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The 25th Seminar on

Mathematical Analysis and Its Applications

Lorestan University, 3-5 May, 2023

$$\left\|\left|A^{\frac{\alpha+\beta}{2}}XB^{1-\frac{\alpha+\beta}{2}}+A^{1-\frac{\alpha+\beta}{2}}XB^{\frac{\alpha+\beta}{2}}\right|\right\|$$

$$\leq |\beta-\alpha|^{-1}\left\|\left|\int_\alpha^\beta (A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu)dv\right|\right\|$$

$$\leq \frac{1}{2}\left\|\left|A^\alpha XB^{1-\alpha}+A^{1-\alpha}XB^\alpha+A^\beta XB^{1-\beta}+A^{1-\beta}XB^\beta\right|\right\|$$

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INVITED SPEAKERS

The 25th Seminar on Mathematical Analysis and its Applications,
May 3-5, 2023, Lorestan University, Lorestan, Iran

Derivations on operator algebras

Shavkat Ayupov*

Abstract

This talk presents a full resolution of the problem stated by Ayupov in 2000, and partly restated in 2014 by Kadison and Liu, concerning derivations on algebras of measurable operators affiliated with von Neumann algebras. First we give preliminaries from the theory of operator algebras, non-commutative integration theory and show the physical background of automorphisms and derivations on operator algebras.

The second part of the talk explains a background of the Ayupov-Kadison-Liu Problem and its connection with general derivation theory in operator algebras starting with fundamental results due to Kaplanski, Kadison, Sakai and others. We shall cite and briefly explain major results concerning derivations on algebras of unbounded operators and list results concerning some special cases of the problem. Finally, the main result yielding the full resolution will be stated.

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Existence and uniqueness results for a nonlinear integral equation related to infectious disease

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Abstract

In the paper [1] the nonlinear integral equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds, \quad t \in \mathcal{R} \tag{1}$$

was interpreted as a model for the spread of certain infectious diseases with periodic contact rate that varies seasonally. In (1), $x(t)$ represents the proportion of infectives in the population at time t, $f(t, x(t))$ (which is periodic in t, i.e. there exists $w > 0$ with $f(t + w, x) = f(t, x)$ for all $t \in \mathcal{R}$ and $x > 0$) is the proportion of new infectives per unit time and $\tau > 0$ is the length of time an individual remains infectious. Authors in [1] were interested in obtaining a nontrivial periodic nonnegative solution to (1). They used Krasnoselskii's fixed point theorem in a cone to achieve this. Over the period of the last 46 years several extensions of (1) have been studied in which the existence of positive solutions, periodic solutions, pseudo-almost-periodic solution, etc. have been examined. In this lecture we shall consider the integral equation

$$x(t) = f(t, x(t)) + \int_{t-\tau}^{t} k(t, s)g(s, x(s))\, ds, \quad t \in \mathcal{R}, \tag{2}$$

to provide sufficient conditions for the existence and uniqueness of positive solutions. We shall also provide an iterative algorithm that converges to the unique solution. Our obtained results are supported by two examples. For this we shall apply a fixed point theorem proved in [2] for the convex-concave and nondecreasing operators defined in a Banach space with a normal solid cone. Our talk is based on a recently published paper(3).

[1]. K.L. Cooke and J.L. Kaplan, A periodicity threshold theorem for epidemics and population growth, Mat. Biosci. 31 (1976) 87-104

[2]. D. Guo, Y.J. Cho, J. Zhu, Partial Ordering Methods in Nonlinear Problems, Nova Science, New York (2004).

[3]. R.P. Agarwal, M. Jleli and B. Samet, An investigation of an integral equation involving convex-concave nonlinearities, *Mathematics*, **2021**(2021), pp. 10, 9, 2372, https://doi.org/10.3390/math9192372

*Speaker

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Some tensorial and hadamard product inequalities for convex functions of selfadjoint operators in Hilbert spaces

Silvestru Sever Dragomir*

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Abstract

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded *selfadjoint operators* on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the *spectral resolution* of A_i for $i = 1, ..., k$; by following Araki and Hansen [1], 2000, we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_k) \, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

In our presentation we show several inequalities for tensorial and Hadamard products of convex functions. Among others we prove that if f is continuously differentiable convex on the open interval I and A, B are selfadjoint operators in $B(H)$ with spectra $\lim funcSp(A)$, $\lim funcSp(B) \subset I$, then

$$\begin{aligned}(f'(A) \otimes 1)(A \otimes 1 - 1 \otimes B) &\geq f(A) \otimes 1 - 1 \otimes f(B) \\ &\geq (A \otimes 1 - 1 \otimes B)(1 \otimes f'(B)).\end{aligned} \tag{2}$$

Also

$$\begin{aligned}(f'(A)\, A) \circ 1 - f'(A) \circ B &\geq (f(A) - f(B)) \circ 1 \\ &\geq A \circ f'(B) - (f'(B)\, B) \circ 1,\end{aligned} \tag{3}$$

where $A \circ B$ the Hadamard product of operators A and B.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying the condition $\langle (A \circ B) e_j, e_j \rangle = \langle Ae_j, e_j \rangle \langle Be_j, e_j \rangle$ for all $j \in \mathbb{N}$, where $\{e_j\}_{j \in \mathbb{N}}$ is an *orthonormal the basis* for the separable Hilbert space H.

For an extended version of this presentation see *RGMIA Res. Rep. Coll.* **25** (2022), Art. 85,

`https://rgmia.org/papers/v25/v25a85.pdf`

For more tensorial and Hadamard product inequalities see **Arts. 74-85** from *RGMIA Res. Rep. Coll.* **25** (2022), available at `https://rgmia.org/v25.php`.

References

[1] H. Araki and F. Hansen, Jensen's operator inequality for functions of several variables, *Proc. Amer. Math. Soc.* **128** (2000), No. 7, 2075-2084.

*Speaker

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The HRT(Heil/Ramanathan/Topiwala) conjecture

Christopher Heil[1,*]

Abstract

The Heil-Ramanathan-Topiwala Conjecture (or HRT Conjecture) states that any finite set of time-frequency translates of a given square-integrable function must be linearly independent. Specifically, if $g \in L^2(\mathbb{R})$ is not the zero function and $\{(a_n, b_n)\}_{n=1}^N$ is a finite set of distinct points in $\mathbb{R}^2$, then $\{e^{2\pi i b_n x} g(x - a_n)\}_{n=1}^N$ is linearly independent. This conjecture was first stated in print in 1996, but remains open today. We will discuss the conjecture, its context, and some of the partial results that are currently known.

[*] Speaker

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Asymptotic analysis and non-convex optimization problems

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Abstract

This talk is based on joint works with A. Amini-Harandi, F.Fakhar and H. R. Hajisharifi.
Let X be a Hausdorff topological space and $f : X \to R \cup \{\infty\}$ be a proper function. Consider the minimization problem

$$(\mathcal{M}) \quad m := \inf_{x \in X} f(x)$$

It is well-known that if f is lower semicontinuous (lsc) and either X is compact or the sublevel sets of f are compact, then by the Weierstrass theorem the problem $(\mathcal{M})$ has a solution. But in many problems which can be modeled by the problem $(\mathcal{M})$ either the function f is not lsc or the space X or the sublevel sets are not compact. Here, we first show that the problem $\mathcal{M}$ has a solution whenever the objective function f is transfer weakly lower semicontinuous and coercive. In order to obtain a solution of the problem $\mathcal{M}$ for noncoercive objective functions on Banach spaces, we introduce a new concept of asymptotic functions and study some their properties. Moreover, we present the concept of nearly quasi-convexity as a generalization of quasi-convexity and prove that the problem $\mathcal{M}$ has solution, in the case where the space X is a reflexive Banach space and the objective function f is nearly quasiconvex, coercive and transfer weakly lower continuous with respect to the weak topology. Furthermore, by asymptotic tools we characterize the solution set of the problem $\mathcal{M}$ for transfer weakly lower continuous and nearly quasiconvex functions on the dual of Banach spaces.

Keywords: Asymptotic functions, transfer weakly lower continuous functions, nearly quasi-convex functions, Weierstrass's theorem.

*Speaker

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On fixed point results for non-expansive mappings

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Abstract

A well-known result of Kirk asserts that a nonexpansive mapping on a non-empty weakly compact convex subset (of a Banach space) possessing a normal structure has a fixed point. In this paper, we prove the existence of fixed points of pointwise eventually non-expansive mappings in modular spaces.

Keywords: pointwise eventually non-expansive mapping, modular spaces, uniformly convex in every direction, fixed point
Mathematics Subject Classification [2023]: 18A32, 18F20, 05C65 (at least 1 and at most 3)

[*]Speaker

The 25th Seminar on Mathematical Analysis and its Applications,
May 3-5, 2023, Lorestan University, Lorestan, Iran

Lie centralizers on some classes of operator algebras

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Abstract

Let $\mathcal{A}$ be an algebra. The Lie brackets is defined by $[a,b]=ab-ba$ for all $a,b\in\mathcal{A}$. Let $L:\mathcal{A}\longrightarrow\mathcal{A}$ be a map. It is called Lie centralizer if $L([a,b])=[L(a),b]=[a,L(b)]$ for all $a,b\in\mathcal{A}$. This presentation is dedicated to Lie centralizers and some studies done about them on some classes of operator algebras. In fact, we first review the motivation of the study of Lie centralizers and then we state and examine some of the results obtained in connection with Lie centralizers on von Neumann algebras, standard operator algebras and nest algebras. The topics mentioned are taken from references [2] to [9].

Keywords: Lie centralizer, von Neumann algebra, standard operator algebra, nest algebra
Mathematics Subject Classification [2023]: 47L10, 47L35, 46L10, 47B99, 16W10

References

[1] R. Behfar and H. Ghahramani, *Lie maps on triangular algebras without assuming unity*, Mediterranean J. Math. 18 (2021), 1–28.

[2] M. Brešar, *Centralizing mappings on von Neumann algebras*, Proc. Amer. Math. Soc. 111 (1991), 501–510.

[3] M. Brešar, *Commuting maps: a survey*, Taiwanese J. Math., 8 (2004), 361–397.

[4] M. Brešar, *Functional identities and zero Lie product determined Banach algebras*, Quart. J. Math. 71 (2020), 649–665.

[5] B. Fadaee and H. Ghahramani, Lie centralizers at the zero products on generalized matrix algebras, J. Algebra and Its Applications, https://doi.org/10.1142/S0219498822501651.

[6] B. Fadaee, A. Fošner and H. Ghahramani, *Centralizers of Lie structure of Lie structure of triangular algebras*, Results in Math, 77, 222 (2022). https://doi.org/10.1007/s00025-022-01756-8.

[7] A. Fošner, H. Ghahramani and F. Wei, Lie centralizers and generalized Lie derivations at zero products, Rocky Mountain J. Math, to appear.

[8] H. Ghahramani and W. Jing, *Lie centralizers at zero products on a class of operator algebras*, Ann. Funct. Anal. 12 (2021), 1–12.

[9] H. Ghahramani, A.H. Mokhtari and F. Wei, Lie centralizers and generalized Lie derivations at zero products, J. Algebra and Its Applications, https://doi.org/10.1142/S0219498824501068.

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GENERAL

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Three critical points for an integral variational inequality

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Abstract

Here, we would like to investigate the existence of at least three solutions for given system $P_{\lambda,\mu}$ such that $\lambda, \mu \in \mathbb{R}$ and $\lambda > 0$

$$-\int_\Omega \nabla u(x).\nabla(v(x)-u(x))dx \leq \lambda \int_\Omega [j^\circ(x,u(x);v(x)-u(x)) + (\mu k)^\circ(x,u(x));v(x)-u(x)]dx.$$

Keywords: Critical points of non-smooth functions; Variational-hemivariational inequalities; Elliptic eigenvalue problems
Mathematics Subject Classification [2023]: 18A32, 18F20, 05C65 (at least 1 and at most 3)

1 Introduction

It is well known and a classical result that a C^1 function $g : R \to R$ with two local minima enjoies to have a third critical point. This result but for functions defined on spaces of multi-dimensional, would be more complicated. Motivated mostly coming from for example mechanics and mathematical physics, the latter case has been studied and treated by several authors. For a short literature:

In [8] "Theorem A" authors showed if X is a separable and reflexive real Banach space; $I \subseteq \mathbb{R}$ an interval; $\phi : X \to \mathbb{R}$ a sequentially weakly lower semicontinuous C^1 functional whose derivative admits a continuous inverse on X^*; $j : X \to \mathbb{R}$ a C^1 functional with compact derivative, Moreover, $\lim_{k\to\infty} \phi(x) + \lambda j(x) = \infty$ for all $\lambda \in I$ and there exists $\rho \in \mathbb{R}$ such that $\sup_{\lambda\in I} \inf_{x\in X}(\phi(x)+\lambda(j(x)+\rho)) < \inf_{x\in X} \sup_{\lambda\in I}(\phi(x)+\lambda(j(x)+\rho))$, then there exists a non-empty open set $A \subseteq I$ and a positive real number r such that for each $\lambda \in I$, the equation $\phi'(x) + \lambda j'(x)$ has at least three solutions in X whose norms are less than r.

In [7], assuming X is a reflexive real Banach space; $I \subseteq \mathbb{R}$ an interval, $\phi : X \to \mathbb{R}$ a sequentially weakly lower semicontinuous C^1 functional which is bounded on each bounded subset of X and it's derivative admits a continuous inverse on X^*, $j : X \to \mathbb{R}$ a C^1 functional with compact derivative,

$$\lim_{k\to\infty} (\phi(x) + \lambda j(x)) = \infty$$

for all $\lambda \in I$ and there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda\in I} \inf_{x\in X} (\phi(x) + \lambda(j(x) + \rho)) < \inf_{x\in X} \sup_{\lambda\in I} (\phi(x) + \lambda(j(x_+\rho)),$$

then, there exist a non-empty open set $A \subseteq I$ and a positive real number r with the following property: For any $\lambda \in \mathbb{R}$ and C^1 functional $\psi(x)$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation $\phi'(x) + \lambda j'(x) + \mu\psi'(x) = 0$ has at least three solutions in X whose norms are less

*Speaker

than r.

In [4], it is given two applications in partial differential inclusions; the first one for a non-homogeneous Neumann boundary value problem and second one for a quasilinear elliptic inclusion problem in $\mathbb{R}^N$. In fact:
For a real reflexive Banach space $(X, ||.||)$ and $X_i, (i = 1, 2)$ two Banach spaces such that the embeddings $X \hookrightarrow X_i$ are compact, Λ a real interval, $h : [0, \infty) \to [0, \infty)$ be a non-decreasing convex function and $\phi_i : X_i \to R (i = 1, 2)$ two locally Lipschitz functions such that $E_{\lambda,\mu} = h(||.||) + \lambda\phi_1 + \mu go\phi_2$ restricted to X which it satisfies in $(PS)_c$-condition for every $c \in R, \lambda \in \Lambda, \mu \in [0, |\lambda| + 1], g \in G_\tau$ and $\tau \geq 0$. Moreover, $h(||.||) + \lambda\phi_1$ is coercive on X for all $\lambda \in \Lambda$ and there exists $\rho \in R$ such that,

$$\sup_{\lambda\in\Lambda} \inf_{x\in X} [h(||x||) + \lambda(\phi_1(x) + \rho)] < \inf_{x\in X} \sup_{\lambda\in\Lambda} [h(||x||) + \lambda(\phi_1(x) + \rho)],$$

then, there exists a non-empty open set $A \subset \Lambda$ and $r > 0$ such that for any $\lambda \in A$ there exists $\mu_0 \in [0, |\lambda|+1]$ that for each $\mu \in [0, \mu_0]$ the function $\xi_{\lambda,\mu} = h(||.||) + \lambda\phi_1 + \mu\phi_2$ has at least three critical points in X whose norms are less than r.

2 Front matter

Here, we focus to 2.2, that is to examine in detail the assumptions of 2.2 for $P_{\lambda,\mu}$. Given a real Banach space $(X, ||.||)$, usually X^* is its dual and $\langle ., . \rangle$ stands for the duality pairing between X^* and X. A map $\phi : X \to R$ is *locally Lipschitz* if to each $z \in X$ there corresponds a neighbourhood U of z and a constant $C_0 \geq 0$ such that

$$|\phi(x) - \phi(y)| \leq C_0||x - y||, \quad \forall x, y \in U.$$

As usual, given $u, v \in X$, the symbol $\phi^\circ(u, v)$ indicates the generalized directional derivative of ϕ at the point u along with the direction v which is defined by

$$\phi^\circ(u, v) := \limsup_{w\to u, t\to 0^+} \frac{\phi(w + tv) - \phi(w)}{t}.$$

The generalized gradient of the function ϕ at u, denoted by $\partial\phi(u)$ and is defined by

$$\partial\phi(u) := \{u^* \in X^* | \langle u^*, v\rangle \leq \phi^\circ(u, v) \forall v \in X\}. \tag{4}$$

Suppose that I is a function on the Banach space X satisfying in

(H) $J = \phi + \psi$, where $\phi : X \to R$ is locally Lipschitz and $\psi : X \to [-\infty, +\infty]$ is proper, convex and lower semicontinuous.

$u \in X$ is called a *critical point* for J if the following inequality fulfils:

$$\phi^\circ(u; v - u) + \psi(v) - \phi(u) \geq 0, \quad \forall v \in X.$$

Theorem 2.1. *[5] Assume that X is a separable reflexive Banach space. If*

(a) for any $\lambda \in \Lambda$, the function $J(., \lambda)$ fulfils $(PS)_c$, $c \in R$ together with $\lim_{||u||\to\infty} J(u, \lambda) = +\infty$, and

(b) there exists a continuous concave function $h : \lambda \to R$ satisfying

$$\sup_{\lambda\in\Lambda} \inf_{u\in X} (J(u, \lambda) + h(\lambda)) < \inf_{u\in X} \sup_{\lambda\in\Lambda} (J(u, \lambda) + h(\lambda)).$$

Then there is an open interval $\Lambda_0 \subseteq \Lambda$ such that for each $\lambda \in \Lambda_0$ the function $J(., \lambda)$ admits at least three critical points in X.

Lemma 2.2. *Let X be refexive and let I fulfit (H). Assume that:*

(j1) *There exist a real Banach space $\hat{X}$ and a locally Lipschitz function $\hat{\phi} : \hat{X} \to \mathbb{R}$ such that X is compactly embedded in $\hat{X}$ and $\hat{\phi}|_X = \phi$.*

(j2) $\lim\limits_{||u||\to\infty} I(u) = \infty$

(j3) *Each sequence $\{u_n\} \subseteq X$ complying with $u_n \rightharpoonup u$ in X; $\psi(u_n) < +\infty$ for every $n \in \mathbb{N}$ and $\limsup_{n\to\infty} \psi(u_n) \leq \psi(u)$ possesses a strongly convergent subsequence.*

Then I satisfies $(PS)_c$ for all $c \in \mathbb{R}$.

Suppose that Ω is a non-empty bounded open subset of the real Euclidean space $(R^N, |.|)$, $N \geq 3$ with smooth boundary $\partial\Omega$ and $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$ with respect to the following norm

$$||u|| := \Big(\int_\Omega |\nabla u(x)|^2 dx\Big)^{\frac{1}{2}}.$$

For $2^* = \frac{2N}{N-2}$ there is a natural embedding $H_0^1 \subseteq L^p(\Omega)$, where $p \in [1, 2^*]$, there is $c_p > 0$ as a constant such that

$$||u||_{L^p(\Omega)} \leq c_p ||u|| \quad \forall u \in H_0^1(\Omega).$$

It is well known that for the case $p \in [1, 2^*)$ this embedding is compact. For a convex closed subset C of $H_0^1(\Omega)$ containing 0 and given two any functions $j, k : \Omega \times R \to R$ the following hypotheses usually will be posited:

(h1) *j and k are measurable with respect to each variable separately.*

(h2) *There are $a > 0, p \in [1, 2^*]$ such that*

$$\max\{|j(x,t), |k(x,t)|\} \leq a(1 + |t|^{p-1}) \quad \forall (x,t) \in w \times R.$$

(h3) *For any $(x, \zeta) \in \Omega \times R$*

$$\min\Big\{\int_0^\zeta j(x,t)dt, -\Big|\int_0^\zeta k(x,t)dt\Big|\Big\} \geq -a(1 + |\zeta|^s),$$

where $0 \leq s < 2$.

(h4) *There is $\gamma > 2$ in which*

$$\liminf_{\zeta\to 0}\Big[\frac{1}{|\zeta|^\gamma} \inf \int_0^\zeta j(x,t)dt\Big] > -\infty.$$

(h5) *There exists $u_0 \in C$ such that*

$$\int_\Omega dx \int_0^{u_0(x)} j(x,t)dt < 0.$$

For $(x, \zeta) \in \Omega \times R$, we set

$$J(x,\zeta) := \int_0^\zeta j(x,t)dt, \quad K(x,\zeta) := \int_0^\zeta k(x,t)dt.$$

3 Main Results

From $(h1)$ and $(h2)$, two functions $j, k : \Omega \times R \to R$ are of course well defined and locally Lipschitz in ζ for each fixed $x \in \Omega$. For this reason, it makes sense the generalized directional derivatives j° and K° of J and K, respectively, with respect to the variable ζ.

Given $\lambda > 0$ if $\lambda, \mu \in \mathbb{R}$ and assume that $M : R \to R$ is a function $\hat{M}(t) := \int_0^t M(s)ds$ such that:

(M1) $\lim_{t\to\infty} M(t) = +\infty$

(M2) there exists $a > 0$ such that for all $t \in \mathbb{R}$ we have $a \leq M(t)$.

Here, we consider the following problem:
Find $u \in C$ fulfilling

$$P_{\lambda,\mu}:$$
$$-M(||u||^2)\Delta u \in \lambda\Big[\partial j(u) + \partial(\mu K)(u)\Big].$$

In fact, there exist $h_1 \in \partial j(u), \quad h_2 \in \partial(\mu K)$ in which

$$-M(||u||^2)\Delta u = \lambda[h_1 + h_2].$$

According to (2.3)

$$-M(||u||^2)\int_\Omega \bigtriangledown(u) \bigtriangledown (u - v) = \lambda\Big[h_1(v - u) + h_2(v - u)\Big]$$
$$\leq \lambda\Big[\int j^\circ(u, v - u) + (\mu K)^\circ(u, v - u)\Big].$$

Theorem 3.1. *Assuming $(h_1) - (h_5)$, there exists $\delta > 0$ in which for any $\mu \in [-\delta, \delta]$ it corresponds to an open interval $\Lambda_\mu \subset [0, +\infty]$ such that $P_{\lambda,\mu}$ has at least three solutions in C which is convex and contains zero and $\lambda \in \Lambda_\mu$.*

Before proving this theorem we need some results.
we choose $X = H_0^1(\Omega)$ and $\hat{X} = L^p(\Omega), \Lambda = [0, +\infty)$ and $\phi_1 = 0$,

$$\phi_2(u) = \int_\Omega j(x, u(x))dx + \mu\int_\Omega K(x, u(x))dx, \quad \forall u \in \hat{X}. \tag{5}$$

and

$$\psi_1(u) = \begin{cases} \hat{M}(||u||^2) & u \in C, \\ \infty & otherwise. \end{cases}$$

By $(h1)$ and $(h2)$ the functions ϕ_1 and ϕ_2 turn out to be locally Lipschitz and

$$f(u, \lambda) = \psi_1(u) + \lambda\phi_2(u), \quad (u, \lambda) \in X \times \Lambda. \tag{6}$$

Lemma 3.2. *$f(u, \lambda)$ is coerciv for any $\lambda \in \Lambda$.*

Proof. we have from $(M2)$:

$$\begin{aligned} f(u,\lambda) = \psi_1(u) + \lambda\phi_2(u) &\geq a||u||^2 - \lambda\Big[\Big|\int_\Omega j(x,u(x))\Big| + |\mu|\int_\Omega \Big|K(x,u(x))\Big|\Big] \\ &\geq a||u||^2 - \lambda\Big[m(\Omega) + |u|_s^s + |\mu|m(\Omega) + |u|_s^s\Big] \\ &\geq a||u||^2 - \lambda[d_1 + d_2||u||^s], \end{aligned}$$

where $d_1 = m(\Omega) + |\mu|m(\Omega)$. Since $s < 2$, assertion holds.

□

Lemma 3.3. *$f(u,\lambda)$ for any $\lambda \in \Lambda$ satisfies in (PS) condition on C.*

Proof. Now, let $\{u_n\} \subseteq C$ satisfy $u_n \rightharpoonup u$ in X besides $\hat{M}(||.||^2)$ is continuos and consequently lower semi-continuous in fact, since

$$||u||^2 \leq \liminf ||u||^2 \tag{7}$$

Because of $M > 0$ so $\hat{M}$ is increasing then

$$\hat{M}(||u||^2) \leq \hat{M}(\liminf ||u||^2) \leq \liminf \hat{M}(||u||^2) \leq \limsup \hat{M}(||u||^2) \leq \hat{M}(||u||^2) \tag{8}$$

we get $\hat{M}(||u_n||^2) \to \hat{M}(||u||)$ and also because since $\hat{M}$ is invertible and its inverse is also continuous, we have $||u_n||^2 \to ||u||^2$. X is uniformly convex so $u_n \to u$. Therefore, by lemma 2.2 for every $\lambda \in \Lambda$ the function $f(u,\lambda)$ complies $(PS)_c$.

□

Lemma 3.4. *(b) of Theorem 2.2 valids.*

Proof. Let $h(\lambda) := \lambda\rho$ for a real number ρ.

$$f(u,\lambda) + h(\lambda) = \psi_1(u) + \lambda\phi_2(u) + \lambda\rho = \psi_1(u) + \lambda(\phi_2(u) + \rho).$$

In the case $\phi_2(u) + \lambda > 0$, (a_2) is obvious, but in the case $\phi_2(u) + \rho \leq 0$ without loss of generality, we may set $\gamma \leq 2^*$. otherwise, we can replace it by any constant $\gamma^* \in [2, 2^*]$. from $(h_3), (h_4)$, and natural embeddings, there exists a constant $b > 0$ in which

$$-b||u||^\gamma \leq \int_\Omega j(x,u(x))dx.$$

On the other hand, since $\gamma > 2$ and $M \geq 1$, from (M2) we have $\hat{M}(||u||^2) \geq ||u||^2$ and by choosing have r small enough with $\psi_1(u) \leq r$ we get

$$\begin{aligned} \int_\Omega j(x,u(x)) > -b||u||^\gamma &\geq -b(\hat{M}(||u||^2)^{\frac{1}{2}} \\ &\geq -b(\psi_1(u))^{\gamma-1} \times (\psi_1(u)) \\ &\geq -b_0(r)\frac{\gamma}{2} - 1 \times r \\ &- \eta r. \end{aligned}$$

according to $h5$ there exists u_0 such that $\int_\Omega j(x,u_0(x)) < 0$ then $0 < -\frac{1}{\hat{M}(||u_0||^2)\int_\Omega j(x,u_0(x))}$ choose η such that

$$0 < \eta < -\frac{1}{\psi_1(u_0)}\int_\Omega j(x,u_0(x)).$$

then for small enough $\varepsilon > 0$,

$$\inf_{\psi_1(u)\leq r}\int_\Omega j(x,u(x)) \geq \frac{r}{\psi_1(u_0)}\int_\Omega j(x,u_0(x)) + \varepsilon.$$

On the other hand, given $\varepsilon > 0$ directly one can find $\delta > 0$ in which

$$\delta\Big(\sup_{\psi_1(u)\leq r}\Big|\int_\Omega j(x,u(x)) + \frac{r}{\psi_1(u_0)}\int_\Omega \mu k(x,u_0(x))\Big|\Big) < \varepsilon.$$

Then,

$$\begin{aligned}
\phi_2(u) &= \int_\Omega j(x,u(x)) + \int_\Omega \mu k(x,u(x)) \\
&\geq \inf_{\psi_1(u)\leq r}\int_\Omega j(x,u(x)) - |\mu| \sup_{\psi_1(u)\leq r}\int_\Omega K(x,u(x)) \\
&> \frac{r}{\psi_1(u_0)}\phi_2(u_0) + \varepsilon - |\mu|\Big(\sup_{\psi_1(u)\leq r}\Big|\int_\Omega K(x,u(x))dx\Big| + \frac{1}{\psi_1(u_0)}\Big|\int_\Omega K(x,u_0(x))dx\Big|\Big) \\
&> \frac{r}{\psi_1(u_0)}\phi_2(u_0).
\end{aligned}$$

Then

$$-\phi_2(u) < -\frac{r}{\psi_1(u_0)}, \tag{9}$$

and so

$$-\phi_2(u) < \rho < -\frac{r}{\psi_1(u_0)}\phi_2(u_0). \tag{10}$$

On the other hand since $\lambda \mapsto \sup_{\lambda\in\Lambda}\inf_{u\in C}[f(u,\lambda)+h(\lambda)$ is upper semicontinuous $\lim_{\lambda\to+\infty} f(u,\lambda)+h(\lambda) = +\infty$ so there exists $\lambda^* \in \Lambda$ in which

$$\begin{aligned}
\sup_{\lambda\in\Lambda}\inf_{u\in C}[f(u,\lambda)+h(\lambda) &= \inf_{u\in C}[f(u,\lambda^*)+h(\lambda^*)] \\
&= \inf_{u\in C}[\psi_1(u) + \lambda^*(\phi_2(u)+\rho)].
\end{aligned}$$

there is some cases:

(i) If $\lambda^*\rho < r$ we get

$$r > \lambda^*\rho > \sup_{\lambda\in\Lambda}\inf_{u\in C}[f(u,\lambda)+h(\lambda).$$

(ii) $\lambda^*\rho > r$ we get

$$\lambda^*(\phi_2(u)+\rho) < \frac{r}{\rho}(\phi_2(u)+\rho) < r - \psi_1(u_0), \tag{11}$$

so

$$\sup_{\lambda\in\Lambda}\inf_{u\in C}[f(u,\lambda)+h(\lambda)] < \psi_1(u_0) + r - \psi_1(u_0) = r. \tag{12}$$

□

Proof of theorem 2.3:
According to hypotheses on M, it is clear that condition (H) satisfies. we have checked coercivity of $P_{\lambda,\mu}$, (PS) and condition (b) of theorem 2.2. therefore, $P_{\lambda,\mu}$ has at least three weak solutions.

In the following we give an example, and show that conditions for M satisfy.

Example 3.5. Let $M(t) = e^t$. Then easily M satisfies $M1$ and $M2$. Therefore, by theorem 2.3 the following problem has at least three solutions

$$-M(||u||^2)\Delta u \in \lambda\Big[\partial j(u) + \partial(\mu k)(u)\Big].$$

References

[1] C. Chen, Y.C. Cheng Kuo, and T.F. Wu, *The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions*, Differential Equations, (2011), 1876-1908.

[2] L. Ding, and C.L. Tang, *Existence and multiplicity of solutions for semilinear elliptic equations with Hardy terms and Hardy Sobolev critical exponents*, Mathematical Analysis and Applications, (2008), 1073-1083.

[3] A. Fiscella, P. Pucci, and B. Zhang, *P-fractional Hardy Schrodinger Kirchhoff systems with critical nonlinearities*, Adv. Nonlinear Anal. (2019), 1111-1131.

[4] A. Kristly, W. Marzantowicz, and C. Varga, A non-smooth three critical points theorem with applications in differential inclusions, Global Optimization. 46(1)(2009), 49-62.

[5] S.A. Marano and D. Motreanu, *On a three critical points theorem for non-differentiable functions and applications to nonlinear boundary value problems*, Nonlinear Analysis, (2002), 37-52.

[6] D. Naimen, *Positive solutions of Kirchhoff type elliptic equations involving the critical Sobolev exponent*, Nonlinear Differential Equations and Applications, (2014), 885-914.

[7] B. Ricceri, *A three critical points theorem revisited*, Nonlinear Anal. (2009), 3084-3089.

[8] B. Ricceri, *On a three critical points theorem*, Arch math. (Basel), (2000), 220-226.

[9] L. Shen, *Multiplicity and asymptotic bahaviour of solutions for Kirchhoff type equations involving the Hardy Sobolev exponent and singular nonlinearity*, Journal of Inequalities and Applications, (2018), 213.

[10] A. Szulkin, *Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems*, Ann Inst Henri Poincare. (1986), 77-109.

[11] Q. Xie, and C.L. Tang, *Existence and multiplicity of solutions for Kirchhoff type problem with critical exponent*, Communications on Pure and Applied Analysis, **12(6)**(2013), 2773-2786.

[12] L. P. Xu and H. Chen, *Ground state solutions for Kirchhoff type equations with a general nonlinearity in the critical growth*, Adv. Non linear Anal. (2018).

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Linear independence of the orbit's members of a J-class vector

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Abstract

Let T be a hypercyclic operator on a Banach space X and x is a hypercyclic vector for T, i.e, $\overline{orb(T,x)} = \overline{\{T^n x; n = 0, 1, 2, \cdots\}} = X$. Beside to hypercyclicity, another concept named "J-class" has been presented as below;
if $x \in X$ is a non-zero vector and

$$J(x) = \{z \in X; \textit{there exist a sequence } \{z_n\} \subset X \textit{ and a strictly increasing sequence of positive integers } \{m_n\}, \textit{ such that } z_n \longrightarrow x \textit{ and } T^{m_n} z_n \longrightarrow z\} = X,$$

then T is called a J-class operator.
In this paper, we want to investigate linear independence of the set $orb(T, x)$ when x is a J-class vector for T.

Keywords: hypercyclicity, J-class operators.
Mathematics Subject Classification [2023]: 47A16

1 Introduction and Preliminaries

Assume that X is a Banach space and $B(X)$ denotes the Banach algebra of all bounded linear operators on X. Consider an operator $T \in B(X)$. If for every pair U, V of nonempty open subsets of X, there is a positive integer n so that

$$T^n(U) \cap V \neq \emptyset,$$

then the operator T is topological transitive. The first example of topological transitive operators was presented by Birkhoff in [4]. If the underlying space is considered as a separable Banach space, then it is a simple matter to see that topological transitivity is equivalent to hypercyclicity. To be more clear, if B is a subset of X, then the orbit of B under T is the set

$$orb(T, B) = \{T^n x;\ x \in B, n = 0, 1, 2, \cdots\}.$$

if B is a singleton $\{x\}$ and $\overline{orb(T, B)} = X$, then T is called a hypercyclic operator and x is a hypercyclic vector for T. If $B = \{\lambda x;\ \lambda \in \mathbb{C}\}$ for some vector $x \in X$ and $orb(T, B)$ is a dense subset of X, then this operator is said to be a supercyclic operator and the vector x is a supercyclic vector for T. Note that, since $\mathbb{C} \cdot \{T^n x;\ x \in X, n = 0, 1, 2, \cdots\} \subseteq T(X)$, so every supercyclic operator and consequently every topological transitive operator T on a separable Banach space has dense range. We recall that, Ansari [1] showed that every infinite-dimensional separable Banach space admits a topological transitive operator. Hence, in the following X is an infinite-dimensional Banach space. It is worth pointing out that when X is an infinite-dimensional non-separable Banach space, then obviously X cannot support hypercyclic operators. However, it is well known that topologically transitive operators may exist in X, see for instance [5].

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As you see, in the above cases the underlying Banach space X should be separable, hence every non-separable Banach space is deleted in hypercyclicity and supercyclicity. Therefore Authors in [3] proposed $J(x)$ for a vector x in Banach space X and $T \in B(X)$ as following:

$$J(x) = \{z \in X; \textit{there exist a sequence } \{z_n\} \subset X \textit{ and a strictly increasing sequence of positive integers } \{m_n\}, \textit{ such that } z_n \longrightarrow x \textit{ and } T^{m_n} z_n \longrightarrow z\}.$$

And they gave the following definition;

Definition 1.1. An operator $T \in B(X)$ is called a J-class operator if there exists a non-zero $x \in X$ such that $J(x) = X$. In this case, the vector x is called a J-class vector for T.

They have proved that on a separable Banach space X, an operator T is hypercyclic if and only if, the underlying Banach space X has a dense subset of the J-class vectors for the operator T, and furthermore they showed that there exists a J-class operator on $\ell^\infty(\mathbb{N})$ which is not hypercyclic because $\ell^\infty(\mathbb{N})$ is not separable. In fact, they have tried to introduce the localized hypercyclicity by the notion of J-class, and for a better explanation of this goal, in [2] we provided an equivalent definition for the $J(x)$ through the use of open sets, as below;

$$J(x) = \{z \in X : \textit{for every pair of neighbourhoods } U,\ V \textit{ of } x,\ z \textit{ respectively, and every } N \in \mathbb{N} \textit{ there exists an integer } n > N \textit{ such that } T^n U \cap V \neq \phi\}.$$

Because checking the linear independence or linear dependence of the members of a certain set is an important issue in mathematics, so in this paper, we want to answer to the below question:

Question. *Assume x is consider as a J-class vector for an operator T. Then which state is correct for the set $orb(T, x)$? linear independent or dependent set?.*

2 Main results

Note that a J-class operator shares some of the same properties as a hypercyclic operator. For example, like hypercyclicity which is a purely infinite-dimensional property [1], J-class aswell as hypercyclicity occur only on infinite-dimensional Banach space. For more clarity, the reader can found the following proposition in [3] as Proposition 5.5, which we will use it in the following.

Proposition 2.1. *Any operator acting on a finite dimensional space can not be a J-class operator.*

Theorem 2.2. *Let $T \in B(X)$ be a J-class operator. Then the orbit of every J-class vector for T forms a linearly independent set.*

Proof. To obtain a contradiction, let $x \in X \setminus \{0\}$ and $J(x) = X$ and also, suppose that

$$\sum_{i=0}^{n} z_i T^i x = 0,$$

for some non-zero scaler $z_i \in \mathbb{C},\ i = 0, 1, \cdots, n$. Then consider $\widehat{X}$ be the linear span of vectors $x, Tx, T^2x, \cdots, T^n x$. Since x is a J-class for T, so it is obvious that x is a J-class for the operator $T_{|\widehat{X}} : \widehat{X} \to \widehat{X}$, which contradicts Proposition 2.1. Therefore the proof is completed.

□

References

[1] S. I. Ansari, *Existence of hypercyclic operators on topological vector spaces*, J. Funct. Anal., **148** (1997), 384–390.

[2] M. Asadipour and B. Yousefi, *On some properties of J-class operators*, Commun. Korean Math. Soc. **34** (2019), no. 1, 145–154.

[3] F. Bayart and E. Matheron, *Dynamics of Linear Operators*, Cambridge University Press, Cambridge, 2009.

[4] G. Birkhoff, *Demonstration dun theoreme sur les fonctions entieres*, C. R. Acad, Sci. Paris, **189** (1929), 473–475.

[5] J. Bonet, L. Frerick, A. Peris and J. Wengenroth, *Transitive and hypercyclic operators on locally convex spaces*, Bull. London Math. Soc., **37** (2005), 254-264.

[6] G. Costakis and A. Manoussos, *J-class operators and hypercyclicity*, J. Operator Theory, 67 (2012), 101–119.

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Some properties of local subspace transitivity

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Abstract

An operator T on Banach space X is called transitive, if for every nonempty open subsets U,V of X, there is a positive integer n, such that $T^n(U) \cap V \neq \phi$. In the present paper, local subspace transitivite operators are introduced. Moreover the local subspace transitivity criterion is stated. Also, we show an operator may satisfies in the local subspace transitivity criterion without being topological transitive.

Keywords: Hypercyclic operators, topologically transitive operators, J-class operators.
Mathematics Subject Classification [2023]: Primary; 47A16, Secondary; 37B99, 54H99

1 Introduction

An operator T for subspace M of a Banach space X over the field $\mathbb{C}$ of complex numbers is called subspace-hypercyclic or M-hypercyclic if there is a vector $x \in X$, so that the intersection $Orb(T,\ x) \cap M = \{T^n x : n \in \mathbb{N} \cup \{0\}\} \cap M$ is dense in M. If $X = M$, then the operator T is called hypercyclic and the vector x is a hypercyclic vector for T. Observe that in this case, the underlying Banach space X should be separable. Then it is well known and easy to show that an operator T is hypercyclic if and only if T is topologically transitive, to be precise, for every pair of nonempty open subsets U, V of X, there exists an integer $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$. Again let M be a pure subspace of X, then the operator T is called M-transitivity if for any pair U, V of nonempty relatively open subsets of M, there exists some integers $n \geq 0$, such that $T^n(U) \cap V \neq \emptyset$ and $T^n(M) \subseteq M$. it was confirmed that M-hypercyclicity is derived from M-transitivity, [6]. Le [5] showed that M-transitivity is not equivalent to M- hypercyclicity when M is a pure subspace of X.
It is worthwhile to mention that, with following remark we can ensure that there is an infinite subset P of $\mathbb{N}$, that T satisfies in M-transitive condition for every member of it.

Remark 1.1. If $T \in B(X)$ is an M-transitive operator and U, V are nonempty relatively open subsets of M and if $n_0 \geq 0$, so that $T^{-n_0}(U) \cap V \neq \emptyset$ and $T^{n_0}(M) \subseteq M$, then two distinct vectors $x, y \in T^{-n_0}(U) \cap V$ and relatively open subsets O_x and O_y of M are assumed, so that $O_x, O_y \subset T^{-n_0}(U) \cap V$ and $O_x \cap O_y = \emptyset$. According to the assumptions, there is an integer $k \geq 1$, such that we have $T^k(M) \subseteq M$ and $T^{-k}(O_x) \cap O_y \neq \emptyset$. Thus, the set $V \cap T^{-(k+n_0)}(U)$ is nonempty and, clearly, $T^{(n_0+k)}(M) \subseteq M$. By repeating the above method, it can be ensured that there is an infinite subset P of $\mathbb{N}$, such that for every $n \in P$, $V \cap T^{-n}(U) \neq \emptyset$ and $T^n(M) \subseteq M$.

The concept of hypercyclicity was localized by introducing certain sets ,which is called J-sets, [3]. To be precise for a given vector $x \in X$ and an operator T;

$$J(x) = \{z \in X; \textit{there exist a sequence } \{z_n\} \subset X \textit{ and a strictly increasing sequence of positive integers } \{m_n\},\ \textit{such that } z_n \longrightarrow x \textit{ and } T^{m_n} z_n \longrightarrow z\}.$$

And an operator $T \in B(X)$ is called a J-class operator if there exists a non-zero $x \in X$ such that $J(x) = X$. In this case, the vector x is called a J-class vector for T.
Similarly, we provide the following definition;

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Definition 1.2. M-extended limit set of vector x under an operator $T \in B(X)$ is the set of all $y \in M$ such that there is a sequence $\{z_n\}$ in subspace M and a strictly increasing sequence $\{k_n\} \subset \mathbb{N}$ such that $z_n \longrightarrow x$ and $T^{k_n} z_n \longrightarrow y$, and for all n, $T^{k_n}(M) \subseteq M$.
The M-extended limit set of vector x under an operator T is denoted by $\mathbb{J}_M(x)$.

See the paper [2] and the good book [4] for some details on hypercyclicity and J-class and related properties.

In this paper, we give an equivalent definition of $\mathbb{J}_M(x)$ and moreover, a sufficient condition is expressed as subspace $\mathbb{J}_M$-class criterion for an operator $T \in B(X)$ to be the subspace $\mathbb{J}_M$-class operator.
In all the following discussions, M is a nontrivial subspace of X.

2 Main results

At the beginning of this section, we state an equivalent definition of $\mathbb{J}_M(x)$.

Theorem 2.1. *Assume that $T \in B(X)$ and $x \in M$. Then we have $\mathbb{J}_M^{tra}(x) = \mathbb{J}_M(x)$ where $\mathbb{J}_M^{tra}(x)$ is the set of all $y \in M$ such that for every relatively open neighborhoods U_x, V_y of vectors x, y in M respectively, and every positive integer n_0, there exists an integer $n > n_0$ such that $T^n(U_x) \cap V_y \neq \emptyset$ and $T^n(M) \subseteq M$.*

Proof. Assume that $y \in \mathbb{J}_M^{tra}(x)$, $k_0 = 1$, $N = k_{n-1}$ and set for every $n \in \mathbb{N}$:

$$U_{(x,n)} = B(x, \frac{1}{n}) \cap M, \qquad V_{(y,n)} = B(y, \frac{1}{n}) \cap M.$$

By the assumption, integer $k_n > N$ and vector $x_n \in U_{(x,n)}$ exist, so that $T^{k_n}(M) \subseteq M$ and $T^{k_n} x_n \in V_{(y,n)}$. Thus the sequences $\{k_n\}$ and $\{x_n\}$ are proposed by induction, so that we have $x_n \longrightarrow x$ and $T^{k_n} x_n \longrightarrow y$. This completes the proof of nontrivial side. □

For an operator $T \in B(X)$, if equality $\mathbb{J}_M(x) = M$ is established for a vector $x \in M$, then x is a subspace $\mathbb{J}$-class vector for subspace M under operator T. We then call T a $\mathbb{J}_M$-class operator.

According to the earlier theorem which in fact defines a localized M-transitive operator, it is concluded that $T \in B(X)$ is an M-transitive operator if and only if for all $x \in M$, the set $\mathbb{J}_M(x)$ is equal to the subspace M.

As hypercyclicity criterion is a useful tool for identifying hypercyclic operators and provides a sufficient condition to ensure that an operator is hypercyclic, we also intend to introduce the local subspace transitivity criterion. Also, we will provide an example that satisfies this criterion. One can see [1] as a useful reference about the following information.

Theorem 2.2 (**Local Subspace Transitivity Criterion**). *Assume that T is an operator on Banach space X and Y is a dense subset of M which is a subspace of X. If for vector $x \in M$, there exists a strictly increasing sequence $\{k_n\} \subset \mathbb{N}$, which satisfies the following conditions;*

1. *There is a sequence $\{z_n\} \subset M$, such that $z_n \longrightarrow x$ and $T^{k_n} z_n \longrightarrow 0$,*
2. *For every $y \in Y$, there is a sequence $\{w_n\} \subset M$, such that $w_n \longrightarrow 0$ and $T^{k_n} w_n \longrightarrow y$,*
3. *For every $n \in \mathbb{N}$, M is an invariant subspace under T^{k_n},*

then $\mathbb{J}_M(x) = M$.

Proof. Assume that $y \in M$ and integer $N \geq 1$; also, for sufficiently small $\varepsilon > 0$, $W_1' := B(0, \frac{\varepsilon}{2})$ is the relatively open ball in M. Now, we set:

$$U_x := W_1' + x, \qquad U_y := W_1' + y.$$

By the assumptions, there is an integer $k_1 \geq N$, such that:

$$T^{k_1}(U_x) \cap W_1' \neq \emptyset \quad and \quad T^{k_1}(W_1') \cap U_y \neq \emptyset.$$

There are vectors $w_1, w_2, w_3, w_4 \in W_1'$, such that:

$$T^{k_1}(w_1 + x) = w_2, \qquad T^{k_1}(w_3) = w_4 + y.$$

Consequently,

$$T^{k_1}(x + w_1 + w_3) = w_2 + w_4 + y$$

and

$$x_1 := x + w_1 + w_3 \in B(x, \varepsilon), \qquad y_1 := y + w_2 + w_4 \in B(y, \varepsilon).$$

Similarly, we can construct the sequences $\{x_n\} \subset M$, $\{y_n\} \subset M$ and $\{k_n\} \subset \mathbb{N}$, $k_n - 1 > k_{n-1}$, such that for each $n \geq 2$;

$$x_n \in B(x, \frac{\varepsilon}{n}), \qquad y_n \in B(y, \frac{\varepsilon}{n})$$

and $T^{k_n} x_n \longrightarrow y$ as $n \longrightarrow \infty$. The above contents along with (3) show $y \in \mathbb{J}_M(x)$ and, since the choice of vector $y \in M$ is arbitrary, equality $\mathbb{J}_M(x) = M$ is obtained, so x is a subspace $\mathbb{J}$-class vector for subspace M under T. □

In the following, an operator will be raised such that it satisfies the local subspace transitivity criterion however it is not a topological transitive operator. In fact, the unilateral weighted backward shift T on $\ell^2(\mathbb{N})$ with the bounded weight sequence $\{w_n \geq 0\}_{n\in\mathbb{N}}$ is topological transitive if and only if $\limsup_n (\prod_{i=1}^{n} w_i) = +\infty$, [4].

Example 2.3. Consider weighted backward shift operator T on $\ell^2(\mathbb{N})$ given by:

$$T(x^1, x^2, \cdots) = (2x^2, \frac{3}{2}x^3, \frac{4}{3}x^4, \cdots).$$

Also let $Y^{'}$ be the set of finite sequences with entries $z \in \mathbb{C}$ that $Re(z) \in \mathbb{Q}$, $Im(z) \in \mathbb{Q}$ and subspace M of $\ell^2(\mathbb{N})$ is considered as:

$$M = \{\{x_n\} \in \ell^2(\mathbb{N}) \ ; \ x_{2k} = 0, \ for \ all \ k \in \mathbb{N}\}.$$

Since $Y := Y^{'} \cap M$ is dense in M, so there are strictly increasing sequence $\{2k\}_k$, sequence $\{x_k\} \subset Y$ that

$$x_k = (x^1, 0, x^3, 0, \cdots, x^{2k-1}, 0, 0, \cdots),$$

$x_k \longrightarrow 0$ as $k \longrightarrow \infty$ and $T^{2k} x_k = 0$.
Now, for the random member $y = (y^1, 0, y^3, 0, \cdots, y^{2m+1}, 0, 0, \cdots) \in Y$ and $k \geq 1$, we set;

$$w_{2k}(y) = (\underbrace{0, \cdots, 0}_{2k-times}, \ \frac{y^1}{2k+1}, 0, \frac{3y^3}{2k+3}, \ 0, \cdots, \frac{(2m+1)y^{2m+1}}{2(k+m)+1}, \ 0, \ 0, \cdots).$$

Clearly, for every $k \in \mathbb{N} \cup \{0\}$, $w_{2k}(y)$ belongs to Y and the sequence $\{w_{2k}(y)\}$ is a sequence in $\ell^2(\mathbb{N})$. Since

$$||w_{2k}(y)||^2 = \sum_{j=1}^{2m} |\frac{j}{2k+j} y^j|^2 \leq \frac{4m^2}{(2k+1)^2} ||y||^2,$$

so $w_2k(y) \longrightarrow 0$, as $k \longrightarrow \infty$. Note that for $n \geq 1$:

$$T^n(x^1, x^2, x^3, \cdots) = \Big((n+1)x^{n+1}, \ \frac{1}{2}(n+2)x^{n+2}, \ \frac{1}{3}(n+3)x^{n+3}, \ \cdots\Big),$$

thus

$$T^{2k} w_{2k}(y) = \Big((2k+1)\frac{1}{2k+1} y^1, 0, (\frac{2k+3}{3})(\frac{3}{2k+3}) y^3, 0, \cdots, (\frac{2(k+m)+1}{(2m+1)})(\frac{2m+1}{2(k+m)+1}) y^{2m+1}, 0, 0, \cdots\Big) = y.$$

Hence the condition (2) holds. For each $k \geq 1$, condition (3) clearly holds; therefore, the operator T satisfies the local subspace transitivity criterion with respect to subspace M and $\mathbb{J}_M(0) = M$.

References

[1] M. Asadipour, *Local subspace transitivity criterion*, Khayyam J. Math., 8 (2022), no. 1, 33–41.

[2] M. Asadipour and B. Yousefi, *On some properties of J-class operators*, Commun. Korean Math. Soc., 34 (2019), no. 1, 145–154.

[3] G. Costakis and A. Manoussos, *J-class operators and hypercyclicity*, J. Operator Theory, 67 (2012), 101–119.

[4] K.G. Grosse-Erdmann and A. Peris, *Linear Chaos*, Springer, 2011.

[5] C.M. Le, *On subspace-hypercyclic operators*, Proc. Amer. Math. Soc. **139** (2011), no. 8, 2847–2852.

[6] B. F. Madore and R. A. Martinez-Avendano, *Subspace hypercyclicity*, J. Math. Anal. Appl. **373** (2011), no. 2, 502–511.

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Adjoint of generalized shift operators over Hilbert spaces

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Abstract

In the following text we compute adjoint of generalized shift $\sigma_\varphi \restriction_{\ell^2(\tau)}: \ell^2(\tau) \to \ell^2(\tau)$ (for $\varphi : \tau \to \tau$ and natural number N such that $\varphi^{-1}(\alpha)$ has at most N elements for each $\alpha \in \tau$).

Keywords: adjoint operator, generalized shiftHilbert space, weighted generalized shift
Mathematics Subject Classification [2023]: 47B37

1 Introduction

Let's recall that for each Hilbert space H there exists nonzero cardinal number τ such that H and $\ell^2(\tau)$ are isomorphic, where $\ell^2(\tau) = \{(x_\alpha)_{\alpha\in\tau} \in \mathbb{C}^\tau : \sum\limits_{\alpha\in\tau} |x_\alpha|^2 < +\infty\}$ equipped with inner product $< x, y >= \sum\limits_{\alpha\in\tau} x_\alpha\overline{y_\alpha}$ (for $x = (x_\alpha)_{\alpha\in\tau}, y = (y_\alpha)_{\alpha\in\tau} \in \ell^2(\tau)$).
Generalized shifts has been introduced for the first time as a generalization of one–sided and two–sided shifts in [4]. For nonempty arbitrary sets X, Γ and self–map $\varphi : \Gamma \to \Gamma$ we call $\sigma_\varphi : X^\Gamma \to X^\Gamma$ with $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) = (x_{\varphi(\alpha)})_{\alpha\in\Gamma}$ (for $(x_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$) a generalized shift. Hence for nonzero cardinal number τ and self–map $\varphi : \tau \to \tau$ we may consider $\sigma_\varphi : \mathbb{C}^\tau \to \mathbb{C}^\tau$ and its restriction to $\ell^2(\tau)$.

Remark 1.1. For nonzero cardinal number τ and $\varphi : \tau \to \tau$ the following statements are equivalent [1]:

- $\sigma_\varphi(\ell^2(\tau)) \subseteq \ell^2(\tau)$,
- $\sigma_\varphi(\ell^2(\tau)) \subseteq \ell^2(\tau)$ and $\sigma_\varphi \restriction_{\ell^2(\tau)}: \ell^2(\tau) \to \ell^2(\tau)$ is continuous,
- there exists $N \geq 1$ such that for each $\alpha \in \tau$, $\varphi^{-1}(\alpha)$ has at most N elements.

On the other hand in Hilbert space H and continuous (bounded) operator $T : H \to H$ one may consider adjoint operator $T^* : H \to H$ such that $< T(x), y >=< x, T^*(y) >$ (for each $x, y \in H$). The main aim of this text is to compute adjoint of "suitable" generalized shift operators over Hilbert spaces.

2 Main results

In this section suppose τ is a nonzero cardinal number, and $\varphi : \tau \to \tau$ is an arbitrary self–map such that $\sigma_\varphi(\ell^2(\tau)) \subseteq \ell^2(\tau)$. By Remark 1.1, $\sigma_\varphi \restriction_{\ell^2(\tau)}: \ell^2(\tau) \to \ell^2(\tau)$ is bounded (continuous) and there exists $N \geq 1$ such that for each $\alpha \in \tau$, $\varphi^{-1}(\alpha)$ has at most N elements. For each $\theta \in \tau$ suppose $\varphi^{-1}(\theta)$ has $N_\theta (\leq N)$ elements. Consider fixed $\gamma \in \tau$. For each $\theta \in \tau$ choose $(\alpha_1^\theta, \cdots, \alpha_N^\theta) \in \tau^N$ such that:

- if $\varphi^{-1}(\theta) = \varnothing$, then $\alpha_1^\theta = \cdots = \alpha_N^\theta = \gamma$,
- if $\varphi^{-1}(\theta) \neq \varnothing$, then $\varphi^{-1}(\theta) = \{\alpha_1^\theta, \ldots, \alpha_{N_\theta}^\theta\}$ and $\alpha_i^\theta = \gamma$ for each $i > N_\theta$.

Also choose $\mathfrak{w}_1^\theta, \ldots, \mathfrak{w}_N^\theta \in \{0, 1\}$ such that:

- if $\varphi^{-1}(\theta) = \varnothing$, then $\mathfrak{w}_1^\theta = \cdots = \mathfrak{w}_N^\theta = 0$,

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- if $\varphi^{-1}(\theta) \neq \varnothing$, then $\mathfrak{w}_1^\theta = \cdots = \mathfrak{w}_{N_\theta}^\theta = 1$ and $\mathfrak{w}_i^\theta = 0$ for each $i > N_\theta$.

Now for each $i \in \{1, \ldots, N\}$ let

$$\mathfrak{w}_i = (\mathfrak{w}_i^\theta)_{\theta\in\tau} \text{ and } \eta_i : \tau \underset{\theta\mapsto\alpha_i^\theta}{\to} \tau \ .$$

Suppose V is a vector space over field F, Γ is a nonempty set, $\eta : \Gamma \to \Gamma$ is arbitrary and $\mathfrak{w} = (\mathfrak{w}_\alpha)_{\alpha\in\Gamma} \in F^\Gamma$, then we may consider weighted generalized shift $\sigma_{\eta,\mathfrak{w}} : V^\Gamma \to V^\Gamma$ with $\sigma_{\eta,\mathfrak{w}}((x_\alpha)_{\alpha\in\Gamma}) = (\mathfrak{w}_\alpha x_{\varphi(\alpha)})_{\alpha\in\Gamma}$ (for $(x_\alpha)_{\alpha\in\Gamma} \in V^\Gamma$) [2, 3].

Remark 2.1. For $\eta : \tau \to \tau$ and $\mathfrak{v} = (\mathfrak{v}_\alpha)_{\alpha\in\Gamma} \in \mathbb{C}^\tau$ the following statements are equivalent (where $\overline{\mathfrak{v}}_\beta = (\mathfrak{v}_\alpha)_{\alpha\in\eta^{-1}(\beta)}$ and $||\overline{\mathfrak{v}}_\beta||_2 = (\sum\limits_{\alpha\in\eta^{-1}(\beta)} |\mathfrak{v}_\alpha|^2)^{1/2}$ for $\beta \in \eta(\tau)$):

- $\sigma_{\eta,\mathfrak{v}}(\ell^2(\tau)) \subseteq \ell^2(\tau)$,
- $\sigma_{\eta,\mathfrak{v}}(\ell^2(\tau)) \subseteq \ell^2(\tau)$ and $\sigma_{\eta,\mathfrak{v}} \restriction_{\ell^2(\tau)} : \ell^2(\tau) \to \ell^2(\tau)$ is continuous,
- $\sup\{||\overline{\mathfrak{v}}_\beta||_2 : \beta \in \eta(\tau)\} < \infty$.

Lemma 2.2. *$\sigma_{\eta_i,\mathfrak{w}_i}(\ell^2(\tau)) \subseteq \ell^2(\tau)$ for each $i \in \{1, \ldots, N\}$.*

Proof. Consider $i \in \{1, \ldots, N\}$ and $\beta \in \eta_i(\tau)$, also let $\overline{\mathfrak{v}}_\beta = (\mathfrak{w}_i^\alpha)_{\alpha\in\eta_i^{-1}(\beta)}$. Then

$$\begin{aligned} ||\overline{\mathfrak{v}}_\beta||_2^2 &= \sum_{\alpha\in\eta_i^{-1}(\beta)} |\mathfrak{w}_i^\alpha|^2 = \sum_{\beta=\eta_i(\theta)} |\mathfrak{w}_i^\theta|^2 = \sum_{\beta=\alpha_i^\theta} |\mathfrak{w}_i^\theta|^2 \\ &= \sum_{\beta=\alpha_i^\theta, i\leq N_\theta} |\mathfrak{w}_i^\theta|^2 = \sum_{\beta=\alpha_i^\theta, i\leq N_\theta} 1 = \sum_{\beta=\alpha_i^\theta, i\leq N_\theta, \varphi(\beta)=\theta} 1 \in \{0,1\} \end{aligned}$$

therefore $\sup\{||\overline{\mathfrak{v}}_\beta||_2 : \beta \in \eta(\tau)\} \in \{0, 1\}$. By Remark 2.1 we have the desired result. □

Theorem 2.3. $\sigma_\varphi \restriction^*_{\ell^2(\tau)} = \sum\limits_{1\leq i\leq N} \sigma_{\eta_i,\mathfrak{w}_i} \restriction_{\ell^2(\tau)}$.

Proof. For $\theta \in \tau$ let $\mathsf{e}_\theta = (\delta_\theta^\alpha)_{\alpha\in\tau}$ where $\delta_\theta^\theta = 1$ and $\delta_\theta^\alpha = 0$ for $\alpha \neq \theta$. Also suppose $\pi_\theta : \mathbb{C}^\tau \to \mathbb{C}$ is the projection map on the θth coordinate.
For each $x, y \in \ell^2(\tau)$ we have $< \sigma_\varphi \restriction_{\ell^2(\tau)} (x), y >=< x, \sigma_\varphi \restriction^*_{\ell^2(\tau)} (y) >$, hence for each $\theta \in \tau$ the following relations are valid:

$$\begin{aligned} \pi_\theta(\sigma_\varphi \restriction^*_{\ell^2(\tau)} (y)) &= < \sigma_\varphi \restriction^*_{\ell^2(\tau)} (y), \mathsf{e}_\theta >=< y, \sigma_\varphi(\mathsf{e}_\theta) >=< y, \sum_{\alpha\in\varphi^{-1}(\theta)} \mathsf{e}_\alpha > \\ &= \sum_{\alpha\in\varphi^{-1}(\theta)} y_\alpha = \sum_{1\leq i\leq N_\theta} y_{\eta_i(\theta)} = \sum_{1\leq i\leq N_\theta} \mathfrak{w}_\theta^i y_{\eta_i(\theta)} = \sum_{1\leq i\leq N} \mathfrak{w}_\theta^i y_{\eta_i(\theta)} \\ &= \sum_{1\leq i\leq N} \pi_\theta(\sigma_{\eta_i,\mathfrak{w}_i}(y)) = \pi_\theta(\sum_{1\leq i\leq N} \sigma_{\eta_i,\mathfrak{w}_i}(y)) \ . \end{aligned}$$

Hence

$$\forall y \in \ell^2(\tau) \ \ \sigma_\varphi^*(y) = (\pi_\theta(\sigma_\varphi^*(y)))_{\theta\in\tau} = (\pi_\theta(\sum_{1\leq i\leq N} \sigma_{\eta_i,\mathfrak{w}_i}(y)))_{\theta\in\tau} = \sum_{1\leq i\leq N} \sigma_{\eta_i,\mathfrak{w}_i}(y) \ ,$$

which leads to the desired result. □

3 Conclusion

For nonzero cardinal number τ and self–map $\varphi : \tau \to \tau$ such that $\sigma_\varphi(\ell^2(\tau)) \subseteq \ell^2(\tau)$ we show the adjoint of generalized shift operator $\sigma_\varphi \restriction_{\ell^2(\tau)} : \ell^2(\tau) \to \ell^2(\tau)$ is sum of finitely many $\{0, 1\}-$weighted generalized shifts.

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References

[1] F. Ayatollah Zadeh Shirazi, F. Ebrahimifar, *Is there any nontrivial compact generalized shift operator on Hilbert spaces?*, Rendiconti del Circolo Matemtico di Palermo, 68, No. 3 (2019), 453–458.

[2] F. Ayatollah Zadeh Shirazi, F. Ebrahimifar, R. Rezavand, *Weighted generalized shift operators on ℓ^p spaces*, Rendiconti del Circolo Matematico di Palermo, Volume 71, Issue 1 (2022), 1–12.

[3] F. Ayatollah Zadeh Shirazi, E. Hakimi, A. Hosseini, R. Rezavand, *Li-Yorke and Devaney chaotic uniform dynamical systems amongst weighted shifts*, Topology and its Applications, 326 (2023), Article ID 108406, 15 p.

[4] F. Ayatollah Zadeh Shirazi, N. Karami Kabir, F. Heydari Ardi, *A note on shift theory*, Mathematica Pannonica, 19, No. 2 (2008), 187-195.

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Strong convergence theorems of the CQ method for a broad class of nonlinear mappings

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Abstract

In this paper, by using CQ method we modify the Manns iterative process and prove a strong convergence theorem for 2-generalized hybrid mappings in a Hilbert space.

Keywords: Fixed point, Hilbert space, CQ method, Strong convergence , Weak convergence
Mathematics Subject Classification [2023]: 47H10, 47H09, 47J25

1 Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H, then a self mapping S of C is said to be: (i) *nonexpansive*, if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$; ; (ii) *nonspreading*, if $2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|Sy - x\|^2$ for all $x, y \in C$; (iii) *hybrid*, if $3\|Sx - Sy\|^2 \leq \|x - y\|^2 + \|Sx - y\|^2 + \|Sy - x\|^2$ for all $x, y \in C$; (iv) *generalized hybrid* [5] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Sx - Sy\|^2 + (1 - \alpha)\|x - Sy\|^2 \leq \beta\|Sx - y\|^2 + (1 - \beta)\|x - y\|^2, \tag{13}$$

for all $x, y \in C$. We call such a mapping an (α, β)-*generalized hybrid* mapping. It easily seen that

- $(1, 0)$-generalized hybrid mapping is nonexpansive mapping;
- $(2, 1)$-generalized hybrid mapping is nonspreading mapping;
- $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mapping is hybrid mapping.

Let us denote by $F(S)$ the set of fixed points of S.

Some iteration algorithms have been introduced which often used to approximate a fixed point of nonlinear mappings. One of them is Mann iteration [4] which introduced in 1953, for a self mapping T of C, as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sx_n, \tag{14}$$

where $x_0 \in C$ chosen arbitrarily, C is a nonempty closed convex subset of a Banach space and $0 \leq \alpha_n \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$. The other one is Ishikawa iteration [3] which introduced in 1974, for a Lipschitzian pseudocontractive self mapping S of K, as follows:

$$\begin{cases} x_0 \in K \text{ chosen arbitrarily}, \\ y_n = \beta_n x_n + (1 - \beta_n)Sx_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sy_n, \end{cases} \tag{15}$$

where K be a nonempty compact convex subset of a Hilbert space H and $0 \leq \beta_n \leq \alpha_n \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and proved strong convergence of the sequence $\{x_n\}$ generated by the above iterative algorithm if $\lim_{n\to\infty} \beta_n = 1$ and $\sum_{n=1}^{\infty}(1 - \alpha_n)(1 - \beta_n) = \infty$. By taking $\beta_n = 1$ for all $n \geq 0$ in (15), Ishikawa iteration process reduces to Mann iteration process.

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Newly, Mann and Ishikawa iterative schemes [3, 4] have been studied by a number of authors (see [1] and references therein) .

Process (15) is indeed more general than process (14). But research has been done on the latter due probably to reasons that the formulation of process (14) is simpler than that of (15) and that a convergence theorem for process (14) may lead to a convergence theorem for process (15) provided that $\{\beta_n\}$ satisfies certain appropriate conditions. On the other hand, the process (14) may fail to converge while process (15) can still converge for a Lipschitz pseudocontractive mapping in a Hilbert space. Actually, Mann and Ishikawa iteration processes have only weak convergence, in general.

Let K be a closed convex subset of H and let P_K be metric (or nearest point) projection from H onto K (i.e., for $x \in H$, $P_K x$ is the only point in K such that $\|x - P_K x\| = inf\{\|x - z\| : z \in H\}$). Let $x \in H$ and $z \in K$, then $z = P_K x$ if and only if:

$$\langle x - z, y - z\rangle \leq 0 \tag{16}$$

for all $y \in K$.

Recently, to obtain strong convergence, many mathematicians have been extensively considered modified processes. Nakajo and Takahashi [6] proposed the following modification of the Mann's iteration for a nonexpansive self mapping S of E in a Hilbert space H:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n) S x_n, \\ C_n = \{u \in E \ : \|y_n - u\| \leq \|x_n - u\|\}, \\ Q_n = \{u \in E \ : \langle x_n - u, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{17}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. The process (17) is called a CQ method for the Mann iteration process since at every step the Mann iterate (denoted y_n in (17)) is used to make the sets C_n and Q_n which are in turn used to make the next iterate x_{n+1} and therefore the name. They proved strong convergence of the sequence $\{x_n\}$ generated by the CQ method (17) if the sequence $\{\alpha_n\}$ is bounded above from one.

Definition 1.1. Let C be a nonempty, closed and convex subset of a Hilbert space H, a self mapping S of C is called 2-*generalized hybrid* [5] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} \alpha_1\|S^2 x - Sy\|^2 + \alpha_2\|Sx - Sy\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Sy\|^2 \\ \leq \beta_1\|S^2 x - y\|^2 + \beta_2\|Sx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2, \end{aligned} \tag{18}$$

for all $x, y \in C$. Such a mapping is called an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping. It is easily seen that an $(0, \alpha_2, 0, \beta_2)$-generalized hybrid mapping is an (α_2, β_2)-generalized hybrid mapping [2]. Also, One can easily show that 2-generalized hybrid mapping is quasi-nonexpansive if the set of it's fixed points is nonempty. In [2], Hojo et al. give two examples of 2-generalized hybrid mapping which are not generalized hybrid mappings. Then, the class of 2-generalized hybrid mappings covers the class of generalized hybrid mappings.

2 Main results

In this section, by using CQ method we modify the Manns iterative process and prove a strong convergence theorem for 2-generalized hybrid mappings in Hilbert space.

Theorem 2.1. *Let C be a nonempty, closed and convex subset of a Hilbert space H and S be a 2-generalized*

hybrid self mapping of C with $F(S) \neq \emptyset$. If $\{x_n\}$ in C be a sequence generated by $x_1 = x \in C$ and

$$\begin{cases} y_n = \frac{1}{n}\sum_{k=0}^{n-1} S^k z_n, \\ z_n = \gamma_n x_n + (1-\gamma_n) S x_n, \\ C_n = \{u \in C : \|y_n - u\|^2 \leq \|x_n - u\|^2 + (\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, u\rangle)\}, \\ Q_n = \{u \in C : \langle x_n - u, x_n - x\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

where $\{\gamma_n\}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty} \gamma_n = 1$. Then $\{x_n\}$ converges strongly to $P_{F(S)}x$.

Theorem 2.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and T be an (α, β)-generalized hybrid self mapping of C with $F(S) \neq \emptyset$. If $\{x_n\}$ in C be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \frac{1}{n}\sum_{k=0}^{n-1} S^k z_n, \\ z_n = \gamma_n x_n + (1-\gamma_n) S x_n, \\ C_n = \{u \in C : \|y_n - u\|^2 \leq \|x_n - u\|^2 + (\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, u\rangle)\}, \\ Q_n = \{u \in C : \langle x_n - u, x_n - x\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

where $\{\gamma_n\}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty} \gamma_n = 1$. Then $\{x_n\}$ converges strongly to $P_{F(S)}x$.

Proof. Since S is an (α, β)-generalized hybrid mapping, hence S is a $(0, \alpha, 0, \beta)$-generalized hybrid mapping. Therefore by Theorem 2.1, we get the desired result. □

Corollary 2.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and S be a 2-generalized hybrid self mapping of C with $F(S) \neq \emptyset$. If $\{x_n\}$ in C be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \frac{1}{n}\sum_{k=0}^{n-1} S^k x_n, \\ C_n = \{u \in C : \|y_n - u\|^2 \leq \|x_n - u\|^2\}, \\ Q_n = \{u \in C : \langle x_n - u, x_n - x\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

then $\{x_n\}$ converges strongly to $P_{F(S)}x$.

Proof. Letting $\gamma_n = 1$ for all $n \in \mathbb{N}$, the result can be obtained from Theorem 2.1 □

Remark 2.4. Since nonexpasive mappings are $(0,1)$-generalized hybrid mappings, nonspreading mappings are $(2,1)$-generalized hybrid mappings and hybrid mappings are $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mappings, then the Theorem 2.2 holds for these mappings.

Remark 2.5. Since nonexpasive mappings are $(0,1)$-generalized hybrid mappings, nonspreading mappings are $(2,1)$-generalized hybrid mappings and hybrid mappings are $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mappings, then the Theorem 2.2 holds for these mappings.

References

[1] S. Alizadeh and F. Moradlou, *Weak and strong convergence theorems and a nonlinear ergodic theorem for N-generalized hybrid mappings* Asian-European Journal of Mathematics. Vol. 10, No. 1 (2017) 1750001.

[2] M. Hojo, W. Takahashi and I. Termwuttipong, *Strong convergence theorem for 2-generalized hybrid mapping in Hilbert spaces*, Nonlinear Anal. 75 (2012) 2166–2176.

[3] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. 40 (1974) 147–150.

[4] W.R. Mann, *Mean value methods in iteration,* Proc. Amer. Math. Soc. 4 (1953) 506–510.

[5] T. Maruyama, W. Takahashi and M. Yao, *Fixed point and mean ergodic theorems for new nonlinear mappings in Hilbert spaces,* J. Nonlinear Convex Anal. 12 (2011) 185–197.

[6] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mapping and nonexpansive mappings*, J. Math. Anal. Appl. 279 (2003) 372–379.

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Bounds for some mappings related to the Hermite-Hadamard inequality for lipschitz functions

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Abstract

In this paper, New bounds for some mappings related to Hermite-Hadamard inequality for Lipschitz functions are introduced.

Keywords: Hermite-Hadamard inequality, Lipschitz functions, convex functions
Mathematics Subject Classification [2023]: 26D15, 26A51

1 Introduction

Let $I = [c, d]$ be an interval in $\mathbb{R}$, $f : I \to \mathbb{R}$ be a convex function and let $a, b \in [c, d], a < b$. We consider the celebrate Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{19}$$

Several refinements and generalizations of Hermite-Hadamard have been found in [1, 2, 3, 4, 5, 6, 7, 15] and references therein. In recent years several extensions have been considered for classical convexity. In order to provide various refinements of this result, S.S. Dragomir introduced the mappings $H, F, G, L : [0, 1] \to \mathbb{R}$, in [3], [5] and established several results in connection to Hermite-Hadamard inequality;

$$\begin{aligned}
H(t) &:= \frac{1}{b-a}\int_a^b f\Big(tx + (1-t)\frac{a+b}{2}\Big)dx,\\
F(t) &:= \frac{1}{(b-a)^2}\int_a^b\int_a^b f\Big(tx + (1-t)y\Big)dxdy,\\
G(t) &:= \frac{1}{2}\Big(f\Big(ta + (1-t)\frac{a+b}{2}\Big) + f\Big(tb + (1-t)\frac{a+b}{2}\Big)\Big)\\
L(t) &:= \frac{1}{2(b-a)}\int_a^b \Big(f\Big(ta + (1-t)x\Big) + f\Big(tb + (1-t)x\Big)\Big)dx,
\end{aligned}$$

for all $t \in [0, 1]$ and $x, y \in I$. Since then numerous articles have appeared in the literature reflecting further applications and properties of these mappings (see [1, 2, 4, 5, 6, 7, 15] and references therein). In [5] S.S. Dragomir by relaxing convexity and utilizing the mapping H, F, introduced several results for Lipschitz functions in related to Hermite-Hadamard inequalities which we recall some of them as follows;

Definition 1.1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be M-Lipschitz on the interval I of if there exits $M \geq 0$ such that for every $x, y \in I$ one has

$$|f(x) - f(y)| \leq M|x - y|.$$

Motivated by the above results in this paper by relaxing the convexity we establish some new bounds for mappings H, F, G, L, in Lipschitz functions setting.

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2 Main results

In this section we shall investigate the cases of the mappings G and L.

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a $M-$Lipschitz function and $a, b \in I$ with $a < b$. Then,*
(i) The mapping G is $\frac{M}{2}(b-a)-$Lipschitz on $[0,1]$.
(ii) For every $t \in [0,1]$ we have the following inequalities

$$\left|G(t) - f\Big(\frac{a+b}{2}\Big)\right| \leq \frac{Mt}{2}(b-a), \tag{20}$$

$$\left|G(t) - \frac{f(a)+f(b)}{2}\right| \leq \frac{M(1-t)}{2}(b-a), \tag{21}$$

$$\left|G(t) - tf\Big(\frac{a+b}{2}\Big) + (1-t)\frac{f(a)+f(b)}{2}\right| \leq M(b-a)t(1-t), \tag{22}$$

$$|F(t) - G(t)| \leq \frac{M(1+t)}{4}(b-a). \tag{23}$$

The following theorem involves some bounds for the mapping L.

Theorem 2.2. *Let $f : I \to \mathbb{R}$ be a $M-$Lipschitz function and $a, b \in I$ with $a < b$. Then,*
(i) The mapping L is $\frac{M}{2}(b-a)-$Lipschitz on $[0,1]$.
(ii) For every $t \in [0,1]$ we have the following inequalities;

$$\left|L(t) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{Mt}{2}(b-a), \tag{24}$$

$$\left|L(t) - \frac{f(a)+f(b)}{2}\right| \leq \frac{M(1-t)}{2}(b-a). \tag{25}$$

(iii) One has the inequalities;

$$\left|\int_0^1 L(t)dt - \frac{1}{2}\Big[\frac{f(a)+f(b)}{2} + \frac{1}{b-a}\int_a^b f(x)dx\Big]\right| \leq \frac{M}{4}(b-a), \tag{26}$$

$$|G(t) - L(t)| \leq \frac{M(1-t)}{4}(b-a), \tag{27}$$

$$|F(t) - L(t)| \leq \frac{M(2+t)}{6}(b-a), \tag{28}$$

$$|H(t) - L(t)| \leq \frac{M(1+t)}{4}(b-a). \tag{29}$$

.

Now we consider the bidimensional interval $\Delta := [a,b] \times [c,d]$ with $a < b$ and $c < d$. The mapping $H : [0,1]^2 \to \mathbb{R}$ defied by

$$H(s,t) := \frac{1}{(b-a)(d-c)} \times \int_a^b \int_c^d f\Big(sx + (1-s)\frac{a+b}{2}, ty + (1-t)\frac{c+d}{2}\Big)dxdy.$$

Theorem 2.3. *Let the function $f : \Delta \to \mathbb{R}$ satisfies the Hölder condition with $M_1, M_2 > 0$, on Δ that is,*

$$|f(x,y) - f(z,w)| \leq M_1|x-z| + M_2|y-w|,$$

Then,
(i) The mapping F satisfies the Hölder condition with $\frac{M_1(b-a)}{4}, \frac{M_2(d-c)}{4}$, on $[0,1]^2$.
(ii) The function $f : [0,1] \to \mathbb{R}$ defined by

$$f(t) := H(t,t),$$

is a $M-$Lipschitz function, where $M := \frac{M_1(b-a)+M_2(d-c)}{4}$.
(iii) The following inequalities hold

$$\left|H(s,t) - f\Big(\frac{a+b}{2}, \frac{c+d}{2}\Big)\right| \leq \frac{M_1(b-a)}{4}s + \frac{M_1(d-c)}{4}t, \tag{30}$$

and

$$\begin{aligned} &\left|H(s,t) - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\ &\leq \frac{M_1(b-a)}{4}(1-s) + \frac{M_1(d-c)}{4}(1-t) \end{aligned} \tag{31}$$

References

[1] S. Banić, Mapping connected with Hermite-Hadamard inequalities for superquadratic functions, *J. Math. Inequal.,* **3** (2009), 557-589.

[2] A. Barani, Generalized Hermite-Hadamard inequality for Lipschtiz functions, (to appear in AIA)

[3] S.S. Dragomir, A mapping in connection to Hadamard,s inequalities, *An. Öster. Akad. Wiss. Math. Natur.,* **128** (1991), 17-20.

[4] S.S. Dragomir and C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and applications, *(RGMIA Monographs http:// rgmia.vu.edu.au/ monographs/ hermite hadamard.html),* Victoria University, 2000.

[5] S.S. Dragomir , Inequalities on Hadamard's type for Lipschitzian mapping and their applications, *J. Math. Anal. Appl.,* **245** (2000), 489-501.

[6] S.S. Dragomir and I. Gomm, Bounds for two mapping associated to the Hermite-Hadamard inequality, *AJMAA.,* **1** (2011), 1-9.

[7] S.S. Dragomir, and I. Gomm, Some new bounds for two mapping related to the Hermite-Hadamard inequality for convex functions, *Numer. Algeb. Cont. Optim.,* **2** (2012), 271-278.

[8] K.L. Tesing, S.R. Hwang and S.S. Dragomir, New Hermite-Hadamard-type inequalities for convex functions (II), *Comput. Math. Appl.,* **62** (2011), 401-418.29-38.

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Bullen's inequality for preinvex functions

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Abstract

In this paper Bullen's inequality for preinvex functions of several variables is introduced.

Keywords: Bullen's inequality, Hermite-Hadamard inequality, preinvex functions
Mathematics Subject Classification [2023]: 26D15, 26A51

1 Introduction

For an interval $I = [c, d]$ in $\mathbb{R}$, let $f : I \to \mathbb{R}$ be a convex function and $a, b \in [c, d], a < b$. We consider the well-known Hadamard's inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{32}$$

Bullen's inequality introduced in [1], assert that the first inequality in (375) is stronger than the second, that is

$$\frac{1}{b-a}\int_a^b f(x)dx - f\Big(\frac{a+b}{2}\Big) \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx, \tag{33}$$

see [3, 4, 15] and references therein for articles in this category. In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of preinvex functions given in [9]. There have been some works in the literature which are devoted to investigating preinvex functions (e.g. see [5] and references therein). A set $S \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : S \times S \to \mathbb{R}^n$, if for every $x, y \in S$ and $t \in [0, 1]$, $y + t\eta(x, y) \in S$. The mapping η is said to be satisfies the condition C if for every $x, y \in S$ and $t \in [0, 1]$,

$$\eta(y, y + t\eta(x, y)) = -t\eta(x, y),$$
$$\eta(x, y + t\eta(x, y)) = (1 - t)\eta(x, y).$$

For every $x, y \in S$ and every $t_1, t_2 \in [0, 1]$ from condition C we have

$$\eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y). \tag{34}$$

Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. Then, the function $f : S \to \mathbb{R}$ is said to be preinvex with respect to η, if for every $x, y \in S$ and $t \in [0, 1]$,

$$f(y + t\eta(x, y)) \leq tf(x) + (1 - t)f(y). \tag{35}$$

Every convex function is preinvex with respect to the map $\eta(x, y) = x - y$ but the converse does not holds.

The Hermite-Hadamard inequality for preinvex functions is introduced by M.A. Noor in [6] as follows,

$$f(a + \frac{1}{2}\eta(b, a)) \leq \frac{1}{\eta(b, a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{36}$$

where $a, b \in S$. Since then numerus articles have been published in this category see, for example [7] and references therein.

The main purpose of this paper is to establish Bullen's inequality for invex functions of several variables.

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2 Main Results

The following theorem is an generalization of Hermite-Hadamard inequality for preinvex functions of several variables.

Theorem 2.1. *Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. Suppose that $f : S \to \mathbb{R}$ is a preinvex integrable function. Then for every $x, y \in S$,*

$$\int_0^1 f(y + t\eta(x, y))dt \leq \frac{f(y) + f(x)}{2}. \tag{37}$$

Moreover, if η satisfies condition C then,

$$f(m) \leq \int_0^1 f(y + t\eta(x, y))dt \leq \frac{f(y) + f(y + \eta(x, y))}{2}, \tag{38}$$

where $m := y + \frac{1}{2}\eta(x, y)$.

The following corollary is an immediate consequence of Theorem 2.1.

Corollary 2.2. *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$ and η satisfies condition C. Assume that f is a preinvex integrable function on S. Suppose that $x, y \in S$ such that $\eta(x, y) \neq 0$. Then,*

$$f(m) \leq \frac{1}{\eta(x, y)} \int_y^{y+\eta(x,y)} f(u)du \leq \frac{f(y) + f(y + \eta(x, y))}{2}, \tag{39}$$

where $m = y + \frac{1}{2}\eta(x, y)$.

Now, we introduce a version of Bullen's theorem for several variable preinvex functions.

Theorem 2.3. *Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. Suppose that η satisfies condition C. Assume that f is a preinvex integrable function on S. Then, the first inequality in (38) is stronger than the second, that is,*

$$\begin{aligned} 0 &\leq \int_0^1 f(y + t\eta(x, y))dt - f(y + \frac{1}{2}\eta(x, y)) \\ &\leq \frac{f(x) + f(y + \eta(x, y))}{2} - \int_0^1 f(y + t\eta(x, y))dt. \end{aligned} \tag{40}$$

The Bullen's theorem for real valued functions defied on invex subsets of $\mathbb{R}$ is as follows.

Corollary 2.4. *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$. Suppose that η satisfies condition C. If $a, b \in S$ with $\eta(a, b) \neq 0$, and f is a preinvex integrable function on S. Then,*

$$\begin{aligned} 0 \leq& \frac{1}{\eta(b, a)} \int_a^c f(x)dx - f\big(a + \frac{1}{2}\eta(b, a)\big) \\ \leq& \frac{f(a) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(b, a)} \int_a^c f(x)dx. \end{aligned} \tag{41}$$

In special case if $\eta(x, y) = x - y$, for every x, y then, the following Bullen's theorem holds for convex functions.

Theorem 2.5. *Let $S \subseteq \mathbb{R}^n$ be a convex set and f be a real valued convex integrable function on S. Then,*

$$\begin{aligned} 0 &\leq \int_0^1 f(tx + (1 - t)y)dt - f(\frac{x + y}{2}) \\ &\leq \frac{f(x) + f(y)}{2} - \int_0^1 f(tx + (1 - t)y))dt. \end{aligned} \tag{42}$$

References

[1] P.S. Bullen, *Error estimates for some elementary quadrature rules*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 602-633 (1978), 97-103.

[2] S.S. Dragomir and C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and applications, (RGMIA Monographs http:// rgmia.vu.edu.au/ monographs/ hermite hadamard.html), Victoria University, 2000.

[3] I. Franjić and J. Pečarić, On Corrected Bullen-Simpson's 3/8 inequality, Tamkang Journal of mathematics, 37 (2006) 135-148.

[4] A. Maria Acu and H. Gonska, On Bullen's and related inequalities, General Mathematics, 22 (2014), 19-26.

[5] S.R. Mohan and S. K. Neogy, On invex sets and preinvex function, J. Math. Anal. Appl. 189 (1995) 901-908.

[6] M.A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, J. Math. Anal. Approx. Theory 2 (2007), 126-131.

[7] M.Z. Sarikaya, N. Alp, H. Bozkurt, On Hermite-Hadamard type integral inequalities for preinvex and log-preinvex functions, Contemporary Analysis and Applied Mathmatics, 1 (2013) 237-252.

[8] K.L. Tseng, S.R. Hwang and K.C. Hsu, Hadamard-type and Bullen-type inequalities for Lipschitzian functions and their applications, Computers and Mathematics with Applications 64 (2012) 651-660.

[9] T. Weir, and B. Mond, Preinvex Functions in multiple Objective Optimization, J. Math. Anal. Appl., 136 (1998) 29-38.

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Spectrum of non-linear schrodinger by sinc self-consistent method

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Abstract

The Sinc self-consistent method is presented to solve non-linear Schrodinger equation and obtain the electronic spectrum of non-linear Schrodinger equations. The electronic spectrum obtained through this and finite difference self-consistent methods are compared. We show that the Sinc self-consistent method is more accurate than finite difference self-consistent.

Keywords: Self-Consistent field iteration, Non-liner eigenvalue problem, Sinc method, Finite Difference method
Mathematics Subject Classification [2023]: 65ZXX, 65Lxx, 35PX

1 Introduction

Sinc numerical methods have been extensively used to solve differential equations because of their exponential convergence rate[1, 2, 8]. The Sinc method, which introduced and developed by F. Stenger, is based on the Whittaker-Shannon-Kotelnikov sampling theorem for entire functions. The aim of this paper is to extend the Sinc-Self Consistent field (SSCF) techniques to calculate the electronic spectrum of non-linear Schrodinger equation via various potential on interval (a, b), within homogeneous boundary conditions $Y(a)$=$Y(b)$=0 as

$$cY''(x) + U(x)Y'(x) + V(x)Y(x) + Q(x)Y^3(x) = EY(x), \quad \int_a^b Y^2(x)dx = 1 \tag{43}$$

where unknown value E and function $Y(x)$ are energy levels and corresponding wave functions, respectively. Also $U(x)$, $V(x)$ and $Q(x)$ are known functions. We call $V(x)$ as potential profile. Non-linear Schrodinger equation is known as a fundamental equation in the mathematical physics, because of their applications in lasers, solar cells, etc. Also, applications of the electronic spectrum appears in absorption coefficient, optical gain, optical rectification, second and third harmonic generations, etc. Yang et al. in [5], investigate the convergence of a numerical method for solving the following type of non-linear eigenvalue problem:

$$H(X)X = X\Lambda \tag{44}$$

where $X \in R^{n\times 1}$, $X^TX = I$, $H(X) \in R^{n\times n}$ is a matrix that has a special structure to be defined below, and $\Lambda \in R$ is a diagonal matrix consisting of the smallest eigenvalues of $H(X)$. They analyse the convergence of SCF iteration in figure 7.

Some properties of the SCF and sinc methods required for our subsequent development are given in the next section. Numerical example is included to demonstrate the validity and applicability of the present technique and a comparison is made with Finite Difference Self-Consistent(FDSCF) method. The method is easy to implement and yields accurate results.

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SCF Iteration

Input: A discrete Laplacian $L \in \mathsf{R}^{n\times n}$; an initial guess $X^{(0)}$ for the eigenvector $X \in \mathsf{R}^{n\times k}$;

Output: $X \in \mathsf{R}^{n\times k}$ such that $X^*X = I_k$ and $H(X)X = X\Lambda_k$, where Λ_k contains the k smallest eigenvalues of $H(X)$.

1. For $i = 1, 2, \ldots$ until convergence
2. construct $H^{(i)} = H(X^{(i-1)})$ using (3);
3. compute $X^{(i)}$ such that $H^{(i)}X^{(i)} = X^{(i)}\Lambda^{(i)}$, and $\Lambda^{(i)}$ contains the k smallest eigenvalues of $H^{(i)}$;
4. end for

Figure 1: The SCF iteration

2 Preliminaries and Notations

In the analysis which we will latter present, the subsequence convergence of the SCF iteration is measured by the distance between two subspaces spanned by columns of $X, Y \in \mathbb{R}^{n\times k}$. We will use the standard distance measure defined in [5] i.e., if $X^TX = Y^TY = I_k$,

$$dist(X, Y) = \|XX^T - YY^T\|_2$$

Theorem 2.1. *Let $X^{(0)} \in \mathbb{R}^{n\times k}$ be the initial guess to the solution of the nonlinear eigenvalue problem 44 that satisfies ${X^{(0)}}^T X^{(0)} = I_k$. If columns of $X^{(i)} \in \mathbb{R}^{n\times k}$ contain eigenvectors associated with the smallest k eigenvalues of $H(X^{(i-1)})$, as we would obtain when applying the SCF iteration, and if the gap between the kth and the k+1st eigenvalues of $H(X^{(i)})$ is greater than or equal to $\delta > 0$ for all i, then*

$$\lim_{i\to\infty} dist^2(X^{i+2}, X^i) = 0.$$

In the next, a brief overview of Sinc functions is presented. Sinc function properties are thoroughly discussed in [8]. The Sinc function is defined on the whole real line as:

$$Sinc(x) = \begin{cases} \frac{sin(\pi x)}{\pi x} & x \neq 0; \\ 1 & x = 0 \end{cases} \tag{45}$$

For $h > 0$, the translated Sinc functions with evenly spaced nodes are given as

$$S(k, h)(x) = Sinc\left(\frac{x - kh}{h}\right) \quad k = 0, \pm1, \pm2, ... \tag{46}$$

The basis functions on (a, b) are then taken to be the composite translated Sinc functions as

$$S_k(x) = S(k, h)o\phi(x) = Sinc\left(\frac{\phi(x) - kh}{h}\right) \quad k = 0, \pm1, \pm2, ... \tag{47}$$

Where $\phi : (a, b) \longrightarrow (-\infty, \infty)$ is a conformal map. We also require the derivatives of the composite translated Sinc function evaluated at the nodes x_i. The following results will be useful to obtain the discrete the system[8]:

$$\delta_{ji}^{(0)} = S_j(x_i) = \begin{cases} 0 & i \neq j, \\ 1 & i = j. \end{cases} \tag{48}$$

$$\delta_{ji}^{(1)} = hS_j'(x_i) = \begin{cases} \frac{(-1)^{(i-j)}}{(i-j)h} & i \neq j, \\ 0 & i = j. \end{cases} \tag{49}$$

$$\delta_{ji}^{(2)} = h^2 S_j''(x_i) = \begin{cases} \frac{-2(-1)^{(i-j)}}{(i-j)^2 h^2} & i \neq j, \\ \\ \frac{-\pi^2}{3h^2} & i = j. \end{cases} \tag{50}$$

Also, for positive integer N and h we have[8]:

$$Y'(x_j) \approx \sum_{k=-N}^{N} Y_k \Big(\phi' S_k'\Big)(x_j) = \sum_{k=-N}^{N} Y_k \phi'(x_j) \delta_{kj}^{(1)} \tag{51}$$

$$\begin{aligned} Y''(x_j) &\approx \sum_{k=-N}^{N} Y_k \left(\phi'^2 S_k'' + \phi'' S_k'\right)(x_j) \\ &= \sum_{k=-N}^{N} Y_k \left(\phi'^2 \delta_{kj}^{(2)} + \phi'' \delta_{kj}^{(1)}\right)(x_j). \end{aligned} \tag{52}$$

3 Main results

By using of the collocation method and relations 2.1 and 52 we have

$$\sum_{k=-N}^{N} Y_k \left(c\phi'^2 \delta_{kj}^{(2)} + \left(c\phi'' + U\phi'\right) \delta_{kj}^{(1)}\right)(x_j) + V(x_j) Y_j + Q(x_j) Y_j^3 = E Y_j. \tag{53}$$

Let

$$p_{kj} = \left(c\phi'^2 \delta_{kj}^{(2)} + \left(c\phi'' + U\phi'\right) \delta_{kj}^{(1)}\right)(x_j).$$

Then, we can write the system 53 as

$$(X + QZ)\, Y = EY \tag{54}$$

where

$$X = \begin{pmatrix} p_{-n,-n} + v_{-n} & \dots & p_{-n,n} \\ \vdots & \ddots & \vdots \\ p_{n,-n} & \dots & p_{n,n} + v_n \end{pmatrix} \quad , Q = \begin{pmatrix} q_{-n} & 0 & \dots & 0 \\ 0 & q_{-n+1} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & q_n \end{pmatrix} \quad , Y = \begin{pmatrix} y_{-n} \\ \vdots \\ y_n \end{pmatrix}$$

and $Z = (DiagY)^2$. Let $Z^0 = 0$. We use the following Algorithm to solve nonlinear eigenvalue problem 54.

1. For $i = 1, 2, ...$ until convergence
2. Construct $H^{(i)} = X + QZ^{(i-1)}$
3. Compute $F^{(i)}$ such that $H^{(i)} F^{(i)} = E^{(i)} F^{(i)}$, and $E^{(i)}$ contains the smallest eigenvalues of $H^{(i)}$;
4. Construct $Y^{(i)}$ such that $Y^{(i)} = \frac{F^{(i)}}{\|F^{(i)}\|_2}$
5. Construct $Z^{(i)} = (DiagY^{(i)})^2$
6. End for

Example 3.1. Consider Eq. 418 on $(-1, 1)$ with $c = -1$, $U(x) = V(x) = 0$. In this example, we set $N = 20$ and $h = \sqrt{\frac{\pi}{N}}$. Table 9 represents the electronic spectrum obtained from Sinc SCF(SSCF), FDSCF methods and the results of reference [4] for various of parameter Q. We denote the absolute errors of SSCF and FDSCF methods in electronic spectrum with ER(SSCF)= E(Ref [4])-E(SSCF) and ER(FDSCF)= E(Ref[4])-E(FDSCF), respectively.

4 Conclusion

In this paper, the electronic spectrum obtained through Sinc Self-Consistent and Finite Difference Self-Consistent methods are compared with each other and also with reference [4]. From the errors in the table 9, Sinc Self-Consistent method for the smaller values of n is more closer to the exact values.

Table 1: Comparison of smallest electronic spectrum of example 3.1 obtained through SSCF and FDSCF.

Q	SSCF	FDSCF	Ref [4]	ER_{SSCF}	ER_{FDSCF}
-0.5	2.092289130	2.084844743	2.0892647	0.003024430	0.004419957
-1	1.709552049	1.702127873	1.7073972	0.002154849	0.005269327
-1.5	1.321319116	1.313792869	1.3192815	0.002037616	0.005488631
-2	0.9272271857	0.919442665	0.92515883	0.0020683557	0.005716165

References

[1] S.M.A. Aleomraninjad, P. Abbasi, *Sinc-Integral method to solve the linear Schrodinger equation* Mathematics and Computational Sciences 1(1) (2020), 43-50.

[2] S.M.A. Aleomraninejada, M. Solaimani, *Electronic spectrum of linear Schrodinger equations with some potentials by Sinc-Galerkin and Sinc-Collocation methods*, 16(3) (2022), 251-260.

[3] J. Lund, K. Bowers, *Sinc Methods for Quadrature and Differential Equations*, SIAM, Philadelphia, (1992).

[4] V.A. Lykah, E.S. Syrkin, *Soft polar molecular layers adsorbed on charged nanowire*, Condensed Matter Physics 7 (2004) 805-812.

[5] C. Yang, W. Gao, J. Meza, *On the convergence of the self-consistent field iteration for a class of non-linear eigenvalue problems*, SIAM J. Matrix Analysis Applications, 30 (2009) 17731788.

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Lie centralizers on nest algebrasy

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Abstract

We characterize Lie centralizers at zero products on nest algebras. Further more, the Lie centralizers will be described for nest algebras.

Keywords: centralizers, Lie centralizers, nest algebras
Mathematics Subject Classification [2023]: 17B40, 47B49, 47L35

1 Introduction

Let $\mathcal{A}$ be an algebra. A linear mapping $\Psi : \mathcal{A} \to \mathcal{A}$ is called a *centralizer* if $\Psi(ab) = \Psi(a)b = a\Psi(b)$ for all $a, b \in \mathcal{A}$. The centralizers on algebras are an important class of mappings which are used in the study of algebra's structure, and have been extensively studied (see [2, 7] and references there in). *Lie product* on $\mathcal{A}$ is defined as $[a, b] = ab - ba$ for any $a, b \in \mathcal{A}$. We say that the linear mapping $\Phi : \mathcal{A} \to \mathcal{A}$ is a *Lie centralizer* if $\Phi([a, b]) = [\Phi(a), b]$ for any $, b \in \mathcal{A}$. It is easily checked that Φ is a Lie centralizer on $\mathcal{A}$ if and only if $\Phi([a, b]) = [a, \Phi(b)]$ for any $a, b \in \mathcal{A}$. Obviously every centralizer is a Lie centralizer but the converse is not necessarily true. Characterizing Lie centralizers in terms of centralizers can be of special interest. Recently in [6] non-additive Lie centralizers on triangular rings (which are not necessarily unital) have been studied and their structures are characterized. In this paper the Lie centralizers on nonunital triangular algebras will be characterized from another point of view.

The study of maps by local properties represents one of the most active research areas in mathematics. One of the interesting issues in mathematics is the characterization of the structure of linear (additive) mappings on algebras (rings) that act through zero products in the same way as certain mappings such as homomorphisms, centralizers, derivations and Lie derivations; for instance, see [1, 2, 3, 5, 8] and the references there in. Using this idea we describe the linear mappings which behaving like Lie centralizers at zero products as follows. Suppose that Φ is a linear mapping on $\mathcal{A}$.

$$a, b \in \mathcal{A}, ab = 0 \Longrightarrow \Phi([a, b]) = [\Phi(a), b] = [a, \Phi(b)] \tag{55}$$

It is clear that every Lie centralizer satisfies in (333), but the converse is not necessarily true. Now the question is that what is the structure of the linear mappings satisfied (333), in terms of Lie centralizers. In this paper we characterize the structure of linear mappings on nest algebras which are satisfied in (333).

Suppose that $\mathcal{H}$ is a Hilbert space on the field of real or complex numbers $\mathbb{F}$. A *nest* on $\mathcal{H}$ is a chain $\mathcal{N}$ of closed subspaces of $\mathcal{H}$ which is closed under intersection and closed linear span and it contains 0 and $\mathcal{H}$. The *nest algebra* corresponding to $\mathcal{N}$ is defined as follows:

$$Alg\mathcal{N} = \{T \in B(\mathcal{H}) \,|\, T(N) \subseteq N \text{ for any } N \in \mathcal{N}\}.$$

$\mathcal{N}$ is said to be a *nontrivial nest* if $\mathcal{N} \neq \{0, \mathcal{H}\}$. If $\mathcal{N}$ is a nontrivial nest and $N \in \mathcal{N}(N \neq 0, N \neq \mathcal{H})$, then the nest algebra $Alg\mathcal{N}$ has a representation as the following triangular algebra.

$$Alg\mathcal{N} = \begin{pmatrix} P_N(Alg\mathcal{N})P_N & P_N(Alg\mathcal{N})(I - P_N) \\ 0 & (I - P_N)(Alg\mathcal{N})(I - P_N) \end{pmatrix}$$

where I is the identity operator and P_N is the projection on $\mathcal{N}$.

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2 Main results

Let $\mathcal{A}$ be an algebra. The centre of $\mathcal{A}$ will be denoted by $Z(\mathcal{A})$. The following theorem is the main result of this paper (see [4]).

Lemma 2.1. *Let $\mathcal{H}$ be a Hilbert space with $dim\mathcal{H} > 2$ and $\mathcal{N}$ be a nontrivial nest on $\mathcal{H}$. Suppose that N is a nontrivial member in $\mathcal{N}$. Then $Z(Alg\mathcal{N}) = \mathbb{F}I$. Also, suppose that $A \in P_N(Alg\mathcal{N})P_N$ and $B \in (I - P_N)(Alg\mathcal{N})(I - P_N)$. If $AT = TB$ for any $T \in P_N(Alg\mathcal{N})(I - P_N)$, then there exists $\lambda \in \mathbb{F}$ such that $A = \lambda P_N$ and $B = \lambda(I - P_N)$.*

The following theorem is the our main result.

Theorem 2.2. *Let $\mathcal{N}$ be a nontrivial nest on the Hilbert space $\mathcal{H}$ with $dim\mathcal{H} > 2$. Suppose that $\Phi : Alg\mathcal{N} \to Alg\mathcal{N}$ is a linear mapping. Then Φ satisfies*

$$A, B \in Alg\mathcal{N}, AB = 0 \Longrightarrow \Phi([A, B]) = [\Phi(A), B] = [A, \Phi(B)],$$

if and only if there exist $\lambda \in \mathbb{F}$ and a linear mapping Λ on $Alg\mathcal{N}$ such that $\Phi(A) = \Lambda(A) + \lambda A$, $\Lambda(A) \in \mathbb{F}I$ for any $A \in Alg\mathcal{N}$ and $\Lambda([A, B]) = 0$ for any $A, B \in Alg\mathcal{N}$ with $AB = 0$.

Sketch of proof: Let Φ satisfies

$$A, B \in Alg\mathcal{N}, AB = 0 \Longrightarrow \Phi([A, B]) = [\Phi(A), B] + [A, \Phi(B)].$$

We decompose Φ in the following form

$$\Phi\left(\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}\right) = \begin{pmatrix} \alpha_1(R) + \beta_1(S) + \tau_1(M) & \alpha_2(R) + \beta_2(S) + \tau_2(M) \\ 0 & \alpha_3(R) + \beta_3(S) + \tau_3(M) \end{pmatrix},$$

where $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \in \begin{pmatrix} P_N(Alg\mathcal{N})P_N & P_N(Alg\mathcal{N})(I - P_N) \\ 0 & (I - P_N)(Alg\mathcal{N})(I - P_N) \end{pmatrix}$.
By applying Φ to

$$\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} = 0;$$

$$\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} = 0$$

;

$$\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} = 0$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} = 0,$$

using Lemma 2.1 and technical calculations, we get the desired result.

The converse is clear. □

By using Theorem 2.2, we describe Lie centralizers on nest algebras.

Corollary 2.3. *Let $\mathcal{N}$ be a nontrivial nest on the Hilbert space $\mathcal{H}$ with $dim\mathcal{H} > 2$. Suppose that $\Phi : Alg\mathcal{N} \to Alg\mathcal{N}$ is a linear mapping. Then Φ is a Lie centralizer if and only if there exist $\lambda \in \mathbb{F}$ and a linear mapping Λ on $Alg\mathcal{N}$ such that $\Phi(A) = \Lambda(A) + \lambda A$, $\Lambda(A) \in \mathbb{F}I$ and $\Lambda([A, B]) = 0$ for any $A, B \in Alg\mathcal{N}$.*

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References

[1] A. Barari, B. Fadaee and H. Ghahramani, *Linear Maps on Standard Operator Algebras Characterized by Action on Zero Products*, Bull. Iran. Math. Soc. 45 (2019), 1573–1583.

[2] R. Behfar and H. Ghahramani, *Lie maps on triangular algebras without assuming unity*, Mediterr. J. Math. 18 (2021), https://doi.org/10.1007/s00009-021-01836-z.

[3] M. Brešar, *Characterizing homomorphisms, derivations and multipliers in rings with idempotents,*, Proc. Roy. Soc. Edinb. Sect. A. 137 (2007), 9–21.

[4] K.R. Davision, *Nest Algebras*, Pitman Research Notes in Mathematics Series, vol. 191, Longman Scientific and Technical, Burnt Mill Harlow, Essex, UK, 1988.

[5] B. Fadaee and H. Ghahramani, *Linear Maps on C^*-Algebras Behaving like (Anti-)derivations at Orthogonal Elements*, Bull. Malays. Math. Sci. Soc. 43 (2020), 2851–2859.

[6] A. Fošner and W. Jing, *Lie centralizers on triangular rings and nest algebras*, Adv. Oper. Theory, 4 (2019), 342–350.

[7] H. Ghahramani, *On centralizers of Banach algebras*, Bull. Malays. Math. Sci. Soc. 38 (2015), 155–164.

[8] P. Ji, W. Qi and X. Sun, *Characterizations of Lie derivations of factor von Neumann algebras*, Linear and Multilinear Algebra, 61 (2013), 417–428.

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Numerical solution of fractional pantograph equations using fractional b-splines

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Abstract

The fractional b-spline collocation method has been used to approximate the solution of fractional pantograph equations. The numerical results shows the applicability and good performance of the proposed method.

Keywords: Fractional b-spline, Fractional pantograph equations
Mathematics Subject Classification [2023]: 41A15, 34A08

1 Introduction

Many physical phenomena in engineering and physics can be modelled by fractional order differential equations. Fractional calculus is used in various areas such as polymer physics [1], viscoelastic materials [2], viscous damping [3], electromagnetism [4]. Consider the following fractional order pantograph differential equation

$$\mathcal{L}u(t) = a_0(t)u(t) + a_1(t)\mathcal{D}^{\beta_1}u(q_1t) + a_2(t)\mathcal{D}^{\beta_2}u(q_2t) = f(t), \quad t \in [0,1], \tag{56}$$

with the initial conditions $u(0) = 0$ where, $0 < q_i < 1, \quad 0 \leq \beta_i < 1, \quad i = 1, 2, \ldots, l$; u is an unknown function, $a_i,\ i = 0, 1, 2$, are the known functions defined in $[0, T]$ and $\mathcal{D}^{\beta}$ is the fractional derivative of order β.

The purpose of the present work is to use the fractional B-spline collocation method for the numerical solution of fractional pantograph differential equation defined in (411) and initial conditions. This idea is based on the assumption that the unknown function $u(t)$ in (411) can be approximated by a linear combination of the fractional B-spline basis functions.

2 Fractional B-spline

In order to define the fractional order b-splines we can use the following truncated power function

$$t_+^{\alpha} = \begin{cases} t^{\alpha} & t \geq 0 \\ 0 & t < 0, \end{cases} \tag{57}$$

where $\alpha > -1$ is a real constant. So, the fractional b-spline of order α can be defined as

$$B^{\alpha}(t) = \frac{\triangle^{\alpha+1} t_+^{\alpha}}{\Gamma(\alpha+1)}\ , \tag{58}$$

where

$$\triangle^{\alpha+1} f(t) = \sum_{k\geq 0} (-1)^k \binom{n}{k} f(t-k), \qquad \alpha > 0 \tag{59}$$

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is the extended difference operator and

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}, \quad k = 0, 1, \ldots, \quad \alpha > 0. \tag{60}$$

Based on the above definition, the fractional b-spline B_α belongs to L^1 for $\alpha > -1$ and belongs to L^2 for $\alpha > \frac{-1}{2}$.

