $i \geq ۰$، $H_I^i(M,N)$، I-هم‌متناهی است پس برای هر $i > t$ نیز $H_I^i(M,N)$، آرتینی و I-هم‌متناهی است. لذا $\tilde{q}_I(M,N) \leq t$ بنابراین $q_I(M,N) = \tilde{q}_I(M,N)$. □

قضیه ۵.۲ ([۲]). فرض کنیم R حلقه نوتری و جابجایی یکدار و I ایده‌آلی از R باشد. فرض کنیم $t \geq ۰$ عدد صحیح و M و N، R-مدول‌های متناهی‌مولد باشند به‌طوری که برای هر $i \neq t$، $H_I^i(M,N)$، I-هم‌متناهی است. دراین‌صورت $H_I^i(M,N)$ برای هر i، I-هم‌متناهی است.

حال نتیجه اصلی این مقاله را بیان می‌کنیم.

قضیه ۶.۲. فرض کنیم R یک حلقه نوتری و جابجایی یکدار و I ایده‌آلی از R باشد. به‌طوری که $q(I,R) \leq ۱$. فرض کنیم M و N، R-مدول‌های متناهی‌مولد باشند به‌طوری که $q_I(M,N) \leqslant ۱$. دراین‌صورت برای هر $i \geqslant ۰$، $H_I^i(M,N)$، I-هم‌متناهی است.

برهان. چون $H_I^۰(M,N) = \mathrm{Hom}_R(M, \Gamma_I N)$ پس $H_I^۰(M,N)$ متناهی‌مولد است و از اینکه I-تابدار نیز می‌باشد لذا نتیجه می‌شود $H_I^۰(M,N)$، I-هم‌متناهی است. از طرفی چون $q_I(M,N) \leq ۱$ و طبق قضیه ۴.۲، $q_I(M,N) = \tilde{q}_I(M,N)$ پس $\tilde{q}_I(M,N) \leq ۱$ لذا طبق تعریف به‌ازای‌هر $i \geq ۲$، $H_I^i(M,N)$، I-هم‌متناهی است. از قضیه ۵.۲ نتیجه می‌شود که $H_I^۱(M,N)$ نیز I-هم‌متناهی است. درنتیجه برای هر $i \geq ۰$، $H_I^i(M,N)$، I-هم‌متناهی است. □

از قضیه ۶.۲، نتایج زیر را داریم.

نتیجه ۷.۲. فرض کنیم R حلقه نوتری و جابجایی یکدار و I ایده‌آلی از R باشد به‌طوری که $\mathrm{cd}(I,R) \leq ۱$. فرض کنیم M و N، R-مدول‌های متناهی‌مولد باشند به‌طوری که $q_I(M,N) \leq ۱$. دراین‌صورت $H_I^i(M,N)$ برای هر $i \geq ۰$، I-هم‌متناهی است.

برهان. چون $q(I,R) \leq \mathrm{cd}(I,R)$ پس $q(I,R) \leq ۱$ لذا حکم از قضیه ۶.۲ نتیجه می‌شود. □

نتیجه ۸.۲. فرض کنیم R حلقه نوتری و جابجایی یکدار باشد به‌طوری که $\dim R \leq ۲$. فرض کنیم M و N، R-مدول‌های متناهی‌مولد باشند به‌طوری که $q_I(M,N) \leq ۱$. دراین‌صورت $H_I^i(M,N)$، برای هر $i \geq ۰$، I-هم‌متناهی است.

برهان. چون $\dim R \leq ۲$ پس $H_I^۲(R)$ آرتینی است و برای هر $i > ۲$، $H_I^i(R) = ۰$ لذا می‌توان نتیجه گرفت $q(I,R) \leq ۱$ پس طبق قضیه ۶.۲، برای هر $i \geq ۰$، $H_I^i(M,N)$، I-هم‌متناهی است. □

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On Linear Systems Over Strongly Algebraically Closed Algebras

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Abstract

In this paper, we define the basic concepts of module theory for A^n, where A^n is as a semimodule over a strongly algebraically closed A and we proved many similar theorems in linear algebra for the space A^n.

Keywords: Semimodule, Linear algebra, Strongly algebraically closed algebra

Mathematics Subject Classification [2010]: Primary: 13BXX,13FXX,05EXX, Secondary: 13HXX, 05EXX

1 Introduction and Preliminaries

Algebraically closed and existentially closed structures concepts have been studied extensively in their general model-theoretic context. A neat summary of the model-theoretic notions used in the sequel may be found in [11]. Specific classes of algebras considered are notably that of groups and division rings. A structure M is algebraically closed if and only if every finite set of equations with coefficients from M, which is solvable in some extension M' of M - of the same type as M- already has a solution in M. Existentially closed structures are defined analogously by allowing also inequalities ([7] , [8], and [10]). By studying subsemimodule generated by a set, we can find a necessary and suffcient condition for consistency of a linear system with coefficients in a strongly algebraically closed algebra.

Semimodules constitute a fairly natural generalization of modules, with broad applications in the mathematical foundations of computer science, so the study of properties content modules that apply in content semimodules is needed [9].

In this paper we obtain a structural criterion of semimodule theory over strongly algebraically closed algebras and we extend some basic definition of linear algebra to concepts of strongly algebraically closed algebras.

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[*] Speaker.

2 Main Results

We recollect the basic definitions and facts about formal concept analysis needed in this work. For further details on this topics we refer the reader to [2], [4], [5], and [6]. Here we need to extend some basic definition of linear algebra to concepts of strongly algebraically closed algebras. A notion of order plays an important role in the theory of algebraic structures. The most important kind of ordering in the general theory of algebras is a lattice ordering, which turns out to be definable by identities in terms of of the least-upper-bound (the join) and greatest-lower-bound (the meet) operations. A lattice is a poset P any pair of elements x, y have a g.l.b. or meet denote by $x \wedge y$, and a l.u.b. or join denote by $x \vee y$.

We say that a lattice L is complete, if every subset of L has a supremum. Let $(L, \leq)$ be a lattice with $\overline{0}$ and $\overline{1}$. We say that x has a complement in $(L, \leq)$ if and only if there exists $b \in L$ such that $a \vee b = \overline{1}$ and $a \wedge b = \overline{0}$. This element b is called a complement of a in $(L, \leq)$ and is denoted by $\overline{a}$. If $\overline{a}$ exists for any $a \in L$, then $(L, \leq)$ is said to be a complemented lattice. A lattice $(L, \leq)$ is called distributive if and only if for all a, b, $c \in L$ we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Any distributive complemented lattice is said to be a Boolean lattice and a Boolean algebra we mean a Boolean lattice together with the unary operation of complementation or a Boolean lattice is a complemented distributive lattice. A lattice L is called algebraically closed, if any finite consistent system of equations with coefficients from L, has a solution in L. A system S with coefficients in L is called consistent, if there is an extension K of L, such that S has a solution in K. One can generalize this definition to an arbitrary class of lattices. Notice that we denote the lattice operations by $\vee$ and $\wedge$.

Definition 2.1. [7] A lattice A in a class $\mathfrak{X}$ is said strongly algebraically closed if every system (not necessarily finite) of equations with parameters in A which has a solution in some extension $B \in \mathfrak{X}$, has already a solution in A.

Suppose A is a strongly algebraically closed algebra and consider A^n as $Mat_{n\times 1}(A)$, the set of all $n \times 1$ matrices over A. Suppose that A is a strongly algebraically closed algebra and $Mat_{n\times 1}(A)$ is the set of all $n \times m$ matrices over the A. We can define a partial order relation on $Mat_{n\times 1}(A)$ as follows: $C \leq D$, $cij \leq d_{ij}$; for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, where C, $D \in Mat_{n\times 1}(A)$. One can see that $(Mat_{n\times 1}(A), \leq)$ is a strongly algebraically closed algebra where its supremum and infimum are defined componentwise on $Mat_{n\times 1}(A)$ induced by the supremum and infimum of strongly algebraically closed algebra A, respectively.

Theorem 2.2. *Suppose A is a strongly algebraically closed algebra. Consider $(A^n, \vee, <)$ as a semimodule over semiring $(A^n, \vee, \wedge)$ with scalar multiplication $\wedge$. Let D, X, y be $m \times n$, $n \times 1$, and $m \times 1$ matrices over A, respectively. The linear system $D \vee X = y$ has a solution if and only if y belongs to the subsemimodule generated by columns of D.*

Let $(L, \leq)$ be a lattice and m, n, $k \in N$. By $L^{m\times n}$ denote the set of all $(m \times n)$-matrices with entries in L. We define the following partial order on the set $L^{m\times n}$, for all $i = 1,, \ldots, m$ and $j = 1, \ldots, n$:

$$D < B \quad \text{if and only if} \quad a_{ij} < b_{ij}.$$

We know that if A is a strongly algebraically closed algebra, then
(a) $D(BC) = (DB)C$ for all $D \in A^{m\times k}$, $B \in A^{k\times 1}$, and $C \in A^{1\times n}$;
(b) $D(C \vee F) = DC \vee DF$ and $(D \vee B)C = DC \vee BC$ for all D, $B \in A^{m\times k}$ and $C, F \in A^{k\times n}$.

Definition 2.3 ([3]). A near semiring is an algebra $(R; +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ such that

(i) $(R; +, 0)$ is a commutative monoid;

(ii) $(R; \cdot, 1)$ is a groupoid satisfying $x \cdot 1 = x = 1 \cdot x$ (a unital groupoid);

(iii) $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$;

(iv) $x \cdot 0 = 0 \cdot x = 0$;

for all x, y, $z \in R$.

We recall from [3] that a semiring is a *near semiring* such that $(R; \cdot, 1)$ is a monoid (i.e. $\cdot$ is also associative) that satisfies left distributivity: $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$, for all x, y, $z \in R$.

We will have the following theorem:

Theorem 2.4. *Let A be a strongly algebraically closed algebra and B, X and Y are $m \times n$, $n \times 1$ and $m \times 1$ matrices over A, respectively. The linear system $BX = Y$ has a solution if and only if b belongs to the subsemimodule generated by columns of B.*

The following corollary coincides with the same in [12].

Corollary 2.5. *Let A be a strongly algebraically closed algebra, $D \in A^{m \times k}$, $B \in A^{m \times n}$, and $C \in A^{k \times n}$.*

(1) The matrix equation $DX = B$ is solvable if and only if $D \cdot (\overline{{}^tD \cdot \overline{B}}) = B$.

(2) The matrix equation $XC = B$ is solvable if and only if $(\overline{B \cdot {}^t C}) \cdot C = B$.

We recall from [1] that the smallest positive integer m such that $D^m = D^t$ for some $t \in \{1, , \ldots, m-1\}$ is said the index and denoted by $index(D)$.

Definition 2.6. For any strongly algebraically closed algebra A, the poset $\mathrm{Idempn}(A, .)$ is a lattice in which the join ($\sqcup$) and the meet ($\sqcap$) are defined by the equalities

$$D \sqcup B = (D \vee B)^{index(A \vee B)}, \qquad D \sqcap B = (D \wedge B)^n.$$

Corollary 2.7. *Let D and B be matrices over a strongly algebraically closed algebra. If $DB = BD$, then $D \sqcup B = D \vee B \vee DB$.*

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Some Graphs Associated to a Hyperring via Hyperideals

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Abstract

In this note we define and study the intersection graph of hyperrings and study the interplay of hyperstructure-theoretic properties of hyperrings with graph-theoretic properties of associated graph. We characterize the hyperring R for which the intersection graph associated to R is disconnected and obtain several necessary and sufficient conditions on a hyperring R such that the intersection graph is complete.

Keywords: Intersection graph, Hyperideals, Hyperrings

Mathematics Subject Classification [2010]: Primary:16Y99,97K30,20N20, Secondary: 16D80, 05C60

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1 Introduction and Preliminaries

In the literature, there are many ways to associate graphs to algebraic structure. Associating a graph with an algebraic structures allows to obtain characterizations and representations of special classes of algebraic structures in terms of graphs and vice versa. One of the classical topics in the theory of graphs is the intersection graph theory [10, 11].

Let $F = \{s_i; i \in I\}$ be a family of sets. The intersection graph ,$G(F)$, of this family is the graph whose set of vertices is F, i.e., the vertices are $s_i, i \in I$ and the vertices s_i and s_j, $(i, j \in I)$ are adjacent (that is s_i and s_j are joined by an edge) if $s_i \neq s_j$ and $s_i \bigcap s_j \neq \varnothing$. For intersection graphs, there is the following theorem:

Theorem 1.1 ([9]). *Every simple graph is an intersection graph, that is, for any simple graph G there exists a family F of sets $s_i, i \in I$ such that G is isomorphic to the intersection graph $G(F)$.*

Various graphs on algebraic structures related to intersection graphs have been defined by many authors since 1964. Bosak [2] took the first step in this direction. In 1969, the intersection graph of non-trivial proper subgroups of a finite group was defined and studied by Cs´ak´any and Poll´ak [5]. Their works were continued by Zelinka [14]. He studied the intersection graph of non-trivial subgroups of finite abelian groups. Chakrabarty et al [3] studied intersection graphs of ideals of rings. Also he studied planarity of intersection graphs of the ring Z_n.
In this note we consider the intersection graph of a family of nontrivial hyperideals of a hyperring R. Let R be a hyperring and $I^*(R)$ be set of all non-trivial hyperideals of R. The intersection

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[*] Speaker.

graph of hyperideals of R denoted by $G(R)$, is a graph whose vertexes is the set $I^*(R)$ and two distinct vertcies I and J are adjacent if $I \bigcap J \neq \langle 0 \rangle$. We notice that the relationship between the intersection graph and algebraic hyperstructures have been already considered by Crosini, Davvaz, Leoranu, krasner, Vougiouklis, and others [4, 6, 8, 12]. First recall some definitions and notations of hyperrings. Then we introduce the intersection graph of hyperideals of hyperrings and characterize the hyperrings R for which the graph $G(R)$ is disconnected and obtain several conditions on a hyperring such that $G(R)$ is complete.

Let H be a nonempty set and $P^*(H)$ the set of nonempty subsets of H. A hyperoperation on H is a map $\circ : H \times H \to P^*(H)$. If $\circ$ is a hyperoperation defined on H, then the couple $(H, \circ)$ is called hypergroupoid. If A and B are nonempty subsets of H, then we define $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, and for $x \in H$, $\{x\} \circ A$ and $A \circ \{x\}$ is denoted by $x \circ A$ and $A \circ x$ respectively. A hypergroupoid $(H, \circ)$ is called semihypergroup if for $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$. An element $e \in H$ is called identity (scalar identity) if for all $a \in H$, $a \in (e \circ a) \bigcap (a \circ e)$, $(\{a\} = (e \circ a) \bigcap (a \circ e))$. A semihypergroup $(H, \circ)$ is called hypergroup if for every $x \in H$, $x \circ H = H \circ x = H$. A subhypergroup $(K, \circ)$ of $(H, \circ)$ is a nonempty subset K of H such that for all $k \in K$, $k \circ K = K \circ k = K$. We say that a hypergroup $(H, \circ)$ is canonical hypergroup if, it is commutative, means $\forall x, y \in H$, $x \circ y = y \circ x$, and has a scalar identity, which means that $\exists e \in H, \forall x \in H, x \circ e = e \circ x = x$, every element has a unique inverse, which means that for each $x \in H$,there exists a unique $x^{-1} \in H$, such that $e \in x \circ x^{-1} \cap x^{-1} \circ x$ and finally if $x \in y \circ z$ then $z \in y^{-1} \circ x$ and $y \in x \circ z^{-1}$. Clearly, the identity of a canonical hypergroup is unique. There are several kinds of hyperrings that can be defined on a nonempty set R. In what follows, we shall consider the hyperring of krasner, multiplicative hyperrings and general hyperrings and H_v-rings [6]. Now, the following definition introduce hyperrings in general form. A hyperring is a triple $(R, +, .)$, where R is a nonempty set, $(+)$ and $(.)$ are hyperoperations such that

1. $(R, +)$ is a commutative canonical hypergroup;
2. $(R, .)$ is a semihypergroup having 0 as a observing element,i.e., $0.a = a.0 = 0$,for all $a \in R$;
3. for all $a, b, c \in R, (a + b).c = (a.c) + (b.c), c.(a + b) = (c.a) + (c.b)$.

The hyperring $(R, +, .)$ is called commutative if for all $a, b \in R, a.b = b.a$. Let $(R, +, .)$ be a hyperring and A a nonempty subset of R. Then A is caled subhyperring of R if $(A, +, .)$ is hyperring.

Let I be a subhyperring of R. I is a hyperideal of R if $a.x \subseteq I$, for all $a \in R$ and $x \in I$.

A hyperideal P is called prime if $P \neq R$ and for $x, y \in R, x.y \subseteq P$ implies that $x \in P$ or $y \in P$.

A hyperideal M of R is called maximal if $M \neq R$ and there is no hyperideal I of R such that $M \subsetneq I \subsetneq R$ and finally a minimal hyperideal of a hyperring R is a non-zero hyperideal which contains no other non-zero hyperideal. Every commutative hyperring with identity has at least one maximal hyperideal and every proper hyperideal of a hyperring is contained in a maximal hyperideal [1].

Definition 1.2. A hyperring R is called hyperfield if $(R-\{0\}, .)$ is an abelian canonical hypergroup.

Definition 1.3. A hyperring R is called hyperintegral domain if for $x, y \in R$, $0 \in xy$, implies that $x = 0$ or $y = 0$.

Definition 1.4. Let $(R_1, +, .)$ and $(R_2, +, .)$ be hyperrings. We define the map

$$\oplus : (R_1 \times R_2) \times (R_1 \times R_2) \to P^*(R_1 \times R_2)$$

such that

$$(a, b) \oplus (a', b') = \{(c, d) | c \in a + a', d \in b + b'\}$$

and the map

$$\otimes : (R_1 \times R_2) \times (R_1 \times R_2) \to P^*(R_1 \times R_2)$$

such that

$$(a, b) \otimes (a', b') = \{(c, d) | c \in a \cdot a', d \in b \cdot b'\}.$$

Then $(R_1 \times R_2, \oplus, \otimes)$ is a hyperring.

Main Results

Throughout this note, R is a general commutative hyperring with idendity. The intersection graph $G(R)$ of hyperideals of R is a simple graph whose vertices are nontrivial proper hyperidals and two vertices I, J are adjacent if $I \neq J$ and $I \cap J \neq \{0\}$. Note that if R is a hyperfield, then $G(R)$ is the null graph which has no vertices .Extending statements to the null graph would introduce unnecessary distractions, so we ignore the null graph, that is all hyperrings are assumed to be nonhyperfields except when stated explicitly. We now find out conditions when the graph $G(R)$ is connected.

Theorem 1.5. *The intersection graph $G(R)$ of a hyperring R is disconnected if and only if R contains at least two minimal hyperideals and every nontrivial hyperideal of R is minimal.*

The above theorem is also true if the hyperideal is maximal.

Simple hyperring is a non-zero hyperring that has no hyperideal except the zero hyperideal and itself.

Theorem 1.6. *Let R be a commutative hyperring. Then the graph $G(R)$ is disconnected if and only if R is a direct product of two simple commutative hyperrings.*

Corollary 1.7. *Let R be a commutative hyperring with identity. Then the graph $G(R)$ is disconnected if only if R is a direct product of two hyperfields.*

Theorem 1.8. *The graph of a hyperintegral domain (but not a hyperfield) is complete.*

A graph is said to be embeddable in the plane, or planer, if it can be drawn in the plane so that its edges intersect only at their ends. We will repeatedly use Kuratowski's theorem [13] which states that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. Let R be a commutative hyperring with identity.Then

Lemma 1.9. *If $G(R)$ is planar, then any chain of hyperideals of R has length at most five.*

Corollary 1.10. *If $G(R)$ is planar, then R is Noetherian and Artinian.*

Lemma 1.11. *If $G(R)$ is null and R contains at least two proper nontrivial distinct hyperideals, then $R \cong R_1 \times R_2$, where R_1, R_2 are hyperfields.*

Theorem 1.12. *$G(R_1 \times R_2)$ is planar if and only if one of $G(R_1), G(R_2)$ is empty, and another is empty or null with at most two vertices.*

Corollary 1.13. *$G(R_1 \times R_2 \times R_3)$ is planar if and only if R_i is a hyperfield for $i = 1, 2, 3$*

Let $Max(R)$ be the set of all maximal hyperideals of R.

Corollary 1.14. *If $G(R)$ is planar, then $|Max(R)| \leqslant 3$.*

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Classification of $(n+4)$-Dimensional Nilpotent n-Lie Algebras of Maximal Class

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Abstract

In this paper, we classify $(n+4)$-dimensional nilpotent n-Lie algebras of maximal class over an arbitrary field, when $n \geq 3$.

Keywords: Filiform n-Lie algebra, nilpotent n-Lie algebra, maximal class

Mathematics Subject Classification [2010]: Primary: 17B05; Secondary: 17B30.

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1 Introduction and Preliminaries

The classification of n-Lie algebras is an important problem in the *n-Lie (Filippov) algebra* [5]. The algebraic classification of Filippov algebra is also discussed in the literature (for example, see [1,3,5]).

An n-Lie algebra A is nilpotent if there exists a non-negative integer k such that $A^k = 0$, where A^{i+1} defines inductively as $A^1 = 1$ and $A^{i+1} = [A^i, A, \ldots, A]$ for $i \geq 1$. The smallest integer k such that $A^{k+1} = 0$ is called the nilindex of A. An n-Lie algebra A of dimension d is maximal class (filiform) if $\dim A^j/A^{j+1} = 1$ for $j = 2, 3, \ldots, c$ and $\dim A/A^2 = n$. For a given filiform n-Lie algebra A, we have $Z_i(A) = A^{c-i+1}$, where $Z_0(A) = 0$ and $Z_i(A)/Z_{i-1}(A) = Z(A/Z_{i-1}(A))$ for all $i \geq 1$.

The filiform n-Lie algebras of dimension at most $n+3$ are known. The only filiform n-Lie algebras of dimension $n+1$ and $n+2$ are $H(n,1)$ and $A_{n,n+2,1}$, respectively. The only filiform n-Lie algebras of dimension $n+3$ are $A_{n,n+3,4}$ and $A_{n,n+3,5}$ (see [2]).

In this paper, we classify $(n+4)$-dimensional nilpotent n-Lie algebras of maximal class over an arbitrary field, when $n \geq 3$.

2 Main Results

In this section, we classify $(n+4)$-dimensional nilpotent n-Lie algebras of maximal class. Let A be a $(n+4)$-dimensional nilpotent n-Lie algebra of maximal class with basis $\{e_1, \ldots, e_{n+4}\}$ and let $Z(A) = \langle e_{n+4}\rangle$. Then $A/\langle e_{n+4}\rangle$ is a $(n+3)$-dimensional nilpotent n-Lie algebra of maximal class. By using Table 1, $A/\langle e_{n+4}\rangle$ is isomorphic to $A_{n,n+3,4}$ or $A_{n,n+3,5}$.

The structure of d-dimensional non-abelian filiform n-Lie algebras with $d \leq n+3$ over arbitrary field are presented in the Table 1.

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[*] Speaker.

Table 1: The structure of d-dimensional non-abelian filiform n-Lie algebras with $d \leq n+3$ over arbitrary field.

Name	Non-zero multiplication
$H(n,1)$	$[e_1,\dots,e_n]=e_{n+1}$
$A_{n,n+2,1}$	$[e_1,\dots,e_n]=e_{n+1},[e_2,\dots,e_{n+1}]=e_{n+2}$
$A_{n,n+3,4}$	$[e_1,\dots,e_n]=e_{n+1},[e_2,\dots,e_{n+1}]=e_{n+2},$ $[e_2,\dots,e_n,e_{n+2}]=e_{n+3}$
$A_{n,n+3,5}$	$[e_1,\dots,e_n]=e_{n+1},[e_2,\dots,e_{n+1}]=e_{n+2},$ $[e_2,\dots,e_n,e_{n+2}]=[e_1,e_3,\dots,e_{n+1}]=e_{n+3}$

Lemma 2.1. *Let A be a $(n+4)$-dimensional nilpotent n-Lie algebra of maximal class and let $A/\langle e_{n+4}\rangle \cong A_{n,n+3,4}$, then A is isomorphic to $A_{n,n+4,10}$.*

Proof. Let $A/\langle e_{n+4}\rangle \cong A_{n,n+3,4}$. In this case, the multiplication table in A can be written as

$$[e_1,\dots,e_n]=e_{n+1}+\alpha e_{n+4}, \qquad [e_2,\dots,e_{n+1}]=e_{n+2}+\beta e_{n+4},$$
$$[e_2,\dots,e_n,e_{n+2}]=e_{n+3}+\gamma e_{n+4}, \qquad [e_{i_1},\dots,e_{i_n}]=\alpha_{i_1,\dots,i_n}e_{n+4},$$

where $1 \leq i_1 < \ldots < i_n \leq n+3$, and $\{i_1,\dots,i_n\} \neq \{1,\dots,n\},\{2,\dots,n+1\},\{2,\dots,n,n+2\}$. Regarding a suitable change of basis, A can be written as

$$\begin{cases} [e_1,\dots,e_n]=e_{n+1},[e_2,\dots,e_{n+1}]=e_{n+2},[e_2,\dots,e_n,e_{n+2}]=e_{n+3},\\ [e_1,\dots,\hat{e}_i,\dots,e_n,e_{n+1}]=\alpha_{1,\dots,\hat{i},\dots,n}e_{n+4}, \quad 2\leq i\leq n,\\ [e_1,\dots,\hat{e}_i,\dots,e_n,e_{n+3}]=\alpha_{1,\dots,\hat{i},\dots,n,n+3}e_{n+4}, \quad 1\leq i\leq n,\\ [e_2,\dots,\hat{e}_i,\dots,e_n,e_{n+1},e_{n+2}]=\alpha_{1,\dots,\hat{i},\dots,n,n+3}e_{n+4}, \quad 2\leq i\leq n. \end{cases}$$

Since e_{n+3} is not belong to $Z(A)$,

at least one of $\alpha_{1,\dots,\hat{i},\dots,n,n+3}$, $\quad 1\leq i\leq n$ is not equal to zero.

Up to isomorphism, we have two possibilities: (a) If $\alpha_{2,\dots,n,n+3}\neq 0$, A can be written as follows:

$$\begin{cases} [e_1,\dots,e_n]=e_{n+1}, \qquad [e_2,\dots,e_{n+1}]=e_{n+2},\\ [e_2,\dots,e_n,e_{n+2}]=e_{n+3}, \quad [e_2,\dots,e_n,e_{n+3}]=\alpha_{2,\dots,n,n+3}e_{n+4},\\ [e_1,\dots,\hat{e}_i,\dots,e_n,e_{n+1}]=\alpha_{1,\dots,\hat{i},\dots,n}e_{n+4}, \quad 2\leq i\leq n. \end{cases}$$

If $\alpha_{1,\dots,\hat{i},\dots,n}=0$, $\quad 2\leq i\leq n$, we have the following algebra:

$$A=\langle e_1,\dots,e_{n+4}:[e_1,\dots,e_n]=e_{n+1},[e_2,\dots,e_{n+1}]=e_{n+2},$$
$$[e_2,\dots,e_n,e_{n+2}]=e_{n+3},[e_2,\dots,e_n,e_{n+3}]=e_{n+4}\rangle.$$

This algebra denoted by $A_{n,n+4,12}$. Otherwise, without loss of generality assume that $\alpha_{1,3,\dots,n}\neq 0$.

Regarding a suitable change of basis,

$$A=\langle e_1,\dots,e_{n+4}:[e_1,\dots,e_n]=e_{n+1},[e_2,\dots,e_{n+1}]=e_{n+2},$$
$$[e_2,\dots,e_n,e_{n+2}]=e_{n+3},[e_2,\dots,e_n,e_{n+3}]=[e_1,e_3,\dots,e_{n+1}]=e_{n+4}\rangle.$$

This algebra is denoted by $A_{n,n+4,13}$.

(b) If $\alpha_{1,3,\dots,n,n+3}\neq 0$, by a suitable change of basis, the following algebra is estimated.

$$A_{n,n+4,14}=\langle e_1,\dots,e_{n+4}:[e_1,\dots,e_n]=e_{n+1},[e_2,\dots,e_{n+1}]=e_{n+2},$$
$$[e_2,\dots,e_n,e_{n+2}]=e_{n+3},[e_3,\dots,e_{n+2}]=[e_1,e_3,\dots,e_n,e_{n+3}]=e_{n+4}\rangle.$$

□

Lemma 2.2. *Let A be a $(n+4)$-dimensional nilpotent n-Lie algebra of maximal class and let $A/\langle e_{n+4}\rangle \cong A_{n,n+3,5}$, then A is isomorphic to $A_{n,n+4,10}$.*

The following theorem is an immediate consequence of lemmas 2.1 and 2.2.

Theorem 2.3. *The $(n+4)$-dimensional nilpotent n-Lie algebras of maximal class for $n>2$ over an arbitrary field are $A_{n,n+4,i}$, for $12\leq i\leq 16$.*

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On Homological Classification of Monoids by Condition $(\mathbf{P_{sc}})$

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Abstract

In 1997, Golchin and Renshaw presented Condition (P_E) and showed that this condition implies weak flatness, but the converse is not true in generally. In this paper, we present Condition (P_{sc}) as a generalization of Condition (P_E). We see also that Condition (P_{sc}) implies weak flatness, but the converse is not true. For left PSF monoids we show that the converse is also true.

Keywords: S-act, Flatness properties, Conditions (P_{sc})

Mathematics Subject Classification [2010]: Primary: 20M30 Secondary: 20M50

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1 Introduction and Preliminaries

Throughout this paper, we use S to denote a monoid and 1 denotes its identity. A non-empty set A is called a *right S-act*, usually denoted by A_S (or simply A), if S acts on A unitarian from the right, that is, there exists a mapping $A \times S \to A$, $(a, s) \mapsto as$, satisfying the conditions $a1 = a$ and $(as)t = a(st)$, for all $a \in A$ and all $s, t \in S$. Left S-act can be defined dually. From now on by S-act we mean right S-act. We refer the reader to [3,4], for basic definitions and terminologies related to semigroups and acts over monoid.

An element s of S is called *right e-cancellable*, for an idempotent $e \in S$, if $s = es$ and $ker\rho_s \leq ker\rho_e$ (ρ_x is the right translation on S, for every $x \in S$, that is, $\rho_x : S \longrightarrow S, t \mapsto tx$, for every $t \in S$). S is called *left PP* if every principal left ideal of S is projective as a left S-act. S is called *left PSF* if every principal left ideal of S is strongly flat as a left S-act. This is equivalent to saying that S is right semi-cancellative, that is, whenever $su = s'u$, for $s, s', u \in S$, there exists $r \in S$ such that $u = ru$ and $sr = s'r$ (see [1,5]).

2 Main Results

In this section we present Condition (P_{sc}) and show that this condition of acts can be transferred to their coproduct and vice versa. Also we show that S_S and the retract of every act satisfy Condition (P_{sc}). We see also that Condition (P_{sc}) implies weak flatness, but the converse is not true. For left PSF monoids we show that the converse is also true.

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[*] Speaker.

Definition 2.1. An S-act A satisfies *Condition* (P_{sc}) if $as = a't$, for $a, a' \in A$ and $s, t \in S$, implies the existence of $a'' \in A$ and $u, v, r, r' \in S$, such that $ar = a''ur$, $a'r' = a''vr'$, $rs = s$, $r't = t$ and $us = vt$.

In the following theorem, all statements are easy consequences of the definition.

Theorem 2.2. *The following statements are true:*

(1) S_S *satisfies Condition* (P_{sc}).

(2) Θ_S *satisfies Condition* (P_{sc}) *if and only if* S *is right reversible.*

(3) *For an idempotent monoid, Conditions* (P_E) *and* (P_{sc}) *are equivalent.*

(4) *Let* $A = \coprod_{i\in I} A_i$, *where each* A_i *is an* S-*act. Then* A *satisfies Condition* (P_{sc}) *if and only if each* A_i *satisfies Condition* (P_{sc}).

(5) *Let* $\{B_i \mid i \in I\}$ *is a chain of subacts of* A *and every* B_i, $i \in I$, *satisfies Condition* (P_{sc}), *then* $\bigcup_{i\in I} B_i$ *satisfies Condition* (P_{sc}).

(6) *If* A *satisfies Condition* (P_{sc}), *then every retract of* A *satisfies Condition* (P_{sc}).

Theorem 2.3. *The following statements are true:*

(1) *Condition* (P_{sc}) *implies weak flatness.*

(2) *For left* PSF *monoid* S, *Condition* (P_{sc}) *and weak flatness property are equivalent.*

(3) *If* S *be left* PP, *then for every* S-*act we have:*

$$(P_E) \Longleftrightarrow (P_{sc}) \Longleftrightarrow \textit{weakly flat}.$$

In the following examples, we show that Condition (P_{sc}) is incomparable with flatness.

Example 2.4. [flatness $\not\Rightarrow$ Condition (P_{sc})] For a proper right ideal I of S, and any $a, b, c \notin S$ set $A(I) := (\{a, b\} \times (S \setminus I)) \cup (\{c\} \times I)$ Then $A(I)$ is a right S-act. By [4, Proposition 3.12.19], $A(I)$ is flat if and only if I satisfies Condition (LU).
Now consider the monoid S with multiplication table

.	0	1	e	x
0	0	0	0	0
1	0	1	e	x
e	0	e	e	0
x	0	x	x	0

and let $I = eS = \{0, e\}$. It is easy to check that $A(I)$ is flat.But, $A(I)$ does not satisfy Condition (P_{sc}).

From the above example, we deduce that weak flatness does not imply Condition (P_{sc}).

Example 2.5 ([2, Example 2]). [Condition $(P_{sc}) \not\Rightarrow$ flatness] Let $U = \{a, b\}$, $V = \{c, d\}$ be left zero semigroups and let $S = U\dot{\cup}V$. Extend the multiplications in U and V to S by letting a and b be left zero elements for S and $cU = \{a\}$, $dU = \{b\}$. It is shown in [2], that all right S^1-acts satisfy Condition (P_E) but not all right S^1-acts are flat. On the other hand, Condition (P_E) implies Condition (P_{sc}). Hence all right S^1-acts satisfy Condition (P_{sc}) but not all right S^1-acts are flat.

3 Classification of Monoids by Condition (P_{sc})

In this section we present some results on homological classifications. We start with questions where all acts satisfy Condition (P_{sc}).

Theorem 3.1. *The following statements are equivalent:*

(1) *All S-acts satisfy Condition (P_{sc}).*

(2) *S is regular and satisfies Condition (R).*
(R) : for any elements $s, t \in S$, there exists $w \in Ss \cap St$ such that $w\rho(s,t)s$.

Proof. (1) $\Rightarrow$ (2). By part (1) of Theorem 2.3 and [4, Theorem 4.7.5], all S-acts, are weakly flat. But, S is regular and satisfies Condition (R).

(2) $\Rightarrow$ (1). Since every regular monoid is left PP, so by [4, Theorem 4.7.5] and part (3) of Theorem 2.3, the result follows. ☐

For fixed elements $u, v \in S$, define a binary relation $P_{u,v}$ on S as follows:

$$(x,y) \in P_{u,v} \quad \Leftrightarrow \quad ux = vy \ (x, y \in S).$$

For $s, t \in S$, let $\mu_{s,t} = \ker \lambda_s \vee \ker \lambda_t$ and for any right ideal I of S, let ρ_I denote the right Rees congruence on S, i.e., for $x, y \in S$,

$$(x,y) \in \rho_I \Leftrightarrow (x = y) \vee (x, y \in I), \qquad L(x,y) = \{(a,b) \in S \times S |\ ax = by\}.$$

Note that $L(x,y)$ is either empty or a subact of ${}_S(S \times S)$. Similarly we define

$$R(x,y) = \{(a,b) \in S \times S |\ xa = yb\}.$$

Therefore $P_{u,v} = R(u,v)$, for every $u, v \in S$.

Theorem 3.2. *The following statements are equivalent:*

(1) *All fg-weakly injective S-acts satisfy Condition (P_{sc}).*

(2) *All cofree S-acts satisfy Condition (P_{sc}).*

(3) *for all $s, t \in S$, there exist $u, v, r, r' \in S$ such that $rs = s$, $r't = t$, $(s,t) \in P_{u,v}$ (or $(s,t) \in P_{ur,vr'}$) and the following conditions hold:*

  (i) $P_{ur,vr'} \subseteq P_{r,s} \circ \mu_{s,t} \circ P_{t,r'}$
  (ii) $\ker \lambda_u \cap (rS \times rS) \subseteq \rho_{sS}$
  (iii) $\ker \lambda_v \cap (r'S \times r'S) \subseteq \rho_{tS}$.

Proof. Implication (1) $\Rightarrow$ (2) is obvious, because cofree $\Rightarrow$ fg-weakly injective.
(2) $\Rightarrow$ (3). Let $s, t \in S$, S_1, S_2 be two sets such that $|S_1| = |S_2| = |S|$ and $\alpha : S \to S_1$, $\beta : S \to S_2$ are bijections. Put $X = S/\mu_{s,t} \ \dot\cup \ S_1 \ \dot\cup \ S_2$. Define the mappings $f, g : S \to X$ as follows:

$$f(x) = \begin{cases} [y]_{\mu_{s,t}} & \text{if there exists } y \in S;\ x = sy \\ \alpha(x) & \text{if } x \in S \setminus sS \end{cases}$$

and

$$g(x) = \begin{cases} [y]_{\mu_{s,t}} & \text{if there exists } y \in S;\ x = ty \\ \beta(x) & \text{if } x \in S \setminus tS. \end{cases}$$

Note thay f, g is well- defined. According to our definition of f and g, we clearly have $fs = gt$. By the assumption, the cofree S-act X^S satisfies Condition (P_{sc}), and so, there exist $u, v, r, r' \in S$ and a map $h : S \to X$, such that $fr = hur, gr' = hvr', rs = s, r't = t, us = vt$. Clearly from $us = vt$, we have $(s, t) \in P_{u,v}$ (or by $rs = s, r't = t$, and $us = vt$ we have $(s, t) \in P_{ur,vr'}$). One could show that the statements $(i), (ii)$ and (iii) are true.
(3) ⇒ (1). Suppose that A is fg-weakly injective, and that $as = a't$, for $a, a' \in A$ and $s, t \in S$. By the assumption, there exist $u, v, r, r' \in S$ such that $rs = s, r't = t, us = vt$ and conditions $(i), (ii), (iii)$ are true. Define a mapping $\varphi : urS \cup vr'S \to A$ by

$$\varphi(x) = \begin{cases} arp & \exists p \in S : \ x = urp \\ \\ a'r'q & \exists q \in S : \ x = vr'q. \end{cases}$$

Note that φ is well-defined. It is clear that φ is an S-homomorphism. Since A is fg-weakly injective, there exists an S-homomorphism $\psi : S_S \to A$ such that $\psi|_{urS \cup vr'S} = \varphi$. Put $a'' = \psi(1)$. Then $ar = \varphi(ur) = \psi(ur) = \psi(1)ur = a''ur$ and $a'r' = \varphi(vr') = \psi(vr') = \psi(1)vr' = a''vr'$, that is, A satisfies Condition (P_{sc}). □

By the proof of Theorem 3.2, we conclude that the above theorem is true for (weakly) injective S-acts.

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Results on Relative Weak Injective (Flat) Modules

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Abstract

Let R be a commutative ring, and let C be a semidualizing R-module. We introduce the notion of finitely presented C-injective modules, finitely presented C-flat modules, weak C-injective modules and weak C-flat modules. Some properties of these modules are investigated.

Keywords: Semidualizing, FP-injective, FP-flat, Weak injective, Weak flat

Mathematics Subject Classification [2010]: 13D05, 13D45, 18G20

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1 Introduction and Preliminaries

Throughout this paper R is a commutative ring and all modules are unital. The notion of semidualizing modules, defined next, was first introduced by Foxby [1]. Then Vasconcelos [9] and Golod [3] rediscovered these modules using different terminology for different purposes.

Definition 1.1. An R-module C is called *semidualizing* if C is super finitely presented, the natural homothety homomorphism $\chi_C^R : R \to \mathrm{Hom}_R(C, C)$ is an isomorphism, and the $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$.

A free R-module of rank one is semidualizing. If R is Noetherian and admits a dualizing module D, then D is a semidualizing. Note that this definition agrees with the established definition when R is Noetherian, in which case condition (i) is equivalent to C being finitely generated. An R-module is *C-projective* (resp. *C-flat* or *C-injective*) if it is isomorphic to a module of the form $P \otimes_R C$ for some projective R-module P (resp. $F \otimes_R C$ for some flat R-module F or $\mathrm{Hom}_R(C, I)$ for some injective R-module I). We let $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$ and $\mathcal{I}_C(R)$ denote the categories of C-projective, C-flat and C-injective R-modules, respectively. The *Auslander class* with respect to C is the class $\mathcal{A}_C(R)$ of R-modules M such that: $\mathrm{Tor}_i^R(C, M) = 0 = \mathrm{Ext}_R^i(C, C \otimes_R M)$ for all $i \geqslant 1$, and the natural map $\gamma_C^M : M \to \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism. The *Bass class* with respect to C is the class $\mathcal{B}_C(R)$ of R-modules M such that: $\mathrm{Ext}_R^i(C, M) = 0 = \mathrm{Tor}_i^R(C, \mathrm{Hom}_R(C, M))$ for all $i \geqslant 1$, and the natural evaluation map $\xi_M^C : C \otimes_R \mathrm{Hom}_R(C, M) \to M$ is an isomorphism.

Let C be a semidualizing R-module. In [4], it is shown that the class $\mathcal{P}_C(R)$ is precovering. So, one can iteratively take precovers to construct an *augmented proper $\mathcal{P}_C$-projective resolution* for any R-module M, that is, a complex $X^+ = \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to M \to 0$ which is

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[*] Speaker.

$\mathrm{Hom}_R(\mathcal{P}_C(R), -)$-exact. The truncated complex $X = \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to 0$ is a *proper $\mathcal{P}_C$-projective resolution* of M. Dually, in [4] it is proved that the class $\mathcal{I}_C(R)$ is enveloping. So, for an R-module N one can construct an *augmented proper $\mathcal{I}_C$-injective coresolution*, that is, a complex $Y^+ = 0 \to N \to \mathrm{Hom}_R(C, I^0) \to \mathrm{Hom}_R(C, I^1) \to \cdots$ which is $\mathrm{Hom}_R(-, \mathcal{I}_C(R))$-exact. Also, in [4] it is shown that the class $\mathcal{F}_C(R)$ is covering. Similarly for an R-module M one can construct an *augmented proper $\mathcal{F}_C$-flat resolution.*

Definition 1.2. Let C be a semidualizing R-module, and let M and N be R-modules. Let J be a proper $\mathcal{I}_C$-coresolution of N. For each i, set $\mathrm{Ext}^i_{\mathcal{M}\mathcal{I}_C}(M, N) := \mathrm{H}_{-i}(\mathrm{Hom}_R(M, J))$. Let G be a proper $\mathcal{F}_C$-resolution of M. For each i, set: $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N) := \mathrm{H}_i(G \otimes_R N)$.

2 Main Results

The notion of FP-injective modules (resp. FP-flat modules) is introduced in [7], as a generalization of injective modules (resp. flat modules). Recently, the weak injective modules (resp. weak flat modules) are introduced in [2] as a generalization of FP-injective modules (resp. FP-flat modules). In this section, we introduce the notion of relative FP-injective modules and weak injective modules with respect to the semidualizing R-module C. Also, we introduce the notion of relative FP-flat modules and weak flat modules and investigate some properties of these modules.

Definition 2.1. Let C be a semidualizing R-module.

(i) An R-module M is called *weak C-injective* (resp. *finitely presented C-injective*) if $\mathrm{Ext}^1_{\mathcal{M}\mathcal{I}_C}(F, M) = 0$ for any super finitely presented R-module F (resp. for any finitely presented R-module F).

(ii) An R-module N is called *weak C-flat* (resp. *finitely presented C-flat*) in the case that $\mathrm{Tor}_1^{\mathcal{F}_C\mathcal{M}}(N, F) = 0$ for any super finitely presented R-module F (resp. for any finitely presented R-module F).

Theorem 2.2. *Let C be a semidualizing R-module. Then the following statements hold.*

(i) *Let R be a weak C-injective R-module, and let M be a super finitely presented R-module such that $M^* = \mathrm{Hom}_R(M, R)$ is super finitely presented R-module and $M^* \in \mathcal{A}_C(R)$. Then M is reflexive.*

(ii) *Let R be a generalized coherent ring, and suppose that every super finitely presented R-module belongs to $\mathcal{A}_C(R)$. Then R is weak C-injective if and only if every super finitely presented module is reflexive.*

Proof. (i) Let M be a super finitely presented R-module. Then there exists the exact sequence $P_1 \to P_0 \to M \to 0$ such that P_0 and P_1 are finitely generated projective R-modules. Assume that $N = \mathrm{Coker}((P_0)^* \to (P_1)^*)$. Then we have the exact sequence $0 \to M^* \to (P_0)^* \to (P_1)^* \to N \to 0$. Therefore, N is super finitely presented and $N \in \mathcal{A}_C(R)$ by [6, Corollary 3.1.8]. On the other hand, we have the exact sequence $0 \to \mathrm{Ext}^1_R(N, R) \to M \to M^{**} \to \mathrm{Ext}^2_R(N, R) \to 0$, by [5, Lemma 2.2]. By [8, Corollary 4.2], $\mathrm{Ext}^1_R(N, R) \cong \mathrm{Ext}^1_{\mathcal{M}\mathcal{I}_C}(N, R) = 0$, since R is a weak C-injective R-module. Also, $\mathrm{Ext}^2_R(N, R) \cong \mathrm{Ext}^2_{\mathcal{M}\mathcal{I}_C}(N, R) = 0$. So, we get the assertion.

(ii) " $\Rightarrow$ " Let R be a weak C-injective R-module, and let M be a super finitely presented R-module. By assumption, M^* is a super finitely presented R-module, and $M^* \in \mathcal{A}_C(R)$. Then M is reflexive by (i), as desired.

" $\Leftarrow$ " Let M be a super finitely presented R-module. It is sufficient to prove that $\mathrm{Ext}^1_{\mathcal{M}\mathcal{I}_C}(M, R) = 0$. By [8, Theorem 4.2], $\mathrm{Ext}^1_{\mathcal{M}\mathcal{I}_C}(M, R) \cong \mathrm{Ext}^1_R(M, R)$, since M and R belong to $\mathcal{A}_C(R)$. As the proof of item (i), we can get the exact sequence $0 \to M^* \to (P_0)^* \to (P_1)^* \to N \to 0$, where P_0 and P_1 are finitely generated projective R-modules. Since R is a generalized coherent ring, we get that N is super finitely presented and $N \in \mathcal{A}_C(R)$. Note that M, P_0, P_1, and N are reflexive, by assumption. Therefore, the exact sequence $P_1 \to P_0 \to M \to 0$ implies that $M \cong \mathrm{Coker}((P_1)^{**} \to (P_0)^{**}) = \mathrm{Coker}((P_1) \to (P_0))$. By [5, Lemma 2.2], we have the exact sequence $0 \to \mathrm{Ext}^1_R(M, R) \to N \to N^{**}$. Since N is reflexive, we get that $\mathrm{Ext}^1_R(M, R) = 0$, as desired. ☐

In the following, we show that if R is self weak C-injective, then every super finitely presented module $M \in \mathcal{A}_C(R)$ is Gorenstein projective, provided some special conditions.

Theorem 2.3. *Let C be a semidualizing R-module, and let R be a weak C-injective module. Then every super finitely presented module $M \in \mathcal{A}_C(R)$ is Gorenstein projective, provided that M^* is a super finitely presented module and $M^* \in \mathcal{A}_C(R)$.*

Proof. Let $M \in \mathcal{A}_C(R)$ be a super finitely presented module such that $M^* \in \mathcal{A}_C(R)$ be a super finitely presented module. Then there exists the long exact sequence $\cdots \to F_n \to \cdots \to F_1 \to F_0 \to M^* \to 0$ of R-modules such that F_i is finitely generated projective for each $i \geqslant 0$. For each $i \geqslant 0$, we set $K_i = \operatorname{Ker}(F_{i+1} \to F_i)$. Then K_i is a super finitely presented module and $K_i \in \mathcal{A}_C(R)$, by [6, Proposition 3.1.7]. By [8, Corollary 4.2], we have $\operatorname{Ext}^1_R(K_i, R) \cong \operatorname{Ext}^1_{\mathcal{MI}_C}(K_i, R) = 0$, since R is weak C-injective. Therefore, we have the long exact sequence $0 \to M^{**} \to F_0^* \to F_1^* \to \cdots \to F_n^* \to \cdots$. By Theorem 2.2, M is reflexive, so we have the long exact sequence $0 \to M \to F_0^* \to F_1^* \to \cdots \to F_n^* \to \cdots$. On the other hand, M is a super finitely presented R-module. Then there exists the long exact sequence $\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ of R-modules such that P_i is finitely generated projective for each $i \geqslant 0$. So we get the following complete projective resolution of M $\cdots \to P_n \to \cdots \to P_1 \to P_0 \to F_0^* \to F_1^* \to \cdots \to F_n^* \to \cdots$, where $M \cong \operatorname{Ker}(P_0 \to F_0^*)$. Assume that L is an arbitrary cosyzygy of this sequence. Then $L \in \mathcal{A}_C(R)$, and L is a super finitely presented R-module. Therefore, by [8, Corollary 4.2], we have $\operatorname{Ext}^1_R(L, R) \cong \operatorname{Ext}^1_{\mathcal{MI}_C}(L, R) = 0$, since R is weak C-injective. This means that for every projective R-module Q, $\operatorname{Hom}_R(-, Q)$ leaves this sequence exact. So, M is Gorenstein projective, as desired. □

3 Conclusion

The notion of a “semidualizing module” is a central notion in relative homological algebra. Among various research areas on semidualizing modules, one sometimes focuses on extending the “absolute” classical notion of homological algebra to the “relative” setting with respect to a semidualizing module. In this paper, we investigate some properties of relative weak injective modules (resp. weak flat modules).

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Some Properties of the Relative Grade of Modules

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Abstract

Let R be a commutative Noetherian ring, and let C be a semidualizing R-module. For R-modules M and N, the notions $\mathrm{grade}_{\mathcal{P}_C}(M,N)$ and $\mathrm{grade}_{\mathcal{I}_C}(M,N)$ are introduced as the relative setting of the notion $\mathrm{grade}(M,N)$ with respect to C. Some properties of the notions $\mathrm{grade}_{\mathcal{P}_C}(M,N)$, $\mathrm{grade}_{\mathcal{I}_C}(M,N)$, and $\mathrm{grade}(M,N)$ are investigated.

Keywords: Semidualizing module, Grade of module, Perfect module

Mathematics Subject Classification [2010]: Primary: 13H10, 13C15

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1 Introduction and Preliminaries

Throughout this paper R is a commutative Noetherian ring and all modules are unital. A finitely generated R-module C is called *semidualizing* if the natural homothety homomorphism $\chi_C^R : R \to \mathrm{Hom}_R(C,C)$ is an isomorphism and $\mathrm{Ext}_R^{\geqslant 1}(C,C) = 0$. An R-module D is called *dualizing* if it is semidualizing and has finite injective dimension. This notion goes back at least to Foxby [2]. For a semidualizing R-module C, we set

$$\mathcal{P}_C(R) = \{P \otimes_R C \mid P \text{ is a projective } R\text{-module}\},$$

$$\mathcal{I}_C(R) = \{\mathrm{Hom}_R(C,I) \mid I \text{ is an injective } R\text{-module}\}.$$

The R-modules in $\mathcal{P}_C(R)$, and $\mathcal{I}_C(R)$ are called *C-projective*, and *C-injective*, respectively. When $C = R$, we omit the subscript and recover the classes of projective, and injective R-modules. For any R-module M one can iteratively take precovers to construct an *augmented proper $\mathcal{P}_C$-projective resolution*. Let M and N be R-modules. Let L be a proper $\mathcal{P}_C$-resolution of M, and let J be a proper $\mathcal{I}_C$-coresolution of N. For each i, set

$$\mathrm{Ext}^i_{\mathcal{P}_C}(M,N) := \mathrm{H}_{-i}(\mathrm{Hom}_R(L,N))$$
$$\mathrm{Ext}^i_{\mathcal{I}_C}(M,N) := \mathrm{H}_{-i}(\mathrm{Hom}_R(M,J)).$$

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[*] Speaker.

These functors are studied in [4] and [5]. We define

$$\begin{aligned}
\operatorname{grade}(M,N) &= \inf\left\{i \mid \operatorname{Ext}^i_R(M,N)\neq 0\right\},\\
\operatorname{grade}_{\mathcal{P}_C}(M,N) &= \inf\left\{i \mid \operatorname{Ext}^i_{\mathcal{P}_C}(M,N)\neq 0\right\},\\
\operatorname{grade}_{\mathcal{I}_C}(M,N) &= \inf\left\{i \mid \operatorname{Ext}^i_{\mathcal{I}_C}(M,N)\neq 0\right\}.
\end{aligned}$$

In this paper, some properties of the notions $\operatorname{grade}(M,N)$, $\operatorname{grade}_{\mathcal{P}_C}(M,N)$, and $\operatorname{grade}_{\mathcal{I}_C}(M,N)$ are investigated.

2 Main Results

Throughout this section, let C be a semidualizing module.

Theorem 2.1. *Let M and N be finitely generated R-modules. Then the following statements hold.*

(i) $\operatorname{grade}(M,N)=\inf\{\operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} \mid \mathfrak{p}\in\operatorname{Supp}_R(M)\}$.

(ii) $\operatorname{grade}_{\mathcal{P}_C}(M,N)=\inf\{\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{Hom}_R(C,N)_\mathfrak{p}) \mid \mathfrak{p}\in\operatorname{Supp}_R(M)\}$.

(iii) $\operatorname{grade}_{\mathcal{I}_C}(M,N)=\inf\{\operatorname{depth}_{R_\mathfrak{p}}((C\otimes_R N)_\mathfrak{p}) \mid \mathfrak{p}\in\operatorname{Supp}_R(M)\}$.

Proof. (i) It is proved in [6, Theorem 1.2].

(ii) By [5, Theorem 4.1], $\operatorname{Ext}^i_{\mathcal{P}_C}(M,N)\cong\operatorname{Ext}^i_R(\operatorname{Hom}_R(C,M),\operatorname{Hom}_R(C,N))$, for every $i\geqslant 0$. Hence

$$\begin{aligned}
\operatorname{grade}_{\mathcal{P}_C}(M,N) &= \operatorname{grade}(\operatorname{Hom}_R(C,M),\operatorname{Hom}_R(C,N))\\
&= \inf\{\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{Hom}_R(C,N)_\mathfrak{p}) \mid \mathfrak{p}\in\operatorname{Supp}_R(\operatorname{Hom}_R(C,M))\}\\
&= \inf\{\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{Hom}_R(C,N)_\mathfrak{p}) \mid \mathfrak{p}\in\operatorname{Supp}_R(M)\}.
\end{aligned}$$

(iii) It is proved in the same argument of (ii). ☐

For a finitely generated R-module M, the *grade of M* is denoted by $\operatorname{grade} M$ and defined by $\operatorname{grade} M=\operatorname{grade}(M,R)$.

Corollary 2.2. *Let M be a finitely generated R-module. Then* $\operatorname{grade} M=\operatorname{grade}(M,C)=\operatorname{grade}_{\mathcal{P}_C}(M,C)=\operatorname{grade}_{\mathcal{I}_C}(M,R)$

Proof. Note that $\operatorname{depth}_{R_\mathfrak{p}} C_\mathfrak{p}=\operatorname{depth}_{R_\mathfrak{p}} R_\mathfrak{p}$ for every $\mathfrak{p}\in\operatorname{Supp}_R(M)$. Now, the assertions follow from Theorem 2.1. ☐

The *$\mathcal{P}_C$-projective dimension* of an R-module M is

$$\mathcal{P}_C\text{-}\operatorname{pd}_R(M)=\inf\{\sup\{n|X_n\neq 0\}|X \text{ is a proper } \mathcal{P}_C\text{-projective resolution of M}\}.$$

The *$\mathcal{I}_C$-injective dimension*, denoted $\mathcal{I}_C\text{-}\operatorname{id}_R(-)$ is defined dually. Let M be a finitely generated R-module. Then

(i) M is called *C-perfect* if $\operatorname{grade} M=\mathcal{P}_C\text{-}\operatorname{pd}_R(M)$.

(ii) M is called *G_C-perfect* if $\operatorname{grade} M=\operatorname{G}_C\text{-}\dim_R(M)$.

If $C=R$, we use the term perfect (G-perfect) instead of R-perfect (G_R-perfect). In [6], the authors proved some properties of G-perfect modules similar to the classical properties of perfect modules. In the following, we investigate some properties of C-perfect and G_C-perfect modules.

Proposition 2.3. *Let $M\neq 0$ be a finitely generated R-module. Then the following statements hold.*

(i) $\operatorname{grade} M=\operatorname{grade}(\operatorname{Hom}_R(C,M))$.

(ii) $\operatorname{grade} M=\operatorname{grade}(C\otimes_R M)$.

(iii) *M is perfect if and only if $C\otimes_R M$ is C-perfect.*

(iv) *M is C-perfect if and only if $\operatorname{Hom}_R(C,M)$ is perfect.*

Proof. *(i)* By Corollary 2.2, we have

$$\begin{aligned}\operatorname{grade} M &= \operatorname{grade}_{\mathcal{P}_C}(M,C)\\ &= \operatorname{grade}(\operatorname{Hom}_R(C,M),R)\\ &= \operatorname{grade}(\operatorname{Hom}_R(C,M)).\end{aligned}$$

So, we get the claim.

(ii) By Corollary 2.2, we have

$$\begin{aligned}\operatorname{grade} M &= \operatorname{grade}_{\mathcal{I}_C}(M,R)\\ &= \operatorname{grade}(C\otimes_R M,C)\\ &= \operatorname{grade}(C\otimes_R M).\end{aligned}$$

Therefore, we get the assertion.

(iii) By [5, Theorem 2.11], $\mathcal{P}_C$-$\operatorname{pd}_R(C\otimes_R M)=\operatorname{pd}_R(M)$. So, we get the result by item *(ii)*.

(iv) By [5, Theorem 2.11], $\mathcal{P}_C$-$\operatorname{pd}_R(M)=\operatorname{pd}_R(\operatorname{Hom}_R(C,M))$. Therefore, M is C-perfect if and only if $\operatorname{Hom}_R(C,M)$ is perfect by item *(i)*. ☐

Theorem 2.4. *Let M be a G_C-perfect R-module such that $\operatorname{grade} M=n$. Then $\operatorname{Ext}^n_R(M,C)$ is G_C-perfect R-module of grade n.*

Proof. The proof is by induction on n. If $n=0$, then the assertion holds by [3, Proposition 5.1.4]. Now let $n>0$. Then there exists an R-regular element x such that $xM=0$. By [1, Lemma 3.1.16], $\operatorname{grade}_{R/xR} M=\operatorname{grade}_{R/xR}(M,C/xC)=n-1$, and $\mathrm{G}_{C/xC}$-$\dim_{R/xR}(M)=n-1$. So M is $G_{C/xC}$-perfect as an (R/xR)-module and hence by induction hypothesis $\operatorname{Ext}^{n-1}_{R/xR}(M,C/xC)$ is $G_{C/xC}$-perfect of grade $n-1$. Therefore, we get the result by [1, Lemma 3.1.16]. ☐

Proposition 2.5. *Let R be a local ring, and let M be a finitely generated R-module such that G_C-$\dim_R M<\infty$. Then the following statements hold.*

(i) *If M is Cohen-Macaulay, then M is G_C-perfect.*

(ii) *If R is Cohen-Macaulay and M is G_C-perfect, then M is Cohen-Macaulay.*

Proof. *(i)* Note that

$$\begin{aligned}\mathrm{G}_C\text{-}\dim_R(M) &= \operatorname{depth} R-\operatorname{depth}_R(M)\\ &= \operatorname{depth} R-\dim_R M\\ &= \operatorname{depth}_R(\operatorname{Hom}_R(C,C))-\dim_R(M)\\ &\leqslant \operatorname{grade}_{\mathcal{P}_C}(M,C).\end{aligned}$$

Also, $\operatorname{grade}_{\mathcal{P}_C}(M,C)=\operatorname{grade} M$. So, we get the assertion.

(ii) It is proved similarly. ☐

Proposition 2.6. *Let R be a local ring, and let C be a dualizing R-module. A finitely generated R-module M is Cohen-Macaulay if and only if it is G_C-perfect.*

Proof. Note that R is a Cohen-Macaulay ring, since R has a dualizing module C. Also, G_C-$\dim_R(M)<\infty$ by [3, Proposition 6.4.6]. Now the assertion follows from Proposition 2.5. ☐

3 Conclusion

It is shown that for a G_C-perfect R-module M of grade n, the R-module $\mathrm{Ext}_R^n(M,C)$ is G_C-perfect of grade n. Also, it is shown that a finitely generated R-module M is Cohen-Macaulay if and only if it is G_C-perfect, provided that C is dualizing for the local ring R.

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Characterization of Monoids by Condition (PWP_S) in Act-S

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Abstract

Valdis Laan introduced Condition (PWP). Then Golchin and Mohammadzadeh introduced Condition (PWP_E), such that Condition (PWP) implies it but the converse is not true in general. In this paper at first we introduce a generalization of Condition (PWP_E), called Condition (PWP_S). Then will give some general properties and a characterization of monoids for which all right acts satisfy this condition.

Keywords: Right S-act, Condition (PWP_S).

Mathematics Subject Classification [2010]: Primary: 20M30, Secondary: 20M50

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1 Introduction and Preliminaries

In this paper at first we introduce a generalization of Condition (PWP_E), called Condition (PWP_S) and will give some general properties also we show that Condition (PWP_E) implies Condition (PWP_S) but the converse is not true. Then, we will give a characterization of monoids S over which all right S-acts satisfy Condition (PWP_S).

2 Characterization of Monoids by Condition (PWP_S) of Right Acts

Recall from [1] as follow:

The right S-act A satisfies Condition (PWP_E), if for all $a, a' \in A$, $s \in S$,

$$as = a's \Rightarrow (\exists a'' \in A)(\exists u, v \in S)(\exists e, f \in E(S))$$
$$(ae = a''ue, a'f = a''vf, es = s = fs \text{ and } us = vs)$$

Definition 2.1. The right S-act A satisfies Condition (PWP_S), if for all $a, a' \in A$, $s \in S$,

$$as = a's \Rightarrow (\exists a'' \in A)(\exists u, v, r, r' \in S)$$
$$(ar = a''u, a'r' = a''v, rs = s = r's \text{ and } us = vs).$$

Clearly, Condition (PWP_E) implies Condition (PWP_S) but the converse is not true, see the following example.

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*Speaker.

Example 2.2. Let $S_1 = \{a^i | i \in \mathbb{N}, a^i a^j = a^{ij}\}$, $S_2 = \{b^i | i \in \mathbb{N}, b^i b^j = b^{ij}\}$, $S_3 = \{d^i | i \in \mathbb{N} \setminus 1, d^i d^j = d^{ij}\}$, $(I, \leq)$ be a totally ordered set which has no the maximum and minimum element, and $S_4 = \{h_i^m | i \in I, m \in \mathbb{N}\}$ such that

$$h_i^m h_j^n = \begin{cases} h_j^n & i < j \\ h_i^{m+n} & i = j. \end{cases}$$

Let $T = S_1 \cup S_2 \cup S_3 \cup S_4$ such that $a^n b^m = b^m a^n = a^n d^m = d^m a^n = b^m d^n = d^n b^m = b^{mn}$, $a^n h_j^m = h_j^m a^n = b^n = b^n h_j^m = h_j^m b^n$ and $d^n h_j^m = h_j^m d^n = d^n$. Clearly T is a semigroup. Let $S = T^1$ and $J = S_2$. Obviously $A(J)$ satisfies Condition (PWP_S). Now we show that $A(J)$ does not satisfy Condition (PWP_E). Since $(a^2, x)d^3 = (b^6, z) = (a^2, y)d^3$, and $e = 1$ is the only idempotent such that $ed^3 = d^3$, there must be $a'' \in A(J)$ and $u, v \in S$ such that $(a^2, x) = a''u$, $(a^2, y) = a''v$ and $ud^3 = vd^3$. But $(a^2, x) = a''u$ implies $a'' = (1, x)$ and $u = a^2$ or $a'' = (a^2, x)$ and $u = 1$, which in every case, $(a^2, y) \neq a''v$, for every $v \in S$.

Recall from [2] that the right S-act Q is injective (Inj), if for any monomorphism $\iota : A \to B$ and any homomorphism $f : A \to Q$ there exists a homomorphism $\overline{f} : B \to Q$ such that $f = \overline{f}\iota$. It is called (fg-) weakly injective ((fg-)WI), if it is injective relative to all embeddings of (finitely generated) right ideals into S.

For elements $u, v \in S$, the relation $P_{u,v}$ is defined on S as

$$P_{u,v} = \big\{(x, y) \in S \times S | ux = vy\big\}.$$

Theorem 2.3. *For any monoid S the following statements are equivalent:*

1. *All fg-weakly injective right S-acts satisfy Condition (PWP_S);*
2. *all weakly injective right S-acts satisfy Condition (PWP_S);*
3. *all injective right S-acts satisfy Condition (PWP_S);*
4. *all cofree right S-acts satisfy Condition (PWP_S);*
5. $(\forall s \in S)$ $(\exists u, v, r, r' \in S)(rs = s = r's \wedge us = vs)$ *and the following conditions hold:*
   (i) $P_{u,v} \subseteq P_{r,s} \circ \ker \lambda_s \circ P_{s,r'}$
   (ii) $\ker \lambda_u \subseteq \ker \lambda_r$
   (iii) $\ker \lambda_v \subseteq \ker \lambda_{r'}$

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious, sine $cofree \Rightarrow Inj \Rightarrow WI \Rightarrow fg-WI$.
(4) ⇒ (5). Let S_1 and S_2 are the separate sets, where $|S_1| = |S_2| = |S|$ and $\alpha : S \to S_1$, $\beta : S \to S_2$ are bijections. Put $X = (S/\ker \lambda_s) \dot{\cup} S_1 \dot{\cup} S_2$, for $s \in S$ and define the mappings $f, g : S \to X$ as follows:

$$f(x) = \begin{cases} [y]_{\ker \lambda_s} & \text{if there exists } y \in S;\ x = sy \\ \alpha(x) & \text{if } x \in S \setminus sS \end{cases}$$

$$g(x) = \begin{cases} [y]_{\ker \lambda_s} & \text{if there exists } y \in S;\ x = sy \\ \\ \beta(x) & \text{if } \ x \in S \setminus sS \end{cases} .$$

Let $sy_1 = sy_2$, for $y_1, y_2 \in S$. Then $(y_1, y_2) \in \ker \lambda_s$, and so, $[y_1]_{\ker \lambda_s} = [y_2]_{\ker \lambda_s}$, that is, $f(sy_1) = f(sy_2)$. So f is well-defined. Similarly, g is well-defined. Since $fs = gs$, and $X^S = \big\{h : S \to X\big\}$ satisfies Condition (PWP_S), there exist mapping $h : S \to X$, $u, v, r, r' \in S$ such that $fr = hu$, $gr' = hv$, $rs = s = r's$ and $us = vs$. Let $(l_1, l_2) \in P_{u,v}$, for $l_1, l_2 \in S$, then

$$f(rl_1) = (fr)l_1 = (hu)l_1 = h(ul_1) = h(vl_2) = (hv)l_2 = (gr')l_2 = g(r'l_2).$$

Thus there exist $y_1, y_2 \in S$ such that $rl_1 = sy_1$ and $r'l_2 = sy_2$, and so $f(rl_1) = [y_1]_{\ker\lambda_s}$ and $g(r'l_2) = [y_2]_{\ker\lambda_s}$, which imply $sy_1 = sy_2$. Also

$$\begin{aligned}
&rl_1 = sy_1 \Rightarrow (l_1, y_1) \in P_{r,s}\\
&sy_1 = sy_2 \Rightarrow (y_1, y_2) \in \ker\lambda_s \Rightarrow (l_1, l_2) \in P_{r,s} \circ \ker\lambda_s \circ P_{s,r'}\\
&sy_2 = r'l_2 \Rightarrow (y_2, l_2) \in P_{s,r'}
\end{aligned}$$

that is, $P_{u,v} \subseteq P_{r,s} \circ \ker\lambda_s \circ P_{s,r'}$, and so (i) is proved.
Now let $(t_1, t_2) \in \ker\lambda_u$, for $t_1, t_2 \in S$. Then $ut_1 = ut_2$ and so

$$f(rt_1) = (fr)t_1 = (hu)t_1 = h(ut_1) = h(ut_2) = (hu)t_2 = (fr)t_2 = f(rt_2).$$

From definition f, we consider two cases as follows:
Case1. If $rt_1, rt_2 \in S \setminus sS$, then $\alpha(rt_1) = \alpha(rt_2)$, which implies $(t_1, t_2) \in \ker\lambda_r$.
Case2. If $rt_1, rt_2 \in sS$ then there exist $y_1, y_2 \in S$ such that $rt_1 = sy_1$ and $rt_2 = sy_2$. Therefore $f(rt_1) = f(rt_2)$ implies $rt_1 = sy_1 = sy_2 = rt_2$, that is $(t_1, t_2) \in \ker\lambda_r$.
Similarly, (iii) is proved.
(5) $\Rightarrow$ (1). Suppose that A is a fg-weakly injective right S-act and $as = a's$, for $a, a' \in A$ and $s \in S$. By assumption, there exist $u, v, r, r' \in S$ such that $rs = s = r's$, $us = vs$ and conditions (i), (ii) and (iii) hold. Define

$$\begin{aligned}
\varphi : uS \cup vS &\to A\\
x &\mapsto \begin{cases} arp & \exists p \in S : \ x = up \\ a'r'q & \exists q \in S : \ x = vq \end{cases}
\end{aligned}$$

First we show that φ is well-defined. If there exist $p, q \in S$ such that $up = vq$, then $(p, q) \in P_{u,v}$. By (i), there exist $y_1, y_2 \in S$ such that $(p, y_1) \in P_{r,s}$, $(y_1, y_2) \in \ker\lambda_s$ and $(y_2, q) \in P_{s,r'}$. Thus $rp = sy_1$, $sy_1 = sy_2$ and $sy_2 = r'q$. Therefore $arp = asy_1 = a'sy_1 = a'sy_2 = a'r'q$. If there exist $p_1, p_2 \in S$ such that $up_1 = up_2$ then $(p_1, p_2) \in \ker\lambda_u$, and so by (ii), $rp_1 = rp_2$, which implies $arp_1 = ae_1p_2$. If there exist $q_1, q_2 \in S$ such that $vq_1 = vq_2$, by (iii), similar to the pervious case, $a'r'q_1 = a'r'q_2$. Thus, φ is well-defined, and obviously it is a homomorphism. Since, by assumption, A is fg-weakly injective, there exists a homomorphism $\psi : S \to A$ such that $\psi|_{uS\cup vS} = \varphi$. Let $a'' = \psi(1)$. Then

$$\begin{cases} ar = \varphi(u) = \psi(u) = \psi(1)u = a''u \\ a'r' = \varphi(v) = \psi(v) = \psi(1)v = a''v \end{cases}$$

that is, A satisfies Condition (PWP_S). □

3 Conclusion

Condition (PWP_E) implies Condition (PWP_S) but the converse is not true. Also so far there is no characterization of monoid for which (fg-weak, weak) injectivity or cofreeness imply Condition (PWP_S). But we have shown it.

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Some Graphs Associated to a Hypermodule

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Abstract

One of the interesting topics in hyperstructures is relation between hyperstructures and graphs. This helps us to study some properties of hyperstructures by associated graph. The purpose of this note is the study of Cayley graphs associated to a general Krasner hypermodule. In this note first we associate two Cayley graphs to every general Krasner hypermodule and then we study the properties of these graphs. Also, the relationship between two associated graphs will be expressed.

Keywords: General Krasner hyperring, General Krasner hypermodule, Unit element, Torsion element, Cayley graph

Mathematics Subject Classification [2010]: Primary: 16Y99,97K30, 20N20, Secondary: 16D80, 05C60

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1 Introduction and Preliminaries

First time, the hyperstructure theory was introduced in 1934 by the French mathematician F. Marty [4]. He introduced the notion of hyperoperations and then hypergroups which are a generalization of groups. These led to many types of hyperrings, for example, general hyperrings, multiplicative hyperrings and Krasner hyperrings. The general hyperring is obtained by considering both addition and multiplication as hyperoperations. The multiplicative hyperring is obtained by considering the multiplication as a hyperoperation, while the addition is a binary operation. The Krasner hyperring [3], a well-known type of hyperrings, was introduced in 1983 by M. Krasner. The Krasner hyperring is obtained by considering the addition as a hyperoperation and the multiplication as a binary operation. Hyperrings have been the starting point for the study of hypermodules, which were considered by R. Ameri, P. Cosini, B. Davvaz, J. Zhan and others, the general Krasner hyperrings and general Krasner hypermodules defined in [1]. The Cayley graph introduction by Arthur Cayley in 1878 is a useful tool for connecting between group theory and the theory of algebraic graphs. In the last 50 years, the theory of Cayley graphs has been growing into a substantial branch in algebraic graph theory [2]. The definition of Cayley graphs of semihypergroup (or hypergroup) was introduced in [5], where some properties of Caley graphs of semihypergroups were studied. In this note we define Cayley graphs associated to an arbitrary general Krasner hypermodule and then we study the properties of these graphs.

First recall some notions of the general Krasner hyperrings and the general Krasner hyprmodules, defined in [1].

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[*] Speaker.

Let R be a nonempty set and $P^{\star}(R)$ denotes the set of all nonempty subsets of R. Every map $+_R : R \times R \longrightarrow P^{\star}(R)$, is said to be a hyperoperation and hyperstructure $(R, +_R)$ is called a hypergroupoid. For all nonempty subsets A, B of R and $x \in R$, define

$$A +_R B = \cup_{a\in A, b\in B}(a +_R b), \qquad x +_R A = \{x\} +_R A, \qquad A +_R x = A +_R \{x\}.$$

Recall that a hypergroupoid $(R, +_R)$ is called semihypergroup, if for all $x, y, z \in R$, $(x+_R y)+_R z = x+_R(y+_R z)$ and a semihypergroup $(R, +_R)$ is called hypergroup, if for $x \in R$, $x+_R R = R+_R x = R$ (reproduction axiom). A hypergroup $(R, +_R)$ is called canonical hypergroup provided that

(i) it is commutative, which means that for all $x, y \in R$, $x +_R y = y +_R x$;

(ii) there exists a unique element $0_R \in R$ such that for all; $x \in R$, $0_R +_R x = \{x\}$;

(iii) for all $x \in R$, there exists a unique element $-x \in R$ such that $0_R \in (x +_R (-x))$;

(iv) for all $x, y, z \in R$, $x \in y +_R z$ implies $y \in x +_R (-z)$ and $z \in x +_R (-y)$.

and we will denote it by $(R, +_R, 0_R)$. A system $(R, +_R, 0_R, \cdot_R)$ is called a general Krasner hyperring whenever

(i) $(R, +_R, 0_R)$ is a canonical hypergroup;

(ii) $(R, \cdot_R)$ is a semihypergroup such that for all $x \in R$ $x \cdot_R 0_R = 0_R \cdot_R x = \{0_R\}$;

(iii) $x \cdot_R (y +_R z) \subseteq (x \cdot_R y) +_R (x \cdot_R z)$ and $(y +_R z) \cdot_R x \subseteq (y \cdot_R x) +_R (z \cdot_R x)$ for all $x, y, z \in R$.

A general Krasner hyperring $(R, +_R, 0_R, \cdot_R)$ is called commutative (with unit element), if for all $x, y \in R$, $x\cdot_R y = y\cdot_R x$ (if there exists an element $1 \in R$ such that for all $x \in R$, $1\cdot_R x = x\cdot_R 1 = \{x\}$). For a given general Krasner hyperring $(R, +_R, 0_R, \cdot_R)$, a canonical hypergroup $(A, +_A, 0_A)$ together with a left external multiplication $* : R\times A \longrightarrow P^{\star}(A)$, is called a left general Krasner hypermodule over general Krasner hyperring R (we say that it is a general Krasner hypermodule and denote it by $(A, +_A, 0_A, *)$), if for all $r, s \in R$ and for all $a, b \in A$,

(i) $r * (a +_A b) \subseteq (r * a) +_A (r * b)$;

(ii) $(r +_R s) * a \subseteq (r * a) +_A (s * a)$;

(iii) $(r \cdot_R s) * a \subseteq r * (s * a)$;

(iv) $0_R * a = \{0_A\}$.

A map $f : A \longrightarrow B$ is called a good R-homomorphism of general Krasner hypermodules if, for all $x, y \in A$ and for all $r \in R$, $f(x +_A y) = f(x) +_B f(y)$ and $f(r *_A x) = r *_B f(x)$. A good R-homomorphism f is called a good R-monomorphism, if f is a one to one map, a good R-epimorphism, if f is an onto map and a good R-isomorphism, if f is a bijective map.

A homomorphism from a general Krasner hyperring $(R, +_R, \cdot_R)$ into a general Krasner hyperring $(S, +_S, \cdot_S)$ is a function $f : R \longrightarrow S$ such that $f(x +_R y) \subseteq f(x) +_S f(y)$ and $f(x \cdot_R y) = f(x) \cdot_R f(y)$, for all $x, y \in R$. A homomorphism f from $(R, +_R, \cdot_R)$ into $(S, +_S, \cdot_S)$ is said to be a good homomorphism if $f(x +_R y) = f(x) +_S f(y)$, for all $x, y \in R$. An isomorphism from $(R, +_R, \cdot_R)$ into $(S, +_S, \cdot_S)$ is a bijective good homomorphism from $(R, +_R, \cdot_R)$ onto $(S, +_S, \cdot_S)$. The general Krasner hyperring $(R, +_R, \cdot_R)$ and $(S, +_S, \cdot_S)$ are said to be isomorphic.

Now, we recall some definitions of graph theory which are needed in this note.

Let X be a graph with the vertex set $V(X)$. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. The categorical product of G and H is the graph, denoted by $G \times H$, and vertex set $V(G) \times V(H)$, such that vertices $\{g, h\}$ and $\{g', h'\}$ are adjacent precisely if $\{g, g'\} \in E(G)$ and $\{h, h'\} \in E(H)$.

2 Main Results

First, we introduce the notion of weak unit and torsion element and after using it, we associate two Cayley graphs to a general Krasner hypermodule.

let R be a general Krasner hyperring with nonezero unit and $(A, +_A, *)$ be a general Krasner hypermodule. Then $\Gamma(A) = \{a \in A : 0_A \in r * a\, for\, some\, 0_R \neq r \in R\}$ and $U(R) = \{b \in R : \exists a \in R\, 1_R \in b \cdot_R a\}$ are the set of torsion elements of A and the set of weak unit elements, respectively. The torsion-unitary Cayley graph of A is a graph with elements of $A \times R$ as vertices and two distinct vertices $(a, r), (b, s)$ are adjacent if $(a, r) - (b, s) \subseteq \Gamma(A) \times U(R)$. This graph is denoted by $\Gamma_R(A)$. The torsion Cayley graph of A, denote by $\Gamma\Gamma_R(A)$, is the graph whose vertices set is A and in which $\{m, n\}$ is an edge if $m - n \subseteq \Gamma(A)$.

Theorem 2.1. *Let $(A, +_A, *)$ be a general Krasner hypermodule. Then:*

(i) If $\Gamma_R(A)$ is complete graph then $A = 0$ and $U(R) = R \setminus 0_R$;

(ii) If $\Gamma(A) = 0$, $A = 0$, $U(R) = R \setminus 0_R$ then $\Gamma_R(A)$ is complete graph.

Theorem 2.2. *Let $(A, +_A, *_A)$ and $(B, +_B, *_B)$ are general Krasner hypermodules over a general Krasner hyperring R and $\Gamma\Gamma_R(A) \cong \Gamma\Gamma_R(B)$. Then $\Gamma_R(A) \cong \Gamma_R(B)$.*

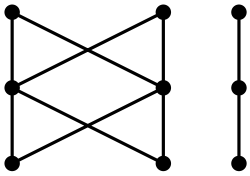
Example 2.3. According to example 4.3 of [1], $(R, +_R, *)$ is a general Krasner hypermodule over general Krasner hyperring $(R, +_R, \cdot_R)$, where $+_R, \cdot_R, *$ is defined by the following tabeles:

$+_R$	0	1	a
0	0	1	a
1	1	$\{0, a\}$	1
a	a	1	$\{0, a\}$

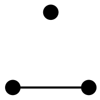
$\cdot_R$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	$\{0, a\}$

$*$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	$\{0, a\}$

Clearly, $\Gamma(R) = \{0, a\}$ and $U(R) = \{1\}$. Cayley graphs $\Gamma\Gamma_R(R)$ and $\Gamma_R(R)$ are as in follows:



$\Gamma(R)$



$\Gamma\Gamma(R)$

Theorem 2.4. *Let $(A, +_A, *_A)$ and $(B, +B, *_B)$ are general Krasner hypermodules over general Krasner hyperring $(R, +_R, 0_R, \cdot_R, 1_R)$ and $\alpha : A \longrightarrow R$ be good R-monomorphism and $U(R) = R \backslash 0_R$ and $\Gamma_R(A) \cong \Gamma_R(B)$ and $\Gamma\Gamma_R(A)$ is a graph that does not have loop. Then $\Gamma\Gamma_R(A) \cong \Gamma\Gamma_R(B)$.*

Definition 2.5. ([1]) Given a collection $\{A_i\}$ of general Krasner hypermodules, the direct product $\prod_{i \in I} A_i$ is just the product of the underlying sets A_i with general Krasner hypermodule hyperstructure given by componentwise hyperaddition and left external multiplication, i.e., for all $(a_i)_{i \in I}, (a'_i)_{i \in I} \in \prod_{i \in I} A_i$ and $r \in R$,

$$\begin{aligned} (a_i)_{i \in I} +' (a'_i)_{i \in I} &= \{(c_i)_{i \in I} : c_i \in a_i + a'_i, i \in I\}, \\ r *' (a_i)_{i \in I} &= \{(t_i)_{i \in I} : t_i \in r * a_i, i \in I\}. \end{aligned}$$

The direct sum $\bigoplus_{i \in I} A_i$ is a subhypermodule of the direct product $\prod_{i \in I} A_i$ consisting of elements $(a_i)_{i \in I}$ such that all but a finitely many a_i are zero.