The fractional b-spline function is not compact support but it decay proportionally to $|x|^{-\alpha-2}$ when x tends to zero. Also B_α is Hölder continues of order α and reproduce the polynomials upto degree $\lceil\alpha\rceil$.

Riemann-Liouville Derivative:
Suppose that $p > 0$ is a real number and $q = \lceil p\rceil$. The fractional Riemann-Louville derivative of order p of the function f is defined as follows:

$$\mathcal{D}_a^{R,p} f(t) = \frac{1}{\Gamma(q-p)} \frac{d^q}{dt^q} \int_a^t (t-\tau) f(\tau) d\tau \tag{61}$$

Caputo Derivative:
This definition was presented by Caputo in 1967. Suppose that $p > 0$ is a real number and $q = \lceil p\rceil$. The fractional Caputo derivative of order p of the function f is defined as follows:

$$\mathcal{D}_a^{C,p} f(t) = \frac{1}{\Gamma(q-p)} \int_a^t (t-\tau)^{q-p-1} f^{(q)}(\tau) d\tau \tag{62}$$

In order to obtain the fractional derivative of fractional b-spline basis functions we use the following relation

$$\mathcal{D}_+^\beta B_\alpha(t) = \frac{\triangle^{\alpha+1} t_+^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}, \quad t \geq 0, \quad 0 < \beta < 1, \tag{63}$$

which then results the following useful relation

$$D_t^\beta B^\alpha(t) = \Delta^\beta B^{\alpha-\beta}(t).$$

3 Implementation of the method

Let $s(x)$ be the fractional b-spline approximation for the solution of problem (411) and initial conditions. Since the space of fractional b-splines is a linear space, it can be written as

$$s(t) = \sum_{j=-m}^{n-1} c_j B_j^\alpha(t). \tag{64}$$

where $B_j^\alpha(t)$ are the fractional b-spline basis functions of degree α and $m = \lceil\alpha\rceil$. Substituting (64) into (411) we have

$$\begin{aligned} \mathcal{L}u(t) = a_0(t) \sum_{j=-m}^{n-1} c_j B_j^\alpha(t) + a_1(t) \sum_{j=-m}^{n-1} c_j \mathcal{D}^{\beta_1} B_j^\alpha(q_1 t) \\ + a_2(t) \sum_{j=-m}^{n-1} c_j \mathcal{D}^{\beta_2} B_j^\alpha(q_2 t) = f(t) \end{aligned} \tag{65}$$

where $a_i(t),\ i = 0, 1, 2$ are known functions and $c_j,\ j = -m, \ldots, n-1$ are unknown coefficients that should be determined and $\mathcal{D}$ represents the fractional Caputo derivative of orders β_1 or β_2.

Let $\triangle \equiv \{t_i\}_{i=0}^{n}$ be a uniform partition in $[a,b]$ with the step size which $h=\frac{b-a}{n}$ divides the interval into n subintervals $[t_{i-1},t_i],\ i=1,2,\ldots,n$. Discretizing (65) at $t=t_i, i=0,1,\ldots,n$, we have

$$\begin{aligned}\mathcal{L}u(t_i)=a_0(t_i)\sum_{j=-m}^{n-1}c_jB_j^{\alpha}(t_i)+a_1(t_i)\sum_{j=-m}^{n-1}c_j\mathcal{D}^{\beta_1}B_j^{\alpha}(q_1t_i)\\ +a_2(t_i)\sum_{j=-m}^{n-1}c_j\mathcal{D}^{\beta_2}B_j^{\alpha}(q_2t_i)=f(t_i),\quad i=0,1,\ldots,n.\end{aligned}\tag{66}$$

The above relation is a system with $n+1$ equations and $n+m$ unknowns, so in order to have a square system we need $m-1$ extra equations. To construct the extra equations we can use the midpoints of the partition near the endpoints of the partition. We define the new partition $\tilde{\triangle}$ on $[a,b]$ in the following form

$$\tilde{\triangle}\equiv\{\tau_i\}_{i=-m}^{n-1}\equiv\triangle\cup\{x_i+\frac{h}{2}\}_{i=0}^{[m/2]}\cup\{x_i-\frac{h}{2}\}_{i=n-[m/2]}^{n}$$

where $x_i+\frac{h}{2}$ are midpoints of the partition chosen near the left and right boundary. So the new partition has $n+m$ points. We rewrite equation (66) at the points τ_i as follows:

$$\begin{aligned}\mathcal{L}u(\tau_i)=a_0(\tau_i)\sum_{j=-m}^{n-1}c_jB_j^{\alpha}(\tau_i)+a_1(\tau_i)\sum_{j=-m}^{n-1}c_j\mathcal{D}^{\beta_1}B_j^{\alpha}(q_1\tau_i)\\ +a_2(\tau_i)\sum_{j=-m}^{n-1}c_j\mathcal{D}^{\beta_2}B_j^{\alpha}(q_2\tau_i)=f(\tau_i),\quad i=-m,\ldots,n-1.\end{aligned}\tag{67}$$

The above system can be written in the following matrix form

$$[A_1+A_2+A_3]C=F$$

where

$$\begin{aligned}&(A_1)_{ij}=a_0(\tau_i)B_j^{\alpha}(\tau_i),\\ &(A_2)_{ij}=a_1(\tau_i)\mathcal{D}^{\beta_1}B_j^{\alpha}(q_2\tau_i),\\ &(A_3)_{ij}=a_2(\tau_i)\mathcal{D}^{\beta_2}B_j^{\alpha}(q_2\tau_i),\quad i,j=-m,\ldots,n-1\\ &C=[c_{-m},c_{-m+1},\ldots,c_{n-1}],\\ &F=[f_{-m},f_{-m+1},\ldots,f_{n-1}].\end{aligned}\tag{68}$$

4 Numerical results

In this section, we apply our proposed method on to approximate the solution of fractional pantograph equations to demonstrate the applicability and accuracy of method.

Example 4.1. Consider the followingr fractional pantograph equation [5]

$$D^{\gamma}u(t)=-u(t)+t^2u(\frac{1}{2}t)+tu(\frac{1}{4}t)+g(t),\quad 0\le t\le 1,\quad .\tag{69}$$

with the initial condition $u(0)=0$, where

$$g(t)=\frac{1}{\Gamma(-\frac{1}{2})}\int_0^t(t-x)^{-\frac{3}{2}}x^2e^{-x}dx+t^2e^{-t}-\frac{t^5}{8}e^{\frac{-t}{2}}-\frac{t^4}{64}e^{\frac{-t}{4}}.\tag{70}$$

For $\gamma=\frac{1}{2}$ the exact solution is t^3e^{-t}. This problem has been solved whit $N=20$ and the results on interval $[0,1]$ are tabulated in table 2. We compare the results with the results in [5]. Also the problem has been solved for $N=10,\ 20,\ 40,80$ and the maximum absolute errors at $t=1$ and the orders of convergence are presented in table 3.

Table 2: The maximum absolute errors with $N = 20$ for example 2.3 for various values of α.

t	$\alpha = 3$	$\alpha = 3.5$	$\alpha = 3.75$	[5]
0.1	$3.14e-7$	$1.29e-6$	$5.82e-6$	$5.65e-9$
0.3	$1.74e-7$	$8.90e-7$	$4.46e-6$	$2.84e-7$
0.5	$2.29e-7$	$7.88e-7$	$4.13e-6$	$6.95e-7$
0.7	$2.48e-7$	$8.20e-7$	$4.35e-6$	$4.33e-7$
0.9	$2.67e-7$	$9.23e-7$	$4.92e-6$	$1.35e-7$

Table 3: The maximum absolute errors and the orders of convergence for example 2.3 with various values of α at $t = 1$

$\frac{1}{h}$	$\alpha = 3$	Order	$\alpha = 3.75$	Order	$\alpha = 4$	Order
10	$1.88e-6$	–	$5.65e-5$	–	$4.33e-5$	–
20	$2.80e-7$	2.8	$5.32e-6$	3.4	$4.36e-6$	3.31
40	$3.27e-8$	3.0	$5.01e-7$	3.4	$4.15e-7$	3.39
80	$3.31e-9$	3.3	$4.58e-8$	3.4	$3.81e-8$	3.45

References

[1] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific, 2000.

[2] R. L. Bagley and P. J. Torvik, *Fractional calculus in the transient analysis of viscoelastically damped structures*, Journal of Computational Physics, 23(6) (1985) 918-925.

[3] S. Aleksandar and P. L. Mihailo, *Electroviscoelasticity of liquid/liquid interfaces: Fractional-order model. journal of colloid and interface science*, Journal of colloid and interface science, 282(1) (2005) 223-230.

[4] N. Engheta, *On fractional calculus and fractional multipoles in electromagnetism*, IEEE Transactions on Antennas and Propagation, 44(4) (1996) 554-566.

[5] Y. Jalilian, M. Ghasemi and J. J. Trujillo, *Existence and numerical simulation of solutions for nonlinear fractional pantograph equations*, International Journal of Computer Mathematics, 94(10) (2017), 2041-2062.

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On computing of transforming matrix for one type of sparse matrices using chebyshev polynomials second kind

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Abstract

Sparse matrices appear frequently in mathematical models. For example, Solving some difference, differential equations and delay differential equations we meet the necessity to compute the arbitrary positive integer of these matrices. We can obtain positive integer powers of the matrices applying the expression $A^m = TJ^mT^{-1}$ where J is the Jordans form of A and T is transforming matrix. In this paper we present the method of computing transforming matrix T for one type of Sparse matrices using Chebyshev polynomials second sind.

Keywords: Chebyshev polynomials, Eigenvalues, Eigenvectors, Jordan's form, Sparse matrices.
Mathematics Subject Classification [2023]: 15B99, 15A18, 65F50

1 Introduction

In order to solve some difference equations, differential equations and delay differential equation we meet the necessity to compute the arbitrary positive integer powers of square matrix [1, 2, 4]. In this paper, we obtain transforming matrix for one type of sparse matrices of order $n = 4p + 3$ ($p \in \mathbb{N}$) of the form

$$A = \begin{bmatrix} 0 & 0 & & & & & & & & & 2 \\ 0 & 0 & -1 & & & & & & & & \\ & -1 & 0 & -1 & & & & & & & \\ & & -1 & 0 & 1 & & & 0 & & & \\ & & & 1 & 0 & 1 & & & & & \\ & & & & 1 & 0 & -1 & & & & \\ & & & & & -1 & 0 & -1 & & & \\ & & & & & & -1 & 0 & & & \\ & & & 0 & & & & & \ddots & & \\ & & & & & & & & & 0 & 0 \\ 2 & & & & & & & & & 0 & 0 \end{bmatrix}. \tag{71}$$

Although these types of matrices may have applications, we investigate these matrices purely from a mathematical point of view. In this article, Interesting formula are presented that will be attractive for math enthusiasts. Now, we are beginning with following definition.

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Definition 1.1. [3] The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial of degree n in x defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \text{when } x = \cos\theta, n = 0, 1, 2, \ldots. \tag{72}$$

With definition above we find that $U_n(x)$ satisfies the recurrence relation

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), U_0(x) = 1, U_1(x) = 2x, n = 2, 3, \ldots \tag{73}$$

The first few Chebyshev polynomialse of the second kind are

$$U_0(x) = 1, U_1(x) = 2x, U_2(x) = 4x^2 - 1, U_3(x) = 8x^3 - 4x, \cdots. \tag{74}$$

The following theorem is useful for the main results of this paper. For the proof we refer the reader to see [2].

Theorem 1.2. *The eigenvalues of A, as given in (71), are as follows*

$$\lambda_k = -2\cos\frac{(k-1)\pi}{n-1}, k = 1, 2, \ldots, n. \tag{75}$$

Since all the eigenvalues λ_k for $k = 1, 2, \ldots, n$ are simple, for each eigenvalue λ_k corresponds single Jordan cell $J_1(\lambda_k)$ in the matrix J. Taking this into account we write down the Jordan's form of the matrix A.

$$J = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n) \tag{76}$$

and applying the relation $\lambda_k = -\lambda_{n-k+1}, k = 1, 2, \cdots, \frac{n-1}{2}$, we can write

$$J = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \cdots, 0, \cdots, -\lambda_3, -\lambda_2, -\lambda_1). \tag{77}$$

2 Main results

In this section we find the transforming matrix T. From relation $A = TJT^{-1}$ follows

$$AT = TJ \tag{78}$$

where, $T = [T_1, T_2, \cdots, T_n]$ and $T_i = [T_{1i}, T_{2i}, \cdots, T_{ni}]^t$, for, $i = 1, 2, \cdots, n$ then we can write

$$A[T_1, T_2, T_3, \cdots, T_n] = [T_1, T_2, T_3, \cdots, T_n]diag(\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n). \tag{79}$$

Now, we have

$$AT_i = \lambda_i T_i, i = 1, 2, 3, \cdots, n. \tag{80}$$

From (80) follows

$$\begin{bmatrix} 2T_{ni} \\ -T_{3i} \\ -T_{2i} - T_{4i} \\ -T_{3i} + T_{5i} \\ T_{4i} + T_{6i} \\ T_{5i} - T_{7i} \\ -T_{6i} - T_{8i} \\ -T_{7i} + T_{9i} \\ T_{8i} + T_{10i} \\ T_{9i} - T_{11i} \\ -T_{10i} - T_{12i} \\ \vdots \\ -T_{n-2i} \\ 2T_{1i} \end{bmatrix} = \begin{bmatrix} \lambda_i T_{1i} \\ \lambda_i T_{2i} \\ \lambda_i T_{3i} \\ \lambda_i T_{4i} \\ \lambda_i T_{5i} \\ \lambda_i T_{6i} \\ \lambda_i T_{7i} \\ \lambda_i T_{8i} \\ \lambda_i T_{9i} \\ \lambda_i T_{10i} \\ \lambda_i T_{11i} \\ \vdots \\ \lambda_i T_{n-1i} \\ \lambda_i T_{ni} \end{bmatrix}, i = 1, 2, 3, \ldots, n. \tag{81}$$

Solving the set of systems (81), we find the eigenvectors of the matrix A

$$T_1 = \begin{bmatrix} \alpha \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ -\alpha \end{bmatrix}, T_n = \begin{bmatrix} \alpha \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ \alpha \end{bmatrix}, T_i = \begin{bmatrix} 0 \\ c_0\beta U_0(\frac{\lambda_i}{2}) \\ c_1\beta U_1(\frac{\lambda_i}{2}) \\ c_2\beta U_2(\frac{\lambda_i}{2}) \\ c_3\beta U_3(\frac{\lambda_i}{2}) \\ c_4\beta U_4(\frac{\lambda_i}{2}) \\ c_5\beta U_5(\frac{\lambda_i}{2}) \\ c_6\beta U_6(\frac{\lambda_i}{2}) \\ c_7\beta U_7(\frac{\lambda_i}{2}) \\ c_8\beta U_8(\frac{\lambda_i}{2}) \\ c_9\beta U_9(\frac{\lambda_i}{2}) \\ \vdots \\ 0 \end{bmatrix}, \ i = 2, 3, \ldots, n-1, \tag{82}$$

here $U_k(x)$ is the kth degree Chebyshov polynomial of the second kind, α and β are arbitrarily numbers and

$$c_i = \begin{cases} -1 & if, i \in \mathbb{X} = \{4p-3 | p = 1, 2, \cdots\} \\ 1 & if, i \in \mathbb{N} \cup \{0\} - X \end{cases} \tag{83}$$

Taking into account (82), (83) and choosing $\alpha = \beta = 1$ arbitrarily we write down the transforming matrix T:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 1 \\ 0 & U_0(\frac{\lambda_2}{2}) & U_0(\frac{\lambda_3}{2}) & U_0(\frac{\lambda_4}{2}) & \cdots & 0 \\ 0 & -U_1(\frac{\lambda_2}{2}) & -U_1(\frac{\lambda_3}{2}) & -U_1(\frac{\lambda_4}{2}) & \cdots & 0 \\ 0 & U_2(\frac{\lambda_2}{2}) & U_2(\frac{\lambda_3}{2}) & U_2(\frac{\lambda_4}{2}) & \cdots & 0 \\ 0 & U_3(\frac{\lambda_2}{2}) & U_3(\frac{\lambda_3}{2}) & U_3(\frac{\lambda_4}{2}) & \cdots & 0 \\ 0 & U_4(\frac{\lambda_2}{2}) & U_4(\frac{\lambda_3}{2}) & U_4(\frac{\lambda_4}{2}) & \cdots & 0 \\ 0 & -U_5(\frac{\lambda_2}{2}) & -U_5(\frac{\lambda_3}{2}) & -U_5(\frac{\lambda_4}{2}) & \cdots & 0 \\ 0 & U_6(\frac{\lambda_2}{2}) & U_6(\frac{\lambda_3}{2}) & U_6(\frac{\lambda_4}{2}) & \cdots & 0 \\ 0 & U_7(\frac{\lambda_2}{2}) & U_7(\frac{\lambda_3}{2}) & U_7(\frac{\lambda_4}{2}) & \cdots & 0 \\ 0 & U_8(\frac{\lambda_2}{2}) & U_8(\frac{\lambda_3}{2}) & U_8(\frac{\lambda_4}{2}) & \cdots & 0 \\ 0 & -U_9(\frac{\lambda_2}{2}) & -U_9(\frac{\lambda_3}{2}) & -U_9(\frac{\lambda_4}{2}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{84}$$

3 Numerical consideration

For example if $n = 7$, from (75) eigenvalues of matrix A can be written for $k = 1, 2, \cdots, 7$ as:
$\lambda_k = -2\cos\frac{(k-1)\pi}{n-1}$, namely, $\lambda_1 = -2$, $\lambda_2 = -\sqrt{3}$, $\lambda_3 = -1$, $\lambda_4 = 0$, $\lambda_5 = 1$, $\lambda_6 = \sqrt{3}$ and $\lambda_7 = 2$. From (84) we can write the transforming matrix T as:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & U_0(\frac{\lambda_2}{2}) & U_0(\frac{\lambda_3}{2}) & U_0(\frac{\lambda_4}{2}) & U_0(\frac{\lambda_5}{2}) & U_0(\frac{\lambda_6}{2}) & 0 \\ 0 & -U_1(\frac{\lambda_2}{2}) & -U_1(\frac{\lambda_3}{2}) & -U_1(\frac{\lambda_4}{2}) & -U_1(\frac{\lambda_5}{2}) & -U_1(\frac{\lambda_6}{2}) & 0 \\ 0 & U_2(\frac{\lambda_2}{2}) & U_2(\frac{\lambda_3}{2}) & U_2(\frac{\lambda_4}{2}) & U_2(\frac{\lambda_5}{2}) & U_2(\frac{\lambda_6}{2}) & 0 \\ 0 & U_3(\frac{\lambda_2}{2}) & U_3(\frac{\lambda_3}{2}) & U_3(\frac{\lambda_4}{2}) & U_3(\frac{\lambda_5}{2}) & U_3(\frac{\lambda_6}{2}) & 0 \\ 0 & U_4(\frac{\lambda_2}{2}) & U_4(\frac{\lambda_3}{2}) & U_4(\frac{\lambda_4}{2}) & U_4(\frac{\lambda_5}{2}) & U_4(\frac{\lambda_6}{2}) & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & \sqrt{3} & 1 & 0 & -1 & -\sqrt{3} & 0 \\ 0 & 2 & 0 & -1 & 0 & 2 & 0 \\ 0 & -\sqrt{3} & 1 & 0 & -1 & \sqrt{3} & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

4 Conclusion

With the help of Chebyshev polynomials of the first and second kind and other polynomials such as Legender, the closed form of the Transforming matrix for other matrices can be obtained.

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References

[1] R.P. Agarwal,*Difference equations and inequalities*, Marcel-Dekker, New York. 1992.

[2] J. Rimas, *Investigation of dynamics of mutually synchronized system*, Telecommunications and Radio Engineering, Part II 32 (2) (1977) 68- 79.

[3] J. Rimas, *Analysis of multidimensional delay system with chain form structure*, In Proceedings of the Seventeenth International Symposium on Mathematical theory of Networks and Systems, Kyoto, Japan, 2006.

[4] J.C. Mason, D.C. Handscomb, Chebyshev Polynomials, CHAPMAN & HALL/CRC, 2003.

[5] M. Beiranvand and M. Ghasemi Kamalvand, Explicit Expression forArbitrary Positive Powers of Special Tridiagonal Matrices, Journal of Applied Mathematics, Volume 2020, Article ID 7290403, 5 pages,https://doi.org/10.1155/2020/7290403.

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Elegant approach for pricing option with tow barriers under time-varying coefficient

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Abstract

In the present paper, an elegant approach is derived for pricing options with two barriers under *Black-Scholes* model with time-varying coefficients. Here, barriers are only checked at specific times, namely *monitoring dates*. By applying suitable transformations in each monitoring interval, the problem is changed to solve the standard Black-Scholes model with constant coefficients which has an analytical solution. Therefore, the option price would be obtained from applying the analytical solution in recursive solving the *Black-Scholes* model in monitoring intervals. Finally, a numerical method is devised to compute the recursive expression in the resulting procedure effectively.

Keywords: Options, Pricing Options with Barriers, Black-Scholes model, Time-varying coefficient
Mathematics Subject Classification [2023]: 65D15, 35E99

1 Introduction

Path-dependent options, *especially European options with two barriers*, are recently traded in the financial markets. Therefore pricing options are considered in financial mathematics by many authors. In this paper, pricing a knock-out European option with two barriers is investigated which would be useless if the price of underlying stock touches one of the two barriers before the expiry time T. Here, the motion of underlying stock is checked only at specific times, namely *monitoring dates* $0 = t_0 < t_1 < \cdots < t_N = T$. Fusai [1] derived an analytical approach for pricing options with only one barrier. Milev and Tagliani [5] devised a numerical method that is similar to the quadrature method. Golbabai et al. [3] developed a finite element method that is more effective than the method of finite difference. In the above-mentioned literature, coefficients including *risk-free rate and volatility* are supposed to be constant. According to some economical reasons, coefficients should be considered time-varied. Lo et al. [4] proposed a simple numerical method that works only for continuous monitoring. In the present paper, by applying some transformations the time-varying Black-Scholes model converts to time-independent one's which could be directly changed to the heat equation. Finally, a simple numerical method is devised for evaluating the recursive integrals yielded from using the analytical solution to the heat equations. This article is arranged as follows. In section 2, the time-dependent Black-Sholes model and the procedure for reducing to the time-independent case are introduced; in section 3 a numerical example is given. Finally, notes and suggestions are expressed in section 4.

2 Main results

In this article, the pricing of the European call option with two barriers on a stock is investigated. In this manner, suppose $\mathcal{P}(x,t,n)$ stands for the price of the call option such that the variable x denotes the stock price in time $t \in [t_n\ t_{n+1}]$. Under the *Black-Scholes* model, the option price $\mathcal{P}(x,t,n)$ verifies the well-known

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Black-Scholes model [1]:

$$-\frac{\partial \mathcal{P}(x,t,n)}{\partial t}+\mu(t)x\frac{\partial \mathcal{P}(x,t,n)}{\partial x}+\tfrac{1}{2}\sigma^2(t)x^2\frac{\partial^2 \mathcal{P}(x,t,n)}{\partial^2 x}-\mu(t)\mathcal{P}(x,t,n)=0. \tag{85}$$

where the functions $\mu(t)$ and $\sigma^2(t)$ are the *risk free rate and the volatility* respectively. The following initial conditions at the monitoring dates t_i are essentially assumed to price options with two barriers:

$$\begin{cases}\mathcal{P}(x,t_0,0)=(x-K)\mathbf{1}_{(U\geq x\geq max(K,L))}; & n=0,\\ \mathcal{P}(x,t_n,n)=\mathcal{P}(x,t_n,n-1)\mathbf{1}_{(U\geq x\geq L)}; & n=1,2,...,N-1.\end{cases} \tag{86}$$

where $\mathcal{P}(x,t_n,n-1)$ is defined as $\mathcal{P}(x,t_n,n-1):=lim_{t\to t_n^-}\mathcal{P}(x,t,n-1)$. Also L, U and K are *lower barrier, upper barrier* and *exercise price* respectively.
Consider the following transformations for option price, stock price and time respectively:

$$\mathcal{P}(x,t,n)=\eta_n(t)\bar{\mathcal{P}}(\bar{x},\bar{t},n)\ ,\ \bar{x}=\phi_n(t)x\ ,\ \bar{t}=\psi_n(t) \tag{87}$$

where the functions $\eta_n(t),\phi_n(t)$ and $\psi_n(t)$ and are defined as follows:

$$\eta_n(t)=exp(\int_{t_n}^{t}(\mu_n\psi_n^{'}(u)-\mu(u))du), \tag{88}$$

$$\phi_n(t)=exp(\int_{t_n}^{t}(\mu(u)-\mu_n\psi_n^{'}(u))du)\ \ , \qquad \psi_n(t)=\frac{1}{\sigma_n^2}\int_{t_n}^{t}\sigma^2(u)du)+t_n \tag{89}$$

with the constants μ_n and σ_n^2 are defined as the *mean value* of the fucnctions $\mu(t)$ and $\sigma^2(t)$ over the monitoring intervals $[t_n,t_{n+1}]$.

After rewriting (85) and the corresponding initial conditions (86) with respect to $\frac{\partial\bar{\mathcal{P}}}{\partial\bar{t}}$, $\frac{\partial\bar{\mathcal{P}}}{\partial\bar{x}}$, $\frac{\partial^2\bar{\mathcal{P}}}{\partial\bar{x}^2}$, the following time-independent Black-Schoels PDE is obtained:

$$-\frac{\partial\bar{\mathcal{P}}}{\partial\bar{t}}+\mu_n\bar{x}\frac{\partial\bar{\mathcal{P}}}{\partial\bar{x}}+\frac{1}{2}\sigma_n^2\bar{x}^2\frac{\partial^2\bar{\mathcal{P}}}{\partial\bar{x}^2}-\mu_n\bar{\mathcal{P}}=0\quad ,\quad (n=0,1,2,...,N-1) \tag{90}$$

$$\begin{cases}\bar{\mathcal{P}}(\bar{x},\bar{t}_0,0)=(\bar{x}-K)\mathbf{1}_{(U\geq\bar{x}\geq max(K,L))}; & n=0,\\ \bar{\mathcal{P}}(\bar{x},\bar{t}_n,n)=\mathcal{P}(\bar{x},\bar{t}_n,n-1)\mathbf{1}_{(U\geq\bar{x}\geq L)}; & n=1,2,...,N-1.\end{cases} \tag{91}$$

where $\bar{\mathcal{P}}(\bar{x},\bar{t}_n,n-1)$ is defined as $\bar{\mathcal{P}}(\bar{x},\bar{t}_n,n-1):=lim_{\bar{t}\to\bar{t}_n^-}\bar{\mathcal{P}}(\bar{x},\bar{t},n-1)$. Now, suppose the following transformation:

$$\bar{\mathcal{P}}(\bar{x},\bar{t},n)=e^{\upsilon_n z+\omega_n\bar{t}}\mathcal{G}(z,\bar{t},n)\quad ,\quad z=ln(\frac{\bar{x}}{L})\quad ,\quad k=ln(\frac{K}{L})\quad ,\quad \theta=ln(\frac{U}{L}). \tag{92}$$

where the coefficients υ_n , ω_n are defined as below:

$$\upsilon_n=-\frac{\mu_n-\frac{\sigma_n^2}{2}}{\sigma_n^2}\quad ,\quad \omega_n=\upsilon_n(\mu_n-\frac{\sigma_n^2}{2})+\upsilon_n^2\frac{\sigma_n^2}{2}-\mu_n. \tag{93}$$

After rewriting (90) and (91) based on $\mathcal{G}(z,\bar{t},n)$, the following heat equations with their corresponding initial conditions result:

$$-\frac{\partial\mathcal{G}}{\partial\bar{t}}+c_n^2\frac{\partial^2\mathcal{G}}{\partial z^2}=0\ ,\ c_n^2=\frac{\sigma_n^2}{2}\quad ,\quad n=0,1,2,..,N-1. \tag{94}$$

$$\begin{cases}\mathcal{G}(z,\bar{t}_0,0)=Le^{-\upsilon_0 z}(e^z-e^k)\mathbf{1}_{(\theta\geq z\geq\delta)} & ,\quad \delta=max\{k,0\}\\ \mathcal{G}(z,\bar{t}_n,n)=\mathcal{G}(z,\bar{t}_n,n-1)exp\{(\upsilon_{n-1}-\upsilon_n)z+(\omega_{n-1}-\omega_n)\bar{t}_n\}\mathbf{1}_{(\theta\geq z\geq 0)}.\end{cases} \tag{95}$$

Using the unique analytical solution to the heat equation in[6], the following expressions for $\mathcal{G}(Z,\bar{t},n)$ is yielded:

$$\mathcal{G}(z,\bar{t},n)=\begin{cases} L\int_0^\infty s_n(z-\xi,\bar{t}-\bar{t}_n)e^{-\upsilon\xi}(e^{\xi}-e^{k})\mathbf{1}_{(\theta\geq\xi\geq\delta)}d\xi\ ,\ n=0,\\ \int_0^\infty s_n(s-\xi,\bar{t}-\bar{t}_n)\mathcal{G}(\xi,\bar{t}_n,n-1)e^{\{-\Delta\upsilon_n\xi-\Delta\omega_n\bar{t}_n\}}\mathbf{1}_{(\theta\geq\xi\geq 0)}d\xi\ ,\ 1\leq n\leq N-1. \end{cases} \tag{96}$$

where: $\Delta\upsilon_n=\upsilon_n-\upsilon_{n-1}$, $\Delta\omega_n=\omega_n-\omega_{n-1}$, and each kernel $s_n(z,\bar{t})$ is the density of Gaussian distribution $N(0,2c_n^2\bar{t})$ as follows:

$$s_n(z,\bar{t})=\frac{1}{\sqrt{4\pi c_n^2\bar{t}}}exp(\frac{-z^2}{4c_n^2\bar{t}})\quad,\quad n=0,1,2,...,N-1$$

Therefore the price of the call option with two barriers is given in the following theorem:

Theorem 2.1. *The price of call option with two barriers L and U at stock price x, exercise price K and at monitoring dates t_{n+1} are evaluated as follow:*

$$\mathcal{P}(x,t_{n+1},n)=\mathcal{G}(ln(\frac{x}{L}),t_{n+1},n)exp\{\upsilon_n ln(\frac{x}{L})+\omega_n t_{n+1}\};\quad,\quad n=0,1,2,...,N-1 \tag{97}$$

where $\mathcal{G}(.,t_{n+1},n)$ is evaluated recursively in (96).

3 Numerical results

In this section, the procedure to compute $\mathcal{G}(z_0,\bar{t}_{N+1},N)$ in (96) is introduced. According to Simpson and Romberg method, computing $\mathcal{G}(z_0,\bar{t}_{N+1},N)$ is done through computing $\mathcal{G}(z_{i_1},\bar{t}_N,N-1)$ in nodes $z_{i_1}=ih(0\leq i\leq 2^n)$ where $h=\frac{\theta}{2^n}$. For computing $\mathcal{G}(z_{i_1},\bar{t}_N,N-1)$, it just enough to compute $\mathcal{G}(z_{i_2},\bar{t}_{N-1},N-2)$ in nodes $z_{i_2}=ih$. Similarly, computing $\mathcal{G}(z_{i_{N-1}},\bar{t}_2,1)$ is done through computing $\mathcal{G}(z_{i_N},\bar{t}_1,0)$ in nodes $z_{i_N}=ih$. The corresponding algorithm is demonstrated below:

Algorithm: pricing option with barrier where there are N monitoring dates.

Input:► $m\in\mathbb{N}$ positive integer, $N\in\mathbb{N}$ number of steps.
Output:◄ $X\in\mathbb{R}^+$, option price.

1 $step\leftarrow 1$
2 $numnode\ \leftarrow\ 2^m$, 3 $h\ \leftarrow\ \theta/numnode$
4 **for** $i=0:numnode$ **do** , 5 $\xi_i\leftarrow\ i.h$, 6 **end**
7 **for** $i=0:numnode$ **do**
8 *Compute* $\mathcal{G}(\xi_i,\bar{t},0)$ *by gaussian quadrature integration*
9 **end**
10 **for** $step=2:N-1$ **do** , 11 **for** $i=0:numnode$ **do**
12 *Compute* $\mathcal{G}(\xi_i,\bar{t}_{step},step-1)$ *by Simpson and Romberg method by* $\mathcal{G}(\xi_j,\bar{t}_{step-1},step-2)\quad 0\leq j\leq numnode$
13 **end** , 14 **end**
15 $\mathbf{X}\ \leftarrow\ \mathcal{G}(z_0,\bar{t}_N,N-1)$ *by Simpson and Romberg method by* $\mathcal{G}(\xi_j,\bar{t}_{N-1},step-2)\quad 0\leq j\leq numnode$

Example: Suppose the values including stock price, exercise price and maturity time as $x=100$, $K=100$ and $T=0.4$ respectively. Also, the time-varying risk free rate and volatility are assumed to be $\mu(t)=0.075+0.05t$ and $\sigma^2(t)=0.03+0.02t$. Results are summarized in table1.

Table 4: Pricing option with two barriers of the example

U	L	Option price N=50	N=125
	80	3.2821222	3.180135
	90	3.1700556	3.048500
120	95	2.413026	2.221869
	99	0.8980312	0.675734
	99.9	0.554516	0.335474

4 Conclusion

In this article, we have studied the problem of pricing options with two barriers under time-varying Black-Scholes models. Checking the motion of the stock is done only at specific times. By the method of the present paper, the time-varying Black-Scholes model is transformed to constant one's which is directly transformed to the heat equations. Finally, a numerical method with a low computational cost is provided.

References

[1] T. Bjork, Arbitrage theory in continuous time, Oxford university press, (2009).

[2] G. Fusai, I. D. Abrahams, C. Sgarra, *An exact analytical solution for discrete barrier options.*, Finance Stochastic, 10 (1) (2006), 1-26.

[3] A. Golbabai, L. Ballestra, D. Ahmadian, *A highly accurate finite element method to price discrete double barrier options*, Comput. Econ., 44 (2) (2014), 153173.

[4] C. F. Lo, H. C. Lee, C. H. Hui, *A simple approach for pricing barrier options with time-dependent parameters*, Quantitative Finance, 3 (2003), 98107.

[5] M. Milev, A. Tagliani, *Numerical valuation of discrete double barrier options*, Comput. Appl. Math., 233 (10) (2010), 24682480.

[6] W. A. Strauss, *Partial differential equations: an introduction*, Wiley (1992).

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A new approach for evaluating the eigenvalues of a matrix

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Abstract

Eigenvalues and eigenvectors of the matrices have a decisive role in many problems. In this paper, we want to evaluate eigenvalues of a matrix using piecewise polynomials and interpolation concept. In fact, we do it by rootfinding the characteristic polynomial of the matrix by piecewise polynomials. This approach can find all the eigenvalues of the matrix on a determined interval. Some numerical tests are given to illustrate the effectiveness of our approach.

Keywords: Eigenvalue, Eigenvector, Matrix
Mathematics Subject Classification [2023]: 65F15, 65H17

1 Introduction

If T is a linear transformation from a vector space V over a field F into itself and v is a nonzero vector in V, then v is an eigenvector of T if $T(v)$ is a scalar multiple of v. This can be written as

$$T(v) = \lambda v$$

where λ is a scalar in F, known as the eigenvalue, characteristic value, or characteristic root associated with v.

There is a direct correspondence between n-by-n square matrices and linear transformations from an n-dimensional vector space into itself, given any basis of the vector space. Hence, in a finite-dimensional vector space, it is equivalent to define eigenvalues and eigenvectors using either the language of matrices.

If V is finite-dimensional, the above equation is equivalent to

$$Ax = \lambda x, \tag{98}$$

where A is the square matrix representation of T of dimension n and x is the nonzero coordinate vector of v. We know that Equation (444) has a nontrivial solution x if and only if the determinant of the matrix $(A - \lambda I)$ is zero. Therefore, the eigenvalues of A are values of λ that satisfy the equation

$$det(A - \lambda I) = 0. \tag{99}$$

Using the Leibniz formula for determinants, the left-hand side of Equation (445) is a polynomial function of the variable λ and the degree of this polynomial is n, the order of the matrix A. Its coefficients depend on the entries of A, except that its term of degree n is always $(-1)^n\lambda^n$. This polynomial is called the characteristic polynomial of A. Equation (445) is called the characteristic equation or the secular equation of A.

The fundamental theorem of algebra implies that the characteristic polynomial of an n-by-n matrix A, being a polynomial of degree n, can be factored into the product of n linear terms,

$$det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\dots(\lambda_n - \lambda). \tag{100}$$

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Table 5: Applying our approach for Example 2.1 with $K = 1$

	root	$\lvert p(root)\rvert$
1	0.000000000306009	2.44×10^{-11}
2	0.200000000014996	4.79×10^{-13}
3	0.399999999754998	1.17×10^{-10}
4	0.999999999843335	7.51×10^{-10}

Table 6: Applying our approach for Example 2.1 with $K = 2$

	root	$\lvert p(root)\rvert$
1	0.000000000000002	1.43×10^{-16}
2	0.199999999999914	2.73×10^{-15}
3	0.400000000000587	2.81×10^{-14}
4	1.000000000000000	5.55×10^{-17}

where each λ_i may be real but in general is a complex number. The numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ which may not all have distinct values, are roots of the polynomial and are the eigenvalues of A.

Eigenvalues and eigenvectors form the basics of computing and mathematics. They are heavily used by scientists. Up to now, several approaches for evaluating the eigenvalues of a matrix have been proposed, see for instance [1, 1, 2, 3, 4]. Here, we want to use a rootfinding technique to evaluate the roots of characteristic polynomial (445) and so evaluating the eigenvalues of A. We do this by applying some piecewise interpolation polynomials.

2 Our approach

As we said in the previous section, we want to obtain eigenvalues of a matrix by a rootfinding technique. For this purpose, for $f : [a, b] \to \mathbb{R}$, we introduce a partition T_h of $[a, b]$ into K subintervals $I_j = [x_j, x_{j+1}]$ of length h_j, with $h = \max_{0 \le j \le K-1} h_j$ and then employ Lagrange interpolation on each I_j using $k + 1$ equally spaced nodes $\{x_j^{(i)}, 0 \le i \le k\}$. Then, for any continuous function f in $[a, b]$, the piecewise interpolation polynomial $\Pi_h^k f$ coincides on each I_j with the interpolating polynomial of $f|_{I_j}$ at the $k + 1$ nodes $\{x_j^{(i)}, 0 \le i \le k\}$.

Our approach is based on the rootfinding of the piecewise interpolation polynomial $\Pi_h^k f$ on each subinterval I_j, $j = 0, 1, ..., K - 1$ and is based on performing the following steps:

Step 1: Divide the interval $[a, b]$ into K subintervals $I_j = [x_j, x_{j+1}]$.

Step 2: Form the piecewise interpolation polynomial $\Pi_h^k f$.

Step 3: Find the roots of the interpolation polynomials on each subinterval I_j.

Step 4: For each root, test for $f(root)$, if it is sufficiently small, stop. Otherwise, refine the meshsize or degree of the piecewise interpolation polynomial.

Example 2.1. The characteristic polynomial of a 4-by-4 matrix is

$$p(x) = x^4 - 1.6x^3 + 0.68x^2 - 0.08x.$$

The exact eigenvalues of this matrix are $\lambda_1 = 0$, $\lambda_2 = 0.2$, $\lambda_3 = 0.4$ and $\lambda_4 = 1$. Now, we want to obtain them using our approach for $K = 1$ and $K = 2$. The obtained results have been displayed in Tables 17 and 18, respectively. As one can see, our approach obtain all the eigenvalues with a nice accuracy. One of the advantages of our approach is that it can find all the roots of the characteristic polynomial simultaneously. By $K = 2$, the obtained results have more accuracy than one obtained by $K = 1$.

Example 2.2. The characteristic polynomial of a 8-by-8 matrix is

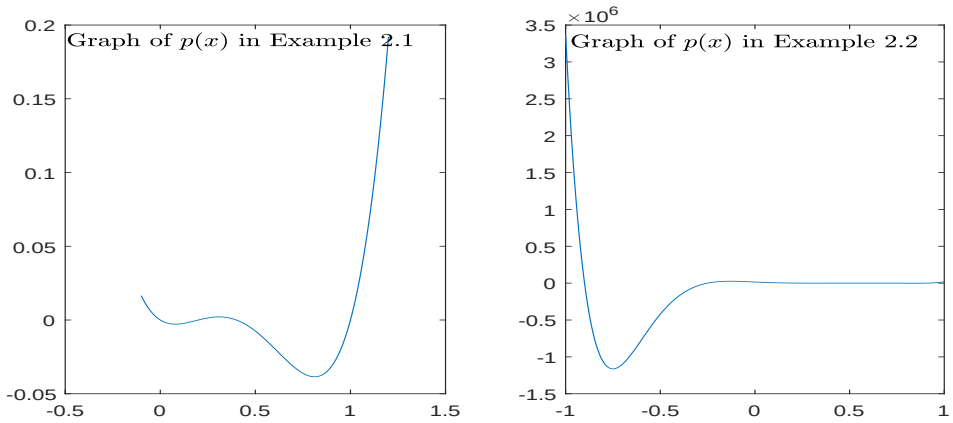


Figure 2: Graphs of the characteristic polynomials in Examples 2.1 and 2.2

Table 7: Applying our approach for Example 2.2 with $K = 1$

	root	$\lvert p(root)\rvert$
1	-0.900000000000000	3.05×10^{-10}
2	-0.250000000000000	0
3	0.250000000000000	0
4	0.449999999999943	3.63×10^{-11}
5	0.500000000000044	4.36×10^{-11}
6	0.600000000000125	8.73×10^{-11}
7	0.699999999999947	1.45×10^{-11}
8	0.899999999999985	8.44×10^{-10}

$$p(x) = 320000x^8 - 7200000x^7 + 3224000x^6 + 4069200x^5 - 4784560x^4 + 1566168x^3 + 40716x^2 - 112023x + 15309.$$

The exact eigenvalues of this matrix are $\lambda_1 = -0.9$, $\lambda_2 = -0.25$, $\lambda_3 = 0.25$, $\lambda_4 = 0.45$, $\lambda_5 = 0.5$, $\lambda_6 = 0.6$, $\lambda_7 = 0.7$ and $\lambda_8 = 0.9$. We evaluate them using our approach for $K = 1$ and $K = 2$. The obtained results have been displayed in Table 19 and 8, respectively. All the eigenvalues have been approximated with an acceptable accuracy. The graph of two characteristic polynomials in the above examples are displayed in Figure 2 which gives a viewpoint of the position of eigenvalues.

3 Conclusion

In this work, a new approach for evaluating the eigenvalues of a matrix was proposed. The proposed approach was based on the interpolating the characteristic polynomial of the matrix on some subintervals by piecewise polynomials and then finding the roots of those polynomials. One of the advantages of our approach is that it can find all the roots of the characteristic polynomial and so all eigenvalues of the matrix, simultaneously.

Table 8: Applying our approach for Example 2.2 with $K = 2$

	root	$\lvert p(root)\rvert$
1	-0.900000000015135	1.02×10^{-4}
2	-0.250000000000000	6.18×10^{-11}
3	0.250000000000000	0
4	0.499999999996817	$1.05\ 5.55 \times 10^{-9}$
5	0.500000000003227	1.09×10^{-9}
6	0.600000000002160	1.35×10^{-9}
7	0.700000000014265	3.13×10^{-8}
8	0.899999999986612	6.22×10^{-7}

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References

[1] A. Böttcher, S.M. Grudsky, *Spectral Properties of Banded Toeplitz Matrices*, SIAM, Philadelphia, 2005.

[2] H.C. Chang, S.E. Liu, R. Burridge, *Exact eigensystems for some matrices arising from discretizations*, Linear Algebra and its Applications, 430 (2009) 999-1006.

[3] C.M. Da Fonseca, S. Kouachi, D.A. Mazilu, I. Mazilu, *A multi-temperature kinetic Ising model and the eigenvalues of some perturbed Jacobi matrices*, Applied Mathematics and Computation, 259 (2015) 205-211.

[4] C.M. Da Fonseca, F. Yilmaz, *Some comments on k-tridiagonal matrices: determinant, spectra, and inversion*, Applied Mathematics and Computation, 270 (2015) 644-647.

[5] S.E. Ekström, C. Garoni, A. Jozefiak, J. Perla, *Eigenvalues and eigenvectors of tau matrices with applications to Markov processes and economics*, Linear Algebra and its Applications, 627 (2021) 41-71.

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On fixed points of contractive iterate mappings

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Abstract

In this article, a new version of contractive iterate is introduced and by using it a fixed point theorem for mappings with closed graph is given. An example is shown that the main result of this paper can be viewed as an improvement of Sehgal's paper [American Mathematical Society, 23(3) (1969), 631-634].

Keywords: Closed graph mapping, Fixed point, Contractive iterate
Mathematics Subject Classification [2023]: 49J40, 54H25, 55M20

1 Introduction

Metric fixed point theory is one of the important branch of nonlinear analysis. In 1922, Banach [1] established the foundation of metric fixed point theory. He gave the first fixed point theorem which guarantee the existence and uniqueness of fixed point and provided a constructive method to find fixed point. In [5], Sehgal initiated the study of fixed points for mappings with contractive iterates at a point. In this article, a new version of contractive iterate is introduced and by using it a fixed point theorem for mappings with closed graph is given. An example is shown that the main result of this paper can be viewed as an improvement of Sehgal's main result [5].

Let (X, d) be a metric space and $T : X \to X$ be a mapping with the following condition (Banach contraction):

$$d(Ty, Tx) \leq kd(y, x), \qquad \forall x, y \in X,$$

where $k \in [0, 1)$.
Banach [1] proved that if X is a complete metric space, then T has unique fixed point u and for each $x \in X$, the iteration $\{T^n(x)\}$ converges to u. The Banach contraction is extended by Sehgal [5]. He presented the following definition which is a generalization of the Banach contraction.

Definition 1.1. Let (X, d) be a metric space and $T : X \to X$ be a mapping. T is called contractive iterate mapping if there exists positive number $k < 1$ such that, for each $x \in X$, there is positive integer $n(x)$ so that $d(T^{n(x)}(y), T^{n(x)}(x)) \leq kd(y, x)$, for all $y \in X$.

It is obvious that every Banach contraction mapping is a contractive iterate mapping while the simple example $T(x) = 1$ if x is rational and $T(x) = 0$ if x is irrational, shows that the converse may fail. It is remarkable that if x is unique fixed point of T^n for some n, then it is the unique fixed point of T and conversely. Therefore, one can put suitable conditions which under them, T^n has unique fixed point, for some n.

Theorem 1.2. *[5] Let (X, d) be a complete metric space and $T : X \to X$ be a contractive iterate mapping. If T is continuous, then T has unique fixed point u and each iteration $\{T^n(x)\}$ converges to u, for each $x \in X$.*

We introduce the following definition which is slightly improvement of the contractive iterate.

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Definition 1.3. Let (X, d) be a metric space and $T : X \to X$ a mapping. We say that T is pointwise contractive iterate if for each $x \in X$, there exists $k(x) \in [0, 1)$ and positive integer $n(x)$ such that $d(T^{n(x)}(x), T^{n(x)}(y)) \leq k(x)d(x, y)$, for all $y \in X$.

Remark that if $k(x) = k$, for each $x \in X$, then Definition 1.2 reduces to the Definition 1.1.
An affirmative answer to the above question given in [3] as follows.

Theorem 1.4. *Let X be a Banach space with uniform normal structure, C a nonempty closed bounded convex subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping. Then T has a fixed point.*

In the next section, we answer the above question for pointwise contractive iterate mappings in the setting of complete metric spaces without the condition uniform normal structure (see Theorem 2.4).

2 Main results

In this section, a fixed point theorem for pointwise contractive iterate mappings which is an improvement of Theorem 2.2 is given. An example, as well the other examples, is provided in order to show that it is real improvement of Theorem 2.2.

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a pointwise contractive mapping such that for any $\{x_n\} \subset X$ with $\sum_{n=1}^{\infty} \prod_{i=1}^{n} k(x_i) < \infty$ and $k(x_i)$ satisfies in Definition 1.2. If T is closed (its graph closed) then T has unique fixed point u and each iteration $\{T^n(x)\}$ converges to u.*

The following example shows that Theorem 2.4 is a real improvement of the main result given in [5].

Example 2.2. Let $X = \mathbb{R}$ and $T : X \to X$ be defined by

$$T(x) = \begin{cases} \frac{1}{2} & x \leq 1 \\ \frac{1}{1-x} & x > 1 \end{cases}$$

It is obvious that T is not continuous at $x = 1$, but its graph is closed. Moreover $T^2(x) = \frac{1}{2}$, $\forall x \in X$. Hence it satisfies all the assumptions of the previous theorem and the unique fixed point is $x = \frac{1}{2}$. But we can not apply Theorem 2.2 for it.

The following example indicates that there are discontinuous (every where) mappings which fulfil all the assumptions of Theorem 2.4.

Example 2.3. Let $T : [0, 1] \to [0, 1]$ be defined by

$$T(x) = \begin{cases} 1, & x \in Q \cap [0, 1]; \\ 0, & x \in Q^c \cap [0, 1]. \end{cases}$$

T is a pointwise contractive mapping and $x = 1$ is unique fixed point of T and each iteration converges to the fixed point. T is a nowhere continuous function and it does not satisfy the hypothesis of Theorem 1.1 and Theorem 1.2.

References

[1] U. Aksoy, I. M. Erhan, A. Fulga and E. Karapnar, *On the fixed points of iterative contractive mappings defined via implicit relation*, Optimization, 2020, https://doi.org/10.1080/02331934.2020.1742122.

[2] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fundamenta Mathematicae, 3 (1922), 133–181.

[3] Tae-Hwa. Kim, Hong-Kun. Xu, *Remarks on asymptotically nonexpansive mappings*, Nonlinear Analysis 41 (2000), 405–415.

[4] W. A. Kirk, *Fixed point theorems for non-lipschitzian mappings of asymptotically nonexpansive type*, Isr. J. Math. 17 (1974), 339–346.

[5] V. M. Sehgal, *A fixed point theorem for mappings with a contractive iterate,* American Mathematical Society, 23(3) (1969), 631–634.

[6] W. Shatanawi, E. Karapinar, H. Aydi, A. Fulga, *Wardowski type contractions with applications on Caputo type nonlinear fractional differential equations*, Univ. Politech. Buchar. Sci. Bull.-Ser. Appl, 82 (2020), 157-170.

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Fixed points for pointwise eventually non-expansive mappings in modular spaces

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Abstract

A well-known result of Kirk asserts that a nonexpansive mapping on a non-empty weakly compact convex subset (of a Banach space) possessing a normal structure has a fixed point. In this paper, we prove the existence of fixed points of pointwise eventually non-expansive mappings in modular spaces.

Keywords: pointwise eventually non-expansive mapping, modular spaces, uniformly convex in every direction, fixed point
Mathematics Subject Classification [2023]: 18A32, 18F20, 05C65 (at least 1 and at most 3)

1 Introduction

Fixed point theory for contractive and nonexpansive mappings defined in Banach spaces has been extensively developed since the mid 1960s. Fixed point theory has been extended to general metric spaces and independently to modular function spaces (see [[6], Ch. 2] or standard texts on metric fixed point theory. Nakano [9] initiated the theory of modular spaces as a generalization of several classes of functions and sequence spaces including $\ell_{p(.)}$, L_p, l_p, and Musielak-Orlicz spaces[8]. Despite essential differences between norms and modulars (the latter are allowed to take infinite values and may not have the triangle property), many surprising analogies have been discovered. To give just a few examples, the property (R) plays a similar role to reflexivity, convergence relates well to the weak topology, modular uniform convexity (though more complex) plays a similar role to norm uniform convexity, and major fixed point theorems like the Banach Contraction Principle and the fixed point theorems of Browder and Kirk have been expressed and proved in the language of modulars (see [6]).

Recently Bachar et al. [1] prepared an analogue to the classical Kirk's fixed point theorem in $\ell_{p(.)}$ by using a version of uniform convexity in every direction ($UCED$), namely $UUCED$. Note that $UCED$ is a special case of normal structure which plays an important role in the study of fixed point properties (see e.g. [5]). Since then the authors extensively started to research the existence of fixed points in modular spaces for non-expansive mappings.

On the other hand, Kirk and Xu [4] introduced the concept of pointwise eventually non-expansive mapping in Banach spaces as follows:

Definition 1.1. Let K be a subset of a Banach space. A mapping $T : K \to K$ is said to be pointwise eventually non-expansive if for each $x \in K$ there exists $N(x) \in \mathbb{N}$ such that for $n \geq N(x)$

$$\|T^n x - T^n y\| \leq \|x - y\|$$

for all $y \in K$.

In the present paper we obtain a fixed point theorem for pointwise eventually non-expansive mappings defined on certain subsets of modular spaces. In particular, we extend the corresponding result presented by Dehaish and Khamsi [2] to pointwise eventually non-expansive mappings without restrictive condition of uniform continuity of the modular.

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2 Preliminaries

We begin by recalling some basic concepts of modular spaces [6].

Definition 2.1. Let X be a vector space over K ($K = \mathbb{C}$ or $\mathbb{R}$). A function $\rho : X \to [0, \infty]$ is called a modular on X if for arbitrary elements x and y of X, it satisfies the following:

1. $\rho(x) = 0$ if and only if $x = 0$,
2. $\rho(\alpha x) = \rho(x)$ for every $\alpha \in K$ with $|\alpha| = 1$,
3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

If we replace (3) by

$$\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y),$$

for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, then the modular ρ is called convex. For a convex modular ρ on X one can associate a modular space X_ρ defined as

$$X_\rho = \left\{x \in X; \lim_{\lambda \to 0} \rho(\lambda x) = 0\right\}.$$

Definition 2.2. Let ρ be a convex modular defined on a vector space X.

1. A sequence $\{x_n\}$ in X_ρ is said to be ρ-convergent to $x \in X_\rho$ if and only if $\rho(x_n - x) \to 0$ as $n \to \infty$.
2. The sequence $\{x_n\}$ in X_ρ is said to be ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
3. X_ρ is said to be ρ-complete if any ρ-Cauchy sequence in X_ρ is a convergent sequence in X_ρ.
4. A subset $B \subset X_\rho$ is said to be ρ-closed if for any sequence $\{x_n\} \subset B$ which ρ-convergent to x, then $x \in B$.
5. A modular ρ is said to satisfy the Fatou property if $\rho(x) \leq \lim_{n\to\infty} \inf \rho(x_n)$ whenever $\{x_n\}$ ρ-converges to x.

3 Main results

We begin this section by introducing the following definition which has a crucial role for proving the main results.

Definition 3.1. Let K be a nonempty, convex and ρ-bounded subset of X_ρ. A function τ from K to $[0, \infty)$ is called ρ-type if there exists a sequence $\{x_n\}$ of K such that for any $z \in K$,

$$\tau(z) = \lim_{n\to\infty} \sup \rho(x_n - z).$$

Definition 3.2. Let ρ be a convex modular. We say that ρ is uniformly convex in every direction if, for any $r > 0$ and $h \in X_\rho \setminus \{0\}$,

$$\delta(r, h) = \inf\left\{1 - \frac{1}{r}\rho(f + \frac{h}{2}) : \rho(f) \leq r, \rho(f + h) \leq r\right\} > 0.$$

We say that ρ is $UUCED$ if for every $s \geq 0$ and $h \neq 0$, there exists $\eta(s, h) > 0$ such that $\delta(r, h) > \eta(s, h)$, for $r > s$.
Notice that in [1] a different version of the above definition has been used but in fact, they are the same.

Definition 3.3. The modular ρ is said to be strictly convex (SC), if for every $x, y \in X_\rho$ such that $\rho(x) = \rho(y)$ and

$$\rho(\frac{x + y}{2}) = \frac{\rho(x) + \rho(y)}{2},$$

we have $x = y$.

We need the following critical lemma in the sequel.

Lemma 3.4. *Let X_ρ be a modular space which has property (R). Assume that ρ is $UUCED$ and satisfies the Fauto property. Let K be nonempty, convex, ρ-closed and ρ-bounded subset of X_ρ and let $\{x_n\}$ be a sequence in K. Then the ρ-type τ from K to $[0,\infty)$ defined by $\tau(u) = \lim\limits_{n\to\infty} \sup \rho(x_n - u)$, for any $u \in K$, has a unique minimum point in K.*

The following lemma is slightly modification of the corresponding result given in [1] from the variable exponent sequence space $\ell_{p(.)}$ to arbitrary modular spaces X_ρ.

Lemma 3.5. *Assume that X_ρ be a modular space which has property (R). Let ρ be (SC) and satisfies Fauto property. Let C be a nonempty, ρ-closed, convex and ρ-bounded subset of X_ρ. Then for any family $\{C_\alpha\}_{\alpha\in\Gamma}$ of ρ-closed convex nonempty subsets of C such that $\bigcap\limits_{\alpha\in\Gamma_f} C_\alpha \neq \emptyset$ for any finite subset $\Gamma_f \subset \Gamma$, we have $\bigcap\limits_{\alpha\in\Gamma} C_\alpha$ is nonempty.*

The concept of pointwise eventually non-expansive mappings was introduced by Kirk and Xu in [4]. The modular version of this definition is introduced as follows:

Definition 3.6. Let K be a subset of X_ρ. A mapping $T : K \to K$ is said to be pointwise eventually non-expansive if for each $x \in K$ there exists $N(x) \in \mathbb{N}$ such that for $n \geq N(x)$

$$\rho(T^n x - T^n y) \leq \rho(x - y)$$

for all $y \in K$.

Now we are ready to state a modular analogue of Kirk' fixed poin theorem.

Theorem 3.7. *Assume that X_ρ is a modular space which has property (R). Let ρ be $UUCED$ and satisfies the Fauto property. Let K be a nonempty, convex, ρ-closed and ρ-bounded subset of X_ρ and $T : K \to K$ be pointwise eventually non-expansive. Then T has a fixed point in K.*

Proof. Suppose $\mathcal{F}$ denotes the set of all nonempty, convex, ρ-closed and ρ-bounded subsets of K which is T -invariant. According to Lemma 3.5 and Zorn's lemma, there exists a subset K_0 of K which is minimal with respect to being nonempty convex, ρ-closed ρ-bounded and invariant under T. If K_0 is a single point then the proof is finished. Otherwise let $\delta_\rho(K_0) > 0$. If T does not have any fixed point in K_0 then for each $x \in K_0$, we assert that the sequence $\{T^n x\}$ does not ρ-converge. Otherwise there exists $y \in K_0$ such that $\lim\limits_{n\to\infty} \rho(T^n x - y) = 0$. Since T is pointwise eventually non-expansive, there exists $N(y) \in \mathbb{N}$ such that for $n \geq N(y)$,

$$\rho(T^n x - T^n y) \leq \rho(x - y).$$

Hence, for $n \geq m \geq N(y)$,

$$\rho(T^n x - T^m y) \leq \rho(T^{n-m} x - y).$$

Thus $\lim\limits_{n\to\infty} \rho(T^n x - T^m y) = 0$ for all $m \geq N(y)$ and by the uniqueness of ρ-limit we get $T^m y = y$ for all $m \geq N(y)$. Therefore $Ty = T(T^{N(y)} y) = y$ which is a contradiction and so the assertion is true. Suppose $x_0 \in K_0$ is an arbitrary element and τ is a ρ-type on K_0 defined by $\tau(y) = \lim\limits_{n\to\infty} \sup \rho(T^n x_0 - y)$ for $y \in K_0$. By Lemma 3.4, τ has a unique minimum point $x_1 \in K_0$. Thus $\tau = \tau(x_1) < \tau(y)$ for all $y \neq x_1$. It follows from the hypothesis on T that $\tau > 0$. By usinig the definition of τ, for any $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that $\rho(T^n x_0 - x_1) \leq \tau + \varepsilon$ for any $n \geq n(\varepsilon)$.
We can assume that there exists a subsequence $\{n_k\}$ such that $T^{n_k} x_1 \neq x_1$. Let us assume that $n_k \geq N(x_0)$ for any k. Since T is pointwise eventually non-expansive we have

$$\rho(T^n x_0 - T^{n_1} x_1) \leq \rho(T^{n-n_1} x_0 - x_1) \leq \tau + \varepsilon$$

for any $n \geq n(\varepsilon) + n_1$. Since ρ is $UUCED$, there exists $\eta(\tau, x_1 - T^{n_1} x_1) > 0$ such that

$$\rho(T^n x_0 - \frac{x_1 + T^{n_1} x_1}{2}) = \rho(T^n x_0 - x_1 + \frac{x_1 - T^{n_1} x_1}{2})$$

$$\leq (\tau+\varepsilon)(1-\delta(\tau+\varepsilon, x_1 - T^{n_1}x_1))$$
$$\leq (\tau+\varepsilon)(1-\eta(\tau, x_1 - T^{n_1}x_1))$$

for any $\varepsilon > 0$. Therefore we must have

$$\tau(\frac{x_1+T^{n_1}x_1}{2}) \leq \tau(1-\eta(\tau, x_1 - T^{n_1}x_1)) < \tau = \tau(x_1)$$

which contradicts the uniqueness of the minimum point of τ. Hence K_0 is singleton and it contains the fixed point of T .

Remark 3.8. (a) Theorem 3.7 generalizes Theorem 3.6 of Dehaish and Khamsi in [2] by replacing monotone non-expansive mappings with eventually non-expansive mappings. It also drops the uniform continuity assumption of the modular ρ.

(b) In [1] it is shown that if the modular ρ on $\ell_{p(.)}$ is $UUCED$, and C be a nonempty, ρ-closed, convex and ρ-bounded subset of $\ell_{p(.)}$ which satisfies the property (R), then every non-expansive mapping $T : C \to C$ has a fixed point in C. This is a special case of Theorem 3.7 by defining $\rho : \ell_{p(.)} \to [0, \infty]$ as follows

$$\rho(x) = \rho((x_n)) = \sum_{n=0}^{\infty} \frac{1}{p(n)}|x_n|^{p(n)}$$

It is easy to verify that ρ is a convex modular on $\ell_{p(.)}$.

References

[1] M. Bachar, M. A. Khamsi, O. Mendez, M. Bounkhel, *A geometric property in $\ell_{p(.)}$ and its applications.* Math. Nachr. **292** (2019), 1931–1940.

[2] B. Bin Dehaish and M. A. Khamsi, *On monotone mappings in modular function spaces,* J. Nonlinear Sci. Appl. **9** (2016), 5219–5228.

[3] M. A. Khamsi, W. M. Kozlowski, *Fixed Point Theory in Modular Function Spaces,* Birkhauser, New York, (2015).

[4] W. A. Kirk, H. K. Xu, *Asymptotic powintwise contraction.* Nonlinear Anal. **69** (2008), 4706-4712.

[5] W. M. Kozlowski, *On modulated topological vector spaces and application.* Bull. Aust. Math. Soc. **101** (2020), 325–332.

[6] K. Kuratowski, *Sur les espaces complets.* Fund. Math. **15** (1930), 301309.

[7] V. Klee, *Summability in $l(p_{11}, p_{21}, ...)$ Spaces,* Studia Math. **25** (1965), 277–280.

[8] J. Musielak, W. Orlicz, *On modular spaces.* Studia Math, **18** (1959), 591-597.

[9] H. Nakano, *Modulared Semi-ordered Linear Spaces,* Maruzen Co., Tokyo, 1950.

[10] W. Orlicz, *Uber konjugierte Exponentenfolgen,* Studia Math. **3** (1931), 200–211.

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A generalization of Ekeland's variational principle by using the τ-distance and its application in equilibrium problem

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Abstract

In this paper, a new version of Ekeland's variational principle by using the concept of τ-distance for bounded from below functions which are not necessarily lower semicontinuous is provided. This new version will be applied to establish an existence theorem for a solution of the equilibrium problem in the complete metric spaces.

Keywords: Lower semicontinuous Regularization, Ekeland's variational principle, Bounded from below, Equilibrium problem, τ-distance
Mathematics Subject Classification [2023]: 18A32, 18F20, 05C65

1 Introduction

To the best of our knowledge, the first appearance of equilibrium problems as we understand them now is due to Muu and Oettli and it was further developed by Blum and Oettli. They are conceptually connected to Ky Fan's minimax inequality Which goes back to the equality result of von Neumann.

Ekeland's variational principle was first expressed by Ekeland and developed by many authors and researchers [2, 3, 4, 5]. Tataru in defined the concept of Tataru's distance and using it proved the generalization of Ekeland's variational principle. Afterwards in 1996, Kada in stated the concept of w-distance and extended Ekeland's variational principle. The concept of τ-distance which a generalization of w-distance and Tataru's distance was first introduced by Suzuki. He also improved the concept of Ekeland's variational principle ([6]).

The purpose of this paper is to study equilibrium problem to get some existence results. In fact, we recall the concepts of τ-distance and w-distance on a complete metric space and then a new version of Ekeland's variational principle by using the concepts of τ-distance and w-distance is proved and, by applying them, a version of existence result of a solution for the equilibrium problem in compact domains in the setting of complete metric spaces are investigated. Finally some examples in order to illustrate the results to this note are given. The results of this paper improves the corresponding results given in [2, 3, 4, 5].

1.1 Preliminaries

In this section we introduce and remind tools that will be useful throughout the paper and we will use standard notations and terminology from real analysis.

By an equilibrium problem (EP)([3]) we understand the problem of finding
$\bar{x} \in C \quad$ such that $\quad f(\bar{x}, y) \geq 0 \quad \forall y \in C,$
where C is a set and $f : C \times C \to \mathbb{R}$ is a bifunction.

Definition 1.1. Given a nonempty subset C of a topological space X, a function $h : C \to \mathbb{R}$ is said to be:

- **lower semicontinuous** if for each $x \in C$ and each $\lambda \in \mathbb{R}$ such that $h(x) > \lambda$, there exists a neighborhood V_x of x such that $h(y) > \lambda$, for all $y \in V_x \cap C$ or equivalently for any net $\{x_\alpha\} \subset C$, $x_\alpha \to x$ implies

$$h(x) \leq \liminf_{\alpha} h(x_\alpha).$$

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- **upper semicontinuous** if for each $x \in C$, $-f$ is lower semicontinuous or equivalently for any net $\{x_\alpha\} \subset C$, $x_\alpha \to x$ implies

$$h(x) \geq \limsup_{\alpha} h(x_\alpha).$$

Also the epigraph of the function h is denoted by $\mathrm{Epi}(h)$ and is defined by $\mathrm{Epi}(h) := \{(x, \lambda) \in C \times \mathbb{R} : h(x) \leq \lambda\}$. The lower sub-level set at level λ of h is defined by $S_h(\lambda) := \{x \in C : h(x) \leq \lambda\}$.

It is well known that a function h is lower semicontinuous if and only if $\mathrm{Epi}(h)$ is closed in $C \times \mathbb{R}$ or equivalently, if and only if $S_h(\lambda)$ is closed in C, for all $\lambda \in \mathbb{R}$.

Given a nonempty subset C of a topological space X, it is a basic fact from real analysis that every function $h : C \to \mathbb{R}$ (not necessarily lower semicontinuous) admits a lower semicontinuous regularization $\overline{h} : C \to \mathbb{R} \cup \{-\infty\}$ defined by $\mathrm{Epi}(\overline{h}) := \overline{\mathrm{Epi}(h)}$, the closure in $C \times \mathbb{R}$, equivalently by

$$\overline{h}(x) = \liminf_{y \to x} h(y) = \sup_{U} \inf_{y \in U \cap C} h(y),$$

where U runs all neighborhoods of x.

It is well known that for any $x \in C$ and any $\lambda \in \mathbb{R}$

1. $\overline{h}(x) = \inf\{\lambda \in \mathbb{R} : x \in \overline{S_h(\lambda)}\}$;
2. $\overline{h}(x) \leq h(x)$.

Example 1.2. Let $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) := \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases},$$

It is clear that h is not lower semicontinuous and its lower semicontinuous regularization is $\overline{h}(x) = x^2$.

Example 1.3. If $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(x) := [x]$, then it is easy to check that

$$\overline{h}(x) = \begin{cases} x - 1 & \text{if } x \in \mathbb{Z} \\ [x] & \text{if } x \notin \mathbb{Z} \end{cases}$$

Definition 1.4. ([5]) Let (X,d) be a metric space. Then a function $p : X \times X \to \mathbb{R}$ is called a τ-distance on X if

(1) $p(x, z) \leq p(x, y) + p(y, z) \quad \forall x, y, z \in X$;

moreover, there exists a function $\eta : X \times \mathbb{R}^+ \to \mathbb{R}^+$ which is concave and continuous in its second argument and satisfying the following conditions:

(2) $\eta(x, 0) = 0$ and $\eta(x, t) \geq t \quad \forall x \in X$, $t \in \mathbb{R}^+$;

(3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply

$$p(w, x) \leq \liminf_{n} p(w, x_n) \quad \forall w \in X;$$

(4) $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$;

(5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

Note that if d is a meter then it is τ-distance, by taking $p = d$ and $\eta(x, t) = t$.

2 The Ekeland Variational Principle for equilibrium problems

In this section stated a new version of Ekeland's variational principle by using the concepts of τ-distance in the setting of complete metric spaces.

Theorem 2.1. *(The Ekeland variational principle) Let C be a nonempty closed subset of a complete metric space (X,d), and $h : C \to \mathbb{R}$ be a lower semicontinuous function bounded from below. For every $\varepsilon > 0$, and for any $x_0 \in C$, there exists $\hat{x} \in C$ such that*
$h(\hat{x}) + \varepsilon d(x_0, \hat{x}) \leq h(x_0)$,
$h(x) + \varepsilon d(x, \hat{x}) > h(\hat{x})$, *for all* $x \in C \setminus \{\hat{x}\}$.

The following theorem is a generalization of Ekeland's variational principle (Theorem 2.2) by replacing a meter by a τ-distance.

Theorem 2.2. *([6]) Let (X,d) be a complete metric space, and let $h : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function, bounded from below. Assume that there exists a τ-distance p on X. Then, for every $\varepsilon > 0$, and for any $u \in X$ with $p(u,u) = 0$, there exists $v \in X$ such that*
$h(v) + \varepsilon p(u, v) \leq h(u)$,
$h(w) + \varepsilon p(v, w) > h(v)$, *for all* $w \in X \setminus \{v\}$.

3 Main results

The next result is a new version of theorem 4.2 by relaxing the lower semicontinuity condition of the function h, and also it is and extension of the corresponding theorem given in [4] for τ-distance functions with a simpler proof than as given in [4].

Theorem 3.1. *Let C be a nonempty closed subset of a complete metric space (X,d), and $h : C \to \mathbb{R}$ be a function bounded from below. Assume that there exists a τ-distance p on X. For every $\varepsilon > 0$, and for any $x_0 \in C$ with $p(x_0, x_0) = 0$, there exists $\hat{x} \in C$ such that*
$\overline{h}(\hat{x}) + \varepsilon p(x_0, \hat{x}) \leq h(x_0)$,
$h(x) + \varepsilon p(x, \hat{x}) > \overline{h}(\hat{x})$, *for all* $x \in C \setminus \{\hat{x}\}$.

The following example illustrates Theorem 4.6.

Example 3.2. Let $X = \mathbb{R}$, $C = [0, \infty)$ and d be the Euclidean metric on X. The function $h : C \to \mathbb{R}$ is defined by $h(x) := [x]$, for all $x \in C$ (Example 1.3) is not lower semicontinuous. Obviously, h satisfies all conditions of Theorem 4.6 and so if we take $x_0 = 1$ and $0 < \varepsilon \leq 1$ then $\hat{x} = 0$ is a candidate which fulfills in the conclusion of Theorem 4.6.

The next theorem is a direct consequence of Theorem 4.6 for bifunctions. Moreover, it provides sufficient conditions for non-emptiness of the solution set of (EP).

Theorem 3.3. *Let C be a nonempty closed subset of a complete metric space (X,d), and $f : C \times C \to \mathbb{R}$ be a bifunction. Assume that there exists a τ-distance p on X. Assume that the following conditions hold*
(i) f is bounded from below and lower semicontinuous with respect to its second argument;
(ii) $f(x,x) = 0$, for all $x \in C$;
(iii) f satisfies the triangle inequality property

$$f(x,y) \leq f(x,z) + f(z,y) \quad \forall x, y, z \in C.$$

Then, for all $\varepsilon > 0$ and all $x_0 \in C$ with $p(x_0, x_0) = 0$, there exists $\hat{x} \in C$ such that
$f(x_0, \hat{x}) + \varepsilon p(x_0, \hat{x}) \leq 0$,
$f(\hat{x}, x) + \varepsilon p(x, \hat{x}) > 0$, *for all* $x \in C \setminus \{\hat{x}\}$.
Furthermore if C be a compact set, and f and p be upper semicontinuous with respect to its first argument, then the solution set of (EP) is nonempty and compact.

Example 3.4. Let $X = \mathbb{R}$, $C = [-1, 1]$, d the Euclidean metric on X and $f : C \times C \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} x^2 + x + y & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

It is easy to verify the solution set (EP) equals to $[\frac{\sqrt{5}-1}{2}, 1]$.

Example 3.5. Let $X = C = \mathbb{R}$ and d be the Euclidean metric on X. Consider the function $p : X \times X \to [0, \infty)$ defined by $p(x, y) = |y|$, for all $x, y \in X$. The function p is a τ-distance. Define the function $h : C \to \mathbb{R}$ by

$$h(x) := \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

, for all $x \in C$ (Example 1.2). Obviously, h satisfies all conditions of Theorem 4.6 and so if we take $x_0 = 0$ then $\hat{x} = 0$ is a candidate which fullfils in the conclusion of Theorem 4.6.

4 Conclusion

A new form of the Ekeland's variational principle for functions which are no necessarily lower semicontinuous in setting of τ-distance spaces. Then, this form of the Ekeland's variational principle is applied for proving and existence result of the solution for the equilibrium problem that an improvement of Theorem 3.2 in [5] and results in [1].

References

[1] MR. Alfuraidan and MA. Khamsi, *Graphicial Ekeland's principle for equilibrium problems.* Proc. Amer, Math. Soc., Ser.B, Volume 9. **2022**, pp 33–40

[2] M. Bianchi, G. Kassay and R. Pini, *Existence of equilibria via Ekeland's principle.* J. Math. Anal. **305** (2005), 502–512.

[3] M. Castellani, G. Massimiliano, *Ekeland's principle for cyclically antimonotone equilibrium problems.* Nonlinear Anal: Real World Appl. **32** (2016), 213–228.

[4] J. Cotrina, M. Thera and J. zuniga, *An Existence Result for Quasi-equilibrium problems via Ekeland's Variational Principle.* J. Optim Theory Appl, **187** (2020), 336–355.

[5] AP. Farajzadeh, S. Plubtieng and A. Hoseinpour, *A generalization of Ekeland variational principle by using the τ-distance with its applications.* J. Inequalities and Appl, (2017): 181.

[6] T. Suzuki, *Generalized distance and existence theorems in complete metric spaces.* J. Math. Anal. Appl.**253** (2001), 440–458.

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On cyclic relatively kannan nonexpasive mappings in Banach spaces

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Abstract

In this article, we present some best proximity point theorems for various classes of cyclic mappings which are nonexpansive in the sense of Kannan. One of the main tools to obtain these existence results is a geometric concept of quasi proximal normal structure. The same conclusions are valid by dropping this geometric notion and considering a stronger condition than Kannan nonexpansiveness.

Keywords: Best proximity point; quasi proximal normal structure; relatively Kannan nonexpansive mapping
Mathematics Subject Classification [2023]: 47H09, 46B20

1 Introduction

Let X be a Banach space, and let A, B be nonempty subsets of X. A mapping $T : A \cup B \to A \cup B$ is said to be *cyclic* provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. For this class of mappings the fixed point equation $x = Tx$ may not have solution.

Definition 1.1. Let T be a cyclic mapping. A point $x \in A \cup B$ is said to be a best proximity point for T provided that $d(x, Tx) = \text{dist}(A, B)$, where $\text{dist}(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$.

Let (A, B) be a nonempty pair in a Banach space X. We say that the pair (A, B) of subsets in a Banach space X satisfies a property if both A and B satisfy that property. For example, (A, B) is convex if and only if both A and B are convex. Moreover, throughout this paper we shall use the following notations and definitions:

$$\begin{aligned}(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C, \quad \text{and} \quad B \subseteq D,\\ \delta_x(A) = \sup\{d(x, y) : y \in A\} \text{ for all } x \in X,\\ \delta(A, B) = \sup\{d(x, y) : x \in A,\ y \in B\}.\end{aligned}$$

Definition 1.2. ([2]) Let (A, B) be a nonempty pair in a normed linear space X. A mapping $T : A \cup B \to A \cup B$ is said to be a cyclic relatively Kannan nonexpansive mapping provided that T is cyclic and

$$\|Tx - Ty\| \leq \frac{1}{2}[\|x - Tx\| + \|y - Ty\|],$$

for all $(x, y) \in A \times B$.

In special case, if $A = B$, then the self-mapping T is called nonexpansive in the sense of Kannan which was first introduced in [4].

In this paper we present some best proximity point theorems for cyclic relatively Kannan nonexpansive mappings.

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2 Quasi proximal normal structure

Definition 2.1. A pair (A, B) of subsets of a linear space X is said to be a proximal pair if for each $(x, y) \in A \times B$ there exists $(\acute{x}, \acute{y}) \in A \times B$ such that

$$\|x - \acute{y}\| = \|\acute{x} - y\| = \text{dist}(A, B).$$

Definition 2.2. ([1]) A convex pair (K_1, K_2) in a Banach space X is said to have proximal normal structure (QPNS for brief) if for any bounded, closed and convex proximal pair $(H_1, H_2) \subseteq (K_1, K_2)$ for which $\text{dist}(H_1, H_2) = \text{dist}(K_1, K_2)$ and $\delta(H_1, H_2) > \text{dist}(H_1, H_2)$, there exits $(p_1, p_2) \in H_1 \times H_2$ such that

$$d(p_1, y) < \delta(H_1, H_2), \quad d(x, p_2) < \delta(H_1, H_2),$$

for all $(x, y) \in H_1 \times H_2$.

It was announced in [3] that every nonempty, bounded, closed and convex pair in a uniformly convex Banach space X has QPNS. Moreover, any compact and convex pair of subsets of a Banach space X has QPNS too.
Here, we present the first existence result.

Theorem 2.3. *([1]) Let (A, B) be a nonempty, weakly compact convex pair in a Banach space X such that (A, B) has* QPNS*. Let $T : A \cup B \to A \cup B$ be a cyclic relatively Kannan nonexpansive mapping such that for any $(x, y) \in A \times B$ if $\|x - y\| = \text{dist}(A, B)$, then $\|Tx - Ty\| = \text{dist}(A, B)$. Then T has a best proximity point.*

3 On dropping of QPNS

Definition 3.1. ([5]) Let (A, B) be a nonempty pair in a Banach space X and $T : A \cup B \to A \cup B$ be a cyclic mapping. We say that T is a cyclic Kannan contraction mapping if

$$\|Tx - Ty\| \leq \alpha\Big\{\|x - Tx\| + \|y - Ty\|\Big\} + (1 - 2\alpha)\text{dist}(A, B),$$

for some $\alpha \in (0, \frac{1}{2})$ and for all $(x, y) \in A \times B$.

It is worth noticing that the class of cyclic relatively Kannan nonexpansive mappings contains the class of cyclic Kannan contraction as a subclass.

In what follows we present a best proximity point theorem for cyclic Kannan contraction mappings without the geometric notion of QPNS.

Theorem 3.2. *([1]) Let (A, B) be a nonempty weakly compact convex pair in a Banach space X. Assume that $T : A \cup B \to A \cup B$ is a cyclic Kannan contraction mapping. Then T has a best proximity point.*

Under some other sufficient conditions and without using QPNS, we can obtain the following existence results.

Theorem 3.3. *([2]) Let (A, B) be a nonempty, weakly compact and convex pair of subsets of a normed linear space X. Suppose $T : A \cup B \to A \cup B$ is a cyclic relatively Kannan nonexpansive mapping such that*

$$\|T^2x - Tx\| < \|x - Tx\|, \quad \textit{for all } x \in A \cup B \textit{ with } \text{dist}(A, B) < \|x - Tx\|. \tag{3.1}$$

Then T has a best proximity point.

Corollary 3.4. *Let A be a nonempty, weakly compact and convex subset of a normed linear space X. Suppose $T : A \to A$ is a Kannan nonexpansive mapping such that*

$$\|T^2x - Tx\| < \|x - Tx\|, \quad \textit{for all } x \in A \textit{ with } \|x - Tx\| > 0.$$

Then T has a unique fixed point.

Theorem 3.5. *([2]) Let (A, B) be a nonempty, weakly compact and convex pair of subsets of a Banach space X. Suppose $T : A \cup B \to A \cup B$ is a cyclic relatively Kannan nonexpansive mapping and suppose that for each nonempty, closed and convex pair $(C_1, C_2) \subseteq (A, B)$ which is T-invariant and such that $\delta(C_1, C_2) > \operatorname{dist}(A, B)$, we have*

$$\inf\{\|x - Tx\| : x \in C_1 \cup C_2\} < \delta(C_1, C_2).$$

Then T has a best proximity point.

The following corollary is a main result of [6].

Corollary 3.6. *Let A be a nonempty, weakly compact and convex subset of a Banach space X and let $T : A \to A$ be a Kannan nonexpansive mapping. Suppose that for each closed and convex subset C of A which is T-invariant and $\operatorname{diam}(C) > 0$, we have*

$$\inf\{\|x - Tx\| : x \in C\} < \operatorname{diam}(C).$$

Then T has a unique fixed point.

4 Conclusion

We have presented some best proximity point results for cyclic relatively Kannan nonexpansive mappings under some sufficient assumptions. In particular cases we have obtained some well-known fixed point theorems due to Kannan ([4]) and Wang ([6]).

References

[1] A. Abkar, M. Gabeleh, *Proximal quasi-normal structure and a best proximity point theorem*, J. Nonlin. Convex Anal., 14 (2013), 653-659.

[2] M. Gabeleh, *Best proximity points and fixed point results for certain maps in Banach spaces*, Numer. Funct. Anal. Optim., 36 (2015), 10131028.

[3] M. Gabeleh, *A characterization of proximal normal structures via proximal diametral sequences*, J. Fixed Point Theory Appl., 19 (2017), 2909–2925.

[4] R. Kannan, *Fixed point theorems in reflexive Banach spaces*, Proc. Amer. Math. Soc., 38 (1973), 111-118.

[5] M.A. Petric, *Best proximity point theorems for weak cyclic kannan contractions*, Filomat, 25:1 DOI: 10.2298/FIL1101145P (2011)

[6] C. Wong, *Fixed points and characterizations of certain maps*, Pacific J. Math., 54 (1974), 305-312.

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A strong convergence theorem for general equilibrium problem and resolvent operator in Banach spaces

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Abstract

In this paper, we introduce a new extragradient algorithm by using generalized metric projection. We prove a strong convergence theorem for finding a common element of the solution set of split feasibility problem and a resolvent operator and the set of solutions of an equilibrium problem.

Keywords: Nonexpansive mapping, Equilibrium problem, Split feasibility problem
Mathematics Subject Classification [2023]: 47H09,47H10

1 Introduction

Let C be a nonempty closed convex subset of a real Banach space X with norm $\|.\|$ and X^* be the dual of X. The duality mapping J_X^p on X is defined by

$$J_X^p(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\},$$

for every $x \in X$. Let X be a smooth Banach space and let J_X be the duality mapping on X. The function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, J_X y\rangle + \|y\|^2, \quad \forall x, y \in X. \tag{101}$$

Clearly, from (101), we can conclude that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2. \tag{102}$$

Also, it is clear from the definition of the function ϕ that the following condition holds for all $x, y \in X$,

$$\phi(x, y) = \langle x, J_X x - J_X y\rangle + \langle y - x, J_X y\rangle \leq \|x\|\|J_X x - J_X y\| + \|y - x\|\|y\|. \tag{103}$$

Now, the function $V : X \times X^* \to \mathbb{R}$ is defined as follows

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2,$$

for all $x \in X$ and $x^* \in X^*$. Moreover, $V(x, x^*) = \phi(x, J_X^{-1}x^*)$ for all $x \in X$ and $x^* \in X^*$. If X is a reflexive strictly convex and smooth Banach space with X^* as its dual, then

$$V(x, x^*) + 2\langle J_X^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*), \forall x \in X, \ \forall x^*, y^* \in X^*. \tag{104}$$

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2 Main results

Theorem 2.1. *Let X_1 and X_2 are real 2-uniformly convex and uniformly smooth Banach spaces. Suppose that C and D are nonempty closed and convex subsets of X_1 and X_2, respectively. Suppose that g is a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies the conditions A1-A4 of [2], $A : X_1 \to X_2$ is a bounded linear operator and $A^* : X_2^* \to X_1^*$ is the adjoint of A. Let $M : X_1 \to 2^{X_1^*}$ be a maximal monotone operator with $M^{-1}0 \neq \emptyset$. Assume that $\psi : C \to X^*$ is an α-inverse strongly monotone operator and f is a relatively nonexpansive mappings from C into itself and $\Gamma = \Omega \cap F(Q_\iota^M) \cap EP(g) \neq \emptyset$. Let $\{x_n\}$ is a sequence generated by $v_1 \in C$ and*

$$\begin{cases} x_n \in C \;\; s.t \;\; g(x_n, y) + \langle \psi x_n, y - x_n \rangle + \frac{1}{r_n}\langle y - x_n, J_{X_1} x_n - J_{X_1} v_n \rangle \geq 0, \\ z_n = \Pi_C J_{X_1}^{-1}(J_{X_1} x_n - \tau A^* J_{X_2}(Ax_n - P_D Ax_n)), \\ w_n = \Pi_C J_{X_1}^{-1}(\beta_n J_{X_1} Q_\iota^M z_n + (1 - \beta_n) J_{X_1} Q_\iota^M f(z_n)), \\ v_{n+1} = \Pi_C J_{X_1}^{-1}[\alpha_n J_{X_1} x_n + (1 - \alpha_n) J_{X_1} w_n], \end{cases} \tag{105}$$

where $r_n \in [a, \infty)$ for some $a > 0$, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[a, b] \subset (0, 1)$, and τ satisfies the following condition:

(i) τ is real number such that $0 < \tau < \frac{2}{c\|A\|^2}$, where c depends on 2-uniformly smoothness of X_1^*.

Then $\{x_n\}$ converges strongly to $q = \Pi_{\Omega \cap F(Q_\iota^M) \cap EP(g)} \circ q$.

Proof. Let $\hat{u} \in \Gamma$. Let $k_n = Ax_n - P_D Ax_n$. From (101), Lemmas 2.2 and 2.7 of [2], we have that

$$\begin{aligned} \phi(\hat{u}, z_n) \leq & \phi(\hat{u}, J_{X_1}^{-1}(J_{X_1} x_n - \tau A^* J_{X_2} k_n)) \\ \leq & \|\hat{u}\|^2 - 2\langle \hat{u}, J_{X_1} x_n \rangle + 2\tau \langle \hat{u}, A^* J_{X_2} k_n \rangle + \|J_{X_1} x_n\|^2 \\ & - 2\tau \langle A^* J_{X_2} k_n, J_{X_1}^* J_{X_1} x_n \rangle + c\tau^2 \|A^* J_{X_2} k_n\|^2 \\ = & \phi(\hat{u}, x_n) + 2\tau \langle A\hat{u} - Ax_n, J_{X_2} k_n \rangle + c\tau^2 \|A\|^2 \|k_n)\|^2. \end{aligned} \tag{106}$$

Since $\langle J_{X_2}(x - P_D x), y - P_D x \rangle \leq 0$ for each $y \in D$ and for each $x \in X_2$, we have that

$$\begin{aligned} \langle J_{X_2} k_n, Ax_n - A\hat{u} \rangle &= \langle J_{X_2} k_n, P_D Ax_n - A\hat{u} \rangle + \langle J_{X_2} k_n, Ax_n - P_D Ax_n \rangle \\ &= \langle J_{X_2} k_n, P_D Ax_n - A\hat{u} \rangle + \|P_D Ax_n - Ax_n\|^2 \geq \|P_D Ax_n - Ax_n\|^2. \end{aligned}$$

From (106), our assumptions and the above, we conclude that

$$\phi(\hat{u}, z_n) = \phi(\hat{u}, x_n) - \tau(2 - c\tau\|A\|^2)\|P_D Ax_n - Ax_n\|^2 \leq \phi(\hat{u}, x_n). \tag{107}$$

It follows from (101), (107), Lemma 2.11 of [2] and the convexity of $\|.\|^2$ that

$$\phi(\hat{u}, w_n) \leq \beta_n \phi(\hat{u}, z_n) + (1 - \beta_n)\phi(\hat{u}, f(z_n)) \leq \phi(\hat{u}, z_n) \leq \phi(\hat{u}, x_n). \tag{108}$$

By (101), (108), Lemmas 2.2 and 2.8 of [2], we have that

$$\phi(\hat{u}, x_{n+1}) = \phi(\hat{u}, K_{r_n} v_{n+1}) \leq \phi(\hat{u}, v_{n+1}) \leq \alpha_n \phi(\hat{u}, x_n) + (1 - \alpha_n)\phi(\hat{u}, w_n) \leq \phi(\hat{u}, x_n). \tag{109}$$

Therefore, $\{\phi(\hat{u}, x_n)\}$ is bounded and $\lim_{n\to\infty} \phi(\hat{u}, x_n)$ exists. Now, by (102), we conclude that $\{x_n\}$ is bounded. It follows from (107), (108) and (2) that the sequences $\{z_n\}$, $\{w_n\}$ and $\{v_n\}$ are bounded. Now, by (108), (2) and Lemma 2.11 of [2], we conclude that

$$\begin{aligned} \phi(\hat{u}, x_{n+1}) \leq & \alpha_n \phi(\hat{u}, x_n) + (1 - \alpha_n)[\beta_n \phi(\hat{u}, Q_\iota^M z_n) + (1 - \beta_n)\phi(\hat{u}, Q_\iota^M f(z_n))] \\ \leq & \alpha_n \phi(\hat{u}, x_n) + (1 - \alpha_n)[\beta_n \big(\phi(\hat{u}, z_n) - \phi(Q_\iota^M z_n, z_n)\big) + (1 - \beta_n)\phi(\hat{u}, z_n)] \\ \leq & \phi(\hat{u}, x_n) - (1 - \alpha_n)\beta_n \phi(Q_\iota^M z_n, z_n). \end{aligned} \tag{110}$$

Now, since $\{\phi(\hat{u}, x_n)\}$ is convergent, it follows from (110) that $\lim_{n\to\infty}\phi(Q_\iota^M z_n, z_n) = 0$. Therefore, from Lemma 2.1 of [2], we have that

$$\lim_{n\to\infty}\|Q_\iota^M z_n - z_n\| = 0. \tag{111}$$

Now, from (107), (108) and (2), we have

$$\begin{aligned}\phi(\hat{u}, x_{n+1}) \leq& \alpha_n\phi(\hat{u}, x_n) + (1-\alpha_n)\phi(\hat{u}, z_n)\\ \leq& \phi(\hat{u}, x_n) - (1-\alpha_n)\tau(2 - c\tau\|A\|^2)\|P_D Ax_n - Ax_n\|^2.\end{aligned} \tag{112}$$

Hence, it follows from (112) that

$$(1-\alpha_n)\tau(2 - c\tau\|A\|^2)\|P_D Ax_n - Ax_n\|^2 \leq \phi(\hat{u}, x_n) - \phi(\hat{u}, x_{n+1}).$$

Then,

$$\lim_{n\to\infty}\|P_D Ax_n - Ax_n\|^2 = 0. \tag{113}$$

Suppose that $r_1 = \sup_n\{\|x_n\|, \|w_n\|\}$. Therefore, from Lemma 2.5 of [2], there exists a continuous strictly increasing convex function $g_1 : [0, 2r_1] \to [0, \infty)$ such that $g_1(0) = 0$ and using (108), Lemmas 2.2 and 2.8 of [2] and the convexity of $\|.\|^2$, we have that

$$\begin{aligned}\phi(\hat{u}, x_{n+1}) =& \phi(\hat{u}, K_{r_n} v_{n+1}) \leq \phi(\hat{u}, v_{n+1}) \leq \alpha_n\phi(\hat{u}, x_n) + (1-\alpha_n)\phi(\hat{u}, w_n)\\ &- \alpha_n(1-\alpha_n)g_1(\|J_{X_1}x_n - J_{X_1}w_n\|)\\ \leq& \phi(\hat{u}, x_n) - \alpha_n(1-\alpha_n)g_1(\|J_{X_1}x_n - J_{X_1}w_n\|).\end{aligned} \tag{114}$$

So, $\alpha_n(1-\alpha_n)g_1(\|J_{X_1}x_n - J_{X_1}w_n\|) \leq \phi(\hat{u}, x_n) - \phi(\hat{u}, x_{n+1})$. Since $\lim_{n\to\infty}\phi(\hat{u}, x_n)$ exists. Therefore, $\lim_{n\to\infty} g_1(\|J_{X_1}x_n - J_{X_1}w_n\|) = 0$. Then,

$$\lim_{n\to\infty}\|x_n - w_n\| = 0. \tag{115}$$

Using (103), (115), the uniformly norm-to-norm continuity of J_{X_1} on bounded sets, the boundedness of the sequences $\{x_n\}$ and $\{u_n\}$, we conclude that

$$\lim_{n\to\infty}\phi(w_n, x_n) = 0. \tag{116}$$

Now, by (104), (113) and Lemmas 2.2 and 2.4 of [2], we conclude that

$$\begin{aligned}\phi(x_n, z_n) \leq& \phi(x_n, J_{X_1}^{-1}(J_{X_1}x_n - \tau A^* J_{X_2}k_n) = V(x_n, J_{X_1}x_n - \tau A^* J_{X_2}k_n)\\ \leq& V(x_n, J_{X_1}x_n) - 2\langle J_{X_1}^{-1}(J_{X_1}x_n - \tau A^* J_{X_2}k_n) - x_n, \tau A^* J_{X_2}k_n)\rangle\\ \leq& 2\|J_{X_1}^{-1}(J_{X_1}x_n - \tau A^* J_{X_2}k_n) - J_{X_1}^{-1}(J_{X_1}x_n)\|\|\tau A^* J_{X_2}k_n\|\\ \leq& \frac{4\tau^2}{c^2}\|A^* J_{X_2}k_n\|^2 \leq \frac{4\tau^2}{c^2}\|A\|^2\|Ax_n - P_D Ax_n\|^2 \to 0 \quad as\ n\to\infty.\end{aligned} \tag{117}$$

Then using Lemma 2.1 of [2], we obtain

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{118}$$

From (116), Lemma 2.2 [2] and our assumptions, it implied that $\phi(w_n, v_{n+1}) \leq \alpha_n\phi(w_n, x_n) + (1-\alpha_n)\phi(w_n, w_n) \to 0$ as $n\to\infty$. Therefore,

$$\lim_{n\to\infty}\|v_{n+1} - w_n\| = 0. \tag{119}$$

Let $r_2 = \sup_n\{\|v_n\|, \|x_n\|\}$. Therefore, by Lemma 2.6 of [2], there exists a continuous, convex and strictly increasing function $g_2 : [0, 2r_2] \longrightarrow [0, \infty)$ such that $g_2(0) = 0$ and $g_2(\|x_n - v_n\|) \leq \phi(x_n, v_n)$. Then, it follows from (2), Lemma 2.8 of [2] and the fact that $x_n = K_{r_n}v_n$, we conclude that

$$g_2(\|x_n - v_n\|) \leq \phi(x_n, v_n) \leq \phi(\hat{u}, v_n) - \phi(\hat{u}, x_n) \leq \phi(\hat{u}, x_{n-1}) - \phi(\hat{u}, x_n) \to 0 \quad as\ \ n\to\infty.$$

Therefore

$$\lim_{n\to\infty} \|x_n - v_n\| = 0, \tag{120}$$

because g_2 is a continuous strictly increasing convex function. Now, by (119), we have $\lim_{n\to\infty} \|w_n - x_{n+1}\| = 0$. From (115), we obtain that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. This shows that $\{x_n\}$ is a Cauchy sequence, so $\{x_n\}$ converges strongly to a point $q \in C$. Therefore, by (118), we imply that $\{z_n\}$ converge strongly to q.

Next, we prove that $q \in F(Q_\iota^M)$. It follows from (111) and uniformly continuity of J_{X_1} on bounded subset of X_1 that $J_{X_1}Q_\iota^M z_n - J_{X_1}z_n \to 0$ as $n \to \infty$. Hence by Definition 2.9 of [2], we have $J_{X_1}Q_\iota^M z_n + \iota M Q_\iota^M z_n = J_{X_1}z_n$. Therefore, there exists $h_n \in MQ_\iota^M z_n$ such that $h_n = \frac{J_{X_1}z_n - J_{X_1}Q_\iota^M z_n}{\iota}$. So, by the above observation, $h_n \to 0$ as $n \to \infty$. On the other hand, Since $z_n \rightharpoonup q$, we can concluded that $Q_\iota^M z_n \rightharpoonup q$. Then, from Lemma 2.10 of [2], $0 \in M_1 q$, i.e., $q \in M^{-1}0 = F(Q_\iota^M)$. Now, we prove that $q \in EP(g)$. Since J_{X_1} is uniformly norm-to-norm continuous on bounded sets, it follows from (120) that

$$\lim_{n\to\infty} \|J_{X_1}x_n - J_{X_1}v_n\| = 0. \tag{121}$$

By $x_n = K_{r_n}v_n$, we conclude that $g(x_n, y) + \langle \psi x_n, y - x_n\rangle + \frac{1}{r_n}\langle y - x_n, J_{X_1}x_n - J_{X_1}v_n\rangle \geq 0$ for all $y \in C$. Moreover, by the condition A2, $g(y, x_n) \leq -g(x_n, y)$ for all $y \in C$. Therefore,

$$g(y, x_n) \leq \langle \psi x_n, y - x_n\rangle + \frac{1}{r_n}\langle y - x_n, J_{X_1}x_n - J_{X_1}v_n\rangle,$$

for all $y \in C$. Using (121), the condition A4 and Letting $n \to \infty$, we have that

$$g(y, q) \leq \langle \psi q, y - q\rangle, \tag{122}$$

for all $y \in C$. Let $y_\lambda = \lambda y + (1-\lambda)q$ for all $y \in C$ and $\lambda \in (0, 1)$. It follows from (122), the conditions A1, A4 and the monotonicity of B that

$$\begin{aligned} 0 =& g(y_\lambda, y_\lambda) + \langle \psi y_\lambda, y_\lambda - y_\lambda\rangle \leq \lambda g(y_\lambda, y) + (1-\lambda)g(y_\lambda, q) + \lambda\langle \psi y_\lambda, y - y_\lambda\rangle \\ &+(1-\lambda)\langle \psi y_\lambda - \psi q, q - y_\lambda\rangle + (1-\lambda)\langle \psi q, q - y_\lambda\rangle \leq \lambda g(y_\lambda, y) + \lambda\langle \psi y_\lambda, y - y_\lambda\rangle, \end{aligned}$$

for all $y \in C$. Therefore, $0 \leq g(y_\lambda, y) + \langle \psi y_\lambda, y - y_\lambda\rangle$. Using the condition A3 and letting $\lambda \to 0$, we obtain that $0 \leq g(q, y) + \langle \psi q, y - q\rangle$ for all $y \in C$. Then $q \in EP(g)$.

Finally, we prove that $q \in \Omega$. From (113), we have that $\|P_D Aq - Aq\| = \lim_{n\to\infty} \|P_D Au_n - Au_n\| = 0$. Therefore, $Aq \in D$, i.e., $q \in \Omega$. Hence, $q = \Pi_{\Omega\cap(F(Q_\iota^M)} \circ q$. □

References

[1] R. P. Agarwal, D. O'Regan, and D. R. Sahu, *Fixed point theory for Lipschitzian-type mappings with applications*, vol. 6, Springer, New York, 2009.

[2] M. Ghadampour, *Extragradient Methods for Solving Split Feasibility Problem and General Equilibrium Problem and Resolvent Operators in Banach Spaces*, Journal of Function Spaces, 2022, https://doi.org/10.1155/2022/4035695

[3] F. Kohsaka, W. Takahashi,*Strong convergence of an iterative sequence for maximal monotone operators in a Banach space*, Abstr Appl Anal, **3**, 239-249, 2004.

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Φ-Connes amenability of l^1-Munn algebras

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Abstract

Generalizing the notion of Connes amenability, we study Φ-Connes amenability of $\mathcal{LM}(\mathcal{A}, P, m, n)$ that Φ is a character on $\mathcal{LM}(\mathcal{A}, P, m, n)$. Among the other things, we study when Φ-Connes amenability of $\mathcal{LM}(\mathcal{A}, P, m, n)$ is equivalent to ϕ-Connes amenablity of $\mathcal{A}$ that ϕ is the unique character on $\mathcal{A}$ associated to Φ. We apply this results to semigroup algebras.

Keywords: Amenability, Banach algebras, Derivation, Group algebra, Invariant mean
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1 Introduction

Let $\mathcal{A}$ be a Banach algebra. A Banach algebra $\mathcal{A}$ is said to be dual if there is a closed submodule $\mathcal{A}_*$ of $\mathcal{A}^*$ such that $\mathcal{A} = (\mathcal{A}_*)^*$. A dual Banach $\mathcal{A}$-bimodule E is called normal Banach $\mathcal{A}$-bimodule if for each $x \in E$, the module maps $a \to a.x$ and $a \to x.a$ $(a \in \mathcal{A})$ are weak*-continuous. Dual Banach algebra $\mathcal{A}$ is called Connes amenable, if for every normal Banach $\mathcal{A}$-bimodule E, every weak*-continuous derivation $D : \mathcal{A} \to E$ is inner. Runde showed that G is amenable if and only if $M(G)$ is Connes amenable. In particular, $l^1(G)$ is amenable if and only if $l^1(G)$ is Connes amenable.

In [1], Eslamzadeh introduced l^1-Munn algebras. He used these algebras to characterize amenable semigroup algebras. Recently it is shown that bounded Hochschild (co)-homology of l^1 -Munn Banach algebras are isomorphic to those of the underlying Banach algebra $\mathcal{A}$ when the related sandwich matrix is invertible over $\mathrm{Inv}(\mathcal{A})$ [4] and in [4] Soroushmehr investigated weak amenability of certain classes of commutative semigroup algebras.

In [2], the authors have introduced the ϕ-version of Connes amenability of dual Banach algebra $\mathcal{A}$ that ϕ is a homomorphism from $\mathcal{A}$ onto $\mathbb{C}$ that lies in predual $\mathcal{A}_*$. We study the Runde s theorem for the case of semigroup algebra. We do further study of the Connes amenability of l^1-Munn algebra, in particular their σwc-virtual diagonal [3]. In this paper, we study ϕ-Connes amenability of l^1-Munn algebras. We use the l^1-Munn algebras to study of ϕ-Connes amenability of semigroup algebras. In order to do this, we follow the argument of [1] and [2].

2 Main results

Let $\mathcal{A}$ be a unital Banach algebra, let I and J be nonempty sets and $P = (p_{ij}) \in M_{J\times I}(\mathcal{A})$ be such that $\|P\|_\infty = sup\{\|p_{ji}\| : j \in J, i \in I\} \leq 1$. The set $M_{I\times J}(\mathcal{A})$ of all $I \times J$ matrices $a = (a_{ij})$ on $\mathcal{A}$ with l^1-norm and the product $A \odot B = APB, (A, B \in M_{I\times J}(\mathcal{A}))$ is a Banach algebra that is called l^1-Munn algebra on $\mathcal{A}$ with sandwich matrix P. It is denoted by $\mathcal{LM}(\mathcal{A}, P, I, J)$ [1]. If index sets I and J are finite with $|I| = m$ and $|J| = n$, then we use the notation $\mathcal{LM}(\mathcal{A}, P, m, n)$. Also ξ_{ij} is denoted the element of $\mathcal{M}_{I\times J}(\mathbb{C})$ with 1 in (i, j)th place and 0 elsewhere.

If I and J are finite, then $\mathcal{LM}(\mathcal{A}, P, I, J)^{**}$ is topologically algebra isomorphic to $\mathcal{LM}(\mathcal{A}^{**}, P, I, J)$. We define $\Gamma : M_{I\times J}(\mathcal{A}_*) \to M_{I\times J}(\mathcal{A})$ by $\langle(\Gamma(f))_{ij}, a\xi_{ij}\rangle \to \langle(f_{ij}), a\xi_{ij}\rangle$, then $M_{I\times J}(\mathcal{A})$ is a dual space with predual $M_{I\times J}(\mathcal{A}_*)$. It is clear that multiplication in $\mathcal{LM}(\mathcal{A}, P, I, J)$ is separately weak*-continuous and then $\mathcal{LM}(\mathcal{A}, P, I, J)$ is a dual Banach algebra. The character space of $\mathcal{A}$ is denoted by $\Delta(\mathcal{A})$. We may write

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$\Delta(\mathcal{LM}(\mathcal{A},P,I,J)) \subset \Delta(\mathcal{A})$. In [6], it is shown that if $\Phi \in \Delta(\mathcal{LM}(\mathcal{A},P,I,J))$ and $P=(p_{ji})$ be a sandwich matrix such that $\{p_{ji}; i \in I, j \in J\} \cap Inv(\mathcal{A}) \neq \emptyset$ and $p_{j_0i_0} \in \{p_{ji}; i \in I, j \in J\} \cap Inv(\mathcal{A})$, then $\phi \in \Delta(\mathcal{A})$ with $\phi(a) = \Phi(p^{-1}_{j_0i_0} a \xi_{i_0j_0})$ is a unique character on $\mathcal{A}$ that $\phi(p_{ji}p_{lk}) = \phi(p_{jk}p_{li})(j,l \in J, i,k \in I)$ and for each $N=(n_{ij}) \in \mathcal{LM}(\mathcal{A},P,I,J)$, Φ is defined by $\Phi(N) = \Sigma_{i\in I j\in J}\phi(n_{ij})\phi(p_{ji})$.

Let $\mathcal{A}$ be a Banach algebra. Let $\mathcal{X}$ be a subspace of $\mathcal{A}^*$ and ϕ be a character, that is a homomorphism from $\mathcal{A}$ onto $\mathbb{C}$ and lies in $\mathcal{X}$. A linear functional $m \in \mathcal{X}^{**}$ is called a mean on $\mathcal{X}$ if $\langle m, \phi\rangle = 1$ and $\langle m, f.a\rangle = \phi(a)\langle m, f\rangle$ for all $a \in \mathcal{A}$ and $f \in \mathcal{X}$. Lau, Kanuith and Pym have introduced the concept of ϕ-amenability of Banach algebra $\mathcal{A}$. A Banach algebra $\mathcal{A}$ is called ϕ-amenable if there exists a left invariant ϕ-mean on $\mathcal{A}^*$. The concept of ϕ-amenability of Banach algebras is characterized in terms of cohomological properties of Banach algebras. A Banach algebra $\mathcal{A}$ is ϕ-amenable if and only if for any Banach $\mathcal{A}$-bimodule $\mathcal{X}$, with left action $a.x = \phi(a)x (a \in \mathcal{A}, x \in X)$, each derivation $D : \mathcal{A} \to \mathcal{X}^*$ is inner. In [2], we introduce the concept of ϕ-Connes amenability of dual Banach algebras. A dual Banach algebra $\mathcal{A}$ is called ϕ-Connes amenable if for every normal Banach $\mathcal{A}$-bimodule $\mathcal{X}$, with left action $a.x = \phi(a)x (a \in \mathcal{A}, x \in X)$, every bounded weak*-continuous derivation $D : \mathcal{A} \to \mathcal{X}$ is inner. We show that dual Banach algebra $\mathcal{A}$ with predual $\mathcal{A}_*$ is ϕ-Connes amenable if and only if there exists a left invariant ϕ-mean on $\mathcal{A}_*$ [Theorem 2.1 [2]]. Our first result is a generalization of Theorem 2.1 in [3].

Theorem 2.1. *Let $\mathcal{LM}(\mathcal{A},P,m,n)$ be a l^1-Munn algebra on Banach algebra $\mathcal{A}$ with sandwich matrix P such that $\{p_{ji}; 1 \leq i \leq m, 1 \leq j \leq n\} \cap Inv(\mathcal{A}) \neq \emptyset$. Let Φ be a character on $\mathcal{LM}(\mathcal{A},P,m,n)$ that $\Phi \in \Delta(\mathcal{LM}(\mathcal{A},P,m,n)) \cap \mathcal{LM}(\mathcal{A}_*,P,m,n)$ with $\phi(p_{ji}p_{lk}) = \phi(p_{jk}p_{li})(1 \leq j,l \leq n, 1 \leq i,k \leq m)$ and $\Phi(N) = \Sigma_{i=1}^m \Sigma_{j=1}^n \phi(n_{ij})\phi(p_{ji})$ for each $N=(n_{ij}) \in \mathcal{LM}(\mathcal{A},P,m,n)$. Then:*
(i) If $\mathcal{LM}(\mathcal{A},P,m,n)$ is Φ-Connes amenable, then $\mathcal{A}$ is ϕ-Connes amenable .
(ii) If $\mathcal{A}$ is ϕ-Connes amenable, then there exist finite index I and J with $|I| = m$ and $|J| = n$ such that l^1-Munn algebra $\mathcal{LM}(\mathcal{A},P,m,n)$ is Φ-Connes amenable.

Theorem 2.2. *Let $\mathcal{LM}(\mathcal{A},P,m,n)$ be a l^1-Munn algebra on Banach algebra $\mathcal{A}$ with sandwich matrix P such that $\{p_{ji}; 1 \leq i \leq m, 1 \leq j \leq n\} \cap Inv(\mathcal{A}) \neq \emptyset$. Let $p_{j_0i_0} \in \{p_{ji}; 1 \leq i \leq m, 1 \leq j \leq n\} \cap Inv(\mathcal{A})$. Let Φ be a character on $\mathcal{LM}(\mathcal{A},P,m,n)$ that $\Phi \in \Delta(\mathcal{LM}(\mathcal{A},P,m,n)) \cap \mathcal{LM}(\mathcal{A}_*,P,m,n)$ with $\phi(p_{ji}p_{lk}) = \phi(p_{jk}p_{li})(1 \leq j,l \leq n, 1 \leq i,k \leq m)$ and*

$$\Phi(N) = \Sigma_{i=1}^m \Sigma_{j=1}^n \phi(n_{ij})\phi(p_{ji})$$

for each $N=(n_{ij}) \in \mathcal{LM}(\mathcal{A},P,m,n)$. Let $\mathcal{A}$ be ϕ-Connes amenable and let E be a normal $\mathcal{A}$-bimodule with left module action

$$[a_{ij}].x = \Phi([a_{ij}])x \quad ([a_{ij}] \in \mathcal{LM}(\mathcal{A},P,m,n), x \in E).$$

Then for every bounded weak*-continuous derivation $\tilde{D} : \mathcal{LM}(\mathcal{A},P,m,n) \to E$, the restriction map $\tilde{D}|_{\mathcal{A}\xi_{i_0j_0}}$ is inner.

Like Connes amenability, the notion of a virtual diagonal adapts naturally to the context of general dual Banach algebras. Let E be a Banach $\mathcal{A}$-bimodule. An element $x \in E$ is called weak*-weakly continuous if the module maps $a \to a.x$ and $a \to x.a$ $(a \in \mathcal{A})$ are weak*-weak continuous. The collection of all weak*-weakly continuous elements of E is denoted by $\sigma wc(E)$. It is shown that, $\sigma wc(E)^*$ is normal. Let $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ be the multiplication map. Then π^* maps $\mathcal{A}_*$ into $\sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$. Consequently, π^{**} drops to a homomorphism $\pi_{\sigma wc} : \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^* \to \mathcal{A}$. An element $M \in \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ is called a σwc-virtual diagonal for $\mathcal{A}$, if $M.u = u.M$ and $u.\pi_{\sigma wc}(M) = u$ for every $u \in \mathcal{A}$. Runde showed that $\mathcal{A}$ is Connes amenable if and only if there is a σwc-virtual diagonal for $\mathcal{A}$. An element $M \in \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ is called a ϕ-σwc-virtual diagonal for $\mathcal{A}$, if $M.u = \phi(u)M$ and $\langle \phi, \pi_{\sigma wc}(M)\rangle = 1$ for every $u \in \mathcal{A}$. It is shown that dual Banach algbra $\mathcal{A}$ is ϕ-Connes amenable if and only if there exists a ϕ-σwc-virtual diagonal for $\mathcal{A}$ [2].

Theorem 2.3. *Let $\mathcal{A}$ be a Banach algebra. Let $\mathcal{LM}(\mathcal{A},P,I,J)$ be a unital dual Banach algebra. Then if $\mathcal{LM}(\mathcal{A},P,I,J)$ has a σwc-diagonal, then $\mathcal{A}$ has a σwc-diagonal*

In Theorem 2.4, sandwich matrix P is invertible and Munn algebra $\mathcal{LM}(\mathcal{A},P,I,J)$ is isomorphic to the matrix algebra $M_n(\mathcal{A})$. It is known that $M_n(\mathcal{A})$ does not have any character when $n > 1$.

Corollary 2.4. *Let $\mathcal{A}$ be a Banach algebra. Let $\mathcal{LM}(\mathcal{A}, P, I, J)$ be a unital dual Banach algebra and let $\mathcal{LM}(\mathcal{A}, P, I, J)$ has a σwc-diagonal, then for each $\phi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$, $\mathcal{A}$ has a ϕ-σwc-diagonal*

Eslamzadeh showed that if P is regular matrix, a nonzero functional $\Phi \in \mathcal{LM}(\mathcal{A}, P, I, J)^*$ is a character if and only if there is a unique non-zero character ϕ of $\mathcal{A}$ associated to Φ via $\phi(a) = \Phi(p_{j_0 i_0}^{-1} a \xi_{j_0 i_0})$. Also it is shown that if P is regular matrix and has a zero entry, then $\mathcal{LM}(\mathcal{A}, P, I, J)$ has no nonzero character.

Example 2.5. let G be a group and consider l^1-Munn algebra $\mathcal{LM}(l^1(G), P, 2, 2)$. Then the three possible forms of regular sandwich matrices P in $\mathcal{LM}(l^1(G), P, 2, 2)$ are

$$P_1 = \begin{pmatrix} \delta_e & 0 \\ 0 & \delta_e \end{pmatrix} \; P_2 = \begin{pmatrix} \delta_e & 0 \\ \delta_e & \delta_e \end{pmatrix} \; P_3 = \begin{pmatrix} \delta_e & \delta_e \\ \delta_e & \delta_e \end{pmatrix}$$

Two matrices P_1, P_2 are invertible and P_3 is not invertible. Then there is not any nonzero character on Munn algebras $\mathcal{LM}(l^1(G), P_1, 2, 2)$ and $\mathcal{LM}(l^1(G), P_2, 2, 2)$. Since P_3 is regular matrix that has not zero entry, then each character Φ on $\mathcal{LM}(l^1(G), P, 2, 2)$ is the form

$$\Phi([a_{ij}]) = \Sigma_{i=1}^{2} \Sigma_{j=1}^{2} \phi(a_{ij}) \phi(p_{ji}) \quad (a_{ij} \in l^1(G))$$

For a semigroup S and $s \in S$, we define maps $L_s, R_s : S \to S$ by $L_s(t) = st, \; R_s(t) = ts, \; t \in S$. If for each $s \in S$, R_s and L_s are finite-to-one maps, then we say that S is weakly cancellative. Before turning our results, we note that if S is a weakly cancellative semigroup, then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$.

Let G be a group, I and J be arbitrary nonempty sets and $G^0 = G \cup \{0\}$. Let $P_G = (a_{ij}) \in M_{J \times I}(G)$. For $x \in G$, let $(x)_{ij}$ be the element of $M_{I \times J}(G^0)$ with x in $(i, j)^{th}$ place and 0 elsewhere. As a set, S consists of collection of all these matrices $(x)_{ij}$. Multiplication in S is given by the formula

$$(x)_{ij}(y)_{kl} = (x a_{jk} y)_{il} \quad (x, y \in G, i, k \in I, j, l \in J).$$

We write $S = \mathcal{M}(G, P, I, J)$. S is called Rees matrix semigroup with sandwich matrix P. Let G be a locally compact group and consider group algebra $l^1(G)$. In the case discrete group G and $\phi \in \Delta(l^1(G)) \cap c_0(G)$, we show that there is a left invariant ϕ-mean on predual $c_0(G)$ if and only if there is a left invariant ϕ-mean on $l^1(G)^*$ [2]. In the following we study this for Rees matrix semigroup algebras.

Theorem 2.6. *Let G be a group and $S = \mathcal{M}(G, P, m, n)$ be a weakly cancellative Rees matrix semigroup with a zero over G. Then $\Phi \in \triangle(l^1(S)) \cap c_0(S)$ if and only if there is a character $\phi \in \triangle(l^1(G)) \cap c_0(G)$ such that for each $f \in l^1(S)$,*

$$\Phi(f) = \Sigma_{i=1}^{m} \Sigma_{j=1}^{n} \phi(f_{ij}) \phi(p_{ji})$$

and

$$\phi(p_{ji} p_{lk}) = \phi(p_{li} p_{jk}) \quad (1 \leq j, l \leq n, 1 \leq i, k \leq m).$$

Theorem 2.7. *Let G be a group. Let $S = \mathcal{M}(G, P, m, n)$ be a weakly cancellative Rees matrix semigroup with a zero over G. If $\Phi \in \triangle(l^1(S)) \cap c_0(S)$ and $l^1(S)$ is Φ-Connes amenable, then there is a character $\phi \in \triangle(l^1(G)) \cap c_0(G)$ such that $l^1(G)$ is ϕ-Connes amenable.*

Theorem 2.8. *Let G be a group. The following are equivalent:*
(i) There exists finite index I and J such that for semigroup $S = \mathcal{M}(G, P, I, J)$, there exists a left invariant Φ-mean on $l^1(S)^*$.
(ii) There exists finite index I and J such that for semigroup $S = \mathcal{M}(G, P, I, J)$, there exists a left invariant Φ-mean on $c_0(S)$.

References

[1] G. H. Esslamzadeh, *Banach algebra structure and amenability of a class of matrix algebras with applications,* J. Funct. Anal., **161** (1999), 364-383.

[2] A. Ghaffari, S. Javadi, *φ-Connes amenability of dual Banach algebras,* Bull. Iran. Math. Soc., **43** (2017), 25-39.

[3] A. Ghaffari, S. Javadi and E. tamimi, *Connes amenability of l^1-Munn algebras,* Tamkang Journal of Mathematics, **53** (2022), 259-266.

[4] F. Habibian, N. Mirzasani and R. Nasr Isfahani, *Homological properties of l^1-Munn algebras,* Revista de Real Academia deiencias Exactas, Fisicasy Naturales. Seri A. Matematicas., **62** (2021), 1-14.

[5] M. Soroushmehr, *Homomorphisms of l^1-Munn algebras and applications to semigroup algebras,* Bull. Austral. Math. Soc., **92** (2015), 115-122.

[6] M. Soroushmehr, *A note on weak amenability of semigroup algebras,* Semigroup forum, **102** (2021), 543-522.

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Weak*-continuous derivations on l^1-Munn algebras and their application to semigroup algebras

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Abstract

In this talk, we study Connes amenability of l^1-Munn algebras. We apply these results to semigroup algebras. We show that for a weakly cancellative semigroup S with finite idempotents, amenability and Connes amenability are equivalent.

Keywords: Amenability, Banach algebra, Munn algebras, semigroup algebras.
Mathematics Subject Classification [2023]: 18A32, 18F20, 05C65

1 Introduction

In [1], Eslamzadeh introduced l^1-Munn algebras. He used these algebras to characterize amenable semigroup algebras. A special case of these algebras was introduced by Munn [6]. l^1 Munn algebras has been studied in some texts. Eslamzadeh in [1] investigated the structure of l^1-Munn algebras. Duncan and Paterson used the l^1-Munn algebras to study of semigroup algebras of completely simple semigroups.

First, amenability appeared in connection with paradoxical decompositions. The BanachTarski paradox shows that the sphere can be partitioned into finitely many sets which may be rotated and reassembled to form two copies of the original sphere. It follows that there is no finitely additive measure on the family of all subsets of the sphere which extends the familiar area measure on Borel sets.

In 1972, Barry Johnson introduced the cohomological notion of an amenable Banach algebra. Amenable Banach algebras have since proved themselves to be widely applicable in modern analysis. In many instances the classical concept of amenability is, however, too strong. For this reason, by relaxing some of the constrains in the definition of amenability, new concepts have been introduced. B. E. Johnson, R. V. Kadison and J. Ringrose introduced a notion of amenability for Von Neumann algebras which modifies Johnson's original definition for general Banach algebras. This notion of amenability was later dubbed Connes amenability.

The motivation to study of the theory of amenable von Neumann algebras stems from the fact that they are dual. It is shown that if A is a von Neumann algebra containing a weak*-dense amenable C*-subalgebra, then for every normal Banach A-bimodule E, every weak*-continuous derivation D : AE is inner. The investigation of Connes amenability for dual Banach algebras which are not von Neumann algebra is interesting for many authors, see [2] and [4]. In this talk, we study Connes amenability of l^1-Munn algebras. We use the l^1-Munn algebras to study of Connes amenability of semigroup algebras of weakly cancellative semigroups. In order to do this, we follow the argument of [1], [3] and [6].

2 Main results

Let A be a dual Banach algebra. A derivation from A to a bimodule E is a linear map $D : A \mapsto E$ satisfying the Leibniz rule $D(ab) = D(a).b + a.D(b)$ $(a, b \in A)$. Let $x \in E$. The map $D_x : A \mapsto E$ is an inner deriviation if $D_x(a) = a.x - x.a$ $(a \in A)$. A is called amenable, if for every normal Banach A-bimodule E, every continuous derivation $D : A \mapsto E^*$ is inner. Now let $\Delta(A)$ be the set of all homomorphisms from A

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onto $\mathbb{C}$ and $\phi \in \Delta(A)$. A is called ϕ-amenable if for any Banach A-bimodule E with left action of the form $a.x = \phi(a)x$ for all $a \in A, x \in E$ every continuous derivation from A into the A-bimodule E^* is an inner derivation.

Let A be a dual Banach algebra with predual A_*. A dual Banach A-bimodule E is called normal Banach A-bimodule if for each $x \in E$, the maps $a \mapsto a.x, a \mapsto x.a$ $(a \in A)$ are weak*-continuous. A is called Connes amenable, if for every normal Banach A-bimodule E, every weak*- continuous derivation $D : A \mapsto E$ is inner. Let E be a Banach A-bimodule. An element $x \in E$ is called weak*-weakly continuous if the module maps $a \mapsto a.x, a \mapsto x.a$ $(a \in A)$ are weak*-weak continuous. The collection of all weak*-weakly continuous elements of E is denoted by $\sigma wc(E)$. It is shown that $\sigma wc(E)^*$ is normal [4]. Let $\pi : A \otimes A \mapsto A$ be the multiplication map. From Corollary 4.6 in [4], π^* maps A_* into $\sigma wc(A \otimes A)^*$. Consequently, π^{**} drops to a homomorphism $\pi_{\sigma wc}$. An element $M \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ is called a σwc-virtual diagonal for A, if $M.u = u.M, u.\pi_{\sigma wc}(M) = u$ for every $u \in A$. In [4], Runde showed that A is Connes amenable if and only if there is a σwc-virtual diagonal for A.

Let A be a unital Banach algebra, let I and J be nonempty sets and $P = (p_{ij}) \in M_{I\times J}(A)$ be such that $\|P\|_\infty = \{sup\|p_{ji}\| : j \in J, i \in I\} \leq 1$. The set $M_{I\times J}(A)$ of all $I \times J$ matrices $a = (a_{ij})$ on A with l^1-norm and the product $A \odot B = APB(A, B \in M_{I\times J}(A))$ is a Banach algebra that is called l^1-Munn algebra on A with sandwich matrix P. It is denoted by $LM(A, P, I, J)$ [1]. Also ξ_{ij} is denoted the element of $M_{I\times J}(C)$ with 1 in (i, j) th place and 0 elsewhere. Throughout we use the notations of [1]. We define

$$\Gamma : M_{I\times J}(A_*) \mapsto M_{I\times J}(A), \langle \Gamma(f)_{ij}, a\xi_{ij}\rangle = \langle (f_{ij}), a\xi_{ij}\rangle.$$

Then $M_{I\times J}(A)$ is a dual space with predual $M_{I\times J}(A_*)$. It is clear that above multiplication in $LM(A, P, I, J)$ is separately weak*-continuous and $LM(A, P, I, J)$ is a dual Banach algebra.

Theorem 2.1. *Let $\mathcal{A}$ be a unital dual Banach algebra. The followings are equivalent:*
(i) $\mathcal{LM}(\mathcal{A}, P, I, J)$ is Connes amenable.
(ii) $\mathcal{A}$ is Connes amenable, I and J are finite and P is invertible.

We apply these results to semigroup algebra $l^1(S)$. Recall that

$$l^1(S) = \{(a_g)_{g\in S} : \|a_g\| = \Sigma_{g\in S}|a_g| < \infty\}.$$

Then each $x \in l^1(S)$ can be written uniquely $x = \Sigma_{g\in S}x_g\delta_g$ for some family $(x_g) \subseteq \mathbb{C}$ such that $\|x_g\| = \Sigma_{g\in S}|x_g| < \infty$. For a semigroup S and $s \in S$, we define maps $L_s, R_s : S \to S$ by $L_s(t) = st,\ R_s(t) = ts,\ (t \in S)$. If for each $s \in S$, R_s and L_s are finite-to-one maps, then we say that S is weakly cancellative. Before turning our results, we note that if S is a weakly cancellative semigroup, then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$. It is known that $(l^1(S)\widehat{\otimes}l^1(S))' = B(l^1(S), l^\infty(S)) = l^1(S \times S)' = l^\infty(S \times S)$, where $T \in B(l^1(S), l^\infty(S))$ is identified with $T \in l^\infty(S \times S)$, where $T(s, t) = \langle T(\delta_s), \delta_t\rangle$. Let G be a group, I and J be arbitrary nonempty sets and $G^0 = G \cup \{0\}$. Let $P_G = (a_{ij}) \in M_{J\times I}(G)$. For $x \in G$, let $(x)_{ij}$ be the element of $M_{I\times J}(G^0)$ with x in $(i, j)^{th}$ place and 0 elsewhere. The set of all $(x)_{ij}$ matrices is denoted by S. Multiplication in S is given by the formula

$$(x)_{ij}(y)_{kl} = (xa_{jk}y)_{il} \quad (x, y \in G, i, k \in I, j, l \in J).$$

We write $S = \mathcal{M}(G, P, I, J)$. S is called Rees matrix semigroup with sandwich matrix P. It is known that S is a completely simple semigroup and each completely simple semigroup is isomorphic to one constructed in this manner. Similarly, we have the semigroup $\mathcal{M}^0(G, P, I, J)$ where the elements of this semigroup are those of $\mathcal{M}(G, P, I, J)$, togheter with the element 0 so that 0 is a matrix with 0 everywhere and $P_G = (a_{ij}) \in M_{J\times I}(G^0)$.

Theorem 2.2. *Let S be a weakly cancellative semigroup and let $l^1(S)$ be unital with unit $e_{l^1(S)}$. If $l^1(S)$ is Connes amenable, then $E(S)$ is finite.*

Example 2.3. Let S be the natural numbers $\mathbb{N}$, with the product

$$(m, n) \to m \vee n = max\{m, n\}.$$

S is a semigroup with identity 1 and weakly cancellative. Clearly $E(S) = S$. Then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$ and $l^1(S)$ is not Connes amenable as $E(S)$ is infinite.

Remark 2.4. It is easy to see that each weakly cancellative semigroup is simple. In fact, suppose that $\mathcal{I}$ be a left ideal of S contaning a nonzero element i, then

$$S = Si \subseteq S\mathcal{I} \subseteq \mathcal{I}$$

and so $\mathcal{I} = S$. Consequently, if S is a weakly cancellative semigroup with $E(S)$ finite, then S is completely simple semigroup and Rees matrix semigroup of the form $S = \mathcal{M}(G, P, I, J)$.

Theorem 2.5. *Let S be a weakly cancellative semigroup and let $l^1(S)$ be unital with unit $e_{l^1(S)}$. If $l^1(S)$ is Connes amenable, then S is a Rees matrix semigroup of the form $S = \mathcal{M}(G, P, I, J)$, $\mathcal{L}M(l^1(G), P, I, J)$ has an identity and $l^1(G)$ is Connes amenable.*

Theorem 2.6. *Let S be a weakly cancellative semigroup with $E(S)$ finite and let $l^1(S)$ be unital with unit $e_{l^1(S)}$. Then S is a Rees matrix semigroup of the form $S = \mathcal{M}(G, P, I, J)$. With above notation, the followings are equivalent:*

(i) $l^1(S)$ is Connes amenable.

(ii) $\mathcal{L}M(l^1(G), P, I, J)$ has an identity and $l^1(G)$ is Connes amenable.

(iii) $l^1(S)$ is amenable.

Theorem 2.7. *Let $S = \mathcal{M}(G, P, I, J)$ be a weakly cancellative semigroup. Let $l^1(S)$ be unital with unit $e_{l^1(S)}$. With above notation, the followings are equivalent:*
(i) $l^1(S)$ is Connes amenable.
(ii) $l^1(S)$ is amenable.

Example 2.8. Let G be an amenable group. Let J be finite of order n. Let $S = \mathcal{M}(G, P, 1, J)$ where P is invertible. Let $(a)_{1j}$ and $(b)_{1l}$ be two non-zero elements of $S = \mathcal{M}(G, P, 1, J)$. It is easy to see that

$$(a)_{1j}(p_{j1}{}^{-1}a^{-1}b)_{1l} = (b)_{1l}.$$

Then S is a weakly cancellative semigroup, and so $l^1(G)$ is amenable. By Proposition 5.6 and Theorem 4.1 in [1], $l^1(S)$ is amenable. Also $l^1(G)$ is Connes amenable and by Theorem 2.1, $l^1(S)$ is Connes amenable.

Most of this talk is taken up with consideration of certain classes of semigroup algebra $l^1(S)$ and the problem of determining when Connes amenability and amenability are equivalent. The following is a generalisation to the case of character amenability.

Theorem 2.9. *Let S be a weakly cancellative semigroup, and let $l^1(S)$ be unital with unit $e_{l^1(S)}$. Then $l^1(S)$ is character Connes amenable if and only if $l^1(S)$ is character amenable.*

3 Conclusion

For a unital weakly cancellative semigroup algebra $l^1(S)$ character Connes amenability and character amenability are equivalent. In the case S is a Rees matrix semigroup of the form $S = M(G, I, J, P)$, Connes amenability and amenability are equivalent.

References

[1] G. H. Esslamzadeh, *Banach algebra structure and amenability of a class of matrix algebras with applications,* J. Funct. Anal., (1999), no. 161, 364383.

[2] A. Ghaffari and S. Javadi, *ϕ-Connes amenability of dual Banach algebras,* Bull. Iran. Math. Soc., (2017), no. 43, 2539.

[3] F. Habibian, N. Mirzasani and R. Nasr Isfahani, *Homological properties of l^1-Munn algebras, Revista de Real Academia deiencias Exactas,* Fisicasy Naturales. Seri A. Matematicas. (2021), no. 62, 1-14.

[4] W. D. Munn, *On semigroup algebras,* Math. Proc. Cambridge Philos. Soc., (1955), no. 51, 115.

[5] V. Runde, *Dual Banach algebras: Connes amenability, normal, virtual diagonals, and injectivity of the predual bimodule,* Math. Scand., (2004), no. 95, 124144.

[6] M. Soroushmehr *A note on weak amenability of semigroup algebras,* Semigroup Forum, (2021), no. 102, 543-52.

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More inequalities on the A-Davis-Wielandt Berezin number of semi Hilbert operators

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Abstract

In [3], we defined the concept of the A-Davis-Wielandt Berezin number for positive operator A. In this paper, we obtain Some upper and lower bounds for the A-Davis-Wielandt Berezin number. Moreover, some inequalities related to the concept of the Davis-Wielandt Berezin number are proved, which are generalizations of known results.

Keywords: A- Berezin number, A-Davis-Wielandt Berezin number, Davis-Wielandt Berezin number, Berezin number, 2×2 operator matrix, Semi Hilbert space.
Mathematics Subject Classification [2023]: Primary: 47A30, 47A12; Secondary 47A63, 47L05.

1 Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the $\mathcal{C}^*$-algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}; \langle \cdot, \cdot \rangle)$ with the identity operator $1_{\mathcal{H}}$ in $\mathcal{B}(\mathcal{H})$. Every where this paper, we suppose that $A \in \mathcal{B}(\mathcal{H})$ is positive operator. Recall that A is called positive, denoted by $A \geq 0$, if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. Such an operator A induces positive semidefinite sesquilinear form as follows:

$$\begin{aligned} \langle ., . \rangle_A : & \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C} \\ & (x, y) \mapsto \langle x, y \rangle_A = \langle Ax, y \rangle, \end{aligned}$$

and $\|x\|_A = \sqrt{\langle x, x \rangle_A}$, $x \in \mathcal{H}$, is the seminorm induced by the above sesquilinear form. This make $\mathcal{H}$ into a semi-Hilbertian space[5].
For $S \in \mathcal{B}(\mathcal{H})$, an operator $R \in \mathcal{B}(\mathcal{H})$ is called an A-adjoint of S if for every $x, y \in \mathcal{H}$, we have $\langle Sx, y \rangle_A = \langle x, Ry \rangle_A$, *i.e.*, $AR = S^*A$.
The set of all operator which admit A-adjoints is denoted by $\mathcal{B}_A(\mathcal{H})$. If $S \in \mathcal{B}_A(\mathcal{H})$, the reduced solution of the equation $AX = S^*A$ is a distinguished A-adjoints operator of S, which is denoted by $S^{\sharp_A}$. An operator $S \in \mathcal{B}(\mathcal{H})$ is called A-selfadjoint if AS is selfadjoint, i.e., $AS = S^*A$ and it is called A-positive if $AS \geq 0$. An operator $S \in \mathcal{B}(\mathcal{H})$ is said to be A-bounded if there exists $c > 0$ such that $\|Sx\|_A \leq c\|x\|_A$, for all $x \in \mathcal{H}$. We denote by $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ is the collection of all A-bounded operators. An operator $S \in \mathcal{B}_A(\mathcal{H})$ is called A-normal if $SS^{\sharp_A} = S^{\sharp_A}S$.
An operator $U \in \mathcal{B}_A(\mathcal{H})$ is called A-unitary if $\|Ux\|_A = \|U^{\sharp_A}x\|_A = \|x\|_A$ for all $x \in \mathcal{H}$.

The function k on $\Omega \times \Omega$ defined by $k(z, \mu) = k_\mu(z)$ is called the reproducing kernel of $\mathcal{H}$, see [4]. It was shown that $k_\mu(z)$ can be represented by

$$k_\mu(z) = \sum_{n=1}^{\infty} \overline{e_n(\mu)} e_n(z)$$

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for any orthonormal basis $\{e_n\}_{n\geq 1}$ of $\mathcal{H}$. Let $\widehat{k}_\mu = \frac{k_\mu}{\|k_\mu\|}$ be the normalized reproducing kernel of the space $\mathcal{H}$. For a given a bounded linear operator S on $\mathcal{H}$, the Berezin symbol(or Berezin transform) of S is the bounded function $\widetilde{S}$ on Ω defined by

$$\widetilde{S}(\mu) = \langle S\widehat{k}_\mu(z), \widehat{k}_\mu(z)\rangle, \ \mu \in \Omega.$$

The Berezin set, the Berezin number, the Berezin norm, the Crawford Berezin number, and the minimum Berezin modulus of a bounded linear operator S on $\mathcal{H}$ are defined, respectively, by

$$\mathrm{Ber}(S) = \left\{\widetilde{S}(\mu) : \mu \in \Omega\right\}, \ \text{and} \quad \mathrm{ber}(S) = \sup\{|\gamma| : \gamma \in \mathrm{Ber}(S)\}.$$

and

$$\|S\|_{\mathrm{Ber}} := \sup_{\mu\in\Omega} \left\|S\widehat{k}_\mu\right\|,$$

$$C_{\mathrm{Ber}}(S) := \inf\{|\widetilde{S}(\mu)| : \mu \in \Omega\}, \ \text{and} \quad m_{\mathrm{Ber}}(S) := \inf\{\|S\widehat{k_\mu}\| : \mu \in \Omega\}.$$

Also, the concepts Davis-Wielandt Berezin set and Davis-Wielandt Berezin number have been introduced in [1], $\mathrm{Ber}_{dw}(S) := \{(\langle S\widehat{k}_\mu, \widehat{k}_\mu\rangle, \langle S\widehat{k}_\mu, S\widehat{k}_\mu\rangle) \ : \ \mu \in \Omega\}$,
and $\mathrm{ber}_{dw}(S) := \sup_{\mu\in\Omega}\{\sqrt{|\langle S\widehat{k}_\mu, \widehat{k}_\mu\rangle|^2 + ||S\widehat{k}_\mu||^4}\}$.

We can clearly see that $\mathrm{ber}_{dw}(S)$ is a generalization of $\mathrm{ber}(S)$, and also $dw(S) \leq \mathrm{ber}_{dw}(S)$. The following properties of $\mathrm{ber}_{dw}(S)$, $S \in \mathbb{B}(\mathcal{H}(\Omega))$ are also known:
(i) $\mathrm{ber}_{dw}(\cdot)$ does not define a norm on $\mathcal{B}(\mathcal{H}(\Omega))$ but it is unitarily invariant, i.e.,
$\mathrm{ber}_{dw}(U^*SU) = \mathrm{ber}_{dw}(S)$ for any unitarily operator $U \in \mathcal{B}(\mathcal{H}(\Omega))$.
(ii)

$$\max\left(\mathrm{ber}(S) \ , \ \|S\|^2_{\mathrm{Ber}}\right) \leq \mathrm{ber}_{dw}(S) \leq \sqrt{\mathrm{ber}^2(S) + ||S||^4_{\mathrm{Ber}}}. \tag{123}$$

In [3], we defined the A-Berezin symbol(or A-Berezin transform) of $S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}(\Omega))$ and positive operator A, which is the bounded function $\widetilde{S}_A$ on $\mathcal{H}(\Omega)$, as follows:

$$\widetilde{S}_A(\mu) = \langle S\widehat{k}_\mu(z), \widehat{k}_\mu(z)\rangle_A, \ \mu \in \Omega.$$

Moreover, the A- Berezin set, the A-Berezin number, the A-Berezin norm, the A-Crawford Berezin number, and the A-minimum Berezin modulus of the operator $S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}(\Omega))$ introduced, respectively, as follows:

$$\mathrm{Ber}_A(S) := \left\{\widetilde{S}_A(\mu) : \mu \in \Omega\right\}, \ \text{and} \quad \mathrm{ber}_A(S) := \sup\{|\gamma| : \gamma \in \mathrm{Ber}_A(S)\}.$$

$$\|S\|_{A-\mathrm{Ber}} := \sup_{\mu\in\Omega} \left\|S\widehat{k}_\mu\right\|_A,$$

$$C_{A-\mathrm{Ber}}(S) := \inf\{|\widetilde{S}_A(\mu)| : \mu \in \Omega\}, \ \text{and} \quad m_{A-\mathrm{Ber}}(S) := \inf\{||S\widehat{k}_\mu||_A : \mu \in \Omega\}.$$

Also, we introduced concepts A-Davis-Wielandt Berezin set and A-Davis-Wielandt Berezin number for $S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}(\Omega))$, as follows:

$$\mathrm{Ber}_{A-dw}(S) = \left\{(\langle S\widehat{k}_\mu, \widehat{k}_\mu\rangle_A, \langle S\widehat{k}_\mu, S\widehat{k}_\mu\rangle_A), \mu \in \Omega\right\},$$

and

$$\mathrm{ber}_{A-dw}(S) = \sup_{\mu\in\Omega}\left\{\sqrt{|\langle S\widehat{k}_\mu, \widehat{k}_\mu\rangle_A|^2 + ||S\widehat{k}_\mu||_A^4}\right\}.$$

Clearly, $\mathrm{ber}_{A-dw}(S)$ is a generalization of $\mathrm{ber}_A(S)$. Also, $\mathrm{ber}_{A-dw}(\cdot)$ is weakly unitarily invariant. Also for the A-Davis-Wielandt Berezin number of operator $S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, we have

$$\max\left(\mathrm{ber}_A(S), ||S||_{A-\mathrm{Ber}}^2\right) \leq \mathrm{ber}_{A-dw}(S) \leq \sqrt{\mathrm{ber}_A^2(S) + ||S||_{A-\mathrm{Ber}}^4}. \tag{124}$$

We establish some new inequalities for the A- Davis–Wielandt Berezin number of Operator matrices. We also show some inequalities for the Davis-Wielandt Berezin number of bounded linear operators on $\mathcal{H}(\Omega)$.

2 Main results

Lemma 2.1. *[2]Let $s, t, e \in \mathcal{H}$ with $\|e\|_A = 1$. Then*

$$|\langle s, e\rangle_A\langle e, t\rangle_A| \leq \frac{1}{2}\left(|\langle s, t\rangle_A| + \|s\|_A\|t\|_A\right).$$

Theorem 2.2. *Suppose that $S \in \mathcal{B}_A(\mathcal{H}(\Omega))$. Then the following inequalities hold:*
(1) $\mathrm{ber}_{A-dw}^2(S) \leq \left\||S|_A^2 + (|S|_A^2)^{\sharp_A}|S|_A^2\right\|_{A-\mathrm{Ber}}$;
(2) $\mathrm{ber}_{A-dw}^2(S) \leq \frac{1}{2}\left(\mathrm{ber}_A(S^2) + \|S\|_{A-\mathrm{Ber}}^2\right) + \|S\|_{A-\mathrm{Ber}}^4$.

Lemma 2.3. *[2]Assume that $s, t, e \in \mathcal{H}$ with $\|e\|_A = 1$. Then*

$$\|s\|_A^2\|t\|_A^2 - |\langle s, t\rangle_A|^2 \geq 2|\langle s, e\rangle_A\langle e, t\rangle_A|(\|s\|_A\|t\|_A - |\langle s, t\rangle_A|).$$

Theorem 2.4. *Let $S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\begin{aligned}\mathrm{ber}_{A-dw}^2(S) \leq 3\left\|(|S|_A^2)^{\sharp_A}|S|_A^2 + |S|_A^2\right\|_{A-\mathrm{Ber}} &- C_{A-\mathrm{Ber}}(|S|_A^2 + S)m_{A-\mathrm{Ber}}(|S|_A^2 + S)\\ &- C_{A-\mathrm{Ber}}(|S|_A^2 - S)m_{A-\mathrm{Ber}}(|S|_A^2 - S).\end{aligned}$$

We determine the exact value of the A-Davis-Wieladt-Berezin number of some 2×2 operator matrices in $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}\bigoplus\mathcal{H})$.

Theorem 2.5. *Assume that $X \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, and $S = \begin{pmatrix} I & X \\ 0 & 0\end{pmatrix}$. Then*

$$\mathrm{ber}_{A-dw}(S) = \begin{cases} \sqrt{2} & \|\mathrm{X}\|_{\mathrm{A-Ber}} = 0 \\ (\cos\theta_0 + \|X\|_{A-\mathrm{Ber}}\sin\theta_0)\left(\cos\theta_0^2 + (\cos\theta_0 + \|X\|_{A-\mathrm{Ber}}\sin\theta_0)^2\right)^{\frac{1}{2}} & \|\mathrm{X}\|_{\mathrm{A-Ber}} \neq 0, \end{cases}$$

where
$\theta_0 = \arctan(\beta + \gamma - \frac{p}{3})$, $\alpha = \frac{1}{27}(2p^3 - 9pq + 27r)$, $\beta = (-\frac{\alpha}{2} + \sqrt{s})^{\frac{1}{3}}$, $\gamma = (-\frac{\alpha}{2} - \sqrt{s})^{\frac{1}{3}}$,
$s = \frac{1}{2^4 3^3 b^6}(8b^8 + 20b^6 + 45b^4 + 61b^2 + 28)$, $r = \frac{3}{2b}$, $q = -\frac{2b^2-2}{b^2}$, $p = -\frac{2b^2-5}{2b}$, *and* $b = \|X\|_{A-\mathrm{Ber}}$.

Theorem 2.6. *Suppose that $X, Y \in \mathcal{B}_A(\mathcal{H}(\Omega))$. Then*

$$\mathrm{ber}_{A-dw}(X + Y) \leq \mathrm{ber}_{A-dw}(X) + \mathrm{ber}_{A-dw}(T) + \mathrm{ber}_A((X^{\sharp_A}Y + Y^{\sharp_A}X)).$$

In particular, if $A(X^{\sharp_A}Y + Y^{\sharp_A}X) = 0$, then

$$\mathrm{ber}_{A-dw}(X + Y) \leq \mathrm{ber}_{A-dw}(X) + \mathrm{ber}_{A-dw}(Y).$$

Theorem 2.6 induces inequality of the A-Davis-Wielandt Berezin number for the off-diagonal parts of 2×2 operator matrix.

Corollary 2.7. *Assume that $X, Y \in \mathcal{B}_A(\mathcal{H}(\Omega))$. Then*

$$\mathrm{ber}_{A-dw}\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \leq \sqrt{\frac{1}{4}\|X\|^2_{A-\mathrm{Ber}} + \|X\|^4_{A-\mathrm{Ber}}} + \sqrt{\frac{1}{4}\|Y\|^2_{A-\mathrm{Ber}} + \|Y\|^4_{A-\mathrm{Ber}}}.$$

In the next result we give inequality of the Davis-Wielandt Berezin number.

Theorem 2.8. *Assume that $S \in \mathcal{B}(\mathcal{H}(\Omega))$. Then*

$$\mathrm{ber}^2_{dw}(S) \leq \inf_{\gamma\in\mathbb{C}}\{(2||Re(\gamma)Re(S) + Im(\gamma)Im(S)|| + ||S^*S - 2Re(\bar{\gamma}S)||)^2 + 2||Re(\bar{\gamma}S)|| - |\gamma|^2 + \mathrm{ber}^2(S - \gamma I)\}. \tag{125}$$

Remark 2.9. By considering $\gamma = 0$ in (125), we have

$$\begin{aligned}\mathrm{ber}_{dw}(S) &\leq \sqrt{\mathrm{ber}^2(S) + ||S^*S||^2} \\ &\leq \sqrt{\mathrm{ber}^2(S) + ||S||^4}.\end{aligned}$$

So (125) is a generalization of (411).

References

[1] M.W. Alomari, M. Hajmohamadi, and M. Bakherad, *Norm-Parallelism of Hilbert space operators and the Davis-Wielandt Berezin number*, *Submmited.*

[2] Aniket Bhanja, Pintu Bhunia and Kallol Paul, *On generalized Davis-Wielandt radius inequalities of semi-Hilbertian space operators*, Math. FA. `ArXive:2006..05069v1`.

[3] F. Goli, R. Lashkaripour and M. Hajmohamadi, *The A-Davis-Wielandt Berezin number of semi Hilbert operators with some related inequalities*, *Accepted in Filomat.*

[4] P.R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., springer, New York, 1982.

[5] A. Zamani, *A-numerical radius inequalities for semi-Hilbertian space operators*, *Linear Algebra Appl.*, **578** 2019, 159–183.

[6] A. Zamani, M.S. Moslehian, M.-T. Chien and H. Nakazato, *Norm-parallelism and the Davis–Wielandt radius of Hilbert space operators*, *Linear Multilinear Algebra*, **67** (11), (2019) 2147–2158.

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Weighted composition operators between weighted dirichlet spaces and weighted Bloch spaces

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Abstract

In this note we study weighted composition operators $\psi C_\varphi f = \psi(f \circ \varphi)$ acting from weighted Dirichlet spaces into weighted Bloch spaces.

Keywords: weighted composition operators, weighted Bergman spaces, weighted Bloch spaces
Mathematics Subject Classification [2023]: 47B33, 47B38

1 Introduction

Through this paper $\mathbb{D}$ will be denote the unit ball in complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ denote the class of all analytic functions on $\mathbb{D}$. For $p > 0$, the weighted Bergman space $A_{w,p}$ defined by

$$A_{w,p} = \left\{ f \in H(\mathbb{D}) : ||f||_{w,p}^p = \int_{\mathbb{D}} w(z)|f(z)|^p \, dA(z) < \infty \right\},$$

where w is a *weight* on $\mathbb{D}$ (strictly positive bounded continuous function). By letting $w_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha$ with $\alpha > -1$, we have the standard weighted Bergman space $A_{w_\alpha,p} = A_\alpha^p$ (if $\alpha = 0$, then $A_\alpha^p = A^p$ and $||.||_{w_\alpha,p} = ||.||_p$). Also, the weighted Dirichlet space $\mathcal{D}_{w,p}$ is the set of all analytic functions f on $\mathbb{D}$ such that $f' \in A_{w,p}$. We define $||f||_{\mathcal{D}_{w,p}} = ||f'||_{w,p}$. Moreover, for a weight v, the weighted Bloch spaces defined as

$$B_v = \{f \in H(\mathbb{D}) : ||f||_{B_v} = \sup_{z\in\mathbb{D}} v(z)|f'(z)| < \infty\},$$

and the weighted Banach spaces of holomorphic functions H_v^∞ consists of all analytic functions f on $\mathbb{D}$ with

$$||f||_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty.$$

Let ψ and φ be two analytic functions on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$. Associated with ψ and φ, we define the weighted composition operator by $\psi C_\varphi(f) = \psi f \circ \varphi$, for $f \in H(\mathbb{D})$. Such operators have been studied on various spaces of analytic functions. In [3], Sharma characterized these operators between Bergman and Bloch spaces. Mirzakarimi and Seddighi [2], studied on the Bergman and Dirichlet spaces. Kumar [1] explored weighted composition operators between different Dirichlet spaces in terms of Carleson measure. Also Wang [4], investigated these operators between different Hilbert Dirichlet spaces. For some other works, we refer to [5, 6].

In this note, we use the integral operator I defined as follows

$$(Ih)(z) = \int_0^z h(t)dt, \quad h \in H(\mathbb{D}), z \in \mathbb{D}.$$

Lemma 1.1. *[7] If $f \in A_{w_\alpha,p}$, then*

$$|f(z)| \leq \frac{||f||_{w_\alpha,p}}{(1-|z|^2)^{(2+\alpha)/p}},$$

for every $z \in \mathbb{D}$.

*Speaker.

Now if $f \in A_{w,p}$, then $w^{1/p} f \in A^p$. So

$$w(z)^{1/p}|f(z)| \leq \frac{||w^{1/p} f||_{w_\alpha,p}}{(1-|z|^2)^{2/p}} = \frac{||f||_{w,p}}{(1-|z|^2)^{2/p}}.$$

Thus

$$|f(z)| \leq \frac{||f||_{w,p}}{(1-|z|^2)^{2/p} w(z)^{1/p}} \tag{126}$$

2 Between weighted Dirichlet spaces and weighted Bloch spaces

In this section we study boundedness and compactness of weighted composition operators between $\mathcal{D}_{w,p}$ and B_v.

Theorem 2.1. Suppose that w and v be two weights and the integral operator $I : A_{w,p} \to H^\infty_w$ be a bonded operator. If

1. $\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{2/p} w(\varphi(z))^{1/p}} < \infty$,
2. $\sup_{z\in\mathbb{D}} \frac{v(z)|\psi'(z)|}{w(\varphi(z))} < \infty$,

then the weighted composition operator $\psi C_\varphi : \mathcal{D}_{w,p} \longrightarrow B_v$ is bounded.

In the following theorem weights for the Bergman spaces are as follow: Let W be an analytic function on $\mathbb{D}$ such that W is non-vanishing, strictly positive and decreasing on $[0,1)$. Then the corresponding weight defined by $w(z) = W(|z|^2)$, for every $z \in \mathbb{D}$. Here are some examples:

- If $W(z) = (1-z)^\alpha$, $\alpha \geq 1$ then $w(z) = (1-|z|^2)^\alpha$ (standard weight).
- If $W(z) = e^{-\frac{1}{(1-z)^\alpha}}$, $\alpha \geq 1$ then $w(z) = e^{-\frac{1}{(1-|z|^2)^\alpha}}$.
- If $W(z) = \sin(1-z)$, then $w(z) = \sin(1-|z|^2)$.

Also we assume that there exists a positive constant C such that

$$\sup_{a\in\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{w(z)}{W(\overline{\varphi(a)}z)} \leq C.$$

Theorem 2.2. Suppose that w is a weight that we defined above. If $\psi C_\varphi : \mathcal{D}_{w,p} \longrightarrow B_v$ be bounded, then

$$\sup_{z\in\mathbb{D}} \frac{v(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}-1} w(\varphi(z))^{1/p}} < \infty \quad \textit{and}$$

$$\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}} w(\varphi(z))^{\frac{1}{p}-1}} < \infty. \tag{127}$$

Theorem 2.3. Suppose that w be a weight of the form $w = |W|$ such that W is an analytic function on $\mathbb{D}$ and $\psi C_\varphi : \mathcal{D}_{w,p} \to B_v$ be bounded. If

1. $\lim_{|\varphi(z)|\to 1} \frac{v(z)|\psi'(z)|}{w(\varphi(z))} = 0$,
2. $\lim_{|\varphi(z)|\to 1} \frac{v(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{2/p} w(\varphi(z))^{1/p}} = 0$,

then the weighted composition operator $\psi C_\varphi : \mathcal{D}_{w,p} \to B_v$ is compact.

Theorem 2.4. Suppose that w is a weight which we defined above. If $\psi C_\varphi : \mathcal{D}_{w,p} \to B_v$ be compact. Then

$$\lim_{|\varphi(z)|\to 1} \frac{v(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}-1} w(\varphi(z))^{\frac{1}{p}}} = 0 \quad \textit{and}$$

$$\lim_{|\varphi(z)|\to 1} \frac{v(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}} w(\varphi(z))^{\frac{1}{p}-1}} = 0. \tag{128}$$

Proof. Let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$. Set

$$f_n(z) = \frac{1-|\varphi(z_n)|^2}{W(\overline{\varphi(z_n)}z)^{\frac{1}{p}}(1-\overline{\varphi(z_n)}z)^{\frac{2}{p}}}.$$

Then f_n is a bounded sequence in $\mathcal{D}_{w,p}$, $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ and

$$|f_n'(\varphi(z_n))| \le \frac{M'}{(1-|\varphi(z_n)|^2)^{\frac{2}{p}} w(\varphi(z_n))^{\frac{1}{p}-1}}.$$

Since $\psi C_\varphi : \mathcal{D}_{w,p} \to B_v$ is compact, $||\psi C_\varphi f_n||_{B_v} \to 0$. On the other hand

$$||\psi C_\varphi f_n||_{B_v} \ge v(z_n)|\psi'(z_n) f_n(\varphi(z_n)) + \psi(z_n)\varphi'(z_n) f_n'(\varphi(z_n))|.$$

So

$$\begin{aligned} ||\psi C_\varphi f_n||_{B_v} + &\frac{v(z_n)|\psi(z_n)\varphi'(z_n)|M'}{(1-|\varphi(z_n)|^2)^{\frac{2}{p}} w(\varphi(z_n))^{\frac{1}{p}-1}} \\ &\ge ||\psi C_\varphi f_n||_{B_v} + v(z_n)|\psi(z_n)\varphi'(z_n) f_n'(\varphi(z_n))| \\ &\ge ||\psi C_\varphi f_n||_{B_v} + v(z_n)|\psi'(z_n) f_n(\varphi(z_n))| \\ &= ||\psi C_\varphi f_n||_{B_v} + \frac{v(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}-1} w(\varphi(z))^{\frac{1}{p}}} \end{aligned}$$

It is enough to prove that the condition (128) holds. We define the following function

$$g_n(z) = \frac{(1-|\varphi(z_n)|^2)^{\frac{p+1}{p}}}{W(\overline{\varphi(z_n)}z)^{\frac{1}{p}-1}(1-\overline{\varphi(z_n)}z)^{\frac{3}{p}}} - \frac{(1-|\varphi(z_n)|^2)^{\frac{p-1}{p}}}{W(\overline{\varphi(z_n)}z)^{\frac{1}{p}-1}(1-\overline{\varphi(z_n)}z)^{\frac{1}{p}}}.$$

Then $\{g_n\}$ is a bounded sequence in $\mathcal{D}_{w,p}$, converges to 0 uniformly on compact subsets of $\mathbb{D}$, $g_n(\varphi(z_n)) = 0$ and

$$g_n'(\varphi(z_n)) = \frac{2}{p} \frac{\overline{\varphi(z_n)}}{w(\varphi(z_n))^{\frac{1}{p}-1}(1-|\varphi(z_n)|^2)^{\frac{2}{p}}}.$$

Since $\psi C_\varphi : \mathcal{D}_{w,p} \to B_v$ is compact,

$$\frac{2}{p} \frac{v(z_n)\ |\varphi(z_n)|\ |\psi(z_n)\varphi'(z_n)|}{w(\varphi(z_n))^{\frac{1}{p}-1}(1-|\varphi(z_n)|^2)^{\frac{2}{p}}} \le ||\psi C_\varphi g_n||_{B_v} \to 0.$$

Thus

$$\lim_{|\varphi(z)|\to 1} \frac{v(z)|\psi(z)\varphi'(z)|}{w(\varphi(z_n))^{\frac{1}{p}-1}(1-|\varphi(z_n)|^2)^{\frac{2}{p}}} = 0.$$

□

References

[1] S. Kumar, *Weighted composition operators between spaces of Dirichlet type*, Rev Mat Complut, 22 (2009), 469488.

[2] G. Mirzakarimi and K. Seddighi, *Weighted composition operators on Bergman and Dirichlet spaces*, Georgian Math. J., 4 (1997), 373383.

[3] A.K. Sharma and S.D. Sharma, *Weighted composition operators betwen Bergman and Bloch spaces*, Commun. Korean Math. Soc., 22(3) (2007), 373-382.

[4] M. Wang, *Weighted composition operators between Dirichlet spaces*, Acta Math. Sci., 31B(2) (2011), 641-651.

[5] E. Wolf, *Weighted composition operators between weighted Bergman spaces and weighted Banach spaces of holomorphic functions*, Rev. Mat. Complut., 21 (2008), 4267-4273.

[6] E. Wolf, *Weighted composition operators between weighted Bloch type spaces*, Mathematical Proceedings of the Royal Irish Academy, 111A (2011), 3747.

[7] K. Zhu, *Spaces of holomorphic functions in the unit ball*, Springer, New York, 2005.

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Gradient estimate for nonlinear equations on almost ricci solitons

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Abstract

In this paper, we study the gradient estimate for positive solutions of the equation $\Delta u + Au = f$ on an almost Ricci soliton M^n. As an application, we derive a gradient estimate for an almost gradient Ricci soliton.

Keywords: Sobolev inequality, Gradient estimate, Almost Ricci soliton
Mathematics Subject Classification [2023]: 53C99, 53C21, 35E20

1 Introduction

Sobolev inequality enables us to obtain many applications in differential geometry. For example, it plays the main role about maximal principle, gradient estimates and consequently upper and lower bounds of the heat kernel. The Cheeger-Colding-Naber theory has been extended to integral Ricci curvature bound in the noncollapsed case and had important results [3, 4]. In fact, Rose in [3] showed that under locally uniformly integral bounds of the negative part of Ricci curvature the heat kernel admits a Gaussian upper bound for small times. After that Dai et al. [1] extended many of the basic estimates for integral curvature to the collapsed case. Zhang and Zhu [5] followed their arguments to prove Sobolev inequality on manifolds with considering a lower bound of the Bakry-Émery Ricci curvature $\mathrm{Ric} + \frac{1}{2}\mathcal{L}_V g \geq -\lambda g$, where $\mathcal{L}_V$ is the Lie derivative along the vector field V and Ric is the Ricci tensor and λ is a positive constant. Afterward they used volume comparison theorem and Sobolev inequalities for elliptic and parabolic gradient estimates. Actually, they considered following equations

$$Ric + \frac{1}{2}\mathcal{L}_V g \geq -\lambda g, \;\; |V|(y) \leq \frac{K}{d(y,O)^\alpha}, \;\; \alpha \in [0,1);$$

and moreover volume noncollapsing condition $Vol(B(a,1)) \geq \rho$, for some constant $\rho > 0$, when $\alpha \neq 0$. Here $d(y,O)$ represents the distance from O to y, and $K \geq 0$ is constant. By these assumptions Zhang and Zhu obtained following upper bound for the solution of $\Delta u = f$ in $B(x,r)$:

$$\sup_{B(x,\frac{1}{2}r)} |\nabla u|^2 \leq C(n,\lambda,K,\alpha,\rho)\Big[r^{-2}(\|u\|^*_{2,B(x,r)})^2 + (\|f\|^*_{2q,B(x,r)})^2\Big],$$

for any $q > \frac{n}{2}$ and moreover

$$\sup_{B(x,\frac{1}{2}r)} u^2 \leq C(n,\lambda,K,\alpha,\rho)\Big[(\|u\|^*_{2,B(x,r)})^2 + r^2(\|f\|^*_{2q,B(x,r)})^2\Big],$$

where

$$\|f\|^*_{q,B(x,r)} = \left(\fint_{B(x,r)} |f|^q dg\right)^{\frac{1}{q}}.$$

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In this paper, we study gradient estimate on almost Ricci solitons for positive solutions of the following equation:

$$\Delta u + Au = f, \tag{129}$$

here A is a smooth function deponds on u.

We say that a Riemannian manifold (M^n, g) is an almost Ricci soliton if there exist a vector field X and a soliton function $\lambda : M^n \longrightarrow \mathbb{R}$ satisfying

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = -\lambda g, \tag{130}$$

where Ric and $\mathcal{L}$ stand, respectively, for the Ricci tensor and the Lie derivative. It called shrinking, steady or expanding, respectively, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. When the vector field X is a gradient of a differentiable function $h : M^n \longrightarrow \mathbb{R}$ the manifold called a gradient almost Ricci soliton; in this case the preceding equation turns out to be

$$\mathrm{Ric} + \nabla^2 h = -\lambda g, \tag{131}$$

where ∇^2 denotes the Hessian of h. Moreover, when either the vector field X is trivial or the potential h is constant, the almost Ricci soliton will be called trivial. We notice that when $n \geq 3$ and X is a Killing vector field, an almost Ricci soliton will be simply a Ricci soliton, since in this case we have an Einstein manifold, which implies that λ is a constant.
Let M^n be an almost Ricci soliton with following conditions

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_V g \geq -\lambda g, \tag{132}$$

and

$$|V|(y) \leq \frac{K}{d(y, O)^\alpha}, \tag{133}$$

for a smooth function λ with upper bound N, a smooth vector field V and any $y \in M$. Here $d(y, O)$ denotes the distance from O to y, K is the positive constant and $0 \leq \alpha < 1$. Especially we consider one more condition named volume non-collapsing condition when $\alpha \neq 0$,

$$Vol(B(x, 1)) \geq \rho, \tag{134}$$

for all $x \in M$ and some constant $\rho > 0$.
Proving our main results, we need to use Sobolev inequality and volume comparison theorem for an almost Ricci soliton which was stablished in [2].

Theorem 1.1 (Volume comparison [2]). *Assume that for an n-dimension almost Ricci soliton (392) and (133) hold. Moreover consider a positive constant N as an upper bound for λ. Suppose in addition that the volume non-collapsing condition (393) holds for positive constants $\rho > 0$, $K \geq 0$ and $0 \leq \alpha < 1$, then for any $0 < r_1 < r_2 \leq 1$, we have the volume ratio bound as follows*

$$\frac{Vol(B(x, r_2))}{r_2^n} \leq e^{C(n,N,K,\alpha,\rho)[N(r_2^2 - r_1^2) + K(r_2 - r_1)^{1-\alpha}]} . \frac{Vol(B(x, r_1))}{r_1^n}. \tag{135}$$

Where $C = C(n, N, K, \alpha, \rho)$ is the constant depends on (n, N, K, α, ρ) and $B(x, r)$ is a ball centered at x with radius r. In particular, this result are true by considering the gradient soliton vector field $V = \nabla f$.

Theorem 1.2 (Sobolev inequality). *Under the same conditions as in the above theorem, we have the following Sobolev inequalities.*

$$\left(\fint_{B(x,r)} |f|^{\frac{n}{n-1}} dg \right)^{\frac{n-1}{n}} \leq C(n) r \fint_{B(x,r)} |\nabla f| dg, \tag{136}$$

and

$$\left(\fint_{B(x,r)} |f|^{\frac{2n}{n-2}} dg \right)^{\frac{n-2}{n}} \leq C(n) r^2 \fint_{B(x,r)} |\nabla f|^2 dg. \tag{137}$$

Moreover, for the case that $V = \nabla f$ for some smooth function f, we get

$$\left(\fint_{B(x,r)} |f|^{\frac{n}{n-1}} dg \right)^{\frac{n-1}{n}} \leq C(n) r \fint_{B(x,r)} |\nabla f| dg. \tag{138}$$

2 Main results

Here is our main result:

Theorem 2.1. *Suppose that on an almost Ricci soliton M^n, (392), (133) and (393) hold. For $q > \frac{n}{2}$, if u and f be smooth functions such that (3.10) holds with $0 \leq u \leq l_1$ and $|A| \leq l_2$ for constants l_1, l_2, and $\lambda \leq N$, for a positive constant N, then there exists a positive constant $r_0 = r_0(n, N, K, \alpha, \rho, l_1, l_2)$ such that for any $x \in M$ and $0 < r \leq r_0$ we have*

$$\sup_{B(x, \frac{1}{2}r)} |\nabla u|^2 \leq C(n, N, K, \alpha, \rho, l_1, l_2) \Big[(\|f\|^*_{2q, B(x,r)})^2 + r^{-2} (\|u\|^*_{2, B(x,r)})^2 \Big].$$

As an application we conclude:

Corollary 2.2. *Suppose that the following condition holds for a gradient almost Ricci soliton*

$$\mathrm{Ric} + Hessh \geq -\lambda g,$$

and more over we have two condition for potential function h as follows

$$|h(y) - h(z)| \leq K_1 d(y,z)^\alpha, \ \text{ and } \ \sup_{x \in M, 0 \leq r \leq 1} \left(r^\beta \|\nabla h\|^*_{q, B(x,r)} \right) \leq K_2.$$

Then there is a constant $r_0 = r_0(n, N, K_1, K_2, \alpha, \beta, l_1, l_2)$, such that by the same conditions as last theorem, the solution of (3.10) satisfies

$$\sup_{B(x, \frac{r}{2})} |\nabla u|^2 \leq C(n, N, K_1, K_2, \alpha, \beta, l_1, l_2) \Big[r^{-2} (\|u\|^*_{2, B(x,r)})^2 + (\|h\|^*_{2q, B(x,r)})^2 \Big],$$

for any $q > \frac{n}{2}$.

References

[1] X. Dai, G. Wei, Z. Zhang, *Local sobolev constant estimate for integral Ricci curvature bounds*, Adv. Math. 325 (2018) 1-33.

[2] S. Hajiaghasi, S. Azami, *New volume comparison with almost Ricci soliton*, Communications of the Korean Mathematical Society 37 (3), (2022), 839-849.

[3] C. Rose, *Heat kernel upper bound on Riemannian manifolds with locally uniform Ricci curvature integral bounds*, J. Geom. Anal. 27 (2017), no. 2, 1737-1750.

[4] G. Wei, R. Ye, *A Neumann type maximum principle for the laplace operator on compact Riemannian manifolds*, Journal of Geometric Analysis 19, no. 3 (2009), 719-736.

[5] Qi S. Zhang, M. Zhu, *New volume comparison results and applications to degeneration of Riemannian metrics*, Advances in Mathematics, 352 (2019) 1096-1154.

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What are transverse Homoclinic points, a Cantor set?

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Abstract

We will first define what a homoclinic orbit is, then we will study some of the properties of homoclinic points, using methods from symbolic dynamics. In particular, we will prove that, under certain assumptions, the existence of one homoclinic point implies the existence of a whole Cantor set of homoclinic points.

Keywords: Homoclinic point, Orbit, Heteroclinic point
Mathematics Subject Classification [2023]: 46B80, 37J12

1 Introduction

The first section gives some basic definitions from the theory of dynamical systems and the mathematical objects necessary for the discussion. The second section presents two of the most important theorems in the theory of nonlinear dynamical system and the definitions of homoclinic and heteroclinic points.

Definition 1.1. A differential dynamical system is a triplet (G, M, ϕ), where M is a differentiable manifold, called the phase space, $G = \mathbb{Z}$ or $\mathbb{R}$, and $\phi : G \times M \longrightarrow M$, $(t, x) \mapsto \phi(t, x) =: \phi^t(x)$, is a differentiable map with the following properties:

- $\phi^t : M \to M, \forall t \in G$, is a diffeomorphism;
- $\phi^0 = id$;
- $\phi^{t+s} = \phi^t \circ \phi^s$ for each $t, s \in G$.

The orbit of $x \in M$ is the set $O(x) = \{\phi^t(x) | t \in G\}$. A point $x^* \in M$ is a fixed point for the dynamical system if $\phi^t(x^*) = x^* \quad \forall t \in G$. A subset $U \subset M$ is said to be invariant if $\phi^t(U) \subseteq U, \quad \forall t \in G$. For more information about dynamical system see [1] or [5].

The discrete case $G = \mathbb{Z}$ is the case of iteration of diffeomorphisms. If $f : M \to M$ is a diffeomorphism, let

$$\begin{array}{cc} \phi^1 = f; & \phi^t = f \circ \cdots \circ f =: f^t; \\ \phi^0 = id; & \phi^{-t} = f^{-1} \circ \cdots \circ f^{-1} =: f^{-t} \quad \forall t \in \mathbb{N}; \end{array}$$

then $(\mathbb{Z}, M, \phi)$ is a discrete dynamical system. The fixed points of f are the fixed points of the dynamical system. Fixed points of the $k-$th composition of f with itself, $f^k(x) = x$, are called periodic points of the dynamical system of period k.

Definition 1.2. A fixed point x of a diffeomorphism f, is said to be hyperbolic if df_x, the linearization of f evaluated in x, has no eigenvalues of modulus 1.

Moreover, two dynamical systems, $(\mathbb{Z}, M, \phi)$ and $(\mathbb{Z}, N, \psi)$ are topologically conjugate if there exists a homeomorphism $h : M \to N$ such that

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$$h \circ \phi^t = \psi^t \circ h \qquad \forall t \in \mathbb{Z}$$

This implies that h takes the orbits of $(\mathbb{Z}, M, \phi)$ to orbits of $(\mathbb{Z}, N, \psi)$.
We restrict ourselves to the case of $M = \mathbb{R}^n$, and consider the dynamical system given by $(G, \mathbb{R}^n, \phi)$, where $G = \mathbb{R}$ or $G = \mathbb{Z}$. If the system has a fixed point p, it is of interest to consider the sets of points that converge or diverge to it.

Definition 1.3. The stable and unstable sets for a fixed point p are

$$W^s(p) = \{x \in \mathbb{R}^n : lim_{t\to+\infty}\phi^t(x) = p\}$$
$$W^u(p) = \{x \in \mathbb{R}^n : lim_{t\to-\infty}\phi^t(x) = p\}$$

In general, these sets do not have a well defined structure. They are invariant, non empty since $p \in W^{s,u}(p)$, and, in the case of flows, connected. If the fixed point is hyperbolic, they have more properties. To state these properties one needs to first consider a local definition of these sets.

Definition 1.4. In a neighborhood U of p the sets

$$W^s_U(p) = \{x \in U : \phi^t(x) \in U \forall t \geq 0 \text{ and } lim_{t\to+\infty}\phi^t(x) = p\}$$
$$W^u_U(p) = \{x \in U : \phi^t(x) \in U \forall t \leq 0 \text{ and } lim_{t\to-\infty}\phi^t(x) = p\}$$

are called the local stable and local unstable sets of p in U.

2 Heteroclinic point

Definition 2.1. [2] Let p a hyperbolic fixed point of a dynamical system $(G, \mathbb{R}^n, \phi)$. A point $q \neq p$ such that

$$q \in W^s(p) \cap W^u(p)$$

is said to be a homoclinic point for p.

If $T_qW^s(p) \oplus T_qW^u(p) = \mathbb{R}^n$ then q is a transverse homoclinic point. See figure 1(a).

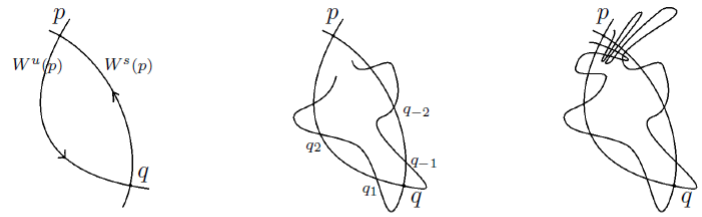


Figure 1: Examples of homoclinic point (a), the first orbit of homoclinic points (b), and formation of other orbits (c)

Definition 2.2. Let $p_1 \neq p_2$ be two hyperbolic fixed points, and $p_1 \neq q \neq p_2$. Then if $q \in W^s(p_1) \cap W^u(p_2)$ the point q is said to be a heteroclinic point.

If $T_qW^s(p_1) \oplus T_qW^u(p_2) = \mathbb{R}^n$ then q is a transverse heteroclinic point. Since W^s and W^u are invariant sets, each point in the orbit of a homoclinic (resp. heteroclinic) point is homoclinic (resp. heteroclinic), and such an orbit is called homoclinic (resp. heteroclinic) orbit.

Theorem 2.3. *(Smale[3], 1963). Let p be a hyperbolic fixed point for a dynamical system $(G, \mathbb{R}^n, \phi)$. Then $W^s(p)$ and $W^u(p)$ are injective immersions of $\mathbb{R}^{n_s}$ and $\mathbb{R}^{n_u}$ in $\mathbb{R}^n$.*

Proof. See [3] or [4, 6]. □

Definition 2.4. A set $\Lambda \subset \mathbb{R}^2$ which

- is closed;
- is perfect, (each point in Λ is a limit point);
- is totally disconnected, ($\nexists$ a non-trivial connected subset of Λ).

is a Cantor set.

Theorem 2.5. *(Main theorem) If p is a hyperbolic fixed point for the dynamical system given by f on $\mathbb{R}^2$, if non-empty, the set of all the transverse homoclinic points for p is a Cantor set.*

Proof. Let U be the unit square $[0,1] \times [0,1] \to \mathbb{R}^2$. Let $H_0 = [0,1] \times [0,1/4]$ and $H_1 = [0,1] \times [3/4,1]$ be two disjoint horizontal strips; let $V_0 = [0,1/4] \times [0,1]$ and $V_1 = [3/4,1] \times [0,1]$ be two disjoint vertical strips. The map $F : U \longrightarrow \mathbb{R}^2$ acts contracting U along x, expanding it along y and folding it, as shown in figure 3.1. The horizontal strips H_0, H_1 are mapped into the vertical strips $V_0 = F(H_0), V_1 = F(H_1)$, while the folded part is mapped outside of U. The orientation of the folding is given by the vertexes A, B, C, D. Thus, $U \cap F(U) = V_0 \cup V_1$; since F is not onto, F^{-1} is not globally defined. However $H_0 \cup H_1$ is the preimage of $U \cap F(U)$, that is $F^{-1}(U \cap F(U)) = H_0 \cup H_1$.

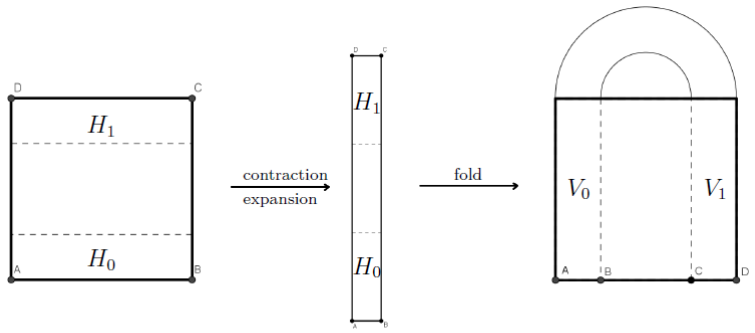


Figure 3.1: On the left is shown U with the subsets H_0, H_1; the contraction and expansion of U, and $\mathcal{F}(U)$ with V_0, V_1 on the right

We want to describe the set of all points in U whose orbits remain in U,

$$\Lambda = \{p | F^i(p) \in U, i \in \mathbb{Z}\}$$

As shown in figure 3.2, applying F to $U \cap F(U)$ we get four vertical strips. Continuing the argument

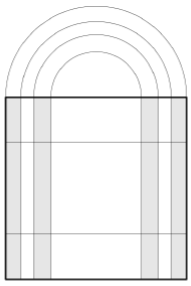


Figure 3.2: The grey area is $U \cap \mathcal{F}(U) \cap \mathcal{F}^2(U)$.

inductively, the set $U \cap F(U) \cap \cdots \cap F^n(U)$ is the union of 2^n vertical strips, that in the limit $n \to +\infty$ become vertical segments and form the set

$$\Lambda_+ = \{p | F^i(p) \in U, i \geq 0\}$$

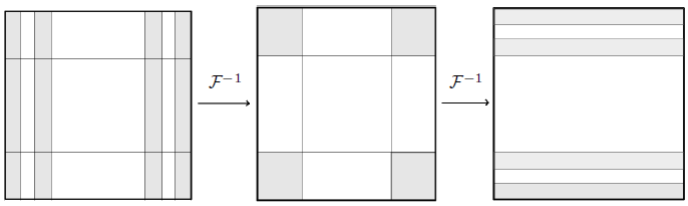


Figure 3: 3.3 [5]

which is a Cantor set (see [3]). If a point $p \in U$, then $F^{-1}(p) \in H_0 \cup H_1$. If $F(p) \in U$ then $F^{-1}(F(p)) \in H_0 \cup H_1 = F^{-1}(U \cap F(U))$. While if $F^2(p) \in U$, $p \in F^{-2}(U \cap F(U) \cap F^2(U))$. The set $F^{-2}(U \cap F(U) \cap F^2(U))$ is shown in figure below, outlined in grey.

Therefore if $(U \cap F(U) \cap \cdots \cap F^n(U))$ is a set of 2^n vertical strips, the $n-$th preimage $F^{-n}(U \cap F(U) \cap \cdots \cap F^n(U))$ is a set of 2^n horizontal strips. When $n \to +\infty$, the strips form the Cantor set of horizontal segments

$$\Lambda_- = \{p | F^i(p) \in U, i \geq 1\}$$

Since $\Lambda = \Lambda_+ \cap \Lambda_-$, a point $p \in \Lambda$ must belong to both a vertical and horizontal segment. Therefore, Λ is a Cantor set. □

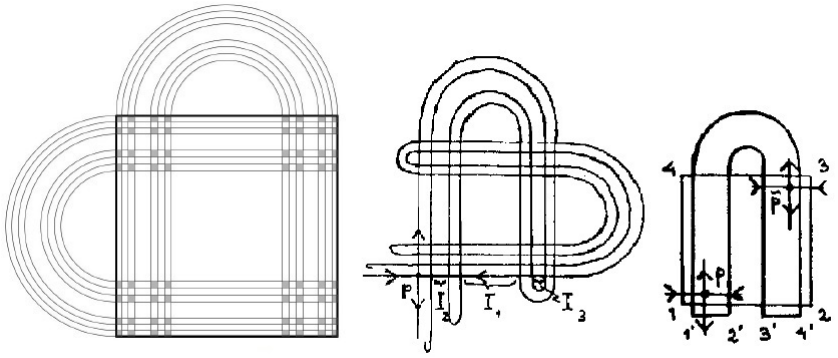


Figure 3.4: Outlined in grey $\bigcap_{n=-2}^{2} \mathcal{F}^n(U)$ to give an idea of the invariant set Λ.

Observe that however far we iterate, the intervals $I_1, I_2, I_3 \subset W^s(p)$ will never be intersected by $W^u(p)$. There is in fact a countably infinite number of such open intervals and $W^s(p) \cap W^u(p)$ consists of boundary points of a Cantor set in $W^s(p)$ which is complement of this intervals.

References

[1] J. Guckenheimer and Philip J Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, volume 42. Springer Science & Business Media, 2013.

[2] J. Palis and Floris Takens. *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations: Fractal Dimensions and Infinitely Many Attractors in Dynamics.* Number 35. Cambridge University Press,1995.

[3] S. Smale. *Differentiable dynamical systems.* Bulletin of the American mathematical Society, 73(6):747817, 1967.

[4] Stephen Smale. *Stable manifolds for differential equations and diffeomorphisms.* In Topologia differenziale, pages 93126. Springer, 2011.

[5] P. R. Tabriziam. *Homoclinic orbits and chaotic behaviour in classical mechanics*. 2008.

[6] S. Wiggins. *Global bifurcations and chaos: analytical methods*, volume 73. Springer Science & Business Media, 2013.

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A strong convergence for finding a fixed point of a nonexpansive mapping in Hilbert spaces

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Abstract

We introduce an implicit method for finding a fixed point of a nonexpansive mapping. Then we prove the strong convergence of the proposed implicit scheme to the fixed point of a nonexpansive mapping.

Keywords: Fixed point, Nonexpansive mapping,sunny nonexpansive retraction.
Mathematics Subject Classification [2023]: 47H09, 47H10

1 Introduction

Let C be a nonempty closed and convex subset of a Banach space E and E^* be the dual space of E. Let $\langle .,. \rangle$ denote the pairing between E and E^*. The normalized duality mapping $J : E \to E^*$ is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

for all $x \in E$. In the sequel, we use j to denote the single-valued normalized duality mapping. Let $U = \{x \in E : \|x\| = 1\}$. E is said to be smooth or said to have a Gâteaux differentiable norm if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in U$, the limit is attained uniformly for all $x \in U$. E is said to be uniformly smooth or said to have a uniformly Fréchet differentiable norm if the limit is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E. A Banach space E is smooth if the duality mapping J of E is single valued. We know that if E is smooth, then J is norm to weak-star continuous; for more details, see [2].

Let C be a nonempty closed and convex subset of a Banach space E. A mapping T of C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$ and a mapping f is an α-contraction on E if $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, $x, y \in E$ such that $0 \leq \alpha < 1$.

In this paper we introduce the following general implicit algorithm for finding a common element of the set of fixed points of a representation $= \{T_t : t \in S\}$ of a semigroup S as nonexpansive mappings from C into itself, with respect to a left regular sequence of means defined on an appropriate subspace of bounded real-valued functions of the semigroup. On the other hand, our goal is to prove that for a sunny nonexpansive retraction P of C onto Fix(T) the following sequence $\{z_n\}$ converges strongly to $P0$:

$$z_n = (1 - \frac{1}{n})Tz_n \quad (n \in \mathbb{N}).$$

Let D be a subset of B where B is a subset of a Banach space E and let P be a retraction of B onto D, that is, $Px = x$ for each $x \in D$. Then P is said to be sunny, if for each $x \in B$ and $t \geq 0$ with $Px + t(x - Px) \in B$, $P(Px + t(x - Px)) = Px$. A subset D of B is said to be a sunny nonexpansive retract of B if there exists a sunny nonexpansive retraction P of B onto D. We know that if E is smooth and P is a retraction of B onto D, then P is sunny and nonexpansive if and only if for each $x \in B$ and $z \in D$, $\langle x - Px\, ,\, J(z - Px) \rangle \leq 0$.
For more details, see [2].

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2 Main results

In this section, we deal with a strong convergence approximation scheme for finding an element of the set of fixed points of a nonexpansive mapping.

Theorem 2.1. *Let C be a nonempty compact convex subset of a real Hilbert space H. Suppose that T is a nonexpansive mapping from C into itself such that* Fix(T) $\neq \emptyset$. *Let P be a unique sunny nonexpansive retraction of C onto* Fix(T) *and $0 \in C$. Suppose that the sequence $\{z_n\}$ is generated by*

$$z_n = (1 - \frac{1}{n})Tz_n \quad (n \in \mathbb{N}), \tag{139}$$

then $\{z_n\}$ strongly converges to $P0$.

Proof. We divide the proof into four steps.

Step 1. The existence of z_n which satisfies (361).

Proof. This follows immediately from the fact that for every $n \in \mathbb{N}$, the mapping N_n given by

$$N_n x := (1 - \frac{1}{n})Tx \qquad (x \in H),$$

is a contraction. Note that

$$\begin{aligned}\|N_n x - N_n y\| =& (1 - \frac{1}{n})\|Tx - Ty\| \\ \leq& (1 - \frac{1}{n})\|x - y\|.\end{aligned}$$

Therefore, by Banach Contraction Principle ([2],p.4), there exists a unique point z_n such that $N_n z_n = z_n$.

Step 2. $\mathfrak{S}\{z_n\} \subset$ Fix(T), where $\mathfrak{S}\{z_n\}$ denotes the set of strongly limit points of $\{z_n\}$.

Proof. Let $z \in \mathfrak{S}\{z_n\}$, and let $\{z_{n_j}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_j} \to z$.

$$\begin{aligned}\|Tz - z\| \leq& \|Tz - Tz_{n_j}\| + \|Tz_{n_j} - z_{n_j}\| + \|z_{n_j} - z\| \\ \leq& 2\|z_{n_j} - z\| + \frac{1}{n_j}\|Tz_{n_j}\|,\end{aligned}$$

then,

$$\|Tz - z\| \leq 2\lim_j \|z_{n_j} - z\| + \lim_j \frac{1}{n_j}\|Tz_{n_j}\| = 0,$$

therefore $z \in$ Fix(T).

Step 3. We prove that

$$\Gamma := \limsup_n \langle -P0\,,\, z_n - P0\rangle \leq 0. \tag{140}$$

Proof. From Proposition 2.10.20 in [1], since P is a sunny nonexpansive retraction of C onto Fix(T) and $0 \in C$,we have that

$$\Gamma := \limsup_n \langle 0 - P0\,,\, z_n - P0\rangle \leq 0.$$

Note that, from the definition of Γ and the fact that C is a compact subset of H, we can select a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ with the following properties:

(i) $\lim_j \langle -P0\,,\, z_{n_j} - P0\rangle = \Gamma$;

(ii) $\{z_{n_j}\}$ converges strongly to a point z;

by Step 3, we have $z \in$ Fix(T). Since H is smooth, we have

$$\Gamma = \lim_j \langle -P0\,,\, z_{n_j} - P0\rangle = \langle -P0\,,\, z - P0\rangle \leq 0.$$

From page 99 in [2], we have, for each $n \in \mathbb{N}$,

$$\begin{aligned}
-\frac{1}{n}\|z_n - P0\|\Big\|^2 & \\
&\geq \Big[(1-\frac{1}{n})\|z_n - P0\|\Big]^2 - \|z_n - P0\|^2 \\
&\geq \Big[\big(1-\frac{1}{n}\big)\|Tz_n - P0\|\Big]^2 - \|z_n - P0\|^2 \\
&\geq 2\Big\langle (1-\frac{1}{n})(Tz_n - P0) - (z_n - P0)\,,\, z_n - P0\Big\rangle \\
&= \frac{2}{n}\langle P0\,,\, z_n - P0\rangle,
\end{aligned}$$

hence,

$$\|z_n - P0\|^2 \leq 2\langle -P0\,,\, z_n - P0\rangle. \tag{141}$$

Step 4. $\{z_n\}$ strongly converges to $P0$.

Proof. Indeed, from (140), (141) and that $P0 \in \mathrm{Fix(T)}$, we conclude

$$\limsup_n \|z_n - P0\|^2 \leq \frac{2}{1-\alpha}\limsup_n \langle -P0\,,\, z_n - P0\rangle \leq 0.$$

That is $z_n \to P0$. □

Acknowledgement

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References

[1] R. P. Agarwal, D. Oregan and D. R. Sahu, *Fixed point theory for Lipschitzian-type mappings with applications, in: Topological Fixed Point Theory and its Applications*, vol. 6, Springer, New York, 2009.

[2] W. Takahashi, *Nonlinear Functional Analysis: Fixed Point Theory and its Applications*, Yokohama Publishers, Yokohama, 2000.

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Remarks on the portion of primes in intervals with similar length

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Abstract

In this paper we study portion of primes in intervals with similar length. Examining several cases, we observe that the interval with smaller values has more primes than the interval with larger values.

Keywords: Landau's inequality, prime number, asymptotic expansion
Mathematics Subject Classification [2023]: 11A41

1 Introduction

Let $\pi(x)$ denotes the number of prime numbers not exceeding x. On the page 216 of his handbook [3], Landau showed that

$$\pi(2x) - 2\pi(x) = -2\log 2\,\frac{x}{\log^2 x} + o\left(\frac{1}{\log^2 x}\right).$$

Thus, for x sufficiently large, we have $\pi(2x) - 2\pi(x) < 0$ or equivalently $\pi(x) > \pi(2x) - \pi(x)$. This implies that the interval $(0, x]$ has more primes than the interval $(x, 2x]$, for x sufficiently large. The inequality

$$\pi(2x) < 2\pi(x), \tag{142}$$

is known as Landau's inequality concerning primes counting function. In this paper, we are motivated by comparing the portion of primes in several intervals with similar lengths. The results of the present paper is part of [2]. We start with comparing the portion of primes in intervals $(0, \lambda\, x]$ and $((1-\lambda)x, x]$ for given $\lambda \in (0, 1)$.

Theorem 1.1. *Let $\lambda \in (0, 1)$ be fixed. The following inequality holds for x sufficiently large,*

$$\pi(x) < \pi(\lambda\, x) + \pi\left((1-\lambda)x\right). \tag{143}$$

The above result shows that although both of intervals $(0, \lambda\, x]$ and $((1-\lambda)x, x]$ has same length, the first interval including smaller integers has more primes than the second one, which includes large integers.

Also, we observe that the inequality (143) compares the portion of primes in intervals far from each other. Naturally, we ask about the portion of primes in intervals with same lengths close to each other. In the following theorem we compare the portion of primes in short intervals $((1-\lambda)x, x]$ and $(x, (1+\lambda)x]$ for given $\lambda \in (0, 1)$.

Theorem 1.2. *Let $\lambda \in (0, 1)$ be fixed. The following inequality holds for x sufficiently large,*

$$\pi(2x) < \pi\left((1-\lambda)x\right) + \pi\left((1+\lambda)x\right) < 2\pi(x). \tag{144}$$

Note that the right hand side of (144) implies that, comparing portion of primes in intervals with similar length, the interval with smaller values has more primes than the interval with larger values.

Finally, in the following result, we compare the portion of primes in the sets $(0, \lambda\, x] \cup (x, (1+\lambda)x]$ and $((1-\lambda)x, x] \cup ((2-\lambda)x, 2x]$, which are zigzag combination of close and far intervals.

Theorem 1.3. *Let $\lambda \in (0, 1)$ be fixed. The following inequality holds for x sufficiently large,*

$$2\pi(x) + \pi(2x) < \pi\left(\lambda\, x\right) + \pi\left((1-\lambda)x\right) + \pi\left((1+\lambda)x\right) + \pi\left((2-\lambda)x\right) < 4\pi(x). \tag{145}$$

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2 An auxiliary expansion

The key to prove the above results is using the following expansion

$$\pi(\alpha\, x) = x\left(\frac{\alpha}{\log x} + \frac{\alpha - \alpha\log\alpha}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big), \tag{146}$$

which holds for any $\alpha > 0$, as $x \to \infty$. More precisely, we prove the following widely applicable expansion.

Proposition 2.1. *Let $\alpha > 0$. For given integer $n \geqslant 0$, as $x \to \infty$ we have*

$$\pi(\alpha\, x) = \alpha\, x \sum_{k=0}^{n} \frac{(-1)^k\, P_k(\log\alpha)}{\log^{k+1} x} + O\left(\frac{x}{\log^{n+2} x}\right), \tag{147}$$

where

$$P_k(t) = \sum_{j=0}^{k} (-1)^j\, \frac{k!}{(k-j)!}\, t^{k-j}. \tag{148}$$

Note that $P_0(t) = 1$ and $P_1(t) = t - 1$, hence implying (146).

Proof of Proposition 2.1. The prime number theorem with error term [1] asserts that

$$\pi(x) = \mathrm{lim}(x) + O\left(x\,\mathrm{e}^{-c\sqrt{\log x}}\right), \tag{149}$$

where $c > 0$ is a computable constant, and $\mathrm{lim}(x) = \int_0^x \frac{1}{\log t}dt$, denotes the logarithmic integral function defined by the Cauchy principal value of integral. Integrating by parts gives

$$\mathrm{lim}(x) = x \sum_{k=0}^{n} \frac{k!}{\log^{k+1} x} + O\left(\frac{x}{\log^{n+2} x}\right), \tag{150}$$

for any integer $n \geqslant 0$. For any integer $n \geqslant 0$ one has $x\,\mathrm{e}^{-c\sqrt{\log x}} = o(\frac{x}{\log^{n+2} x})$ as $x \to \infty$. Thus, by using (149) we get the expansion

$$\pi(x) = x \sum_{k=0}^{n} \frac{k!}{\log^{k+1} x} + O\left(\frac{x}{\log^{n+2} x}\right), \tag{151}$$

and so

$$\pi(\alpha\, x) = \alpha\, x \sum_{k=0}^{n} \frac{k!}{(\log x + \log\alpha)^{k+1}} + O\left(\frac{x}{\log^{n+2} x}\right).$$

The binomial expansion asserts that

$$(1+t)^{-(k+1)} = \sum_{m=0}^{n} (-1)^m c_m\, t^m + O(t^{n+1}),$$

as $t \to 0$, where $c_0 = 1$ and $c_m = \frac{1}{m!}\prod_{i=1}^{m}(k+i)$ for $m \geqslant 1$. Thus

$$\begin{aligned}\sum_{k=0}^{n} \frac{k!}{(\log x + \log\alpha)^{k+1}} &= \sum_{k=0}^{n} \frac{k!}{\log^{k+1} x}\left(1 + \frac{\log\alpha}{\log x}\right)^{-(k+1)} \\ &= \sum_{k=0}^{n} \frac{k!}{\log^{k+1} x} \sum_{m=0}^{n} \frac{(-1)^m c_m \log^m\alpha}{\log^m x} + O\left(\frac{1}{\log^{n+2} x}\right).\end{aligned}$$

Diagonal collecting terms of the above double sum gives

$$\sum_{k=0}^{n} \frac{k!}{\log^{k+1} x} \sum_{m=0}^{n} \frac{(-1)^m c_m \log^m\alpha}{\log^m x} = \sum_{k=0}^{n} \frac{r_k}{\log^{k+1} x} + O\left(\frac{1}{\log^{n+2} x}\right),$$

where

$$r_k = \sum_{j=0}^{k} (-1)^{k-j} j! c_{k-j} \log^{k-j}\alpha.$$

This completes the proof. □

3 Proof of Theorems

Proof of Theorem 1.1. The expansion (146) with $\alpha = \lambda$ and $\alpha = 1 - \lambda$ gives

$$\pi(\lambda\, x) + \pi\left((1-\lambda)x\right) = x\left(\frac{1}{\log x} + \frac{C_2(\lambda)}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big), \tag{152}$$

where $C_2(\lambda) = 1 - \lambda\log\lambda - (1-\lambda)\log(1-\lambda)$. The function $C_2(\lambda)$ is strictly increasing for $\lambda \in (0, \frac{1}{2})$ and admits limit conditions $\lim_{\lambda\to 0^+} C_2(\lambda) = \lim_{\lambda\to 1^-} C_2(\lambda) = 1$. Thus, symmetry of $C_2(\lambda)$ with respect to $\lambda = \frac{1}{2}$ implies $1 < C_2(\lambda) \leqslant C_2(\frac{1}{2}) = 1 + \log 2$. Comparing the coefficients of (152) and the expansion (161) with $n = 1$ completes the proof. □

Proof of Theorem 1.2. The expansion (161) with $n = 1$ gives

$$2\pi(x) = x\left(\frac{2}{\log x} + \frac{2}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big). \tag{153}$$

Also, we use the expansion (146) with $\alpha = 2$, $\alpha = 1 - \lambda$ and $\alpha = 1 + \lambda$ to obtain

$$\pi(2x) = x\left(\frac{2}{\log x} + \frac{2 - 2\log 2}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big), \tag{154}$$

and

$$\pi\left((1-\lambda)x\right) + \pi\left((1+\lambda)x\right) = x\left(\frac{2}{\log x} + \frac{D_2(\lambda)}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big),$$

where $D_2(\lambda) = 2 - (1-\lambda)\log(1-\lambda) - (1+\lambda)\log(1+\lambda)$. The function $D_2(\lambda)$ admits limit values $\lim_{\lambda\to 0^+} D_2(\lambda) = 2$ and $\lim_{\lambda\to 1^-} D_2(\lambda) = 2 - 2\log 2$. Also, for $\lambda \in (0,1)$, we observe that $\frac{d}{d\lambda} D_2(\lambda) = \log\frac{1-\lambda}{1+\lambda} < 0$. Hence, $2 - 2\log 2 < D_2(\lambda) < 2$. Comparing the coefficients completes the proof. □

Proof of Theorem 1.3. The expansion (161) with $n = 1$ implies

$$4\pi(x) = x\left(\frac{4}{\log x} + \frac{4}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big). \tag{155}$$

Also, the expansions (153) and (154) give

$$2\pi(x) + \pi(2x) = x\left(\frac{4}{\log x} + \frac{4 - 2\log 2}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big).$$

We use the expansion (146) with $\alpha = \lambda$, $\alpha = 1 - \lambda$, $\alpha = 1 + \lambda$ and $\alpha = 2 - \lambda$ to get

$$\pi\left(\lambda\, x\right) + \pi\left((1-\lambda)x\right) + \pi\left((1+\lambda)x\right) + \pi\left((2-\lambda)x\right) = x\left(\frac{4}{\log x} + \frac{E_2(\lambda)}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big),$$

with $E_2(\lambda) = 4 - \lambda\log\lambda - (1-\lambda)\log(1-\lambda) - (1+\lambda)\log(1+\lambda) - (2-\lambda)\log(2-\lambda)$. Note that $E_2(\lambda) = E_2(1-\lambda)$, and $\lim_{\lambda\to 0^+} E_2(\lambda) = \lim_{\lambda\to 1^-} E_2(\lambda) = 4 - 2\log 2$. For $\lambda \in (0, \frac{1}{2})$, we observe that $\frac{d}{d\lambda} E_2(\lambda) = \log\frac{(1-\lambda)(2-\lambda)}{\lambda(1+\lambda)} > 0$. Thus,

$$4 - 2\log 2 < E_2(\lambda) \leqslant E_2\Big(\frac{1}{2}\Big) = 4 + 4\log 2 - 3\log 3 < 4.$$

Comparing the coefficients completes the proof. □

4 Conclusion

In this paper we compare portion of primes in several intervals with similar length. We prove if I_1 and I_2 are two intervals with similar Lebesgue measure, then the closer interval to 0, eventually will consider more primes than the far interval.

References

[1] H. Davenport, *Multiplicative Number Theory*, Springer, 1980.

[2] M. Hassani, *Portion of primes in short intervals and Landau's inequality*, Journal of Zankoy Sulaimani - Part A, 22 (2020), 345-352.

[3] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, AMS Chelsea Publishing, 1974.

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Analysing an inequality of Ramanujan concerning prime counting function

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Abstract

Let $\pi(x)$ be the number of prime numbers not exceeding x. Ramanujan's inequality concerning prime counting function asserts that $\pi(x)^2 < \frac{\mathrm{e}\,x}{\log x}\,\pi\left(\frac{x}{\mathrm{e}}\right)$ for x sufficiently large. In this paper we study sharpness and structure of this inequality.

Keywords: Ramanujan's inequality, prime number, asymptotic expansion
Mathematics Subject Classification [2023]: 11A41

1 Introduction

Let $\pi(x)$ denotes the number of prime numbers not exceeding x. Among several conjectures and results concerning distribution of prime numbers, on page 310 (lines -4 and -3) of his second notebook [6], Ramanujan asserts that as $N \to \infty$,

the number of primes less than N is less than $\sqrt{\frac{\mathrm{e}N}{\log N}}$ the number of primes less than $\frac{N}{\mathrm{e}}$.

Berndt [1, page 112] rewrites this inequality as follows

$$\pi(x)^2 < \frac{\mathrm{e}\,x}{\log x}\,\pi\left(\frac{x}{\mathrm{e}}\right) \qquad \text{(for } x \text{ sufficiently large).} \tag{156}$$

This inequality is known as Ramanujan's inequality concerning prime counting function in the literature [2, 3, 4, 5]. In 2012 the author [4] initiated studying structure of (156), by proving that the choose of the number e in this inequality is very critical (Theorem 3.3 in the following). In 2013 the author [3] studied the following generalization of (156) for a given positive integer n,

$$\pi(x)^{2^n} < \frac{\mathrm{e}^n}{\prod_{k=1}^{n}\left(1-\frac{k-1}{\log x}\right)^{2^{n-k}}}\left(\frac{x}{\log x}\right)^{2^n-1}\pi\left(\frac{x}{\mathrm{e}^n}\right).$$

For $n = 1$ the above generalization coincides with (156). In 2020, the author [5] continued his study of Ramanujan's inequality by giving several results about its "sharpness" and "structure". In this lecture, we review these results of the author. Because of limitation in the pages of this report (limited to at most 4 pages), we refer the interested audience to [5] for proofs and other detailed remarks.

2 Sharpness of Ramanujan's inequality

To work on the sharpness of Ramanujan's inequality we give full asymptotic expansions of its left and right hand sides expressions, providing some very nice corollaries.

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Theorem 2.1. *Let* $\ell_k = \sum_{j=0}^{k} j!(k-j)!$ *and* $r_k = \sum_{j=0}^{k} j!\binom{k}{j}$. *Then, for a given integer* $n \geqslant 0$ *we have*

$$\pi(x)^2 = x^2 \sum_{k=0}^{n} \frac{\ell_k}{\log^{k+2} x} + O\left(\frac{x^2}{\log^{n+3} x}\right), \tag{157}$$

and

$$\frac{\mathrm{e}\, x}{\log x}\pi\left(\frac{x}{\mathrm{e}}\right) = x^2 \sum_{k=0}^{n} \frac{r_k}{\log^{k+2} x} + O\left(\frac{x^2}{\log^{n+3} x}\right). \tag{158}$$

As an immediate corollary, we obtain full asymptotic expansions of $\pi(x)^2 - \frac{\mathrm{e}\, x}{\log x}\pi\left(\frac{x}{\mathrm{e}}\right)$ as $x \to \infty$.

Corollary 2.2. *Let* $d_k = \ell_k - r_k = \sum_{j=0}^{k} j!\Big((k-j)! - \binom{k}{j}\Big)$. *For a given integer* $n \geqslant 4$ *we have*

$$\pi(x)^2 - \frac{\mathrm{e}\, x}{\log x}\,\pi\left(\frac{x}{\mathrm{e}}\right) = x^2 \sum_{k=4}^{n} \frac{d_k}{\log^{k+2} x} + O\left(\frac{x^2}{\log^{n+3} x}\right). \tag{159}$$

Note that $d_0 = d_1 = d_2 = d_3 = 0$, and some more initial values of d_k are $d_4 = -1$, $d_5 = -14$, $d_6 = -145$, $d_7 = -1412$, $d_8 = -13985$ and so on. The sequences $(\ell_n)_{n\geqslant 0}$ and $(r_n)_{n\geqslant 0}$ are known in OEIS as A003149 and A000522, respectively. The recurrence $\ell_n = n! + \frac{n+1}{2}\ell_{n-1}$ holds for each $n \geqslant 1$. Thus, by induction we obtain

$$2\, n! \leqslant \ell_n \leqslant \frac{8}{3}\, n!, \tag{160}$$

for each $n \geqslant 1$. Moreover, the above recurrence implies that $\ell_n \sim 2\, n!$ as $n \to \infty$. On the other hand, we observe that if $\mathrm{e}_n = \sum_{k=0}^{n} \frac{1}{k!}$, then

$$0 < \mathrm{e} - \mathrm{e}_n = \sum_{k=1}^{\infty} \frac{1}{(n+k)!} = \frac{1}{n!} \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{n+j} < \frac{1}{n!} \sum_{k=1}^{\infty} \frac{1}{(n+1)^k} = \frac{1}{n\, n!}.$$

Thus, for each $n \geqslant 1$ we obtain $r_n = n!\, \mathrm{e}_n = \lfloor \mathrm{e}\, n! \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. Meanwhile, for $n \geqslant 4$ we have $n! \geqslant 24 > 1/(\mathrm{e} - \frac{8}{3})$. Thus, $\frac{8}{3} n! < \mathrm{e}\, n! - 1$, and considering the right hand side of (160) we obtain $\ell_n < r_n$ for each $n \geqslant 4$. Thus, we obtain the following refinement of Ramanujan's inequality (156).

Corollary 2.3. *Let* $m \geqslant 4$ *is a fixed integer. Then, for* x *sufficiently large*

$$\pi(x)^2 < \frac{\mathrm{e}\, x}{\log x}\,\pi\left(\frac{x}{\mathrm{e}}\right) + x^2 \sum_{k=4}^{m} \frac{d_k}{\log^{k+2} x}.$$

3 Structure of Ramanujan's inequality

The prime number theorem with error term asserts that $\pi(x) = \mathrm{lim}(x) + O(x\mathrm{e}^{-c\sqrt{\log x}})$, where lim is the logarithmic integral, and $c > 0$ is a computable constant (for example, we can take $c = \frac{1}{3}$, and this approximation remains valid for $x > 1.2$ with the coefficient 1 for the Oh term). This gives the expansion

$$\pi(x) = x \sum_{k=0}^{n} \frac{k!}{\log^{k+1} x} + O\left(\frac{x}{\log^{n+2} x}\right), \tag{161}$$

for any integer $n \geqslant 0$. Using (161) with $n = 4$ implies

$$\pi(x)^2 - \frac{\mathrm{e}\, x}{\log x}\,\pi\left(\frac{x}{\mathrm{e}}\right) = -\frac{x^2}{\log^6 x} + O\left(\frac{x^2}{\log^7 x}\right),$$

and this proves (156) for x sufficiently large.

3.1

The first observation concerns the factor $\frac{x}{\log x}$ on the right hand side of (156). This factor basically comes from the true size of $\pi(x)$, which is $\frac{x}{\log x}$. Although, we know that $\frac{x}{\log x-1}$ is much closer approximation for $\pi(x)$, the following result asserts that the factor $\frac{x}{\log x}$ on the right hand side of (156) is the best choice among several approximations of the form $\frac{x}{\log x-h}$.

Theorem 3.1. *Let h is a given real number. If $h \geqslant 0$ then for x sufficiently large*

$$\pi(x)^2 < \frac{\mathrm{e}\, x}{\log x - h}\, \pi\Big(\frac{x}{\mathrm{e}}\Big).$$

If $h < 0$ then the above inequality reverses.

Remark 3.2. More precisely, the expansion (161) with $n = 4$ gives

$$\pi(x)^2 = x^2 \left(\frac{1}{\log^2 x} + \frac{2}{\log^3 x} + \frac{5}{\log^4 x} + \frac{16}{\log^5 x} + \frac{64}{\log^6 x} + O\Big(\frac{1}{\log^7 x}\Big)\right), \tag{162}$$

and

$$\frac{\mathrm{e}\, x}{\log x - h}\, \pi\Big(\frac{x}{\mathrm{e}}\Big) = x^2 \left(\frac{1}{\log^2 x} + \frac{2 + A_1(h)}{\log^3 x} + \frac{5 + B_1(h)}{\log^4 x} + \frac{16 + C_1(h)}{\log^5 x} + \frac{65 + D_1(h)}{\log^6 x} + O\Big(\frac{1}{\log^7 x}\Big)\right),$$

where $A_1(h) = h$, $B_1(h) = h(h+2)$, $C_1(h) = h(h^2+2h+5)$, $D_1(h) = h(h^3+2h^2+5h+16)$. Comparing the coefficients shows that $h = 0$ is the most critical choice, as in Ramanujan's inequality (156).

3.2

The second observation on the structure of Ramanujan's inequality (156) concerns the factor e. The following result is an expression of Theorem 1.3 of [4], however, in [5] we give a much simpler proof.

Theorem 3.3. *If $\alpha \geqslant \mathrm{e}$ then for x sufficiently large*

$$\pi(x)^2 < \frac{\alpha\, x}{\log x}\, \pi\Big(\frac{x}{\alpha}\Big).$$

If $0 < \alpha < \mathrm{e}$ then the above inequality reverses.

Remark 3.4. Assume that $\alpha > 0$, and let $u = \log \alpha$. Then, the expansion (161) with $n = 4$ gives

$$\frac{\alpha\, x}{\log x}\, \pi\Big(\frac{x}{\alpha}\Big) = x^2 \left(\frac{1}{\log^2 x} + \frac{A_2(u)}{\log^3 x} + \frac{B_2(u)}{\log^4 x} + \frac{C_2(u)}{\log^5 x} + \frac{D_2(u)}{\log^6 x} + O\Big(\frac{1}{\log^7 x}\Big)\right),$$

where $A_2(u) = u+1$, $B_2(u) = u^2+2u+2$, $C_2(u) = u^3+3u^2+6u+6$, $D_2(u) = u^4+4u^3+12u^2+24u+24$. Note that for $\alpha = \mathrm{e}$ one has $A_2(1) = 2$, $B_2(1) = 5$, $C_2(1) = 16$ and $D_2(1) = 65$. Comparing these values and the coefficients in (162) shows that $\alpha = \mathrm{e}$ is the most critical choice, as in Ramanujan's inequality (156).

3.3

As another observation on the structure of Ramanujan's inequality, in (156) we replace x by $\mathrm{e}\, x$. Thus Ramanujan's inequality takes the following equivalent form

$$\pi(\mathrm{e}\, x)^2 < \frac{\mathrm{e}^2\, x}{1 + \log x}\, \pi(x). \tag{163}$$

The expansion (161) with $n = 4$ gives

$$\pi(\mathrm{e}\,x)^2 = \mathrm{e}^2\,x^2\left(\frac{1}{\log^2 x} + \frac{2}{\log^4 x} + \frac{4}{\log^5 x} + \frac{19}{\log^6 x} + O\Big(\frac{1}{\log^7 x}\Big)\right),$$

and

$$\frac{\mathrm{e}^2\,x}{1+\log x}\,\pi(x) = \mathrm{e}^2\,x^2\left(\frac{1}{\log^2 x} + \frac{2}{\log^4 x} + \frac{4}{\log^5 x} + \frac{20}{\log^6 x} + O\Big(\frac{1}{\log^7 x}\Big)\right).$$

Thus, we obtain (163) for x sufficiently large. Here, the point is missing the factor $\frac{1}{\log^3 x}$ in both expansions, for which to see the reason, for a given h we have

$$\frac{\mathrm{e}^2\,x}{h+\log x}\,\pi(x) = \mathrm{e}^2\,x^2\left(\frac{1}{\log^2 x} + \frac{A_3(h)}{\log^3 x} + \frac{B_3(h)}{\log^4 x} + \frac{C_3(h)}{\log^5 x} + \frac{D_3(h)}{\log^6 x} + O\Big(\frac{1}{\log^7 x}\Big)\right),$$

where $A_3(h) = -h+1$, $B_3(h) = h^2 - h + 2$, $C_3(h) = -h^3 + h^2 - 2h + 6$, $D_3(h) = h^4 - h^3 + 2h^2 - 6h + 24$. Comparing the coefficients shows that $h = 1$ is the most critical choice, as in (163). Moreover, we get the following result.

Theorem 3.5. *Let h is a given real number. If $h \leqslant 1$ then for x sufficiently large*

$$\pi(\mathrm{e}\,x)^2 < \frac{\mathrm{e}^2\,x}{h+\log x}\,\pi(x).$$

If $h > 1$ then the above inequality reverses.

4 Conclusion

Ramanujan's inequality in the prime number theory is one of challenging inequalities, because to study it we need to use very sharp inequalities about prime counting function. In this paper we reviewed some results to show how much its structure is sharp, and also we analysed its structure by examining several variants of its form. In brief, our observations show that Ramanujan was very clever in critical building of his inequality.

References

[1] B. C. Berndt, *Ramanujan's Notebooks* (Part IV), Springer-Verlag, 1994.

[2] A. W. Dudek and D. J. Platt, *On solving a curious inequality of Ramanujan*, Exp. Math., 24 (2015), no. 3, 289-294.

[3] M. Hassani, *Generalizations of an inequality of Ramanujan concerning prime counting function*, Appl. Math. E-Notes, 13 (2013), 148-154.

[4] M. Hassani, *On an inequality of Ramanujan concerning the prime counting function*, Ramanujan J., 28 (2012), 435-442.

[5] M. Hassani, *Remarks on Ramanujan's inequality concerning the prime counting function*, Commun. Math., 29 (2021), 473-482.

[6] S. Ramanujan, *Notebooks (2 volumes)*, Tata Institute of Fundamental Research, Bombay, 1957.

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Approximately orthogonality prserving pair of operators on Hilbert C^*-modules

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Abstract

In this note, we will study approximately-orthogonality preserving pair of operators on Hilbert C^*-modules over a commutative C^*-algebra.

Keywords: Orthogonality Preserving, Hilbert C^*-modules, Continuous Fields of Hilbert Spaces
Mathematics Subject Classification [2023]: 16L08, 47B49.

1 Introduction

Orthogonality preserving property is one of the most interesting subjects of linear preservers problems. In the setting of Hilbert spaces, Wigner's theorem is the starting point of considerations on orthogonality preserving mappings [18]. A mapping $T : H \longrightarrow K$ between Hilbert spaces is called orthogonality preserving if

$$x \perp y \quad \Rightarrow \quad T(x) \perp T(y), \quad (x, y \in H),$$

where $x \perp y$ means that $\langle x, y\rangle = 0$.

As a generalization of standard orthogonality relation $\perp$, the notion of approximately orthogonality is introduced. We say two vectors $x, y \in H$ are approximately orthogonal or ε-orthogonal and denote it by $x \perp^{\varepsilon} y$, for $\varepsilon \in [0, 1)$, if

$$|\langle x, y\rangle| \leq \varepsilon \|x\| \|y\|.$$

Let $\delta, \varepsilon \in [0, 1)$. A mapping $T : H \longrightarrow K$, between Hilbert spaces is called approximately orthogonality preserving or (δ, ε)-orthogonality preserving, if

$$x \perp^{\delta} y \quad \Rightarrow \quad T(x) \perp^{\varepsilon} T(y), \quad (x, y \in H).$$

Approximately orthogonality preserving mappings between Hilbert spaces have been studied in [3, 4, 19, 21]. In [19], it has been shown that approximately orthogonality mapping between Hilbert spaces is necessarily continuous and injective. A mapping $T : H \to K$, between Hilbert spaces is called ε-orthogonality preserving if

$$x \perp y \quad \Rightarrow \quad T(x) \perp^{\varepsilon} T(y) \quad (x, y \in H).$$

The properties of such map has been investigated in [21].

Recently, some authors have studied orthogonality preserving and approximately orthogonal mappings in the framework of Hilbert C^*-modules [5, 9, 11, 12, 15]. We recall that in a Hilbert C^*-module $(E, \langle\cdot,\cdot\rangle)$, elements x, y are said orthogonal, denoted by $x \perp y$, if $\langle x, y\rangle = 0$, and also for a given $\varepsilon \in [0, 1)$, they are called ε-orthogonal, denoted by $x \perp^{\varepsilon} y$, if $\|\langle x, y\rangle\| \leq \varepsilon \|x\| \|y\|$. Now, similar to the setting of Hilbert

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spaces, we can propose the notions of orthogonality preserving, ε-orthogonality preserving and approximately orthogonality preserving for a map T between Hilbert C^*-modules.