Definition 2.6. ([1]) Let $(A, +_A, *)$ be a general Krasner hypermodule over general Krasner hyperring R. It is called a trivial general Krasner hypermodule, if for all $r \in R$ and for all $a \in A$, we have $|r * a| = 1$.

Definition 2.7. Let $(A, +_A, *)$ be a general Krasner hypermodule over general Krasner hyperring R. It is called an ultraassociative general Krasner hypermodule, if for all $r \in R$, $a, b \in A$, $r * (a +_A b) = (r * a) +_A (r * b)$.

Now, by the above definitions we have the following propositions.

Proposition 2.8. *Let* $(A, +_A, *_A)$ *and* $(B, +_B, *_B)$ *are general Krasner hypermodules. Then there exists a bijective homomorphism from* $\Gamma_R(A \bigoplus B)$ *onto* $\Gamma\Gamma_R(A) \times \Gamma_R(B)$.

Proposition 2.9. *If in the above proposition,* R *is a commutative Krasner hyperring and* A, B *are trivial ultraassociative general Krasner hypermodules. Then*

$$\Gamma_R(A \bigoplus B) \cong \Gamma\Gamma_R(A) \times \Gamma_R(B).$$

3 Conclusion

The torsion-unitary Cayley graph and torsion Cayley graph of a general Krasner hypermodue is presented and the properties of these graphs were studied.

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بیست و هشتمین سمینار جبر ایران
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تأثیر یادگیری الکترونیکی بر یادگیری مبحث جبر و معادله درس حسابان ۱ یازدهم دوره متوسطه دوم سال تحصیلی ۱۴۰۲-۱۴۰۳ ناحیه ۲ زاهدان

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چکیده. هدف این تحقیق تأثیر کاربرد یادگیری الکترونیکی بر آموزش مبحث جبر و معادله درس حسابان ۱ یازدهم رشته ریاضی فیزیک است. در این مقاله تأثیر یادگیری الکترونیکی بر یادگیری مبحث جبر و معادله درس حسابان ۱ دانش‌آموزان سال یازدهم ریاضی فیزیک متوسطه دوم موردبررسی قرار گرفته است. تحقیق حاضر پیرو طرح شبه آزمایشی از نوع پیش‌آزمون و پس‌آزمون است. جامعه آماری، دانش‌آموزان سال یازدهم رشته ریاضی فیزیک متوسطه دوم در سال تحصیلی ۱۴۰۲-۱۴۰۳ ناحیه ۲ زاهدان هستند که از بین آن‌ها چهار کلاس (دو کلاس دخترانه و دو کلاس پسرانه) به‌عنوان گروه آزمایش و چهار کلاس (دو کلاس دخترانه و دو کلاس پسرانه) به‌عنوان گروه گواه با روش نمونه‌گیری خوشه‌ای در دسترس انتخاب‌شده‌اند. ابزار مورداستفاده (در پیش‌آزمون و پس‌آزمون)، آزمون محقق ساخته شامل پنج سؤال با پایایی (۰/۸۱، ۰/۷۹، ۰/۸۰، ۰/۷۸، ۰/۸۱) و(۰/۸۰، ۰/۷۹، ۰/۷۹، ۰/۸۱، ۰/۸۳) بوده است. برای تجزیه‌وتحلیل یافته‌ها از میانگین، انحراف استاندارد، آزمون t گروه‌های مستقل استفاده‌شده است. نتایج نشان داد که تفاضل میانگین نمرات گروه‌های آزمایش و گواه معنادار است. بنابراین یادگیری الکترونیکی در بهبود یادگیری دانش‌آموزان اثر دارد و گروه‌های آزمایشی که از فناوری اطلاعات در یادگیری خود بهره گرفته‌اند بازخورد آن را دریافت کرده‌اند و در یادگیری پیشرفت بیشتری نشان داده‌اند.

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۱. مقدمه

فناوری اطلاعات، امکان نوینی است که فرایندهای گردآوری، ذخیره، پردازش، بازیابی و اشاعه اطلاعات را دچار تحول نموده و موجب آسان شدن و سرعت گرفتن این فرایندها به نفع بشر شده است. حوزه آموزش نیز از این پدیده متاثر شده و تحولات وسیعی در این عرصه به وقوع پیوسته است. یادگیری الکترونیکی، به عنوان رویکردی تازه در ارایه محیط یادگیری مجهز، خوش طرح، تعاملی و یادگیرنده محور برای هر کس، در هر جا و در هر زمان با بکارگیری منابع و مشخصه‌های فناوری‌های مختلف دیجیتالی و همسو با شکل‌های دیگر محیط‌های آموزشی برای ایجاد نظامی آزاد، منعطف و توزیع شده در آموزش تعریف می‌شود [۱]. سیستم‌های مبتنی بر رایانه کاربرد گسترده‌ای در آموزش دارند. این سیستم‌ها محدودیت زمان و مکان را برای آموزش از بین برده‌اند. در سیستم مدیریت یادگیری، رایج ترین سیستم رایانه‌ای برای آموزش، معلم دانش را با ابزارهایی مانند متن یا چند رسانه‌ای در اختیار دانشجو قرار می‌دهد و با وی

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ارتباط برقرار می‌نماید، هزینه پایین این نوع سیستم موجب استفاده گسترده از آن شده است [۲] یادگیری الکترونیکی به عنوان بارزترین کاربرد فناوری اطلاعات و ارتباطات، وجهی به منشور آموزش در سطح پایه و عالی افزوده است که به عنوان الگویی جدید، حوزه آموزش را دگرگون ساخته است و در آن امکان ارائه آموزش‌های اختصاصی شده وجود دارد. کلارک و مایر[1]، یادگیری الکترونیکی را نوعی یادگیری می‌دانند که توسط رایانه از طریق لوح فشرده، اینترنت یا اینترانت صورت می‌گیرد. این نوع یادگیری ویژگی‌های زیر را در بردارد: شامل محتوایی متناسب با اهداف آموزشی است؛ جهت تسهیل یادگیری از روش‌های مختلف آموزشی نظیر مثال‌ها و تمرین استفاده می‌کند؛ برای انتقال محتوا و روش‌ها عناصر رسانه‌ای نظیر تصاویر و واژه‌ها را به‌کار می‌گیرد؛ می‌تواند توسط مربی آموزش داده شود (یادگیری الکترونیکی همزمان) یا برای مطالعات خودآموز فردی طراحی گردد (یادگیری الکترونیکی غیرهمزمان)؛ بر مبنای اطلاعات و مهارت‌های جدید که با اهداف یادگیری فردی ارتباط دارد، تشکیل شده و یا عملکرد سازمانی را بهبود می‌بخشد. یادگیری الکترونیکی با استفاده از اصول آموزش از دور و تعلیم و تربیت، فرصت ارتباطات همزمان و غیر همزمان را فراهم می‌کند. همچنین این نوع یادگیری از سویی با استفاده از مزیت اطلاعاتی، فناوری محاسباتی و ارتباطی و از سوی دیگر با کاربرد طیف گسترده‌ای از چند رسانه‌ای‌های الکترونیکی، به طور فزاینده‌ای در جهان آموزش، گسترش می‌یابد. سازمان جهانی یونسکو در سال ۲۰۱۴ طی اعلامیه‌ای بیان کرد یادگیری الکترونیکی تنها یک ابزار برای آموزش نیست؛ بلکه به سنگ بنای ساخت و ساز جوامع دانش فراگیر، تبدیل شده است. یادگیری الکترونیکی به عنوان عاملی که فرایند یادگیری را تسهیل می‌بخشد و منجر به توسعه دستاوردهای دانشجویان با استفاده از فناوری می‌شود، شناخته شده است. یادگیری الکترونیکی یا آموزش برخط راهبردهای کلاس درس سنتی را از طریق توسعه دادن دامنه و مقیاس آن، منقلب کرده است. به این طریق که یک روش خردمندانه برای آنهایی که به منابع ضروری برای تمام کردن دوره درسی‌شان دسترسی ندارند فراهم می‌کند [۴]. استفاده از این محیط‌های چندرسانه‌ای با امکان انتقال اطلاعات به‌صورت صوت، تصویر، متن، نقاشی و با استفاده از اصول طراحی وب موجب ایجاد علاقه و انگیزه در یادگیرندگان گردیده است [۳]. با توجه به موارد اشاره شده این تحقیق در نظر دارد تأثیر یادگیری الکترونیکی بر یادگیری جبر و معادله درس حسابان ۱ یازدهم رشته ریاضی فیزیک دوره متوسطه دوم سال تحصیلی ۱۴۰۳-۱۴۰۲ ناحیه ۲ زاهدان را مورد بررسی قرار دهد. بر این اساس فرضیه‌ای به شرح زیر تدوین گشت: آیا یادگیری الکترونیکی بر یادگیری مبحث جبر و معادله درس حسابان ۱ یازدهم رشته ریاضی فیزیک دوره متوسطه دوم تأثیر مثبتی دارد.

۲. جامعه، نمونه و روش نمونه‌گیری

جامعه آماری تحقیق شامل دانش‌آموزان دختر و پسر سال یازدهم متوسطه دوم سال تحصیلی ۱۴۰۳-۱۴۰۲ ناحیه ۲ زاهدان است. از جامعه فوق با روش نمونه‌گیری خوشه‌ای در دسترس چهار کلاس به‌عنوان گروه آزمایش (دو کلاس دخترانه و دو کلاس پسرانه) و چهار کلاس به‌عنوان گروه گواه (دو کلاس دخترانه و دو کلاس پسرانه) انتخاب شد.

۳. ابزار پژوهش

برای اینکه نظریه‌های حاصله مورد آزمون قرار گیرند و به اثرات یادگیری الکترونیکی پی ببریم، از آزمون پیشرفت تحصیلی (هم در پیش‌آزمون و هم در پس‌آزمون) استفاده شد. محتوای آزمون‌ها را سرگروه‌های آموزشی ریاضی استان سیستان و بلوچستان از مباحث کتاب حسابان ۱ سال یازدهم متوسطه دوم رشته ریاضی‌فیزیک (فصل جبر و معادله) استخراج کردند. آزمون شامل پنج سؤال تشریحی بوده است که دانش‌آموز با دادن جواب بین ۰ تا ۱ نمره را از هر سؤال دریافت می‌کند. طراحی نهایی به‌صورت پیش‌آزمون در ابتدای هر درس و پس‌آزمون در انتهای هر درس صورت گرفت (مبحث جبر و معادله شامل پنج درس است). روایی محتوایی آزمون‌ها از نظر سرگروه‌های آموزشی و کارشناسان سنجش و اندازه‌گیری

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[1] meyre & Clarlk

مطلوب بود. پایایی پیش‌آزمون‌ها و پس‌آزمون‌ها با استفاده از ضریب پایایی کودر– ریچاردسون ۲۱ به ترتیب (۸۱/۰، ۷۹/۰، ۸۰/۰، ۷۸/۰، ۸۱/۰) و (۸۰/۰، ۷۹/۰، ۷۹/۰، ۸۱/۰، ۸۳/۰) به‌دست‌آمده که کاملاً مطلوب است.

۴. روش تحقیق

برای اجرای آزمایش چهار معلم و کلاس به‌عنوان گروه آزمایش و چهار معلم و کلاس به‌عنوان گروه گواه از مدارس ناحیه ۲ زاهدان انتخاب شدند. معلمان انتخاب‌شده طی چندین جلسه درباره اهداف پژوهش، یادگیری الکترونیکی و فرایند اجرای آن توسط دانش‌آموز توجیه شدند. اجرای طرح با در اختیار قرار دادن ابزار یادگیری الکترونیکی ازجمله لوح فشرده آموزشی و کمک‌آموزشی، ماشین‌حساب ، نرم‌افزارهای ساده ریاضی، نرم‌افزارهای صفحات گسترده، نرم‌افزار جئوجبرا، کلیپ‌ها و گیف‌های مربوط به مبحث و منابع اینترنتی برای درس حسابان ۱ یازدهم ریاضی فیزیک مبحث جبر و معادله برای گروه‌های آزمایشی آغاز گردید. پس از اجرا معلمان نتایج را تجزیه‌وتحلیل، و نقاط قوت و ضعف دانش‌آموزان در حوزه یادگیری الکترونیکی را تعیین کردند تا بر اساس آن به رفع نواقص بپردازند. در مورد گروه گواه به همان روش سنتی که فاقد استفاده از فناوری اطلاعات است عمل کرده‌ایم.

۵. یافته‌ها

فرضیه این تحقیق عبارت است از : آیا یادگیری الکترونیکی بر یادگیری مبحث جبر و معادله درس حسابان ۱ یازدهم رشته ریاضی فیزیک دوره متوسطه دوم تأثیر مثبتی دارد. داده‌های حاصل از آزمون‌ها (پیش‌آزمون و پس‌آزمون) در نرم‌افزار SPSS وارد شد و نتایج به این صورت بود. با انجام تفاضل پیش‌آزمون از پس‌آزمون از روش t گروه‌های مستقل استفاده شد که نتایج زیر به‌دست‌آمده است.

گروه‌ها	N	میانگین	انحراف استاندارد	T	سطح معنی‌داری
آزمایش	۴۵	۸۹۳/۲	۱۵۱۹/۳	**۷۹۴/۱	۰۰۰/۰
گواه	۵۰	۰۱۰/۱			

$*P \leq ۰/۰۵$ $**P \leq ۰/۰۱$

جدول ۱: نتایج t مربوط به تفاضل میانگین‌های گروه‌های گواه و آزمایش.

نتایج جدول ۱ نشان می‌دهد که تفاوت معنی‌داری در سطح $P \leq ۰/۰۱$ بین دو گروه وجود دارد. در این مقایسه، میانگین گروه آزمایش $M = ۲/۸۹$ بیشتر از میانگین گروه گواه $M = ۱/۰۱$ بوده است بنابراین یادگیری الکترونیکی بر یادگیری مبحث جبر و معادله درس حسابان ۱ یازدهم رشته ریاضی فیزیک دوره متوسطه دوم تأثیر مثبتی دارد که دلیل بر تائید فرضیه تحقیق است.

۶. بحث و نتیجه‌گیری

در فرضیه این تحقیق تأثیر مثبت یادگیری الکترونیکی بر یادگیری مبحث جبر و معادله درس حسابان ۱ سال یازدهم رشته ریاضی فیزیک دوره متوسطه دوم مورد بررسی قرار گرفت که فرضیه تائید شد؛ که بر اساس نتایج حاصل بین میزان استفاده دانش‌آموزان از فناوری اطلاعات و ارتباطات و میزان یادگیری آن‌ها در مبحث جبر و معادله درس حسابان ۱ یازدهم ریاضی فیزیک رابطه‌ای مستقیمی وجود دارد؛ بنابراین می‌توان به‌وسیله یادگیری الکترونیکی میزان یادگیری دانش‌آموزان را در مبحث جبر و معادله درس حسابان ۱ یازدهم ریاضی فیزیک افزایش داد. همچنین استفاده از فناوری‌های اطلاعات و ارتباطات می‌تواند فهم

بهتری از مبحث جبر و معادله در دانش‌آموزان ایجاد کند که این امر خود باعث ایجاد انگیزه و رغبت در دانش‌آموز نسبت به درس حسابان ۱ یازدهم رشته ریاضی فیزیک می‌شود.

مراجع

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On Non-Commutative Graph of a Polygroup

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Abstract

In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. A polygroup $(P, \circ)$ is an special type of a hypergroup, which satisfies the axioms similar to a group. In this regards we associate to P a graph Γ_P, whose vertices are elements of $P \setminus \zeta(P)$, where $\zeta(P)$ is the center of P; and x connected to y by edge in case $x \circ y\omega \neq yx\omega$, where ω is the heart of P. In this regards, we obtain some properties of graph Γ_P, and study the relationship between the graph properties Γ_P and algebraic properties of P. In particular, we prove if $\zeta(P) \neq P$, then $dim(\Gamma_P) = 2$ and Γ_P is a connected graph.

Keywords: Polygroup, Fundamental relation, Fundamental group, Noncommutative graph.

Mathematics Subject Classification [2010]: Primary: 20N20, Secondary: 05C25

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1 Introduction and Preliminaries

Finding the relationship between hyperstructures and properties of its associated graphs is an interesting topic in the last years. There are many papers on assigning a graph to a ring or a group and investigation of algebraic properties of ring or group using the associated graph, for instance see, ([4], [3]). Also, in algebraic hyperstructure theory, some authors study the relations between an algebraic hyperstructure, such as a semihypergroup, hypergroup or a hyperrings, and their associated graphs(for more details see [6, 7, 10, 14, 16]).

In this paper we consider the class of polygroups as an important subclasses of hypergroups, which were studied by Comer [5]. In this paper we consider a polygroup P and associate to P a graph $\Gamma_H = (H, E)$, via the heart of a P. In this regards, we introduce various kinds of graphs to a (resp. hypergroupoid) hypergroup via the relation β, and its transitive closure β^*, which is called the fundamental relation. Precisely, consider a hypergroupoid $(H, \circ)$, we associated to H a graph $\Gamma_H = (H, E)$, with vertices as the elements of H, and x is adjacent to y if x and y belongs to a finite hyperproduct of the elements of H. In this regards, we investigate the relationship between graph properties of Γ_H and algebraic properties of H. In particular, we study the relationship between algebraic properties of a given semihypergroup(resp. hypergroup) H and study the graph

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[*] Speaker.

properties of Γ_H. A hyperstructure (or hypergroupoid) is a nonempty set H with a hyperoperation $\circ$ defined on H, that is, a mapping from $H \times H$ into $P^*(H)$, the family of all non-empty subsets of H. If $(x, y) \in H \times H$, its image under $\circ$ is denoted by $x \circ y$, or xy. If A, B are non-empty subsets of$H$ then $A \circ B$ is given by $A \circ B = \bigcup\{x \circ y | x \in A, y \in B\}$. $x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$. Generally, the singleton a is identified with its member a. The structure $(H, \circ)$ is called a semihypergroup if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$, and a semihypergroup $(H, \circ)$ is a hypergroup if $x \circ H = H \circ x = H, \forall x \in H$, which is called the reproduction axiom. This axiom means that for any $x, y \in H$ there exist $u, v \in H$ such that $y \in x \circ u$ and $y \in v \circ x$.

Definition 1.1. [9] A polygroup is a hypergroup $\langle P, ., e, -1\rangle$, where $e \in P$, -1, is an unitary operation on P and the following axioms hold for all $x, y, z \in P$, the following hold:

(i) $e.x = x.e = x$;

(ii) $x \in y \circ z \Longrightarrow y \in x \circ z^{-1} \Longrightarrow z \in y^{-1} \circ x$.

A polygroup P in which $x.y = y.x$ for all $x, y \in P$, is called commutative polygroup. An exhaustive review updated to 1992 of hypergroups theory appears in [6]. Also, for a good review of polygroups refer to [9]. A recent book and [7] contains a wealth of applications of algebraic hyperstructures. Let H and K be hypergroups, a function.

For all $n \geq 1$, we define the relation β_n on a semihypergroup H, as follows:
$a\beta_n b \Longleftrightarrow \exists(x_1, \ldots, x_n) \in H, \{a, b\} \subseteq \prod_{i=1}^{n} x_i$, and $\beta = \cup_{n\geq 1}\beta_n$,
where $\beta_1 = \{(x, x) \mid x \in H\}$, is the diagonal relation on H. Suppose that β^* is the transitive closure of β, the relation β^* is a strongly regular relation [12].

Theorem 1.2 ([11]). *If H is hypergroup, then $\beta = \beta^*$.*

2 Main Results

The kernel of the canonical map $\varphi : P \longrightarrow P/\beta^*$ is called the core , or the heart of P and is denoted by ω_P (or ω). For any group G, the subgroups Z_i for $i \in \{0, 1, 2, \ldots\}$ is defined by (here abbreviate $Z_i(G)$ as Z_i) $Z_0 = \{e\}$, where e is the identity element of G, and for $i > 0$, Z_i is the subgroup of G corresponding to $Z(\frac{G}{Z_{i-1}})$ by the correspondence theorem $\frac{Z_i}{Z_{i-1}} = Z(\frac{G}{Z_{i-1}})$. Clearly, $[Z_i, G] \subseteq Z_{i-1}$. The sequence of subgroups $Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq ...$ is called the upper central series of G. its i-th term Z_i is called the i-th center of G. A group G is said to be nilpotent if $Z_m(G) = G$ for some integer m; in the case, the smallest integer c exists such that $Z_c(G) = G$, it is called the class of G. It is easy to see that $x \in Z_i$ if and only if $[x, y_1, \ldots, y_i] = 1$; $y \in G$.
Let G be a group and $Z(G)$ be its center. A graph Γ_G, whose vertices are the elements of $G \setminus Z(G)$ and x connected to y by edge in case $xy \neq yx$, was first considered by Paul Erdös. We denote by $V(G)$ the set of all vertices of Γ_G. A path ρ is a sequence $v_0e_1v_1...e_kv_k$, whose terms are alternately distinct vertices and distinct edges, such that for any i, $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i. In this case ρ is called a path between v_0 and v_k. The number k is called the length of ρ. If v_0 and v_k are adjacent in Γ by an edge e_{k+1}, then $\rho \cup \{e_{k+1}\}$ is called a cycle. The length of a cycle defined the number of its edges. The length of the shortest cycle in a graph Γ is called girth of Γ and denoted by $girth(\Gamma)$. If v and w are vertices in Γ, then $d(v, w)$ denotes the length of the shortest path between v and w. The largest distance between all pairs of the vertices of Γ is called the diameter of Γ, and is denoted by $diam(\Gamma)$. A graph is connected if there is a path between each pair of the vertices of Γ. A subset S of the vertices of a connected graph Γ is called a cut set if $T \backslash S$ is not a connected graph. For a graph Γ and a subset S of the vertex set $V(\Gamma)$, denoted by $N_\Gamma[S]$ the set of vertices in Γ which are in S or adjacent to a vertex in S. If $N_\Gamma[S] = V(\Gamma)$, then S is said to be a dominating set.
In the sequel first we define center of a polygroup, then we associate a graph to a polygroup, and we study some properties of this graph. Let P be a polygroup and $\zeta(P)$ denotes its center. We will associate to every polygroup P, a graph Γ_P, whose vertices are the elements of $P \setminus \zeta(P)$ and x connected to y by edge in case $xy\omega \neq yx\omega$. We denote by $V(P)$ the set of vertices of graph Γ_P.

Let $\{x \mid xy\omega = yx\omega$ for all $y \in P$ $\}$, we show that this set ia a subpolygroup of P. Also, a new definition for the center of P is obtained, which it is denoted it by $\zeta(P)$. Also, the notions of upper central series of a polygroup P, and centralizer an element x in P are introduced. In a polygroup P, the commutator of two elements $x, y \in P$ is defined by $[x, y] = \{t \mid t \in x^{-1}y^{-1}xy\}$. If $A \subseteq P$, then $[A, y] = \{t \mid t \in A^{-1}y^{-1}Ay\}$. Therefore, $[[x, y], y] = \{t \mid t \in [x, y]^{-1}y^{-1}[x, y]y\}$. And inductively is defined as $[x,_n y] = [[x,_{n-1} y], y] = \{t \mid t \in [x,_{n-1} y]^{-1}y^{-1}[x,_{n-1} y]y\}$. Also, $A^x = \{t \mid t \in x^{-1}Ax\}$.

Theorem 2.1. *Let P be a polygroup. Then $\beta^*(xy) = \beta^*(yx)$ if and only if $xy\omega = yx\omega$.*

Definition 2.2. Let P be a polygroup. Define $\zeta(P)$, center of P by

$$\zeta(P) = \{x | xy\omega = yx\omega, \forall y \in P\}.$$

Theorem 2.3. *If P is a polygroup, then $\zeta(P)$ is a normal subpolygroup of P.*

The next result immediately follows from Theorem 2.1.

Corollary 2.4. *Let P be a polygroup. Then $x \in \zeta(P)$ if and only if $\beta^*(x) \in Z(P/\beta^*)$.*

Definition 2.5. Let P be a polygroup.

(i) $\zeta_0(P) = \omega$; and

(ii) $\zeta_n(P) = \{x \mid xy\zeta_{n-1}(P) = yx\zeta_{n-1}(P), \forall x, y \in P\}$, for $n \geq 1$.

Note that $\omega \subseteq \zeta_1(P) \subseteq ...$, which is called the ascending central series of P.

Theorem 2.6. *Let P be a polygroup and $x \in P$. Then for all $y \in P$, on has*

$$\beta^*(xy)Z_n(P/\beta^*) = \beta^*(yx)Z_n(P/\beta^*) \quad \Leftrightarrow \quad xy\zeta_n(P) = yx\zeta_n(P).$$

Moreover, $x \in \zeta_{n+1}(P)$ if and only if $\beta^*(x) \in Z_{n+1}(P/\beta^*)$.

Definition 2.7. For two polygroups P, H, the graphs Γ_P, Γ_H are isomorphic($\Gamma_P \cong \Gamma_H$) if there is a one-to-one correspondence $\varphi : P\backslash\zeta(P) \longrightarrow H\backslash\zeta(H)$ preserving edges, i.e. $x, y \in P\backslash\zeta(P)$, $xy\omega \neq yx\omega$ if and only if $\varphi(x)\varphi(y)\omega \neq \varphi(y)\varphi(x)\omega$.

Theorem 2.8. *Suppose P and H are polygroups and $\zeta(P) \neq P, \zeta(H) \neq H$. If $\Gamma_P \cong \Gamma_H$, then $\Gamma_{P\times A} \cong \Gamma_{H\times B}$, for any two commutative polygroups A, B with the same order.*

Definition 2.9. For two polygroups P, H, their associated graphs Γ_P, Γ_H are isomorphic, and we write($\Gamma_P \cong \Gamma_H$) if there is a one-to-one correspondence mapping $\varphi : P\backslash\zeta(P) \longrightarrow H\backslash\zeta(H)$ preserving edges, i.e.

$$x, y \in P\backslash\zeta(P), xy\omega \neq yx\omega \quad \Leftrightarrow \quad \varphi(x)\varphi(y)\omega \neq \varphi(y)\varphi(x)\omega.$$

Theorem 2.10. *Let P and H be polygroups with $\zeta(P) \neq P, \zeta(H) \neq H$. If $\Gamma_P \cong \Gamma_H$, then $\Gamma_{P\times A} \cong \Gamma_{H\times B}$, for any two commutative polygroups A and B of the same order.*

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P-Semiprime Submodules

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Abstract

Let R be a commutative ring with identity and M be a unitary R-module. This article introduces the concept of P-semiprime submodules which are a generalization of semiprime submodules and characterizes a certain class of semiprime submodules.

Keywords: Semiprime submodule, Radical of a submodule, Multiplication module
Mathematics Subject Classification [2010]: Primary: 13C13 Secondary: 13C99

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1 Introduction and Preliminaries

In this paper, all rings are commutative with identity and all modules are unitary. For the sake of completeness, we begin by giving some notions and notations which will be used throughout the paper.

Prime submodules play an important role in module theory over commutative rings with identity, where a proper submodule P of an R- module M is called prime, if whenever $rm \in P$ for $r \in R$, $m \in M$ implies either $m \in P$ or $r \in (P :_R M) = \{s \in R | sM \subseteq P\}$. The notion of the prime submodule has been widely studied in many papers. See, for example, [5], [7].

Let K be a proper submodule of an R- module M. If there exist a prime submodule of M which contains K, then the intersection of all prime submodules containing K is called the radical of K and is denoted by $rad(K)$, see [3], [6] . Also, the envelope of K, denoted by $E(K)$, is defined as $E(K) = \{a \in M | a = rm \ for \ some \ r \in R, m \in M \ and \ r^n m \in K \ for \ some \ positive \ integer \ n\}$.

A proper submodule S of M is said to be semiprime; if $r^2 m \in S$ where $r \in R$ and $m \in M$ then $rm \in S$, see [4], [10]. By definition, every prime submodule is semiprime, but the converse is not true in general. For example, the proper submodule $6\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}$ is semiprime. However, $2 \times 3 \in \mathbb{Z}$, but $2 \notin \mathbb{Z}$ and $3 \notin \mathbb{Z}$.

Let C be a multiplicatively closed subset of R. Then there is a surjective map from the set of all the semiprime submodules S of M such that $(S :_R x) \cap C = \emptyset$ for some $x \in M$ to the set of all the semiprime submodules of the R_C -module M_C.

Let $\{S_i\}$ be a non-empty family of semiprime submodules of M. Then $\bigcap S_i$ is a semiprime submodule of M . Furthermore, if $A = \bigcup S_i$ is totally ordered (by inclusion), then A is also a semiprime submodule whenever $A \neq M$.

The torsion subset of an R-module M, denoted by $T(M)$, is defined as

$$T(M) = \{m \in M | rm = 0, \text{ for some nonzero elemen } r \in R\} = \bigcup_{r \in R} Ann(r).$$

Note that $T(M)$ is not a submodule of M in general.

An R-module M is called multiplication, whenever for every proper submodule N of it there exists an ideal I of R such that $N = IM$. I is called presentation ideal of N. Examples of multiplication modules are every ring, every cyclic module, every ideal in a Dedekind domain, see [1]. It can be shown that $N = (N : M)M$. Clearly, M is a multiplication module if and only if for each $m \in M$, $Rm = (Rm :_R M)M$, that is,

$$M = \Sigma_{m \in M} Rm = \Sigma_{m \in M}(Rm :_R M)M.$$

Proposition 1.1 ([9]). *Let M be an R- module and S be a semiprime submodule of M. Then $(N :_R M)$ is a semiprime ideal of R.*

The converse of Priposition 1.1 is not true in general. Let $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}$ and $S = \langle (9, 0) \rangle$. Then it is clear that $(S :_R M) = 0$. Since $\mathbb{Z}$ is an integral domain, $(S :_R M)$ is a prime ideal and hence a semiprime ideal of $\mathbb{Z}$. But S is not a semiprime submodule of M. Because, $3^2 \times 2 \in S$ but $3 \times 2 \notin S$.

It was shown [9] that if M is a multiplication R-module, then a proper submodule N of M is semiprime if and only if $(N :_R M)$ is a semiprime ideal of R.

Proposition 1.2 ([9]). *If M is a finitely generated R- module, then every proper submodule of M is contained in a semiprime submodule.*

Theorem 1.3 ([4]). *Let M be a Noetherian R-module with $T(M) \neq M$. Then $T(M)$ is a union of semiprime submodules of M.*

We recall that if M is a multiplication R-module and $K_1 = IM$, $K_2 = JM$, for some ideals I and J of R, are submodules of M then the product of K_1 and K_2, denoted by K_1K_2, is defined by IJM.

Proposition 1.4. *Let M be a multiplication R- module and F a prime submodule of M. If K_1, K_2 are submodules of M such that $K_1^n K_2 \subseteq F$ for some positive integer n, then $K_1 \subseteq F$ or $K_2 \subseteq F$.*

Proof. Suppose that $K_1^n K_2 \subseteq F$, but $K_1 \nsubseteq F$. Since M is multiplication, there exist ideals I, J of R such that $K_1 = IM$ and $K_2 = JM$. Therefore, $K_1^n K_2 = I^n JM \subseteq F$. Hence, $I^n J \subseteq (F :_R M)$. Now, $(F :_R M)$ prime implies that $I \subseteq (F :_R M)$ or $J \subseteq (F :_R M)$. Since $I \nsubseteq (F :_R M)$ we get $J \subseteq (F :_R M)$. □

We follow [2] and [8] for terminologies and notations not defined here.

2 Main Results

The purpose of this section is to generalize the notion of semiprime submodules.

Definition 2.1. The semiprime submodule S of an R-module M is calles P-semiprime whenever $(S :_R M) = P$ is a prime ideal of R.

Lemma 2.2. *Let P be a prime ideal of the ring R and S be a P-semiprime submodule of a finitely generated R-module M. Then*

$$\{m \in M | rm \in S \text{ for some } r \in R - P\}$$

is P-semiprime.

Theorem 2.3. *Let P be a prime ideal of the ring R and S be a P-semiprime submodule of an R-module M. Then*

$$E(S) + PM \subseteq rad(S).$$

Proof. Since S is a semiprime submodule of M, we get $E(S) = S$ is a submodule of M. So, $E(S) + PM = S + PM$. Now, let N be an arbitrary prime submodule of M containing S. For each $x \in PM$ we get $x = \Sigma(p_i m_i)$, where $p_i \in P$, $m_i \in M$. Hence, $p_i \in (S : M)$ and $S \subseteq N$ implies that $p_i M \subseteq N$. Therefore, $\Sigma(p_i m_i) \in N$ which means $PM \subseteq N$. Since N is an arbitrary prime submodule of M containing S, we concclude that $PM + S \subseteq rad(S)$. □

Theorem 2.4. *Let P be a prime ideal of a ring R and M be an R-module. Let*

$$\tilde{P} = \{m \in M | rm \in PM \; for \; some \; r \notin P\}.$$

Then $\tilde{P} = M$ or $\tilde{P}$ is a P-semiprime submodule of M.

Proof. Let $\tilde{P} \neq M$ and $r \in R - P$, $m \in M$ where $r^2m \in \tilde{P}$. So there exists $x \notin P$ such that $x(r^2m) \in PM$. Now, P prime implies that $x^{n-1}r^{n-1} \notin P$, that is, $rm \in \tilde{P}$.

Now let $r \in P$. So for any $m \in M$, $rm \in PM$. Let $r^2m \in \tilde{P}$. Hence $rm \in PM \subseteq \tilde{P}$ shows in any case $\tilde{P}$ is a semiprime submodule of M. It remains to prove that $(\tilde{P} :_R M) = P$. $\tilde{P} \neq M$ implies that there exists $m \in M - \tilde{P}$. Let $r \in (\tilde{P} :_R M)$ so $rm \in \tilde{P}$, therefore there exists $a \notin P$ such that $ram \in PM$. If $r \notin P$ then $ra \notin P$. But this is a contradiction, because this implies that $m \in \tilde{P}$. Therefore $r \in P$ and so $(\tilde{P} :_R M) \subseteq P$. From $PM \subseteq \tilde{P}$ we find that $(\tilde{P} :_R M) = P$. □

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On Free Multialgebras

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Abstract

As it is well known from universal algebra, for every type $\mathbb{F}$, there exist algebras over $\mathbb{F}$, which are free, i.e., do not satisfy any identities or, alternatively, satisfy the universal mapping property for the class of $\mathbb{F}$-algebras. In this regards, consider a type $\mathbb{F}$ of the generating set, they are, up to isomorphisms, unique and equal to algebras of term functions, or equivalently, there is a forgetful functor, from the category of $\mathbb{F}$-algebras to **Sets**, the category of sets, has a left adjoint. We will prove that this result does not extend to multialgebras, in fact in this category the free multialgebra do not exists on arbitrary set.

Keywords: Multialgebra, Free multialgebra, Category of multialgebras, Algebras of term functions.

Mathematics Subject Classification [2010]: 20N20, 16N20

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1 Introduction and Preliminaries

One of the important approach to *multialgebras*, which is called also *hyperalgebras*, is from abstract algebraic logic. However, these logics can be semantically characterized by means of non-deterministic algebraic structures such as *Nmatrices, RNmatrices* and swap structures. These structures are based on multialgebras, which generalize algebras by allowing the result of an operation to assume a non-empty set of values. This leads to an interest in exploring the foundations of multialgebras applied to the study of logic systems. It is well known from universal algebra that for a type $\mathbb{F}$, there exists algebras of type $\mathbb{F}$, which are absolutely free, meaning that they do not satisfy any identities or, alternatively, satisfy the universal mapping property for the class of $\mathbb{F}$-algebras. At the following we will show that this result does not extend to multialgebras. Not only multialgebras satisfying the universal mapping property do not exist, but the forgetful functor, from the category of $\mathbb{F}$-multialgebras to category sets, **Sets**, does not have a left adjoint. A multialgebra or a *hyperalgebra* $\langle H, (\beta_i, \mid i \in I)\rangle$ is a set H with together a collection $(\beta_i, \mid i \in I)$ of hyperoperations on H.

In the sequel H is a fixed nonvoid set, $P^*(H)$ is the family of all nonvoid subsets of H, and for a positive integer n we denote by H^n the set of n-tuples over H (for more details see [1–9, 11–13]). For a positive integer n an n-ary *hyperoperation* β on H is a function $\beta : H^n \rightarrow P^*(H)$. Also, we say that n is the *arity* of β. A *nullary hyperoperation* on H is just an element of $P^*(H)$; i.e. a nonvoid subset of H. A *hyperalgebraic system* or a *multialgebra* $\langle H, (\beta_i, \mid i \in I)\rangle$ is the set H with together a collection $(\beta_i, \mid i \in I)$ of hyperoperations on H. A subset S of H is *closed* under the n-ary hyperoperation β if $(x_1, \ldots, x_n) \in S^n$ implies that $\beta(x_1, \ldots, x_n) \subseteq S$. A subset S of a multialgebra

$\mathbb{H} = \langle H, (\beta_i, \mid i \in I)\rangle$ is a *submultialgebra* of $\mathbb{H}$ if S is closed under each hyperoperation β_i, for all $i \in I$. The *type* of $\mathbb{H}$ is the map τ from I into the set $\mathbb{N}^*$ of nonnegative integers assigning to each $i \in I$ the arity of β_i. Instead of $\tau(i)$ we write n_i. Two multialgebras of the same type are similar. For a nullary operation β set $\overline{\beta} = \beta$. It is easy to see that $\overline{\mathbb{H}} = \langle P^*(H), (\overline{\beta}_i, \mid i \in I)\rangle$ is an algebra of the same type as $\mathbb{H}$. Whenever possible we write a instead of the the singleton $\{a\}$; e.g. for a binary hyperoperation $\circ$ and $a, b, c \in H$ we write $a \circ (b \circ c)$ for $\{a\} \circ (\{b\} \circ \{c\}) = \bigcup\{a \circ u | u \in b \circ c\}$. An equivalence relation on A compatible (resp. *strongly compatible*) with a multialgebra $\mathbb{H}$ on A is *congruence* (resp. *strongly congruence*) of $\mathbb{H}$. Denote by $Con(\mathbb{H})$ (resp. $Cons(\mathbb{H})$) the set of all congruences (resp. strongly congruences) of $\mathbb{H}$.

Definition 1.1. Let $\mathbb{H} = \langle H, (\beta_i, \mid i \in I)\rangle$ and $\overline{\mathbb{H}} = \langle \overline{H}, (\overline{\beta}_i, \mid i \in I)\rangle$ be two similar multialgebras. A map h from H into $\overline{H}$ is called a

(i) A *homomorphism* if for every $i \in I$ and all $(a_1, \ldots, a_{n_i}) \in H^{n_i}$ we have that

$$h(\beta_i((a_1, \ldots, a_{n_i}))) \subseteq \overline{\beta}_i(h(a_1), \ldots, h(a_{n_i}));$$

(ii) a *good homomorphism* if for every $i \in I$ and all $(a_1, \ldots, a_{n_i}) \in H^{n_i}$ we have that

$$h(\beta_i((a_1, \ldots, a_{n_i}))) = \overline{\beta}_i(h(a_1), \ldots, h(a_{n_i})).$$

Definition 1.2. A universal algebra(or algebra) is a multialgebra whose hyperoperations are singleton valued(i.e. $\mid \beta(a_1, ..., a_n) \mid = 1$ are called operations.

Proposition 1.3. *Let $\mathbb{H}$ be a multialgebra and $S \subseteq H$. If $a \in Sg^{\mathbb{H}}(S)$, then there is a finite set T such that $T \subseteq S$ and $a \in Sg^{\mathbb{H}}(T)$.*

Note that on the contrary of universal algebra, for multialgebras, there are various kinds of categories of multialgebras, since there are various kinds of homomorphisms of multialgebras, such as homomorphisms, good homomorphisms, multivalued homomorphisms, good multivalued homomorphisms and etc.. The author in [1] introduced and studied the category of hyperalgebras and investigate its relationship to category of algebras. In the following for technical reason we consider multialgebras together multivalued homomorphisms as morphisms. Denote this category by **MA**. Then the following hold:

(1) The objects of **MA** are multialgebras of a fix type;

(2) For the objects A, B of **MA**, the set of all morphisms from A to B, are all multivalued homomorphisms from A to B, and is denoted by $Hom(A, B)$.

(3) The composition gf of morphism $f : A \to B$ and $g : B \to A$ is defined by $gf : A \to C(gf(x) \mapsto g(f(x))$, where $(g \circ f)(a) = \bigcup_{b \in f(a)} g(b), \quad \forall a \in A$;

(4) For any object A, the morphism $1_A : A \to A(x \mapsto x)$, is the identity morphism.

Note that in the part (2) in above if $Hom(A, B)$ is replaced by $Hom_g(A, B)$, the set of all strong(or good) multivalued homomorphisms, then a new category will be obtained, which it is denoted by $\mathbf{MA}_S$. In fact, $\mathbf{MA}_S \preceq \mathbf{MA}$ is a subcategory of **MA**, is denoted by $\mathbf{MA}_S$, which is not a full subcategory. Moreover, if we replace (resp. strong)multivalued homomorphisms with (resp. strong)homomorphisms we obtain new categories of multialgebras denoted (resp. $\mathbf{mA}_S$) **mA**. The subcategory relations between these categories is as follows:

$$\mathbf{mA}_S \preceq \mathbf{mA} \preceq \mathbf{MA}_S \preceq \mathbf{MA}.$$

2 Main Results

As it is well known the study of free algebra is an important topics in this field. Thus a free algebra over $\mathbb{K}$ must belong to $\mathbb{K}$. However, in a class $\mathbb{K}$, there need not be any free algebra. However, under some condition for a class $\mathbb{K}$, the free algebra exists. for example if $\mathbb{K}$ is an equational class, then all free algebras over $\mathbb{K}$ exist. But we will prove that for the classes of (nontrivial) multialgebras the free multialgebra dose not exist(for instance see [12]). At the following we introduce the notion of free multialgebras.

Definition 2.1. Let K be a family of multialgebra of type $\mathbf{F}$. Given a set X of variables defined the congruence $\theta_K(X)$ on $\mathbf{T}(X)$ by

$$\theta_K(X) = \bigcap \Phi_K(X),$$

where

$$\boldsymbol{\Phi}_k(X) = \{\varphi \in Con\ \mathbf{T}(X);\ \mathbf{T}(X)/\varphi \in IS(K)\}.$$

Define $F_k(\bar{X})$, the weak free multialgebra over $\bar{X}$, by

$$\mathbf{F}_K(\bar{X}) = \mathbf{T}(X)/\theta_K(X), \qquad \bar{X} = X/\theta_K(X).$$

For $x \in X$ we write $\bar{x}$ for $x/\theta_K(X)$ and let $p = p(x_1, \ldots, x_n) \in \mathbf{T}(X)$, we write $\bar{p}$ for $p^{F_k(\bar{X})}(\bar{x}_1, \ldots, \bar{x}_n)$.

If X is finite, say $X = \{x_1, \ldots, x_n\}$, we often write $\mathbf{F}_K(\bar{x}_1, \ldots, \bar{x}_n)$ for $\mathbf{F}_K(\bar{X})$. Note that $\mathbf{F}_K(\bar{X})$ is the universe of $\mathbf{F}_K(\bar{X})$.

Remark 2.2. (1) $\mathbf{F}_K(\bar{X})$ exists if and only if $\mathbf{T}(X)$ exists if and only if $X \neq \varnothing$, or $\mathcal{F}_0 \neq \varnothing$.

(2) If $\mathbf{F}_K(\bar{X})$ exists, then $\bar{X}$ is a set of generators of $\mathbf{F}_K(\bar{X})$ as X generates $\mathbf{T}(X)$.

(3) If $\mathcal{F}_0 \neq \varnothing$, then the algebra $\mathbf{F}_K(\bar{\varnothing})$ is often referred to as an initial object by category theorists and computer scientists.

(4) If $K = \varnothing$ or K consists solely of trivial multialgebras, the $\mathbf{F}_K(\bar{X})$ is a trivial multialgebra as $\theta_K(X) = \nabla$.

(5) If K has nontrivial multialgebra $\mathbf{A}$ and $\mathbf{T}(X)$ exists, then $X \cap \big(x/\theta_k(X)\big) = \{x\}$ as distinct members x, y of X can be separated by some homomorphism $\alpha : \mathbf{T}(X) \to \mathbf{A}$. In this case

$$|\bar{X}| = |X|.$$

Theorem 2.3. *Let A, B, C be multialgebras and $\alpha : A \to B$, $\beta : A \to C$ be two multivalued homomorphism. If $\ker\beta \subseteq \ker\alpha$ and β is onto, then there exists a multivalued homomorphism $\gamma : C \to B$, such that $\alpha = \gamma \circ \beta$.*

Definition 2.4 (See [12])**.** Let $\mathbb{K}$ be a class of multialgebras of type $\mathbf{F}$ and $I(X)$ be a multialgebra of type $\mathcal{F}$ which is generated by X. If for every $A \in \mathbb{K}$ and for every (multivalued) map $\alpha : X \to I$ there is a maximum homomorphism (with respect to inclusion of homomorphisms) $\beta : I(x) \to A$, which extends α (i.e., $\beta(x) = \alpha(x)$ for $x \in X$), then we say $I(X)$ has the quasi universal mapping property for $\mathbb{K}$ over X, X is called a set of free generators of $I(X)$, and $I(X)$ is said to be freely generated by X.

Theorem 2.5 (See [12])**.** *Suppose $\mathbb{K}$ be a family of multialgebras of type $\mathbf{F}$ and suppose $\mathbf{T}(X)$ exists. Then $\mathbf{F}_K(\bar{X})$ has the quasi universal mapping property for $\mathbb{K}$ over $\bar{X}$, that is for every (multivalued) map $\alpha : X \to I$ there is a maximum homomorphism(with respect to inclusion of homomorphisms) $\beta : I(x) \to A$, which extends α (i.e., $\beta(x) = \alpha(x)$ for $x \in X$).*

Theorem 2.6. *Let $\mathbb{K}$ be a family of multialgebra of type $\mathbf{F}$ and $\mathbf{F}_K(\bar{X})$, $\mathbf{F}_K(\bar{Y})$ be two K-quasi free multialgebra over $\bar{X}, \bar{Y}$. If $|X| = |Y|$ and $\mathbf{T}(x)$ exists, then $\mathbf{F}_K(\bar{X}) \cong \mathbf{F}_K(\bar{Y})$.*

Corollary 2.7. *If $\mathbb{K}$ is a class of multialgebras of type $\mathbf{F}$ and $A \in |K|$, then for sufficiently large X, $A \in \big(H(\mathbf{F}_K(\bar{X})\big)$.*

Open Problem. Consider the fundamental functor

$$F : \mathbf{MA} \to \mathbf{A},$$

from category of multialgebras of type $\mathbf{F}$ to category of algebras, which assign to every multialgebra A its fundamental algebra A/α^*(for more see [4]). What objects in $\mathbf{MA}$ are preserving under F? for example dose this functor is continuous? or preserve direct products and weak free objects?

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Grey Injective S-Acts

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Abstract

In this paper, we introduce the concept of grey injective S-act and strongly grey injective in the category of grey S-acts over monoids. We study some properties of these notions and prove that the category of grey S-acts over a commutative monoid has enough strongly grey injective S-acts. Also, we investigate the Skornjakov' Theorem for grey S-acts.

Keywords: Grey injective, Grey S-act, Monoid

Mathematics Subject Classification [2010]: Primary: 18D35, 20M30, 18A20, 20M50

1 Introduction and Preliminaries

Injectivity is one of the notions which is studied in several categories. Many authors studied the concept on this notion. For more see, [1], [2], [9], [11], [10]. The Grey system is one of the most important scientific achievements in the field of how to use uncertain information, which was presented by Deng [4]. For example see, [3], [8], [13]. In this paper, we define grey injective S-acts and study some properties of this notion. Also, we show that any grey S-act on a commutative monoid can be embedded into grey injective S-act. We recall the following definition needed in the sequel.

We recall from [13], the definition of grey numbers. Let $\mathbb{R}$ be set of real number and $g^{\pm}$ be a union set of closed or open intervals $g^{\pm} = \bigcup_{i=1}^{n}[a_i^-, a_i^+]$, which $i = 1, 2, 3, \cdots, n$, n is an integer and $0 < n < \infty$, $a_i^-, a_i^+ \in \mathbb{R}$ and $a_{i-1}^+ \leq a_i^- \leq a_i^+ \leq a_{i+1}^+$. For any interval $[a_i^-, a_i^+]$, p_i is probability for $g \in [a_i^-, a_i^+]$. If the following conditions hold for

(i) $p_i > 0$ if and only if $[a_i^-, a_i^+] \in g^{\pm}$

(ii) $p_i = 0$ if and only if $[a_i^-, a_i^+] \notin g^{\pm}$

(iii) $\Sigma_{i=1}^{n} p_i = 1$

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[*] Speaker.

then we call g a grey number represented by $g^{\pm}$. $g^{-}{=}inf_{a_i^- \in g^{\pm}} a_i^-$ and $g^{+}{=}sup_{a_i^+ \in g^{\pm}} a_i^+$ are called the lower and upper of $g^{\pm}$.

If $g^- = g^+$, then $g^{\pm}$ is a white number. It is clear that the special case of the grey set is the white set and the fuzzy set is the special case of the white set. If we replaced the characteristic function with a fuzzy membership function, then the white set become a fuzzy set. We recall the grey lattice operation from [12], which for grey numbers $x^{\pm} = [x^-, x^+]$ and $y^{\pm} = [y^-, y^+]$, the *Join* and *Meet* of these grey numbers are defined as $x^{\pm} \vee y^{\pm} = [\ \min\{x^-, y^-\}, \max\{x^+, y^+\}]$ and $x^{\pm} \wedge y^{\pm} = [\ \max\{x^-, y^-\}, \min\{x^+, y^+\}]$, respectively. Now according to the definition of Join and Meet, the partial order $\preceq$ on grey set (X, χ_A) is shown as below:

$$x^{\pm} \preceq y^{\pm} \quad \Longleftrightarrow \quad x^+ \leq y^+ and\ y^- \leq x^-$$

We recall from [5], category *GSet*, in which objects are grey sets and, morphism between two grey sets $A = (U, \chi)$ and $B = (V, \mu)$, which $\chi : U \longrightarrow D[0,1]^{\pm}$ and $\mu : V \longrightarrow D[0,1]^{\pm}$, is a function f such that $\chi^+(x) \leq \mu^+(f(x))$ and lower $\mu^-(f(x)) \leq \chi^-(x)$, for any $x \in U$,where $D[0,1]^{\pm}$ is the set of all grey numbers within the interval $[0,1]$.

Let S be a monoid. A *(right) S-act* is a non-empty set A together with a map $A \times S \to A, (a,s) \mapsto as$, such that for all $a \in A, s,t \in S$, $(as)t = a(st)$ and $a1 = a$. A non-empty subset $B \subseteq A$ is called a *subact* of A if $bs \in B$ for all $b \in B$ and $s \in S$. An element θ in an S-act A is said to be a *zero* or *fixed element* if $\theta s = \theta$ for all $s \in S$. Let A and B be two S-acts. A mapping $f : A \to B$ is called an *S-homomorphism* if $f(as) = f(a)s$, for all $a \in A, s \in S$. The category of all S-acts and homomorphisms between them is denoted by **Act-S**. We recall category **Act$_0$-S** in which all monoids contain zero 0.

From [6], the category $\mathbf{Act} - \mathbf{GS}$, in which, the objects are the function χ_A, which is called (right) grey S-acts, such that there exists a characteristic function $\chi : A \longrightarrow D[0,1]^{\pm}$ from S-act with the following properties:

(i) $\chi_A^+(a) \leq \chi_A^+(as)$, $\chi_A^-(as) \leq \chi_A^-(a)$ for any $a \in A$, $s \in S$.

(ii) $\chi_A^+(\theta_A) = 1$ and $\chi_A^-(\theta_A) = 0$.

(iii) If we consider S as an S-act, $\chi_S^+(1_S) = 1$, $\chi_S^-(1_S) = 0$.

The morphisms between two grey S-acts χ_A and μ_B, which is called GS-morphism and denoted by $\widetilde{f}\colon \chi_A \longrightarrow \mu_B$ is an S-homomorphism $f : A \longrightarrow B$ such that $\chi_A^+(a) \leq \mu_B^+(f(a))$ and $\mu_B^-(f(a)) \leq \chi_A^-(a)$, for any $a \in A$.

A GS-morphism $\widetilde{f}\colon \chi_A \longrightarrow \eta_B$ is (an epimorphism) a monomorphism if and only if f is an (epimorphism) a monomorphism. A GS-morphism $\widetilde{f}\colon \chi_A \longrightarrow \eta_B$ is an isomorphism if f is an isomorphism, and for any $a \in A$, $\chi_A(a) = \eta_B(\widetilde{f}\ (a))$. Define $Im(\widetilde{f}) = \{f(a) | a \in A\}$.

Throughout this paper, unless otherwise stated, all monoids have zero 0 and all S-acts are centered.

2 Main Results

In this section, we study injective objects in the category of grey S-acts and investigate Skornjakov' Theorem for grey S-acts. We show that the product of the family of grey S-acts is grey injective S-act if and only if any of this family is grey injective S-act. Also, we define the notion of strongly grey injective S-act and show that any grey S-act can be embedded into strongly grey injective S-act.

Definition 2.1. Let χ_A and ν_B be two grey S-acts. Then χ_A is a grey subact of ν_B (ν_B is extension of χ_A) if A is a subact of S-act B and for any $a \in A$, $\chi_A^{\pm}(a) \preceq \nu_B^{\pm}(a)$, which is denoted $\chi_A \preceq \nu_B$.

Also, we call a grey S-act χ_A is a strongly grey subact of grey S-act ν_B if A is subact of B and $\nu_{B_{|A}} = \chi_A$.

Definition 2.2. Consider two grey S-acts χ_A and ν_B. We said χ_A is inclusion of grey S-act ν_B, denoted by $\chi_A \subseteq \nu_B$, if $A \subseteq B$ and $\chi_A^{\pm}(a) \preceq \nu_B^{\pm}(a)$ for any $a \in A$.

Definition 2.3. A grey S-act μ_C is called a grey injective S-act if for any grey monomorphism $\widetilde{f}$: $\chi_A \longrightarrow \nu_B$ and for any grey GS-morphism $\widetilde{g}$: $\chi_A \longrightarrow \mu_C$ there exists a GS-morphism $\widetilde{h}$: $\nu_B \longrightarrow \mu_C$ such that $\widetilde{h}\widetilde{f}=\widetilde{g}$. It is called strongly grey injective S-act, if it is injective respect to all inclusions of strongly grey subacts into grey S-acts.

Clearly, if μ_C is a grey injective S-act, then C is an injective S-act. However, the conversely is not true, always.

Example 2.4. Consider monoid $S = \{0, 1\}$ and characteristic function $\chi_S : S \longrightarrow D[0, 1]^{\pm}$ which $\chi(s) = t_0^1$ for any $s \in S$. Consider the following diagram

$$\begin{tikzcd}
\nu_A \arrow[r, "\iota"] \arrow[d, "\widetilde{f}"'] & \mu_B \\
\chi_S &
\end{tikzcd}$$

Since S is injective by [7], there exists an S-homomirphism $h : B \longrightarrow S$ such that $h_{|A} = f$. Clearly, $\widetilde{h}$ is a GS-morphism and so χ_S is a grey injective S-act.

We recall a grey S-act χ_A is said to be cyclic, abbreviation C-grey S-act, if there exists $x \in A$ such that $\chi_A = \langle x_t \rangle$ which $\chi_A(x) = t_{g^-}^{g^+}$ and

$$\langle x_t \rangle (a) = \begin{cases} t_{g^-}^{g^+} & a \in x(S - 0) \\ \dfrac{g^- + g^+}{2} & a \notin xS \\ t_0^1 & a = \theta_A \end{cases}$$

Lemma 2.5. *A grey S-act is grey injective if and only if it is injective relative to the inclusions of grey subacts.*

In the following, we prove the Skornjakov' Theorem for grey S-acts. For more about this theorem, see [7].

Theorem 2.6. *(Skornjakov' Theorem For Grey S-acts) A grey S-act μ_Q is grey injective S-act if and only if it is injective relative to all inclusions into C- grey S-act and all inclusions to grey cyclic S-acts.*

Definition 2.7. Consider grey S-act ν_A and element $a \in A$. We recall S-homomorphism $\lambda_a : S \longrightarrow A$ such that $\lambda_a(s) = as$ for any $s \in S$. Let $\Lambda = \{\lambda_a\}_{a \in A}$. Clearly, Λ_A is an S-act with mapping $\Lambda_A \times S \to \Lambda_A, (\lambda_a, s) \mapsto \lambda_a s,$, which $(\lambda_a.s)(t) = \lambda_a(st)$, for any $\lambda_a \in \Lambda, s \in S$. Now let S be a commutative monoid and define grey function $\eta : \Lambda_A \longrightarrow D[0, 1]^{\pm}$ such that $\eta(\lambda_a) = \bigvee\{\lambda(as))|s \in S\}$, for any $\lambda_a \in \Lambda$. We show that η_{Λ_A} is a grey S-act. Consider $\lambda_a \in \Lambda_A$ and $t \in S$. We have $\eta(\lambda_a) = \bigvee\{\lambda(as))|s \in S\} \le \bigvee\{\lambda((as)t)|s \in S\} = \bigvee\{\lambda(a(ts))|s \in S\} = \eta(\lambda_a.t)$.

Theorem 2.8. *Let μ_Q be a grey S-act on a commutative monoid. Then the grey S-act η_{Λ_Q} is a strongly grey injective S-act.*

Theorem 2.9. *Let Sbe a commutative monoid. Any grey S-act can be embedded into strongly grey injective S-act.*

Proposition 2.10. *Let $\{\chi_{i_{A_i}} | i \in I\}$ be a family of grey S-acts. The grey S-act $\prod_{i \in I} \chi_{i_{A_i}}$ is a grey injective S-act if and only if for any $i \in I$, $\chi_{i_{A_i}}$ is grey injective S-act.*

Theorem 2.11. *(i) Any grey injective S-act is a retract of its extension.*

(ii) If grey S-act, on a commutative monoid, is a retract of its strongly grey extension, then it is strongly grey injective S-act.

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Intersection of Parametric Polynomial Ideals

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Abstract

In this paper, we study the intersection of polynomial ideals with parametric coefficients using the concept of Gröbner systems. Expanding on the intersection of polynomial ideals with numeric coefficients, we introduce the notion of an intersection system for two (or more) parametric polynomial ideals by decomposing the parameter space into a finite set of cells, and for each cell, we can compute the intersection of the mentioned ideals. Additionally, we provide an algorithm for computing this system and implement it in `Maple`.

Keywords: Gröbner system, Parametric polynomial, Intersection system, Specialization

Mathematics Subject Classification [2010]: 13P10, 13P15, and 13F20

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1 Introduction and Preliminaries

When considering the intersection of two polynomial ideals $I, J \subset R = \mathbb{K}[x_1, \ldots, x_n]$, finding a Gröbner basis for the intersection can help simplify calculations and make it easier to analyze the properties of the ideals. One common method for finding the Gröbner basis of the intersection of two polynomial ideals I and J is to introduce a new indeterminate t, and one uses an elimination ordering such that the first block contains only t and the other block contains all the other variables (this means that a monomial containing t is greater than every monomial that does not contain t). With this monomial ordering, a Gröbner basis of $I \cap J$ consists of the polynomials that do not contain t, in the Gröbner basis of the ideal. More precisely, If $I = \langle f_1, \ldots, f_m \rangle$ and $J = \langle g_1, \ldots, g_n \rangle$ are two polynomial ideals in R, then $I \cap J = \langle tf_1, \ldots, tf_m, (1-t)g_1, \ldots, (1-t)g_n \rangle \cap \mathbb{K}[x_1, \ldots, x_n]$. This intersection has been computed by using Gröbner bases. Gröbner basis is a useful generator, a key computational tool for studying polynomial ideals introduced and developed by Buchberger in 1965 (see his PhD thesis [1]). Many engineering and scientific problems can be represented by a suitable set of polynomials with parametric coefficients and need to be analyzed repeatedly for different parameter values. Therefore, this article focuses solely on the intersection of parametric polynomial ideals which has not, to our knowledge, been explored in the literature. The following example shows that the mentioned traditional approach for computing the intersection may not be used for such ideals.

Example 1.1. Let us consider $I = \langle ax(x-1), y^2(cy-a+2) \rangle$ and $J = \langle (x-b)^2 y^3 \rangle$ are two polynomial ideals in $\mathbb{K}[a, b, c][x, y]$ where $a, b, c \in \mathbb{C}$ are parameters and x, y are variables. The intersection of I and J is computed by the function `PolynomialIdeals:-Intersect(I, J)` of `Maple 18`:

$$\langle y^3(ab^2x^2 - 2abx^3 + ax^4 - ab^2x + 2abx^2 - ax^3), y^4(b^2c - 2bcx + cx^2) - y^3(ab^2 + 2abx - ax^2 + 2b^2 - 4bx + 2x^2) \rangle$$

By substitution of $a = 2, b = 0$ and $c = 0$ in the above $I \cap J$ we have $\langle 2x^4y^3 - 2x^3y^3\rangle$. However, this is wrong for these values since when $I|_{a=2,b=0,c=0} = \langle 2x(x-1)\rangle$ and $J|_{a=2,b=0,c=0} = \langle x^2y^3\rangle$ we obtain $I \cap J|_{a=2,b=0,c=0} = \langle x^3y^3 - x^2y^3\rangle$.

One may be interested in determining the values of parameters s.t.

$$\sigma(I \cap J) = \sigma(I) \cap \sigma(J)$$

where $\sigma : \mathbb{K}[\mathbf{a}] \to \mathbb{K}' \supseteq \mathbb{K}$ is a specialization involving parameter assignments. To achieve this, we introduce the concept of an *intersection system* for two (or more) parametric polynomial ideals and present an algorithm for its computation. In this process, we utilize the notion of Gröbner systems. This concept, along with its algorithms, was initially introduced by Weispfenning [5], representing an extension of Gröbner bases to polynomials with parametric coefficients. Essentially, for a parametric ideal, calculating its Gröbner system allows us to divide the parameter space into a finite set of cells. Each cell is associated with a set of polynomials. Given specific parameter values, we can identify the corresponding cell and the set of polynomials within it, which forms a Gröbner basis of the ideal for those particular parameter values. First, let's recall the concept of Gröbner bases and so refer to [2, 3] for further details. Let $R = \mathbb{K}[\mathbf{x}]$ be the polynomial ring over the field $\mathbb{K}$ on $\mathbf{x} = x_1, \ldots, x_n$. Below, we denote a monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in R$ by $\mathbf{x}^\alpha$ where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a sequence of non-negative integers. A monomial order on R is a total order $\prec$ on the set of all monomials which has the following two properties:

1. For all $\alpha, \beta, \gamma \in \mathbb{N}^n$, if $\mathbf{x}^\alpha \prec \mathbf{x}^\beta$ then $\mathbf{x}^{\alpha+\gamma} \prec \mathbf{x}^{\beta+\gamma}$.
2. The constant monomial is the smallest; i.e., $1 \prec \mathbf{x}^\alpha$ for all $\alpha \in \mathbb{N}^n$.

A classical example of a monomial ordering is the lexicographical monomial ordering, denoted by $\prec_{lex}$. For two monomials $\mathbf{x}^\alpha, \mathbf{x}^\beta \in R$, we have $\mathbf{x}^\beta \prec_{lex} \mathbf{x}^\alpha$ if the left-most non-zero entry of $\alpha - \beta$ is positive. Let $f \in R$ and $\prec$ be a monomial order on R. The leading monomial of f is the greatest monomial (w.r.t. $\prec$) appearing in f, where we denote it by $\mathrm{LM}(f)$. The leading monomial ideal of $I = \langle f_1, \ldots, f_k\rangle \subset R$ is defined to be $\mathrm{LM}(I) = \langle \mathrm{LM}(f) | f \in I\rangle$. A finite subset $\{g_1, \ldots, g_k\} \subset I$ is called a Gröbner basis for I w.r.t. $\prec$ if $\mathrm{LM}(I) = \langle \mathrm{LM}(g_1), \ldots, \mathrm{LM}(g_k)\rangle$.

Now consider $S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$ where $\mathbb{K}$ is an arbitrary field, $\mathbf{a} = a_1, \ldots, a_m$ is a sequence of parameters and $\mathbf{x} = x_1, \ldots, x_n$ is a sequence of variables. Let $\prec_{\mathbf{x}}$ be a monomial order on the variables and $\prec_{\mathbf{a}}$ be a monomial order on the parameters. The product of $\prec_{\mathbf{x}}$ and $\prec_{\mathbf{a}}$ gives rise to an ordering on S, denoted by $\prec_{\mathbf{x},\mathbf{a}}$ which is defined as follows: For all $\alpha, \beta \in \mathbb{N}^n$ and $\gamma, \delta \in \mathbb{N}^m$, $\mathbf{x}^\alpha\mathbf{a}^\gamma \prec_{\mathbf{x},\mathbf{a}} \mathbf{x}^\beta\mathbf{a}^\delta \iff \mathbf{x}^\alpha \prec_{\mathbf{x}} \mathbf{x}^\beta$or ($\mathbf{x}^\alpha = \mathbf{x}^\beta$and $\mathbf{a}^\gamma \prec_{\mathbf{a}} \mathbf{a}^\delta$).

Definition 1.2. Let $F \subset S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$ be a finite set and $G = \{(G_i, N_i, W_i)\}_{i=1}^{\ell}$ be a finite triple set where $N_i, W_i \subset \mathbb{K}[\mathbf{a}]$ and $G_i \subset S$. The set G is called a Gröbner system for $\langle F\rangle$ w.r.t. $\prec_{\mathbf{x},\mathbf{a}}$ if for any i and for any homomorphism $\sigma : \mathbb{K}[\mathbf{a}] \to \mathbb{K}' \supseteq \mathbb{K}$, we have

- $\sigma(G_i) \subset \mathbb{K}'[\mathbf{x}]$ is a Gröbner basis for $\sigma(\langle F\rangle) \subset \mathbb{K}'[\mathbf{x}]$ w.r.t. $\prec_{\mathbf{x}}$
- $\sigma(p) = 0 \quad \forall p \in N_i \subset K[a]$
- $\sigma(q) \neq 0 \quad \forall q \in W_i \subset K[a]$

The GES-GVW-CGS algorithms [4] is one of the most efficient algorithms for computing Gröbner systems.

2 Main Results

In this section, we introduce the concept of an intersection system for two (or more) parametric polynomial ideals, and we describe an algorithm to compute it.

Definition 2.1. Let I and J are two parametric polynomial ideals in $S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$. A finite triple set $L = \{(N_i, W_i, L_i)\}_{i=1}^{\ell}$ is called an *intersection system* of $I \cap J$ if for each i and any specialization $\sigma : \mathbb{K}[\mathbf{a}] \to \mathbb{K}' \supseteq \mathbb{K}$ satisfying the parametric constraint (N_i, W_i), we have $\sigma(L_i) = \sigma(I) \cap \sigma(J)$.

Based on this definition we can present the Intersect algorithm to compute an intersection system. In doing so, Let $I = \langle f_1, \dots, f_m \rangle$ and $J = \langle g_1, \dots, g_n \rangle$ are two polynomial ideals in $S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$. We consider the ideal

$$H = \langle tf_1, \dots, tf_m, (1-t)g_1, \dots, (1-t)g_n \rangle \subset S[t] = \mathbb{K}[\mathbf{a}, t, \mathbf{x}]$$

and compute a Gröbner system of H w.r.t. the monomial order $\mathbf{a} \prec_{lex} \mathbf{x} \prec_{lex} t$ (the lexicographical order in which t is greater than the $\mathbf{x}$ and $\mathbf{a}$). The polynomials of this system which do not contain the variable t will form a system (a Gröbner system) of $I \cap J$.

Algorithm 1 Intersect

```
Require: $I = \langle f_1, \dots, f_m \rangle$ and $J = \langle g_1, \dots, g_n \rangle$ : two polynomial ideals in $S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$
Ensure: An intersection system of $I \cap J$
  $H := \langle tf_1, \dots, tf_m, (1-t)g_1, \dots, (1-t)g_n \rangle$
  $L := \{(N_i, W_i, L_i)\}_{i=1}^{\ell}$ a Gröbner system of $H$ with respect to $\mathbf{a} \prec_{lex} \mathbf{x} \prec_{lex} t$
  for $(N_i, W_i, L_i) \in L$ do
    if $L_i \neq \{1\}$ then
      $L_i := L_i \cap \mathbb{K}[\mathbf{a}, \mathbf{x}]$; (Eliminate those elements of $L_i$ contains the variable $t$)
    end if
  end for
  Return $\{(N_i, W_i, L_i)\}_{i=1}^{\ell}$
```

Theorem 2.2. Intersect *algorithm terminates in finitely many steps and is correct.*

Proof. The termination of the Intersect algorithm is ensured by the termination of Gröbner system computations. The correctness of the algorithm stems directly from the standard method for computing the intersection of two ideals in $R = \mathbb{K}[\mathbf{x}]$; the polynomial ring with constant coefficients [3, page 187, Theorem 11]. More precisely, let us consider (N_i, W_i, L_i) as a triple of Gröbner system L at the i-th step of the execution of the **for**-loop. If $L_i \neq \{1\}$ then since all parameters in L_i are non-zero, we can apply the Elimination Theorem [3, page 116, Theorem 2]. Therefore any L_i is the intersection of I and J satisfying the parametric conditions (N_i, W_i). ☐

Example 2.3. Let us consider $I = \langle ax(x-1), y^2(cy-a+2) \rangle$ and $J = \langle (x-b)^2y^3 \rangle$ in $\mathbb{K}[a,b,c][x,y]$ the mentioned ideals at example 1.1. Using our `Maple` implementation of the Intersect algorithm, we obtain the following intersection of $I \cap J$ as follows:

$$\begin{cases}
([a],[c], & [b^2cy^4 - 2bcxy^4 + cx^2y^4 + 2b^2y^3 - 4bxy^3 + 2x^2y^3]) \\
([c,a],[1], & [b^2y^3 - 2bxy^3 + x^2y^3]) \\
([c],[a,a-2], & [ab^2y^3 - 2abxy^3 + ax^2y^3 - 2b^2y^3 + 4bxy^3 - 2x^2y^3]) \\
([c,a-2],[b,b-1], & [b^2x^2y^3 - 2bx^3y^3 + x^4y^3 - b^2xy^3 + 2bx^2y^3 - x^3y^3]) \\
([a^2-2a],[a,b,c,b-1], & [ab^2cy^3 - 2abcxy^3 + acx^2y^3]) \\
([ab^4 - 2ab^3 + ab^2, a^2-2a],[a,c], & [ab^2cy^3 - 2abcxy^3 + acx^2y^3]) \\
([c, b^4 - 2b^3 + b^2, a-2],[1], & [y^3(-2b^3x^2 + 2b^3x + 3b^2x^2 - 3b^2x - 2bx^2 + x^3 + 2bx - x^2)]) \\
 & \\
([\,],[ab^2c(b-1)^2(a-2)], & [b^2cy^4 - 2bcxy^4 + cx^2y^4 - ab^2y^3 + 2abxy^3 - ax^2y^3 + \\
 & 2b^2y^3 - 4bxy^3 + 2x^2y^3, ab^2x^2y^3 - 2abx^3y^3 + ax^4y^3 - \\
 & ab^2xy^3 + 2abx^2y^3 - ax^3y^3]) \\
 & \\
([ab^4 - 2ab^3 + ab^2],[a,c,a-2], & [b^2cy^4 - 2bcxy^4 + cx^2y^4 - ab^2y^3 + 2abxy^3 - ax^2y^3 + \\
 & 2b^2y^3 - 4bxy^3 + 2x^2y^3, -2ab^3x^2y^3 + 2ab^3xy^3 + 3ab^2x^2y^3 - \\
 & 3ab^2xy^3 - 2abx^2y^3 + ax^3y^3 + 2abxy^3 - ax^2y^3]).
\end{cases}$$

For instance, if $a = 2, b = 0$ and $c = 0$ then the seventh branch corresponds to these values of parameters (these values satisfy (N_v, W_7)). Therefore, $G_7|_{a=2,b=0,c=0} = [y^3(-2b^3x^2 + 2b^3x + 3b^2x^2 - 3b^2x - 2bx^2 + x^3 + 2bx - x^2)]|_{a=2,b=0,c=0}$ will be the intersection for the ideal $I\,|_{a=2,b=0,c=0} \cap J\,|_{a=2,b=0,c=0}$. This means that for any specialization σ holding (N_7, W_7) we have $\sigma(I) \cap \sigma(J) = \sigma(G_7)$. Consequently, $I|_{a=2,b=0,c=0} \cap J|_{a=2,b=0,c=0} = \langle y^3(x^3 - x^2) \rangle$.

It is worth noting that a similar method can be applied to compute the intersection of more than two polynomial ideals with numeric coefficients [2, Proposition 6.19]. The following theorem is a generalization of this proposition in parametric coefficient cases.

Theorem 2.4. *Let* $I_1, I_2, \dots, I_r$ *be parametric polynomial ideals of* $S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$ *and let*

$$J = \langle t_1 I_1, \dots, t_r I_r, 1 - \Sigma_{i=1}^{r} t_i \rangle \subset \mathbb{K}[a_1, \dots, a_m, x_1, \dots, x_n, t_1, \dots, t_r] = \mathbb{K}[\mathbf{a}, \mathbf{x}, \mathbf{t}]$$

where $\mathbf{t} = t_1, \dots, t_r$ *are new variables. Assume that* $L := \{(N_i, W_i, L_i)\}_{i=1}^{\ell}$ *be a Gröbner system of* J *with respect to* $\mathbf{a} \prec_{lex} \mathbf{x} \prec_{lex} \mathbf{t}$*. Then, the polynomials in this system that do not include the variables* t_i *will constitute a system (a Gröbner system) for* $I_1 \cap I_2 \cap \cdots \cap I_r$*.*

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بعضی سری‌های توانی صوری لورانت روی میدان‌ها

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چکیده. حلقه سری‌های توانی صوری لورانت روی میدان L بسیار جذاب است. به‌ویژه از منظر نظریه میدان و نظریه گالوا. کلاس‌های توابع گویا، توابع جبری، توابع مشتق‌پذیر-جبری یا (به اختصار D-جبری) یک زنجیر تشکیل داده که مطالعه هر کدام از اجزاء آن به‌تنهایی نیز جذاب است، به‌ویژه زمانی که علاوه بر توجه به عمل دوتایی جمع و ضرب روی آن‌ها به یک عمل دیگری که پیچش یا ضرب هدمارد و یا حتی ”ضرب بچه‌ها“ نیز گفته می‌شود، در صورت امکان تجهیز شود. در این مقاله ضمن مطالعه نتایجی از مفاهیم فوق که توسط چندین ریاضیدان مشهور بررسی شده‌اند، یکی از سوالات باز را پاسخ داده و چندین سوال باز جدید نیز مطرح می‌نمائیم.

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۱. مقدمه

در سرتاسر این مقاله L یک میدان است. حلقه چندجمله‌ای‌ها روی L را با $L[x]$، میدان کسرهای آن را با $L(x)$ (که به میدان توابع گویا مشهور است)، حلقه سری‌های توانی صوری روی L را با $L[[x]]$ و میدان کسرهای آن را با $L((x))$ (که به میدان سری‌های توانی لورانت مشهور است) نمایش می‌دهیم و هر کدام را می‌توان در حالت چندمتغیره نیز معرفی نمود.

یادآوری می‌کنیم که عناصر $L(x)$ لزوما در $L[[x]]$ نیستند زیرا که هر تابع گویا لزوماً یک بسط سری توانی صوری ندارد و عناصر $L((x))$ دارای بسطی به شکل $\sum_{i=-N}^{\infty} a_i x^i$ هستند.

تعریف ۱.۱. عنصر $f \in L((x))$ را روی L یک تابع جبری گوییم هرگاه f در یک چندجمله‌ای ناصفر با ضرایب در $L(x)$ صدق کند. یک عنصر غیرجبری را متعالی گوییم.

مثال ۲.۱. عنصر $f = \sum_{n=0}^{\infty} \binom{2n}{n} x^n = (1-4x)^{-1/2}$ روی هر میدان دلخواه L جبری است و عنصر $g = \sum_{n=0}^{\infty} x^{n!}$ روی هر میدان دلخواه L متعالی است ([۲، ۷]).

در ادبیات ریاضی عملگرهای متعددی روی حلقه سری‌های توانی (لورانت) تعریف می‌شود. یکی از این عملگرها، عملگر مشتق است. این عملگر زمانی کارایی دارد که مشخصه میدان L صفر باشد.
پس از این به بعد مقصود از L، یک میدان از مشخصه صفر است.

تعریف ۳.۱. فرض کنید $f \in L((x))$ و f'، f''، f'''، ...، $f^{(n)}$، ... مشتقات صوری f باشند. تابع f را یک تابع مشتق‌پذیر-جبری (و یا به اختصار D-جبری) گوییم هرگاه f'، f''، f'''، ...، $f^{(n)}$، ... در یک چندجمله‌ای ناصفر صدق کنند.

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واژگان کلیدی. سری توانی صوری، میدان توابع گویا، عناصر جبری و متعالی، عناصر مشتق‌پذیر-جبری.

به عبارت دیگر، f را D-جبری گوییم هرگاه

$$tr.deg._{L(x)}L\left(x, f, f', f'', \ldots, f^{(n)}, \ldots\right) < \infty$$

که در آن $tr.deg._B A$ به معنی درجه پایه متعالی A روی B است. چنانچه f، D-متناهی نباشد، آن را متعالیاً متعالی (یا به اختصار [1]) گوییم.

مثال ۴.۱. سری‌های $f = \sum_{n=0}^{\infty} x^{n^2}$، $g = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ و $h = \sum_{n=0}^{\infty} n!x^n$ همگی D-جبری‌اند ولی سری‌های $w = \sum_{n=0}^{\infty} x^{2^n}$ و $y = \sum_{n=0}^{\infty} \frac{1}{(n^2)!}x^{n^2}$ هر کدام TT هستند. [۷، ۴]

۲. قطر سری‌های توانی صوری

عملگری دیگری که در مبحث سری‌های توانی صوری چندمتغیره مشاهده می‌کنیم، عملگر قطر است که به صورت زیر تعریف می‌شود.

$$d : L[[x_1, x_2, \ldots, x_k]] \to L[[x]]$$

$$d(f) = d\left(\sum_{\substack{n_j \geq 0 \\ j=1,2,\ldots,k}} a_{n_1 n_2 \cdots n_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}\right) = \sum_{n=0}^{\infty} a_{nn\cdots n} t^n$$

که در آن $a_{nn\cdots n} \in L$.

در حقیقت قطر یک سری چندمتغیره، یک سری یک متغیره است. این عملگر توسط افراد زیادی مورد مطالعه قرار گرفته است ([۱، ۲، ۳، ۴، ۵]).

اولین سوالی که به ذهن خواننده می‌آید این است که آیا قطر هر سری توانی صوری گویای چندمتغیره، گویاست؟

مثال ۱.۲.

$$f = \sum_{n,m=0}^{\infty} \binom{n+m}{n} x^n y^m = \frac{1}{1-x-y}$$

یک تابع گویاست ولی

$$d(f) = \sum_{n=0}^{\infty} \binom{2n}{n} t^n = (1-4t)^{-1/2}$$

که یک تابع غیرگویا (و در حقیقت جبری) است.

این اتفاق شانسی نیست. در حقیقت قضیه زیر ابتدا توسط Furstenberg در [۲] به صورت تحلیلی و سپس توسط Gessel در [۳] با توجه به جبری‌بسته بودن میدان اعداد مختلط $\mathbb{C}$، با استفاده از تجزیه کسرها اثبات شد.

قضیه ۲.۲. فرض کنید $\mathbb{C}$ میدان اعداد مختلط و $f(x,y) = \sum a_{mn}x^m y^n \in \mathbb{C}[[x,y]]$ یک تابع گویا باشد. در این صورت $d(f) = \sum_{n=0}^{\infty} a_{nn}t^n$ روی $\mathbb{C}$ جبری است.

تذکر. قضیه فوق برای حالت n متغیره که $n > 2$ برقرار نیست.