Also, some people have studied the orthogonality preserving property for pair of mappings between Hilbert spaces[14]. Recently, Frank et al has generalized this notion to Hilbert C^*-modules [6].

In this note, we will consider the notion of (δ, ε)-orthogonality preserving or approximately orthogonality preserving for pair of operators on Hilbert spaces, also its generalization to the context of Hilbert C^*-modules. Moreover, we will establish a property of (δ, ε)-orthogonality preserving for a pair of operators on Hillbert C^*-modules over a commutative C^*-algebra.

Indeed, we will use a categorical approach to Hilbert C^*-modules over a commutative C^*-algebra and the fact that corresponding to every Hilbert C^*-module over a commutative C^*-algebra, there is a continuous field of Hilbert spaces.

2 Continuous Fields of Hilbert Spaces

First we recall that the category of (left)Hilbert C^*-modules over a commutative C^*-algebra $\mathcal{A} = C_0(Z)$ is equivalent to the category of continuous field of Hilbert spaces over the locally compact Hausdorff space Z [8, 16], that is defined as follows.

Definition 2.1. Let the pair $((H_z)_{z\in Z}, \Gamma)$, where Z is a locally compact Hausdorff space, $(H_z)_{z\in Z}$ is a family of Hilbert spaces and Γ is a subset of $\prod_{z\in Z} H_z$. Also, we set

$$C_0 - \prod_{z\in Z} H_z = \{x \in \prod_{z\in Z} H_z : \ [z \mapsto \|x(z)\|] \in C_0(Z)\}.$$

The pair $((H_z)_{z\in Z}, \Gamma)$ satisfying the following properties is said to be a continuous filed of Hilbert spaces.

1) Γ is a linear subspace of $C_0 - \prod_{z\in Z} H_z$.
2) The set $\{x(z) : x \in \Gamma\}$ equals to H_z, for every $z \in Z$.
3) If $x \in C_0 - \prod_{z\in Z} H_z$ and for every $z \in Z$ and every $\epsilon > 0$ there is a $x' \in \Gamma$ such that $\|x(s) - x'(s)\| < \epsilon$ in some neighbourhood of z, then $x \in \Gamma$.

If there is no confusion, we denote a continuous field of Hilbert spaces $((H_z)_{z\in X}, \Gamma)$ by Γ.

If Γ is a continuous field of Hilbert spaces, then for every $x, y \in \Gamma$, the function $z \mapsto \langle x(z), y(z)\rangle$ is an element of $C_0(Z)$, In fact, Γ is a (left)Hilbert $C_0(Z)$-module equipped with the following pointwise multiplication and $C_0(Z)$-valued inner product

$$(f \cdot x)(z) = f(z)x(z) \qquad \& \qquad \langle x, y\rangle(z) = \langle x(z), y(z)\rangle,$$

for all $f \in C_0(Z)$, $x, y \in \Gamma$ and $z \in Z$. Moreover, corresponding to every Hilbert $C_0(Z)$-module $\mathcal{M}$, there is a unique continuous field of Hilbert spaces isomorphic to $\mathcal{M}$.

Some facts about continuous fields of Hilbert spaces, which are needed to prove our main result. The following can be concluded directly from [7, Lemma 2.3].

Lemma 2.2. *Let $z_0 \in Z$ and $y \in \Gamma$. If $y(z_0) \neq 0$, then there is $g \in C_0(Z)$ such that $\|gy\| = \|g(z_0)y(z_0)\| = 1$. Consequently, if $H_{z_0} \neq \{0\}$, then for any unit vector $h \in H_{z_0}$, there is $x \in \Gamma$ such that $\|x\| = 1$ and $x(z_0) = h$.*

Lemma 2.3. *Let $\psi : (\{H_z\}_{z\in Z}, \Gamma) \to (\{K_z\}_{z\in Z}, \Gamma')$ be a non-zero $C_0(Z)$-linear mapping. For every $z \in Z$, the map $\psi_z : H_z \longrightarrow K_z$ defined by $\psi_z(x(z)) = (\psi(x))(z)$ is well-defined and linear. Moreover the $C_0(Z)$-linear map ψ is bounded if and only if for every $z \in Z$, ψ_z is linear and bounded and $\sup_{z\in Z} \|\psi_z\| < \infty$. Indeed, $\|\psi\| = \sup_{z\in Z} \|\psi_z\|$ [8].*

Proof. By [13, Proposition 1.3.10], for every $x \in \Gamma$ and $\alpha \in (0, \frac{1}{2})$ there is $y \in \Gamma$ such that $x = \langle x, x\rangle^\alpha y$ and so $\psi(x) = \langle x, x\rangle^\alpha \psi(y)$. Hence, obviously $x(z) = 0$ implies that $\psi(x)(z) = 0$. The rest of the proof is straightforward. □

3 Approximately-Orthogonality Preserving Pair of Operators on Hilbet C^*-modules

In this section, we will define approximately-orthogonality preserving or (δ, ε)-orthogonality preserving mappings on Hilbert spaces and also Hilbert C^*-modules. Then, using a categorical approach to Hilbert C^*-modules over a commutative C^*-algebra, we will investigate (δ, ε)-orthogonality preserving Pair of operators on Hilbet C^*-modules over a commutative C^*-algebra $C_0(Z)$, for some locally compact Hausdorff space Z. Indeed, we will show that if (Phi, ψ) a (δ, ε)-orthogonality preserving pair of operators on a Hilbert $C_0(Z)$-module, then for every $z \in Z$, (Φ_z, ψ_z) is a (δ, ε)-orthogonality preserving pair of operators.

Definition 3.1. We say a pair (Φ, ψ) of Linear maps between Hilbert spaces H, K are (δ, ε)-orthogonality preserving if for every $x, y \in H$, $x \perp^{\delta} y$ implies $\Phi(x) \perp^{\varepsilon} \psi(y)$. (δ, ε)-orthogonality preserving mapping between Hilbert C^*-modules can be defined in a similar way.

Lemma 3.2. *Suppose that V is an open subset of Z and $x, y \in \Gamma$. If $x(z) \perp^{\delta} y(z)$, for every $\delta > 0$ and $z \in V$. Then $\psi(x)(z) \perp^{\varepsilon} \Phi(y)(z)$, for all $z \in V$.*

Proof. Let $z_0 \in V$. We can assume that $\psi(x)(z_0) \neq 0$ and $\psi(y)(z_0) \neq 0$. By Lemma 1.4, we can choose $g, h \in C_0(Z)$ such that

$$\|g\psi(x)\| = \|g(z_0)\psi(x)(z_0)\| = 1 \qquad \|h\Phi(y)\| = \|h(z_0)\Phi(y)(z_0)\| = 1.$$

Also, there is a $f \in C_0(Z)$ such that

$$\|f\| = 1, \quad f(z_0) = 1 \quad f|_{V^c} = 0. \tag{164}$$

For every $z_0 \in Z$,

$$|\langle fgx(z_0), hy(z_0)\rangle||fg(z_0)||h(z_0)||\langle x(z_0), y(z_0)\rangle| \leq |fg(z_0)|||h(z_0))|||\langle x, y\rangle\|.$$

Hence, according to $x(z_0) \perp^{\delta} y(z_0)$, for every $z_0 \in V$, we have

$$vert\langle fgx(z_0), hy(z_0)\rangle| \leq \delta\|fgx(z_0)\|\|hy(z_0)\| \leq \delta\|fgx\|\|hy\|. \tag{165}$$

By Equation 164,

$$\|\langle fgx, hy\rangle\| = \sup_{z_0 \in V}, |\langle fgx(z_0), hy(z_0)\rangle|$$

Now, Equation 165 yields that $\|\langle fgx, hy\rangle \leq \delta\|fgx\|\|hy\|$. In other words, $fgx \perp^{\delta} hy$. Moreover, since the pair (ψ, Φ) is (δ, ε)-orthogonality preserving pair $\psi(fgx) \perp^{\varepsilon} \Phi(hy)$. Hence,

$$\|\langle \psi(fgx), \Phi(hy)\rangle\| \leq \varepsilon\|\psi(fgx)\|\|\Phi(hy)\|.$$

Now, by the choice of f, g and $h \in C_0(Z)$, we have

$$|f(z_0)||g(z_0)||\langle \psi(x)(z_0), \Phi(y)(z_0)\rangle\| \leq \varepsilon\|f\|\|g\psi(x)\|\|h\Phi(y)\| \leq \varepsilon\|f\|\|f\psi(x)\|\|h\Phi(y)\|$$

Hence,

$$|f(z_0)||g(z_0)||\langle \psi(x)(z_0), \Phi(y)(z_0)\rangle\| \leq \varepsilon|g(z_0)|\|\psi(x)(z_0)\||h(z_0)|\|\Phi(y)(z_0)\|.$$

Consequently,

$$|\langle \psi(x)(z_0), \Phi(y)(z_0)\rangle| \leq \varepsilon\|\psi(x)(z_0)\|\|\Phi(y)(z_0)\|,$$

which is the desired result. □

Theorem 3.3. *For every $z \in Z$, the pair (Φ_z, ψ_z) is (δ, ε)-orthogonality pre*

Proof. Let $z_0 \in Z$, $x, y \in \Gamma$. Consider $0 < \delta' < \delta$ and let $x(z_0) \perp^{\delta'} y(z_0)$. By the definition,

$$|\langle x(z_0), y(z_0)\rangle| \leq \delta' \|x(z_0)\| \|y(z_0)\|.$$

Moreover, since $\delta' < \delta$, $|\langle x(z_0), y(z_0)\rangle| \leq \delta' \|x(z_0)\| \|y(z_0)\|$.

Now, according to the continuity of the map $z \mapsto \langle x(z), y(z)\rangle$, there is some open neighborhood V_{z_0} that for every $z \in V_{z_0}$, we have

$$|\langle x(z), y(z)\rangle \leq \delta\delta \|x(z)\| \|y(z)\|.$$

Hence, for every $z \in V_{z_0}$, $x(z) \perp^{\delta} y(z)$. Now, by the previous lemma, for every $z \in V_{z_0}$, $\psi(x)(z) \perp^{\epsilon} \Phi(y)(z)$. In particular, $\psi(x)(z_0) \perp^{\varepsilon} \Phi(y)(z_0)$. Consequently, $\psi_{z_0}(x(z_0)) \perp^{\varepsilon} \Phi(y(z_0))$. Consequently, $(\psi_{z_0}(x(z_0))\varepsilon\Phi_{z_0}(y(z_0))$ is (δ', ε)-orthogonality periserving. Moreover, since it holds every $\delta' < \delta$, consider the limit of δ' as converges to δ, hence we have (ψ_{z_0}, Φ_{z_0}) is (δ, ε)-orthogonality preserving.

□

References

[1] M.B. Asadi, F. Olyaninezhad, Orthogonality preserving pairs of operators on Hilbert $C_0(Z)$-modules, Linear and Multilinear algebra, https://doi.org/10.1080/03081087.2020.1825610.

[2] J. Chmieliński, Linear mappings approximately preserving orthogonality, J. Math. Anal. Appl. 304 (2005) 158-169.

[3] J. Chmieleński, Orthogonality equations with two unknown functions. Aequationes Math., 90 (2016), 11-23.

[4] J. Chmieliński, Stability of the orthogonality preserving property in finite-dimensional inner product spaces, J. Math. Anal. Appl. 318 (2006) 433-443.

[5] M. Frank, A.S. Mishchenko, A.A. Pavlov, Orthogonality preserving, C^*-conformal and conformal module mappings on . J. Func. Anals., 260 (2011) 327-339.

[6] M. Frank, M.S. Moslehian, A. Zamani., Orthogonality preserving property for pairs of operators on Hilbert C^*-modules. arXive: 1711.04724v2 [Math.OA] 2019. 01-109.

[7] O. Hatori, T. Miura, H. Takagi, Characterization of isometric isomorphisms between uniform algebras via non-linear range-preserving properties, Proc. Amer. Math. Soc., 134 (2006) 2923-2930.

[8] C. Heunen, M.L. Reyes, Frobenius Structures Over Hilbert C^*-modules, Commun. Math. Phys. 361 (2018) 787-824.

[9] D. Ilišević, A. Turnšek, Approximately orthogonality preserving mappings on C^*-modules, J. Math. Anal. Appl. 341 (2008) 298-308.

[10] E. C. Lance, *Hilbert C*-modules. A toolkit for operator algebraists*, London Mathematical Society Lecture Note Series, 210, Cambridge University Press, Cambridge, 1995.

[11] C.-W. Leung, C.-K. Ng and N.-C. Wong, Linear orthogonality preservers of Hilbert bundles, J. Aust. Math. Soc. 89 (2010) 245-254.

[12] C.-W. Leung, C.-K. Ng and N.-C. Wong, Linear orthogonality preservers of Hilbert C^*- modules, J. Operator Theory 71 (2014) 571-584.

[13] V. M. Manuilov and E. V. Troitsky, Hilbert C^*-modules, Amer. Math. Soc., Translations of Mathematical Monographs, vol. 226, 2005.

[14] L. Molnár, Generalizations of Wigner's unitary- antiunitary theorem for indefinite inner product spaces. Comm. Math. Phys. 201 (2000) 785-791.

[15] M.S. Moslehian, A. Zamani, Mapping preserving approximately orthogonality in Hilbert C^*-modules, Math. Scand. 122 (2018) 257-276.

[16] R.G. Swan, Vector bundles and projective modules, Trans. Amer. Math. Soc. 105 (1962) 264-277.

[17] A. Turnšek, On mappings approximately preserving orthogonality, J. Math. Anal. Appl. 336 (2007) 625-631.

[18] E. Wigner, Gruppentheorie und ihre Anwendung auf die Quantenmechanik Atomspek-tren, Vieweg, Braunschweig, 1931.

[19] P. Wójcik, On certain basis connected with operator and its applications, J. Math. Anal. Appl. 423 (2015) 1320-1329.

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Furstenberg booundary for pair of groups

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Abstract

In this note, we will study Furstenberg boundary for pair of topological discrete groups.

Keywords: Discrete Group, Furstenberg Boundary, Hamana Boundary
Mathematics Subject Classification [2023]: 46L35, 20F65, 37A20.

1 Introduction

In [6], considering the minimal C^*-subalgebra of $l^\infty(G)$ that arises as the image of a positive G-equivaiant projection on $l^\infty(G)$, for an arbitrary discrete group G, it has been shown that this algebra exists and is unique up to isomorphism. Indeed, this fact follows from Hamana's work on G-injective envelopes [3], [4]. Morerver, it can be identified with $C(\partial_F G)$, the space of all continuous functions on the Ferstenberg's universal G-boundary. This fact yields to some results on rigidity for certain G-equivarint maps
In this note, we consider Furstenberg boundary of a pair of groups, introduced by [5] and show that the preceding classical fact can be generalized to a more general case of Furstenberg boundary of a pair of groups G, H.

2 Furstenberg Boundary of Pair of Groups

We recall that for a discrete topological group G and a compact Hausdorff space X, group G is said to act on X if there is a continuous group homeomorphism from G into $homeo(X)$, the group of all homomorphisms of X onto itself. In this case, we say that X is a G-flow. Morphisms in the category of G-flows consist of G-equivariant maps. i.e. $\phi : X \to Y$ s.t.

$$(\forall g \in G, \forall x \in \mathcal{X}) \qquad \phi(gx) = g\phi(x).$$

For given G-flow X and $x \in X$, the space $O(x) = \{gx : g \in G\}$ is called the orbit of $x \in X$. A G-flow X is said to be minimal if $\overline{O(x)} = X$, for every $x \in X$. Also, a G-flow X is said strongly proximal if $P(X)$, the space probability measures on X is proximal, equivalently, for every probability measure $\nu \in P(X)$, the weak*-closure of the G-orbit $G\nu$ contains the point mass $\delta_x \in P(X)$, for some $x \in X$. Every minimal and strongly proximal G-flow X is called G-boundary. Furstenberg proved in [1] that every group G has a unique G-boundary that is universal. In the sense that every G-boundary is a continuous G-equivariant image of $\partial_F(G)$. For more details on G-flows, one can see [2].

An affine (G, H)-flow is an affine G-flow with an H-fixed point. It is an irreducible affine (G, H)-flow if it does not contain smaller affine (G, H)-flow. In [5], it has been shown that there is an irreducible affine (G, H)-flow that is universal in the sense that it admits a G-morphism onto every irreducible affine (G, H)-flow. Moreover, this universal flow is unique. The unique universal irreducible affine (G, H)-flow is denoted by $\Delta(G, H)$.

Furstenberg boundry of the pair (G, H) is the set of extremal points, i.e.

$$\partial_F(G, H) = Ext(\Delta(G, H)).$$

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Indeed, [5, Proposition 1.4] implies that

$$P(\partial_F(G,H)) = \Delta(G,H).$$

Definition 2.1. A G-flow X is called a (G,H)-boundary if $P(X)$ is (G,H)-irreducible.

By [5, Proposition 1.7], a G-flow X is (G,H)-boundary iff H fixes a measure ν in $P(X)$ and every such measure in $P(X)$ is G-contracted to every point of X, i.e $\delta_x \in \overline{G\nu}$, for every $x \in X$.

Remark 2.2. In the following, we mention some main properties of Furstenberg boundary of pair of groups (G,H) (See [5]).

- In general, $\partial_F(G,H)$ does not satisfy universality. In [M, Proposition 1.8], providing a concrete example, it is shown that there is not always a G-map from $\partial_F(G,H)$ to every (G,H)-boundary, even for discrete groups.
- What remains true is that for every (G,H)-boundary X, there is a G-map $\partial_F(G,H) \to P(X)$ whose image contains X, when X is identified with a subspace of X.
- However, the Furstenberg boundary is unique up to unique identification.

3 (G,H)-operator systems and Hamana boundary

Let G be a discrete group and H a subgroup of G. Also, let $\mathcal{S}$ be an operator system. We say that $\mathcal{S}$ is a G-operator system if there is a homomorphism from $\mathcal{S}$ into the group of order isomorphisms on $\mathcal{S}$ that sends the identity element of G to the unit of $\mathcal{S}$. We refer to such a homomorphism map as the G-action on $\mathcal{S}$.

Definition 3.1. A G-operator system that

$$(\forall h \in H, \forall x \in X), \qquad h.x = x$$

is called a (G,H)-operator system.

In the following, we require Hamana's theory of injective envelopes for (G,H)-operator systems. For the general theory of injective envelopes of operator systems and G-operator systems, we refer to [3] and [6].

Let $\mathcal{S}$ be a (G,H)-operator system. One can consider $\lambda^\infty(G,\mathcal{S})$ as a (G,H)-operator system. Moreover, $I_{(G,H)}(\mathbb{C}) \subseteq l^\infty(G,\mathcal{S})$ equipped with the Choi-Effross product is a commutative C^*-algebra. Hence, there is a compact Hausdorff space, called Hamana boundary of pairs of groups, denoted by $\partial_H(G,H)$, such that $C(\partial_H(G,H)) = I_{(G,H)}(\mathbb{C})$. Applying a similar argument of existence of injective envelope of operator systems, the following holds.

Proposition 3.2. *Let $\mathcal{S}$ be a (G,H)-operator system. Then $\mathcal{S}$ has a (G,H)-injective envelope $(I_{(G,H)}(\mathcal{S}),\kappa)$. This injective envelope is unique in the sense that for every (G,H)-injective envelope (U,i) of $\mathcal{S}$, there is a completely isometric G-equivariant map $\phi : I_{(G,H)}(\mathcal{S}) \to U$ such that $\phi\kappa = \mathrm{i}$.*

In the classical case, where we have a G-operator system, in other words, $H = \{e\}$, the unit of group G. [6] show that $\partial_H(G)$ satisfies the following properties:

1. Minimality;
2. Strong Proximality;
3. Universality.

Hence, by uniqueness of a universal, minimal and strongly proximal G-flow, we have $\partial_H(G) = \partial_F(G)$. In the case of Hamana boundary of pair of groups, we show that $\partial_H(G,H) = \partial_F(G,H)$. In other words, $C(\partial_H(G,H)) = I_{(G,H)}(\mathbb{C})$.

In the following, we show that $\partial_H(G,H) = \partial_F(G,H)$. Consequently,

$$C(\partial_F(G,H)) = I_{G,H}(\mathbb{C}).$$

In order to do that, we apply the following lemmas. which are generalizations of [1, Proposition 4.2] and [6, Lemma 3.2].

Lemma 3.3. *Let M be an irreducible affine (G,H)-flow and B be a (G,H)-boundary. Then any equivariant map from M to $P(B)$ has δ_M as its range. Moreover, there is at most one equivariant map from M' to δ_M.*

Lemma 3.4. *Let M be an irreducible affine (G,H)-flow. Let B be a (G,H)-boundary. There is at most one unital positive (G,H)-equivariant map*

$$\phi : C(B) \to C(M).$$

If such a map exists, then ϕ is a unital $*$-homoemorphism.

Proposition 3.5. *Let G be a discrete group and $H \leq G$. Then,*

$$P(\partial_H(G,H)) = \Delta(G,H).$$

In other words,

1. *$P(\partial_H(G,H))$ is an irreducible affine (G,H)-flow, equivalently, $\partial_H(G,H)$ is a (G,H)-boundary.*
2. *$P(\partial_H(G,H))$ satisfies the universality property.*

Proof. Equivalent to the concept of being a (G,H)-boundary, we need to show the following:

- H fixes a measure in $P(X)$.
  Let ν be measure in $P(X)$. We show that H fixes a measure in $P(X)$. In other words, for every $h \in H$, $h\nu = \nu$. Note that, for every $f \in C(X)$,

  $$h\nu(f) = \int f d(h\nu) = \nu(f^h) = \int_X f d\nu = \nu(f).$$

  Hence, the desired result holds.

- The fact that every measure in $P(X)$ is G-contracted to every point of X can be proved similar to the classical case (See [6]).

In the following we show that $P(\partial_H(G,H)$ stisfies the universality property. In other wordrs, Let X be a (G,H)-irreducible affine flow. There is a (G,H)-morphism $\phi : P(\partial_H(G,H)) \to X$. This can be proved with a slight modification of [6, Theorem 3.11] and considering Lemma 357 and Lemma 368.

□

References

[1] H. Furstenberg, *Boundary Theory and Stochastic Process on Homogeneous Spaces*, Harmonic analysis on homogeneous spaces, Proceedings of the Symposium of Pure Mathematics, vol. 26, American Mathematiacl Society, Providence, RI., 1973.

[2] S. Glasner, *Proximal Flows*, Lecture Notes in Mathematics, vol. 517, Springer Verlag, Berlin, Heidelberg, New York, 1976.

[3] M. Hamana, *Injective envelopes of Banach modules*, Tohoku Mathematical Journal, **30** (1978), 439-453.

[4] *Injective Envelopes of C^*-dynamical Systems*, Tohoku Mathematical Journal, **37**, (1985), 463-487. M. Hamana,

[5] N. Monod, Furstenberg boundaries for pairs of groups.

[6] M. Kalantar and M. Kennedy, Boundaries of reduced group C^*-algebras of discrete groups.

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Contractive mappings and some fiexed point theorems on S-metric spaces

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Abstract

In this paper, we extend the notion of Rhoades mapping to S-metric spaces and define a new type of contractive mappings. Then, we present the notion of a cluster point on an S-metric space and study some properties of cluster points. We give some fixed point theorems by means of cluster points on S-metric spaces.

Keywords: C_S-mapping;L_S-mapping; cluster point.
Mathematics Subject Classification [2023]: 4E35, 54E40, 54E45, 54E50

1 Introduction

Metric spaces are very important in various areas of mathematics such as analysis, topology, applied mathematics etc. So various generalizations of metric spaces have been studied and several fixed point results were obtained (for example, see [1]). Recently, Sedghi, Shobe and Aliouche have defined the concept of an S-metric space as follows:

Definition 1.1. [4] Let X be a nonempty set and $S : X^3 \to [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$:

(1) $S(x, y, z) = 0$ if and only if $x = y = z$,

(2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then S is called an S-metric on X and the pair (X, S) is called an S-metric space.

Let (X, d) be a complete metric space and T be a self-mapping on X. In [10], the following condition was introduced for a self-mapping T: for each $x, y \in X$, $x \neq y$:

$$(R.25) \qquad d(Tx, Ty) < max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

(such mappings were later called Rhoades mappings). However, no fixed point theorem was given in [6] for mappings satisfying (R.25). Chang presented the notion of a C-mapping and obtained some fixed point theorems using such mappings under (R.25) in [1]. Also, Liu, Xu and Cho defined the concept of an L-mapping and studied some fixed point theorems using such mappings under (R.25) in [3]. On the other hand, Park introduced a new contractive mapping using the diameter of $U_x \cup U_y$ where $U_x = \{T^n x : n \in \mathbb{N}\}$. He presented also the relationships between these contractive mappings and the condition (R.25) and then obtained some fixed point theorems.

Motivated by the above studies, we extend the notion of Rhoades mapping to S-metric spaces and define a new type of contractive mappings. Then, we present the notion of a cluster point on an S-metric space and study some properties of cluster points. We give some fixed point theorems by means of cluster points on S-metric spaces.

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2 Main results

First, we define some new contractive mappings and the notions of a C_S-mapping and an LS-mapping on an S-metric space. Also we investigate their relationships with each other and give counterexamples.

Now we recall some definitions, lemmas, a remark and a corollary which are needed in the sequel. The following can be found in the papers referred to.

Definition 2.1. [4] Let (X, S) be an S-metric space and AX.

(1) A subset A of X is called S-bounded if there exists $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$.

(2) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, there exists $n_0 \in N$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \varepsilon$. We denote this by $\lim_{n\to\infty} x_n = x$ or $\lim_{n\to\infty} S(x_n, x_n, x) = 0$.

(3) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, there exists $n_0 \in N$ such that for all $n, m \geq n_0$, $S(x_n, x_n, x_m) < \varepsilon$ for each $\varepsilon > 0$.

(4) The S-metric space (X, S) is called complete if every Cauchy sequence is conovergent.

Lemma 2.2. *[4] Let (X, S) be an S-metric space. Then*

$$S(x, x, y) = S(y, y, x), \tag{2.1}$$

for all $x, y \in X$.

Lemma 2.3. *[4] Let (X, S) be an S-metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n \to x$ and $y_n \to y$, then $S(x_n, x_n, y_n) \to S(x, x, y)$.*

Remark 2.4. [5] Every S-metric space is topologically equivalent to a B-metric space.

Corollary 2.5. *[5] Let $T : X \to Y$ be a map from an S-metric space X to an S-metric space Y . Then T is continuous at $x \in X$ if and only if $Tx_n \to Tx$ whenever $x_n \to x$.*

Now we consider the Rhoades condition (R25) for S-metric spaces and define a C_S-mapping (or an L_S-mapping).

Definition 2.6. Let (X, S) be an S-metric space and T be a self-mapping of X. We define

$$(S25) \qquad S(Tx, Tx, Ty) < max\{S(x, x, y), S(Tx, Tx, x), S(Ty, Ty, y), S(Ty, Ty, x), S(Tx, Tx, y)\},$$

for each $x, y \in X$, $x \neq y$.

Definition 2.7. Let (X, S) be an S-metric space and T be a self-mapping on X. T is called a C_S-mapping on X if for each $x \in X$ and each positive integer $n \geq 2$ satisfying T

$$T^i x \neq T^j x, 0 \leq i, j \leq n - 1, \tag{2.2}$$

we have

$$S(T^n x, T^n x, T^i x) < max_{1\leq j\leq n}\{S(T^j x, T^j x, x)\}, \quad i = 1, 2, ..., n - 1. \tag{2.3}$$

Definition 2.8. Let (X, S) be an S-metric space and T be a self-mapping on X. T is called an L_S-mapping on X if for each $x \in X$ and each positive integer $n \geq 2$ with the condition (2.2) we have

$$S(T^n x, T^n x, T^i x) < max_{0\leq p\leq q\leq n}\{S(T^p x, T^p x, T^q x)\}, \quad i = 1, 2, ..., n - 1. \tag{2.4}$$

Proposition 2.9. *Let (X, S) be an S-metric space and T be a self-mapping on X. If T satisfies the condition (S25), then T is a C_S-mapping.*

Proposition 2.10. *Let (X, S) be an S-metric space. Then the notions of a C_S-mapping and an L_S-mapping are equivalent.*

Now we give the definition of the notion of diameter on an S-metric space.

Definition 2.11. Let (X, S) be an S-metric space and A be a nonempty subset of X. We define

$$diam\{A\} = sup\{S(x, x, y) : x, y \in A\}.$$

Then $diamA$ is called the diameter of A. If A is an S-bounded set, then we will write $diamA$.

Definition 2.12. Let (X, S) be an S-metric space, T be a self-mapping on X, $U_x = T^n x : n \in \mathbb{N}$, $diam\{U_x\} < \infty$ and $diam\{U_y\} < \infty$. We define

$$(S25a) \qquad S(Tx, Tx, Ty) < diam\{U_x \cup U_y\},$$

for each $x, y \in X$ with $x \neq y$.

Proposition 2.13. *Let (X, S) be an S-metric space and T be a self-mapping on X. If T satisfies the condition (S25), then T satisfies the condition (S25a).*

Now, we obtain new fixed point theorems by means of cluster points on an S-metric space.

Definition 2.14. [4] Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$, the open ball $B_S(x, r)$ is defined as follows:

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\}.$$

Definition 2.15. Let (X, S) be an S-metric space and $A \subset X$ be any subset. A point $x \in X$ is a cluster point of A if

$$(B_S(x, r)\{x\}) \cap A \neq \emptyset,$$

for every $r > 0$.

Theorem 2.16. *Let (X, S) be an S-metric space and $A \subset X$. Then x is a cluster point of A if and only if there exist $x_i \in S(i = 1, 2, ..., n, ...)$ such that $x_i \neq x_j$ for each $i \neq j$ and $lim_{n\to\to\infty} S(x_n, x_n, x) = 0$.*

Theorem 2.17. *Let (X, S) be an S-metric space, T be a continuous C_S - mapping on X and x be a point in X for which $\{T^n x\}_{n=0}^{\infty}$. has a cluster point x_0. Then $T^n x_0$, $n = 0, 1, 2, \ldots$ are cluster points of $\{T^n x\}_{n=0}^{\infty}$.*

Theorem 2.18. *Let (X, S) be an S-metric space, T be a continuous C_S-mapping on X and x be a point in X for which $\{T^n x\}_{n=0}^{\infty}$ has a cluster point x_0. Then T has a fixed point in $\{T^n x\}_{n=0}^{\infty}$ if and only if one of the following is satisfied:*

(1) $\{T^n x\}_{n=0}^{\infty} = 0$ is a convergent sequence.

(2) There existsa a positive integer q such that $T^q z = z$, where z is some point in $\{T^n x\}_{n=0}^{\infty}$.

Corollary 2.19. *Let (X, S) be an S-metric space, T be a continuous L_S-mapping (or T satisfies the condition (S25)) on X and x be a point in X for which $\{T^n x\}_{n=0}^{\infty}$ has a cluster point x_0. Then T has a fixed point in $\{T^n x\}_{n=0}^{\infty}$ if and only if one of the following is satisfied:*

(1) $\{T^n x\}_{n=0}^{\infty}$ is a convergent sequence.

(2) There exists a positive integer q such that $T^q z = z$, where z is some point in $\{T^n x\}_{n=0}^{\infty}$.

References

[1] S. S. Chang, *On Rhoades open questions and some fixed point theorems for a class of mappings*, Proc. Amer. Math. Soc. 97 (2) (1986) 343346.

[2] A. Gholidahneh, S. Sedghi, T. Dosenovic and S. Radenovic, *Ordered S-metric spaces and coupled common fixed point theorems of integral type contraction*, to appear in Math. Interdisciplinary Res.

[3] A.A. Jafarian, H. Radjavi, P. Rosenthal and A.R. Sourour, *Simultaneous triangularizability, near commutativity and Rota's theorem*, Trans. of American Math. Soc., 347 (1995), No. 6, 2191–2199.

[4] Z. Liu, Y. Xu and Y. J. Cho, *On characterizations of fixed and common fixed points*, J. Math. Anal. Appl. 222 (1998) 494504.

[5] S. Sedghi, N. Shobe and A. Aliouche, *A generalization of fixed point theorems in S-metric spaces*, Mat. Vesnik 64 (3) (2012) 258266.

[6] S. Sedghi and N. V. Dung, *Fixed point theorems on S-metric spaces*, Mat. Vesnik 66 (1) (2014) 113124.

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Fixed-Point theorems in complete Metric spaces

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Abstract

In this paper, we have extended and improved some existent results in a complete metric space. Then, we present a slight improvement on some well-known lemmas that are related to nonexpansive mappings and nonincreasing distances mappings.

Keywords: fixed point, complete metric space, reflexive Banach space
Mathematics Subject Classification [2023]: 4E35, 54E40, 54E45, 54E50

1 Introduction

In mathematics and related elds, the xed point theory plays a signicant role in several elds, see [1]. In this recent work, we continue studying some existent results by many researchers on xed point theorems in Banach and metric spaces. In 2014, a general xed point theorem for multivalued mappings that is not always a contraction was proposed by Abdul Latif [1]. In addition, he derived some modern studies on contraction mappings. In 2020, Singh and Anthony studied and extended the concept of more than one xed point into mappings on a cone Sb-metric space. Moreover, they provided some robust examples to validate their results e structure of our proposed paper is organized as follows: In following, we present our rst generalization of results by Ray [3], where we provide some well-known results consisting of multivalued mappings in a complete metric space (X, d). Moreover, we present the same study for the results provided by Taskovic [4]. Then, we provide some results about the xed point theorem of three mappings f_1, f_2, and f_3 in a complete metric space (X, d). Furthermore, a brief discussion of few common fixed point theorems is going to be presented. Also, we will give a study of the three mappings f_1, f_2, and f_3 that satisfy the following equation:

$$d(f_1x, f_2y)^2\phi(d(f_3x, f_1x), d(f_3y, f_2y), d(f_3x, f_2y), d(f_3y, f_1x), d(f_3x, f_1x), d(f_3x, f_2y), d(f_3y, f_1y), d(f_3y, f_2y)) \tag{166}$$

where $x, y \in X$; $f_1f_3 = f_3f_1$, $f_2f_3 = f_3f_2$, $f_1(x) \subset f_3(x)$; $f_2(x) \subset f_3(x)$; $and \phi$ are continuous function that satisfy the following condition:

$$\phi(s, s, a_1s, a_2) < s \quad for \quad any \quad s > 0 \quad and \quad a_i \in \{0, 1, 2\} \quad such \quad that \quad a_1 + a_2 = 2. \tag{2}$$

More studies have been done and considered by many authors regarding theorems that satisfy different functional inequalities and in terms of fixed point theorems in different spaces as well as several applications such as solving nonlinear and differential equations, see [5]. In following, we mainly extend the results that have been obtained by Kirk in [1]. In fact, we work on modifying the given lemmas that are related to multivalued mappings in order to extend the results for the sake of generalization.

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2 Main results

Jleli and Bessem introduced a novel definition of generalized metric spaces, where they worked on extending several existing fixed-point findings along with the Banach contraction fundamental theorem [4]. In addition, Taskovic presented and proved a fixed-point theorem on a metric space (X, d) along with a mapping T, which is not necessarily continuous and satisfy a condition of he type,

$$ad(Tx,Ty)+bd(x,Tx)+cd(y,Ty)-min\{d(x,Ty),(y,Tx)\} \leq qd(x,y), \qquad \text{for all} \quad x,y \in X \tag{3}$$

In the following theorem, we present the first result for two mappings f_1 and f_2 that generalizes the abovementioned result.

Theorem 2.1. *Suppose that f_1 and f_2 are a couple selffunctionals of a complete metric space (X, d) with the following condition:*

$$ad(f_1x, f_2y) + bd(x, f_1x) + cd(y, f_2y) - min\{d(x, f_2y), (y, f_1x)\} \leq qd(x, y), \tag{4}$$

for all $x, y \in X$ such that the numbers $a, b, c \geq 0$ and $q > 0$ with $a > q + 1$ and $a + c > 0$. Then f_1 and f_2 will contain a unique common fixed point.

Now, we introduce the below genera theorem of Ray [3] for a couple of self-mapping on a metric space.

Theorem 2.2. *Suppose that f_1 is a continuous functional in a complete metric space (X, d) and suppose that $f_2 \subset f_1$ is another mapping in the same space where the two mappings commute with each other, and f_2 satisfies the following condition:*

$$\begin{aligned} d(f_2(x), f_2(y)] \leq\ & \alpha(d(f_1(x), f_1(y)))d(f_1(x), f_1(y)) \\ & +\beta(d(f_1(x), f_1(y))[d(f_1(x), f_2(x))d(f_1(y), f_2(y))]\gamma(d(f_1(x), f_1(y))] \\ & \gamma(f_1(x), f_1(y))[d(f_1(x)g(y)) + d(f_1(y), f_2(x))], \end{aligned} \tag{5}$$

for each $f_1(x) \neq f_1(y)$, where α, β, and γ are monotonically decreasing functions from $(0, \infty)$ into $(0, 1)$ with $\alpha(t) + 2\beta(s) + 2\gamma(s) < 1$ and $s \in (0, \infty)$. Then, the two mappings f_1 and f_2 have only one common fixed point in X.

Another generalization of Theorem 2 of Ray goes as follows:

Theorem 2.3. *Suppose that f_1 and f_2 are two self-functionals in a complete metric space (X, d), where the two mappings satisfy the following condition:*

$$\begin{aligned} d(f_1x, f_2y)] \leq\ & \alpha(d(x, y))d(x, y) + \beta(d(x, y)) \\ & [d(x, f_1x) + d(y, f_2y)] \\ & + \gamma(d(x, y)[d(x, f_2y) + d(y, f_1x)], \end{aligned} \tag{6}$$

for every $x \neq y \in X$, such that α, β, and γ are monotonically decreasing functions from $(0, \infty)$ into $[0, 1)$ with $\alpha(s) + 2\beta(s) + 2\gamma(s) < 1$, $s \in (0, \infty)$ Then the two mappings f_1 and f_2 have only one common fixed point in X.

Let us assume that ϕ represent an upper semicontinuous function from the space R^{+4} to the space R^+, where this function is monotonically increasing in every variable and for each number $s > 0$ the following condition is satisfied:

$$\phi(s, s, a_1s.a_2s) < s, \tag{7}$$

where $a_i \in \{0, 1, 2\}$ such that $a_1 + a_2 = 2$.

In the following, we present the main theorem of this work.

Theorem 2.4. *Suppose that f_1, f_2, and f_3 are three selffunctionals of a complete metric space (X, d), which satisfy the following conditions:*

$$f_1 f_3 = f_3 f_1, f_2 f_3 = f_3 f_2, f_1(x) \subset f_3(x) \quad and \quad f_2(x) \subset f_3(x)$$

$$[d(f_1 x, f_2 y)]^2 \leq \phi[d(f_3 x, f_1 y), d(f_3 y, f_2 y), d(f_3 x, f_2 y), d(f_3 y, f_1 x), d(f_3 x, f_1 x), d(f_3 x, f_2 y), d(f_3 y, f_1 x) d(f_3 y, f_2 y)].$$

for all $x, y \in X$. Further, let f_3 be continuous, then f_1, f_2, and f_3 have only one common fixed point in X. The proof of this theorem goes in a similar way to the one by Devi Prasad . Therefore, we omit the proof.

Now, we present the following three theorems.

Theorem 2.5. *Suppose that f_1, f_2, and f_3 are three selffunctionals of a complete metric space (X, d) sin such a way that,*

$f_1 f_3 = f_3 f_1, f_2 f_3 = f_3 f_2, f_1(x) \subset f_3^r(x)$, and $f_2(x) \subset f_3^r(x)$, then there is a non-negative and nonzero integersp, q, and r satisfying the fhe following condition:

$$d[(f_1^p x, f_2^q y)]^2 = \phi \begin{bmatrix} (f_3^r x, f_1^p x) d(f_3^r y, f_2^q y) d(f_3^p x, f_2^q y) d(f_3^p y, f_1^p x), \\ d(f_3^r x, f_1^p x) d(f_3^r y, f_2^q y) d(f_3^r x, f_1^p y) d(f_3^r y, f_2^q x), \end{bmatrix}, \tag{9}$$

for all $x, y \in X$, and if f_3^r is continuous, then $f_1, f_2, and f_3$ have only one common fixed point.

Theorem 2.6. *Suppose that $\{f_{1n}\}$, $\{f_{2n}\}$, and $\{f_{3n}\}$ are sequences of self-functionals of a complete metric space (X, d) such that $\{f_{1n}\}$, $\{f_{2n}\}$, and $\{f_{3n}\}$ converge uniformly to selfmappings f_1, f_2, and f_3 on X with f_3 continuous. Suppose that for each $n \geq 1$, x_n is a common fixed point of f_{1n} and f_{3n}, and y_n is a common fixed point of f_{2n} and f_{3n}. Furthermore, let f_1, f_2, and f_3 satisfy conditions (8) and (7). If x_0 is the common fixed point of f_1, f_2, and f_3, and $supd(x_n, x_0) < \infty$ and $supd(y_n, x_0) < \infty$. Then $x_n \to x_0$ and $y_n \to y_0$.*

Theorem 2.7. *Let $\{f_{1n}\}$, $\{f_{2n}\}$, and $\{f_{3n}\}$ be sequences of selfmappings of a complete metric space (X, d) such that f_{3n} is continuous for each n and f_{1n}, f_{2n}, and f_{3n} satisfy conditions (8) and (7) for each $n \geq 1$. If f_1, f_2, and f_3 are uniform limits of f_{1n}, f_{2n}, and f_{3n} respectively, then,f_1, f_2, and f_3 also satisfy conditions (8) and (7). Also, let $\{x_n\}$ be a sequence of the only one common fixed point of f_{1n}, f_{2n}, and f_{3n} converges to the only one common fixed point x_0 of f_1, f_2, and f_3, whenever $supd(x_n, x_0) < \infty$.*

Suppose that (Ω, d) is a metric space and let Q be a class of countably compact subsets of Ω, and suppose that this class is stable under intersections, normal, and contains all the closed balls in . Also, suppose that $\{\Lambda_i, (i \in N_0)\}$ are subsets of . Moreover, we define ζ as

$$\zeta(\Lambda) = \begin{cases} r(\Lambda)/\delta(\Lambda) & \delta(\Lambda) > 0 \\ 1 & \delta(\Lambda) = 0 \end{cases},$$

such that $\delta(\Lambda) = sup\{d(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 \in \Lambda\}$ $r_u(\Lambda) = sup\{d(\lambda_1, \lambda_2) : \lambda_2 \in \Lambda\}(\lambda_1 \in \Lambda) r(\Lambda) = inf\{r_{\lambda_1}(\Lambda) : \lambda_1 \in \Lambda\}$

We shall take this assumption in order to initiate our proposed theorems.

Theorem 2.8. *If $T : \Omega \to F(x)$ be a mapping such that,*

$$T(x) \cap \Omega \neq \emptyset \qquad forallx \in \Omega, \tag{10}$$

and for all $x \in \Omega$.

$$T(x) \cap Q = \{T(x) \cap \Lambda : \Lambda \in Q\}, \tag{11}$$

is a compact class of which each nonempty member is a compact subset of X. For any $G \in Q$ satisfying $T(\xi) \cap G \neq \emptyset$, for all $\in G$,

$$H(T(x) \cap G, T(y) \cap G) \leq d(x, y), forallx, y \in G. \tag{12}$$

Then, T has a fixed point in Ω.

Remark 2.9. In the following, we mention two theorems by Kirk and Samanta in [1].

Corollary 2.10. *(see [1]). Suppose that (Ω, d) is a metric space, which is bounded and nonempty and let Q be a countably compact class of subsets in Ω, which is normal and stable under random intersections. In addition, suppose that we have closed balls of Ω that are contained in Q. 9en, there exists a fixed point in each nonexpansive mapping T of Ω into itself.*

Corollary 2.11. *Let S be bounded and a closed convex subset in a reflexive Banach space X, and if $\psi : S \longrightarrow 2^x$ is a mapping such that,*

$$\psi(x) \cap S \neq \phi, forall x \in S, \tag{13}$$

for any closed convex subset G of S satisfying $l(\xi) \cap G \neq \emptyset, forall \xi \in G$

$$H(\psi(x) \cap G, , \psi(y) \cap G) \leq \|x - y\| \quad whenever \quad x, y(\neq x) \in G. \tag{14}$$

Then, l has a fixed point.

References

[1] W. A. Kirk, *An abstract fixed point theorem for nonexpansive mappings*, Proceedings of the American Mathematical Society, vol. 82, no. 4, pp. 640642, 1981.

[2] A. Latif, *A general fixed point theorem for multivalued mappings that are not necessarily contractions and applications*, in Abstract and Applied Analysisvol. 2014, Hindawi, 2014.

[3] B. K. Ray, *On Cirics fixed point theorem*,Fundamenta Mathematicae, vol. 94, no. 3, pp. 221229, 1977.

[4] M. R. Taskovic, *Monotonic mappings on ordered sets, a class of inequalities with finite differences and fixed points*, Publications de lInstitut Mathematique, vol. 17, pp. 163172, 1974.

[5] M. Almahalebi and S. Kabbaj, *A fixed point approach to stability of the quartic equation in 2-Banach spaces*, 9e Journal of Mathematics and Computer Science, vol. 3, no. 4, pp. 972984, 2013.

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Amenability of certain Banach algebras related to locally compact groups

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Abstract

In this note, we characterize generalized notions of amenibility for the Banach algebra $L_0^\infty(G)^*$ according to algebraic and topological properties of the locally compact G.

Keywords: Amenability, Banach algebra, derivation, locally compact group, vanishing at infinity
Mathematics Subject Classification [2023]: 22D15, 46H05, 47B47

1 Introduction

Let G be a locally compact group with a fixed left Haar measure ds. The group algebra $L^1(G)$, Lebesgue space $L^\infty(G)$, measure algebra $M(G)$ and $L_0^\infty(G)$ have their usual meanings; see for example [3]. We recall that $L_0^\infty(G)^*$ with the following product is a Banach algebra which is in relevance to the group algebra $L^1(G)$ of G.

$$\langle m \odot n, g\rangle = \langle m, ng\rangle \quad (m,\, n \in L_0^\infty(G)^*, g \in L_0^\infty(G)),$$

where

$$\langle ng, \phi\rangle = \langle n, \frac{1}{\Delta}\, \widetilde{\phi} * g\rangle,$$

for all $\phi \in L^1(G)$, where $\widetilde{\phi}(s) = \phi(s^{-1})$, Δ denotes the modular function of G, and $*$ is the convolution product. For each $\phi \in L^1(G)$, let ϕ also denote the functional in $L_0^\infty(G)^*$ defined by

$$\langle \phi, g\rangle := \int_G \phi(s)\; g(s)\; ds,$$

for all $g \in L_0^\infty(G)$. Note that this duality defines a linear isometric embedding of $L^1(G)$ into $L_0^\infty(G)^*$, and that $L^1(G) = L_0^\infty(G)^*$ if and only if G is discrete. Moreover, $L_0^\infty(G)^*$ has an identity if and only if G is discrete.

The Banach algebra $L_0^\infty(G)^*$ has been introduced and studied extensively by Lau and Pym [3] and many authors has been investigated several aspects of harmonic analysis on $L_0^\infty(G)^*$.Furthermore, various concepts of amenability have been described for Banach algebras related to locally compact groups; see for example [1], [4] and [5].

Here, the investigations give us several characterizations of algebraic and topological properties of G in terms of amenability notions of $L_0^\infty(G)^*$; such as amenability, discreteness, and finiteness.

2 The results

Suppose that $\mathcal{U}$ is a Banach algebra. Then a *Banach $\mathcal{U}$-bimodule* is a Banach space E which is algebraically $\mathcal{U}$-bimodule for which there is a constant C such that for each $a \in \mathcal{U}$ and $x \in E$, $\|a \cdot x\| \leq C\|a\|\, \|x\|$ and $\|x \cdot a\| \leq C\|a\|\, \|x\|$. A *derivation* $D : \mathcal{U} \to E$ means a linear map satisfying $D(ab) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in \mathcal{U}$. Furtheremore, a derivation D is said to be *inner* if there is $x \in E$ such that $D = \mathrm{ad}_x$, where

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the derivation $\mathrm{ad}_x : \mathcal{U} \longrightarrow E$ is defined by $\mathrm{ad}_x(a) = a \cdot x - x \cdot a$ for all $a \in \mathcal{U}$. Note that E^* is also a Banach $\mathcal{U}$-bimodule with the actions induced via $\langle a \cdot \xi, x\rangle = \langle \xi, x \cdot a\rangle$ and $\langle \xi \cdot a, x\rangle = \langle \xi, a \cdot x\rangle$ for all $a \in \mathcal{U}$, $x \in E$ and $\xi \in E^*$.

According to Johnson's definition [2], $\mathcal{U}$ is called *amenable* if every continuous derivation from $\mathcal{U}$ into the dual Banach $\mathcal{U}$-bimodule E^* is inner for all Banach $\mathcal{U}$-bimodules E. Also, $\mathcal{U}$ is called *contractible* if for every Banach $\mathcal{U}$-bimodule E, every continuous derivation $D : \mathcal{U} \to E$ is inner.

Ghahramani and et.al. in [1] have introduced and studied the notion of approximate amenability. They called $\mathcal{U}$ *approximately amenable* when for each Banach $\mathcal{U}$-bimodule E, every continuous derivation D from $\mathcal{U}$ into E^* is approximately inner; that is, there exists a net $(\xi_\alpha) \subseteq E^*$ such that $D(a) = \lim_\alpha \mathrm{ad}_{\xi_\alpha}(a)$ in E^* for all $a \in \mathcal{U}$.

Moreover, they called $\mathcal{U}$ *boundedly approximately amenable* if, in addition, the net (x_α) can be chosen such that the net (ad_{x_α}) is uniformly bounded.

Furthermore, they proved that $\mathcal{U}$ is amenable if and only if $\mathcal{U}$ is uniformly approximately amenable; i.e., every continuous derivation from $\mathcal{U}$ into any dual Banach $\mathcal{U}$-bimodule maybe approximated uniformly on the unit ball of $\mathcal{U}$ by inner derivations.

We commence with a characterization of approximate amenability for $L_0^\infty(G)^*$. Before stating, let us recall that an element $u \in L_0^\infty(G)^*$ is called a mixed identity if $\phi \odot u = u \odot \phi = \phi$ for all $\phi \in L^1(G)$. Denote by $\mathcal{E}_1(G)$ the nonempty set of all mixed identities u with norm one in $L_0^\infty(G)^*$.It is known that $u \in \mathcal{E}_1(G)$ if and only if it is a weak*-cluster point of an approximate identity in $L^1(G)$ bounded by one; or equivalently, it is a right identity of $L_0^\infty(G)^*$ with norm one. Let us also recall that G is called *amenable* if there exists a left-invariant mean on $L^\infty(G)$; that is, a positive norm one functional m in $L^\infty(G)^*$ for which $\langle m, L_x g\rangle = \langle m, g\rangle$ for all $g \in L^\infty(G)$ and $x \in G$, where $(L_x g)(y) = g(x^{-1}y)$ for all $y \in G$.

Theorem 2.1. *Let G be a locally compact group. Then the following assertions are equivalent.*
(a) *$L_0^\infty(G)^*$ is amenable.*
(b) *$L_0^\infty(G)^*$ is boundedly approximately amenable.*
(c) *$L_0^\infty(G)^*$ is approximately amenable.*
(d) *G is discrete and amenable.*

Proof. That (d) implies (a) follows from the classical result of Johnson that $L^1(G)$ is amenable whenever G is amenable by Theorem 2.5 of [2]. Since, (a) implies (b) and (b) implies (c) trivially, to complete the proof, we need only to show that (c) implies (d); to that end, suppose that $L_0^\infty(G)^*$ is approximately amenable. It follows from Lemma 2.2 in [1] that $L_0^\infty(G)^*$ has a left approximate identity (u_γ). For every $u \in \mathcal{E}_1(G)$ we have

$$\|u_\gamma - u\| = \|u_\gamma \odot u - u\| \to 0.$$

This shows that for any $m \in L_0^\infty(G)^*$,

$$\begin{aligned} \|u \odot m - m\| &\leq \|(u - u_\gamma) \odot m\| + \|u_\gamma \odot m - m\| \\ &\leq \|u - u_\gamma\|\,\|m\| + \|u_\gamma \odot m - m\| \to 0. \end{aligned}$$

That is, u is a left identity for $L_0^\infty(G)^*$. Since $L_0^\infty(G)^*$ has always a right identity, $L_0^\infty(G)^*$ has an identity. Now, invoke Proposition 3.1 from [4] to conclude that G is discrete. Furthermore, $L^1(G) = L_0^\infty(G)^*$, and thus $L^1(G)$ is approximately amenable. Hence, G is also amenable. □

In our next result, we characterize essential amenability of $L_0^\infty(G)^*$. First let us recall that a Banach algebra A is called *essentially amenable* (resp. *approximately essentially amenable*) if every continuous derivation $D : A \longrightarrow X^*$ is inner (resp. approximately inner) for all A-bimodules X which is neo-unital; that is,

$$X = \{a \cdot y \cdot b : a, b \in A, y \in X\}.$$

Theorem 2.2. *Let G be a locally compact group. Then the following assertions are equivalent.*
(a) *$L_0^\infty(G)^*$ is essentially amenable.*
(b) *$L_0^\infty(G)^*$ is approximately essentially amenable.*
(c) *G is discrete and amenable.*

Proof. That (c) implies (a) follows from Theorem 3.1. Also, (a) implies (b) trivially. Now, suppose that $L_0^\infty(G)^*$ is essentially approximately amenable. Choose $u \in \mathcal{E}_1(G)$, and note that the map $m \mapsto u \odot m$ is a continuous epimorphism from $L_0^\infty(G)^*$ onto $u \odot L_0^\infty(G)^*$. Next recall from Theorem 2.11 of [3] that $u \odot L_0^\infty(G)^*$ is isometrically isomorphic to the measure algebra of $M(G)$. Thus, there exists a continuous epimorphism $\tau : L_0^\infty(G)^* \to M(G)$. For each $M(G)$-bimodule X, let Y be the Banach space X considering as a neo-unital $L_0^\infty(G)^*$-bimodule with actions

$$m \bullet \xi = \tau(m) \cdot \xi \ \text{ and } \ \xi \bullet m = \xi \cdot \tau(m)$$

for all $\xi \in Y$ and $m \in L_0^\infty(G)^*$. So, if $D : M(G) \to X^*$ is a continuous derivation, then $D \circ \tau : L_0^\infty(G)^* \to Y^*$ is a continuous derivation, and there exists a net $(\xi_\alpha) \subseteq X^*$ such that

$$\begin{aligned} D(\tau(m)) &= (D \circ \tau)(m) \\ &= \lim_\alpha (m \bullet \xi_\alpha - \xi_\alpha \bullet m) \\ &= \lim_\alpha (\tau(m) \cdot \xi_\alpha - \xi_\alpha \cdot \tau(m)) \end{aligned}$$

for all $m \in L_0^\infty(G)^*$. Therefore, D is approximately inner. This shows that $M(G)$ is approximately amenable, and (a) holds by Theorem 3.1 of [1]. □

We next state notions of contractibility of $L_0^\infty(G)^*$.

Theorem 2.3. *Let G be a locally compact group. Then the following assertions are equivalent.*
(a) *$L_0^\infty(G)^*$ is contractible.*
(b) *$L_0^\infty(G)^*$ is uniformly approximately contractible.*
(c) *$L_0^\infty(G)^*$ is uniformly approximately essentially contractible.*
(d) *$L_0^\infty(G)^*$ is essentially contractible.*
(e) *G is finite.*

Proof. Suppose that $L_0^\infty(G)^*$ is essentially contractible. Then $L_0^\infty(G)^*$ is essentially amenable, and therefore G is discrete by Theorem 419. This shows that $L_0^\infty(G)^*$ is equal to $L^1(G)$, and $L^1(G)$ is contractible. It follows that G is finite; see for example [6]. We have shown that (d) implies (e). The other implications are trivial. □

We end the note by the following result.

Theorem 2.4. *Let G be a locally compact group. Then $L_0^\infty(G)^*$ is approximately essentially contractible if and only if G is discrete and amenable.*

References

[1] F. Ghahramani and R. J. Loy, *Generalized notions of amenability*, J. Funct. Anal., 208 (2004), 229-260.

[2] B. E. Johnson, *Cohomology in Banach Algebras*, Mem. Amer. Math. Soc., 127, 1972.

[3] A. T. Lau and J. Pym, *Concerning the second dual of the group algebra of a locally compact group*, J. London Math. Soc., 41 (1990), 445-460.

[4] M. J. Mehdipour and R. Nasr-Isfahani,*Compact left multipliers on Banach algebras related to locally compact groups*, Bull. Austral. Math. Soc. 79 (2009), 227-238.

[5] R. Nasr-Isfahani and S. Soltani Renani, *Character injectivity and projectivity of Banach modules*, Quart. J. Math. Oxford., 62 (2014), 665-676.

[6] Yu. V. Selivanov, *On Banach algebras of small global dimension zero*, Usp. Mat. Nauk, 31 (1976), 227-228.

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Property (X^+) in planar system of Liénard type

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Abstract

In this work we use qualitative theory of differential equations to study the qualitative behavior of the solutions of a generalized Liénard system. Under quite general assumptions we present some sharp conditions under which the solutions of the system to have property (X^+).

Keywords: Liénard system, differential equation, planar system
Mathematics Subject Classification [2023]: 34A12, 34C10

1 Introduction

It is well known that the Liénard system

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x), \tag{167}$$

is of grate importance in various applications. Hence, asymptotic and qualitative behavior of this system and some of its extensions have been widely studied by many authors. Various questions on the stability, boundedness, oscillation and periodicity of solutions of (167) and its generalizations have received a considerable amount of attention in the last four decades. We consider the system of two differential equations

$$\dot{x} = h(k(y) - F(x)), \quad \dot{y} = -g(x), \tag{168}$$

which is a generalized Liénard system, where F, g , k and h are continuous functions which ensure the existence of a unique solution to the initial value problem. Moreover, the following assumptions hold.

($\mathbf{C}_1$) $F(x)$ and $g(x)$ are continuous on $\mathbb{R}$ with $F(0) = 0$, $xg(x) > 0$ for $x \neq 0$, $h(u)$ is continuously differentiable and strictly increasing with $h(0) = 0$ and $h(\pm\infty) = \pm\infty$ and $k(u)$ is continuously differentiable and strictly increasing with $k(0) = 0$ and $k(\pm\infty) = \pm\infty$.

Under these assumptions, the origin is the unique critical point for system (168).

In order to study the global asymptotic stability of the zero solution, oscillation problem and existence of periodic solutions of system (168) the significant point is to find conditions for deciding whether all trajectories intersect the vertical isocline $k(y) = F(x)$.

In [1], the authors proved a proposition about the existence of a unique solution for initial value problem corresponding to system (167). In the following, we have the same result for system (168).

Proposition 1.1. *If (C_1) is satisfied, then for any initial point $p(x_0, y_0)$, system* (168) *has a unique orbit passing through p.*

Definition 1.2. System (168) has property (X^+) in the right half-plane (resp., in the left half-plane) if for every point (x_0, y_0) with $k(y_0) > F(x_0)$ and $x_0 \geq 0$ (resp., $k(y_0) < F(x_0)$ and $x_0 \leq 0$), the positive semi-orbit of (168) passing through (x_0, y_0) crosses the vertical isocline $k(y) = F(x)$.

In this paper we will find conditions for deciding whether system (168) has property (X^+) in the right and left half-planes. Our results extend the results of Aghajani et al. [4] with $k(y) = y$, Villari and Zanolin [2], and Hara and Sugie [3] for this system with $h(x) = x$ and $k(y) = y$ and improve the results presented by Sugie et al. [2] and Gyllenberg and Yan [1].

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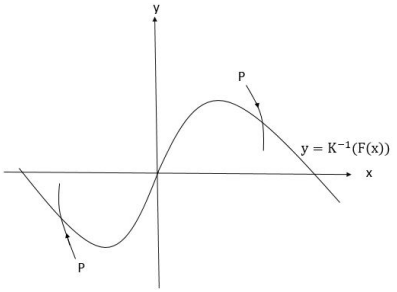


Figure 4: Property (X^+)

2 Sufficient conditions for property (X^+)

Note: Hereafter we assume that the condition $\mathbf{(C_1)}$ holds.

Let $G(x) = \int_0^x g(\eta)d\eta$. First, we introduce a system which is equivalent to (168). Define function $\phi(x)$ by $\phi(x) = \sqrt{2G(x)}sgn(x)$, and map $\Phi : R^2 \to R^2$ by $\Phi(x, y) = (\phi(x), y) \equiv (u, v)$. Changing variables $u = \sqrt{2G(x)}sgn(x)$, $v = y$, $d\tau = \frac{g(x)sgn(x)}{\sqrt{2G(x)}}dt$ and denoting τ by t again, we can transform system (168) into the following system

$$\begin{aligned} \dot{u} &= h(k(v) - F^*(u)) \\ \dot{v} &= -u, \end{aligned} \tag{169}$$

where F^* is a continuous function defined by $F^*(u) = F(G^{-1}(\frac{1}{2}u^2 sgn(u)))$, and $G^{-1}(w)$ is the inverse function to $G(x)sgn(x)$.

Consider the following two conditions on function h.

$\mathbf{(A_1)}$ For every $y < x < 0$, assume that h satisfies the following condition.

$$h(x) - h(y) \geq h(x - y). \tag{170}$$

$\mathbf{(A_2)}$ For every $0 < x < y$, assume that h satisfies the following condition.

$$h(x) - h(y) \leq h(x - y). \tag{171}$$

Remark 2.1. Suppose that $h(x)$ is an odd function. Then $\mathbf{(A_1)}$ and $\mathbf{(A_2)}$ are equivalent.

In the following the main results about property (X^+) will be stated.

Theorem 2.2. *Suppose that* $\mathbf{(A_1)}$ *holds and* $h'(x)$ *and* $k'(x)$ *are increasing for* $x < 0$. *If there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \geq \alpha > 0$ *and* $a(\beta) \geq 0$ *for some* $\beta \geq \alpha$, *such that*

$$\limsup_{u \to +\infty} \left(\int_b^u \frac{h(F^*(s))a'(s) + 2\sqrt{a'(s)}\sqrt{sa(s)h'(F^*(s))k'(k^{-1}(F^*(s)))}}{a^2(s)} ds + \frac{h(F^*(u))}{a(u)} \right) = +\infty, \tag{172}$$

for some $b > 0$. *Then, system* (169) *has property* (X^+) *in the right half-plane.*

Notice that if h satisfies $\mathbf{(A_1)}$, then by dividing two sides of (170) by $x - y$ and then $y \to x$ we conclude that $h'(x) \geq h'(0) \geq 0$ for $x < 0$. Now by the same way as in the proof of the theorem above we can prove the following theorem which does not need the condition $h'(x)$ be increasing.

Theorem 2.3. *Suppose that* $(\mathbf{A}_1)$ *holds and* $l = h'(0) \neq 0$ *and* $k'(x)$ *is increasing for* $x < 0$. *If there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \geq \alpha > 0$ *and* $a(\beta) \geq 0$ *for some* $\beta \geq \alpha$, *such that*

$$\limsup_{u\to+\infty}\left(\int_b^u \frac{h(F^*(s))a'(s) + 2\sqrt{a'(s)}\sqrt{sa(s)k'(k^{-1}(F^*(s)))l}}{a^2(s)}ds + \frac{h(F^*(u))}{a(u)}\right) = +\infty, \tag{173}$$

for some $b > 0$, *then system* (169) *has property* (X^+) *in the right half-plane.*

The following corollary is obtained by straightforward application of the theorem above.

Corollary 2.4. *Assume* $(\mathbf{A}_1)$ *holds and* $l = h'(0) \neq 0$ *and* $k'(x)$ *is increasing for* $x < 0$. *If there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \geq \alpha > 0$ *and* $a(\beta) \geq 0$ *for some* $\beta \geq \alpha$, *such that*

$$\liminf_{u\to+\infty}\frac{h(F^*(u))}{a(u)} > -\infty \text{ and}$$
$$\limsup_{u\to+\infty}\int_b^u \frac{h(F^*(s))a'(s) + 2\sqrt{a'(s)}\sqrt{sa(s)k'(k^{-1}(F^*(s)))l}}{a^2(s)}ds = +\infty,$$

for some $b > 0$. *Then, system* (169) *has property* (X^+) *in the right half-plane.*

The following theorem is useful in applications.

Theorem 2.5. *Suppose that* $h_1(x) \leq h_2(x)$ *for* $x > 0$. *If system* (168) *with* $h_2(x)$ *as* $h(x)$ *has property* (X^+) *in the right half-plane, then it has property* (X^+) *in the right half-plane with* $h_1(x)$ *as* $h(x)$ *too.*

The following analogous results are obtained with respect to property (X^+) in the left half-plane.

Theorem 2.6. *Suppose that* $(\mathbf{A}_2)$ *holds and* $h'(x)$ *is increasing and* $k'(x)$ *is decreasing for* $x > 0$. *If there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \leq \alpha < 0$ *and* $a(\beta) \leq 0$ *for some* $\beta \leq \alpha$, *such that*

$$\liminf_{u\to-\infty}\left(\int_u^b \frac{h(F^*(s))a'(s) - 2\sqrt{a'(s)}\sqrt{sa(s)h'(F^*(s))k'(k^{-1}(F^*(s)))}}{a^2(s)}ds - \frac{h(F^*(u))}{a(u)}\right) = -\infty, \tag{174}$$

for some $b < 0$, *then system* (169) *has property* (X^+) *in the left half-plane.*

Theorem 2.7. *Suppose that* $(\mathbf{A}_2)$ *holds and* $l = h'(0) \neq 0$ *and* $k'(x)$ *is decreasing for* $x > 0$. *If there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \leq \alpha < 0$ *and* $a(\beta) \leq 0$ *for some* $\beta \leq \alpha$, *such that*

$$\liminf_{u\to-\infty}\left(\int_u^b \frac{h(F^*(s))a'(s) - 2\sqrt{a'(s)}\sqrt{sa(s)k'(k^{-1}(F^*(s)))l}}{a^2(s)}ds - \frac{h(F^*(u))}{a(u)}\right) = -\infty \tag{175}$$

for some $b < 0$, *then system* (169) *has property* (X^+) *in the left half-plane.*

Corollary 2.8. *Assume* $(\mathbf{A}_2)$ *holds and* $l = h'(0) \neq 0$ *and* $k'(x)$ *is decreasing for* $x > 0$. *If there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \leq \alpha < 0$ *and* $a(\beta) \leq 0$ *for some* $\beta \leq \alpha$, *such that*

$$\liminf_{u\to-\infty}\frac{h(F^*(u))}{a(u)} > -\infty,$$
$$\liminf_{u\to-\infty}\int_u^b \frac{h(F^*(s))a'(s) - 2\sqrt{a'(s)}\sqrt{sa(s)k'(k^{-1}(F^*(s)))l}}{a^2(s)}ds = -\infty,$$

for some $b < 0$. *Then system* (169) *has property* (X^+) *in the left half-plane.*

3 Conclusion

To conclude, we considered generalised Liénard system (169) and proved an existence and uniqueness result for initial value problem corresponding to the generalised Linard system. Also, we presented some new and sharp conditions under which all trajectories of this system intersect the vertical isocline (property (X^+)) in the both half-planes.

References

[1] A. Aghajani, Y. Jalilian, and V. Roomi, *Oscillation theorems for the generalized Lienard system*, Mathematical and Computer Modelling, 54 (2011), 24712478.

[2] M. Gyllenberg, P. Yan, *New conditions for the intersection of orbits with the vertical isocline of the Liénard system*, Mathematical and Computer Modelling, 49 (2009), 906-911.

[3] J. Sugie, et al., *On global asymptotic stability of systems of Lienard type*, J.Math. Anal. Appl.219 (1998), 140-164.

[4] T. Hara, J. Sugie, *When all trajectories in the Liénard plane cross the vertical isocline?* , Nonlinear Differential Equations Appl. 2(4)(1995), 527-551.

[5] G. Villari, F. Zanolin, *On a dynamical system in the Liénard plane. Necessary and sufficient conditions for the intersection with the vertical isocline and applications*, Funkcial. Ekvac. 33 (1990), 19-38.

[6] P. Yan, J. Jiang, *On global asymptotic stability of second order nonlinear differential systems*, Appl. Anal. 81 (2002), 681-703.

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Existence of Homoclinc orbit in equivalent system of Liénard equations with relativistic acceleration

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Abstract

This paper is a study of one of the most beautiful phenomena in dynamical system: homoclinic orbit. Homoclinic orbits have been introduced by Poincaraé more than a century ago, and since then, they became a fundamental tool in the study of chaos. In this work we study the existence of homoclinic orbits of the planar system of Liénard type.

Keywords: Liénard System, Homoclinic Orbit, planar system
Mathematics Subject Classification [2023]: 34C37, 34A12, 34C10

1 Introduction

Since the pioneering work of Liénard [4] in 1928, the literature devoted to the study of periodic solutions of Liénard differential equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{176}$$

under various conditions upon the continuous functions f and g is enormous.

In recent years there has been a growing interest toward the study of generalizations of the equation, such as

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + f(x)\dot{x} + g(x) = 0, \tag{177}$$

which Newtons acceleration $\ddot{x}$ is replaced by relativistic acceleration [1, 2].

In this work by transforming Eq. (177) to planar system the existence of homoclinic orbit for equivalent system will be studied. Another approach, inspired by the use of the Liénard plane in the classical case, is to write Eq. (177) in the form

$$\frac{d}{dt}\left[\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + F(x)\right] + g(x) = 0 \tag{178}$$

where $F(x) = \int_0^x f(\eta)d\eta (x \in \mathbb{R})$, and make change of variable

$$y = \frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + F(x), \quad |\dot{x}| < 1, \tag{179}$$

which is equivalent to

$$\dot{x} = \frac{y - F(x)}{\sqrt{1 + [y - F(x)]^2}}, \quad y \in \mathbb{R}. \tag{180}$$

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Consider the planar system

$$\begin{aligned} \dot{x} &= \frac{y - F(x)}{\sqrt{1 + [y - F(x)]^2}} \\ \dot{y} &= -g(x), \end{aligned} \tag{181}$$

which is equivalent system of (176), F and g are continuous functions satisfying suitable assumptions in order to ensure the existence of a unique solution to the initial value problems. Moreover, suppose that $F(0) = 0$ and $xg(x) > 0$ for $x \neq 0$. Under this assumptions origin is the unique equilibrium point of system (181).

Definition 1.1. In system (444), a trajectory is said to be a homoclinic orbit if its $\alpha-$ and $\omega-$limit sets are the origin.

Recently, Authors in [1, 5] studied existence of Homoclinic orbit for other type of Liénard system. The main purpose of this paper is to give an implicit necessary and sufficient condition and some explicit sufficient conditions on $F(x)$ and $g(x)$ under which system (181) has homoclinic orbits.

The curve $\Gamma = \{(x, y)|y = F(x)\}$ is called the characteristic curve of (181). Let

$$\Gamma_1 = \{(x, y)|y = F(x) \ \ and \ \ x > 0\},$$

and

$$\Gamma_2 = \{(x, y)|y = F(x) \ \ and \ \ x < 0\}.$$

Then, $\Gamma = \Gamma_1 \bigcup \Gamma_2 \bigcup (0, 0)$. Positive and negative orbits of (181) passing through $p \in \mathbb{R}^2$ are shown by $\mathrm{O}^+(p)$ and $\mathrm{O}^-(p)$, respectively.

The following definitions are presented to state our main results.

Definition 1.2. System (181) has property (Z_1^+) (resp., (Z_3^+)) if there exists a point $p(x_0, y_0) \in \Gamma_1$ (resp., $p(x_0, y_0) \in \Gamma_2$), such that the $\mathrm{O}^+(p)$ of (181) starting at p approaches the origin through only the first (resp., third) quadrant (see Fig. 1.2).

Definition 1.3. System (181) has property (Z_2^-) (resp., (Z_4^-)) if there exists a point $p(x_0, y_0) \in \Gamma_2$ (resp., $p(x_0, y_0) \in \Gamma_1$), such that the $\mathrm{O}^-(p)$ of (181) starting at p approaches the origin through only the second (resp., fourth) quadrant.

If system (181) has both properties (Z_1^+) and (Z_2^-), then a homoclinic orbit exists in the upper half-plane. Similarly, if system (181) has both properties (Z_3^+) and (Z_4^-), then a homoclinic orbit exists in the lower half-plane.

2 Necessary and Sufficient Conditions for Property of (Z_1^+)

In this section we will give necessary and sufficient conditions for system (181) to have properties (Z_1^+) and (Z_2^-). First, consider the following lemma about asymptotic behavior of solutions of (181).

Lemma 2.1. *For each point $H(c, F(c))$ with $c > 0$ or $c < 0$, the positive or negative semi-orbit of* (181) *starting at H crosses the negative y-axis if the following condition hold.*
($\mathbf{A}_1$) *There exists a $\delta > 0$ such that $F(x) < 0$ for $-\delta < x < \delta$ or $F(x)$ has an infinite number of positive zeroes clustering at $x = 0$.*

Hereafter we assume that there exists a $\delta > 0$ such that $F(x) > 0$ for $-\delta < x < \delta$.

Theorem 2.2. *System* (181) *has property (Z_1^+) if and only if there exist a constant $\delta > 0$ and a continuous function $\phi(x)$ such that*

$$0 \leq \phi(x) < F(x) \quad \ and \quad \ \int_0^x \frac{g(\eta)\sqrt{1 + [\phi(\eta) - F(\eta)]^2}}{F(\eta) - \phi(\eta)} d\eta \leq \phi(x) \tag{182}$$

for $0 < x < \delta$.

Corollary 2.3. *If there exists $\delta > 0$ such that*

$$\frac{1}{F(x)}\int_0^x \frac{g(\eta)\sqrt{1+F(\eta)^2}}{F(\eta)}d\eta \leq \frac{1}{4} \quad for \quad 0 < x < \delta. \tag{183}$$

Then, system (181) *has property* (Z_1^+).

3 Homoclinc Orbit

In this section some results will be presented about the existence of homoclinic orbit in the upper half-plane for system (181).

Theorem 3.1. *System* (181) *has homoclinic orbit in the upper half-plane if and only if there exist a constant $\delta > 0$ and a continuous function $\phi(x)$ such that*

$$0 \leq \phi(x) < F(x) \quad and \quad \int_0^x \frac{g(\eta)\sqrt{1+[\phi(\eta)-F(\eta)]^2}}{F(\eta)-\phi(\eta)}d\eta \leq \phi(x) \tag{184}$$

for $0 <\mid x \mid< \delta$.

The following corollary are obtained from Theorem 3.1, which provide explicit conditions for system (181) to have homoclinic orbit in upper half-plane.

Corollary 3.2. *If there exists $\delta > 0$ such that*

$$\frac{1}{F(x)}\int_0^x \frac{g(\eta)\sqrt{1+F(\eta)^2}}{F(\eta)}d\eta \leq \frac{1}{4} \quad for \quad 0 <\mid x \mid< \delta. \tag{185}$$

Then, system (181) *has homoclinic orbit in the upper half-plane.*

Remark 3.3. Suppose that F is an even and g is an odd function. It is easy to see that system (181) has property (Z_1^+) if and only if it has property (Z_2^-). Therefore, if system (181) has property (Z_1^+), then it has a homoclinic orbit in the upper half-plane.

Example 3.4. Consider system (181) with

$$F(x) = x^2 \quad \text{and} \quad g(x) = \frac{x^3}{2\sqrt{1+x^4}}. \tag{186}$$

From Corollary 3.2, we have

$$\frac{1}{F(x)}\int_0^x \frac{g(\eta)\sqrt{1+F(\eta)^2}}{F(\eta)}d\eta = \frac{1}{2x^2}\int_0^x \eta d\eta = \frac{1}{4}$$

Therefore, this system has property (Z_1^+). Since F is even and g is odd, Remark 3.3 implies that this system has a homoclinic orbit in the upper half-plane.

4 Conclusion

To conclude, we considered generalised Liénard system (181) and studied an existence of Homoclinic orbit for equivalent system of Liénard equations with relativistic acceleration. Also, we presented some new and sharp sufficient conditions for system (181) to has special property (property (Z_1^+)) on vector field of system (181) which needed to existence of Homoclinic orbit.

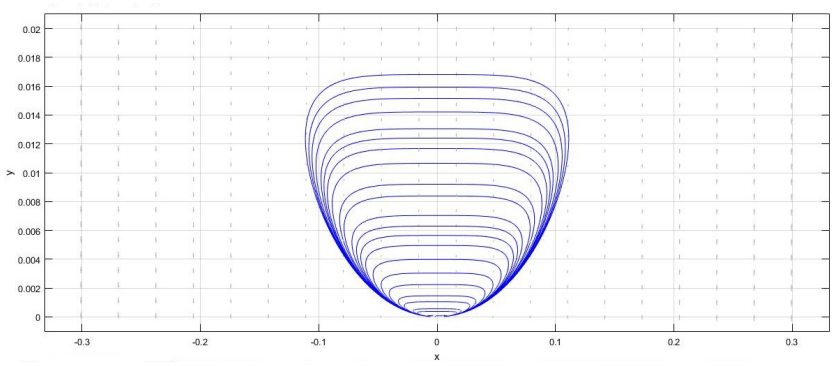


Figure 5: Phase portrait for system (186).

References

[1] A. Aghajani and A. Moradifam, *On the homoclinic orbits of the generalized Liénard equations,* Appl. Math. Lett. 20 (2007), 345-351.

[2] M. Hayashi, *Homoclinic orbits for the planar system of Liénard,* Qual. Theory Dyn. Syst. 12(2013), No. 2, 315-322.

[3] A. Liénard, *Étude des oscillations entretenues,* Revue générale délectricité 23 (1928) 901912, 946954.

[4] J. Mawhin and G. Villari, *Periodic solutions of some autonomous Liénard equations with relativistic acceleration,* Nonlinear Anal. 160 (2017) 1624.

[5] S. Pérez-González, J. Torregrosa and P.J. Torres, *Existence and uniqueness of limit cycles for generalized ϕ-Laplacian Liénard equations,* J. Math. Anal. Appl. 439 (2016) 745765.

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Positive solution to fractional difference equations

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Abstract

In this paper, we deal with the existence of a non-trivial solution for the following fractional discrete boundary-value problem

$$\begin{cases} {}_{T+1}\nabla_k^{\alpha}\,({}_k\nabla_0^{\alpha}(u(k))) = f(k,u(k)), \quad k \in [1,T]_{\mathbb{N}_0}, \\ u(0) = u(T+1) = 0, \end{cases}$$

where $0 \leq \alpha \leq 1$ and ${}_0\nabla_k^{\alpha}$ is the left nabla discrete fractional difference and ${}_k\nabla_{T+1}^{\alpha}$ is the right nabla discrete fractional difference and $f : [1,T]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ is a positive continuous function. The technical approach is based on the matrix theory. An example is included to illustrate the main results.

Keywords: Discrete fractional calculus; Discrete nonlinear boundary value problem; Non trivial solution; Matrix theory.
Mathematics Subject Classification [2023]: 39A10, 34B15

1 Introduction

The boundary value problems with fractional difference operator plays a fundamental role in different fields of research, such as mechanical engineering, control systems, economics, computer science, physics, artificial or biological neural networks, cybernetics, ecology and many others. Important tools in the study of nonlinear difference equations are fixed point methods and upper and lower solution techniques. In [3] discrete boundary value problems have been investigated by adopting matrix approach.
The aim of this paper is to establish the positivity of non-trivial solution for the following discrete boundary-value problem

$$\begin{cases} {}_{T+1}\nabla_k^{\alpha}\,({}_k\nabla_0^{\alpha}(u(k))) = f(k,u(k)), \quad k \in [1,T]_{\mathbb{N}_0}, \\ u(0) = u(T+1) = 0, \end{cases} \tag{187}$$

where $0 \leq \alpha \leq 1$ and ${}_0\nabla_k^{\alpha}$ is the nabla discrete fractional difference and ${}_k\nabla_{T+1}^{\alpha}$ is the nabla discrete fractional difference and $\nabla u(k) = u(k) - u(k-1)$ is the backward difference operator $f : [1,T]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ is a positive continuous function and $T \geq 2$ is fixed positive integer and $\mathbb{N}_1 = \{1,2,3,\cdots\}$ and ${}_T\mathbb{N} = \{\cdots T-2, T-1, T\}$ and $[1,T]_{\mathbb{N}_0}$ is the discrete set $\{1,2,\cdots,T-1,T\} = \mathbb{N}_1 \bigcap {}_T\mathbb{N}$.

Definition 1.1. [2] For any real number, the α rising function is increasing on $\mathbb{N}_0$ and

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad \text{such that } t \in \mathbb{R}\backslash\{\cdots,-2,-1,0\}, 0^{\overline{\alpha}} = 0.$$

Definition 1.2. let f be defined on $\mathbb{N}_{a-1} \bigcap {}_{b+1}\mathbb{N}$, $a < b$, $\alpha \in (0,1)$, then the left nabla discrete fractional difference is defined by

$$\left({}_k\nabla_{a-1}^{\alpha} f\right)(k) \;=\; \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{k} f(s)(k-\rho(s))^{\overline{-\alpha-1}}, \quad k \in \mathbb{N}_a,$$

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and the right one by

$$\left({}_{b+1}\nabla_k^{\alpha} f\right)(k) \;=\; \frac{1}{\Gamma(-\alpha)}\sum_{s=k}^{b} f(s)(s-\rho(k))^{\overline{-\alpha-1}}, \quad k \in {}_b\mathbb{N},$$

where $\rho(k) = k-1$ be the backward jump operator.

As in [4] one can show that, for $\alpha \to 0$, one has ${}_a\nabla_k^{\alpha}(f(k)) \to f(t)$ and for $\alpha \to 1$, one has ${}_a\nabla_k^{\alpha}(f(k)) \to \nabla f(t)$.
We note that there exist two the nabla Riemann fractional differences and the nabla Caputo fractional differences, which for $0 < \alpha < 1$, coincide when f vanishes at the end points that is $f(a-1) = 0 = f(b+1)$ [1]. So, for convenience, we use the symbol ∇^{α} instead of them.

2 Preliminary results

Now, we establish variational framework. Define the finite $T-$dimensional Hilbert space

$$W := \{u : [0, T+1]_{\mathbb{N}_0} \to \mathbb{R} : u(0) = u(T+1) = 0, u = (u(1), u(2), ..., u(T))^{\dagger}\},$$

which $u^{\dagger}$ denotes the transpose of u and W is equipped with the usual inner product and the norm

$$\langle u, v\rangle = \sum_{k=1}^{T} u(k)v(k), \qquad \|u\|_2 := \left(\sum_{k=1}^{T} |u(k)|^2\right)^{\frac{1}{2}}.$$

It is known that the following norm

$$\|u\| = \left\{\sum_{k=1}^{T} |\left({}_k\nabla_0^{\alpha} u\right)(k)|^2\right\}^{\frac{1}{2}},$$

is an equivalent norm in W. A direct computation shows that,

$$\|u\| = u^{\dagger}\mathbb{A}u, \quad \forall u \in W.$$

we let

$$z_u(k) := \frac{1}{\Gamma(1-\alpha)}\sum_{s=1}^{k} u(s)(k-\rho(s))^{\overline{-\alpha}}, \quad w_u(k) := \frac{1}{\Gamma(1-\alpha)}\sum_{s=k}^{T} u(s)(s-\rho(k))^{\overline{-\alpha}},$$

$$\widetilde{z_u}(k) := ({}_k\nabla_0^{\alpha} u)(k) = \left(\nabla_k z_u\right)(k) = z_u(k) - z_u(k-1),$$
$$\widetilde{w_u}(k) := \left({}_{T+1}\nabla_k^{\alpha} u\right)(k) = \left(-\Delta_k w_u\right)(k) = w_u(k) - w_u(k+1).$$

thus, one has

$${}_{T+1}\nabla_k^{\alpha}\left({}_k\nabla_0^{\alpha}(u(k))\right) \;=\; {}_{T+1}\nabla_k^{\alpha}\left(z_u(k) - z_u(k-1)\right), \quad \forall k \in [1, T]$$

where

$$w_{z_u}(k) := \frac{1}{\Gamma(1-\alpha)}\sum_{s=k}^{T} z_u(s)(s-\rho(k))^{\overline{-\alpha}}$$

$$= \frac{1}{\Gamma^2(1-\alpha)}\sum_{s=k}^{T}\sum_{t=1}^{s} u(t)(s-\rho(t))^{\overline{-\alpha}}(s-\rho(k))^{\overline{-\alpha}}$$

$$z_u = (z_u(1), z_u(2), ..., z_u(T))^{\dagger}, \quad w_u = (w_u(1), w_u(2), ..., w_u(T))^{\dagger}$$

$$\widetilde{z_u} = (\widetilde{z_u}(1), \widetilde{z_u}(2), ..., \widetilde{z_u}(T))^\dagger, \quad \widetilde{w_u} = (\widetilde{w_u}(1), \widetilde{w_u}(2), ..., \widetilde{w_u}(T))^\dagger$$

then by Definition 1.1, and $C := DB$ and $\gamma_i := \frac{-\alpha\Gamma(i-1-\alpha)}{\Gamma(i).\Gamma(1-\alpha)}$ for any $i = 1, 2, 3, 4, ..., T$, hence

$$C = \begin{bmatrix} \gamma_1 & 0 & 0 & \cdots & 0 \\ \gamma_2 & \gamma_1 & 0 & \cdots & 0 \\ \gamma_3 & \gamma_2 & \gamma_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_T & \gamma_{T-1} & \gamma_{T-2} & \cdots & \gamma_1 \end{bmatrix}.$$

It is clear that $\gamma_1 = 1$ and $\gamma_2 = -\alpha$ and $\gamma_{i+1} = \frac{i-1-\alpha}{i}\gamma_i$ and $-\alpha < \gamma_{i-1} < \gamma_i < 0$ for any $i = 3, 4, ..., T$.

Remark 2.1. The matrix $C = [c_{ij}]$ has all the elements below the main diagonal as negative. The diagonal elements of the matrix C are 1 that is $c_{ii} = 1$, $i = 1, 2, ..., T$ and the determinant of the matrix $C = [c_{ij}]$ is denoted by the symbols $|C|$ is 1 and then is non-singular or invertible.

Remark 2.2. The matrix C^{-1} is lower triangular and it is obtained from the identity matrix I_n by performing the same sequence of elementary row operations as were used to convert C^{-1} to I_n[6]. By elementary row operations method, the diagonal elements of the matrix $C^{-1} = [c_{ij}]$ are 1 and it has all the elements below the main diagonal as positive.

Let $\mathbb{A} := C^\dagger C$. So, the matrix $\mathbb{A}$ is real symmetric matrixes.
Let $\lambda_{\min}$ and $\lambda_{\max}$ denote respectively the minimum and the maximum eigenvalues of $\mathbb{A}$, for any $u \in W$, we have,

$$\sqrt{\lambda_{\min}}\|u\|_2 < \|u\| < \sqrt{\lambda_{\max}}\|u\|_2, \tag{188}$$

Therefor from (188), $\|u\| \to +\infty$ if and only if $\|u\|_2 \to +\infty$.

Taking the definition $\mathbb{A}$ into account, (357) converts to

$$\mathbb{A}\begin{bmatrix} u(1) \\ u(2) \\ u(3) \\ \vdots \\ u(T) \end{bmatrix} = \begin{bmatrix} f(1, u(1)) \\ f(2, u(2)) \\ f(3, u(3)) \\ \vdots \\ f(T, u(T)) \end{bmatrix}.$$

We obtain the next result which guarantees the same conclusion of the Strong maximum principle.

Theorem 2.3. *If $u \in W$ be a non-trivial solution to* (357)*, then $u > 0$.*

Proof. By taking account Remark 2.1, one can conclude that $|\mathbb{A}| = |C||C^\dagger| = 1$, hence the matrix $\mathbb{A}$ is non-singular or invertible and the equation (357) becomes

$$\begin{bmatrix} u(1) \\ u(2) \\ u(3) \\ \vdots \\ u(T) \end{bmatrix} = \mathbb{A}^{-1}\begin{bmatrix} f(1, u(1)) \\ f(2, u(2)) \\ f(3, u(3)) \\ \vdots \\ f(T, u(T)) \end{bmatrix},$$

where $\mathbb{A}^{-1} = [a_{ij}]$, $i, j = 1, 2, ..., T$ is the inverse of $\mathbb{A}$ and $\mathbb{A}^{-1} = C^{-1}\{C^{-1}\}^\dagger$. Taking account Remark 2.2, C^{-1} has all the elements below the main diagonal as positive and $\{C^{-1}\}^\dagger$ has all the elements above the main diagonal as positive. So $\mathbb{A}^{-1}$ has all the elements as positive. Since $f(k, u(k)) > 0$ for every $k \in [1, T]_{\mathbb{N}_0}$ and $a_{ij} > 0$, $i, j = 1, 2, ..., T$, one can conclude that

$$u(k) = (a_{k1} + a_{k2} + \cdots + a_{kT})f(k, u(k)) > 0, \quad \text{for every } k \in [1, T]_{\mathbb{N}_0}.$$

□

We now present an example to illustrate the result.

Example 2.4. If u be a non-trivial solution to the problem

$$\begin{cases} {}_k\nabla_4^{0.5}\left({}_0\nabla_k^{0.5}(u(k))\right) = u^2(k)(\ln\frac{k+1}{k}), \quad k \in [1,3], \\ u(0) = u(4) = 0. \end{cases}$$

Then $u > 0$. Indeed, $T = 3$ and $\alpha = 0.5$, so

$$C = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ -0.125 & -0.5 & 1 \end{bmatrix}, \quad \mathbb{A} = \begin{bmatrix} 1.265625 & -0.4375 & -0.125 \\ -0.4375 & 1.25 & -0.5 \\ -0.125 & -0.5 & 1 \end{bmatrix},$$

$$\mathbb{A}^{-1} = \begin{bmatrix} 1 & 0.5 & 0.375 \\ 0.5 & 1.25 & 0.6875 \\ 0.375 & 0.6875 & 1.390625 \end{bmatrix}.$$

and $f(k, u(k)) = u^2(k)(\ln\frac{k+1}{k}) > 0$ for every $k \in [1, 3]$.

References

[1] T. Abdeljawad, *On delta and nabla Caputo fractional differences and dual identities.* Discrete Dynamics in Nature and Society 2013 (2013).

[2] F.M. Atici, PW. Eloe, *Discrete fractional calculus with the nabla operator.* Electron. J. Qual. Theory Differ. Equ. Spec. Ed. I 2009, 3, (2009).

[3] G. Bonanno, P. Candito and G. D'Aguì, *Positive solutions for a nonlinear parameter-depending algebraic system,* Electron. J. Differential Equations, Vol. 2015 (2015), No. 17, pp. 1–14.

[4] M. Caputo, M. Fabrizio, *A new definition of fractional derivative without singular kernel.* Prog. Fract. Differ. Appl. 1(2), 73-85 (2015).

[5] K. Matthews, *Elementary linear algebra,*(1996), http://www.maths.uq.edu.au/0krm/ela.html.

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On approximate n-Jordan derivations

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Abstract

In this paper, we consider conditions under which approximate n-Jordan derivations on Banach algebras are exact n-Jordan derivations.

Keywords: Approximate n-Jordan derivation, nth power property, n-dimensional Jensen type functional equation, Banach algebra
Mathematics Subject Classification [2023]: 47B47, 39B82

1 Introduction and preliminaries

It is a well-known result due to Bourgin [1] that every approximate ring homomorphism from a Banach algebra onto a unital Banach algebra is necessarily a ring homomorphism. More precisely, he proved the following: If f is a mapping of a Banach algebra $\mathcal{A}$ onto a unital Banach algebra $\mathcal{B}$ satisfying

$$\|f(a+b)-f(a)-f(b)\| \leq \varepsilon,$$

$$\|f(ab)-f(a)f(b)\| \leq \delta$$

for some $\delta, \varepsilon \geq 0$ and all $a, b \in \mathcal{A}$, then f is a ring homomorphism of $\mathcal{A}$ onto $\mathcal{B}$.

Let us mention that a Banach algebra $\mathcal{A}$ is not without order if there exist nonzero elements a_0 and b_0 in $\mathcal{A}$ such that $a_0\mathcal{A} = \mathcal{A}b_0 = \{0\}$. For example, both semisimple Banach algebras and Banach algebras with approximate identities are without order. On approximate derivations Miura et al. in [6] proved the following.

Theorem 1.1. *Let $\mathcal{A}$ be a without order Banach algebra, $\varepsilon \geq 0$ and $p \in [0,\infty) \setminus \{1\}$. If $f : \mathcal{A} \to \mathcal{A}$ is a mapping satisfying*

$$\|f(a+b)-f(a)-f(b)\| \leq \varepsilon\left(\|a\|^p + \|b\|^p\right),$$

$$\|f(ab)-af(b)-f(a)b\| \leq \varepsilon\|a\|^p\|b\|^p$$

for all $a, b \in \mathcal{A}$, then f is a ring derivation.

A mapping $f : \mathcal{R} \to \mathcal{R}$, where $\mathcal{R}$ is an arbitrary ring, is called an n-Jordan derivation if f is additive and satisfies

$$f\left(a^n\right) = \sum_{i=1}^{n} a^{i-1} f\left(a\right) a^{n-i} \tag{189}$$

for any a in $\mathcal{R}$, where $a^0 r = r = r a^0$ for any element r in $\mathcal{R}$. The notion of n-Jordan derivations was introduced by I.N. Herstein [4, p. 528]. In the literature, (189) is known as the nth power property; see, e.g., [5]. Recall that in the case when $\mathcal{R}$ is an algebra over a field $\mathbb{F}$, we define n-Jordan derivations as $\mathbb{F}$-linear (i.e., linear over the field $\mathbb{F}$) mappings satisfying the nth power property.

In this paper, we study n-Jordan derivations and approximate n-Jordan derivations on Banach algebras.

First of all, notice that a 2-Jordan derivation is a Jordan derivation, in the usual sense, on a ring. It is easy to show that if f is a Jordan derivation, then f is an n-Jordan derivation for all $n > 2$, but the converse is not true, in general. For illustration, we present the following interesting example.

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Example 1.2. Suppose that $n \geq 3$ is a fixed integer and

$$\mathcal{A} = \left\{ \begin{bmatrix} 0 & \mu_{1,2} & \mu_{1,3} & \cdots & \mu_{1,n} & \mu_{1,n+1} \\ 0 & 0 & \mu_{2,3} & \cdots & \mu_{2,n} & \mu_{2,n+1} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \mu_{n-1,n} & \mu_{n-1,n+1} \\ 0 & 0 & \cdots & 0 & 0 & \mu_{n,n+1} \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} : \ \mu_{1,2}, \mu_{1,3}, \ldots, \mu_{n,n+1} \in \mathbb{R} \right\}.$$

Then $\mathcal{A}$ is a Banach algebra equipped with the usual matrix-like operations and with the norm given by the sum of all absolute values of entries. Define the mapping $f \colon \mathcal{A} \longrightarrow \mathcal{A}$ via

$$f(\mathbf{a}) = \begin{bmatrix} 0 & \cdots & 0 & \mu_{1,n} & 2\mu_{1,n+1} \\ 0 & \cdots & 0 & \mu_{2,n} & \mu_{2,n+1} \\ \vdots & \cdots & \vdots & \vdots & \\ 0 & \cdots & 0 & \mu_{n-1,n} & \mu_{n-1,n+1} \\ 0 & \cdots & 0 & 0 & \mu_{n,n+1} \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

where $\mathbf{a}$ is an arbitrary element of $\mathcal{A}$. Then f is a bounded linear mapping on $\mathcal{A}$ and

$$\mathbf{a}^k = \begin{bmatrix} 0 & \cdots & 0 & \eta_{1,k+1} & \eta_{1,k+2} & \cdots & \eta_{1,n} & \eta_{1,n+1} \\ 0 & \cdots & 0 & 0 & \eta_{2,k+2} & \cdots & \eta_{2,n} & \eta_{2,n+1} \\ \vdots & \cdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \ddots & 0 & \eta_{n-k,n} & \eta_{n-k,n+1} \\ 0 & \cdots & 0 & 0 & \ddots & 0 & 0 & \eta_{n-k+1,n+1} \\ 0 & \cdots & 0 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \quad (2 \leq k \leq n),$$

where

$$\eta_{i,j} = \sum_{i_1=i+1}^{j-k+1} \sum_{i_2=i_1+1}^{j-k+2} \cdots \sum_{i_{k-1}=i_{k-2}+1}^{j-1} \prod_{\ell=1}^{k} \mu_{i_{\ell-1},i_\ell} \ \ (1 \leq i \leq n-k+1, \ i+k \leq j \leq n+1, \ i_0 = i, \ i_k = j).$$

Hence,

$$f(\mathbf{a}^n) = \begin{bmatrix} 0 & \cdots & 0 & 2\prod_{i=1}^{n} \mu_{i,i+1} \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(n+1)\times(n+1)} = \sum_{i=1}^{n} \mathbf{a}^{i-1} f(\mathbf{a}) \mathbf{a}^{n-i}.$$

Thus f is an n-Jordan derivation, but it is not an m-Jordan derivation for all $m = 2, 3, \ldots, n-1$.

2 Main results

In what follows, we assume that $\mathcal{A}$ is a normed algebra, $\mathcal{B}$ is a Banach algebra, ε and δ are nonnegative real numbers and Ω stands for a connected subset of $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ such that $1 \in \Omega$ and $\Omega \setminus \{1\} \neq \emptyset$. For a mapping $f : \mathcal{A} \to \mathcal{B}$, we define an operator $\omega_\gamma f : \mathcal{A}^n \to \mathcal{B}$ by

$$\omega_\gamma f(a_1, \ldots, a_n) := f\left(\sum_{i=1}^{n} \gamma r_i a_i\right) - \sum_{i=1}^{n} \gamma r_i f(a_i)$$

for all $a_1, \ldots, a_n \in \mathcal{A}$ and all $\gamma \in \Omega$, where $r_1, \ldots, r_n$ are positive rational numbers with $\sum_{i=1}^{n} r_i = 1$.

Lemma 2.1. *Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$. Then*

$$\omega_\gamma f(a_1, \ldots, a_n) = 0 \tag{190}$$

for all $a_1, \ldots, a_n \in \mathcal{A}$ and all $\gamma \in \Omega$ if and only if f is $\mathbb{C}$-linear.

Proof. Substituting $a_i = a$ $(1 \leq i \leq n)$ in (190), we obtain

$$f(\gamma a) = \gamma f(a) \tag{191}$$

for all $a \in \mathcal{A}$ and all $\gamma \in \Omega$. Hence

$$\omega_1 f(a_1, \ldots, a_n) = 0 \tag{192}$$

for all $a_1, \ldots, a_n \in \mathcal{A}$. In view of (192), it follows $f(r_i a) = r_i f(a)$ for all $a \in \mathcal{A}$ and $i \in \{1, \ldots, n\}$. Letting $a_1 = a$, $a_2 = b$ and $a_i = 0$ $(3 \leq i \leq n)$ in (190), we observe that

$$f(a + b) = f(a) + f(b) \tag{193}$$

for all $a, b \in \mathcal{A}$. Now, it follows from (191), (193) and Lemma 1 in [2] that f is $\mathbb{C}$-linear. The converse is trivial. □

From Lemma 2.1 and Theorems 2.1 and 2.2 of [3], we deduce the following result.

Lemma 2.2. *Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$. Then f is $\mathbb{C}$-linear if one of the following two assertions holds.*

(i) There exist $p_1, \ldots, p_n \in \mathbb{R}$ with $\sum_{i=1}^n p_i \neq 0, 1$ such that

$$\|\omega_\gamma f(a_1, \ldots, a_n)\| \leq \varepsilon \prod_{i=1}^n \|a_i\|^{p_i}$$

for all $a_1, \ldots, a_n \in \mathcal{A} \setminus \{0\}$ and all $\gamma \in \Omega$.

(ii) There exists $p < 0$ such that

$$\|\omega_\gamma f(a_1, \ldots, a_n)\| \leq \varepsilon \sum_{i=1}^n \|a_i\|^p$$

for all $a_1, \ldots, a_n \in \mathcal{A} \setminus \{0\}$ and all $\gamma \in \Omega$.

Theorem 2.3. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping with $f(0) = 0$. Then f is an n-Jordan derivation if one of the following two assertions holds.*

(i) There exists $q \in \mathbb{R} \setminus \{0, 1\}$ such that

$$\left\| f\left(a^n\right) - \sum_{i=1}^n a^{i-1} f\left(a\right) a^{n-i} \right\| \leq \delta \|a\|^{nq} \tag{194}$$

for all $a \in \mathcal{A} \setminus \{0\}$ and the condition (i) of Lemma 2.1 is valid.

(ii) There exists $q < 0$ such that the inequality (194) and the condition (ii) of Lemma 2.1 are valid.

3 Conclusion

In this paper, we have studied a new class of Jordan derivations, which are called n-Jordan derivations. We have considered approximate n-Jordan derivations on Banach algebras connected with the n-dimensional Jensen functional equation (192).

References

[1] D.G. Bourgin, *Approximately isometric and multiplicative transformations on continuous function rings*, Duke Math. J. 16 (1949), 385-397.

[2] J. Brzdęk and A. Fŏsner, *Remarks on the stability of Lie homomorphisms*, J. Math. Anal. Appl. 400 (2013), 585-596.

[3] Iz. EL-Fassi, *Hyperstability of an n-dimensional Jensen type functional equation*, Afr. Mat. 27 (2016), 1377-1389.

[4] I. N. Herstein, *Lie and Jordan structures in simple, associative rings*, Bull. Amer. Math. Soc. 67 (1961), 517-531.

[5] C. Lanski, *Generalized derivations and nth power maps in rings*, Commun. Algebra 35 (2007), 3660-3672.

[6] T. Miura, G. Hirasawa and S.E. Takahasi, *A perturbation of ring derivations on Banach algebras*, J. Math. Anal. Appl. 319 (2006), 522-530.

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Ćirić's quasi-contractions via wt-distances

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Abstract

In this talk, we will generalize the known fixed-point theorem of Ćirić [2] as well as the recent results by D. Kocev et al. [12] and E. Karapinar et al. [10] using the concept of wt-distance.

Keywords: Fixed point, wt-distance, quasi-contraction.
Mathematics Subject Classification [2023]: 47H10,47H09

1 Introduction

Bakhtin in [1] and Czerwik in [5] and [6] introduced b-metric spaces (as a generalization of metric spaces) and proved the contraction principle in this framework.

Definition 1.1. Let X be a set and let $d : X \times X \longrightarrow [0, \infty)$ be a map that satisfies the following:

(1) $d(x, y) = 0 \Longleftrightarrow x = y \forall x, y \in X$;

(2) $d(x, y) = d(y, x) \forall x, y \in X$;

(3) $d(x, y) \leq s[d(x, z) + d(z, y)] \forall x, y, z \in X$ for some constant $s \geq 1$.

The function d is called a b-metric with coefficient s and a triplet (X, d, s) is called a b-metric space.

The class of b-metric spaces is larger than the class of metric spaces which is shown in the following examples.

Example 1.2. Let X be the set of Lebesgue measurable functions on $[0, 1]$ such that $\int_0^1 |f(x)|^2 dx < \infty$. Define $d : X \times X \to [0, \infty)$ by $d(f, g) = \int_0^1 |f(x) - g(x)|^2 dx$. Then the ordered pair (X, d) is a b-metric space with $s = 2$.

Example 1.3. Let $\mathbb{R}$ be the set of real numbers. Define

(1) $d_1(x, y) = |x - y|^2$ for any $x, y \in \mathbb{R}$;

(2) $d_2(x, y) = |x - y|^2 + |\frac{1}{x} - \frac{1}{y}|^2$, for any $x, y \in \mathbb{R}$.
Then $(\mathbb{R}, d_i), i = 1, 2$ are b-metric spaces with $s = 2$.

The following example shows that b-metric is not continuous and that in b-metric spaces open ball is not always open set.

Example 1.4. Let $X = \{0, 1, 1/2, ..., 1/n, ...\}$ and
$d(x, y) = 0$ if $x = y$, $d(x, y) = 1$ if $x \neq y \in \{0, 1\}$, $d(x, y) =| x - y |$ if $x \neq y \in \{0, \frac{1}{2n}, \frac{1}{2m}\}$ and $d(x, y) = 4$ otherwise.
Then it holds that:
(1): d is a b-metric on X with coefficient $s = \frac{8}{3}$;

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(2): d is not a metric on X;
(3): d is not continuous in each variable because we have that
$\lim_{n\to\infty} d(0, \frac{1}{2n}) = \lim_{n\to\infty} \frac{1}{2n} = 0$ and
$\lim_{n\to\infty} d(1, \frac{1}{2n}) = 4 \neq 1 = d(1, 0)$.
(4): $0 \in B(1, 2)$, but $B(0, r) \nsubseteq B(1, 2)$ for every $r > 0$, so $B(1, 2)$ is not an open set in topology induced by b-metric d.

The concept of w-distance was introduced by Kada, Suzuki and Takahashi in [9]. They gave examples of w-distance and improved Caristi's fixed point theorem, Ekeland's variational's principle and the nonconvex minimization theorem according to Takahashi.

Definition 1.5. Let X be a metric space with metric d. Then a function $p : X \times X \to [0, \infty)$ is called a w-distance on X if the following are satisfied:

(1) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$,

(2) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous,

(3) for any $\epsilon > 0$, there exist $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Recently, Hussain et al. [8] introduced the concept of the wt-distance in generalized b-metric spaces, which is a generalization of the w-distance, and also proved some fixed point theorems in a partially ordered b-metric space by using the wt-distance.

Definition 1.6. Let (X, d) be a b-metric space with constant $s \geq 1$. Then a function $p : X \times X \longrightarrow [0, \infty)$ is called wt-distance on X if the following conditions are satisfied:

(a) $p(x, z) \leq s[p(x, y) + p(y, z)] \forall x, y, z \in X$;

(b) $\forall x \in X$, $p(x, \cdot) : X \longrightarrow [0, \infty)$ is s-lower semicontinuous;

(c) $\forall \epsilon > 0 \exists \delta > 0$ so that $p(z, x) \leq \delta \wedge p(z, y) \leq \delta \Longrightarrow d(x, y) \leq \epsilon$.

Let us recall that a real-valued function f defined on a b-metric space X is said to be s-lower semicontinuous at a point x_0 in X if either $\liminf_{x_n \to x_0} f(x_n) = \infty$ or $f(x_0) \leq \liminf_{x_n \to x_0} sf(x_n)$, whenever $x_n \in X$ and $x_n \to x_0$.

Here are some examples of wt-distance.

Example 1.7. Let (X, d, s) be a b-metric space. Then the b-metric d is a wt-distance on X.

Example 1.8. Let $X = \mathbb{R}$ and $d_1(x, y) = (x - y)^2$. Then the function
$p : X \times X \to [0, \infty)$ defined by $p(x, y) = |x|^2 + |y|^2$ for every $x, y \in X$ is a wt-distance on X.

Example 1.9. Let $X = \mathbb{R}$ and $d_1(x, y) = (x - y)^2$. Then the function
$p : X \times X \to [0, \infty)$ defined by $p(x, y) = |y|^2$ for every $x, y \in X$ is a wt-distance on X.

The convergence in b-metric spaces is defined in [11] in the following way:

Definition 1.10. Let (X, d) be a b-metric space.

(1) The sequence $\{x_n\}$ converges to $x \in X \Longleftrightarrow lim_{n\to\infty} d(x_n, x) = 0$;

(2) The sequence $\{x_n\}$ is Cauchy $\Longleftrightarrow lim_{n,m\to\infty} d(x_n, x_m) = 0$.

We say that (X, d) is complete if and only if any Cauchy sequence in X is convergent.

The following useful lemma was proved in [9] for w-distance and the proof is similar for wt-distance.