---

[1] transcendental Transcendentally

مثال ۳.۲.

$$\begin{aligned} f &= \sum_{n_i\geq ۰}\binom{n_۱+n_۲+n_۳}{n_۱,n_۲,n_۳}x_۱^{n_۱}x_۲^{n_۲}x_۳^{n_۳} \\ &= \sum_{n\geq ۰}(x_۱+x_۲+x_۳)^n = \frac{۱}{۱-x_۱-x_۲-x_۳} \end{aligned}$$

یک تابع گویاست، اما $d(f)=\sum_{n\geq ۰}\binom{۳n}{n,n,n}t^n$ یک سری متعالی است. این مطلب و تعمیمی از آن را در [۶] ارائه دادیم. در واقع نشان دادیم که برای $k,t\in\mathbb{N}$ که $t\geq ۱$ و $k\geq ۳$، تابع

$$g_t=\sum_{m=۰}^{\infty}\binom{km}{m,m,\cdots,m}^t x^m$$

روی هر میدان از مشخصه صفر متعالی است.

در اینجا یک حدس هم به‌عنوان سوال باز مطرح می‌کنیم.
حدس. با توجه به سری فوق، چناچه $s\in\mathbb{N}$ که $s\geq ۱$ و $s\neq t$، در این صورت g_t و g_s به‌طور جبری مستقل‌اند. به عبارت دیگر

$$tr.deg._{L(x)}L(x,g_s,g_t)>۱.$$

تذکر. بنابر مثال ۳.۲، قطر هر تابع جبری سه متغیره نیز لزوما جبری نیست.

آیا می‌توان گفت که D−جبری است؟
در این‌جا پاسخی منفی به این سوال حتی برای حالت دومتغیره ارائه می‌دهیم. قبل از آن به نکته زیر توجه می‌کنیم.
نکته. مجموعه تمام سری‌های توانی صوری D−جبری تحت ضرب بسته است.
حال نشان می‌دهیم که قطر یک سری D−جبری (و حتی یک تابع جبری) دو متغیره، D−جبری نیست.

مثال ۴.۲. فرض کنید $f(x)=\sum_{n=۰}^{\infty}x^{n^۲}$ و $g(y)=\sum_{k=۰}^{\infty}\frac{۱}{k!}y^k$ که D−جبری‌اند. حال

$$h(x,y)=(\sum_{n\geq ۰}x^{n^۲})\left(\sum_{k\geq ۰}\frac{۱}{k!}y^k\right)$$

یک سری دومتغیره D−جبری است. داریم

$$d(f)=\sum\frac{۱}{(n^۲)!}t^{n^۲}$$

که بنابر مثال ۴.۱، یک سری D−جبری نیست.

در ادامه بحث عملگر d را با استفاده از ضرب هدمارد و یا پیچش دو سری توانی صوری ارائه خواهیم داد و نتایج فوق را در مورد سری‌های توانی صوری روی میدان‌های با مشخصه مثبت که در ادامه نتایج [۵] می‌باشد را پی می‌گیریم.
در خاتمه با معرفی نرم ناارشمیدسی (non-Archimedean) ویژگی بعضی از میدان‌های میانی بین $L(x)$ و $L((x))$ را بررسی می‌کنیم.

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Baer Criterion for Injectivity in Abelian Categories

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Abstract

It is well-known that the Baer Criterion for injectivity of R-modules, for a ring R with unit, is not true in a general category, even in a general abelian category. In this paper, we prove some results analogous to the Baer Criterion for injectivity in abelian categories and Grothendieck categories. In particular, we generalize the known fact that G-injectivity is the same as injectivity, if G is a generator in a Grothendieck category. Furthermore, some Baer type theorems for general abelian categories is proved. Fianlly, equivalent conditions to satisfying a classical kind of Baer criterion are found in (locally presentable) abelian categories.

Keywords: Injectivity, Baer criterion, abelian category, Grothendieck category.

Mathematics Subject Classification [2010]: 18E10, 18G05, 16D50.

1 Introduction and Preliminaries

Let R be a ring with unit. If a left unitary R-module A is R-injective (Definition 1.1), then it is injective. This is known as Baer criterion (see [2]). One of the important consequences of the Baer criterion is that the category of R-modules has enough injectives and the other its main result is that for the case where R is a principal ideal domain, the injective R-modules and so particularly injective abelian groups, are characterized as divisible ones.

Although the Baer criterion is not true in general, seeking for Baer criterion type results for injectivity of objects in some categories of algebraic systems, abelian categories and localic topoi have been investigated in many papers (see for example, [3], [4], [5], [6], [7], [9], [12], [14], [15]). As the original Baer criterion was proved for unital modules and since abelian categories are natural generalizations of the category of R-modules, we decided to consider the Baer type criterions in abelian categories.

It is known that if G is a generator in an Grothendieck category, then G-injectivity is equivalent to injectivity (see for example lemma 1 [10]). In Section 2 of [13], we generalize this fact by showing that G-injectivity for a generator G is equivalent to having no proper essential extension (Theorem 2.1). Then, it is concluded that F-injectivity is equivalent to injectivity, where F is the free object over the singleton set (Theorem 2.3).

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[*] Speaker.

A general Baer type criterion has been proved in [11] (see [11], Theorem 9.5.5)) for abelian categories. In Section 3 of [13], we also investigate a situation such that the conditions of this theorem are fulfilled and so a Baer criterion type result obtains for abelian categories (see Theorem 3.2). Moreover another Baer criterion type result for abelian categories is proved by utilizing Freyd-Mitchell's Embedding Theorem (see Theorem 3.3).

Finally, in Section 4 of [13], introducing the notion of satisfying the classical Baer criterion, an equivalent condition to have it being satisfied is found for both abelian categories and locally presentable abelian categories. Following those results, it is concluded that the subcategory of injective objects of a Grothendieck category is accessibly embedded in it(Theorem 4.3).

In this paper the results that we proved in [13] are briefly mentioned. The most basic notions of abelian categories needed in this paper are mainly from [8] and [13]. We just recall the following definitions.

Definition 1.1 ([13, Definition 2.1]). Let $\mathcal{A}$ be a category and A be an object of $\mathcal{A}$. An object E is said to be A-*injective*, if for each monomorphism $i : B \to A$ and for each morphism $f : B \to E$ there exists a morphism $\overline{f} : A \to E$ such that $\overline{f} \circ i = f$.

Definition 1.2 ([13, Definition 2.2]).

1. Let $\mathcal{A}$ be a category. An object K is said to be injective with respect to a morphism $m : A \to A^{'}$ provided that for each morphism $f : A \to K$, there exists a morphism $f^{'} : A^{'} \to K$ such that $f^{'} \circ m = f$.
2. If $\mathcal{M}$ is a class of morphisms in a category $\mathcal{A}$ then an object K is said to be $\mathcal{M}$-injective provided that K is injective with respect to each morphism in $\mathcal{M}$.

Definition 1.3 ([13, Definition 4.1]). If there exists an object A in a category $\mathcal{A}$ such that A-Injectivity coincides with injectivity, then we say that the category $\mathcal{A}$ satisfies the *classical Baer criterion.*

2 Baer Criterion in Grothendieck Categories

In this section we generalize the Baer criterion for Grothendieck categories as follows.

Theorem 2.1 ([13, Proposition 2.6]). *Let $\mathcal{A}$ be an abelian category with a generator G. If an object A is G-injective, then A has no proper essential extension.*

As a corollary of the above theorem, we have "*If $\mathcal{A}$ is an abelian category with a generator G in which injectivity coincides with not having proper essential extension, then in $\mathcal{A}$, G-injectivity coincides with injectivity.*" Therefore, as a corollary of Theorem 2.1, we have:

Theorem 2.2 ([13, Corollary 2.7]). *If $\mathcal{A}$ is a Grothendieck category with a generator G, then in $\mathcal{A}$, G-injectivity coincides with injectivity.*

Theorem 2.3 ([13, Corollary 2.8]). *Let $\mathcal{A}$ be a Grothendieck category which is concrete and F be the free object over the singleton set. Then an object A is F-injective if and only if it is injective.*

3 Some Baer Type Criterions for Injectivity in Abelian Categories

In this section we prove some Baer criterion for abelian categories. First, we prove a Baer criterion in abelian categories with generator. Notice that in the proof of this we do not use any equivalent version of axiom of choice. for more details see [13] proof of theorem 3.5.

Theorem 3.1 ([13, Theorem 3.5]). *Let $\mathcal{A}$ be an abelian category and G be a generator of $\mathcal{A}$. If $\mathcal{M}_1$ is the class of monomorphisms f such that Codomain$(f) = G$ and $\mathcal{M}_2$ is the class of monomorphisms g such that Codomain(g) is finitely generated, then an object A is $\mathcal{M}_1$-injective if and only if it is $\mathcal{M}_2$-injective.*

Now, we give situation such that the conditions of theorem 3.10 [13] are fulfilled and so a Baer criterion type result obtains for abelian categories.

Theorem 3.2 ([13, Theorem 3.10]). *Let $\mathcal{A}$ be an abelian category and $\mathcal{A}_0$ a subcategory of $\mathcal{A}$ such that $\mathcal{A}_0$ admits small filtrant inductive limits and the inclusion functor $i : \mathcal{A}_0 \to \mathcal{A}$ commutes with such limits and $\mathcal{M} \subseteq Mor(\mathcal{A}_0)$. Suppose that any morphism in $\mathcal{A}_0$ is monic and pushout transfers $\mathcal{M}$-morphisms. Furthermore, assume that $\mathcal{M}$ is right cancellable and if $s : U \to V$, $u : U \to X$ is a pullback for $v : V \to Y$, $f : X \to Y$ with $f \in Mor(\mathcal{A}_0)$, $s \in \mathcal{M}$ such that s is an isomorphism, then f is an isomorphism. Then any $Y \in \mathcal{A}$ which is $\mathcal{M}$-injective is $Mor(\mathcal{A}_0)$-injective.*

Theorem 3.3 ([13, Proposition 3.11]). *Let $\mathcal{A}$ be an small abelian category and $\mathcal{L} \subseteq Fun(\mathcal{A}, \mathbf{Ab})$ be the category of left exact functors. If $\mathcal{L}$ has a representable injective co-generator I and the image of Freyd-Mitchel's embedding is a Serre subcategory of the associated* **R-Mod**, *then an object A is B-injective if and only if it is injective, where $Hom_{\mathcal{A}}(B, -) = I$.*

4 Classical Baer Criterion in Abelian Categories

In this section we study classical Baer criterion in locally presentable abelian categories. For the basic concepts cencerning locally presentable categories one can see [1].

Theorem 4.1 ([13, Proposition 4.2]). *If $\mathcal{A}$ is an abelian category, then the classical Baer criterion holds in $\mathcal{A}$ if and only if there exists a set $\mathcal{M}$ of monomorphisms such that $\mathcal{M}$-injectivity coincides with injectivity.*

Now, we characterize locally presentable abelian categories which satisfy the classical Baer criterion.

Theorem 4.2 ([13, Proposition 4.6]). *For a locally presentable abelian category $\mathcal{A}$ which has enough injectives, the following are equivalent:* (i) *The classical Baer criterion holds in $\mathcal{A}$.* (ii) *The subcategory of injective objects is accessibly embedded in $\mathcal{A}$, that is, there exists a regular cardinal λ such that the subcategory of injective objects is closed under λ-directed colimits.*

Now, by using Theorem 4.2 and Theorem 2.1, we can prove the following theorem.

Theorem 4.3 ([13, Corollary 4.7]). *If $\mathcal{A}$ is a Grothendieck category, then the subcategory of its injective objects is accessibly embedded in $\mathcal{A}$.*

Theorem 4.4 ([13, Example 4.8]). *Let R be a unitary ring and λ a regular cardinal such that $\lambda > cardR$. Then the subcategory of injective modules in* **R-Mod** *is closed under λ-directed colimits.*

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Minimaxness and Cominimaxness of Local Cohomology Modules

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Abstract

Let I be an ideal of a commutative Noetherian ring R. It is shown that the R-modules $H^i_I(M)$ are I-cominimax, for all finitely generated R-modules M and all $i \in \mathbb{N}_0$, if the R-modules $H^i_I(R)$ are I-cominimax with dimension not exceeding 1, for all integers $i \geq 2$.

Keywords: Local cohomology module, Minimax module,I-cominimax module

Mathematics Subject Classification [2010]: Primary: 13D45,14B15; Secondary: 13E05

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1 Introduction and Preliminaries

Throughout this paper, R denotes a commutative Noetherian ring (with non-zero identity) and I will denote an ideal of R. The symbol $\mathbb{Z}$ denotes the set of integers; in addition, $\mathbb{N}$ (respectively $\mathbb{N}_0$) will denote the set of positive (respectively non-negative) integers. For each R-module L, the set of minimal elements of $\mathrm{Ass}_R L$ with respect to inclusion is denoted by $\mathrm{mAss}_R\, L$; also, $\mathrm{Assh}_R\, L$ denotes the set $\{\mathfrak{p} \in \mathrm{Ass}_R L : \dim R/\mathfrak{p} = \dim L\}$. We denote $\mathrm{Supp}\, R/I = \{\mathfrak{p} \in \mathrm{Spec}\, R : \mathfrak{p} \supseteq I\}$ by $V(I)$.
For an R-module M, the ith local cohomology module of M with support in $V(I)$ is defined as:

$$H^i_I(M) = \varinjlim_{n \geq 1} \mathrm{Ext}^i_R(R/I^n, M).$$

We refer the reader to [6] or [11] for more details about local cohomology.

Recall that for an R-module M, the notion $\mathrm{cd}(I, M)$, *the cohomological dimension of M with respect to I*, is defined as:

$$\mathrm{cd}(I, M) = \sup\{i \in \mathbb{N}_0 : H^i_I(M) \neq 0\}$$

and the notion $q(I, M)$, which for the first time was introduced by Hartshorne, is defined as:

$$q(I, M) = \sup\{i \in \mathbb{N}_0 : H^i_I(M) \text{ is not Artinian}\},$$

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[*] Speaker.

with the usual convention that the supremum of the empty set of integers is interpreted as $-\infty$. These two notions have been studied by several authors (see [3, 4, 8–10, 12, 13]).

In the sequel the symbol $\mathscr{C}(R, I)_{com}$ denotes the category of all I-cominimax R-modules and $\mathscr{C}^1(R, I)_{com}$ denotes the category of all R-modules $M \in \mathscr{C}(R, I)_{com}$ such that $\dim M \leq 1$. An R-module M is called $\mathfrak{a}$-cominimax if the support of M is contained in $V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is minimax for all $i \geq 0$. The concept of the $\mathfrak{a}$-cominimax modules is introduced in [2]. Also, throughout this paper, let $\mathscr{I}'(R)$ denote the class of all ideals I of R such that $H^i_I(M) \in \mathscr{C}(R, I)_{com}$, for all finitely generated R-modules M and all $i \in \mathbb{N}_0$.

Recall that the *I-transform functor*, denoted by $D_I(-)$ is defined as:

$$D_I(-) = \varinjlim_{n \geq 1} \operatorname{Hom}_R(I^n, -).$$

In general, the R-module $D_I(R)$ has an R-algebra structure (see [6, Exercise 2.2.3]). In fact, with this structure $D_I(R)$ is a commutative ring with identity. Also, it is well known that if $D_I(-)$ is an exact functor then $D_I(R)$ is a finitely generated R-algebra. But, in general we don't know when the ring $D_I(R)$ is Noetherian.

Throughout this paper, for each pair of the sets X and Y, the expression $X \subseteq Y$ means that X is a subset of Y and the expression $X \subset Y$ means that $X \subseteq Y$ and $X \neq Y$. For an Artinian R-module A, the set of attached prime ideals of A is denoted by $\operatorname{Att}_R A$. Also, for any non-nilpotent element x of R and any R-module M, the localization of M at the multiplicatively closed subset $S = \{1_R, x, x^2, x^3, \dots\}$ of R is shown by M_x. For each ideal I of a Noetherian ring R and each R-module M, we denote the submodule $\bigcup_{n=1}^{\infty}(0 :_M I^n)$ of M by $\Gamma_I(M)$. Furthermore, for any ideal I of a commutative ring T, we denote the set of minimal prime ideals over I by $\operatorname{Min} I$. Also, we show set of all maximal ideals of a ring T by $\operatorname{Max}(T)$. Finally, for any ideal J of T, *the radical of* J, denoted by $\operatorname{Rad}(J)$, is defined to be the set $\{x \in T : x^n \in J \text{ for some } n \in \mathbb{N}\}$. For any unexplained notation and terminology we refer the reader to [6, 7, 14].
Let R be a Noetherian ring and I be an ideal of R. Recall that a subcategory $\mathscr{M}$ of the category of all R-modules is said to be a *Serre category* if in any short exact sequence of R-modules and R-homomorphisms, the middle module is in $\mathscr{M}$ if and only if the two other modules are in $\mathscr{M}$. Let $\mathscr{C}^1(R, I)$ be the Serre category of all I-torsion R-modules M with $\dim M \leq 1$. We want to emphasize at the outset, that two categories $\mathscr{C}^1(R, I)$ and $\mathscr{C}^1(R, I)_{cof}$ are different. In fact always $\mathscr{C}^1(R, I)_{com}$ is a proper subcategory of $\mathscr{C}^1(R, I)$. Now, for any R-module N, we define the notation $\mathrm{c}^1(I, N)$ as the greatest integer i such that $H^i_I(N)$ is not in $\mathscr{C}^1(R, I)$ if there exist such i's and $-\infty$ otherwise. Finally, we recall that in [3] the notion $\widetilde{q}(I, N)$ is defined as the greatest integer i such that $H^i_I(N)$ is not an Artinian I-cofinite module if there exist such i's and $-\infty$ otherwise.

Lemma 1.1. *Let I be an ideal of a ring R. Assume that M and N are two finitely generated R-modules such that* $\operatorname{Supp} M \subseteq \operatorname{Supp} N$. *Then the following statements hold:*

1. $\mathrm{c}^1(I, M) \leq \mathrm{c}^1(I, N)$.
2. $q(I, M) \leq q(I, N)$.
3. $\widetilde{q}(I, M) \leq \widetilde{q}(I, N)$.
4. $\operatorname{cd}(I, M) \leq \operatorname{cd}(I, N)$.

Proof. (1) Considering the fact that $\mathscr{C}^1(R, I)$ is a Serre category, the assertion follows immediately from [3, Theorem 2.3].

(2) See [8, Theorem 3.2].

(3) See [3, Theorem 2.6].

(4) See [9, Theorem 2.2].

□

Lemma 1.2 (See [1, Lemma 2.3]). *Let I be an ideal of a ring R and $\mathscr{M}$ be a Serre subcategory of the category of R-modules. Let $n \in \mathbb{N}_0$ and M be an R-module such that $\operatorname{Ext}^j_R(R/I, H^i_I(M)) \in \mathscr{M}$, for all $0 \leq i < n$ and all $j \in \mathbb{N}_0$. If the R-modules $\operatorname{Ext}^n_R(R/I, M)$ and $\operatorname{Ext}^{n+1}_R(R/I, M)$ are in $\mathscr{M}$, then the R-modules $\operatorname{Hom}_R(R/I, H^n_I(M))$ and $\operatorname{Ext}^1_R(R/I, H^n_I(M))$ are in $\mathscr{M}$.*

Lemma 1.3 (See [5, Proposition 2.6]). *Suppose that I is an ideal of a ring R such that $\Gamma_I(R) = 0$ and $q(I, R) \leq 1$. Let N be a finitely generated R-module. Then the R-modules $\operatorname{Tor}^R_i(N, D_I(R))$ are Artinian and I-cofinite, for all $i \in \mathbb{N}$, and the R-modules $\operatorname{Ext}^j_R(R/I, N \otimes_R D_I(R))$ are finitely generated, for all $j \in \mathbb{N}_0$.*

2 Main Results

Theorem 2.1. *Let I be an ideal of a ring R such that $\Gamma_I(R) = 0$ and $H^i_I(R) \in \mathscr{C}^1(R, I)_{com}$, for all integers $i \geq 2$. Then, for each finitely generated R-module N and each integer $i \in \mathbb{N}$, the R-module $\operatorname{Tor}^R_i(N, D_I(R))$ is I-cominimax and the R-modules $\operatorname{Ext}^j_R(R/I, N \otimes_R D_I(R))$ are minimax, for all $j \in \mathbb{N}_0$.*

Theorem 2.2. *Let I be an ideal of a ring R such that $H^i_I(R) \in \mathscr{C}^1(R, I)_{com}$ for each integer $i \geq 2$. Then $I \in \mathscr{I}'(R)$.*

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Some New Results About Local Cohomology Modules

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Abstract

Let (R, m, k) be a commutative Noetherian local ring and I be an ideal of R. In this paper we are going to introduce a new invariant for any non-zero finitely generated R-module M of dimension n, which is shown with $\lambda_R(M)$. Then we will obtain an upperbound for $\lambda_R(M)$ and some other results concerning the $\lambda_R(M)$.

Keywords: Local cohomology module, Associated primes, Krull dimension

Mathematics Subject Classification [2010]: Primary: 13D45,14B15; Secondary: 13E05

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1 Introduction and Preliminaries

Throughout this paper, R will always denote a non-trivial commutative Noetherian ring with identity. For an R-module M, the i^{th} local cohomology module of M with respect to an ideal I is defined as

$$H_I^i(M) = \varinjlim_{n \geq 1} \mathrm{Ext}_R^i(R/I^n, M).$$

For each R module L ,we denote by $\mathrm{Assh}_R(L)$ the set $\left\{\mathfrak{p} \in \mathrm{Ass}_R(L) : \dim \frac{R}{\mathfrak{p}} = \dim L\right\}$. For any ideal $\mathfrak{b}$ of R, we denote $\{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{b}\}$ by $V(\mathfrak{b})$ and the *radical* of $\mathfrak{b}$, denoted by $\sqrt{\mathfrak{b}}$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b} \text{ for some } n \in \mathbb{N}\}$. Recall that, for an R- module M, the set $\mathrm{MinAss}_R(M)$ is defined as

$$\{\mathfrak{p} \in \mathrm{Ass}_R(M) : \nexists \mathfrak{q} \in \mathrm{Ass}_R(M), \mathfrak{q} \subsetneqq \mathfrak{p}\}.$$

The reader is referred to [1] for more details about local cohomology. Recall that, for an R-module M, the *cohomological dimension* of M *with respect* to I is defined as

$$\mathrm{Cd}(I, M) := \mathrm{Sup}\left\{i \in \mathbb{Z} \,:\, H_I^i(M) \neq 0\right\}.$$

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[*] Speaker.

Also for any proper ideal I of R, the *arithmetic rank* of I, denoted by $\operatorname{ara}(I)$, is the least number of elements of R required to generate an ideal which has the same radical as I.

Theorem 1.1 (Non-Vanishing Theorem)**.** *Assume that $(R, \mathfrak{m})$ is local and let M be a non-zero finitely generated R-module of dimension n. Then $H^n_{\mathfrak{m}}(M) \neq 0$.*

Proof. See [1, Theorem 6.1.4]. □

As an application of Theorem 1.1, the following set

$$\{I : H^n_I(M) \neq 0 \text{ and } I \text{ is a proper ideal of } R\}$$

is not empty. In the following Definition we introduce an invariant, called $\lambda_R(-)$, which is base of our paper.

Definition 1.2. Let $(R, \mathfrak{m})$ be *a* Noetherian local ring of dimension d and M be a non-zero finitely generated R-module of dimension n. The invariant $\lambda_R(-)$ relative to M is defined by

$$\lambda_R(M) = \sup\left\{\dim \frac{R}{I} : H^n_I(M) \neq 0\right\}.$$

2 Main Results

Theorem 2.1. *Let $(R, \mathfrak{m})$ be a Noetherian complete regular local ring of dimension d and M a non-zero finitely generated R- module of dimension n. Then $\lambda_R(M) = d - n$.*

Theorem 2.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M, N be two finitely generated R-modules such that* $\operatorname{Supp} M = \operatorname{Supp} N$. *Then $\lambda_R(M) = \lambda_R(N)$.*

Theorem 2.3. *Let (R, m, k) be a Noetherian local ring of dimension d and $M \neq 0$ be a finitely generated R-module, of dimension n. Then*

$$\lambda_R(M) \leq \lambda_{\hat{R}}(M \otimes_R \hat{R}) \leq 1 + \dim_k \mathrm{m}/\mathrm{m}^2 - n.$$

Theorem 2.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence of non-zero finitely generated R- modules. Then

$$\lambda_R(M) \leq \operatorname{Max}\{\lambda_R(L), \lambda_R(N)\}.$$

Theorem 2.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and M be a non-zero finitely generated R-module of dimension n. Then*

$$\lambda_R(M) = \sup\left\{\lambda_R\left(\frac{R}{\mathfrak{p}}\right) : \mathfrak{p} \in \operatorname{Assh}_R(M)\right\}.$$

Theorem 2.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$ and $\lambda_R(R) = 0$. Then* $|\operatorname{Assh}_R(R)| = 1$.

Theorem 2.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$. Suppose that* $\operatorname{ara}(\mathfrak{p}) < d$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. *Then $\lambda_R(R) = 0$.*

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Planarity of the Intersection Graph of the Idealization

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Abstract

Let R be a commutative ring with identity. The intersection graph of ideals of a ring R is an undirected simple graph denoted by $\Gamma(R)$ whose vertices are in a one-to-one correspondence with non-zero proper ideals and two distinct vertices are joined by an edge if and only if the corresponding ideals of R have a non-zero intersection. Let M be a unitary R-module and let $R \ltimes M$ be the idealization of M in R. In this paper, we obtain necessary and sufficient conditions on a ring R and a module M such that $\Gamma(R \ltimes M)$ is planar. In fact, we prove that $\Gamma(R \ltimes M)$ is planar if and only if $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$, $\mathfrak{m}_1 \cap \mathfrak{m}_2 = 0$, $\mathfrak{m}_2 M = 0$ and $\dim_{R/\mathfrak{m}_2}(M) = 1$ or M is a simple module and R has only one non-trivial ideal.

Keywords: Idealization, Intersection graph

Mathematics Subject Classification [2010]: Primary: 13A15, Secondary: 05C75

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1 Introduction and Preliminaries

One of the classical topics in the theory of graphs is the intersection graph theory. Let R be a commutative ring and M be a unitary R-module. The *intersection graph of ideals* of a ring R is an undirected simple graph denoted by $\Gamma(R)$ whose vertices are in a one-to-one correspondence with non-zero proper ideals and two distinct vertices are joined by an edge if and only if the corresponding ideals of R have a non-zero intersection. The intersection graph of ideals of a ring R was introduced [6] and studied in several papers, see [1,2,8,9].

Idealization of M in R, denoted by $R \ltimes M$, is the ring whose additive structure is that of the external direct sum $R \oplus M$ and whose multiplication is defined by $(r, m)(r', m') := (rr', rm' + r'm)$ for all $r, r' \in R$ and all $m, m' \in M$.

Other area of interest in recent years is the theory of idealization of M in R, see [3,4,7]. In [8], authors studied idealization by combinatorial methods and investigated the ideals in $R \ltimes M$ using graph-theoretic concepts. In this paper, we study the graph-theoretic properties of the intersection graph of idealization and we are especially interested in the planarity of the intersection graph of idealization.

Throughout this paper, all graphs are simple with no loops and multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For distinct vertices $x_1, \ldots, x_n \in V(G)$, $x_1 - x_2 - \cdots - x_n$ denotes a *path* from x_1 to x_n. A *cycle* of length n is a path of the form $x_1 - x_2 - \cdots - x_n - x_1$ where $x_i \neq x_j$ when $i \neq j$ and denoted by C_n. A graph in which each pair of distinct vertices is joined by an edge is called a *complete* graph. By K_n, we mean the complete graph over n vertices. A graph is *bipartite* if its vertex set can be partitioned into two subsets X and Y such that every edge has one end in X and one end in Y. By $K_{r,s}$, we mean the complete bipartite graph which $|X| = r$ and $|Y| = s$. If $r = 1$, then it is called *star graph*. Let $S \subseteq V$ be any subset of vertices of G. Then the *induced subgraph* on S is the graph whose vertex set is S and edge set contains all edges of $E(G)$ connecting pairs of vertices in S and denoted by $\langle S \rangle$. A *clique* of G is a complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G.

A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A *subdivision* of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths.

All over the paper let $S := R \ltimes M$. We denote the set of all *maximal ideals* of R by $\operatorname{Max}(R)$. The non-zero module M is called a *simple module* if the only submodules of M are 0 and M.

2 Main Results

A simple characterization of planar graphs was given by Kuratowski in 1930 [5, Page 153].

Theorem 2.1. *A graph is planar if and only if it contains no subdivision of* $K_{3,3}$ *or* K_5*.*

A natural question that arises is that under which conditions the intersection graph of idealization is planar? We study this question in this paper.

The next theorem provides a characterization for the planarity of the intersection graph $R \ltimes M$ when R is a field.

Theorem 2.2 ([10]). *Let* $S = F \ltimes M$*, where* F *is a field. Then* $\Gamma(S)$ *is planar if and only if* $\dim_F(M) \leq 2$.

A maximal ideal of S is of the form $\mathfrak{m} \ltimes M$ for a maximal ideal $\mathfrak{m}$ of R [3, Theorem 3.2]. It follows that all maximal ideals of S make a clique with $0 \ltimes M$ and one has the following lemma.

Lemma 2.3 ([10]). *Let* $S = R \ltimes M$*. Then* $\omega(\Gamma(S)) \geq |\operatorname{Max}(R)| + 1$.

Using the above lemma, one can show that $\Gamma(S)$ is not planar whenever $|\operatorname{Max}(R)| \geq 3$.

Lemma 2.4 ([10]). *Let* $S = R \ltimes M$ *and assume that* $|\operatorname{Max}(R)| \geq 3$*. Then* $\Gamma(S)$ *is not planar.*

In the next two lemmas, we consider the case where $|\operatorname{Max}(R)| = 2$

Lemma 2.5 ([10]). *Let* $S = R \ltimes M$ *and assume* $\operatorname{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ *such that* $\mathfrak{m}_1 \cap \mathfrak{m}_2 \neq 0$*. Then* $\Gamma(S)$ *is not planar.*

Lemma 2.6 ([10]). *Let* $S = R \ltimes M$ *and assume* $\operatorname{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ *such that* $\mathfrak{m}_1 \cap \mathfrak{m}_2 = 0$*. Then the following statements hold.*

1. *If* $\mathfrak{m}_1 M \neq 0$ *and* $\mathfrak{m}_2 M \neq 0$*, then* $\Gamma(S)$ *is not planar.*
2. *If* $\mathfrak{m}_1 M \neq 0$, $\mathfrak{m}_2 M = 0$ *and* $\dim_{R/\mathfrak{m}_2}(M) \geq 2$*, then* $\Gamma(S)$ *is not planar.*
3. *If* $\mathfrak{m}_1 M \neq 0$, $\mathfrak{m}_2 M = 0$ *and* $\dim_{R/\mathfrak{m}_2}(M) = 1$*, then* $\Gamma(S)$ *is planar.*

Example 2.7. Let $S = \mathbb{Z}_{pq} \ltimes M$ where $M = \mathbb{Z}_{pq}/p\mathbb{Z}_{pq}$. We have $\operatorname{Max}(\mathbb{Z}_{pq}) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ where $\mathfrak{m}_1 = q\mathbb{Z}_{pq}$ and $\mathfrak{m}_2 = p\mathbb{Z}_{pq}$. It is obvious that $\mathfrak{m}_1 M \neq 0, \mathfrak{m}_2 M = 0$ and $\dim_{\mathbb{Z}_{pq}/p\mathbb{Z}_{pq}}(M) = 1$. Then S has four non-trivial ideals and it is a planar graph, by Lemma 2.6.

In the sequel, we consider $|\operatorname{Max}(R)| = 1$.

Lemma 2.8 ([10]). *Let* $S = R \ltimes M$ *and assume that* R *has only one non-zero proper ideal* $\mathfrak{m}$ *and that* M *is a simple* R*-module. Then* $\Gamma(S)$ *is planar.*

Proof. By [8, Theorem 11], $\Gamma(S)$ is a star graph so it is planar. □

The minimal number of generators of a module M will be denoted by $\mu(M)$. In the following lemmas, we consider the values of $\mu(\mathfrak{m})$ carefully.

Lemma 2.9 ([10]). *Let* $S = R \ltimes M$ *and assume that* $(R, \mathfrak{m})$ *is a local ring such that* $\mu(\mathfrak{m}) \geq 3$. *Then* $\Gamma(S)$ *is not planar.*

Lemma 2.10 ([10]). *Let* $S = R \ltimes M$ *and assume that* $(R, \mathfrak{m})$ *is a local ring such that* $\mu(\mathfrak{m}) = 2$. *Then* $\Gamma(S)$ *is not planar.*

We use the following simple lemma in the continue.

Lemma 2.11 ([10]). *Assume* $(R, \mathfrak{m})$ *is a local ring such that* $\mathfrak{m} = \langle r \rangle$ *and* $r^4 = 0$. *Then all non-trivial ideals of* R *are* $\mathfrak{m}, \mathfrak{m}^2$ *and* $\mathfrak{m}^3$.

Lemma 2.12 ([10]). *Let* $S = R \ltimes M$ *and assume that* $(R, \mathfrak{m})$ *be a local ring such that* $\mathfrak{m} = \langle r \rangle$. *Then* $\Gamma(S)$ *is not planar unless that* $\mathfrak{m}^2 = 0$ *and* M *is a simple module.*

Combining the above lemmas, we get the following theorem.

Theorem 2.13 ([10]). *Let* $S = R \ltimes M$. *Then* $\Gamma(S)$ *is planar if and only if* $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$, $\mathfrak{m}_1 \cap \mathfrak{m}_2 = 0$, $\mathfrak{m}_2 M = 0$ *and* $\dim_{R/\mathfrak{m}_2}(M) = 1$ *or* M *is a simple module and* R *has only one non-trivial ideal.*

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Some Results on the Intersection Graph

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Abstract

Let R be a commutative ring with identity. The intersection graph of ideals of a ring R is an undirected simple graph denoted by $\Gamma(R)$ whose vertices are in a one-to-one correspondence with non-zero proper ideals and two distinct vertices are joined by an edge if and only if the corresponding ideals of R have a non-zero intersection. Let M be a unitary R-module and let $R \ltimes M$ be the idealization of M in R. In this paper, we study some graph-theoretic properties of the intersection graph of idealization. We determine the set of ideals of the $\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$ and $\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, for distinct prime numbers p and q.

Keywords: Idealization, Intersection graph

Mathematics Subject Classification [2010]: Primary: 13A15, Secondary: 05C75

1 Introduction and Preliminaries

One area of interest in recent years is to associate graphs on algebraic structures. Associating a graph to a ring is a research subject in this area and has attracted considerable attention. Let R be a commutative ring and M be a unitary R-module. The *intersection graph of ideals* of a ring R is an undirected simple graph denoted by $\Gamma(R)$ whose vertices are in a one-to-one correspondence with non-zero proper ideals and two distinct vertices are joined by an edge if and only if the corresponding ideals of R have a non-zero intersection. The intersection graph of ideals of a ring R was introduced [6] and studied in several papers, see [1,2,9,10]. *Idealization of M in R*, denoted by $R \ltimes M$, is the ring whose additive structure is that of the external direct sum $R \oplus M$ and whose multiplication is defined by $(r, m)(r', m') := (rr', rm' + r'm)$ for all $r, r' \in R$ and all $m, m' \in M$.

This construction introduced in 1956 by Nagata [11] and has been extensively studied, see [4,7,8].

Other area of interest in recent years is the theory of idealization of M in R, see [3,5,7]. In this paper, we study some properties of idealization $\mathbb{Z}_n$ in $\mathbb{Z}_n$ where $\mathbb{Z}_n$ is the ring of integers modulo n. We determine the set of all ideals of the $\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$ and $\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, for distinct prime numbers p and q.

Throughout this paper, all graphs are simple with no loops and multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For distinct vertices $x_1, \dots, x_n \in V(G)$, $x_1 - x_2 - \cdots - x_n$ denotes a *path* from x_1 to x_n. A *cycle* of length n is a path of the form $x_1 - x_2 - \cdots - x_n - x_1$ where $x_i \neq x_j$ when $i \neq j$ and denoted by C_n. A graph in which each pair of distinct vertices is joined by an edge is called a *complete* graph. By K_n, we mean the complete graph over n vertices.

2 Main Results

There is a closed relation of an ideal L of $R \ltimes M$ with some ideal of the form $I \ltimes N$ where I is an ideal of R and N is a submodule of M.

Lemma 2.1. *Let L be an ideal of $R \ltimes M$. Then there exist an ideal I_L of R and a submodule N_L of M such that $I_L \ltimes N_L$ is an ideal of S and $L \subseteq I_L \ltimes N_L$.*

In fact, one has $I_L = \{r \in R | \exists m \in M; (r,m) \in L\}$ and $N_L = \{m \in M | \exists r \in R; (r,m) \in L\}$.

The next theorem determines the ideals of the idealization $S = \mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$, for each prime number p.

Theorem 2.2. *Let $S = \mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$, where p is a prime number. Then the non-trivial ideals of S are $0 \ltimes p\mathbb{Z}_{p^2}$, $0 \ltimes \mathbb{Z}_{p^2}$, $p\mathbb{Z}_{p^2} \ltimes p\mathbb{Z}_{p^2}$, $p\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$ and $L_s := \{(ips, jp+i) \mid 0 \leq i,j \leq p-1\}$ for each $1 \leq s \leq p-1$.*

Proof. Assume that L is an ideal of S. Then, by Lemma 2.1, there exist an ideal I_L of $\mathbb{Z}_{p^2}$ and a submodule N_L of $\mathbb{Z}_{p^2}$ such that $L \subseteq I_L \ltimes N_L$. If $I_L = 0$, then $L = 0 \ltimes N_L = 0 \ltimes 0$, $L = 0 \ltimes p\mathbb{Z}_{p^2}$ or $L = 0 \ltimes \mathbb{Z}_{p^2}$, by [9, Lemma 2]. Assume now that $I_L \neq 0$. Hence $I_L = \mathbb{Z}_{p^2}$ or $I_L = p\mathbb{Z}_{p^2}$. If $I_L = \mathbb{Z}_{p^2}$, then $L = \mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$, by [9, Lemma 2].

So we may assume that $I_L = p\mathbb{Z}_{p^2}$. There is an element $m \in \mathbb{Z}_{p^2}$ such that $(p,m) \in L$. Hence $(0,p) = (p,m)(0,1) \in L$; so that one has $0 \ltimes p\mathbb{Z}_{p^2} \subseteq L \subseteq I_L \ltimes N_L = p\mathbb{Z}_{p^2} \ltimes N_L$. Thus $p\mathbb{Z}_{p^2} \subseteq N_L$ and we have $N_L = p\mathbb{Z}_{p^2}$ or $N_L = \mathbb{Z}_{p^2}$. Assume first that $N_L = p\mathbb{Z}_{p^2}$. This gives $0 \ltimes p\mathbb{Z}_{p^2} \subseteq L \subseteq p\mathbb{Z}_{p^2} \ltimes p\mathbb{Z}_{p^2}$ which yields $p| \mid L \mid |p^2$. These imply that $L = 0 \ltimes p\mathbb{Z}_{p^2}$ or $L = p\mathbb{Z}_{p^2} \ltimes p\mathbb{Z}_{p^2}$.

Assume now that $N_L = \mathbb{Z}_{p^2}$. This in conjunction with $0 \ltimes p\mathbb{Z}_{p^2} \subset L$ gives $0 \ltimes p\mathbb{Z}_{p^2} \subset L \subseteq p\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$ which yields $p| \; |L| \; |p^3$; so that $|L| = p^2$ or $|L| = p^3$. If $|L| = p^3$, then $L = p\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$. In the rest of the proof, we assume that $|L| = p^2$. Since $N_L = \mathbb{Z}_{p^2}$, one has an element $r \in \mathbb{Z}_{p^2}$ such that $(r,1) \in L$ and then $r \in I_L = p\mathbb{Z}_{p^2}$. If $r = 0$, then $(0,1) \in L$. Thus $0 \ltimes \mathbb{Z}_{p^2} \subset L$ and $L = I_L \ltimes \mathbb{Z}_{p^2} = p\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$, by [9, Lemma 2], which is a contradiction. So we can assume that $0 \neq r \in p\mathbb{Z}_{p^2}$ and $r = ps$, for some $1 \leq s \leq p-1$. Hence $(ps,1) \in L$. If there is $1 \leq s' \leq p-1$ such that $s \neq s'$ and $(ps',1)$, then $(p(s-s'),0) \in L$. Since $s - s'$ is invertible element in $\mathbb{Z}_{p^2}$, one has $((s-s')^{-1},0)(p(s-s'),0) = (p,0) \in L$. Thus L contains the set $p\mathbb{Z}_{p^2} \ltimes 0$. On the other hand, we know that $0 \ltimes p\mathbb{Z}_{p^2} \subset L$, so we have $p\mathbb{Z}_{p^2} \ltimes p\mathbb{Z}_{p^2} \subseteq L$ which implies $L = p\mathbb{Z}_{p^2} \ltimes p\mathbb{Z}_{p^2}$.

Therefore, we can assume that there is only one element $1 \leq s \leq p-1$ such that $(ps,1) \in L$. On the other hand, we know that $0 \ltimes p\mathbb{Z}_{p^2} \subset L$, i.e., $(0,jp) \in L$, for all $1 \leq j \leq p-1$. By adding the element $(ps,1)$ to these elements, we get $\{(ips, jp+i) \mid 0 \leq i,j \leq p-1\} \subseteq L$. It follows from $|L| = p^2$ that $L = \{(ips, jp+i) \mid 0 \leq i,j \leq p-1\} = L_s$. Note that one can check easily the L_s is an ideal of S. Assume that $(ps,1) = (ips', jp+i)$ for $1 \leq s \neq s' \leq p-1$. So we have $1 \equiv jp+i \pmod{p^2}$ and this implies that $i = 1$. On the other $ps \equiv ips'$ in conjunction with $i = 1$ yields that $p \mid s - s'$ which is a contradiction. Therefore $(ps,1) \in L_s \setminus L_{s'}$ for $1 \leq s \neq s' \leq p-1$. This completes the proof. $\square$

It was shown in [9, Lemma 8] that $\Gamma(\mathbb{Z}_p \ltimes \mathbb{Z}_p) \cong K_1$, for each prime number p. The following corollary which is an immediate result of Theorem 2.2, gives us the structure of the graph $\Gamma(\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2})$.

Corollary 2.3. *Let $S = \mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$, where p is a prime number. Then $\Gamma(\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}) \cong K_{p+3}$.*

The following theorem determines the ideals of $S = \mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, for distinct prime numbers p and q.

Theorem 2.4. *Let $S = \mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, where p and q are distinct prime numbers. Then the non-trivial ideals of S are $0 \ltimes p\mathbb{Z}_{pq}$, $0 \ltimes q\mathbb{Z}_{pq}$, $0 \ltimes \mathbb{Z}_{pq}$, $p\mathbb{Z}_{pq} \ltimes p\mathbb{Z}_{pq}$, $p\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, $q\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$ and $q\mathbb{Z}_{pq} \ltimes q\mathbb{Z}_{pq}$.*

Proof. Assume that L is an ideal of S. Then by Lemma 2.1, there exist an ideal I_L of $\mathbb{Z}_{pq}$ and a submodule N_L of $\mathbb{Z}_{pq}$ such that $L \subseteq I_L \ltimes N_L$. If $I_L = 0$, then $L = 0 \ltimes N_L = 0 \ltimes 0$, $L = 0 \ltimes p\mathbb{Z}_{pq}$, $L = 0 \ltimes q\mathbb{Z}_{pq}$ or $L = 0 \ltimes \mathbb{Z}_{pq}$, by [9, Lemma 2]. If $I_L = \mathbb{Z}_{pq}$, then $L = \mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, by [9, Lemma 2]. So we may assume that $I_L = p\mathbb{Z}_{pq}$ or $I_L = q\mathbb{Z}_{pq}$ and, by symmetry, it is sufficient to determine the ideal L for which $I_L = p\mathbb{Z}_{pq}$. Since $p \in I_L$, there is an element $x \in \mathbb{Z}_{pq}$ such that $(p,x) \in L$. So $(0,p) = (p,x)(0,1) \in L$ and, hence one has $0 \ltimes p\mathbb{Z}_{pq} \subseteq L \subseteq I_L \ltimes N_L = p\mathbb{Z}_{pq} \ltimes N_L$. Thus $p\mathbb{Z}_{pq} \subseteq N_L$ and this implies that $N_L = p\mathbb{Z}_{pq}$ or $N_L = \mathbb{Z}_{pq}$.

Assume first that $N_L = p\mathbb{Z}_{pq}$. Then we have $q|\ |L|\ |q^2$ and so $|L|= q$ or $|L|= q^2$. If $|L|= q$, then $L = 0 \ltimes p\mathbb{Z}_{pq}$ which contradicts $I_L = p\mathbb{Z}_{pq}$. If $|L|= q^2$, then $L = p\mathbb{Z}_{pq} \ltimes p\mathbb{Z}_{pq}$.

Assume now that $N_L = \mathbb{Z}_{pq}$. In this case, we have $0 \ltimes p\mathbb{Z}_{pq} \subseteq L \subseteq p\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$. Hence $q|\ |L|\ |pq^2$. If $|L|= q$, then $L = 0 \ltimes p\mathbb{Z}_{pq}$.

Assume next that $|L|= pq^2$. Since $N_L = \mathbb{Z}_{pq}$, one has an element $r \in I_L = p\mathbb{Z}_{pq}$ such that $(r,1) \in L$. If $r = 0$, then $(0,1) \in L$; so $0 \ltimes \mathbb{Z}_{pq} \subset L$ and $L = I_L \ltimes \mathbb{Z}_{pq} = p\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, by [9, Lemma 2]. If $0 \neq r \in p\mathbb{Z}_{pq}$, then $r = ps$, for some $1 \leq s \leq pq-1$. Thus we have $(ps,1) \in L$. On the other hand, we know that $0 \ltimes p\mathbb{Z}_{pq} \subset L$, i.e., $(0,jp) \in L$, for all $1 \leq j \leq q-1$. By adding the element $(ps,1)$ to these elements, we have $\{(ips, jp+i) \mid 0 \leq i \leq pq-1 \ \ and \ \ 0 \leq j \leq q-1\} \subseteq L$. This implies that $L = \{(ips, jp+i) \mid 0 \leq i \leq pq-1 \ \ and \ \ 0 \leq j \leq q-1\} = p\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$.

Assume finally that $|L|= pq$. Since $N_L = \mathbb{Z}_{pq}$, one has an element $r \in p\mathbb{Z}_{pq}$ such that $(r,1) \in L$. If $r = 0$, then $(0,1) \in L$; so $0 \ltimes \mathbb{Z}_{pq} \subseteq L$ and $L = I_L \ltimes \mathbb{Z}_{pq} = p\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, by [9, Lemma 2], which is a contradiction. If $0 \neq r \in p\mathbb{Z}_{pq}$, then $r = ps$, for some $1 \leq s \leq pq-1$. Similar above argument, we can see $\{(ips, jp+i) \mid 0 \leq i \leq pq-1 \ \ and \ \ 0 \leq j \leq q-1\} \subseteq L$, which is a contradiction, since $|L|= pq$.

Similarly, for $I_L = q\mathbb{Z}_{pq}$, we have $L = q\mathbb{Z}_{pq} \ltimes q\mathbb{Z}_{pq}$, $L = 0 \ltimes q\mathbb{Z}_{pq}$ or $L = q\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$ and we are done. □

As an immediate result, we obtain the structure of the graph $\Gamma(\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq})$.

Corollary 2.5. *Let $S = \mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, where p, q are distinct prime numbers. Then we have $\Gamma(\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}) \cong K_7 \setminus C_4$, where $C_4 := 0 \ltimes p\mathbb{Z}_{pq} - q\mathbb{Z}_{pq} \ltimes q\mathbb{Z}_{pq} - p\mathbb{Z}_{pq} \ltimes p\mathbb{Z}_{pq} - 0 \ltimes q\mathbb{Z}_{pq} - 0 \ltimes p\mathbb{Z}_{pq}$.*

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When the Intersection Graph of the Idealization is a Star Graph?

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Abstract

Let R be a commutative ring with identity. The intersection graph of ideals of a ring R is an undirected simple graph denoted by $\Gamma(R)$ whose vertices are in a one-to-one correspondence with non-zero proper ideals and two distinct vertices are joined by an edge if and only if the corresponding ideals of R have a non-zero intersection. Let M be a unitary R-module and let $R \ltimes M$ be the idealization of M in R. In this paper, we investigate the interplay between the algebraic properties of $R \ltimes M$ and the graph-theoretic properties of $\Gamma(R \ltimes M)$. Under some conditions on the ring R and the module M, we determine the exact form of ideals of $R \ltimes M$ and characterize all rings R and modules M for which $\Gamma(R \ltimes M)$ is a star graph.

Keywords: Idealization, Intersection graph

Mathematics Subject Classification [2010]: Primary: 13A15, Secondary: 05C75

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1 Introduction and Preliminaries

There are a lot of papers on assigning a graph to a ring and many authors have studied ring-theoretic properties in terms of graph-theoretic properties. Zero-divisor graphs, annihilating-ideal graphs and intersection graphs are examples of such definition, see for instance [1,2,4,8,9]. When one assigns a graph to an algebraic structure numerous interesting algebraic problems arise from the translation of some graph-theoretic parameters such as clique number, chromatic number, independence number and so on.

Let R be a commutative ring with identity and let M be a unitary R-module. *Idealization of M in R*, denoted by $R \ltimes M$, is the ring whose additive structure is that of the external direct sum $R \oplus M$ and whose multiplication is defined by $(r, m)(r', m') := (rr', rm' + r'm)$ for all $r, r' \in R$ and all $m, m' \in M$. Note that if N is a submodule of M, then $0 \ltimes N$ is an ideal of $R \ltimes M$ and $0 \ltimes M$ is nilpotent ideal of $R \ltimes M$ of index 2. Idealization introduced by Nagata in 1956 [13] and has been extensively studied, see [6,7,11].

intersection graph of ideals of a ring R is an undirected simple graph denoted by $\Gamma(R)$ whose vertices are in a one-to-one correspondence with non-zero proper ideals and two distinct vertices are joined by an edge if and only if the corresponding ideals of R have a non-zero intersection. The intersection graph of ideals of a ring R was introduced in 2009 [10] and studied in several papers, see [2,3,12,14].The main purpose of this paper is to investigate the intersection of ideals in $R \ltimes M$ using graph-theoretic concepts. Also, we will characterize when the intersection graph of the idealization is a star graph.

Throughout this paper, all graphs are simple with no loops and multiple edges. Let $G = (V, E)$ be a graph with vertex set V and edge set E. For two distinct vertices x, $y \in V$, $\mathrm{d(x, y)}$ denotes the length of a shortest path from x to y, if such a path exists; otherwise, $\mathrm{d(x, y)} = \infty$. A graph in which each pair of distinct vertices is joined by an edge is called a *complete* graph. By K_n, we mean the complete graph over n vertices. A graph is *bipartite* if its vertex set can be partitioned into two subsets X and Y such that every edge has one end in X and one end in Y. A complete bipartite graph which $|X| = r$ and $|Y| = s$ is denoted by $K_{r,s}$. If $r = 1$, then it is called *star graph*. Let $S \subseteq V$ be any subset of vertices of G. Then the *induced subgraph* $\langle S\rangle$ is the graph whose vertex set is S and edge set contains all edges of $E(G)$ connecting pairs of vertices in S.

2 Main Results

First, we give some algebraic results of idealization which will be useful in the rest of the paper.

Let I be an ideal of R and N be a submodule of M. It was shown in [5, Theorem 3.1] that $I \ltimes N$ is an ideal of S if and only if $IM \subseteq N$. The following lemma provides a closed relation of any ideal L of S with some ideal of the form $I \ltimes N$. In continue, we will use this lemma frequently throughout the paper.

Lemma 2.1. *Let L be an ideal of S. Then there exist an ideal I of R and a submodule N of M such that $I \ltimes N$ is an ideal of S and $L \subseteq I \ltimes N$.*

Proof. Set $I = \{r \in R | \exists m \in M \textit{ such that } (r, m) \in L\}$ and $N = \{m \in M | \exists r \in R \textit{ such that } (r, m) \in L\}$. It is not hard to see that I is an ideal of R, N is a submodule of M and $L \subseteq I \ltimes N$. Now, we show that $I \ltimes N$ is an ideal of S. Let $i \in I$ and $m \in M$. It is enough to show that $im \in N$, by [5, Theorem 3.1]. There is $n \in N$ such that $(i, n) \in L$ and so $(i, n)(1, m) = (i, n + im) \in L \subset I \times N$. Thus $im \in N$ and the result follows. □

In the rest of the paper, the ideal I and the submodule L which are obtained in Lemma 2.1 will be denoted by I_L and N_L, respectively.

It was shown in [5, Theorem 3.1] that the ideals of $R \ltimes M$ containing $0 \ltimes M$ are of the form $J \ltimes M$ for some ideal J of R. In the third part of the next lemma, we determine the exact form of J. Also, the following lemma is useful in the paper.

Lemma 2.2 ([15]). *Suppose that L is an ideal of S. The following statements hold:*

(i) If $I_L = 0$, then $L = 0 \ltimes N_L$,

(ii) If $I_L = R$, then $L = R \ltimes M$,

(iii) If $0 \ltimes M \subseteq L$, then $L = I_L \ltimes M$.

In the sequel, we will characterize when the intersection graph of the idealization is a star graph.

Lemma 2.3 ([15]). *Let $S = F \ltimes M$, where F is a field. Then the proper ideals of S have the form $0 \ltimes N$ where N is a submodule of M.*

Theorem 2.4 ([15]). *Let $S = F \ltimes M$, where F is a field. Then $\Gamma(S)$ is a star graph if and only if $\dim_F(M) = 2$.*

It was shown in [5, Theorem 3.2] that the maximal ideals of $R \ltimes M$ have the form $\mathfrak{m} \ltimes M$ where $\mathfrak{m}$ is a maximal ideal of R. Next, we will use this statement without referring.

Lemma 2.5 ([15]). *$\Gamma(S) = K_1$ if and only if $S = R \ltimes R$ where R is a field.*

Now we continue the characterization, when R is not a field. First, we determine the ideals of S when R has only one non-zero proper ideal and M is a simple R-module. Also, the following lemma is needed to state the main theorem of this section.

Lemma 2.6 ([15]). *Let L be a non-zero ideal of S. Assume that R has only one non-zero proper ideal $\mathfrak{m}$ and M is a simple R-module. Then the following statements hold.*

(i) If $0 \ltimes M \subseteq L$*, then* $L = 0 \ltimes M$ *or* $L = \mathfrak{m} \ltimes M$ *or* $L = R \ltimes M$*.*

(ii) There exists a non-zero element r *in* R *and a non-zero element* x *in* M *such that* $\mathfrak{m} = \langle r \rangle$*,* $r^2 = 0$*,* $M = Rx$ *and* $0 : x = \mathfrak{m}$*.*

(iii) If $0 \ltimes M \nsubseteq L$*, then* $L = \mathfrak{m} \ltimes 0$ *or* $L = \langle (r, sx) \rangle$ *where* s *is a unit element of* R*.*

(iiii) If $\langle (r, sx) \rangle \cap \langle (r, s'x) \rangle \neq 0$ *where* s, s' *are unit elements of* R*, then* $\langle (r, sx) \rangle = (r, s'x) \rangle$*.*

Example 2.7. Let $S = \mathbb{Z}_4 \ltimes 2\mathbb{Z}_4$ and L be an ideal of S. Assume first that $0 \ltimes 2\mathbb{Z}_4 \subseteq L$. Then $L = 0 \ltimes 2\mathbb{Z}_4$ or $L = 2\mathbb{Z}_4 \ltimes 2\mathbb{Z}_4$ or $L = \mathbb{Z}_4 \ltimes 2\mathbb{Z}_4$ by Lemma 2.6 (i). Suppose that $0 \ltimes 2\mathbb{Z}_4 \nsubseteq L$. Then $L = 2\mathbb{Z}_4 \ltimes 0$ or $\langle (2, 2s) \rangle$, where $s = 1$ or $s = 3$ by Lemma 2.6 (iii). But $s = 1$ or $s = 3$ creates only one ideal $\langle (2, 2) \rangle$.

Theorem 2.8 ([15]). *Let* R *not be a field. Then* $\Gamma(S)$ *is a star graph if and only if* R *has only one non-zero proper ideal and* M *is a simple* R*-module.*

Corollary 2.9 ([15]). *Let* $\Gamma(S)$ *is a star graph. Then the order of* $\Gamma(S)$ *is at least 4.*

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درباره رابطه میانگی در مجموعه‌های جزئاً مرتب

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چکیده. در این نوشته ضمن بررسی رابطه میانگی در مجموعه‌های جزئاً مرتب، مجموعه جزئاً مرتبی ارائه دهیم که فاقد برش است. این مثال نتیجه‌ای از مقاله اخیر برونو کورسله را رد می‌کند.

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۱. مقدمه

رابطه مابینی یک مفهوم استاندارد در مطالعه ساختارهایی مانند درخت، ترتیب جزئی و گراف می‌باشد. رابطه مابینی یک رابطه سه تایی $B(x, y, z)$ است که بیان می‌کند عضو y مابین اعضای x و z قرار دارد. این مابینی (قرارگرفتن بین دو عضو) بستگی به ساختار مورد نظر دارد که در آن تعریف می‌شود. این رابطه در منطق مرتبه اول برای ترتیب‌های خطی اصل موضوعی می‌شود. در حالت خاص، هر ترتیب خطی X به طور یکتایی از رابطه مابینی متناظر، خودش بدست می‌آید. اما مفهوم «مابینی جزئی» منجر به برخی مسائل منطقی و الگوریتمی می‌شود.

مابینی ترتیب‌های جزئی در [۲] اصل‌بندی (اصل موضوعی) شده است. این اصل‌بندی توسط یک مجموعه نامتناهی صورت گرفته است.

ما در اینجا ضمن بررسی برخی نتایج [۱] نشان می‌دهیم که لم ۸.۳ و در نتیجه قضیه ۶.۳ آن نادرست هستند.

تعریف ۱.۱. فرض کنید که $S = (V, B)$ یک ساختار سه گانه باشد، یعنی B یک رابطه سه موضعی روی مجموعه V باشد. B را یک رابطه میانگی روی V گویند هرگاه در اصول

$B_1 : B(x, y, z) \Rightarrow \neq (x, y, z).$

$B_2 : B(x, y, z) \Rightarrow B(z, y, x).$

$B_3 : B(x, y, z) \Rightarrow \neg B(x, z, y).$

صدق کند. برای رابطه ترتیبی جزئی $P = (V, <)$ رابطه میانگی متناظر با آن به صورت زیر تعریف می‌شود:

$Bet(P)(x, y, z) :\Longleftrightarrow x < y < z \text{ or } z < y < x$

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تعریف ۲.۱. (الف) $S = (V, B)$ را که $B \subseteq V^۳$، یک ساختار سه گانه می‌گیریم. گراف گایفمن ساختار S عبارت است از $Gf(S) := (V, E)$، که در آن E متشکل از یال‌های بی‌سوی $u - v$ است به طوری که $u \neq v$ و u و v متعلق به یک سه تایی در B باشد.
(ب) اگر $P = (V, \leq)$ یک مجموعه جزئا مرتب باشد آنگاه گراف مقایسه‌پذیر آن که با $Comp(P)$ نشان داده می‌شود متشکل از مجموعه رئوس V و یال‌های $u - v$ می‌باشد که $u \neq v$ و مقایسه پذیرند، یعنی $u < v$ یا $v < u$. در واقع $Comp(P)$ گراف گایفمن ساختار دوگانه $P = (V, \leq)$ می‌باشد.

طبق تعریف، $Gf(Bet(P)) \subseteq Comp(P)$. مثال زیر نشان می‌دهد که رابطه شمول ممکن است اکید باشد. اگر $Gf(Bet(P))$ همبند باشد، آنگاه $Comp(P)$ نیز همبند است اما عکس این مطلب درست نیست، زیرا اگر P دارای زنجیری به طول ۳ نباشد آنگاه $Gf(Bet(P))$ فاقد یال است.

مثال ۳.۱. در این مثال، یک مجموعه جزئا مرتب P ارائه می‌شود که برای آن داریم
$Gf(Bet(P)) \subset Comp(P).$
قرار دهید: $P = (V, \leq)$، که در آن $V = \{a, b, c, d, e, f\}$ و رابطه ترتیبی جزئی $\leq$ توسط روابط $a < b < c < d$، $e < c$، $e < f$، $b < f$ و ویژگی‌های تعدی و بازتابی تولید می‌شود. در این صورت یال $e - f$ از $Comp(P)$ در $Gf(Bet(P))$ قرار ندارد، زیرا رئوس e و f به هیچ زنجیر به طول ۳ متعلق نیستند. اگر شرط $e < f$ را حذف کنیم، آنگاه رابطه ترتیبی حاصل که آن را با P' نشان می‌دهیم، همان رابطه مابینی متناظر با P را دارد و $Gf(Bet(P')) = Comp(P')$.

تعریف ۴.۱. رابطه ترتیب جزئیP، $B-$مینیمال گفته می‌شود هرگاه $Gf(Bet(P)) = Comp(P)$. معادلا، هر دو عضو مقایسه‌پذیر متعلق به یک زنجیر، حداقل سه عضوی باشند. واضح است که هر ترتیب جزئی بدیهی $B-$مینیمال است. همچنین هر ترتیب جزئی فاقد عضو مینیمال یا فاقد عضو ماکزیمال یک ترتیب جزئی $B-$مینیمال است.
ترتیب جزئی P، $B-$مینیمال قوی گفته می‌شود هرگاه برای هر $x \in Min(P)$ و هر $y \in Max(P)$ عضو z در V موجود باشد به طوری که $x < z < y$.

اگر P، $B-$مینیمال قوی باشد آنگاه $B-$مینیمال نیز است. زیرا اگر دو عضو نامساوی x و y قابل مقایسه باشند آنگاه متعلق به یک زنجیر با طول حداقل ۳ می‌باشد یا این که یکی متعلق به $Min(P)$ و دیگری متعلق به $Max(P)$ می‌باشند. در حالت اول شرط $B-$مینیمالیتی فراهم می‌شود. در حالت دوم، طبق $B-$مینیمالیتی قوی، عضو z موجود است که بین x و y قرار می‌گیرد لذا در این حالت نیز x و y متعلق به یک زنجیر با طول حداقل ۳ می‌باشند. در نتیجه شرط $B-$مینیمالیتی برقرار است.
اگر $Min(P)$ یا $Max(P)$ تهی باشد آنگاه P، $B-$مینیمال قوی است. اما مثال زیر نشان می‌دهد که $B-$مینیمال قوی واقعا قوی‌تر است.

مثال ۵.۱. فرض کنید $P = (V, \leq)$ که $V = \{a, b, c, d, e, f\}$ و رابطه $\leq$ تولید شده توسط رابطه‌های $a < b < c$، $d < e < f$، و رابطه‌های حاصل از خاصیت‌های انعکاسی و انتقال می‌باشد. در این صورت $Gf(Bet(P)) = Comp(P)$. لذا P، $B-$مینیمال است. امّا P، $B-$مینیمال قوی نیست؛ زیرا $d \in Min(P)$ و $c \in Max(P)$؛ در حالی که هیچ عضو $x \in V$ موجود نیست که $d < x < c$.

قضیه ۶.۱ ([۱]). فرض کنید $P = (V, \leq)$ یک ترتیب جزئی و $\leq'$ ترتیب جزئی دیگری روی V حاصل از $<'$ باشد به طوری که $x <' y$ اگر و تنها اگر $x < y$ و اگر $x \in Min(P)$ و $y \in Max(P)$ آنگاه $z \in V$ موجود باشد که $x < z < y$. در این صورت $\tilde{P} := (V, \leq') \subseteq P$ یک ترتیب جزئی $B-$مینیمال بوده و $Bet(\tilde{P}) = Bet(P)$. در واقع، ترتیب جزئی تعریف شده یکتا ترتیب جزئی $B-$مینیمال $Q = (V, \leq_Q)$ می‌باشد که $Bet(Q) = Bet(P)$ و $Q \subseteq P$.

برهان. برای اثبات ترتیب جزئی بودن رابطه $\leq'$ روی V، از آنجا که طبق تعریف $\leq'$، $\tilde{P} \subseteq P$، کافی است نشان دهیم که رابطه $<'$ دارای خاصیت تعدی یا انتقالی است. برای این منظور، فرض کنید که $x <' y <' z$. در این صورت $x < y < z$، پس $x < z$.

حال اگر $x \in Min(P)$ و $z \in Max(P)$ (در غیر این صورت، $x <' z$) آنگاه از آنجا که $x < y < z$ (یعنی عضوی بین x و z موجود است) پس طبق تعریف داریم: $x <' z$.
چون $\tilde{P} \subseteq P$، پس $Bet(\tilde{P}) \subseteq Bet(P)$. برعکس اگر $x < y < z$ آنگاه طبق تعریف $\tilde{P}$ داریم: $x <' y <' z$. در واقع برای بدست آوردن $\tilde{P}$ از P فقط زنجیرهای دو عضوی آن را حذف کرده‌ایم. در نتیجه P و $\tilde{P}$ دارای زنجیرهایی به طول حداقل ۳ یکسان می‌باشند. به ویژه، $Bet(\tilde{P}) = Bet(P)$.
اگر $\tilde{P}$، $B-$مینیمال نباشد آنگاه اعضای x و y موجودند که $x <' y$ و x و y متعلق به هیچ زنجیر با طول حداقل ۳ در $\tilde{P}$ و در نتیجه در P نیستند. در این صورت $x \in Min(P)$ و $y \in Max(P)$ و از آنجا که $x <' y$، پس طبق تعریف عضو z موجود است که $x < z < y$. این یک تناقض است.

برای اثبات یکتایی $\tilde{P}$، فرض کنید که $Q = (V, \leq_Q)$ یک زیر رابطه ترتیبی جزئی از P باشد که $Bet(Q) = Bet(P)$. نشان می‌دهیم که $\tilde{P} \subseteq Q$. برای این منظور، فرض کنید که $x <' y$. در این صورت طبق تعریف $<'$، عضو $z \in V$ موجود است که $x < y < z$ یا $z < x < y$ یا $x < z < y$.
در حالت اول داریم : $(x, y, z) \in Bet(P) = Bet(Q)$؛ لذا $x <_Q y <_Q z$ (توجه کنید که حالت $z <_Q y <_Q x$، با توجه به این که $Q \subseteq P$ و $x < y < z$ غیرممکن است). پس $x <_Q y$. در دو حالت دیگر به طور مشابه نتیجه می‌شود که $x <_Q y$. در نتیجه $\tilde{P} \subseteq Q$، بنابراین، $\tilde{P}$ یکتا ترتیب جزئی $B-$مینیمال مشمول در P می‌باشد که $Bet(\tilde{P}) = Bet(P)$. □

۲. نتایج اصلی

در این بخش با ارائه یک مثال نشان می‌دهیم که لم ۸.۳ و قضیه ۶.۳ حاصل از آن در [۱] نادرست هستند.

تعریف ۱.۲. یک برش از یک ترتیب جزئی $P = (V, \leq)$ یک افراز (L, U) از V می‌باشد به طوری که
(الف) زیرمجموعه L از پائین بسته و زیرمجموعه U از بالا بسته باشند،
(ب) هر زنجیر ماکزیمال در P هر دو زیرمجموعه L و U را قطع کند.

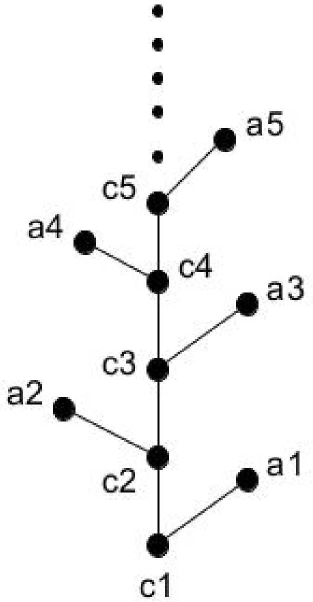
توجه کنید که اگر (L, U) یک برش در P باشد آنگاه (U, L) یک برش در P^{rev} می‌باشد.

لم ۲.۲ ([۱]). هر مجموعه جزئا مرتب P که فاقد عضو ایزوله باشد، دارای برش است.

در واقع ایراد برهان ارائه شده برای لم در [۱] در این فرض اشتباه است که هر زنجیر ماکزیمال شامل عضوی از پادزنجیر ماکزیمال است. توجه کنید که لم فوق برای ترتیب‌های جزئی متناهی درست است.

قضیه ۳.۲. مجموعه جزئا مرتب همبند موجود است که فاقد برش می‌باشد.

برهان. P را ترتیب جزئی می‌گیریم که نمودار هاسه آن به صورت زیر باشد:



در این مثال ترتیب جزئی شمارای P همبند می‌باشد و لذافاقد نقطه ایزوله است. امّا P دارای هیچ برشی نیست. برش ارائه شده در برهان لم فوق برای P به صورت زیر می‌باشد که در واقع برش نیست.
مجموعه $A = \{a_۱, a_۲, \ldots\}$ یک پادزنجیر در P می‌باشد و برای آن

$$U = \{x \in V - Min(P) | \exists y \in A, y \leq x\} = A$$

و $L = V - U = C$. امّا برای زنجیر ماکزیمال C داریم: $C \cap U = \emptyset$. بنابراین (L, U) برش نیست. □

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آشوب آموزش مجازی، نظریه گراف یاریگر پویایی خودآموزی خلاقانه و تغییر پارادایم

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چکیده. آموزش مجازی یک راه حل عالی برای پاسخگویی به نیازهای آموزشی است.
اما در شرایط امروز که تهدیدی نامرئی و فراگیر، زندگی ما را دوباره تعریف می‌کند و همه آنچه را که داریم، در بر می‌گیرد؛ نادیده گرفتن مزایای آموزش مجازی، ناکافی و ناکارآمدی نسبی آن و ...، موجب چالشی در نظام آموزشی گردیده است:

«آشوبی در آموزش مجازی مدارس در حال رخ دادن است ».

اگر نتوانیم چاره‌ای بیاندیشیم سردرگمی در این فضای آشوبناک موجب کند شدن گسترش برنامه‌های یادگیری آنلاین مدارس یا حتی از بین رفتن کامل آن می‌شود.
بنابراین، با توجه به نظریه گراف، به عنوان یکی از جذاب‌ترین مفاهیم علم داده و توانایی آن در نظم بخشیدن به آشوب‌هایی این چنین و نیز مفهوم لبه آشوب به عنوان ناحیه گذار بین نظم و آشوب ؛ این باور به ذهن متبادر می‌گردد که:
آینده از آن خودآموزی خلاقانه است و البته راهکاری مؤثر برای تغییر پارادایم و نظریه گراف یک روش کمی مناسب برای توجیه این ضرورت.

۰.۱ مقدمه

همه‌گیری جهانی بیماری کووید – ۱۹ ، تغییرات قابل توجهی در نحوه زندگی، کار و یادگیری ما ایجاد کرده است. با تعطیلی مدارس و دانشگاه‌ها برای جلوگیری از انتشار ویروس، یادگیری الکترونیکی به عنوان راه حلی مناسب برای ادامه آموزش ظاهر شد که تغییر ناگهانی به سمت آموزش مجازی ، باعث ایجاد هرج و مرج در نظام آموزشی گردید، زیرا بسیاری از مدارس و دانشگاه‌ها برای گذار آماده نبودند که منجر به نارسایی و ناکارآمدی نسبی آنها در دستیابی به اهداف مختلف آموزشی گردید. این پدیده باعث ایجاد چالش در سیستم آموزشی شد و اکنون نیز که آن دوران سخت را سپری کرده‌ایم، توجه به این نکته ضروری است که : «اگر نتوانیم چاره ای بیندیشیم، سردرگمی در این فضای آشفته باعث کاهش سرعت گسترش برنامه‌های یادگیری آنلاین محیط‌های آموزشی شده یا حتی موجب حذف کامل آن خواهد شد» [۱]. چاره‌ای که با یاری جستن از توانایی ذاتی ریاضیات و زیر شاخه‌هایش مانند جبر و گراف در یافتن پاسخ پرسش‌ها؛ ممکن و دست یافتنی است.

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واژگان کلیدی. نظریه گراف، لبه آشوب، آموزش مجازی، تغییر پارادیم، خودآموزی خلاقانه.

۲. یادگیری الکترونیکی و آموزش مجازی، پارادایم و تغییر پارادایم

اصطلاح آموزش الکترونیکی نخستین بار در سال ۱۹۹۹ در سمیناری در مورد سیستم‌های آموزش مبتنی بر کامپیوتر شنیده شد که الیوت ماسی ، میزبان و متصدی سمینار آن را به کار برد. تعاریف گوناگونی از آموزش مجازی و یادگیری الکترونیکی، مطرح شده است که هر تعریف، براساس نوع نگرش و دیدگاه ارائه‌کنندگان، قابل بیان است [۲]. اما با جمع‌بندی نظرات و تعاریف گوناگون، می‌توان گفت: یادگیری الکترونیکی، سیستم یادگیری مبتنی بر با کمک آموزش رسمی منابع الکترونیکی است. در حالی که تدریس می‌تواند در داخل یا خارج از کلاس باشد و استفاده از کامپیوتر و اینترنت جزء اصلی این نوع از آموزش را تشکیل می‌دهد.

در دهه اخیر، گسترش سریع به کارگیری فناوری اینترنت، تأثیر شایسته‌ای بر جنبه‌های پرشمار زندگی افراد داشته است و در پی به کارگیری فناوری روز در سیستم‌های اطلاعات، سیستم آموزشی نیز دست خوش دگرگونی‌های زیادی شده است. یادگیری الکترونیکی به عنوان الگوی جدید آموزشی مدرن، مطرح می‌شود و به کارگیری یادگیری الکترونیکی، امکان مشارکت بیشتر دانشجویان و استادان و دسترسی به دامنه گسترده‌تری از منابع را فراهم می‌کند. بنابراین مفهوم پیشین یادگیری در حال تغییر اساسی است و دیگر به کلاس‌های حضوری، محدود نمی‌شود.

۲.۱. پارادایم و تغییر پارادایم، ضرورتی راهگشا. در ابتدا لازم به ذکر است در این نوشتار، ماهیت پارادایم‌ها از منظر تعریف توماس کوهن مورد بحث قرار می‌گیرد. از نظر کوهن، پارادایم عبارت است از: «آنچه اعضای یک جامعه علمی و به تنهایی آنها را به اشتراک می‌گذارند» و « وقتی پارادایم‌ها تغییر می‌کنند، خود جهان نیز با آنها تغییر کند» [۳] و [۴]. استنباط این است که پارادایم‌ها زمانی تغییر می‌کنند که یک جامعه به عنوان یک کل، تغییراتی را که به ارمغان می‌آورد، بپذیرد. پارادایم‌های آموزشی، چارچوب‌هایی هستند که نحوه آموزش، یادگیری و تفکر ما را در مورد آموزش راهنمایی می‌کنند. تغییر پارادایم در آموزش الکترونیکی شامل حرکت از روش‌های سنتی تدریس به سیستمی پویاتر و سازگارتر است. این تغییر شامل چندین عنصر کلیدی شامل حرکت از رویکرد معلم محور به رویکرد یادگیرنده محور، حرکت از رویکردی با اندازه مناسب به رویکردی شخصی‌تر و در نهایت شامل حرکت از برنامه درسی ثابت به برنامه درسی انعطاف‌پذیرتر است که دانش آموزان در مورد آنچه می‌آموزند و چگونه یاد گیری آن، انتخاب بیشتری دارند. با این وصف، آینده آموزش الکترونیکی روشن است و با تغییر پارادایم به سمت سیستم پویاتر و سازگارتر، آموزش الکترونیکی پتانسیل ایجاد انقلابی در آموزش را دارد. زیرا از مصادیق این تغییر پارادایم، با حل چالش اصلی نظام آموزشی، یعنی جلوگیری از کاهش آمار آموزش گیرندگان، توقف آموزش تمام دانش آموزان با تغییر مفهوم تدریس، معنای ارزشیابی، شرایط یادگیری و نقش معلمان و دانش آموزان همراه است. یعنی ورود به دنیای پس از کرونا با چشم اندازی جدید نیازمند تبیین‌های نوآورانه فلسفی و آموزشی است [۵]. هرچند این تغییر پارادیم، چالش‌هایی در پی خواهد داشت اما با اتخاذ راهبردهایی دانش مدار، می‌توان ضمن غلبه بر آن از مزایای حاصله استفاده نمود. یکی از راهبردها، استفاده از نظریه گراف در ساماندهی آشوب‌های آموزش مجازی با توجه به مفهوم نظریه آشوب برای تحقق یادگیری خودراهبر و خودآموزی خلاق است که به کمک آنها می‌توان سیستمی ایجاد کرد که شخصی‌تر، جذاب تر و مؤثرتر باشد.

۳. نظریه آشوب، لبه آشوب در آموزش و نوپیدایی خودآموزی خلاق

پیشگام نظریه آشوب، ادوارد لورنز است، هرچند رگه‌هایی از مطالعات در این زمینه را در کارهای دیگران مانند هنری پوانکاره پیر لاپلاس و حتی حکیم عمر خیام نیشابوری نیز می‌توان یافت. با مروری اجمالی، می‌یابیم مؤلفه اصلی نظریه آشوب، ایده‌هایی است که سیستم‌ها، هر چقدر هم که پیچیده باشند به یک نظریه اساسی متکی هستند و آن این است که سیستم‌ها و وقایع بسیار ساده یا کوچک می‌توانند باعث رفتارها یا حوادث بسیار پیچیده شوند.

ایده « وابستگی حساس به شرایط اولیه »، توسط ادوارد لورنز در سال ۱۹۶۱ به صورت گرافیکی کشف و « اثر پروانه‌ای » نامیده شد که یکی از ویژگی‌های اصلی نظریه آشوب است. نظریه آشوب توسط پریگوژین و استنگرز به صورت نظری تجزیه و تحلیل و توسط جیمز گلیک رایج گردید [۶، ۷]. مفاهیم پیچیدگی، متغیر بودن و غیرقابل پیش بینی بودن جایگزین مفاهیم سادگی، منظم بودن و پیش بینی‌پذیری شدند. در طی گسترش این نظریه، واژه نخستین نظریه آشوب جای خود را به واژه‌ای فراخ‌تر به نام نظریه پیچیدگی داد. نظریه آشوب به ریاضیات غیرخطی در سیستم‌های طبیعی محدود می‌شد و این در حالی است که نظریه پیچیدگی با توانایی کاربرد آن در رفتار سیستم‌های پیچیده اجتماعی علاوه بر سیستم‌های طبیعی بازنمایی می‌شود. به کارگیری نظریه آشوب در سیستم‌های پیچیده امروزی مانند مدارس و دانشگاه‌ها، به وضوح فرصت قابل توجهی برای تغییر و تحول فراهم می‌کند، زیرا یادگیری و تفکر فرآیندهای خطی نیستند. مدیران کلاس باید بدانند که پیچیدگی و تناقضات فضایی را فراهم می‌کند که برای خلاقیت بسیار مساعد است. توانایی یک سازمان برای حفظ پویایی و خلاقیت با تلاشی که در جهت بهبود مستمر، سازماندهی، ریسک کردن، تحول و توسعه می‌کند؛ نسبت مستقیم دارد. وقتی انیشتین گفت: « مهم است که دائماً سؤال بپرسی » او بر اهمیت پشتکار تأکید می‌کرد. وقتی صحبت از آموزش شخصی یک فرد می‌شود، مهم است که به یاد داشته باشیم که موفقیت ثابت نیست و گاهی اوقات سریع‌تر از زمان‌های دیگر پیشرفت می‌کند. به این ترتیب، برای مقابله با استبداد آموزشی، نظریه آشوب پیشنهاد کند که سیستم‌های آموزشی نباید دانش آموزان « ناموفق » را طرد یا حذف کنند، زیرا ممکن است در آینده پیشرفت قابل توجهی داشته باشند و دانش آموزانی که در ابتدا ناموفق تلقی می‌شوند ممکن است پس از مدتی به دانش آموزان موفق تبدیل شوند. بیکر و آلتون کاربرد نظریه آشوب را در سیستم آموزشی بررسی کرده‌اند و طبق یافته‌های آنان، مدارس و سیستم‌های آموزشی، مناطقی هستند که به شدت برای اثر پروانه‌ای مساعد هستند [۸، ۹]. در زندگی روزمره موقعیت‌های بحرانی وجود دارد همان طور که در مطالعات علمی وجود دارد، به نحوی که حتی تغییرات کوچک در رویدادهای متوالی، موجب مشکلات فزاینده بزرگ‌تری می‌گردد. نظریه آشوب به ما اطلاع دهد که این موقعیت‌ها در همه جا وجود دارد. براساس یافته‌های پژوهشی می‌توان گفت که اثر پروانه‌ای نشانه‌هایی از آینده به ما می‌دهد و این تأثیر را می‌توان در سیستم‌های آموزشی نیز مشاهده کرد [۱۰].