Lemma 1.11. *Let (X,d) be a b-metric space and let p be a wt-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,+\infty)$ converging to 0, and let x, y, $z \in X$. Then the following hold:*

(i) *If $p(x_n,y) \le \alpha_n$ and $p(x_n,z) \le \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$;*

(ii) *if $p(x_n,y_n) \le \alpha_n$ and $p(x_n,z) \le \beta_n$ for any $n \in \mathbb{N}$, then y_n converges to z;*

(iii) *if $p(x_n,x_m) \le \alpha_n$ for any n, $m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*

(iv) *if $p(y,x_n) \le \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

In [14], Sehgal proved the following generalization of Banach theorem:

Theorem 1.12. *Let (X,d) be a complete metric space and $T : X \to X$ a continuous mapping satisfying the condition: there exists a $k < 1$ such that for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$*
$d(T^{n(x)}y, T^{n(x)}x) \le k \cdot d(y,x)$

In 1983, Ćirić [2], among other things, proved the following interesting generalization of Sehgal result [14].

Theorem 1.13. *Let (X,d) be a complete metric space, $q \in [0,1)$ and $T : X \mapsto X$. If for each $x \in X$ there exists a positive integer $n = n(x)$ such that*

$$d(T^n x, T^n y) \le q \cdot \max\Big\{d(x,y), d(x,Ty), \dots, d(x,T^n y), d(x,T^n x)\Big\}, \tag{195}$$

holds for all $y \in X$, then T has a unique fixed point $u \in X$. Moreover, for every $x \in X$, $u = \lim_m T^m x$.

Two generalizations of Ćirić's theorem are published recently. In the framework of w-distance, D. Kocev et all. [12] proved the following theorem:

Theorem 1.14. *Let (X,d) be a complete metric space, $T : X \longrightarrow X$ a mapping and let p be a ω-distance on (X,d).If for every $x \in X$ there exists $n = n(x) \in \mathbb{N}$ such that*

$$p(T^n x, T^n y) \le q \max\Big\{p(x,y), p(x,Ty), p(x,T^2y), ..., p(x,T^n y), p(x,T^n x)\Big\} \tag{196}$$

holds for every $y \in X$ and some $q \in [0,1)$, and if for every $y \in X$ such that $y \ne T^{n(y)}y$ we have that

$$\inf\{p(x,y) + p(x,Tx) + p(x,T^2x) + ... + p(x,T^{n(y)}x) : x \in X\} > 0, \tag{197}$$

then T has a unique fixed point $z \in X$. Moreover, $p(z,z) = 0$ and for every $x \in X$, $z = \lim_m T^m x$.

On the other side, E. Karapinar et al. [10] proved the generalization of Ćirić's theorem in b-metric spaces:

Theorem 1.15. *Let (X,d) be a complete b-metric space with $s \ge 1$, and*
$T : X \mapsto X$ a mapping. If for each $x \in X$ there exists a positive integer $n = n(x)$ such that

$$d(T^n x, T^n y) \le \alpha \cdot \max\Big\{d(x,y), d(x,Ty), \dots, d(x,T^n y), d(x,T^n x)\Big\}, \tag{198}$$

holds for some $\alpha \in [0,1/s)$ and all $y \in X$, then T has a unique fixed point $x^* \in X$. Moreover, for every $x \in X$, we have that $\lim_m T^m x = x^*$.

The function T that satisfies condition from Theorem 1.3. we will call b-quasi-contraction.

2 Ćirić's quasi-contraction and wt-distance

In this section we will present some fixed-point results in the context of wt-distance which are generalizations of the results regarding Ćirić's quasi-contraction in metric space and b-metric space. Let us recall first the notion of s-orbitally continuous mapping.

Definition 2.1. Let (X,d) be a b-metric space with constant $s \geq 1$. For $x \in X$, $T : X \to X$, $O(x;\infty) = \{x, Tx, T^2x, ...\}$ is called the T-orbit of x. For metric spaces see . Mapping T is s-orbitally continuous at x if for any sequence $\{x_n\}$ in $O(x;\infty)$ which is convergent to u, then Tx_n is convergent to Tu, as $n \to \infty$.[7] A function $\mathfrak{F} : X \to \mathbb{R}^+$ is said to be the (T,s)-orbitally lower semi-continuous mapping at x if $\{x_n\}$ is a sequence in $O(x;\infty)$ and $x_n \to u$ implies

$$\mathfrak{F}(u) \leq \liminf_{n\to\infty} s\mathfrak{F}(x_n).$$

For metric spaces see [3].

We now define what it means that a mapping defined on a b-metric space obeys the condition $(C;k,s)$.

Definition 2.2. Mapping $T : X \to X$ on a b-metric space (X,d) with constant $s \geq 1$, obeys the condition $(C;k,s)$ if there is a constant $k \geq 0$ such that for every sequence $x_n \in X$,

$$x_n \to x_0 \in X \quad \Rightarrow \quad \mathfrak{D}(x_0) \leq k \cdot \limsup s\mathfrak{D}(x_n),$$

where $\mathfrak{D}(x) = d(x,Tx)$, $x \in X$.

We point out that the condition $(C;1)$ in metric spaces was introduced and studied by Ćirić [4] and the case $(C;k)$ in metric spaces was introduced and studied by [13].

The following result is the generalization of results in [2],[12] and [10] which are mentioned in the previous section.

Theorem 2.3. *Let (X,d) be a complete b-metric space with constant $s \geq 1$, $T : X \longrightarrow X$ a mapping and let p be a wt-distance on (X,d) . If for every $x \in X$ there exists $n = n(x) \in \mathbb{N}$ such that*

$$p(T^nx, T^ny) \leq q \max\Big\{p(x,y), p(x,Ty), p(x,T^2y), ..., p(x,T^ny), p(x,T^nx)\Big\} \tag{199}$$

holds for every $y \in X$ and some $q \in [0,1/s)$. Assume that we have one of the following conditions:

(i) if for every $y \in X$ such that $y \neq T^{n(y)}y$ we have that

$$\inf\{p(x,y) + p(x,Tx) + p(x,T^2x) + ... + p(x,T^{n(y)}x) : x \in X\} > 0, \tag{200}$$

;

(ii) $\mathfrak{D}(x) = d(x,Tx)$ is (T,s)-orbitally lower semi-continuous;

(iii) T obeys the condition $(C;k,s)$.

Then T has a unique fixed point $z \in X$. Moreover, $p(z,z) = 0$ and for every $x \in X$, $z = \lim_m T^mx$.

Example 2.4. Let $X = \{(x,0) : 0 \leq x \leq 1\} \cup \{(0,x) : 0 \leq x \leq 1\}$ be a metric space with the Euclidean metric. Then a function $p : X \times X \mapsto [0,\infty)$ defined by
$p((x,0),(y,0)) = p((x,0),(0,y)) = p((0,x),(y,0)) = p((0,x),(0,y)) = y$
is a w-distance on X, so it is also a wt-distance on X.
Let $T : X \to X$ be a mapping such that
$T((x,0)) = (0,x)$ and $T((0,x)) = (\frac{1}{2}x, 0)$.

It is obvious that T has a unique fixed point $(0,0)$, but it is not a contraction in the Euclidean metric. On the other side, we have that
$p(T^2x, T^2y) = \frac{1}{2}p(x,y)$ for every $x, y \in X$, so the function p satisfies the inequality (199).

Corollary 2.5. *Let (X,d) be a b-metric space with constant $s \geq 1$ and let T be a b-quasi-contraction from X into itself with constant $q \in [0,1/s)$. Then T has a unique fixed point $u \in X$. Moreover, for every $x \in X$, $u = \lim_m T^m x$.*

Theorem 2.6. *Let (X,d) be a complete b-metric space with constant $s \geq 1$, $T : X \longrightarrow X$ a continuous mapping and let p be a wt-distance on (X,d). If for every $x \in X$ there exists $n = n(x) \in \mathbb{N}$ such that*

$$p(T^n x, T^n y) \leq q \max\Big\{p(x,y), p(x,Ty), p(x,T^2y), ..., p(x,T^n y), p(x,T^n x)\Big\} \tag{201}$$

holds for every $y \in X$ and some $q \in [0,1/s)$, then T has a unique fixed point $z \in X$. Moreover, $p(z,z) = 0$ and for every $x \in X$, $z = \lim_m T^m x$.

References

[1] I. A. Bakhtin, The contraction mapping principle in quasimetric spaces, Funct. Anal. Ulianowsk Gos. Ped. Inst. **30** (1989), 26-37 .

[2] Lj. B. Ćirić, On Sehgal's map with a contractive iterate at a point, Publ. Inst. Math. **33** (1983), 59–62.

[3] Lj. B. Ćiric, On some maps with non-unique fixed points, Publ. Inst. Math. **17(31)**, (1974), 52-58.

[4] Lj. B. Ćirić, *On mappings with contractive iteration*, Publ. Inst. Math.(N.S), **26(40)** (1979), 79-82.

[5] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostrav. **1** (1993), 5-11.

[6] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, Atti Sem. Mat. Fis. Univ. Modena **46** (1998), 263-276.

[7] T. L. Hicks and B. E. Rhoades, A Banach type fixed-point theorem, Math. Japon. **24** (1979), 327-330.

[8] N. Hussain, R. Saadati and P. Agrawal, On the topology and *wt*-distance on metric type spaces, Fixed Point Theory Appl. **88** (2014).

[9] O. Kada, T. Suzuki and W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metrix spaces, Math. Japonica **44** (1996), 381–391.

[10] E. Karapinar, Z.D. Mitrović, A. Öztürk and S. Radenović, On a theorem of Ćirić in b-metric spaces, Rend. Circ. Mat. Palermo, II. Ser (2020).

[11] M.A. Khamsi and N. Hussain, KKM mappings in metric type spaces, Nonlinear Anal. **73** (2010), 3123–3129.

[12] D. Kocev, E. Karapinar and V. Rakočević, On quasi-contraction mappings of Ćirić and Fisher type via *w*-distance, Quae. Math. **42** (2019), 1–14.

[13] D. Kocev and V. Rakočević, On a theorem of Brian Fisher in the framework of w-distance, Carpatian Journal of Mathematics,**33 (2)**, (2017), 199-205.

[14] V.M. Sehgal, A fixed point theorem for mappings with a contractive iterate, Proc. Amer. Math. Soc. **23** (1969), 631-634.

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A reverse inequality to the power means and the Karcher means

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Abstract

Let f be an operator convex function on $(0,\infty)$, and Φ be a unital positive linear maps on $B(H)$. we give a complementary inequality to Davis-Choi-Jensen's inequality as follows

$$f(\Phi(A)) \geq \frac{4R(A,B)}{(1+R(A,B))^2}\Phi(f(A)),$$

where $R(A,B) = \max\{r(A^{-1}B), r(B^{-1}A)\}$ and $r(A)$ is the spectral radius of A. We also prove that if $G(A_1,\ldots,A_n)$ is the generalized geometric mean defined by Ando-Li-Mathias for n positive definite operators, then

$$\Phi(G(A_1,\ldots,A_n)) \geq \left(\frac{2h^{\frac{1}{2}}}{1+h}\right)^{n-1} G(\Phi(A_1),\ldots,\Phi(A_n)),$$

where $h = \max\limits_{1\leq i,j\leq n} R(A_i,A_j)$.

Keywords: Geometric means, Positive linear maps
Mathematics Subject Classification [2023]: 47A63, 47A64,47A65

1 Introduction

Let $B(H)$ denote the set of all bounded linear operators on a complex Hilbert space H. An operator $A \in B(H)$ is positive definite (resp. positive semi-definite) if $\langle Ax,x\rangle > 0$ (resp. $\langle Ax,x\rangle \geq 0$) holds for all nonzero $x \in H$. If A is positive semidefinite, we denote $A \geq 0$. Let $\mathcal{PS}, \mathcal{P} \subset B(H)$ be the sets of all positive semi-definite operators and positive definite operators, respectively. To reach inequalities for bounded self-adjoint operators on Hilbert space, we shall use the following monotonicity property for operator functions: If $X \in B(H)$ is self adjoint with a spectrum $Sp(X)$, and f,g are continuous real valued functions on an interval containing $Sp(X)$, then

$$f(t) \geq g(t),\ t \in Sp(X) \Rightarrow\ f(X) \geq g(X). \tag{202}$$

For $A,B \in \mathcal{P}$ the geometric mean $A\sharp B$ of A and B is defined by $A\sharp B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$. For $A,B \in \mathcal{P}$, let

$$R(A,B) = \max\{r(A^{-1}B), r(B^{-1}A)\}$$

where $r(A)$ means the spectral radius of A and we have

$$r(B^{-1}A) = \inf\{\lambda > 0 : A \leq \lambda B\} = \|B^{\frac{-1}{2}}AB^{\frac{-1}{2}}\|.$$

$R(A,B)$ was defined in [2], and many nice properties of $R(A,B)$ were shown as follows: If $A,B,C \in \mathcal{P}$, then

(i) $R(A,C) \leq R(A,B)R(B,C)$ (triangle inequality)
(ii) $R(A,B) \geq 1$, and $R(A,B) = 1$ iff $A = B$

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$$(iii)\ \|A-B\| \leq (R(A,B)-1)\|A\|.$$

The Thompson distance $d(A,B)$ on the convex cone Ω of positive definite operators is defined by

$$d(A,B) = \log R(A,B) = \max\{\log r(A^{-1}B), \log r(B^{-1}A)\},$$

see [2, 3]. we know that Ω is a complete metric space with respect to this metric and the corresponding metric topology on Ω agree with the relative norm topology.

As a basic inequality with respect to the metric, the following inequality for a weighted geometric mean of two operators hold [2, 3]:

$$d(A_1 \sharp_\nu A_2, B_1 \sharp_\nu B_2) \leq (1-\nu)d(A_1,B_1) + \nu d(A_2,B_2)$$

for $A_1, A_2, B_1, B_2 \in \Omega$ and $\nu \in (0,1)$.

Let A and B be two positive definite operators on a Hilbert space H and Φ be a unital positive linear map on $B(H)$. Ando [1, Theorem 3] showed the following property of a positive linear map in connection with the operator geometric mean.

$$\Phi(A\sharp B) \leq \Phi(A)\sharp\Phi(B). \tag{203}$$

Inequality (2.1) is extended to an operator mean σ in Kubo-Ando theory as follows:

$$\Phi(A\sigma B) \leq \Phi(A)\sigma\Phi(B),$$

In particular for the weighted geometric mean, we have

$$\Phi(A\sharp_\nu B) \leq \Phi(A)\sharp_\nu\Phi(B), \tag{204}$$

where ν is a real number in $(0,1]$. A complementary inequality to inequality (2.2), is the following important inequality:
Let $0 < m_1 I \leq A \leq M_1 I$ and $0 < m_2 I \leq B \leq M_2 I$ and $0 < \nu \leq 1$, then

$$K(h,\nu)\Phi(A)\sharp_\nu\Phi(B) \leq \Phi(A\sharp_\nu B), \tag{205}$$

where $h = \frac{M_1 M_2}{m_1 m_2}$ and $K(h,\nu)$ is the generalization Kantorovich constant defined by

$$K(h,\nu) = \frac{h^\nu - h}{(\nu-1)(h-1)}\left(\frac{(\nu-1)(h^\nu-1)}{\nu(h^\nu-h)}\right)^\nu.$$

The special case, $K(h,2) = K(h,-1) = \frac{(1+h)^2}{4h}$ is called the Kantorovich constant. The generalization Kantorovich constant $K(h,\nu)$ has the following properties:

$$K(h,\nu) = K(h,1-\nu)$$
$$0 < K(h,\nu) \leq 1 \text{ for } 0 < \nu \leq 1,$$

and $K(h,\nu)$ is decreasing for $\nu \leq \frac{1}{2}$ and increasing for $\nu > \frac{1}{2}$ therefore for all $\nu \in \mathbb{R}$, $K(h) = K(h,\frac{1}{2}) = \frac{2h^{\frac{1}{4}}}{1+h^{\frac{1}{2}}} \leq K(h,\nu)$.

It can easily be seen that if A and B are two positive definite operators in $\mathcal{P}$, then

$$K\left(R(A,B)^2,\nu\right)\Phi(A)\sharp_\nu\Phi(B) \leq \Phi(A\sharp_\nu B). \tag{206}$$

2 The power means and the Karcher means

The Karcher mean, also called the Riemannian mean, has long been of interest in the field of differential geometry. Recently it has been used in a diverse variety of settings: diffusion tensors in medical imaging and radar, covariance matrices in statistics, kernels in machine learning and elasticity. Power means for positive definite matrices and operators have been introduced in [5, 6]. The Karcher mean and power means have recently become an important tool for studying of positive definite operators and an interesting subject for matrix analysts and operator theorists. We would like to refer the reader to [4, 5, 6] and references therein for more information.

Geometric and power means of two operators can be extended over more than 3operators via the solution of operator equations as follows. Let n be a natural number, and let $\triangle_n$ be a set of all probability vectors, i.e.,

$$\triangle_n = \{\omega = (w_1, \ldots, w_n) \in (0,1)^n | \sum_{i=1}^{n} w_i = 1\}.$$

Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathcal{P}^n$ and $\omega = (\mathrm{w}_1, \ldots, w_n) \in \triangle_n$. Then the weighted Karcher mean $\Lambda(\omega; \mathbb{A})$ is defined by a unique positive solution of the following operator equation;

$$\sum_{i=1}^{n} \mathrm{w}_i \log(X^{\frac{-1}{2}} A_i X^{\frac{-1}{2}}) = 0. \tag{207}$$

The weighted power mean $P_t(\omega; \mathbb{A})$ is defined by a unique positive solution of the following operator equation;

$$I = \sum_{i=1}^{n} \mathrm{w}_i (X^{\frac{-1}{2}} A_i X^{\frac{-1}{2}})^t \text{ for } t \in (0,1],$$

or equivalently

$$X = \sum_{i=1}^{n} \mathrm{w}_i (X \sharp_t A_i) \text{ for } t \in (0,1]. \tag{208}$$

For $t \in [-1, 0)$, it is defined by

$$P_t(\omega; \mathbb{A}) = (P_{-t}(\omega; \mathbb{A}^{-1}))^{-1},$$

where $\mathbb{A}^{-1} = (A_1^{-1}, \ldots, A_n^{-1})$.

Lawson and Lim in [5, Corollary 6.7] gave an important connection between the weighted Karcher means and the weighted power means in strong operator topology, as follows:

$$\Lambda(\omega, \mathbb{A}) = \lim_{t \to 0} P_t(\omega, \mathbb{A}). \tag{209}$$

Moreover, they also showed that the following property holds for these means:

$$P_{-t}(\omega; \mathbb{A}) \leq \Lambda(\omega; \mathbb{A}) \leq P_t(\omega; \mathbb{A}) \text{ for all } t \in (0,1]. \tag{210}$$

Let Φ be a unital positive linear map on $B(H)$. In [6] is proved that if $t \in (0,1]$, then

$$\Phi(P_t(\omega; \mathbb{A})) \leq P_t(\omega; \Phi(\mathbb{A})), \tag{211}$$

where $\Phi(\mathbb{A}) = (\Phi(A_1), \ldots, \Phi(A_n))$.

We recall the Kadison's Schwarz inequalities

$$\Phi(A^2) \geq \Phi(A)^2, \quad \Phi(A^{-1}) \geq \Phi(A)^{-1} \tag{212}$$

and two complementary inequalities to them, whenever $0 < m \leq A \leq M$:

$$\frac{(m+M)^2}{4mM} \Phi(A)^2 \geq \Phi(A^2), \quad \frac{(m+M)^2}{4mM} \Phi(A)^{-1} \geq \Phi(A^{-1}). \tag{213}$$

The following inequality unifies Kadison's Schwarz inequalities into a single form.

$$\Phi(BA^{-1}B) \geq \Phi(B)\Phi(A)^{-1}\Phi(B). \tag{214}$$

If $0 < m \leq A, B \leq M$, then the following inequality is a complementary inequality to (2.10)

$$\frac{(m+M)^2}{4mM}\Phi(B)\Phi(A)^{-1}\Phi(B) \geq \Phi(BA^{-1}B). \tag{215}$$

In the following Theorem, we give a reverse inequality to the inequality (3.2).

Theorem 2.1. *Let Φ be a unital positive linear map on $B(H)$ and $n \geq 2$ be a positive integer. If $0 < t \leq 1$ and $(A_1, \ldots, A_n) \in \mathcal{P}^n$. Then*

$$\Phi(P_t(\omega; \mathbb{A})) \geq K\left(h_0, \frac{1}{2}\right)^{\frac{1}{t}} P_t(\omega; \Phi(\mathbb{A})), \tag{216}$$

where $h_0 = \max\limits_{1\leq i,j\leq n} R^2(A_i, A_j)$.

Let X be a positive definite operator on a Hilbert space H. Then $m = \lambda_{min}(X) \leq X \leq \lambda_{max}(X) = M$, where $\lambda_{min}(X)$(resp. $\lambda_{max}(X)$) is the minimum (resp. maximum) of the specrum of X.

Theorem 2.2. *Let Φ be a unital positive linear map on $B(H)$ and $n \geq 2$ be a positive integer. If $(A_1, \ldots, A_n) \in \mathcal{P}^n$. Then*

$$\Lambda(\omega; \Phi(\mathbb{A})) \geq \Phi(\Lambda(\omega; \mathbb{A})) \geq \frac{4\hbar}{(1+\hbar)^2}\ \Lambda(\omega; \Phi(\mathbb{A})), \tag{217}$$

where $m_i = \lambda_{min}(A_i)$, $M_i = \lambda_{max}(A_i)$ and $\hbar = \max\limits_{1\leq i\leq n} \frac{M_i}{m_i}$.

References

[1] T. Ando, *Concavity of certain maps on positive definite matrices and applications to Hadamard products*, Linear Algebra Appl. 26 (1979) 203-241.

[2] E. Andruchow, G. Corach, D. Stojanoff, *Geometrical significance of the Lowener-Heinz inequality*, Proc. Amer. Math. Soc.128 (1999) 1031-1037.

[3] G. Corach, H.Porta, L. Recht, *Convexity of the geodesic distance on space of positive operators* , Illinois J. Math. 38 (1994) 7-94.

[4] J. Lawson and Y. Lim, *Monotonic properties of the least squares mean*, Math. Ann. 351 (2011) 267-279.

[5] J. Lawson and Y. Lim, *Karcher means and Karcher equations of positive definite operators*, Trans. Amer. Math. Soc. Ser. B, 1 (2014), 1-22.

[6] Y. Lim and M. Plfia, *Matrix power means and the Karcher mean*, J. Funct. Anal., 262 (2012), 1498-1514.

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Legendre spectral element method for solving time-fractional Burgers equation

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Abstract

In this paper, we study the Legendre spectral element method for solving the time-fractional Burgers equation. To this end, we use the Legendre polynomials for interpolation and obtain the discrete scheme of the problem. To evaluate the efficiency and accuracy of the method, we present the results of our method and compare them with other methods for one example.

Keywords: spectral element method, time-fractional Burgers equation, Caputo fractional derivative
Mathematics Subject Classification [2023]: 65M60, 42C10, 34K37

1 Introduction

In 1984, Patera with the division of domains, applied a spectral method to a greater number of subdomains. He proposed the spectral element method by combining the spectral method and the finite element method [4]. Patera in his innovative method, use the Chebyshev polynomials as the interpolation basis function. Legendre spectral element method (LSEM) were developed by Maday and Patera [1]. In this paper, we consider the time-fractional Burgers equation (T-FBE) [2]

$$\begin{aligned} &\frac{\partial^\theta y(x,t)}{\partial t^\theta} + y(x,t)\frac{\partial y(x,t)}{\partial x} - \nu\frac{\partial^2 y(x,t)}{\partial x^2} = g(x,t), \quad x \in (0,1), \quad t \in (0,T], \\ &y(x,0) = \phi(x), \quad x \in [0,1], \, y(0,t) = \psi_1(t), \quad t \in (0,T], \quad y(1,t) = \psi_2(t), \quad t \in (0,T]. \end{aligned} \tag{218}$$

where $\frac{\partial^\theta y(x,t)}{\partial t^\theta}$ is the time fractional differential of order $\theta \in (0,1)$, $\nu > 0$ is viscosity parameter and g is a known function.

2 Preliminaries

Definition 2.1. If $u(X,t)$ be differentiable function, $m \in \mathbb{N}$, $t > 0$, and θ be the order of derivative, then Caputo fractional derivative is defined as follows:

$$D_t^\theta u(X,t) = I^{m-\theta} D_t^m u(X,t) = \begin{cases} \frac{1}{\Gamma(m-\theta)} \int_0^t \frac{u^{(m)}(s)}{(t-s)^{1-m-\theta}} ds, & m-1 < \theta < m, \\ u_t^{(m)}(X,t), & \theta = m, \end{cases}$$

LSEM

In the LSEM, we first divide the domain Ω into N_e non-overlapping subdomains Ω_e, $\bar{\Omega} = \bigcup_{e=1}^{N_e} \bar{\Omega}_e$, $\bigcap_{e=1}^{N_e} \Omega_e = \{\}$. Define the approximation space U^h as $U^h = \{u \in U : \quad u|_{\Omega_e} \in P_N\}$, where P_N is a polynomial space of less than or equal to N. Now on each element Ω_e, we define the approximate solution of the order N as follows

$$u^e(x,t) = \sum_{j=0}^{N} u_j^e(t)\,\varphi_j(x), \qquad 1 \le e \le N_e, \tag{219}$$

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where φ_j is the j^{th} Lagrange polynomial of order N on the Gauss-Legendre-Lobatto points $\{\xi_i\}_{i=0}^{N}$

$$\varphi_j(\xi) = \frac{1}{N(N+1)L_N(\xi_j)} \frac{(\xi^2-1)L'_N(\xi)}{\xi-\xi_j}, \qquad 0 \le j \le N, \quad -1 \le \xi \le 1,$$

and L_N is a Legendre polynomials of order N. If $h_e = x_e - x_{e-1}$, the stiffness and mass matrices on each element are calculated as follows

$$S_{ij}^e = \int_{x_{e-1}}^{x_e} \varphi'_i(x)\varphi'_j(x)\,dx = \frac{2}{h_e}\int_{-1}^{1} \varphi'_i(\xi)\varphi'_j(\xi)\,d\xi = \frac{2}{h_e}\sum_{k=0}^{N} d_{ik}d_{jk}w_k$$

$$M_{ij}^e = \int_{x_{e-1}}^{x_e} \varphi_i(x)\varphi_j(x)\,dx = \frac{h_e}{2}\int_{-1}^{1} \varphi_i(\xi)\varphi_j(\xi)\,d\xi = M_{ij}^e = \frac{h_e}{2}\delta_{ij}w_i$$

$$w_k = \frac{2}{N(N+1)[L_N(t_k)]^2}, \quad 0 \leqslant k \leqslant N, \quad d_{ik} = \frac{L_N(\xi_k)}{L_N(\xi_i)}\frac{1}{\xi_k-\xi_i}, \ i \neq k, \quad d_{ii} = \frac{L'_N(\xi_i)}{2L_N(\xi_i)}.$$

3 LSEM for time-fractional Burgers equation

Let us discretize the time interval with step size $\tau = \frac{T}{M}$ as $t_k = k \times \tau$, $k = 0, 1, ..., M$. Now, using the definition 1.1, we approximate the Caputo time-fractional derivative term as follows

$$\frac{\partial^\theta y(x,t_{k+1})}{\partial t^\theta} \approx \sum_{r=0}^{k} \vartheta_r \left(y(x,t_{k-r+1}) - y(x,t_{k-r})\right), \ \vartheta_r = \frac{\tau^{-\theta}}{\Gamma(2-\theta)}\left((r+1)^{1-\theta} - r^{1-\theta}\right).$$

So, the Equation (218) can be written as follows

$$\sum_{r=0}^{k} \vartheta_r \left(y^{k-r+1}?-?y^{k-r}\right) + y^{k+1}y_x^{k+1} + \nu y_{xx}^{k+1} = g^{k+1},$$
$$y(x,0) = \phi(x), \ x \in [0,1], \quad y(0,t) = \psi_1(t), \ y(1,t) = \psi_2(t), \quad t \in (0,T],$$

where $y^k = y(x,t_k)$. In order to homogenize the boundary conditions, we consider $y(x,t) = u(x,t) + \hat{y}(x,t)$. Because of $\hat{y}_{xx}^k = 0$, new problem with the new initial and boundary conditions is derived as

$$\sum_{r=0}^{k} \vartheta_r \left(u^{k-r+1}\hat{y}^{k-r+1} - u^{k-r}\hat{y}^{k-r}\right) + u^{k+1}\hat{y}^{k+1}u_x^{k+1}\hat{y}_x^{k+1} + \nu u_{xx}^{k+1} = g^{k+1}$$

$$u(x,0) = y(x,0) - \hat{y}(x,0) = \phi(x) - \hat{y}(x,0), \quad x \in [0,1], \ u(0,t) = u(1,t) = 0, \quad t \in (0,T].$$

We set $\mathcal{W}_N = span\{\varphi_0, \ \varphi_1, \ ..., \ \varphi_N\}$, then the standard LSEM approximation to (218) is

$$\begin{aligned} &\vartheta_0\left(u^{k+1}\hat{y}^{k+1},\omega\right) + \left(u^{k+1}\hat{y}^{k+1}u_x^{k+1}\hat{y}_x^{k+1},\omega\right) + \nu\left(y_x^{k+1},\omega_x\right) \\ &= \left(g^{k+1},\omega\right) + \vartheta_0\left(u^k\hat{y}^k,\omega\right) - \left(\sum_{r=1}^{k}\vartheta_r\left(u^{k-r+1}\hat{y}^{k-r+1} - u^{k-r}\hat{y}^{k-r}\right),\omega\right) \end{aligned} \tag{220}$$

By placing the approximate solution $u^\alpha = \sum_{i=0}^{N-2} U_i^\alpha \varphi_i$ in (220) instead of u^l, $l = 0, 1, ..., k$, and using the test function $\omega = \varphi_j, j = 0, 1, ..., k$, we have

$$\sum_{i=0}^{N-2} U_i^{k+1}\left[\left(\hat{y}^{k+1}\hat{y}_x^{k+1}\varphi_i(\varphi_i)_x,\varphi_j\right) + \nu\left((\varphi_i)_x,(\varphi_j)_x\right) + \vartheta_0\left(\hat{y}^{k+1}\varphi_i,\varphi_j\right)\right] = \left(g^{k+1},\varphi_j\right)$$

$$+ \sum_{i=0}^{N-2} U_i^{k1}\vartheta_0\left(\hat{y}^k\varphi_i,\varphi_j\right) - \sum_{r=1}^{k}\left[\vartheta_r\sum_{i=0}^{N-2}U_i^{k-r+1}\left(\hat{y}^{k-r+1}\varphi_i,\varphi_j\right) - U_i^{k-r}\left(\hat{y}^{k-r}\varphi_i,\varphi_j\right)\right].$$

The matrix form of the above equation is

$$\left(\mathbf{A} + \nu\mathbf{S} + \vartheta_0\mathbf{M}^1\right)\mathbf{U}^{k+1} = \mathbf{G}^{k+1} + \vartheta_0\mathbf{M}^2\mathbf{U}^k - \sum_{r=1}^{k}\left[\vartheta_r\left(\mathbf{M}^3\mathbf{U}^{k-r+1} - \mathbf{M}^4\mathbf{U}^{k-r}\right)\right],$$

Table 9: Comparison of approximate solution of Example 4.1 at $t = 1$ with $N = 10$ and $\tau = 0.001$, and some methods in [1, 2, 3]

x	Exact	LSEM	[1]	[2]	[3]
0.2	1.221403	1.221443	1.2222	1.221462	1.222203
0.4	1.491825	1.491912	1.4932	1.491934	1.493437
0.6	1.822119	1.822249	1.8238	1.822258	1.824294
0.8	2.225541	2.225647	2.2269	2.225666	2.227650

Table 10: Error norms in Example 4.1 with $\tau = 2^{-4}$ and $N = 4, 8, 12$.

	LSEM		[1]	
N	L_2	L_∞	L_2	L_∞
4	3.3506×10^{-2}	2.5024×10^{-2}		
8	3.1782×10^{-3}	2.1207×10^{-3}		
12	7.5203×10^{-4}	7.0245×10^{-4}	1.768×10^{-2}	7.731×10^{-2}

$$\mathbf{A}_{ij} = \left(\hat{y}^k \hat{y}_x^k \varphi_i (\varphi_i)_x, \varphi_j\right), \qquad \mathbf{S}_{ij} = \left((\varphi_i)_x, (\varphi_j)_x\right), \qquad \mathbf{G}_{ij}^{k+1} = \left(g^{k+1}, \varphi_j\right)$$
$$\mathbf{M}_{ij}^1 = \left(\hat{y}^{k+1}\varphi_i, \varphi_j\right), \ \mathbf{M}_{ij}^2 = \left(\hat{y}^k \varphi_i, \varphi_j\right), \quad \mathbf{M}_{ij}^3 = \left(\hat{y}^{k-r+1}\varphi_i, \varphi_j\right), \ \mathbf{M}_{ij}^4 = \left(\hat{y}^{k-r}\varphi_i, \varphi_j\right)$$

4 Numerical results

Example 4.1. Consider the T-FBE (218) with

$$g(x,t) = \frac{2t^{2-\theta}e^x}{\Gamma(3-\theta)} + t^4 e^{2x} - \nu t^2 e^x, \ \ \nu = 1$$
$$y(x,0) = 0, \ x \in [0,1], \quad y(0,t) = t^2, \ t \in (0,1] \quad y(1,t) = et^2, \ \ t \in (0,1].$$

The exact solution is $y(x,t) = t^2 e^x$ [3]. Using interpolation, we have $\hat{y}(x,t) = \left(t^2(e-1)\right)x + t^2$, after homogenization, the new equation will be

$$\sum_{r=0}^{k} \vartheta_r \left(u^{k-r+1}\hat{y}^{k-r+1} - u^{k-r}\hat{y}^{k-r}\right) + u^{k+1}\hat{y}^{k+1}u_x^{k+1}\hat{y}_x^{k+1} + \nu u_{xx}^{k+1} = g^{k+1}$$

$$u(x,0) = 0 - \hat{y}(x,0) = -\left(t^2(e-1)\right)x - t^2, \ \ x \in [0,1], \ u(0,t) = u(1,t) = 0, \quad t \in (0,T].$$

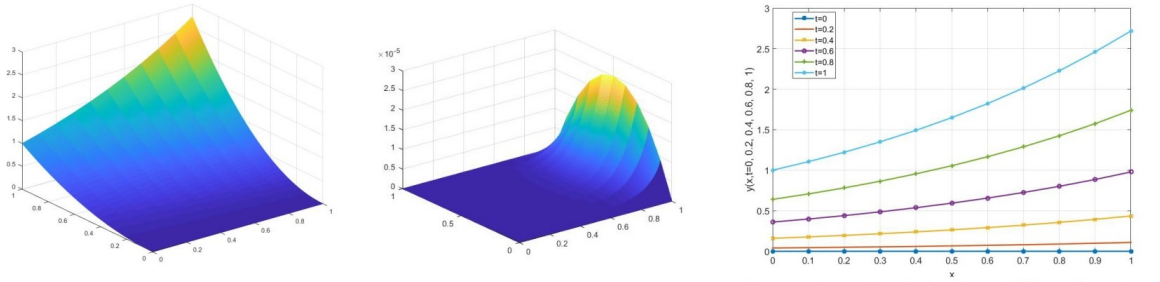
Tables 9 and 10 shows the comparison of LSEM with $N = 8$ and $T = 1$, already existing in the literatures. It is seen from this table that the LSEM gives satisfactory results for error norms when compared with the other ones. Figure 7, show the graphs of LSEM solutions, absolute error and LSEM solutions, $y(x,t)$, for different times, respectively. From the Figure, one can see that exact and approximate solutions are in good agreement.

5 Conclusion

In this paper, we proposed a LSEM for the study of T-BEs. From computations, we noticed that the presented method works well even when the degree of polynomials is small. Moreover, the convergence of our method based on error norms has been discussed and observed that it converges exponentially. Numerical results have been compared with exact solution as well as earlier work reported in the literature. In light of the results, it is clear that the LSEM is suitable to apply for such problems.

References

[1] A. Duangpan, R. Boonklurb, T. Treeyaprasert, *Finite Integration Method with Shifted Chebyshev Polynomials for Solving Time-Fractional Burgers Equations*, Mathematics 7 (12) (2019), 1-24.



LSEM approximation

Absolute error of LSEM approximation

The graphical results $y(x,t)$ of Example 4.1 at different times

Figure 6: LSEM approximation with $N = 10$ and $\tau = 0.001$.

[2] A. Esen, O. Tasbozan, *Numerical solution of time fractional Burgers equation*, Acta Universitatis Sapientiae, Mathematica, vol.7, no.2, (2016), 167-185.

[3] A.T. Patera, *A spectral element method for fluid dynamics: laminar flow in a channel expansion*, J. Comput. Phys. 54, (1984), 468488.

[4] I. Ali, S. Haq, S.F. Aldosary, K.S. Nisar and F. Ahmad, *Numerical solution of one- and two-dimensional time-fractional Burgers equation via Lucas polynomials coupled with Finite difference method*, Alexandria Engineering Journal, Volume 61, Issue 8, (2022), 6077–6087.

[5] Y. Maday, A.T. Patera, *Spectral Element Methods for the Incompressible Navier-Stokes Equations*, Surveys on Computational Mechanics. ASME, New York, 1989.

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Some recent developments on majorization inequalities

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Abstract

Comparison of two vector quantities often leads to interesting inequalities that can be expressed succinctly as "majorization" relations. Majorization is one of the most basic concepts in matrix theory, first considered over a century ago as a way to address set of problems from economics, engineering and physics. More recently, it has become a central mathematical tool in quantum information theory, beginning with work of Nielsen. In this work we introduce majorization and provide a review on some recent developments on majorization inequalities.

Keywords: Majorization, Singular values, τ-Measurable operator.
Mathematics Subject Classification [2023]: 47A63, 47A30, 46L05

1 Introduction

The notion of majorization was first defined for elements of $\mathbb{R}^n$, n-tuples of real numbers. For $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$, suppose $x^\downarrow = (x_1^\downarrow, \dots, x_n^\downarrow)$ and $y^\downarrow = (y_1^\downarrow, \dots, y_n^\downarrow)$ are obtained from x and y by rearranging their components in decreasing order. Then x is said to be majorized by y, denoted $x \prec y$, if $\sum_{i=1}^k x_i^\downarrow \leq \sum_{i=1}^k y_i^\downarrow$, for all $1 \leq k \leq n$ and $\sum_{i=1}^n x_i^\downarrow = \sum_{i=1}^n y_i^\downarrow$. It is a well-known that for $x, y \in \mathbb{R}^n$, $x \prec y$ if and only if there is a doubly stochastic operator $D : \mathbb{R}^n \to \mathbb{R}^n$ for which $x = Dy$. Recall that an operator $D : \mathbb{R}^n \to \mathbb{R}^n$ is called doubly stochastic if $D\mathbf{1} = \mathbf{1}$ and $D^*\mathbf{1} = \mathbf{1}$, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$ and D^* is the adjoint operator of D. Equivalently, D is doubly stochastic if each row and each column of its corresponding matrix sums to one. For two $m \times n$ matrices A and B, A is majorized by B in symbols $A \prec B$ if there is a doubly stochastic $m \times m$ matrix D such that $A = DB$.

This notion was soon extended to the continuous case by several authors [1, 5]. In their setting, the vectors of $\mathbb{R}^n$ are replaced by integrable functions on a finite measure space $(X, \mathcal{A}, \mu)$. Generally if $f : X \to \mathbb{R}$ is an integrable function on a sigma finite measure space (X, A, μ), then the decreasing rearrangement of f, denoted $f^\downarrow$, is defined for each $t \in [0, \mu(X)]$ as follows.

$$f^\downarrow(t) = \inf\{s \mid d_f(s) \leq t\} = \sup\{s \mid d_f(s) > t\},$$

where $d_f(s) := \mu\big(\{x \in X; f(x) > s\}\big)$ is the distribution function of f. Based on this rearrangement of integrable function, the notion of majorization is now easily extended to $L^1(X, \mu)$, the space of all real valued integrable functions on X.

Definition 1.1. [1] Let $(X, \mathcal{A}, \mu)$ be a sigma finite measure space. Then for two functions $f, g \in L^1(X, \mu)$, f is said to be majorized by g, denoted $f \prec g$, if

$$\int_0^t f^\downarrow \, dm \leq \int_0^t g^\downarrow \, dm,$$

for all $t \in [0, \mu(X)]$, and

$$\int_0^{\mu(X)} f^\downarrow \, dm = \int_0^{\mu(X)} g^\downarrow \, dm.$$

Here dm is the Lebesque measure on the interval $[0, \mu(X)]$.

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Definition 1.2. [1] Let $(X, \mathcal{A}, \mu)$ be σ finite measure space. A positive operator T on $L^1(X)$ is called semi-doubly stochastic if

$$\int_X Tf\, d\mu = \int_X f\, d\mu, \quad \forall f \in L^1,$$

$$\int_X T^*\chi_E\, d\mu \leq \mu(E), \quad \forall E \in \mathcal{A} \ \ with \ \ \mu(E) < \infty.$$

The set of all semi-doubly stochastic operators on $L^1(X, \mu)$ is denoted by $\mathcal{SDS}(L^1(X, \mu))$. It is clear that $\mathcal{DS}(L^1(X, \mu)) \subset \mathcal{SDS}(L^1(X, \mu))$. But the converse is not true in general.

Corollary 1.3. [1] Let X be a σ-finite measure space. Then for $f, g \in L^1(X, \mu)$, $g \prec f$ if and only if there exists sequence $(S_n)_{n\in\mathbb{N}} \in S\mathcal{DS}\big(L^1(X, \mu)\big)$ such that $S_n f \xrightarrow{L^1} g$.

2 Main results

In this section we establish an analogue for majorization type of Young and Heinz inequalities for τ-measurable operators. Let $\mathcal{H}$ be an infinite dimensional Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators in $\mathcal{H}$. In what follows, $\mathcal{N}$ is a von Neumann. The identity in $\mathcal{N}$ is denoted by 1. A von Neumann algebra is said to be σ-finite if it admits at most countably many orthogonal projections. We are only interested in semi-finite von Neumann algebras, that is, those which admit a faithful normal semi-finite trace τ. We fix a couple $(\mathcal{M}, \tau)$ for semi-finite von Neumann algebra $\mathcal{M}$ with semi-finite trace τ. The cone of positive operators, the identity and the projection lattice in $\mathcal{M}$ are denoted by $\mathcal{M}^+$, 1 and $\mathcal{P}(\mathcal{M})$ respectively.

An (unbounded) operator x with domain $\mathcal{D}(x) \subseteq \mathcal{H}$ is densely defined, if $\mathcal{D}(x)$ is dense in $\mathcal{H}$. The operator x is called closed whenever its graph is a closed subspace of $\mathcal{H} \times \mathcal{H}$. If $x : \mathcal{D}(x) \to \mathcal{H}$ is a closed densely defined linear operator, then it can be shown that the operator x^*x is self-adjoint and positive. A linear operator $x : \mathcal{D}(x) \to \mathcal{H}$ is called affiliated with $\mathcal{M}$, if $ux = xu$ for all unitary $u \in \mathcal{M}'$. Note that if $x \in B(\mathcal{H})$, then x is affiliated with $\mathcal{M}$ if and only if $x \in \mathcal{M}$.

A closed and densely defined linear operator $x : \mathcal{D}(x) \to \mathcal{H}$ is said to be τ-measurable if x affiliated with $\mathcal{M}$, and there exists $\lambda \geq 0$ such that $\tau(e^{|x|}(\lambda, \infty)) < \infty$. The collection of all τ-measurable operators is denoted by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a complex $*$-algebra with unit element 1. The von Neumann algebra $\mathcal{M}$ is a $*$-subalgebra of $L_0(\mathcal{M})$.

Let $x \in L_0(\mathcal{M})$ and $t > 0$. The t-th singular value of x (or generalized s-numbers) is the number denoted by $\mu_t(x)$ and for each $t \in \mathbb{R}_0^+$ is defined by

$$\mu_t(x) = \inf\{\| xe \| : e \in \mathcal{P}(\mathcal{M}), \tau(1 - e) \leq t\}.$$

The notation of generalized s-numbers for τ-measurable operators was carefully developed by T. Fack and H. Kosaki[2]. For every $x \in \mathcal{M}$, $\mu_t(x)$ is non increasing and right continuous and

$$\mu_t(|x|) = \mu_t(x^*) = \mu_t(x) \quad \text{and} \quad \tau(|x|) = \int_0^\infty \mu_t(x)\, dt.$$

Proposition 2.1. *[2] For τ-measurable operator x, the following conditions are equivalent:*

1. *$\tau(p^x(s, \infty)) < \infty$ for all $s > 0$,*
2. *$\lim_{t\to\infty} \mu_t(x) = 0$,*
3. *there exists a sequence of bounded operators $x_n \in L_1(\mathcal{M})$ such that $x_n \to x$ in the measure topology.*

With any of these three characterizations, we say that x is τ-compact operator. These operators form a complete bilateral ideal in $L_0(\mathcal{M})$ that we will denote by $\mathcal{K}(L_0(\mathcal{M}))$. It is known that for every positive operator $x \in \mathcal{K}(L_0(\mathcal{M}))$,

$$\sigma(x) = \overline{\{\mu_t(x) : \quad t > 0\}}.$$

Note that a τ-compact operator is not necessarily bounded.

For $0 < p < \infty$, $L_p(\mathcal{M}, \tau)$ is defined as the set of all τ-measurable operators x such that

$$\| x \|_p = \tau(|x|^p)^{\frac{1}{p}} < \infty. \tag{221}$$

Moreover, we put $L_\infty(\mathcal{M}, \tau) = \mathcal{M}$ and denote by $\| \cdot \|_\infty$ the usual operator norm. For simplicity from now on $L_p(\mathcal{M}, \tau)$ will denoted by $L_p(\mathcal{M})$. Let $1 \leq p < \infty$, an operator $x \in \mathcal{M}$ is said to be locally integrable if there exists $\delta > 0$ such that

$$\int_0^\delta \mu_t(x)^p \, dt < \infty.$$

The set containing all these operators is denoted by $\mathfrak{L}^p_{loc}(\mathcal{M})$. Note that in particular, all bounded operators $a \in \mathcal{M}$ are of this class. Moreover,

$$\int_0^\delta \mu_t(x)^p \, dt \geq \mu_\delta(x)^{p-1} \int_0^\delta \mu_t(x) \, dt$$

implies that $\mathfrak{L}^p_{loc}(\mathcal{M}) \subset \mathfrak{L}^1_{loc}(\mathcal{M})$ for each $p \geq 1$ [4].

Definition 2.1. Let a, b be positive τ-measureable operators. We say that a is submajorized (weakly majorized) by b in symbol $a \prec_w b$, if $\int_0^s \mu_t(a)dt \leq \int_0^s \mu_t(b)dt$ for all $s > 0$. Moreover a is said to be majorized by b and is indicated by $a \prec b$, if $a \prec_w b$ and $\int_0^\infty \mu_t(a)dt = \int_0^\infty \mu_t(b)dt$.

Theorem 2.2. *[5] Let a, b be positive bounded operators and z be a bounded operator in $L_0(\mathcal{M})$. Then for $p, q, r \in \mathbb{R}^+$ that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$,*

$$\frac{1}{r}|az^2b|^r \prec_w \frac{1}{p} \mid bz^* \mid^p + \frac{1}{q} \mid az \mid^q . \tag{222}$$

Moreover, $\frac{1}{r}|az^2b|^r \prec \frac{1}{p} \mid bz^* \mid^p + \frac{1}{q} \mid az \mid^q$ if

$$\mid bz^* \mid^p = \mid az \mid^q .$$

Han and Shao in [3], proved that if x, y be τ-measurable operators and $0 \leq \nu \leq 1$, then

$$\mu_t(x^\nu y^{1-\nu} + x^{1-\nu} y^\nu) \leq \mu_t(x + y). \tag{223}$$

Theorem 2.3. *Let x, y be operators in $L_0(\mathcal{M})$. Then for every $0 \leq \nu \leq 1$*

$$\mid x^\nu y^{1-\nu} + x^{1-\nu} y^\nu \mid \prec_w \mid x + y \mid . \tag{224}$$

Moreover, if $x, y \in L_1(\mathcal{M})$ are bounded operators or $xy \in \mathfrak{L}^2_{loc}(\mathcal{M})$, then

$$\mid x^\nu y^{1-\nu} + x^{1-\nu} y^\nu \mid \prec \mid x + y \mid, \tag{225}$$

if and only if $\mid x \mid = \mid y \mid$.

3 Conclusion

In this article, we present a number of majorization inequalities. These inequalities depend on the properties of singular values. With a similar method in the proof of [5, Theorem 3.3], we can prove the condition of equality in inequality (223). It is not clear for us as to whether or not the following majorization inequality for τ-measurable operators is true

$$\mid a^\nu z b^{1-\nu} + b^\nu z a^{1-\nu} \mid \prec \mid az + zb \mid .$$

We hope that the obtained results will be useful to study the majorization theory in the future.

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References

[1] F. Bahrami, S.M. Manjegani, and S. Moein, *Semi-doubly stochastic operators and majorization of integrable functions*, Bulletin of the Malaysian Mathematical Sciences Society series 44 (2) (2021), 693-703.

[2] T. Fack and H. Kosaki, *Generalized s-numbers of τ-measurable operators*, Pac. J. Math. 123 (1986), 269-300.

[3] Y. Han and J. Shao, *More results on generalized singular number inequalities of τ-measurable operators*, Journal of Inequalities and Appl. 144 (2016). https://doi.org/10.1186/s13660-016-1085-1.

[4] G. Larotonda, *The case of equality in Young's inequality for the s-numbers in semi-finite von Neumann algebras*, J. Operator Theory 81 (2019), 157-173.

[5] Maleki Khouzani, Z. Maleki Khouzani, S. M. Manjegani, *Some inequalities and majorization for products of τ-measurable operators*, Journal of Inequalities and Appl. 114 (2022). https://doi.org/10.1186/s13660-022-02850-9.

[6] S. M. Manjegani, *On young and Heinz inequalities for faithful tracial states*, lin. and Mult. Alg. 65 (2017), 2432-2456.

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Some simple proofs on the recent algorithms

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Abstract

In this paper, some simple proofs of the convergence of the recent iterative algorithms by relaxed (u, v)-cocoercive mappings due to some recent algorithms is presented.

Keywords: Relaxed (u, v)-cocoercive mapping; Strong convergence; α-expansive mapping
Mathematics Subject Classification [2023]: 47H10

1 Introduction

In this paper, a simple proof for the convergence of some iterative algorithms is presented which improves and refines the original proofs.

Suppose that C is a nonempty closed convex subset of a real normed linear space E and E^* is its dual space. Suppose that $\langle ., . \rangle$ denotes the pairing between E and E^*. The normalized duality mapping $J : E \to E^*$ is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

for each $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is called smooth if for all $x \in U$, there exists a unique functional $j_x \in E^*$ such that $\langle x, j_x \rangle = \|x\|$ and $\|j_x\| = 1$ (see [1]).

For a map T from E into itself, we denote by Fix(T) := {x ∈ E : x = Tx}, the fixed point set of T.

Recall the following well known concepts:

1. Suppose that C is a nonempty closed convex subset of a real Banach space E. A mapping $B : C \to E$ is called relaxed (u, v)-cocoercive [2], if there exist two constants $u, v > 0$ such that

   $$\langle Bx - By, j(x - y) \rangle \geq (-u)\|Bx - By\|^2 + v\|x - y\|^2,$$

   for all $x, y \in C$ and $j(x - y) \in J(x - y)$.

2. Suppose that C is a nonempty closed convex subset of a real Banach space E and B is a self mapping on C. If there exists a positive integer α such that

   $$\|Bx - By\| \geq \alpha\|x - y\|.$$

   for all $x, y \in C$, then B is called α-expansive.

2 Main results

First we consider some iterative algorithm about Relaxed (u, v)-cocoercive mappings.

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2.1 Relaxed (u, v)-cocoercive mappings

G. Cai and S. Bu, W. Chantarangs, C. Jaiboon, and P. Kumam, J.S. Jung, P. Kumam and C. Jaiboon, X. Qin, M. Shang and H. Zhou, X. Qin, M. Shang and Y. Su, X. Qin, M. Shang and Y. Su, considered some iterative algorithms for finding a common element of the set of fixed points of nonexpansive mappings and the set of solutions of a variational inequality $VI(C, A)$, where A is a relaxed (u, v)-cocoercive mapping of C into H(to see their paper, one can refer to [2] and the references therein).

Lemma 2.1. *Let A be a relaxed (m, v)-cocoercive mapping and ϵ-Lipschitz continuous such that $v - m\epsilon^2 > 0$ and $VI(C, A) \neq \emptyset$. Then A is a $(v - m\epsilon^2)$-expansive mapping and $VI(C, A)$ is a singleton.*

To see an example, Theorem 3.1 from P. Kumam and C. Jaiboon in [2] is considered. Next, in the following remark, a simple proof for some similar results is presented: **A simple Proof:** (i). Consider Theorem 3.1 from P. Kumam and C. Jaiboon in [2] and the ξ-Lipschitz continuous and relaxed (m, v)-cocoercive mapping B. We may assume that $(v - m\xi^2) > 0$, and hence from Lemma 4.1, B is $(v - m\epsilon^2)$-expansive, i.e,

$$\|Bx - By\| \geq (v - m\xi^2)\|x - y\|, \tag{226}$$

and $VI(E, B)$ is singleton i.e. there exists an element $p \in E$ such that $VI(E, B) = \{p\}$, hence $\Gamma = \{p\}$ in the theorem. The authors prove that

$$\lim_n \|Bz_n - Bp\| = 0. \tag{227}$$

Now, put $x = z_n$ and $y = p$ in (226), and from (226) and (227), we have

$$\lim_n \|z_n - p\| = 0.$$

Hence, $z_n \to p$. As a result one of the main claims of the theorem is established (note $\Gamma = VI(E, B) = \{p\}$).

(ii). A similar remark applies to the main results of other references of [2].

3 Inverse strongly monotone mappings

J. Chen, L. Zhang, T. Fan, S. Takahashi and W. Takahashi, H. Iiduka and W.Takahashi, K. R. Kazmi, Rehan Ali and Mohd Furkan, X. Qin, M. Shang and Y. Su, M. Zhang, S. Shan and N. Huang, T. Jitpeera and P. Kumam, S. Peathanom and W. Phuengrattana, Piri, X. Qin, M. Shang and Y. Su and M. Lashkarizadeh Bami and E. Soori considered some iterative methods for finding a common element of a set of fixed points of nonexpansive mapping and the set of solutions of a variational inequality $VI(C, A)$, where A is an α-inverse strongly monotone mapping of C into H. In this section, a spacial case similar to cocoercive mappings is studied (to see their paper, one can refer to [2] and the references therein).

Lemma 3.1. *Let A be an α-inverse strongly monotone mapping and $VI(C, A) \neq \emptyset$. Suppose that A is also an γ-expansive mapping. Then $VI(C, A)$ is a singleton.*

Theorem 3.2. ***A simple proof in a spacial case:*** *(i). Consider Theorem 3.1 in Iiduka and Takahashi in [2]; note A is an α-inverse strongly monotone mapping there. If we consider an extra condition of γ-expansiveness (or A being one to one) in the theorem then from Lemma 3.1, we have that $VI(C, A)$ is a singleton and hence in the theorem $F(S) \cap VI(C, A)$ is a singleton, i.e, $F(S) \cap VI(C, A) = VI(C, A) = \{u\}$ for an element $u \in C$. The authors prove that*

$$\lim_n \|Ax_n - Au\| = 0. \tag{228}$$

Now, put $x = x_n$ and $y = u$ and then from (228) and (2), we have

$$\lim_n \|x_n - u\| = 0.$$

Hence, $x_n \to u$. Therefore we have the main claim of Theorem 3.1 (note $F(S) \cap VI(C, A) = VI(E, B) = \{u\}$).

(ii). A similar comment applies to the main results in the another references in [2].

4 A simple proof for the theorem

S. Imnang (see the reference in [3]) considered an iterative algorithm for finding a common element of the set of fixed points of nonexpansive mappings and the set of solutions of a variational inequality. Our argument will rely on the following lemma.

Lemma 4.1. *Suppose that C is a nonempty closed convex subset of a Banach space E. Suppose that $A : C \to E$ is a relaxed (m, v)-cocoercive mapping and ϵ-Lipschitz continuous with $v - m\epsilon^2 > 0$. Then A is a $(v - m\epsilon^2)$-expansive mapping.*

The following theorem is due to S. Imnang that solves the viscosity iterative problem for a new general system of variational inequalities in Banach spaces:

Theorem 4.2. *Suppose that X is a Banach space which is uniformly convex and 2-uniformly smooth with the 2-uniformly smooth constant K, C is a nonempty closed convex subset of X and Q_C is a sunny nonexpansive retraction from X onto C. Assume that $A_i : C \to X$ is relaxed (c_i, d_i)-cocoercive and L_i-Lipschitzian with $0 < \lambda_i < \frac{d_i - c_i L_i^2}{K^2 L_i^2}$ for each $i = 1, 2, 3$. Suppose that f is a contraction mapping with the constant $\alpha \in (0, 1)$ and $S : C \to C$, a nonexpansive mapping such that $\Omega = F(S) \cap F(G) \neq \emptyset$, where G is defined as in Lemma 1.1. Suppose that $x_1 \in C$ and $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are the following sequences:*

$$\begin{cases} z_n = Q_C(x_n - \lambda_3 A_3 x_n), \\ y_n = Q_C(z_n - \lambda_2 A_2 z_n), \\ x_{n+1} = a_n f(x_n) + b_n x_n + (1 - a_n - b_n) S Q_C(y_n - \lambda_1 A_1 y_n, \end{cases}$$

where $\{a_n\}$ and $\{b_n\}$ are two sequences in (0, 1) such that

1. $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;
2. $0 < \lim inf_{n\to\infty} b_n \leq \lim sup_{n\to\infty} b_n < 1$.

Then $\{x_n\}$ converges strongly to $q \in \Omega$, which solves the following variational inequality:

$$\langle q - f(q), J(q - p)\rangle \leq 0, \forall f \in \Pi_C, p \in \Omega.$$

A simple Proof: Let $i = 1, 2, 3$. Consider Theorem 4.2 and the L_i-Lipschitz continuous and relaxed (c_i, d_i)-cocoercive mapping A_i in Theorem 4.2. From the condition that $0 < \lambda_i < \frac{d_i - c_i L_i^2}{K^2 L_i^2}$, we have that $0 < 1 + 2(\lambda_i c_i L_i^2 - \lambda_i d_i + K^2 \lambda_i^2 L_i^2) < 1$. Then we have that $I - \lambda_i A_i$ is nonexpansive when $0 < 1 + 2(\lambda_i c_i L_i^2 - \lambda_i d_i + K^2 \lambda_i^2 L_i^2)$. Then applying the coefficients $\alpha_i = 1 + 2(\lambda_i c_i L_i^2 - \lambda_i d_i + K^2 \lambda_i^2 L_i^2)$ we have that $I - \lambda_i A_i$ is α_i-contraction, for each $i = 1, 2, 3$. Also, note that Q_C is nonexpansive and $I - \lambda_i A_i$ is α_i-contraction, for each $i = 1, 2, 3$. Hence, we conclude that

and since $0 < \alpha_1 \alpha_2 \alpha_3 < 1$ then G is an α-contraction with $\alpha = \alpha_1 \alpha_2 \alpha_3$, hence from Banach's contraction principle $F(G)$ is a singleton set and hence, Ω is a singleton set i.e., there exists an element $p \in X$ such that $\Omega = \{p\}$. Since $(d_i - c_i L_i^2) > 0$, from Lemma 4.1, A_i is $(d_i - c_i L_i^2)$-expansive, i.e,

$$\|A_i x - A_i y\| \geq (d_i - c_i L_i^2)\|x - y\|, \tag{229}$$

in Theorem 4.2. The authors proved that

$$\lim_n \|A_3 x_n - A_3 p\| = 0, \tag{230}$$

for $x^* = p$. Now, put $x = x_n$ and $y = p$ in (229), and from (229) and (230), we have

$$\lim_n \|x_n - p\| = 0.$$

Hence, $x_n \to p$. As a result one of the main claims of Theorem 4.2 is established (note $\Omega = \{p\}$).

Note that the main aim of theorem 3.1 are $x_n \to p$ and

$$\langle q - f(q), J(q-p)\rangle \leq 0, \forall f \in \Pi_C, p \in \Omega.$$

Next, we show that the main aim of theorem 3.1 can be concluded from the relations (3.12) in page 11 and the proof in Theorem 4.2 can be simplified even further using the above. Note, the part of the proof between the relations (3.12) to the end of the proof of Theorem 3.1 can be removed from the proof. Indeed, since immediately from (3.12), we conclude that $x_n \to p$, i.e., the first aim of Theorem 3.1 is concluded. The second aim of the theorem i.e.,

$$\langle q - f(q), J(q-p)\rangle \leq 0, \forall f \in \Pi_C, p \in \Omega,$$

is clear, because $p = q$ ($\Omega = \{p\}$) and $J(0) = \{0\}$. Consequently, the relations between (3.12) in page 11 to the end of the proof of Theorem 3.1 in page 11 can be removed.

One can see another simple proof for the algorithm due to K. R. Kazmi et al, published in the journal Numerical Algorithms in [4].

References

[1] R. P. Agarwal, D. O'Regan and D. R. Sahu, *Fixed point theory for Lipschitzian-type mappings with applications, in: Topological Fixed Point Theory and its Applications*, vol. 6, Springer, New York, 2009.

[2] R. P. Agarwal, E. Soori and D. ORegan, *simple proof for the algorithms of relaxed (u, v)-cocoercive mappings and -inverse strongly monotone mappings*, Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 327-333.

[3] E. Soori, R. P. Agarwal, *A simple proof for Kazmi et al.'s iterative scheme*, Advances in the Theory of Nonlinear Analysis and its Application, 2021.

[4] E. Soori, D. ORegan and R. P. Agarwal, *A simple proof for Imnangs algorithms*, Journal of Inequalities and Applications (2022) 2022:165.

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Essential norm of substitution integral operators on Orlicz spaces

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Abstract

In this paper, We investigate the problem of conditions on the generating Young functions, the functions u, φ and $h = d(\mu \circ \varphi^{-1})/d\mu$, under which operator T^u_φ is of closed range or finite rank. Finally, we determine the lower and upper estimates for the essential norm of T^u_φ on Orlicz spaces under certain conditions.

Keywords: Orlicz spaces, Substitution vector-valued integral operator, Essential norm, Conditional expectation operators
Mathematics Subject Classification [2023]: 47B20; 47B38

1 Introduction

Let $\theta : \mathbb{R} \to \mathbb{R}^+$ be a continuous convex function such that
(1) $\theta(x) = 0$ if and only if $x = 0$.
(2) $\lim_{x\to\infty} \theta(x) = \infty$.
(3) $\lim_{x\to\infty} \frac{\theta(x)}{x} = \infty$.
The convex function θ is called Young's function. With each Young's function θ, one can associate another convex function $\theta^* : \mathbb{R} \to \mathbb{R}^+$ having similar properties, which is defined by

$$\theta^*(y) = \sup\{x|y| - \theta(x) : x \geq 0\}.$$

The convex function θ^* is called complementary Young function to θ. Let $X = (X, \Sigma, \mu)$ be a σ-finite complete measure space. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. If θ is a Young function, then the set of Σ-measurable functions

$$L^\theta(\mu) = \{f : X \to \mathbb{C} : \exists \alpha > 0, \int_X \theta(\alpha|f|)d\mu < \infty\}$$

is a Banach space, with respect with the Luxemburg norm defined by

$$\|f\|_\theta = \inf\{\delta > 0 : \int_X \theta(\frac{|f|}{\delta})d\mu \leq 1\}.$$

$(L^\theta(\mu), \|.\|_\theta)$ is called Orlicz space. A Young function θ is said to satisfy the Δ_2-condition(globally) if $\theta(2x) \leq k\theta(x), x > x_0(x_0 = 0)$, for some constant$k > 0$. constant

Proposition 1.1. *[4](Holder's inequality). For all $f \in L^\theta(\mu)$ and $g \in L^{\theta^*}(\mu)$,*

$$\int_X |fg|d\mu \leq 2\|f\|_\theta \|g\|_{\theta^*}.$$

Let θ_1, θ_2 be two Young functions, then θ_1 is called stronger than θ_2, which $\theta_1 \succ \theta_2$[or $\theta_2 \prec \theta_1$], if

$$\theta_2(x) \leq \theta_1(ax), \quad x \geq x_0 \geq 0,$$

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for some $a \geq 0$ and x_0, if $x_0 = 0$ then this condition is said to hold globally. Throughout this note we assume that θ satisfies Δ_2-condition.
The Orlicz norm on $L^\theta(\mu)$ is given by

$$\|f\|_\theta := \{\int_X fgd\mu; g \in L^{\theta^*}(\mu), \|g\|_{\theta^*} \leq 1\}.$$

If the Young function satisfies Δ_2-condition, then the dual space of the Orlicz space equipped with the Luxemburg norm is isometrically isomorphic th the Orlics space generated by the complementary function and equipped with the Orlicz norm. Let θ and θ^* be a pair of complementary Orlicz functions. Then each $g \in L^{\theta^*}(\mu)$ defines a bounded linear functional F_g on $L^\theta(\mu)$ by $F_g(f) = \int fgd\mu, f \in L^\theta(\mu)$.

Simple functions are not necessarily dense in $L^\theta(\mu)$. But, if θ satisfies Δ_2-condition, then simple functions are dense in $L^\theta(\mu)$. It is well-known that if $A \in \Sigma$ and $0 < \mu(A) < \infty$ then $\|\chi_A\|_\theta = 1/\theta^{-1}(1/\mu(A))$ where $\theta^{-1}(t) = \inf\{\delta > 0, \theta(s) > t\}$ is the right continuous inverse of θ. The usual convergence in the Orlicz space $L^\theta(\mu)$ can be introduced in term of the Orlicz norm $\|.\|_\theta$ as $x_n \to x$ in $L^\theta(\mu)$ means $\|x_n - x\|_\theta \to 0$. Also, a sequence $\{x_n\}_{n=1}^\infty$ in $L^\theta(\mu)$ is said to converges in θ-mean to $x \in L^\theta(\mu)$, if

$$\lim_{n\to\infty} I_\theta(x_n - x) = \lim_{n\to\infty} \int_X \theta(|x_n - x|)d\mu = 0.$$

For more details on Orlicz spaces, we refer to [4].

Let $\varphi : X \to X$ be a non-singular measurable transformation; i.e. $\mu \circ \varphi^{-1} \ll \mu$. It is assumed that the Radon-Nikodym derivative $h = d\mu \circ \varphi^{-1}/d\mu$ is almost everywhere finite-valued, or equivalently $\varphi^{-1}(\Sigma) \subseteq \Sigma$ is a sub-σ-finite algebra [5]. We have the following change of variable formula:

$$\int_{\varphi^{-1}(A)} f \circ \varphi d\mu = \int_A hfd\mu \qquad A \in \Sigma, f \in L^0(\Sigma).$$

Any nonsingular measurable transformation φ induces a linear operator (composition operator) C_φ from $L^0(\mu)$ into itself defined by

$$C_\varphi(f)(x) = f(\varphi(x)) \quad ; x \in X, \quad f \in L^0(\mu),$$

where $L^0(\Sigma)$ denotes the linear space of all equivalence classes of Σ-measurable functions on X. Here non-singularity of φ guarantees that the operator C_φ is well defined as a mapping from $L^0(\Sigma)$ into itself. If C_φ maps on Orlicz space $L^\theta(\mu)$ into itself , then C_φ is called composition operator on $L^\theta(\mu)$. Note that, in this case C_φ is bounded. The support of a measurable function f is defined by $\sigma(f) = \{x \in X; f(x) \neq 0\}$. For a given complex Hilbert space $\mathcal{H}$, let $u : X \to \mathcal{H}$ be a mapping. We say that u is weakly measurable if for each $g \in \mathcal{H}$ the mapping $x \mapsto \langle u(x), g\rangle$ of X to $\mathbb{C}$ is measurable. We will denote this map by $\langle u, g\rangle$.

Definition 1.2. Let $\varphi : X \to X$ be a non-singular measurable transformation and $\mathcal{C}_\varphi$ be a composition operator on $L^\theta(X)$. Also let $u : X \to \mathcal{H}$ be a weakly measurable function. Then the pair (u, φ) induces a substitution vector-valued integral operator $T_u^\varphi : L^\theta(\mu) \to \mathcal{H}$ defined by

$$\langle T_u^\varphi f, g\rangle = \int_X \langle u, g\rangle f \circ \varphi d\mu, \quad f \in L^\theta(\mu).$$

It is easy to see that T_u^φ is well defined and linear.

From [2], we have that if $u : X \to \mathcal{H}$ be a weakly measurable function. We say that $(u, \varphi, \mathcal{H})$ has absolute property, if for each $f \in L^\theta(X)$, there exists $g_f \in \mathcal{D}$ such that $\sup_{g\in\mathcal{D}} \int_X |\langle u, g\rangle||C_\varphi f|d\mu = \int_X |\langle u, g_f\rangle||C_\varphi f|d\mu$, and $\langle u, g_f\rangle = e^{i(-\arg C_\varphi f + \beta_f)}|\langle u, g_f\rangle|$, for a constant β_f.

Proposition 1.3 ([2]). *Assume that $(u, \varphi, \mathcal{H})$ has the absolute property. Then*

$$\sup_{g\in\mathcal{D}} |\int_X \langle u, g\rangle C_\varphi f d\mu| = \sup_{g\in\mathcal{D}} \int_X |\langle u, g\rangle||C_\varphi f|d\mu.$$

Throughout of this paper we assume that $(u, \varphi, \mathcal{H})$ has the absolute property.

For a sub-σ-finite algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \to E^{\mathcal{A}} f$, defined for all non-negative f as well as for all $f \in L^p(\Sigma), 1 \leq p \leq \infty$, where $E^{\mathcal{A}} f$, by Radon-Nikodym Theorem, is the unique $\mathcal{A}$-measurable function satisfying

$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \quad \forall A \in \mathcal{A}.$$

For more details on the properties of $E^{\mathcal{A}}$ see [4]. Throughout this paper, we assume that $\mathcal{A} = \varphi^{-1}(\Sigma)$ and $E^{\varphi^{-1}(\Sigma)} = E$.

2 Main results

Definition 2.1. Let $u : X \to \mathcal{H}$ be a weakly measurable function. We say that u is a semi-weakly bounded function if for some $M > 0$,

$$\|\langle u, \lambda \rangle\|_{\theta^*} \leq M \|\lambda\|, \quad \text{for each} \quad \lambda \in \mathcal{H}.$$

Theorem 2.2. *Let $u : X \to \mathcal{H}$ be a weakly measurable function. If u is a semi-weakly bounded function and $h \in L^\infty(\Sigma)$, then $T_u^\varphi : L^\theta(\mu) \to \mathcal{H}$ is bounded.*

Proposition 2.3. *Let $u : X \to \mathcal{H}$ be a weakly measurable function. Then*

(i) If for each $\lambda \in \mathcal{H}_1$, the functions $\langle u, \lambda \rangle$ are conditionable and T_u^φ is bounded. Hence for all $\lambda \in \mathcal{H}_1$, $hE(\langle u, \lambda \rangle) \circ \varphi^{-1} \in L^{\theta^*}(\mu)$.

(ii) T_u^φ is bounded if and only if $\sup_{\lambda \in \mathcal{H}_1} \|hE(\langle u, \lambda \rangle) \circ \varphi^{-1}\|_{\theta^*} < \infty$.

Theorem 2.4. *The substitution vector-valued integral operator T_u^φ is a compact operator on $L^\theta(\mu)$ if and only if for any $\epsilon > 0$ the set*

$$N_\epsilon := \{x \in X : \sup_{\lambda \in \mathcal{H}_1} hE|\langle u, \lambda \rangle| \circ \varphi^{-1} \geq \epsilon\}$$

consists of finitely many atoms.

First, we characterize the closedness of range of a substitution vector-valued integral operator from $L^\theta(\mu)$ to $\mathcal{H}$.
We start by the following lemma. Put $J := \cup_{\lambda \in \mathcal{H}_1} \sigma(hE(|\langle u, \lambda \rangle| \circ \varphi^{-1})$.

Lemma 2.5. *Let T_u^φ be a bounded substitution vector-valued integral operator from $L^\theta(\mu)$ to $\mathcal{H}$ and there is a constant $c > 0$ such that $\sup_{\lambda \in \mathcal{H}_1} hE(|\langle u, \lambda \rangle| \circ \varphi^{-1} \geq c$ on J, then $T_u^\varphi \mid_J$ is injective..*

Theorem 2.6. *Let T_u^φ be a bounded substitution vector-valued integral operator from $L^\theta(\mu)$ to $\mathcal{H}$. Then, the following statements are hold.*

(i) Suppose T_u^φ from $L^\theta(\mu)$ to $\mathcal{H}$ has closed range and $x \prec \theta$ then there is a constant $c > 0$ such that $\sup_{\lambda \in \mathcal{H}_1} hE(|\langle u, \lambda \rangle| \circ \varphi^{-1} \geq c$ on J.

(ii) If there is a constant $c > 0$ such that $\sup_{\lambda \in \mathcal{H}_1} hE(|\langle u, \lambda \rangle| \circ \varphi^{-1} \geq c$ on J and $\theta \prec x$, then T_u^φ from $L^\theta(\mu)$ to $\mathcal{H}$ has closed range

Let $\mathcal{B}$ be a Banach space and $\mathcal{K}$ be the set of all compact operators on $\mathcal{B}$, the essential norm of T means the distance from T to $\mathcal{K}$ in the operator norm, namely

$$\|T\|_e = \inf\{\|T - S\| : S \in \mathcal{K}\}.$$

Clearly, T is compact if and only if $\|T\|_e = 0$.

Theorem 2.7. *Let T_u^φ be a bounded operator on $L^\theta(\mu)$. Also let $\alpha = inf\{r > 0, N_r consists\ of\ finitely\ many\ atoms\}$ and $N_r = \{x \in X := \sup_{\lambda\in\mathcal{H}_1} \|hE(|\langle u, \lambda\rangle|) \circ \varphi^{-1} \geq r\}$. Then we obtain that*
(i) $\|T_u^\varphi\|_e = 0$ if and only if $\alpha = 0$.
(ii) $\|T_u^\varphi\|_e \geq \frac{1}{a}\alpha$ where $\theta \prec x$(i.e. for some $a > 0$ we have $\theta(x) \leq ax$. In particular if $a \leq 1$ we have $\|T_u^\varphi\|_e \geq \alpha$
(iii) $\|T_u^\varphi\|_e \leq a\alpha$ where $x \prec \theta$(i.e. for some $a > 0$ we have $\theta(x) \leq ax$.In particular if $a \leq 1$ we have $\|T_u^\varphi\|_e \leq \alpha$

References

[1] Y. Cui, H. Hudzik, R. Kumar and L. Maligranda, *Composition operators in Orlicz spaces*, J. Aus. Math. Soc. **76**(2004), 189-206.

[2] H. Emamalipour, M. R. Jabbarzadeh and Z. Moayyerizadeh, *A substitution vector-valued integral operator*, J .Math. Anal. Appl, **431**(2015), 812-821.

[3] B. S. Komal, S . Gupta and N. Suri, *Weighted composition operators on Orlicz spaces*, Int. J. Contemp. Math. Sci. **1**(2010), 11-20.

[4] M. M. Rao and Z. D. Ren, *Applications of Orlicz spaces*, Marcel Dekker, Inc. New York, 2002.

[5] R. K. Singh and J. S. Manhas, *Composition operators on function spaces*, North Holland Math. Studies 179, Amsterdam 1993.

[6] A. C. Zaanen, *Integration*, 2nd ed., North-Holland, Amsterdam, 1967.

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Fidelity of density operators in a C*-algebra

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Abstract

Joszas definition of fidelity for a pair of (mixed) quantum states is studied in the context of C*-algebras. In the unital C*-algebra $\mathcal{A}$ with a faithful trace functional τ, the role of quantum states in the classical quantum-mechanical framework is assumed by positive elements $\rho \in \mathcal{A}$ for which $\tau(\rho) = 1$. We will introduce α-Fidelity for density operators.

Keywords: pure state, density operator, fidelity, τ-state, α-fidelity.
Mathematics Subject Classification [2023]: 47A50 81P15

1 Introduction

Fidelity is a central concept to quantum information. The notion of fidelity provides a quantitative measure for the qualitative assessment of how well data or information has been preserved through some type of transmission procedure or information processing task. This concept appears in quantum information theory as well, with the aim of providing a similar quantitative measure. In a rather different form (namely, in the guise of the Bures metric), the notion of fidelity also occurs in operator algebra theory [1]. In one of the most important settings, fidelity is a numerical measure of how close one state σ of a quantum system is to another state ρ. For pure states ξ and η that are unit vectors in a separable Hilbert space $\mathcal{H}$, the fidelity is measured by $|\langle \xi, \eta \rangle|$, the modulus of the inner product of ξ and η. Therefore, high fidelity occurs when the numerical measure is close to 1, and at exactly 1 the vectors ξ and η are identical up to a phase factor. For mixed states are density operators σ and ρ acting on $\mathcal{H}$ there is a very satisfactory and useful notion of fidelity suggested by Josza [2] denoted by $\mathcal{F}(\sigma, \rho)$ and defined by $\mathrm{tr}\,(|\sigma^{\frac{1}{2}}\rho^{\frac{1}{2}}|)$, where tr denotes the canonical trace on the algebra $\mathcal{B}(\mathcal{H})$ of a bounded linear operators acting on $\mathcal{H}$. In the case where σ and ρ are rank-1 density operators, then the fidelity measure $\mathrm{tr}\,(|\sigma^{\frac{1}{2}}\rho^{\frac{1}{2}}|)$ coincides with $|\langle \xi, \eta \rangle|$, where $\xi, \eta \in \mathcal{H}$ are unit vectors that span the ranges of σ and ρ respectively. Let $\mathcal{T}(\mathcal{H})$ be the Banach space of trace-class operators on $\mathcal{H}$. Recall that a channel, or quantum channel, is a bounded linear map $\mathcal{E} : \mathcal{T}(\mathcal{H}) \longrightarrow \mathcal{T}(\mathcal{H})$, where such that the dual map $\mathcal{E}^* : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ is normal, unital, and completely positive and

$$\mathrm{tr}\,(\mathcal{E}(s)x) = \mathrm{tr}\,(s\mathcal{E}^*(x)), \quad \text{for all} \quad s \in \mathcal{T}(\mathcal{H}) \quad \text{and} \quad x \in \mathcal{B}(\mathcal{H}).$$

Using Stinespring Theorem to $\mathcal{E}^*$ and by using the fact that $\mathcal{E}^*$ is unital and normal, it was shown in [3, 4] that every channel $\mathcal{E}$ of $\mathcal{T}(\mathcal{H})$ has the form

$$\mathcal{E}(s) = \sum_{n=1}^{\infty} v_n s v_n^*, \qquad s \in \mathcal{T}(\mathcal{H}),$$

where $\{v_n\} \subset \mathcal{B}(\mathcal{H})$ is a sequence such that $\sum_{n=1}^{\infty} v_n^* v_n = 1$. It is well known that if $\mathcal{E} : \mathcal{T}(\mathcal{H}) \longrightarrow \mathcal{T}(\mathcal{H})$ is a channel, then

$$\mathcal{F}(\sigma, \rho) \leq \mathcal{F}(\mathcal{E}(\sigma), \mathcal{E}(\rho))$$

for all density operators σ and ρ acting on $\mathcal{H}$. Channel $\mathcal{E}$ is a unitary channel if there exists a unitary operator u on $\mathcal{H}$ such that $\mathcal{E}(s) = usu^*$ for every $s \in \mathcal{T}(\mathcal{H})$. It is clear that a unitary channel $\mathcal{E}$ preserves fidelity in the sense that $\mathcal{F}(\sigma, \rho) = \mathcal{F}(\mathcal{E}(\sigma), \mathcal{E}(\rho))$. It important and natural to determine whether unitary channels are only channels in which fidelity is preserved. Molnár [6] proved that if $\mathcal{E} : \mathcal{T}(\mathcal{H}) \longrightarrow \mathcal{T}(\mathcal{H})$ is a surjective channel that preserves fidelity, then $\mathcal{E}$ is a unitary channel.

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2 Main results

In this section we study fidelity for τ-state operators in a C*-algebra $\mathcal{A}$. In what follows, $\mathcal{A}$ is a unital C*-algebra. The real vector space of selfadjoint (or hermitian) elements of $\mathcal{A}$ is given by $\mathcal{A}_{sa}$, and the cone of positive elements of $\mathcal{A}$ is denoted by $\mathcal{A}_+$. We shall also assume that $\mathcal{A}$ admits at least one faithful trace; that is, there exists a continuous linear map $\tau : \mathcal{A} \longrightarrow \mathbb{C}$ such that, for all $x, y \in \mathcal{A}$, (i) $\tau(xy) = \tau(yx)$, (ii) $\tau(x^*x) \geq 0$, and (iii) $\tau(x^*x) = 0$ only if $x = 0$. For elements $h, k \in \mathcal{A}$, the notation $h \leq k$ indicates that $k - h \in \mathcal{A}_+$. Two elements $x, y \in \mathcal{A}$ are said to be orthogonal if $xy = yx = x^*y = xy^* = 0$. The group of invertible elements of $\mathcal{A}$ is denoted by $GL(\mathcal{A})$ and the set $GL(\mathcal{A})_+$ of positive invertible elements of $\mathcal{A}$ is defined by $GL(\mathcal{A})_+ = GL(\mathcal{A}) \cap \mathcal{A})$.

Proposition 2.1. [1] If $a\sigma, \rho \in \mathcal{A}_+$, then

$$\tau(a) = \frac{1}{2} \inf_{y \in GL(\mathcal{A})_+} \left(\tau(ay) + \tau(ay^{-1}\right), \tag{231}$$

$$\tau\left(|\sigma^{\frac{1}{2}}\rho^{\frac{1}{2}}|\right) = \frac{1}{2} \inf_{y \in GL(\mathcal{A})_+} \left(\tau(\rho y) + \tau(\sigma y^{-1})\right), \tag{232}$$

and

$$\tau\left(|\sigma^{\frac{1}{2}}\rho^{\frac{1}{2}}|\right) = \frac{1}{4} \inf_{y \in GL(\mathcal{A})_+} \left[\tau(\sigma y) + \tau(\sigma y^{-1}) + \tau(\rho y) + \tau(\rho y^{-1})\right]. \tag{233}$$

We use the term "τ-state" for density operators in C*-algebra $\mathcal{A}$ and the set of all such operators is denoted by S_τ and defied by

$$S_\tau = \{\rho \in \mathcal{A}_+ : \quad \tau(\rho) = 1\}.$$

Definition 2.2. [1] The τ-fidelity of pair of $\sigma, \rho \in S_\tau$ is quantity denoted by $\mathcal{F}_\tau(\sigma, \rho)$ and defined by $\mathcal{F}_\tau(\sigma, \rho) = \tau\left(|\sigma^{\frac{1}{2}}\rho^{\frac{1}{2}}|\right)$.

The essential properties of τ-fidelity are describe in the following Theorem.

Theorem 2.3. [1] If $\sigma, \rho \in S_\tau$, then

1. $\mathcal{F}_\tau(\sigma, \rho) = \mathcal{F}_\tau(\rho, \sigma)$,
2. $0 \leq \mathcal{F}_\tau(\sigma, \rho) \leq 1$.
3. $\mathcal{F}_\tau(\sigma, \rho) = 0$ if and only if $\sigma \perp \rho$, and
4. $\mathcal{F}_\tau(\sigma, \rho) = 1$ if and only if $\rho = \sigma$.

Furthermore, if $\mathcal{E} : \mathcal{A} \longrightarrow \mathcal{A}$ is a positive linear map such that $\tau \circ \mathcal{E} = \tau$, then

$$\mathcal{F}_\tau(\sigma, \rho) \leq \mathcal{F}_\tau\left(\mathcal{E}(\sigma), \mathcal{E}(\rho)\right).$$

Now, we define α-fidelity for operators $\sigma, \rho \in S_\tau$. With similar argument in the proof of Theorem 2.1 and using Young's inequality [5, Theorem 4.4] we can prove the properties of α-fidelity in Theorem 3.2.

Definition 2.4. The α-fidelity of a pair of density operators $\sigma, \rho \in\in S_\tau$ and for $0 \leq \alpha < 1$, is the nonnegative real number $F_\alpha(\sigma, \rho)$ defined by

$$\mathcal{F}_\alpha(\sigma, \rho) = \frac{1}{2}\tau\left(\mid \sigma^\alpha \rho^{1-\alpha} \mid + \mid \rho^\alpha \sigma^{1-\alpha} \mid\right).$$

According to part (1) Theorem 2.1, it is clear that if $\alpha = \frac{1}{2}$, then $\mathcal{F}_{\frac{1}{2}}(\sigma, \rho) = \mathcal{F}_\tau(\sigma, \rho)$.

Theorem 2.5. Let σ, ρ be operators in S_τ, then

1. $\mathcal{F}_\alpha(\sigma, \rho) = \mathcal{F}_\alpha(\rho, \sigma)$,

2. $0 \leq \mathcal{F}_\alpha(\sigma, \rho) \leq 1$,
3. $\mathcal{F}_\alpha(\sigma, \rho) = 0$ if and only if $\sigma \perp \rho$, and
4. $\mathcal{F}_\alpha(\sigma, \rho) = 1$ if and only if $\sigma = \rho$.

Furthermore, if $\mathcal{E} : \mathcal{A} \longrightarrow \mathcal{A}$ is a positive linear map such that $\tau \circ \mathcal{E} = \tau$, then

$$\mathcal{F}_\alpha(\sigma, \rho) \leq \mathcal{F}_\alpha\left(\mathcal{E}(\sigma), \mathcal{E}(\rho)\right).$$

It is well known that if $\mathcal{E} : \mathcal{T}(\mathcal{H}) \longrightarrow \mathcal{T}(\mathcal{H})$ is a channel, then

$$\mathcal{F}_\alpha(\sigma, \rho) \leq \mathcal{F}_\alpha(\mathcal{E}(\sigma), \mathcal{E}(\rho)).$$

for all density operators σ and ρ acting on $\mathcal{H}$.

Definition 2.6. A positive linear map $\mathcal{E} : \mathcal{A} \longrightarrow \mathcal{A}$ preserves α-fidelity if

1. $\tau \circ \mathcal{E} = \tau$, and
2. $\mathcal{F}_\alpha\left(\mathcal{E}(\sigma), \mathcal{E}(\rho)\right) = \mathcal{F}_\alpha(\sigma, \rho)$. for all operators $\sigma, \rho \in S_\tau$.

It is clear that if $\mathcal{E}$ is a unitary channel, then

$$\mathcal{F}_\alpha(\sigma, \rho) = \mathcal{F}_\alpha(\mathcal{E}(\sigma), \mathcal{E}(\rho)).$$

Definition 2.7. A linear map $\mathcal{E} : \mathcal{A} \longrightarrow \mathcal{A}$ is an order isomorphism, if $\mathcal{E}$ is a bijection and if both $\mathcal{E}$ and $\mathcal{E}^{-1}$ are positive.

Proposition 2.8. [1] If a positive linear map $\mathcal{E} : \mathcal{A} \longrightarrow \mathcal{A}$ preserves τ-fidelity, then $\mathcal{E}$ is an injection. Moreover, if $\mathcal{E}$ is also surjective, then $\mathcal{E}$ is an order isomorphism.

By using similar argument that used in the proof of proposition 322, we can prove the following proposition

Proposition 2.9. If a positive linear map $\mathcal{E} : \mathcal{A} \longrightarrow \mathcal{A}$ preserves α-fidelity, then $\mathcal{E}$ is an injection. Moreover, if $\mathcal{E}$ is also surjective, then $\mathcal{E}$ is an order isomorphism.

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References

[1] D. R. Farenick, S. Jaques, and M. Rahaman, *The Fidelity of Density Operators in an Operator-Algebraic Framework*, J. Math. Phys. 57, 102202 (2016); https://doi.org/10.1063/1.4965876.

[2] R. Jozsa, *Fidelity for mixed quantum states*, J. Modern Opt., 41(12) (1994), 23152323.

[3] K. Kraus, *General state changes in quantum theory*, Ann. Physics, 64-2(1971):,311335.

[4] K. Kraus, *States, effects, and operations*, volume 190 of Lecture Notes in Physics. Springer-Verlag, Berlin, 1983. Fundamental notions of quantum theory, Lecture notes edited by A. Bohm, J. D. Dollard and W. H. Wootters.

[5] S.M. Manjegani, *On young and Heinz inequalities for faithful tracial states*, lin. and Mult. Alg. 65 (2017), 2432-2456.

[6] L. Molnár, *Fidelity preserving maps on density operators*, Rep. Math. Phys., 48(3) (2001), 299303.

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On Jleli-Samet-Mizogochi-Takahashi contractions in θ_b-metric spaces and related fixed point results

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Abstract

Motivated by Jleli-Samet [J. Inequal. Appl. **2014**:38 (2014)] we introduce and study a new generalized b-metric space called θ_b-metric space and a new contraction called Jleli-Samet-Mizogochi-Takahashi contraction to prove a fixed point result as a generalization of the Banach contraction principle. Moreover, we discuss some illustrative contractions to highlight the realized improvements.

Keywords: Jleli-Samet-Mizogochi-Takahashi , θ_b-metric, fixed point
Mathematics Subject Classification [2023]: 54H25, 47H10

1 Introduction

After Bakhtin [5] and Czerwik [6, 7] introduced b-metric spaces, many authors attempted to generalize this practical concept. Kamran and others [8] presented controlled b-metric spaces. Following that, double-controlled b-metric spaces were introduced. In this article, we present a new category of generalized b-metric spaces, which will certainly be of interest to many researchers in the field of fixed point theory.

The most well-known conclusion in fixed point theory, the Banach contractive principle or Banach fixed point theorem, states that every contractive mapping in a complete metric space has a unique fixed point. By applying various types of contractive conditions in different spaces, this result can be generalized in a huge number of ways. A fixed point conclusion was demonstrated recently as a generalization of the Banach contraction principle by Jleli-Samet [2], who also established a new contraction known as the Jleli-Samet-contraction.

One of the interesting results which also generalizes the Banach contraction principle was given by Samet *et al.* [3] by defining α-ψ-contractive and α-admissible mappings.

Definition 1.1. [3] Let T be a self-mapping on a set X and let $\alpha : X \times X \to [0, \infty)$ be a function. We say that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Definition 1.2. [4] Let $f : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that f is a triangular α-admissible mapping if

(T1) $\alpha(x, y) \geq 1 \quad$ implies $\quad \alpha(fx, fy) \geq 1, \quad x, y \in X;$

(T2) $\begin{cases} \alpha(x, z) \geq 1 \\ \alpha(z, y) \geq 1 \end{cases}$ implies $\alpha(x, y) \geq 1, \quad x, y, z \in X.$

Lemma 1.3. *[4] Let f be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Define sequence $\{x_n\}$ by $x_n = f^n x_0$. Then*

$$\alpha(x_m, x_n) \geq 1 \text{ for all } m, n \in \mathbb{N} \text{ with } m < n.$$

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Definition 1.4. [6] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is a b-metric if, for all $x, y, z \in X$,

(b_1) $d(x, y) = 0$ iff $x = y$,

(b_2) $d(x, y) = d(y, x)$,

(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

Denote by $\mathfrak{B}$ the set of all functions $\beta : [0, +\infty) \to [0, 1)$ satisfing $\limsup_{t \to r^+} \beta(t) < 1$, for all $r \in [0, +\infty)$.

Theorem 1.5. *(Mizoguchi and takahashi [1]) Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a set-valued mapping such that*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y),$$

for all $x, y \in X$, where $\alpha : [0, +\infty) \to [0, 1)$ satisfies $\limsup_{t \to r^+} \alpha(t) < 1$, for all $r \in [0, +\infty)$. Then T has a fixed point.

Remark 1.6. Note that the conclusion of Theorem 2.4 remains true if we take $\alpha : [0, +\infty) \to (0, 1)$ and $\limsup_{t \to r^+} \alpha(t) < 1$, for all $r \in (0, +\infty)$.

Consistent with [2] we denote by Θ_0 the set of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

θ_1. θ is increasing;

θ_2. for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim\limits_{n \to \infty} \theta(t_n) = 1$ if and only if $\lim\limits_{n \to \infty} t_n = 0$;

θ_3. there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim\limits_{t \to 0^+} \frac{\theta(t)-1}{t^r} = \ell$.

Recall the following result.

Theorem 1.7. *[2, Corollary 2.1] Let (X, d) be a complete metric space and let $T : X \to X$ be a given map. Suppose that there exist $\theta \in \Theta_0$ and $k \in (0, 1)$ such that*

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \ \Rightarrow \ \theta(d(fx, fy)) \leq \theta(d(x, y))^k.$$

Then T has a unique fixed point.

Note that the Banach contraction principle is a special case of the above Theorem.

Now, we will consider a new class Θ containing all nonrecreasing continuous functions $\theta : [0, \infty) \to [1, \infty)$ and for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim\limits_{n \to \infty} \theta(t_n) = 1$ if and only if $\lim\limits_{n \to \infty} t_n = 0$. In fact we have a wider range of functions than that introduced in [2].

Remark 1.8. It is clear that $f(t) = e^t$ does not belong to Θ_0, but it belongs to Θ. Other examples are $f(t) = \cosh t$, $f(t) = \frac{2 \cosh t}{1+\cosh t}$, $f(t) = 1 + \ln(1 + t)$, $f(t) = \frac{2+2 \ln(1+t)}{2+\ln(1+t)}$, $f(t) = e^{te^t}$ and $f(t) = \frac{2e^{te^t}}{1+e^{te^t}}$, for all $t \geq 0$.

In this paper, we introduce and study a new generalized b-metric space called θ_b-metric space and a new contraction called Jleli-Samet-Mizogochi-Takahashi contraction to prove a fixed point result as a generalization of the Banach contraction principle. Our results extend those of Jleli-Samet and several other authors.

2 Main Results

We have the following extension of the concept of b-metric space.

Definition 2.1. Let X be a (nonempty) set. A function $\varrho : X \times X \to [0, \infty)$ is a θ_b-metric if there exist a $\theta \in \Theta$ and a constant $s \geq 1$ such that for all $x, y, z \in X$,

(θb_1) $\varrho(x, y) = 0$ iff $x = y$,

(θb_2) $\varrho(x, y) = \varrho(y, x)$,

(θb_3) $\theta(\varrho(x, z)) \leq [\theta(\varrho(x, y))\theta(\varrho(y, z))]^s$.

In this case, the pair (X, ϱ) is called a θ_b-metric space.

Note that a b-metric need not be a continuous function. So, we have this fact about θ_b-metric spaces. Also note that taking $\theta(t) = e^t$, the concept of θ_b-metric reduces to a b-metric. Thus the class of θ_b-metric space is larger than the class of b-metric space.

Lemma 2.2. *Let (X, d) be a θ_b-metric and suppose that $\{x_n\}$ and $\{y_n\}$ are θ_b-convergent to x and y, respectively. Then we have*

$$[\theta(d(x, y))]^{\frac{1}{s^2}} \leq \theta[\liminf_{n\to\infty} d(x_n, y_n)] \leq \theta[\limsup_{n\to\infty} d(x_n, y_n)] \leq [\theta(d(x, y))]^{s^2}.$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have,

$$[\theta(d(x, z))]^{\frac{1}{s}} \leq \liminf_{n\to\infty} \theta[d(x_n, z)] \leq \limsup_{n\to\infty} \theta[d(x_n, z)] \leq [\theta(d(x, z))]^s.$$

Definition 2.3. Let (X, d) be a θ_b-metric space with constant $s \geq 1$ and let T be a self-mapping on X. Also suppose that $\alpha : X \times X \to [0, \infty)$ be a function. We say that T is an α-β-Jleli-Samet-Mizogochi-Takahashi contraction if there exist a $\beta \in \mathfrak{B}$ and a constant $s \geq 1$ so that for all $x, y \in X$ with $1 \leq \alpha(x, y)$ we have

$$\theta\big(d(Tx, Ty)\big)^s \leq \big(M_{s,\theta}(x, y)\big)^{\beta(d(x,y))}, \tag{234}$$

where

$$M_{s,\theta}(x, y) = \max\left\{\theta(d(x, y)), \theta(d(x, Tx)), \theta(d(y, Ty)), \sqrt[2s]{\theta(d(x, Ty))\theta(d(y, Tx))}\right\}. \tag{235}$$

Now we state and prove our main result of this section.

Theorem 2.4. *Let (X, d) be a complete θ_b-metric-metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is a triangular α-admissible mapping;

(ii) T is an α-β-Jleli-Samet-Mizogochi-Takahashi contraction;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iv) T is α-continuous.

Then T has a fixed point. Moreover, T has a unique fixed point if $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$.

Taking $\theta(t) = e^t$, $\beta(x) = k$ where $k \in (0, 1)$ in the above theorem, we obtain a generalization of the Banach contraction principle in the setup of b-metric spaces.

Corollary 2.5. *Let (X, d) be a complete b-metric space. Let T be a self-mapping on X and $\alpha : X \times X \to [0, \infty)$ be a function such that the mapping T satisfies the following assertions:*

(i) T is a triangular α-admissible mapping;

(ii)

$$sd(Tx, Ty) \leq k.M_s(x, y),$$

for all $x, y \in X$ with $1 \leq \alpha(x, y)$ where

$$M_s(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s} \right\};$$

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iv) T is α-continuous.

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$.

In the following theorem we replace the α-continuity of T by another condition (iv').

Theorem 2.6. *Let (X, d) be a complete θ_b-metric-metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is a triangular α-admissible mapping;

(ii) T is an α-β-Jleli-Samet-Mizogochi-Takahashi contraction;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iv') if $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ with $x_n \to x$ as $n \to \infty$, then $1 \leq \alpha(Tx_n, Tx)$ holds for all $n \in \mathbb{N}$.

Then T has a fixed point. Moreover, T has a unique fixed point whenever $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$.

References

[1] N. Mizoguchi and W. Takahashi, *Fixed point theorems for multi-valued mappings on complete metric space*, J. Math. Anal. Appl., 141 (1989) 177-188.

[2] M. Jleli, B. Samet, *A new generalization of the Banach contraction principle*, J. Inequal. Appl. **2014**:38 (2014).

[3] B. Samet, C. Vetro and P. Vetro, Fixed point theorem for α-ψ-contractive type mappings, Nonlinear Anal., 75 (2012) 2154–2165.

[4] E. Karapınar, P. Kumam and P. Salimi, On α-ψ-Meir-Keeler contractive mappings, Fixed Point Theory Appl., 2013, 2013:94.

[5] I. A. Bakhtin, *The contraction principle in quasimetric spaces* [in Russian], Funk. An. Ulianowsk Fos. Ped. Inst. 30 (1989), 26–37.

[6] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inf. Univ. Ostrav., 1 (1993), 5–11.

[7] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Mat. Fis. Univ. Modena, 46 (1998), 263–276.

[8] Mlaiki, N.; Aydi, H.; Souayah, N.; Abdeljawad, T. Controlled metric type spaces and the related contraction principle. Mathematics 2018, 6, 194. [Google Scholar] [CrossRef]

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Löwner-Heinz inequality for sector matrices

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Abstract

Let $A, B \in S_\theta$ such that $o < m \leq \Re A, \Re B \leq M$, σ be an arbitrary mean and Φ be a positive unital linear map. Then

$$\Phi^{2p}\left(\Re(A\sigma B)\right) \leq K^{3p}\sec^{6p}\theta\ \Phi^{2p}\left(\Re(A\sigma^* B)\right)$$

and for operator means σ_1 and σ_2 between σ and σ^*, we have

$$\Phi^p(\Re(A\sigma_1 B)) \leq \alpha^p \Phi^p(\Re(A\sigma_2 B)),$$

where $\alpha = max\{K, 4^{1-\frac{2}{p}}K\}$ and $0 < p < 1$.

Keywords: Kantorovich constant; Operator mean; Operator inequality.
Mathematics Subject Classification [2023]: 47A63, 47A64, 15B48

1 Introduction

The numerical range of $A \in M_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ \|x\| = 1\}.$$

Let $W(A) \subset S_\theta$ for some $0 \leq \theta < \frac{\pi}{2}$, where S_θ denote the sector region in the complex plane as follows:

$$S_\theta = \{z \in \mathbb{C} : \Re z > 0, |\Im z| \leq (\Re z)\tan\theta\}.$$

In this case, A is called a secrotial matrix and will write $A \in \mathcal{S}_\theta$. Since $0 \notin S_\theta$, then all $A \in \mathcal{S}_\theta$ are invertible.

Let σ be an operator mean with representing function f. The operator mean with representing function $f(t^{-1})^{-1}$ is called the adjoint of σ and denoted by σ^*. It followes from the definition that

$$A\sigma^* B = (A^{-1}\sigma B^{-1})^{-1}.$$

In particular $!_\nu^* = \nabla_\nu$.

For operator mean of accretive operators, we have the following results.

Lemma 1.1. *[1] Let $A, B \in \mathcal{S}_\theta$. Then $A\sigma B \in \mathcal{S}_\theta$ and*

$$\Re A\sigma\Re B \leq \Re(A\sigma B) \leq \sec^2\theta(\Re A\sigma\Re B).$$

For two positive operators A and B, the Löwner-Heinz inequality states that, if $A \leq B$, then for all $0 < p < 1$,

$$A^p \leq B^p. \tag{236}$$

But $A \leq B \Rightarrow A^2 \leq B^2$ is not true in general. However we have the following useful Lemma.

Lemma 1.2. *[2] If $A, B \in \mathbb{M}_n$ satisfy $0 \leq A \leq B$ and $o < m \leq A \leq M$ with $h = \frac{M}{m}$, then we have $A^2 \leq K(h)B^2$.*

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A linear map $\Phi : B(H) \to B(H)$ is called positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. If $\Phi(I) = I$, where I denoted the identity operator, then we say that Φ is unital.

Khosravi and et al. In [3] gave an inequality for the means as follows:
Let $o < m \leq A, B \leq M$, σ be an arbitrary mean, Φ be a positive unital linear map and $p > 0$. Then

$$\Phi^p(A\sigma B) \leq \alpha^p \Phi^p(A\sigma^* B), \tag{237}$$

where $\alpha = max\{K, 4^{1-\frac{2}{p}}K\}$.

Lemma 1.3. *[3] Let $o < m \leq A, B \leq M$, σ be an arbitrary mean. Then for every positive unital linear map Φ and operator means σ_1 and σ_2 between σ and σ^*, it holds that*

$$\Phi^p(A\sigma_1 B) \leq \alpha^p \Phi^p(A\sigma_2 B),$$

where $\alpha = max\{K, 4^{1-\frac{2}{p}}K\}$ and $p \geq 0$.

2 main results

Theorem 2.1. *Let $A, B \in S_\theta$ such that $o < m \leq \Re A, \Re B \leq M$, σ be an arbitrary mean and Φ be a positive unital linear map. Then*

$$\Phi^2\left(\Re(A\sigma B)\right) \leq K^3 \sec^6\theta \ \Phi^2\left(\Re(A\sigma^* B)\right), \tag{238}$$

where $K = \frac{(h+1)^2}{4h}$ and $h = \frac{M}{m}$.

Proof. Compute

$$\begin{aligned}
\cos^2\theta \ \Phi\left(\Re(A\sigma B)\right) &\leq \Phi\left(\Re A \sigma \Re B\right) && \text{(by Lemma 368)} \\
&\leq K\Phi(\Re A \sigma^* \Re B) && \text{(by (2.5))} \\
&\leq K\Phi\left(\Re(A\sigma^* B)\right) && \text{(by Lemma 368),}
\end{aligned}$$

therefore

$$\Phi\left(\Re(A\sigma B)\right) \leq K \sec^2\theta \ \Phi\left(\Re(A\sigma^* B)\right).$$

Note that if $o < m \leq \Re A, \Re B \leq M$, then

$$o < m \leq \Re A, \Re B \leq \Re(A\sigma B)$$

and

$$\Re(A\sigma B) \leq \sec^2\theta \ (\Re A \sigma \Re B) \leq M \sec^2\theta,$$

therefore we have

$$o < m \leq \Phi\left(\Re(A\sigma B)\right) \leq M \sec^2\theta.$$

Hence by Lemma 3.1 we have

$$\begin{aligned}
\Phi^2\left(\Re(A\sigma B)\right) &\leq \frac{(m + \sec^2\theta M)^2}{4mM \sec^2\theta} K^2 \sec^4\theta \ \Phi^2\left(\Re(A\sigma^* B)\right) \\
&\leq \frac{(\sec^2\theta m + \sec^2\theta M)^2}{4mM} K^2 \sec^2\theta \ \Phi^2\left(\Re(A\sigma^* B)\right) \\
&= K^3 \sec^6\theta \ \Phi^2\left(\Re(A\sigma^* B)\right).
\end{aligned}$$

□

Corollary 2.2. *Let $A, B \in S_\theta$ such that $o < m \leq \Re A, \Re B \leq M$, σ be an arbitrary mean and Φ be a positive unital linear map. Then*

$$\Phi^2\left(\Re(A\sharp_\nu B)\right) \leq K^3 \sec^4\theta \ \Phi^2\left(\Re(A\nabla_\nu B)\right). \tag{239}$$

Proof. If $\sigma =!_\nu$, then $\sigma^* = \nabla_\nu$ and (239) is true by Theorem 2.1. □

Utilizing Lemma 1.1, similar to the proof of Theorem 2.1, one can conclude the following observation.

Theorem 2.3. *Let $A, B \in S_\theta$ such that $o < m \leq \Re A, \Re B \leq M$ and σ be an arbitrary mean. Then for every positive unital linear map Φ and operator means σ_1 and σ_2 between σ and σ^*, it holds that*

$$\Phi^2\left(\Re(A\sigma_1 B)\right) \leq K^3 \sec^4\theta \; \Phi^2\left(\Re(A\sigma_2 B)\right). \tag{240}$$

Corollary 2.4. *Let $A, B \in S_\theta$ such that $o < m \leq \Re A, \Re B \leq M$ and σ be an arbitrary mean. Then for every positive unital linear map Φ,*

$$\Phi^2\left(\Re(A\sharp_\nu B)\right) \leq K^3 \sec^6\theta \; \Phi^2\left(\Re(A\nabla_\nu B)\right) \tag{241}$$

and

$$\Phi^2\left(\Re(A!_\nu B)\right) \leq K^3 \sec^6\theta \; \Phi^2\left(\Re(A\sharp_\nu B)\right). \tag{242}$$

Proof. If $\sigma_1 = \sharp_\nu$ and $\sigma_2 = \nabla_\nu$ are replaced in Theorem 2.3, we get (241) and by replacing $\sigma_1 =!_\nu$ and $\sigma_2 = \sharp_\nu$ in it obtain (242). □

Remark 2.5. In [3], the authors proved that for two positive operators $o < m \leq A, B \leq M$, positive unital linear map Φ and operator means σ_1 and σ_2, it holds that

$$\Phi^p(\Re(A\sigma_1 B)) \leq \alpha^p \Phi^p(\Re(A\sigma_2 B)),$$

where $\alpha = max\{K, 4^{1-\frac{2}{p}}K\}$ and $p > 0$. Now By using (236) and Theorems 2.1 and 2.3, we get

$$\Phi^{2p}\left(\Re(A\sigma B)\right) \leq K^{3p} \sec^{6p}\theta \; \Phi^{2p}\left(\Re(A\sigma^* B)\right)$$

and

$$\Phi^{2p}\left(\Re(A\sigma B)\right) \leq K^{3p} \sec^{6p}\theta \; \Phi^{2p}\left(\Re(A\sigma^* B)\right),$$

where $\alpha = max\{K, 4^{1-\frac{2}{p}}K\}$ and $0 < p < 1$.

References

[1] Y. Bedrani, F. Kittaneh and M. Sababeh, *From positive to accretive matrices*, *Positivity* https://doi.org/10.1007/s11117-021-00831-8(2021).

[2] M. Fujii, S. Izumino, R. Nakamoto and Y. Seo, *Operator inequalities related to Cauchy-Schwarz and Holder- McCarthy inequalities*, Nihonkai Math. J.,8(1997), 117122.

[3] M. khosravi, M. S. Moslehian, A. Sheikhhosseini, *Some operator inequalities involving operator means and positive linear maps*, Linear Multilinear Algebra.**66**,1186-1198 (2018).

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Accurate numerical method based on modified moving least squares for of solving nonlinear quadratic integral equations

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Abstract

In this article, an idea based on modified moving least squares (MMLS) and spectral collocation method is used to estimate the solution of nonlinear quadratic integral equations (NQIEs). We used the shifted Chebyshev polynomials as basis functions. Applying the proposed method leads to the conversion of the problem into a system of algebraic equations.

Keywords: Nonlinear quadratic integral equations , modified moving least squares, Numerical integration, Spectral collocation method
Mathematics Subject Classification [2023]: 45G10, 65L60, 65D30

1 Introduction

The nonlinear quadratic integral equations provide an important tool for modeling numerous phenomena and processes of engineering. Numerical solution of these equations have been studied by many researchers [1, 2, 3**?**]. The fundamental problem that this paper focuses on is as follows:

$$\psi(x) = \xi(x) + (\int_0^x \Upsilon_1(x,t)N_1(t,\psi(t))dt)(\int_0^x \Upsilon_2(x,t)N_2(t,\psi(t))dt), \tag{243}$$

where $\psi(x) \in C^3(D)$ is an unknown function and $x \in D = [0,1]$, $\xi(x) \in C^3(D)$, $\Upsilon_1(x,t), \Upsilon_2(x,t) \in C^3(D \times D)$ and $N_1(t,\psi(t)), N_2(t,\psi(t)) \in C^3(D \times \mathbb{R})$ are known functions. In the present paper, a new method based on the MMLS method and spectral collocation method has been suggested to obtain the numerical solution of NQIEs. Finally, error analysis are checked theoretically and numerically to confirm the accuracy and efficiency of the proposed method.

2 The MLS approximation

The MLS method as a practical numerical method has been introduced by Shepard [5]. Suppose that $\psi : [a,b] \to \mathbb{R}$ be a continuous real function and points $(t_i, \psi_i), i = 0, 1, \ldots, n$ are available. Assume that P_q is the space of polynomials of degree $q \ll n$.
The MLS approximation of $\psi(t)$ for all $t \in \bar{\Omega}$ is denoted by $\hat{\psi}(t)$ and is computed as $\hat{\psi}(t) = h^T(t)a(t)$, where $h(t) = [h_0(t), h_1(t), \ldots, h_m(t)]^T$ and $a(t) = [a_0(t), a_1(t), \ldots, a_m(t)]^T$. Also, $\{h_i(t)\}_{i=0}^m$ is an ordered basis for P_q and $a_j(t), j = 0, \ldots, m$ are unknown coefficients that we need to determine them. The MLS method employs a specific class of derivative functions and presents the approximate function $\hat{\psi}(t)$ in this class. For this purpose, we solve the following optimization problem $J\left[\hat{\psi}\right] = \sum_{i=0}^n w_i(t)(\hat{\psi}(t_i) - \psi_i)^2$, where the weight function $w_i(t)$ is associated with the node i, n is the number of nodes in domain $\bar{\Omega}$ with $w_i(t) > 0$, and ψ_i's are the artificial nodal values and may not be equal with the nodal values of the unknown trial function $\hat{\psi}(t)$, i.e. $\hat{\psi}(t_i) \neq \psi_i$. The MLS shape functions can be created using different weight functions. In this paper, we use the spline weight function of third degree. We let $\Delta J = 0$ to minimize the optimization problem $J\left[\hat{\psi}\right]$. Therefore we have

$$\hat{\psi}(t) = h^T(t)A^{-1}(t)H(t)\Psi = \sum_{i=0}^n \phi_i(t)\psi_i, \qquad t \in \bar{\Omega},$$

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where

$$\phi_i(t) = \sum_{k=0}^{m} h_k(t) \left[A^{-1}(t)H(t)\right]_{ki}, \qquad a(t) = A^{-1}(t)H(t)\Psi,$$

$$A(t) = \sum_{i=0}^{n} w_i(t)h(t_i)h^T(t_i), \qquad \Psi = [\psi_0, \psi_1, \ldots, \psi_n]^T,$$

$$H(t) = [w_0(t)h(t_0), w_1(t)h(t_1), \ldots, w_n(t)h(t_n)],$$

that the matrix $A(t)$ is called moment matrix and is of order $m \times m$. Also, $\phi_i(t)$ are called the shape functions of the MLS approximation corresponding to nodal point t_i.
Generally, different bases can be used for the above method. In this paper, we use the shifted Chebyshev polynomials as basis functions.

Theorem 2.1. *[6] Suppose that $E \subseteq \mathbb{R}^d$ is compact and satisfies an interior cone condition with angle $\alpha \in (0, \frac{\pi}{2})$ and radius $\iota > 0$. Then there exists a constant $Q > 0$ that can be computed explicitly such that for all $\psi \in C^{q+1}(E^*)$, where E^* is defined as the closure of $\cup_{t\in E} B(t, 2\kappa h_0)$, and all quasi-uniform $T \subset E$ with $h_{T,E} \leq h_0$, where $h_0 = \frac{\iota}{\kappa}$ with $\kappa = \frac{16(1+sin(\alpha))^2 q^2}{3sin^2(\alpha)}$, the approximation error is bounded as follows:*

$$\| \psi - \hat{\psi} \|_{L^\infty(E)} \leq Q h_{T,E}^{q+1} \mid \psi \mid_{C^{q+1}(E^*)},$$

where $\mid \psi \mid_{C^{q+1}(E^*)} = \max_{|\beta|=q+1} \| D^\beta \psi \|_{L^\infty(E^*)}$.

Remark 2.2. In order to create the modified MLS method as a more stable scheme, we use the shifted and scaled polynomial basis function.

3 Description of the method

In this section, we implement the MLS method for solving NQIEs of the form (357). The $n+1$ nodal points x_i are selected over the interval $[0, 1]$. We have selected the $n+1$ roots of the $T_n^*(x)$ as nodal points $x_i = \frac{1}{2}\left(1 - \cos\left(\frac{(2i+1)\pi}{2n+2}\right)\right), \quad i = 0, 1, \ldots, n$. We employ the shifted Chebyshev polynomial as basic functions. We insert $\hat{\psi}(x) = \sum_{i=0}^{n} \psi_i \phi_i(x)$ as the MMLS approximation of $\psi(x)$ and $\hat{\xi}(x)$ as the MMLS approximation $\xi(x)$ of into Eq.(357) and obtain:

$$\hat{\psi}(x) = \hat{\xi}(x) + \left(\int_0^x \Upsilon_1(x,t) N_1(t, \hat{\psi}(t))dt\right)\left(\int_0^x \Upsilon_2(x,t) N_2(t, \hat{\psi}(t))dt\right), \tag{244}$$

where $R_n(x)$, $x \in [0, 1]$ is residual error. In order to use the Gauss-Legendre quadrature, we change the integration distance $[0, x]$ into the fixed distance $[-1, 1]$. Then Eq.(368) becomes

$$\begin{aligned}\hat{\psi}(x) = \hat{\xi}(x) &+ \left(\frac{x}{2}\int_{-1}^{1} \Upsilon_1(x, \frac{x}{2}(s+1)) N_1(\frac{x}{2}(s+1), \hat{\psi}(\frac{x}{2}(s+1)))ds\right) \\ &\times \left(\frac{x}{2}\int_{-1}^{1} \Upsilon_2(x, \frac{x}{2}(s+1)) N_2(\frac{x}{2}(s+1), \hat{\psi}(\frac{x}{2}(s+1)))ds\right).\end{aligned} \tag{245}$$

Suppose that Eq.(371) is exact for the collocation points $\{\zeta_r = \frac{r}{n}\}_{r=0}^{n}$. In order to approximate the integral terms in the Eq.(371), we use the $l-$points the Gauss-Legendre quadrature. Suppose that $\{\gamma_k\}_{k=1}^{l}$ and $\{w_k\}_{k=1}^{l}$ denote the quadrature points and the quadrature weights, respectively. Therefore, for $r = 0, 1, \ldots, n$, the Eq.(371) can be rewritten as follows:

$$\begin{aligned}&\sum_{j=0}^{n} \phi_j(\zeta_r)\hat{\psi}_j - \left(\frac{\zeta_r}{2}\sum_{k=1}^{l} w_k \Upsilon_1(\zeta_r, \frac{\zeta_r}{2}(\gamma_k+1)) N_1(\frac{\zeta_r}{2}(\gamma_k+1), \sum_{j=0}^{n} \phi_j(\frac{\zeta_r}{2}(\gamma_k+1))\hat{\psi}_j)\right) \\ &\times \left(\frac{\zeta_r}{2}\sum_{k=1}^{l} w_k \Upsilon_2(\zeta_r, \zeta_r, \frac{\zeta_r}{2}(\gamma_k+1)) N_2(\zeta_r, \frac{\zeta_r}{2}(\gamma_k+1), \sum_{j=0}^{n} \phi_j(\zeta_r, \frac{\zeta_r}{2}(\gamma_k+1))\hat{\psi}_j)\right) \simeq \hat{\xi}(\zeta_r).\end{aligned} \tag{246}$$

These equations provide an $(n+1) \times (n+1)$ system of equations that can be solved by an appropriate method to obtain the unknowns $\hat{\psi}_j$.

4 Error analysis

In this section, we discuss the error of the proposed method and provide an upper error bound. We define

$$\|\psi\| = \sup_{x\in D} |\psi(x)|, \qquad E_n = \|\psi(x) - \hat{\psi}(x)\|.$$

Consider that for Eq.(357), the following assumptions $A1 - A4$ are satisfied:

(A1) $\|\mathscr{N}_1(s,\psi_1(s)) - \mathscr{N}_1(s,\psi_2(s))\| \vee \|\|\mathscr{N}_2(s,\psi_1(s)) - \mathscr{N}_2(s,\psi_2(s))\|\| \leq L\|\psi_1(s) - \psi_2(s)\|$.

(A2) $\|\mathscr{N}_1(s,\psi(s))\| + \|\mathscr{N}_2(s,\psi(s))\| \leq \eta(1 + \|\psi(s)\|)$,
where L, η are positive constants for $s \in [0,T]$.

(A3) $\|\psi(s)\| \leq \mathscr{M} < \infty, \quad s \in [0,T]$.

(A4) $\|\Upsilon_i(x,s)\| \leq K_i, \quad i = 1,2, \qquad (x,s) \in [0,T]\times[0,T]$,
where $K_i, \quad i = 1,2$, are positive constants.

Theorem 4.1. *The approximate solution of Eq. (357) by using MMLS method satisfies in the following relation:*

$$E_n \leq \frac{Qh^3_{T,E} \mid \psi \mid_{C^3(E^*)}}{1 - \eta(1 + \|\psi(s)\|)},$$

if $\eta(1 + \|\psi(s)\|) < 1$.

Proof. We subtract Eq.(357) from Eq.(368). Also, the following relationship holds

$$\begin{gathered} E_n(x) = \xi(x) - \hat{\xi}(x) + \\ (\int_0^x \Upsilon_1(x,t)N_1(t,\psi(t))dt) \times \left((\int_0^x \Upsilon_2(x,t)N_2(t,\psi(t))dt) - (\int_0^x \Upsilon_2(x,t)N_2(t,\hat{\psi}(t))dt)\right) \\ +(\int_0^x \Upsilon_2(x,t)N_2(t,\hat{\psi}(t))dt)\left((\int_0^x \Upsilon_1(x,t)N_1(t,\psi(t))dt)(\int_0^x \Upsilon_1(x,t)N_1(t,\hat{\psi}(t))dt)\right). \end{gathered}$$

Therefore, by taking norm and by using Lipschitz condition, we have

$$E_n \leq K_1K_2LE_n\left(\|N_1(t,\psi(t))\| + \|N_2(t,\hat{\psi}(t))\|\right).$$

Also, we can write

$$\|N_2(t,\hat{\psi}(t))\| = \|N_2(t,\psi(t)) - N_2(t,\hat{\psi}(t))\| + \|N_2(t,\psi(t))\|.$$

Therefore, by using the above relations, Theorem 2.1 and ignoring the phrases that are included E_n^2, the proof is complete. □

5 Numerical results

Example 1: Consider the following nonlinear quadratic integral equations [2]:

$$\psi(x) = x - (\exp^x - 1)(\frac{x^3}{30} + \frac{x^5}{50}) + (\int_0^x \frac{t^2+1}{10}\psi^2(t)dt)(\int_0^x \exp(\psi(t))dt), \quad x \in [0,1],$$

where the exact solution is $\Psi(t) = t$. The numerical results of this method have been reported in Table 11.

6 Conclusion

In this paper, we surveyed an appropriate approach based on MLS and the spectral collocation method for solving the NQIEs. We used the shifted Chebyshev polynomials as basis functions and collocation method. So, we reduced the problem into a nonlinear system of algebraic equations and solved it by using Newtons method. The most important advantage of the proposed method is that it is free from domain elements because our approach is a meshless method. This method is a simple scheme. So the proposed method is easy to implement. This method provides an accurate numerical solution with a small volume of computations.

Table 11: The comparison of the obtained results for Example 1

Methods	Coditions	Maximum error
Proposed method	$n = 6, l = 4$	1.175E-11
Proposed method	$n = 11, l = 4$	7.306E-12
RT method [2]	$M = 100$	8.44E-07
AD method [2]	$m = 100$	1.27E-08
MHFs method [2]	$m = 15$	2.511E-04

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References

[1] E.A.A. Ziada, *Adomian solution of a nonlinear quadratic integral equation*, J. Egyptian Math. Soc. 21 (2013) 52–56.

[2] F. Mirzaee, E. Hadadian, *Application of modified hat functions for solving nonlinear quadratic integral equations*, Iranian J. Numer. Anal. Optim. 6 (2) (2016) 65–84.

[3] A.M.A. El-Sayed, H.H.G. Hashem, E.A.A. Ziada, *Picard and Adomian decomposition methods for a quadratic integral equation of fractional order*, Comput. Appl. Math. 33 (1) (2014), 95–109.

[4] A. Benkerrouche, M. S. Souid, G. Stamov , I. Stamova, *On the Solutions of a Quadratic Integral Equation of the Urysohn Type of Fractional Variable Order*, entropy, 24 (2022), 886.

[5] D. Shepard, *A two-dimensional interpolation function for irregularly spaced points*, Proc. 23rd Nat. Conf. ACM Press New York (1968) 517–524.

[6] H. Wendland, *Local polynomial reproduction and moving least squares approximation*, IMA J. Numer. Anal. 21.1 (2001) 285–300.

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A class of Finsler modules

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Abstract

Let A and B be C^*-algebras, $\varphi : A \to B$ be a linear $*$-homomorphism, and let E and F be Finsler modules over C^*-algebras A and B, respectively. In this talk, we state the definition of homomorphisms between Finsler modules and demonstrate the notion of φ-positively full Finsler modules. In particular, we show that for a $*$-isomorphism $\varphi : A \to B$ and a surjective φ-homomorphism $T : E \to F$, F is φ-positively full if and only if E is φ^{-1}-positively full. Finally, introducing the concepts of the implemented Finsler module and unitary equivalence, we show that the Finsler B-module F and the implemented Finsler A-module $E^{(\varphi,T)}$ are unitary equivalent.

Keywords: Finsler module, unitary operator
Mathematics Subject Classification [2023]: Primary 46L08, Secondary: 46L05

1 Introduction

A (left) Hilbert C^*-module over a C^*-algebra A is an algebraic left A-module E equipped with an A-valued inner product $\langle .,. \rangle$ which is A-linear in the first and conjugate linear in the second variable such that E is Banach space with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. The Hilbert A-module E is called *full* if $A_E := span\{\langle x, x\rangle : \quad x, y \in E\}$ is dense in A. Note that A_E is an ideal in E, called the *range ideal* of E. We denote by $\langle E, E\rangle$ the closure of A_E and call it the *support* of E. Therefore, E is a full Hilbert A-module if $\langle E, E\rangle$ is equal to A.

We recall that a linear map $T : E \to E$ is adjointable if there exists a map $T^* : E \to E$ that fulfills $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in E$. Such a map is called the adjoint map of T. This definition implies that each adjointable map is a bounded A-linear operator (see [4, p. 8]). An adjointable operator $T : E \to E$ is called unitary if $TT^* = T^*T = I$.

Hilbert C^*-modules are a generalization of Hilbert spaces, but there are some differences between these two classes. For example, each bounded operator on a Hilbert space has an adjoint, but a bounded A-linear map on a Hilbert A-module is not adjointable in general. For this and other general facts concerned with Hilbert C^*-modules we refer to [4].

In 1995, Phillips and Weaver [6] demonstrated an interesting generalization of Hilbert C^*-modules entitled "Finsler modules".

Let A be a C^*-algebra and A^+ be the set of all positive elements of A. Suppose that E is complex linear space which is a left A-module (and $\lambda(a.x) = (\lambda a).x = a.(\lambda x)$ where $\lambda \in \mathbb{C}, a \in A$ and $x \in E$) and there exists a map $\rho_A : E \to A^+$ such that

(i) The map $\|.\|_E : x \to \|\rho_A(x)\|$ is a norm on E which makes E into a Banach space; and

(ii) $\rho_A(a.x)^2 = a.\rho_A(x)^2.a^*$ for all $a \in A$ and $x \in E$.

Then, E is called a *Finsler A-module* under the map ρ_A. A Finsler A-module E is said to be *full* if the linear span $\{\rho_A(x)^2 : x \in E\}$, denoted by $\langle \rho_A(E)\rangle$, is dense in A.

As an example, suppose that A is a C^*-algebra and take $E := A$. Then, A with respect to its product as the usual action, is a left A-module. Additionally, A equipped with the map $\rho_A : A \to A^+$ defined by $\rho_A(a) := (a.a^*)^{\frac{1}{2}}$ is a full Finsler A-module.

In 2002, Bakic and B. Guljas [2] represent a new interpretation of morphisms on Hilbert C^*-modules, as a class of module maps. Applying the aforementioned morphisms, they extended the notion of unitaries on

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Hilbert C^*-modules. One year later, Amyari and Niknam [1], modified these interpretations of morphisms and unitary operators under the frame of Finsler modules.

Let $\varphi : A \to B$ be a linear $*$-homomorphism of C^*-algebras. Suppose that E and F are Finsler modules over A and B, respectively. A linear map $T : E \to F$ is called a *φ-module map* if $T(a.x) = \varphi(a).T(x)$ for all $a \in A$, $x \in E$.

Following [1], a φ-module map $T : E \to F$ is called a *φ-homomorphism* if $\rho_B(T(x)) = \varphi(\rho_A(x))$ for all $a \in A$, $x \in E$.

It is trivial that if $T : E \to F$ is a φ-homomorphism, then $\varphi(\rho_A(x))$ is a positive element of B for every $x \in E$.

The above consideration motivate us to define a new class of Finsler modules as follows.

Definition 1.1. Let A and B be C^*-algebras, $\varphi : A \to B$ be a linear $*$-homomorphism, and let E be a Finsler A-module. A Finsler B-module F is called *φ-positively full* if for every positive element $b \in B$, there exists an element $x \in E$ such that $\varphi(\rho_A(x)) = b$.

In the case when $B := A$, $F := E$, and $\varphi := I_A$ (the identity operator on A), we call E is a *I_A-positively full* or briefly a *positively full* Finsler module (see [3]).

Example 1.2. Suppose that $\varphi : A \to B$ be a linear $*$-isomorphism of C^*-algebras. Then, A and B are Finsler modules over C^*-algebras A and B, respectively. Also, for any arbitrary positive element $b \in B$, there is a unique positive element $a \in A$ such that $\varphi(a) = b$. Let $a^{\frac{1}{2}}$ as the positive square square root of a and put $x := a^{\frac{1}{2}}$. Therefore, $b = \varphi(\rho_A(x))$.

We end this section with the following useful theorem which can be found in [1].

Theorem 1.3. *Let E and F be Finsler modules over C^*-algebras A and B, respectively. Let $\varphi : A \to B$ be a linear $*$-homomorphism and $T : E \to F$ be a φ-homomorphism. If φ is an injection, then T is an isometry, hence also injective. If F is full and T is surjective, then φ is also a surjection.*

2 Main results

The following lemma states that φ-positively full Finsler modules are a class of full Finsler modules.

Lemma 2.1. *Let E and F be Finsler modules over C^*-algebras A and B, respectively. Let $\varphi : A \to B$ be a linear $*$-homomorphism and $T : E \to F$ be a φ-homomorphism. If F is φ-positively full, then is full.*

The following lemma determines some conditions under which a Finsler A-module is I_A-positively full.

Lemma 2.2. *Let A and B be C^*-algebras, and let E be a Finsler A-module. If $\varphi : A \to B$ is a linear $*$-isomorphism and F is a φ-positively full Finsler B-module, then E is I_A-positively full and so is full.*

The next result shows that for a linear $*$-isomorphism $\varphi : A \to B$, the range of a surjective φ-homomorphism is φ-positively full if and only if its domain is φ^{-1}-positively full.

Theorem 2.3. *Let E and F be Finsler modules over C^*-algebras A and B, respectively. Let $\varphi : A \to B$ be a linear $*$-isomorphism and $T : E \to F$ be a surjective φ-homomorphism. Then, F is φ-positively full if and only if E is φ^{-1}-positively full.*

Before we state the next corollary, we need the following useful theorem which can be found in [5].

Theorem 2.4. *Let E and F be Finsler B-modules, let $\varphi : A \to B$ be a linear $*$-isomorphism of C^*-algebras, and let $T : E \to F$ be a bijective linear operator. Define the module action $a.x := T^{-1}(\varphi(a).T(x))$ on E. Then, E equipped with the map $\rho_A(x) := \varphi^{-1}(\rho_B(T(x)))$ can also be regarded as a Finsler A-module.*

We call the above alternative Finsler A-module E the Finsler A-module implemented by (φ, T) or briefly the implemented Finsler A-module and denote it by $E^{(\varphi,T)}$.

Corollary 2.5. *The Finsler B-module F is φ-positively full if and only if the implemented Finsler A-module $E^{(\varphi,T)}$ is φ^{-1}-positively full.*

The following definition, which can be found in [1], demonstrates a different interpretation of a unitary operator between Finsler modules.

Definition 2.6. Let E and F be Finsler modules over C^*-algebras A and B, respectively. A linear map $T : E \to F$ is said to be a *unitary operator* if there exists an injective homomorphism $\varphi : A \to B$ such that T is a surjective φ-homomorphism.

Definition 2.7. Two Finsler modules E and F are said to be *unitary equivalent* if there is a unitary operator from E to F.

Following the method has been used in [5], we have the two next theorems.

Theorem 2.8. *Unitary equivalence in the set of positively full Finsler modules is an equivalence relation.*

Theorem 2.9. *The Finsler B-module F and the implemented Finsler A-module $E^{(\varphi,T)}$ are unitary equivalent. Conversely, if E and F are positively full unitary equivalent Finsler modules over C^*-algebras A and B, respectively, then A and B are isomorphic C^*-algebras.*

The following theorem shows that positively fullness of a Finsler module requires positively fullness of its unitary equivalent.

Theorem 2.10. *Let E and F be Finsler modules over C^*-algebras A and B, respectively. Suppose that E and F are unitary equivalent. Then, E is I_B-positively full if and only if F is I_A-positively full.*

We end this paper with the following corollary which is deduced from Theorems 2.9,2.10.

Corollary 2.11. *The Finsler B-module F is I_B-positively full if and only if the implemented Finsler A-module $E^{(\varphi,T)}$ is I_A-positively full.*

3 Conclusion

Let E and F be Finsler modules over C^*-algebras A and B, respectively. Let $\varphi : A \to B$ be a linear $*$-isomorphism and $T : E \to F$ be a surjective φ-homomorphism. Then, F is φ-positively full if and only if E is φ^{-1}-positively full.

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References

[1] M. Amyari and A. Niknam, *On homomorphisms of Finsler modules*, Intern. Math. Journal, 3 (2003), 277–281.

[2] D. Bakic and B. Guljas, *On a class of module maps of Hilbert C^*-modules*, Math. Commun., 7 (2002), pp. 177–192.

[3] M. Imaninezhad and M. Amyari, *A class of Finsler modules*, In Proceedings of the 44^{th} Annual Iranian Mathematics conference, Mashhad, Iran, 2013.

[4] E. C. Lance, *Hilbert C^*-Modules*, Lecture Note Series 210, Cambridge Univ. Press, 1995.

[5] M. Mosadeq, *Dynamical systems implemented by isomorphic groups of unitaries*, Sahand Commun. Math. Anal. (Articles in Press, Available Online from 12 March 2023).

[6] N. C. Phillips and N. Weaver, *Modules with norms which take values in a C^*-algebra* , Pacific J. Math, 185 (1998), 163-181.

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q–analogue of Sălăgean operator on univalent functions related to Hölder inequality

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Abstract

q–analogue of Sălăgean type operator has been applied in this paper to investigate a new subfamily of univalent functions defined in the open unit disk. We point out coefficient bounds, convolution preserving property. The Hölder inequality is also indicated.

Keywords: Univalent function, Coefficient inequality, Convolution, Hölder inequality.
Mathematics Subject Classification [2023]: 30C45, 30C50

1 Introduction

Let $\mathcal{N}$ denote the set of all functions $f(z)$ which are analytic and univalent in the open unit disk:

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\},$$

and have series expansion of the type:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k. \tag{247}$$

For any non-negative integer n, the q–integer n $(0 < q < 1)$ is defined by:

$$[n]_q = \frac{1-q^n}{1-q}, \qquad ([0]_q = 0). \tag{248}$$

Also the q–number shift factorial is introduced by:

$$[n]_q! = [1]_q[2]_q \dots [n]_q, \qquad ([0]_q! = 1). \tag{249}$$

It is obvious that, when $q \to 1$, $[n]_q$ is the classical factorial.

For $f \in \mathcal{N}$, in [2], q–derivative is defined as follows:

$$D_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \tag{250}$$

see also [3].

It can easily be shown that:

$$D_q z^k = [k]_q z^{k-1}, \qquad D_q\Big(\sum_{k=1}^{\infty} a_k z^k\Big) = \sum_{k=1}^{\infty} [k]_q a_k z^{k-1}. \tag{251}$$

For functions f and g that are analytic in $\mathbb{U}$ and have the form (247), we define that convolution (or Hadamard product) of f and g by:

$$f(z) * g(z) = (f * g)(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad (z \in \mathbb{U}). \tag{252}$$

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Now, we consider the differential operator $\mathcal{S}_q : \mathcal{N} \to \mathcal{N}$ by:

$$\begin{aligned}\sqrt{f}(z) &= \Phi(q, m, \lambda, z) * f(z) \\ &= z - \sum_{k=2}^{\infty} \big([1 + (k-1)\lambda]_q\big)^m a_k z^k, \qquad (m \in \mathbb{N} \cup \{0\}),\end{aligned} \tag{253}$$

where

$$\Phi(q, m, \lambda, z) = z - \sum_{k=2}^{\infty} \big(1 + (k-1)\lambda\big)^m z^k. \tag{254}$$

For more details see [4].

When $q \to 1$, the operator $\sqrt{}$ reduces to the generalized Sălăgean differential operator [3].

This operator and special cases have been studied by various authors in earlier papers. For example see [2] and [1].

Now by applying q–analogue of Sălăgean type differential operator given by (252), we introduce a new subfamily $\sqrt{}(\alpha, \beta, \gamma, t)$ of univalent analytic functions $\mathcal{N}$ which satisfying the condition:

$$\left| \frac{z\big(\mathcal{S}_q^{m,\lambda} f(z)\big)' - f_t(z)}{\big(\gamma - (\alpha - \gamma)(1 - \beta)\big) f_t(z) - \gamma z\big(\mathcal{S}_q^{m,\lambda} f(z)\big)'} \right| < 1, \tag{255}$$

where $-1 \leqslant \gamma < \alpha$, $0 < \beta < 1$, $0 \leqslant t \leqslant 1$ and:

$$f_t(z) = (1 - t)z + t f(z). \tag{256}$$

In this paper, we obtain coefficient estimate, convolution structure and Hölder inequality for the subfamily $\sqrt{}(\alpha, \beta, \gamma, t)$.

2 Main results

In this section, we find a sharp coefficient bound for functions in $\sqrt{}(\alpha, \beta, \gamma, t)$.

Theorem 2.1. *A function $f(z)$ given by (247) is in the class $\sqrt{}(\alpha, \beta, \gamma, t)$ if and only if:*

$$\sum_{k=2}^{\infty} \Big(\big(k\big([1 + (k-1)\lambda]_q\big)^m - t\big)(1 - \gamma) + t(\alpha - \gamma)(1 - \beta)\Big) a_k \leqslant (\alpha - \gamma)(1 - \beta). \tag{257}$$

Remark 2.2. We note that the function:

$$F(z) = z - \sum_{k=2}^{\infty} \frac{(\alpha - \gamma)(1 - \beta)}{\big(k\big([1 + (k-1)\lambda]_q\big)^m - t\big)(1 - \gamma) + t(\alpha - \gamma)(1 - \beta)} z^k, \tag{258}$$

shows that the inequality (304) is sharp.

Also if $f(z) \in \sqrt{}(\alpha, \beta, \gamma, t)$, then:

$$a_k \leqslant \frac{(\alpha - \gamma)(1 - \beta)}{\big(k\big([1 + (k-1)\lambda]_q\big)^m - t\big)(1 - \gamma) + t(\alpha - \gamma)(1 - \beta)}, \qquad (k \geqslant 2). \tag{259}$$

3 Convolution preserving and Hölder inequality

In this section we show that the family $\sqrt{}(\alpha, \beta, \gamma, t)$ is closed under convolution and verify application of Hölder inequality.

Theorem 3.1. *Let* $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ *and* $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ *be in the class* $\sqrt{(}\alpha, \beta, \gamma, t)$, *then* $(f * g)(z)$ *defined by* (252) *belongs to* $\sqrt{(}\alpha, \beta^*, \gamma, t)$, *where:*

$$\beta^* \leqslant 1 - \frac{X(\alpha-\gamma)(1-\beta)^2}{\big(X + t(\alpha-\gamma)(1-\beta)\big)^2 - t(\alpha-\gamma)^2(1-\beta)^2}, \tag{260}$$

and

$$X = \big(k\big([1+(k-1)\lambda]_q\big)^m - t\big)(1-\gamma). \tag{261}$$

Theorem 3.2. *If* $f_j(z) \in \sqrt{(}\alpha, \beta_j, \gamma, t)$, $(j = 1, 2, \ldots, s)$, *then* $\mathcal{H}_z(z) \in \sqrt{(}\alpha, \beta_0, \gamma, t)$, *where:*

$$\beta_0 \leqslant 1 - \frac{X}{Y - t(\alpha-\gamma)}, \tag{262}$$

$$Y = \frac{\prod_{j=1}^{s} \big(X + t(\alpha-\gamma)(1-\beta_j)\big)^{p_j}}{\prod_{j=1}^{s} (\alpha-\gamma)^{p_j-1}(1-\beta_j)^{p_j}}, \tag{263}$$

and X *is given in* (261).

Proof. Since $f_j(z) \in \sqrt{(}\alpha, \beta_j, \gamma, t)$, $(j = 1, 2, \ldots, s)$, so by (247), we have:

$$\sum_{k=2}^{\infty} \frac{\big(k\big([1+(k-1)\lambda]_q\big)^m - t\big)(1-\gamma) + t(\alpha-\gamma)(1-\beta_j)}{(\alpha-\gamma)(1-\beta_j)} a_{k,j} \leqslant 1,$$

so

$$\left(\sum_{k=2}^{\infty} \frac{\big(k\big([1+(k-1)\lambda]_q\big)^m - t\big)(1-\gamma) + t(\alpha-\gamma)(1-\beta_j)}{(\alpha-\gamma)(1-\beta_j)} a_{k,j}\right)^{\frac{1}{q_j}} \leqslant 1,$$

where $q_j > 1$, $j = 1, 2, \ldots, s$ and $\sum_{j=1}^{s} \frac{1}{q_j} = 1$. By using the Hölder inequality (262), we get:

$$\sum_{k=2}^{\infty} \left[\prod_{j=1}^{s} \left(\frac{\big(k\big([1+(k-1)\lambda]_q\big)^m - t\big)(1-\gamma) + t(\alpha-\gamma)(1-\beta_j)}{(\alpha-\gamma)(1-\beta_j)}\right)^{\frac{1}{q_j}} a_{k,j}^{\frac{1}{q_j}}\right] \leqslant 1.$$

Hence, we have to find the largest β_0 such that:

$$\sum_{k=2}^{\infty} \frac{\big(k\big([1+(k-1)\lambda]_q\big)^m - t\big)(1-\gamma) + t(\alpha-\gamma)(1-\beta_0)}{(\alpha-\gamma)(1-\beta_0)} \Big(\prod_{j=1}^{s} a_{k,j}^{p_j}\Big) \leqslant 1.$$

But

$$\sum_{k=2}^{\infty} \frac{\big(k\big([1+(k-1)\lambda]_q\big)^m - t\big)(1-\gamma) + t(\alpha-\gamma)(1-\beta_0)}{(\alpha-\gamma)(1-\beta_0)} \Big(\prod_{j=1}^{s} a_{k,j}^{p_j}\Big) \leqslant$$

$$\sum_{k=2}^{\infty} \left[\prod_{j=1}^{s} \left(\frac{\big(k\big([1+(k-1)\lambda]_q\big)^m - t\big)(1-\gamma) + t(\alpha-\gamma)(1-\beta_0)}{(\alpha-\gamma)(1-\beta_0)}\right)^{\frac{1}{q_j}} a_{k,j}^{\frac{1}{q_j}}\right].$$

Therefore, we need to obtain the largest β_0 such that:

$$\frac{\big(k\big([1+(k-1)\lambda]_q\big)^m - t\big)(1-\gamma) + t(\alpha-\gamma)(1-\beta_0)}{(\alpha-\gamma)(1-\beta_0)} \Big(\prod_{j=1}^{s} a_{k,j}^{p_j}\Big) \leqslant$$

$$\prod_{j=1}^{s}\left[\frac{\Big(k\Big([1+(k-1)\lambda]_q\Big)^m-t\Big)(1-\gamma)+t(\alpha-\gamma)(1-\beta_0)}{(\alpha-\gamma)(1-\beta_0)}\right]^{\frac{1}{q_j}} a_{k,j}^{\frac{1}{q_j}}.$$

Since

$$\prod_{j=1}^{s}\left[\frac{\Big(k\Big([1+(k-1)\lambda]_q\Big)^m-t\Big)(1-\gamma)+t(\alpha-\gamma)(1-\beta_j)}{(\alpha-\gamma)(1-\beta_j)}\right]^{p_j-\frac{1}{q_j}} a_{k,j}^{p_j-\frac{1}{q_j}} \leqslant 1,$$

where $p_j-\frac{1}{q_j}\geqslant 0$, $j=1,2,\ldots,s$, we conclude that:

$$\prod_{j=1}^{s} a_{k,j}^{p_j-\frac{1}{q_j}} \leqslant \frac{1}{\prod\limits_{j=1}^{s}\left[\frac{\Big(k\Big([1+(k-1)\lambda]_q\Big)^m-t\Big)(1-\gamma)+t(\alpha-\gamma)(1-\beta_j)}{(\alpha-\gamma)(1-\beta_j)}\right]^{p_j-\frac{1}{q_j}}}.$$

This inequality implies:

$$\frac{\Big(k\Big([1+(k-1)\lambda]_q\Big)^m-t\Big)(1-\gamma)+t(\alpha-\gamma)(1-\beta_0)}{(\alpha-\gamma)(1-\beta_0)} \leqslant \frac{\prod\limits_{j=1}^{s}\Big[\Big(k\Big([1+(k-1)\lambda]_q\Big)^m-t\Big)(1-\gamma)+t(\alpha-\gamma)(1-\beta_j)\Big]^{p_j}}{\Big(\prod\limits_{j=1}^{s}(\alpha-\gamma)(1-\beta_j)\Big)^{p_j}}.$$

So

$$\Big(\Big(k\Big([1+(k-1)\lambda]_q\Big)^m-t\Big)(1-\gamma)+t(\alpha-\gamma)(1-\beta_0)\Big)\prod_{j=1}^{s}(\alpha-\gamma)^{p_j}(1-\beta_j)^{p_j} \leqslant \prod_{j=1}^{s}\Big[\Big(k\Big([1+(k-1)\lambda]_q\Big)^m-t\Big)(1-\gamma)+t(\alpha-\gamma)(1-\beta_j)\Big]^{p_j}(\alpha-\gamma)(1-\beta_0).$$

Thus

$$\frac{\Big(k\Big([1+(k-1)\lambda]_q\Big)^m-t\Big)(1-\gamma)+t(\alpha-\gamma)(1-\beta_0)}{1-\beta_0} \leqslant Y,$$

when Y is defined in (263).

After a simple calculation, we get the required result (262), so the proof is complete. □

References

[1] G. E. Andrews, R. Askey, and R. Roy. *Special functions*, volume 71, Cambridge university press, 2000.

[2] J. H. Choi, Y. C. Kim, and S. Owa. *Generalizations of Hadamard products of functions with negative coefficients.* Journal of mathematical analysis and applications, 199(2) (1996), 495–501.

[3] G. Gasper and M. Rahman. *Basic hypergeometric series*, volume 35. Cambridge university press Cambridge, UK, Vol. 35 of Encyclopedia of Mathematics and its Applications 1990.

[4] M. Govindaraj and S. Sivasubramanian. *On a class of analytic functions related to conic domains involving q–calculus*, Analysis Mathematica, 43(3) (2017), 475–487.

[5] S. Hussain, S. Khan, M. A. Zaighum, and M. Darus. *Certain subclass of analytic functions related with conic domains and associated with salagean q–differential operator.* AIMS Mathematics, 2 (2017), 622–634.

[6] S. Hussain, S. Khan, M. A. Zaighum, and M. Darus. *Applications of a q–Sălăgean type operator on multivalent functions.* Journal of Inequalities and Applications, 301(1) (2018), 2018.

[7] G. S. Salagean. *Subclasses of univalent functions In Complex Analysis*, Springer, Complex Analysis–Fifth Romanian-Finnish Seminar: Part 1 (Bucharest 1981), Lecture Notes is Mathematics(1983) 362–372.

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On the plane stochastic pattern for non-negative tensors

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Abstract

In this paper we extend the theory of non-negative tensors. We present a condition that ensures the existence of at least k zero entries on every diagonal of a non-negative tensor. Also, we give a sufficient condition for non-negative tensors to have a plane stochastic pattern.

Keywords: Permanent, Plane stochastic, Tensor.
Mathematics Subject Classification [2023]: 15A15, 15A69

1 Introduction

A *tensor* $\mathscr{A} = (a_{i_1\dots i_d})_{n_1\times\cdots\times n_d}$ of order d and dimension $(n_1,\dots,n_d)$ is a multi-array of entries $a_{i_1\dots i_d} \in \mathbb{F}$, where $i_j = 1,\dots,n_j$ for $j = 1,\dots,d$ and $\mathbb{F}$ is a field. When $n_1 = n_2 = \cdots = n_d = n$, we say that $\mathscr{A}$ is a square tensor of order d and dimension n. A tensor $\mathscr{A}$ is called non-negative if all its entries are non-negative [4]. Similar to the permanent of a matrix, the permanent of a tensor is defined as the sum of all diagonal products of the tensor. Some properties of permanents were extended from matrices to tensors by Dow and Gibson [3]. We refer the interested reader to [5] for more information on the applications of the permanents of tensors in practical fields. A non-negative tensor of order d and dimension n that the sum of all elements on each of its hyperplanes is equal to 1 is called plane stochastic tensots [1]. In [2], Christensen and Fischer used the plane stochastic tensors in error-correcting codes.

This paper is organized as follows. In Section 2, we review some definitions that will use in other sections. In section 3, we give some theorem about non-negative tensors.

2 Preliminary

In this section, we first give some definitions that will be used in the proofs of our theorems.

Definition 2.1 ([3])**.** Given $k \in \{0,1,\dots,d\}$, a *k-dimensional plane* in $\mathscr{A}$ is a subtensor obtained by fixing $d-k$ indices and letting the other k indices vary from 1 to n. A 1-dimensional plane is said to be a *line*, and a $(d-1)$-dimensional plane is a *hyperplane*.

Definition 2.2 ([3])**.** Let $\mathscr{A} = (a_{i_1\dots i_d})$ be a tensor of order d and dimension $n_1 \times n_2 \times \cdots \times n_d$. The *permanent* of $\mathscr{A}$ is defined by

$$per(\mathscr{A}) := \sum_{\sigma_k} \prod_{i=1}^{n_1} a_{i\sigma_2(i)\dots\sigma_d(i)},$$

where the summation runs over all one-to-one functions σ_k from $\{1,\dots,n_1\}$ to $\{1,\dots,n_k\}$ and $k = 2,\dots,d$, with $per(\mathscr{A}) = 0$ if $n_1 > n_k$ for some k.

Definition 2.3 ([3])**.** For a tensor $\mathscr{A} = (a_{i_1\dots i_d})_{n_1\times\cdots\times n_d}$, the hyperplanes obtained by fixing $i_k, 1 \le k \le d$, are said to be *hyperplanes of type k*.

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Theorem 2.4 ([5]). *If $\mathscr{A}$ is a d-dimensional tensor of order n, then*

$$per(\mathscr{A}) = \sum_{i_2,\dots,i_d=1}^{n} a_{1,i_2,\dots,i_d} per(\mathscr{A}(1|i_2|\dots|i_d)),$$

where $\mathscr{A}(1|i_2|\dots|i_d)$ is a d-dimensional tensor of order $n-1$, which is obtained from $\mathscr{A}$ by removing hyperplanes i_k of type k for $k = 1, i_2, \dots, i_d$.

3 The permanent of a non-negative plane stochastic tensor

The following theorem is a generalization of the Frobenius–König theorem from matrices to tensors.

Theorem 3.1 ([3]). *Let $\mathscr{A}$ be a tensor of order d and dimension $n_1 \times n_2 \times \cdots \times n_d$. If $\mathscr{A}$ contains an $m_1 \times m_2 \times \cdots \times m_d$ zero subtensor such that*

$$\sum_{i=1}^{d} m_i = 1 + \sum_{i=2}^{d} n_i,$$

then $per(\mathscr{A}) = 0$.

Now, we present a condition that ensures the existence of at least k zero entries on every diagonal of $\mathscr{A}$.

Theorem 3.2. *Let $\mathscr{A} = (a_{i_1\dots i_d})_{n_1\times\cdots\times n_d}$ be a non-negative tensor of order d and dimension $n_1 \times \cdots \times n_d$ such that $n_1 = \min_i n_i$. If $\mathscr{A}$ has an $m_1 \times \cdots \times m_d$ zero subtensor with the property*

$$\sum_{i=1}^{d} m_i = k + \sum_{i=2}^{d} n_i,$$

then every diagonal of $\mathscr{A}$ contains at least k zero entries.

Proof. We extend $\mathscr{A}$ to a tensor $\mathscr{B}$ of order d and dimension $n_1 \times (n_2+1) \times (n_3+1) \times \cdots \times (n_k+1) \times n_{k+1} \times \cdots \times n_d$. To do so, we add one hyperplane of type s to the hyperplanes of $\mathscr{A}$ of type s, for $s = 2, 3, \dots, k$, such that the entries on the added hyperplanes are positive. (For example, if $\mathscr{A} = (a_{ijk})_{3\times3\times3}$ is a tensor and $k = 2$, then $\mathscr{B}$ is a tensor of order 3 and dimension $3 \times 4 \times 3$ such that $a_{i4k} > 0$ for $i, k = 1, 2, 3$.) Now, if $\mathscr{A}$ has an $m_1 \times \cdots \times m_d$ zero subtensor with the property

$$\begin{aligned} \sum_{i=1}^{d} m_i &= k + \sum_{i=2}^{d} n_i \\ &= 1 + (\sum_{i=2}^{d} n_i) + (k-1), \end{aligned} \tag{264}$$

then $\mathscr{B}$ also contains a zero subtensor with property (264). Hence, Theorem 3.1 implies that $per(\mathscr{B}) = 0$. Thus, every diagonal of $\mathscr{B}$ contains at least one 0. We claim that every diagonal of $\mathscr{A}$ contains at least k entries equal to 0. To see this, note that if a diagonal of $\mathscr{A}$ contains t entries equal to 0, where $t \le k-1$, then $n_1 - t$ nonzero entries of this diagonal together with suitable t nonzero entries that we added to $\mathscr{A}$ create a positive diagonal for $\mathscr{B}$, contradicting the equality $per(\mathscr{B}) = 0$. □

In what follows, we consider non-negative tensors of order d and dimension n with the property

$$per(\mathscr{A}(i_1|i_2|\dots|i_d)) > 0, \qquad \forall i_1, \dots, i_d = 1, \dots, n, \tag{265}$$

where $\mathscr{A}(i_1|i_2|\dots|i_d)$ is a subtensor of order d and dimension $n-1$, obtained from $\mathscr{A}$ by deleting the hyperplanes i_k of type k for $k = i_1, \dots, i_d$. In the following examples, we consider some tensors with the property mentioned in (265).

Example 3.3. Let $\mathscr{A}$ be a non-negative tensor of order 3 and dimension 2. If all entries of $\mathscr{A}$ are positive, then $\mathscr{A}$ satisfies (265).

Example 3.4. If $\mathscr{A}$ is a non-negative tensor of order 3 and dimension 3 with positive entries

$$a_{111}, a_{113}, a_{121}, a_{122}, a_{132}, a_{133}, a_{221}, a_{222}, a_{231}, a_{232}, a_{311}, a_{313}, a_{331}, a_{333}$$

whose all other entries are equal to 0, then $\mathscr{A}$ satisfies (265).

In the following theorem, we study the permanent of a tensor for which (265) is true.

Theorem 3.5. *If $\mathscr{A}$ is a non-negative tensor of order d and dimension n with property (265), $c \in \mathbb{R}$ and $c \neq 0$, then*

$$per(\mathscr{A} + c\mathscr{E}_{i_1\dots i_d}) > per(\mathscr{A}), \qquad c > 0,$$

or

$$per(\mathscr{A} + c\mathscr{E}_{i_1\dots i_d}) < per(\mathscr{A}), \qquad c < 0.$$

Here, $\mathscr{E}_{i_1\dots i_d}$ is the tensor of order d and dimension n which has 1 in position $(i_1, \dots, i_d)$ and 0 in the others.

Proof.

$$\begin{aligned} per(\mathscr{A} + c\mathscr{E}_{i_1\dots i_d}) &= \sum_{\sigma_k} (\prod_{t \neq i_1} a_{t\sigma_2(t)\dots\sigma_d(t)}).(a_{i_1\dots i_d} + c) \\ &= per(\mathscr{A}) + cper\mathscr{A}(i_1|\dots|i_d). \end{aligned}$$

Now, the proof is complete because of (265). □

In the next theorem, a stronger result is obtained for $(0,1)$-tensors with property (265).

Theorem 3.6. *Let $\mathscr{A}$ be a $(0,1)$-tensor of order d and dimension n. If $\mathscr{A}$ satisfies (265), then*

$$per(\mathscr{A} + \sum_{t=1}^{m} \mathscr{E}_{i_{1_t}\dots i_{d_t}}) \geqslant per(\mathscr{A}) + m.$$

Proof. It is clear by using induction on m. □

Definition 3.7 ([1])**.** A non-negative tensor $\mathscr{A}$ of order d and dimension n has a plane stochastic pattern if there exists a non-negative plane stochastic tensor $\mathscr{B}$ of order d and dimension n such that

$$a_{i_1\dots i_d} = 0 \Longleftrightarrow b_{i_1\dots i_d} = 0, \qquad i_1, \dots, i_d = 1, \dots, n.$$

Example 3.8. Let $\mathscr{A}$ be a tensor of order 3 and dimension 2 with entries $a_{111} = a_{122} = a_{212} = a_{221} = 1$ and other entries of it are zero. Clearly, $\mathscr{A}$ has a plane stochastic pattern since there exists a non-negative plane stochastic tensor $\mathscr{B}$ of order 3 and dimension 2 with entries $a_{111} = a_{122} = a_{212} = a_{221} = \frac{1}{2}$ and other entries of it be equal to zero.

Example 3.9. Let A be a tensor of order 2 and dimension 3 (or a matrix of size 3×3) with entries $a_{11} = a_{12} = a_{13} = a_{21} = a_{22} = a_{33} = 1$ and other entries of it be equal to zero. Since there is not a plane stochastic matrix B of size 3×3 that satisfies in Definition 3.7 thus A does not has a plane stochastic pattern.

In the following theorem, we consider the relationship between tensors with property (265) and tensors with a plane stochastic pattern.

Theorem 3.10. *If a non-negative tensor $\mathscr{A}$ of order d and dimension n satisfies (265), then $\mathscr{A}$ has a plane stochastic pattern.*

Proof. Consider the tensor $\mathscr{S} = (s_{i_1\dots i_d})_{n\times\dots n}$ defined by

$$s_{i_1\dots i_d} = a_{i_1\dots i_d}\frac{per(\mathscr{A}(i_1|\dots|i_d))}{per(\mathscr{A})}, \qquad i_1,\dots,i_d = 1,\dots,n.$$

It is clear that $\mathscr{S}$ is a non-negative tensor, and that $a_{i_1\dots i_d} = 0$ implies $s_{i_1\dots i_d} = 0$. Also, if $s_{i_1\dots i_d} = 0$, then $a_{i_1\dots i_d} = 0$, since $per(\mathscr{A}(i_1|\dots|i_d)) > 0$ for all $i_1,\dots,i_d = 1,\dots,n$. Hence, the tensor $\mathscr{S}$ has the same zero pattern as $\mathscr{A}$. Now, we claim that $\mathscr{S}$ is a plane stochastic tensor. To see this, we compute the summation of the entries of the hyperplane i_j, for all $i_j = 1,\dots,n$ and $j \in \{1,\dots,d\}$:

$$\begin{aligned}\sum_{i_1,\dots,i_{j-1},i_{j+1},\dots,i_d=1}^{n} s_{i_1\dots i_d} &= \frac{1}{per(\mathscr{A})}\sum_{i_1,\dots,i_{j-1},i_{j+1},\dots,i_d=1}^{n} a_{i_1\dots i_j\dots i_d}per(\mathscr{A}(i_1|\dots|i_{j-1}|i_j|i_{j+1}|\dots|i_d))\\ &= \frac{1}{per(\mathscr{A})}per(\mathscr{A})\\ &= 1.\end{aligned}$$

Similarly, for each $j \in \{1,\dots,d\}$, the summation of the entries of the hyperplane i_j equals 1. Thus, $\mathscr{S}$ is a plane stochastic tensor, showing that $\mathscr{A}$ has a plane stochastic pattern. □

References

[1] M. Che, C. Bu, L. Qi, Y. Wei, *Nonnegative tensors revisited: plane stochastic tensors*, Linear Multilinear Algebra., 67(7) (2019) 1364-1391.

[2] J. P. R. Christensen, P. Fischer, *Multidimensional stochastic matrices and error-correcting codes*, Linear Algebra Appl., 183 (1993) 255-276.

[3] S.J. Dow, P.M. Gibson, *Permanents of d-dimensional matrices*, Linear Algebra Appl., 90 (1987) 133-145.

[4] L. Qi, Z. Luo, *Tensor Analysis: Spectral Theory and Special Tensors, Society for Industrial and Applied Mathematics*, Philadelphia, 2017.

[5] A.A. Taranenko, *Permanents of multidimensional matrices: properties and applications*, J. Appl. Ind. Math., 10(4) (2016) 567-604.

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An operator inequality for multiplicative functions

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Abstract

In this paper, we prove some operator inequality for operator monotone and multiplicative functions. Indeed, we give operator perspective inequalities equivalent to the Loewner-Heinz inequality in the view point of perspective and generalized perspective of operator monotone and multiplicative functions.

Keywords: multiplicative function, operator perspective, operator monotone
Mathematics Subject Classification [2023]: 46L05, 46L60, 81T05

1 Introduction and preliminaries

The non-commutative perspective of the continuous one variable function f defined in [2] by setting

$$\mathcal{P}_f(A,B) = B^{1/2} f(B^{-1/2}AB^{-1/2})B^{1/2},$$

where A is a self-adjoint operator and B is an invertible positive operator on a Hilbert space $\mathcal{H}$ with spectra in a closed interval [2]. We proved the necessary and sufficient conditions for joint convexity (resp. concavity) of the perspective and generalized perspective functions [2]. Let $h > 0$ be a real continuous function. We defined [2] the non-commutative generalized perspective of two variables (associated with f and h) by

$$\mathcal{P}_{f\Delta h}(A,B) = h(B)^{1/2} f(h(B)^{-1/2}Ah(B)^{-1/2})h(B)^{1/2},$$

where A is a self-adjoint operator and B is an invertible positive operator on a Hilbert space $\mathcal{H}$ with spectra in a closed interval.

The celebrated Loewner–Heinz inequality asserts that, $f(t) = t^\alpha$ with $0 < \alpha \leq 1$ is operator monotone on $[0,\infty)$. The Loewner–Heinz inequality means that $A \leq B$ implies $A^\alpha \leq B^\alpha$ for $0 < \alpha \leq 1$. Uchiyama [6] showed that $A \leq B$ if and only if $(A+\lambda)^\alpha \leq (B+\lambda)^\alpha$ for every $\lambda > 0$.

The concept of α-geometric mean of two invertible positive operators for $0 \leq \alpha \leq 1$ introduced by Kubo and Ando [1] by

$$A\#_\alpha B := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}.$$

The α-geometric mean is perspective of the function $f(t) = t^\alpha$. The concept of operator (α,β)-geometric mean as a generalization of the concept of operator α-geometric mean introduced by Nikoufar et al. [3] for $0 \leq \alpha, \beta \leq 1$ as follows:

$$A\#_{(\alpha,\beta)}B := A^{\frac{\beta}{2}}(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^\alpha A^{\frac{\beta}{2}}.$$

Every operator α-geometric mean is an operator $(\alpha,1)$-geometric mean. The generalized perspective of the functions $f(t) = t^\alpha, h(t) = t^\beta$ is the (α,β)-geometric mean of two invertible positive operators [4].

Let A and B be bounded self-adjoint operators on a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$. Recall that $A \leq B$ means $\langle Ah,h\rangle \leq \langle Bh,h\rangle$ for every $h \in \mathcal{H}$. By $\mathcal{B}(\mathcal{H})$ we mean the set of all bounded linear operators on $\mathcal{H}$ and we denote by $\mathcal{B}(\mathcal{H})^+$ the set of positive operators in $\mathcal{B}(\mathcal{H})$. A real continuous function $f(t)$ defined on a real interval is said to be operator monotone, provided that $A \leq B$ implies that $f(A) \leq f(B)$ for any two self–adjoint operators A and B whose spectra are in the interval.

Throughout this paper A, B, C, D stand for positive operators and for a real number λ we write $A + \lambda$, for short, instead of $A + \lambda I$.

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2 In the view point of perspective and generalized perspective

Let $f:(0,\infty)\to\mathbb{R}$ be a continuous function. The transpose $\tilde{f}$ of f is defined by

$$\tilde{f}(x)=xf(x^{-1}),\quad x>0.$$

Note that the transpose $\tilde{f}$ of an operator monotone function f on $(0,\infty)$ is operator monotone again and the transpose $\tilde{f}$ of an operator convex function f on $(0,\infty)$ is operator convex again. We say that f is multiplicative if $f(ts)=f(t)f(s)$ for every $t,s>0$. Note that if f is multiplicative, then $f(\lambda A)=f(\lambda)f(A)$ for every $\lambda>0$ and $A\in\mathcal{B}(\mathcal{H})^+$. We show that the converse of Loewner–Heinz inequality in the view point of perspective of operator monotone and multiplicative functions.

Lemma 2.1. *[5] Let $f:(0,\infty)\to\mathbb{R}$ be any function and $\tilde{f}$ the transpose of f. Then*

$$\mathcal{P}_{\tilde{f}}(A,B)=\mathcal{P}_f(B,A)$$

for every $A,B\in\mathcal{B}(\mathcal{H})^+$.

Corollary 2.2. *Let $f:(0,\infty)\to\mathbb{R}$ be a function.*

(i) If f is non-negative and operator monotone, then so is $\tilde{f}$.

(ii) If f is multiplicative, then so is $\tilde{f}$.

Theorem 2.3. *[5] If $A\le B$, $C\le D$ and f is operator monotone, then*

$$\mathcal{P}_f(A,C)\le\mathcal{P}_f(B,D).$$

Theorem 2.4. *[6] Let $f(t)$ be a non-constant operator monotonic function in a neighbourhood of $t=a$. Then $A\le B$ if and only if there exists a sequence $\{t_n\}$ such that $t_n\to 0$ and*

$$f(a+t_nA)\le f(a+t_nB).$$

Lemma 2.5. *[5] Let $f:(0,\infty)\to\mathbb{R}$ be multiplicative and $A,B\in\mathcal{B}(\mathcal{H})^+$. Then*

(i) $\mathcal{P}_f(A+\lambda,B+\mu)=f(\lambda)\mathcal{P}_f(\frac{A}{\lambda}+1,B+\mu)$ for every $\lambda,\mu>0$,

(ii) $\mathcal{P}_f(A+\lambda,B+\mu)=\tilde{f}(\mu)\mathcal{P}_f(A+\lambda,\frac{B}{\mu}+1)$ for every $\lambda,\mu>0$.

Theorem 2.6. *[5] Let $f:(0,\infty)\to(0,\infty)$ be a non-constant multiplicative and operator monotone function and $A,B,C,D\in\mathcal{B}(\mathcal{H})^+$. Then the following statements are equivalent:*

(i) $A\le B$ and $C\le D$,

(ii) $\mathcal{P}_f(A+\lambda,C+\mu)\le\mathcal{P}_f(B+\lambda,D+\mu)$ for every $\lambda,\mu>0$,

(iii) $\mathcal{P}_f(sA+1,tC+1)\le\mathcal{P}_f(sB+1,tD+1)$ for every $s,t>0$.

The following result is an extension of [6, Theorem 3.1] for $f(t)=t^{1/2}$.

Corollary 2.7. *For $A,B,C,D\in\mathcal{B}(\mathcal{H})^+$ and $0\le\alpha\le 1$ the following statements are equivalent:*

(i) $A\le B$ and $C\le D$,

(ii) $(A+\lambda)\#_\alpha(C+\mu)\le(B+\lambda)\#_\alpha(D+\mu)$ for every $\lambda,\mu>0$,

(iii) $(sA+1)\#_\alpha(tC+1)\le(sB+1)\#_\alpha(tD+1)$ for every $s,t>0$.

We show that the converse of Loewner–Heinz inequality in the view point of generalized perspective of operator monotone and multiplicative functions.

Theorem 2.8. *If $A \leq B$, $C \leq D$ and f, h are operator monotone, then*

$$\mathcal{P}_{f\Delta h}(A, C) \leq \mathcal{P}_{f\Delta h}(B, D).$$

Lemma 2.9. *Let $f : (0, \infty) \to \mathbb{R}$ and $h : (0, \infty) \to (0, \infty)$ be multiplicative and $A, B \in \mathcal{B}(\mathcal{H})^+$. Then*

(i) $\mathcal{P}_{f\Delta h}(A + \lambda, B + \mu) = \tilde{f}(h(\mu))\mathcal{P}_{f\Delta h}(A + \lambda, \frac{B}{\mu} + 1)$ *for every* $\lambda, \mu > 0$,

(ii) $\mathcal{P}_{f\Delta h}(A + \lambda, B + \mu) = f(\lambda)\mathcal{P}_{f\Delta h}(\frac{A}{\lambda} + 1, B + \mu)$ *for every* $\lambda, \mu > 0$.

Theorem 2.10. *Suppose that $f, h : (0, \infty) \to (0, \infty)$ are non-constant multiplicative and operator monotone functions and $A, B, C, D \in \mathcal{B}(\mathcal{H})^+$. Then the following statements are equivalent:*

(i) $A \leq B$ *and* $C \leq D$,

(ii) $\mathcal{P}_{f\Delta h}(A + \lambda, C + \mu) \leq \mathcal{P}_{f\Delta h}(B + \lambda, D + \mu)$ *for every* $\lambda, \mu > 0$,

(iii) $\mathcal{P}_{f\Delta h}(sA + 1, tC + 1) \leq \mathcal{P}_{f\Delta h}(sB + 1, tD + 1)$ *for every* $s, t > 0$.

The following result is a generalization of [6, Theorem 3.1] and Corollary 2.7.

Corollary 2.11. *For $A, B, C, D \in \mathcal{B}(\mathcal{H})^+$ and $0 \leq \alpha, \beta \leq 1$ the following statements are equivalent:*

(i) $A \leq B$ *and* $C \leq D$,

(ii) $(A + \lambda)\#_{(\alpha,\beta)}(C + \mu) \leq (B + \lambda)\#_{(\alpha,\beta)}(D + \mu)$ *for every* $\lambda, \mu > 0$,

(iii) $(sA + 1)\#_{(\alpha,\beta)}(tC + 1) \leq (sB + 1)\#_{(\alpha,\beta)}(tD + 1)$ *for every* $s, t > 0$.

Conclusions

We applied the converse of Loewner-Heinz inequality to the perspective and generalized perspective of an operator monotone and multiplicative function and we then gave the perspective type inequalities equivalent to Loewner-Heinz inequality.

References

[1] F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann. 246, (1979-1980), 205–224.

[2] A. Ebadian, I. Nikoufar, and M. Eshagi Gordji, *Perspectives of matrix convex functions*, Proc. Natl. Acad. Sci. 108 (2011), No. 18, 7313–7314.

[3] I. Nikoufar, A. Ebadian, and M. Eshagi Gordji, *The simplest proof of Lieb concavity theorem*, Adv. Math. 248 (2013), 531–533.

[4] I. Nikoufar, *On operator inequalities of some relative operator entropies*, Adv. Math. 259 (2014), 376–383.

[5] I. Nikoufar and M. Shamohammadi, *The converse of the LoewnerHeinz inequality via perspective*, Linear Algebra Appl. 66 (2018), No. 2, 243–249.

[6] M. Uchiyama, *A converse of the Loewner-Heinz inequality, geometric mean and spectral order*, Proc. Edinb. Math. Soc. 57 (2014), 565–571.

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On some improved operator inequalities

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Abstract

In this paper, we provide an operator version of the Jensen's inequality for P-class functions. We provide some applications and improve some operator inequalities.

Keywords: P-class function, s-convex function, convex function
Mathematics Subject Classification [2023]: 47A63, 47A60, 26D15

1 Introduction and preliminaries

Jensen's inequality for convex functions is one of the most important result in the theory of inequalities. Mond and Pečarić established an operator version of the Jensen inequality for a convex function in [2] as follows:

Theorem 1.1. *Let $f : [m, M] \to \mathbb{R}$ be a continuous convex function. If $x \in \mathcal{H}$, $\langle x, x\rangle = 1$, then for every self-adjoint operator C such that $mI \leq C \leq MI$,*

$$f(\langle Cx, x\rangle) \leq \langle f(C)x, x\rangle. \tag{266}$$

for each $x \in \mathcal{H}$ with $\langle x, x\rangle = 1$.

As a special case of Theorem 1.1 we have the following Hölder-MacCarthy inequality.

Theorem 1.2. *[1, Theorem 2] Let C be a self-adjoint positive operator on a Hilbert space $\mathcal{H}$. Then*

(i) $\langle C^r x, x\rangle \geq \langle Cx, x\rangle^r$ for all $r > 1$ and $x \in \mathcal{H}$ with $\langle x, x\rangle = 1$;

(ii) $\langle C^r x, x\rangle \leq \langle Cx, x\rangle^r$ for all $0 < r < 1$ and $x \in \mathcal{H}$ with $\langle x, x\rangle = 1$;

(i) If C is invertible, then $\langle C^r x, x\rangle \geq \langle Cx, x\rangle^r$ for all $r < 0$ and $x \in \mathcal{H}$ with $\langle x, x\rangle = 1$.

Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. The spectrum of an operator $A \in B(\mathcal{H})$ is denoted by $Sp(A)$. A function $f : \mathbb{I} \to \mathbb{R}$ is a P-class function on $\mathbb{I}$ if

$$f(\lambda x + (1 - \lambda)y) \leq f(x) + f(y), \tag{267}$$

where $x, y \in \mathbb{I}$ and $\lambda \in [0, 1]$. The set of all P-class functions contains the set of all convex functions and the set of all nonnegative monotone functions. Every non-zero P-class function is nonnegative valued.

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2 Main Results

We investigate the corresponding results similar to that of Theorem 1.1 for the case of P-class functions and its special cases.

Theorem 2.1. *Let C be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and assume that $Sp(C) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuous P-class function on $[m, M]$, then*

$$f(\langle Cx, x\rangle) \leq 2\langle f(C)x, x\rangle \tag{268}$$

for each $x \in \mathcal{H}$ with $||x|| = 1$.

Theorem 2.2. *[3] Let the conditions of Theorem 2.1 be satisfied. Then*

$$\langle f(C)x, x\rangle \leq f(m) + f(M). \tag{269}$$

Theorem 2.3. *Let the conditions of Theorem 2.1 be satisfied. Let J be an interval such that $f([m, M]) \subset J$. If $F(u, v)$ is a real function defined on $J \times J$ and non–decreasing in u, then*

$$\begin{aligned} F(\langle f(C)x, x\rangle, f(\langle Cx, x\rangle)) &\leq \max_{t\in[m,M]} F(2(f(m) + f(M)), f(t)) \\ &= \max_{\theta\in[0,1]} F(2(f(m) + f(M)), f(\theta m + (1-\theta)M)). \end{aligned} \tag{270}$$

Corollary 2.4. *Let the conditions of Theorem 2.1 be satisfied. Then,*

(i) $\langle f(C)x, x\rangle \leq 2\lambda f(\langle Cx, x\rangle)$, *where* $\lambda = \frac{f(m)+f(M)}{\min_{t\in[m,M]} f(t)}$,

(ii) $\langle f(C)x, x\rangle \leq 2\lambda + f(\langle Cx, x\rangle)$, *where* $\lambda = f(m) + f(M) - \min_{t\in[m,M]} f(t)$.

For instance, when $f(t) = \ln t$, $t \in [m, M] \subseteq [1, \infty)$, f is P-class and we have

$$\frac{\ln m}{2(\ln M + \ln m)}\langle \ln(C)x, x\rangle \leq \ln(\langle Cx, x\rangle) \leq 2\langle \ln(C)x, x\rangle.$$

In the next corollary, we obtain the Hermite-Hadamard's type inequality for P-class functions.

Corollary 2.5. *Let the conditions of Theorem 2.1 be satisfied and let p and q be nonnegative numbers, with $p + q > 0$, for which*

$$\langle Cx, x\rangle = \frac{pm + qM}{p + q}.$$

Then

$$\frac{1}{2}f\Big(\frac{pm + qM}{p + q}\Big) \leq \langle f(C)x, x\rangle \leq f(m) + f(M).$$

We can improve the Hölder-MacCarthy inequality by providing an upper bound. We use the fact that the function t^r, $0 < r < 1$, is P-class, in addition to being concave.

Lemma 2.6. *Let $\alpha, \beta > 0$ and $0 < r < 1$. Then, $(\alpha + \beta)^r \leq \alpha^r + \beta^r$.*

Corollary 2.7. *[3] Let C be a self-adjoint positive operator on a Hilbert space $\mathcal{H}$. Then*

(i) for all $0 < r < 1$ and $x \in \mathcal{H}$ with $||x|| = 1$,

$$\langle C^r x, x\rangle \leq \langle Cx, x\rangle^r \leq 2\langle C^r x, x\rangle, \tag{271}$$

(ii) for all $r > 1$ and $x \in \mathcal{H}$ with $||x|| = 1$,

$$\langle Cx, x\rangle^r \leq \langle C^r x, x\rangle \leq 2^r\langle Cx, x\rangle^r. \tag{272}$$

Let w_i, x_i be positive numbers with $\sum_{i=1}^{n} w_i = 1$. Then the weighted power means are defined by

$$M_n^{[r]}(x;w) = \Big(\sum_{i=1}^{n} w_i x_i^r\Big)^{1/r}, \quad r \neq 0$$

and

$$M_n^{[0]}(x;w) = \prod_{i=1}^{n} x_i^{w_i}$$

is called weighted geometric mean and denoted by G_w. It is well-known that if $s \leq r$, then

$$M_n^{[s]}(x;w) \leq M_n^{[r]}(x;w). \tag{273}$$

The weighted arithmetic mean of a non-empty sequence of data $\{x_1, x_2, ..., x_n\}$ and corresponding non-negative weights $\{w_1, w_2, ..., w_n\}$ with $\sum_{i=1}^{n} w_i = 1$ is defined by

$$A_w = \sum_{i=1}^{n} w_i x_i$$

and the weighted harmonic mean of them is defined by

$$H_w = \Big(\sum_{i=1}^{n} w_i x_i^{-1}\Big)^{-1}.$$

The arithmetic-geometric-harmonic mean inequality is a well-known inequality as follows:

$$H_w \leq G_w \leq A_w.$$

According to improved Hölder-MacCarthy inequality we identify the following relation between the weighted arithmetic mean and the weighted power mean.

Corollary 2.8. *Let w_i, x_i be positive numbers with $\sum_{i=1}^{n} w_i = 1$. Then*

(i) for all $0 < r < 1$,

$$M_n^{[r]}(x;w) \leq A_w \leq 2^{1/r} M_n^{[r]}(x;w),$$

(ii) for all $r > 1$,

$$A_w \leq M_n^{[r]}(x;w) \leq 2A_w.$$

Some refinements of the arithmetic-geometric-harmonic mean inequality are of interest.

Remark 2.9. Let w_i, x_i be positive numbers with $\sum_{i=1}^{n} w_i = 1$.

(i) For all $0 < r < 1$,

$$\begin{aligned} 2^{-1/r} M_n^{[-r]}(x;w) \leq H_w &\leq M_n^{[-r]}(x;w) \\ &\leq G_w \leq M_n^{[r]}(x;w) \leq A_w \leq 2^{1/r} M_n^{[r]}(x;w). \end{aligned}$$

Replacing x_i^{-1} with x_i in Corollary 2.8(i) and applying the monotonically decreasing function t^{-1} to both sides of the inequalities we get the first and second inequalities. The third and forth inequalities obtain by (273). We deduce the last two inequalities by Corollary 2.8(i).

(ii) For all $r > 1$,

$$\frac{1}{2} H_w \leq M_n^{[-r]}(x;w) \leq H_w \leq G_w \leq A_w \leq M_n^{[r]}(x;w) \leq 2A_w.$$

Similar to that of part (i) and Corollary 2.8(ii) we reach the first and second inequalities. The third and forth inequalities are well-known inequalities. The last two inequalities are obtained in Corollary 2.8(ii).

Conclusions

We proved an operator version of the Jensen's inequality for P-class functions. We also provided some applications and improved some operator inequalities. An operator version of the Jensen's inequality can be proved for s-convex functions [4].

References

[1] R. P. Agarwal and S. S. Dragomir, *A survey of Jensen type inequalities for functions of self-adjoint operators in Hilbert spaces*, Comput. Math. Appl. 59 (2010), No. 12, 3785-3812.

[2] B. Mond, J. Pečarić, *Convex inequalities in Hilbert space*, Houston J. Math. 19 (1993), 405-420.

[3] I. Nikoufar and D. Saeedi, *Some Inequalities for P-Class Functions*, Filomat 34 (2020), No. 13, 4555-4566.

[4] I. Nikoufar and D. Saeedi, *An operator version of the Jensen inequality for s-convex functions*, Complex Anal. Oper. Theory 15 (2021), No. 5, 18 pages.

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Duality and alpha-duality of fusion frames in Hilbert spaces

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Abstract

In this paper, we get some results about duals and α-duals of fusion frames in Hilbert spaces. Especially, the direct sums and tensor products for α-duals of fusion frames are considered.

Keywords: Hilbert space, fusion frame, direct sum, tensor product, α-dual.
Mathematics Subject Classification [2023]: 42C15

1 Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [2] in 1952 to study some problems in nonharmonic Fourier series.

Let H be a separable Hilbert space and let I be a finite or countable index set. A family $\mathcal{F} = \{f_i\}_{i\in I} \subseteq H$ is a *frame* for H, if there exist two positive numbers A and B such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2,$$

for each $f \in H$. A and B are the *lower* and *upper* frame bounds, respectively.

Many generalizations of frames have been introduced. One of the most important of them is fusion frame introduced in [3].

Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of a Hilbert space H, and $\{\omega_i\}_{i\in I}$ be a family of weights, i.e., $\omega_i > 0$ for each $i \in I$. Then $\mathcal{W} = \{(W_i, \omega_i)\}_{i\in I}$ is a *fusion frame*, if there are two positive numbers A and B such that for each $f \in H$,

$$A\|f\|^2 \leq \sum_{i\in I} \omega_i^2 \|\pi_{W_i}(f)\|^2 \leq B\|f\|^2,$$

where π_{W_i} is the orthogonal projection onto the subspace W_i. If only the right-hand inequality is required, then $\mathcal{W}$ is called a *Bessel fusion sequence*. If $A = B$, then $\mathcal{W}$ is called a *tight fusion frame*.

It is easy to see that if $\mathcal{W} = \{(W_i, \omega_i)\}_{i\in I}$ is a Bessel fusion sequence, then the operator $S_{\mathcal{W}}$ defined on H by $S_{\mathcal{W}} f = \sum_{i\in I} \omega_i^2 \pi_{W_i} f$ is well-defined, bounded and positive. Also, if $\mathcal{W}$ is a fusion frame, then $S_{\mathcal{W}}$ is invertible.

Let $\mathcal{W} = \{(W_i, \omega_i)\}_{i\in I}$ and $\mathcal{V} = \{(V_i, \upsilon_i)\}_{i\in I}$ be two Bessel fusion sequences. Then, $\mathcal{V}$ is called a dual of $\mathcal{W}$ if $\sum_{i\in I} \upsilon_i \omega_i \pi_{W_i} \pi_{V_i} f = f$, for each $f \in H$, see [4].

Direct sums and tensor products of fusion frames in Hilbert spaces have been studied by some authors recently (for more information, see [5, 6] and the references stated therein). In this note, we get some results for the tensor product and direct sum of α-duals of fusion frames. Especially, some obtained results for duals are generalized to α-duals.

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2 Main results

In this section, $\mathcal{W}$ and $\mathcal{V}$ are supposed to be $\{(W_i,\omega_i)\}_{i\in I}$ and $\{(V_i,\upsilon_i)\}_{i\in I}$, respectively. Also, here, I, J and I_k, for each $1\leq k\leq n$, are finite or countable index sets. H, H_j, $H_{i(k)}$ and $H_{i(k)j}$ are separable Hilbert spaces for each $j\in J$, $k\in\{1,\dots,n\}$ and $i(k)\in I_k$. $\mathcal{W}_j=\{(W_{ij},\omega_i):i\in I\}$, $\mathcal{V}_j=\{(V_{ij},\upsilon_i):i\in I\}$, $\mathcal{W}^{(k)}=\{(W_{i(k)},\omega_{i(k)})\}_{i(k)\in I_k}$, $\otimes_{k=1}^n\mathcal{W}^{(k)}$ is

$$\{(W_{i(1)}\otimes\ldots\otimes W_{i(n)},\omega_{i(1)}\ldots\omega_{i(n)})\}_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)},$$

and $\mathcal{W}_j^{(k)}=\{(W_{i(k)j},\omega_{i(k)})\}_{i(k)\in I_k}$, where W_{ij},V_{ij} are closed subspaces of H_j, $W_{i(k)}$ is a closed subspace of $H_{i(k)}$ and $W_{i(k)j}$ is a closed subspace of $H_{i(k)j}$. Note that if M_k is a closed subspace of H_k, for each $1\leq k\leq n$, then it is easy to see that $\pi_{\otimes_{k=1}^n M_K}=\otimes_{k=1}^n\pi_{M_k}$.

Recall that if H and H' are Hilbert spaces, then the tensor product $H\otimes H'$ is a Hilbert space. The inner product for simple tensors is defined by $\langle(f\otimes f'),(g\otimes g')\rangle=\langle f,g\rangle\langle f',g'\rangle$, where $f,g\in H$ and $f',g'\in H'$. If U and U' are bounded linear operators on H and H', respectively, then the tensor product $U\otimes U'$ is a bounded linear operator on $H\otimes H'$. Also $(U\otimes U')^*=U^*\otimes U'^*$ and $\|U\otimes U'\|=\|U\|\|U'\|$.

Tensor products have important applications, for example tensor products are useful in the approximation of multi-variate functions of combinations of univariate ones.

The concept of α-duality was introduced for g-frames in [1]. Similarly, this notion can be considered for fusion frames.

Definition 2.1. Let $\alpha\in\mathbb{Z}$ and $\mathcal{W}$ and $\mathcal{V}$ be two fusion frames for H. Then, $\mathcal{V}$ is called an α-dual of $\mathcal{W}$ if $\sum_{i\in I}\upsilon_i\omega_i\pi_{W_i}\pi_{V_i}f=S_{\mathcal{W}}^{\alpha}f$, for each $f\in H$.

Example 2.2. (i) Since $\sum_{i\in I}\omega_i\omega_i\pi_{W_i}\pi_{W_i}f=S_{\mathcal{W}}f$, $\mathcal{W}$ is a 1-dual of itself.

(ii) If $\mathcal{V}$ is a dual of $\mathcal{W}$, then $\mathcal{V}$ is a 0-dual of $\mathcal{W}$.

Theorem 2.3. *$\mathcal{W}^{(k)}$ is a fusion frame for each $1\leq k\leq n$ if and only if $\otimes_{k=1}^n\mathcal{W}^{(k)}$ is a fusion frame. In this case, $S_{\otimes_{k=1}^n\mathcal{W}^{(k)}}=\otimes_{k=1}^n S_{\mathcal{W}^{(k)}}$. If A_k and B_k are lower and upper bounds of $\mathcal{W}^{(k)}$, respectively, then $\otimes_{k=1}^n\mathcal{W}^{(k)}$ is an $(\Pi_{k=1}^n A_k,\Pi_{k=1}^n B_k)$ fusion frame.*

Now we get the following result for α-duals.

Proposition 2.4. *Suppose that $\mathcal{W}^{(k)}$'s and $\mathcal{V}^{(k)}$'s are Bessel fusion sequences. If $\mathcal{V}^{(k)}$ is an α-dual of $\mathcal{W}^{(k)}$, for each $k\in\{1,\dots,n\}$, then $\otimes_{k=1}^n\mathcal{V}^{(k)}$ is an α-dual of $\otimes_{k=1}^n\mathcal{W}^{(k)}$.*

Corollary 2.5. *Suppose that $\mathcal{W}^{(k)}$'s are A_k-tight fusion frames. If $\mathcal{V}^{(k)}$ is an α-dual of $\mathcal{W}^{(k)}$, for each $k\in\{1,\dots,n\}$, then $\left\{\left(W_{i(1)}\otimes\ldots\otimes W_{i(n)},\frac{\omega_{i(1)}\ldots\omega_{i(n)}}{A_1^{\alpha}}\right)\right\}_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)}$ and $\left\{\left(V_{i(1)}\otimes\ldots\otimes V_{i(n)},\frac{\upsilon_{i(1)}\ldots\upsilon_{i(n)}}{(\Pi_{k=2}^n A_k)^{\alpha}}\right)\right\}_{(i(1),\ldots,i($ are duals.*

Let $\mathcal{W}_j=\{(W_{ij},\omega_i):i\in I\}$ be a Bessel fusion sequence for H_j, $j\in J$, with upper bound B_j such that $B=sup\{B_j:j\in J\}<\infty$. Then $\{\mathcal{W}_j\}_{j\in J}$ is called a *B-Bounded family of Bessel fusion sequences* or shortly *B-BFBFS*.

Let $\mathcal{W}_j=\{(W_{ij},\omega_i):i\in I\}$ be an (A_j,B_j) fusion frame for H_j, $j\in J$, such that $A=inf\{A_j:j\in J\}>0$ and $B=sup\{B_j:j\in J\}<\infty$. Then, we say that $\{\mathcal{W}_j\}_{j\in J}$ is an *(A,B)-bounded family of fusion frames* or shortly *(A,B)-BFFF*.

Theorem 2.6. *$\{\mathcal{W}_j\}_{j\in J}$ is a BFBFS (resp. BFFF) if and only if $\oplus_{j\in J}\mathcal{W}_j:=\{(\oplus_{j\in J}W_{ij},\omega_i):i\in I\}$ is a Bessel fusion sequence (resp. fusion frame).*

Theorem 2.7. *Let $\{\mathcal{W}_j\}_{j\in J}$ and $\{\mathcal{V}_j\}_{j\in J}$ be BFBFS. If $\mathcal{V}_j$ is an α-dual for $\mathcal{W}_j$, for each $j\in J$, then $\oplus_{j\in J}\mathcal{V}_j:=\{(\oplus_{j\in J}V_{ij},\upsilon_i):i\in I\}$ is an α-dual for $\oplus_{j\in J}\mathcal{W}_j:=\{(\oplus_{j\in J}W_{ij},\omega_i):i\in I\}$.*

Proposition 2.8. *Let $\{\mathcal{W}_j^{(k)}\}_{j\in J}$ and $\{\mathcal{V}_j^{(k)}\}_{j\in J}$ be BFBFS, for each $1\leq k\leq n$ and let $\mathcal{V}_j^{(k)}$ be an α-dual of $\mathcal{W}_j^{(k)}$, for each $j\in J$ and $k\in\{1,\dots,n\}$. Then $\otimes_{k=1}^n(\oplus_{j\in J}\mathcal{V}_j^{(k)})$ is an α-dual of $\otimes_{k=1}^n(\oplus_{j\in J}\mathcal{W}_j^{(k)})$.*

References

[1] M. R. Abdollahpour and A. Najati, *G-frames and Hilbert-Schmidt operators*, Bull. Iranian Math. Soc., **4** (2011) 141–155.

[2] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., **72** (1952) 341–366.

[3] P. Casazza and G. Kutyniok, *Frames of subspaces*, Contemp. Math. Amer. Math. Soc., **345** (2004) 87–113.

[4] P. Gavruta, *On the duality of fusion frames*, J. Math. Anal. Appl., **333** (2007) 871–879.

[5] A. Khosravi and M. S. Asgari, *Frames of subspaces and approximation of the inverse frame operator*, Houston J. Math., **33** (2007) 907–920.

[6] A. Khosravi and M. Mirzaee Azandaryani, *Fusion frames and g-frames in tensor product and direct sum of Hilbert spaces*, Appl. Anal. Discrete Math., **6** (2012) 287–303.

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Topics on numerical range of tensors

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Abstract

In this paper, the numerical range of an even-order tensor is defined using the norm of its square matrix unfolding. The basic results of the numerical range of a matrix, are proved to hold for the numerical range of an even-order tensor. We get a relation between tenasor and matrix numerical ranges, then numerical range of Toeplitz tensor obtain using its unfolding matrix.

Keywords: numerical range, Toeplitz tensor, unfolding operators, even-order tensor
Mathematics Subject Classification [2023]: 15A69, 15A60

1 Introduction

A complex $2m^{th}$-order n-dimensional tensor can be defined as $\mathcal{A} = (a_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m})$, where $1 \leq i_1, i_2, \ldots, i_m, j_1, j_2, \ldots, j_m \leq n$ (or $\mathcal{A} \in \mathbb{C}^{(2m,n)}$). In the following discussion, we also use $\mathcal{A}(i_1, i_2, \ldots, i_m, j_1, j_2, \ldots, j_m)$ to denote the $(i_1, i_2, \ldots, i_m, j_1, j_2, \ldots, j_m)^{th}$ entry of $\mathcal{A}$.

Definition 1.1. Let $\mathcal{A}$ be a $2m^{th}$-order n-dimensional tensor. Furthermore, assume that a nonzero m^{th}-order n-dimensional tensor $\mathcal{X}$ and $\lambda \in \mathbb{C}$ satisfy

$$\mathcal{A}.\mathcal{X} = \lambda\mathcal{X}, \tag{274}$$

where

$$(\mathcal{A}.\mathcal{X})_{i_1,i_2,\ldots,i_m} = \sum_{k_1,k_2,\ldots,k_m=1}^{n} a_{i_1,i_2,\ldots,i_m,k_1,k_2,\ldots,k_m} x_{k_1,k_2\ldots,k_m},$$

and $1 \leq i_l \leq n, 1 \leq l \leq m$. Then, we refer to λ as an eigenvalue of $\mathcal{A}$, and to $\mathcal{X}$ as its corresponding eigentensor.

For $m = 1$, $\mathcal{A}$ is a square matrix and Definition 368 reduces to the definition of matrix eigenvalues and eigenvectors. In this paper, *eigenvalue* is always meant to be in the sense of Definition 368. The unfolding matrix of a tensor is a useful tool for the study of such tensor problems as those concerning eigenvalues and the numerical range.

Definition 1.2. [1] Let $\mathcal{A}$ be a $2m^{th}$-order n-dimensional tensor. The square matrix unfolding of $\mathcal{A}$ with an ordering P is an n^m-by-n^m matrix A_P whose $(k, h)^{th}$ entry is given by

$$A_P(k, h) = \mathcal{A}(i'_1, i'_2, \ldots, i'_m, j'_1, j'_2, \ldots j'_m),$$

where

$$k = n^{m-1}(i'_1 - 1) + n^{m-2}(i'_2 - 1) + \cdots + n(i'_{m-1} - 1) + i'_m,$$
$$h = n^{m-1}(j'_1 - 1) + n^{m-2}(j'_2 - 1) + \cdots + n(j'_{m-1} - 1) + j'_m,$$

and $1 \leq i'_l, j'_l \leq n, 1 \leq l \leq m$. Here, P is the permutation matrix corresponding to ordering P that satisfies

$$(i'_1, i'_2, \ldots, i'_m) = (i_1, i_2, \ldots, i_m)P, \qquad (j'_1, j'_2, \ldots, j'_m) = (j_1, j_2, \ldots, j_m)P.$$

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Recently, we defined the numerical range of even-order tensors using the even-order tensor unfolding matrix norms. We showed that the basic properties of the numerical range of a matrix, such as *compactness* and *convexity*, are valid for the numerical range of an even-order tensor. It is useful to estimate Toeplitz tensor eigenvalues in the process of image restoration[2].

Definition 1.3. If $\|.\|$ is the matrix norm, the function $|||\ .\ |||_P$ defined on $\mathbb{C}^{(2m,n)}$ by

$$|||\ \mathcal{A}\ |||_P = \|A_P\|$$

Tensor P_2-norm: $|||\ \mathcal{A}\ |||_{P_2} = \|A_P\|_2$. for any $i_1, i_2, \ldots, i_m, j_1, j_2, \ldots, j_m$ and $(k_1, k_2, \ldots, k_m)$ being any permutation of $(i_1, \ldots, i_m)$. It is clear $\mathbb{F}$ is a vector space and the set of even-order real symmetric tensors is a subset of $\mathbb{F}$.

Let $|||\ .\ |||_{max_2}$ be a norm on $\mathbb{F}$ defined by

$$|||\ \mathcal{A}\ |||_{max_2} = \max_{\mathcal{X}\neq 0} \frac{|||\ \mathcal{A}.\mathcal{X}\ |||_{p_2}}{|||\ \mathcal{X}\ |||_{p_2}},$$

Definition 1.4. Let $\mathcal{A}$ be a $2m^{th}$-order n-dimensional tensor. The numerical range of $\mathcal{A}$ is defined by

$$W_{|||.|||_P}(\mathcal{A}) = \bigcap_{\lambda\in\mathbb{C}} \{\mu \in \mathbb{C} :\mid \mu - \lambda \mid \leq |||\ \mathcal{A} - \lambda\mathcal{I}_E\ |||_P\},$$

where $|||\ .\ |||_P$ is a tensor P-norm consistent with the tensor norm, and $\mathcal{I}_E$ is a $2m^{th}$-order n-dimensional tensor.

Our idea is to generalize Theorems of the matrix numerical range to the numerical range of an even-order tensor, one that contains the tensor eigenvalues in the sense of Definition 368.

2 Main results

In this section, we consider the numerical ranges of even-order tensors based on the tensor norm. We prove some the results of tensor numerical range. In the following lemma, we obtain a relationship between the numerical range of an even-order tensor and that of its matrix unfolding.

Lemma 2.1. Let $\mathcal{A}$ be a $2m^{th}$-order n-dimensional tensor. Then

$$W_{|||.|||_{P_2}}(\mathcal{A}) = F(A_P),$$

where A_P is the matrix unfolding of $\mathcal{A}$ with the ordering P, and

$$F(A_P) = \{x^* A_P x : x \in \mathbb{C}^{n^m}, \|x\|_2 = 1\}.$$

We define an inner product $\langle .,. \rangle_P$ on $\mathbb{C}^{(m,n)}$ by

$$\langle \mathcal{X}, \mathcal{Y} \rangle_P = \sum_{j=1}^{n^m} x_P(j)\overline{y}_P(j) = \langle x_P, y_P \rangle,$$

where P is a permutation matrix, $\langle .,. \rangle$ is the Euclidean inner product, and x_P, y_P are the vectorizations of $\mathcal{X}, \mathcal{Y} \in \mathbb{C}^{(m,n)}$ with the ordering P, respectively.

In the following theorem, we find a relationship between $F(\mathcal{A})$ and $W_{|||.|||_2}(\mathcal{A})$, where

$$F(\mathcal{A}) = \{\langle \mathcal{A}\mathcal{X}, \mathcal{X} \rangle_P : \mathcal{X} \in \mathcal{C}^{(m,n)}, |||\ \mathcal{X}\ |||_2 = 1\},$$

and P is the permutation matrix. It is easy to verify that

$$F(\mathcal{A}) = \{x_P^* A_P x_P : x_P \in \mathbb{C}^{n^m}, \|x_P\|_2 = 1\}.$$

Theorem 2.2. For every $\mathcal{A} \in \mathbb{C}^{(2m,n)}$,

$$F(\mathcal{A}) \subset W_{|||.|||_{max_2}}(\mathcal{A}) \subset W_{|||.|||_{P_2}}(\mathcal{A}),$$

where $|||\ .\ |||_{max_2}$ is the norm induced by $|||\ .\ |||_2$.

In Theorem 2.4 below, we find a relationship between the numerical range of an even-order tensor $\mathcal{A}$ and the convex hull of $F(\mathcal{A})$.

Lemma 2.3. If $\mathcal{A} \in \mathbb{C}^{(2m,n)}$, then

$$\lim_{\lambda \to +\infty} |||\ \mathcal{A} + \lambda \mathcal{I}_E\ |||_{max_2} - \lambda = \sup Re(F(\mathcal{A})),$$

where $F(\mathcal{A}) = \{\tilde{x}_P^* A_P \tilde{x}_P : \|\tilde{x}_P\|_2 = 1\}$ and $|||\ .\ |||_{max_2}$ is the norm induced by the tensor norm.

Recall that for a set X contained in a complex vector space, $Co(X)$ denotes the convex hull of X, which is the set of all convex combinations of finitely many points of X. It is the smallest closed convex set that contains X.

Theorem 2.4. For every $\mathcal{A} \in \mathbb{C}^{(2m,n)}$,

$$W_{|||.|||_{max_2}}(\mathcal{A}) = Co(F(\mathcal{A})).$$

Now, we show that the numerical range of an even-order tensor contains the numerical ranges of its subtensors.

Theorem 2.5. Let $\mathcal{A}=(a_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m})$ and consider an index set $J \subset \{1, 2, \ldots, n\}$. Then,

$$W_{|||.|||_P}(\mathcal{A}(J)) \subset W_{|||.|||_P}(\mathcal{A})$$

, where $|||\ .\ |||_P$ is a norm with the property that

$$|||\ \mathcal{B}(J)\ |||_P \leq |||\ \mathcal{B}\ |||_P, \tag{275}$$

for any $\mathcal{B} \in \mathbb{C}^{(2m,n)}$.

For a real matrix $A \in \mathbb{R}^{n \times n}$, $F(A)$ is symmetric about the real axis, that is,

$$F(A) = \overline{F(A)}$$

. This result still holds for the numerical range of an even-order tensor.

Theorem 2.6. For $\mathcal{A} \in \mathbb{R}^{(2m,n)}$,

$$W_{|||.|||_P}(\mathcal{A}) = \overline{W_{|||.|||_P}(\mathcal{A})}$$

.

It is well-known that the eigenvalues of a matrix are included in its numerical range. We show that the numerical ranges of even-order tensors also contain the eigenvalues. It is clear that $\lambda \mathcal{I}_E \cdot \mathcal{X} = \lambda \mathcal{X}$, where $\mathcal{X} \in \mathbb{C}^{(m,n)}$ and $\mathcal{I}_E \in \mathbb{C}^{(2m,n)}$ is the identity tensor. This note leads in the following theorem.

Theorem 2.7. Let $\mathcal{A}$ be a $2m^{th}$-order n-dimensional tensor, and $|||\ .\ |||_P$ be a tensor P-norm which is consistent with a tensor norm $\|.\|$. Then, all eigenvalues of $\mathcal{A}$ are included in $W_{|||.|||_P}(\mathcal{A})$.

Theorem 2.8. Let $|||\ .\ |||_P$ be a consistent tensor P-norm on $\mathbb{C}^{(2m,n)}$. Then, for every $\mathcal{A} \in \mathbb{C}^{(2m,n)}$, $\frac{1}{n}$ $tr(\mathcal{A}) \in W_{|||.|||_P}(\mathcal{A})$.

Remark 2.9. Theorem 2 and Theorem 2.8 show that $W_{|||.|||_P}$ (.) is non-empty for any tensor in $\mathbb{C}^{(2m,n)}$.

It is well-known that the restoration of an image is an ill-conditioned problem [1]. It is effective to estimate the eigenvalues of a Toeplitz tensor (convolution operator) $\mathcal{T}$ so that a suitable regularization method can be used in the image restoration process [1]. Therefore, we study the numerical ranges of Toeplitz tensors.

Definition 2.10. A $2m^{th}$-order n-dimensional tensor $\mathcal{T} = (t_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m})$ is called a Toeplitz tensor if

$$t_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = r_{j_1-i_1,j_2-i_2,\ldots,j_m-i_m},$$

where $\mathcal{R}$=$(r_{i_1,i_2,\ldots,i_m})$ is an m^{th}-order $(2n-1)$-dimensional tensor.

It is clear that when $m = 1$, the above definition coincides with that of a Toeplitz matrix. We remark that the square matrix unfolding of a Toeplitz tensor is a Toeplitz matrix. It is proved that the numerical range of the square matrix unfolding of a Toeplitz tensor is a closed interval [1]. According to Lemma 2.1,

$$W_{|||.|||_{P_2}}(\mathcal{T}) = F(T_P),$$

where $\mathcal{T}$ is the Toeplitz tensor, and T_P is the matrix unfolding of $\mathcal{T}$. This implies that the numerical range of a Toeplitz tensor is a closed interval.

References

[1] L.B. Cui, C. Chen, W. Li M. K. Ng. *An eigenvalue problem for even order tensors with its applications*, Linear and Multilinear Algebra, 64((2016)), 602-621.

[2] M.Pakmanesh and H. Afshin, *Numerical ranges of even-order tensor*, Banach J. Math. Anal. 15(59) (2021). https://doi.org/10.1007/s43037-021-00142-w.

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Remarks on subclasses of univalent functions

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Abstract

In this paper, we introduced new subclasses of analytic functions. Also, we give a characterization of the subclasses of univalent functions by using the method of Loewner chains.

Keywords: Bi-univalent functions, Loewner chains, Almost uniformly starlike functions, Almost bi-starlike functions of order α, Bounded turning functions.
Mathematics Subject Classification [2023]: 30C45

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{276}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. The Koebe one-quarter theorem [12] states that the image of $\mathbb{U}$ under every function f from $\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}).$$

and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f) \qquad , \qquad r_0(f) \geq \frac{1}{4}).$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + ... \quad .$$

which implies that f^{-1} is analytic. A function $f \in A$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{U}$.
Although, the familiar Koebe function is not in the class of σ, there are some examples of functions member of σ, such as

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log(\frac{1+z}{1-z})$$

so on. Other common examples of functions in $\mathcal{S}$ for example

$$z - \frac{z^2}{2} \qquad \text{and} \qquad \frac{z}{1-z^2}$$

are also not members of σ.[5]

Definition 1.1. Let $\mathcal{P}$ denote the class of holomorphic functions p in $\mathbb{U}$ such that $p(0) = 1$ and Re $p(z) > 0, z \in \mathbb{U}$.
This class is usually called the Caratheodory class.

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Let $\mathcal{R}$ denote the class of bounded turning functions in $\mathcal{A}$, which are univalent in $\mathbb{U}$. Also, the class $\mathcal{R}$ is analytically defined as,

$$\mathcal{R} = \{f \in \mathcal{S} : f^{'}(z) \in \mathcal{P}\}, \quad z \in \mathbb{U}. \tag{277}$$

For $\alpha \in [0,1)$, let $S^*(\alpha)$ be the subclass of $\mathcal{S}$, consisting starlike functions order of α in $\mathbb{U}$. The class $S^*(\alpha)$ introduced by Robertson in 1936 [4]. A function $f(z) \in \mathcal{S}$ is in the class $S^*(\alpha)$ in $\mathbb{U}$, if it satisfies

$$Re\,\frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathbb{U}.$$

For $k \in [0,\infty)$, let $k - US^*$ be the subclass of $\mathcal{S}$, consisting $k-$uniformly starlike functions in $\mathbb{U}$. The class $k - US^*$ introduced and studied by Kanas and Wisniowska in 1998 [3]. A function $f(z) \in \mathcal{S}$ is in the class $k - US^*$ in $\mathbb{U}$, if it satisfies

$$Re\,\frac{zf'(z)}{f(z)} > k|\frac{zf'(z)}{f(z)} - 1|, \quad z \in \mathbb{U},$$

note that for $k = 1$, $k - US^* = US^*$, the class of uniformly starlike functions.

Recently, Xu and Liu [6] introduced almost starlike functions of order α that $0 \leq \alpha < 1$. The class consisting of almost starlike functions order of α,$(0 \leq \alpha < 1)$ denoted by $\check{\mathcal{S}}^*(\alpha)$.
Especially the class $\check{\mathcal{S}}^*(0) = \check{\mathcal{S}}^*$.

Definition 1.2. A function $f(z)$ given by 281 on $\mathbb{U}$ is said to be in the class $\check{\mathcal{S}}^*(\alpha)$, if the following condition is satisfied:

$$Re\,\frac{f(z)}{zf'(z)} > \alpha,$$

where $z \in \mathbb{U}, \quad 0 \leq \alpha < 1$.

for $0 \leq \alpha < 1$,
a function $f \in \sigma$ is in the class $\check{\mathcal{S}}^*_\sigma(\alpha)$ of almost bi-starlike functions of order α, if both f and its inverse map f^{-1} is almost starlike of order α. Especially the classes $\check{\mathcal{S}}^*_\sigma(0) = \check{\mathcal{S}}^*_\sigma$ is almost bi-starlike functions.

In the lemma and theory of Loewner chains, if f is a function which depends holomorphically on $z \in \mathbb{U}$ and is also a function of other real variables, it is customary to write $f'(z,.)$ instead of $\frac{\partial f}{\partial z}(z,.)$.

Lemma 1.3. *[2] The function $f : \mathbb{U} \times [0,\infty) \to \mathbb{C}$ with $f(0,t) = 0, f'(0,t) = e^t$, is a Löwner chain if and only if the following conditions hold:*
(i) There exist $r \in (0,1)$ and a constant $M \geq 0$ such that $f(.,t)$ is holomorphic on $\mathbb{U}_r$ for each $t \geq 0$, where $\mathbb{U}_r = \{z \in \mathbb{C} : |z| < r\}$, locally absolutely continuous in $t \geq 0$ locally uniformly with respect to $z \in \mathbb{U}_r$, and

$$|f(z,t)| \leq Me^t, \qquad |z| \leq r, \quad t \geq 0.$$

(ii) There exists a function $p(z,t)$ such that $p(.,t) \in \mathcal{P}$ for each $t \geq 0, p(z,.)$ is measurable on $[0,\infty)$ for each $z \in \mathbb{U}$, and for all $z \in \mathbb{U}_r$,

$$\frac{\partial f}{\partial t}(z,t) = zf'(z,t)p(z,t), \qquad a.e. \quad t \geq 0.$$

(iii) For each $t \geq 0, f(.,t)$ is the analytic continuation of $f(.,t)|_{\mathbb{U}_r}$ to $\mathbb{U}$, and furthermore this analytic continuation exists under the assumptions (i) and (ii).

Lemma 1.4. *Let $f(z,t)$ be a Loewner chain. Then there exists a function $p(z,t)$ such that $p(.,t) \in \mathcal{P}$, $t \geq 0$, $p(z,t)$ is measurable in $t \in [0,\infty)$ for each $z \in mathbbU$, and*

$$\frac{\partial f}{\partial t}(z,t) = zf'(z,t)p(z,t), \quad z \in \mathbb{U}, t \geq 0. \tag{278}$$

In the present paper, we introduce new subclasses of the univalent functions and characterization the classes by using of the Loewner chains method are given.

2 Main results

In the section by using method of Loewner chains, characterization of subclasses of univalent functions are obtained.

Theorem 2.1. *Suppose f is a normalized univalent function on $\mathbb{U}$, f be in the class $\mathcal{R}$, if and only if*

$$f(z,t) = f(z) + (e^t - 1)z, \quad z \in \mathbb{U}, t \geq 0, \tag{279}$$

is a Loewner chain.

For $0 \leq \alpha < 1$, a function $f \in \sigma$ is in the class $\check{\mathcal{S}}_\sigma^*(\alpha)$ of almost bi-starlike functions of order α, if both f and its inverse map f^{-1} is almost starlike of order α. Especially the class $\check{\mathcal{S}}_\sigma^*(0) = \check{\mathcal{S}}_\sigma^*$ is almost bi-starlike functions.

Definition 2.2. A function $f(z)$ given by 281 on $\mathbb{U}$ is said to be in the class $\check{\mathcal{S}}_\sigma^*(\alpha)$, if the following conditions are satisfied:

$$Re\,\frac{f(z)}{zf'(z)} > \alpha,$$

and

$$Re\,\frac{g(w)}{wg'(w)} > \alpha,$$

where $w = f(z), \quad g = f^{-1}, \quad 0 \leq \alpha < 1$.

Definition 2.3. A function $f(z)$ given by 281 on $\mathbb{U}$ is said to be in the class $k - U\check{\mathcal{S}}^*$, if the following condition is satisfied:

$$Re\,\frac{f(z)}{zf'(z)} > k|\frac{f(z)}{zf'(z)} - 1|,$$

where $z \in \mathbb{U}, \quad k \in [0, \infty)$.
Note that for $k = 1$, $k - U\check{\mathcal{S}}^* = U\check{\mathcal{S}}^*$, the class of uniformly almost starlike functions.

Theorem 2.4. *Suppose f is a normalized univalent function on $\mathbb{U}$, f be in the class $\check{\mathcal{S}}_\sigma^*(\alpha)$, if and only if the following conditions hold:*

$$i : f(z,t) = e^{\frac{1}{1-\alpha}t} f(e^{\frac{\alpha}{\alpha-1}t} z),\ \textit{is a Loewner chain},\ (0 \leq \alpha < 1, z \in \mathbb{U}, t \geq 0.),$$
$$ii : g(w,t) = e^{\frac{1}{1-\alpha}t} g(e^{\frac{\alpha}{\alpha-1}t} w),\ \textit{is a Loewner chain},\ (0 \leq \alpha < 1, w \in \mathbb{U}, t \geq 0.),$$

where $g = f^{-1}$.

Corollary 2.5. *Suppose f is a normalized univalent function on $\mathbb{U}$, f be in the class $\check{\mathcal{S}}_\sigma^*$, if and only if the following conditions hold:*

$$i : f(z,t) = e^t f(z),\ \textit{is a Loewner chain},\ (z \in \mathbb{U}, t \geq 0.),$$
$$ii : g(w,t) = e^t g(w),\ \textit{is a Loewner chain},\ (, w \in \mathbb{U}, t \geq 0.),$$

where $g = f^{-1}$.

Theorem 2.6. *Suppose f is a normalized univalent function on $\mathbb{U}$, f be in the class $k - U\check{\mathcal{S}}^*$, if and only if,*

$$f(z,t) = e^{\frac{1}{1-\beta}t} f(e^{\frac{\beta}{\beta-1}t} z), (z \in \mathbb{U}, t \geq 0.),$$

is Loewner chain, where $\beta = k|\frac{f(z)}{zf'(z)} - 1| \leq 1$.

3 Conclusion

The investigation of the subclasses of univalent functions by using the method of Loewner chains has been the main interest for researchers to study the Univalent and bi-Univalent classes. Many studies related to this problem are around analytic normalized functions. Here, we introduce new subclasses of the univalent functions and study the classes by using of the Loewner chains method In future research, we give a characterization for classes of functions which are defined in terms of the quantum derivative operators.

References

[1] P.L. Duren, *Univalent functions*, Grundlehern der Mathematischen Wissenschaften, Springer, New York, (1983).

[2] I. Graham, G. Kohr, *Geometric function theory in one and higher dimensions*, Marcel Dekker, New York, (2003).

[3] S. Kanas, A. Wisniowska, *Conic regions and k-uniform convexity II*, Folia. Sci. Univ. Tech. Resov., 170, (1998), 65-78.

[4] M. S. Robertson, *On the theory of univalent functions*, Ann. Math., 37, (1936), 374-408.

[5] H. M. Srivastava,A. K. Mishra, P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*,, Appl. Math. Lett, 23, 10, (2010) 1188-1192.

[6] Q.- H. Xu, T.-S. Liu, *Loewner chains and a subclass of biholomorphic mappings*, J. Math. Anal. Appl., 334, (2007), 10961105.

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Self-localized fusion frames

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Abstract

Fusion frames are a very active area of research today because of their myriad of applications in pure mathematics, applied mathematics, engineering,medicine, signal and image processing and much more. Polynomially and exponentially localized frames with respect to a Riesz basis and also self-localized frames studied for improving the efficiency of frames. In this note, we define the concept of self-localized fusion frame and we show that the canonical dual fusion frame possesses the same intrinsic localization as the original fusion frame. We investigate the conditions under which a fusion frame is self-localized.

Keywords: frame, localized frame, self-localized, fusion frame
Mathematics Subject Classification [2023]: 42C15

1 Introduction

A new direction in frame theory was launched by Grochenig under the name of "localization of frames". One of the motivations was to find a way to describe what could be called "good frames" . Whether a frame is good or not clearly depends on the context , but one of the typical requirements is that the canonical dual frame inherits attractive properties from the frame itself. Frames were developed as a powerful tool in signal processing, coding theory, sampling theory, communication theory, and many other fields. In the real life, the new applications of frames motivated the researchers to generalize frame theory to "frame of subspaces" which introduced by P.G. Casazza and G. Kutyniok and after renamed to "fusion frames" [1, 2]. Actually the fusion frames is the development of simulating the structure of sensor networks. Sensors in any process have severe restrictions and restrictions, which are often measured and checked carefully to have maximum power. Sensors in each network are simulated as frame vectors to form the frame for a subspace in the Hilbert space. In fact, the subspaces are used as reconstructed frames, and this local reconstruction is used as a background for fusion frames to completely reconstruct the original signal. And also in the discussion of positioning a frame, which is a new criterion for the quality of frames, it has created a new path in the theory of frames. In fact, frames provide non-orthogonal and redundant expansions in Hilbert space and are useful whenever the redundancy of a fact is real. In the following, we will state the main definition of frame, frame self-localization, and frame fusion, and we will provide a new definition for self-localized frame fusion and show that conventional double fusion frame has the same localization. Throughout, $\mathcal{H}$ is a separable finite or infinite dimensional Hilbert space, $\mathbf{I}$ an index set which may finite or countable set. For more treatment on frame theory see[5].