۳.۱. سیستم‌های دینامیکی - آشوبناک، ویژگی‌ها و نقش آن در آموزش. سیستم پویا (دینامیک)، متشکل از مجموعه عناصری است که وضعیت آن با گذر زمان و مطابق با قواعد مشخص تغییر می‌کند. در ریاضیات، این نوع سیستم و توابع معرف آن، به منظور نمایش وابستگی زمانی یک نقطه در فضای هندسی مورد استفاده قرار می‌گیرند این سیستم‌ها، بخشی جدایی‌ناپذیر از پدیده‌های اطراف ما هستند و در محیط‌های آموزشی نیز سایه گسترانیده‌اند، آن هم به شیوه‌ای آشوبناک! زیرا نسبت به شرایط اولیه حساسند، چرخش متناوب آنها متراکم است و از نظر توپولوژیکی نیز با هم ترکیب می‌شوند. در زمینه آموزش، نظریه آشوب بینش تازه‌ای در مورد یک سیستم آموزشی ارائه می‌دهد که متشکل از یک دنیای مکانیکی است که در آن بسیاری از افراد تلاش می‌کنند تا اطلاعات را تحت هر فلسفه آموزشی که در آن زمان رایج است، بیاموزند. ساخت مدل‌های غیرخطی که یادگیری را تسهیل می‌کنند، بسیار مهم است.

همان طور که لوری و استوپکا استدلال می‌کنند، دستیابی به نتایج ثابت در مورد سطح موفقیت دانش آموزان با استفاده از ابزارهای ارزشیابی سنتی، غیر ممکن است. آنها در عوض پیشنهاد می‌کنند که پارادایم‌های جدیدی که توانایی‌های یادگیری مستقل را توسعه می‌دهند، اجرا و دنبال شوند [۱۱]. تورمن با تجزیه و تحلیل جنبه‌های مختلف نظریه آشوب، تلاش کرد دیدگاه جدیدی را ایجاد کند و نظریه آشوب را از نظر سازمان، سیستم‌های آموزشی و مدیران امروزی روشن کند [۱۲]. آلتون و ارتورک جنبه‌های مختلف کاربرد نظریه آشوب در مدیریت آموزش را بررسی کردند [۱۳، ۱۴]. در این نظریه، آشوب و نوسانات در سیستم‌ها می‌توانند به عنوان منابع اصلی تحول و یادگیری در مقیاس‌های مختلف عمل کنند و آشوب به

عنوان عامل تحول، پویایی و تغییر، یادگیری به عنوان پاسخ به آشوب، تعامل و همگرایی و مفهوم گشتاور به معنای انتخاب و بهره گیری از فرصت‌های مختلفی است که در پیوند با آشوب و تحولات موجود در محیط فراهم می‌شوند؛ مورد توجه قرار می‌گیرند.

۲.۳. خودآموزی و لبه آشوب در آموزش. خودآموزی، نوعی فرآیند یادگیری است که یادگیرنده و یاددهنده در آن یک نفر هستند و خود فرد، مقرون به صرفه شناخته می‌شود و یادگیرنده در آن آزادی کامل در انتخاب موضوع، زمان و فضای آموزشی و انگیزشی خود دارد و کلاس‌های درسی و دیگر فعالیت‌های اجتماعی می‌توانند مکملی برای خودآموزی باشند. در زمینه آموزشی، تحقیقات نشان داد که یادگیری خودراهبر مستقیماً بر توانایی حل مسئله که با تفکر انتقادی مرتبط است، تأثیر می‌گذارد. در این میان، تفکر انتقادی طور مثبت با تفکر خلاق مرتبط است. بنابراین از آنجا که یادگیری خودراهبر با خلاقیت مرتبط است، پدید آمدن وضعیت خلاق نیازمند ایجاد تعادل میان ثبات، نظم و یکنواختی از یک سو و بی‌نظمی، آشوب و پراکندگی از سویی دیگر است. این ناحیه در واقع همان لبه آشوب است که در نظریه آشوب، مطرح شده است .

۴. کاربرد نظریه گراف در تحلیل فضای آشوبناک

از آنجا که آشوب سیستم‌های پیچیده‌ای مانند آموزش مجازی، می‌تواند از منابع مختلف پیچیدگی ساختاری، پویایی، شبکه تنوع و تغییرات نشأت بگیرد، تحلیل و مدل‌سازی آشوب به عنوان ابزاری برای شناسایی و پیش بینی قطعیت‌ها، نوسانات و نقاط ضعف سیستم استفاده می‌شود تا بهبود پایداری، عملکرد و کیفیت آموزش مجازی را فراهم کرده و باعث کاهش هزینه‌ها و افزایش رضایت دانشجویان و مدرسان گردد. استفاده از نظریه گراف در تجزیه و تحلیل شبکه‌های اجتماعی دانشجویان در سیستم آموزش مجازی، در شناسایی ارتباطات بین مدرسان و دانشجویان، تحلیل ساختار شبکه‌های اجتماعی، پیدا کردن جوامع و گروه‌های موجود با استفاده از الگوریتم‌های شناسایی اجتماعات و کاوش جوامع، تحلیل و ارزیابی اثربخشی فعالیت‌ها و رویدادها پیش‌بینی رفتار و عملکرد دانشجویان به مدیران آموزش مجازی کمک کند تا بهبودهای مداومی در فرآیند آموزش و یادگیری اعمال نمایند.

۵. نتایج

۱. آشوب سیستم‌های دینامیکی پیچیده با استفاده از مفاهیم گراف، قابل مدل‌سازی تحلیلی است.

۲. در آموزش مجازی، شبکه اجتماعی دانشجویان را می‌توان با گراف وزن‌دار جهت‌داری که گره‌ها دانشجویان و یال‌ها میزان و نوع ارتباط متفاوت بین آنان را نشان می‌دهد؛ مدل‌سازی و سپس با مفاهیم چگالی خوشه‌بندی ، همپوشانی ، زیرگراف‌ها ، معیارهای مرکزیت و احاطه‌گری بررسی و در نهایت توسط الگوریتم‌های بهینه‌سازی دایکسترا بلمن-فورد ، فلویید وارشال پریم و کراسکال نیومن انتشار برچسب جستجوی اول عمق و ...؛ بهینه‌سازی و تحلیل نمود [۱۵، ۱۶].

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Some Applications of Discrete Groups in Nonlinear Mechanics

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Abstract

The main purpose of this research is to express the applications of group theory, especially Lie groups and discrete groups. These groups are of great importance in the study of many applied sciences and nonlinear mechanics and mathematical models based on some classes of differential equations and integral equations. In addition, by introducing the theory of discrete groups and its fundamental properties, we will express some of the applications of these groups in solving practical problems.

Keywords: Discrete group, Differential equations, Integral equations

Mathematics Subject Classification [2010]: Primary: 35C05; 92F05; 34L30; 47E05; 35D35.

1 Introduction and Preliminaries

The group analysis is the most popular theoretical and analytical method for solving differential equations. In this section, we introduce some useful transformations by using the concept of the discrete group theory and its applications in exact solution of differential equations. Note that, all of these transformations are invertible and this invertibility allows us to avoid some lengthy computations for the conversion of the initial and boundary condition. Also, under these transformations, the solution of the transformed equation can be converted into the solution of the reference equation [5]- [6].

Let D be a class of ordinary differential equation and

$$D(x, y, a) = 0,$$

be an equation in this class, where a is a vector parameters. We shall seek the transformations F_i that are closed in the class $D(x, y, a) = 0$, i.e., they change only the vector a:

$$F_i : D(x, y, a) \to D(t, u, b_i).$$

If each F_i has an inverses, then the collection $\{F_i\}$ defines a discrete transformation group on the class $D(x, y, a) = 0$.

All the existing methods of exact solution of ordinary differential equations can be conditionally divided into two groups:

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[*] Speaker.

(I) a search for transformation of the original ordinary differential equations in class D to some other class of ordinary differential equations D_1, which belongs to one of the standard classes of ordinary differential equations having known solutions;

(II) a search for a transformation leaving the original ordinary differential equation in D invariant, that is, transformation into "itself", that gives independent information about the solution.

The discrete group method does not operate with a single equation as in the applications of the Lie method, but operates with a class of equations D, depending on a vector a of parameters containing the investigated equation; but contrary to approach (I), one considers the transformations of the given class D which are closed in themselves on a chosen class of ordinary differential equations.

In the literature, there are two methods for searching discrete group transformations, namely, point transformations and Backlund transformations. Therefore, discrete group transformations are related to point and Backland transformations. In this article, we have introduced a number of useful transformations based on Backland and point transformations. We show that these transformations have all the properties of the discrete group.

The class of generalized Emden-Fowler equations

$$y''(x) = Ax^n y^m (y'_x)^l, \tag{1}$$

is determined by a three-dimensional parametre vector $a = (n, m, l) \in \mathbb{R}^3$. Application of $RF-pair(X, X)$ to the generalized Emden-Fowler equations (1), we obtain a transformation

$$g : (n, m, l) \longrightarrow \left(\tfrac{1}{1-l}, -\tfrac{n}{n+1}, \tfrac{2m+1}{m} \right), \tag{2}$$

$$\left\{ \begin{array}{rl} y'_x = & t^{\frac{1}{1-l}}, \\ y = & (u'_t)^{\frac{-1}{m}}, \\ x = & u^{\frac{1}{n+1}}, \end{array} \right. \qquad \left\{ \begin{array}{rl} u'_t = & y^{-m}, \\ u = & x^{n+1}, \\ t = & (y'_x)^{l-1}, \end{array} \right. \tag{3}$$

where $g^3 = E$ and

$$g^{-1} : (n, m, l) \longrightarrow \left(-\tfrac{m}{m+1}, -\tfrac{1}{l-2}, \tfrac{n-1}{n} \right), \tag{4}$$

$$\left\{ \begin{array}{rl} y'_x = & u^{\frac{1}{2-l}}, \\ y = & t^{\frac{1}{m+1}}, \\ x = & (u'_t)^{\frac{1}{n}}, \end{array} \right. \qquad \left\{ \begin{array}{rl} u'_t = & x^n, \\ u = & (y'_x)^{2-l}, \\ t = & y^{m+1} \end{array} \right. \tag{5}$$

which defines the group

$$G_3\{g|\ g^3 = E\},$$

where E unity (the identity transformation). For further details see [3,6] and references therein.

The parameter subspace $a = (n, m, 0)$ defines the set of classical Emden-Fowler equations. It is well known that the point transformation

$$S : (n, m, 0) \longrightarrow (-m - n - 3, m, 0), \tag{6}$$

$$y = \frac{u}{t}, \qquad x = \frac{1}{t},$$

which defines the group

$$G_2\{S \Big| \ S^2 = E\}.$$

It is not hard to show that for $n = 1$, the transformation T_1 represents a composition Sg^{-1},

$$Sg^{-1} \equiv T_1 : (1, m, l) \longrightarrow \left(-\frac{2ml + 3l - 3m - 5}{(m+1)(l-2)}, \frac{1}{l-2}, 0 \right),$$

$$\left\{\begin{array}{rl} y'_x = & (\frac{u}{t})^{\frac{1}{2-l}}, \\ y = & t^{-\frac{1}{m+1}}, \\ x = & tu'_t - u, \end{array}\right. \qquad \left\{\begin{array}{rl} u'_t = & (y'_x)^{2-l} - \frac{2-l}{m+1}Axy^{m+1}, \\ u = & y^{-m-1}(y'_x)^{2-l}, \\ t = & y^{-m-1}. \end{array}\right. \tag{7}$$

where g^{-1} and S are defined in (4) and (6) respectively. For further details see [?] and references therein.

Theorem 1.1. *The class of generalized Emden-Fowler equations $a = (1, m, l)$ admits a general group*

$$G_2\Big\{S\Big|\ S^2 = E\Big\} \otimes G_3\Big\{g\Big|\ g^3 = E\Big\} \sim D_3\Big\{(S, g^{-1})\Big|\ (g^{-1})^3 = S^2 = E\Big\},$$

which is maximal in the Backlund transformation class defined by means of the $RF-pair$ method [3]. This group may be given by the graph depicted in figure 1. The graph represented in figure 1 is valid for all values m and l, except for the singular points $m = -1$ and $l = 2$.

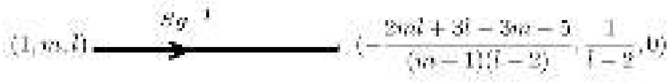

Figure 1: Group D_3

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Geodesic Vectors of Square-Root Metric on Generalized Symmetric Space of Type 7 with $\lambda \neq 0$ on Some Lie Algebras

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Abstract

In this paper, we obtain homogeneous geodesic vectors of square-root metric $F = \sqrt{\alpha(\alpha+\beta)}$ on 5-dimensional generalized symmetric spaces of type 7 with $\lambda \neq 0$ on some Lie algebras.

Keywords: Geodesic vector, Lie algebra, Lie group, Square-root metric.

Mathematics Subject Classification [2010]: Primary: 22E15, 22E60, 53C60.

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1 Introduction

In [1], the author introduced the notion of generalized symmetric spaces or regular s-spaces. Let (M, F) be a connected Finsler manifold. A symmetry at $x \in M$ is an isometry of (M, F) for which x is an isolated fixed point. A s-structure on (M, F) is a family $\{s_x\}_{x\in M}$ such that for each $x \in M$, s_x is a symmetry at $x \in M$. We note that an s-structure is called regular if for any two points $x, y \in M$, $s_x \circ s_y = s_z \circ s_x$ and $z = s_x(y)$. We call the s-structure $\{s_x\}_{x\in M}$ of order k if $(s_x)^k = id_M$ for all $x \in M$ and k is the minimal number with this property. If (M, F) admits an s-structure of order two then it is a usual symmetric Finsler space [4].

One of the family of Finsler metrics is the (α, β)-metrics. The notion of (α, β)-metrics are introduced by Matsumoto. If $F = \alpha + \beta$, then we get the Randers metric. Suppose $(M = \frac{G}{H}, F)$ be a connected homogeneous Finsler space, G is a connected transitive group of isometries of M and H is the isotropy subgroup at a point $o \in M$. We note that any homogeneous Finsler manifold $M = G/H$ is a reductive homogeneous space. Then for Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of G and H respectively we have a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ such that $Ad(H)\mathfrak{m} \subset \mathfrak{m}$. In this paper we study homogeneous geodesics of left invariant square-root metrics on five dimensional generalized symmetric spaces.

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*Speaker.

2 Preliminaries

Definition 2.1 ([4]). Let M be a (connected) smooth manifold. A Finsler metric on M is a function $F : TM \to [0,\infty)$ such that F is C^∞ on the slit tangent bundle $TM\backslash\{0\}$ and the restriction of F to any T_xM, $x \in M$, is a Minkowski norm. The pair (M,F) is called a Finsler space. Also for Finsler space (M,F) we have the following bilinear symmetric form $g_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(x,y+su+tv)|_{s=t=0}$.

Let $\alpha = \sqrt{\widetilde{a}_{ij}\left(x\right)y^i y^j}$ be a norm induced by a Riemannian metric $\widetilde{a}$ and $\beta\left(x,y\right) = b_i(x)y^i$ be a 1-form on an n-dimensional manifold M. Let $b := \|\beta(x)\|_\alpha = \sqrt{\tilde{a}(x)b_i(x)b_j(x)}$ and the function F is defined as $F := \alpha\varphi(s), s = \frac{\beta}{\alpha}$, where φ is a positive C^∞ function on $(-b_0, b_0)$ satisfying $\varphi\left(s\right) - s\varphi'\left(s\right) + \left(b^2 - s^2\right)\varphi''\left(s\right) > 0$ and $|s| \leq\ b < b_0$. Then F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form $F = \alpha\varphi(\beta/\alpha)$ is called an (α,β)-metric. A Finsler space having the Finsler function $F = \sqrt{\alpha(\alpha+\beta)}$, is called a square-root space. It is known that for an arbitrary 1-form β on a Riemannian manifold $(M,\tilde{a})$ there exists a unique vector field X on M such that $\tilde{a}(y,\tilde{X}(x)) = \beta(x,y)$ for every $x \in M, y \in T_xM$. So we can write square-root metric as $F(x,y) = \sqrt{\tilde{a}(y_x,y_x) + \sqrt{\tilde{a}(y_x,y_x)}\tilde{a}(X_x,y_x)}$.

Now consider the Chern connection on π^*TM whose coefficients are denoted by Γ^i_{jk}. Let $\gamma(t)$ be a smooth regular curve in M with velocity field V. Suppose $W(t) := W^i(t)\frac{\partial}{\partial x^i}$ be a vector field along γ. Then the covariant derivative D_VW with reference vector V have the form $\left(\frac{dW^i}{dt} + W^jV^k(\Gamma^i_{jk})_{(\gamma,V)}\right]\frac{\partial}{\partial x^i}|_{\gamma(t)}$. A curve $\gamma(t)$ with the velocity $V = \dot{\gamma}(t)$, is a Finslerian geodesic if $D_V\left(\frac{V}{F(V)}\right) = 0$ with reference vector V.

Definition 2.2. Suppose $(G/H,F)$ be a homogeneous Finsler manifold with a fixed origin o. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebra of G and H respectively and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ a reductive decomposition. Therefore, a homogeneous geodesic through the $o \in G/H$ is a geodesic $\gamma(t)$ of the form $\gamma(t) = \exp(tZ)(o)$ for every $t \in \mathbb{R}$ and $0 \neq Z \in \mathfrak{g}$.

In Riemannian setting the authors in [2], proved that a $X \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if

$$\langle [X,Y]_{\mathfrak{m}}, X_{\mathfrak{m}}\rangle = 0, \quad \forall Y \in \mathfrak{m}. \tag{1}$$

After this, the second author in Finsler setting shown that:

Lemma 2.3 ([3]). *Suppose $(G/H,F)$ be a homogeneous Finsler space with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Therefore, $Y \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if $g_{Y_{\mathfrak{m}}}(Y_{\mathfrak{m}}, [Y,Z]_{\mathfrak{m}}) = 0$ for every $Z \in \mathfrak{m}$ where the subscript $\mathfrak{m}$ indicates the projection of a vector from $\mathfrak{g}$ to $\mathfrak{m}$.*

3 Homogeneous Geodesics of Generalized Symmetric Space of Type 7 with $\lambda \neq 0$

The 5-dimensional homogeneous generalized symmetric spaces M of type 7 indeed are real matrix groups as follows:

$$\begin{bmatrix} e^{\lambda t} & 0 & 0 & 0 & x \\ 0 & e^{-\lambda t} & 0 & 0 & y \\ te^{\lambda t} & 0 & e^{\lambda t} & 0 & u \\ 0 & -te^{-\lambda t} & 0 & e^{-\lambda t} & v \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We note that, M is also $\mathbb{R}^5(x,y,u,v,t)$, equipped with a Riemannian metric

$$\widetilde{a} = dt^2 + e^{-2\lambda t}(tdx - du)^2 + e^{2\lambda t}(tdy + dv)^2 + a^2(e^{-2\lambda t}dx^2 + e^{2\lambda t}dy^2) + 2\gamma(dydu - dxdv),$$

where $\lambda, a, \gamma \in \mathbb{R}$, $\lambda \geq 0, a > 0$ and $\gamma^2 < a^2$.

In this paper we assume that $\lambda \neq 0$. From [1] we have $\mathfrak{h} = 0$. In the following, there exists a basis $\{X_1, X_2, Y_1, Y_2, W\}$ of $\mathfrak{g}$ such that for $[,]$ we have [1]:

$$[W, X_1] = \lambda X_1 + X_2, \quad [W, X_2] = \lambda X_2, \quad [W, Y_1] = -\lambda Y_1 - Y_2, \quad [W, Y_2] = -\lambda Y_2, \quad [W, W] = 0,$$

and the other multiplication are zero. Also for $\tilde{a}$ we have:

$$\tilde{a}(X_1, X_1) = \tilde{a}(Y_1, Y_1) = a^2, \qquad \tilde{a}(X_2, X_2) = \tilde{a}(Y_2, Y_2) = \tilde{a}(W, W) = 1,$$

$$\tilde{a}(X_1, Y_2) = -\gamma, \quad \tilde{a}(X_2, Y_1) = \gamma.$$

Now we construct an orthonormal frame field $\{e_1, e_2, e_3, e_4, e_5\}$, by setting

$$e_1 = \frac{X_1}{a}, \quad e_2 = X_2, \quad e_3 = \frac{Y_1 - \gamma X_2}{\sqrt{a^2 - \gamma^2}}, \quad e_4 = \frac{a^2 Y_2 + \gamma X_1}{a\sqrt{a^2 - \gamma^2}}, \quad e_5 = W. \tag{2}$$

By using the above definition of Lie bracket and equations (2) we have:

$$[e_1, e_5] = -\lambda e_1 - \frac{e_2}{a}, \qquad [e_2, e_5] = -\lambda e_2,$$

$$[e_3, e_5] = -\frac{\gamma e_1}{a\sqrt{a^2 - \gamma^2}} + \frac{2\gamma\lambda e_2}{\sqrt{a^2 - \gamma^2}} + \lambda e_3 + \frac{e_4}{a}, \quad [e_4, e_5] = -\frac{2\gamma\lambda e_1}{\sqrt{a^2 - \gamma^2}} - \frac{\gamma e_2}{a\sqrt{a^2 - \gamma^2}} + \lambda e_4.$$

Now we consider homogeneous geodesics of left invariant square-root metrics defined by the Riemannian metric $\tilde{a}$ and vector field $X = \sum_{i=1}^{5} x_i e_i$ on 5-dimensional generalized symmetric spaces of type 7 with $\lambda \neq 0$. By using the definition 2.1 and some computations, for the square-root metric we have:

$$g_y(u, v) = \tilde{a}(u, v) - \frac{1}{2}\frac{\tilde{a}(y, v)\tilde{a}(u, y)\tilde{a}(X, y)}{\tilde{a}(y, y)^{\frac{3}{2}}} + \frac{1}{2}\frac{\tilde{a}(u, v)\tilde{a}(X, y) + \tilde{a}(X, v)\tilde{a}(u, y) + \tilde{a}(y, v)\tilde{a}(y, u)}{\tilde{a}(y, y)^{\frac{1}{2}}}. \tag{3}$$

So for all $z \in \mathfrak{m}$, we have:

$$g_y(y, [y, z]) = \tilde{a}(y, [y, z]) \left[1 + \frac{1}{2}\frac{\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}}\right] + \frac{1}{2}\tilde{a}(X, [y, z])\sqrt{\tilde{a}(y, y)}. \tag{4}$$

So from equation (1) and lemma 2.3, $Y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if

$$\tilde{a}\left(\sum_{i=1}^{5} y_i e_i, \left[\sum_{i=1}^{5} y_i e_i, e_j\right]\right) = 0 \;\text{ and }\; \tilde{a}\left(\sum_{i=1}^{5} x_i e_i, \left[\sum_{i=1}^{5} y_i e_i, e_j\right]\right) = 0, \tag{5}$$

for every $j = 1, 2, 3, 4, 5$. So we get:

$\mathbf{j = 1:}$ $a\lambda y_5 y_1 + y_5 y_2 = 0$ and $a\lambda y_5 x_1 + y_5 x_2 = 0$, $\mathbf{j = 2:}$ $\lambda y_5 y_2 = 0$ and $\lambda y_5 x_2 = 0$,

$$\mathbf{j = 3:}\quad y_5\left(\lambda y_3 + \frac{2\gamma\lambda}{\sqrt{a^2 - \gamma^2}} y_2 + \frac{1}{a} y_4 - \frac{\gamma}{a\sqrt{a^2 - \gamma^2}} y_1\right) = 0 \;\text{ and }$$
$$y_5\left(\lambda x_3 + \frac{2\gamma\lambda}{\sqrt{a^2 - \gamma^2}} x_2 + \frac{1}{a} x_4 - \frac{\gamma}{a\sqrt{a^2 - \gamma^2}} x_1\right) = 0,$$

$$\mathbf{j = 4:}\quad y_5\left(\lambda y_4 - \frac{2\gamma\lambda}{\sqrt{a^2 - \gamma^2}} y_1 - \frac{\gamma}{a\sqrt{a^2 - \gamma^2}} y_2\right) = 0 \;\text{ and }$$
$$y_5\left(\lambda x_4 - \frac{2\gamma\lambda}{\sqrt{a^2 - \gamma^2}} x_1 - \frac{\gamma}{a\sqrt{a^2 - \gamma^2}} x_2\right) = 0,$$

$$\mathbf{j=5}: -y_1\left(\lambda y_1+\frac{1}{a}y_2\right)-\lambda y_2^2+y_3\left(\lambda y_3+\frac{2\gamma\lambda}{\sqrt{a^2-\gamma^2}}y_2+\frac{1}{a}y_4-\frac{\gamma}{a\sqrt{a^2-\gamma^2}}y_1\right)$$
$$+y_4\left(\lambda y_4-\frac{2\gamma\lambda}{\sqrt{a^2-\gamma^2}}y_1-\frac{\gamma}{a\sqrt{a^2-\gamma^2}}y_2\right)=0,\quad \text{and}$$
$$-y_1\left(\lambda x_1+\frac{1}{a}x_2\right)-\lambda y_2x_2+y_3\left(\lambda x_3+\frac{2\gamma\lambda}{\sqrt{a^2-\gamma^2}}x_2+\frac{1}{a}x_4-\frac{\gamma}{a\sqrt{a^2-\gamma^2}}x_1\right)$$
$$+y_4\left(\lambda x_4-\frac{2\gamma\lambda}{\sqrt{a^2-\gamma^2}}x_1-\frac{\gamma}{a\sqrt{a^2-\gamma^2}}x_2\right)=0,$$

Now for $y_5\neq 0$, from the above equations we have $y_1=y_2=y_3=y_4=0$ and $x_1=x_2=x_3=x_4=0$. if $y_5=0$ and $X=x_5e_5$ then we have

$$\begin{aligned}-y_1\left(\lambda y_1+\frac{1}{a}y_2\right)-\lambda y_2^2+y_3\left(\lambda y_3+\frac{2\gamma\lambda}{\sqrt{a^2-\gamma^2}}y_2+\frac{1}{a}y_4-\frac{\gamma}{a\sqrt{a^2-\gamma^2}}y_1\right)&\\ +y_4\left(\lambda y_4-\frac{2\gamma\lambda}{\sqrt{a^2-\gamma^2}}y_1-\frac{\gamma}{a\sqrt{a^2-\gamma^2}}y_2\right)&=0.\end{aligned}\tag{6}$$

Therefore, we proved that $Y=\sum_{i=1}^{5}y_ie_i$ is a geodesic vector of a generalized symmetric space of type 7 with $\lambda\neq 0$ equipped with a left invariant square-root metric defined by Riemannian metric $\tilde{a}$ and vector field $X=\sum_{i=1}^{5}x_ie_i$ if and only if

$$Y=y_5e_5+x_5e_5,\quad \textbf{or}\quad Y=\sum_{i=1}^{4}y_ie_i\ \text{ and the equation }\ (6)\ \text{ holds.}$$

Theorem 3.1. *Let (M,F) be a 5-dimensional generalized symmetric spaces of type 7 with $\lambda\neq 0$ equipped with a left invariant square-root metric defined by the Riemannian metric $\tilde{a}$ and vector field $X=x_5e_5$. Then $Y\in\mathfrak{g}$ is a geodesic vector of (M,F) if and only if Y is a geodesic vector of $(M,\tilde{a})$.*

Proof. From equation (4), lemma 2.3 and equation (1) we get assert. □

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Exactness of the Tensor Functor and Flatness Properties of Acts Over Monoids

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Abstract

In this talk, we delve into the notion of flatness in S-acts where S is a monoid, inspired by module theory. While in some cases like in acts with unique zero over a monoid with zero, flatness coincides to the exactness of the tensor functor analogous to modules, we show that flatness is weaker than the exactness of the tensor functor in general acts over monoids. It is proved that for a right S-act A_S, the tensor functor $A_S \otimes {}_S-$ is exact if and only if A_S is flat as well as indecomposable. We investigate conditions under which exactness of the tensor functors $A_S \otimes {}_S-$ and $C_S \otimes {}_S-$ in a Rees short exact sequence $A_S \xrightarrow{f} B_S \xrightarrow{g} C_S$ is transferred to $B_S \otimes {}_S-$ and vice versa. Some classifications of monoids based on the exactness of the tensor functor of their corresponding acts are also presented.

Keywords: S-Act, Flat, Tensor functor, Rees short exact sequence

Mathematics Subject Classification [2010]: Primary: 20M30, 20M15, Secondary: 20M50, 18G99

1 Introduction and Preliminaries

Monoid actions on sets, denoted as S-acts, have permeated various mathematical domains. Over the past thirty years, significant attempts have been made in understanding the properties of S-acts. Kilp et al. [4] offered a comprehensive survey of this field in 2000. In module theory, an R-module M_R is classified as flat if the tensor functor $M_R \otimes {}_R-$ is exact, indicating preservation of short exact sequences or, equivalently, preservation of all monomorphisms. The concept of flat acts, defined based on the preservation of embeddings by the tensor functor of acts, was initially introduced by Kilp [5] in 1970. The discussions on the flatness of S-acts up to 2000 were extensively compiled in [4]. Chen [1] introduced the notion of exact sequences for S-acts to investigate projectivity within the category of S-acts through exact functors. Furthermore, Chen and Shum [2] delved into the split Rees short exact sequences of S-acts, uncovering outcomes distinct from known results concerning split short exact sequences of modules. It is a natural question whether the flatness of acts corresponds to the exactness of the tensor functor. While this correspondence holds true in the category of acts with a unique zero over a monoid with zero and also in the category of unitary acts with a unique zero over a semigroup with zero (see [3,6]) like modules, a unique scenario arises in general acts over monoids. Specifically, in the category of right S-acts over a monoid S, flatness is demonstrated to be strictly weaker than the exactness of the tensor functor. We thoroughly characterize acts A_S for which $A_S \otimes {}_S-$ is exact as flat as well as indecomposable ones and identify certain classes of such acts. Additionally, we investigate conditions under which exactness of $A_S \otimes {}_S-$ and $C_S \otimes {}_S-$ in a Rees short exact sequence $A_S \xrightarrow{f} B_S \xrightarrow{g} C_S$ is transferred to $B_S \otimes {}_S-$ and vice versa. Furthermore, we obtain some classification of monoids S through exactnes of the tensor functor of S-acts.

Let S be a monoid. A non-empty set A is called a *right S-act* (or *right act over S*), usually denoted by A_S, if there exists a mapping $A \times S \longrightarrow A$, $(a, s) \mapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A$ and $s, t \in S$. The notion of a *left S-act* denoted by ${}_SA$ can be analogously defined. An element $\theta \in A$ is said to be a *zero* of A_S if $\theta s = \theta$ for all $s \in S$. The one-element S-act is denoted by Θ_S. A non-empty subset $B \subseteq A_S$ is called a *subact* of A_S if $bs \in B$ for all $s \in S$ and $b \in B$. In this case, B_S is called an *extension* of A_S. Each non-empty set A can be made into an S-act with *trivial action*: $as = a$ for all $a \in A$ and $s \in S$. Let A_S and B_S be two S-acts. A mapping $f : A_S \longrightarrow B_S$ is called a *homomorphism* if $f(as) = f(a)s$ for all $a \in A_S, s \in S$. We denote the category of all right S-acts with homomorphisms between them by **Act-S**. Also the categories of all left S-acts with homomorphisms and all sets with maps are denoted by **S-Act** and **Set**, respectively. In these categories, monomorphisms (epimorphisms) are exactly injective (surjective) morphisms, and isomorphisms and bijective morphisms coincide.

A non-empty subset I of a monoid S is called a (*left*) *right ideal* of S if $(SI \subseteq I)$ $IS \subseteq I$. A monoid S is called (*left*) *right reversible* if every two (right) left ideals of S have a non-empty intersection.

Let A_S be an S-act. An equivalence relation ρ on A_S is called a *congruence* on A_S if $a\rho a'$ implies that $as\rho a's$ for $a, a' \in A_S$, $s \in S$. Let ρ be a congruence on A_S. The *factor act* $A_S/\rho = \{[a]_\rho \mid a \in A\}$ is given by the action $[a]_\rho s = [as]_\rho$ for every $a \in A, s \in S$. Moreover, the *canonical epimorphism* $\pi_\rho : A_S \longrightarrow A_S/\rho$ is defined by $\pi_\rho(a) = [a]_\rho$. The *diagonal relation* $\{(a, a) \mid a \in A\}$ and the *universal relation* $A \times A$ on A_S are denoted by Δ_A and ∇_A, respectively. Any subact $B_S \subseteq A_S$ defines the *Rees congruence* $\rho_B := \nabla_B \cup \Delta_A$ on A_S and the resulting factor act is denoted by A_S/B_S. In the case that A is a set and $B \subseteq A$, the equivalence relation ρ_B on A is called the *Rees equivalence* and the factor set is denoted by A/B. Let $f : A_S \longrightarrow B_S$ be a homomorphism. Then the *kernel congruence* $\ker f$ on A_S is defined by $a(\ker f)a'$ if and only if $f(a) = f(a')$ for $a, a' \in A_S$. If A and B are sets and $f : A \longrightarrow B$ is a map, then $\ker f$ is an equivalence relation on A called the *kernel equivalence*.

Let A_S be a right S-act and ${}_SB$ be a left S-act. The *tensor product* $A_S \otimes {}_SB$ can be taken as the factor set $(A \times B)/\rho$, where ρ is the equivalence relation on the set $A \times B$ generated by all pairs $((as, b), (a, sb))$ for $a \in A, b \in B, s \in S$. The $[(a, b)]_\rho$ is denoted by $a \otimes b$. Let A_S be an S-act. Recall that the *tensor functor* $A_S \otimes {}_S- : S\textbf{-Act} \longrightarrow \textbf{Set}$ mapping each left S-act ${}_SB$ to the set $A_S \otimes {}_SB$ and each homomorphism $f : {}_SB \longrightarrow {}_SB'$ to the map $1_A \otimes f : A_S \otimes {}_SB \longrightarrow A_S \otimes {}_SB'$ defined by $(1_A \otimes f)(a \otimes b) = a \otimes f(b)$ for any $a \in A, b \in B$, is a covariant functor. If this functor preserves all monomorphisms, then A_S is called *flat*; and if it preserves all embeddings of (principal) left ideals into S, then A_S is called (*principally*) *weakly flat*.

The reader is referred to [4] for more information and undefined terms and notations about acts over monoids. Throughout S stands for a monoid.

A sequence of S-acts and homomorphisms (or sets and maps) of the form

$$\cdots \longrightarrow A_{-1} \xrightarrow{f_{-1}} A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow \cdots$$

is said to be *exact* at A_n provided that $\nabla_{\mathrm{Im} f_{n-1}} \cup \Delta_{A_n} = \ker f_n$, in which $\nabla_{\mathrm{Im} f_{n-1}} \cup \Delta_{A_n}$ is in fact the Rees congruence (equivalence) $\rho_{\mathrm{Im} f_{n-1}}$ on A_n. Clearly, the inclusion $\Delta_{A_n} \subseteq \ker f_n$ always holds. The above sequence is called *exact* if it is exact at A_i for every integer i. An exact sequence $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ at A_2 is called *Rees short left exact* if f_1 is a monomorphism, *Rees short right exact* if f_2 is an epimorphism, and *Rees short exact* if it is Rees short left and right exact (see [1,2]). A functor between two concrete categories is said to be (*left, right*) *exact* if it preserves all Rees short (left, right) exact sequences.

2 Main Results

It is well-known that the tensor functor is always right exact in the category of modules. There is the same situation in the category of acts with a unique zero over a monoid with zero and also in the category of unitary acts with a unique zero over a semigroup with zero (see [3,6]). In contrast to these categories, the following example shows that the tensor functor is not necessarily right exact in the category of acts over monoids.

Example 2.1. Let S be a monoid. Take the S-act $A_S = \Theta_S \sqcup \Theta_S = \{\theta_1, \theta_2\}$. Then the tensor functor $A_S \otimes {}_S-$ is not right exact.

Remark 2.2. As stated in [6, Example 4], analogously to the case of modules, the tensor functor of acts is not left exact in general.

Definition 2.3. Let A_S be an S-act. Then we say that the tensor functor $A_S \otimes {}_S-$ is *exact* if it preserves Rees short exact sequences, that is, for any Rees short exact sequence ${}_SB \xrightarrow{f} {}_SC \xrightarrow{g} {}_SD$ in S-**Act**, the sequence $A_S \otimes {}_SB \xrightarrow{1\otimes f} A_S \otimes {}_SC \xrightarrow{1\otimes g} A_S \otimes {}_SD$ is Rees short exact in **Set**, i.e. $1 \otimes f$ is injective, $\rho_{\mathrm{Im}(1\otimes f)} = \ker(1 \otimes g)$ and $1 \otimes g$ is surjective.

Note that the functor $A_S \otimes {}_S-$ preserves all epimorphisms. So in the above definition, the surjectivity of $1 \otimes g$, and also the inclusion $\Delta_{A_S \otimes {}_SC} \subseteq \ker(1 \otimes g)$ always hold.

Theorem 2.4. *Let A_S be an S-act. Then $A_S \otimes {}_S-$ is exact if and only if A_S is flat and indecomposable.*

As in the case of modules, a sequence $A_S \xrightarrow{f} B_S \xrightarrow{g} C_S$ is called *left* (*right*) *split* if there exists a homomorphism $f' : B_S \longrightarrow A_S$ ($g' : C_S \longrightarrow B_S$) with $f'f = 1_A$ ($gg' = 1_C$), where 1_A is the identity map on A. Note that left and right splitting of a sequence $A_S \xrightarrow{f} B_S \xrightarrow{g} C_S$ means that A_C and C_S are retracts of B_S, respectively.

Proposition 2.5. *Any right split Rees short exact sequence of S-acts is left split.*

Theorem 2.6. *Let $A_S \xrightarrow{f} B_S \xrightarrow{g} C_S$ be a right split Rees short exact sequence and B_S is injective (projective), then A_S and C_S are injective (projective).*

Proposition 2.7. *Let A_S be an S-act.*

- *(i) If for each extension B_S of A_S, the S-act B_S/A_S is projective, then S contains a left zero and A_S is injective.*
- *(ii) If B_S is an injective extension of A_S for which the S-act B_S/A_S is projective, then A_S is injective.*

Proposition 2.8. *Let F_S be a free S-act. Then $F_S \otimes {}_S-$ is exact if and only if $F_S \cong S_S$.*

Lemma 2.9. *Let a sequence $A_S \xrightarrow{f} B_S \xrightarrow{g} C_S$ in* **Act**-*S satisfy $\rho_{\mathrm{Im} f} = \ker g$ and let g be surjective. If A_S and C_S are indecomposable, then so is B_S.*

Theorem 2.10. *Let $A_S \xrightarrow{f} B_S \xrightarrow{g} C_S$ be a left split Rees short exact sequence and $A_S \otimes {}_S-$ and $C_S \otimes {}_S-$ are exact. Then $B_S \otimes {}_S-$ is exact.*

Corollary 2.11. *Let $A_S \xrightarrow{f} B_S \xrightarrow{g} C_S$ be a right split Rees short exact sequence. Then $A_S \otimes {}_S-$ and $C_S \otimes {}_S-$ are exact if and only if $B_S \otimes {}_S-$ is exact.*

Theorem 2.12. *For any monoid S the following statements are equivalent:*
(i) *For any indecomposable right S-act A_S, $A_S \otimes {}_S-$ is exact.*
(ii) *All right S-acts are flat.*
(iii) *S is regular and satisfies Condition*
(R) *for any $s, t \in S$ there exists $w \in Ss \cap St$ such that $w\rho(s,t)s$.*

An S-act A_S is called *strongly flat* if the functor $A_S \otimes_S -$ preserves pullbacks and equalizers. It is well-known that A_S is strongly flat if and only if it satisfies Conditions (P) and (E) (see [4, Theorem 3.16.8]).

Theorem 2.13. *For any monoid S, the following statements are equivalent:*
(i) *For any right S-act A_S, if $A_S \otimes {}_S-$ is exact, then A_S is strongly flat.*
(ii) *All flat right S-acts are strongly flat.*
(iii) $S = \{1\}$.

Theorem 2.14. *For any monoid S, the following statements are equivalent:*
(i) *For any right S-act A_S, if $A_S \otimes {}_S-$ is exact, then A_S is regular.*
(ii) *All flat right S-acts are regular.*
(iii) *All weakly flat right S-acts are regular.*
(iv) *All indecomposable weakly flat right S-acts are regular.*
(v) *Every element of S different from 1 is a right zero.*

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Trivial Morphisms in spNom

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Abstract

Within the context of nominal sets, the notion of support allows us to associate each nominal set X with a preorder relation $\preceq$. This particular preorder is commonly referred to as the support-preorder. In this paper, our focus lies in investigating and describing the behavior of trivial morphisms within the category of support-preordered nominal sets.

Keywords: Nominal set, Support-preordered nominal set, Trivial morphism.

Mathematics Subject Classification [2010]: 08A30, 08C05, 18A20, 20M30, 68Q70.

1 Introduction and preliminaries

Nominal set theory provides a mathematical framework for studying semantics, modifying variables, and much more in computer science. Indeed, Fraenkel presented nominal sets in [4] as an alternative model of set theory in 1922. In this context Mostowski studied further, which is why nominal sets are sometimes referred to as Fraenkel-Mostowski sets. In the 1990s, Gabbay and Pitts [7] rediscovered nominal sets for the computer science community, and this notion sparked a lot of interest in semantics [1, 2, 5, 6, 8].

Every nominal set can be viewed as a preordered set equipped with the support preorder relation based on the notion of support. Here considering support-preordered nominal sets, we define and the notion of trivial morphism and charactrize and characterize such morphisms. Additionally, we explore the concepts of prekernel and precokernel, investigating when a morphism possesses a prekernel (or precokernel).

Suppose $\mathbb{D}$ is a set, then a permutation π of $\mathbb{D}$ is a bijective map from $\mathbb{D}$ to itself. The permutations of $\mathbb{D}$ with composition and identity form a group, called the *symmetric group* on the set $\mathbb{D}$ and denoted by $\mathrm{Sym}\mathbb{D}$. A permutation $\pi \in \mathrm{Sym}\mathbb{D}$ is finitary if the set $\{d \in \mathbb{D} \mid \pi d \neq d\}$ is a finite subset of $\mathbb{D}$. The set of all finitary permutations is denoted by $\mathrm{Perm}(\mathbb{D})$ and clearly it is a subgroup of symmetric group on $\mathbb{D}$. We fix a countable infinite set $\mathbb{D}$ whose elements are denoted

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[*] Speaker.

by $a, b, c, ...$ and called *atomic names*. A a $\mathrm{Perm}(\mathbb{D})$-*set* is a set X equipped with an action of the group $\mathrm{Perm}(\mathbb{D})$, $\mathrm{Perm}(\mathbb{D}) \times X \to X$ mapping (π, x) to πx subject to the rules:

$$(1)\, \pi_1(\pi_2 x) = (\pi_1 o \pi_2)x, \text{ and } (2)\, id\, x = x,$$

for every $\pi_1, \pi_2 \in \mathrm{Perm}(\mathbb{D})$ and every $x \in X$. $\mathrm{Perm}(\mathbb{D})$-sets are the objects of a category, denoted by $\mathrm{Perm}(\mathbb{D})$-**Set** whose morphisms are equivariant maps, i.e. maps subject to the rule $f(\pi x) = \pi f(x)$, for all $x \in X, \pi \in \mathrm{Perm}(\mathbb{D})$. A subset Y of a $\mathrm{Perm}(\mathbb{D})$-set X is called *equivariant* if $\pi y \in Y$, for all $\pi \in \mathrm{Perm}(\mathbb{D})$ and $y \in Y$. An element x of a $\mathrm{Perm}(\mathbb{D})$-set X is called a *zero* element if $\pi x = x$, for all $\pi \in \mathrm{Perm}(\mathbb{D})$. The set of all zero elements of the $\mathrm{Perm}(\mathbb{D})$-set X is denoted by $\mathcal{Z}(X)$. A $\mathrm{Perm}(\mathbb{D})$-set all of whose elements are zero is called *discrete*. Given a $\mathrm{Perm}(\mathbb{D})$-set X, a set of atomic names $D \subseteq \mathbb{D}$ is a *support* for an element $x \in X$ if for all $\pi \in \mathrm{Perm}(\mathbb{D})$ and for every $d \in D$, $\pi(d) = d$ implies $\pi x = x$. Given a $\mathrm{Perm}(\mathbb{D})$-set X, we say an element $x \in X$ is finitely supported, if there is some finite set of atomic names that is, a support for the element x.

Example 1.1. Given a $\mathrm{Perm}(\mathbb{D})$-set X, the power set of X, $\mathcal{P}(X)$, with the action $\mathrm{Perm}(\mathbb{D}) \times \mathcal{P}(X) \to \mathcal{P}(X), (\pi, S) \rightsquigarrow \{\pi x : x \in S\}$ is a $\mathrm{Perm}(\mathbb{D})$-set. A set of atomic names D supports $S \in \mathcal{P}(X)$ if and only if

$$\forall \pi \in \mathrm{Perm}(\mathbb{D}), \forall d \in D\ \pi(d) = d \Longrightarrow \forall\ x \in S \quad \pi x \in S.$$

Definition 1.2. [9] A *nominal set* is a $\mathrm{Perm}(\mathbb{D})$-set all of whose elements are finitely supported. Nominal sets are the objects of a category, denoted by **Nom**, whose morphisms are equivariant maps and whose compositions and identities are as in the category of $\mathrm{Perm}(\mathbb{D})$-**Set**.

Remark 1.3. Suppose X is a nominal set and $x \in X$. Intersection of two finite supports of x is a (finite) support of x, [9, Propositions 2.1 and 2.3]. So each $x \in X$ has the least (finite) support which is denoted by $\mathrm{supp}_X\, x$, and when there is no possibility of error, we denote it by $\mathrm{supp}\, x$. In fact, $\mathrm{supp}\, x = \bigcap\{C : C \text{ is a finite support of } x\}$.

Example 1.4. (i) The set $\mathbb{D}$ is a nominal set with the natural action $\pi d = \pi(d)$.

(ii) The action of $\mathrm{Perm}(\mathbb{D})$ on $\mathbb{D}$ extends pointwise to action of $\mathrm{Perm}(\mathbb{D})$ on tuples $\mathbb{D}^n$ and $\mathbb{D}^{(n)}$. So, the sets $\mathbb{D}^n = \{(d_1, d_2, \ldots, d_n) \in \mathbb{D}^n \mid d_i \in \mathbb{D}\}$ and $\mathbb{D}^{(n)} = \{(d_1, d_2, \ldots, d_n) \in \mathbb{D}^n \mid d_i \neq d_j \text{ for } i \neq j\}$ are nominal sets.

Notation 1.5. We will frequently write $\mathcal{P}_{\mathrm{fs}}(X)$ for the set consisting of all finitely supported subsets of a given nominal set X. By $\mathrm{Fix}\, C$ we mean the set $\{\pi \in \mathrm{Perm}(\mathbb{D}) | \ \pi a = a, \ \textit{for every}\ a \in C\}$, where $C \subseteq \mathbb{D}$. We also denote by $\mathcal{P}_{\mathrm{f}}(\mathbb{D})$ the set consisting of all finite subsets of $\mathbb{D}$, and by $\mathcal{P}_{\mathrm{cof}}(\mathbb{D})$ the set consisting of all subsets of $\mathbb{D}$ with finite complement.

2 Trivial Morphisms in the Category spNom

Every nominal set together with the support-preorder given in Definition 2.1 can be considered as a preordered set. We direct our attention to the category **spNom** of support-preordered nominal sets, in this section, with a view to investigating the properties of its objects and morphisms.

Definition 2.1. By the *support-preorder* on a nominal set X, we mean the binary relation $\preceq$ on X defined by:

$$x \preceq y \Leftrightarrow \mathrm{supp}\, x \subseteq \mathrm{supp}\, y.$$

Since $\preceq$ is a preorder (i.e. reflexive and transitive), the pair $(X, \preceq)$ is called a *support-preordered nominal set* or briefly *sp-nominal set*.

It can be easily seen that the support-preorder is equivariant (or action preserving); meaning that:

$$x_1 \preceq x_2 \Rightarrow \pi x_1 \preceq \pi x_2,$$

for each $x_1, x_2 \in X, \pi \in \mathrm{Perm}(\mathbb{D})$.

Lemma 2.2 ([3, Lemma 3.4]). *Let X be an sp-nominal set and $x, x' \in X$. Then, there exists π with $\pi x \preceq x'$ or $\pi x' \preceq x$.*

Definition 2.3. Suppose X and Y are two sp-nominal sets. An equivariant map $f : X \to Y$ is called *support-preorder preserving* or for convenience *sp-preserving* whenever $f(x_1) \preceq f(x_2)$, for all $x_1 \preceq x_2 \in X$.

In [3], we consider the category of support-preordered nominal sets and sp-preserving maps between them denoted by **spNom**.

Lemma 2.4 ([3, Lemma 3.13]). *Suppose X and Y are two sp-nominal sets, $f : X \to Y$ is an sp-preserving map, and $x \in X$ with $\operatorname{supp} f(x) \neq \emptyset$. Then, $\operatorname{supp} f(x) = \operatorname{supp} x$.*

Definition 2.5. (i) An sp-preserving map $f : X \to Y$ is called *trivial* if for each $x, x' \in X$ with $x \preceq x'$ we have $f(x) = f(x')$.

(ii) Suppose $f : X \to Y$ is an sp-preserving map. We say that an sp-preserving map $k : X_1 \to X$ is a *prekernel* of f if the following properties are satisfied:

  (1) fk is a trivial morphism.

  (2) Whenever $\lambda : A \to X$ is an sp-preserving map, and $f\lambda$ is trivial, then there exists a unique sp-preserving map $\lambda' : A \to X_1$ such that $\lambda = k\lambda'$.

(iii) Suppose $f : X \to Y$ is an sp-preserving map. We say that an sp-preserving map $p : Y \to Y_1$ is a *precokernel* of f if the following properties are satisfied:

  (1) pf is a trivial morphism.

  (2) Whenever $\lambda : Y \to Y'$ is an sp-preserving map, and λf is trivial, then there exists a unique sp-preserving map $\lambda_1 : Y_1 \to Y'$ such that $\lambda = \lambda_1 p$.

Theorem 2.6. *An sp-preserving map $f : X \to Y$ is trivial if and only if $f(x) = \theta$ for some $\theta \in \mathcal{Z}(Y)$ or there exists a presentation $X = \bigcup_{i \in I} \mathrm{Perm_f}(\mathbb{D})x_i$ such that $\operatorname{supp} x_i = \operatorname{supp} x_j$ and $\mathrm{Im} f \cong \mathrm{Perm_f}(\mathbb{D})(\{d_1, \ldots, d_n\})$ for some $n \in \mathbb{N}$.*

Theorem 2.7. *An sp-preserving map $f : X \to Y$ has a prekernel if and only if we have at least one of the following cases:*

(i) f is trivial.

(ii) $\mathcal{Z}(X) \neq \emptyset$.

(iii) there is a unique $n \in \mathbb{N}^0$ such that $\mathrm{Im} f|_{X_1} \cong \langle \{d_1, \ldots, d_n\} \rangle$ for some equivariant subset X_1 of X.

Theorem 2.8. *Every sp-preserving map $f : X \to Y$ has a precokernel.*

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Some Results in T-Fuzzy Implicative Filters of BE-Algebras

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Abstract

The concept of triangular normed fuzzy implicative filters (T-fuzzy implicative filters) is introduced in BE-algebras. Some sufficient conditions are established for every T-fuzzy filter of a BE-algebra to become a T-fuzzy implicative filter.

Keywords: BE-algebra, T-fuzzy filter, T-fuzzy implicative filter

Mathematics Subject Classification [2010]: Primary: 06F35, 03G25, Secondary: 08A30

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1 Introduction and preliminaries

In 2007, H. S. Kim and Y. H. Kim ([5]) introduced the notion of a BE-algebra. In 2011, S. S. Ahn et al. ([1]) investigated fuzzy BE-algebras and characterized fuzzy BE-algebras in terms of level subalgebras of fuzzy BE-algebras. Since filters are important substructures and play an important role in algebraic structures, A. Borumand Saeid et al. ([2]) defined various filters of BE-algebras and they have shown the relationship between them. In 2013, G. Dymek and A. Walendziak ([3]) discussed some properties of fuzzy filters in BE-algebras. In 2013, M. S. Rao ([7]) introduced the notion of implicative fuzzy filters and studied the properties of these filters in BE-algebras. In 2023, Y. B. Jun ([4]) introduced the notion of positive implicative Lukasiewicz fuzzy filters in BE-algebras and investigated the relationship between fuzzy positive implicative filter and positive implicative Lukasiewicz fuzzy filter in BE-algebras. In this paper, the notion of T-fuzzy implicative filter in a BE-algebra is introduced, and some of the properties of these filters is studied. It is shown that every fuzzy implicative filter in a BE-algebra is a T-fuzzy implicative filter, but the converse is not true. Further, some sufficient conditions are derived for a T-fuzzy filter of a BE-algebra to become a T-fuzzy implicative filter. An extension property of T-fuzzy implicative filters is also studied.

In the following we recall some aspects which are necessary in the sequel.

Definition 1.1 ([5]). An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following properties:

(1) $x * x = 1$,

(2) $x * 1 = 1$,

---

[*] Speaker.

(3) $1 * x = x$,

(4) $x * (y * z) = y * (x * z)$, for all $x, y, z \in X$.

Theorem 1.2 ([5]). *Let $(X, *, 1)$ be a BE-algebra. Then we have the following:*

(1) $x * (y * x) = 1$,

(2) $x * ((x * y) * y) = 1$.

We introduce a relation $\leq$ on a BE-algebra X by $x \leq y$ implies $x * y = 1$.

Definition 1.3 ([5]). Let $(X, *, 1)$ be a BE-algebra. A non-empty subset F of X is called a filter of X if, for all $x, y \in X$, it satisfies the following properties:

(1) $1 \in F$,

(2) $x \in F$ and $x * y \in F$ imply that $y \in F$.

Definition 1.4 ([8]). For any set X, a fuzzy set in X is a function $\mu : X \to [0, 1]$.

Definition 1.5 ([3]). Let X be a BE-algebra. A fuzzy set μ of X is called a fuzzy filter if it satisfies the following properties:

(1) $\mu(1) \geq \mu(x)$,

(2) $\mu(y) \geq min\{\mu(x), \mu(x * y)\}$, for all $x, y \in X$.

Definition 1.6 ([3]). Let μ be a fuzzy set in a BE-algebra X. For any $\alpha \in [0, 1]$, the set $\mu_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}$ is called a level subset of X.

Definition 1.7 ([7]). A fuzzy set μ of a BE-algebra X is called a fuzzy implicative filter of X if it satisfies the following properties:

(1) $\mu(1) \geq \mu(x)$,

(2) $\mu(x * z) \geq min\{\mu(x * (y * z)), \mu(x * y)\}$, for all $x, y, z \in X$.

Definition 1.8 ([6]). Let $I = [0, 1]$. Then by a t-norm T, we mean a function $T : I \times I \to I$ satisfying the following:

(1) $T(x, 1) = x$,

(2) $y \leq z$ implies $T(x, y) \leq T(x, z)$,

(3) $T(x, y) = T(y, x)$,

(4) $T(x, T(y, z)) = T(T(x, y), z)$, for all $x, y, z \in I$.

A few t-norms which are frequently encountered are T_m, $Prod$ and min defined by $T_m(x, y) = max\{x + y - 1, 0\}$, $Prod(x, y) = xy$ and

$$\min\{x, y\} = \begin{cases} x & \text{if } x \leq y, \\ y & \text{if } y < x. \end{cases}$$

For any t-norm T on I, it can be easily observed that $T(\alpha, \beta) \leq \min\{\alpha, \beta\}$ for all $\alpha, \beta \in I$.

2 Main Results

In this section, the notion of triangular normed fuzzy implicative filters is introduced in BE-algebras. Some sufficient conditions are derived for every triangular normed fuzzy filter of a BE-algebras to become a triangular normed fuzzy implicative filter.

Definition 2.1. A fuzzy set μ of a BE-algebra X is called a fuzzy filter of X with respect to a t-norm T (simply called T-fuzzy filter) if it satisfies:
(1) $\mu(1) \geq \mu(x)$,
(2) $\mu(y) \geq T(\mu(x), \mu(x * y))$, for all $x, y \in X$.

Definition 2.2. A fuzzy set μ of a BE-algebra X is called a fuzzy implicative filter of X with respect to a t-norm T (simply called T-fuzzy implicative filter) if it satisfies:
(1) $\mu(1) \geq \mu(x)$ for all $x \in X$,
(2) $\mu(x * z) \geq T(\mu(x * (y * z)), \mu(x * y))$, for all $x, y, z \in X$.

Proposition 2.3. *Every T-fuzzy implicative filter of a BE-algebra is a T-fuzzy filter.*

The following example shows that the converse of proposition 2.3 is not true in general.

Example 2.4. Let $X = \{1, a, b, c, d\}$ be a non-empty set. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

Then it can be easily verified that $(X, *, 1)$ is a BE-algebra. Define a fuzzy set μ on X as follows:

$$\mu(x) = \begin{cases} .9, & \text{if } x = a, 1 \\ .2, & \text{otherwise} \end{cases}$$

for all $x \in X$. Then clearly μ is a T_m-fuzzy filter of X, but μ is not a T_m-fuzzy implicative filter of X since

$$\begin{aligned} \mu(b * c) &< \max\{\mu(b * (d * c)) + \mu(b * d) - 1, 0\} \\ &= T_m(\mu(b * (d * c)), \mu(b * d)). \end{aligned}$$

Theorem 2.5. *Assume that X is transitive and μ is a T-fuzzy filter with $\mu(1) = 1$. then the following are equivalent:*

(1) μ is a T-fuzzy implicative filter.

(2) $\mu(x * y) = \mu(x * (x * y))$.

(3) $\mu((x * y) * (x * z)) \geq \mu(x * (y * z))$.

In the following, we derive some sufficient conditions for every T-fuzzy filter of a transitive BE-algebra to become a T-fuzzy implicative filter.

Theorem 2.6. *Every T-fuzzy filter μ of a transitive BE-algebra X is a T-fuzzy implicative filter if satisfies the following condition for all $x, y \in X$.*

$$(TF_1) \qquad \mu(y) \geq \mu(x * (x * y))$$

Theorem 2.7. *Every T-fuzzy filter μ of a transitive BE-algebra X is a T-fuzzy implicative filter if satisfies the following condition for all $x, y, z \in X$.*

$$(TF_2) \qquad \mu((x * y) * z) \geq \mu(x * (y * z)).$$

Proposition 2.8. *Every fuzzy implicative filter is a T-fuzzy implicative.*

The converse of the above proposition is not true. However, we derive a sufficient condition for every T-fuzzy implicative filter to become a fuzzy implicative filter.

Theorem 2.9. *Let* F *be an implicative filter of* X*. Then there exists a T-fuzzy implicative filter* μ *such that* $\mu_t = F$*, for some* $t \in [0,1]$*.*

Theorem 2.10. *(Extension property for T-fuzzy implicative filters) let* μ *and* ν *be two T-fuzzy filters of transitive BE-algebra* X *such that* $\mu \subseteq \nu$ *and* $\mu(1) = \nu(1) = 1$*. If* μ *is a T-fuzzy implicative filter, then* ν *is also a T-fuzzy implicative filter.*

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Basic Commutators in n-Lie Algebras and the Relationship Between the Number of Basic Commutators in Lie and n-Lie Algebras

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Abstract

In this paper, we give the structure of free n-Lie algebras. Next, we introduce basic commutators in n-Lie algebras and generalize the Witt formula to calculate the number of the basic commutators. Also, we prove that the set of all of the basic commutators of weight w and length $n+(w-2)(n-1)$ is a basis for F^w, where F^w is the wth term of the lower central series in the free n-Lie algebra F.

Keywords: n-Lie algebra, Basic commutators, Free n-Lie algebras, Basic product, String.

Mathematics Subject Classification [2010]: Primary: 17B05, 17B10, Secondary: 17B99

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1 An Introduction to Basic Commutators

The consept of basic commutators is defined in groups and Lie algebras and there is also a way to construct and identify them. Moreover, a formula for calculating their number is obtained. In 1985, Filippov [1] introduced the concept of *n-Lie algebras*, as an n-ary multilinear and skew-symmetric operation $[x_1,\dots,x_n]$, which satisfies the following generalized Jacobi identity $[[x_1,\dots,x_n],y_2,\dots,y_n]=\sum_{i=1}^{n}[x_1,\dots,[x_i,y_2,\dots,y_n],\dots,x_n]$. Clearly, such an algebra becomes a ordinary Lie algebra when $n=2$. In 1962, A.I. Shirshov [4] gives a method that generalizes Halls method for choosing a basis in a free Lie algebra. Basic commutators is of particular importance in calculating the dimensions of different spaces and is therefore highly regarded. P. Niroomand and M. Parvizi in [2], investigate to obtain some more results about 2-nilpotent multiplier $\mathcal{M}^{(2)}(L)$ of a finite dimensional nilpotent Lie algebra L and by using the Witt formula, calculate its dimension. Moreover, A. Salemkar, B. Edalatzadeh, and M. Araskhan in [3] introduce thte concept of c-nilpotent multiplier $\mathcal{M}^{(c)}(L)$ of a finite dimensional Lie algebra L and obtain some bounds for $\mathcal{M}^{(c)}(L)$ by using the Witt formula and basic commutators.

In this section, our aim is to define basic commutators of n-Lie algebras. In the previous sections, we completely and precisely define the basic commutators by using the words. Indeed in this section, we are going to briefly introduce the basic commutators for n-Lie algebra L (especially) with the free representation $\frac{F}{R}$ over the field $\mathbb{F}$, in the bracket form. In other words, we introduce a method to determine the basic commutators in n-Lie algebras. Note that two properties of n-Lie algebras, as “skew-symmetric” and “Jacobian identity”, are important and necessary to determine the basic commutators.

In general, it can be said that the set of all basic commutators A is the smallest subset of the set of all commutators of L, named W, such that A can be produced W.

The basic commutators are characterized by two concepts as "*weight*" and "*length*". In the case when $n = 2$, these two concepts are the same. The basic commutator $0 \neq c \in F$ has weight w_c, if $c \in F^{w_c}$ and $c \notin F^{w_c+1}$, where F^{w_c} is w_cth terms of lower central series, and we denote it by w_c. Also, the length of c is the number of its components that are in X, and we denote it by m_c and calculate it as follows:

$$m = n + (w_c - 2)(n-1) = n + w_c n - 2n - w_c + 2 = (w_c - 1)n - (w_c - 2).$$

In the general case, weight and length are denoted by w and m, respectively. If in any n-Lie algebras (for each $n \in \mathbb{N}$), we denote the number of length of the basic commutator of weight w_0 by $m^n_{w_0}$, then we can obtain the following results:

- $m^n_{w_0} = m^{w_0}_n, \qquad n \geq 2$, and $w_0 \geq 2$.
- $m^3_{w_0} = m^2_{w_0} + m^2_{w_0-1}$.

Let $X = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a basis set of n-Lie algebra L. We define the relation "$<$" on X as $x_1 < x_2 < \cdots < x_n < \cdots$. In what follows, we define the basic commutators in L as follows:

1. Every basic element $c^1_i = x_i$ in X is the basic commutator of weight 1.
2. The element $c^2_s = [x_{i_1}, x_{i_2}, \ldots, x_{i_n}]$ (where $x_{i_j} \in X$, for all $j = 1, \ldots, n$) is a basic commutator of length n and weight 2. It belongs to F^2, if $x_{i_1} > x_{i_2} > \cdots > x_{i_n}$; in other words, $i_1 > i_2 > \cdots > i_n$. We can compare every two basic commutators c^2_s and c^2_k with weight 2 and length n. Suppose that $c^2_s = [x_{i_1}, x_{i_2}, \ldots, x_{i_n}]$ and that $c^2_k = [x'_{j_1}, x'_{j_2}, \ldots, x'_{j_n}]$, and let r_0 be the first index (in moving from right to left) of x_{i_r}'s and x'_{j_r}'s which $x_{i_{r_0}} \neq x'_{j_{r_0}}$. If $x_{i_{r_0}} > x'_{j_{r_0}}$ or, equivalently, $i_{r_0} > j_{r_0}$, then we say $c^2_s > c^2_k$, and otherwise, we say $c^2_s < c^2_k$.
3. Let $c^w_t = [c^{w_1}_{j_1}, \ldots, c^{w_n}_{j_n}]$, where all of $c^{w_1}_{j_1}, \ldots, c^{w_n}_{j_n}$ are basic commutators of weights $w_1, \ldots, w_n$ and lengths $n + (w_1 - 2)(n-1), \ldots, n + (w_n - 2)(n-1)$, respectively. Then c^w_t is called a basic commutator if the following conditions hold:
    - $w > w_1 \geq w_2 \geq \cdots \geq w_n$.
    - Whenever $w_s = w_{s+1}$, then $c^{w_s}_{j_s} > c^{w_{s+1}}_{j_{s+1}}$, equivalently, $j_s > j_{s+1}$.
    - Whenever $w_s > w_{s+1}$ and $c^{w_s}_{j_s} = [c^{w'_1}_{j_{s_1}}, \ldots, c^{w'_n}_{j_{s_n}}]$, then $c^{w'_n}_{j_{s_n}} \leq c^{w_n}_{j_n} < \cdots < c^{w_{s+2}}_{j_{s+2}} < c^{w_{s+1}}_{j_{s+1}}$.

    In this case, c^w_t is an basic element of F^w of weight w and length $n + (w-2)(n-1)$.

Note that every basic commutator of weight w is smaller than every basic commutator of weight larger than w. Hence $c^1_i < c^2_j < c^3_k < \cdots$, for all indices $i, j, k, \ldots$.

Now, we state the theorems that are among the main results and our goals in this paper.

Theorem 1.1. *Let F be a free n-Lie algebra with the ordered basis set $X = \{x_1, x_2, x_3, \ldots\}$, where $x_1 < x_2 < x_3 < \ldots$. The set of all defined basic commutators of weight w (denoted by $B(w)$) on X is linearly independent.*

The following theorem is one of the main results of this paper, which is similarly discussed in group theory and Lie algebras.

Theorem 1.2. *Let F be a free Lie n-algebra. Then $\dfrac{F^i}{F^{i+c}}$ is abelian Lie n-algebras whose basis is the set of all basic commutators of weights $i, i+1, i+2, i+3, \ldots, i+c-1$ and lengths $n + (i-2)(n-1), n + (i-1)(n-1), \ldots, n + (i+c-3)(n-1)$.*

2 Counting of Basic Commutators

It is necessary to note that counting of basic commutators on every arbitrary basis set X is not possible, in general. In fact, although it is possible to define the concept of basic commutators for any countable set, counting their number is only possible if the set X is finite. The following formula, known as the *Witt formula*, is provided in group theory and Lie algebras for counting of all of the basic commutators:

$$l_d(w) = \frac{1}{w}\sum_{r|w}\mu(r)d^{\frac{w}{r}}, \tag{1}$$

where d is the number of generators of given group/Lie algebra (the number of members of X), μ is the *Mbius function*, which is defined for positive integers by the rule $\mu(1) = 1$, and for $w = p_1^{a_1}p_2^{a_2}\dots p_s^{a_s}$ that $p_1, p_2, \dots, p_s$ are distinct primes numbers, by $\mu(p_1p_2\dots p_s) = (-1)^s$ and $\mu(p_1^{a_1}p_2^{a_2}\dots p_s^{a_s}) = 0$, if any $a_i > 1$.

In group theory, the strategy of the proof of the above formula is to remove nonbasic commutators from all the commutators. That is, it first counts all possible commutators on X and then removes nonbasic members and their periods. Indeed in n-Lie algebras, with this method, only a bound can be found for the number of basic commutators.

Now, let $X = \{x_1, x_2, \dots, x_d\}$ with the ordered relation $x_d > x_{d-1} > \dots > x_2 > x_1$. We are going to determine the number of all of basic commutators of weight w and length $m = n + (w-2)(n-1)$.

Theorem 2.1. *Let L be an n-Lie algebra of dimension d. Then $l_d^n(2) = \binom{d}{n}$ and $l_n^n(w) = \sum_{i=1}^{w-2} a_i\binom{n}{i}$, and in Table 1 (for $i \geq 1$), see the order between the coefficients.*

Table 1: coefficients $\binom{n}{i}$'s

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
$l_n^n(4)$	1	1	–	–	–	–	–	–
$l_n^n(5)$	1	2	1	–	–	–	–	–
$l_n^n(6)$	1	3	3	1	–	–	–	–
$l_n^n(7)$	1	4	6	4	1	–	–	–
$l_n^n(8)$	1	5	10	10	5	1	–	–
$l_n^n(9)$	1	6	15	20	15	6	1	–
$l_n^n(10)$	1	7	21	35	35	21	7	1

Theorem 2.2. *Let F be a free n-Lie algebra over the ordered set $X = \{x_i | x_{i+1} > x_i : i = 1, 2, \dots, d\}$ and let w be a positive integer number. Then the number of basic commutators of weight w is*

$$l_d^n(w) = \sum_{j=1}^{\alpha_0}\beta_{j^*}\left(\sum_{i=2}^{w-1}\alpha_i\binom{\binom{d}{n-1}}{w-i}\right), \tag{2}$$

where $\alpha_0 = \binom{d-1}{n-1}$, α_i, $(2 \leq i \leq w-1)$ is the coefficient of the $(i-2)$th sentence in Newton's binomial expansion $(a+b)^{w-3}$, and if $\binom{k-1}{n-1} + 1 \leq j \leq \binom{k}{n-1}$, (for $k = n-1, n, n+1, n+2, \dots, d-1$), then $j^* = \binom{k-1}{n-1} + 1$ and $\beta_{j^*} = (d - n - j^* + 2)$.

Theorem 2.3. *The number of basic commutators of weight 2 on d generators in n-Lie algebras can be concluded by the number of basic commutators of weights $w \leq n$ in Lie algebras, and we have*

$$\binom{d}{n} = \sum_{s=2}^{n}(-1)^{n-s}\left(\frac{a_s}{(s+1)(s+2)\dots(n-1)n}\right)l_d(s) = \sum_{s=2}^{n}(-1)^{n-s}\left(\frac{a_s}{\prod_{\substack{r=3\\ r\neq s}}^{n} r}\right)l_d(s). \tag{3}$$

The following corollary expresses the relationship between the number of basic commutators in Lie algebras and n-Lie algebras. The following formula is Witt's generalized formula.

Corollary 2.4. *The number of basic commutators of arbitrary weight w on d generators in n-Lie algebras, can be concluded by the number of basic commutators in Lie algebra. We have*

$$l_d^n(w) = \sum_{j=1}^{\alpha_0} \beta_{j^*} \left(\sum_{i=2}^{w-1} \alpha_i \left(\sum_{s=2}^{w-i} (-1)^{w-i-s} \left(\frac{a_s}{\prod\limits_{\substack{r=3 \\ r\neq s}}^{w-i} r} \right) l_{d^*}(s) \right) \right),$$

where $d^* = \binom{d}{n-1}$, $\alpha_0 = \binom{d-1}{n-1}$, α_i $(2 \leq i \leq w-1)$ is the coefficient of the $(i-2)$th sentence in Newton's binomial expansion $(a+b)^{w-3}$. If $\binom{k-1}{n-1} + 1 \leq j \leq \binom{k}{n-1}$, (for $k = n-1, n, n+1, n+2, \ldots, d-1$), then $j^* = \binom{k-1}{n-1} + 1$ and $\beta_{j^*} = (d - n - j^* + 2)$.

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Tools for Working with Cyclic Cohomology of Nondegenerate Algebras

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Abstract

We give general tools for working with cyclic cohomology of nondegenerate algebras. Using our way of approach in this paper, at least in general when we impose some extra conditions, one always has to make sure that it is natural to look at the cyclic cohomology theory of algebras with an identity and to try to extend it to the algebras without identity but with a nondegenerate product.

Keywords: Nondegenerate algebras, Cyclic cohomology.

Mathematics Subject Classification [2010]: 17B37, 19D55, 20G42.

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1 Introduction and Preliminaries

The aim of this paper mainly is to give general tools for working with cyclic cohomology of nondegenerate algebras. As far as we know, there are no general methods or practical tools to study this subject of nondegenerate algebras (of course, in the Van Daele approach to the algebras without identity). There are many different ways where our approach can be applied and maybe we can even apply it in different fields of mathematics. To construct a cyclic module structure to each field of mathematics, we need to choose the appropriate spaces with three special types of operators (cyclic, face and degeneracy operators) satisfy certain rules (see [1]). Although in general, we will only work on cyclic operators (because it is sufficiently general to illustrate our approach), but we feel it is possible to define face and degeneracy operators at the same time, of course we know that it is a difficult work. An advantage of our method in this paper is that we can apply it directly on (weak) multiplier Hopf algebras (see [2], and see also Appendix of [3] for a cyclic module structure on weak Hopf algebras). Maybe the main idea of this paper comes from the basic work of Van Daele in [4] on extended modules where we get a strictly bigger module in general, but the paper [5], where the Connes-Moscovici cyclic module is generalized to regular multiplier Hopf algebras, has been the motivating source for writing this subsection. It is important to notice that we get nothing new when our algebra has an identity (of course, in most cases). Throughout this paper $\mathbb{C}$ denotes the field of complex numbers. Let A be an associative algebra over $\mathbb{C}$. We say that A is a *nondegenerate algebra* if the product in A is nondegenerate, that is, if $ab = 0$ for all $b \in A$ implies $a = 0$ and $ab = 0$ for all $a \in A$ implies $b = 0$. It is clear that any unital algebra is a nondegenerate algebra. A linear map z on A is called a *left multiplier* of A if $z(ab) = z(a)b$ for all a, b in A. The *left multiplier algebra* $L(A)$ is the vector space of all left multipliers of A. The composition of maps makes $L(A)$ into a unital algebra and the identity map $\iota : A \to A$ is clearly its unit. We can consider A as a subset of $L(A)$, using the nondegeneracy of the product in A. In fact the map $\varphi : A \to L(A)$ defined by $\varphi(a)(b) = ab$ is an injective map.

2 Main Result

Let A be a nondegenerate algebra and n be an integer greater or equal 1. Assume that A acting on its left multiplier algebra $L(A)$ in n ways and from right. Denote these n actions by $z \cdot_1 a, \cdots, z \cdot_n a$, whenever a is in A and acts on $z \in L(A)$. We always assume that any action is *nondegenerate*, in the sense that $z = 0$ if $z \cdot_k a = 0$ for all $a \in A$. Assume also that $z \cdot_k a \in A$ when z, a are both in A.

Definition 2.1. To each integer $1 \leq k \leq n$, we denote by $L_k(A)$ the vector space of all linear maps $l : A \to A$ satisfying

$$l(ab) = l(a) \cdot_k b,$$

for all $a, b \in A$.

Remark that the right hand side of the above equality is an element in A because it is assumed that $A \cdot_k A \subset A$. The map $a \mapsto l_{k,a}$ from A to $L_k(A)$ defined by $l_{k,a}(b) = a \cdot_k b$ is injective because the module is assumed to be nondegenerate. So we can view A as sitting in $L_k(A)$ for any $1 \leq k \leq n$. If A has an identity then for any $l \in L_k(A)$ we have

$$l_{k,l(1)}(b) = l(1) \cdot_k b = l(1b) = l(b),$$

by the property of the map l. So in this case, we get $A \equiv L_k(A)$. However, the point is that in the case of a non-unital algebra, the space of such maps is in general strictly bigger than A itself (see [4] for examples). In some cases, we can even find elements in the left multiplier algebra $L(A)$ and not of A that can be considered in $L_k(A)$, by the property that the module action $L(A)$ is nondegenerate. The following definition can be used for this purpose.

Definition 2.2. We denote by $C^n(A)$ the vector space of all $z \in L(A)$ such that

$$z \cdot_k a \in A$$

for all $1 \leq k \leq n$ and all $a \in A$.

Clearly A is a subset of $C^n(A)$, again because $A \cdot_k A \subset A$. We also can view $C^n(A)$ as sitting in any $L_k(A)$, although some care is required because it is a subset of $L(A)$. The map $z \mapsto l_{k,z}$ from $C^n(A)$ to $L_k(A)$ defined by $l_{k,z}(a) = z \cdot_k a$ is injective because the module $L(A)$ is nondegenerate. Take

$$\widetilde{C}^n(A) := L(A) \oplus L_1(A) \oplus \cdots \oplus L_n(A),$$

as the direct sum of $L(A)$ with these spaces $L_k(A)$. In this paper, we will often use the symbol $z \cdot_0 a$ for the element za, when $z \in L(A)$ and $a \in A$. The map

$$z \mapsto (l_{0,z}, l_{1,z}, \cdots, l_{n,z}) \tag{1}$$

from $C^n(A)$ to $\widetilde{C}^n(A)$ such that $l_{k,z}(a) = z \cdot_k a$ is clearly injective. So, we can also view $C^n(A)$ as sitting in $\widetilde{C}^n(A)$. In this paper, sometimes we identify an element z in $C^n(A)$ with $(l_{0,z}, l_{1,z}, \cdots, l_{n,z})$. But before we continue, let us first look at an important remark.

Remark 2.3. The map (1) plays an important role in this paper because we have imbeddings of A in $C^n(A)$ and of $C^n(A)$ in $\widetilde{C}^n(A)$, and so one might guess that a map on A can be extended to a map on $C^n(A)$ and then on $\widetilde{C}^n(A)$. And conversely, for any given $\widetilde{\tau}_n : \widetilde{C}^n(A) \to \widetilde{C}^n(A)$ it seems natural to try to have $\widetilde{\tau}_n(A) \subset A$ and $\widetilde{\tau}_n(C^n(A)) \subset C^n(A)$, by using imbedding (1).

With this approach in mind, we are now ready to define our operators on A, $C^n(A)$ and $\widetilde{C}^n(A)$ (although we first define our operators on A, $\widetilde{C}^n(A)$ and then on $C^n(A)$). But, however it is obvious that extra assumptions are needed because (maybe) there are the different actions. Assume that there is an element $\gamma \in C^n(A)$ with the conditions

$$\underbrace{\gamma \cdot_1 (\gamma \cdot_1 (\cdots (\gamma \cdot_1 a) \cdots))}_{n+1\text{-times}} = a, \tag{2}$$

$$\gamma \cdot_1 (a \cdot_{k-1} b) = (\gamma \cdot_1 a) \cdot_k b, \tag{3}$$

for all $a, b \in A$ and all $1 \leq k \leq n$. The left hand side of (2) makes sense and it is an element in A, since γ is assumed to be an element in $C^n(A)$. For a better understanding of condition (2), if $n = 1$ we have $\gamma \cdot_1 (\gamma \cdot_1 a) = a$ and if $n = 2$ we have $\gamma \cdot_1 (\gamma \cdot_1 (\gamma \cdot_1 a)) = a$. Similarly for any $n \geq 3$. Note also that, the both sides of (3) are in A because $\gamma \in C^n(A)$ and $A \cdot_k A \subset A$.

Remark 2.4. In (3) above, if $k = 1$ then (using the module property) one has automatically that

$$\gamma \cdot_1 (a \cdot_0 b) = \gamma \cdot_1 (ab) = (\gamma \cdot_1 a) \cdot_1 b.$$

Thus the case when $n = 1$ we only have condition (2).

We are now ready for the following, an important consequence of γ.

Lemma 2.5. *For all $a, b \in A$ we have*

$$\gamma \cdot_1 (a \cdot_n b) = (\gamma \cdot_1 a)b. \tag{4}$$

We have for all $a \in A$ that $\gamma \cdot_1 a \in A$ since γ is an element in $C^n(A)$. Hence:

Definition 2.6. Define the linear map $\tau_n : A \to A$ by $\tau_n(a) = \gamma \cdot_1 a$.

Indeed $\tau_n^{n+1}(a) = \gamma \cdot_1 (\gamma \cdot_1 (\cdots (\gamma \cdot_1 a) \cdots)) = a$, by using condition (2). So $\tau_n^{n+1} = \iota$. We now can extend τ_n to $\widetilde{C}^n(A)$ (see Remark 2.3).

Definition 2.7. Define an operator $\widetilde{\tau}_n$ from $\widetilde{C}^n(A)$ to $\widetilde{C}^n(A)$ by

$$\widetilde{\tau}_n(l_0, l_1, \cdots, l_n) = (\tau_n \circ l_n, \tau_n \circ l_0, \tau_n \circ l_1, \cdots, \tau_n \circ l_{n-1}),$$

for all $(l_0, l_1, \cdots, l_n) \in \widetilde{C}^n(A)$.

Using (3) and Lemma 2.5, it is not so hard to see that the range of the map $\widetilde{\tau}_n$ lies in $\widetilde{C}^n(A)$ and that $\widetilde{\tau}_n(a) = \tau_n(a)$ for all $a \in A$. Moreover

$$\widetilde{\tau}_n^{n+1}(l_0, l_1, \cdots, l_n) = (\tau_n^{n+1} \circ l_0, \tau_n^{n+1} \circ l_1, \cdots, \tau_n^{n+1} \circ l_n) = (l_0, l_1, \cdots, l_n),$$

since $\tau_n^{n+1} = \iota$. So $\widetilde{\tau}_n^{n+1} = \iota$.

By Lemma 2.5, one always has to make sure that there exists an operator from $C^n(A)$ to the left multiplier algebra $L(A)$ so that extends the map $\tau_n : A \to A$, denoted again by τ_n. But it is not possible to show that the range of this map is in $C^n(A)$ in general. In fact, it depending on the situation (see Proposition 2.10 below).

Definition 2.8. Define the linear map $\tau_n : C^n(A) \to L(A)$ by

$$\tau_n(z)(a) = \tau_n(z \cdot_n a), \tag{5}$$

when $z \in C^n(A)$ and $a \in A$.

Here we used the map $\tau_n : A \to A$ in the right hand side of (5). Remark that $z \cdot_n a$ belongs to A because $z \in C^n(A)$. So $\tau_n(z \cdot_n a)$ makes sense and it is an element in A. To prove that $\tau_n(z)$ is a left multiplier of A we have

$$\tau_n(z)(ab) = \tau_n(z \cdot_n ab) = \gamma \cdot_1 (z \cdot_n ab) = \gamma \cdot_1 ((z \cdot_n a) \cdot_n b).$$

Since $z \cdot_n a \in A$ we can use Lemma 2.5 and so

$$\tau_n(z)(ab) = (\gamma \cdot_1 (z \cdot_n a))b = \tau_n(z \cdot_n a)b = \tau_n(z)(a)b,$$

as we wanted to prove. Note also that $\tau_n : C^n(A) \to L(A)$ extends the map $\tau_n : A \to A$, again using Lemma 2.5.

Remark 2.9. As explained above, we do not know that the range of the map τ_n is in $C^n(A)$. But, this will follow when

$$\tau_n(z) \cdot_k a = \tau_n(z \cdot_{k-1} a), \tag{6}$$

for all $1 \leq k \leq n$ and all $z \in C^n(A)$, $a \in A$. Of course, this is true for when z, a are both in A, using (3) and (4).

Proposition 2.10. *Suppose τ_n satisfies (6), for all $1 \leq k \leq n$ and all $z \in C^n(A)$, $a \in A$. Then its range is in $C^n(A)$. Moreover $\widetilde{\tau}_n$ extends τ_n. Also $\tau_n^{n+1} = \iota$.*

Remark 2.11. It can be a good problem to make the collection $\{C^n(A)\}_{n\geq 0}$ (or the collection $\{\widetilde{C}^n(A)\}_{n\geq 0}$) into a cyclic module structure. Of course, first maybe the main difficulty is to give a meaning to the space $C^0(A)$. We must also define face and degeneracy operators at the same time. However this is a difficult work.

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On Injectivity of Generalized Hyper S-Acts

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Abstract

In the study of S-acts, numerous significant discoveries have highlighted the relationship between regularity in a monoid S and the injectivity of S-acts, along with related concepts. In this talk, we explore the concept of strongly regular elements within hypermonoids as a generalization of regularity in monoids. We extend several foundational results to both hypermonoids and generalized hyper S-acts, or briefly GHS-acts, over hypermonoids. Specifically, we establish that a monoid S qualifies as a strongly regular and injective GHS-act if and only if all its right ideals are C-injective.

Keywords: Hypermonoid, Generalized hyper S-act, Injectivity, W-Injectivity, PW-Injectivity, C-Injectivity, Strong regularity

Mathematics Subject Classification [2010]: 14L17, 20N20, 20M50

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1 Introduction and Preliminaries

Consider a set S. On S, one can define a binary operation that associates each pair of elements in S with another element within S. Moreover, it is possible to define an operation that maps each pair to a non-empty subset of S. The first case pertains to algebraic structures such as semigroups, monoids and groups, whereas the second case pertains to algebraic hyperstructures

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[*] Speaker.

including hypersemigroups, hypermonoids and hypergroups. The concept of monoid actions is generalized to hyperactions in hyperstructures. Initial discussions of hyperactions on hypermonoids appear in [4,6,7], and hyperactions on monoids were investigated in [3,5]. The study of C-injective S-acts is detailed in [8], while [1] extends C-injectivity to GHS-acts and introduces the concept of semi-injectivity.

Classical results have demonstrated linkages between injectivity (and its variations such as W-injectivity and PW-injectivity) and the regularity of elements within monoids. This note introduces a generalization of regularity, termed strong regularity, and extends numerous classical findings to hypermonoids and GHS-acts.

In the following, we provide some fundamental concepts related to hyperstructures needed in the sequel.

Let H be a semigroup. An element $h \in H$ is called *regular* if there exists an element $x \in H$ such that $h = hxh$. If all elements of a semigroup are regular, then H is called a *regular semigroup*. We note that if $h \in H$ is regular and $h = hxh$ for some $x \in H$, then hx is an idempotent. For an arbitrary set A, we denote the powerset of A by $\mathcal{P}(A)$, and as a convention, we let $\mathcal{P}^*(A) = \mathcal{P}(A) \setminus \{\emptyset\}$. A *hypersemigroup* is a pair $(S, \circ)$ where $\circ : S \times S \to \mathcal{P}^*(S)$ is a map satisfying $a \circ (b \circ c) = (a \circ b) \circ c$ for any $a, b, c \in S$, and for any $S' \subseteq S$ and $s \in S$, we define $S' \circ s = \bigcup_{s' \in S'} s' \circ s$ and $s \circ S' = \bigcup_{s' \in S'} s \circ s'$. If there exists $e \in S$ such that $a \circ e = e \circ a$ and $a \in a \circ e$ for any $a \in S$, then S is called a *hypermonoid*, and e is an *identity element* of S. An identity element e of a hypermonoid S is called a *pure identity* if $a \circ e = \{a\}$ for any $a \in S$. Let S be a hypermonoid and I be a non-empty subset of S. Then, I is called a *right ideal* of S if $I \circ S = \bigcup_{s \in S} I \circ s \subseteq I$. The concept of being regular is generalized for hypersemigroups, and an element h of a hypersemigroup $(H, \circ)$ is regular if $h \in h \circ H \circ h$. An *idempotent* element of a hypersemigroup S is an element $s \in S$ such that $s \circ s = \{s\}$. For basic concepts about S-acts, we recommend [2].

Let $(S, \circ)$ be a hypermonoid and X a non-empty set. Define $* : X \times S \to \mathcal{P}^*(X)$ such that for any $x \in X$ and $s, t \in S$:

1. $x * (s \circ t) = (x * s) * t$,
2. $x \in x * e$,

where e is the identity in S. This structure $(X_S, *)$ is called a *generalized hyper S-act* (GHS-act). If a subset $X' \subseteq X$ satisfies $X' * S \subseteq X'$, then $(X'_S, *)$ forms a GHS-subact. A GHS-act is *pure* if $x * \{e\} = \{x\}$ for all $x \in X$. Homomorphisms between GHS-acts are maps $\phi : X_S \to Y_S$ that satisfy $\phi(x * s) = \phi(x) * s$. A *retraction* is a GHS-homomorphism that is also a right-inverse to the inclusion of a GHS-subact, making the subact a *retract* of the original act. Let X_S be a GHS-act and Y_S be a GHS-subact of X_S. A GHS-homomorphism $f : X_S \to Y_S$ is called a *retraction* if $fi = id_{Y_S}$, where $i : Y_S \hookrightarrow X_S$ is the inclusion map. If there exists a retraction $f : X_S \to Y_S$, then Y_S is called a *retract* of X_S.

Let X_S be a GHS-act. Then X_S is called *injective* if for any one-to-one GHS-homomorphism $i : Y_S \hookrightarrow Z_S$ and any GHS-homomorphism $f : Y_S \to X_S$, there exists a GHS-homomorphism $g : Z_S \to X_S$ commuting the following diagram:

$$\begin{tikzcd}
Y_S \arrow[r, hook] \arrow[d, "f"'] & Z_S \arrow[dl, dotted, "g"] \\
X_S &
\end{tikzcd}$$

Considering the above assumptions, X_S is called *F-injective* (C-injective) if Y_S is finitely generated (cyclic). Further, X_S is called PW-injective if $Y_S = I$ is a cyclic right ideal of S, $Z_S = S$ and i is the inclusion map. Note that any cyclic right ideal of S is of the form $s \circ S$ for some $s \in S$.

2 Main Results

In this section, we delve into the relationships between injectivity and its derived notions in terms of strongly regularity.

First we introduce a new concept extending the notion of regularity in hypersemigroups.

Definition 2.1. Let $(S, \circ)$ be a hypersemigroup. An element $x \in S$ is called *strongly regular* whenever there exists an idempotent element $z \in x \circ S$ for which $z \circ x = \{x\}$, and for any $a \in S$, $z \circ a$ is a singleton.

If any element of S is strongly regular, then we call S a *strongly regular hypersemigroup.* Moreover, if S is a hypermonoid, then it is said to be a *strongly regular hypermonoid.*

For instance, the hypermonoid $S = \{1, a\}$ where $a \circ a = \{1, a\}$ is clearly strongly regular.

Consider a hypersemigroup S. It can be observed that any element s in S that is strongly regular is also regular. Therefore, a hypersemigroup that is strongly regular must also be regular.

Let S be a semigroup. Define $s \circ t = \{s, t, st\}$ for any $s, t \in S$. Then $(S, \circ)$ is a hypersemigroup. The following example shows that the concepts of regularity and strong regularity are actually different in hypersemigroups.

Example 2.2. Consider $\mathbb{N}$ with the above hypersemigroup structure (i.e. $a \circ b = \{a, b, ab\}$ for any $a, b \in \mathbb{N}$). Let $a \in \mathbb{N} \backslash \{1\}$. Then $a \circ 1 \circ a = \{a, 1\} \circ a = a \circ a \cup 1 \circ a = \{a, a^2\} \cup \{1, a\} = \{1, a, a^2\}$ and so $a \in a \circ 1 \circ a \subseteq a \circ \mathbb{N} \circ a$, which means that a is regular. But a is not strongly regular, otherwise, there exists an idempotent $b \in \mathbb{N}$ with $b \in a \circ \mathbb{N}$ and $\{a\} = b \circ a = \{b, a, ab\}$. This follows that $ab = a = b$, and hence $a = 1$ which is a contradiction.

Recall from [2] that an S-act A_S over a monoid S is PW-injective if and only if for any $s \in S$ and any homomorphism $f : sS \to A_S$, there exists $z \in A_S$ with $f(x) = zx$ for any $x \in sS$. In the following, we study this property for GHS-acts where S is a hypermonoid.

Proposition 2.3. *Let S be a hypermonoid. A GHS-act X_S is PW-injective if and only if for any $s \in S$ and any GHS-homomorphism $f : s \circ S \to X_S$, there exists $z \in X_S$ such that $z * a$ is a singleton for any $a \in S$, and $\{f(x)\} = z * x$ for any $x \in s \circ S$.*

Corollary 2.4. *Let S be a hypermonoid and $s \in S$. If $s \circ S$ is PW-injective, then s is a strongly regular element of S.*

The following theorem characterizes all hypermonoids S for which any GHS-act is PW-injective.

Theorem 2.5. *Let S be a hypermonoid. Then the following are equivalent:*

1. *All GHS-acts are PW-injective.*
2. *All right ideals of S are PW-injective.*
3. *All finitely generated right ideals of S are PW-injective.*
4. *All principal right ideals of S are PW-injective.*
5. *S is strongly regular.*

Corollary 2.6. *Let S be a hypermonoid. Then the following are equivalent:*

1. *All principal right ideals of S are injective.*
2. *S is strongly regular and injective.*

A classification of monoids in terms of the C-injectivity or injectivity of a certain class of their right ideals can be found in [8, Theorem 9]. The following result generalizes this result to hypermonoids.

Theorem 2.7. *Let S be a hypermonoid. The following statements are equivalent:*

1. *All right ideals of S are C-injective.*
2. *All finitely generated right ideals of S are C-injective.*
3. *All principal right ideals of S are C-injective.*
4. *All principal right ideals of S are injective.*
5. *All principal right ideals of S are F-injective.*
6. *S is strongly regular and injective.*

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Random Matrices and Some Properties

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Abstract

In this article, we have proposed a new model for a real lifetime, which can be used in topics such as vehicle speed, asphalt, etc. We discuss the distributional properties of this model, and it has been used to the generalization of Nadarajah and Kotz distribution.

Keywords: Real Lifetime, Random Stochastic Matrices, Randomly Weighted Averages, Cauchy Composition Test

Mathematics Subject Classification [2010]: Primary: 13BXX,13FXX,05EXX, Secondary: 13HXX, 05EXX

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1 Introduction

Multiple stochastic matrices have long been the focus of many researchers in applied and theoretical disciplines, and significant results have been published in books and journals, but recently the close relationship between this discussion and average dynamics has been suggested by some authors. This has intensified attention to the issue. They see average dynamics as an essential role in studying changes in distributed systems and algorithms, as well as providing examples. On the other hand, paralleled, in the last few decades, authors have also argued about finding the distribution of a mixed random variable that does not seem to be irrigated to multiplication of random matrices. Example given in these works are lifetime. So the results will undoubtedly be in line with assumptions that are consistent with the examples. In this article, by changing our perspective on the interesting work of Homei and Nadarajah(2018) and accepting their assumptions, we have discussed and obtained the results for multiplying random matrices of aspects which that are very different from their work and those of others.

In this paper $\mathbf{S}$ is a random combination of random matrices and $\overline{\mathbf{S}}$ is randomly weighted averages of random matrices.(for symbols see Homei 2012, Homei 2015, , Homei and Nadarajah 2018, Homei 2021)

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2 Product of Random Stochastic Matrices

Theorem 2.1. *Let* $\mathbf{X}$ *be any random matrix with bounded support and* Y *be independent random variable of* $\mathbf{X}$ *with* $Ga(\sum_{j=1}^{n}\alpha_j)$ *distribution. If*

$$\mathbf{S}\left(\alpha_{\mathbf{1}},\dots,\alpha_{\mathbf{n}};\widetilde{\beta}_{\mathbf{1}},\dots,\widetilde{\beta}_{\mathbf{n}}\right)\overset{\mathrm{d}}{=}Y\mathbf{X},$$

then $\mathbf{X}$ *and* $\overline{\mathbf{S}}\left(\alpha_{\mathbf{1}},\dots,\alpha_{\mathbf{n}};\widetilde{\beta}_{\mathbf{1}},\dots,\widetilde{\beta}_{\mathbf{n}}\right)$ *have identical distribution.*

Proof. At first we define $Y^+=\sum_{i=1}^{n}Y_i$ (and $\alpha^+=\sum_{j=1}^{n}\alpha_j$), which has $Ga(\alpha^+)$ distribution, then by use of one of assumptions we have

$$Y^+.\frac{\sum_{i=1}^{n}Y_i\mathbf{X}_i}{Y^+}\overset{\mathrm{d}}{=}Y\mathbf{X},$$

the fraction $\frac{\sum_{i=1}^{n}Y_i\mathbf{X}_i}{Y^+}$ has the same distribution as $\overline{\mathbf{S}}(\alpha_{\mathbf{1}},\dots,\alpha_{\mathbf{n}};\widetilde{\beta}_{\mathbf{1}},\dots,\widetilde{\beta}_{\mathbf{n}})$, so we can rewrite the mentioned expression in the form of

$$Y^+\overline{\mathbf{S}}\overset{\mathrm{d}}{=}Y\mathbf{X},$$

then both sides have the same moments.

$$E\left((Y^+)^{k_1}S_{11}^{k_{11}}.(Y^+)^{k_{12}}S_{12}^{k_2}\cdots Y^{+k_n}S_{nn}^{k_{nn}}\right)=E\left(Y^{k_{11}}X_{11}^{k_{11}}.Y^{k_{12}}X_{12}^{k_{12}}\cdots Y^{k_{nn}}X_{nn}^{k_{nn}}\right)$$

and so $E\left((Y^+)^{k^+}\right)E\left(S_{11}^{k_{11}}.S_{12}^{k_{12}}\cdots S_{nn}^{k_{nn}}\right) = E(Y^{k^+})E(X_{11}^{k_{11}}.X_{12}^{k_{12}}\cdots X_{nn}^{k_{nn}})$ where $k^+=\sum_{j=1}^{n}\sum_{i=1}^{n}k_{ij}$.

Considering the same distribution of Y^+ and Y, we can omit the first expectations from both sides of the equation

$$E\left((Y^+)^{k^+}\right)E\left(S_{11}^{k_{11}}.S_{12}^{k_{12}}\cdots S_{nn}^{k_{nn}}\right)=E(Y^{k^+})E\left(X_{11}^{k_{11}}.X_{12}^{k_{12}}\cdots X_{nn}^{k_{nn}}\right)$$

As a result of having bounded support variables, the equation of the same moments of two variables conduces to the same distribution, so the proof is completed and $\mathbf{X}$ and $\overline{\mathbf{S}}(\alpha_{\mathbf{1}},\dots,\alpha_{\mathbf{n}};\widetilde{\beta}_{\mathbf{1}},\dots,\widetilde{\beta}_{\mathbf{n}})$ have identical distribution. ☐

Theorem 2.2. *If random coefficient environment* $\mathbf{X}_i$*'s are identically distributed independent random matrices, then components*

$$\mathbf{S}\left(\mathbf{n}\alpha,\dots,\mathbf{n}\alpha;\widetilde{\beta}_{\mathbf{1}},\dots,\widetilde{\beta}_{\mathbf{n}}\right)$$

have independent gamma(α) *distribution if and only if* $\widetilde{\beta}_1=\cdots=\widetilde{\beta}_n=(\alpha,\cdots,\alpha)$.

Proof.

$$\begin{aligned}
E\left(e^{t_1'\mathbf{S}^{(\mathbf{1})}+t_2'\mathbf{S}^{(\mathbf{2})}+\cdots+t_n'\mathbf{S}^{(\mathbf{n})}}\right)&=E\left(e^{t_1'\sum_{i=1}^{n}Y_i\mathbf{X}_{\mathbf{i}}^{(\mathbf{1})}}\right)\cdots E\left(e^{t_n'\sum_{i=1}^{n}Y_i\mathbf{X}_{\mathbf{i}}^{(\mathbf{n})}}\right)\\
&=\prod_{i=1}^{n}E\left(e^{t_1'Y_i\mathbf{X}_{\mathbf{i}}^{(\mathbf{1})}}\right)\prod_{i=1}^{n}\cdots\prod_{i=1}^{n}E\left(e^{t_n'Y_i\mathbf{X}_{\mathbf{i}}^{(\mathbf{n})}}\right)\\
&=\prod_{i=1}^{n}E\left(E\left(e^{t_1'Y_i\mathbf{X}_{\mathbf{i}}^{(\mathbf{1})}}|\mathbf{X}_i\right)\right)\prod_{i=1}^{n}\\
&\quad\cdots\prod_{i=1}^{n}E\left(E\left(e^{t_n'Y_i\mathbf{X}_{\mathbf{i}}^{(\mathbf{n})}}|\mathbf{X}_i\right)\right)\\
&=\prod_{i=1}^{n}E((\frac{1}{1-t_1'\mathbf{x}_{\mathbf{i}}^{(\mathbf{1})}})^{\alpha})\prod_{i=1}^{n}\cdots\prod_{i=1}^{n}E\left(\left(\frac{1}{1-t_n'\mathbf{x}_{\mathbf{i}}^{(\mathbf{n})}}\right)^{\alpha}\right)
\end{aligned}$$

$$= \left(E\left(\left(\frac{1}{1-t_1'\mathbf{x}_{\mathbf{i}}^{(\mathbf{1})}}\right)^{\alpha}\right)\right)^{n} \cdots \left(E\left(\left(\frac{1}{1-t_n'\mathbf{x}_{\mathbf{i}}^{(\mathbf{n})}}\right)^{\alpha}\right)\right)^{n}$$

The last statement is the Stieltjes transformation which is unique, so leads to the prove of both if part and only if part. □

Theorem 2.3. $\overline{\mathbf{S}}\left(\sum_{\mathbf{i=1}}^{\mathbf{k}} \alpha_{\mathbf{i}}^{(\mathbf{1})}, \ldots, \sum_{\mathbf{i=1}}^{\mathbf{k}} \alpha_{\mathbf{i}}^{(\mathbf{n})}; \underline{\alpha}^{(\mathbf{1})}, \cdots, \underline{\alpha}^{(\mathbf{n})}\right)$ *has the*

$$MDirichlet\left(\sum_{j=1}^{n} \alpha_1^{(j)}, \cdots, \sum_{j=1}^{n} \alpha_k^{(j)}\right)$$

matrix distribution, where $\underline{\alpha}^{(j)} = \left\langle \alpha_1^{(j)}, \cdots, \alpha_k^{(j)} \right\rangle$ *(*$j = 1, \cdots, n$*).*

Proof. Let Y_j $(j = 1, \cdots, n)$ be independent random variables independent from $(\mathbf{X}_1, \cdots, \mathbf{X}_n)$ that have the distribution $Ga(\alpha_j)$, respectively. It can be seen, by some classic ways (e.g. $E(e^{t'T}) = [\Psi(t)]^{\sum_j \alpha_j}$, that the distribution of $\mathbf{T} = \sum_j \mathbf{T}_j = \sum_j Y_j \mathbf{X}_j$ is the same distribution of $\mathbf{T}_j$ with the parameter $(\sum_j \alpha_j, \cdots, \sum_j \alpha_j)$. We can also write $\mathbf{T} \overset{\mathrm{d}}{=} Y\mathbf{X}$ in which Y has the gamma distribution with the parameter $(\sum_{i=1}^{k} \alpha_j)$ and $\mathbf{T}$ has the $MDirichlet(\sum \alpha_i, \cdots, \sum \alpha_i)$ distribution, and Y and $\mathbf{X}$ are independent from each other. By using theorem 2.1 the proof is completed.

□

Theorem 2.4. $\overline{\mathbf{S}}(\mathbf{k}\alpha, \ldots, \mathbf{k}\alpha; \widetilde{\beta}_{\mathbf{1}}, \ldots, \widetilde{\beta}_{\mathbf{n}})$ *has* $MDirichlet(n\alpha, \cdots, n\alpha)$ *distribution if and only if* $\widetilde{\beta}_1 = \cdots = \widetilde{\beta}_n = (\alpha, \cdots, \alpha)$*.*

Proof. Supposing the following equality both if part and only if part are proved.

$$\mathbf{T} = \sum_{i=1}^{n} Y_i \mathbf{X}_i = Y^+ . \frac{\sum_{i=1}^{n} Y_i \mathbf{X}_i}{Y^+} = Y^+ \sum_{i=1}^{n} W_i \mathbf{X}_i$$

. □

Theorem 2.5. $\overline{\mathbf{S}}\left(\alpha_{\mathbf{1}}, \ldots, \alpha_{\mathbf{n}}; \frac{\mathbf{1}}{\mathbf{2}} + \underline{\alpha}^{(\mathbf{1})}, \cdots, \frac{\mathbf{1}}{\mathbf{2}} + \underline{\alpha}^{(\mathbf{n})}\right)$ *has the* $MDirichlet(\frac{1}{2} + \sum_{i=1}^{n} \alpha_i, ..., \frac{1}{2} + \sum_{i=1}^{n} \alpha_i)$ *matrix distribution, where* $\underline{\alpha}^{(j)} = \langle \alpha_j, \cdots, \alpha_j \rangle$ *(*$j = 1, \cdots, n$*).*

Proof. Let Y_j $(j = 1, \cdots, n)$ be independent random variables independent from $(\mathbf{X}_1, \cdots, \mathbf{X}_n)$ that have the distribution $Ga(\frac{1}{2} + \alpha_j)$, respectively. The distribution of $\mathbf{T} = \sum_j \mathbf{T}_j = \sum_j Y_j \mathbf{X}_j$ is the same distribution of $\mathbf{T}_j$ with the parameter $(\frac{1}{2} + \sum_j \alpha_j, \cdots, \frac{1}{2} + \sum_j \alpha_j)$. We have $\mathbf{T} = Y\mathbf{X}$ in which Y has the gamma distribution with the parameter $(\frac{1}{2} + \sum_{i=1}^{k} \alpha_j)$ and $\mathbf{T}$ has the $Dirichlet(\frac{1}{2} + \sum \alpha_i, \cdots, \frac{1}{2} + \sum \alpha_i)$ distribution, and Y and $\mathbf{X}$ are independent from each other. □

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Characterization of Identities With Four Object Variables In Quasigroups

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Abstract

In this note, the set of some identities with four object variables of an arbitrary fixed set Q are investigated. We reprove a theorem of V.D.Belousov with new technique. From these results we obtain new identities that characterized quasigroups isotopic to groups and when added to a quasigroup form a group.

Keywords: Quasigroups, Primitive quasigroup, Group, Balanced identity, Variety, Isotope

Mathematics Subject Classification [2010]: 20N05

1 Introduction and Preliminaries

V. D. Belousov's papers, Etta Falconer, Jaroslav Ježek and Tomaš Kepka, Yu. M. Movsisyan, and many other authors have articles in this matter. An ultimate goal, the determination of all such varieties, seem at the moment far from being completed. Although the description of all identities that characterize quasigroups which are isotopic to groups is an open problem. In particular, the possibility of existence of identities with three variables which characterize quasigroups isotopic to groups until this time is unknown. In this paper we specialize on characterizing quasigroups which are isotopic to groups, with four object variables.

First recall some definitions from quasigroup theory.

Definition 1.1. A quasigroup $(Q, \cdot)$ is a set Q with a binary operation $\cdot : Q \times Q \to Q$, denoted by juxtaposition, such that for each $a, b \in Q$, the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x, y \in Q$.

2 Main Results

Theorem 2.1. *The quasigroup $Q(\cdot)$ is an isotope of group $G(*)$ if and only if one of the following identities holds for $Q(\cdot)$*

(a) $x\,[y \setminus \{[(yy)/z]\,u\}] = [\{x\,[y \setminus (yy)]\}\,/z]\,u$

(b) $x\,[y \setminus \{[(yz)/y]\,u\}] = [\{x\,[y \setminus (yz)]\}\,/y]\,u$

(c) $x\,[z \setminus \{[(yy)/y]\,u\}] = [\{x\,[z \setminus (yy)]\}\,/y]\,u$

Consequently the variety of quasigroups isotopic to groups characterized by mentioned four-variable identities.
Finally we will have the following theorem that is showed identities with four object variable when added to a quasigroup form a group.
Let's proof a theorem that is useful for proof of next theorems.

Theorem 2.2. *Every one of the identities:*

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

$$x \cdot (y/z) = (x \cdot y)/z$$

$$x \cdot (y \setminus z) = (x/y) \cdot z$$

$$x \setminus (y \cdot z) = (x \setminus y) \cdot z$$

$$x \setminus (y/z) = (x \setminus y)/z$$

when added to a quasigroup form a group.

The following theorems are analogical results of this theorem .

Theorem 2.3. *Every one of the identities:*

$$(x \cdot y) \cdot (z \cdot u) = ((x \cdot y) \cdot z)) \cdot u$$

$$(x \setminus y) \cdot (z \cdot u) = ((x \setminus y) \cdot z) \cdot u$$

$$(x/y) \cdot (z \cdot u) = ((x/y) \cdot z) \cdot u$$

$$x \cdot ((y \cdot z) \cdot u) = (x \cdot (y \cdot z)) \cdot u$$

$$x \cdot ((y \setminus z) \cdot u) = (x \cdot (y \setminus z)) \cdot u$$

$$x \cdot ((y/z) \cdot u) = (x \cdot (y/z)) \cdot u$$

$$x \cdot (y \cdot (z \cdot u)) = (x \cdot y) \cdot (z \cdot u)$$

$$x \cdot (y \cdot (z \setminus u)) = (x \cdot y) \cdot (z \setminus u)$$

$$x \cdot (y \cdot (z/u)) = (x \cdot y) \cdot (z/u)$$

when added to a quasigroup form a group.

Proof: It is sufficient let $A(x, y) = w$ or $B(y, z) = v$ or $C(z, u) = t$, where $A, B, C \in \Sigma = \{\cdot, /, \setminus\}$. Therefor we will have associative law.

Theorem 2.4. *Every one of the identities:*

$$(x \cdot y) \cdot (z \setminus u) = ((x \cdot y)/z)) \cdot u$$

$$(x/y) \cdot (z \setminus u) = ((x/y)/z) \cdot u$$

$$(x \setminus y) \cdot (z \setminus u) = ((x \setminus y)/z) \cdot u$$

$$x \cdot ((y \cdot z) \setminus u) = (x/(y \cdot z)) \cdot u$$

$$x \cdot ((y \setminus z) \setminus u) = (x/(y \setminus z)) \cdot u$$

$$x \cdot ((y/z) \setminus u) = (x/(y/z)) \cdot u$$

$$x \cdot (y \setminus (z \cdot u)) = (x/y) \cdot (z \cdot u)$$

$$x \cdot (y \setminus (z \setminus u)) = (x/y) \cdot (z \setminus u)$$

$$x \cdot (y \setminus (z/u)) = (x/y) \cdot (z/u)$$

when added to a quasigroup form a group.

Theorem 2.5. *Every one of the identities:*

$$(x \cdot y) \cdot (z/u) = ((x \cdot y) \cdot z))/u$$

$$(x/y) \cdot (z/u) = ((x/y) \cdot z)/u$$

$$(x \setminus y) \cdot (z/u) = ((x \setminus y) \cdot z)/u$$

$$x \cdot ((y \cdot z)/u) = (x \cdot (y \cdot z))/u$$

$$x \cdot ((y/z)/u) = (x \cdot (y/z))/u$$

$$x \cdot ((y \setminus z)/u) = (x \cdot (y \setminus z))/u$$

$$x \cdot (y/(z \cdot u)) = (x \cdot y)/(z \cdot u)$$

$$x \cdot (y/(z \setminus u)) = (x \cdot y)/(z \setminus u)$$

$$x \cdot (y/(z/u)) = (x \cdot y)/(z/u)$$

when added to a quasigroup form a group.

Theorem 2.6. *Every one of the identities:*

$$(x \cdot y) \setminus (z \cdot u) = ((x \cdot y) \setminus z)) \cdot u$$

$$(x/y) \setminus (z \cdot u) = ((x/y) \setminus z) \cdot u$$

$$(x \setminus y) \setminus (z \cdot u) = ((x \setminus y) \setminus z) \cdot u$$

$$x \setminus ((y \cdot z) \cdot u) = (x \setminus (y \cdot z)) \cdot u$$

$$x \setminus ((y/z) \cdot u) = (x \setminus (y/z)) \cdot u$$

$$x \setminus ((y \setminus z) \cdot u) = (x \setminus (y \setminus z)) \cdot u$$

$$x \setminus (y \cdot (z \cdot u)) = (x \setminus y) \cdot (z \cdot u)$$

$$x \setminus (y \cdot (z \setminus u)) = (x \setminus y) \cdot (z \setminus u)$$

$$x \setminus (y \cdot (z/u)) = (x \setminus y) \cdot (z/u)$$

when added to a quasigroup form a group.

Theorem 2.7. *Every one of the identities:*

$$(x \cdot y) \setminus (z/u) = ((x \cdot y) \setminus z))/u$$

$$(x/y) \setminus (z/u) = ((x/y) \setminus z)/u$$

$$(x \setminus y) \setminus (z/u) = ((x \setminus y) \setminus z)/u$$

$$x \setminus ((y \cdot z)/u) = (x \setminus (y \cdot z))/u$$

$$x \setminus ((y/z)/u) = (x \setminus (y/z))/u$$

$$x \setminus ((y \setminus z)/u) = (x \setminus (y \setminus z))/u$$

$$x \setminus (y/(z \cdot u)) = (x \setminus y)/(z \cdot u)$$

$$x \setminus (y/(z \setminus u)) = (x \setminus y)/(z \setminus u)$$

$$x \setminus (y/(z/u)) = (x \setminus y)/(z/u)$$

when added to a quasigroup form a group.

3 Coclusion

In this paper we characteraized quasigroup identities with four object variable, without repetation, isotope closure to groups. Althouh one can, useing isotope, investigate other identities that are isotopic to groups by means of belousov's theorem and our proof of this theorem .

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Exploring Group Theory with First-Order Logic

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Abstract

First-order logic, a powerful tool of formal logic, provides a lens to analyze mathematical structures and unlock their properties. In this note, we investigate groups within the framework of first-order logic.

Keywords: First-order language, Algebraic structure, First-order theory

Mathematics Subject Classification [2010]: Primary: 03G27

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1 Introduction

First-order logic provides a powerful framework for reasoning about objects, their properties, and relationships. However, sometimes we want to go beyond basic logical statements and explore more complex structures with specific rules. This logic system allows us to build complex statements using variables, quantifiers (like ”for all” and ”there exists”), logical operators (AND, OR, NOT), and predicates, functions and constants [2].

In the vast landscape of mathematics, we encounter systems governed by well-defined operations. These operations, like addition and multiplication, combine elements to produce new ones, forming the core rules of the system. Enter algebraic structures, a captivating realm where we delve into these operations and their properties [3]. Algebraic structures are focusing on:
Set of Objects: It could be the set of integers for arithmetic operations or vectors for linear algebra.
Operations: These are the actions performed on the objects, defining the rules of interaction. Familiar examples include addition $+$, multiplication $\times$, or taking inverses -1.
On other words algebraic structures focus on operations and their properties.

First-order logic acts as a bridge, allowing us to translate the abstract world of mathematical structures into a precise symbolic language. By focusing on operations and their properties, algebraic structures become a particularly elegant example of this translation. As we delve deeper into this fascinating intersection, we unlock a powerful toolkit for analyzing and comprehending the intricate worlds of mathematics.

The connection between these two areas arises when we use first-order logic to describe the properties and relationships within an algebraic structure. Here are some key aspects to understand:
Language: We define a specific first-order language with symbols representing the elements, operations, and relations within the chosen algebraic structure.
Interpretation: We assign specific meanings to these symbols within a particular structure. For example, the addition symbol ”+” in a group might be interpreted as the actual addition operation on real numbers.

Axioms: We express the defining properties of the algebraic structure using first-order formulas. These formulas become the "axioms" that the structure must satisfy.

By studying algebraic structures in first-order logic, we gain several advantages:

a) We can precisely capture the essence of an algebraic structure using a formal language.

b) We can use the tools of first-order logic to prove theorems about the structure, demonstrating relationships between its elements and operations.

c) We can analyze different types of algebraic structures using a unified framework within first-order logic.

In this paper, we discuss the application of first-order logic in group theory. This paper is outlined as follows. In section 2 we provides an introduction to preliminaries about first-order logic and in section 3 we give complete description of ordered Abelian group theory in first-order logic.

2 The Foundation of Formalization

In this section, we introduce the key concepts of first-order logic [1,2,4].

Definition 2.1 (First-order language)**.** A first-order language $\mathcal{L}$ is given by following data:

1. A set of function symbols $\mathcal{F}$ and positive integers n_f for each $f \in \mathcal{F}$.
2. A set of relation symbols $\mathcal{R}$ and positive integers n_R for each $R \in \mathcal{R}$.
3. A set of constants symbols $\mathcal{C}$.
4. Variable symbols $v, w, v_1, v_2, \cdots$, the Boolean connectives $\wedge, \vee, \neg$, the quantifier $\forall$ and $\exists$, and parantheses (and).

In first-order logic, we use symbols to construct abstract objects mainly formulas and sentences.

The set of terms is the smallest set that includes all variables, all constant symbols, and any expression of the form $f(t_1, \ldots, t_{n_f})$, where f is a function symbol and $t_1, \ldots, t_{n_f}$ are terms themselves.

We say that ϕ is atomic formula if it is either of form $t_1 = t_2$ for two terms t_1, t_2 or it is of form $R(t_1, \ldots, t_{n_R})$ where R is relation symbol and $t_1, \ldots, t_{n_R}$ are terms.

Definition 2.2. The set of formulas is the smallest set $\mathcal{W}$ containing the atomic formulas such that

1. if ϕ is in $\mathcal{W}$, then $\neg\phi$, $\exists w\phi$ and $\forall w\phi$ are in $\mathcal{W}$.
2. if ϕ and ψ are in $\mathcal{W}$, then $\phi \wedge \psi$ and $\phi \vee \psi$ are in $\mathcal{W}$.

A variable, denoted by w, is considered free in a formula, denoted by ϕ, if it's not within the scope of an existential quantifier $\exists w$ or a universal quantifier $\forall w$. Otherwise, it's considered bound. A formula that lacks free variables is called a sentence.

For example, the formula $\exists v(\forall w(v = 1 + w))$ is a sentence because the variable 'v' is bound by the outer existential quantifier and 'w' is bound by the inner universal quantifier. In contrast, $\exists v(v = 1 + w)$ is not a sentence since 'v' remains free. Finally, a collection of sentences is referred to as a theory.

For each first-order language $\mathcal{L}$, we can define different interpretations as $\mathcal{L}$-structures.

Definition 2.3. An $\mathcal{L}$-structure is given by the following data:

1. A nonempty set M is called universe of structure.
2. A function $f^{\mathcal{M}} : M^{n_f} \to M$ for each function symbol $f \in \mathcal{L}$.
3. A set $R^{\mathcal{M}} \subseteq M^{n_R}$ for each relatin symbol $R \in \mathcal{L}$.
4. An element $c^{\mathcal{M}}$ for each constant $c \in \mathcal{L}$.

This structure is denoted by $\mathcal{M} = (M, (f^{\mathcal{M}} : f \in \mathcal{F}), (R^{\mathcal{M}} : R \in \mathcal{R}), (c^{\mathcal{M}} : c \in \mathcal{C}))$.

For each $\mathcal{L}$-structure $\mathcal{M}$, the set of all sentences that are true in $\mathcal{M}$ is called the full theory of $\mathcal{M}$ and is denoted by $Th(\mathcal{M})$. Finding an axiom system T for $Th(\mathcal{M})$ is an important subject in this field. We describe an axiom system for $Th(\mathbb{Q}, +, -, <, 0)$ in the section 3.

3 Theory of Groups in First-Order Logic

To explore the group theory using first-order logic, we must first establish a suitable language to describe and analyze groups. This language will serve as the foundation for expressing axioms, formulating theorems, and delving into the intricacies of group structures.

- A language of group is $\mathcal{L} = \{+, -, 0\}$ that $+$ is a 2-ary function symbol, $-$ is 1-ary function symbol.
- The set of terms in group language is the smallest set of expressions that contains variables and the constant symbol 0 and is closed under the function symbols $+$ and $-$. For example, $v + w$, $-v + 0$ are terms atomic formula. Atomic formulas in group language are simply equations between terms. An example of an atomic formula is $v + w = 0$. The set of formulas in group language is built upon atomic formulas. It is closed under the logical connectives $\wedge$, $\vee$ and $\neg$, existential quantifier $\exists$ and universal quantifier $\forall$. For eample $\forall v \exists w(v + w = 0 \wedge v = v - w)$ and $\forall v(v + v = w)$ are formulas.
- $\mathcal{M}_G = (M, +^{\mathcal{M}}, -^{\mathcal{M}}, 0^{\mathcal{M}})$ is an $\mathcal{L}_G$-structure where:
  - * M is a nonempty set.
  - * $+^{\mathcal{M}} : M \times M \to M$ is a function representing addition.
  - * $-^{\mathcal{M}} : M \to M$ is a function representing the inverse operation.
  - * $0^{\mathcal{M}} \in M$ is an element of M.

By considering $+^{\mathcal{N}}(m, n) = m + n$, $-^{\mathcal{N}}(m) = 1 + m$ and $0^{\mathcal{N}} = 0$, $\mathcal{N}_G = (\mathbb{N}, +^{\mathcal{N}}, -^{\mathcal{N}}, 0^{\mathcal{N}})$ is an $\mathcal{L}_G$-structure. Also $\mathcal{Z}_G = (\mathbb{Z}, +, -, 0)$ is an $\mathcal{L}_G$- structure. Not every $\mathcal{L}_G$-structure becomes a group. We will now introduce axioms (sentences in group language) that, if true within an $\mathcal{L}_G$-structure, guarantee that structure is a group:

1. Associativity Axiom: $\sigma_1 = \forall x \forall y \forall z(x + (y + z) = (x + y) + z)$
2. Identity Axiom: $\sigma_2 = \forall x(x + 0 = 0 + x = x)$
3. Inverse Axiom: $\sigma_3 = \forall x(x - x = 0 = -x + x)$

Consider a theory T_G consisting of axioms σ_1,σ_2, and σ_3.
Soundness and Completeness theorem(informally): Any structure that satisfies all the axioms in T_G is guaranteed to be a group (Soundness). Conversely, every group can be interpreted as a model of T_G (Completeness).
Universal Theory and Subgroups: T_G is a universal theory. This means that if $\mathcal{M}_G$ is a model of T_G and $\mathcal{N}_G$ is a substructure of $\mathcal{M}_G$ (i.e., a subset inheriting the relevant structure from $\mathcal{M}_G$), then $\mathcal{N}_G$ must also be a model of T_G. In simpler terms, if a larger structure satisfies the axioms of T_G, then any of its substructures automatically inherit those properties and become groups themselves. This can be seen as an abstract criterion for identifying subgroups within a group.
Incompleteness of T_G: Because $(\mathbb{Z}, +, -, 0)$ and $(\mathbb{Z}_2, +, -, 0)$ are the models of T_G and

$$(\mathbb{Z}_2, +, -, 0) \models \exists x(x \neq 0 \wedge x + x = 0)$$
$$(\mathbb{Z}, +, -, 0) \models \neg(\exists x(x \neq 0 \wedge x + x = 0))$$

So $T_G \not\models \exists x(x \neq 0 \wedge x + x = 0)$ and $T_G \not\models \neg(\exists x(x \neq 0 \wedge x + x = 0))$. Hence it is not complete.

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A Graph of Left Submodules

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Abstract

In this Paper, we introduce a graph $\Gamma_+(M)$, whose vertices are the nonzero proper left submodules of a left R-module M and two vertices N and K are adjacent if $N + K = M$. We investigate some properties of the graph $\Gamma_+(M)$. For example, the diameter, girth, clique number and chromatic number of the graph $\Gamma_+(M)$ are studied, when M is a finitely generated left R-module.

Keywords: Left submodules, Finitely generated modules, Clique number, Chromatic number.

Mathematics Subject Classification [2010]: Primary: 05C07, 05C10, 05C38, 16D80

1 Introduction and Preliminaries

Throughout this paper, all rings are associative with unity, and all modules are unital.
In 1988, Istvan Beck [5] addressed the issue of a graph with the algebraic structure ring and introduced the zero divisor graph of a commutative ring. Later on, this introduction was slightly modified by Anderson and Naseer in 1993 [3]. The concept of the zerodivisor graph was further modified. Many authors studied the zero divisor graph in the sense of Anderson and Livingston as in 1999 [2]. Since then, the concept of the zero divisor graph of a ring has played a vital role in its expansion. Motivating from this well-expanded idea of Beck, several correspondences of a graph with algebraic structures have been introduced with a various applications. The total graph of a commutative ring by Anderson and Badawi in 2008 [4], and the intersection graph of ideals of a ring by Chakrabarty et al. [6].

Recently, this subject has received a good deal of attention from several authors assigning a graph to a ring or a group and then studying the algebraic properties of these objects via their associated graphs [1].

In the following, we present some definitions, notations, and results about elementary graph theory. By a *graph* $G = (V, E)$, the authors mean a nonempty set V and a symmetric binary relation (possibly empty) E on V. The set V is called *the set of vertices*, and E is called *the set of edges* of G. Two elements x and y in V are said to be *adjacent* if $(x, y) \in E$ and we write $x \sim y$. Here, $H = (W, F)$ is called a *subgraph* of G if H itself is a graph and $\emptyset \neq W \subseteq V$ and $F \subseteq E$. If all the vertices of G are pairwise adjacent, then G is said to be *complete*. A complete subgraph of a graph G is called a clique. A maximal clique is a clique and is maximal with respect to inclusion. The *clique number* of G, written as $\omega(G)$, is the maximum size of a clique in G. The *chromatic number* of G, denoted as $\chi(G)$, is the minimum number of colors needed to label the vertices so that the adjacent vertices receive different colors. A *path* of length k in a graph is an alternating sequence of vertices and edges, $x_0, e_0, x_1, e_1, x_2, \ldots, x_{n-1}, e_{n-1}, x_n$, where x_i's are distinct (except possibly the first and last vertices) and e_i is the edge joining x_i and x_{i+1}. The authors call this a path joining x_0 and x_k. A *cycle* is a path with $x_0 = x_k$. A cycle of length 3 is called a *triangle*. A graph is *connected* if there exists a path joining x and y for any pair of vertices $x, y \in V$. A graph is said to be *triangulated* if for any vertex x in V, there exist y, z in V, such that $x \sim y \sim z$ is a triangle. Two vertices x and y of graph G are *orthogonal*, written $x \perp y$, if x and y are adjacent and there is no vertex z of G that is adjacent to both x and y. A graph G is called complemented if, for each vertex x of G, there is a vertex y of G (called a *complement* of x) such that $x \perp y$. For vertices x and y of G, $d(x, y)$ is the length of the shortest path from x to y. If there is no path, then $d(x, y) = \infty$. Then, the *diameter* of G is $diam(G) := sup\{d(x, y) \mid x \text{ and } y \text{ are vertices of } G\}$. The *girth* of G, denoted by $gr(G)$, is the length of the shortest cycle in G ($gr(G) = \infty$ if G contains no cycles) [7].

In the paper, our main objective is to establish connections between module theoretic properties and associated graph's properties. For a left R-module M, we associate a graph $\Gamma_+(M)$ whose vertices are nonzero proper left submodules of M and two vertices in $\Gamma_+(M)$ are adjacent if they are comaximal. We find $\omega(\Gamma_+(M))$ and $\chi(\Gamma_+(M))$ and see that they are equal to the cardinal number of the set of maximal left submodules of M (Proposition 2.6). We show that under which conditions the graph $\Gamma_+(M)$ is a forest or a tree (Theorem 2.7).

2 Main Results

Definition 2.1. Let M be a left R-module and Γ be the family of all proper nonzero left submodules of M. On Γ we define the following adjacency and denote the resulting graph by $\Gamma_+(M)$. Two vertices N and K are adjacent if $N + K = M$, that is, they are comaximal left submodules.

Recall that a left submodule N of a left R-module M is said to be left *small* if for a left submodule K of M, $N + K = M$ implies $K = M$. It is well-known that $Rad(M)$, is small in M. In fact, $Rad(M)$ is the sum of all left small submodules of M. The following preposition is easily deduced.

Proposition 2.2. *Let M be a left R-module. Then the following statements hold.*

1) *A nonzero proper left submodule N of M is left small if and only if N is an isolated vertex in $\Gamma_+(M)$.*
2) *Let M be a finitely generated left R-module. Then M has only one maximal left submodule if and only if $\Gamma_+(M)$ has no edges.*

Proposition 2.3. *Let M be a left R-module, N and K be two nonzero proper left submodules of M such that $N \subseteq K$. Then $deg(N) \leq deg(K)$.*

Proof. Suppose that N and K be two nonzero proper left submodules of M such that $N \subseteq K$. If for arbitrary left submodule L of M, $N + L = M$, then $K + L = M$. Therefore $deg(N) \leq deg(K)$. □

Lemma 2.4. *Let M be a finitely generated left R-module with at least two nonzero proper left submodules. Then $\Gamma_+(M)$ is connected if and only if $Rad(M) = 0$.*

Proof. Assume that $Rad(M) = 0$ and $N, K \in \Gamma_+(M)$. Then there are maximal left submodules $\mathcal{L}$ and $\mathcal{L}'$ such that $N \nsubseteq \mathcal{L}$ and $K \nsubseteq \mathcal{L}'$. Either $\mathcal{L} = \mathcal{L}'$ or $\mathcal{L} \neq \mathcal{L}'$. In either case, there exists a path from N to K in $\Gamma_+(M)$. Hence N and K are connected. Conversely, assume that $\Gamma_+(M)$ is connected. If $Rad(M) \neq 0$, then $Rad(M)$ cannot be adjacent to any other vertex, a contradiction. □

Corollary 2.5. *Let M be a finitely generated left R-module with $Rad(M) = 0$. Then $diam(\Gamma_+(M)) \leqslant 3$.*

Proof. It follows from the proof of Lemma 2.4. □

Proposition 2.6. *Let M be a finitely generated left R-module. Then the following statements hold.*

1) $\omega(\Gamma_+(M)) = \chi(\Gamma_+(M))$ *is equal to the number of maximal left submodules of* M.

2) $gr(\Gamma_+(M)) = 3$ *except when* M *has at most two maximal left submodules.*

Proof. (1) Suppose that H be a complete subgraph of $\Gamma_+(M)$. Hence choose a maximal left submodule $\mathcal{L}_N$ with $N \subseteq \mathcal{L}_N$, for any vertice N of H. Since $N + K = M$, $\mathcal{L}_N + \mathcal{L}_K = M$ for any distinct vertices N and K of H. Hence $\mathcal{L}_N \neq \mathcal{L}_K$. Thus the subgraph of $\Gamma_+(M)$ induced by $\{\mathcal{L}_N \mid N \text{ is a vertex of } H\}$ is a complete graph having as many vertices as H. Now, since the set of all maximal left submodules of M is a complete subgraph of $\Gamma_+(M)$, $\omega(\Gamma_+(M))$ is the cardinal number of the set of maximal left submodules of M. Now, let
$\{\mathcal{L}_i | i \in I\}$ be the set of all maximal left submodules of M and suppose that $<$ is a well ordering on I. Let $H_i = \{N \leq M | 0 \neq N \subseteq \mathcal{L}_i \text{ and } N \notin \bigcup_{j<i} H_j\}$ for each $i \in I$. Then $\mathcal{L}_i \in H_i$ for each $i \in I$, and so $H_i \neq \emptyset$. Also $\{H_i | i \in I\}$ form a partition for the set of vertices of $\Gamma_+(M)$. Since any two vertices in H_i are not adjacent for every $i \in I$, all the vertices in H_i can have the same color. However, the $\mathcal{L}_i$'s must have different colors. Therefore $\chi(\Gamma_+(M)) = |I|$.
(2) If $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ are three distinct maximal left submodules of M, then they are the vertices of a triangle in $\Gamma_+(M)$. □

Theorem 2.7. *Let* M *be a finitely generated left* R*-module. Then* $\Gamma_+(M)$ *is a forest (tree) if and only if either* M *has exactly one maximal left submodule or* M *has exactly two maximal left submodules such that all non-maximal left submodules of* M *are contained in only one of them.*

Proof. Assume that M has exactly one maximal left submodule. Hence $\Gamma_+(M)$ is a graph with no edges. Therefore $\Gamma_+(M)$ is a forest. Now, suppose that M has only two maximal left submodules $\mathcal{L}_1$ and $\mathcal{L}_2$ such that all non-maximal left submodules of M are contained either in $\mathcal{L}_1$ or in $\mathcal{L}_2$, then $\Gamma_+(M)$ is a forest. In this case $\Gamma_+(M)$ is the union of a star graph and some isolated vertices. Conversely, Let M be a module with exactly two maximal left submodules $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\Gamma_+(M)$ is a forest. If there exist left submoules $N_1 \subset \mathcal{L}_1$ with $N_1 \nsubseteq \mathcal{L}_2$ and $N_2 \subset \mathcal{L}_2$ with $N_2 \nsubseteq \mathcal{L}_1$, then we have the cycle $N_1 \sim N_2 \sim \mathcal{L}_1 \sim \mathcal{L}_2 \sim N_1$ in $\Gamma_+(M)$. Then non-maximal left submodules of M are all contained either in $\mathcal{L}_1$ or in $\mathcal{L}_2$. □

Example 2.8. Let M be a finitely generated left R-module. It is clear that if $\Gamma_+(M)$ is a forest, then M has at most two maximal left submodules. However, $\mathbb{Z}_4 \times \mathbb{Z}_4$-module $\mathbb{Z}_4 \times \mathbb{Z}_4$ shows that even for a module with exactly two maximal submodules, $\Gamma_+(M)$ is not necessarily a forest.

Definition 2.9. Let M be a left R-module. Then two vertices N and K are orthogonal in $\Gamma_+(M)$ if we have $N + K = M$, and for any vertex $H \in \Gamma_+(M)$ either $N + H \neq M$ or $K + H \neq M$, in this cace we write $N \perp K$.

Theorem 2.10. *Let* M *be a finitely generated left* R*-module. Then the following statements are equivalent.*

(1) $\Gamma_+(M)$ *has no triangle.*

(2) Any two adjacent left submodules are orthogonal.

(3) M *has at most two maximal left submodules.*

Proof. (1) $\Longrightarrow$ (2) Assume that N and K be two adjacent vertices that are not orthogonal. Then there exists another vertex H which is adjacent to both N and K, by the definition of orthogonality, a contradiction.
(2) $\Longrightarrow$ (3) If there exist at least three maximal left submodules, then they cannot be orthogonal to each other.
(3) $\Longrightarrow$ (1) Suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ are the only two maximal left submodules of M. Then for any three vertices N, K and H of $\Gamma_+(M)$, at least two of them are contained in either $\mathcal{L}_1$ or $\mathcal{L}_2$, and hence they are not adjacent. Therefore there is no triangle in $\Gamma_+(M)$. □

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Some Notes on n-Submodules

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Abstract

The present study investigates the properties of n-submodules. The results demonstrate that a necessary and sufficient condition for a proper submodule is an n-submodule. In a certain condition, it was determined that there exist n-submodules of $\mathrm{Hom}_R(\_,\_)$.

Keywords: n-Submodule, Secondary module

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1 Introduction and Preliminaries

In this paper, R denotes a commutative ring with identity, and all modules are assumed to be unitary.

Let M be an R-module, and N a submodule of M. It is then evident that $(N :_R M) = \{r \in R | rM \subseteq N\}$ is an ideal of R. If $r \in R$ and $x \in M$, with $rx \in N$ implying $r \in \sqrt{(N :_R M)}$ or $x \in N$, then a proper submodule N of a R-module M is said to be primary. The dual notion of primary R-modules, secondary R-modules, was first proposed by MacDonald (see [3]). If the homomorphism $f_a : M \to M$, which works by multiplication, is either surjective or nilpotent for each $a \in R$, then M, a non-zero R-module, is secondary. In [4], the concepts of n-ideals and n-submodules were initially introduced.

Let M and L be R-modules. Given that N is an n-submodule of M, it can be shown in Theorem 2.9 that an n-submodule of $\mathrm{Hom}_R(M : L)$ can be constructed using an appropriate method. It is shown in Theorem 2.11, let M be a secondary R-module and L be an R-module. The zero submodule of $\mathrm{Hom}_R(M, L)$ is an n-submodule.

2 n-Submodule

Definition 2.1. [4] Let M be an R-module. A proper submodule N of M is called to be an n-submodule if, for any $a \in R$ and $m \in M$, the condition that $am \in N$ and $a \notin \sqrt{\mathrm{Ann}_R(M)}$ implies that $m \in N$ is satisfied.

Lemma 2.2. *Let M be an R-module, and let N be a proper submodule of M. Then the following statements are equivalent:*

(1) N is an n-submodule.

(2) For every $a \in R$, the map $f_a : \frac{M}{N} \to \frac{M}{N}$ defined by $f_a(x+N) = ax+N$ for every $x+N \in \frac{M}{N}$ is either injective or nilpotent. Furthermore, we have that $\sqrt{(N :_R M)} = \sqrt{Ann_R(M)}$.

Proof. (1) $\Longrightarrow$ (2) Since N is an n-submodule, it follows from Proposition 3.2 in [1] that $(N :_R M) \subseteq \sqrt{Ann_R(M)}$. Since $Ann_R(M) \subseteq (N :_R M)$, it implies that $\sqrt{(N :_R M)} = \sqrt{Ann_R(M)}$. Let $a \in R$ and $f_a(x+N) = 0$. Then, $ax \in N$. Consequently, either $x \in N$ or $a \in \sqrt{Ann_R(M)}$. Therefore, f_a is either injective or nilpotent.
(2) $\Longrightarrow$ (1) Let $a \in R$, $x \in M$ be such that $ax \in N$. If f_a is injective, then $x \in N$. If f_a is nilpotent, then $f_a^k(x+N) = a^k x + N = N$, $a^k \in \sqrt{(N :_R M)}$, and $a \in \sqrt{Ann_R(M)}$. Therefore N is an n-submodule. □

It can be demonstrated that a proper submodule N of an R-module M is primary if and only if, for every $a \in R$, the multiplication-based homomorphism $f_a : M/N \to M/N$ is either injective or nilpotent. Consequently, every n-submodule is a primary submodule. In the following example, we will demonstrate that there exists a primary submodule such that it is not an n-submodule.

Example 2.3. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}$. It can be demonstrated that $8\mathbb{Z}$ is a primary submodule of M. However, it is not an n-submodule.

The following are examples of n-submodules.

Example 2.4. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{16}$ as R-module. It can be shown that $\langle \bar{2} \rangle$, $\langle \bar{4} \rangle$ and $\langle \bar{8} \rangle$ are n-submodules of M.

Example 2.5. It can be demonstrated that every proper subspace of a vector space V is an n-submodule.

The following example illustrates the existence of modules that lack n-submodules.

Example 2.6. We demonstrate that the quotient group $\frac{\mathbb{Q}}{\mathbb{Z}}$ as a $\mathbb{Z}$-module lacks an n-submodule. It is clear that $\sqrt{Ann_{\mathbb{Z}}(\frac{\mathbb{Q}}{\mathbb{Z}})} = 0$. Let $\frac{L}{\mathbb{Z}}$ be a proper submodule of $\frac{\mathbb{Q}}{\mathbb{Z}}$. Therefore, there exists an element of the form $\frac{a}{b} + \mathbb{Z}$ in the quotient group $\frac{\mathbb{Q}}{\mathbb{Z}}$ that is not in the submodule $\frac{L}{\mathbb{Z}}$. Consequently, $b \notin \sqrt{Ann_{\mathbb{Z}}(\frac{\mathbb{Q}}{\mathbb{Z}})} = 0$. To illustrate this, consider the function $f_b : \frac{\mathbb{Q}}{L} \to \frac{\mathbb{Q}}{L}$ be a function such that $f_b(x+L) = bx + L$. Then, $f_b(\frac{a}{b} + L) = 0$ and $\frac{a}{b} + L \neq 0$. Consequently, f_b is not injective.

Definition 2.7 ([2]). Let M be a module over a commutative ring R. A proper submodule N of M is said to be a generalization of an n-submodule ($G.n$-submodule) if the condition $am \in N$ implies that $m \in N$ or $a \in \sqrt{\mathrm{Ann}_R(N)}$ for any $a \in R$ and $m \in M$.

The subsequent example illustrates a $G.n$-submodule that is not an n-submodule.

Example 2.8. In $\mathbb{Z}_8 \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module, $\langle \bar{4} \rangle \oplus \langle 0 \rangle$ is a $G.n$-submodule but not an n-submodule.

In the following theorems, we will demonstrate that n-submodules may be found in $\mathrm{Hom}_R(\_,\_)$.

Theorem 2.9. *Let R be a commutative ring and M, L be R-modules. Let N be an n-submodule of M and let*

$$\hat{N} = \{f \in Hom_R(L, M) | f(L) \subseteq N\}.$$

If $\hat{N}$ is a proper submodule of $Hom_R(L, M)$, then $\hat{N}$ is an n-submodule of $Hom_R(L, M)$.

Proof. Let $a \in R$ and $f \in \mathrm{Hom}_R(L, M)$ be such that $af \in \hat{N}$. Therefore, $af(l) \in \hat{N}$ for all $l \in L$. Since N is an n-submodule of M and $af(l) \in N$, it follows that either $a \in \sqrt{\mathrm{Ann}_R(M)}$ or $f(l) \in N$. Consequently, a is an element of the radical of the annihilator of the module $\mathrm{Hom}_R(N, M)$, or $f \in \hat{N}$. Therefore, $\hat{N}$ is an n-submodule of $\mathrm{Hom}_R(L, M)$. □

Proposition 2.10. *Let M, L be R-modules and N be a $G.n$-submodule of M. Let*

$$\hat{N} = \{f \in Hom_R(L, M) | f(L) \subseteq N\}.$$

If $\hat{N}$ is a proper submodule of $Hom_R(L, M)$, then $\hat{N}$ is a $G.n$-submodule of $Hom_R(L, M)$.

Theorem 2.11. *Let M be a secondary R-module and L be an R-module. Then the zero submodule of $Hom_R(M, L)$ is an n-submodule.*

Proof. Let $af = 0$. Then, $af(x) = 0$ for all $x \in M$. Consequently, $f(ax) = 0$. Since M is a secondary R-module, there exists an integer k such that either $a^kM = 0$ or $aM = M$. If $a \in \sqrt{\mathrm{Ann}_R(M)}$, then $a \in \sqrt{\mathrm{Ann}_R(\mathrm{Hom}_R(M, L))}$. If $aM = M$, then for $m \in M$, there exists $x \in M$ such that $m = ax$. Therefore, $f(m) = f(ax) = 0$. This implies that $f = 0$, which in turn implies that $\langle 0 \rangle$ is an n-submodule of $\mathrm{Hom}_R(M, L)$. □

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نقش زیرگروه‌های نرمال و زیرگروه‌های مشخصه در ناوردایی آماری

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چکیده. ناوردایی یکی از شاخه‌های استنباط آماری است که در آن از نظریه گروه‌ها استفاده می‌شود. در این مقاله کاربردهای زیرگروه‌های نرمال و مشخصه در یک مدل ناوردای آماری به ویژه در یافتن آماره‌های ماکسیمال و بسنده ساختاری ضعیف مطرح می‌شود.

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۱. مقدمه

نظریه گروه‌ها یکی از شاخه‌های مهم ریاضیات است که در رشته‌های گوناگون علوم از جمله فیزیک، شیمی، اقتصاد، مکانیک و کوانتوم کاربرد دارد. نظریه گروه‌ها در برخی شاخه‌های آمار و احتمال به‌ویژه در استنباط آماری مورد استفاده قرار می‌گیرد. به عنوان نمونه کاربرد نظریه گروه‌ها در مسئله برآورد به‌ویژه محاسبه بهترین برآوردگرهای هموردای ضعیف با کمترین مخاطره در [۴] و کاربرد آن در بسندگی ساختاری ضعیف در [۳] مورد تحلیل و بررسی قرار گرفته شده است. در این بین در برخی مدل‌های ناوردا تحت یک گروه توپولوژیکی که روی فضای متغیر تصادفی عمل می‌کند، زیرگروه‌های نرمال و زیرگروه‌های مشخصه آن گروه در رسیدن به اهداف استنباط آماری در مسئله برآورد مفید هستند که در این مقاله به ذکر چند مورد از کاربردهای این زیرگروه‌ها در استنباط آماری اشاره می‌شود. در حالتی که عمل گروه به‌طور یکتا انتقالی باشد، به‌راحتی تابع هموردای ضعیف به برآوردگر تبدیل می‌شود و به کمک آن آماره ناوردای ماکسیمال پیدا خواهد شد. در حالتی که گروه به‌طور یکتا انتقالی نیست ولی شامل یک زیرگروه با این خاصیت است که مشخصه نیز هست می‌توان این روش را به کار برد. ابتدا چند مفهوم مقدماتی تعریف می‌شوند.

زیرگروه $G \leq N$ را یک زیرگروه نرمال G گویند و می‌نویسند $N \trianglelefteq G$، هرگاه برای هر $a \in G$، $a^{-۱}Na = N$. به عنوان مثال، گروه خطی ویژه از درجه n، شامل تمام ماتریس‌های وارون‌پذیر $n \times n$ با دترمینان ۱، یعنی $Sl_n = \{A \in Gl_n : \det(A) = ۱\}$ زیرگروه نرمال گروه خطی عمومی از درجه n شامل تمام ماتریس‌های وارون‌پذیر $n \times n$، یعنی Gl_n است. به‌عبارت دیگر $Sl_n \trianglelefteq Gl_n$.
زیرگروه $K \leq G$ را یک زیرگروه مشخصه از G گویند هرگاه به ازای هر $\alpha \in Aut(G)$، $\alpha(K) \subseteq K$ که در آن $Aut(G)$ مجموعه تمام خودریختی‌های گروه G است. اما چون برای هر $\alpha \in Aut(G)$، $\alpha^{-۱} \in Aut(G)$، در این صورت می‌توان گفت زیرگروه $K \leq G$ یک زیرگروه مشخصه از G است، هرگاه به ازای هر $\alpha \in Aut(G)$، $\alpha(K) = K$. ثابت می‌شود هر زیرگروه مشخصه یک زیرگروه نرمال است ولی عکس آن درست نیست. همچنین زیرگروه مشتق G یعنی $G' = \{xyx^{-۱}y^{-۱} : x, y \in G\}$ یک زیرگروه مشخصه از G است. اگر G یک گروه باشد و تابع $xy^{-۱} \to (x, y)$ از $G \times G$ به G پیوسته باشد، G را یک گروه توپولوژیکی نامند. گروه G روی مجموعه ناتهی $\mathcal{X}$ عمل می‌کند، هرگاه یک تابع $G \times \mathcal{X} \to \mathcal{X}$ (که تصویر (g, x) تحت این تابع را با gx نشان می‌دهند)

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وجود داشته باشد به‌قسمی که به ازای هر $x \in \mathcal{X}$ و هر $a, b \in G$، $(ab)x = a(bx)$ و $ex = x$. در این حالت $\mathcal{X}$ را G-فضا می‌نامند. اگر گروه G روی $\mathcal{X}$ عمل کند، رده‌های هم‌ارزی را مدارهای عمل G روی $\mathcal{X}$ گویند، یعنی $Gx = \{gx : g \in G\}$. گردایه تمام مدارها را فضای خارج‌قسمت گویند و با $G/\mathcal{X} = \{Gx : x \in \mathcal{X}\}$ نمایش می‌دهند. اگر $\mathcal{X}$ یک G-فضا باشد، زیرگروه $G_x = \{g : gx = x\}$ پایدارساز G در x نامیده می‌شود. در ضمن عمل G روی $\mathcal{X}$ آزاد نامند، هرگاه برای هر $x \in \mathcal{X}$، $G_x = \{e\}$ و آن را انتقالی نامند، هرگاه برای هر $x, x' \in \mathcal{X}$ یک $g \in G$ وجود دارد به‌طوری که $x' = gx$ و اگر $g \in G$ منحصر به فرد وجود داشته باشد، عمل G روی $\mathcal{X}$ را یکتا انتقالی گویند [۱]. فرض کنید $\mathcal{X}$ و $\mathcal{Y}$ دو G-فضا باشند. تابع $f : \mathcal{X} \to \mathcal{Y}$ را ناوردا گویند اگر برای هر $g \in G$ و $x \in \mathcal{X}$، $f(gx) = f(x)$. تابع ناوردای f را ناوردای ماکسیمال گویند، هرگاه $f(x_1) = f(x_2)$ نتیجه دهد برای یک $g \in G$، $x_1 = gx_2$ [۲]. تابع اندازه‌پذیر $f : \mathcal{X} \to \mathcal{Y}$ را G-هم‌وردای ضعیف گویند، هرگاه یک خودریختی $\alpha_f \in Aut(G)$ وجود داشته باشد به‌طوری که برای هر $g \in G$ و $x \in \mathcal{X}$، $f(gx) = \alpha_f(g)f(x)$ [۴].

در بخش دوم و سوم مقاله، به‌ترتیب کاربردهای زیرگروه‌های نرمال و زیرگروه‌های مشخصه در یافتن آماره‌های ماکسیمال و همچنین آماره‌های بسنده ساختاری ضعیف مطرح می‌شود.

۲. کاربرد زیرگروه‌های نرمال و مشخصه در یافتن آماره‌های ناوردای ماکسیمال

در این بخش با محدود کردن روی گروه‌های به‌طور یکتا انتقالی، فضای پارامتر را می‌توان به عنوان یک گروه با عمل دوتایی جدید در نظر گرفت. بر این اساس توابع هم‌وردای ضعیف را می‌توان به برآوردگرهای G-هم‌وردای ضعیف تبدیل کرد و از این طریق می‌توان توابع ناوردای ماکسیمال را توسط این برآوردگرها پیدا کرد. مزیت این است که چون در آمار معمول است که به جای توابع هم‌وردای ضعیف از برآوردگرهای هم‌وردای ضعیف استفاده شود، در حالت گروه به‌طور یکتا انتقالی پیدا کردن این توابع ساده‌تر خواهد بود. در حالت خاص که گروه شامل یک زیرگروه به‌طور یکتا انتقالی و مشخصه است مسأله بررسی خواهد شد. بعد از آن یک راه ساده برای یافتن آماره ناوردای ماکسیمال بر اساس برآوردگرهای G-هم‌وردای ضعیف ارائه می‌شود.

مثال ۱.۲. برای گروه آبلی G و تابع G-هم‌وردای $\tau : \mathcal{X} \to G$ با خودریختی $\alpha \in Aut(G)$، تابع $f : \mathcal{X} \to \mathcal{X}$ با ضابطه $f(x) = (\tau(x))^{-1}x$ در نظر گرفته می‌شود. در این صورت برای $g \in G$ و $x \in \mathcal{X}$، $f(gx) = (\tau(x))^{-1}(\alpha(g))^{-1}gx = \beta(g)f(x)$ که در آن $\beta(g) = (\alpha(g))^{-1}g$. اما f هم‌وردای ضعیف نیست، زیرا در حالت کلی $\beta \notin Aut(G)$. در حالت خاص اگر τ، G-هم‌وردا باشد (یعنی $\alpha(g) = g$)، در این صورت $\beta(g) = e$ و توسط گزاره ۳.۲، f ناوردای ماکسیمال است. همچنین اگر τ، G-ناوردا باشد $(\alpha(g) = e)$،در این صورت $\beta = 1_G \in Aut(G)$ و بنابراین f، G-هم‌ورداست.

مثال ۲.۲. اگر عمل $H \trianglelefteq G$ روی $\mathcal{X}$ بدیهی باشد و $\tau : \mathcal{X} \to G$ برای هر $g \in G$ و $x \in \mathcal{X}$ و یک $h \in H$ در تساوی $\tau g(x) = h^{-1}gh\tau(x)$ صدق کند، $f(x) = (\tau(x))^{-1}x$ ناوردای ماکسیمال است.

از دو مثال اخیر، ایده ساختن محاسبه آماره ناوردای ماکسیمال پیدا می‌شود.

گزاره ۳.۲. ([۴]). اگر برای تابع هم‌وردای ضعیف $\tau : \mathcal{X} \to G$ که $\tau(gx) = \alpha(g)\tau(x)$ و برای هر $g \in G$ و $x \in \mathcal{X}$ و یک $\alpha \in Aut(G)$، $\beta(g) = (\alpha(g))^{-1}g \in G_x$ وجود داشته باشد، در این صورت $f(x) = (\tau(x))^{-1}x$ ناوردای ماکسیمال است.

در مثال ۲.۲ برای هر $g \in G$ و $x \in \mathcal{X}$ و برخی مقادیر $h, h' \in H = H_x$،

$$\beta(g) = (\alpha(g))^{-1}g = h^{-1}g^{-1}hg = h^{-1}g^{-1}gh' = h^{-1}h' \in G_x$$

و بنابراین طبق گزاره ۳.۲، $f(x) = (\tau(x))^{-1}x$ ناوردای ماکسیمال است.

زمانی که گروه G روی G-فضای Θ به‌طور یکتا انتقالی عمل کند می‌توان G را توسط Θ اندیس‌گذاری کرد. می‌توان یک نقطه پایه دلخواه $\theta_۰ \in \Theta$ و یک خودریختی $\alpha \in Aut(G)$ را در نظر گرفت و هر عضو $\theta \in \Theta$ را به‌طور یکتا به صورت $\theta = \alpha(g_\theta)\theta_۰$ نوشت. به وضوح e متناظر با $\theta_۰$ خواهد بود. از این که G روی Θ به‌طور یکتا انتقالی است، G روی Θ به‌طور آزاد عمل می‌کند و لذا تساوی $\alpha(g_{h\theta})\theta_۰ = h\theta = h\alpha(g_\theta)\theta_۰$ نتیجه می‌دهد که برای هر $\theta \in \Theta$ و $h \in G$، $\alpha(g_{h\theta}) = h\alpha(g_\theta)$. همچنین $\eta_\alpha : \Theta \to G$ با ضابطه $\eta_\alpha(\theta) = \alpha(g)$ یک تابع دوسویی G-هم‌ورداست و به‌طور مشابه $\eta(\theta) = g_\theta = \alpha^{-۱}(\eta_\alpha(\theta))$ یک تابع دوسویی G-هم‌وردای ضعیف با $\alpha^{-۱} \in Aut(G)$ خواهد بود. قضیه زیر در حالتی که گروه به‌طور یکتا انتقالی نیست ولی شامل یک زیرگروه با این خاصیت است که مشخصه نیز هست، برای یافتن آماره ناوردای ماکسیمال مورد استفاده قرار می‌گیرد.

قضیه ۴.۲. ([۴]). اگر G شامل یک زیرگروه مشخصه H که روی Θ به‌طور یکتا انتقالی است باشد و برای هر $h_۰ \in H$، $\alpha(h_۰) = h_۰$. همچنین یک تابع $G_{\theta_۰}$-هم‌وردای ضعیف $\tau_۰ : \Theta \to G_{\theta_۰}$ با $\alpha \in Aut(G)$ وجود داشته باشد به‌طوری که برای هر $g_۰ \in G_{\theta_۰}$ و $\theta \in \Theta$، $(\alpha(g_۰))^{-۱}g_۰ \in G_\theta$. علاوه بر آن $\tau : \Theta \to \Theta$ یک تابع $G_{\theta_۰}$-هم‌وردا باشد که τ روی هر مدار $H\theta$ یک‌به‌یک باشد. در این صورت $f'(\theta) = (\tau_۰ \circ \tau(h_\theta^{-۱}\theta))^{-۱}\tau(h_\theta^{-۱}\theta)$ روی Θ ناوردای ماکسیمال است که h_θ عضو یکتای H است که برای ثابت $\theta_۰ \in \Theta$ و $\alpha \in Aut(G)$، $\alpha(h_\theta)\theta_۰ = \theta$.

۳. کاربرد زیرگروه‌های نرمال و مشخصه در بسندگی ساختاری ضعیف

برآوردگر $t : \mathcal{X} \to \Theta$ را یک MLE برای θ گویند، هرگاه $\sup_{\theta\in\Theta} f_\theta(\mathbf{x}) = f_t(\mathbf{x})$ که در آن $\mathbf{x} = (x_۱, \cdots, x_n)$ [۲]. اگر t یک MLE یکتا باشد، G-هم‌وردا نیز هست. فرض کنید گروه توپولوژیکی G روی فضای نمونه $\mathcal{X}$ و فضای پارامتر Θ عمل می‌کند. با تعریف تصویر کانونی $\pi : \mathcal{X} \to G/\mathcal{X}$، $\pi(x) = Gx$ و عمل قطری G روی $\mathcal{X} \times \Theta$، به صورت $g(x,\theta) = (gx, g\theta)$ می‌توان تصویر کانونی جدید $\tilde{\pi} : \mathcal{X} \times \Theta \to G/(\mathcal{X} \times \Theta)$ را مطرح کرد. با توجه به فرض‌های بالا، زوج (t,π) را بسنده ساختاری ضعیف گویند، هرگاه برای هر $\theta \in \Theta$ تابع $\tilde{\pi}_\theta : \mathcal{X} \to G/(\mathcal{X} \times \Theta)$ با ضابطه $\tilde{\pi}_\theta(x) = \tilde{\pi}(x,\theta)$ یک تابع از (t,π) باشد. لازم به ذکر است که اگر (t,π) بسنده ساختاری ضعیف باشد، آن‌گاه بسنده نیز هست. بنابراین بسندگی ساختاری ضعیف یک تعمیم از بسندگی معمول است [۳].

قضیه ۱.۳. ([۳]). اگر G روی Θ به طور انتقالی عمل کند و تابع $t : \mathcal{X} \to \Theta$ هم‌وردای ضعیف باشد به طوری که برای هر $g \in G$ و $\theta \in \Theta$، $(\alpha_t(g))^{-۱}g \in G_\theta$ و G_θها نرمال باشند، آن‌گاه (t,π) بسندگی ساختاری ضعیف است. اگر G روی $\mathcal{X}$ به طور آزاد عمل کند و t پوشا باشد، آن‌گاه شرط لازم وکافی برای بسندگی ساختاری (t,π)، نرمال بودن G_θهاست.

مثال ۲.۳. ([۳]). اگر G روی Θ به طور انتقالی عمل کند و برای $\theta \in \Theta$، $G_\theta \lhd G$ و تابع $t : \mathcal{X} \to \Theta$ برای هر $g \in G$ و $x \in \mathcal{X}$ و یک $h \in G_\theta$، $t(gx) = h^{-۱}ght(x)$ وجود داشته باشد، آن‌گاه برای هر $\theta \in \Theta$، $g \in G$ و برخی مقادیر $h, h' \in G_\theta$،

$$(\alpha_t(g))^{-۱}g = h^{-۱}g^{-۱}hg = h^{-۱}g^{-۱}gh' = h^{-۱}h' \in G_\theta$$

و بنابراین با توجه به قضیه (۱.۳)، (t,π) بسنده ساختاری ضعیف است.

اگر G روی Θ به طور یکتا انتقالی عمل کند و برای هر $\theta \in \Theta$، $G_\theta = [G,G]$، آن‌گاه (t,π) برای تمام توابع هم‌وردای $t : \mathcal{X} \to G$ بسنده ساختاری ضعیف است [۳]. دو فضای $\mathcal{X}$ و Θ دارای نوع مدار یکسان هستند، هرگاه G_xها و G_θها با هم مزدوج باشند، یعنی برای هر $\theta \in \Theta$ و $x \in \mathcal{X}$ و یک $g \in G$، $G_x = gG_\theta g^{-۱}$. همچنین $\mathcal{X}$ را یک TT-فضا گویند، هرگاه $\mathcal{X}$ همسان‌ریخت با فضای حاصل‌ضرب $\mathcal{X}_\infty \times \mathcal{X}_\in$ باشد به گونه‌ای که G روی $\mathcal{X}_\infty$ به طور بدیهی عمل کند و G روی $\mathcal{X}_\in$ به طور انتقالی عمل کند.

نتیجه ۳.۳. ([۳]). فرض کنید G_θها نرمال باشند و $\mathcal{X} \cong \mathcal{X}_\infty \times \mathcal{X}_\in$ یک TT-فضا باشد و یک تابع هموردای ضعیف دوسویی $\tau : \mathcal{X}_\in \to \Theta$ وجود داشته باشد به قسمی که برای هر $g \in G$ و $\theta \in \Theta$، $(\alpha_t(g))^{-۱}g \in G_\theta$، در این صورت $\mathcal{X}$ و Θ دارای نوع مدار یکسان‌اند.

نتیجه ۴.۳. ([۳]). اگر $\mathcal{X}$ و Θ دارای نوع مدار یکسان باشند و G روی Θ به طور انتقالی عمل کند و $t : \mathcal{X} \to \Theta$ یک تابع هموردای ضعیف باشد، آنگاه اگر G_θها نرمال بوده و برای هر $g \in G$ و $x \in \mathcal{X}$، $(\alpha_t(g))^{-۱}g \in G_x$، آنگاه (t, π) بسنده ساختاری ضعیف است.

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Some Results on the Relations Between Cohomological Dimensions

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Abstract

Let R be a commutative Noetherian ring, I, J be two ideals of R, and M, N be two R-modules. We study the cohomological dimension of N, M with respect to (I, J), denoted by $\operatorname{cd}(I, J, N, M)$, and we get some relations between this invariant and other types of cohomological dimensions.

Keywords: Cohomological dimension, Local cohomology

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1 Introduction and Preliminaries

Throughout this paper, R is a commutative Noetherian ring with a non-zero identity, I, J are two ideals of R, and M, N are two R-modules.

The local cohomology theory has been an significant tool in commutative Algebra and Algebraic Geometry. There are some generalizations of the this theory. For any integer i, Herzog [5] introduced the i-th generalized local cohomology functor $H^i_I(-,-)$ as

$$H^i_I(N, M) = \varinjlim_{t \in \mathbb{N}} \operatorname{Ext}^i_R(N/I^tN, M)$$

for all R-modules N and M. If $N = R$, then $H^i_I(N, -) = H^i_I(-)$.

Another generalization of local cohomology theory has been given by Takahashi et al. [8]. Let $\mathrm{W}(I, J) = \{\mathfrak{p} \in \operatorname{Spec}(R) : I^t \subseteq J + \mathfrak{p} \text{ for some positive integer } t\}$. The set of elements x of M such that $\operatorname{Supp}(Rx) \subseteq \mathrm{W}(I, J)$, is denoted by $\Gamma_{I,J}(M)$. Let $\tilde{\mathrm{W}}(I, J) = \{\mathfrak{a} \leq R : I^t \subseteq J + \mathfrak{a} \text{ for some positive integer } t\}$. Then $x \in \Gamma_{I,J}(M)$ if and only if $\operatorname{Ann}_R(x) \in \tilde{\mathrm{W}}(I, J)$. It is easy to see that $\Gamma_{I,J}(M)$ is a submodule of M, and $\Gamma_{I,J}(-)$ is a covariant, R-linear functor from the category of R-modules to itself. For any non-negative integer i, the local cohomology functor $H^i_{I,J}(-)$ with respect to (I, J), is defined to be the i-th right derived functor of $\Gamma_{I,J}(-)$. If $J = 0$, then $H^i_{I,J}(-) = H^i_I(-)$.

Nam et al. [7] introduced a common generalization of these theories as follows. The module $\Gamma_{I,J}(\operatorname{Hom}_R(N, M))$ is denoted by $\Gamma_{I,J}(N, M)$. It is easy to see that $\Gamma_{I,J}(N, -)$ is a left exact, covariant functor from the category of R-modules to itself. For an integer i, the i-th generalized local cohomology functor with respect to (I, J), denoted by $H^i_{I,J}(N, -)$, is defined as the i-th right derived functor of $\Gamma_{I,J}(N, -)$. If $N = R$, then $H^i_{I,J}(N, -) \cong H^i_{I,J}(-)$. It is easy to see that

$$H^i_{I,J}(N, M) \cong \varinjlim_{\mathfrak{a} \in \tilde{\mathrm{W}}(I,J)} \operatorname{Ext}^i_R(N/\mathfrak{a}N, M).$$

It follows that if $J = 0$, then $H^i_{I,J}(N, -) \cong H^i_I(N, -)$. When N is finitely generated, then $H^i_{I,J}(N, M)$ is just the i-th generalized local cohomology of N and M relative to (I, J), which is defined by Zamani [10]. Note that the generalized local cohomology with respect to a pair of ideals, is a special case of the generalized local cohomology with respect to a system of ideals, which was introduced by Bijan-Zadeh [2].