Definition 1.1. A family of vectors $\mathcal{F} = \{f_i\}$ is called a frame for $\mathcal{H}$ if there exist constants such that $0 < A \leq B < \infty$

$$A \parallel f \parallel^2 \leq \sum_{i \in \mathbf{I}} \mid \langle f, f_i \rangle \mid^2 \leq B \parallel f \parallel^2 \qquad for\ all \qquad f \in \mathcal{H}$$

The constant A and B are called frame bounds. If we only have the right-hand inequality, we call $\mathcal{F}$ a Bessel sequence.

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The representation space associated with a frame is $l^2(\mathbf{I})$. If $\mathcal{F} = \{f_i\}_{i\in\mathbf{I}}$ is a Bessel sequence, the synthesis operator for $\mathcal{F}$ is the bounded linear operator $T_F : l^2(\mathbf{I}) \to \mathcal{H}$, given by $T_F(\{c_i\}_{i\in\mathbf{I}}) = \sum_{i\in\mathbf{I}} c_i f_i$. The analysis operator for $\mathcal{F}$ is $T_F^* : \mathcal{H} \to l^2(\mathbf{I})$ and satisfies $T_F^* f = \{\langle f, f_i\rangle\}_i \in\in \mathbf{I}$. By composing T_F and T_F^* we obtain the frame operator

$$S_F : \mathcal{H} \to \mathcal{H}, \qquad S_F f = T_F T_F^* f = \sum_{i\in\mathbf{I}} \langle f, f_i\rangle f_i,$$

which is a positive, self-adjoint and invertible operator and the following reconstruction formula holds for all $f \in \mathcal{H}$:

$$f = \sum_{i\in\mathbf{I}} \langle f, \tilde{f}_i\rangle f_i = \sum_{i\in\mathbf{I}} \langle f, f_i\rangle f_i \tilde{f}_i,$$

where $\tilde{f}_i = S_F^{-1} f_i$ $(i \in I)$. Also $\tilde{F} = \{\tilde{f}_i\}_{i\in I}$ is a frame for $\mathcal{H}$ and called the canonical dual frame of $\mathcal{F} = \{f_i\}_{i\in I}$.

We first define the concept for localization of sets in a Hilbert space. We refer for the general theory of frames and Riesz bases and to [2, 3].

Definition 1.2. The frame $\{f_k\}_{k=1}^{\infty}$ is called self-localized with decay rate $s > 1$ if there exists a constant $C > 0$ such that

$$| \langle f_k, f_j\rangle | \leq C(1+ | k - j |)^{-s}, \forall k, j \in \mathbb{N}.$$

Definition 1.3. Let $\{W_i\}_{i\in I}$ be a sequence of closed subspaces of $\mathcal{H}$, and let $\{v_i\}_{i\in I} \in \mathcal{L}^{\infty}(\mathbf{I})$ such that $v_i > 0$ for every $i \in \mathbf{I}$. The family $\{(\mathcal{W}_i, v_i)\}_{i\in\mathbf{I}}$is said a fusion frame for $\mathcal{H}$ if there exist numbers $0 < C \leq D < \infty$ such that

$$C \parallel f \parallel^2 \leq \sum_{i\in\mathbf{I}} v_i^2 \parallel P_{W_i}(f) \parallel^2 \leq D \parallel f \parallel^2 \qquad \forall f \in \mathcal{H},$$

where P_{W_i} is the orthogonal projection on W_i. We call C and D the fusion frame bounds. A fusion frame $\{(\mathcal{W}_i, v_i)\}_{i\in\mathbf{I}}$ is called a tight fusion frame if the constants C and D can be chosen so that $C = D$. If $C = D = 1$, we say that it is a Parseval fusion frame.

Let $\{(W_i, v_i)\}_{i\in\mathbf{I}}$ be a fusion frame for $\mathcal{H}$ and let

$$\left(\sum_{i\in\mathbf{I}} \oplus \mathcal{W}_i\right)_{l^2} = \left\{\{f_i\}_{i\in\mathbf{I}} | f_i \in \mathcal{W}_i, \forall i \in I, \sum_{i\in\mathbf{I}} \parallel f_i \parallel^2 \leq \infty\right\}$$

The analysis operator of $\{(\mathcal{W}_i, v_i)\}_{i\in\mathbf{I}}$ is defined by

$$T_{\mathcal{W}} : \mathcal{H} \longmapsto \left(\sum_{i\in\mathbf{I}} \oplus \mathcal{W}_i\right)_{l^2}, \; T_{\mathcal{W}}(f) = \{v_i P_{W_i}(f)\}_{i\in\mathbf{I}}.$$

The synthesis operator of $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ is defined by

$$T_{\mathcal{W}}^* : \left(\sum_{i\in\mathbf{I}} \oplus \mathcal{W}_i\right)_{l^2} \to \mathcal{H}, \; T_{\mathcal{W}}^*(\{f_i\}) = \sum_{i\in\mathbf{I}} v_i P_{W_i}(f_i).$$

The operator

$$S_{\mathcal{W}} : \mathcal{H} \longmapsto \mathcal{H}, \;\; S_{\mathcal{W}}(f) = T_{\mathcal{W}}^* T_{\mathcal{W}}(f) = \sum_{i\in I} v_i^2 P_{\mathcal{W}_i}(f).$$

is called the fusion frame operator of $\{(\mathcal{W}_i, v_i)\}_{i\in\mathbf{I}}$.

2 Localized fusion frames

Improving and extending of the notion of localized frames, in this section we define the concept of localization for fusion frame and we show some of its properties.

Definition 2.1. Let $\{(w_i, v_i)\}_{i\in\mathbf{I}}$ is a fusion frame for $\mathcal{H}$ then it is called a self-local fusion frame if :

$$K_{ij} = v_i^2 v_j^2 trace(P_{w_i} P_{w_j}) \leq C(1+ \mid i-j \mid)^{-2s}$$

for any $i, j \in \mathbf{I}$.

The following proposition shows that if a frame is self-localized the obtained fusion frame is also is self-localized.

Proposition 2.2. *Let $\{f_k\}_{k=1}^{\infty}$ be a self-localized frame and also $v_i = \| f_i \|, W_i = \langle f_i \rangle$ then $\{(W_i, v_i)\}_{i\in\mathbf{I}}$ is a self-localized fusion frame .*

It is known that a self-localized frame is invariant under the action of an invertible operator. The following theorem shows that the image of a localized fusion frame under an invertible operator is still a fusion frame and localized.

Theorem 2.3. *Let $\{(\mathcal{W}_i, v_i)\}_{i\in\mathbf{I}}$ be a self-localized fusion frame for $\mathcal{H}$ with associated fusion frame operator S_W and let U be an invertible operator on $\mathcal{H}$ satisfying $U^*U(W_i) \subset W_i$ for all $i \in \mathbf{I}$, then $\{(UW_i, v_i)\}_{i\in I}$ is a self-localized fusion frame for $\mathcal{H}$ with fusion frame operator US_WU^{-1}.*

References

[1] P.G. Casazza and G. Kutyniok, *Frames of subspaces," wavelets, frames and operator theory*, (College Park, MD, 2003), Contemp. Math. , Amer. Math. Soc., Providence, RI, 345, (2004), 87–113.

[2] P. G. Casazza , G. Kutyniok and Sh. Li, *Fusion frame and distributed processing*, Appl. Comput. Harmon. Anal. 25 (2008) 114–132.

[3] O. Christensen, *An introduction to frames and Riesz Basis,* Birkhäuser, Boston, 2016.

[4] K. Grochenig, *Localization of Frames, Banach Frames, and the Invertibility of the Frame Operator.* J. Fourier Anal. Appl. 10, (2004), 105–132.

[5] K. Grochenig, *Localized frames are finite unions of Riesz sequences*, Adv. Comp. Math. 18, (2003), 149–157.

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Self-localized fusion frames

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Abstract

One of the major problems in quantum detection problem is the injectivity problem which studied via frame theory. It has shown that a quantum injective frame is a frame that can be used to distinguish density operators (states) from their frame measurements, and the frame quantum detection problem asks to characterize all such frames. In this note, we will answer to some of the problems in quantum detection using g-frames.

Keywords: Quantum detection, frames, g-frames)
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0.1 Frames and g-frames in Hilbert spaces

Frames in Hilbert spaces were first introduced by Duffin and Schaeffer to deal with nonharmonic Fourier series in 1952 [4] and widely studied from 1986 since the great work by Daubechies, Grossmann and Meyer. Frames are basis- like building blocks that span a vector space but allow for linear dependency, which is useful to reduce noise, find sparse representations, spherical codes, compressed sensing, signal processing, wavelet analysis etc.

Throughout this paper, $\mathcal{H}$ is a separable Hilbert space, $\{\mathcal{H}\}_{i\in\mathbb{I}}$ a family of Hilbert spaces, $\mathbb{I}$ a finite or countable index set, $\mathcal{B}(\mathcal{H})$ the C^*-algebra of bounded operators on $\mathcal{H}$ and $\{e_i\}_{i\in\mathbb{I}}$ is an orthonormal basis (ONB) for $\mathcal{H}$. The operator T is a Hilbert-Schmidt operator if $\|T\|_2 := (\sum_i \|Te_i\|^2)^{\frac{1}{2}} < \infty$ and it is a trace class operator if $\|T\|_1 := \sum_i \langle |T|e_i, e_i\rangle < \infty$, in this case the trace of T is $tr(T) = \sum_i \langle Te_i, e_i\rangle$ which is finite and independent of the orthonormal basis. It is known that trace induces an inner product by $\langle T, S\rangle_{HS} = tr(TS^*)$ for the Hilbert -Schmidt operators T, S. Let $Sym(\mathcal{H}) := \{T : T \in \mathcal{B}(\mathcal{H}), T = T^*\}$ denote the real Banach space of self-adjoint operators on $\mathcal{H}$ and $Sym^+(\mathcal{H}) := \{T \in Sym(\mathcal{H}), T = T^* \geq 0\}$. The family of trace class operators on $\mathcal{H}$ denoted by $\mathcal{S}_1$ and Hilbert-Schmidt by $\mathcal{S}_2$.

Definition 0.4. A countable family of elements $\{f_i\}_{i\in\mathbb{I}}$ in $\mathcal{H}$ is a frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that:

$$A\|x\|^2 \leq \sum_{i\in\mathbb{I}} |\langle x, f_i\rangle|^2 \leq B\|x\|^2, \quad \forall x \in \mathcal{H}. \tag{280}$$

The numbers A and B are celled the lower and upper frame bounds, respectively. The frame $\{f_i\}_{i\in\mathbb{I}}$ is called tight, if $A = B$ and is called Parseval, if $A = B = 1$. Also the sequence $\{f_i\}_{i\in\mathbb{I}}$ is called Bessel sequence, if the upper inequality in (357) holds. A Riesz basis for $\mathcal{H}$ is a family of the form $\{Ve_i\}_{i\in\mathbb{I}}$, where $\{e_i\}_{i\in\mathbb{I}}$ is an orthonormal basis for $\mathcal{H}$ and the operator $V : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded and invertible operator.

For the Bessel sequence $\{f_i\}_{i\in\mathbb{I}}$, the analysis operator $T : \ell^2(\mathbb{I}) \longrightarrow \mathcal{H}$ is defined by $T(x) = \{\langle x, f_i\rangle\}_{i\in\mathbb{I}}$, for all $x \in \mathcal{H}$. Its adjoint operator $T^* : \mathcal{H} \to \ell^2(\mathbb{I})$ called synthesis operator and $T\{c_i\} = \sum_{i\in\mathbb{I}} c_i f_i$, for all $\{c_i\} \in \ell^2$. The operator $S : \mathcal{H} \longrightarrow \mathcal{H}$, which is defined by $S(x) = TT^*(x) = \sum_{i\in\mathbb{I}} \langle x, f_i\rangle f_i$, for all $x \in \mathcal{H}$, is called the frame operator. For a frame $\{f_i\}_{i\in\mathbb{I}}$, the operator T^* is onto, T is one-to-one and S is positive, self adjoint and invertible.

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A dual of the Bessel sequence $\{f_i\}_{i\in\mathbb{I}} \subseteq \mathcal{H}$ is a Bessel sequence $\{g_i\}_{i\in\mathbb{I}}$ in $\mathcal{H}$, such that

$$x = \sum_{i\in\mathbb{I}} \langle x, g_i\rangle\, f_i, \quad x \in \mathcal{H}.$$

For a frame $\{f_i\}_{i\in\mathbb{I}}$, the sequence $\{S^{-1}f_i\}_{i\in\mathbb{I}}$ is a frame which is called the canonical dual of $\{f_i\}_{i\in\mathbb{I}}$. For a more comprehensive treatment of frame theory, we highly recommend the excellent book by Christensen [5].

Over the years, various extensions of the frame theory have been investigated. Several of these are contained as special cases of the elegant theory for g-frames that was introduced by W. Sun in [6]. For example, one can consider: bounded quasi-projectors, fusion frames, pseudo-frames, oblique frames, outer frames and etc.

Definition 0.5. We call $\Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) \ : \ i \in \mathbb{I}\}$ a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$, or simply, a g-frame for $\mathcal{H}$, if there exist two positive constants A, B such that

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

The positive numbers A and B are called the lower and upper g-frame bounds, respectively. We call Λ a tight g-frame if $A = B$ and we call it a Parseval g-frame if $A = B = 1$. If only the second inequality holds, we call it a g-Bessel sequence. If Λ is a g-frame, then the g-frame operator S_Λ is defined by

$$S_\Lambda f = \sum_{i\in\mathbb{I}} \Lambda_i^* \Lambda_i f, \quad f \in \mathcal{H}$$

which is a bounded, positive and invertible operator such that

$$AI \leq S_\Lambda \leq BI$$

and for each $f \in \mathcal{H}$, we have

$$f = S_\Lambda S_\Lambda^{-1} f = S_\Lambda^{-1} S_\Lambda f = \sum_{i\in I} S_\Lambda^{-1} \Lambda_i^* \Lambda_i f = \sum_{i\in I} \Lambda_i^* \Lambda_i S_\Lambda^{-1} f.$$

The canonical dual g-frame for Λ is defined by $\{\Lambda_i S_\Lambda^{-1}\}_{i\in\mathbb{I}}$ with bounds $\frac{1}{B}$, $\frac{1}{A}$. In other words, $\{\Lambda_i S_\Lambda^{-1}\}_{i\in\mathbb{I}}$ and $\{\Lambda_i\}_{i\in\mathbb{I}}$ are dual g-frames with respect to each other.

0.2 Positive Operator-Valued Measures (POVM) and Quantum Detection Problem

Quantum theory tries to predict the probability of observing outcomes from a sequence of measurements of the system in an unknown state. This process is called quantum state tomography. The outcome statistics are described by a positive operator-valued measure (POVM). In fact, quantum measurement extracts the transmitted information from received quantum signals and therefore performs an important role of quantum communications, [7]. The simplest quantum measurement is the projection-valued measure (PVM), also called standard measurement or von Neumann measurement, where elementary projectors are usually used. Sometimes the positive-operator valued measure (POVM) is more efficient of obtaining information about the state of a quantum system than a standard measurement. Let X denote a set of outcomes (e.g. this could be a finite or infinite subset of $\mathbb{Z}^d$ or $\mathbb{R}^d$). Let β denote a sigma algebra of subsets of X.

Definition 0.6. A positive operator-valued measure (POVM) is a function $\Pi : \beta \to Sym^+(\mathcal{H})$ satisfying:

1. $\Pi(\phi) = 0$.
2. For every at most countable disjoint family $\{V_i\} \subset \beta$ and $x, y \in \mathcal{H}$ we have $\langle \Pi(\bigcup V_i)x, y\rangle = \sum_i \langle \Pi(V_i)x, y\rangle$.
3. $\Pi(X) = I$ (the identity operator).

A quantum system is defined as a von Neumann algebra $\mathcal{A}$ of operators acting on $\mathcal{H}$. The set of states on $\mathcal{H}$ is $\mathbb{S}(\mathcal{H}) := \{T \in \mathcal{S}_1, T = T^* \geq 0, tr(T) = 1\}$. It is known that [1], the set of quantum states are in one-to-one correspondence with the linear functionals on $\mathcal{A}$ of the form:

$$\rho : \mathcal{A} \to \mathbb{C}, \quad for \quad S \in \mathbb{S}(\mathcal{H}), \rho(T) = tr(ST), \forall T \in \mathcal{A}.$$

Let $L(\beta, \mathbb{R})$ denote the set of real-valued bounded functions defined on β. Given a POVM Π associated to a von Neumann algebra $\mathcal{A}$, the quantum detection problem asks if there is a unique quantum state $\rho \in \mathbb{S}(\mathcal{A})$ compatible with the set of quantum measurements performed by the POVM Π? The quantum detection problem asks two questions:

1. Is the following map injective $M : \mathbb{S}(\mathcal{A}) \to L(\beta, \mathbb{R}), M(\rho)(U) = \rho(\Pi(U))$? (Injectivity Problem)
2. Range analysis, or state estimation: Assume M is injective. Then, given a map $p \in L(\beta, \mathbb{R})$, determine if p is in the range of M, hence is of the form $p = M(\rho)$ for some unique $\rho \in \mathbb{S}(\mathcal{A})$.

POVMs have a natural, and very valuable, subset which comes from Hilbert space frame theory.

1 Injectivity Problem via frames

The quantum detection problem with discrete frame coefficient measurements was recently settled by Botelho-Andrade et al. for both finite and infinite dimensional Hilbert spaces in [1, 2], where the characterization was given in terms the spanning properties of some derived sequences from the frame vectors. It is natural to study this problem and results for frame's extensions. For continuous frames it has been investigated by Deguang Han and et.al. [3]. In this note, we will study g-frames which follow the similar results for fusion frames.

Definition 1.1. A family of vectors $\mathcal{X} = \{x_k\}_{k\in\mathbb{I}}$ in a Hilbert space $\mathcal{H}$ is said to be injective if given a Hilbert- Schmidt self-adjoint operator T satisfying $\langle Tx_k, x_k\rangle = 0$ for all $k \in \mathbb{I}$, then $T = 0$.

It is known [1] that if a family of vectors gives injectivity in a Hilbert space $\mathcal{H}^n$, then it is a frame for $\mathcal{H}^n$ and in case $\{x_k\}_{k\in\mathbb{I}}$ a frame for $\mathcal{H}$ which gives injectivity. If F is a bounded invertible operator on $\mathcal{H}$, then $\{Fx_k\}_{k\in\mathbb{I}}$ also gives injectivity.

Definition 1.2. Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) \ : \ i \in \mathbb{I}\}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ for $\mathcal{H}$, where the bounded operators $\Lambda_i, i \in \mathbb{I}$ are Hilbert-Schmidt operators. Λ is said to be injective if given a Hilbert-Schmidt self-adjoint operator T satisfying $\langle \Lambda_i T, \Lambda_i\rangle_{\mathcal{HS}} = 0$ for all $k \in \mathbb{I}$, then $T = 0$.

Theorem 1.3. *Let $\Lambda = \{\Lambda_i : \mathcal{H} \to \mathcal{H}_i, i \in \mathbb{I}\}$ be g-frame for $\mathcal{H}$. The following statements are equivalent.*

1. *Whenever T_1, T_2 are Hilbert-Schmidt, positive, self adjoint and $\langle \Lambda_i T_1, \Lambda_i\rangle_{HS} = \langle \Lambda_i T_2, \Lambda_i\rangle_{HS}$ for all $i \in \mathbb{I}$, then $T_1 = T_2$.*
2. *Whenever T_1, T_2 are Hilbert-Schmidt self adjoint and $\langle \Lambda_i T_1, \Lambda_i\rangle_{HS} = \langle \Lambda_i T_2, \Lambda_i\rangle_{HS}$ for all $i \in \mathbb{I}$, then $T_1 = T_2$.*
3. *$\Lambda = \{\Lambda_i : \mathcal{H} \to \mathcal{H}_i, i \in \mathbb{I}\}$ is injective.*

References

[1] S. Botelho-Andrade, P. G. Casazza, D. Cheng and Tin T. Tran, *The Solution to the Frame Quantum Detection Problem*, J. Fourier. Anal. Appl. **25**(2019) 2268-2323.

[2] S. Botelho-Andrade, P. G. Casazza, D. Cheng and Tin T. Tran, *The quantum detection problem: a survey*. Springer Proc. Math. Stat. **255**(2017) 337-352.

[3] D. Han, Q. Hu and R. Liu, *Injective continuous frames and quantum detections*, Banach J. Math. Anal. **15**:12(2021) https://doi.org/10.1007/s43037-020-00086-7

[4] R. J. Duffin, A. C. Schaeffer, *A class of nonharmonic Fourier series*, J. Trans. Amer. Math. Soc., **72**(1952) 341-366 .

[5] O. Christensen, *An introduction to frames and Riesz Basis,* Birkhäuser, Boston, 2016.

[6] W. Sun, *G-frames and g-Riesz bases*, *J. Math. Anal. Appl.*, **322(1)**, (2006) 437-452.

[7] M. Paris and J. Eehacek (Eds) *Quantum state estimation*, Springer-Verlag, Berlin , 2004.

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Critical local equation with variable exponent

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Abstract

This paper is devoted with the existence and multiplicity of solutions of parametric of a local equation with variable exponent and Dirichlet boundary condition. The approach is based on the variational method.

Keywords: $p(x)$-Laplacian, genus theory, Sobolev space
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1 Introduction

During the last two decades p(x)-Laplacian problems have deserved the attention of many researchers.

In [1] studied the Kierchhoff type equation

$$-M(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)\triangle_{p(x)}u = f(x,u) \qquad in\ \Omega,$$

was studied with Dirichlet boundary conditions

$$u = 0 \quad on \ \ \partial\Omega.$$

The operator $-div[\phi(x,|\nabla u|)\nabla u]$ was studied by I. H. Kim and Y. H. Kim in [2], by many authors after them.

Let $\Omega \subseteq \mathbb{R}^N$ be abounded domain with smooth enough boundary.

For the more strictly case when $\phi(x,t) = |t|^{p(x)-2}$, the operator reduces to the p(x)- Laplacian, i.e., this operator was

$$-\Delta_{p(x)}u = -div(|\nabla u|^{p(x)-2}\nabla u)$$

For more details we refer to [4] and [3].

Our main result is focuses on the study of solutions for the following boundary value problem:

$$\begin{cases} -div[\phi(x,|\nabla u|)\nabla u] = \lambda(g(x)|u|^{q(x)-2})u & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases} \tag{281}$$

Let λ be a positive real parameter and $\phi(x,t)$ is, and $|t|^{p(x)-2}$, where p and s are continuous functions on $\bar{\Omega}$ in which

$$2 \le p(x) < q(x) < p^*(x), \tag{282}$$

where $p^*(x) = \dfrac{Np(x)}{N-p(x)}$ and $p(x) < N$ for all $x \in \bar{\Omega}$.

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2 Preliminaries

As usual Ω is a bounded domain in $\mathbb{R}^N$ and

$$C_+(\Omega) = \{p \in C(\bar{\Omega}) : \min_{x\in\Omega} p(x) > 1\}$$

For any (Lebesgue) continuous function $p : \Omega \to (1, \infty)$, set For any $p \in C_+(\bar{\Omega})$ the associated *variable exponent Lebesgue space* is defined by

$$L^{p(x)}(\Omega) := \{u : \Omega \to \mathbb{R} \ \ measurable \ \ function \ \ \int_\Omega |u|^{p(x)}dx < \infty\}$$

which is equipped with the so called Luxemburg norm:

$$|u|_{p(x)} := \inf\left\{\mu > 0 : \ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}$$

It is well known that $L^{p(x)}(\Omega)$ is a Banach space.[3]

Proposition 2.1. *[5] $(L^{p(x)}(\Omega), |.|_{p(x)})$ is a separable, uniformly convex, reflexive space and its conjugate space is $L^{q(x)}(\Omega)$, where $q(x)$ is the conjugate function of $p(x)$, i.e.,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad \forall x \in \Omega$$

Moreover, for any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ the Holder's type inequality

$$|\int_\Omega uvdx| \le (\frac{1}{p^-} + \frac{1}{q^-})|u|_{p(x)}|v|_{q(x)} \le 2|u|_{p(x)}|v|_{q(x)} \tag{283}$$

holds.

The modular of the $L^{p(x)}(\Omega)$ space is defined by the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$,

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}dx,$$

has an important role in manipulating the generalized Lebesgue spaces.

If $p(x)$ is constant on $\Omega, u \in L^{p(x)}(\Omega)$ and $(u_n) \subseteq L^{p(x)}_{(\Omega)}$ then the following relations hold:

$$|u|_{p(x)} < 1 \Rightarrow |u|^{p^+}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^-}_{p(x)}, \tag{284}$$

$$|u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^+}_{p(x)}, \tag{285}$$

$$|u|_{p(x)} = 1 \Rightarrow \rho_{p(x)}(u) = 1, \tag{286}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \rho_{p(x)}(u_n - u) \to 0. \tag{287}$$

We have

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : \ \ |\nabla u| \in L^{p(x)}(\Omega)\right\}.$$

On $W^{1,p(x)}(\Omega)$ we may consider the following equivalent norms:

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}$$

and

$$\|u\| = \inf\{\mu > 0 : \quad \int_\Omega (\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} dx \le 1\}.$$

$W^{1,p(x)}_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the norm $\|.\|_{p(x)}$.

It is well known that

$$W^{1,p(x)}_0(\Omega) = \left\{u; \ u\Big|_{\partial\Omega} = 0, \ u \in L^{p(x)}(\Omega), \ |\nabla u| \in L^{p(x)}(\Omega)\right\}.$$

Remark 2.2. [5] If $p, q : \Omega \to (1, \infty)$ are Lipschitz continuous $p^+ < N$ and $p(x) \leq q(x) \leq p^+(x)$, for any $x \in \Omega$, where $p^*(x) = \dfrac{Np(x)}{N - p(x)}$, the embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is continuous and compact.

Taking account of [2] for $p \in C^+(\bar{\Omega})$, the $p(.)-$ Poincare type inequality valid

$$|u|_{p(x)} \leq C\|\nabla u\|_{p(x)} \tag{288}$$

where $C > 0$ is a constant which depends on p and Ω.

For a bounded domain $\Omega \subset \mathbb{R}^N$ and p as a global Log-Holder continuous function, on $W_0^{1,p(x)}(\Omega)$ we can use $|\nabla u|_{p(x)}$ instead of $\|u\|_{p(x)}$.

In the next section, we recall an important result concerning the embedding between $W_0^{1,p(x)}(\Omega)$ and $L^{q(x)}(\Omega)$.

3 Main results

In order to state more precisely our result we consider the following conditions:

(H_1) $\phi : \Omega \times [0, \infty) \to [0, \infty)$ fulfills the following assumptions: $\phi(., t)$ is measurable on Ω for all $t \geq 0$ and $\omega(x, .)$ is locally absolutely continuous on $[0, \infty)$ for almost all $x \in \Omega$.

(H_2) $\phi : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$c_1 t^{s(x)-2} \leq \phi(x, t) \leq c_2 t^{q(x)-2}$$

for all $(t, x) \in]0, +\infty[\times\bar{\Omega}$, where c_1, c_2 are positive constants and $s, q \in C(\bar{\Omega})$ such that $1 < s(x) < q(x) < p^*(x) < \dfrac{Np(x)}{N - p(x)}$ for all $x \in \bar{\Omega}$.

Definition 3.1. Let E be a real Banach space and A be a symmetric subset of $\mathbb{R}^n \backslash \{0\}$ which is closed in E.

Set $\mathcal{R} = \{A \subset E - \{0\};\ A \textit{ is compact and } A = -A\}$. Let $A \subset \mathcal{R}$ and we define the *genus* of A as,

$$\gamma(A) = \inf \left\{m;\ \exists f \in C^0(A, \mathbb{R}^m \backslash \{0\});\ f(-u) = f(u)\right\}$$

and $\gamma(A) = \infty$ if does not exist such a map f.

It is clear that $\gamma(\phi) = 0$.

Moreovere if A is a subset, which consists of finitely many pairs of points, then $\gamma(A) = 1$.

Definition 3.2. $u \in W_0^{1,p(x)}(\Omega)$ is called a weak solution for (281) if:

$$\int_\Omega \phi(x, |\nabla u(x)|)\nabla u(x).\nabla v(x)dx = \lambda \int_\Omega g(x)|u(x)|^{q(x)-2}u(x)v(x)dx$$

for all $v \in W_0^{1,p(x)}(\Omega)$. In what follows

$$\Phi_0(x, s) = \int_0^s \phi(x, t)tdt,$$

and we define the functional

$$\Phi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$$

by

$$\Phi(u) = \int_\Omega \Phi_0(x, |\nabla u(x)|)dx.$$

Lemma 3.3. *[8] Assume that* (H_1) *and* (H_2) *hold. Then* $\Phi \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ *and its Frechet derivative is given by*

$$\langle \Phi'(u), v\rangle = \int_\Omega \phi(x, |\nabla u(x)|)\nabla u(x).v(x)dx.$$

Lemma 3.4. *[8] Assume that* (H_1) *-* (H_2) *hold. Then the operator* $\Phi' : W_0^{1,p(x)}(\Omega) \to W_0^{1,q(x)}(\Omega)$ *is a strictly monotone on* $W_0^{1,p(x)}(\Omega)$ *and a mapping of type* (S_+)*, i.e.,*

if $u_n \rightharpoonup u$ *in* $W_0^{1,p(x)}(\Omega)$ *as* $n \to \infty$ *and* $\limsup_{n\to\infty}\langle \Phi'(u_n) - \Phi(u), u_n - u\rangle \le 0$ *then* $u_n \to u$ *in* $W_0^{1,p(x)}(\Omega)$ *as* $n \to \infty$ *.*

Denote the following functional:

$$J(u) = \int_\Omega \Phi_0(x, |\nabla u|)dx - \lambda \int_\Omega \frac{g(x)}{q(x)}|u|^{q(x)}dx$$

the energy functional.

Using the properties founded in [2] and in the Chapter 3, section 3.2 of [5], $J(u)$ is well defined on $W_0^{1,p(x)}(\Omega)$, a C^1 functional and for every $v \in W_0^{1,p(x)}(\Omega)$

$$\begin{aligned} J'(u)(v) = &\int_\Omega \phi(x, |\nabla u(x)|)\nabla u(x).\nabla v(x)dx \\ &- \lambda \int_\Omega g(x)|u(x)|^{q(x)-2}u(x).v(x)dx. \end{aligned}$$

Therefore, critical points of this energy functional are weak solutions for the problem (281).

We consider $\Omega \subset \mathbb{R}^N (N > 3)$ as a bounded domain with a smooth boundary, $\lambda > 0$ is a real number and $p, q, r \in C_+(\Omega)$ such that:

$$1 < p^- \le p(x) \le p^+ < q^- \le q(x) \le q^+ < p^*(x) = \frac{Np(x)}{N - p(x)} \tag{289}$$

and $p(x) < N$ for all $x \in \bar{\Omega}$.

Furthermore, we assume that the function $g(x)$ satisfies the hypotheses:

(g_1) $g : \bar{\Omega} \to [0, \infty)$, $g \in L^\infty(\bar{\Omega})$.

Definition 3.5. The functional J satisfies the Palais- Smale condition (PS) if for every sequence $(u_n) \subset Y$ which is

$$|J(u_n)| \le C \quad and \quad J'(u_n) \to 0 \quad as \quad n \to \infty,$$

there is a convergence subsequence of (u_n).

The following result obtianed by Clark [2] is the main idea.

Theorem 3.6. *[2] Let* $J \in C^1(W_0^{1,p(x)}, \mathbb{R})$ *and satisfies in* (PS) *condition. We assume that following conditions*

(i) J *is bounded from below and even;*

(ii) There is a compact set $T \in R$ *such that* $\gamma(T) = k$ *and* $\sup_{x\in T} J(x) < J(0)$*, holds.*

Then J *possesses at least* k *pairs of distinct critical points, and their corresponding critical values are less than* $J(0)$*.*

Theorem 3.7. *Assuming that* (H_1) *and* (H_2) *hold. If* $p(x) < q(x) < p^*(x)$ *for all* $x \in \bar{\Omega}$*, then there are at least* k *pairs of different critical points for* (281)*.*

Corollary 3.8. *Assuming that* (H_1) *and* (H_2) *hold. If* $p(x) < q(x) < p^*(x)$ *for all* $x \in \bar{\Omega}$*, then there are infinitely solution for* (281)*.*

Lemma 3.9. *We assume that* (H_1) *and* (H_2) *hold. Then* J *is coercive on* X *and bounded from below.*

Lemma 3.10. *Assume that* (H_1) *and* (H_2)*,* (g_1) *and* (458)*, then* J *satisfies the* (PS) *condition for all* $\lambda > 0$*.*

References

[1] M. Avci, B. Cokic and R. Mashiyev, *Existence and multiplicity of the solution of the $p(x)$-Kirchhoff type equation via genus theory,* Math. App. SCI. **(34)** (2011), 1751- 1759.

[2] Dc. Clarke, *A variant of the Lusternik Schnirelman theory.* Indiana University Mathematics Journal **22**(1972) 6514.

[3] I. Dorotheea. Stircu, *An existence result for quasilinear elliptic equations with variable exponents,* Annals of the University of Craiova, Mathematics and Computer Seience Series, Volume **44(2)**. 2017, 299-315.

[4] V. Florin. UTA, *Groud state solution and concestration phenomena in nonlinear eigenvalue problems with variable exponents,* Annals of the University of Craiova, Mathematics and Computer Seience Series, Volume **45(1)**. 2018, 122-136.

[5] I. H. Kim, Y. H. Kim, *Mountation pass type solutions and positivity of the infimum eigenvalue for quasi linear elliptic equations with variable exponents,* Manuscripta Math. **147**(2015), 169-191.

[6] K. Ho, I. Sim, *On degenerate $p(x)$-Laplace equations involving critical growth with two parameters,* Nonlinear Analysis **(132)** (2016), 95- 114.

[7] V. Radulescu, D. Repovs, *Partial Differential Equations with variable Exponents: Variational Methods and Rualitative Analysis,* CRC Press, Taylor Francis Group Boca Raton FL, 2015.

[8] D. Repovs, *Stationary waves of Schrodinger-type equations with variable exponent,* Anal. (Singap) **13** (2015), no. 6, 645-661.

[9] K. Saoudi, M. Kratou, and S. Alsadhan, *Multiplicity Results for the p(x)-Laplacian equation with singular nonlinearities and nonlinear Neumann boundary condition,* Hindawi Publishing Corporation International Journal of Differntial Equations Volume 2016, Article ID3149482, 14 pages.

[10] S. Taarabti, Z. El Allali and K. Ben Haddouch, *Eignvalues of the p(x)-Biharmonic operator with indefinite weight under Neumann boundary conditions,* Bol. Soc. paran. Mat. **(35)** V. 36 1(2018), 195-213.

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Estimation type results for a class of fractional integrals

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Abstract

Some Hermite-Hadamard's mid-point type inequalities related to Katugampola's fractional integrals are given where the first derivative of considered mappings is Lipschitzian or convex. Also some applications of obtained results are presented.

Keywords: Fractional integrals, Hermite-Hadamard's Mid-point type inequalities, Lipschitzian mappings, Convex mappings, Special means
Mathematics Subject Classification [2023]: 26A33, 26A51, 26D10

1 Introduction

Recently U. N. Katugampola in [2], introduced an Erdélyi-Kober type fractional integral operator which now is known as Katugampola fractional integral. The Katugampola fractional integral is a generalization of Riemann-Liouville and Hadamard fractional integrals simultaneously. The following definition is modified version of Definition 4.3 in [2].

Definition 1.1. [1, 3] Let $[a,b] \subset \mathbb{R}$ be a finite interval. The left and right side Katugampola fractional integrals of order $\alpha > 0$ are defined respectively by

$$ {}^{\rho}I^{\alpha}_{a^+} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t)dt, $$

and

$$ {}^{\rho}I^{\alpha}_{b^-} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t)dt, $$

where $a < x < b$, $\rho > 0$, $\Gamma(\alpha)$ is Gamma function and the integrals exist.

2 Mid-Point Type Inequalities

In this section we obtain three Hermite-Hadamard's mid-point type theorems related to Katugampola's fractional integrals by considering the concepts of Lipschitzian and convex mappings. The following lemma is of important to achieve our main results.

Lemma 2.1. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°. For $0 \le a < b$ and $\rho > 0$, suppose that $a^\rho, b^\rho \in I^\circ$ and $f' \in L[a^\rho, b^\rho]$. Then for $\alpha > 0$, the following identities for fractional integrals hold:*

$$ -\rho(b^\rho - a^\rho)\Bigg\{ \int_0^{\frac{1}{\sqrt[\rho]{2}}} t^{(\alpha+1)\rho-1} f'\Big(t^\rho a^\rho + (1-t^\rho)b^\rho\Big)dt \tag{290} $$
$$ + \int_{\frac{1}{\sqrt[\rho]{2}}}^1 (t^{\alpha\rho} - 1)t^{\rho-1} f'\Big(t^\rho a^\rho + (1-t^\rho)b^\rho\Big)dt\Bigg\} $$

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$$= f\Big(\frac{a^\rho + b^\rho}{2}\Big) - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}{}^\rho I^\alpha_{a^+} f(b^\rho),$$

and

$$\begin{aligned}&\rho(b^\rho - a^\rho)\Big\{\int_0^{\frac{1}{\sqrt[\rho]{2}}} t^{(\alpha+1)\rho-1} f'\big(t^\rho b^\rho + (1-t^\rho)a^\rho\big)dt \\ &+ \int_{\frac{1}{\sqrt[\rho]{2}}}^1 (t^{\alpha\rho}-1)t^{\rho-1} f'\big(t^\rho b^\rho + (1-t^\rho)a^\rho\big)dt\Big\} \\ &= f\Big(\frac{a^\rho + b^\rho}{2}\Big) - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}{}^\rho I^\alpha_{b^-} f(a^\rho).\end{aligned} \tag{291}$$

Furthermore

$$\begin{aligned}&\frac{\rho(b^\rho - a^\rho)}{2}\Big\{\int_0^{\frac{1}{\sqrt[\rho]{2}}} t^{(\alpha+1)\rho-1}\Big[f'\big(t^\rho b^\rho + (1-t^\rho)a^\rho\big) - f'\big(t^\rho a^\rho + (1-t^\rho)b^\rho\big)\Big]dt \\ &+ \int_{\frac{1}{\sqrt[\rho]{2}}}^1 (t^{\alpha\rho}-1)t^{\rho-1}\Big[f'\big(t^\rho b^\rho + (1-t^\rho)a^\rho\big) - f'\big(t^\rho a^\rho + (1-t^\rho)b^\rho\big)\Big]dt\Big\} \\ &= f\Big(\frac{a^\rho + b^\rho}{2}\Big) - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\big[{}^\rho I^\alpha_{a^+} f(b^\rho) + {}^\rho I^\alpha_{b^-} f(a^\rho)\big].\end{aligned} \tag{292}$$

Definition 2.2. [5] A function $f : [a,b] \to \mathbb{R}$ is said to satisfy a Lipschitz condition on interval $[a,b]$ (M-Lipschitzian) if there is a constant M so that, for any two points $x, y \in [a,b]$,

$$|f(x) - f(y)| \le M|x - y|.$$

By using Lemma 2.1, we can obtain a mid-point type theorem in the case that first derivative of considered function is Lipschitzian.

Theorem 2.3. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°. For $0 \le a < b$ and $\rho > 0$, suppose that $a^\rho, b^\rho \in I^\circ$ and f' satisfies a Lipschitz condition on $[a^\rho, b^\rho]$ with respect to M. Then for $\alpha > 0$, the following mid-point type inequality holds:*

$$\Big|f\Big(\frac{a^\rho + b^\rho}{2}\Big) - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\big[{}^\rho I^\alpha_{a^+} f\big(b^\rho\big) + {}^\rho I^\alpha_{b^-} f(a^\rho)\big]\Big| \le \frac{M(b^\rho - a^\rho)^2(\alpha^2 - \alpha + 2)}{8(\alpha+1)(\alpha+2)}. \tag{293}$$

Example 2.4. Consider $f(x) = \sin x$, $x \in [a,b]$ with $0 \le a < b$. From the fact that $|\cos x - \cos y| \le |x - y|$, we have that $f'(x) = \cos x$ satisfies a Lipschitz condition with respect to $M = 1$. Then from Theorem 2.3 in a special case we have

$$\Big|\sin\Big(\frac{a+b}{2}\Big) - \frac{\cos a - \cos b}{b - a}\Big| \le \frac{(b-a)^2}{24}.$$

Theorem 2.5. *Suppose that for $\rho > 0$ and $0 \le a < b$, the function $f : [a^\rho, b^\rho] \to \mathbb{R}$ satisfies a Lipschitz condition on $[a^\rho, b^\rho]$ with respect to M. Then the following mid-point type inequality holds:*

$$\begin{aligned}&\Big|f\Big(\frac{a^\rho + b^\rho}{2}\Big) - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}\Big[\Big(\frac{1}{2}\Big)^\alpha {}^\rho I^\alpha_{a^+} f\big(b^\rho\big) + \Big(1 - (\frac{1}{2})^\alpha\Big){}^\rho I^\alpha_{b^-} f(a^\rho)\Big]\Big| \\ &\le \frac{M(b^\rho - a^\rho)\Big((\frac{1}{2})^{\alpha-1} + \alpha - 1\Big)}{2(\alpha+1)}.\end{aligned} \tag{294}$$

Example 2.6. Consider $f(x) = \tan x$, $x \in [\frac{-\pi}{3}, \frac{\pi}{3}]$. For any $a, b \in [\frac{-\pi}{3}, \frac{\pi}{3}]$, there exists $t \in (a, b)$ such that

$$1 + \tan^2 t = \frac{\tan b - \tan a}{b - a},$$

showing that

$$|\tan b - \tan a| \leq 4|b - a|.$$

So for $0 \leq a < b \leq \frac{\pi}{3}$ we have Now if we set $\alpha = 1$, then we get

$$\left| \tan\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a} \ln \frac{\sec b}{\sec a} \right| \leq b - a.$$

Theorem 2.7. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°. For $0 \leq a < b$ and $\rho > 0$, suppose that $|f'|$ is convex and integrable on $[a^\rho, b^\rho]$. Then in the case that $0 < \alpha\rho \leq 1$, the following mid-point type inequality holds:*

$$\left| f\Big(\frac{a^\rho + b^\rho}{2}\Big) - \frac{\alpha\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} \Big[{}^\rho I^\alpha_{a^+} f\big(b^\rho\big) + {}^\rho I^\alpha_{b^-} f(a^\rho) \Big] \right| \leq \frac{b^\rho - a^\rho}{2^{\alpha+1}(\alpha+1)} \Big(|f'(a^\rho)| + |f'(b^\rho)| \Big).$$

3 Special Means

In this section as an application of our results we obtain some generalized inequalities related to two well known special means:

$$A(a,b) = \frac{a+b}{2} \qquad \textit{arithmetic mean},$$

$$L_n(a,b) = \Big[\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \Big]^{\frac{1}{n}} \qquad \textit{generalized log}-\textit{mean}, \ n \in \mathbb{N}, \ a < b.$$

Consider $f(t) = t^n$ for $t \geq 0$, $n \in \mathbb{N}$. Now $M = nb^{(n-1)\rho}$, and so from Theorem 2.5 we have

$$\left| \Big(\frac{a+b}{2}\Big)^n - \frac{\alpha^2}{(b-a)^\alpha} \Big[\Big(\frac{1}{2}\Big)^\alpha \sum_{m=0}^{n} \frac{a^{n-m}(b-a)^{\alpha+m} P(n,m)}{\prod_{i=0}^{m}(\alpha+i)} \right. \tag{295}$$

$$\left. + \Big(1 - (\frac{1}{2})^\alpha\Big) \sum_{m=0}^{n} \frac{(-1)^m b^{n-m}(b-a)^{\alpha+m} P(n,m)}{\prod_{i=0}^{m}(\alpha+i)} \Big] \right| \leq \frac{nb^{(n-1)}(b-a)\Big((\frac{1}{2})^{\alpha-1} + \alpha - 1\Big)}{2(\alpha+1)}.$$

where

$$P(n,m) = \frac{n!}{(n-m)!},$$

which is the number of possible permutations of k objects from a set of n.
In special case if we consider $\alpha = 1$, then it is not hard to see that

$$J^1_{a^+} f(b^n) + J^1_{b^-} f(a^n) = \frac{2(b^{n+1} - a^{n+1})}{n+1}.$$

So from inequality (295) we obtain that

$$\left| A^n(a,b) - L^n_n(a,b) \right| \leq \frac{nb^{n-1}(b-a)}{4}. \tag{296}$$

Also with similar argument as above, from Theorem 2.7 we have

$$\left| \Big(\frac{a^\rho + b^\rho}{2}\Big)^n - \frac{\alpha\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} \Big[{}^\rho I^\alpha_{a^+}\big(b^{n\rho}\big) + {}^\rho I^\alpha_{b^-}(a^{n\rho}) \Big] \right| \leq \frac{n(b^\rho - a^\rho)}{2^{\alpha+1}(\alpha+1)} \Big(a^{(n-1)\rho} + b^{(n-1)\rho} \Big), \tag{297}$$

and if $\rho \to 1$, then

$$\left| \Big(\frac{a+b}{2}\Big)^n - \frac{\alpha^2}{2(b-a)^\alpha} \sum_{m=0}^{n} \frac{[a^{n-m} + (-1)^m b^{n-m}](b-a)^{\alpha+m} P(n,m)}{\prod_{i=0}^{m}(\alpha+i)} \right| \leq \frac{n(b-a)(a^{n-1}+b^{n-1})}{2^{\alpha+1}(\alpha+1)}. \tag{298}$$

Now if in (298) we consider $\alpha = 1$, then we recapture inequality (3.1) in [4]:

$$\left| A^n(a,b) - L_n^n(a,b) \right| \leq \frac{n(b-a)}{4} A(a^{n-1}, b^{n-1}), \tag{299}$$

showing that (297) and (298) generalize (299).

References

[1] H. Chen and U. N. Katugampola, *Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals*, J. Math. Anal. Appl. **446** (2017), 1274–1291.

[2] U. N. Katugampola, *New approach to a generalized fractional integral*, Appl. Math. Comput. **218**(3) (2011), 860–865.

[3] U. N. Katugampola, *New approach to generalized fractional derivatives*, Bull. Math. Anal. Appl. **6**(4) (2014), 1–15.

[4] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comp. **147**(1) (2004), 137–146.

[5] A. W. Robert and D. E. Varbeg, *Convex Functions*, Academic Press, New York and London (1973).

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Two mappings in connection with Fejér's inequality

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Abstract

Two mappings in connection with Fejér's inequality are considered for the convex and nonsymmetric monotone functions. Some properties and results for Fejér's inequality are obtained. As applications, some special means type inequalities are given.

Keywords: 26D15, 26A51, 52A01
Mathematics Subject Classification [2023]: Convex functions; Fejér's inequality; special means

1 Introduction

In 1906, L. Fejér [1] proved the following integral inequalities known in the literature as Fejér's inequality:

$$f\Big(\frac{a+b}{2}\Big)\int_a^b w(x)dx \le \int_a^b f(x)w(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b w(x)dx, \tag{300}$$

where $f:[a,b]\to\mathbb{R}$ is convex and $w:[a,b]\to\mathbb{R}^+=[0,+\infty)$ is integrable and symmetric to $x=\frac{a+b}{2}\big(w(x)=w(a+b-x),\forall x\in[a,b]\big)$. If in (300) we consider $w\equiv 1$, we recapture the classic Hermite–Hadamard inequality [2, 3]:

$$f\Big(\frac{a+b}{2}\Big)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}.$$

We introduce two difference mappings, L_w and P_w, related to Fejér's inequality:

$$L_w:[a,b]\to\mathbb{R},\qquad L_w(t)=\frac{f(a)+f(t)}{2}\int_a^t w(s)ds-\int_a^t f(x)w(x)dx,$$
$$P_w:[a,b]\to\mathbb{R},\qquad P_w(t)=\int_a^t f(x)w(x)dx-f\Big(\frac{a+t}{2}\Big)\int_a^t w(x)dx,$$

and we obtain some properties for L_w and P_w that imply some refinements for Fejér's inequality in the case that w is a nonsymmetric monotone function.

To obtain our respected results, we need the modified version of Theorem 5 in [4] which includes the left and right part of Fejér's inequality in the monotone nonsymmetric case.

Theorem 1.1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a convex function on the interval I and differentiable on I°. Consider $a,b\in I^\circ$ with $a<b$ such that $w:[a,b]\to\mathbb{R}$ is a nonnegative, integrable and monotone function. Then*

(1) If $w'(x)\le 0$ $(w'(x)\ge 0)$, $a\le x\le b$ and $f(a)\le f(b)$ $\big(f(a)\ge f(b)\big)$, then

$$\int_a^b f(x)w(x)dx\le\frac{f(a)+f(b)}{2}\int_a^b w(x)dx. \tag{301}$$

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(2) If $w'(x) \geq 0$ $(w'(x) \leq 0)$, $a \leq x \leq b$ *and* $f(a) \leq f(\frac{a+b}{2})$ $\big(f(a) \geq f(\frac{a+b}{2})\big)$, *then*

$$f\Big(\frac{a+b}{2}\Big)\int_a^b f(x)w(x)dx \leq \int_a^b f(x)w(x)dx. \tag{302}$$

The main point in Theorem 1.1 (1) $(w'(x) \leq 0)$, is that we have (301) for any $x, y \in [a,b]$ with $f(x) \leq f(y)$ without the need for w to be symmetric with respect to $\frac{x+y}{2}$. Also similar properties hold for other parts of the above theorem.

2 Main Results

The first result of this section is about some properties of the mapping L_w where the function w is nonincreasing.

Theorem 2.1. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a convex function on the interval* I *and differentiable on* I°. *Consider* $a, b \in I^\circ$ *with* $a < b$ *such that* $w : [a,b] \to \mathbb{R}$ *is a nonnegative and differentiable function with* $w'(x) \leq 0$ *for all* $a \leq x \leq b$. *Then*

(i) The mapping L_w *is nonnegative on* $[a,b]$, *if* $f(a) \leq f(t)$ *for all* $t \in [a,b]$.

(ii) The mapping L_w *is convex on* $[a,b]$, *if* f *is non-decreasing. Also* L_w *is monotonically non-decreasing on* $[a,b]$.

(iii) The following refinement of (301) holds:

$$\begin{aligned}&\int_a^b f(x)w(x)\int_y^b f(x)w(x)dx + \frac{f(a)+f(y)}{2}\int_a^y w(x)dx \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b w(x)dx,\end{aligned} \tag{303}$$

for any $y \in [a,b]$ *with* $f(a) \leq f(y)$.

(iv) If f *is nondecreasing, then the following inequality holds:*

$$\begin{aligned}&t\int_a^u f(x)w(x)dx + (1-t)\int_a^v f(x)w(x)dx - \int_a^{tu+(1-t)v} f(x)w(x)dx \\ &\leq t\frac{f(u)+f(a)}{2}\int_a^u w(x)dx + (1-t)\frac{f(v)+f(u)}{2}\int_a^v w(x)dx \\ &- \frac{f(tu+(1-t)v)+f(a)}{2}\int_a^{tu+(1-t)v} w(x)dx,\end{aligned}$$

for any $u, v \in [a,b]$ *and each* $t \in [0,1]$.

(v) If $f' \in L([a,b])$, *then for each* $t \in [a,b]$ *we have*

$$|L_w(t)| \leq \frac{(t-a)^2}{2}\int_a^t w(x)|f'(x)|dx.$$

when $|f'|$ *is convex on* $[a,b]$, *then:*

$$|L_w(t)| \leq \frac{t-a}{2}\Big[|f'(a)|\int_a^t (t-x)w(x)dx + |f'(t)|\int_a^t (x-a)w(x)dx\Big].$$

The following result is including some properties of the mapping P_w in the case that w is nondecreasing.

Theorem 2.2. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a convex function on the interval* I *and differentiable on* I°. *Consider* $a, b \in I^\circ$ *with* $a < b$ *such that* $w : [a,b] \to \mathbb{R}$ *is a nonnegative and continuous function with* $w'(x) \geq 0$ *for all* $a \leq x \leq b$. *Then*

(i) P_w is nonnegative, if $f(a) \le f\big(\frac{a+t}{2}\big)$ for any $t \in [a,b]$.

(ii) If for any $x < y$ we have $f(x) \le f(\frac{x+y}{2})$, then P_w is nondecreasing on $[a,b]$.

(iii) If $f' \in L([a,b])$, then for each $t \in [a,b]$ we have

$$|P_w(t)| \le (t-a)\bigg[\int_a^{\frac{a+t}{2}} w(x)(x-a)|f'(x)|dx + \int_{\frac{a+t}{2}}^t w(x)(t-x)|f'(x)|dx\bigg].$$

Furthermore when $|f'|$ is convex on $[a,b]$, then:

$$\begin{aligned}|P_w(t)| \le &\bigg[\int_a^{\frac{a+t}{2}} w(x)(t-x)(x-a)dx + \int_{\frac{a+t}{2}}^t w(x)(t-x)^2dx\bigg]|f'(a)| \\ &+ \bigg[\int_a^{\frac{a+t}{2}} w(x)(x-a)^2dx + \int_{\frac{a+t}{2}}^t w(x)(t-x)(x-a)dx\bigg]|f'(t)|.\end{aligned}$$

(iv) The following inequality holds:

$$P_w(t) - L_w(t) \le \int_a^t f(x)w(x)dx,$$

provided that $f(a) \le f\big(\frac{a+t}{2}\big)$ for all $t \in [a,b]$.

(v) If for any $x < y$ we have $f(x) \le f(\frac{x+y}{2})$, then the following refinement of (302) holds:

$$\begin{aligned}&f\Big(\frac{a+b}{2}\Big)\int_a^b w(x)dx \\ &\le \int_a^t f(x)w(x)dx + f\Big(\frac{a+b}{2}\Big)\int_a^b w(x)dx - f\Big(\frac{a+t}{2}\Big)\int_a^t w(x)dx \\ &\le \int_a^b f(x)w(x)dx,\end{aligned} \tag{304}$$

for all $t \in [a,b]$.

3 Applications

The following means for real numbers $a, b \in \mathbb{R}$ are well known:

$$\begin{aligned}A(a,b) &= \frac{a+b}{2} && \textit{arithmetic mean},\\ L_n(a,b) &= \Big[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\Big]^{\frac{1}{n}} && \textit{generalized log}-\textit{mean},\ n \in \mathbb{R},\ a < b.\end{aligned}$$

The following result holds between the two above special means:

Theorem 3.1. *For any $a, b \in \mathbb{R}$ with $0 < a < b$ and $n \in \mathbb{N}$ we have*

$$A^n(a,b) \le L_n^n(a,b) \le A(a^n, b^n). \tag{305}$$

Here we give some refinements for the inequalities mentioned in (305).
Consider $a, b \in (0,\infty)$ with $a < b$. Define

$$\begin{cases} f(x) = x^n, & x \in [a,b] \text{ and } n \ge 1; \\ w(x) = x^{-s}, & x \in [a,b] \text{ and } s \in [0,1) \cup (1,\infty). \end{cases}$$

From (303) with some calculations, we have

$$\begin{aligned}(b-a)L_{n-s}^{n-s}(a,b) &\leq (b-t)L_{n-s}^{n-s}(t,b) + A(a^n,t^n)\Big(\frac{t^{1-s}-a^{1-s}}{1-s}\Big) \\ &\leq A(a^n,b^n)\Big(\frac{b^{1-s}-a^{1-s}}{1-s}\Big).\end{aligned} \tag{306}$$

Inequality (306) gives a refinement for the right part of (305).

In the case that $s=0$, we have:

$$L_n^n(a,b) \leq \Big(\frac{b-t}{b-a}\Big)L_n^n(t,b) + \Big(\frac{t-a}{b-a}\Big)A(a^n,t^n) \leq A(a^n,b^n),$$

for all $t \in [a,b]$.

Now with the same assumption for f and w as was used to obtain (306), by the use of (304) we get:

$$\begin{aligned}&A^n(a,b)\Big(\frac{b^{1-s}-a^{1-s}}{1-s}\Big) \\ &\leq A^n(a,b)\Big(\frac{b^{1-s}-a^{1-s}}{1-s}\Big) + (t-a)L_{n-s}^{n-s}(t,a) - A^n(a,t)\Big(\frac{t^{1-s}-a^{1-s}}{1-s}\Big) \\ &\leq (b-a)L_{n-s}^{n-s}(b,a),\end{aligned} \tag{307}$$

for all $t \in [a,b]$ and $s \in [0,1)\cup(1,\infty)$. Inequality (307) gives a refinement for the left part of (305). In the special case, if we set $s=0$, then we get:

$$A^n(a,b) \leq A^n(a,b) + \Big(\frac{t-a}{b-a}\Big)\big[L_n^n(t,a) - A^n(a,t)\big] \leq L_n^n(b,a),$$

for all $t \in [a,b]$.

References

[1] L. Fejér, *Über die fourierreihen, II.* Math. Naturwise. Anz Ungar. Akad. Wiss. **24** (1906), 369–390.

[2] D.S. Mitrinović and I.B. Lacković, *Hermite and convexity*, Aequ. Math. **28** (1985), 229–232.

[3] S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[4] S. Abramovich and L.E. Persson, *Extensions and Refinements of Fejér and Hermite-Hadamard type inequalities*, Math. Inequal. Appl. **21** (2018), 759–772.

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An approximate notion of Banach homology with respect to a multiplicative linear functional

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Abstract

For a Banach algebra A and a non-zero multiplicative linear functional ϕ on A, we define a concept of approximate left ϕ-biflatness. We study some concrete Banach algebras like group algebras and semigroup algebras. To show the differences between this concept and classical ones some examples to illustrate in Banach homology are given.

Keywords: Biflatness, Amenability, Banach algebras
Mathematics Subject Classification [2023]: 46M10, 43A20

1 Introduction

The concept of biflat Banach algebras were studied and introduced by Helemskii in [2] (see also [3]). In fact a Banach algebra A is biflat if there exists a bounded A-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ such that $\pi_A^{**} \circ \rho(a) = a$, where $\pi_A : A \otimes_p A \longrightarrow A$ is the product morphism given by $\pi_A(a \otimes b) = ab$ for every $a, b \in A$. The basic properties of biflatness can be found in [2]. It is worth mentioning that the biflatness has direct relation to the amenability of a Banach algebra. Note that a Banach algebra A is amenable if and only if A is biflat and posses a bounded approximate identity (see [3]).

A new generalization of biflatness like approximate biflatness has been introduced in [3]. A Banach algebra A is called *approximately biflat* if there exists a (not necessarily uniformly bounded) net of bounded A-bimodule morphisms $\theta_\alpha : (A \otimes_p A)^* \longrightarrow A^*$ such that $W^*OT - \lim_\alpha \theta_\alpha \circ \pi_A^* = \mathrm{id}_{A^*}$, where W^*OT stands for the weak-star operator topology and id_{A^*} is denoted for the identity map on A^*, the dual space of A. Here W^*OT is the locally convex topology on $B(A^*)$ determined by the family of seminorms $\{p_{f,a} : f \in A^*, a \in A\}$, where $p_{f,a}(T) = |Tf(a)|$. As a main result in [3], it is shown that if the Segal algebra $S(G)$ is approximately biflat, then the underlying group G is amenable, where G is a locally compact group.

Definition 1.1. Let A be a Banach algebra and $\phi \in \Delta(A)$. Then, we say that A is *approximately left ϕ-biflat* if there exists a net of bounded linear maps from A into $(A \otimes_p A)^{**}$, say $(\rho_\alpha)_{\alpha \in I}$, such that

(i) $a \cdot \rho_\alpha(b) - \rho_\alpha(ab) \xrightarrow{||\cdot||} 0$,

(ii) $\rho_\alpha(ba) - \phi(a)\rho_\alpha(b) \xrightarrow{||\cdot||} 0$,

(iii) $\tilde{\phi} \circ \pi_A^{**} \circ \rho_\alpha(a) - \phi(a) \to 0$,

for every $a, b \in A$, here $\tilde{\phi}$ is denoted for the unique extension of ϕ to A^{**} which is given by $\tilde{\phi}(F) = F(\phi)$ for all $F \in A^{**}$.

Suppose that A is a Banach algebra and $\phi \in \Delta(A)$. A Banach algebra A is called (*approximately*) *left ϕ-amenable*, if there exists a bounded (not necessarily bounded) net (m_α) in A such that

$$am_\alpha - \phi(a)m_\alpha \to 0, \quad \phi(m_\alpha) \to 1, \qquad (a \in A),$$

respectively. For further information about classical cases, we refer to [1], [4] and [5].

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2 Genereal results and their applications to concrete algebras

Proposition 2.1. *Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that A is approximately left ϕ-biflat which posses an element a_0 such that $aa_0 = a_0a$ for all $a \in A$ with $\phi(a_0) = 1$. Then, A is approximately left ϕ-amenable.*

Lemma 2.2. *Let A be a Banach algebra and $\phi \in \Delta(A)$. If A is approximately left ϕ-amenable, then A is approximately left ϕ-biflat.*

The following result asserts that approximate left ϕ-biflatness is weaker than approximate biflatness.

Proposition 2.3. *Suppose that A is a Banach algebra and $\phi \in \Delta(A)$. If A is approximately biflat, then A is approximately left ϕ-biflat.*

Our next result describes an interaction between the approximate left ϕ-biflatness of a Banach algebra and of its closed ideals.

Proposition 2.4. *Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that I is a closed ideal of A which $\phi|_I \neq 0$. If A is approximately left ϕ-biflat, then I is approximately left $\phi|_I$-biflat.*

Let A and B be Banach algebras and $\phi \in \Delta(A)$ and $\psi \in \Delta(B)$. It is known that the functional $\phi \otimes \psi$ given by $\phi \otimes \psi(a \otimes b) = \phi(a)\psi(b)$ is a multiplicative linear functional on $A \otimes_p B$.

Theorem 2.5. *Let A and B be Banach algebras with $\phi \in \Delta(A)$ and $\psi \in \Delta(B)$. Suppose that A has an identity and B has an idempotent x_0 which $\psi(x_0) = 1$. If $A \otimes_p B$ is approximately left $\phi \otimes \psi$-biflat, then A is approximately left ϕ-amenable.*

Theorem 2.6. *Let G be a SIN group. Then, $L^1(G)$ is approximately left ϕ-biflat if and only if G is amenable.*

Remark 2.7. Let $A = C^1[0, 1]$, be the set of all continuously differentiable functions on $[0, 1]$. Then, A is a Banach algebra with the following norm

$$\|f\| = \|f\|_\infty + \|f'\|_\infty.$$

It is known that $\Delta(A) = \{\phi_t : t \in [0, 1]\}$, where $\phi_t(f) = f(t)$ for all $f \in A$. We claim that A is not approximately left ϕ_t-biflat. To see this, we suppose in contradiction that A is approximately left ϕ_t-biflat. Since A is commutative, there exists f_0 in A such that $ff_0 = f_0f$ and $\phi_t(f_0) = 1$ for all $f \in A$. By Proposition 2.1, A is approximately left ϕ_t-amenable (or ϕ-pseudo amenable). But [5, Example 2.9(c)] asserts that A is not approximately left ϕ_t-amenable which is a contradiction.

Theorem 2.8. *Let G be a locally compact group. Then $M(G) \otimes_p l^1(S)$ is approximately left $\phi \otimes \psi$-biflat if and only if G is amenable, where S is a right zero semigroup, that is, a discrete semigroup which $st = t$ for all $s, t \in S$.*

In the upcoming example, we give a Banach algebra which is neither left ϕ-biflat nor left ϕ-amenable but is approximately left ϕ-biflat.

Example 2.9. Let $A = \ell^1 = l^1(\mathbb{N})$ be the set of all complex sequences $a = (a_n)$ such that $||a|| = \sum_{n=1}^{\infty} |a_n| < \infty$. With the product

$$(a * b)(n) = \begin{cases} a(1)b(1) & \text{if} \quad n = 1 \\ a(1)b(n) + b(1)a(n) + a(n)b(n) & \text{if} \quad n > 1 \end{cases}$$

for every $a, b \in \ell^1$, A becomes a Banach algebra. It is verified that $\Delta(A) = \{\phi_1\} \cup \{\phi_1 + \phi_n : n \geq 2\}$, where $\phi_n(a) = a_n$ for all $a \in A$.

We wish to show that A is not left ϕ_1-biflat. On contrary, suppose that A is left ϕ_1-biflat. Since A is commutative, by similar argument as in the proof of Proposition 2.1, A is left ϕ_1-amenable which contradicts with [5, Example 2.9(a)]. Once more, by using [5, Example 2.9(a)] A is approximately left ϕ_1-amenable. Now, Lemma 2.2 implies that A is approximately left ϕ_1-biflat.

References

[1] H. P. Aghababa, L. Y. Shi and Y. J. Wu, *Generalized notions of character amenability*, Acta Math. Sin. (Engl. Ser.) **29** (2013), no. 7, 1329-1350.

[2] A. Ya. Helemskii, *The homology of Banach and topological algebras*, Kluwer Academic Publishers, Dordrecht, 1986.

[3] V. Runde, *Lectures on Amenability*, Springer, New York, 2002.

[4] E. Kaniuth, A. T. Lau and J. Pym, *On ϕ-amenability of Banach algebras*, Math. Proc. Camb. Philos. Soc., **44** (2008), 85-96.

[5] R. Nasr-Isfahani and M. Nemati, *Character pseudo-amenability of Banach algebras*, Colloquium Mathematicum, **132** (2013), 177-193.

[6] E. Samei, N. Spronk and R. Stokke, *Biflatness and pseudo-amenability of Segal algebras*, Canad. J. Math., **62** (2010), 845-869.

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On amenable-like properties of ℓ^p-Matrix algebras

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Abstract

We study a class of ℓ^p-matrix algebras, namely $\mathcal{LM}_I^p(\mathbb{C})$. We show that $\mathcal{LM}_I^p(\mathbb{C})$ is pseudo-contractible if and only if the index set I is finite. Also, pseudo-amenability of $\mathcal{LM}_I^p(\mathbb{C})$ is investigated. For a subclass of ℓ^p-matrix algebras, say ℓ^p-upper triangular algebras Amenability, approximate biprojectivity and approximate biflatness are discussed here, where $1 \leq p \leq 2$.

Keywords: Biprojectivity, pseudo-amenability, Banach algebras
Mathematics Subject Classification [2023]: 46M10, 43A20

1 Introduction

A Banach algebra $\mathcal{A}$ is called amenable if there exists a bounded net (m_α) in $\mathcal{A}\otimes_p\mathcal{A}$ such that $a\cdot m_\alpha - m_\alpha\cdot a \to 0$ and $\pi_{\mathcal{A}}(m_\alpha)a \to a$ for all $a \in \mathcal{A}$. For a locally compact group G, the group algebra $L^1(G)$ is amenable if and only if G is amenable.

By relaxing the boundedness in the definition of amenability, Ghahramani and Zhang gave two amenable-like notions of pseudo-amenability and pseudo-contractibility. A Banach algebra $\mathcal{A}$ is called pseudo-amenable (pseudo-contractible) if there exists a net (m_α) in $\mathcal{A}\otimes_p\mathcal{A}$ such that $a\cdot m_\alpha - m_\alpha\cdot a \to 0$ ($a\cdot m_\alpha = m_\alpha\cdot a$) and $\pi_{\mathcal{A}}(m_\alpha)a \to a$ for all $a \in \mathcal{A}$, respectively.

G. H. Esslamzadeh introduced and studied a class of matrix algebras, named ℓ^1-Munn algebras. The rationale behind this nomination is that the algebraic form of these matricial algebras with finite index sets was first introduced by Munn. Then Esslamzadeh extended Munn works and studied their general analytic form and applied them to solve some harmonic analysis. Also, he investigated some cohomological properties of these algebras such as amenability and the existence of bounded approximate identity. M. Lashkarizadeh Bami and S. Naseri extended the notion of ℓ^1-Munn algebras to ℓ^p-Munn algebras. They showed that ℓ^p-Munn algebras are Banach algebras if and only if $1 \leq p \leq 2$. Also, amenability and the existence of bounded approximate identity for ℓ^p-Munn algebras were discussed. For more information on the amenability-like concepts, we encourage the readers to study[2].

In this paper, we study some notions of amenability like pseudo-amenability and pseudo-contractibility for special types of Esslamzadeh-Munn algebras, namely $\mathcal{LM}_I^p(\mathbb{C})$. We show that $\mathcal{LM}_I^p(\mathbb{C})$ with some mild assumptions is pseudo-contractible if and only if I is finite. We prove that $\mathcal{LM}_I^p(\mathbb{C})$ for an arbitrary set I is pseudo-amenable. Also, we study approximate biprojectivity and approximate biflatness of some triangular Banach subalgebras of $\mathcal{LM}_I^p(\mathbb{C})$.

In the following, we remind some notations and definitions. Let I be any non-empty set. Then we denote $\mathcal{LM}_I^p(\mathbb{C})$ as the vector space of all $I\times I$ matrices $A=[a_{ij}]$ over $\mathbb{C}$ such that $||A||_p = (\sum_{i,j\in I}(|a_{ij}|^p)^{\frac{1}{p}} < \infty$. With the matrix operations and $||\cdot||_p$ as a norm, $\mathcal{LM}_I^p(\mathbb{C})$ becomes a Banach algebra, provided that $1 \leq p \leq 2$. In the case $1 \leq p \leq 2$ and I is a totally ordered set, we denote

$$\mathcal{UP}_I^p(\mathbb{C}) = \{[a_{ij}] \in \mathcal{LM}_I^p(\mathbb{C}) | a_{ij} = 0 \quad \text{whenever} \quad i > j\}.$$

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Let $\mathcal{A}$ be a Banach algebra. The projective tensor product $\mathcal{A} \otimes_p \mathcal{A}$ is a Banach $\mathcal{A}$-bimodule with the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \qquad (b \otimes c) \cdot a = b \otimes ca, \qquad (a, b, c \in \mathcal{A}).$$

The product morphism $\pi_{\mathcal{A}} : \mathcal{A} \otimes_p \mathcal{A} \to \mathcal{A}$ is given by $\pi_{\mathcal{A}}(a \otimes b) = ab$, for all $a, b \in \mathcal{A}$.

2 Main results

Theorem 2.1. *Let I be a non-empty index set and $1 \leq p \leq 2$. Then $\mathcal{LM}_I^p(\mathbb{C})$ posses a central approximate identity if and only if I is finite.*

$\mathcal{A}$ is called contractible if there exists $m \in \mathcal{A} \otimes_p \mathcal{A}$ such that $a \cdot m = m \cdot a$ and $\pi_{\mathcal{A}}(m)a = a$ for all $a \in \mathcal{A}$. For more details see [2].

Theorem 2.2. *Let I be a non-empty index set and $1 \leq p \leq 2$. Then $\mathcal{LM}_I^p(\mathbb{C})$ is pseudo-contractible if and only if I is finite.*

A Banach algebra $\mathcal{A}$ is called biprojective if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \to \mathcal{A} \otimes_p \mathcal{A}$ such that $\pi_{\mathcal{A}} \circ \rho(a) = a$, for all $a \in \mathcal{A}$ [1].

Theorem 2.3. *Let I be a non-empty index set and $1 \leq p \leq 2$. Then $\mathcal{LM}_I^p(\mathbb{C})$ is biprojective.*

Corollary 2.4. *Let I be a non-empty index set and $1 \leq p \leq 2$. Then $\mathcal{LM}_I^p(\mathbb{C})$ has an approximate identity.*

Lemma 2.5. *Suppose that $\mathcal{A}$ is a biprojective Banach algebra with an approximate identity. Then $\mathcal{A}$ is pseudo-amenable.*

Corollary 2.6. *Let I be a non-empty index set and $1 \leq p \leq 2$. Then $\mathcal{LM}_I^p(\mathbb{C})$ is pseudo-amenable.*

A Banach algebra $\mathcal{A}$ is called approximately biprojective if there exists a net of $\mathcal{A}$-bimodule morphisms from $\mathcal{A}$ into $\mathcal{A} \otimes_p \mathcal{A}$ such that $\pi_{\mathcal{A}} \circ \rho_\alpha(a) \to a$ for all $a \in \mathcal{A}$, see [4].

Theorem 2.7. *Suppose that I is a totally ordered set with a smallest element and $1 \leq p \leq 2$. Then $\mathcal{UP}_I^p(\mathbb{C})$ is not approximately biprojective unless it is trivial.*

A Banach algebra $\mathcal{A}$ is called approximately biflat if there exists a net (ρ_α) of $\mathcal{A}$-bimodule morphisms from $(\mathcal{A} \otimes_p \mathcal{A})^*$ into $\mathcal{A}^*$ such that $\rho_\alpha \circ \pi_{\mathcal{A}}^* \xrightarrow{W^*OT} id_{\mathcal{A}^*}$. Here W^*OT stands for the weak* operator topology, see [3].

Theorem 2.8. *Suppose that I is a totally ordered set with a smallest element and $1 \leq p \leq 2$. Then $\mathcal{UP}_I^p(\mathbb{C})$ is not approximately biflat unless it is trivial.*

Theorem 2.9. *Let I be a totally ordered set and $1 \leq p \leq 2$. Then $\mathcal{UP}_I^p(\mathbb{C})$ is not amenable unless it is trivial.*

Theorem 2.10. *Let I be a totally ordered set and $1 \leq p \leq 2$. Then $\mathcal{UP}_I^p(\mathbb{C})$ is not pseudo-contractible unless it is trivial.*

Remark 2.11. As an application of this paper, we turn our attention toward a simplest triangular matrix algebras. That is $\mathcal{A} = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{C}\}$ with matrix operations and the ℓ^1-norm, $\mathcal{A}$ becomes a Banach algebra. Using above theorem $\mathcal{A}$ is neither approximately biprojective nor is approximately biflat.
As another application, we should remind that the semigroup algebra $\ell^1(S)$ has a direct relation with $\mathcal{LM}_I^p(\mathbb{C})$, so we can characterize amenable-like properties of some semigroup algebras.

References

[1] A. Ya. Helemskii, *The Homology of Banach and Topological Algebras*, Kluwer, Academic Press, Dordrecht, (1989).

[2] V. Runde, *Lectures on Amenability*,Panorama, Lecture Notes in Mathematics, Vol. 1774, Springer-Verlag, Berlin, (2020).

[3] E. Samei, N. Spronk, and R. Stokke, *Biflatness and pseudo-amenability of Segal algebras*, Canad. J. Math., **62**, 845-869 (2010).

[4] Y. Zhang, *Nilpotent ideals in a class of Banach algebras*, Proc. Amer. Math. Soc., **127**, 3237-3242 (1999).

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The linear functionals on ample topological algebras with bounded elements

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Abstract

A normed subspace of continuous linear functionals are introduced for topological algebras with bounded elements, and several theorems are extended for topological algebras with bounded elements. For example, some conditions are obtained for a metrisable infrasequential topological algebra to be an ample topological algebra.

Keywords: topological algebra with bounded elements, infrasequential topological algebra, ample topological algebra
Mathematics Subject Classification [2023]: 46H05, 46H20

1 Introduction

A topological algebra is an algebra which is also a Hausdorff topological vector space in such a way that the ring multiplication is separately continuous. As a common favourite, the normed algebras have been studied intensively by mathematicians. Later, some theorems from normed algebras extended to locally bounded topological algebras and many of the results obtained in this area use p-normed for these classes of topological algebras. Also, it contained some results for seminorm on locally convex topological algebras. Allan [6] provided the definition of the boundedness radius β to develop locally convex topological algebras. By β, many of famous theorems were considered in many papers for several classes of topological algebras which are neither locally bounded nor locally convex.In [1] Ansari introduced a norm $v(.)$ on a subspace $\hat{A}$ of $L(A,\mathbb{C}) = A'$ (that is the algebraic dual a topological algebra A) by applying some conditions. Also, he proved that each multiplicative linear functional on A belongs to $\hat{A}$. In [2], some theorems were extended by Ansari and Sabet for β-subadditive algebra. Moreover, a new format of the concept of the numerical range was introduced by applying the norm $v(.)$.

2 Main results

In this paper, we provide a necessary and sufficient condition that $\hat{A}$ separates points on A. Also, the above results lead us to generalize the norm $v(.)$ which is from $L(A,\mathbb{C})$ to $L(A,B)$ for a suitable topological algebra B. We denote this normed subspace with $\hat{L}(A,B)$.

3 A norm on a subspace of the algebraic dual space of an algebra with bounded elements

Let X and Y be complex topological vector spaces, and let $L(X,Y)$ be the vector space of all linear mappings of X into Y with the pointwise addition and scalar multiplication. $BL(X,Y)$ denotes the collection of all bounded linear mappings of X into Y and $CL(X,Y)$ denotes the collection of all continuous linear mappings

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of X into Y. It is obvious that if X is metrisable, then $CL(X,Y) = BL(X,Y)$.
If X and Y are normed spaces, then $BL(X,Y)$ is as a normed space with the norm given by

$$\|T\| = \sup\{\|Tx\| : \|x\| \leq 1, x \in X\} \quad (T \in BL(X,Y)).$$

In particular, X^* (that is the set of all continuous linear functionals on X) is a normed subspace of X' (that is the space of all linear functionals on X). Furthermore, if X is a locally bounded space and Y is a normed space, then $BL(X,Y)$ is again the normed subspace in $L(X,Y)$. We say that a complex topological algebra A is a fundamental topological algebra if there exists $b>1$ such that for every sequence (a_n) in A the convergence of $b^n(a_n - a_{n-1})$ to zero in A implies that (a_n) is a Cauchy sequence. Moreover, a fundamental topological algebra A is said to be locally multiplicative (see [1]), if there exists a neighbourhood U_0 of zero in A such that for every neighbourhood V of zero of A, the sufficiently large powers of U_0 lie in V. In [1], such an algebra was called an FLM algebra. It is shown that if A is a metrisable FLM algebra, then there exists a norm on a subspace of $A^{'}$ ([1], Theorem 4.1). By this norm, some results from Banach algebra are extended to some classes of topological algebras.A is said to be strongly sequential topological algebra if there exists a neighborhood U of 0 such that for all $x \in U$, $x^n \to 0$ as $n \to \infty$ and A is said to be infrasequential topological algebra, if for each bounded set $B \subseteq A$ there is $\lambda > 0$ such that for all $x \in B, (\lambda x)^n \to 0$ as $n \to \infty$.

An elemnent x of a topological algebra A is called to be bounded if there exists $\lambda>0$ such that the set $\{(\frac{x}{\lambda})^n : n \in \mathbb{N}\}$ is bounded in A. If every element of A is bounded, then A is called a topological algebra with bounded elements.
The radius of boundedness of a bounded element x denoted by $\beta(x)$ and defined by

$$\beta(x) = \inf\{\lambda > 0 \ : \ (\frac{x^n}{\lambda^n})_n \ is \ bounded\}.$$

We say $E \subseteq A$ is β-bounded if there is $M>0$ such that for every $x \in E$, we have $\beta(x) \leq M$. It is proved that

$$\beta(x) = \inf\{\lambda>0 : \left(\frac{x^n}{\lambda^n}\right) \to 0\}.$$

Every infrasequential topological algebra is a topological algebra with bounded elements.
In the rest of this paper, all of topological algebras are assumed to be with bounded elements and to contain the unit element.

Theorem 3.1. *Let A and B be topological algebras, let $f \in L(A,B)$, and let $v(f) = \sup\{\beta(f(x)) : \beta(x)<1\}$, then we have*

(1) $v(f) = \sup\{\beta(f(x)) : \beta(x) \leq 1\} = \sup\{\beta(f(x)) : x \in \beta^{-1}(\{0,1\})\}$.

(2) Let $x \in A$. If $\beta(x) \neq 0$, then $\beta(f(x)) \leq v(f)\beta(x)$, if $\beta(x) = 0$, then $\beta(f(x)) \leq v(f)$.

(3) Let $\beta_B^{-1}(0) = \{0\}$, then $f = 0$ if and only if $v(f) = 0$.

(4) If (B, β_B) is a normed algebra and if

$$\hat{L}(A,B) = \{f \in L(A,B) : v(f)<\infty\},$$

then $v(.)$ is a norm on $\hat{L}(A,B)$ which makes $(\hat{L}(A,B), v(.))$ into a normed space. Moreover, if (B, β_B) is a complete space, then $\hat{L}(A,B)$ is a Banach space.

(5) Let $\beta_B^{-1}(0) = \{0\}$. If $f \in \hat{L}(A,B)$ and $\beta_A(x) = 0$, then $f(x) = 0$ and consequently, $f \in \hat{L}(A,B) \Rightarrow (\beta(f(x)) \leq v(f)\beta(x)$ for all $x \in A)$.

(6) Let A be a metrisable infrasequential algebra. If B is a β-subadditive algebra and every β-bounded subset of B is bounded, then $\hat{L}(A,B) \subset CL(A,B)$.

(7) If $\beta_B^{-1}(0) = \{0\}$ *and* $f \in \hat{L}(A,B)$, *then*

$$v(f) = \inf\{M : \beta(f(x)) \leq M\beta(x) \ , \ for\ all\ x \in A\}.$$

Proof. (1) Let $r = \sup\{\beta(f(x)) : \beta(x)<1\}$, and $r^{'} = \sup\{\beta(f(x)) : \beta(x) \leq 1\}$, then it is clear that $r \leq r^{'}$. Let $n \in \mathbb{N}$. If $\beta(x) \leq 1$, then $\beta((1-\frac{1}{n})x)<1$. So $\beta(f((1-\frac{1}{n})x)) \leq r$. Hence $(1-\frac{1}{n})\beta(f(x)) \leq 1$, consequently, $r^{'} \leq r$.
On the other hand, if $r'' = \sup\{\beta(f(x)) : x \in \beta^{-1}(\{0,1\})\}$, then it is clear that $r'' \leq r^{'}$. Let $0 < \beta(x) \leq 1$, and let $\beta(f(\frac{x}{\beta(x)})) \leq r''$, then $r^{'} \leq r''$.

(2) Let $x \in A$, and $\beta(x) \neq 0$. Since $\beta(\frac{x}{\beta(x)}) = 1$, so $\beta(f(\frac{x}{\beta(x)})) \leq v(f)$ and hence $\beta(f(x)) \leq v(f)\beta(x)$. The case $\beta(x) = 0$ is obvious.

(3) Part (3) can be yielded from Part (2) directly.

(4) It readily follows that $v(\alpha f) = |\alpha|v(f)$ and $v(f+g) \leq v(f) + v(g)$
for each $\alpha \in \mathbb{C}$ and $f, g \in \hat{L}(A,B)$. For completing the proof, we show that if (B, β_B) is complete, then $(\hat{L}(A,B), v(.))$ is complete. Let $(f_n)_n$ is a Cauchy sequence of $(\hat{L}(A,B), v(.))$, and let $x \in A$. By Part (2), $(f_n(x))_n$ is a Cauchy sequence of (B, β_B). Therefore there exists $f : A \to B$ such that for every $x \in A$, $f_n(x) \to f(x)$ in (B, β_B). Let $\epsilon>0$, and let $x \in A$ such that $\beta(x)<1$, then there exists $N \in \mathbb{N}$ such that

$$\beta(f_n(x) - f_m(x)) \leq v(f_n - f_m)<\frac{\epsilon}{2}$$

whenever $m, n \geq N$. Now $m \to \infty$ gives us

$$\beta(f_n(x) - f(x)) \leq \frac{\epsilon}{2}$$

Thus $v(f_n - f)<\epsilon$ whenever $n \geq N$.

(5) Let $n \in \mathbb{N}$. If $\beta(x) = 0$, By Part (2) and since $\beta(nf(x)) \leq v(f)$, then $\beta(f(x)) \leq \frac{v(f)}{n}$ which implies $f(x) = 0$.

(6) Let $f \in \hat{L}(A,B)$, and let E be a bounded subset of A. Since A is an infrasequential algebra, then E is β-bounded and by Part (5), $f(E)$ is bounded and consequently $f \in CL(A,B)$.

(7) Take $G = \{M : \ \beta(f(x)) \leq M\beta(x) \ , \ for\ all\ x \in A\}$. If $\beta(x) \leq 1$ and if $M \in G$, then $\beta(f(x)) \leq M$, for all $x \in A$. Indeed, $v(f) \leq M$, for all $M \in G$. Therefore $v(f) \leq \inf G$. On the contrary, by Part (5) and since $\beta(f(x)) \leq v(f)\beta(x)$, for any $x \in A$, we have $v(f) \in G$. Hence $\inf G \leq v(f)$.

□

Corollary 3.2. *(see [2], Theorem 3.1) Let* A *be a topological algebra, let* $f \in A^{'}$, *and let* $v(f) = \sup\{|f(x)| : \beta(x)<1\}$, *then we have*

(1) $v(f) = \sup\{|f(x)| : \beta(x) \leq 1\} = \sup\{|f(x)| : x \in \beta^{-1}(\{0,1\})\}$.

(2) Let $x \in A$. *If* $\beta(x) \neq 0$, *then* $|f(x)| \leq v(f)\beta(x)$. *If* $\beta(x) = 0$, *then* $|f(x)| \leq v(f)$.

(3) $f = 0$ *if and only if* $v(f) = 0$.

(4) If $f \in \hat{A} = \{f \in A^{'} : v(f)<\infty\}$, *then* $v(.)$ *is a norm on* $\hat{A}$ *and therefore* $(\hat{A}, v(.))$ *is a Banach space.*

(5) If $f \in \hat{A}$ *and if* $\beta(x) = 0$, *then* $f(x) = 0$, *consequently,*

$$f \in \hat{A} \Rightarrow (|f(x)| \leq v(f)\beta(x) \ for\ all\ x \in A).$$

(6) Let A *be a metrisable infrasequential algebra, then* $\hat{A} \subset A^*$.

(7) If $f \in \hat{A}$ then $v(f) = \inf\{M : |f(x)| \leq M\beta(x) \ , \ for \ all \ x \in A\}$.

Theorem 3.3. *Suppose that A is a topological algebra. If $\hat{A}$ separates points on A, then $\beta^{-1}(0) = \{0\}$.*

Proof. Let $\beta(x) = 0$. If $x \neq 0$, then there exists $\Lambda \in \hat{A}$ such that $\Lambda x \neq 0$. Since $\Lambda \in \hat{A}$, so $|\Lambda x| \leq v(\Lambda)\beta(x)$. Hence $\Lambda(x) = 0$ which is impossible.

□

Corollary 3.4. *If (A, τ) is a β-subadditive topological algebra, then $\hat{A}$ separates points on A if and only if β_A is a normed algebra.*

Proof. It is sufficient to observe that $(A, \beta_A)^* = \widehat{(A, \tau)}$. □

A topological vector space X is called ample if X^* separates points of X. A topological algebra A is called ample if it is ample as a topological vector space.

Corollary 3.5. *If A is a metrisable infrasequential topological algebra that β is subadditive and if $\beta^{-1}(0) = \{0\}$, then A is ample.*

Proof. Since $\hat{A} \subseteq A^*$, by Corollary 3.4, we get the result. □

Theorem 3.6. *Let (A, τ) be a metrisable infrasequential topological algebra such that $\hat{A}$ separates points on A, and let $\hat{\tau}$ be the $\hat{A}$-topology on A, then $\hat{\tau}$ is a locally convex topology on A such that $\hat{\tau} \subseteq \tau$. Moreover, if A is infinite dimensional, then $(A, \hat{\tau})$ is not a locally bounded space.*

4 Conclusion

In fact, we obtain a necessary and sufficient condition that $\hat{A}$ separates points on A. Also Corollary 3.2 and in [2], Theorem 3.10 give us Theorem 3.6.

References

[1] G. R. Allan, *A spectral theory for locally convex algebras*, Proc. London Math. Soc. 115 (1965), 399-421.

[2] E. Ansari-piri, *The linear functionals on fundamental locally multiplicative topo- logical algebras*, Turk J Math. 34 (2010), 385-391.

[3] E. Ansari-piri and M. Sabet, *The numerical range of an element of a class of topological algebras*, Italian J. Pure app. mathematics. 41 (2019), 58-64.

[4] W. Rudin, *Functional Analysis*, McGRAW-HILL, Inc. (1991).

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A priori error estimate for approximating common fixed point for a pair of self or nonself mappings in complete metric spaces

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Abstract

In this article, we find a priori error estimate of the common fixed point for a pair of self or nonself mappings, which is defined on a complete metric space. Our results extend and improve some recent results in the literature

Keywords: Fixed point, Self mappings, Priori error estimate, Complete metric spaces
Mathematics Subject Classification [2023]: 41A25, 47H10

1 Introduction

Fixed point theory is an important tool to solve the equation $Tx = x$ for mappings T defined on subsets of metric or normed spaces. A basic result in fixed point theory is the Banach contraction principle. One of the advantages of Banach's fixed point theorem is the estimation of the error of successive iterations and the rate of convergence. There are equations $Tx = x$ for which the exact solution is not easy to find or even is not possible to find. The error estimate is very useful in these cases. An extensive study about approximations of fixed points for self-maps can be found in [2, 3, 6, 6].

Let A and B be nonempty subsets of a metric space (X, d). One kind of generalization of the Banach Contraction Principle is the notation of cyclical maps, i.e. $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. In 2016, Boyon Zlatanov obtained a priori and a posteriori error estimates for approximating the best proximity points for cyclic contraction maps. One other kind of generalization of the Banach contraction principle is the notation of noncyclical maps, i.e. $T : A \cup B \to A \cup B$ such that $T(A) \subseteq A$ and $T(B) \subseteq B$.

In this article, in both cyclical and noncyclical cases, we have obtained a "priori error estimate" to approximate the common fixed point for a pair of nonself mappings and a pair of self-mappings, respectively.

2 Preliminaries

In this section, we recall some definitions and facts that will be used hereafter.

Theorem 2.1. [6, Theorem 2.1] *Let T be a self mapping on complete metric space (X, d) satisfying*

$$d(Tx, Ty) \leq cd(x, y),$$

for each $x, y \in X$ where $c \in [0, 1)$. Then the Picard iteration associated with T, i.e., the sequence $\{x_n\}_{n=0}^{\infty}$, defined by

$$x_{n+1} := Tx_n = T^n x_0 \quad for \quad n \geq 0,$$

converges to ξ, for any initial guess $x_0 \in X$. Also, priori error estimate of the Picard iteration is given by

$$d(T^n x_0, \xi) \leq \frac{c^n}{1-c} d(x_0, Tx_0).$$

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Theorem 2.2. [4, Theorem 2-8] *Let A and B be nonempty and closed subsets of a complete metric space (X, d). Let T be a cyclic mapping on $A \cup B$ such that*

$$d(Tx, Ty) \leq c \max\big\{d(x, y), d(x, Tx), d(y, Ty)\big\},$$

for all $x \in A$ and $y \in B$ where $c \in [0, 1)$. Then T has a unique fixed point x^* in $A \cap B$ and the Picard iteration $\{T^n x_0\}$ converges to x^* for any starting point $x_0 \in A \cup B$.

From the previous theorem, we immediately obtain the following common fixed point result.

Corollary 2.3. [4, Theorem 2.10] *Let (X, d) be a complete metric space and let $T\colon X \to X$ and $S\colon X \to X$ are mappings satisfying*

$$d(Tx, Sy) \leq c \max\big\{d(x, y), d(x, Tx), d(y, Sy)\big\},$$

for each $x, y \in X$ where $c \in [0, 1)$. Then T and S have a unique common fixed point in X.

Theorem 2.4. [4, Theorem 3.6] *Let A and B be nonempty and closed subsets of a complete metric space (X, d). Let T be a noncyclic mapping on $A \cup B$ satisfying*

$$d(Tx, Ty) \leq c \max\big\{d(x, y), d(x, Ty), d(y, Tx)\big\},$$

for each $x \in A$ and $y \in B$ where $c \in [0, 1)$. Then T has a unique fixed point x^* in $A \cap B$ and the Picard iteration $\{T^n x_0\}$ converges to x^* for any starting point $x_0 \in A \cup B$.

From Theorem 2.4, we immediately obtain the following common fixed point result.

Corollary 2.5. [4, Theorem 3.7] *Let (X, d) be a complete metric space and let $T\colon X \to X$ and $S\colon X \to X$ are two mappings satisfying*

$$d(Tx, Sy) \leq c \max\big\{d(x, y), d(x, Sy), d(y, Tx)\big\}$$

for all $x, y \in X$ where $c \in [0, 1)$. Then S and T have a unique common fixed point in X.

3 Main results for a pair of nonself mappings

In this section, we begin with the following Theorem as a result of Lemma 3.2 of [5] which will be used later.

Theorem 3.1. *Let A and B be nonempty subsets of a metric space (X, d) and let T be a cyclic mapping on $A \cup B$ satisfying*

$$d(Tx, Ty) \leq c \max\big\{d(x, y), d(x, Tx), d(y, Ty)\big\},$$

for each $x \in A$ and $y \in B$ where $c \in [0, 1)$. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then for every $m, n \in \mathbb{N}$, we have

$$d(x_{2n}, x_{2m+1}) \leq \frac{c^{l_{n,m}}}{1 - c}\, M_{x_0},$$

where $l_{n,m} = \min\{2n, 2m\}$ and

$$M_{x_0} = \max\big\{d(x_0, Tx_0), d(x_0, T^2 x_0), d(Tx_0, T^2 x_0)\big\}.$$

Theorem 3.2. *Let the assumptions in Theorem 3.1 be satisfied. Then T has a unique fixed point $\xi \in A \cap B$ and a priori error estimate of the Picard iteration is given by*

$$d(T^{2n} x_0, \xi) \leq \frac{c^{2n}}{1 - c} M_{x_0}.$$

Corollary 3.3. *Let A and B be nonempty subsets of a metric space (X,d) and let T be a cyclic mapping on $A\cup B$ satisfying*

$$d(Tx,Ty)\leq cd(x,y),$$

for each $x\in A$ and $y\in B$ where $c\in[0,1)$. For $x_0\in A$, define $x_{n+1}:=Tx_n$ for each $n\geq 0$. Then T has a unique fixed point $\xi\in A\cap B$ and a priori error estimate of the Picard iteration is given by

$$d(T^nx_0,\xi)\leq\frac{c^n}{1-c}d(x_0,Tx_0).$$

Theorem 2.1 is a special case of the next Corollary.

Corollary 3.4. *Let (X,d) be a complete metric space and let $T:A\to B$ and $S:B\to A$ are two mappings satisfying*

$$d(Tx,Sy)\leq cd(x,y),$$

for all $x\in A$ and $y\in B$ where $c\in[0,1)$. Then T and S have a unique common fixed point $\xi\in A\cap B$ and a priori error estimate of the Picard iteration is given by

$$d(T^nx_0,\xi)\leq\frac{c^n}{1-c}d(x_0,Tx_0).$$

4 Main results for a pair of self-mappings

In this section, we begin with the following lemma as a result of Lemma 3.2 of [4] which will be used later.

Lemma 4.1. *Let A and B be nonempty subsets of a metric space (X,d) and let T be a noncyclic mapping on $A\cup B$ satisfying*

$$d(Tx,Ty)\leq c\max\big\{d(x,y),d(x,Ty),d(y,Tx)\big\},$$

for each $x\in A$ and $y\in B$ where $c\in[0,1)$. For $x_0\in A$ and $y_0\in B$, define $x_{n+1}:=Tx_n$ and $y_{n+1}:=Ty_n$ for each $n\geq 0$. Then for every $m,n\in\mathbb{N}$, we have

$$d(x_n,y_m)\leq c^{k_{n,m}}\ M_{x_0,y_0},$$

where $k_{n,m}=\min\{n,m\}$ and

$$M_{x_0,y_0}=\frac{1}{1-c}\max\big\{d(x_0,y_0),d(x_0,Ty_0),d(y_0,Tx_0),d(x_0,Tx_0),d(y_0,Ty_0)\big\}.$$

Theorem 4.2. *Let the assumptions in Lemma 4.1 be satisfied. Then T has a unique fixed point $\xi\in A\cap B$ and a priori error estimate of the Picard iteration is given by*

$$d(x_n,\xi)\leq\frac{c^n}{1-c}M_{x_0,y_0}.$$

Corollary 4.3. *Let A and B be nonempty subsets of a metric space (X,d) and let T be a noncyclic mapping on $A\cup B$ satisfying*

$$d(Tx,Ty)\leq cd(x,y),$$

for each $x\in A$ and $y\in B$ where $c\in[0,1)$. For $x_0\in A$ and $y_0\in B$, define $x_{n+1}:=Tx_n$ and $y_{n+1}:=Ty_n$ for each $n\geq 0$. Then T has a unique fixed point $\xi\in A\cap B$ and a priori error estimate of the Picard iteration is given by

$$d(x_n,\xi)\leq\frac{c^n}{1-c}d(x_0,y_0).$$

Theorem 2.1 is a special case of the next Corollary.

Corollary 4.4. *Let (X, d) be a complete metric space and let $T : A \to A$ and $S : B \to B$ are two mappings satisfying*

$$d(Tx, Sy) \leq cd(x, y),$$

for all $x \in A$ and $y \in B$ where $c \in [0, 1)$. Then T and S have a unique common fixed point $\xi \in A \cap B$ and a priori error estimate of the Picard iteration is given by

$$d(T^n x_0, \xi) \leq \frac{c^n}{1-c} d(x_0, Tx_0).$$

References

[1] V. Berinde, *Iterative Approximation of Fixed Points*, Springer, Berlin, 2007.

[2] K. Goebel and W.A Kirk, *Topics in metric fixed point theory*, Cambridge University Press, 1990.

[3] A. Ilchev, *Error estimates for approximating best proximity points for Kannan cyclic contractive maps*, AIP Conference Proceedings, 2048(1) (2018), 050002.

[4] A. Safari-Hafshejani, *Generalizations of Banach's contraction principle and Kannan and Chatterjea's theorems for cyclic and noncyclic mappings*, J. Mahani Math. Res., 12(2) (2023), 349-362.

[5] A. Safari-Hafshejani, A. Amini-Harandi and M. Fakhar *Best proximity points and fixed points results for noncyclic and cyclic Fisher quasi-contractions*, Numer. Funct. Anal. Optim., 40(5) (2019), 603-619.

[6] B. Zlatanov, *Error estimates for approximating best proximity points for cyclic contractive maps*, Carpathian J. Math., 32(2) (2016), 265-270.

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Some existence and stability results for Hilfer-fractional implicit q-differential equations under nonlocal conditions

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Abstract

In this paper, we establish the existence and uniqueness results for implicit q-differential equations of Hilfer-type fractional order via Schaefer's fixed point theorem and Banach contraction principle. Next, we establish the equivalent mixed-type integral for nonlocal condition. Further, we prove the Ulam stability results. The Grownwall's lemma for singular kernels plays an important role to prove our results. We verify our results by providing examples.

Keywords: Hilfer fractional derivative, Implicit differential equation, Fixed point, Existence, Ulam stability
Mathematics Subject Classification [2023]: 26A33, 34K40, 34K14

1 Introduction

The topic of fractional calculus is as old as the differential calculus and it has been developed up to nowadays [5, 8]. Fractional q-differential and integral equations have recently been applied in different areas of engineering, mathematics, physics and bio-engineering and so on [2].

The aim of this paper is to study the implicit q-differential equation under nonlocal condition involving Hilfer q-fractional derivative of the form

$$\begin{cases} \mathcal{D}^{\sigma,\xi}_{q;0^+}\mathrm{u}(r) = \mathfrak{v}\left(r, \mathrm{u}(r), \mathcal{D}^{\sigma,\xi}_{q;0^+}\mathrm{u}(r)\right), & r \in \mathrm{J} := [0,T], \\ \mathcal{I}^{1-\nu}_{q;0^+}\mathrm{u}(0) = \sum_{i=1}^{m} c_i \mathrm{u}(\tau_i), & \sigma \le \nu = \sigma + \xi - \sigma\,\xi < 1,\ \tau_i \in \mathrm{J}, \end{cases} \tag{308}$$

where $\mathcal{D}^{\sigma,\xi}_{q;0^+}$ is the Hilfer fractional q-derivative, $0 < \sigma < 1$, $0 \le \xi \le 1$, $\mathfrak{v}\colon \mathrm{J}\times\mathbb{R}^2 \to \mathbb{R}$ is a given continuous function, $\mathcal{I}^{1-\nu}_{q;0^+}$ is the left-sided Riemann-Liouville fractional integral of order $1-\nu$, c_i are real numbers and $\tau_i = 1, 2, \ldots, m$ are prefixed points satisfying $0 < \tau_1 \le \tau_2 \le \cdots \le \tau_m < T$.

2 Prerequisites

We consider the fractional q-calculus on the specific time scale $\mathbb{T} = \mathbb{R}$ where $\mathbb{T}_{s_0} = \{0\} \cup \{r : r = s_0 q^n\}$, for nonnegative integer n, $s_0 \in \mathbb{R}$ and $q \in J$. Let $a \in \mathbb{R}$. Define [7] $[a]_q = (1-q^a)/(1-q)$. The power function $(r-s)^n_q$ with $n \in \mathbb{N}_0$ is defined by $(r-s)^{(n)}_q = \prod_{k=0}^{n-1}(r-sq^k)$, for $n \ge 1$ and $(r-s)^{(0)}_q = 1$, where r and s are real numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ [1]. If $s = 0$, then it is clear that $r^{(\sigma)} = r^\sigma$ [3]. The q–Gamma function is given by $\Gamma_q(\nu) = ((1-q)^{(\nu-1)})/((1-q)^{\nu-1})$, where $\nu \in \mathbb{R} \setminus \{0, -1, -2, \cdots\}$ [7]. Note that, $\Gamma_q(\nu+1) = [\nu]_q \Gamma_q(\nu)$. The q-derivative of function h, is defined by $\mathcal{D}_q y(\nu) = (y(\nu) - y(q\nu))/((1-q)\nu)$, and $\mathcal{D}_q y(0) = \lim_{\nu\to 0}\mathcal{D}_q y(\nu)$ [1]. Furthermore, the higher order q–derivative of a function y is defined by $\mathcal{D}^n_q y(\nu) = \mathcal{D}_q(\mathcal{D}^{n-1}_q y)(\nu)$, for $n \ge 1$, where $\mathcal{D}^0_q y(\nu) = y(\nu)$ [1]. The q–integral of a function y is defined on $[0,\delta]$ by $\mathcal{I}_q y(\nu) = \int_0^\nu y(\xi)\,\mathrm{d}_q\xi = \nu(1-q)\sum_{k=0}^{\infty} q^k y(\nu q^k)$, for $0 \le \nu \le \delta$, provided the series is absolutely converges [1]. If ν in $[0,\tau]$, then $\int_\nu^\tau y(\nu)\,\mathrm{d}_q r = (1-q)\sum_{k=0}^{\infty} q^k[\tau y(\tau q^k) - \nu y(\nu q^k)]$, whenever the series exists. The operator $\mathcal{I}^n_q$ is given by $\mathcal{I}^0_q y(\nu) = y(\nu)$ and $\mathcal{I}^n_q y(\nu) = \mathcal{I}_q(\mathcal{I}^{n-1}_q y)(\nu)$ for $n \ge 1$ and $y \in C([0,\tau])$ [1]. It has been proved that $\mathcal{D}_q(\mathcal{I}_q y)(\nu) = y(\nu)$ and $\mathcal{I}_q(\mathcal{D}_q y)(\nu) = y(\nu) - y(0)$ whenever y is continuous at $\nu = 0$ [1]. The fractional Riemann–Liouville type q-integral of the function y on $\mathrm{I} = (0,1)$ for $\sigma \ge 0$ is defined by $\mathcal{I}^0_q y(r) = y(r)$ and

$$\mathcal{I}^\sigma_q \mathrm{u}(r) = \frac{1}{\Gamma_q(\sigma)}\int_0^r (r-q\xi)^{(\sigma-1)} y(\xi)\,\mathrm{d}_q\xi = r^\sigma(1-q)^\sigma \sum_{k=0}^{\infty} q^k \frac{\prod_{i=1}^{k-1}\left(1-q^{\sigma+i}\right)}{\prod_{i=1}^{k-1}\left(1-q^{i+1}\right)}\mathrm{u}(rq^k), \tag{309}$$

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for $r \in \mathrm{I}$ [4]. Also, the Caputo fractional q-derivative of a function y is defined by

$$ {}^{c}\mathcal{D}_q^{\sigma} y(r) = \mathcal{I}_q^{[\sigma]-\sigma}(\mathcal{D}_q^{[\sigma]} y)(r) = \frac{1}{\Gamma_q([\sigma]-\sigma)} \int_0^r (r-q\xi)^{([\sigma]-\sigma-1)} \mathcal{D}_q^{[\sigma]} y(\xi)\, \mathrm{d}_q\xi, \tag{310} $$

where $r \in J$ and $\sigma > 0$ [4]. The left-sided Hilfer fractional derivative of order $0 < \sigma < 1$ and $0 \le \xi \le 1$ of function $\mathrm{u}(r)$ is defined by $\mathcal{D}_{0^+}^{\sigma,\xi}\mathrm{u}(r) = (\mathcal{I}_{q;0^+}^{\xi(1-\sigma)} D(\mathcal{I}_{q;0^+}^{(1-\xi)(1-\sigma)}\mathrm{u}))(r)$, where $D := \frac{\mathrm{d}}{\mathrm{d}r}$ [5]. The Hilfer fractional derivative is considered as an interpolator between the Riemann-Liouville and Caputo derivative, then the following remarks can be presented to show the relation with Caputo and Riemann-Liouville operators. The operator $\mathcal{D}_{q;0^+}^{\sigma,\xi}$ also can be written as $\mathcal{D}_{0^+}^{\sigma,\xi} = \mathcal{I}_{q;0^+}^{\xi(1-\sigma)} D \mathcal{D}_{q;0^+}^{(1-\xi)(1-\sigma)} = \mathcal{I}_{0^+}^{\xi(1-\sigma)} \mathcal{D}_{q;0^+}^{\nu}$, $\nu = \sigma + \xi - \sigma\xi$ [5]. Let $\xi = 0$, the left-sided Riemann-Liouville fractional q-derivative can be presented as $\mathcal{D}_{q;0^+}^{\sigma} := \mathcal{D}_{q;0^+}^{\sigma,0}$ [5]. Let $\xi = 1$, the left-sided Caputo fractional q-derivative can be presented as ${}^{C}\mathcal{D}_{q;0^+}^{\sigma} := \mathcal{I}_{q;0^+}^{1-\sigma} D$ [5]. For $0 \le \nu < 1$, we denote the space $C_\nu(\mathrm{J})$ as $C_\nu(\mathrm{J}) := \{\mathrm{u}(r) : (0,T] \to \mathbb{R} \,|\, r^\nu \mathrm{u}(r) \in C(\mathrm{J})\}$, where $C_\nu(\mathrm{J})$ is the weighted space of the continuous functions u on the finite interval J. Obviously, $C_\nu(\mathrm{J})$ is the Banach space with the norm $\|\mathrm{u}\|_{C_\nu} = \|r^\nu \mathrm{u}(r)\|_C$. Meanwhile, $C_\nu^n(\mathrm{J}) := \left\{\mathrm{u} \in C^{n-1}(\mathrm{J}) : \mathrm{u}^{(n)} \in C_\nu(\mathrm{J})\right\}$ is the Banach space with the norm $\|\mathrm{u}\|_{C_\nu^n} = \sum_{i=0}^{n-1} \left\|\mathrm{u}^{(k)}\right\|_C + \left\|\mathrm{u}^{(n)}\right\|_{C_\nu}$, $n \in \mathbb{N}$. Moreover, $C_\nu^0(\mathrm{J}) := C_\nu(\mathrm{J})$.