The notion of cohomological dimension of M with respect to I, denoted by $\operatorname{cd}(I, M)$, is defined as the supremum of all non-negative integers i for which $H^i_I(M) \neq 0$. Amjadi and Naghipour [1] introduced the cohomological dimension of two modules N, M with respect to an ideal I as $\operatorname{cd}(I, N, M) = \sup\{i \in \mathbb{N}_0 : H^i_I(N, M) \neq 0\}$. It is clear that if $N = R$, then $\operatorname{cd}(I, N, M) = \operatorname{cd}(I, M)$. As another generalization of this concept, Chu and Wang [4] introduced the cohomological dimension of M with respect to a pair of ideals (I, J), denoted by $\operatorname{cd}(I, J, M)$, equal to the supremum of all non-negative integers i for which $H^i_{I,J}(M) \neq 0$. It is obvious that if $J = 0$, then $\operatorname{cd}(I, J, M) = \operatorname{cd}(I, M)$. As a generalization of all these notions, the cohomological dimension of N, M with respect to (I, J) is defined as $\operatorname{cd}(I, J, N, M) = \sup\{i \in \mathbb{N}_0 : H^i_{I,J}(N, M) \neq 0\}$. It is clear that if $J = 0$, then $\operatorname{cd}(I, J, N, M) = \operatorname{cd}(I, N, M)$, and if $N = R$, then $\operatorname{cd}(I, J, N, M) = \operatorname{cd}(I, J, M)$. In this paper, we study the cohomological dimension of two modules with respect to a pair of ideals, and we obtain some relations between this invariant and other types of cohomological dimensions.

2 Main Results

Recall that R is a Noetherian ring, I, J are two ideals of R, and M, N are two R-modules.

Definition 2.1. Cohomological dimension of N, M with respect to (I, J), denoted by $\operatorname{cd}(I, J, N, M)$, is defined as

$$\operatorname{cd}(I, J, N, M) = \sup\{i \in \mathbb{N}_0 : H^i_{I,J}(N, M) \neq 0\}.$$

If $J = 0$, then $\operatorname{cd}(I, J, N, M) = \operatorname{cd}(I, N, M)$, which was defined in [1]. If $N = R$, then $\operatorname{cd}(I, J, N, M) = \operatorname{cd}(I, J, M)$, which was defined in [4].

In the following result, we get some basic properties of this invariant.

Proposition 2.2. *Let I', J' be two ideals of R.*

(i) $\operatorname{cd}(I, J, N, M) = \operatorname{cd}(\sqrt{I}, J, N, M) = \operatorname{cd}(I, \sqrt{J}, N, M)$.

(ii) $\operatorname{cd}(II', J, N, M) = \operatorname{cd}(I \cap I', J, N, M)$.

(iii) $\operatorname{cd}(I, JJ', N, M) = \operatorname{cd}(I, J \cap J', N, M)$.

(iv) If $J' \subseteq J$, *then* $\operatorname{cd}(I + J', J, N, M) = \operatorname{cd}(I, J, N, M)$. *In particular,* $\operatorname{cd}(I + J, J, N, M) = \operatorname{cd}(I, J, N, M)$.

Now, we study the relations between the cohomological dimension of two modules with respect to a pair of ideals, and the cohomological dimension of a module with respect to a pair of ideals.

Proposition 2.3. *Let I', J' be two ideals of R, M be a finitely generated R-module, and let N be a finitely generated R-module of finite projective dimension.*

(i) $\operatorname{cd}(I + I', J, N, M) \leq \operatorname{cd}(I, J, N, M) + \operatorname{cd}(I', J, M)$.

(ii) $\operatorname{cd}(I, J \cap J', N, M) \leq \operatorname{cd}(I, J, N, M) + \operatorname{cd}(I, J', M)$.

Note that $\operatorname{Min}(J)$ denotes the set of the minimal elements of $\operatorname{V}(J) = \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq J\}$.

Corollary 2.4. *Let M be a finitely generated R-module, and let N be a finitely generated R-module of finite projective dimension. If $\mathfrak{q} \in \operatorname{Min}(J)$, then*

$$\operatorname{cd}(I, J, N, M) \leq \operatorname{cd}(I, \mathfrak{q}, N, M) + \sum_{\mathfrak{p} \in \operatorname{Min}(J) - \{\mathfrak{q}\}} \operatorname{cd}(I, \mathfrak{p}, M).$$

Lemma 2.5. *If N is a projective R-module, then* $\operatorname{cd}(I, J, N, M) = \operatorname{cd}(I, J, M)$.

Theorem 2.6. *Let R be a local ring, and let N be a finitely generated R-module of finite projective dimension. Then* $\operatorname{cd}(I,J,N,M) \leq \operatorname{pd}(N) + \operatorname{cd}(I,J,M)$.

In the next result, we get some upper bounds for cohomological dimensions.

Proposition 2.7. *Let N be a finitely generated R-module of finite projective dimension.*

(i) $\operatorname{cd}(I,J,N,M) \leq \operatorname{pd}(N) + \operatorname{ara}(I\bar{R})$, *where* $\bar{R} = R/\sqrt{J+\operatorname{Ann}(M)}$.

(ii) $\operatorname{cd}(I,J,N,M) \leq \operatorname{pd}(N) + \operatorname{ara}(I)$.

(iii) $\operatorname{cd}(I,J,N,M) \leq \operatorname{pd}(N) + \dim(M)$.

(iv) If M is a finitely generated R-module, then

$$\operatorname{cd}(I,J,N,M) \leq \operatorname{pd}(N) + \dim(M/JM) + 1.$$

(v) If M is a finitely generated R-module of finite Krull dimension, then

$$\operatorname{cd}(I,J,N,M) \leq \operatorname{pd}(N) + \dim(M \otimes_R N).$$

Corollary 2.8. *Let I', J' be two ideals of R, and let N be a finitely generated R-module of finite projective dimension. Set* $\bar{R} = R/\sqrt{J+\operatorname{Ann}(M)}$.

(i) $\operatorname{cd}(I+I',J,N,M) \leq \operatorname{pd}(N) + \operatorname{ara}(I\bar{R}) + \operatorname{cd}(I',J,M)$.

(ii) $\operatorname{cd}(I,J\cap J',N,M) \leq \operatorname{pd}(N) + \operatorname{ara}(I\bar{R}) + \operatorname{cd}(I,J',M)$.

Corollary 2.9. *Let J' be an ideal of R, and N be a finitely generated R-module of finite projective dimension. Then*

$$\operatorname{cd}(I,J\cap J',N,M) \leq \operatorname{pd}(N) + \dim(R/(J+\operatorname{Ann}(M))) + 1 + \operatorname{cd}(I,J',M).$$

Proposition 2.10. *Let R be a local ring, and let N be a finitely generated R-module of finite projective dimension.*

(i) If M is a finitely generated R-module, and $J \neq R$, then

$$\operatorname{cd}(I,J,N,M) \leq \operatorname{pd}(N) + \dim(M/JM).$$

(ii) $\operatorname{cd}(I,J,N,M) \leq \operatorname{pd}(N) + \dim(R/J)$.

Corollary 2.11. *Let R be a local ring, J' be an ideal of R, and let N be a finitely generated R-module of finite projective dimension. If $J \neq R$, then*

$$\operatorname{cd}(I,J\cap J',N,M) \leq \operatorname{pd}(N) + \dim(R/(J+\operatorname{Ann}(M))) + \operatorname{cd}(I,J',M).$$

Finally, we get some formulas on the relations between the cohomological dimension of two modules with respect to a pair of ideals, and the cohomological dimension of two modules with respect to an ideal.

Lemma 2.12. *For each non-negative integer i,*

$$H^i_{I,J}(N,M) \cong \varinjlim_{\mathfrak{a}\in\tilde{\mathrm{W}}(I,J)} H^i_{\mathfrak{a}}(N/\mathfrak{a}N,M).$$

Proposition 2.13. *(i)* $\operatorname{cd}(I,J,N,M) \leq \sup\{\operatorname{cd}(\mathfrak{a},N,M) : \mathfrak{a}\in\tilde{\mathrm{W}}(I,J)\}$.

(ii) $\operatorname{cd}(I,J,N,M) \leq \sup\{\operatorname{cd}(\mathfrak{a},N/\mathfrak{a}N,M) : \mathfrak{a}\in\tilde{\mathrm{W}}(I,J)\}$.

(iii) If N is a projective R-module, then

$$\operatorname{cd}(I,J,N,M) \leq \sup\{\operatorname{cd}(\mathfrak{a},R/\mathfrak{a},\operatorname{Hom}_R(N,M)) : \mathfrak{a}\in\tilde{\mathrm{W}}(I,J)\}.$$

(iv) $\operatorname{cd}(I,J,M) \leq \sup\{\operatorname{cd}(\mathfrak{a},R/\mathfrak{a},M) : \mathfrak{a}\in\tilde{\mathrm{W}}(I,J)\}$.

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A Route to Quantum Computing Through the Theory of Quantum Graphs

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Abstract

Based on our previous works, and in order to relate them with the theory of quantum graphs and the quantum computing principles, we once again try to introduce some newly developed technical structures just by relying on our toy example, i.e. the coordinate ring of $n \times n$ quantum matrix algebra $M_q(n)$, and the associated directed locally finite graphs $\mathcal{G}(\Pi_n)$, and the Cuntz-Krieger C^*-graph algebras. Meaningly, we introduce a $(4i-6)$-quantum entangled system by using the Cuntz-Krieger $\mathcal{G}(\Pi_i)$-families associated to the $4i-6$ distinct Hamiltonian paths of $\mathcal{G}(\Pi_i)$, for $i \in \{2, \dots, n\}$.

Keywords: Quantum computing, Quantum information theory, Quantum state, Quantum system, Qubit, Quantum graph, Cuntz-Krieger C^*-graph algebra, Quantum permutation group, Hamiltonian path.

Mathematics Subject Classification [2010]: Primary 46L05, 68Q09; Secondary 46L67, 81P40, 46L55, 17B81, 81P45.

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1 Introduction and Preliminaries

The field of quantum error correction is dealing with protecting fragile quantum information from the unexpected errors in an effort to build a super efficient quantum computer.

In February 2019, Adrian Chapman was looking some new approaches in order to tackle a long-standing search for the Holy Grail of quantum error correction. A way of encoding quantum information that is resistant to errors by constructions and doesn't require active correction.

In order to get a little bit closer to the above explained wondering, our idea and our proposed approach begins by a question by Alain Connes, asking if "there are quantum permutation groups, and what would be they look like?".

In late nineties, Shuzhou Wang came with an answer, saying that "the quantum permutation group S_n^+ could be defined as the largest compact quantum group acting on the set $\{1, \dots, N\}$".

---

[*] Speaker.

His approach was by looking at it as the compact set $X_N := \{x_1, \dots, x_N\}$ consisting of a finite set of points (pointwise isomorphic), and studying its function space

$$C(X_N) \cong C^* \left(p_1, \dots, p_N \text{ projections} \middle| \sum_{i=1}^{N} p_i = 1 \right).$$

Finally, this has led him to define something like

$$C(S_n^+) := \left(u_{ij} | u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{k=1}^{n} u_{kj} = \sum_{k=1}^{n} u_{ik} = 1 \right),$$

for $i, j \in \{1, \dots, n\}$, and calling $S_n^+ = (C(S_n^+)$ the quantum symmetric (permutation) group as the quantum automorphism group of X_N.

The matrix $u = (u_{ij})_{i,j}$ with entries u_{ij} satisfying the relations of the quantum permutation groups, will be called a magic unitary matrix.

Recently, Rollier and Vaes [5] put one step forward and applied the above constructions to the connected locally finite graphs. Following their results, and by using the $n^2 - 2$ connected locally finite graphs $\mathcal{G} := \{\mathcal{G}(\pi_i) | i \in \{1, \dots, n\}\}$, we were able to show that the set $\mathcal{G}$ is possessing a nondegenerate $*$-monoid algebra structure equipped with the binary operations $\pi_i \overset{+}{\to} \pi_j := (V_i \cup V_j, E_i \cup E_j)$, together with the identity element π_2, on which has triggered us in proposing Theorem 2.1.

2 First Initial Set of Multiplier Hopf $*$-Graph Algebras

Already, inspired by results presented in [5], we succeeded in proving an almost identical result, but we been not satisfied, hence we came up with the following result. [3]

Theorem 2.1 (First initial set of multiplier Hopf $*$-Graph Algebras). *For the graph C^*-algebra $C^*(S, P) := C^*(\pi_n) = M_{n^2}(\mathbb{C})$, and the Cuntz-Krieger $\mathcal{G}(\pi_n)$-family $S = \{S_{e_{ii}} = E_{i+1,1}, S_{e_{ij}} = E_{j,1}$ forj $\geq i, S_{e_{ij}} = E_{1,i}$ fori $\geq j\}$, define*

$$\Delta : \mathcal{O}(M_{n^2}(\mathbb{C})) \left[t^{-1}\right] \to M \left(\mathcal{O}(M_{n^2}(\mathbb{C})) \left[t^{-1}\right] \otimes \mathcal{O}(M_{n^2}(\mathbb{C})) \left[t^{-1}\right] \right),$$

taking $E_{i,j}$ to $E_{k,h} \otimes E_{o,r} := E_{\ell,m}$, for $\ell = P_o^k$ and P_r^h, expanded linearly on whole of $\mathcal{O}(M_{n^2}(\mathbb{C})) \left[t^{-1}\right]$.

Then Δ is a coproduct on $\mathcal{O}(M_{n^2}(\mathbb{C})) \left[t^{-1}\right] = \mathcal{O}(GL(n))$, and $(\mathcal{O}(GL(n)), \Delta)$ is a multiplier Hopf $*$-graph algebra, for $i, j, k, h, o, r \in \{1, \dots, n^2\}$ and $\ell, m \in \{1, \dots 2n^2\}$.

3 A Route to Quantum Computing Through the Theory of Quantum Graphs

At this point, we already have what we need in order to enter to the quantum computing and the quantum information theory!

So, consider $\mathcal{G}(\Pi_2)$, the directed graph associated with $\mathbb{K}[M_q(2)]$, and its adjacency matrix Π_2. The interested reader might have a look at [3], in order to obtain more information. Consider its set of Cuntz-Krieger $\mathcal{G}(\Pi_2)$-family

$$\begin{aligned} S = \Bigg\{ & S_e := \sum_{n=1}^{\infty} E_{6n,3n-2}, S_f := \sum_{n=1}^{\infty} E_{6n-4,3n-2}, S_h := \sum_{n=1}^{\infty} E_{6n-3,3n}, \\ & S_g := \sum_{n=1}^{\infty} E_{6n-4,3n-1}, S_i := \sum_{n=1}^{\infty} E_{6n-1,3n}, S_j := \sum_{n=1}^{\infty} E_{6n-3,3n-1} \Bigg\}, \end{aligned} \tag{1}$$

note that $\mathcal{G}(\Pi_2)$ consists of two distinct Hamiltonian paths. Let us call them $\mathcal{P}_1$ and $\mathcal{P}_2$.

Then by employing the defining relations (1), it will not be too difficult to see that $S_e^*S_e + S_i^*S_i + S_g^*S_g = I$, and $S_f^*S_f + S_j^*S_j + S_h^*S_h = I$ are true statements within $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. We might be interested in defining a completely positive trace-preserving map $\Psi_{\Pi_2} : M_m \to M_m$ (m will be specified later), in a way that we have $\Psi_{\Pi_2}(\mathcal{X}) = \sum_m S_m \mathcal{X} S_m^*$ for all $\mathcal{X} \in M_m$, and we will call it our quantum channel. After that, for $\ell \in \{e, f, h, g, i, j\}$, the generating partial (matrix) isometries S_ℓ, will be called the Choi-Kraus operators, and

the noncommutative (confusability) graph of Ψ_{Π_2} will be the operator system

$$\mathcal{S}_{\Psi_{\Pi_2}} = \mathsf{span}\{S_\ell^* S_\ell\} \subseteq M_m, \tag{2}$$

in a way that completely characterizes the number of zero-error messages on which one can send through the quantum channel Ψ_{Π_2}.

In [4], we proved that $\mathcal{G}(\Pi_n)$ consists of $4n-6$ distinct Hamiltonian paths, and the idea of using (distinct) Hamiltonian paths, came from the famous traveling salesman problem (TSP) and its generalization, i.e. the traveling salesman path problem ($TSPP$), and trying to find an optimized algorithm in order to solve that.

Postulate 3 (State Space Postulate)**.** [1] In any isolated (physical) system, there is a complex space together with an inner product known as the state space of the system.

Remark 3.1. Any physical system will completely be described by its unit vector from its state space, on which the vector will be called the state vector of the system.

Depending on the underlying (directed) graph of the quantum system, the state space will be different. But let us just consider $\mathbb{C}^p$ to be the state space of our pre-assumed quantum system. Note that, depending on the choice, the state vectors of the state space, will completely describe the space. By a state vector, we mean the unit vector in $\mathbb{C}^p$, which will be called a qubit, in the language of the information theory, and we have the following definition.

Definition 3.2. A linear combination $\phi = c_1|0\rangle + \cdots + c_p|p-1\rangle$ of the basis vectors of $\mathbb{C}^p$, for $c_1, \ldots, c_p \in \mathbb{C}$ will be called a qubit.

Remark 3.3. In Definition 3.2, c_is, for $i \in \{1, \ldots, p\}$, will be called amplitudes, and they are satisfying in the normalization condition $|c_1|^2 + \cdots + |c_p|^2 = 1$. It is easy to see that this normalization condition will imply $\||\phi\rangle\| = \sqrt{\langle\phi|\phi\rangle} = 1$.

At this point, we need the second postulate, on which could be stated as follows and is concerned with qubit multiples.

Postulate 4. [1] If a system ϕ is a combination of two different (physical) systems ϕ_1 and ϕ_2, with the corresponding states $\mathbb{E}_1$ and $\mathbb{E}_2$, then the state of ϕ will be $\mathbb{E}_1 \otimes \mathbb{E}_2$, correspondent to $\phi_1 \otimes \phi_2$.

Definition 3.4. The state of a q-qubit quantum system is a unit vector in $(\mathbb{C}^2)^{\otimes q}$.

3.0.1 2-Qubit Quantum System

Now, getting back to the main constructions, let $\{0,1\}^q$ be the binary strings of length q. Then by using the binary representation of the natural number $n = 2^{k-1}a_{k-1} + \cdots + 2a_1 + a_0$, for $a_j \in \{0, 1\}$, for all $0 \le j \le k-1$, represented as $s = a_{k-1}a_{k-2}\cdots a_1 a_0$, it is known that we can define a q-qubit system as [1] $|\phi\rangle = \sum_{j\in\{0,1\}^q} \alpha_j |j\rangle$, if we have $\sum_{j\in\{0,1\}^q} |\alpha_j|^2 = 1$, for $\alpha_j \in \mathbb{C}$.

Now, consider $\mathcal{G}(\Pi_2)$, and take $m = 4$. Here, our quantum system will be a combination of two subsystems $\mathbb{E}_1$ and $\mathbb{E}_2$ with the correspondent states ϕ_1 and ϕ_2, and since in this case the Hamiltonian paths are a combination of three edges, hence let $\mathbb{C}^3$ be our state space ϕ. Consider $\phi_1 = \alpha_1|0\rangle + \alpha_2|1\rangle + \alpha_3|2\rangle$ and $\phi_2 = \beta_1|0\rangle + \beta_2|1\rangle + \beta_3|2\rangle$ such that we have $\sum_{i=1}^3 \alpha_i^2 = \sum_{i=1}^3 \beta_i^2 = 1$. Not that the only possible cases for α_is and β_is are $\pm\frac{1}{\sqrt{3}}$.

Now we can define a 2-qubit system associated to the pre-assumed quantum system as follows, for $\sum_{j\in\{0,1\}^2}|\alpha_j|^2=1$,

$$\begin{aligned}|\phi\rangle &= \sum_{j\in\{0,1\}^2}\alpha_j|j\rangle = \alpha_{00}|00\rangle+\alpha_{01}|01\rangle+\alpha_{10}|10\rangle+\alpha_{11}|11\rangle\\ &= \alpha_0|0\rangle+\alpha_1|1\rangle+\alpha_2|2\rangle+\alpha_3|3\rangle, \end{aligned}\tag{5}$$

It is easy to see that, in (5), the only possible options for α_ks are $\pm\frac{1}{2}$, and as a result we have the following proposition.

Proposition 3.5 ([4]). *Our pre-assumed 2-qubit quantum system is entangled!*

Proof. The proof of this proposition, is almost a trivial conclusion of the Definition 3.6, because, in any case, $|\phi\rangle$ will never be equal to the tensor product $|\phi_1\rangle\otimes|\phi_2\rangle$. ☐

Definition 3.6 ([1]). A quantum state $|\phi\rangle\in(\mathbb{C}^m)^{\otimes q}$ is a product state if it can be expressed as a tensor product $|\phi_1\rangle\otimes\cdots\otimes|\phi_q\rangle$ of q 1-qubit states. Otherwise, it is entangled.

3.0.2 $4n-6$-Qubit Quantum System

Claim 6 ([4]). In the case of $\mathcal{G}(\Pi_n)$, the pre-assumed $4n-6$-qubit quantum system is entangled.

Note that in order to prove or disprove this claim, the person first needs to prove or disprove Claim 2.7., from [3].

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On the Pure Injective Dimension in the Category of N-Complexes of Sheaves

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Abstract

Let $(\mathcal{A}, \text{-}\otimes\text{-})$ be a symmetric monoidal closed Grothendieck category. A generalization of the Lambek Theorem ([12]) will be proved in the category of N-complexes in $\mathcal{A}$. In addition, if $\mathcal{A}$ is sufficiently nice, we will give a characterization of pure injective flat N-complexes and prove that any flat N-complex in $\mathcal{A}$ has finite pure injective dimension.

Keywords: Grothendieck category, Scheme, Sheaf, N-Complex, Flat N-complex, Pure injective dimension

Mathematics Subject Classification [2010]: Primary: 18G20, 18G35, 18E15, 14F05.

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1 Introduction and Preliminaries

In this article we will assume that $(\mathcal{A}, \text{-}\otimes\text{-})$ is a symmetric monoidal closed Grothendieck category in the sense of [9] and $\mathcal{J}$ is the **injective cogenerator** of $\mathcal{A}$. It is known that there exists a bifunctor $\mathcal{H}om_{\mathcal{A}}(\text{-},\text{-}) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathcal{A}$ such that for any object $\mathcal{G}$ in $\mathcal{A}$, $\text{-}\otimes\mathcal{G} : \mathcal{A} \longrightarrow \mathcal{A}$ is the left adjoint of $\mathcal{H}om_{\mathcal{A}}(\mathcal{G},\text{-}) : \mathcal{A} \longrightarrow \mathcal{A}$, i.e. for any pair of objects $\mathcal{F}$ and $\mathcal{K}$ in $\mathcal{A}$,

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{F} \otimes \mathcal{G}, \mathcal{K}) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{H}om_{\mathcal{A}}(\mathcal{G}, \mathcal{K})) \qquad (1)$$

is an isomorphism of abelian groups (see [9] for more details). Let us recall the definition of purity and flatness in $\mathcal{A}$.

Definition 1.1. Let $\mathcal{F}$ be an object in $\mathcal{A}$.

(a) $\mathcal{F}$ is said to be *flat* if $\mathcal{F}\otimes$- preserves exact sequences in $\mathcal{A}$.

(b) A short exact sequence $\mathcal{E}$ in $\mathcal{A}$ is called *pure* if for any object $\mathcal{G}$ in $\mathcal{A}$, $\mathcal{E} \otimes \mathcal{G}$ remains a exact.

(c) An monomorphism in $\mathcal{A}$ is called *pure* if its corresponding short exact sequence in $\mathcal{A}$ is pure.

Recall that an object $\mathcal{T}$ in $\mathcal{A}$ is pure injective if it is injective with respect to pure exact sequences. The existence of pure injective preenvelopes in $\mathcal{A}$ is the main subject of [9]. Assume that $(\text{-})^{+} := \mathcal{H}om_{\mathcal{A}}(\text{-}, \mathcal{J})$. Remember from [9] that, for any object $\mathcal{X}$ in $\mathcal{A}$, $\mathcal{X}^{+}$ is pure injective and $\mathcal{X} \longrightarrow \mathcal{X}^{++}$ is a pure monomorphism. Furthermore, the following conditions are hold in $\mathcal{A}$.

(1) Let $\mathcal{F}$ be an object in $\mathcal{A}$.

  - (a) $\mathcal{F}$ is a flat object.
  - (b) $\mathcal{F}^+$ is injective.
  - (c) Any short exact sequence ending in $\mathcal{F}$ is pure.

(2) Let $\mathcal{E}$ be short exact sequence in $\mathcal{A}$. Then, $\mathcal{E}$ is pure if and only if $\mathcal{E}^+$ splits (see also [8]).

Let $N \geq 2$ be a fix positive integer. N-complexes are a generalization of complexes which are first introduced by Mayer in [13]. The homological properties of N-complexes were studied by Kapranov and Dubois-Violette in [11], [2]. Besides their applications in theoretical physics [1], [6], the homological properties of N-complexes have become a subject of study for many authors as in, [3], [5], [4], [15]. An N-complex in $\mathcal{A}$ is a sequence

$$\cdots \xrightarrow{d^{n-1}_{\mathbf{X}}} X^n \xrightarrow{d^n_{\mathbf{X}}} X^{n+1} \xrightarrow{d^{n+1}_{\mathbf{X}}} \cdots$$

of objects and morphisms in $\mathcal{A}$ such that composing of any N-consecutive maps gives 0, i.e. $d^N = 0$. A morphism $f : \mathbf{X} \to \mathbf{Y}$ of N-complexes is a collection of morphisms $f^n : \mathbf{X}^n \to \mathbf{Y}^n$ making all the rectangles commute. The category of N-complexes in $\mathcal{A}$ is denoted by $\mathbf{C}_N(\mathcal{A})$. Kernel, image and colimits are defined componentwise. So, $\mathbf{C}_N(\mathcal{A})$ is a Grothendieck category. The aim of this work is to show that the main results of [8] and [9] are hold in $\mathbf{C}_N(\mathcal{A})$ and will deduce that under certain conditions, the class of flat N-complexes is closed under pure injective preenvelopes. This provides an answer to a question raised in [14]. Moreover, we show that any flat N-complex in $\mathcal{A}$ has finite pure injective dimension.

Definition 1.2. Let $\mathbf{X} \in \mathbf{C}_N(\mathcal{A})$.

- (a) A short exact sequence $\mathcal{E}$ in $\mathbf{C}_N(\mathcal{A})$ is called *pure* if $\mathcal{E}^+$ splits.
- (b) A monomorphism f in $\mathbf{C}_N(\mathcal{A})$ is called *pure* if its corresponding short exact sequence in $\mathbf{C}_N(\mathcal{A})$ is pure.
- (c) $\mathbf{X}$ is called pure injective if it is injective with respect pure exact sequences in $\mathbf{C}_N(\mathcal{A})$.

Definition 1.3. A morphism $f : \mathbf{X} \longrightarrow \mathbf{Y}$ of N-complexes is called null-homotopic if there exists $s^n \in \mathrm{Hom}_{\mathcal{A}}(X^n, Y^{n-N+1})$ such that

$$f^n = \sum_{i=0}^{N-1} d_{\mathbf{Y}}^{i-(N-1)} s^{n+i} d_{\mathbf{X}}^{i}$$

that $d^0_{\mathbf{X}} = 1_{\mathbf{X}}$, $d^1_{\mathbf{X}} = d^n_{\mathbf{X}}$,$d^{-1}_{\mathbf{X}} = d^{n-1}_{\mathbf{X}}$,$d^2_{\mathbf{X}} = d^{n+1}d^n$,$d^{-2}_{\mathbf{X}} = d^{n-1}d^{n-2}$ and etc .

Definition 1.4. Let $0 \leq r < N$, $i \in \mathbb{Z}$ and $\mathbf{X}$ be an N-complex of objects in $\mathcal{A}$. We define

$$Z^i_r(\mathbf{X}) := \mathrm{Ker}\left(d^{i+r-1}_{\mathbf{X}} \cdots d^i_{\mathbf{X}}\right), \qquad B^i_r(\mathbf{X}) := \mathrm{im}\left(d^{i-1}_{\mathbf{X}} \cdots d^{i-r}_{\mathbf{X}}\right)$$

Definition 1.5. An N-complex $\mathbf{X}$ is called N-acyclic if $Z^i_r(\mathbf{X}) = B^i_r(\mathbf{X})$ for each $i \in \mathbb{Z}$ and all $r = 1, 2, \ldots, N-1$.

2 Main Results

Lemma 2.1. *For any N-complex $\mathbf{X}$ in $\mathcal{A}$, $\mathbf{X}^+$ is pure injective and $\mathbf{X} \longrightarrow \mathbf{X}^{++}$ is a pure monomorphism.*

An N-acyclic complex $\mathbf{Y}$ in $\mathcal{A}$ is called *N-pure acyclic* if for any object $\mathcal{G} \in \mathcal{A}$, $\mathbf{X} \otimes \mathcal{G}$ is N-acyclic.

Definition 2.2. Let $\mathbf{F}$ be an N-complex in $\mathcal{A}$. $\mathbf{F}$ is called *flat* if it is an N-pure acyclic N-complex of flat objects in $\mathcal{A}$.

Proposition 2.3. *Let $\mathbf{F}$ be an N-complex in $\mathcal{A}$.*

(a) **F** *is a flat object.*

(b) $\mathbf{F}^+$ *is injective.*

(c) Any short exact sequence ending in **F** *is pure.*

Let $(X, \mathcal{O}_X)$ be a quasi-compact and quasi-separated scheme. X is called *coherent* if for any affine open subset U of X $\mathcal{O}_X(U)$ is a coherent ring. In the reminder of this work, we assume that X is coherent and $\mathcal{A}$ is the category of all quasi-coherent sheaves of $\mathcal{O}_X$-modules. The next result provides an answer to a question raised in [14].

Theorem 2.4. *The class of all flat N-complexes is closed under pure injective envelope.*

Theorem 2.5. *Let* **X** *and* **Y** *be N-pure acyclic N-complexes of pure injective flat objects in $\mathcal{A}$. Any morphism from* **X** *into* **Y** *is null homotopic.*

Theorem 2.6. *If X has finite Krull dimension then any flat N-complex in $\mathcal{A}$ has finite pure injective dimension.*

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Some Quotients of Homotopy Categories

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Abstract

Let $(X, \mathcal{O}_X)$ be a scheme and $\mathbb{K}(X)$ be the homotopy category of complexes of quasi-coherent sheaves of $\mathcal{O}_X$-modules. In this work, we talk about derived and co-derived categories of X. Especially, we present the relation between the complete hereditary cotorsion theories and these quotient categories..

Keywords: Triangulated category, Homotopy category, Derived category, Quasi-coherent sheaf.

Mathematics Subject Classification [2010]: Primary: 18G20, 18G35, 18E15, 14F05.

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1 Introduction

Let $(X, \mathcal{O}_X)$ be a scheme and $\mathfrak{Qcoh}X$ be the category of all quasi-coherent sheaves of $\mathcal{O}_X$-modules. The homotopy category $\mathbb{K}(X)$ of X is a framework for working with chain homotopies and homotopy equivalences. The objects in $\mathbb{K}(X)$ are chain complexes and morphisms are equivalence classes of chain maps up to homotpies. A *chain complex* in $\mathfrak{Qcoh}X$ is a sequence

$$\mathbf{G} : \cdots \longrightarrow G^{n-1} \xrightarrow{\delta_{\mathbf{G}}^{n-1}} G^n \xrightarrow{\delta_{\mathbf{G}}^{n}} G^{n+1} \longrightarrow \cdots$$

of quasi-coherent sheaves of $\mathcal{O}_X$-modules such that, $\forall n \in \mathbb{Z}$, $\delta_{\mathbf{G}}^{n}\delta_{\mathbf{G}}^{n-1} = 0$. For each $n \in \mathbb{Z}$, the n-th cohomology of $\mathbf{G}$, denoted by $\mathrm{H}^n(\mathbf{G})$, is defined by

$$\mathrm{H}^n(\mathbf{G}) = \mathrm{Ker}\delta_{\mathbf{G}}^{n}/\mathrm{Im}\delta_{\mathbf{G}}^{n-1}.$$

Let $\mathbf{C}(X)$ be the category of all complexes of quasi-coherent sheaves of $\mathcal{O}_X$-modules. It is well-known that $\mathbf{C}(X)$ is a locally Grothendieck category. Recall that a complex $\mathbf{G}$ is said to be *acyclic* if all cohomologies are trivial. An acyclic complex G is said to be *pure* if for any quasi-coherent sheaf K of $\mathcal{O}_X$-modules, $K \otimes_{\mathcal{O}_X} \mathbf{G}$ remains acyclic. Notice that $\mathrm{H}^n(.) : \mathbf{C}(X) \longrightarrow \mathbf{C}(X)$ is a covriant additive functor. A morphism $f : \mathbf{G} \longrightarrow \mathbf{H}$ of chain complexes is said to be *quasi-isomorphism* if for each $n \in \mathbb{Z}$, $\mathrm{H}^n(f)$ is an isomorphism of quasi-coherent sheaves of $\mathcal{O}_X$-modules. Let $f, g : \mathbf{G} \longrightarrow \mathbf{H}$ be morphisms of chain complexes of quasi-coherent sheaves of $\mathcal{O}_X$-modules, f and g are called homotopic (or $f - g$ is null-homotopic) if $\forall n \in \mathbb{Z}$, there exists a morphism $s^n : G^n \longrightarrow H^{n-1}$such that

$$\delta_{\mathbf{H}}^{n-1}s^n + s^{n+1}\delta_{\mathbf{G}}^{n} = f^n - g^n.$$

The homotopy category $\mathbb{K}(X)$ is defied as follows; objects in $\mathbb{K}(X)$ are chain complexes of quasi-coherent sheaves of $\mathcal{O}_X$ and morphisms are defied by, for each pair $\mathbf{G}, \mathbf{H}$ of chain complexes in $\mathfrak{Qcoh}X$,

$$\mathrm{Hom}_{\mathbb{K}(X)}(\mathbf{G}, \mathbf{H}) = \mathrm{Hom}_{\mathbf{C}(X)}(\mathbf{G}, \mathbf{H})/\mathrm{Ht}(\mathbf{G}, \mathbf{H})$$

where $\mathrm{Ht}(\mathbf{G}, \mathbf{H})$ is the subgroup of $\mathrm{Hom}_{\mathbf{C}(X)}(\mathbf{G}, \mathbf{H})$ consisting of homotopic morphisms.

2 Main Results

Let $\mathbb{K}_{\mathrm{ac}}(X)$ (resp. $\mathbb{K}_{\mathrm{pac}}(X)$ be the triangulated subcategory of $\mathbb{K}(X)$ consisting of acyclic (resp.pure acyclic) chain complexes in $\mathfrak{Qcoh}X$. Let $\mathbb{D}(X)=\mathbb{K}(X)/\mathbb{K}_{\mathrm{ac}}(X)$ (resp. $\mathbb{D}_{\mathrm{pac}}(X)=\mathbb{K}(X)/\mathbb{K}_{\mathrm{pac}}(X)$) be the derived (resp, pure derived) category of X. Recall that X is called *locally noetherian* (resp. *coherent*) if for any affine open subset U of X $\mathcal{O}_X(U)$ is a noetherian (resp. coherent) ring (see [2], [3–5]).

Proposition 2.1. *There exists an equivalence of triangulated categories between* $\mathbb{D}(X)$ *and the homotopy category of dg-injective complexes of quasi-coherent sheaves of* $\mathcal{O}_X$*.*

Proposition 2.2. *There exists an equivalence of triangulated categories between* $\mathbb{D}_{\mathrm{pac}}(X)$ *and the homotopy category of dg-pure injective complexes of quasi-coherent sheaves of* $\mathcal{O}_X$*.*

Definition 2.3. Let $\mathbf{F}$ be an chain complex of $\mathcal{O}_X$-modules. $\mathbf{F}$ is called *flat* if it is an pure acyclic complex of flat objects in $\mathfrak{Qcoh}X$.

Lemma 2.4. *Let* $\mathbf{F}=(F^n,\delta^n)$ *be a chain complex of quasi-coherent sheaves of* $\mathcal{O}_X$*-modules. Then* $\mathbf{F}$ *is flat if and only it is an acyclic chain complex such that for each integer* n, $Ker\delta^n$ *is a flat quasi-coherent sheaves of* $\mathcal{O}_X$*-modules.*

Let $\mathbb{K}(\mathrm{Flat}X)$ be the homotopy category of flat quasi-coherent sheaves of $\mathcal{O}_X$-modules and $\mathbb{K}_{\mathrm{pac}}(\mathrm{Flat}X)$ its subcategory consisting of pure acyclic complexes. The pure derived category of flat quasi-coherent sheaves is defined by the quotient $\mathbb{D}_{\mathrm{pac}}(\mathrm{Flat}X)=\mathbb{K}(\mathrm{Flat}X)/\mathbb{K}_{\mathrm{pac}}(\mathrm{Flat}X)$ (see [1] for more details).

Proposition 2.5. *There exists an equivalence of triangulated categories between* $\mathbb{D}_{\mathrm{pac}}(\mathrm{Flat}X)$ *and the homotopy category of dg-cotorsion complexes of flat quasi-coherent sheaves of* $\mathcal{O}_X$*.*

Let R be an associative ring with identity and R-Mod be the category of all left R-modules. A left R-module M is called *absolutely pure* if any short exact sequence starting in M is pure. Let $\mathbb{K}(\mathrm{Abs}R)$ be the homotopy category of absolutely pure left R-modules and $\mathbb{K}_{\mathrm{pac}}(\mathrm{Abs}R)$ its subcategory consisting of pure acyclic complexes. The co-derived category of R is defined by the quotient $\mathbb{D}_{\mathrm{pac}}(\mathrm{Abs}R)=\mathbb{K}(\mathrm{Abs}R)/\mathbb{K}_{\mathrm{pac}}(\mathrm{Abs}R)$.

Proposition 2.6. *There exists an equivalence of triangulated categories between* $\mathbb{D}_{\mathrm{pac}}(\mathrm{Abs}R)$ *and the homotopy category of dg-injective complexes.*

The next result has been proved in [6].

Corollary 2.7. *If* R *coherent then there exists an equivalence* $\mathbb{D}_{\mathrm{pac}}(\mathrm{Abs}R)$ *and the homotopy category of injective left* R*-modules.*

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Injectivity Versus Ideal Injectivity in the Category of S-Systems

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Abstract

This paper aims to investigate the Baer-type criteria for the injectivity of S-acts. In contrast to the case of modules, where the Baer Criterion for injectivity is valid for modules over a ring, it is an open problem for acts over semigroups. Every injective S-act is ideal injective, but the converse is not generally true. Here we give some conditions under which every ideal injective S-act is injective.

Keywords: Injectivity, Ideal injectivity, S-Act

Mathematics Subject Classification [2010]: Primary:20M30, 08A60, Secondary: 08B30

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1 Introduction and preliminaries

For the subclasses $\mathcal{M}_1$ and $\mathcal{M}_2$ of monomorphisms in a category $\mathcal{C}$, the investigation between $\mathcal{M}_1$-injectivity and $\mathcal{M}_2$-injectivity, is important. Note that if $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then $\mathcal{M}_2$-injectivity implies $\mathcal{M}_1$-injectivity. The converse of this fact may be called the **Baer type criteria**.

Although the Baer Criterion for injectivity (weak injectivity implies injectivity) is true for modules over a ring (with an identity), it is an open problem for acts over a semigroup S (with or without identity). The closest notion about the Baer criterion for injectivity in the category of S-acts is the Skornjakov-Baer Criterion, which says that an S-act A is injective if and only if it is injective with respect to subacts of cyclic acts. In this article, we try to get as close to Baer criterion as possible.

Throughout the paper, unless otherwise stated, S is a semigroup with or without left identity 1. A semigroup S with a left identity is called a *left monoid*. One of the notions closely related to weakly injectivity in the category of S-acts is ideal injectivity. Every injective S-act is ideal injective and each ideal injective S-act is weakly injective. These concepts are not equal in general, but, for example, if S is a left monoid then weakly injectivity and ideal injectivity are equivalent. V.Gould in [8] defined the notions α-injectivity and α-Baer criterion on monoids, for any cardinal α greater than 1 as follow,

An S-act A is called *α-injective* if any homomorphism $f : I \to A$, where I is a right ideal of S with a generating set of fewer than α elements, can be extended to $g : S \to A$. Also a right S-act A satisfies the *α-Baer criterion* if, for any right ideal I of S with a generating set of fewer than α elements and any homomorphism $f : I \to A$, there exists an element a in A such that $f(s) = as$ for all $s \in I$. Let γ be a cardinal such that any right ideal of a monoid S has a generating set with fewer than γ elements. Then the γ-injective S-act is called *weakly injective* and ,in this article, if A satisfies the γ-Baer criterion, then A is said to be *ideal injective* S-act. It is clear that an injective S-act is α-injective for any cardinal α. But the converse is not generally true. In [8, Prop. 3.3] it is shown that for a monoid S, an S-act A is α-injective if and only if it satisfies the α-Baer criterion. This meanse , in the case where S is a monoid, ideal injectivity is equivalent to weakly injectivity. In this paper we give some conditions over which all ideal injective S-acts are injective and consequently if S is a monoid under which conditions weakly injectivity implies injectivity. To this observation the main results of the paper are Theorems 2.2 and 2.7. It is worth noting that the internal characterisations of completely right injective monoids (i.e If all S-acts are injective) have been developed independently by Fountain [6] and Shoji [10].

Here we give some preliminaries about S-acts needed in the sequel. A *right S-act A* is a nonempty set on which S acts unitarily, i.e. for each $a \in A$ and $s, t \in S$, $(as)t = a(st)$ and if S has left identity 1, $a1 = a$. A map $f : A \to B$ between S-acts A and B is called a *homomorphism or S-map* if for each $a \in A$ and $s \in S$, $f(as) = f(a)s$. An inclusion $i : A \to B$ is called a *retraction* if there exists a homomorphism $g : B \to A$ such that for each $a \in A$, $g(a) = a$. The category of all (right) S-acts and homomorphisms between them is denoted by **Act-S**.

A *right congruence* on A is an equivalence relation on A that is compatible with the right S-action. In particular for a subact K of an S-act A a binary relation ρ_K defined by $(a, b) \in \rho_K$ if and only if $a = b$ or $\{a, b\} \subseteq K$ is a right congruence on A which is called a *Rees congruence*. The set of all right congruences on A is denoted by $Con(A)$. An S-act A is called *injective* if for any subact C of an S-act B, every homomorphism $f : A \to C$ can be lifted to a homomorphism $g : B \to A$. An extension B of A is said to be *essential extension* if every homomorphism $f : B \to C$ is a monomorphism whenever $f\mid_A$ is a monomorphism. An injective essential extension of an S-act A, denoted by $E(A)$, is called an *injective hull* of A. The existence of injective hull is shown in [3].

For undefined terms and notations concerning S-acts, one may consult [4, 9].

2 Main Results

An S-act A is said to be *ideal injective* if for every right ideal I of S, every homomorphism $f : I \to A$ is of the form λ^I_a for some $a \in A$, where $\lambda^I_a : I \to A$ is defined by $\lambda^I_a(s) = as$. It is not difficult to check that an S-act A is ideal injective if and only if every homomorphism $f : I \to A$ can be extended to $\bar{f} : S^1 \to A$. Also an S-act A is called *weakly injective* if every S-map $f : I \to A$ can be extended to $\bar{f} : S \to A$. Clearly, ideal injectivity implies weakly injectivity and if S has a left identity, these two notions coincide. For the semigroup $S = (\mathbb{N}, min)$, since $id : S \to S$ is not of the form λ^S_a, S is not ideal injective. But it is weakly injective, for, every homomorphism $f : n_0 \downarrow \to S$ can be extended to the homomorphism $g : S \to S$ given by $g(m) = f(m.n_0)$, where $n_0 \downarrow = \{n \in S \mid n \leq n_0\}$.

Every injective S-act is ideal injective, but the converse is not generally true. For instance, any group S is an ideal injective as an S-act but it is not injective S-act. Also the monoid $S = (\mathbb{N}, max)$ is an ideal injective S-act which is not injective.

The Baer type problems are about the converse of this fact. In this section, we use ideal injectivity to give some Baer type results about injectivity of S-acts. We introduce some semigroups over all of which every ideal injective S-act is injective.

As a simple result, since every S-map $f : I \to S$ can be extended to $g : S \to E(S)$, we have: *The right S-acts S is ideal injective if and only if S has a left identity.*

Lemma 2.1. *If A is ideal injective, B is a proper essential extensionof A and $b \in B \setminus A$, then:*
(i) *$I_b \neq S$.*
(ii) *if A has a fixed element, then $\emptyset \neq I_b \neq S$*

For a homomorphism $f : A \to B$, $ker(f) = \{(a_1, a_2) \mid f(a_1) = f(a_2)\}$ is a right congruence on A. Also for every right ideal I of S and $a \in A$, $Ker(\lambda^I_a) = ker(\lambda^I_a) \cup \{(s, s) \mid s \in S \setminus I\}$, is a right congruence on S. For every right ideal K of S and $s \in S$, $K_s = \{u \in S \mid su \in K\}$ is either empty or a right ideal of S.

Theorem 2.2. *Suppose that S has a zero element. Whenever the following condition holds, then each ideal injective S-act is injective.*

For each right ideal K of S, $\theta \in Con(S)$ and $s, t \in S$, if $K_s = K_t$ and $su\theta tu\,(\forall u \in K_s)$, then $s\theta t$.

Now we look at another closure operator using by ideal injectivity. For an extension B of A, we define a closure operator $C_B(A) = \{b \in B \mid I_b \neq \emptyset\}$. Note that every injective S-act has a fixed element and If A does not have a fixed element, then $E(A)$ has only one fixed element, see [2, Prop. 1]. In the following Lemma part (ii), $E(A)$ has only one fixed element, denoted by 0.

Lemma 2.3. (i) *If A has a fixed element, then $C_{E(A)}(A) = E(A)$.*
(ii) *If A does not have fixed element, then $C_{E(A)}(A) = E(A) \setminus \{0\}$.*

Theorem 2.4. *For an S-act A, the S-act $C_{E(A)}(A)$ is an ideal injective S-act.*

As a consequence of Lemma 2.3 and Theorem 2.4, we have, if an S-act A does not have a fixed element, then $C_{E(A)}(A) = E(A) \setminus \{0\}$ is an ideal injective which is not injective.

Theorem 2.5. *If each ideal injective S-act is injective, then S has a left zero element.*

Let I be a right ideal of S, A a right S-act and $a \in A$. Far each $s \in S$, consider $I_s = \{u \in S^1 \mid su \in I\}$. It is not difficult to check that, a relation $\rho(I, a)$ on S given by:

$$s\rho(I,a)t \text{ if and only if } I_s = I_t \text{ and } \lambda^{I_s}_{as} = \lambda^{I_s}_{at}$$

is a right congruence on S.

Lemma 2.6. *For a right ideal I of S and $a, b \in A$, if $\lambda^I_a = \lambda^I_b$, then $\rho(I, a) = \rho(I, b)$.*

Theorem 2.7. *An S-act A is injective if and only if A is ideal injective with a fixed element and for each right ideal I of S and $a \in A$, there exists $b \in A$ such that $\lambda^I_a = \lambda^I_b$ and $\rho(I, b) \subseteq ker\lambda^S_b$.*

An S-act A is called *s-complete* if every homomorphism $f : S \to A$ is of the form λ^S_a for some $a \in A$, see [1]. Clearly every ideal injective S-act is s-complete, so we have the following results from [1].

Theorem 2.8. *In the following cases, every ideal injective S-act with at least one fixed element is injective.*
(i) *For every nontrivial right ideal I of S, $\bar{I} \neq I$. In particular,*
  1) *Monogenic semigroup.*
  2) *There exists an element $s_0 \in S$, such that for all $s, t \in S$, $st = s_0$.*
(ii) *Every nonempty proper right ideal of S generates by a central idempotent element.*
(iii) *the semigroup S is a Clifford semigroup and every proper nonempty right ideal of S is principal.*
(iv) *If for every proper nonempty right ideal I of S there exists a nonempty right ideal J of S such that $I \cap J = \emptyset$. In particular, if S is a left zero semigroup.*

An important special case of Clifford semigroups is the commutative chain with the relation $(x \leq y \Leftrightarrow xy = x)$ or $(x \leq y \Leftrightarrow xy = y)$, such as commutative bands, $S = (\mathcal{N}, max)$ and $S = (\mathcal{N}, min)$. The category of all S-acts for the semigroup $S = (\mathcal{N}, min)$, so called *projection algebras* and mostly used in computer science, has been studied by Ebrahimi and Mahmoudi [5] and Giuli [7].

There is still an open question concerning ideal injectivity:

Is there a necessary and sufficient condition on S such that all ideal injective S-acts with at least one fixed element are injective?

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New Bounds on the Energy, Laplacian Energy and Sombor Index of a Graph

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Abstract

The *energy* of a graph G, denoted by $\varepsilon(G)$, is defined as the sum of the absolute values of all eigenvalues of G. One of the novel topological indices, named as *Sombor index* was introduced by Gutman, defined as $SO(G) = \sum_{uv\in E(G)} \sqrt{d^2(u)+d^2(v)}$, where d_u and d_v are the degree of vertices u and v in G, respectively. It was proved that if G is a graph of order at least 3, then $\varepsilon(G) \leq So(G)$ and if G is a connected graph of order n which is not $P_n (n \leq 8)$, then $\varepsilon(G) \leq \frac{So(G)}{2}$. In this paper, we have strengthen these results and obtain several lower and upper bounds between the graph energy, Laplacian energy and Sombor index.

Keywords: Energy of graph, Laplacian energy, Sombor index.
Mathematics Subject Classification [2010]: 05C20

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1 Introduction and Preliminaries

Let $G = (V(G), E(G))(\mid V(G) \mid= n, \mid E(G) \mid= m)$ be a simple undirected graph, where $V(G)$ and $E(G)$ denote the set of vertices and edges. The numbers n and m are called the *order* and the *size* of the graph G respectivily. Also, the degree of vertex $u \in G$ is denoted by d_u. Complete graphs with order n, edgeless graphs with order n, complete bipartite graphs with order $n = a + b$, the paths with order n are denote by K_n, $\bar{K_n}$, $K_{a,b}$ and P_n, respectivily.

The *adjacency matrix* of a graph G, which is denoted by $A(G)$, is defined by its entries as $a_{ij} = 1$ if the vertices v_i and v_j are adjacent and 0 otherwise. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the *eigenvalues* of $A(G)$. The *energy* of a graph G, denoted by $\varepsilon(G)$, was introduced by Gutman in 1978 and is defined as the sum of the absolute values of all eigenvalues of its adjacency matrix. After that, many mathematicians studied about graph energy and obtained many upper and lower bounds for it in term of m and n. Another type of bounds is obtaining bounds for graph energy in term of other topological indices such as Randik and Sombor index. Also, the energy of a vertex is developed by Arizmendi et al ing [2].

For a graph G, let $D = D(G)$ be a degree matrix of G which its entries are as $a_{ii} = d_{u_i}$ and 0 otherwise. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. The eigenvalues of $L(G)$ are all nonnegative real number. If $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0$ are the sequence of eigenvalues of $L(G)$, then the *Laplacian energy* of G is defined as $L\varepsilon(G) = \Sigma_{i=1}^{n} \mid \mu_i - \frac{2m}{n} \mid$.

One of the novel topological indices, named as *Sombor index* was introduced by Gutman, defined as $SO(G) = \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}$, where d_u and d_v are the degree of vertices u and v in G, respectively.

It was proved that if G is a graph of order at least 3, then $\varepsilon(G) \leq So(G)$, see [4,5] and if G is a connected graph of order n which is not $P_n(n \leq 8)$, then $\varepsilon(G) \leq \frac{So(G)}{2}$, see [1]. In this paper, we have strengthen these results and obtain several lower and upper bounds between the graph energy, Laplacian energy and Sombor index.

2 Main Results

This section has devoted to the bound results of graph energy and Laplacian energy in term of its Sombor index.

The following results gives some known inequality between energy and Sombor index of a graph.

Theorem 2.1 ([4]). *For every graph G with minimum vertex degree $\delta \geq 2$, $\varepsilon(G) \leq So(G)$.*

Theorem 2.2 ([5]). *Let G be a connected graph with n vertices. If $n \geq 3$, $\varepsilon(G) \leq So(G)$.*

Theorem 2.3 ([1, Th. 3]). *If G is a connected graph of order n which is not $P_n(n \leq 8)$, then $\varepsilon(G) \leq \frac{So(G)}{2}$.*

Here, we improve the bounds given in these theorems. We start by the following two lemmas which are nedded in the sequel.

Lemma 2.4 ([3, Cor 3.1]). *Let G be a simple graph with $n \geq 2$ vertices and m edges. Then $So(G) \geq \frac{2\sqrt{2}m^2}{n}$.*

Lemma 2.5 ([6, Th. 5.4.1]). *For every graph G with n vertices and m edge, $\varepsilon(G) \leq \sqrt{2mn}$.*

Theorem 2.6. *For every simple graph G with n vertices and m edge,*

$$\varepsilon(G) \leq \frac{So(G)}{2(\frac{m}{n})^{\frac{3}{2}}}$$

The equality holds if and only if $G = \bar{K_n}$ or $\bigcup_{i=1}^{m} K_2$.

Corollary 2.7. (i) *If a graph G has at least one cyclic, then $\varepsilon(G) \leq \frac{So(G)}{\frac{2m}{n}}$.*

(ii) *Since $\frac{2m}{n} \geq \delta$, so $2(\frac{m}{n})^{\frac{3}{2}} \geq \frac{\delta^{\frac{3}{2}}}{\sqrt{2}}$ and hence for every simple graph G, $\varepsilon(G) \leq \frac{So(G)}{\frac{\delta^{\frac{3}{2}}}{\sqrt{2}}}$.*

By appling Lemma 2.3 and Theorem 2.6, the following result is obtained.

Theorem 2.8. *If G is a connected graph of order n which is not $P_n(n \leq 8)$, then*

$$\varepsilon(G) \leq \frac{So(G)}{max\{2, 2(\frac{m}{n})^{\frac{3}{2}}\}}$$

In the following, we present some lower bounds on the graph energy versus the Sombor index. In [4] Ülker et al obtained some lower bounds for graph energy in term of Sombor index.

Theorem 2.9 ([4]). *Let G be a graph with maximum degree $\Delta(G)$. Then $\varepsilon(G)\Delta(G)^3 \geq So(G)$.*

Theorem 2.10 ([4]). *Let G be a Δ-regular graph. Then $\varepsilon(G)\Delta(G)^2 \geq So(G)$.*

Now we want to improve the above bounds by introducing some new bounds.

The energy of the graph vertex is stated in reference [2] and for the vertex x_i, the energy of vertex x_i is denoted by $\varepsilon(x_i)$. We refrain from detailind it here. The energy of a graph is the sum of its vertex energies, i.e, $\varepsilon(G) = \varepsilon(x_1) + \varepsilon(x_2) \cdots \varepsilon(x_n)$.

Theorem 2.11 ([2]). *Let G be a graph with at least one edge. Then for all $x_i \in V(G)$, $\varepsilon(x_i) \geq \frac{d_i}{\Delta_G}$. Equality holds if and only if $G \cong K_{d,d}$.*

Lemma 2.12. *Let G be a graph with maximum degree Δ. Then*

$$\varepsilon(G) \geq \frac{2m}{\Delta}$$

with equality if and only if $G \cong K_{d,d}$.

Theorem 2.13. *Let G be a graph with maximum degree Δ. Then*

$$\varepsilon(G) \geq \frac{So(G)}{\frac{\Delta^2}{\sqrt{2}}}$$

The equality holds if and only if G is a regular complete bipartite graph.

Theorem 2.14. *For every simple graph G with n vertices and m edge,*

$$\varepsilon(G) \geq \frac{So(G)}{\frac{\sqrt{m}\Delta}{\sqrt{2}}}$$

with equality if and only if G is a regular complete bipartite graph.

Finally, we present an upper and lower bounds for Laplacian energy in term of Sombor index.

Theorem 2.15. *For every simple graph G with n vertices and m edge, $L\varepsilon(G) \leq \frac{So(G)}{\frac{\sqrt{2}m}{n}}$.*

Corollary 2.16. *For every simple graph G with n vertices and m edge,*
(i) *If G is a tree, then $\frac{\sqrt{2}m}{n} \geq 1$ and hence $L\varepsilon(G) \leq So(G)$.*
(ii) *If G is not a tree, then $m \geq n$ and hence $L\varepsilon(G) \leq \frac{So(G)}{\sqrt{2}}$.*
(iii) *Since $2m \geq n\delta$, then $L\varepsilon(G) \leq \frac{So(G)}{\frac{\delta}{\sqrt{2}}}$.*

Theorem 2.17. *For every simple graph G with n vertices and m edge, $L\varepsilon(G) \geq \frac{So(G)}{\frac{m}{\sqrt{2n}}}$.*

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The Relationship Between Zero-Divisor Graphs and Annihilator Graphs with Less than Five Vertices

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Abstract

In this paper, we study the relationship between zero-divisor graphs and annihilator graphs a commutative semigroup that $Z(S) \neq S$ and $|Z(S)| \leqslant 5$ and we show that $AG(S)$ is never isomorphic to the graph P_3.

Keywords: Zero-divisor graph, Annihilator graph, Isolated vertex, End vertex

Mathematics Subject Classification [2010]: Primary: 13BXX,13FXX,05EXX, Secondary: 13HXX, 05EXX

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1 Introduction and Preliminaries

For any commutative semigroup S with zero element 0, there is a simple undireceted graph, which is called the zero-divisor graph and is denoted by $\Gamma(S)$ (cf. [6]). The vertex set of $\Gamma(S)$ is $Z(S)^* = Z(S) \setminus \{0\}$ and x is adjacent to y in $\Gamma(S)$ if and only if $xy = 0$, for each two distinct elements x and y in $Z(S)^*$. It was proved that $\Gamma(S)$ is connected and the diameter of $\Gamma(S)$ is less than or equal to three. Also if $\Gamma(S)$ contains a cycle, then its girth is less than or equal to four.

In [1], we introduced the annihilator graph for a commutative semigroup S, which is denoted by $AG(S)$. The graph $AG(S)$ is an undirected graph with vertex set $Z(S)^*$ and two distinct vertices x and y are adjacent if and only if $ann_S(xy) \neq ann_S(x) \cup ann_S(y)$, where $ann_S(x) = \{s \in S \mid xs = 0\}$. Some basic properties of $AG(S)$ are investigated in [1]. For example, it was proved that if $Z(S) \neq S$, then $\Gamma(S)$ is a subgraph of $AG(S)$.

Clearly $K_{1,2}$ and K_3, are all of the connected graphs with tree vertices. Now suppose that S is a commutative semigroup with $Z(S) \neq S$ and $|Z(S)| = 4$. Then since $AG(S)$ is a connected graph and $\Gamma(S)$ is a subgraph of $AG(S)$ therefore if $AG(S)$ is isomorphic to $K_{1,2}$, then $\Gamma(S)$ is isomorphic to $K_{1,2}$ and if $AG(S)$ is isomorphic to K_3, then $\Gamma(S)$ is isomorphic to $K_{1,2}$ or K_3.

In the continue, S is a commutative semigroup, with non-zero identity 1, such that $Z(S) = \{0, x, y, z, w\}$ and $Z(S) \neq S$. Then S has at least six elements. Without loss of generality, we may assume that $S = \{0, 1, x, y, z, w\}$. Since all of the graphs P_3, C_4, $K_{1,3}$, $K_3 + \{wx\}$, $K_4 \setminus \{wy\}$ and K_4, are connected graphs with four vertices, and they satisfy in conditions (1-4) of [4, Theorem 1], by [4, Theorem 2], for any one of these graphs, there is a semigroup S such that $\Gamma(S)$ is isomorphic to one of the graphs P_3, C_4, $K_{1,3}$, $K_3 + \{wx\}$, $K_4 \setminus \{wy\}$ or K_4.

Lemma 1.1. *If S is a commutative semigroup with $Z(S) \neq S$ and $\Gamma(S)$ is isomorphic to P_3, then $AG(S)$ is isomorphic to C_4.*

Proof. Suppose that $\Gamma(S)$ is isomorphic to P_3 with $w \sim x \sim y \sim z$. Then $wx = xy = yz = 0$, $wz \neq 0$, $wy \neq 0$ and $zx \neq 0$. If $wy = w$, then $wz = (wy)z = w(yz) = 0$, which is impossible. Also if $wy = x$ or $wy = z$, then $zx = 0$, which is again impossible. Thus $wy = y$, and so $w^2y = wy = y$. This implies that $w^2 \neq 0$. Similarly, $zx = x$ and $z^2 \neq 0$. Therefore w is not adjacent to y and also z is not adjacent to x in $AG(S)$. Now if $wz = w$ or $wz = z$, then $wy = 0$ or $zx = 0$, which is impossible. Hence $wz = x$ or $wz = y$. First suppose that $wz = x$. Then $x^2 = wzx = 0$. Since $w^2 \neq 0$, $wz = x$ and $wx = 0$, we have $w \in ann_S(x) = ann_S(wz)$ and $w \notin ann_S(w) \cup ann_S(z)$, which implies that w is adjacent to z in $AG(S)$. Similarly if $wz = y$, then $y^2 = (wz)y = w(zy) = 0$, $z^2 \neq 0$, and since $zy = 0$, we have w is again adjacent to z in $AG(S)$. Also we have $\Gamma(S)$ is a subgraph of $AG(S)$. Therefore $AG(S)$ is isomorphic to C_4. ☐

Lemma 1.2. *Let S be a semigroup with $Z(S) \neq S$. Then $AG(S)$ is isomorphic to C_4 if and only if we either have $\Gamma(S)$ is isomorphic to P_3 or C_4.*

Proof. If $\Gamma(S)$ is isomorphic to P_3, then, by Lemma 1.1, $AG(S)$ is isomorphic to C_4. Now suppose that $\Gamma(S)$ is isomorphic to C_4 with $wx = xy = yz = zw = 0$. So $zx \neq 0$ and $wy \neq 0$. If $wy = x$, then $xz = (wy)z = w(yz) = 0$, and if $wy = z$, then $xz = 0$, which are impossible. Hence $wy = w$ or $wy = y$, therefore w is not adjacent to y in $AG(S)$. By a similar argument, $zx = x$ or $zx = z$, and so z is not adjacent to x in $AG(S)$. Since C_4 is isomorphic to $\Gamma(S)$ and $\Gamma(S)$ is a subgraph of $AG(S)$ then $AG(S)$ is isomorphic to C_4.

Conversely, suppose on the contrary that $AG(S)$ is isomorphic to C_4 and $\Gamma(S)$ is isomorphic to non of the graphs P_3 or C_4. Then there exists a vertex x such that $d(x) = 3$ in $\Gamma(S)$. Since $\Gamma(S)$ is a subgraph of $AG(S)$, we have $d(x) = 3$ in $AG(S)$ is isomorphic to C_4, which is a contradiction. Therefore $\Gamma(S)$ is isomorphic to C_4 or P_3. ☐

If $\Gamma(S)$ is isomorphic to P_3 or C_4, then, by Lemma 1.2, $AG(S)$ is isomorphic to C_4. Thus if $AG(S)$ is isomorphic to K_4, then $\Gamma(S)$ is isomorphic to non of the graphs P_3 or C_4. Therefore if $AG(S)$ is isomorphic to K_4, then $\Gamma(S)$ is isomorphic to one of the graphs K_4, $K_4 \setminus \{wy\}$, $K_{1,3}$ or $K_3 + \{wx\}$. If $Z(S) \neq S$ and $\Gamma(S)$ is isomorphic to K_4, then $\Gamma(S)$ is a subgraph of $AG(S)$ and $AG(S)$ is isomorphic to K_4.

Proposition 1.3. *Let S be a commutative semigroup with $Z(S) \neq S$, and let $\Gamma(S)$ is isomorphic to $K_{1,3}$ with center x. Then $AG(S)$ is isomorphic to non of the graphs P_3, C_4 or $K_4 \setminus \{wy\}$.*

Proof. First assume on the contrary that $AG(S)$ is isomorphic to $K_4 \setminus \{wy\}$. Since $AG(S)$ is isomorphic to $K_4 \setminus \{wy\}$ and $\Gamma(S)$ is isomorphic to $K_{1,3}$ with center x, so $wz \neq 0$, $wy \neq 0$ and $yz \neq 0$. Also both of the vertices w and y are adjacent to z in $AG(S)$. Since z is adjacent to y in $AG(S)$, we have $zy = x$ or $zy = w$. Similarly, we have $wz = x$ or $wz = y$. Now if $zy = x$ or $wz = x$, then $wyz = 0$, and hence we have the following cases:

(a) If $wy = y$ or $wy = w$, then $yz = wyz = 0$ or $wz = wyz = 0$, which is impossible.

(b) If $wy = x$, then, since $z \notin ann_S(w) \cup ann_S(y)$ and $z \in ann_S(wy) = ann_S(x)$, we have w is adjacent to y in $AG(S)$, which is impossible.

(c) If $wy = z$, then $z^2 = wyz = 0$. Thus $z \in ann_S(z) = ann_S(wy)$ and $z \notin ann_S(w) \cup ann_S(y)$. Hence w is adjacent to y in $AG(S)$, which is again impossible.

Therefore $wz \neq x$ and $yz \neq x$, which implies that $wz = y$ and $yz = w$. Since $wz = y$, we have $wz^2 = zy \neq 0$ and $w^2z = wy \neq 0$. Thus $z^2 \neq 0$ and $w^2 \neq 0$. Now if $y^2 \neq 0$, then $ann_S(y) = ann_S(wz) = \{0, x\} = ann_S(w) \cup ann_S(z)$. This implies that z is not adjacent to w in $AG(S)$, which is impossible. Hence $y^2 = 0$. On the other hand, if $y^2 = 0$, then, since $zy = w$, we have $wy = (zy)y = zy^2 = 0$, which is a contradiction. Therefore $AG(S)$ is not isomorphic to $K_4 \setminus \{wy\}$. Also, since $\Gamma(S)$ is isomorphic to $K_{1,3}$ and $\Gamma(S)$ is a subgraph of $AG(S)$, we have $AG(S)$ is not isomorphic to P_3 or C_4. ☐

Proposition 1.4. *Let S be a commutative semigroup with $Z(S) \neq S$, and let $AG(S)$ is isomorphic to $K_4 \setminus \{wy\}$. Then $\Gamma(S)$ is isomorphic to non of the graphs P_3, C_4, K_4 or $K_{1,3}$.*

Proof. Suppose on the contrary that $\Gamma(S)$ is isomorphic to one of the graphs P_3, C_4, K_4 or $K_{1,3}$. Since $\Gamma(S)$ is a subgraph of $AG(S)$, if $\Gamma(S)$ is isomorphic to K_4, then $AG(S)$ is isomorphic to K_4, which is a contradiction. Now assume that $\Gamma(S)$ is isomorphic to P_3 or C_4. Then, by Lemma 1.2, $AG(S)$ is isomorphic to C_4, which is a contradiction. Also, if $\Gamma(S)$ is isomorphic to $K_{1,3}$, then, by Proposition 1.3, $AG(S)$ is not isomorphic to $K_4 \setminus \{wy\}$, which is again a contradiction. Thus $\Gamma(S)$ is isomorphic to non of the graphs P_3, C_4, K_4 or $K_{1,3}$. ☐

In [2, proposition 3.19], we have shown that S is a commutative semigroup with $Z(S) \neq S$, and if $\Gamma(S)$ is isomorphic to $K_{1,3}$ with center x, then $AG(S)$ is not isomorphic to $K_3 + \{wx\}$ with $w \sim x \sim y \sim z \sim x$. Therefore we have the following lemma.

Lemma 1.5. *Let S be a commutative semigroup with $Z(S) \neq S$, and let $AG(S)$ is isomorphic to $K_3 + \{wx\}$. Then $\Gamma(S)$ is isomorphic to $K_3 + \{wx\}$.*

Proof. Suppose that $Z(S) \neq S$ and $AG(S)$ is isomorphic to $K_3 + \{wx\}$. Then $\Gamma(S)$ is a subgraph of $AG(S)$, and so $\Gamma(S)$ is isomorphic to non of the graphs K_4 or $K_4 \setminus \{wy\}$. On the other hand, if $\Gamma(S)$ is isomorphic to P_3 or C_4, then, by Lemma 1.2, $AG(S)$ is isomorphic to C_4. Also if $\Gamma(S)$ is isomorphic to $K_{1,3}$, then, by [2, proposition 3.19], $AG(S)$ is not isomorphic to $K_3 + \{wx\}$. Therefore $\Gamma(S)$ is isomorphic to $K_3 + \{wx\}$. ☐

Proposition 1.6. *Let S be a commutative semigroup with $Z(S) \neq S$, and let $AG(S)$ is isomorphic to $K_{1,3}$. Then $\Gamma(S)$ is isomorphic to $K_{1,3}$.*

Proof. Suppose that $AG(S)$ is isomorphic to $K_{1,3}$. Since $\Gamma(S)$ is connected with four vertices, if $\Gamma(S)$ is not isomorphic to $K_{1,3}$, then $\Gamma(S)$ contains at least a path of length three. Also $Z(S) \neq S$, and so $\Gamma(S)$ is a subgraph of $AG(S)$. Thus $AG(S)$ contains at least a path of length three, which is impossible. Therefore $\Gamma(S)$ is isomorphic to $K_{1,3}$. ☐

2 Main Results

Corollary 2.1. *There is no semigroup S such that $Z(S) \neq S$ and $AG(S)$ is isomorphic to P_3.*

Proof. Suppose on the contrary that there is a semigroup S such that $AG(S)$ is isomorphic to P_3 with $w \sim x \sim y \sim z$. If $\Gamma(S)$ is not isomorphic to P_3 or C_4, then $\Gamma(S)$ has at least one vertex, say x, with $d(x) = 3$ in $\Gamma(S)$, and Since $\Gamma(S)$ is a subgraph of $AG(S)$, we have $d(x) = 3$ in $AG(S)$, which is a contradiction. Also if $\Gamma(S)$ is isomorphic to P_3 or C_4, then, by Lemma 1.2, $AG(S)$ is isomorphic to C_4, which is again a contradiction. Therefore there is no semigroup S such that $AG(S)$ is isomorphic to P_3. ☐

Theorem 2.2. *Suppose that S is a commutative semigroup with $Z(S) \neq S$ and $|Z(S)| = 5$. Then we have one of the following statements.*

1. *If $AG(S)$ is isomorphic to C_4, then $\Gamma(S)$ is isomorphic to P_3 or C_4.*
2. *If $AG(S)$ is isomorphic to K_4, then $\Gamma(S)$ is isomorphic to $K_{1,3}$, or $K_3 + \{wx\}$, or $K_4 \setminus \{wy\}$ or K_4.*
3. *If $AG(S)$ is isomorphic to $K_4 \setminus \{wy\}$, then $\Gamma(S)$ is isomorphic to $K_3 + \{wx\}$, or $K_4 \setminus \{wy\}$.*
4. *If $AG(S)$ is isomorphic to $K_3 + \{wx\}$, then $\Gamma(S)$ is isomorphic to $K_3 + \{wx\}$.*
5. *If $AG(S)$ is isomorphic to $K_{1,3}$, then $\Gamma(S)$ is isomorphic to $K_{1,3}$.*

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On Lifting of a Subset of the Fundamental Group

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Abstract

In this paper, by reviewing the concept of covering maps and semicovering maps, we define the lifting of a subset of the fundamental group of topological space X with respect to a (semi)covering map. Also we investigate the properties of this subset. For example, suppose $p : \tilde{X} \longrightarrow X$ is a (semi)covering map. A lifting subset H of $\pi_1(X, x_0)$ is a subgroup of $\pi_1(\tilde{X}, \tilde{x}_0)$ if and only if H is a subgroup of $\pi_1(X, x_0)$.

Keywords: Fundamental group, (Semi)covering map, Lifting path, Normal subgroup

Mathematics Subject Classification [2010]: Primary: 57M10, 57M12, Secondary: 57M05

1 Introduction and Preliminaries

Recall that a continuous map $p : \tilde{X} \longrightarrow X$ is called a covering of X, if for every $x \in X$ there is an open subset U of X with $x \in U$ such that U is evenly covered by p i.e. $p^{-1}(U)$ is a disjoint union of open subsets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p.
Assume that X and $\tilde{X}$ are topological spaces and $p : \tilde{X} \longrightarrow X$ is a continuous map. Let $f : (Y, y_0) \longrightarrow (X, x_0)$ be a continuous map and $\tilde{x}_0 \in p^{-1}(x_0)$. If there exists a continuous map $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ such that $p \circ \tilde{f} = f$, then $\tilde{f}$ is called a *lifting* of f.
The map p has *path lifting property* if for every path f in X, there exists a lifting $\tilde{f} : (I, 0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ of f. Also, the map p has *unique path lifting property* if for every path f in X, there is at most one lifting $\tilde{f} : (I, 0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ of f (see [3].)

Brazas [1, Definition 3.1] generalized the concept of covering map by the phrase *"A semicovering map is a local homeomorphism with continuous lifting of paths and homotopies"*. Note that a map $p : Y \to X$ has *continuous lifting of paths* if $\rho_p : (\rho Y)_y \to (\rho X)_{p(y)}$ defined by $\rho_p(\alpha) = p \circ \alpha$ is a homeomorphism, for all $y \in Y$, where $(\rho Y)_y = \{\alpha : I = [0, 1] \to Y | \alpha(0) = y\}$. Also, a map $p : Y \to X$ *has continuous lifting of homotopies* if $\Phi_p : (\Phi Y)_y \to (\Phi X)_{p(y)}$ defined by $\Phi_p(\phi) = p \circ \phi$ is a homeomorphism, for all $y \in Y$, where elements of $(\Phi Y)_y$ are endpoint preserving homotopies of paths starting at y. He also simplified the definition of semicovering maps by showing that having continuous lifting of paths implies having continuous lifting of homotopies.

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[*] Speaker.

Lemma 1.1. *Let $p : \tilde{X} \to X$ be a local homeomorphism with unique path lifting and path lifting properties. Let $x_0, x_1 \in X$ and $f, g : I \to X$ be paths such that $f(0) = g(0) = x_0$, $f(1) = g(1) = x_1$ and $\tilde{x}_0 \in p^{-1}(x_0)$. If $F : f \simeq g$ rel $\dot{I}$ and $\tilde{f}, \tilde{g}$ are the lifting of f and g, respectively, with $\tilde{f}(0) = \tilde{x}_0 = \tilde{g}(0)$, then $\tilde{F} : \tilde{f} \simeq \tilde{g}$ rel $\dot{I}$.*

The following theorem can be found in [1, Corollary 2.6 and Proposition 6.2].

Theorem 1.2. *(Lifting Criterion Theorem for Semicovering Maps).*
If Y is connected and locally path connected, $f : (Y, y_0) \to (X, x_0)$ is continuous and $p : \tilde{X} \to X$ is a semicovering map where $\tilde{X}$ is path connected, then there exists a unique $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ such that $p \circ \tilde{f} = f$ if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

The following theorem can be concluded from [2, Theorem 2.4].

Theorem 1.3. *A map $p : \tilde{X} \to X$ is a semicovering map if and only if it is a local homeomorphism with unique path lifting and path lifting properties.*

In this paper, we introduce the lifting of a subset of $\pi_1(X, x_0)$ with respect to p. Also we investigate the properties of this subset. For example, suppose $p : \tilde{X} \longrightarrow X$ is a (semi)covering map. A lifting subset $\tilde{H}$ of $\pi_1(X, x_0)$ is a subgroup of $\pi_1(\tilde{X}, \tilde{x}_0)$ if and only if H is a subgroup of $\pi_1(X, x_0)$. Also let $p : \tilde{X} \longrightarrow X$ be a (semi)covering map. Then p is a universal covering map if and only if $H = \pi_1(X, x_0)$ and A lifting subset $\tilde{H}$ of $\pi_1(X, x_0)$ is trivial.

2 Main Results

Let $p : \tilde{X} \longrightarrow X$ be a (semi)covering map. We know lifting of path with respect to p has the important role in studying (semi)covering map. We study concept of lifting of a subset of $\pi_1(X, x_0)$ with respect to p and its role on p. Supoose $H \subseteq \pi_1(X, x_0)$ and $\tilde{x}_0 \in \tilde{X}$, we define $\tilde{H}_{p,\tilde{x}_0} := \{[g] \in \pi_1(\tilde{X}, \tilde{x}_0) | [p \circ g] \in H\}$. In order not to make a mistake, we denote $\tilde{H}_{p,\tilde{x}_0}$ by $\tilde{H}$.

Theorem 2.1. *Let $p : \tilde{X} \longrightarrow X$ be a (semi)covering map. $H \leq \pi_1(X, x_0)$ if and only if $\tilde{H} \leq \pi_1(\tilde{X}, \tilde{x}_0)$.*

Lemma 2.2. *If $p : \tilde{X} \longrightarrow X$ is a (semi)covering map and $H' \subseteq H$, then $\tilde{H'} \subseteq \tilde{H}$.*

The following corollary is a consequence of the above lemma.

Corollary 2.3. *If $p : \tilde{X} \longrightarrow X$ is a(semi)covering map, then $< \tilde{H} >= \tilde{K}$ where $K =< H >$.*

In the following theorem, we show that if $H = \pi_1(X, x_0)$, $\tilde{H} = 1$, then p is a universal covering map and the converse is true.

Theorem 2.4. *Let $p : \tilde{X} \longrightarrow X$ be a (semi)covering map. Then p is a universal covering map if and only if $H = \pi_1(X, x_0)$ and $\tilde{H} = 1$.*

Proof. Let $[\alpha] \in \pi_1(\tilde{X}, \tilde{x_0})$, so by definition of coveing map $[p \circ \alpha] = p_*([\alpha]) \in \pi_1(X, x_0) = H$. So $[\alpha] = \tilde{H} = 1$, thus $\pi_1(\tilde{X}, \tilde{x_0}) = 1$. Therefore by definition of universal covering map, p is a universal covering map and the converse is trivial. □

The following corollary is a consequence of the above theorem.

Corollary 2.5. *If $p : (\tilde{X}, \tilde{x_0}) \longrightarrow (X, x_0)$ is a (semi)covering map and $K = H \cap p_*(\pi_1(\tilde{X}, \tilde{x_0}))$, then $\tilde{H} = \tilde{K}$.*

Proof. By Lemma 2.2, $\tilde{K} \subseteq \tilde{H}$. Suppose α is an arbitrary element of $\tilde{H}$, so there exist $\beta \in H$ such that $p \circ \beta = \alpha$. thus $\beta \in p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ and therefore $\beta \in K$. So $\alpha \in \tilde{K}$. □

In the following example, we intruduced a (semi)covering map such that if $H = 2\mathbb{Z}$ then $\tilde{H} = 1$ and $H = 4\mathbb{Z}$ then $\tilde{H} = \mathbb{Z}$.

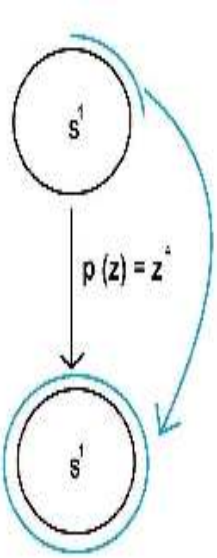

Figure 1: $p : S^1 \longrightarrow S^1$ defined by $p(z) = z^4$

Example 2.6. Consider the famous covering map $p : S^1 \longrightarrow S^1$ defined by $p(z) = z^4$ (see Figure 1), if $H = 2\mathbb{Z}$ then $\tilde{H} = 1$ and $H = 4\mathbb{Z}$ then $\tilde{H} = \mathbb{Z}$.

In the definition (semi)covering map, we see that $\tilde{H} \subseteq \pi_1(\tilde{X}, \tilde{x_0})$. Note that $\tilde{H} \subseteq \pi_1(\tilde{X}, \tilde{x_0})$ is not a subset of $\pi_1(\tilde{X}, \tilde{x})$ for any point $\tilde{x} \neq \tilde{x_0}$. To present a similar fact, we can consider subsets corresponding to $\tilde{H}$ in $\pi_1(\tilde{X}, \tilde{x})$ by the isomorphism $\psi_\alpha : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(\tilde{X}, \tilde{x})$ for every path α from $\tilde{x_0}$ to $\tilde{x}$. We denote $[\alpha]^{-1}\tilde{H}[\alpha]$ by $\alpha^{-1}\tilde{H}\alpha$.

Lemma 2.7. *If $p : \tilde{X} \longrightarrow X$ is a (semi)covering map and α is a path in $\tilde{X}$ with starting at $\tilde{x_0}$ and $\alpha(1) = x$ and $H_{p\circ\alpha} := (p \circ \alpha)^{-1}H(p \circ \alpha)$, then $\tilde{H}_{p\circ\alpha} = \alpha^{-1}\tilde{H}\alpha$.*

The following corollary is a consequence of the above lemma.

Corollary 2.8. *If $p : \tilde{X} \longrightarrow X$ is a (semi)covering map and α is a path in $\tilde{X}$ with starting at $\tilde{x_0}$ and $\alpha(1) = x$ such that $p \circ \alpha$ is a loop and H is a normal subgroup of $\pi_1(\tilde{X}, \tilde{x})$, then $H_{p\circ\alpha} = H$ and $\tilde{H}_{p\circ\alpha} = \alpha^{-1}\tilde{H}\alpha$.*

Proof. Let $p : \tilde{X} \longrightarrow X$ is a (semi)covering map and α is a path in $\tilde{X}$ with starting at $\tilde{x_0}$ and $\alpha(1) = x$ such that $p \circ \alpha$ is a loop. Since $[p \circ \alpha]$ is a loop, $[p \circ \alpha] \in \pi_1(\tilde{X}, \tilde{x})$ and H is a normal subgroup of $\pi_1(\tilde{X}, \tilde{x})$, so $(p \circ \alpha)^{-1}H(p \circ \alpha) = H$. Thus by Lemma 2.7 $\tilde{H}_{p\circ\alpha} = \alpha^{-1}\tilde{H}\alpha$. □

In the following corollary, we show that for a (semi)covering map $p : \tilde{X} \longrightarrow X$ where α is a loop in $\tilde{X}$ at $\tilde{x_0}$ and $\tilde{H}_{p\circ\alpha}$ is a normal subgroup of $\pi_1(\tilde{X}, \tilde{x_0})$, then $\tilde{H}_{p\circ\alpha} = \tilde{H}$ and $H_{p\circ\alpha} = (p \circ \alpha)^{-1}H(p \circ \alpha)$.

Corollary 2.9. *If $p : \tilde{X} \longrightarrow X$ is a (semi)covering map and α is a loop in $\tilde{X}$ at $\tilde{x_0}$ and $\tilde{H}_{p\circ\alpha}$ is a normal subgroup of $\pi_1(\tilde{X}, \tilde{x_0})$, then $\tilde{H}_{p\circ\alpha} = \tilde{H}$ and $H_{p\circ\alpha} = (p \circ \alpha)^{-1}H(p \circ \alpha)$.*

Proof. Let $p : \tilde{X} \longrightarrow X$ is a (semi)covering map and α is a loop in $\tilde{X}$ at $\tilde{x_0}$. Since $\tilde{H}_{p\circ\alpha}$ is a normal subgroup of $\pi_1(\tilde{X}, \tilde{x_0})$, $\alpha^{-1}\tilde{H}\alpha = \tilde{H}$. So by the Lemma 2.7, $H_{p\circ\alpha} = H$ and $\tilde{H}_{p\circ\alpha} = \alpha^{-1}\tilde{H}\alpha$. □

Corollary 2.10. *If $p : \tilde{X} \longrightarrow X$ is a (semi)covering map and λ is a path in X with starting at x and $\tilde{\lambda}$ is lifting of λ with starting at $\tilde{x_0}$ and $H_\lambda = \lambda^{-1}H\lambda$, then $\tilde{H}_\lambda = \lambda^{-1}\tilde{H}\lambda$.*

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تجزیه مقدار تکین در یادگیری ماشین

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چکیده. در مسائل کلاسیک یادگیری ماشین، ماتریس‌ها و بردارها برای نمایش داده‌ها و پارامترها به کار می‌روند. تجزیه‌های ماتریسی هم در حوزه یادگیری ماشین، بسیار قابل ملاحظه و حائز اهمیت هستند. در این میان، تجزیه مقدار تکین که غالبا با نام SVD نیز معرفی می‌شود، یک ابزار ریاضی بسیار قدرتمند در دنیای علم داده و یادگیری ماشین به حساب می‌آید. SVD در درجه اول برای کاهش بعد، استخراج ویژگی، کاهش نویز و فشرده سازی داده‌ها استفاده می‌گردد. در این مقاله، نگاهی موشکافانه تر به این تجزیه جالب داشته و با مثالی تصویرسازی شده از دنیای واقعی، کاربرد آن را در یادگیری ماشین توضیح خواهیم داد.

۱. مقدمه

بسیاری از مسائل علم داده و یادگیری ماشین از جمله پیش پردازش داده ها، استخراج ویژگی، داده کاوی، تجزیه و تحلیل داده ها، پیش بینی و دسته بندی داده ها، به زبان ریاضی و با استفاده از مفاهیم ریاضیاتی حل می‌شوند. مزایای کاهش بعد، کاهش حافظه و حجم محاسبات، تشخیص داده‌های پرت (نوفه)، و کشف روابط و عامل‌های پنهان در ماتریس اولیه است.
کاهش بعد درمرحلهٔ پیش پردازش اطلاعات انجام می‌شود. برخی از روش‌های کاهش بعد، یک تقریب کم رتبه را از داده‌های با بعد بالا استخراج می‌کنند. از بهترین انواع روش‌های کاهش بعد می‌توان به تجزیهٔ مقدار تکین اشاره کرد.

قضیه ۱.۱ ([۱]). ماتریس $A \in \mathbb{R}^{m\times n}$ با رتبه r را می‌توان به صورت $U\Sigma V^T$ تجزیه کرد که در آن $U \in \mathbb{R}^{m\times m}$ و $V \in \mathbb{R}^{n\times n}$ ماتریس هایی متعامد و $\Sigma = diag(\sigma_۱, \sigma_۲, \dots \sigma_p)$ با $p = min\{m, n\}$ ماتریسی قطری است به طوری که

$$\sigma_۱ \geq \sigma_۲ \geq \cdots \geq \sigma_r > ۰, \qquad \sigma_r + ۱ = \cdots \sigma_p = ۰.$$

تجزیهٔ ماتریس A به صورت $A = U\Sigma V^T$ را تجزیهٔ مقدار تکین A می نامیم.

2020 Mathematics Subject Classification. Primary: 13HXX, 05EXX; Secondary: 16WXX, 05EXX.

واژگان کلیدی. یادگیری ماشین، تجزیه مقدار تکین.

برای ما، U جهت داده‌ها در A را اخذ می‌کند، Σ بزرگی یا اهمیت داده‌ها را نشان می‌دهد و در نهایت V همبستگی داده‌ها در A را ضبط می‌کند.

فرض کنید $A = U\Sigma V^T$ با رتبهٔ r، $k < r$ عدد صحیح مثبت و $A_k = U_k\Sigma_k V_k^T$ باشد که در آن U_k و V_k به ترتیب شامل k ستون اول U و V و Σ_k ماتریس قطری $k \times k$ شامل k سطر و k ستون اول Σ است. ماتریس A_k یک تقریب برای ماتریس A با رتبهٔ k است که آن را می‌توان به صورت مجموعی از ماتریس‌های رتبه یک نوشت:

$$A_k = \sigma_1 u_1 v_1^T + \cdots + \sigma_k u_k v_k^T, \quad k < r,$$

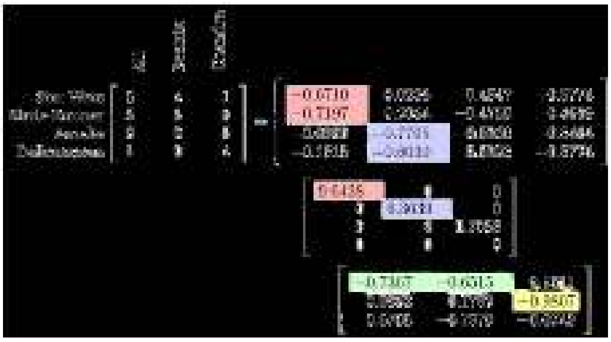
که در اینجا، u_j و v_j به ترتیب، ستون‌های U و V را نشان می‌دهند. ماتریس A_k بهترین تقریب از رتبهٔ k در نرم‌های دو و فروبنیوسی برای ماتریس A است.

به عنوان مثال، ماتریس U_k بیانگر اطلاعات مربوط به کاربران (تعداد سطرهای آن برابر m، تعداد کاربران) و ماتریس V_k بیانگر اطلاعات مربوط به کالاها (تعداد ستون‌های آن برابر n، تعداد کالاها) می تواند باشد. از طرفی، ماتریس $A_k A_k^T$ بیانگر تشابه بین کاربران و ماتریس $A_k^T A_k$ بیانگر تشابه بین کالاهاست. [۱]

۲. یک مثال کاربردی [۲]: یافتن ویژگی در رتبه بندی فیلم و مصرف کنندگان

می خواهیم یک تفسیر عملی از تجزیه مقدار تکین با تحلیل داده هایی درباره مردم و فیلم‌های مورد علاقه شان ارایه کنیم.

سه بیننده به نام‌های علی، بئاتریس و چاندرا را در نظر بگیرید که چهار فیلم جنگ ستارگان، بلید رانر، املی و اغذیه فروشی را امتیازدهی کرده اند. این امتیازات مقادیری بین صفر (بدترین) و پنج (بهترین) بوده و در یک ماتریس داده به نام $A \in \mathbb{R}^{4\times 3}$ مطابق شکل ۱ کدگذاری شده اند. هر سطر، یک فیلم و هر ستون یک کاربر را نشان می‌دهد. بنابراین، بردار ستون‌های رتبه بندی فیلم، هر کدام متعلق به یک بیننده از بین علی، بئاتریس و چاندرا است.



شکل ۱: ماتریس امتیازدهی

تجزیه A به کمک SVD راهی را برای استخراج روابطی در اختیار ما می‌گذارد که طی آن متوجه می‌شویم مردم چگونه به فیلم‌ها امتیاز می‌دهند؛ به ویژه اگر ساختاری وجود داشته باشد که نشان دهد چه کسی چه فیلمی را می‌بیند.

اعمال SVD به ماتریس داده A چند فرضیه را مطرح می‌کند:

الف) همه بینندگان با بهره گیری از یک نگاشت خطی یکسان به فیلم‌ها امتیاز می‌دهند.
ب) هیچ خطا یا نویزی در رتبه بندی وجود ندارد.
پ) بردارهای تکین چپ u_i را می‌توان به عنوان فیلم‌های کلیشه ای و بردارهای تکین راست v_j را به عنوان بینندگان کلیشه ای تعبیر کرد.

بعد از این، فرض را بر این می‌گذاریم که ترجیحات سینمایی خاص هر بیننده ای را می‌توان به عنوان یک ترکیب خطی از v_j بیان کرد. به طور مشابه، قابلیت همانندی فیلم را می‌توان به صورت ترکیبی خطی از u_i نوشت. از این رو، یک بردار در دامنه SVD را می‌توان به عنوان یک بیننده در فضای بینندگان کلیشه ای و یک بردار در حوزه هم دامنه SVD را هم می‌توان به طور متناظر به عنوان یک فیلم در فضای فیلم‌های کلیشه ای متصور شد.

حال، اجازه دهید تجزیه SVD ماتریس کاربرفیلم خود را بررسی کنیم:

- اولین بردار تکین سمت چپ یعنی $u_۱$ حاوی مقادیر بزرگی برای دو فیلم علمی تخیلی و اولین مقدار تکین بزرگ است. بنابراین، $u_۱$ نوعی از کاربران مجموعه خاصی از فیلم‌ها یعنی علمی تخیلی را دسته بندی می‌کند. به طور مشابه، اولین مقدار تکین سمت راست یعنی $v_۱$ مقادیر بزرگی را برای علی و بئاتریس، یعنی افرادی که امتیاز بالایی به فیلم‌های علمی تخیلی داده اند، نشان می‌دهد. این دریافت‌ها این فکر را تلقین می‌کنند که گویی $v_۱$ ادراکی از یک فرد عاشق فیلم‌های علمی تخیلی را منعکس می‌کند.
- از طرفی، به نظر می‌رسد $u_۲$ تم فیلم خانه هنری فرانسوی را به تصویر می‌کشد و $v_۲$ نشان می‌دهد که چاندرا به یک عاشق ایده‌آل چنین فیلم هایی شبیه است.
- یک عاشق ایده‌آل فیلم‌های علمی تخیلی، فردی متعصب است که تنها فیلم‌های علمی تخیلی را دوست دارد و لذا عاشق علمی تخیلی v_j به همه چیز جز فیلمی که تم علمی تخیلی دارد، امتیاز صفر می‌دهد. این منطق توسط ساختار قطری ماتریس مقدار تکین Σ هم تلویحا تأیید می‌شود.
- بنابراین، یک فیلم خاص، براساس اینکه چگونه به طور خطی به فیلم‌های کلیشه ای تجزیه می‌شود، نمایانده می‌شود. همین طور، یک فرد هم از طریق اینکه چگونه با یک ترکیب خطی تم‌های سینمایی را دسته بندی می‌کند، نمایش داده می‌شود.

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Generalization of Sandor Type Inequality

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Abstract

Sandor type inequality for two classes of pseudo-integrals are shown. One of them deals with pseudo-integrals where pseudo-operations ($\odot$ and $\otimes$) are defined via a monotone continuous generator function. The other one concerns the pseudo-integrals based on a semiring with an idempotent addition and a generated pseudo-multiplication.

Keywords: Sandor type inequality, Fuzzy integral inequality, Pseudo-integral, Semiring

Mathematics Subject Classification [2010]: Primary:35A23, 26E50, 28A25.

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1 Introduction and Preliminaries

The theory of fuzzy measures and fuzzy integral (Sugeno integral) has introduced by Sugeno [5] in his Ph.D. theses on 1974. The properties and applications of fuzzy integral have been studied by many authors.

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a, b] \subseteq [-\infty, +\infty]$ endowed with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$ ([3]). Based on this structure, there were developed the concepts of $\oplus$ measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc.

Sandor inequality in classical case is the following form.

Theorem 1.1 ([1]). *Let $f : [a, b] \to \mathbb{R}$ be a convex and non-negative function. Then*

$$\frac{1}{b-a}\int_a^b f^2(x)dx \leq \frac{1}{3}\left[f^2(a) + f(a)f(b) + f^2(b)\right], \tag{1}$$

holds.

Sandor's inequality is proved in some versions for Sugeno integrals, for more details of these versions, we refer reader to [1].

Now, we are going to review some well known results of pseudo-operations, pseudo-analysis and pseudo-additive measures and integrals in details.

Let $[a, b]$ be a closed (in some cases can be considered semi-closed) subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$.

Definition 1.2 (Wang and Klir [4]). The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a,b] \times [a,b] \to [a,b]$ which is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$, i.e., for each $x \in [a,b], \mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either a or b).

Let $[a,b]_+ = \{x | x \in [a,b], \mathbf{0} \preceq x\}$.

Definition 1.3 (Wang and Klir [4]). The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a,b] \times [a,b] \to [a,b]$ which is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a,b]_+$, associative and for which there exists a unit element $\mathbf{1} \in [a,b]$, i.e., for each $x \in [a,b], \mathbf{1} \odot x = x$.

We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e., $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

We shall consider the semiring $([a,b], \oplus, \odot)$ for two important (with completely different behavior) cases. The first case is when pseudo-operations are generated by a monotone and continuous function $g : [a,b] \to [0,\infty)$, i.e., pseudo-operations are given with:

$$x \oplus y = g^{-1}(g(x) + g(x)) \qquad \text{and} \qquad x \odot y = g^{-1}(g(x)g(x)). \tag{2}$$

Then, the pseudo-integral for a function $f : [c,d] \to [a,b]$ reduces on the $g-$integral

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\left(\int_c^d g(f(x))dx\right). \tag{3}$$

More details on this structure as well as corresponding measures and integrals can be found in [2]. The second class is when $x \oplus y = \max(x,y)$ and $x \odot y = g^{-1}(g(x)g(y))$, the pseudo-integral for a function $f : \mathbb{R} \to [a,b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = \sup_{x \in \mathbb{R}} \left(f(x) \odot \psi(x)\right),$$

where function ψ defines sup-measure m. Any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-additine.

Theorem 1.4. *(Mesiar and Pap [2]). Let $([0,\infty], \sup, \odot)$ be a semiring , when $\odot$ is a generated with g, i.e., we have $x \odot y = g^{-1}(g(x)g(y))$ for every $x, y \in (0,\infty)$. Let m be the same as in Theorem* **??**, *Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$ -measures, where $\oplus_\lambda$ is a generated by $g^\lambda, \lambda \in (0,\infty)$ such that for every continuous function $f : [0,\infty] \to [0,\infty]$,*

$$\begin{aligned} \int^{\sup} f \odot dm &= \lim_{\lambda \to \infty} \int^{\oplus_\lambda} f \odot dm_\lambda \\ &= \lim_{\lambda \to \infty} (g^\lambda)^{-1}\left(\int g^\lambda(f(x))dx\right). \end{aligned} \tag{4}$$

2 Main Result

In this section, we express and prove Sandor's inequality for pseudo-integrals.

Theorem 2.1. *Let $f : [a,b] \to [c,d]$ be a continuous, convex and non-negative function and $g : [c,d] \to [0,\infty)$ be a continuous and increasing function. Then*

$$\left(\frac{1}{b-a}\right) g\left(\int_{[a,b]}^{\oplus} f_{\odot}^2(x)dx\right) \leq \frac{1}{3} g\Big(\left[f_{\odot}^2(a) \oplus f(a) \odot f(b) \oplus f_{\odot}^2(b)\right]\Big), \tag{5}$$

holds.

Proof. From the right side of inequality, we have

$$\begin{aligned}
&\frac{1}{3}\left[f_{\odot}^{2}(a)\oplus f(a)\odot f(b)\oplus f_{\odot}^{2}(b)\right]\\
&\quad=\frac{1}{3}\left\{\left[g^{-1}(g(f)(a)\cdot g(f)(a))\oplus g^{-1}\left(g(f)(a)\cdot g(f)(b)\right)\right]\right.\\
&\quad\quad\left.\oplus\left[g^{-1}\left(g(f)(b)\cdot g(f)(b)\right]\right\}\right.\\
&\quad=\frac{1}{3}\left\{g^{-1}\Big(g(f)^{2}(a)+g(f)(a)\cdot g(f)(b)+g(f)^{2}(b)\Big)\right\}.
\end{aligned}$$

Continuting left side of the above mentioned inequality follows that:

$$\begin{aligned}
&\frac{1}{3}\left(g^{-1}\left[\left(\frac{3}{b-a}\right)\int_{a}^{b}g(f)(x)\cdot g(f)(x)dx\right]\right)\\
&\quad=\frac{1}{3}\left(g^{-1}\left[\left(\frac{3}{b-a}\right)\int_{a}^{b}gg^{-1}\left(g(f)(x)\cdot g(f)(x)dx\right)\right]\right)
\end{aligned}$$

It follows that:

$$\begin{aligned}
&\frac{1}{3}\left(g^{-1}\left[\left(\frac{3}{b-a}\right)g\left(\int_{[a,b]}^{\oplus}f_{\odot}^{2}(x)dx\right)\right]\right)\\
&\quad\leq\frac{1}{3}\left[f_{\odot}^{2}(a)\oplus f(a)\odot f(b)\oplus f_{\odot}^{2}(b)\right].
\end{aligned}$$

Then

$$\left(\frac{1}{b-a}\right)g\left(\int_{[a,b]}^{\oplus}f_{\odot}^{2}(x)dx\right)\leq\frac{1}{3}g\left(f_{\odot}^{2}(a)\oplus f(a)\odot f(b)\oplus f_{\odot}^{2}(b)\right).$$

Thereby, the proof is complete. ☐

When, we restrict our argument to semiring $([0,1],\oplus,\odot)$, we obtain the following theorem.

Corollary 2.2. *Let $f:[0,1]\to[c,d]$ be a continuous, convex and non-negative function and $g:[c,d]\to[0,\infty)$ be a continuous and increasing function. Then*

$$\left(\frac{1}{b-a}\right)g\left(\int_{[0,1]}^{\oplus}f_{\odot}^{2}(x)dx\right)\leq\frac{1}{3}g\left[f_{\odot}^{2}(0)\oplus f(0)\odot f(1)\oplus f_{\odot}^{2}(1)\right],\tag{6}$$

holds.

Example 2.3. Let f and g are defined from $[0,1]$ to $[0,1]$ by $f(x)=x^2$ and $g(x)=\sqrt{x}$. Then we have

$$\frac{1}{4}=\frac{1}{1-0}\int_{[0,1]}^{\oplus}f_{\odot}^{2}(x)dx\leq\frac{1}{3}g\left[f_{\odot}^{2}(0)\oplus f(0)\odot f(1)\oplus f_{\odot}^{2}(1)\right]=\frac{1}{3}.$$

The following example shows that convexity of f in Theorem 2.1 is necessary.

Example 2.4. Suppose that $f(x)=\sqrt{x}$ and $g(x)=x^2$. Them simple calculaion show that

$$\frac{2}{3}=\frac{1}{1-0}\int_{[0,1]}^{\oplus}f_{\odot}^{2}(x)dx\not\leq\frac{1}{3}g\left[f_{\odot}^{2}(0)\oplus f(0)\odot f(1)\oplus f_{\odot}^{2}(1)\right]=\frac{1}{3}.$$

We can not remove the assumption g is increasing in Theorem 2.1. The following example shows this fact.

Example 2.5. Let $f(x)=\sqrt{x}$ and $g(x)=\sqrt{1-x}$. Then we have

$$\frac{5}{6}=\frac{1}{1-0}\int_{[0,1]}^{\oplus} f_{\odot}^{2}(x)dx \nleq \frac{1}{3}g\left[f_{\odot}^{2}(0)\oplus f(0)\odot f(1)\oplus f_{\odot}^{2}(1)\right]=\frac{1}{3}.$$

Corollary 2.6. *In the Theorem 2.1, if we suppose that $g(x)=x$, then the Inequality (5) follows that Inequality (1).*

Now, we generalize the Sandor type inequity by the semiring $([a,b],\sup,\odot)$.

Theorem 2.7. *Let $f:[a,b]\to[a,b]$ be a measurable comonotone function and $([a,b],\sup,\odot)$ be a simiring and m be the same as Theorem 1.4. If g is a continuous and increasing function, then the following inequality*

$$\left(\frac{1}{b-a}\right)g\left(\int_{[a,b]}^{\sup} f_{\odot}^{2}(x)dx\right)\leq\frac{1}{3}g\left[f_{\odot}^{2}(a)\oplus f(a)\odot f(b)\oplus f_{\odot}^{2}(b)\right], \tag{7}$$

holds.

Proof. The proof is similar to the Theorem 2.1. □

3 Conclusion

In this paper, we have proved Sandor type inequality for pseudo integrals. More precisely: Let $f:[a,b]\to[c,d]$ be a continuous, convex and non-negative function and $g:[c,d]\to[0,\infty)$ be a continuous and increasing function. Then

$$g\left(\frac{1}{b-a}g\left(\int_{[a,b]}^{\oplus} f_{\odot}^{2}(x)dx\right)\right)\leq\frac{1}{3}\left(g\left[f_{\odot}^{2}(a)\oplus f(a)\odot f(b)\oplus f_{\odot}^{2}(b)\right]\right),$$

holds. Also we have given some illustrate examples.

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A Note on Frames in Quaternionic Hilbert Spaces

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Abstract

Quaternions combine a real scalar with a 3D vector containing real coefficients. Many concepts in functional analysis and operator theory have been studied in quaternionic spaces. In this note, we will review quaternionic Hilbert spaces and then explore frame theory from the perspective of quaternions.

Keywords: Quaternionic Hilbert spaces, Frames, Bessel sequences

Mathematics Subject Classification [2010]: 42C15, 42A38

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1 Quaternionic Hilbert Spaces

Quaternions can be considered as the combination of a real scalar and a 3D vector that has real coefficients. This vector forms the imaginary part of the quaternion. Quaternionic number systems are division rings. Other division rings are real numbers and complex numbers. Octonions do not form a division ring. As remarked by Birkhoff and von Neumann in their celebrated seminal work on Quantum Logic in 1936 [1], Quantum Mechanics may alternatively be formulated on a Hilbert space where the ground field of complex numbers is replaced for the division algebra of quaternions. Nowadays, the picture is more clear on the one hand and more strict on the other hand, after the efforts started in 1964 by Piron [3] and concluded in 1995 by Sol‘er [5].

Irish mathematician William Rowan Hamilton introduced quaternions in 1843. He aimed to extend the idea of complex numbers, which represent points in plane (with addition and multiplication properties), to points in space. While points in space are represented as coordinates of triplets, adding and subtracting them was known, but multiplying and dividing them posed a challenge unlike complex numbers. The great breakthrough in quaternions finally came, when Hamilton carved the formula for the quaternions given by

$$i^2 = j^2 = k^2 = ijk = -1.$$

Quaternions are used in pure as well as applied mathematics, especially in calculations involving three-dimensional rotations, as seen in computer graphics, crystallographic texture analysis, robotics and physics. Rotations represented using quaternions are more concise, effective, and numerically stable for computation as compared to representations using matrices. In modern terms, quaternions are a four dimensional non-commutative extension of the complex numbers over the set of real numbers. It was the first non-commutative four-dimensional associative normed division algebra over the real numbers, and therefore a ring, being a skew-field.

Quaternions are a four dimensional non-commutative extension of the complex numbers over $\mathbb{R}$ and are generally denoted by $\mathbb{H}$. The set of quaternions $\mathbb{H}$ contain elements of the form

$$q = q_0 + iq_1 + jq_2 + kq_3, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}.$$

These elements, called quaternions, form a four-dimensional vector space over $\mathbb{R}$ under component-wise addition and component-wise scalar multiplication with basis set $\{1, i, j, k\}$. Moreover, the product of any two quaternions (also called Hamilton product) is determined by the multiplication of the basis elements and the distributive law. Within the set $\mathbb{H}, 0$ represents the null element, while 1 represents the identity element with respect to multiplication. This enables $\mathbb{H}$ to form a non-commutative ring with unity, thereby forming a skew field.

For any $q = q_0 + iq_1 + jq_2 + kq_3 \in \mathbb{H}$. q_0 is called the real (or scalar part) and $iq_1 + jq_2 + kq_3$ is called the imaginary (or vector part) of q. The conjugate of q is given by $\overline{q} := q_0 - iq_1 - jq_2 - kq_3$. The norm of q is defined as $|q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$

For any non-zero quaternion $q = q_0 + iq_1 + jq_2 + kq_3 \in \mathbb{H}$. , there exists a unique inverse q^{-1} given by $q^{-1} = \frac{\overline{q}}{|q|^2}$

Right quaternionic vector spaces are basically right modules over the ring $\mathbb{H}$. The formal definition of a right quaternionic vector space is as follows:

Definition 1.1 ([4]). A right quaternionic vector space $\mathcal{H}_{\mathbb{H}}$ is a linear vector space with the right multiplication of scalars which satisfies the following properties:

(i) $(h_1 + h_2)q = h_1q + h_2q, \quad h_1, h_2 \in \mathcal{H}_{\mathbb{H}}, q \in \mathbb{H}.$

(ii) $h(p + q) = hp + hq, \quad h \in \mathcal{H}_{\mathbb{H}}, p, q \in \mathbb{H}.$

(iii) $h(pq) = (hp)q, \quad h \in \mathcal{H}_{\mathbb{H}}, p, q \in \mathbb{H}.$

Definition 1.2 ([4]). A right quaternionic pre-Hilbert space (or right quaternionic inner product space) $\mathcal{H}_{\mathbb{H}}$is a right quaternionic vector space endowed with the inner product $\langle .|. \rangle : \mathcal{H}_{\mathbb{H}}\mathcal{H}_{\mathbb{H}}$ which satisfies the following properties:

(i) $\langle h|h_1 + h_2\rangle = \langle h|h_1\rangle + \langle |h_2\rangle, \quad h, h_1, h_2 \in \mathcal{H}_{\mathbb{H}}.$

(ii) $\langle h|h\rangle > 0, \quad h \neq 0.$

(iii) $\overline{\langle h_1|h_2\rangle} = \langle h_2|h_1\rangle, \quad h_1, h_2 \in \mathcal{H}_{\mathbb{H}}.$

(iv) $\langle h_1|h_2q\rangle = \langle h_1|h_2\rangle q, \quad h_1, h_2 \in \mathcal{H}_{\mathbb{H}}, q \in \mathbb{H}.$

A right quaternionic Hilbert space is a right quaternionic pre-Hilbert space which is complete with respect to the norm induced by the above defined inner product. Define the quaternionic norm $\|.\| : \mathcal{H}_{\mathbb{H}} \to \mathbb{H}^+$ on $\mathcal{H}_{\mathbb{H}}$ by $\|h\| = \sqrt{\langle h|h\rangle}$.

2 Frames in Complex and Quaternionic Hilbert Spaces

Traditionally, most of the mathematical and analytical topics have been studied on complex or real Hilbert spaces. In the last few decades, many concepts of functional analysis and operator theory on quaternion Hilbert spaces have been investigated and studied, and by using the ideas in complex or real Hilbert has tried to generalize those concepts in quaternion Hilbert spaces, and interestingly, almost similar proofs have been presented for most of the relevant. theorems. In this category, we intend to study and examine the concept of frames and continuous frames in these spaces.

Frames for Hilbert spaces have been first introduced by Duffin and Scheaffer in the study of some problems in nonharmonic Fourier series in 1952, [2]. A discrete frame is a countable family of elements in a separable Hilbert space which allows for a stable, not necessarily unique, decomposition of an arbitrary element into an expansion of the frame elements.

Recall that for a Hilbert space $\mathcal{H}$ and a countable index set I, a family of vectors $\{f_i\}_{i\in I} \subseteq \mathcal{H}$ is called a discrete frame for $\mathcal{H}$, if there exist constants $0 < A \leq B < +\infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H},$$

the constants A and B are called frame bounds. The frame $\{f_i\}_{i\in I}$ is called tight if $A = B$ and Parseval if $A = B = 1$. The frame decomposition is the most important frame result. It shows for the frame $\{f_i\}_{i\in I}$, every element in $\mathcal{H}$ has a representation as an infinite linear combination of the frame elements; i.e., there exist coefficients $\{c_i(f)\}_{i\in I}$ such that $f = \sum_{i\in I} c_i(f) f_i$, where $f \in \mathcal{H}$ is arbitrary.

The notion of frames has been extended to quaternionic Hilbert spaces by S. K. Sharma and S. Goel [4].

Definition 2.1. Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in I}$ be a sequence in $V_R(\mathfrak{Q})$. Then fuigi2I is said to be a frame for $V_R(\mathfrak{Q})$ if there exist two finite constants with $0 < A \leq B < \infty$ such that

$$A\|u\|^2 \leq \sum_{i\in I} |\langle u|u_i\rangle|^2 \leq B\|u\|^2, \qquad u \in V_R(\mathfrak{Q}).$$

Inspired by the above ideas, the concept of continuous frames for quaternionic Hilbert spaces can be presented as follows.

Definition 2.2. Let (Ω, μ) be a measure space with positive measure μ and $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space. A weakly-measurable mapping $F : \Omega \to V_R(\mathfrak{Q})$ is called a continuous frame for $V_R(\mathfrak{Q})$ with respect to (Ω, μ) if there exist constants $0 < A \leq B < \infty$ such that

$$A\|u\|^2 \leq \int_\Omega |\langle u|F(\omega)\rangle|^2 d\mu(\omega) \leq B\|u\|^2, \qquad u \in V_R(\mathfrak{Q}).$$

The constants A and B are called continuous frame bounds. This mapping F is called tight continuous frame if $A = B$ and if $A = B = 1$ it called a Parseval continuous frame. The mapping is called Bessel if the second inequality holds. In this case, B is called Bessel constant.

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On Lie Superalgebras with Supercomplex Structure

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Abstract

In this talk we introduce a supercomplex structure on a given Lie superalgebra. Using this structure one can define supercomplex structure on Lie supergroups. We study Lie supergroups admitting supercomplex structure. Also, we paid attention on the solvability of Lie superalgebras.

Keywords: Supercomplex structure, Lie superalgebra, Super-symmetry

Mathematics Subject Classification [2010]: Primary: 53D37, Secondary: 81R40

1 Introduction and Preliminaries

Supercomplex structures on manifolds has physical source in the theory of super-symmetry. The theory of Lie superalgebra is a mathematical model of super-symmetry. The main question is "which Lie superalgebras do admit a notion of supercomplex structures?". In this paper, we introduce a supercomplex structure on a given Lie superalgebra and then we extend such an structure to Lie supergroups. Also, we give some examples of Lie supergroups which admit supercomplex structure.

In general, there is a relationship between normed division algebras and certain super-symmetry theories which plays main role in the following patterns:

1. The only normed division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{Q}$ of dimensions 1, 2, 4 and 8, respectively.
2. The classical superstring makes sense only in space-times of dimensions 3, 4, 6 and 10.
3. The classical super-2-brane makes sense only in space-times of dimensions 4, 5, 7 and 11.

This subject describe some conditions that imply the existence of certain super structures. For instance, in the superstring of dimensions 3, 4, 6 and 10, one can use the normed division algebras to construct a Lie 2-superalgebra superstring which extends the Poincar´e Lie superalgebra in these dimensions. Also, in the super-2-brane of dimensions 4, 5, 7 and 11, one can use the normed division algebras to construct a Lie 3-superalgebra 2-brane which extends the Poincar´e Lie superalgebra in these dimensions.

Form geometric points of view, Borel, Samelson and Wang have studied some manifolds equipped with homogeneous complex structures ([2, 5, 7]). In 1988, Spindel et. al. ([6]) have interested an extended supersymmetry on some Lie groups. Then, Joyce and others have considered some Lie groups equipped with a left invariant hypercomplex structure ([3]) and Barberies has given a classification of the four dimensional Lie groups admitting hypercomplex structures ([1]). Recently, kalus ([4]) et. al. have extended the concept of complex structure to Lie supergroups.

Definition 1.1. A *supercomplex structure* on a real Lie algebra g is a pair $\{J_1, J_2\}$ of endomorphisms of g satisfying the conditions $J_1^2 = J_2^2 = -I$, $J_1J_2 = -J_2J_1$ and $N_1 = N_2 = 0$ where I is the identity and N_k is the Nijenhuis tensor associated to J_k as $N_k(X,Y) = [J_kX, J_kY] - J_k[J_kX, Y] - J_k[X, J_kY] - [X,Y]$ for $X, Y \in g$.

A given supercomplex structure on the Lie algebra associated to a Lie group G induces an invariant supercomplex structure on G by left translations.

Definition 1.2. A supermanifold (graded manifold) of dimension $n|m$ is a ringed space $M = (|M|, O_M)$, where $|M|$ is a topological space (Hausdorff, countable base) and the structural sheaf O_M is a sheaf of super $\mathbb{R}$-algebras with unity, locally isomorphic to $\mathbb{R}^{n|m}$.

Lie supergroups are the best useful particular examples of supermanifolds, which have both compatible supermanifold and Lie group structures.

The common definition of a Lie supergroup is a generalization of the classical group axioms to morphisms of supermanifolds .

Definition 1.3. A *Lie supergroup* is a group object in the category of supermanifolds, i.e. a supermanifold G together with morphisms $m : G \times G \to G$,$i : G \to G$ and $1 : \mathbb{R}^{0|0} \to G$.

Definition 1.4. A superalgebra is a $Z_2 - graded$ algebra $A = A_0 \oplus A_1$
(that is, if $a \in A_\alpha$,$b \in A_\beta$,$\alpha, \beta \in$, $Z_2 = \bar{0}, \bar{1}$, then $ab \in A_{\alpha+\beta}$
A Lie superalgebra is a superalgebra $g = g_0 \oplus g_1$ with an operation $[,]$ satisfying the following axioms:

1. $[a,b] = -(-1)^{\alpha\beta}[b,a]$ for $a \in g_\alpha, b \in g_\beta$,
2. $[a,[b,c]] = [[a,b],c] + (-1)^{\alpha\beta}[b,[a,c]]$ for $a \in g_\alpha, b \in g_\beta$.

The *Lie superalgebra* is the linear models of the Lie supergroup. Its classifications can be found in works of V. Kac and others. We remember a definition which generalize the notion of a Lie algebra. To a Lie supergroup G one can associate a Lie superalgebra which is the subspace of Der(A_G) consisting of left invariant vector fields of G.

Definition 1.5. Given $g = g_0 \oplus g_1$ a real Lie superalgebra, a supercomplex structure on g is a family $\{J_k\}_{k=1,2}$ of endomorphisms of g satisfying the conditions:

1. $J_k^2X = -X$, ($k = 1, 2$ and $X \in g$),
2. $J_1J_2X = -J_2J_1X$,
3. $N_1 = N_2 = 0$,

where N_k is the Nijenhuis tensor corresponding to J_k.

2 Main Results

Here we give some theorems on lie superalgebras equipped with a supercomplex structure.

Proposition 2.1. *A Lie superalgebra $g = g_0 \oplus g_1$ is solvable if and only if the Lie algebra g_0 is solvable.*

Theorem 2.2. *Lie superalgebra $g = g_0 \oplus g_1$, admits a supercomplex structure if and only if Lie algebra g_0 admits a supercomplex structure.*

we can extend some theorems of hypercomplex structured Lie algebra to Lie superalgebras with supercomplex structure. One of the important results is as follows.

Theorem 2.3. *If g_0 is not solvable and $g = g_0 \oplus g_1$ admits an supercomplex structure, then $g_0 = R \oplus so(3)$, Moreover, the supercomplex structure on $R \oplus so(3)$ is unique up to equivalence.*

When g_0 is solvable, we consider four cases based on the possible values of dimension of the derivative of g_0 as dim ${g'}_0 = 0, 1, 2, 3$. When dim ${g'}_0 = 0$, clearly g_0 is abelian. A supercomplex structure on g can be obtained by choosing two endomorphisms satisfying condition 1. In this situation, the integrability condition 2 is automatically satisfied and one can show that, there is a one to one correspondence between supercomplex structures on g and points in the space $GL(4n, R)/GL(n, H)$. The correspondence is established by fixing a supercomplex structure $\{J^0_\alpha\}_{\alpha=1,2}$ and sending $GL(4n, R) \ni T \to TJ^0_\alpha T^{-}1_{\alpha=1,2}$. In particular, it follows that every supercomplex structure is equivalent to $\{J^0_\alpha\}_{\alpha=1,2}$.

In case dim $\acute{g}_0 = 1$, we have the following proposition:

Proposition 2.4. *Let $g = g_0 \oplus g_1$ be a Lie superalgebra, and dim ${g'}_0 = 1$. Then g does not admit any supercomplex structure.*

Proof. Assume that g admits a supercomplex structure, we consider its restriction on g_0. By Remark 3.2 in [1], its center is $\Im = O$. Let X be a nonzero element of ${g'}_0$. There exists $Y \in g$ such that $[Y, X] = X$. Then g_0 decomposes,

$$g_0 = ker(adx) \cap ker(ady) \oplus RX \oplus RY.$$

By applying the Jacobi identity to U, V, Y, where $U, V \in ker(adx) \cap ker(ady)$, we get that $[U, V] = 0$; hence $\Im = ker(adx) \cap ker(ady)$, a contradiction. $\square$

Theorem 2.5. *Let $g = g_0 \oplus g_1$ be a Lie superalgebra and dim ${g'}_0 = 2$.*
(i)If g admits a supercomplex structure, then $g_0 \cong aff(C)$;
(ii) the equivalence classes of supercomplex structure on g are parametrized by the space RP^2.

Theorem 2.6. *Let $g = g_0 \oplus g_1$ be a Lie superalgebra and dim ${g'}_0 = 3$ and g_0 is solvable then one of the following holds: (a) ${g'}_0$ is abelian, in this case if g admits a supercomplex structure, then g_0 corresponds to the space RH^4. (b) ${g'}_0$ is a Heisenberg algebra, in this case if g admits a supercomplex structure, then g_0 corresponds to the space CH^2.*

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Invariant Supercomplex Structures on Some Super Lie Groups

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Abstract

In this talk we study the invariant supercomplex structure (ISS) on super Lie groups. As usual, one may prefer to study ISS first on the super Lie algebra associated to a given super Lie group. We manage to use a such method to introduce the mentioned structure.

Keywords: Supercomplex structure,Super Lie algebra, Supersymmetry

Mathematics Subject Classification [2010]: Primary: 53D37, Secondary: 81R40

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1 Introduction

Supercomplex structure on manifolds as an extension of complex structure has been interested in mathematics and physics. The supersymmetry theory may be modeled by supercomplex structure. In this content, Spindel, Sevrin, Troot and Van Proeyen discovered the left ISS on compact Lie groups. Also, Joyce ([3]) studied homogeneous Supercomplex structure and homogeneous quaternion manifold which have root in the works of Borel ([2]), Samelson ([5]) and Wang ([7]). Joyce et al. found that the group $T^{2n-r} \times G$ has a left ISS, where n is the number of $sp(1)$ subalgebras in the Lie algebra of the Lie group G. ISS On 4-dimensional real Lie groups are classified by Barberis ([1]).

Definition 1.1. (i) A *supercomplex structure* on a manifold M is a pair $\{J_1, J_2\}$ of anticommuting complex structures on M.

(ii) Given $g = g_0 \oplus g_1$ a real Lie algebra, a supercomplex structure on g is a family $\{J_1, J_2\}$ of endomorphisms of g satisfying the conditions:

1. $J_1^2 = J_2^2 = -I$,
2. $J_1 J_2 = -J_2 J_1$,
3. $N_k = 0, \quad (k = 1, 2)$,

where I is the identity and N_k is the Nijenhuis tensor corresponding to J_k as:

$$N_k(X, Y) = [J_k X, J_k Y] - J_k([X, J_k Y] + [J_k X, Y]) - [X, Y] \quad (\forall X, Y \in g).$$

Clearly, if G is a Lie group with Lie algebra g, a supercomplex structure on g induces by left translations of an ISS on G.

The second key notion is *supermanifold*. In physics, a supermanifold is a manifold with both bosonic and fermionic coordinates. This coordinates are usually denoted by $(x, \theta, \bar{\theta})$, where x is the spacetime vector and θ and $\bar{\theta}$ are Grassmann spinors. From geometric point of view, a supermanifold is an special case of the non-commutative manifold. Moreover, both approaches are equivalent in the categorical sense, as was shown in [1].

Lie supergroups are the best useful particular examples of supermanifolds, which have both compatible supermanifold and Lie group structures. The common definition of a Lie supergroup is a generalization of the classical group axioms to morphism of supermanifolds. The common definition of a Lie supergroup is a generalization of the classical group axioms to morphisms of supermanifolds .

Definition 1.2. A *super Lie group* is a group object in the category of supermanifolds, i.e. a supermanifold G together with morphisms $m : G \times G \to G$,$i : G \to G$ and $1 : \mathbb{R}^{0|0} \to G$.

Definition 1.3. (i) A superalgebra is a $\mathbb{Z}_2 - graded$ algebra $\mathbb{A} = \mathbb{A}_0 \oplus \mathbb{A}_1$. It means that, if $a \in \mathbb{A}_\alpha$, $b \in \mathbb{A}_\beta$ where $\alpha, \beta \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, then $ab \in \mathbb{A}_{\alpha+\beta}$.

(ii) A super Lie algebra is a superalgebra $\eth = \eth_0 \oplus \eth_1$ with an operation $[,]$ satisfying the following axioms:

1. $[a, b] = -(-1)^{\alpha\beta}[b, a]$ for $a \in \eth_\alpha, b \in \eth_\beta$,
2. $[a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]]$ for $a \in \eth_\alpha, b \in \eth_\beta$.

The *super Lie algebra* is the linear models of the super Lie group. Its classifications can be found in works of V. Kac and others. To a super Lie group G one can associate a super Lie algebra which is the subspace the space of left invariant vector fields on G.

2 Main Results

The main aim is to give a supercomplex structure on some super Lie groups. To our knowledge, this is completely a new and useful work.

Theorem 2.1. A real super Lie group G with almost complex structure J induces a complex super Lie group if and only if J preserves left-invariance of super derivations and the Lie super bracket is J-linear in both arguments. This means that J comes from a complex structure on the super Lie algebra g.

Proof. Proof. If J satisfies the conditions above, then it is integrable due to the graded version of the Newlander-Nierenberg theorem. Since J is compatible with the adjoint action of G on $\eth$, so J can be continued to $R(G)\sharp E(g)$ compatible with multiplication, inverse $(g\sharp X \to -(g^{-1})Ad(g)(X)$ for $g \in G$ and $X \in g$ and unity. This includes that the corresponding morphisms are morphisms of complex supermanifolds. □

Theorem 2.2. Let $g = g_0 \oplus g_1$ where g_0 is not solvable. Then g_0 admits a super complex structure if and only if $g_0 = \mathbb{R} \oplus so(3)$. Moreover, the supercomplex structure on $R \oplus so(3)$ is unique up to equivalence.

Proof. The first assertion follows by exhibiting a super complex structure $\{J_1, J_2\}$ on $R \oplus so(3)$. Let $\{Z, X, Y, W\}$ be a basis of $\mathbb{R} \oplus so(3)$ such that $Z \in \mathbb{R}$, $[X, Y] = W$, $[Y, W] = X$, and $[W, X] = Y$. Assume that, $\{J_1, J_2\}$ is defined as $J_1(Z) = X$, $J_1(Y) = W$, $J_1^2 = -I$, $J_2(Z) = Y$, $J_2(W) = X$, $J_2^2 = -I$. It is clear that J_1 and J_2 do not commute. Also,

$$N_1(Z, Y) = [J_1(Z), J_1(Y)] - J_1[Z, J_1(Y)] - J_1[J_1(Z), Y] - [Z, Y] = [Z, W] - J_1[X, Y] = 0.$$

By a lemma in [1], J_1 is integrable. A similar computation shows that $N_2(Z, W) = 0$ and then $H = \{J_1, J_2\}$ defines a super complex structure on $\mathbb{R} \oplus so(3)$.

Now, we assume that g_0 admits a super complex structure $\{J_1, J_2\}$. Since g_0 is not solvable, by the Levi decomposition of g_0 and the classification of real simple 3-dimensional Lie algebras, we get that $g_0 \cong \mathbb{I} \oplus so(3)$ or $g_0 \cong \mathbb{I} \oplus sl(2, \mathbb{R})$, a direct sum of ideals. If Z is a nonzero element in $\mathbb{I}$ we define $X = J_1(Z)$, $Y = J_2(Z)$, $W = J_l J_2(Z)$. Then Z, X, Y, W is a basis of g_0. We compute $[X, Y], [Y, W], [W, X]$. Write $[X, Y] = aZ + bX + cY + dW$. Since $N_1(Z, Y) = O = N_2(Z, X)$ we have: $J_1[X, Y] = [X, W], J_2[X, Y] = [Y, W]$. Thus

$$[W, X] = bZ - aX + dY - cW, \qquad [Y, W] = -cZ + dX + aY - bW. \tag{1}$$

Now, the coefficient of Z in $[[X, Y], W] + [[Y, W], X] + [[W, X], Y]$ is $a^2 + b^2 + c^2$; hence $a = b = c = 0$ and $[X, Y] = dW, [Y, W] = dX, [W, X] = dY$. Since $d \neq 0$ (g_0 is not abelian) then X, Y, W generates a three dimensional Lie algebra isomorphic to $so(3)$. If $\{J'_1, J'_2\}$ is another super complex structure on g_0, setting $J'_3 = J'_1 J'_2$, it follows from the above procedure that there exists $d' \neq 0$ such that $[J'_\alpha Z, J'_\beta Z] = d' J'_\gamma Z$, where (α, β, γ) is a cyclic permutation of (1,2,3). Let$\phi \in \mathrm{End}(g_0)$ be defined by $\phi Z = d/d' Z$ and $\phi J_\alpha = J'_\alpha \phi$, for $\alpha = 1, 2$. One verifies that ϕ is an automorphism of g_0; hence $\{J_1, J_2\}$ and $\{J'_1, J'_2\}$ are equivalent. This concludes the proof of the theorem. ☐

Theorem 2.3. Let G be a real super Lie group associated to the Harish-Chandra super-pair (G, g) and let $\zeta : G \to G^{\mathbb{C}}$ be the universal complexification of G. Then the complex Harish-Chandra super-pair $(G^{\mathbb{C}}, g^{\mathbb{C}})$ is associated to a universal complexi?cation $\eth^{\mathbb{C}}$ of $\eth$.

Proof. . Let $(H, \eqsim)$ be a complex Harish-Chandra super-pair and $(\lambda, \Lambda_*) : (G, \eth) \to (H, \eqsim)$ be a morphism of real Harish-Chandra super-pairs. Let $\lambda^{\mathbb{C}} : G^{\mathbb{C}} \to H$ be the underlying complexification and put $\Lambda_*^{\mathbb{C}} := D_e(\lambda^{\mathbb{C}}) \oplus \mu$, where $\mu : \eth_1 \otimes \mathbb{C} \to \eqsim_1$ is the complex linear continuation of $\Lambda_*|_{g_1}$. Then $(\lambda^{\mathbb{C}}, \Lambda_* \mathbb{C})$ is unique with the required properties. ☐

The following proposition shows that if g_0 admits a super complex structure, then it has to be of dimension greater than one.

Proposition 2.4. If g_0 admits a super complex structure, then dim $g'_0 \neq 1$.

Proof. . If g_0 admits a super complex structure we may assume that $h = \{O\}$. Let X be a nonzero element of g'_0. There exists $Y \in g$ such that $[Y, X] = X$. Then g_0 decomposes $g_0 = ker(adx) \cap ker(ady) \oplus RX \oplus RY$. By applying the Jacobi identity to U, V, Y, where $U, V \in ker(adx) \cap ker(ady)$, we get that $[U, V] = 0$; hence $\Im = ker(adx) \cap ker(ady)$, a contradiction. ☐

Theorem 2.5. If g_0 admits a super complex structure and dim $g'_0 = 2$ then $g_0 \cong aff(\mathbb{C})$.

Proof. First, we expand the super complex structure on $aff(\mathbb{C})$. Let $H = J_{\alpha\alpha} = 1, 2$ be the following family of endomorphisms of $aff(\mathbb{C})$. $J_1(X) = -W$, $J_1(Y) = Z$, $j_1^2 = -I$, $J_2(X) = Y$, $J_2(Z) = -W$ and $J_2^2 = -I$. It is easy to check the integrability of J_1 and J_2 by using Lemma (2.2) from [1]. Conversely, assume now that $J_{\alpha\alpha} = 1, 2$ defines a super complex structure on g_0. Changing $J_{\alpha\alpha} = 1, 2$, if necessary , we may assume that $J_2 : g'_0 \to g'_0$; hence $g_0 = g'_0 \oplus J_i(g'_0)$. Let X', Y' be a basis of g'_0 where $Y' = J_2(X')$. It follows that $X', Y', J_1(X'), J_i(Y')$ is a basis of g_0. There exist two skew-symmetric, bilinear forms α, β on g_0 such that $[V, W] = \alpha(V, W)X' + \beta(V, W)Y' \ \forall V, W \in g_0$ The integrability condition $N_1(X', Y') = 0$ and the fact that g'_0 is abelian (since g_0 is solvable) yield $[J_1(X'), J_1(Y')] = 0, [X', J_1(Y')] = [Y', J_1(X')]$, and from $N_2(X', J_1(X') = 0$ we obtain $[X', J_1(X')] = -[Y', J_1(Y')]$. The Jacobi identity gives $\alpha(X', J_1(X')) = \beta(X', J_1(Y')), \alpha(X', J_1(Y')) = -\beta(X', J_1(X'))$, and therefore the bracket in g_0, determined by $c = \alpha(X', J_1(X'))$ and $d = \alpha(X', J_1(Y'))$, looks as follows: $[X', J_1 X'] = cX' - dY'$, $[Y', J_1(X')] = dX' + cY'$, $[X', J_1(Y')] = dX' + cY'$, $[Y', J_1(Y')] = -cX' + dY'$, and we must have $c \neq 0 ord \neq 0$ because dim $g'_0 = 2$. Taking $X = (C^2 + d^2)^{-1}(dX' + cY'), Y = (c^2 + d^2)^{-1}(-cX' + dY') \ Z = (c^2 + d^2)^{-1}(cJ_l X' + dJ_1 Y'), W = (c^2 + d^2)^{-l}(-dJ_1 X' + cJ_1 Y')$ it is easy to see that $[X, Z] = X, [Y, Z] = Y, [X, W] = Y, [Y, W] = -X$, so that $g_0 \cong aff(C)$, as asserted. We note that J_1 and J_2 take the following form relative to the basis $\{X, Y, Z, W\}$: $J_1(X) = aZ - bW, J_1(Y) = bZ + aW, j_1^2 = -I$, $J_2(X) = Y, J_2(Z) = -W, J_2^2 = -I$, where $a = 2cd(c^2 + d^2)^{-1}$ and $b = (d^2 - c^2)(c^2 + d^2)^{-1}$; hence $a^2 + b^2 = 1$. Let J_α^0 for $\alpha = 1, 2$ be the super complex structure obtained by setting $J_1^0(X) = -W, J_1^0(Y) = Z, (J_1^0)^2 = -I$ and $J_2^o = J_2$.

We claim that J^0_α and J_α for $\alpha = 1, 2$ are equivalent. In fact, one shows that $\phi \in End(g_0)$ defined by $\phi X = bX + aY$, $\phi Y = -aX + bY, \phi Z = Z$ and $\phi W = W$ gives an automorphism of g_0 such that $\phi J_\alpha = J^0_\alpha(\varphi), \alpha = 1, 2$.

Therefore, the equivalence classes of super complex structure on g_0 are in one to one correspondence with points in the space $O(2)/SO(3) = RP^2$. □

Theorem 2.6. If dim $g'_0 = 3$ and g_0 is solvable then one of the following holds: (a) g'_0 is abelian or (b) g'_0 is a Heisenberg algebra. In case (a) g_0 admits a super complex structure if and only if g_0 corresponds to the space RH^4. More over, g_0 has a unique super complex structure, up to equivalence. In case (b) g_0 admits a super complex structure if and only if it corresponds to the space CH^2. The equivalence classes of super complex structure on g_0 are parametrized by the space RP^2.

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پوسترهای ارائه شده در بیست و هشتمین سمینار جبر ایران

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Serreness of the Category of Cofinite Modules and Cofiniteness of Local Cohomology Modules

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Abstract

Let R be a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of R, X an arbitrary R-module, and n, t two non-negative integers. In this paper, we prove that the category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite $\mathrm{D}_{<n+1}$ R-modules and the category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite $\mathrm{FD}_{<n+1}$ R-modules are two Serre subcategories of the category of R-modules. We also show that if $\dim(R/\mathfrak{a}) \leq n$, then $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i \leq t$ if and only if $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq t$.

Keywords: Cofinite modules, Local cohomology modules, Serre subcategories

Mathematics Subject Classification [2010]: 13D07, 13D45

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1 Introduction and Preliminaries

Throughout, let R denote a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of R, M a finite (i.e., finitely generated) R-module, X an arbitrary R-module which is not necessarily finite, and n, t two non-negative integers. For basic results, notations, and terminology not given in this paper, readers are referred to [4,5].

In [6], Hartshorne defined an $\mathfrak{a}$-torsion R-module X to be $\mathfrak{a}$-*cofinite* if $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all i and asked the following questions:

Question 1.1. Under what hypotheses, is the category of $\mathfrak{a}$-cofinite R-modules an Abelian category?

Question 1.2. Under what hypotheses, is $\mathrm{H}^i_{\mathfrak{a}}(M)$ an $\mathfrak{a}$-cofinite R-module for all i?

Recall that a subcategory of the category of R-modules is said to be *Serre* if it is closed under taking submodules, quotients, and extensions. Note that every Serre category is Abelian. Recall also that X is said to be an $\mathrm{FD}_{<n}$ (or *in dimension* $< n$) R-module if there exists a finite submodule X' of X such that $\dim_R(X/X') < n$ [1,3]. We say that X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-*cofinite* R-module if X is an $\mathfrak{a}$-torsion R-module and $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all i [2, Definition 4.1]. Note that, by [10, Theorem 2.3], the class of $\mathrm{FD}_{<n}$ R-modules forms a Serre subcategory of the category of R-modules. Also, X is a finite R-module if and only if X is an $\mathrm{FD}_{<0}$ R-module, and so X is an $\mathfrak{a}$-cofinite R-module if and only if X is an $(\mathrm{FD}_{<0}, \mathfrak{a})$-cofinite R-module. Thus, it is natural to raise the following questions as generalizations of Questions 1.1 and 1.2.

Question 1.3. Under what hypotheses, is the category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules a Serre category (and so an Abelian category)?

Question 1.4. Under what hypotheses, is $\mathrm{H}^i_{\mathfrak{a}}(M)$ an $(\mathrm{FD}_{<n},\mathfrak{a})$-cofinite R-module for all i?

In this paper, we study the above questions. We prove that the category of $(\mathrm{FD}_{<n},\mathfrak{a})$-cofinite $\mathrm{D}_{<n+1}$ R-modules and the category of $(\mathrm{FD}_{<n},\mathfrak{a})$-cofinite $\mathrm{FD}_{<n+1}$ R-modules are two Serre subcategories of the category of R-modules. Here, the class of all R-modules X with $\dim_R(X) < n$ is denoted by $\mathrm{D}_{<n}$. We also show that if $\dim(R/\mathfrak{a}) \leq n$, then $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n},\mathfrak{a})$-cofinite R-module for all $i \leq t$ if and only if $\mathrm{Ext}^i_R(R/\mathfrak{a},X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq t$.

2 Serreness of the Category of Cofinite Modules

The following lemma is needed in this paper. Recall that an arbitrary R-module X is said to be *minimax* if there exists a finite R-submodule X' of X such that X/X' is an Artinian R-module [11].

Lemma 2.1. *Suppose that X is an $\mathfrak{a}$-cofinite R-module. Then X is an Artinian (resp. a minimax) R-module if and only if X is a $\mathrm{D}_{<1}$ (resp. an $\mathrm{FD}_{<1}$) R-module.*

Proof. Assume that X is an $\mathfrak{a}$-cofinite $\mathrm{D}_{<1}$ (resp. $\mathrm{FD}_{<1}$) R-module. Thus $\mathrm{Hom}_R(R/\mathfrak{a},X)$ (resp. there exists a finite R-submodule X' of X such that $\mathrm{Hom}_R(R/\mathfrak{a},X/X')$) is a finite R-module and $\dim_R(X) < 1$ (resp. $\dim_R(X/X') < 1$). Hence $\mathrm{Hom}_R(R/\mathfrak{a},X)$ (resp. $\mathrm{Hom}_R(R/\mathfrak{a},X/X')$) is an Artinian R-module and so X (resp. X/X') is an Artinian R-module from [4, Theorem 7.1.2], as we desired. □

Melkersson, in [7, Corollary 1.7] and [8, Corollary 4.4], proved that the category of $\mathfrak{a}$-cofinite Artinian R-modules and the category of $\mathfrak{a}$-cofinite minimax R-modules are two Serre subcategories of the category of R-modules. By Lemma 2.1, this means that the category of $\mathfrak{a}$-cofinite $\mathrm{D}_{<1}$ R-modules and the category of $\mathfrak{a}$-cofinite $\mathrm{FD}_{<1}$ R-modules are two Serre subcategories of the category of R-modules. The following theorem generalizes Melkersson's results [7, Corollary 1.7] and [8, Corollary 4.4].

Theorem 2.2. *Let n be a non-negative integer. Then the category of $(\mathrm{FD}_{<n},\mathfrak{a})$-cofinite $\mathrm{D}_{<n+1}$ R-modules and the category of $(\mathrm{FD}_{<n},\mathfrak{a})$-cofinite $\mathrm{FD}_{<n+1}$ R-modules are two Serre subcategories of the category of R-modules.*

Proof. Let

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

be a short exact sequence of R-modules. We show that X' and X'' are two $(\mathrm{FD}_{<n},\mathfrak{a})$-cofinite $\mathrm{D}_{<n+1}$ (resp. $\mathrm{FD}_{<n+1}$) R-modules if and only if X is an $(\mathrm{FD}_{<n},\mathfrak{a})$-cofinite $\mathrm{D}_{<n+1}$ (resp. $\mathrm{FD}_{<n+1}$) R-module.

(⇒). This follows from the long exact sequence

$$\begin{array}{rllll}
0 & \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a},X') & \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a},X) & \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a},X'') \\
 & \longrightarrow \cdots & & \\
 & \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{a},X') & \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{a},X) & \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{a},X'') \\
 & \longrightarrow \mathrm{Ext}^{i+1}_R(R/\mathfrak{a},X') & \longrightarrow \mathrm{Ext}^{i+1}_R(R/\mathfrak{a},X) & \longrightarrow \mathrm{Ext}^{i+1}_R(R/\mathfrak{a},X'') \\
 & \longrightarrow \cdots & &
\end{array}$$

and [10, Theorem 2.3].

(⇐). It is clear that X' and X'' are $\mathrm{D}_{<n+1}$ (resp. $\mathrm{FD}_{<n+1}$) R-modules (resp. by [10, Theorem 2.3]). Since X is an $(\mathrm{FD}_{<n},\mathfrak{a})$-cofinite R-module, $\mathrm{Hom}_R(R/\mathfrak{a},X')$ is an $\mathrm{FD}_{<n}$ R-module from the above long exact sequence. Thus X' is an $(\mathrm{FD}_{<n},\mathfrak{a})$-cofinite R-module by [9, Lemma 2.1] because X' is an $\mathfrak{a}$-torsion $\mathrm{FD}_{<n+1}$ R-module. Hence X'' is an $(\mathrm{FD}_{<n},\mathfrak{a})$-cofinite R-module from the above long exact sequence and [10, Theorem 2.3]. □

Corollary 2.3. (see [7, Corollary 1.7] and [8, Corollary 4.4]) *The category of $\mathfrak{a}$-cofinite Artinian R-modules and the category of $\mathfrak{a}$-cofinite minimax R-modules are two Serre subcategories of the category of R-modules.*

Proof. By considering Lemma 2.1, put $n = 0$ in Theorem 2.2. □

Let $\mathcal{M}$ be a Serre subcategory of the category of R-modules. We say that $\mathcal{M}$ is a *Melkersson subcategory with respect to* $\mathfrak{a}$ if for any $\mathfrak{a}$-torsion R-module X, $(0:_X \mathfrak{a}) \in \mathcal{M}$ implies that $X \in \mathcal{M}$. $\mathcal{M}$ is called a *Melkersson subcategory* when it is a Melkersson subcategory with respect to all ideals of R [2, Definition 3.1]. The category of $\mathfrak{a}$-cofinite Artinian R-modules is a Melkersson subcategory with respect to $\mathfrak{a}$ and the category of Artinian R-modules is a Melkersson subcategory (see [8, Proposition 4.1] and [4, Theorem 7.1.2]). In the next result, we show that the category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules is a Melkersson subcategory with respect to $\mathfrak{a}$ when $\dim(R/\mathfrak{a}) \leq n$.

Corollary 2.4. *Let n be a non-negative integer with $\dim(R/\mathfrak{a}) \leq n$. Then the category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules is a Melkersson subcategory with respect to $\mathfrak{a}$.*

Proof. From Theorem 2.2, the category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules is a Serre subcategory of the category of R-modules and so is a Melkersson subcategory with respect to $\mathfrak{a}$ by [9, Lemma 2.1]. ☐

Corollary 2.5. *Suppose that $\dim(R/\mathfrak{a}) = 0$. Then the category of $\mathfrak{a}$-cofinite R-modules is a Melkersson subcategory with respect to $\mathfrak{a}$.*

Proof. Take $n = 0$ in Corollary 2.4. ☐

3 Cofiniteness of Lcal Cohomology Modules

Theorem 3.1. *Suppose that n is a non-negative integer such that $\dim(R/\mathfrak{a}) \leq n$. Suppose also that X is an arbitrary R-module and t is a non-negative integer. Then $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i \leq t$ if and only if $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq t$.*

Proof. We argue by induction on t. The case $t = 0$ follows from Corollary 2.4 because we have $\mathrm{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(X)) \cong \mathrm{Hom}_R(R/\mathfrak{a}, X)$. Suppose that $t > 0$ and that $t-1$ is settled. Let $\overline{X} := X/\Gamma_{\mathfrak{a}}(X)$. By the short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(X) \longrightarrow X \longrightarrow \overline{X} \longrightarrow 0,$$

we get the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(X)) \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{a}, X) \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{a}, \overline{X}) \longrightarrow \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(X)) \longrightarrow \cdots$$

which shows that if $\Gamma_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module or equivalently $\mathrm{Hom}_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module, then, for all $i \geq 0$, $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module if and only if $\mathrm{Ext}^i_R(R/\mathfrak{a}, \overline{X})$ is an $\mathrm{FD}_{<n}$ R-module. Now, let $E := \mathrm{E}_R(\overline{X})$ and $Q := E/\overline{X}$. By the induction hypothesis on Q, $\mathrm{H}^i_{\mathfrak{a}}(Q)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i \leq t-1$ if and only if $\mathrm{Ext}^i_R(R/\mathfrak{a}, Q)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq t-1$. Since $\Gamma_{\mathfrak{a}}(\overline{X}) = 0$, we have $\Gamma_{\mathfrak{a}}(E) = 0$ from the fact that E is an essential extension of $\overline{X}$. Thus $\mathrm{Hom}_R(R/\mathfrak{a}, \overline{X}) = 0$ and $\mathrm{Hom}_R(R/\mathfrak{a}, E) = 0$. Hence, by applying the derived functors of $\Gamma_{\mathfrak{a}}(-)$ and $\mathrm{Hom}_R(R/\mathfrak{a}, -)$ to the short exact sequence

$$0 \longrightarrow \overline{X} \longrightarrow E \longrightarrow Q \longrightarrow 0,$$

we get $\mathrm{H}^i_{\mathfrak{a}}(Q) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(\overline{X}) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(X)$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, Q) \cong \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, \overline{X})$ for all $i \geq 0$. Therefore $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i \leq t$ if and only if $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq t$. This terminates the induction argument. ☐

Corollary 3.2. *Suppose that $\dim(R/\mathfrak{a}) = 0$, X is an arbitrary R-module, and t is a non-negative integer. Then $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathfrak{a}$-cofinite R-module for all $i \leq t$ if and only if $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all $i \leq t$.*

Proof. Take $n = 0$ in Theorem 3.1. ☐

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Hamiltonian Groups with Perfect Order Classes

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Abstract

A finite group is said to have **perfect order classes** if the number of elements of any given order is either zero or the divisor of the order of the group. A group is **Hamiltonian** if it is non-abelian and every one of its subgroups are normal! The purpose of this note is to classify the Hamiltonian groups which have the perfect order classes.In fact we will show that finite Hamiltonian groups have perfect order classes, if and only if, they are isomorphic to the direct product of the quaternion group of order 8, a non-trivial cyclic 3-group and a group of order at-most 2

Theorem. A finite Hamiltonian group G has perfect order classes if and only if, it is isomorphic to $Q \times C_{3^K}$ or $Q \times C_2 \times C_{3^K}$ for some positive integer k

Keywords: Order Class, Perfect Order Classes, Hamiltonian group, Hall normal subgroup

Mathematics Subject Classification [2010]: Primary: 20D10, 20F16, Secondary: 20B10

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1 Introduction and Preliminaries

Let G be a finite group and define and equivalence relation on the elements of G by declaring that two elements in G are equivalent if and only if , the have the same order in G . the equivalence class of g in G is called its **order class** and is denoted by $g^{[G]}$, in other words:

$$g^{[G]} = \{x \in G||x| = |g|\}$$

the cardinality of $g^{[G]}$ will be denoted by $f_g(G)$. For a positive integer k, let $f_k(G)$ be the number of elements of G with order k .In any group G, the order class $e^{[G]}$ pf the identity element if always a singleton , so $f_e(G) = 1$ however the subscripts is interpreted

we say that G has **perfect order classes** if $f_g(G)$ is a divisor of the order of G, for any g in G. Groups of this property seem first to have studied in [1]. see also [2,3].
the symmetric group S_3 has perfect order classes because it has 2 elements of order 3 and 2 is a

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[*] Speaker.

divisor of the order of S_3 and also has 3 elements of order 2, which is also a divisor of the order of S_3

A group is Hamiltonian if it is non-abelian in which all of the subgroups are normal. In other words a non-Abelian Dedekind group is called Hamiltonian! Now we will state the main theorem of this note!

Main Theorem

Theorem 1.1. *A finite Hamiltonian group G has perfect order classes if and only if, it is isomorphic to* $Q \times C_{3^k}$ *or* $Q \times C_2 \times C_{3^k}$ *for some positive integer k*

it follows from this that the smallest Hamiltonian group with perfect order classes is the group $Q \times C_3$ *of order 24.In particular, despite being involved in every Hamiltonian group , the quaternion group Q of order 8 does not itself have perfect order classes(It has a total of 6 elements of order 4 and 6 does not divide 8).*

All groups will be considered to be finite and will be written multiplicatively. we will denote the identity element of any group by e as well as a group of order 1. The quaternion group of order 8 will be denoted as Q. If G is a group and n a non-negative integer, we will denote G^n *the direct product of n copies of G.Now we are ready to state some fundamental results.*

Proposition 1.2. *If g is an element of a finite group G, then* $f_g(G)$ *is divisible by* $\varphi(|g|)$

Proof. define an equivalence relation as $a \sim b$ iff $< a >=< b >$. Now put [a] the equivalence class of a under this relation. now for all $g \in G$ and all $a \in g^{[G]}$ we have that $[a] \subset g^{[G]}$ hence $|[a]|$ divides $f_g(G)$. hence $|[a]| = \varphi(|g|)$ and we are done !!! □

Lemma 1.3. *A normal Hall subgroup(a subgroup whose order is a hall divisor of the order of the group) of a finite group contains the complete order class of each of its members*

Proof. let H be a normal hall subgroup of a finite group G and let h be a member of H. since $|H|$ and $|G : H|$ are coprime , there are integers a and b such that $a|H| + b|G : H| = 1$. This follows from Bezout's theorem. If g is any member of $h^{[G]}$, then $|g| = |h|$ is a divisor of $|H|$, so we have $g = g^{a|g|+b|G:H|} = g^{b|G:H|}$, then in the quotient group G/H , we have $g^{b|G:H|}H = (gH)^{b|G:H|} = H$. hence g belongs to H. since g was arbitrary , we get $h^{[H]} \subset H$ and we are done!!! □

Theorem 1.4 (5,5.3.7)**.** *A finite Hamiltonian group G is a direct product of Q,an elementary abelian 2-group and an abelian group of odd order*

Lemma 1.5 (4,lemma 2.5)**.** *If H is a normal Hall subgroup of a finite group G , then for each divisor d of H, we have* $f_d(G) = f_d(H)$

2 Main Results

Lemma 2.1 (4,lemma 3.1,6, theorem 2.1)**.** *let* $G = Q \times E \times A$ *be a finite Hamiltonian group, where* $E \cong C_2^e$*, for e a non-negative integer, A is an abelian group of odd order. For each divisor d of the order of G, we have :*

$$f_d(G) = f_d(A) \tag{1}$$

$$f_{2d}(G) = (2^{e+1} - 1).f_d(A) \tag{2}$$

$$f_{4d}(G) = 3.2^{e+1}.f_d(A) \tag{3}$$

Lemma 2.2 (4,lemma 3.3)**.** *If k is a positive integer, then the groups* $G = Q \times C_2 \times C_{3^k}$ *and* $G = Q \times C_{3^k}$*, have perfect order classes.*

suppose that G is a finite Hamiltonian group with perfect order classes, then

$$G = Q \times E \times T \times P$$

where $E \cong C_2^e$ *is an elementary abelian 2-group of rank* $e \geq 0$ *, T is a non-trivial abelian 3-group and P is an abelian group of such that* $gcd(|P|, 3) = 1$

Lemma 2.3. *The order of G is divisible by at-most one prime number greater than 3. In particular, P is a p-group, for some prime number* $p > 3$

Proof. suppose, by contradiction, that G is divisible by two odd primes p and q, both greater than 3, then

$$f_{12pq}(G) = 3.2^{e+1}.f_3(G).f_p(G).f_q(G)$$

now each $f_3(G), f_p(G)$ and $f_q(G)$ nis even, fo $f_{12pq}(G)$ is divisible by 2^{e+4}, which is a contradiction. ☐

Lemma 2.4 (4,lemma 3.5)**.** *let* $G = Q \times C_2^e \times T \times P$*, where T is a non-trivial abelian 3-group,* $e \geq 0$*, P is an abelian p-group, for some prime* $p > 3$*. If G has perfect order classes, then P is trivial*

Lemma 2.5 (4,lemma 3.7)**.** *let* $G = Q \times C_2^e \times T$*, where T is an abelian group of order* 3^k*, for a positive integer k. If G has perfect order classes, then* $e \in \{0, 1\}$ *and T is cyclic*

Proof of the main theorem. from lemma 3.1 we see that Hamiltonian groups of the form $Q \times C_{3^K}$ and $Q \times C_2 \times C_{3^K}$ have perfect order classes. the converse follows from lemmas 3.2,3.3 and 3.4

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Grey Divisible S-Acts

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Abstract

In this paper, we introduce the concept of grey divisible S-acts in the category of grey S-acts over monoids. We study some properties of this notion and show that any grey injective S-act is a grey divisible S-act.

Keywords: Grey divisible, Grey S-act, Monoid

Mathematics Subject Classification [2010]: Primary: 18D35, 20M30, 18A20, 20M50

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1 Introduction and Preliminaries

The Grey system is one of the most important scientific achievements in the field of how to use uncertain information, which was presented by Deng [2]. For example see, [1], [6], [8]. In this paper, we define grey divisible S-act and investigate some properties of this notion. We proved any grey injective S-act is grey divisible S-acts.

We recall the following definition needed in the sequel.

We recall from [8], the definition of grey numbers. Let $\mathbb{R}$ be set of real number and $g^{\pm}$ be a union set of closed or open intervals $g^{\pm} = \bigcup_{i=1}^{n}[a_i^-, a_i^+]$, which $i = 1, 2, 3, \cdots, n$, n is an integer and $0 < n < \infty$, $a_i^-, a_i^+ \in \mathbb{R}$ and $a_{i-1}^+ \leq a_i^- \leq a_i^+ \leq a_{i+1}^+$. For any interval $[a_i^-, a_i^+]$, p_i is probability for $g \in [a_i^-, a_i^+]$. If the following conditions hold for

(i) $p_i > 0$ if and only if $[a_i^-, a_i^+] \in g^{\pm}$

(ii) $p_i = 0$ if and only if $[a_i^-, a_i^+] \notin g^{\pm}$

(iii) $\Sigma_{i=1}^{n} p_i = 1$

then we call g a grey number represented by $g^{\pm}$. $g^- = inf_{a_i^- \in g^{\pm}} a_i^-$ and $g^+ = sup_{a_i^+ \in g^{\pm}} a_i^+$ are called the lower and upper of $g^{\pm}$.

If $g^- = g^+$, then $g^{\pm}$ is a white number. It is clear that the special case of the grey set is the white set and the fuzzy set is the special case of the white set. If we replaced the characteristic function with a fuzzy membership function, then the white set become a fuzzy set. We recall the grey lattice operation from [7], which for grey numbers $x^{\pm} = [x^-, x^+]$ and $y^{\pm} = [y^-, y^+]$, the *Join* and *Meet* of these grey numbers are defined as $x^{\pm} \vee y^{\pm} = [\min\{x^-, y^-\}, \max\{x^+, y^+\}]$ and

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[*] Speaker.

$x^{\pm} \wedge y^{\pm} = [\ \max\ \{x^-, y^-\},\ \min\ \{x^+, y^+\}]$, respectively. Now according to the definition of Join and Meet, the partial order $\preceq$ on grey set (X, χ_A) is shown as below:

$$x^{\pm} \preceq y^{\pm} \Longleftrightarrow x^+ \leq y^+ and\ y^- \leq x^-$$

We recall from [3], category $GSet$, in which objects are grey sets and, morphism between two grey sets $A = (U, \chi)$ and $B = (V, \mu)$, which $\chi : U \longrightarrow D[0,1]^{\pm}$ and $\mu : V \longrightarrow D[0,1]^{\pm}$, is a function f such that $\chi^+(x) \leq \mu^+(f(x))$ and lower $\mu^-(f(x)) \leq \chi^-(x)$, for any $x \in U$,where $D[0,1]^{\pm}$ is the set of all grey numbers within the interval $[0,1]$,.

Let S be a monoid. A (*right*) *S-act* is a non-empty set A together with a map $A \times S \to A, (a,s) \mapsto as$, such that for all $a \in A, s, t \in S$, $(as)t = a(st)$ and $a1 = a$. A non-empty subset $B \subseteq A$ is called a *subact* of A if $bs \in B$ for all $b \in B$ and $s \in S$. An element θ in an S-act A is said to be a *zero* or *fixed element* if $\theta s = \theta$ for all $s \in S$. Let A and B be two S-acts. A mapping $f : A \to B$ is called an *S-homomorphism* if $f(as) = f(a)s$, for all $a \in A, s \in S$. The category of all S-acts and homomorphisms between them is denoted by **Act-**S. We recall category $\mathbf{Act_0\text{-}S}$ in which all monoids contain zero 0. Fore more see [5]

From [4], the category $\mathbf{Act} - \mathbf{GS}$, in which, the objects are the function χ_A, which is called (right) grey S-acts, such that there exists a characteristic function $\chi : A \longrightarrow D[0,1]^{\pm}$ from S-act with the following properties:

(i) $\chi_A^+(a) \leq \chi_A^+(as)$, $\chi_A^-(as) \leq \chi_A^-(a)$ for any $a \in A$, $s \in S$.

(ii) $\chi_A^+(\theta_A) = 1$ and $\chi_A^-(\theta_A) = 0$.

(iii) If we consider S as an S-act, $\chi_S^+(1_S) = 1$, $\chi_S^-(1_S) = 0$.

The morphisms between two grey S-acts χ_A and μ_B, which is called GS-morphism and denoted by $\tilde{f}\colon \chi_A \longrightarrow \mu_B$ is an S-homomorphism $f : A \longrightarrow B$ such that $\chi_A^+(a) \leq \mu_B^+(f(a))$ and $\mu_B^-(f(a)) \leq \chi_A^-(a)$, for any $a \in A$.

A GS-morphism $\tilde{f}\colon \chi_A \longrightarrow \eta_B$ is (an epimorphism) a monomorphism if and only if f is an (epimorphism) a monomorphism. A GS-morphism $\tilde{f}\colon \chi_A \longrightarrow \eta_B$ is an isomorphism if f is an isomorphism, and for any $a \in A$, $\chi_A(a) = \eta_B(\tilde{f}\ (a))$. Define $Im(\tilde{f}) = \{f(a) | a \in A\}$.
Throughout this paper, unless otherwise stated, all monoids have zero 0 and all S-acts are centered.

2 Main Results

In this section, we define grey divisible S-acts, which is denoted by grey divisible S-act. We investigate some properties of this notion and show that any grey injective S-act is a grey divisible S-act.

We recall [5] an S-act A is divisible it for any $a \in A$ and cancellable element $s \in S$ there exists $b \in S$ such that $a = bs$.

Analogous to definition grey point in category of fuzzy sets [**?**], we present the following definition.

Definition 2.1. Let A be an S-act. Consider element $a \in A$ and grey number $t_{g^-}^{g^+} \in D[0,1]^{\pm}$. Define mapping $a_t : A \longrightarrow D[0,1]^{\pm}$ such that

$$a_t(b) = \begin{cases} t_{g^-}^{g^+} & a = b \\ t_0^1 & o.w \end{cases}$$

We call a_t is a grey point with support a and value t. It χ_A is a grey S-act, we say $a_t \in \chi$ if $t_{g^-}^{g^+} \preceq \chi_A(a)$.

Definition 2.2. A characteristic function χ_A on S-act A is called weakly grey divisible, briefly w-grey divisible, if for any $a_t \in \chi$ and cancellable element $s \in S$, there exist $b_{t'} \in \chi$ such that $\chi_A(a) = \chi_A(bs)$. Also, the characteristic function χ_A is called grey divisible, briefly grey divisible, if $a = bs$. Clearly, any grey divisible is weakly grey divisible, but the converse is not true, always.

Example 2.3. Consider monoid $\mathbb{N}_0$ as an $\mathbb{N}_0$-act with usual addition as the action. Define characteristic function $\chi_{\mathbb{N}_0} : \mathbb{N}_0 \longrightarrow D[0,1]^{\pm}$ such that for any $n \in \mathbb{N}_0$, $\chi_{\mathbb{N}_0}(n) = t_{0.1}^{0.2}$, which is w-grey divisible and is not grey divisible.

Definition 2.4. Let χ_A be a grey S-act. It is called a (w-) grey divisible S-act if A is divisible S-act, and χ_A is a (w-) grey divisible.

Proposition 2.5. *The following statements are equivalent:*

(i) Every characteristic function χ on any S-act A is grey divisible.

(ii) Every characteristic function χ on S-act S is grey divisible.

(iii) Any element of S is left invertible.

Proof. $(i) \Rightarrow (ii)$, is clear. $(ii) \Rightarrow (iii)$, consider $s \in S$ and grey point $1_{t_0^1}$ which is belongs to χ. By assumption, there exists $s'_{t'^{g^+}_{g^-}} \in \chi$ such that $1 = s's'$. So the result is true.

$(iii) \Rightarrow (i)$, consider grey point $a_{t^{g^+}_{g^-}} \in \chi$ and cancellable element $s \in S$. By assumption there exists $s' \in S$ such that $a = a.1 = a.s's = (a.s')s$. Now let $a' = a.s$, $t' = t$ and define $a'_{t'^{g^+}_{g^-}} : A \longrightarrow D[0,1]^{\pm}$. Since $t^{g^+}_{g^-} \preceq \chi(a) \preceq \chi(as') = \chi(a')$, we have $a'_{t^{g^+}_{g^-}} \in \chi$. ☐

Theorem 2.6. (i) *The grey S-act χ_Θ, which $\Theta = \{\theta\}$ is grey divisible S-act.*
(ii) *Any homomorphism image of a grey divisible S-act is a grey divisible S-act.*

Corollary 2.7. *Retract of a grey divisible S-act is grey divisible S-act.*

Proof. Consider GS-morphism $\widetilde{f}\colon \chi_A \longrightarrow \nu_B$ which, ν_B is a grey divisible S-act. Since χ_A is retract of ν_B, there exists a GS-morphism $\widetilde{g}\colon \nu_B \longrightarrow \chi_A$ such that $\widetilde{g}\widetilde{f}=\widetilde{\iota}$. Now by Theorem 2.6, the proof is complete. ☐

Let χ_A and ν_B be two grey S-acts. Then χ_A is a grey subact of ν_B (ν_B is extension of χ_A) if A is a subact of S-act B and for any $a \in A$, $\chi_A^{\pm}(a) \preceq \nu_B^{\pm}(a)$, which is denoted $\chi_A \preceq \nu_B$.

Definition 2.8. Consider two grey S-acts χ_A and ν_B. We said χ_A is inclusion of grey S-act ν_B, denoted by $\chi_A \subseteq \nu_B$, if $A \subseteq B$ and $\chi_A^{\pm}(a) \preceq \nu_B^{\pm}(a)$ for any $a \in A$.

Definition 2.9. A grey S-act μ_C is called a grey injective S-act if for any grey monomorphism $\widetilde{f}\colon \chi_A \longrightarrow \nu_B$ and for any grey GS-morphism $\widetilde{g}\colon \chi_A \longrightarrow \mu_C$ there exists a GS-morphism $\widetilde{h}\colon \nu_B \longrightarrow \mu_C$ such that $\widetilde{h}\widetilde{f}=\widetilde{g}$.

Theorem 2.10. *Any grey injective S-act is a grey divisible S-act.*

Proof. Let χ_A be a grey injective S-act. We show that χ_A is a grey divisible S-act. Consider element $a \in A$ and cancellable element $s \in S$. Define characteristic function $\nu_{\langle s\rangle} : \langle s\rangle \longrightarrow D[0,1]^{\pm}$ such that $\nu_{\langle s\rangle}(st) = \chi_A(at)$ for any $t \in S$ and $\mu_S : S \longrightarrow D[0,1]^{\pm}$ such that $\mu_S(s) = t_0^1$. Also, consider S-homomorphism $f : \langle s\rangle \longrightarrow A$ such that $f(st) = at$, for any $t \in S$.

Consider the following diagram

$$\begin{tikzcd}
\nu_{\langle s\rangle} \arrow[r, "\widetilde{\iota}"] \arrow[d, "\widetilde{f}"'] & \mu_S \\
\chi_A &
\end{tikzcd}$$

Now since χ_A is a grey injective S-act, there exists a GS-morphism $\widetilde{h}\colon \mu_S \longrightarrow \chi_A$ such that $\widetilde{h}\widetilde{\iota}=\widetilde{f}$. We have $a =\widetilde{f}\ (s) =\widetilde{h}\widetilde{\iota}\ (s) =\widetilde{h}\ (s) =\widetilde{h}\ (1.s) =\widetilde{h}\ (1)s$. Now let $\widetilde{h}\ (1) = b$. Hence $a = bs$, and so A is divisible S-act. We show that χ_A is grey divisible. Consider grey point $a_t \in \chi$. Since $\widetilde{h}$ is a GS-morphism, we have $\mu_S(1) \preceq \chi_A(\widetilde{h}\ (1)) = \chi_A(b)$. Now by considering $t' = \mu_S(1)$ and grey point $b_{t'}$, the assertion is hold. ☐

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ساختارهای جبری خاکستری

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چکیده. نظریه سیستم‌های خاکستری در سال ۱۹۸۲ به وسیله دنگ ابداع شد. ارتباط بین نظریه سیستم‌های خاکستری و سیستم‌های جبری در این مقاله مورد بررسی قرارگرفته است. ساختارهای جبری مهم مانند گروه ها، حلقه‌ها و مدول‌ها مورد بررسی قرار می‌گیرد. مدول‌های فازی و مدول‌های خشن قبلا مورد بررسی قرار گرفته است. در این مقاله، ما یک $R-$ مدول را به عنوان مجموعه مرجع در نظر می‌گیریم و مفهوم زیر مدول خاکستری را توجه به زیر مدولی از یک R − مدول معرفی می‌کنیم که یک مفهوم توسعه یافته از یک زیر مدول در یک R − مدول است.

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۱. مقدمه

در سال‌های اخیر ابزارهای ریاضی که روی مفاهیم مبهم بنا شده اند توسعه فراوانی گرفته اند. این مفاهیم مانند مجموعه‌های فازی، سیستم‌های خاکستری و مجموعه‌های خشن به سرعت برای ساختن سیستم‌های کاربرپسند ابداع شده اند. سیستم‌های خاکستری یک مفهوم منحصر به فردی است که با سیستم‌های پیوسته به همراه داده‌های نامشخص سر و کار دارد. مفهوم سیستم‌های خاکستری توسط جالونگ دنگ [۳] در سال ۱۹۸۲ ایجاد شد که روش جدیدی برای مطالعه مسایلی بود که داده‌های کم و اطلاعات ضعیف دارند.

به کار بردن مفاهیم مبهم برای ساختارهای جبری مانند گروه، حلقه و مدول از سال ۱۹۷۱ توسط رزنفلد [۵] آغاز شد. او اولین بار تعریف زیر گروه‌های فازی یک گروه را ارایه نمود. بیسواس و ناندا [۱] در سال ۱۹۹۴ مفهوم زیر گروه‌های خشن را مطرح کردند. دواز [۲] در سال ۲۰۰۴ زیرحلقه‌ها (ایده آل ها) ی خشن را معرفی نمودند. کاروکی و مردسون [۴] ساختار مجموعه‌های خشن و گروه‌های خشن را مورد بررسی قرار دادند. در سال ۲۰۰۶ دواز و مهدوی پور [؟] زیر مدول‌های خشن از یک R− مدول را معرفی کردند.

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*سخنران

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واژگان کلیدی. مدول، گروه، سیستم‌های خاکستری، حلقه .

در این مقاله می‌خواهیم ارتباط میان مجموعه‌های فازی، خاکستری و خشن را ببینیم و ساختارهای جبری تولید شده روی آنها را با هم مقایسه کنیم. در بخش بعدی تعاریف مربوط به مجموعه‌های فازی، خشن و خاکستری را خواهیم دید و در بخش دوم مدول بنا شده روی مجموعه‌های فازی، خشن و خاکستری را خواهیم دید.

۲. مجموعه‌های فازی، خشن و خاکستری

تعریف ۱.۲. قرار دهید U یک مجموعه جهانی باشد. آن گاه مجموعه A در U را مجموعه فازی گویند که به صورت مجموعه زوج مرتب $A = \{< x, \mu_A(x) >: x \in U\}$ خواهد بود و در آن $\mu_A : U \to [۰, ۱]$ تابع عضویت A می‌باشد و $\mu_A(x)$ درجه تعلق x به A است.

تعریف ۲.۲. قرار دهید زوج $apr = (U, b)$ یک فضای تقریبی روی U باشد و $U \mid B$ مجموعه همه کلاس‌های هم ارزی از B باشد. B یک رابطه هم ارزی روی U است. یک مجموعه که اجتماع مجموعه تهی و عناصر $U \mid B$ را مجموعه قابل تعریف گوییم. خانواده همه مجموعه‌های قابل تعریف در فضای تقریبی apr را با $Def(apr)$ نشان می‌دهیم. دو زیر مجموعه $\underline{A}, \overline{A} \in Def(apr)$ که $\underline{A} \subseteq \overline{A}$ باشد را در نظر بگیرید. زوج $(\underline{A}, \overline{A})$ را مجموعه خشن گویند.

تعریف ۳.۲. یک عدد خاکستری عددی است که کران‌های بالا و پایین مشخصی دارد اما مکان نامشخصی در بین کران هایش دارد. یک عدد خاکستری برای یک سیستم به صورت زیر تعریف می‌شود: $g^{\pm} \in [g^-, g^+] = \{g^- \leq t \leq g^+\}$ که در آن $g^{\pm}$ یک عدد خاکستری است، t اطلاعات است و g^- و g^+ کران‌های پایین و بالای اطلاعات است. محاسبات اعداد خاکستری خیلی شبیه به مقادیر بازه ای است.

تعریف ۴.۲. برای یک مجموعه $A \subseteq U$ ، اگر مقدار تابع مشخصه هر x نسبت به A را بتوان با یک عدد منفرد $v \in [۰, ۱]$ بیان کرد $\chi_A : U \to [۰, ۱]$ در این صورت A را مجموعه سفید گوییم.

در واقع مجموعه‌های فازی می‌توانند به عنوان یک مجموعه سفید در نظر گرفته شوند. مجموعه کلاسیک به وضوع یک مجموعه سفید است اما به هیچ وجه فازی نیست.

تعریف ۵.۲. برای یک مجموعه $A \subseteq U$ ، اگر مقدار تابع مشخصه هر x نسبت به A را بتوان با یک عدد سیاه بیان کرد، در این صورت A را مجموعه سیاه گوییم.

تعریف ۶.۲. برای یک مجموعه $A \subseteq U$ ، اگر بتوان مقدار تابع مشخصه هر x نسبت به A را با یک عدد خاکستری $g_A^{\pm}(x) \in \bigcup_{i=۱}^{n} [a_i^-, a_i^+] \in D[۰, ۱]^{\pm}$ بیان کرد که $\chi_A : U \to D[۰, ۱]^{\pm}$، آن گاه A یک مجموعه خاکستری است.

در اینجا $D[۰, ۱]^{\pm}$ به مجموعه همه اعداد خاکستری در بازه $[۰, ۱]$ اشاره دارد. تابع مشخصه در اینجا یک عبارت کلی است و می‌توان آن را با تابع عضویت، تابع احتمال و تابع امکان جایگزین کرد. اگر تابع مشخصه را با تابع عضویت فازی جایگزین کنیم، مجموعه سفید به یک مجموعه فازی تبدیل می‌شود.

تعریف ۷.۲. فرض کنید A یک مجموعه خاکستری و $A \subseteq U$ باشد. برای $x \in U$، مقدار $g^{\pm}$ تابع مشخصه x نسبت به A باشد.

- اگر $g_A^{\pm}$ عدد سفید باشد، آن گاه x را یک عنصر سفید گوییم.
- اگر $g_A^{\pm}$ عدد سیاه باشد، آن گاه x را یک عنصر سیاه گوییم.
- اگر $g_A^{\pm}$ عدد خاکستری باشد، آن گاه x را یک عنصر خاکستری گوییم.

مشابه مورد یک عدد خاکستری، عدم قطعیت ناشی از ناقص بودن اطلاعات را می‌توان با استفاده از درجه ای از خاکستری اندازه گیری کرد. با توجه به ویژگی خاص مجموعه‌های خاکستری، درجه خاکستری برای یک عنصر و یک مجموعه در اینجا تعریف می‌شود:

تعریف ۸.۲. قرار دهید U مجموعه جهانی باشد و x یک عنصر باشد و $x \in U$. برای مجموعه خاکستری $A \subseteq U$ مقدار تابع مشخصه x نسبت به A برابر $g_A^{\pm}(x) \in D[۰, ۱]^{\pm}$ است. درجه خاکستری بودن عنصر x برای مجموعه A را به صورت $g_A^{\circ}(x) =\mid g^{+} - g^{-} \mid$ بیان می‌کنند.

با استفاده از درجه خاکستری بودن یک عنصر درجه خاکستری بودن یک مجموعه را به صورت زیر تعریف می‌کنند:

تعریف ۹.۲. فرض کنید U مجموعه متناهی جهانی باشد. A یک مجموعه خاکستری و $A \subseteq U$ باشد. x_i یک عنصر مرتبط با A باشد و $x_i \in U$ با $i = ۱, ۲, ۳, \dots, n$ و n اندازه مجموعه U باشد. درجه خاکستری بودن مجموعه A را به صورت $g_A^{\circ} = \frac{\sum_{i=۱}^{n} g_A^{\circ}(x_i)}{n}$ تعریف می‌کنیم.

قضیه ۱۰.۲ ([۶]). فرض کنید U یک مجموعه جهانی باشد. A یک مجموعه خاکستری باشد و $A \subseteq U$. x یک عنصر باشد و $x \in U$. همچنین $g_A^{\pm}$ مقدار تابع مشخصه x نسبت به A باشد و $g_A^{\circ}(x)$ درجه خاکستری بودن $g_A^{\pm}$ باشد. g_A° نیز درجه خاکستری بودن A باشد. خواص زیر برای x و A برقرار است:

- مجموعه A یک مجموعه سفید است اگر و تنها اگر $g_A^{\circ} = ۰$.
- مجموعه A یک مجموعه سیاه است اگر و تنها اگر $g_A^{\circ} = ۱$.
- مجموعه A یک مجموعه کلاسیک است اگر و تنها اگر $g_A^{\circ} = ۰$ و برای هر $x \in U$ داشته باشیم $g_A^{\pm}(x) \in \{۰, ۱\}$.
- مجموعه A یک مجموعه فازی است اگر و تنها اگر $g_A^{\circ} = ۰$ و برای هر $x \in U$ داشته باشیم
  $g_A^{\pm}(x) \in [۰, ۱]$.

قضیه ۱۱.۲ ([۶]). A یک مجموعه خشن است اگر و تنها اگر $g_A^{\circ} > ۰$ و برای هر $x \in U$ داشته باشیم $g_A^{\pm}(x) \subseteq \{۰, ۱\}$. $g_A^{\pm}(x)$ به مقدار تابع مشخصه x نسبت به $A \subseteq U$ اشاره دارد. g_A° درجه خاکستری بودن A است.

نتیجه ۱۲.۲. برش‌ها در مجموعه خاکستری یعنی $A_{\alpha} = \{x \in A \mid g_A^{\pm}(x) \geq \alpha\}$ یک مجموعه خشن است.

مجموعه‌های خاکستری شامل مجموعه‌های خشن به عنوان یک مورد خاص در مورد مجموعه جهانی متناهی هستند. یک مجموعه خشن یک مجموعه خاکستری خاص است.

۳. مدول‌های خاکستری

در این بخش می‌خواهیم ساختار جبری R - مدول روی حلقه R را که بر اساس مفهوم‌های فازی، خشن و خاکستری ایجاد شده است را ببینیم. ابتدا ساختار مدول فازی را می‌بینیم:

تعریف ۱.۳. قرار دهید M یک R- مدول باشد. مجموعه فازی X از M را مدول فازی از یک R -مدول M گوییم، هرگاه:

- برای هر $x, y \in M$ داشته باشیم $X(x, y) \geq min\{X(x), X(y)\}$.
- برای هر $x \in M$ و $r \in R$ داشته باشیم $X(rx) \geq X(x)$.
- $X(۰) = ۱$

گزاره ۲.۳. فرض کنید A یک مجموعه فازی از R- مدول M باشد. آن‌گاه زیر مجموعه تراز A_t به ازای $t \in [۰, ۱]$ یک زیر مدول از M است اگر و تنها اگر A زیر مجموعه فازی از X باشد که X یک مدول فازی از R- مدول M است.

تعریف ۳.۳. فرض کنید M یک R - مدول و S یک زیرمدول از M باشد. همچنین $Apr_S(A) = (\underline{Apr}_S(A), \overline{Apr}_S(A))$ یک مجموعه خشن در فضای تقریبی (M, S) باشد. اگر $\underline{Apr}_S(A)$ و $\overline{Apr}_S(A)$ دو زیرمدول از M باشند، آن‌گاه $Apr_S(A)$ را مدول خشن گوییم.

گزاره ۴.۳. فرض کنید A و B دو زیر مدول از M باشند به طوری که $A \subseteq B$ ، آن‌گاه $Apr_A(B)$ زیر مدول خشن از M است.

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حل عددی دستگاه معادلات غیرخطی با الگوریتم فراابتکاری ARO و مقایسه آن با حل جبری با پایه‌ی گروبنر

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چکیده. در محاسبات عددی، حل یک دستگاه معادلات غیرخطی به عنوان یکی از سخت‌ترین مسائل شناخته می‌شود. روش‌های عددی سنتی مانند روش نیوتن و نسخه‌های مختلف آن، برای حل اینگونه دستگاه‌ها به یک حدس اولیه مناسب نیاز دارند. حدس اولیه نامناسب می‌تواند تأثیر منفی بر عملکرد و همگرایی این روش‌ها داشته باشد. در عمل، یافتن یک حدس اولیه مناسب ممکن است مشکل و هزینه‌بر باشد. برای غلبه بر این چالش‌ها، این مقاله پیشنهاد استفاده از الگوریتم فراابتکاری ARO برای حل عددی دستگاه‌های معادلات غیرخطی مطرح شده است. با توجه به اینکه حل یک دستگاه معادلات غیرخطی می‌تواند به حل یک مسئله بهینه‌سازی تبدیل شود، الگوریتم فراابتکاری ARO توانایی خوبی در یافتن جواب اینگونه مسائل را دارد. همچنین یکی از شناخته‌ترین روش موجود برای حل تحلیلی دستگاه معادلات غیر خطی استفاده از پایه گروبنر است که محاسبه آن از پیچیدگی نمایی برخوردار است. برای ارزیابی عملکرد روش پیشنهادی، دو دستگاه معادلات غیرخطی پیچیده با استفاده از این الگوریتم و روش جبری پایه گروبنر حل شده و نتایج آنها با یکدیگر مقایسه شده است که عملکرد برتر روش پیشنهادی را نسبت به روش جبری پایه گربنر نشان می‌دهد. با استفاده از الگوریتم فراابتکاری ARO، نیاز به حدس اولیه دقیق برای حل دستگاه معادلات غیرخطی کاهش می‌یابد و این امکان را فراهم می‌کند که مسائل پیچیده را با هزینه و زمان کمتر حل کنیم.

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واژگان کلیدی. حل عددی معادلات غیرخطی، پایه گروبنر، ARO، فراابتکاری.

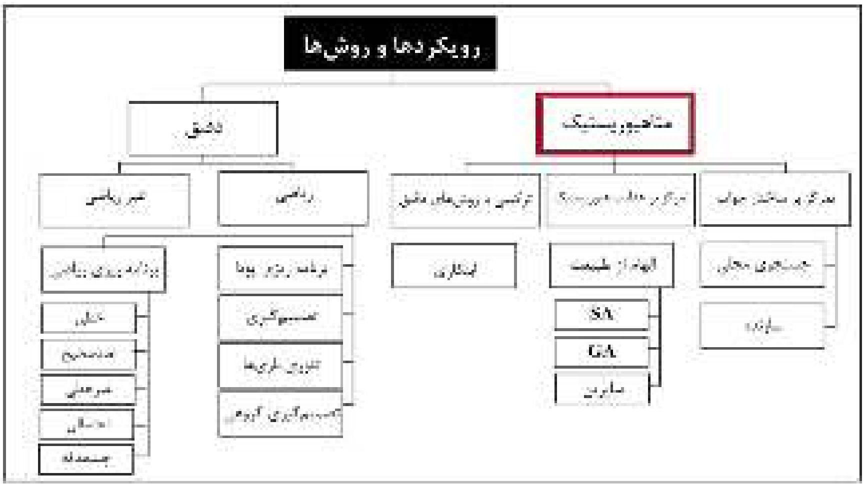
۱. مقدمه

گسترش فناوری در دهه‌های اخیر، به ویژه ارتقاء چشمگیر سرعت و قدرت پردازش رایانه ها، بر پیشرفت علوم مختلف سایه افکنده است. شاخه جبری محاسباتی نیز از این قاعده مستثنا نیست. امروزه، شاهد ارائه الگوریتم‌های متنوعی برای حل مسائل هندسه جبری به وسیله نرم افزارهای مختلف هستیم.

حل دستگاه معادلات غیرخطی یکی از مسائل بحرانی در حوزه محاسبات عددی، به ویژه در کاربردهای مختلف مهندسی، محسوب می‌شود. تلاش‌های فراوانی برای یافتن روش‌های کارا جهت حل دستگاه‌های معادلات غیرخطی توسط محققان صورت گرفته است و تئوری‌ها و الگوریتم‌های متعددی در این زمینه ارائه شده‌اند. با این حال، همچنان چالش‌هایی در روش‌های سنتی مانند روش نیوتن وجود دارد؛ زیرا این روش‌ها بسیار حساس به انتخاب درست حدس اولیه برای راه‌حل می‌باشند. علاوه بر این، انتخاب حدس اولیه مناسب برای حل دستگاه معادلات غیرخطی یک چالش بزرگ است. اگر حدس اولیه نادرست باشد، الگوریتم ممکن است با شکست مواجه شده و یا نتایج نامناسبی ارائه دهد.[۱]

در چند دهه گذشته، رویکردهای متعددی برای بهینه‌سازی مسائل طراحی شده‌اند. با امکانات و فرآیندهای پیشرفته جامعه انسانی و صنعت مدرن در سال‌های اخیر، پیچیدگی مسائل بهینه‌سازی در دنیای واقعی افزایش یافته است. این واقعیت چالش‌های بیشتری را برای روش‌های بهینه‌سازی ایجاد کرده است و نیاز به روش‌های نوین برای حل این مسائل را احساس می‌کنیم.

روش‌های بهینه‌سازی را می‌توان به طور کلی به دو دسته قطعی و فراابتکاری تقسیم کرد. الگوریتم‌های قطعی بر اساس توابع ریاضی خاصی عمل می‌کنند و فرآیندی قطعی، تکراری و بدون تصادف دارند. این الگوریتم‌ها می‌توانند در حل مسائل غیرخطی موثر باشند، اما ممکن است به اطلاعات مشتق نیاز داشته باشند و در مسائل پیچیده با جواب‌های بهینه محلی محدود شوند. به همین دلیل، این روش‌ها در مواجهه با مسائل پیچیده با قیود و پیک‌های متعدد ناکارآمد هستند.[۲]



شکل ۱: رویکردها و روش‌های بهینه سازی

در این مقاله دو دستگاه معادله غیر خطی با روش جبری پایه گروبنر و روش پیشنهادی این مقاله حل شده است. در بخش ۲ روش پیشنهادی معرفی شده است و در بخش بعدی نتایج آن با روش جبری پایه گروبنر مقایسه شده است که برتری روش پیشنهادی در آن مشهود است.

۲. روش پیشنهادی

در این بخش رویکرد مقاله جهت حل دستگاه معادلات غیر خطی شرح داده شده است. دستگاه معادلات غیر خطی زیر را در نظر بگیرید:

$$\begin{cases} f_۱(x_۱, x_۲, x_۳, ...) = ۰ \\ f_۲(x_۱, x_۲, x_۳, ...) = ۰ \\ \vdots \\ f_n(x_۱, x_۲, x_۳, ...) = ۰ \end{cases} \qquad (۱.۲)$$

که در آن n تعداد متغیرها و m تعداد معادلات دستگاه معادلات غیرخطی است. برای حل دستگاه معادلات غیرخطی، این دستگاه را به صورت مسئله بهینه‌سازی بدون قیود، به صورت زیر بازنویسی می‌کنیم.

$$\begin{cases} Find : X = (x_۱, x_۲, \dots, x_n), X \in R^n; \\ Min : F(X) = \sum_{i=۱}^{n} f_i^۲ \end{cases} \qquad (۲.۲)$$

یا

$$\begin{cases} Find : X = (x_۱, x_۲, \dots, x_n), X \in R^n; \\ Min : F(X) = \sqrt{\sum_{i=۱}^{n} f_i^۲} \end{cases} \qquad (۳.۲)$$

هر الگوریتم فراابتکاری نیازمند یک تابع هدف است که در هر مرحله، جواب‌های به دست آمده را ارزیابی کند. با تبدیل دستگاه معادلات غیرخطی به یک مسئله بهینه‌سازی، ارتباطی بین الگوریتم‌های فراابتکاری و روش‌های حل دستگاه معادلات غیرخطی ایجاد می‌شود. برای حل دستگاه معادلات ۱.۲ ، آن را به شکل ۲.۲ یا ۳.۲ نمایش می‌دهیم و از آن‌ها به عنوان توابع هدف برای الگوریتم‌های فراابتکاری استفاده می‌کنیم.

در این مقاله الگوریتم فرا ابتکاری ARO(Artificial rabbits optimization) برای حل مسله بهینه سازی جهت حل دستگاه معادلات غیر خطی پیشنهاد داده شده است. خرگوش‌ها در محیط طبیعی، رفتارهای خاصی را برای تغذیه، فرار و پنهان شدن از شکارچیان اجرا می‌کنند. الگوریتم ARO بر پایه این رفتارهای خرگوش‌ها طراحی شده است. به منظور جلوگیری از شناسایی محل لانه خود و یا فرار از دست شکارچیان از استراتژی‌های متنوعی استفاده می‌کنند که برای بررسی این الگوریتم می‌توانید به منبع [۳] مراجعه کنید.

به کمک پایه‌ی گروبنر نیز می‌توانیم به صورت جبری دستگاه معادلات غیرخطی را حل کنیم. این روش مخصوص دستگاه معادلات متشکل از چند جمله‌ها است. بنابراین برای حل دستگاه معادلات غیر خطی با استفاده از سری تیلور به چند جمله‌ای‌ها تبدیل کنیم که باعث می‌شود جواب جبری نیز باخطا همراه باشد.

۳. نتایج

در این قسمت، توسط روش پیشنهادی، دو دستگاه معادلات حل شده است تا عملکرد آن بررسی شود. این دستگاه‌های معادلات غیرخطی، از حوزه‌های مختلف مهندسی شیمی در دنیای واقعی مورد بررسی قرار گرفته‌اند.

مورد اول:

$$\begin{cases} e^{x_۱^۲} - ۸x_۱ sin(x_۲) \\ x_۱ + x_۲ - ۱ \\ (x_۳ - ۱)^۳ \end{cases} \qquad (۱.۳)$$

که در آن $-۱۰ \leq x_i \leq ۱۰, i = ۱, \dots, ۳$ است. با روش پیشنهادی جواب‌های تقریبی زیر برای دستگاه بالا بدست آمده است. و همچنین در جدول زیر مقدار تابع هدف نیز آورده شده است.

Method	$x_۱$	$x_۲$	$x_۳$
ARO	۱۷۵۵۹۵.۰	۸۲۴۴۱۲.۰	۰۰۰۰۰۹.۱
Gröbner basis	۱۷۶۴۵۴۸۲۶۲.۰	۸۲۳۵۴۵۱۷۳۸.۰	۱
Approximate solution	۱۷۵۵۹۹.۰	۸۲۴۴۰۱.۰	۱

مورد دوم:

$$\begin{cases} ۳x_۱ - cos(x_۲x_۳) - ۰٫۵ \\ x_۱^۲ - ۶۲۵x_۲^۲ - ۰٫۵ \\ x^{(} - x_۱x_۲) + ۲۰x_۳ + \frac{۱۰\pi-۳}{۳} \end{cases} \tag{۲.۳}$$

که در آن $-۲۰ \leq x_i \leq ۲۰, i = ۱, \dots, ۳$ است. با روش پیشنهادی جواب‌های تقریبی زیر برای دستگاه بالا بدست آمده است. و همچنین در جدول زیر مقدار تابع هدف نیز آورده شده است.

Method	$x_۱$	$x_۲$	$x_۳$
ARO	۰.۵ E-۱	۵۰۲۲۷۲.۵ E-۸	۲۳۵۹۸.۵- E-۱
Gröbner basis	۰.۵ E-۱	۰	۵۲۳۵۹۸.۰- E-۱
exact solution	۰.۵ E-۱	۹۲۳۶۲.۸ E-۱۱	۵۲۳۵۹۸.۰- E-۱

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Euler Operational Matrix Method for Functional Integral Equations

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Abstract

This study investigates functional Volterra integral equations. Specific conditions are imposed to guarantee the existence and uniqueness of the solution to this equation in the space of square integrable functions. Subsequently, employing the Euler operational matrix and the collocation method, an approximation to the solution is derived. This methodology converts the functional integral equation into a system of nonlinear algebraic equations, which are then solved using conventional numerical methods or iterative methods.

Keywords: Functional Volterra integral equation, Euler polynomials, Operational matrix, Function approximation, Collocation method.

Mathematics Subject Classification [2010]: Primary: 65R20.

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1 Introduction and Preliminaries

During recent decades, the investigation of functional integral equations has attracted the attention of numerous scholars, owing to their significance in the formulation of real-world problem models. While several studies have addressed the existence and uniqueness of solution to functional integral equations (see, e.g., [3, 4]), the development of computational techniques for such equations has remained relatively nascent and has only commenced recently. For example, the collocation method with piecewise continuous basis functions and the spectral element method with Gauss-Lobatto-Legendre collocation points were introduced respectively in [7] and [2] for the numerical solution of functional integral equations of the form

$$u(x) = f(x) + g\left(x, \int_a^b k(x,y)u(y)dy\right), \quad x \in [a,b]. \tag{1}$$

In this paper, we consider the following functional integral equation

$$u(x) = f(x) + g\left(x, \int_0^x k(x,y)u(y)dy\right), \quad x \in [0,1], \tag{2}$$

---

[*] Speaker.

where f, k and g are known functions, and u is the solution to be determined.

Throughout the paper, we consider the following conditions on the functions $f(x)$, $g(x,y)$ and $k(x,y)$:

($\mathbf{C_1}$) $f \in \mathfrak{L}^2([0,1])$;

($\mathbf{C_2}$) $k \in \mathfrak{L}^2([0,1] \times [0,1])$;

($\mathbf{C_3}$) $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory's conditions and there exist positive constant α and nonnegative function $\theta \in \mathfrak{L}^2([0,1])$ such that:

$$|g(x,y)| \le \theta(x) + \alpha|y|, \quad x \in [0,1],\ y \in \mathbb{R}.$$

($\mathbf{C_4}$) g fulfills the Lipschitz condition with respect to its second argument with the Lipschitz constant μ, i.e., $|g(x,u) - g(x,v)| \le \mu|u-v|, \quad x \in [0,1],\ u,v \in \mathbb{R}$.

($\mathbf{C_5}$) The kernel k and constant μ satisfy $\mu\|k\|_2 < 1$.

Lemma 1.1. *If* ($\mathbf{C_1}$)-($\mathbf{C_5}$) *hold, then* (2) *has a unique solution in* $\mathfrak{L}^2([0,1])$.

Definition 1.2. The Euler polynomials and numbers (introduced by Euler in 1740) have various applications in number theory, contributing to a rich literature on the subject [8]. The Euler polynomials and numbers are constructed from the following relations:

$$\begin{aligned}
&(i) \quad E_n(x) = \textstyle\sum_{m=0}^{n} \frac{1}{2^m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (x+k)^n,\\
&(ii) \quad \textstyle\sum_{k=0}^{n} \binom{n}{k} E_k(x) + E_n(x) = 2x^n, \quad n \ge 0 \quad [5],\\
&(iii) \quad \textstyle\sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = \frac{2}{e^t + e^{-t}} = \frac{1}{\cosh(t)},\\
&(iv) \quad E_n = 2^n E_n(\tfrac{1}{2}) \ [1].
\end{aligned} \tag{3}$$

From part (iii) in (3), it follows that Euler numbers are coefficients of the Maclaurin series expansion of the hyperbolic cosine inverse. It is mentionable that $E_n \neq E_n(0)$ for $n \ge 1$.
Euler polynomials satisfy the following relation [6]:

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_{n-k}(0) x^k, \quad n \ge 0. \tag{4}$$

Let N be a fixed nonnegative integer and define

$$\begin{aligned}
T(x) &= \left[1, x, x^2, \ldots, x^N\right]^T,\\
E(x) &= \left[E_0(x), E_1(x), E_2(x), \ldots, E_N(x)\right]^T.
\end{aligned}$$

By varing n from 0 to N in (4) and expressing the result in the matrix form, we obtain

$$E(x) = \widehat{D}T(x), \tag{5}$$

where $\widehat{D}$ is a lower triangular matrix of order $N+1$ of the form

$$D = [\widehat{D}_{ij}]_{i,j=0}^{N}, \qquad \widehat{D}_{ij} = \begin{cases} 0, & i < j,\\ \binom{i}{j} E_{i-j}(0), & i \ge j. \end{cases} \tag{6}$$

Using (5), we define the dual matrix of $E(x)$ as

$$DU = \int_0^1 E(x)E^T(x)dx = \widehat{D}G(1)\widehat{D}^T,$$

where $\widehat{D}$ is the matrix defined by (6) and $G(x)$ is the Gram matrix defined as:

$$G(x) = \int_0^x T(y)T^T(y)dy = [G_{ij}(x)]_{i,j=0}^{N}, \qquad G_{ij}(x) = \frac{x^{i+j-1}}{i+j-1}.$$

2 Main Results

2.1 Function Approximation

Theorem 2.1. *Let* $u \in \mathfrak{L}^2([0,1])$ *be an arbitrary function and be approximated in terms of Euler polynomials by* $\bar{u}_N(x) = \sum_{k=0}^{N} u_k E_k(x) = E^T(x)U$. *Then, the Euler coefficients* u_k *in* $\bar{u}_N(x) = \sum_{k=0}^{N} u_k E_k(x) = E^T(x)U$, *for all* $k = 0, 1, 2, \ldots, N$, *may be calculated using the following relation*

$$u_k = \frac{1}{k!}\int_0^1 u^{(k)}(x)dx + \sum_{r=1}^{N-k} \frac{2}{r+1}\binom{k+r}{r} u_{k+r}E_{r+1}(0) \quad k = N, \ldots, 0. \tag{7}$$

Theorem 2.2. *Let* $k \in \mathfrak{L}^2([0,1]\times[0,1])$ *be an arbitrary function and be approximated in terms of Euler polynomials by* $\bar{k}_N(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} k_{m,n}E_m(x)E_n(x)$. *Then, the coefficients* $k_{m,n}$ *in* $\bar{k}_N(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} k_{m,n}E_m(x)E_n(x)$, *for all* $m, n = 0, 1, 2, \ldots, N$, *may be calculated using the following relation*

$$\begin{aligned} k_{m,n} = {} & \frac{1}{m!n!}\int_0^1\int_0^1 \frac{\partial^{m+n}k(x,t)}{\partial x^m \partial t^n}dxdt + \sum_{j=1}^{N-n}\frac{2}{j+1}\binom{n+j}{j}k_{m,n+j}E_{j+1}(0) \\ & - \sum_{i=1}^{N-m}\frac{2}{i+1}\binom{m+i}{i}E_{i+1}(0)\sum_{j=0}^{N-n}\frac{2}{j+1}\binom{n+j}{j}k_{m+i,n+j}E_{j+1}(0), \quad m, n = N, \ldots, 0. \end{aligned} \tag{8}$$

2.2 Euler Operational Matrix of Integration

The integration of Euler vector $E(t)$ over $[0,x]$ can be computed as follows

$$\int_0^x E(t)dt = OE(x) + \frac{1}{N+1}E_{N+1}(x)e_{N+1},$$

where

$$O = [O_{ij}]_{i,j=0}^{N}, \quad O_{ij} = \begin{cases} -\frac{1}{i+1}E_{i+1}(0), & j = 0, \\ \frac{1}{j}, & j = i+1, \\ 0, & \text{otherwise}, \end{cases}$$

and e_{N+1} denotes the last column of the identity matrix I of order $N+1$.

2.3 Numerical Solution of Functional Integral Equations

Consider the functional integral equation (2). Let f, g and k satisfy conditions $(\mathbf{C_1})$-$(\mathbf{C_5})$. We approximate $f(x)$ and $k(x,y)$ in terms of Euler polynomials as described in Theorems 2.1 and 2.2, respectively. By substituting these approximations in (2) and applying the operational matrix of integration, we get

$$E^T(x)U \simeq E^T(x)F + g\Big(x, E^T(x)K\widehat{U}OE(x)\Big). \tag{9}$$

Then, by collocating (9) at Newton-Cotes nodes $x_l = \frac{2l+1}{2(N+1)}$, $l = 0, 1, \ldots, N$, we obtain

$$E^T(x_l)U = E^T(x_l)F + g\Big(x_l, E^T(x_l)K\widehat{U}OE(x_l)\Big), \quad l = 0, 1, \ldots, N, \tag{10}$$

which is a system of nonlinear algebraic equations for unknowns $u_0, u_1, \ldots, u_N$. Then, the approximate solution to (2) will be calculated by $u_N(x) = \sum_{i=0}^{N} u_iE_i(x) = E^T(x)U$.

3 Conclusion

This study indicates that this approach exhibits satisfactory performance in solving functional integral equations. These findings imply that this method has the capability to analyze further issues across different domains.

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Conditions that the Power Series Ring Over a Nil Reversible Ring is Nil Reversible

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Abstract

In this article, we study conditions under which the power series ring over a nil reversible ring is nil reversible. For this purpose, we show that in a semiprime ring, the concepts of central reduced ring and nil reversible ring are equivalent. Also, we prove that a semiprime ring R is nil reversible if and only if the power series ring $R[[x]]$ is a central reduced. Finally, we see that a semiprime ring R is nil reversible if and only if $R[[x]]$ is nil reversible.

Keywords: Nil reversible ring, Power series ring, Semiprime ring.

Mathematics Subject Classification [2010]: Primary 16U80; Secondary 16Y99, 16N60.

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1 Introduction and Preliminaries

In this paper, all rings are associative with identity. An ideal I of a ring R is said to be *reduced ideal*, if for each $x \in I$, $x^n = 0$ leads to $x = 0$, for $n \in \mathbb{N}$. If R is a reduced ideal, then R is called a *reduced ring*. An *abelian ring* is a ring in which all idempotent elements are central. *Semiprime ring*, is a ring in which for any ideal J of R, $J^2 = 0$ implies $J = 0$. A ring R is named *reversible,* if for any $r, s \in R$, $rs = 0$ implies $sr = o$. A ring R is *semicommutative*, if $rs = 0$ yields $rRs = 0$, for all $r, s \in R$.

Nil reversible ring intoduced in [4]. If for any r of a ring R and nilpotent element $s \in R$, $rs = 0$ iff $sr = 0$, then R is called *nil reversible ring*. In this paper, after studying some properties of nil reversible rings, we study conditions under which the power series ring over a nil reversible ring is nil reversible. *Central reduced ring* is a ring in which every nilpotent element is central. It is obvious that reduced rings and central reduced rings are nil reversible. We see that in a semiprime ring R the concepts of reduced, central reduced and nil reversible are equivalent. Further, we prove that a semiprime ring R is nil reversible if and only if the power series ring $R[[x]]$ is a central reduced. Finaly, we see that a semiprime ring R is a nil reversible ring if and only if $R[[x]]$ is nil reversible.

1.1 Some Properties of Nil Reversible Rings

In this part, we peruse some features of nil reversible rings. It is clear that every subring of a nil reversible ring is nil reversible. Also, a reversible ring is nil reversible. In the next example we see that a nil reversible ring need not be reversible.

Example 1.1. Consider the ring $R = \mathbb{Z}_2[x, y]$ where $xy \neq yx$ and

$$I = \langle yx^2, y^2x, xy \rangle .$$

Let $S = \frac{\mathbb{Z}_2[x,y]}{I}$. Clearly, S is not reversible. On the other hand, the set of all nilpotent elements of S is $\{0 + I, yx + I\}$. From this, it is easy to see that S is nil reversible.

Proposition 1.2. *Every nil reversible ring is an abelian ring.*

The question that arises here, is whether every abelian ring is nil reversible. The following Example state that each abelian ring is not necessarily nil reversible.

Example 1.3. Let R be the ring of matrices of the form

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix}$$

where $m, n, p, q \in \mathbb{Z}$ and $m \equiv q \pmod 2$ and $n \equiv p \pmod 2$. By [3, Example 2.7], we see R is an abelian ring. But R is not a nil reversible ring. Because $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ is a nilpotent element of R and

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

But

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$$

Recall that a ring R is named *semicommutative* if $rs = 0$ yields $rRs = 0$, where $r, s \in R$. It is clear that reversible rings are semicommutative. If for each s, t of a ring R, $st = 0$ leads to srt is a nilpotent of R, for all $r \in R$, then R is called *weakly semicommutative.*

Proposition 1.4. *Every nil reversible ring is a weakly semicommutative ring.*

2 Main Results

In this part, we study power series ring over nil reversible rings. We study conditions under which the power series ring over a nil reversible ring is nil reversible. We see that a semiprime ring R, is a nil reversible ring if and only if $R[[x]]$ is a nil reversible ring.

In [2, P.P 361], it has been proved that reduced rings are reversible. The next Lemma is stated to refer to this fact.

Lemma 2.1. *Every reduced ring is a reversible ring.*

We know that each reduced ring is nil reversible. In the following, we see that a semiprime nil reversible ring is reduced.

Proposition 2.2. *For a semiprime ring R, the following conditions coincide:*

1. *R is a nil reversible ring.*
2. *R is a reduced ring.*

Proof. If R is a reduced ring, then Lemma 2.1 tells us that R is reversible and so R is nil reversible. Conversely, assume R is nil reversible and for $a \in R$, $a^2 = 0$. Then $ra = ar$, by nil reversibility of R. So $0 = ra^2 = raa = ara$, for all $r \in R$. From this $a = 0$, since R is semiprime. Therefore R is reduced. ☐

Recall that a ring R is called *central reduced* if every nilpotent element of R is central. It is clear that reduced ring and central reduced ring are nil reversible. Proposition 2.2 and [1, Theorem 2.17] shows that in a semiprime ring the notions reduced, central reduced and nil reversible are equivalent. From this we have the following result.

Theorem 2.3. *Let R be a semiprime ring. Then the following conditions are equivallent:*

1. *R is reduced ring.*
2. *R is nil reversible ring.*
3. *R is central reduced ring.*

The following Theorem affirm that the power series ring over a semiprime ring R is central reduced iff R is central reduced.

Theorem 2.4. *Let R be a semiprime ring. Then the following sets are equivallent:*

1. *R is reduced ring.*
2. *R is central reduced ring.*
3. *The power series ring $R[[x]]$ is a central reduced ring.*

Theorems 2.4 and 2.3 lead to the next result.

Theorem 2.5. *For a semiprime ring R, the following conditions are equivallent:*

1. *R is reduced ring.*
2. *R is nil reversible ring.*
3. *R is central reduced ring.*
4. *The power series ring $R[[x]]$ is a central reduced ring.*

Corollary 2.6. *In any semiprime ring R, the following conditions are equivalent:*

1. *R is a nil reversible ring.*
2. *The power series ring $R[[x]]$ is a nil reversible ring.*

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میدان سیگمایی ناوردا و بسنده از دیدگاه گروه توپولوژیکی

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چکیده. در این مقاله، از دیدگاه گروه‌های توپولوژیکی نشان داده می‌شود که اشتراک یک میدان سیگمایی ناوردا با یک میدان سیگمایی بسنده لزومی ندارد بسنده باشد. همچنین میدان سیگمایی ناوردا و تقریباً ناوردا معادل هستند اگر و تنها اگر میدان سیگمایی ناوردا مستقل از یک میدان سیگمایی بسنده باشد.

۰. مقدمه

در آمار ریاضی، بسندگی یا ناوردایی در کاهش بعد مسأله و فشرده کردن داده‌ها موفق هستند. هال و همکاران [۵] نشان دادند که تحت شرایط خاص این تقلیل می‌تواند توسط به‌کارگیری هر دو اصل به‌طور همزمان انجام شود. برای خلاصه کردن داده‌ها تا آنجا که ممکن است قبل از به‌کارگیری ناوردایی از بسندگی استفاده می‌شود که این رهیافت تنها زمانی معقول است که مسأله کاهش یافته نیز ناوردا باشد و برای هر قاعده هم‌وردا براساس نمونه اصلی، یک قاعده هم‌وردا بر اساس آمار بسنده وجود داشته باشد که به خوبی قاعده اولی باشد. لازم به ذکر است که این دو شرط در بیشتر مسائل آماری برقرارند. در مقالاتی نظیر [۳، ۲، ۴، ۵] به ارتباط مفاهیم بسندگی و ناوردایی پرداخته شده است. در این مقاله ارتباط بین میدان‌های سیگمایی ناوردا و تقریباً ناوردا و میدان‌های سیگمایی بسنده مورد بررسی قرار می‌گیرد. سپس یک شرط کافی برای بسندگی اشتراک یک میدان سیگمایی ناوردا با یک میدان سیگمایی بسنده ارائه شده و با یک مثال نقض، نشان داده می‌شود که اشتراک یک میدان سیگمایی ناوردا با یک میدان سیگمایی بسنده لزومی ندارد بسنده باشد، مگر این که میدان سیگمایی ناوردا و میدان سیگمایی تقریباً ناوردا به مفهومی که تعریف می‌شود با یکدیگر هم‌ارز باشند و خانواده توزیع‌های احتمال مغلوب باشد. در انتها نشان می‌دهیم شرط لازم و کافی برای معادل بودن دو میدان سیگمایی ناوردا و تقریباً ناوردا آن است که میدان سیگمایی ناوردا از یک میدان سیگمایی بسنده مستقل باشد.

فرض کنید $(\mathcal{X}, \mathcal{A})$ یک فضای اندازه‌پذیر، $\mathcal{P}$ یک خانواده از توزیع‌های احتمال روی میدان سیگمایی $\mathcal{A}$ و G یک گروه توپولوژیک باشد که روی $\mathcal{X}$ عمل کند و $\mathcal{P}$ تحت G ناوردا است، هرگاه برای هر $P \in \mathcal{P}$ و $g \in G$، $Pg^{-1} \in \mathcal{P}$. پس $\{Pg^{-1} : P \in \mathcal{P}, g \in G\} = \mathcal{P}$ که Pg^{-1} اندازه احتمال روی $\mathcal{A}$ به صورت $Pg^{-1}(A) := P\left(g^{-1}A\right), A \in \mathcal{A}$ تعریف می‌شود. برای زیرمیدان‌های سیگمایی $\mathcal{B}, \mathcal{C} \subset \mathcal{A}$، $\mathcal{B} \subset \mathcal{C}(\mathcal{P})$ اگر و تنها اگر برای هر $B \in \mathcal{B}$ یک $C \in \mathcal{C}$ که برای هر $P \in \mathcal{P}$، $P(B\triangle C) = 0$ وجود داشته باشد و $\mathcal{B} \sim \mathcal{C}(\mathcal{P})$ اگر و تنها اگر $\mathcal{B} \subset \mathcal{C}(\mathcal{P})$ و $\mathcal{C} \subset \mathcal{B}(\mathcal{P})$.
فرض کنید $\mathcal{A}_I := \{A \in \mathcal{A} : gA = A, \forall g \in G\}$ و $(\mathcal{A}_{I^*} := \{A \in \mathcal{A} : P(A\triangle gA) =$

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واژگان کلیدی. گروه توپولوژیکی، میدان سیگمایی بسنده، میدان سیگمایی ناوردا، میدان سیگمایی تقریباً ناوردا.

$0, \forall P \in \mathcal{P}, g \in G\}$) زیرمیدان سیگمایی از مجموعه‌های G-ناوردا ($\mathcal{P}$-تقریباً G-ناوردا) از $\mathcal{A}$ و $\mathcal{A}_{\mathrm{S}}$ یک زیرمیدان سیگمایی بسنده برای $\mathcal{A} \mid \mathcal{P}$ تعریف شود. هال و همکاران [۵] نشان دادند اگر روابط

$$\forall g \in G : g\mathcal{A}_S = \mathcal{A}_S \tag{۱.۱}$$

و

$$\mathcal{A}_S \cap \mathcal{A}_I \sim \mathcal{A}_S \cap \mathcal{A}_{I^*} \tag{۲.۱}$$

برقرار باشند، آن‌گاه $\mathcal{A}_S \cap \mathcal{A}_I$ برای $\mathcal{A}_I \mid \mathcal{P}$ بسنده است. همچنین مطرح کردند که آیا شرط اول به تنهایی برای اثبات این نتیجه کافی است یا خیر؟ با ذکر مثال ۱.۲ در بخش دوم نشان می‌دهیم که جواب منفی است مگر اینکه $\mathcal{A}_I \sim \mathcal{A}_{I^*}(\mathcal{P})$ و $\mathcal{A} \mid \mathcal{P}$ مغلوب باشد. نشان می‌دهیم، برای خانواده‌های مغلوب همواره $\mathcal{A}_S \cap \mathcal{A}_{I^*}$ برای $\mathcal{A}_{I^*} \mid \mathcal{P}$ بسنده است که این نتیجه برای خانواده‌های دیگر معتبر نیست. مشاهده می‌شود $\mathcal{A}_I \supset \mathcal{A}_{I^*}$. برای $A, B \in \mathcal{A}$ می‌نویسیم $A \sim B$ اگر A و B، $\mathcal{P}$-معادل باشند. قرار می‌دهیم $\mathcal{A}_{SI} = \{A \in \mathcal{A}_I : \exists B \in \mathcal{A}_S, B \sim A\}$. به وضوح $\mathcal{A}_{SI} \subset \mathcal{A}_I$ ولی $\mathcal{A}_{SI} \subset \mathcal{A}_S$ لزوماً برقرار نیست. در بیشتر مثال‌های مورد علاقه، $\mathcal{A}_S$ طوری انتخاب می‌شود که $\mathcal{A}_{SI} = \mathcal{A}_S \cap \mathcal{A}_I$. اشتراک اساسی $\mathcal{A}_{SI^*} \subset \mathcal{A}$ به طور مشابه تعریف می‌شود. یک آماره (تابع اندازه‌پذیر از $\mathcal{X}$) S هموردا برای G است اگر برای هر $x, y \in \mathcal{X}$ و $g \in G$ و $Sx = Sy$ نتیجه دهد $Sgx = Sgy$. (در مرجع [۱] گفته می‌شود S با G جابه‌جا می‌شود). یک تابع هموردا روی برد خودش یک گروه از تبدیلات یک‌به‌یک $\bar{G}$ تعریف شده توسط $\bar{g}S = Sg$ القا می‌کند و متناظر یک همریختی است. از این به بعد فرض می‌شود S برای یک میدان سیگمایی داده شده $\mathcal{B}$ روی برد S اندازه‌پذیر است و باید اعضای $\bar{G}$، $(\mathcal{B}, \mathcal{B})$ اندازه‌پذیر و در پی آن دواندازه‌پذیر باشند. S را تقریباً هموردا گوییم اگر برای هر $g \in G$ یک تبدیل دواندازه‌پذیر $\bar{g}$ روی برد S وجود داشته باشد و از این رو $Sg = \bar{g}S$.

۲. بسندگی برای خانواده‌های مغلوب

در این بخش با ذکر یک مثال نقض نشان می‌دهیم نتیجه هال و همکاران [۵] مبنی بر بسندگی $\mathcal{A}_S \cap \mathcal{A}_{I^*}$ برای $\mathcal{A}_{I^*} \mid \mathcal{P}$ در خانواده‌های مغلوب برقرار است و نتیجه برای خانواده‌های دیگر معتبر نیست.

مثال ۱.۲. فرض کنید $\mathcal{X} = \{0, 1\}^{\mathbb{R}}$ و $\mathcal{A}$ مجموعه توانی از $\mathcal{X}$ و G یک گروه از جایگشت‌های مختصات فضای حاصل‌ضرب $\{0, 1\}^{\mathbb{R}}$ باشد و $\mathcal{A} \mid P_i$ یک اندازه احتمال متمرکز در $x_i \in \mathcal{X}$ با مختصات مساوی ۱ $i = 0, 1$ باشد. قرار دهید $\mathcal{P} := \{P_0, P_1\}$. از این که برای هر $g \in G$ و هر $i = 0, 1$، $P_i g^{-1} = P_i$، برای هر $g \in G$ داریم $\mathcal{P}g^{-1} = \mathcal{P}$. فرض کنید $\mathcal{A}_s$، میدان سیگمایی حاصل‌ضرب $\mathcal{A}_r$، $r \in \mathbb{R}$ باشد که برای هر $r \in \mathbb{R}$، $\mathcal{A}_r$ مجموعه توانی $\{0, 1\}$ تعریف می‌شود. به وضوح برای هر $g \in G$، $g\mathcal{A}_S = \mathcal{A}_s$ و $\mathcal{A}_I \sim \mathcal{A}_{I^*}$. از این که $A \in \mathcal{A}_S$ با $x_0 \in A$ و $x_1 \notin A$ وجود دارد، $\mathcal{A}_S$ برای $\mathcal{A} \mid \mathcal{P}$ بسنده است. همچنین از این که $\varnothing$ و $\mathcal{X}$ تنها مجموعه‌های G-ناوردا شامل $\mathcal{A}_S$ هستند، $\mathcal{A}_S \cap \mathcal{A}_I$ برای $\mathcal{A}_I \mid \mathcal{P}$ بسنده نیست.

اگر $\mathcal{A}_S \cap \mathcal{A}_{I^*}$ برای $\mathcal{A}_{I^*} \mid \mathcal{P}$ بسنده باشد، آن‌گاه شرط ۲.۱ تضمین می‌کند که به‌صورت بدیهی $\mathcal{A}_S \cap \mathcal{A}_I$ برای $\mathcal{A}_{I^*} \mid \mathcal{P}$ و در پی آن برای $\mathcal{A}_I \mid \mathcal{P}$ بسنده است. بنابراین باید شرایط بسندگی $\mathcal{A}_S \cap \mathcal{A}_{I^*}$ برای $\mathcal{A}_{I^*} \mid \mathcal{P}$ بررسی شود. برک [۲]، شرط ۱.۱ را برای برقراری بسندگی $\mathcal{A}_S \cap \mathcal{A}_{I^*}$ برای $\mathcal{A}_{I^*} \mid \mathcal{P}$ پیشنهاد کرد. از این که میدان سیگمایی بسنده مینیمال در این شرط صدق می‌کند و هر خانواده مغلوب یک میدان سیگمایی بسنده مینیمال را اختیار می‌کند و همچنین از این که در خانواده‌های مغلوب، هر میدان سیگمایی شامل یک میدان سیگمایی بسنده، بسنده است مسئله حل می‌شود.

توجه ۲.۲. اگر $\mathcal{A} \mid \mathcal{P}$ مغلوب و $\mathcal{A}_S$ برای $\mathcal{A} \mid \mathcal{P}$ بسنده باشد، آن‌گاه $\mathcal{A}_S \cap \mathcal{A}_{I^*}$ برای $\mathcal{A}_{I^*} \mid \mathcal{P}$ بسنده است و همچنین $\mathcal{A}_S \cap \mathcal{A}_I \sim \mathcal{A}_S \cap \mathcal{A}_{I^*}$ نتیجه می‌دهد $\mathcal{A}_S \cap \mathcal{A}_I$ برای $\mathcal{A}_I \mid \mathcal{P}$ بسنده است.

مثال ۳.۲ نشان می‌دهد که در حالت کلی ادعای توجه ۲.۲ برای خانواده‌های $\mathcal{P}\mid\mathcal{A}$ که مغلوب نیستند برقرار نیست، مگر این که $\mathcal{P}\mid\mathcal{A}_{I^*}$ مغلوب باشد.

مثال ۳.۲. فرض کنید $\mathcal{A}$ میدان بورل روی $\mathcal{X}=\mathbb{R}-\{۰\}$ و G گروه ضربی αx، $\alpha>۰$ باشد. همچنین $P|\mathcal{A}$ و $Q|\mathcal{A}$ اندازه‌های احتمال تعریف شده به صورت $P(-۱)=P(۱)=\frac{۱}{۲}$ و $Q(A):=\lambda(A\cap(۰,۱))$، $A\in\mathcal{A}$ باشند که λ اندازه لبگ هست. اگر $\mathcal{P}:=\{Pg^{-۱},Qg^{-۱}:g\in G\}$ در این صورت برای هر $g\in G$، $\mathcal{P}g^{-۱}=\mathcal{P}$. از این که مجموعه تهی تنها مجموعه $\mathcal{P}\mid\mathcal{A}$-پوچ است، داریم $\mathcal{A}_{I^*}=\mathcal{A}_I=\{\varnothing;(-\infty,۰);(۰,\infty);\mathcal{X}\}$. میدان سیگمایی $\mathcal{A}_S:=\{A\in\mathcal{A}:$ یا $\{-۱,۱\}\subset A$ یا $\{-۱,۱\}\subset\bar{A}\}$ برای $\mathcal{P}\mid\mathcal{A}$ بسنده است. اما $\mathcal{A}_S\cap\mathcal{A}_{I^*}=\{\varnothing,\mathcal{X}\}$ برای $\mathcal{P}\mid\mathcal{A}_{I^*}$ بسنده نیست، زیرا $P\mid\mathcal{A}_{I^*}\neq Q\mid\mathcal{A}_{I^*}$.

مثال ۱.۲ نشان می‌دهد که در حالت کلی، اگر شرط $\mathcal{A}_S\cap\mathcal{A}_I\sim\mathcal{A}_S\cap\mathcal{A}_{I^*}$ با $\mathcal{A}_I\sim\mathcal{A}_{I^*}$ عوض شود، ادعای دوم توجه ۲.۲ برقرار نیست.

۳. بسندگی و ناوردایی

اگر f یک آماره کران‌دار با مقادیر حقیقی باشد، برای $g\in G$ و $P\in\mathcal{P}$، $E_P(f|g\mathcal{A}_S)\sim E(fg|\mathcal{A}_S)g^{-۱}$. همچنین اگر $\mathcal{A}_S$ آماره بسنده مینیمال یا معادل با میدان سیگمایی القایی توسط یک آماره تقریباً هم‌وردای S باشد، آن‌گاه برای هر $g\in G$، $g\mathcal{A}_S\sim\mathcal{A}_S$.

لم ۱.۳. فرض کنید برای هر $g\in G$، $g\mathcal{A}_S\sim\mathcal{A}_S$. در این صورت گزاره‌های زیر هم‌ارزاند:

(الف) اگر f آماره کران‌دار با مقادیر حقیقی تقریباً ناوردا باشد، $E(f|\mathcal{A}_S)$ تقریباً ناورداست.
(ب) $\mathcal{A}_S$ و $\mathcal{A}_{I^*}$ به طور شرطی به شرط $\mathcal{A}_{SI^*}$ مستقل‌اند.

یک نتیجه این لم این است که اگر برای هر $g\in G$، $g\mathcal{A}_S\sim\mathcal{A}_S$، آن‌گاه $\mathcal{A}_S$ برای $\mathcal{A}_{I^*}$ بسنده است. تحت فرض اضافی $\mathcal{A}_{SI^*}\sim\mathcal{A}_{SI}$، داریم $\mathcal{A}_{SI}$ برای $\mathcal{A}_{I^*}$ بسنده است.

اکنون برای جمع‌بندی موضوع، شرایط و گزاره‌های زیر را در نظر بگیرید:

۱) $\mathcal{A}_{SI^*}\sim\mathcal{A}_{SI}$; ۲) $\mathcal{A}_{SI}\sim\{\varnothing,\mathcal{X}\}$; ۳) $\mathcal{A}_{SI^*}\sim\{\varnothing,\mathcal{X}\}$; ۴) $\mathcal{A}_S\perp\mathcal{A}_{I^*}$;
۵) $\mathcal{A}_{I^*}\perp\mathcal{A}_I$; ۶) $\mathcal{A}_{SI^*}$ برای $\mathcal{A}_{I^*}$ بسنده است; ۷) $\mathcal{A}_S\perp\mathcal{A}_{I^*}|\mathcal{A}_{SI^*}$;
۸) $\forall g\in G, g\mathcal{A}_S\sim\mathcal{A}_S$; ۹) $\mathcal{A}_S$ به‌طور کامل کران‌دار است;
۱۰) زیرمیدان سیگمایی از $\mathcal{A}_{I^*}$ برای $\mathcal{A}_{I^*}$ بسنده است;
۱۱) P تولید می‌کند G; ۱۲) $\mathcal{A}_S\vee\mathcal{A}_I\sim\mathcal{A}$.

در انتها می‌خواهیم روابط بین این گزاره‌ها را بررسی کنیم.
طبق لم ۱.۳، داریم ۱ $\Rightarrow$ ۷ $\Rightarrow$ ۸ و ۱, ۲ $\Leftrightarrow$ ۳.

اگر $\mathcal{A}_S$ توسط یک آماره هم‌وردا مثل S تولید شود، آن‌گاه ۲ اساساً لازم است که $\bar{G}$ روی برد S انتقالی باشد. به وضوح ۱۰ $\Rightarrow$ ۱۱، اما عکس آن برقرار نیست. (اختیار کنید $X\sim N(\mu,\sigma^۲)$ و $Y\sim N(\mu,۱-\sigma^۲)$ که $\mu\in\mathbb{R}$ و $۱>\sigma>۰$ و $X\perp Y$ گروه تبدیلات $\mathcal{P}$ را ناوردا نگه می‌دارد و آماره ناوردای ماکسیمال برابر با $X-Y$ است. $\mathcal{P}$ روی $\mathcal{A}_I$ متلاشی می‌شود. از این که $\mathcal{A}_I\sim\mathcal{A}_{I^*}$ برای این گروه، $\mathcal{P}$ روی $\mathcal{A}_{I^*}$ متلاشی می‌شود. با این حال G، $\mathcal{P}$ را تولید نمی‌کند.

قضیه ۲.۳. داریم ۳, ۷ $\Leftrightarrow$ ۴. لذا اگر ۷ برقرار باشد ۳ $\Leftrightarrow$ ۴. همچنین ۷, ۱۲, ۳, ۵ $\Leftrightarrow$ ۴. لذا اگر ۷, ۱۲ برقرار باشد، ۳, ۵ $\Leftrightarrow$ ۴ $\Leftrightarrow$ ۳.

اگر $\mathcal{A}_S$ طوری انتخاب شود که ۷ برقرار باشد (به طور مثال $\mathcal{A}_S$ آماره بسنده مینیمال باشد)، آن‌گاه برای $\mathcal{A}_{SI^*}$ بدیهی داریم $\mathcal{A}_S\perp\mathcal{A}_{I^*}$ و به عبارت دیگر $\mathcal{A}_{SI^*}\sim\mathcal{A}_{SI}$ و $\mathcal{A}_{SI}\sim\{\varnothing,\mathcal{X}\}$.
همچنین داریم ۴ $\Rightarrow$ ۱, ۲, ۸, ۱۱.
با توجه به قضیه ۲.۳، به نظر می‌رسد شرط ۱۱ اضافی باشد. به هر حال ۱۰ $\Rightarrow$ ۶, ۳.

همچنین ۳,۸ ⇔ ۴ ⇔ ۳,۷ و قضیه باسو به صورت ۴ ⇒ ۹,۱۰ بیان می‌شود.
در [۳] ثابت شده ۵ ⇒ ۱۲,۴ که از ۲.۳ نیز به دست می‌آید.

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Numerical Solution of Functional Integral Equation Using Sigmoidal Functions

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Abstract

In this paper, we study the numerical approximation of a class of nonlinear integral equations of Volterra type called nonlinear functional Volterra Urysohn integral equations. Using a technique based on the Picard iterative method, the existence and uniqueness of the solution to the equation was proved. For the numerical approximation of the solution, we used the collocation method with sigmoidal functions. By using a Gronwall inequality, the convergence of the method is proved. At the end, some numerical examples are presented to show the effectiveness of the method.

Keywords: Sigmoidal functions, Volterra-Urysohn integral equations, Collocation method

Mathematics Subject Classification [2010]: Primary: 65R20

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1 Introduction and Preliminaries

Many authors have studied functional integral equations in the past years. Functional integral equations have great applications in physics, engineering and biology [1].

Various studies have been done about the existence of solution of functional integral equations [2,3].

In this paper, our study is devoted to nonlinear functional Volterra-Urysohn integral equation

$$z(t) = p(t) + g\left(t, \int_0^t k(t,s,z(s))ds\right), \quad t \in [0, \bar{X}]. \tag{1}$$

By considering some assumptions, the existence and uniqueness of the solution to equation (1) has been proved in [4]. Our idea in this article is using the collocation method with sigmoidal functions to find a numerical approximation to the solution of fnctional integral equation (1).

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[*] Speaker.

Definition 1.1. Function $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ is called a sigmoidal function if the following conditions are satisfied

$$\lim_{x\to-\infty} \sigma(x) = 0 \qquad \textit{and} \qquad \lim_{x\to\infty} \sigma(x) = 1.$$

For example, the Heviside function $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ for $x < 0$, is a sigmoidal function. Also, $\sigma(x) := (1 + e^{-x})^{-1}$, which is known as the logistic function, is a sigmoidal function. Any continuous function on the interval $(-\infty, \infty)$ can be approximated using sigmoidal functions [5].

2 Numerical Method

In this section, we describe a collocation method for solving (1). We look for an approximate soluation $(G_N z)(t)$ to (1) in the form

$$(G_N z)(t) = \sum_{k=1}^{N} z_k H(t - t_k) + z_0 H(t - t_{-1}), \quad t \in [0, \bar{X}], \tag{2}$$

where $H(t)$ is Heviside function and $t_j := hk,\ k = -1, 0, 1, \ldots, N, h = \frac{\bar{X}}{N}$. In (2) coefficients $z_0, \ldots, z_N$ are unknowns.

Remark 2.1. The coefficients z_k in (2) do not depend on the choice of sigmoid functions. It means that, different approximations of z can be obtained using different sigmoidal functions while the coefficients are the same [5].

By substituting (2) in (1), we obtain

$$(G_N z)(t) = p(t) + g\left(t, \int_0^t k(t, s, (G_N z)(s))ds\right). \tag{3}$$

If $\Pi_N := \{t_0, t_1, \ldots, t_N\}$ is the set of collocation points, then the coefficients $z_0, \ldots, z_N$ can be obtained by the following collocation conditions:

$$(G_N z)(t_i) = p(t_i) + g\left(t_i, \int_0^{t_i} k(t_i, s, (G_N z)(s))ds\right). \tag{4}$$

Now, using (2) in equation (4), the coefficients z_i can be evaluated from the following system of nonlinear algebraic equations

$$z_i = p(t_i) + g\left(t_i, \sum_{m=0}^{i-1} \int_{t_m}^{t_{m+1}} k\Big(t_i, s, \sum_{j=0}^{m} z_j\Big) ds\right) - \sum_{m=0}^{i-1} z_m, \quad i = 0, 1, \ldots, N. \tag{5}$$

3 Convergence

In this section, we study the convergence of the proposed numerical method.

The following assumptions are considered:

(i) function p satisfies the Lipschitz condition with Lipschitz constant $\tau_1 > 0$.

(ii) function g satisfies the Lipschitz condition

$$\begin{aligned} |g(t_1, v_1) - g(t_2, v_2)| &\leq \tau_2 \|(t_1, v_1) - (t_2, v_2)\|_2 \\ &= \tau_2 \sqrt{(t_1 - t_2)^2 + (v_1 - v_2)^2}, \quad (t_1, v_1), (t_2, v_2) \in [0, \bar{X}] \times \mathbb{R}. \end{aligned}$$

(iii) kernel k is continuously differentiable with respect to t and z and it satisfies the Lipschitz conditions:

$$|k(t_1, s, z) - k(t_2, s, z)| \leq \tau_3 |t_1 - t_2|, \quad t_1, t_2, s \in [0, \bar{X}],\ z \in \mathbb{R},$$

$$|k(t,s,z_1)-k(t,s,z_2)| \le \tau_4|z_1-z_2|, \quad t,s\in[0,\bar{X}],\ z_1,z_2\in\mathbb{R}.$$

(iiii) for any continously differentiable function z, there exsists a $C=C(z)>0$ such that

$$|k(t,s,z(s)| \le C, \quad \text{for all } (t,s)\in D,$$

where $D=\{(t,s):\ 0\le t,s\le\bar{X}\}$.

Theorem 3.1. *If (i)-(iiii) hold, then the collocation method (4), (5) convergence to the unique soluation (1) with the first order of convergence.*

4 Numerical Results

In this section, to show the effectiveness of the presented method, we have provided some examples. We define $E_1(t)=z_N^1(t)-z(t)$ and $E_2(t)=z_N^2(t)-z(t)$, where $z(t)$ denotes the exact solution and $z_N^1(t)$ and $z_N^2(t)$ are approximate solutions obtained by our method with the superposition of Heviside and logestic sigmoidal functions, respectively. The norm of the error is defined as follows

$$\|E_1\|_\infty=\mathbf{max}\left\{|z_N^1(t_i)-z(t_i)|, \quad 1\le i\le N\right\}, \tag{6}$$

and

$$\|E_2\|_\infty=\mathbf{max}\left\{|z_N^2(t_i)-z(t_i)|, \quad 1\le i\le N\right\}. \tag{7}$$

The numrical results are tabulated in Tables. The computations were performed in MATHEMATICA 10.

Example 4.1. We consider the following functional integral equation

$$z(t)=\sin(t)-e^{\sin(t)}+1+\left(\int_0^t \cos(s)e^{z(s)}ds\right), \quad t\in[0,1]. \tag{8}$$

where the exact solution $z(t)=\sin(t)$. The equation (8) was analyzed in [6]. Table 1 shows the numerical results obtained by our method. Also, in Figures 1 and 2, the exact soltion $z(t)$ and the approximate solutions $z_N^1(t)$ and $z_N^2(t)$ for N=50 and N=20 are plotted. Numerical results given in tables and figures show the presented numerical method convergence to the exact solution.

N	Heviside function	Logistic function
	$\|E_1\|_\infty$	$\|E_2\|_\infty$
20	5.51203×10^{-2}	6.71246×10^{-2}
30	3.78377×10^{-2}	4.61362×10^{-2}
40	2.88112×10^{-2}	3.51559×10^{-2}
50	2.32634×10^{-2}	2.84×10^{-2}
60	1.95076×10^{-2}	2.38229×10^{-2}
70	1.67963×10^{-2}	2.05168×10^{-2}
200	5.98466×10^{-3}	7.31783×10^{-3}

Table 1: Error norms for Example 4.1.

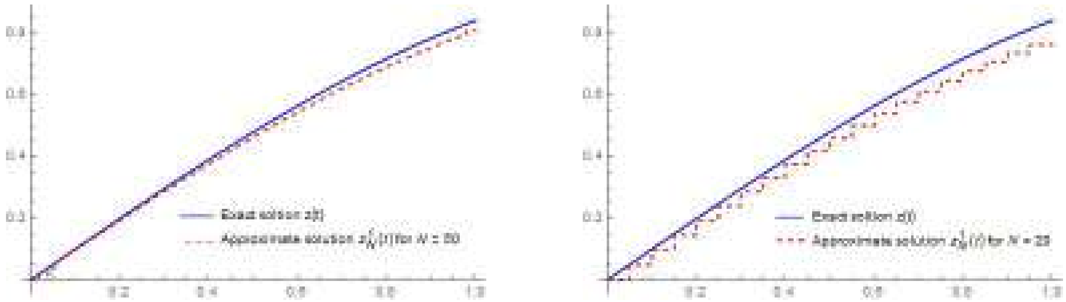

Figure 1: Approximate soloution $z_N^1(t)$ of Example 4.1; left: N=50, right: N=20.

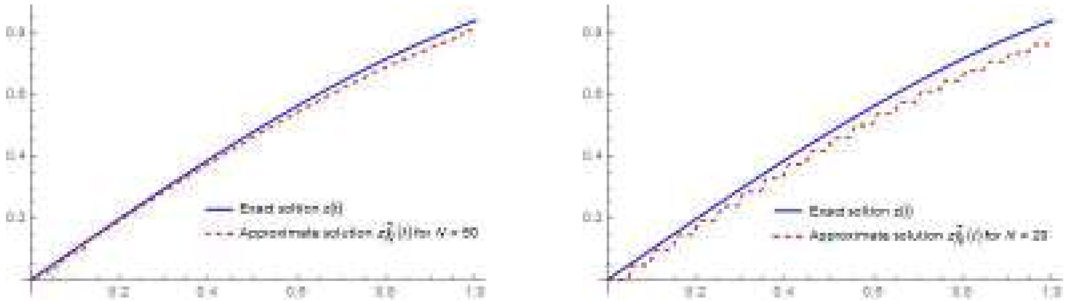

Figure 2: Approximate soloution $z_N^2(t)$ of Example 4.1; left: N=50, right: N=20

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Generalization of the Bellman inequality for Pseudo-integrals

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Abstract

In this paper, we prove Bellman's type inequality for pseudo-integral. Bellman type inequality for two classes of pseudo-integral are shown.one of them concerning the pseudo-integrals based on a function reduces on the g-integral where pseudo-operations are defined by a monotone and continuous function g. Another one concerns the pseudo-integrals based on a semiring $([a,b], max, \bigodot)$ where $\bigodot$ is generated.

Keywords: Bellman type inequality, Pseudo-integral, Pseudo-addition, Pseudo-multiplication, Fuzzy integrals inequality

Mathematics Subject Classification [2010]: Primary: 13BXX,13FXX,05EXX, Secondary: 13HXX, 05EXX

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1 Introduction and Preliminaries

The classical Bellman's integral inequality ([5]) holds : if f,g are non-negative and concave on [0,1] then

$$\int_0^1 fg \geq \frac{1}{2}\left(\int_0^1 f^2\right)^{\frac{1}{2}}\left(\int_0^1 g^2\right)^{\frac{1}{2}}.$$

The purpose of this paper is to prove a Bellman type inequality for the pseudo integrals.

Firstly,we introduce some basic notation and properties of the pseudo integrals.

. Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a,b] \subset [-\infty,\infty]$ endowed with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$ ([1,4]). Let $[a,b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty,\infty]$. The full order on $[a,b]$ will be denoted by $\preceq$.

The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a,b] \times [a,b] \to [a,b]$ which is commutative, nondecreasing (with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$, i.e., for each $x \in [a,b], \mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either a or b). Let $[a,b]_+ = \{x | x \in [a,b], \mathbf{0} \preceq x\}$

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[*]speaker.

Definition 1.1. The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and for which there exists a unit element $\mathbf{1} \in [a, b]$, i.e., for each $x \in [a, b], \mathbf{1} \odot x = x$. We assume also $\mathbf{0} \odot x = \mathbf{0}$ that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e., $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$. The structure $([a, b], \oplus, \odot)$ is a semiring ([2]). In this paper, we will consider semirings with the following continuous operations:

Case I: The pseudo-addition is idempotent operation and the pseudo-multiplication is not.

(a) $x \oplus y = \sup(x, y), \odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = a$ and the idempotent operation sup induces a full order in the following way: $x \preceq y$ if and only if $\sup(x, y) = y$.

(b) $x \oplus y = \inf(x, y), \odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0}= b$ and the idempotent operation *inf* induces a full order in the following way: $x \preceq y$ if and only if $\inf(x, y) = y$.

Case II: The pseudo-operations are defined by a monotone and continuous function $g : [a, b] \to [0, \infty]$, i.e., pseudo operations are given with $x \oplus y = g^{-1}(g(x) + g(x))$ and $x \odot y = g^{-1}(g(x)g(x))$. If the zero element for the pseudo-addition is a, we will consider increasing generators. Then $g(a) = 0$ and $g(b) = \infty$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = \infty$. If the generator g is increasing (respectively decreasing), then the operation $\oplus$ induces the usual order (respectively opposite to the usual order) on the interval $[a, b]$ in the following way: $x \preceq y$ if and only if $g(x) \leq g(y)$.

Case III: Both operations are idempotent. We have

(a) $x \oplus y = \sup(x, y), x \odot y = \inf(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = a$ and $\mathbf{1} = b$. The idempotent operation *sup* induces the usual order ($x \preceq y$ if and only if $\sup(x, y) = y$).

(b) $x \oplus y = \inf(x, y), x \odot y = \sup(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = b$ and $\mathbf{1} = a$. The idempotent operation *inf* induces an order opposite to the usual order ($x \preceq y$ if and only if $\inf(x, y) = y$).

Let X be a non-empty set. Let $\mathbb{A}$ be a σ-algebra of subsets of a set X.

We shall consider the semiring $([a, b], \oplus, \odot)$, when pseudo-operations are generated by a monotone and continuous function $g : [a, b] \to [0, \infty]$, i.e., pseudo-operations are given with $x \oplus y = g^{-1}(g(x) + g(y))$ and $x \odot y = g^{-1}(g(x)g(y))$.

Then the pseudo-integral for a function $f : [c, d] \to [a, b]$ reduces on the g-integral([6])

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\left(\int_{c}^{d} g(f(x))dx\right). \tag{1}$$

More on this structure as well as corresponding measures and integrals can be found in ([3]) . The second class is when $x \oplus y = \max(x, y)$ and $x \odot y = g^{-1}(g(x)g(y))$, the pseudo-integral for a function $f : \mathbb{R} \to [a, b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = \sup\left(f(x) \odot \psi(x)\right),$$

where function ψ defines sup-measure m. Any sup-measure generated as essential supremum of a continuouse denisty can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-additive . For any continuouse function $f : [0, \infty] \to [0, \infty]$ the integral $\int^{\oplus} f \odot dm$ can be obtained as a limit of g-integrals, We denoted by μ the usual Lebesgue measure on $\mathbb{R}$. We have

$$m(A) = ess\sup(x|x \in A) = \sup\{a|\mu(x|x \in A, x > a) > 0\}.$$

Theorem 1.2 ([6]). *Let m be a* sup*-measure on $([0, \infty], \mathbb{B}[0, \infty])$, where $\mathbb{B}([0, \infty])$ is the Borel σ-algebra on $[0, \infty]$, $m(A) = ess\sup_{\mu}(\psi(x)|x \in A)$, and $\psi : [0, \infty] \to [0, \infty]$ is a continuous density. Then for any pseudo-addition $\oplus$ with a generator g there exists a family m_λ of $\oplus_\lambda$-measure on $([0, \infty], \mathbb{B})$, where $\oplus_\lambda$ is a generated by g^λ (the function g of the power λ), $\lambda \in (0, \infty)$, such that $\lim_{\lambda \to \infty} m_\lambda = m$.*

2 Main Results

Theorem 2.1 ([6]). *Let $([0,\infty],\sup,\odot)$ be a semiring, when $\odot$ is a generated with g, i.e., we have $x\odot y=g^{-1}(g(x)g(y))$ for every $x,y\in(0,\infty)$. Let m be the same as in Theorem 2.1., Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$ -measures, where $\oplus_\lambda$ is a generated by $g^\lambda,\lambda\in(0,\infty)$ such that for every continuous function $f:[0,\infty]\to[0,\infty]$,*

$$\int^{\sup} f\odot dm=\lim_{\lambda\to\infty}\int^{\oplus_\lambda} f\odot dm_\lambda=\lim_{\lambda\to\infty}(g^\lambda)^{-1}\left(\int g^\lambda(f(x))dx\right).$$

Theorem 2.2. *(Bellman's inequality for pseudo-integrals) Let $u,v:[0,1]\to[a,b]$ be two measurable functions and let a generator $g:[a,b]\to[0,\infty)$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing and non-negative function.*
if u and v are comonotone, then the inequality

$$g^{\frac12}\left(\int_{[0,1]}^{\oplus}(u\odot v)dx\right)\ge\frac12\left(\int_{[0,1]}^{\oplus}u^2dx\right)^{\frac12}\odot\left(\int_{[0,1]}^{\oplus}v^2dx\right)^{\frac12}.$$

holds

Proof. Know that

$$\int_{[0,1]}^{\oplus}(u\odot v)dx=g^{-1}\left(\int_0^1 g(u\odot v)\right)dx=g^{-1}\left(\int_0^1 g(u)g(v)dx\right). \tag{2}$$

if u and v are comonotone functions, then from (2) and using the Bellman integral inequality and Jensen inequality for concave functions ([3]) , we have

$$\begin{aligned}
\int_{[0,1]}^{\oplus}(u\odot v)dx &\ge \frac12 g^{-1}\left[g\left(g^{-1}\left(\int_0^1 g(u^2)dx\right)^{\frac12}\right)\times g\left(g^{-1}\left(\int_0^1 g(v^2)dx\right)^{\frac12}\right)\right]\\
&=\frac12 g^{-1}\left(g\left(\int_{[0,1]}^{\oplus}u^2dx\right)^{\frac12}\times g\left(\int_{[0,1]}^{\oplus}v^2dx\right)^{\frac12}\right)\\
&=\frac12\left(\int_{[0,1]}^{\oplus}u^2dx\right)^{\frac12}\odot\left(\int_{[0,1]}^{\oplus}v^2dx\right)^{\frac12}\\
&=\frac12\left(\left(\int_{[0,1]}^{\oplus}u^2dx\right)\odot\left(\int_{[0,1]}^{\oplus}v^2dx\right)\right)^{\frac12}\\
&=\frac12\left[g^{-1}\left(g\int_{[0,1]}^{\oplus}u^2dx.g\int_{[0,1]}^{\oplus}v^2dx\right)\right]^{\frac12}\\
&=\frac12\left[g^{-1}\left(g\left(g^{-1}\int_0^1 g(u^2)dx\right).g\left(g^{-1}\int_0^1 g(v^2)dx\right)\right)\right]^{\frac12}\\
&=\frac12\left[g^{-\frac12}\left(\int_0^1 g(u^2)dx\right)^{\frac12}\left(\int_0^1 g(v^2)dx\right)^{\frac12}\right]\\
&=\frac12\left[g^{-\frac12}\left(\left(\int_0^1 g(u)\right)^2dx\right)^{\frac12}.\left(\int_0^1(g(v))^2dx\right)^{\frac12}\right]\\
&\le\frac12 g^{-\frac12}\times 2\int_0^1 g(u)g(v)dx\\
&=g^{-\frac12}\int_0^1 g(u.v)dx
\end{aligned}$$

$$
\begin{aligned}
&= g^{-\frac{1}{2}} \int_{[0,1]}^{\oplus} g(u.v)dx \\
&\leq g^{-\frac{1}{2}} \left(g \int_{[0,1]}^{\oplus} (u \odot v)dx \right) \\
&= g^{\frac{1}{2}} \left(\int_{[0,1]}^{\oplus} (u \odot v)dx \right).
\end{aligned}
$$

Which complete the proof. □

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