Lemma 2.1. *(see [6], Lemma 20) Let $\sigma > 0$, $\xi > 0$, and $\nu = \sigma + \xi - \sigma\xi$, then*

$$ \mathcal{I}_{q;0^+}^{\nu} \mathcal{D}_{q;0^+}^{\nu} \mathrm{u} = \mathcal{I}_{q;0^+}^{\sigma} \mathcal{D}_{q;0^+}^{\sigma,\xi} \mathrm{u}, \ \mathcal{D}_{q;0^+}^{\nu} \mathcal{I}_{q;0^+}^{\sigma} \mathrm{u} = \mathcal{D}_{q;0^+}^{\xi(1-\sigma)} \mathrm{u}(r), \qquad \forall\, \mathrm{u} \in C_{1-\nu}^{\nu}(\mathrm{J}). $$

Lemma 2.2. *(see [6], Lemma 21) Let $\mathrm{u} \in L^1(\mathrm{J})$ and $\mathcal{D}_{q;0^+}^{\xi(1-\sigma)} \mathrm{u} \in L^1(\mathrm{J})$ existed, then*

$$ \mathcal{D}_{q;0^+}^{\sigma,\xi} I_{q;0^+}^{\sigma} \mathrm{u} = \mathcal{I}_{q;0^+}^{\xi(1-\sigma)} \mathcal{D}_{q;0^+}^{\xi(1-\sigma)} \mathrm{u}. $$

Lemma 2.3. *[6, Theorem 23] Let $\mathfrak{v} : \mathrm{J} \times \mathbb{R} \to \mathbb{R}$ be a function such that $\mathfrak{v} \in C_{1-\nu}(\mathrm{J})$ for any $\mathrm{u} \in C_{1-\nu}(\mathrm{J})$. A function $\mathrm{u} \in C_{1-\nu}^{\nu}(\mathrm{J})$ is a solution of fractional initial value problem $\mathcal{D}_{q;0^+}^{\sigma,\xi}\mathrm{u}(r) = \mathfrak{v}(r, \mathrm{u}(r))$, for $0 < \sigma < 1$, $0 \le \xi \le 1$, under condition $\mathcal{I}_{q;0^+}^{1-\nu}\mathrm{u}(0) = \mathrm{u}_0$, here $\nu = \sigma + \xi - \sigma\xi$, iff u satisfies the following Volterra integral equation:* $\mathrm{u}(r) = \frac{\mathrm{u}_0 r^{\nu-1}}{\Gamma_q(\nu)} + \mathcal{I}_{q;0^+}^{\sigma} \mathfrak{v}(r, \mathrm{u}(r))$.

According to Lemma 2.3, a new and important equivalent mixed type integral equation for our Problem (308) can be established. We adopt some ideas from (see [9, Lemma 2.12] to establish an equivalent mixed type integral equation:

$$ \mathrm{u}(r) = \frac{Z r^{\nu-1}}{\Gamma(\sigma)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_{\mathrm{u}}(s) ds + \frac{1}{\Gamma(\sigma)} \int_0^r (r-s)^{\sigma-1} K_{\mathrm{u}}(s) ds, \tag{311} $$

where $Z := \frac{1}{\Gamma(\nu) - \sum_{i=1}^{m} c_i (\tau_i)^{\nu-1}}$, if $\Gamma(\nu) \neq \sum_{i=1}^{m} c_i (\tau_i)^{\nu-1}$. For brevity, we shall take $K_{\mathrm{u}}(r) := \mathcal{D}_{q;0^+}^{\sigma,\xi}\mathrm{u}(r) = \mathfrak{v}(s, \mathrm{u}(r), K_{\mathrm{u}}(r))$.

Lemma 2.4. *Let $\mathfrak{v} : \mathrm{J} \times \mathbb{R}^2 \to \mathbb{R}$ be a function such that $\mathfrak{v} \in C_{1-\nu}(\mathrm{J})$ for any $\mathrm{u} \in C_{1-\nu}(\mathrm{J})$. A function $\mathrm{J} \in C_{1-\nu}^{\nu}(\mathrm{J})$ is a solution of the problem* (308) *if and only if* u *satisfies the mixed type integral* (3.3).

3 Main results

In this section, we are concerned with the existence of solutions for the problem (308).

Theorem 3.1. *Assume that*

(H1) There exist $l, p, q \in C_{1-\nu}(\mathrm{J})$ with $l^* = \sup_{r \in \mathrm{J}} l(r) < 1$ such that

$$ |\mathfrak{v}(r, \mathrm{u}, \acute{\mathrm{u}})| \le l(s) + p(r)\,|\mathrm{u}| + q(r)\,|\acute{\mathrm{u}}|, \qquad \forall\, r \in \mathrm{J},\ \mathrm{u}, \acute{\mathrm{u}} \in \mathbb{R}. $$

Then, the problem (308) *has at least one solution in $C_{1-\nu}^{\nu}(\mathrm{J}) \subset C_{1-\nu}^{\sigma,\xi}(\mathrm{J})$, where $C_{1-\nu}^{\nu} = \{\mathrm{u} \in C_{1-\nu}(\mathrm{J}), \mathcal{D}_{q;0^+}^{\nu}\mathrm{u} \in C_{1-\nu}(\mathrm{J})\}$ and $C_{1-\nu}^{\sigma,\xi} = \{\mathrm{u} \in C_{1-\nu}(\mathrm{J}), \mathcal{D}_{q;0^+}^{\sigma,\xi}\mathrm{u} \in C_{1-\nu}(\mathrm{J})\}$.*

The following arguments are based on the Banach contraction principle.

Theorem 3.2. *Assume that hypothesis*

(H2) Let $\mathfrak{v} : \mathrm{J} \times \mathbb{R}^2 \to \mathbb{R}$ *be a function such that* $\mathfrak{v} \in C_{1-\nu}^{\xi(1-\sigma)}(\mathrm{J})$ *for any* u *in* $C_{1-\nu}^{\nu}(\mathrm{J})$ *and there exist positive constants* $K > 0$ *and* $L > 0$ *such that*

$$\left|\mathrm{u}(r,\mathrm{u},\acute{\mathrm{u}}) - \mathfrak{v}(r,\overline{\mathrm{u}},\overline{\acute{\mathrm{u}}})\right| \leq K\left|\mathrm{u}-\overline{\mathrm{u}}\right| + L\left|\acute{\mathrm{u}}-\overline{\acute{\mathrm{u}}}\right|, \quad \mathrm{u},\acute{\mathrm{u}},\overline{\mathrm{u}},\overline{\acute{\mathrm{u}}} \in \mathbb{R},\ r \in \mathrm{J}.$$

If $\left(\frac{K}{1-L}\right)\frac{B(\nu,\sigma)}{\Gamma_q(\sigma)}\left(|Z|\sum_{i=1}^{m} c_i(\tau_i)^{\sigma+\nu-1} + T^{\sigma}\right) < 1$, *then the system* (308) *has a unique solution.*

Now, we prove four different types of Ulam stability results for Problem (308). We consider the Ulam-Hyers stability (UH) for the problem

$$\mathcal{D}_{q;0^+}^{\sigma,\xi}\mathrm{u}(r) = \mathfrak{v}\left(r,\mathrm{u}(r),\mathcal{D}_{q;0^+}^{\sigma,\xi}\mathrm{u}(r)\right), \qquad r \in \mathrm{J}. \tag{312}$$

Let $\epsilon > 0$ and $\varphi : \mathrm{J} \to [0,\infty)$ be a continuous function. We consider the following inequalities, for $r \in \mathrm{J}$,

$$\left|D_{q;0^+}^{\alpha,\xi}\mathrm{z}(r) - \mathfrak{v}(r,\mathrm{z}(r),D_{q;0^+}^{\alpha,\xi}\mathrm{z}(r))\right| \leq \epsilon, \qquad \left|D_{q;0^+}^{\sigma,\xi}\mathrm{z}(r) - \mathfrak{v}(r,\mathrm{z}(r),D_{q;0^+}^{\sigma,\xi}\mathrm{z}(r))\right| \leq \epsilon\varphi(r), \tag{313}$$

$$\left|D_{q;0^+}^{\sigma,\xi}\mathrm{z}(r) - \mathfrak{v}(r,\mathrm{z}(r),D_{q;0^+}^{\sigma,\xi}\mathrm{z}(r))\right| \leq \varphi(r). \tag{314}$$

Definition 3.3. The equation (308), for $r \in \mathrm{J}$ is

- UH stable if there exists a real number $C_{\mathfrak{v}} > 0$ such that for each $\epsilon > 0$ and for each solution $\mathrm{z} \in C_{1-\nu}^{\nu}(\mathrm{J})$, of the inequality (313) there exists a solution $\mathrm{u} \in C_{1-\nu}^{\nu}(\mathrm{J})$ of equation (308) with $|\mathrm{z}(r) - \mathrm{u}(r)| \leq C_{\mathfrak{v}}\epsilon$;
- generalized UH stable if there exists $\psi_{\mathfrak{v}} \in C([0,\infty),[0,\infty))$, $\psi_{\mathfrak{v}}(0) = 0$ such that for each solution $\mathrm{z} \in C_{1-\nu}^{\nu}(\mathrm{J})$, of the inequality (313) there exists a solution $\mathrm{u} \in C_{1-\nu}^{\nu}(\mathrm{J})$ of Equation (308) with $|\mathrm{z}(r) - \mathrm{u}(r)| \leq \psi_{\mathfrak{v}}\epsilon$;
- UHR stable with respect to $\varphi \in C_{1-\nu}(\mathrm{J})$, if there exists a real number $C_{\mathfrak{v}} > 0$ such that for each $\epsilon > 0$ and for each solution $\mathrm{z} \in C_{1-\nu}^{\nu}(\mathrm{J})$ of the inequality (313) there exists a solution $\mathrm{u} \in C_{1-\nu}^{\nu}(\mathrm{J})$ of Equation (308) with $|\mathrm{z}(r) - \mathrm{u}(r)| \leq C_{\mathfrak{v}}\epsilon\varphi(r)$;
- generalized UHR stable with respect to $\varphi \in C_{1-\nu}(\mathrm{J})$ if there exists a real number $C_{\mathfrak{v},\varphi} > 0$ such that for each solution $\mathrm{z} \in C_{1-\nu}^{\nu}(\mathrm{J})$ of the inequality (314) there exists a solution $\mathrm{u} \in C_{1-\nu}^{\nu}(\mathrm{J})$ of Eq. (308) with $|\mathrm{z}(r) - \mathrm{u}(r)| \leq C_{\mathfrak{v},\varphi}\varphi(r)$.

Lemma 3.4. *Let* $0 < \sigma < 1$, $0 \leq \xi \leq 1$, *if a function* $z \in C_{1-\nu}^{\nu}(\mathrm{J})$ *is a solution to the inequality* (313), *then* z *is a solution of the integral inequality*

$$\left|z(r) - A_z - \frac{1}{\Gamma(\sigma)}\int_0^r (r-s)^{\sigma-1}K_z(s)ds\right| \leq \left(\frac{|Z|\,(mc)T^{\nu+\sigma-1}}{\Gamma(\sigma+1)} + \frac{T^{\sigma}}{\Gamma(\sigma+1)}\right)\epsilon, \tag{315}$$

where, $A_z = \frac{Zr^{\nu-1}}{\Gamma(\sigma)}\sum_{i=1}^{m} c_i\int_1^{\tau_i}(\tau_i-\sigma)^{\sigma-1}K_z(s)ds$.

Lemma 3.5. *Let* $0 < \sigma < 1$, $0 \leq \xi \leq 1$, *if a function* $z \in C_{1-\nu}^{\nu}(\mathrm{J})$ *is a solution of the inequality* (313), *then* z *is a solution of the integral inequality*

$$\left|z(r) - A_z - \frac{1}{\Gamma(\sigma)}\int_0^r (r-s)^{\sigma-1}K_z(s)ds\right| \leq \left(Zr^{\gamma-1}(mc)+1\right)\epsilon\lambda_{\varphi}\varphi(r). \tag{316}$$

Theorem 3.6. *If the hypothesis* (H2) *is satisfied, then the problem* (308) *is UH stable.*

Theorem 3.7. *Assume that* (H2),

(H3) *There exists an increasing function* $\varphi \in C_{1-\nu}(\mathrm{J})$ *and there exists* $\lambda_{\varphi} > 0$ *such that for any* $r \in \mathrm{J}$, $\mathcal{I}_{q;0^+}^{\sigma}\varphi(r) \leq \lambda_{\varphi}\varphi(r)$,

are satisfied, then the problem (308) *is UHR stable.*

Example 3.8. Consider Hilfer-fractional implicit q-differential equations

$$\begin{cases} \mathcal{D}^{\sigma,\xi}_{q;1^+}\mathrm{u}(r) = \frac{1}{5e^{r+2}\left(1+|\mathrm{u}(r)|+\left|\mathcal{D}^{\sigma,\xi}_{q;1^+}\mathrm{u}(r)\right|\right)}, & r\in \mathrm{J}=(1,2],\ 0<q<1,\\ \mathcal{I}^{1-\nu}_{q;1^+}\mathrm{u}(1) = 2\mathrm{u}\left(\frac{2}{3}\right), & \nu=\sigma+\xi-\sigma\xi. \end{cases} \tag{317}$$

Now $\mathfrak{u}(t,u,\acute{u}) = \frac{1}{5e^{r+2}(1+|u|+|\acute{u}|)}$, $r\in \mathrm{J}$, $u,\acute{u}\in[0,\infty)$, and we see that $\sigma=\frac{1}{2}$, $\xi=\frac{1}{2}$ and $\nu=\frac{3}{4}$. Clearly, the function $\mathfrak{v}$ is continuous, and for $\mathrm{u},\acute{u},\overline{\mathrm{u}},\overline{\acute{u}}\in[0,\infty)$, and $r\in \mathrm{J}$,

$$\left|\mathfrak{u}(r,\mathrm{u},\acute{u})-\mathfrak{v}(r,\overline{\mathrm{u}},\overline{\acute{u}})\right| \le \frac{1}{5e^3}\left(|u-\overline{u}|+\left|\acute{u}-\overline{\acute{u}}\right|\right), \qquad K=L=\frac{1}{5e^3}.$$

Hence, the condition (H2) is satisfied. Thus, $\left(\frac{K}{1-L}\right)\frac{B(\nu,\sigma)}{\Gamma(\sigma)}\left(|Z|\sum_{i=1}^{m}c_i(\tau_i)^{\sigma+\nu-1}+T^{\sigma}\right)<1$. Theorem 3.2 implies that the problem (317) has a unique solution on J. Let $\varphi(r)=r$. Since $\mathcal{I}^{\frac{1}{2}}_{q;1^+}\varphi(r)\frac{1}{\Gamma(\frac{1}{2})}\int_1^r(r-s)^{\frac{1}{2}-1}(s)ds \le \frac{r}{\Gamma(\frac{1}{2})}\int_1^r(r-s)^{\frac{1}{2}-1}ds \le \frac{2\varphi(r)}{\sqrt{\pi}}$, condition (H3) is satisfied with $\lambda_\varphi=\frac{2}{\sqrt{\pi}}$. It follows from Theorem 3.7 that the problem (317) is UHR stable.

4 Conclusion

The implicit q-differential equation under nonlocal condition involving Hilfer q-fractional derivative, has been explored in this work. The examination of this specific condition gives us a capable device for modeling most logical wonders without the ought to remove most parameters which have a basic part within the physical translation of the considered marvels.

References

[1] C.R. Adams, *The general theory of a class of linear partial q–difference equations*, Transactions of the American Mathematical Society **26** (1924), 283–312.

[2] M.H. Annaby and Z.S. Mansour, *q–fractional calculus and equations*, Springer Heidelberg, Cambridge, 2012.

[3] F. Atici and P. W. Eloe, *Fractional q–calculus on a time scale*, Journal of Nonlinear Mathematical Physics **14** (2007), no. 3, 341–352.

[4] R.A.C. Ferreira, *Nontrivials solutions for fractional q–difference boundary value problems*, Electronic journal of qualitative theory of differential equations **70** (2010), 1–101.

[5] R. Hilfer, *Application of fractional calculus in physics*, World Scientific, Singapore, 1999.

[6] R. Hilfer, Y. Luchko, and Z. Tomovski, *Operational method for the solution of fractional differential equations with generalized riemann-lioville fractional derivative*, Fractional calculus and Applications Analysis **12** (2009), 289–318.

[7] F.H. Jackson, *q–difference equations*, American Journal of Mathematics **32** (1910), 305–314.

[8] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science, North-Holland Mathematics Studies, 2006.

[9] J. Wang and Y. Zhang, *Nonlocal initial value problems for differential equations with hilfer fractional derivative*, Applied Mathematics and Computation **266** (2015), 850–859.

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P-hyponormal classes of weighted composition operators

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Abstract

In this note, we discuss measure theoretic weighted composition operators on $L^2(\Sigma)$ in some operator classes that are weaker than p-hyponormal.

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Keywords: Conditional expectation, weighted composition operator, hyponormal, paranormal

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1 Introduction

Let (X, Σ, μ) be a σ-finite measure space and let $\varphi : X \to X$ be a measurable transformation such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ. It is assumed that the Radon-Nikodym derivative $h = (d\mu \circ \varphi^{-1})$ is finite valued or equivalently $(X, \varphi^{-1}(\Sigma), \mu)$ is σ-finite. We use the notation $L^2(\varphi^{-1}(\Sigma))$ for $L^2(X, \varphi^{-1}(\Sigma), \mu|_{\varphi^{-1}(\Sigma)})$ and henceforth we write μ in place of $\mu|_{\varphi^{-1}(\Sigma)}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote that the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of $f \in L^0(\Sigma)$ is defined by $\sigma(f) := \{x \in X : f(x) \neq 0\}$. For a finite valued function $u \in L^0(\Sigma)$, the weighted composition operator W on $L^2(\Sigma)$ induced by φ and u is given by $W = M_u \circ C_\varphi$ where M_u is a multiplication operator and C_φ is a composition operator on $L^2(\Sigma)$ defined by $M_u f := uf$ and $C_\varphi f := f \circ \varphi$, respectively.

Let $1 \leq p \leq \infty$ and $\mathcal{A} = \varphi^{-1}(\Sigma)$. If $\varphi^{-1}(\Sigma) \subseteq \Sigma$, there exists an operator $E := E^{\varphi^{-1}(\Sigma)} : L^p(\Sigma) \to L^p(\mathcal{A})$ which is called conditional expectation operator. $\mathcal{D}(E)$, the domain of E, contains the set of all non-negative measurable functions $f \in L^p(\Sigma)$ which satisfies

$$\int_A f d\mu = \int_A E(f) d\mu, \qquad A \in \varphi^{-1}(\Sigma).$$

Recall that $E : L^2(\Sigma) \to L^2(\mathcal{A})$ is a surjective, positive and contractive orthogonal projection. For more details on the properties of E see [5, 6, 8]. Since by the change of variable formula,

$$\int_X f \circ \varphi d\mu = \int_X h f d\mu, \qquad f \in L^1(\Sigma),$$

we have $\|Wf\|_2 = \|\sqrt{hE(|u|^2) \circ \varphi^{-1}} f\|_2$. Put $J := hE(|u|^2) \circ \varphi^{-1}$. It follows that W is bounded on $L^2(\Sigma)$ if and only if $J \in L^\infty(\Sigma)$ (see [9] and also [3] for a discussion about $E(\cdot) \circ \varphi^{-1}$ when φ is not invertible). Thus, W^n is a bounded operator precisely when $J_n := h_n E_n(|u_n|^2) \circ \varphi^{-n} \in L^\infty(\Sigma)$, where $n \geq 0$, $h_n = d\mu \circ \varphi^{-n}/d\mu$, $u_n = u(u \circ \varphi)(u \circ \varphi^2) \cdots (u \circ \varphi^{n-1})$ and $E_n = E^{\varphi^{-n}(\Sigma)}$. From now on, we assume that $J \in L^\infty(\Sigma)$ and $u \geq 0$. Put $h_0 = 1$, $J_1 = J$, $h_1 = h$ and $E_1 = E$.

Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on the infinite-dimensional complex Hilbert space $\mathcal{H}$. Let $T = U|T|$ be the polar decomposition for $T \in \mathcal{B}(\mathcal{H})$, where U is a partial isometry and $|T| = (T^*T)^{1/2}$.

Definition 1.1. Let $n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$ and let $p > 0$. We denote by $\mathbb{H}(p, n, k)$ the set of all operators T on $\mathcal{H}$ such that $T^{*k}(T^{*n}T^n)^p T^k \geq T^{*k}(T^n T^{*n})^p T^k$.

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2 Main results

In the following two theorems, to avoid tedious calculations, we investigate only $\mathbb{H}(p,1,k)$ and $\mathbb{H}(p,n,0)$ classes of weighted composition operators. Recall that $T \in \mathbb{H}(p,1,k)$ if and only if $T^{*k}(T^*T)^pT^k \geq T^{*k}(TT^*)^pT^k$, and $T \in \mathbb{H}(p,n,0)$ if and only if $(T^{*n}T^n)^p \geq (T^nT^{*n})^p$. We assume throughout this note that W is a bounded weighted composition operator on $L^2(\Sigma)$.

Theorem 2.1. *Let $k \in \mathbb{N}$. Then $W \in \mathbb{H}(p,1,k)$ if and only if*

$$E_k({u_k}^2 J^p) \geq E_k(u_k u(h^p \circ \varphi)(E(u^2))^{p-1}E(uu_k)), \qquad \text{on } \sigma(h_k).$$

Corollary 2.2. *Let $C_\varphi \in \mathcal{B}(L^2(\Sigma))$. Then $C_\varphi \in \mathbb{H}(p,1,k)$ if and only if $E_k(h^p) \geq E_k(h^p \circ \varphi)$ on $\sigma(h_k)$. In particular (see [1]), if $k=1$, then C_φ is p-quasihyponormal if and only if $E(h^p) \geq h^p \circ \varphi$ on $\sigma(h)$.*

Lemma 2.3. *[7] Let α and β be nonnegative and measurable functions. Then for every $f \in L^2(\Sigma)$,*

$$\int_X \alpha|f|^2 d\mu \geq \int_X |E_n(\beta f)|^2 d\mu$$

if and only if $\sigma(\beta) \subseteq \sigma(\alpha)$ and $E_n(\frac{\beta^2}{\alpha}\chi_{\sigma(\alpha)}) \leq 1$.

Theorem 2.4. *$W \in \mathbb{H}(p,n,0)$ if and only if $\sigma(u_n) \subseteq \sigma(J_n)$ and*

$$(h_n^p \circ \varphi^n)(E_n({u_n}^2))^{p-1}E_n(\frac{u_n^2}{J_n^p}\chi_{\sigma(J_n)}) \leq 1.$$

Corollary 2.5. *[4] W is p-hyponormal if and only if $\sigma(u) \subseteq \sigma(J)$ and*

$$E\left((\frac{J\circ\varphi}{J})^p \frac{u^2}{E(u^2)}\right) \leq 1.$$

Example 2.6. Let $\{m_n\}_{n=1}^\infty$ be a sequence of positive real numbers. Consider the space $\ell^2(m) = L^2(\mathbb{N}, 2^\mathbb{N}, m)$, where $2^\mathbb{N}$ is the power set of natural numbers and m is a measure on $2^\mathbb{N}$ defined by $m(\{n\}) = m_n$. Let $u = \{u_n\}_{n=1}^\infty$ be a sequence of non-negative real numbers. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a non-singular measurable transformation; i.e. $\mu \circ \varphi^{-1} \ll \mu$. Direct computation shows that (see [6])

$$h_n(k) = \frac{1}{m_k}\sum_{j\in\varphi^{-n}(k)} m_j; \qquad E_n(f)(k) = \frac{\sum_{j\in\varphi^{-n}(\varphi^n(k))} f_j m_j}{\sum_{j\in\varphi^{-n}(\varphi^n(k))} m_j};$$

and

$$J_n(k) = \frac{1}{m_k}\sum_{j\in\varphi^{-n}(k)} (u_n(j))^2 m_j,$$

for all non-negative sequence $f = \{f_n\}_{n=1}^\infty \in \ell^2(m)$ and $k \in \mathbb{N}$. Let X be the set of nonnegative integers, let Σ be the σ-algebra of all subsets of X, take m to be the point mass measure determined by the

$$m = (1, 1, 1, c, d, c^2, d^2, c^3, d^3, \cdots)$$

where c and d are fixed positive real numbers. Our point transformation φ is defined by

$$\varphi(n) = \begin{cases} 0 & n = 0, 1, \\ n-1 & n \geq 2. \end{cases}$$

Note that this example was used in [1] and [2] to show that composition operators can separate almost all weak hyponormality classes. Define u by

$$u(n) = \begin{cases} 1 & n = 0; \\ c & n \geq 1 \text{ (odd)}; \\ d & n \geq 2 \text{ (even)}. \end{cases}$$

Then by Theorem 2.1, $W \in \mathbb{H}(p,1,2)$ if and only if

$$d^2 + (\frac{1+c^2}{d^2})^p - d^2(\frac{c^3}{d^2})^p \leq 1. \tag{318}$$

And by Theorem 2.4, $W \in \mathbb{H}(p,2,0)$ if and only if

$$(1+c^2+(cd)^2)^{p-1}\{\frac{1}{(1+c^2+(cd)^2)^p} + \frac{c^2}{c^p(cd)^{2p}} + \frac{(cd)^2}{d^p(cd)^{2p}}\} \leq 1. \tag{319}$$

In particular, if $p = q = 2$ and $c = d \in (0, 0.5)$, then by (2.1) and (2.2), W neither in $\mathbb{H}(p,1,2)$ nor in $\mathbb{H}(p,2,0)$. Also if $c = d \in (0.5, 2.3)$, then W is in $\mathbb{H}(p,1,2) \cap \mathbb{H}(p,2,0)$. Now let

$$u(n) = \begin{cases} 1 & n = 0,1; \\ n & n \geq 2. \end{cases}$$

Then by Theorem 2.4, $W \in \mathbb{H}(p,2,0)$ if and only if

$$(\frac{4}{6c})^p + 4(\frac{1}{6d})^p \leq 5 \times 4^p. \tag{320}$$

and by Theorem 2.1, $W \in \mathbb{H}(p,1,2)$ if and only if

$$4(\frac{9}{4}c)^p - (\frac{1}{2})^p \geq 3, \tag{321}$$

According to (2,3) and (2,4), if $p = q = r = 2$ and $c = d \in (0, 0.11)$, then $W \in \mathbb{H}(p,2,0) \setminus \mathbb{H}(p,1,2)$. However if $c = d \in (15, \infty)$, then W is in $\mathbb{H}(p,1,2) \setminus \mathbb{H}(p,2,0)$.

References

[1] C. Burnap and I. Jung, Composition operators with weak hyponormalities, J. Math. Anal. Appl. **337** (2008), 686-694.

[2] C. Burnap, I. Jung and A. Lambert, Separating partial normality classes with composition operators, J. Operator Theory **53** (2005), 381-397.

[3] J. Campbell and J. Jamison, On some classes of weighted composition operators, Glasgow Math. J. **32** (1990), 87-94.

[4] M. Cho and T. Yamazaki, Characterizations of p-hyponormal and weak hyponormal weighted composition operators. Acta Sci. Math. (Szeged) **76** (2010), 173-181.

[5] J. Herron, Weighted conditional expectation operators, Oper. Matrices **5** (2011), 107-118.

[6] Alan Lambert, Localising sets for sigma-algebras and related point transformations, Proc. Roy. Soc. Edinburgh Sect. A **118** (1991), 111-118.

[7] A. Lambert, Hyponormal composition operators, Bull. London Math. Soc. **18** (1986), 395-400.

[8] M. M. Rao, Conditional measure and applications, Marcel Dekker, New York, 1993.

[9] T. Hoover, A. Lambert and J. Queen, The Markov process determined by a weighted composition operator, Studia Math. (Poland) **LXXII** (1982), 225-235.

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Some characterization of approximate proper solutions via nonlinear scalarization with variable ordering structure

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Abstract

In this paper, approximate proper efficient (nondominated and minimal) solutions of vector optimization problems with variable ordering structures in different senses are studied. The relationships between the introduced notions are investigated. Some scalarization results based on new functionals defined by elements from the dual cones are given. Moreover, some existing results are addressed.

Keywords: Approximate proper solutions, Variable ordering structure, Scalarization, Vector optimization
Mathematics Subject Classification [2023]: 90C29, 90C30, 90C48

1 Introduction and Preliminaries

In recent decades, approximate solutions to a multi-objective problem (MOP) have been analyzed in several frameworks with different goals. In applied optimization, the models characterize simplifed versions of real problems, and moreover, numerical algorithms may produce approximate solutions. On the other hand, vector optimization problems with variable ordering structures have been investigated by various scholars in the recent years. Let X and Y be two real normed spaces. We assume that $C : Y \rightrightarrows 2^Y$ is a given set-valued mapping which satisfies $0 \in bdC(y)$, and $C(y)$ is an ordering cone for each $y \in Y$. Furthermore, assume that $k^0 \in Y$ is a nonzero vector satisfying $C(y) + [0, \infty)k^0 \subseteq C(y)$ for all $y \in Y$. Moreover, let $\varepsilon > 0$ be given. Let $f : X \to Y$ and $\emptyset \neq \Omega \subset X$, consider the following vector optimization problem

$$Min\{f(x) : x \in \Omega\}, \tag{322}$$

The following definition is known in the literature; see [1, 4, 5]. In order to introduce some of the optimality notions, the following ordering relations are required. For $y^1, y^2 \in Y$:

$$y^1 \leq_1 y^2 \quad if \quad y^2 \in y^1 + \big(C(y^1) \setminus \{0\}\big),$$
$$y^1 \leq_2 y^2 \quad if \quad y^2 \in y^1 + \big(C(y^2) \setminus \{0\}\big).$$

Definition 1.1. [4] Let $\bar{x} \in \Omega$.
a) $\bar{x}$ is called an εk^0-minimal solution to (322) with respect to the mapping $C(.)$, $(\varepsilon k^0 - M(\Omega, f, C))$, if $\Big(f(\bar{x}) - \varepsilon k^0 - \big(C(f(\bar{x})) \setminus \{0\}\big)\Big) \cap f(\Omega) = \emptyset$.
b) $\bar{x}$ is called an εk^0-nondominated solution to (322) with respect to the mapping $C(.)$, $(\varepsilon k^0 - N(\Omega, f, C))$, if $\Big(f(\bar{x}) - \varepsilon k^0 - \big(C(f(x)) \setminus \{0\}\big)\Big) \cap \{f(x)\} = \emptyset$ for all $x \in \Omega$.

In the current work, we introduce approximate properly optimal solutions in different senses (Henig, Benson and Borwein) for problems with a VOS. The relationships between the introduced notions are studied. Furthermore, we give necessary and sufficient conditions for various approximate proper optimal notions.

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2 Proper Approximate Solutions

A very important solution concept in vector optimization, which plays an important role from both theoretical and practical points of view, is properly efficiency. This concept has been studied in many publications to eliminate the situations in which the trade-off between criteria is unbounded[6]. Introduction of the properly optimal solution in vector optimization with a VOS in different senses , including Borwein , Benson and Henig were first done by Eichfelder and Kasimbeyli [2]. After that, Hartley properly optimal solutions, super optimal solutions and robust solutions in vector optimization with a VOS were investigated [3]. In the section, we define and study proper approximate solutions of vector optimization problems with variable ordering structure in the sense of Henig, Benson, Borwein.

Definition 2.1. Let $\bar{x} \in \Omega$.
a) $\bar{x}$ is called a properly εk^0-minimal solution to (322) with respect to the mapping $C(.)$ in the sense of Henig ($\varepsilon k^0 - MHe(\Omega, f, C)$), if there is a set-valued mapping $C' : Y \rightrightarrows 2^Y$ with $C'(f(x))$ a convex cone and $C(f(x)) \setminus \{0\} \subseteq \text{int}C'(f(x))$ for all $x \in \Omega$ such that $\bar{x} \in \varepsilon k^0 - M(\Omega, f, C')$.
b) $\bar{x}$ is called a properly εk^0-minimal solution to (322) with respect to the mapping $C(.)$ in the sense of Benson ($\varepsilon k^0 - MBe(\Omega, f, C)$), if $f(\bar{x})$ is an εk^0-minimal element of the set $A_1 := \{f(\bar{x}) - \varepsilon k^0\} + clcone\Big(f(\Omega) + C(f(\bar{x})) - \{f(\bar{x}) - \varepsilon k^0\}\Big)$ with respect to the mapping $C(.)$.
c) $\bar{x}$ is called a properly εk^0-minimal solution to (322) with respect to the mapping $C(.)$ in the sense of Borwein ($\varepsilon k^0 - MBo(\Omega, f, C)$), if $f(\bar{x})$ is an εk^0-minimal element of the set $A_2 := \{f(\bar{x}) - \varepsilon k^0\} + T\Big(f(\Omega) + C(f(\bar{x})), f(\bar{x}) - \varepsilon k^0\Big)$ with respect to the mapping $C(.)$.
d) $\bar{x}$ is called a properly εk^0-nondominated solution to (322) with respect to the mapping $C(.)$ in the sense of Henig ($\varepsilon k^0 - NHe(\Omega, f, C)$), if there is a set-valued mapping $C' : Y \rightrightarrows 2^Y$ with $C'(f(x))$ a convex cone and $C(f(x)) \setminus \{0\} \subseteq \text{int}C'(f(x))$ for all $x \in \Omega$ such that $\bar{x} \in \varepsilon k^0 - N(\Omega, f, C')$.
e) $\bar{x}$ is called a properly εk^0-nondominated solution to (322) with respect to the mapping $C(.)$ in the sense of Benson ($\varepsilon k^0 - NBe(\Omega, f, C)$), if $f(\bar{x})$ is an εk^0-nondominated element of the set $A_3 := \{f(\bar{x}) - \varepsilon k^0\} + clcone\Big(\bigcup_{\omega\in\Omega}\big(f(\omega) + C(f(\omega))\big) - \{f(\bar{x}) - \varepsilon k^0\}\Big)$ with respect to the mapping $C(.)$.
f) The element $\bar{x}$ is a properly εk^0-nondominated solution to (322) with respect to the mapping $C(.)$ in the sense of Borwein ($\varepsilon k^0 - NBo(\Omega, f, C)$), if $f(\bar{x})$ is an εk^0-nondominated element of the set $A_4 := \{f(\bar{x}) - \varepsilon k^0\} + T\Big(\bigcup_{\omega\in\Omega}\big(\{f(\omega)\} + C(f(\omega))\big), f(\bar{x}) - \varepsilon k^0\Big)$ with respect to the mapping $C(.)$.

Theorem 2.2. *Let $\bar{x} \in \Omega$.*
a) If $\bar{x} \in \varepsilon k^0 - MHe(\Omega, f, C)$ or $\bar{x} \in \varepsilon k^0 - MBe(\Omega, f, C)$ or $\bar{x} \in \varepsilon k^0 - MBo(\Omega, f, C)$ then $\bar{x} \in \varepsilon k^0 - M(\Omega, f, C)$
b) If $\bar{x} \in \varepsilon k^0 - NHe(\Omega, f, C)$ or $\bar{x} \in \varepsilon k^0 - NBe(\Omega, f, C)$ or $\bar{x} \in \varepsilon k^0 - NBo(\Omega, f, C)$, then $\bar{x} \in \varepsilon k^0 - N(\Omega, f, C)$

Theorem 2.3. *Let $\bar{x} \in \Omega$. Then the following hold:*
a) $\varepsilon k^0 - MBe(\Omega, f, C) \subseteq \varepsilon k^0 - MBo(\Omega, f, C)$,
$\varepsilon k^0 - NBe(\Omega, f, C) \subseteq \varepsilon k^0 - NBo(\Omega, f, C)$.
b) If $f(\Omega) + C(f(\bar{x}))$ is starshaped with respect to $f(\bar{x})$, then $\varepsilon k^0 - MBo(\Omega, f, C) \subseteq \varepsilon k^0 - MBe(\Omega, f, C)$.
c) If $\bigcup_{\omega\in\Omega}(\{f(\omega)\} + C(f(\omega)))$ is starshaped with respect to $f(\bar{x})$, then $\varepsilon k^0 - NBo(\Omega, f, C) \subseteq \varepsilon k^0 - NBe(\Omega, f, C)$.

Theorem 2.4. *Let $\bar{x} \in \Omega$. Then the following hold:*
a) $\varepsilon k^0 - MHe(\Omega, f, C) \subseteq \varepsilon k^0 - MBe(\Omega, f, C)$.
b) If $C(f(\bar{x}))$ has a weakly compact base, then $\varepsilon k^0 - MBe(\Omega, f, C) \subseteq \varepsilon k^0 - MHe(\Omega, f, C)$.
c) If $C(f(x))$ has a weakly compact base for all $x \in X$, then $\varepsilon k^0 - NBe(\Omega, f, C) \subseteq \varepsilon k^0 - NHe(\Omega, f, C)$.

3 scalarization

In this section, some scalarization results are provided to characterize the properly εk^0 minimal, nondominated solution of 322. Let Y^* denotes the topological dual space of Y and it is equipped with the norm $\|.\|_*$. The dual cone and the positive dual cone of a set $C \subseteq Y$ is defined as $C^* := \{\zeta^* \in Y^* : \langle \zeta^*, c\rangle \geq 0,\ \forall c \in C\}$, and $C^{*^0} := \{\zeta^* \in Y^* : \langle \zeta^*, c\rangle > 0,\ \forall c \in C \setminus \{0\}\}$.

Let a map $y^* : Y \to Y^*$ and an element $\bar{x} \in \Omega$ be given. We consider two functionals $\varphi^\varepsilon_{\bar{x}}, \psi^\varepsilon_{\bar{x}} : Y \to \mathbb{R}$ with $\varphi^\varepsilon_{\bar{x}}(y) := \langle y^*(f(\bar{x})), y - (f(\bar{x}) - \varepsilon k^0)\rangle$ and $\psi^\varepsilon_{\bar{x}}(y) := \langle y^*(y), y - (f(\bar{x}) - \varepsilon k^0)\rangle$.

Theorem 3.1. *Let $\bar{x} \in \Omega$.*
a) Let $y^* : Y \to Y^*$ be a map such that $y^*(f(\bar{x})) \in C(f(\bar{x}))^*$. If

$$\varphi^\varepsilon_{\bar{x}}(y) > \varphi^\varepsilon_{\bar{x}}(f(\bar{x}) - \varepsilon k^0) = 0 \qquad for\ all\ \ y \in f(\Omega) \setminus \{f(\bar{x})\}, \tag{323}$$

then $\bar{x} \in \varepsilon k^0 - M(\Omega, f, C)$.
b) If $\bar{x} \in \varepsilon k^0 - M(\Omega, f, C)$ then there is a map $y^* : Y \to Y^*$ with $y^*(f(\bar{x})) \in C(f(\bar{x}))^*$ such that

$$\psi^\varepsilon_{\bar{x}}(y) > \psi^\varepsilon_{\bar{x}}(f(\bar{x}) - \varepsilon k^0) = 0 \qquad for\ all\ \ y \in f(\Omega) \setminus \{f(\bar{x})\},$$

c) If $intC(f(\bar{x}))^* \neq \emptyset$ then $\bar{x} \in \varepsilon k^0 - M(\Omega, f, C)$ if and only if there is a map $y^* : Y \to Y^*$ with $y^*(f(\bar{x})) \in C(f(\bar{x}))^{*^0} \setminus \{0_{Y^*}\}$ such that

$$\varphi^\varepsilon_{\bar{x}}(y) \geq \varphi^\varepsilon_{\bar{x}}(f(\bar{x}) - \varepsilon k^0) = 0 \qquad for\ all\ \ y \in f(\Omega) \setminus \{f(\bar{x})\}, \tag{324}$$

Theorem 3.2. *Let $\bar{x} \in \Omega$.*
a) If there exists a map $y^* : Y \to Y^*$ such that $y^*(f(\bar{x})) \in C^{*^0} = C(f(\bar{x}))^{*^0}$ and

$$\varphi^\varepsilon_{\bar{x}}(y) \geq 0 \qquad for\ all\ \ y \in f(\Omega) \setminus \{f(\bar{x})\}, \tag{325}$$

then $\bar{x} \in \varepsilon k^0 - MBe(\Omega, f, C)$ and $\bar{x} \in \varepsilon k^0 - MBo(\Omega, f, C)$.
b) If the topology give Y as the topological dual space of Y^*, $intC(f(\bar{x}))^* \neq \emptyset$ and $f(\Omega) + C(f(\bar{x}))$ is convex, then $\bar{x} \in \varepsilon k^0 - MBe(\Omega, f, C) = \varepsilon k^0 - MBo(\Omega, f, C)$ if and only if there exists a map $y^* : Y \to Y^*$ with $y^*(f(\bar{x})) \in C(f(\bar{x}))^{*^0}$ such that (325) holds.
c) If the topology give Y as the topological dual space of Y^*, $intC(f(\bar{x}))^* \neq \emptyset$ and $\bar{x} \in \varepsilon k^0 - MBe(\Omega, f, C)$, then there exists a map $y^* : Y \to Y^*$ with $y^*(y) \in C(f(\bar{x}))^{*^0}$ for all $y \in f(\Omega) \setminus \{f(\bar{x})\}$ such that

$$\psi^\varepsilon_{\bar{x}}(y) \geq 0 \qquad for\ all\ \ y \in f(\Omega). \tag{326}$$

Theorem 3.3. *Let $\bar{x} \in \Omega$.*
a) If $C(f(\bar{x}))$ have a weakly compact base, $y^* : Y \to Y^*$ be a map such that $y^*(f(\bar{x})) \in C(f(\bar{x}))^{*^0}$ and (325) holds, then $\bar{x} \in \varepsilon k^0 - MHe(\Omega, f, C)$.
b) Suppose that Y is a locally convex Hausdorff topological vector space and there is a locally convex topology compatible with the dual pairing in Y^*. If there exists a map such that $y^*(f(\bar{x})) \in C(f(\bar{x}))^{*^0}$ and (325) holds, then $\bar{x} \in \varepsilon k^0 - MHe(\Omega, f, C)$.
c) If the topology give Y as the topological dual space of Y^*, $intC(f(\bar{x}))^* \neq \emptyset$, $f(\Omega) + C(f(\bar{x}))$ is convex and $\bar{x} \in \varepsilon k^0 - MHe(\Omega, f, C)$, then there exists a map $y^* : Y \to Y^*$ with $y^*(f(\bar{x})) \in C(f(\bar{x}))^{*^0}$ such that (325) holds.
d) If $\bar{x} \in \varepsilon k^0 - MHe(\Omega, f, C)$, then there exists a map $y^* : Y \to Y^*$ with $y^*(y) \in C(f(\bar{x}))^{*^0}$ for all $y \in f(\Omega) \setminus \{f(\bar{x})\}$ such that

$$\psi^\varepsilon_{\bar{x}}(y) > 0 \qquad for\ all\ \ y \in f(\Omega) \setminus \{f(\bar{x})\}. \tag{327}$$

Remark 3.4. There are similar results for the the properly εk^0 nondominated solution of 322 respectively.

4 Conclusion

In this paper, some new approximate properly efficient solutions as Henig, Benson and Borwein minial/nondominated solutions, in vector optimization with a VOS, are introduced, and theorems establishing relationship between these concepts are proved. Necessary and sufficient conditions for these solutions are presented.

References

[1] G. Eichfelder, *Optimal elements in vector optimization with a variable ordering structure*, J. Optim. Theory Appl., 151 (2011), 217-240.

[2] G. Eichfelder, R. Kasimbeyli, *Properly optimal elements in vector optimization with variable ordering structure*, J. Global Optim., 60 (2014), 689-712.

[3] S. Shahbeyk, M. Soleimani-damaneh, and R. Kasimbeyli, *Hartley properly and super nondominated solutions in vector optimization with a variable ordering structure*, J. Global Optim., 71 (2018), 383-405.

[4] B. Soleimani, *Characterization of Approximate Solutions of Vector Optimization Problems with a Variable Order Structure*, J. Optim. Theory Appl., 162 (2014), 605-632.

[5] B. Soleimani, C. Tammer, *Concepts for approximate solutions of vector optimization problems with variable order structure*, Vietnam J. Math., 42 (2014), 43-566.

[6] M. Zamani, M. Soleimani-damaneh, *Proper efficiency, scalarization and transformation in multi-objective optimization: unified approaches*, Optimization, (2020), 1-22.

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Inverse problem for Sturm–Liouville operators with a transmission and parameter dependent boundary conditions

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Abstract

In this paper, we consider the inverse problem for Sturm–Liouville operator with eigenparameter dependent boundary and discontinuity conditions inside a finite closed interval. By defining a new Hilbert space and using spectral data of a kind, we prove that the potential function can be uniquely determined by a set of value of eigenfunctions at an interior point and part of two sets of eigenvalues.

Keywords: Inverse Sturm–Liouville equation, Jump condition, Hilbert space.
Mathematics Subject Classification [2023]: 34B24, 34L10, 31A25.

1 Introduction

We consider the boundary value problem $L = L(q(x); h, H; d)$

$$\ell y := -y'' + qy = \lambda y, \quad x \in [0,\, d) \cup (d,\, \pi] \tag{328}$$

$$U(y) := y'(0) - hy(0) = 0, \tag{329}$$
$$V(y) := Hy'(\pi) + \lambda y(\pi) = 0, \tag{330}$$

with the jump conditions

$$\begin{aligned} U_1(y) &:= y(d+0) - ay(d-0) = 0, \\ U_2(y) &:= y'(d+0) - by'(d-0) - cy(d-0) = 0, \end{aligned} \tag{331}$$

where $q \in L^2[0,\pi]$, $h, H \in \mathbb{R}$, $0 < H < \infty$. $d \in (0,\pi)$, a, b, c are real, and $ab > 0$.

The method of separation of variables for solving PDEs with discontinuous boundary conditions naturally led to ODE with discontinuities inside of the interval which often appear in mathematics. Inverse spectral problem consists in recovering operators from their spectral characteristics. For example the mathematical formulation of a large variety of technical and physical problem led to inverse problems such as identifying the density of the vibrating string from data collected from the sets of frequencies of oscillations of the vibrating string with barrier.

The inverse spectral Sturm–Liouville problem can be regarded as three aspects, e.g., existence, uniqueness and reconstruction of the coefficients given specific properties of eigenvalues and eigenfunctions. Here we want to look at the question of uniqueness for the above problem using two sets of spectra, or one spectrum plus part of a set of value of eigenfunctions at an interior point. We refer to the somewhat complementary surveys in [2] for further aspects of this field. One can find the similar works for continuous and discontinuous conditions in [1, 3, 4, 5, 6].

Remark 1.1. The same result can be obtained by the same method in the more general case of the parameter dependent boundary conditions

$$\begin{aligned} &y'(0) - hy(0) = 0, \\ &\lambda(y'(\pi) + H_1 y(\pi)) - H_2 y(\pi) - H_3 y'(\pi) = 0. \end{aligned}$$

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2 Asymptotic form of solutions and eigenvalues

In this section, we introduce the special inner product in the Hilbert space $(L_2(0,\pi)\oplus\mathbb{C}$ and define a linear operator A in it such that the considered problem (357)–(371) can be interpreted as the eigenvalue problem of A. So, we define a new Hilbert space inner product on $T := L_2(0,\pi)\oplus\mathbb{C}$ by

$$\langle F,G\rangle_T = \int_0^\pi f\bar{g}w\,dx + \frac{w(\pi)}{H}f_1\bar{g_1},$$

where

$$w(x) = \begin{cases} 1, & 0\le x<d,\\ \frac{1}{ab}, & d<x\le\pi, \end{cases}$$

is the weight function and $F = \begin{pmatrix} f(x)\\ f_1\end{pmatrix}$, $G = \begin{pmatrix} g(x)\\ g_1\end{pmatrix} \in T$ and $\bar{g}$, $\bar{g_1}$ are the conjugate of g, g_1 respectively. Define $R_1(f) := f(\pi) = f_1$ and $R_1'(f) := Hf'(\pi) = f_1'$. We now define the operator

$$A : T \to T$$

by

$$AF = \begin{pmatrix} \ell f\\ -R_1'(f)\end{pmatrix} \quad \text{where } F = \begin{pmatrix} f(x)\\ R_1(f)\end{pmatrix} \in D(A),$$

with domain

$$D(A) = \left\{ F = \begin{pmatrix} f(x)\\ f_1\end{pmatrix} \;\middle|\; \begin{array}{l} f(x), f'(x)\in AC[0,d)\cup(d,\pi],\ f(d\pm 0),\\ f'(d\pm 0) \text{ is defined},\ \ell f\in L^2(0,\pi),\\ U(f)=U_1(f)=U_2(f)=0, f_1=R_1(f). \end{array} \right\}$$

Thus, we can change the boundary value problem (357)–(371) of the form

$$AY = \lambda Y, \qquad Y := \begin{pmatrix} y(x)\\ R_1(y)\end{pmatrix} \in D(A),$$

in the Hilbert space T. It is easy that verified the eigenvalues of the operator A coincide with those of the problem (357)–(371). Suppose that the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of (357) under the initial conditions:

$$\varphi(0,\lambda) = 1, \qquad \varphi'(0,\lambda) = h, \tag{332}$$

and

$$\psi(\pi,\lambda) = H, \qquad \psi'(\pi,\lambda) = -\lambda. \tag{333}$$

From the linear differential equations we obtain that the Wronskian

$$\Delta(\lambda) := W(\varphi(\lambda),\psi(\lambda))$$

are independent on $x\in[0,d)\cup(d,\pi]$ respectively.

Corollary 2.1. *The zeros of $\Delta(\lambda)$ are coincide, and eigenvalues of the problem* (357)–(371) *coincide with the zeros of the function $\Delta(\lambda)$.*

Corollary 2.2. *By self–adjointness of A and Corollary 430, all eigenvalues of the problem* (357)–(371) *are real and simple.*

Theorem 2.3. *Let $\lambda=\rho^2$, $\tau := \mathrm{Im}\rho$. For equation* (357) *with boundary conditions* (368) *and jump conditions* (371) *as $|\lambda|\to\infty$, the following asymptotic formulas hold.*

$$\varphi(x;\lambda) = \begin{cases} \cos\rho x + \left(h+\frac{1}{2}\int_0^x q(t)dt\right)\frac{\sin\rho x}{\rho} + O\left(\frac{\exp(|\tau|x)}{\rho^2}\right), & x<d,\\ (b_1\cos\rho x + b_2\cos\rho(2d-x)) + f_1(x)\frac{\sin\rho x}{\rho} & x>d,\\ \quad +f_2(x)\frac{\sin\rho(2d-x)}{\rho} + O\left(\frac{\exp(|\tau|x)}{\rho^2}\right), & \end{cases}$$

$$\varphi'(x;\lambda)=\begin{cases} -\rho\sin\rho x+\left(h+\frac{1}{2}\int_0^x q(t)dt\right)\cos\rho x+O\left(\frac{\exp(|\tau|x)}{\rho}\right) & x<d,\\ \rho[(-b_1\sin\rho x+b_2\sin\rho(2d-x))]+f_1(x)\cos\rho x & x>d,\\ \quad -f_2(x)\cos\rho(2d-x)+O\left(\frac{\exp(|\tau|x)}{\rho}\right), \end{cases}$$

where

$$b_1:=\frac{a+b}{2},\qquad b_2:=\frac{a-b}{2}$$

and

$$f_1(x)=b_1\left(h+\frac{1}{2}\int_0^x q(t)dt\right)+\frac{c}{2},$$
$$f_2(x)=b_2\left(h-\frac{1}{2}\int_0^x q(t)dt+\int_0^d q(t)dt\right)+\frac{c}{2}.$$

Then the characteristic function is

$$\begin{aligned}\Delta(\lambda) &=\rho^2(b_1\cos\rho\pi+b_2\cos\rho(2d-\pi))+\rho[(f_1(\pi)-Hb_1)\sin\rho\pi\\ &\quad+(f_2(\pi)+Hb_2)\sin\rho(2d-\pi)]+O(\exp(|\tau|\pi)).\end{aligned}\tag{334}$$

Theorem 2.4. *The corresponding eigenvalues $\{\lambda_n\}$ of the boundary value problem L admit the following asymptotic form as $n\to\infty$:*

$$\sqrt{\lambda_n}=n-1+\eta_n+O\left(\frac{1}{n}\right),$$

where $\eta_n\in(0,1)$.

3 Main results

In this section, together with L we consider the boundary value problem $\tilde{L}=L(\tilde{q}(x);h,H;d)$ of the same form but a different coefficient $\tilde{q}$.

Theorem 3.1. *If*

$$\lambda_n=\tilde{\lambda}_n,\qquad W(\varphi_n,\tilde{\varphi}_n)_{(d-0)}=0$$

for any $n\in\mathbb{N}$ and $d\le\frac{\pi}{2}$, then $q(x)=\tilde{q}(x)$ a.e. on $[0,d)$.

Theorem 3.2. *Let $d\in(\frac{\pi}{2},\pi)$ be a jump point. Let $\lambda_n=\tilde{\lambda}_n$, and $W(\varphi_n,\tilde{\varphi}_n)_{(d-0)}=0$, for each $n\in\mathbb{N}$. Then $q(x)=\tilde{q}(x)$ almost everywhere on $(d,\pi]$.*

Let $l(n)$ be a subsequence of natural numbers such that

$$l(n)=\frac{n}{\sigma}(1+\epsilon_n),\quad 0<\sigma\le 1,\quad \epsilon_n\to 0$$

and let μ_n and $\tilde{\mu}_n$ be the eigenvalues of problems $L(q;h,H_1;d)$ and $L(\tilde{q};h,H_1;d)$ with the jump conditions (371), respectively, and

$$H_1y'(\pi)+\lambda y(\pi)=0\tag{335}$$

where $H\neq H_1$.

Theorem 3.3. *Let $d\in(\frac{\pi}{2},\pi]$ be a jump point and $\sigma>\frac{2d}{\pi}-1$. Let $\lambda_n=\tilde{\lambda}_n$, $\mu_{l(n)}=\tilde{\mu}_{l(n)}$ and $W(\varphi_n,\tilde{\varphi}_n)_{(d-0)}=0$, for each $n\in\mathbb{N}$. Then $q(x)=\tilde{q}(x)$ almost everywhere on $[0,d)\cup(d,\pi]$.*

Let $m(n)$ and $r(n)$ be subsequences of natural numbers such that

$$m(n)=\frac{n}{\sigma_1}(1+\epsilon_{1n}),0<\sigma_1\le 1,\epsilon_{1n}\to 0$$

and

$$r(n)=\frac{n}{\sigma_2}(1+\epsilon_{2n}),0<\sigma_2\le 1,\epsilon_{2n}\to 0.$$

Corollary 3.4. *Let $d \in (0, \frac{\pi}{2}]$ be a jump point and fix $\sigma_1 > \frac{2d}{\pi}$. Let $\lambda_{m(n)} = \tilde{\lambda}_{m(n)}$ for each $n \in \mathbb{N}$ and $W(\varphi_{m(n)}, \tilde{\varphi}_{m(n)})_{(d-0)} = 0$. Then $q(x) = \tilde{q}(x)$ almost everywhere on $[0, d)$.*

Corollary 3.5. *Let $d \in (\frac{\pi}{2}, \pi)$ be a jump point, fix $\sigma > \frac{2d}{\pi} - 1$ and $\sigma_2 > 2 - \frac{2d}{\pi}$. If for each $n \in \mathbb{N}$*

$$\lambda_n = \tilde{\lambda}_n, \qquad \mu_{l(n)} = \tilde{\mu}_{l(n)}, \qquad W(\varphi_{r(n)}, \tilde{\varphi}_{r(n)})_{(d-0)} = 0,$$

then $q(x) = \tilde{q}(x)$ almost everywhere on $[0, \pi]$.

References

[1] R. Kh. Amirov, *On Sturm–Liouville operators with discontinuity conditions inside an interval*, J. Math. Anal. Appl., 317 (2006), 163–176.

[2] G. Freiling, V. A. Yurko, *Inverse Sturm–Liouville problems and their applications*, (NOVA Science Publishers, New Yurk), 2001.

[3] Yu Pingwang, *An interior inverse problem for Sturm–Liouville operator with eigenparameter dependent boundary conditions*, Tamkang Journal of Mathematics, 42(3) (2011) 395–403.

[4] M. Shahriari, *Inverse Sturm–Liouville problems using three spectra with finite number of transmissions and parameter dependent conditions*. Bull. Iran. Math. Soc., 43(5) (2017), 1341-1355.

[5] M. Shahriari, A. Jodayree Akbarfama, G. Teschl, *Uniqueness for inverse Sturm–Liouville problems with a finite number of transmission conditions*, J. Math. Anal. Appl., 395 (2012), 19–29.

[6] M. Shahriari and H. Mirzaei, *Inverse Sturm–Liouville problem with conformable derivative and transmission conditions*, Hacet. J. Math. Stat. doi.org/10.15672/hujms.1080599.

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An efficient algorithm for solving Sturm-Liouville differential operators with a constant delay

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Abstract

In this manuscript, we present a simple and efficient computational algorithm for solving eigenvalue problems of regular second-order differential operators with a constant delay inside the interval. The numerical results demonstrate reliability and efficiency of the proposed algorithm.

Keywords: Functional differential equation, non-self-adjoint operator, constant delay.
Mathematics Subject Classification [2023]: 34B24, 34B27

1 Introduction

The method of separation of variables for solving PDEs with naturally leads to ODE inside of the interval which often appear in mathematics, physics and other fields of science and engineering. The Sturm–Liouville problems (SLPs) play a significant role in many areas of science, engineering, and mathematics [6]. A standard form of the Sturm–Liouville equation is given as

$$\ell y := -\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = \lambda w(x)y, \tag{336}$$

where $p(x)$, $q(x)$ and $w(x)$ are specified functions, which, depending on the studies considered, are required to satisfy additional conditions. The above differential equation, in conjunction with the separated boundary conditions of the form

$$y'(a) - hy(a) = 0, \quad y'(b) + Hy(b) = 0, \qquad h, H \in \mathbb{R}$$

is known as a regular SLP if $p(x)$, $w(x) > 0$, and $p'(x)$, $q(x)$, and $w(x)$ are continuous functions over the finite interval $[a, b]$. A λ for which the above problem has a non-trivial solution is called an eigenvalue, and the corresponding solution, an eigenfunction.

In this paper, we introduce the notion second-order differential operators with a constant delay. We present a simple and efficient computational algorithm for solving eigenvalues of Sturm–Liouville problem with a constant delay. For more developments on Sturm–Liouville functional problems, we refer to [2, 5].

2 Formulas and method

We consider the boundary value problem

$$\sum_{i=0}^{2} q_i(x)y^{(i)}(x) + q(x)y(x-d) = \lambda w(x)y(x), \qquad x \in (a, b) \tag{337}$$

with the separated boundary conditions

$$\sum_{i=0}^{1} p_{1i}y^{(i)}(a) = 0, \qquad \sum_{i=0}^{1} p_{2i}y^{(i)}(b) = 0, \tag{338}$$

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where $q_i(x) \in L(a,b)$, $w(x) > 0$ on (a,b), $q(x) \in L(d,b)$, and $q(x) = 0$ for $x < d$; $d \in (a,b)$ are real. Actually, the problem becomes solving a set of the differential equation (337) with the boundary conditions (338). Unfortunately, it is difficult to obtain exact eigenvalues of Eqs. (337)-(338). Therefore, the numerical methods must be proposed to solve such an eigenvalue problem. In this section, avoiding solving Eq. (337) directly, by using the similar methods of [3] and [4], we introduce a simple method to determine the eigenvalues of the Sturm-Liouville equation with constant delay. For this purpose, we expand the solution $y(x)$ of Eq. (337) in the following polynomial form:

$$y(x) = \sum_{i=0}^{N} c_i x^i + R_N(x) \tag{339}$$

where c_i are unknown coefficients, R_N is the rest, and N is a certain positive integer which is chosen large enough such that the rest has a negligible error. We further assume that our solution can be approximated by

$$y(x) \approx \sum_{i=0}^{N} c_i x^i. \tag{340}$$

Ultimately, The main idea is to obtain a homogeneous system of equation in the unknowns c_i, $i = 0, 1, \ldots N$, the roots of whose characteristic equation constitute the eigenvalues of the problem. First, from the boundary conditions (338) and using (340) we obtain the following equations

$$\sum_{i=0}^{N} (f_{m,i} - \lambda k_{m,i}) c_i = 0, \qquad m = 0, 1, \tag{341}$$

where

$$f_{0,i} = \sum_{m=0}^{N} \sum_{n=0}^{1} D_m^n P_{1n} a^{m-n} = 0, \quad k_{0,i} = 0, \qquad i = 1, 2, \ldots N \tag{342}$$

$$f_{1,i} = \sum_{m=0}^{N} \sum_{n=0}^{1} D_m^n P_{2n} b^{m-n} = 0, \quad k_{1,i} = 0, \qquad i = 1, 2, \ldots N \tag{343}$$

with

$$D_0^n = 1, \qquad D_n^m = \prod_{i=m-n+1}^{m} i. \tag{344}$$

Next, substituting (340) in (337), using the linearity property, we get

$$\sum_{i=0}^{N} \sum_{j=0}^{2} c_i D_j^i q_j(x) x^{i-j} + \sum_{i=0}^{N} c_i q(x)(x-d)^i - \lambda \sum_{i=0}^{N} c_i w(x) x^i = 0 \tag{345}$$

We note that to obtain other linear algebraic equations of unknown coefficients c_i, we multiply both sides of (345) by $x^l (l = 0, 1, \ldots, N-2)$ and then integrate with respect to x between a and b we get,

$$\sum_{i=0}^{N} (f_{ji} - \lambda k_{ji}) c_i = 0, \qquad j = 2, 3, \ldots, N \tag{346}$$

where

$$f_{ji} = \int_a^b \left(\sum_{n=0}^{2} q_n(x) x^{i+j-n-2} \right) dx + \int_a^b q(x)(x-d)^{i+j-2} dx, \tag{347}$$

$$k_{ji} = \int_a^b w(x) x^{i+j-2} dx$$

Therefore, from Eqs. (342), (343) and (345) we get the following linear system of $N+1$ equations.

$$\sum_{i=0}^{N}(f_{ji}-\lambda k_{ji})c_i=0, \qquad j=0,1,\ldots,N. \tag{348}$$

For simplicity, the system (348) can be written in matrix form

$$(\mathbf{F}-\lambda\mathbf{K})\mathbf{C}=0 \tag{349}$$

where $\mathbf{F}$ and $\mathbf{K}$ are square $(N+1)(N+1)$ matrices with $F_{mi}=f_{mi}$ and $K_{mi}=k_{mi}$, and $\mathbf{C}=(c_0,c_1,\ldots,c_N)^t$. To obtain a nontrivial solution of the system of equations, the determinant of the coefficient matrix of the system must be vanish; then we get a characteristic function in eigenvalues λ:

$$\det(\mathbf{F}-\lambda\mathbf{K})=0 \tag{350}$$

such that $\det(\mathbf{F}-\lambda\mathbf{K})$ is a polynomial of degree $N-1$ in λ. The eigenvalues of the original problem would be those that satisfy (350).

3 Numerical results

In this section, we consider three examples to demonstrate the performance and efficiency of the present algorithm.

Example 3.1. Consider the Sturm-Liouville differential operator (337) with a constant delay subject to Dirichlet boundary conditions

$$-y''+q(x)y(x-d)=\lambda y, \qquad y(0)=0, \quad y(1)=0. \tag{351}$$

Remark 3.2. Let us consider the function $S(x,\lambda)$ be the solution of the integral equation

$$S(x,\lambda)=\frac{\sin\rho x}{\rho}+\int_0^x\frac{\sin\rho(x-t)}{\rho}q(t)S(t-d,\lambda)dt \tag{352}$$

where $\lambda=\rho^2$. Let $M\in\mathbb{N}$ be such that $dM<1\le d(M+1)$, applying the method of [2], using successive approximations, we get

$$S(x,\lambda)=S_0(x,\lambda)+S_1(x,\lambda)+\cdots+S_M(x,\lambda), \quad \text{where } S_0(x,\lambda)=\frac{\sin\rho x}{\rho}, \quad x>0$$

$$S_k(x,\lambda)=\int_{kd}^x\frac{\sin\rho(x-t)}{\rho}q(t)S_{k-1}(t-d,\lambda)dt,\ x\ge kd \quad \text{and } S_k(x,\lambda)=0, \quad x\le kd.$$

The characteristic function of the problem (353) in $d=0.5$ is

$$\Delta(\lambda)=S(1,\lambda)=\frac{\sin\rho}{\rho}+\frac{1}{\rho}\int_{0.5}^1\sin\rho(1-t)q(t)\sin\rho(t-0.5)dt.$$

So, by applying the Maple we get the exact eigenvalues.

Example 3.3. Consider the following Sturm-Liouville differential operator with a constant delay with the following Dirichlet boundary conditions

$$-4(x^2-1)y''-2(2x-1)y'+q(x)y(x-d)=\lambda y, \qquad y(0)=0, \quad y(1)=0. \tag{353}$$

We note that in this problem the eigenvalues in the case $d=1$ is $\lambda_n=(2n)^2$ (see [1]).

Table 12: The numerical and exact values of the first 5 eigenvalues with $d = 0.5$ and $N = 14$, 24 in example 3.1

k	$q(x) = x$, λ_k	Λ_k^{14}	$\lvert\lambda_k - \Lambda_k^{14}\rvert$	Λ_k^{24}	$\lvert\lambda_k - \Lambda_k^{24}\rvert$
1	10.11118838067	10.11118843541	$5.4738E-08$	10.11118840108	$2.0410E-08$
2	39.10068836107	39.10068811753	$2.4353E-07$	39.10068827422	$8.6851E-08$
3	88.74759370468	88.74759431228	$6.0760E-07$	88.74759400045	$2.9577E-07$
4	158.2880086008	158.2880563179	$4.7717E-05$	158.2880090765	$4.7566E-07$
5	246.7881361529	246.7915371381	$3.4009E-03$	246.7878834185	$2.5273E-04$

Table 13: Approximate solution and errors in computed solution of Example 3.3

k	$\Lambda_k^{14},\ d=1$	$\Lambda_k^{14},\ d=0.9$	$\Lambda_k^{14},\ d=0.8$	$\Lambda_k^{14},\ d=0.5$
1	4	4.04999376522492	4.13338935220738	4.40712468546217
2	16	15.8418025604232	15.6703687270841	15.7203506505346
3	36	36.2414522447466	36.2671066982499	35.5766341677692
4	64	63.7816937825459	64.0399319574756	64.0320208123033
5	100	100.083773089782	99.7077197057886	100.359287053023

References

[1] W. N. Everitt, *A catalogue of Sturm-Liouville differential equations. Sturm–Liouville Theory: Past and Present*, (2005), 271–331.

[2] G. Freiling and V. A. Yurko, *Inverse problems for Sturm–Liouville differential operators with a constant delay*. Applied Mathematics Letters, 25(11) (2012), 1999–2004.

[3] M. A., Hajji, Q. M. Al-Mdallal, and F. M. Allan, *An efficient algorithm for solving higher-order fractional Sturm–Liouville eigenvalue problems*, Journal of Computational Physics, 272 (2014), 550–558.

[4] M. Shahriari and J. Manafian, *An efficient algorithm for solving the fractional Dirac differential operator*. Adv. Math. Models. Appl., 5(3) (2020), 289–297.

[5] M. Shahriari, B. N. Saray, and J. Manafian, *Reconstruction of the Sturm–Liouville differential operators with discontinuity conditions and a constant delay*. Indian Journal of Pure and Applied Mathematics, 51 (2020), 659–668.

[6] A. Zettl, *Sturm-liouville theory*, (No. 121). American Mathematical Soc. (2012).

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Weak amenability of commutative Fréchet algebras

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Abstract

The aim of this paper is to investigate the weak amenability of commutative Fréchet algebras and some consequences that may be derived from it. We give some examples of weakly amenable and non-weakly amenable commutative non-normable Fréchet algebras.

Keywords: Weak amenability, commutative Fréchet algebra
Mathematics Subject Classification [2023]: Primary 46H20, 46A04, 46M40; Secondary 43A10, 43A20, 46A45.

Introduction and Preliminary

For a Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-module X the first Hochschild cohomology group $\mathcal{H}^1(\mathcal{A}, \mathcal{X})$ plays a special role in the characterization of some important properties of Banach algebras. For example, the Banach algebra $\mathcal{A}$ is amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{X}^*) = \{0\}$, for all Banach $\mathcal{A}$-module X. This definition was introduced by Johnson [3]. The Banach algebra $\mathcal{A}$ (commutative or non-commutative) is also called weakly amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$ [4]. This generalizes that was introduced by Bade, Curtis and Dales in [1], that a commutative Banach algebra $\mathcal{A}$ is weakly amenable if $\mathcal{H}^1(\mathcal{A}, X) = \{0\}$ for every symmetric Banach $\mathcal{A}$-module X. Many important Banach algebras was studied in this concept for example every C^*-algebras are weakly amenable[2]. A Fréchet space is a complete metric topological vector space whose topology is given by a countable separating family of seminorms $\{p_n : n \in \mathbb{N}\}$. A Fréchte space $(\mathcal{A}, \{p_n\}_{n\in\mathbb{N}})$ is called a Fréchet algebra if it is an algebra and the seminorms are submultiplicative; that is,

$$p_n(xy) \leq p_n(x)p_n(y), \qquad x, y \in \mathcal{A}, n \in \mathbb{N}.$$

We also denote the unit linked of Fréchet algebra $\mathcal{A}$ by $\mathcal{A}^\sharp$ that is $\mathcal{A} \oplus \mathbb{C}$ and its product is defined by:

$$(a + \alpha e)(b + \beta e) := ab + \alpha b + \beta a + \alpha\beta e$$

where $e = (0, 1)$; $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$. $\mathcal{A}^\sharp$ is also a Fréchet algebra with family of semi-norms $\{P_n^*\}$ related to the above semi-norms that is defined by $P_n^*(a + \lambda e) := P_n(a) + |\lambda|$ for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Recall that the definition of weak amenability of Banach algebras involves dual spaces and so, if we want to find idea for Fréchet algebras, we need to establish clear what we mean by the dual space $\mathcal{X}^*$ of a Fréchet space $\mathcal{X}$. We consider dual space $\mathcal{X}^*$ will be endowed with the strong topology, which means the topology induced by family of seminorms $\{p_B\}$ where B is a bounded subset of $\mathcal{X}$ such that:

$$p_B(f) = \sup\{|f(x)| : x \in B\}.$$

Note that this coincides with the usual norm topology if $\mathcal{X}$ is a normed space. Let $\mathcal{A}$ be a Fréchet algebras, a Fréchet space $\mathcal{X}$ is a Fréchet $\mathcal{A}$-module provided that $\mathcal{A}$-module actions, (right-left), on $\mathcal{X}$ are separately continuous. Also note that given a Fréchet $\mathcal{A}$-module $\mathcal{X}$, $\mathcal{X}^*$ with strong topology is a locally convex $\mathcal{A}$-module with continuous actions in the following way:

$$\langle x, a \cdot \phi\rangle = \langle x \cdot a, \phi\rangle \quad \textit{and} \quad \langle x, \phi \cdot a\rangle = \langle a \cdot x, \phi\rangle,$$

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where $a \in \mathcal{A}$, $\phi \in \mathcal{X}^*$ and $x \in \mathcal{X}$. In particular, the dual space $\mathcal{A}^*$ of $\mathcal{A}$ is a locally convex $\mathcal{A}$-module. An A-module X is called symmetric, whenever $a \cdot x = x \cdot a$ for all $a \in \mathcal{A}$ and $x \in X$. A derivation $D : \mathcal{A} \to \mathcal{X}$, is a bounded linear map satisfying $D(ab) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in \mathcal{A}$. Let Δ be the space of characters on Fréchet algebra $\mathcal{A}$ and $\phi \in \Delta \cup \{0\}$ then $\mathbb{C}$ the space of complex numbers becomes symmetric Fréchet $\mathcal{A}$-module by product:

$$a \cdot x = x \cdot a = \phi(a)x, \qquad (a \in \mathcal{A}, x \in \mathbb{C}),$$

which is denoted by $\mathbb{C}_\phi$. A point derivation is a derivation from $\mathcal{A}$ into $\mathbb{C}_\phi$ such that:

$$d(ab) = \phi(a)d(b) + d(a)\phi(b), \qquad (a, b \in \mathcal{A}).$$

Suppose that $\mathcal{A}$ is an involutive algebra, a seminorm p on $\mathcal{A}$ is called C^*-seminorm if $p(x^*x) = p(x)^2$ for all $x \in \mathcal{A}$. An involutive topological algebra, whose topology is induced by a family of C^*-seminorms, is called C^*-convex algebra. A complete C^*-convex algebra is called locally C^* -algebra. A Fréchet locally C^*-algebra is clearly a metrizable complete C^*-convex algebra, that equivalently means metrizable locally C^*-algebra.

1 Weak amenability of commutative Fréchet algebra

Let $\mathcal{A}$ be a commutative Fréchet algebra and X be a symmetric Fréchet $\mathcal{A}$-module. The following is a straightforward extension of weak amenability of commutative Banach algebras is given by [1] to commutative Frechet algebras.

Definition 1.1. A commutative Fréchet algebra $\mathcal{A}$ is called weakly amenable if X is a symmetric Fréchet $\mathcal{A}$-module and $D : \mathcal{A} \to X$ is a continuous derivation, then D is necessarily zero.

For Banach algera $\mathcal{A}$, Bade, Curtis and Dales in [1] observed that this definition is equivalent to every derivations from $\mathcal{A}$ into $\mathcal{A}'$ is zero. Next theorem shows that the similar result holds for Fréchet algebra. Recall that $\mathcal{A}'$ with strong topology is locally convex $\mathcal{A}$-module.

Lemma 1.2. *Let $\mathcal{A}$ be a commutative Fréchet algebra, if $\mathcal{A}^\sharp$ is weakly amenable then $\mathcal{A}$ is weakly amenable.*

Proof. Let X be a symmetric Fréchet $\mathcal{A}$-module and $D : \mathcal{A} \to X$ be a derivation. Then X is $\mathcal{A}^\sharp$-module with following actions

$$(a + \lambda) \cdot x := a \cdot x + \lambda x, \quad x \cdot (a + \lambda) := x \cdot a + \lambda x,$$

where $x \in X, a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. If $D^\sharp : \mathcal{A}^\sharp \to X$ given by $D^\sharp(a + \lambda e) := D(a)$ where e is the unit linked of $\mathcal{A}$ and $a \in \mathcal{A}$, then $D^\sharp$ is derivation and by weak amenability of $\mathcal{A}^\sharp$ we have $D^\sharp \equiv 0$. So this implies that $D \equiv 0$. □

Proposition 1.3. *Let $\mathcal{A}$ be a commutative Fréchet algebra then $\mathcal{A}$ is weakly amenable if and only if $\bar{\mathcal{A}^2} = \mathcal{A}$.*

Proof. Let $\mathcal{A}$ be weakly amenable then if $\bar{\mathcal{A}^2} \neq \mathcal{A}$ by [6, Corollary 3 of Theorem 18.1] there exists a $\lambda \in \mathcal{A}'$ with $\lambda \neq 0$ such that $\lambda(\mathcal{A}^2) = 0$. Now define $D(a) = \lambda(a)\lambda$ which is a non-zero continuous operator from $\mathcal{A}$ into $\mathcal{A}'$ and as $D(ab) = 0$ for all $a, b \in \mathcal{A}$ by assumption, we have:

$$(a \cdot Db)(c) = \lambda(b)\lambda(ac) = 0, \qquad c \in \mathcal{A}$$

so $a \cdot D(b) = 0$ hence D is a non-zero derivation which contradict with weak amenability of $\mathcal{A}$.

For the reverse let $\bar{\mathcal{A}^2} = \mathcal{A}$ we prove that $\mathcal{A}^\sharp$ is weakly amenable. Suppose that $K_\sharp = \{a \otimes b : ab = 0, a, b \in \mathcal{A}^\sharp\}$ and put $K_0 = K_\sharp \cap (\mathcal{A}\hat{\otimes}\mathcal{A})$. For a Freéchet $\mathcal{A}^\sharp$-module X consider a derivation $D : \mathcal{A}_\sharp \to X$ and for arbitrary $\lambda \in X'$ define $R_x(a) = \lambda(a \cdot x), (a \in A_\sharp)$ then $R_x \in (\mathcal{A}^\sharp)'$ and $R : x \to R_x$ from X into $(\mathcal{A}^\sharp)'$ is a continuous operator such that $R(a \cdot x) = a \cdot R(x)$ for all $a \in \mathcal{A}_\sharp$ and $x \in X$. Now let $\tilde{D} = R \circ D$, this operator is continuous derivation from $\mathcal{A}^\sharp$ into $(\mathcal{A}_\sharp)'$, $\tilde{D}$ vanishes on $\bar{K_\sharp^2}$ and therefore $\tilde{D}$ vanishes on $\bar{K_0^2} = \overline{K_\sharp(\mathcal{A}\hat{\otimes}\mathcal{A})}$. In particular $\tilde{D}$ vanishes on elements of the form $(1 \otimes c - c \otimes 1)a \otimes b = a \otimes bc - ac \otimes b$ where $a, b, c \in \mathcal{A}$ and 1 is unit of $\mathcal{A}^\sharp$ so we have $\langle bc, \tilde{D}(a)\rangle$. using the derivation on the right-hand side we get $\langle ab, \tilde{D}(ac)\rangle = 0$. Since $\bar{\mathcal{A}^2} = \mathcal{A}$ we see that $\tilde{D} \equiv 0$ on $\mathcal{A}$ and hence on $\mathcal{A}^\sharp$ but in the other side since λ and hence R is arbitrary so $D \equiv 0$ which shows $\mathcal{A}^\sharp$ is weakly amenable and by the above lemma $\mathcal{A}$ is weakly amenable. □

Theorem 1.4. *Let $\mathcal{A}$ be a commutative Fréchet algebra then $\mathcal{A}$ is not weakly amenable if and only if there is non-zero derivation $D : \mathcal{A} \to \mathcal{A}'$.*

Proof. Suppose that $\mathcal{A}$ is not weakly amenable then by the above proposition $\bar{\mathcal{A}^2} \neq \mathcal{A}$ and hence there is a non-zero derivation as in the first part of the proof of this proposition.

For the convers, let $D : \mathcal{A} \to A'$ be a non-zero continuous derivation, so there is a $x_0 \in \mathcal{A}$ and semi-norm p on $\mathcal{A}'$ such that $p(D(x_0)) \neq 0$. Let Y_p be the completion of $\frac{\mathcal{A}'}{ker\ p}$ with norm $P(x + ker\ p) := p(x)$, for all $x \in \mathcal{A}$. Clearly Y_p is a symmetric Fréchet $\mathcal{A}$-module and

$$\pi_p \circ D : \mathcal{A} \to Y_p,$$

is a continuous derivation with $\pi_p \circ D(x_0) \neq 0$, where π_p is canonical projection. So this implies that $\mathcal{A}$ is not weakly amenable Fréchet algebra. □

Example 1.5. Let $C^\infty[0,1]$ be the space of all complex-valued functions on $[0,1]$ which are smooth functions. It is commutative Fréchet algebra with pointwise production and countable family of semi-norms $\{p_n\}$ as follows.

$$p_n(f) = \sum_{i=0}^{n} \sup\left\{\left|D^i f(x)\right|,\ x \in [0,1]\right\}, \quad f \in C^\infty[0,1], n \in \mathbb{N}.$$

If $\phi : C^\infty[0,1] \to \mathbb{C}$ given by $\phi(f) := f(0)$ for all $f \in C^\infty[0,1]$, then ϕ is homomorphism. And we have the non-zero point derivation $D : C^\infty[0,1] \to \mathbb{C}_\phi$ by $D(f) := f'(0)$. since every point derivation is a derivation so $C^\infty[0,1]$ is not weakly amenable.

Example 1.6. Let X be a hemicompact and completely regular space and let $C(X)$ denote the algebra of continuous, complex-valued functions on X and for $f \in C(X)$; let f^* denote the function, $f^*(x) = \overline{f(x)}$. Then $C(X)$ equipped with the family of countable seminorms $\{\rho_K\}$, $\rho_K(f) = \sup\{|f(x)| : x \in K\}$, where K is a compact subset of X, is a commutative locally C^*-algebra [5]. Since X is union of increasing sequence of compact sets like $\{K_n\}_{n\in\mathbb{N}}$ so if D is a non-zero derivation from $C(X)$ into a $C(X)$-module then for a $f \in C(X)$, $Df \neq 0$ but $Df_{|_{K_n}} = 0$ for all $n \in \mathbb{N}$ because $C(K_n)$ is C^*-algebra and by [2, Theorem 2.1] every C^*-algebra is weakly amenable which contradict with our hypothesis that D is non-zero hence $C(X)$ is commutative weakly amenable Fréchet algebra.

References

[1] W.G. Bade, P.C. Curtis Jr. and H.G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras,* Proc. London Math. Soc. 55 (1987), 359377.

[2] H. G. Dales, F. Ghahramani and N. Grønbæk, *Derivations into iterated duals of Banach algebras,* Studia Math. 128 (1998), 19-54.

[3] B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. 127 (1972), 96 pp.

[4] B. E. Johnson, *Derivations from $L^1(G)$ into $L^1(G)$ and $L^\infty(G)$,* Lecture Notes in Math. 1359, Springer-Verlag, Berlin (1988), 191-198.

[5] N. C. Phillips, *Inverse limits of C*-algebras,* J. Operator Theory 19 (1988), 159 195. (1975), 298 pp

[6] F. Treves, *Topological vector spaces, distributions and kernels,* Academic Press, New York-London, 1967.

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Some alternative Tychonoff fixed point theorems

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Abstract

In this paper, we give several fixed point theorems in two terms. First we present an alternative version of the Tychonoff fixed point theorem. Then we show some generalizations of the alternative Tychonoff fixed point theorem by the concept of measure of noncompactness in Fréchet spaces.

Keywords: Measure of noncompactness, Tychonoff fixed point theorem, An alternative Tychonoff fixed point theorem
Mathematics Subject Classification [2023]: 47H09, 39B62, 34A12

1 Introduction

Assume that E is a given Banach space with the norm $\| \cdot \|$. Denote by B(x, r) the closed ball in E centered at x and with radius r. If X is a subset of E, then the symbols $\bar{X}$ and $ConvX$ stand for the closure and the convex hull of X, respectively. Moreover, we denote by $\mathcal{M}_E$ the family of all nonempty bounded subsets of E and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact sets. In 1930, Kuratowski [1] introduced the first measure of noncompactness in the following way:

$$\alpha(X) := \inf\{\delta > 0 | X = \bigcup_{i=1}^{n} X_i \text{ for some } X_i \text{ with } diam(X_i) < \delta \text{ for } 1 \leq i \leq n < \infty\}$$

for $X \in \mathcal{M}_E$ of E, where diam(X) denotes the diameter of a set X

$$diam(X) := \sup\{d(x, y) | x, y \in X\}.$$

Another important measure is the so-called Hausdorff (or ball), defined as follows [4]

$$\chi(X) := \inf\{\epsilon > 0 | \text{X can be covered by finitely many balls with radius} \leq \epsilon \text{ in E}\}.$$

Now, we provide some definitions and auxiliary facts which will be needed in the sequel.

Definition 1.1. [2] A mapping $\mu : \mathcal{M}_E \to \mathbb{R}_+ = [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1. The family $ker\mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathcal{N}_E$.
2. $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$.
3. $\mu(X) = \mu(\bar{X})$.
4. $\mu(ConvX) = \mu(X)$.
5. $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
6. If $\{X_n\}$ is a nested sequence of closed sets from $\mathcal{M}_E$ such that $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

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Definition 1.2. [3] A map $F : X \subseteq E \to E$ is said to be a condensing operator if $\alpha(F(Y)) < \alpha(Y)$, for all bounded sets $Y \subset X$ with $\alpha(Y) \neq 0$, where α is Kuratowskii measure of noncompactness.

Now, we define k-set contractive map which is used in the following theorems.

Definition 1.3. A map $F : X \subset E \to E$ is said to be a k-set contractive (here $k \geq 0$ is a constant) if $\alpha(F(Y)) \leq k\alpha(Y)$ for all bounded sets $Y \subset X$.

We turn our attention to establishing a fixed point theory for nonself maps, called Alternative Leray-Schauder fixed point theorem.

Theorem 1.4 (Alternative Leray-Schauder fixed point theorem)**.** *Let E be a Banach space, C a closed, convex subset of E, U is an open subset of C and $p \in U$. Suppose that $F : U \to C$ is a continuous, compact (that is, $F(U)$ is a relatively compact subset of C) map. Then either*
(A1) F has a fixed point in U, or
(A2) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u) + (1\lambda)p$.

In 1980, *Mönch* generalized Schauder's fixed point theorem and we present his result here.

Theorem 1.5 (*Mönch*'s fixed point theorem)**.** *[3] Let C be a closed, convex subset of a Banach space E with $x_0 \in C$. Suppose there is a continuous map $F : C \to C$ with the following property:*

$$D \subset C \text{ countable and } D \subset co(x_0 \cup F(D)) \text{ imply that } D \text{ is relatively compact.}$$

Then F has a fixed point in C.

Theorem 1.6. *Let E be a Banach space, C a closed, convex subset of E, U is an open subset of C and $p \in U$. Suppose that $F : U \to C$ is a continuous, condensing map with $F(U)$ a bounded set in C and assume that*

$$x \neq tF(x) + (1 - t)p \, for \, x \in \partial U \, and \, t \in (0, 1) \tag{354}$$

holds. Then F has a fixed point in U.

Theorem 1.7. *Let E be a Banach space, C a closed, convex subset of E, U is an open subset of C and $p \in U$. Suppose that $F : U \to C$ is a continuous, k-set contractive (here $0 \leq k < 1$ is a constant) map with $F(U)$ a bounded set in C. Then either*
(A1) F has a fixed point in U, or
(A2) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u) + (1 - \lambda)p$.

Definition 1.8. [2] A Fréchet space is a locally convex space that is complete with respect to a translation-invariant metric.

In 1934, Tychonoff proved the following theorem.

Theorem 1.9 (Tychonoff fixed point theorem)**.** *[3] Let E be a Hausdorff locally convex linear topological space, C be a compact, and convex subset of E, and $F : C \to C$ be a continuous mapping. Then F has at least one fixed point.*

2 Main results

In this section, we extend the above theorems in the Fréchet space by using Theorem 1.9. The following theorem is extended of Theorem 1.4.

Theorem 2.1 (Alternative Tychonoff fixed point theorem)**.** *Let E be a Fréchet space, C a closed, convex subset of E, U is an open subset of C and $p \in U$. Suppose that $F : U \to C$ is a continuous, compact (that is, $F(U)$ is a relatively compact subset of C) map. Then either*
(A1) F has a fixed point in U, or
(A2) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u) + (1 - \lambda)p$.

Also *Mönch* Theorem 1.5 in Banach space can be extended in Fréchet space as follows.

Theorem 2.2. *Let C be a closed, convex subset of a Fréchet space E with $x_0 \in C$. Suppose there is a continuous map $F : C \to C$ with the following property:*

$$D \subset C \text{ countable and } D \subset co(x_0 \cup F(D)) \text{ imply that } D \text{ is relatively compact.}$$

Then F has a fixed point in C.

Theorem 2.3. *Let E be a Fréchet space, C a closed, convex subset of E, U is an open subset of C and $p \in U$. Suppose that $F : U \to C$ is a continuous map which satisfies Mönch's condition (that is, if $D \subset U$ is countable and $D \subset co(p \cup F(D))$, then D is compact) and assume that*

$$x \neq tF(x) + (1-t)p \, for \, x \in \partial U \, and \, t \in (0,1) \tag{355}$$

holds. Then F has a fixed point in U.

Next theorem is an extension of Theorem 1.6.

Theorem 2.4. *Let E be a Fréchet space, C a closed, convex subset of E, U is an open subset of C and $p \in U$. Suppose that $F : U \to C$ is a continuous, condensing map with $F(U)$ a bounded set in C and assume that*

$$x \neq tF(x) + (1-t)p \, for \, x \in \partial U \, and \, t \in (0,1) \tag{356}$$

holds. Then F has a fixed point in U.

Finally, we extend Theorem 2.5 as follows.

Theorem 2.5. *Let E be a Fréchet space, C a closed, convex subset of E, U is an open subset of C and $p \in U$. Suppose that $F : U \to C$ is a continuous, k-set contractive (here $0 \leq k < 1$ is a constant) map with $F(U)$ a bounded set in C. Then either*
(A1) F has a fixed point in U, or
(A2) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u) + (1-\lambda)p$.

3 Conclusion

We establish some generalizations of the alternative Tychonoff fixed point theorem via the measure of noncompactness which has an application in existence of solutions of integral equations in the Fréchet space.

References

[1] A. Aghajani, J. Banas and N. Sabzali, *Some generalizations of Darbo fixed point theorem and applications*, Bull. Belg. Math. Soc. Simon Stevin, 20 (2013) 345-358.

[2] R.P. Agarwal and D. O'Regan, *Fixed point theory and applications*, Cambridge University Press, 2004.

[3] R.R. Akmerov, M.I. Kamenski, A.S. Potapov, A.E. Rodkina, B.N. Sadovskii, *Measures of Noncompactness and Condensing Operators*, Birkhauser-Verlag, Basel, 1992.

[4] J.B. Conway, *A Course in Functional Analysis*, 2nd edition, Springer-Verlag, 1994, ISBN 0-387-97245-5.

[5] K. Kuratowski, *Sur les espaces complets*, Fund. Math. 5 (1930) 301309.

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Existence of infinitely many solutions for a perturbed mixed boundary value problem involving the one-dimensional p-Laplacian

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Abstract

The existence of infinitely many weak solutions for perturbed mixed boundary value problem involving the one-dimensional p-Laplacian operator is established under suitable assumptions on the nonlinear term. our approach is based on recent variational methods for smooth functionals defined on reflexive Banach spaces.

Keywords: Perturbed mixed boundary value problem; Variational methods; Critical point theory.
Mathematics Subject Classification [2023]: 18A32, 18F20, 05C65 (at least 1 and at most 3)

1 Introduction

Consider the following perturbed mixed boundary value problem

$$\begin{cases} -(\rho(x)|u'|^{p-2}u')' + s(x)|u|^{p-2}u = \lambda f(x,u) + \mu g(x,u) & x \in (a,b), \\ u(a) = u'(b) = 0, \end{cases} \tag{357}$$

where $p > 2$, $\lambda > 0$ and $\mu \geq 0$ are real numbers, $a, b \in \mathbb{R}$ with $a < b$, ρ, $s \in L^{\infty}([a,b])$ with $\rho_0 = essinf_{x\in[a,b]}\rho(x) > 0$, $s_0 = essinf_{x\in[a,b]}s(x) \geq 0$ and $f, g : [a,b] \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory function.

2 Preliminaries

Our main tool is Theorem 2.1 consequences of the existence result of a local minimum [6, Theorem 3.1] which is inspired by Ricceri's variational principle (see [20]).

Theorem 2.1. *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous, For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\left(\sup_{v\in\Phi^{-1}((-\infty,r))} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional*

$$I_\lambda = \Phi - \lambda\Psi$$

to $\Phi^{-1}((-\infty, r))$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

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(b) *If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*

(1) *I_λ possesses a global minimum, or*

(2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.*

(c) *If $\delta < +\infty$, then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either*

(1) *there is a global minimum of Φ which is a local minimum of I_λ, or*

(2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ, that converges weakly to a global minimum of Φ.*

In order to study the problem (357), the variational setting is the space

$$X := \{u \in W^{1,p}([a,b]); u(a) = 0\}$$

endowed with the norm

$$||u|| := \left(\int_a^b \rho(x)|u'(x)|^p dx + \int_a^b s(x)|u(x)|^p dx \right)^{1/p}.$$

We observe that the norm $||.||$ is equivalent to the usual one.
It is well known that $(X, ||.||)$ is compactly embedded in $(C^0([a,b]), ||.||_\infty)$ and

$$||u||_\infty \leq \left(\frac{(b-a)^{p-1}}{\rho_0} \right)^{\frac{1}{p}} ||u||, \tag{358}$$

for every $u \in X$.
We need the following proposition in the proof of Theorem 3.1.

Proposition 2.2. *[13, Proposition 2.3] Let $T : X \to X^*$ be the operator defined by*

$$T(u)v = \int_a^b \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_a^b s(x)|u(x)|^{p-2}u(x)v(x)dx$$

for every $u, v \in X$. Then T admits a continuous inverse on X^*.

We use the following notations:

$$||\rho||_\infty := esssup_{x\in[a,b]}\rho(x)$$
$$||s||_\infty := esssup_{x\in[a,b]}s(x)$$

Corresponding to f and g we introduce the functions $F : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $G : [a,b] \times \mathbb{R} \to \mathbb{R}$, respectively, as follows

$$F(x,t) := \int_0^t f(x,\xi)d\xi, \quad \text{for all } (x,t) \in [a,b] \times \mathbb{R}$$

and

$$G(x,t) := \int_0^t g(x,\xi)d\xi, \quad \text{for all } (x,t) \in [a,b] \times \mathbb{R}.$$

We mean by a (weak) solution of problem (357), any function $u \in X$ such that

$$\int_a^b \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_a^b s(x)|u(x)|^{p-2}u(x)v(x)dx$$

$$-\lambda \int_a^b f(x,u(x))v(x)dx - \mu \int_a^b g(x,u(x))v(x)dx = 0,$$

for every $v \in X$.

3 Main results

Put

$$k_2 := \frac{2(p+1)\rho_0}{2^p(p+1)||\rho||_\infty + (p+2)(b-a)^p||s||_\infty},$$

$$f_\infty := \liminf_{\xi\to+\infty} \frac{\int_a^b sup_{|v|<\xi}F(x,v(x))dx}{\xi^p},$$

$$g_\infty := \lim_{\xi\to+\infty} \frac{\int_a^b sup_{|v|<\xi}G(x,v(x))dx}{\xi^p},$$

$$\beta := \limsup_{\xi\to+\infty} \frac{\int_{\frac{a+b}{2}}^b F(x,\xi)dx}{\xi^p},$$

$$\lambda_1 := \frac{\rho_0}{p\beta k_2(b-a)^{p-1}}$$

and

$$\lambda_2 := \frac{\rho_0}{pf_\infty(b-a)^{p-1}}$$

We now formulate our main result as follows.

Theorem 3.1. *Assume that*

(A_1) $F(x,t) \geq 0$ *for every* $(x,t) \in [a,b]\times[0,+\infty)$;

(A_2) $f_\infty < \beta k_2$;

Then, for each $\lambda \in (\lambda_1,\lambda_2)$ *and for every* L^1*-Carathéodory function* $g : [a,b]\times\mathbb{R}\to\mathbb{R}$ *whose potential* $G(x,t) := \int_0^t g(x,\xi)d\xi$ *for all* $(x,t)\in[a,b]\times[0,+\infty)$, *is a non-negative function satisfying the condition*

$$g_\infty < +\infty \tag{359}$$

if we put

$$\mu_{g,\lambda} := \frac{\rho_0 - \lambda p(b-a)^{p-1}f_\infty}{p(b-a)^{p-1}g_\infty}$$

where $\mu_{g,\lambda} = +\infty$ *when* $g_\infty = 0$, *the problem* (357) *has an unbounded sequence of weak solutions for every* $\mu \in [0,\mu_{g,\lambda})$.

Proof. Our aim is to apply Theorem 2.1(b) to problem (357). X be reflexive real Banach space and subset $W^{1,p}([a,b])$. To this end, fix $\overline{\lambda}\in(\lambda_1,\lambda_2)$ and g satisfying our assumptions. Since $\overline{\lambda}<\lambda_2$, we have

$$\mu_{g,\overline{\lambda}} := \frac{\rho_0 - \overline{\lambda}p(b-a)^{p-1}f_\infty}{p(b-a)^{p-1}g_\infty} > 0.$$

Now fix $\overline{\mu}\in(0,\mu_{g,\overline{\lambda}})$ and the functionals $\Phi,\Psi : X\to\mathbb{R}$ given by

$$\Phi(u) := \frac{||u||^p}{p} = \frac{1}{p}\left(\int_a^b \rho(x)|u'(x)|^p dx + \int_a^b s(x)|u(x)|^p dx\right), \tag{360}$$

$$\Psi(u) := \int_a^b \left(F(x,u(x)) + \frac{\overline{\mu}}{\overline{\lambda}}G(x,u(x))\right)dx \tag{361}$$

and put

$$I_{\overline{\lambda}}(u) := \Phi(u) + \overline{\lambda}\Psi(u)$$

for each $u\in X$.
Not that the weak solutions of (357) are exactly the critical points of $I_{\overline{\lambda}}$. Then functionals Φ,Ψ satisfy

the regularity assumptions of Theorem 2.1. Indeed, by standard arguments, we have that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_a^b \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_a^b s(x)|u(x)|^{p-2}u(x)v(x)dx$$

for any $v \in X$. Furthermore, the differential $\Phi' : X \to X^*$ admits a continuous inverse with Proposition 2.2. On the other hand, the fact that X is compactly embeddedb into $C^0([a,b])$ implies that the functional Ψ is well defined, continuously Gâteaux differentiab and with compact derivative, whose Gâteaux derivative is given by

$$\Psi'(u)(v) = \int_a^b f(x,u(x))v(x)dx + \frac{\overline{\mu}}{\overline{\lambda}}\int_a^b g(x,u(x))v(x)dx.$$

Furthermore, $\lim_{||u||\to+\infty}\Phi(u) = +\infty$ for all $u \in X$ and so is coercive.
First of all, we will show that $\overline{\lambda} < \frac{1}{\gamma}$. Hence, let $\{\xi_n\}$ be a sequence of positive numbers suh that $\lim_{n\to+\infty}\xi_n = +\infty$. Put $r_n := \frac{\rho_0\xi_n^p}{p(b-a)^{p-1}}$ for all $n \in \mathbb{N}$. Then, for all $u \in X$ with $\Phi(u) \leq r_n$ taking (358) into account, one has

$$\begin{aligned}\Phi^{-1}(-\infty, r_n) &= \{u \in X; \Phi(u) < r_n\}\\ &= \{u \in X; \frac{||u||^p}{p} < \frac{\rho_0\xi_n^p}{p(b-a)^{p-1}}\}\\ &= \{u \in X; |u| < \xi_n\}\end{aligned}$$

Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\Phi^{-1}((-\infty,r_n))}\frac{\Big(\sup_{v\in\Phi^{-1}((-\infty,r_n))}\Psi(v)\Big) - \Psi(u)}{r_n - \Phi(u)}\\ &\leq \frac{\sup_{v\in\Phi^{-1}((-\infty,r_n))}\Psi(v)}{r_n}\\ &\leq \frac{p(b-a)^{p-1}}{\rho_0}\Big(\frac{\int_a^b sup_{|v|<\xi_n}F(x,v(x))dx}{\xi_n^p} + \frac{\overline{\mu}}{\overline{\lambda}}\frac{\int_a^b sup_{|v|<\xi_n}G(x,v(x))dx}{\xi_n^p}\Big)\end{aligned}$$

Moreover, from the assumption (A_2) and (359)

$$\begin{aligned}\gamma &= \liminf_{r\to+\infty}\varphi(r) \leq \liminf_{n\to+\infty}\varphi(r_n)\\ &\leq \frac{p(b-a)^{p-1}}{\rho_0}\Big(f_\infty + \frac{\overline{\mu}}{\overline{\lambda}}g_\infty\Big) < +\infty.\end{aligned}$$

the assumption $\overline{\mu} \in (0, \mu_{g,\overline{\lambda}})$ immediately yields $\gamma < \frac{1}{\overline{\lambda}}$. Let $\overline{\lambda}$ be fixed. We claim that the functional $I_{\overline{\lambda}}$ is unbounded from below. Since

$$\frac{1}{\overline{\lambda}} < \frac{p\beta k_2(b-a)^{p-1}}{\rho_0}$$

there exist a sequence $\{d_n\}$ and a positive constant d such that $\lim_{n\to+\infty} d_n = +\infty$ and

$$\frac{1}{\overline{\lambda}} < d < \frac{pk_2(b-a)^{p-1}\int_{\frac{a+b}{2}}^b F(x,d_n)dx}{\rho_0 d_n^p} \tag{362}$$

Let $\{w_n\}$ be a sequence in X defined by

$$w_n(x) = \begin{cases}\frac{2d_n}{b-a}(x-a) & \text{if } x \in \left[a, \frac{a+b}{2}\right[\\ d_n & \text{if } x \in \left[\frac{a+b}{2}, b\right]\end{cases} \tag{363}$$

for any $n \in \mathbb{N}$, it is easy to see that $w_n \in X$ and one has

$$\Phi(w_n) \leq \frac{\rho_0 d_n^p}{pk_2(b-a)^{p-1}} \tag{364}$$

from (A_1) and since G is nonnegative, for the definition of Ψ, we infer

$$\Psi(w_n) \geq \int_{\frac{a+b}{2}}^{b} F(x, d_n)dx, \tag{365}$$

so

$$I_{\overline{\lambda}}(w_n) = \Phi(w_n) + \overline{\lambda}\Psi(w_n) \leq \frac{\rho_0 d_n^p}{pk_2(b-a)^{p-1}} - \overline{\lambda}\int_{\frac{a+b}{2}}^{b} F(x, d_n)dx < \frac{\rho_0 d_n^p}{pk_2(b-a)^{p-1}}(1 - \overline{\lambda}d), \tag{366}$$

for every $n \in \mathbb{N}$ large enogh. Since $\overline{\lambda}d > 1$ and $\lim_{n\to+\infty} d_n = +\infty$, we have

$$\lim_{n\to+\infty} I_{\overline{\lambda}}(w_n) = -\infty.$$

Then, the functional $I_{\overline{\lambda}}$ is unbounded from below, and it follows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, by (2.1)(b), there exists a sequence $\{u_n\}$ of critical points of $I_{\overline{\lambda}}$ such that $\lim_{n\to+\infty} ||u_n|| = +\infty$ and the conclusion is achieved. □

4 Applications

In this section, we point out some consequences and applications of the results previously obtained. First, we present the following consequence of Theorem 3.1 with $\mu = 0$.

Theorem 4.1. *Assume that all the assumptions in the Theorem 3.1 hold. Then, for each $\lambda \in \Lambda_1$ the problem*

$$\begin{cases} -(\rho(x)|u'|^{p-2}u')' + s(x)|u|^{p-2}u = \lambda f(x,u) & x \in (a,b), \\ u(a) = u'(b) = 0, \end{cases} \tag{367}$$

has an unbounded sequence of weak solutions in X.

Corollary 4.2. *Assume that the assumption (A_1) in Theorem 3.1 holds. Suppose that*

$$f_\infty < \frac{\rho_0}{p(b-a)^{p-1}} < \beta k_2.$$

Then, the problem

$$\begin{cases} -(\rho(x)|u'|^{p-2}u')' + s(x)|u|^{p-2}u = f(x,u) & x \in (a,b), \\ u(a) = u'(b) = 0, \end{cases} \tag{368}$$

has an unbounded sequence of weak solutions in X.

Remark 4.3. We notice that instead of Assumption (A_2) in Theorem 3.1 we are allowed to assume the more general condition

(A_3) there exist two sequence $\{\alpha_n\}$ and $\{\beta_n\}$ with

$$\alpha_n^p < k_2\beta_n^p$$

for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} \beta_n = +\infty$ such that

$$\liminf_{n\to+\infty} \frac{\int_a^b sup_{|v|<\beta_n} F(x, v(x))dx - \int_{\frac{a+b}{2}}^{b} F(x, \alpha_n)dx}{k_2\beta_n^p - \alpha_n^p} < \limsup_{n\to+\infty} \frac{\beta_n^P}{\int_{\frac{a+b}{2}}^{b} F(x, \alpha_n)dx}.$$

Obviously, Assumption (A_2) follows from Assumption (A_3), by choosing $\alpha_n = 0$ for all $n \in \mathbb{N}$. Moreover, if we assume (A_3), instead of (A_2) and set $r_n = \frac{\rho_0 \beta_n^p}{p(b-a)^{p-1}}$ for all $n \in \mathbb{N}$, by the same reasoning as in Theorem 3.1, we obtain

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\Phi^{-1}((-\infty,r_n))} \frac{\left(\sup_{v\in\Phi^{-1}((-\infty,r_n))}\Psi(v)\right)-\Psi(u)}{r_n-\Phi(u)}\\ &\leq \frac{\sup_{v\in\Phi^{-1}((-\infty,r_n))}\Psi(v)-\Psi(w_n)}{r_n-\Phi(w_n)}\\ &\leq (\frac{pk_2(b-a)^{p-1}}{\rho_0})\frac{\int_a^b sup_{|v|<\beta_n}F(x,v(x))dx-\int_{\frac{a+b}{2}}^b F(x,\alpha_n)dx}{k_2\beta_n^p-\alpha_n^p},\end{aligned}$$

where $w_n(x)$ is defined as given in (363), for $x \in [a,b]$ with α_n insetead of d_n. We then have the same conclusion as in Theorem 3.1 with λ_2 replaced by

$$\lambda_2' := \left((\frac{pk_2(b-a)^{p-1}}{\rho_0})\liminf_{n\to+\infty}\frac{\int_a^b sup_{|v|<\xi_n}F(x,v(x))dx-\int_{\frac{a+b}{2}}^b F(x,\alpha_n)dx}{k_2\beta_n^p-\alpha_n^p}\right)^{-1}.$$

We want to point out a simple consequence of Theorem 3.1, in which the function f has separated variables.

Corollary 4.4. *Let $f_1 \in L^1([a,b])$ and $f_2 \in C(\mathbb{R})$ be two functions. Put $\tilde{F}(t) = \int_a^b f_2(\xi)d\xi$ for all $t \in \mathbb{R}$ and assume that*

(A_4) $f_1(x) > 0$ *for each* $x \in [a,b]$ *and* $f_2(t) \geq 0$ *for each* $t \in \mathbb{R}$;

(A_5) $\liminf_{n\to+\infty}\frac{\tilde{F}(\xi_n)}{\xi_n^p} < +\infty$;

(A_6) $\limsup_{n\to+\infty}\frac{\tilde{F}(\xi_n)}{\xi_n^p} = +\infty$.

Then, for every $\lambda \in \left(0, \frac{\rho_0}{p(b-a)^{p-1}\int_a^b f_1(x)dx \liminf_{n\to+\infty}\frac{\tilde{F}(\xi_n)}{\xi_n^p}}\right)$ *the problem*

$$\begin{cases} -(\rho(x)|u'|^{p-2}u')' + s(x)|u|^{p-2}u = \lambda f_1(x)f_2(t) & x \in (a,b),\\ u(a) = u'(b) = 0, \end{cases} \tag{369}$$

has an unbounded sequence of weak solutiona in X.

Proof. Set $f(x,u) = f_1(x)f_2(u)$ for each $(x,u) \in [a,b]\times\mathbb{R}$. Since

$$F(x,t) = f_1(x)\tilde{F}(t)$$

from (A_4), (A_5) and (A_6) we obtain (A_1) and (A_2), respectively. □

Here, we present a simple consequence of Theorem 3.1 in the case when f does not depend upon x.

Theorem 4.5. *Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that:*

(A_7) $H(t) = \int_0^t h(\xi)d\xi \geq 0$ *for every* $t \in [0,+\infty)$.

(A_8) *Putting* $f_\infty' := \liminf_{t\to+\infty}\frac{\max_{|\xi|\leq t}H(\xi)}{t^p}$ *and* $\beta' := \limsup_{t\to+\infty}\frac{H(t)}{t^p}$ *one has* $f_\infty' < k_2\beta'$.

Then, for each $\lambda \in \left(\frac{\rho_0}{p\beta' k_2(b-a)^{p-1}}, \frac{\rho_0}{pf'_\infty(b-a)^{p-1}}\right)$ *and for every* $q \in \mathbb{C}^0(\mathbb{R})$ *such that*

$$Q(t) = \int_0^t q(\xi)d\xi \geq 0 \tag{370}$$

for every $t \in [0, +\infty)$,

$$Q_\infty := \limsup_{t\to+\infty} \frac{\max_{|\xi|\leq t} Q(\xi)}{t^p} < +\infty,$$

if we put $\mu^* := \frac{\rho_0 - \lambda p(b-a)^{p-1} f'_\infty}{p(b-a)^{p-1} Q_\infty}$, *for every* $mu \in [0, \mu^*)$ *the problem*

$$\begin{cases} -(\rho(x)|u'|^{p-2}u')' + s(x)|u|^{p-2}u = \lambda h(u) + \mu q(u) & x \in (a,b), \\ u(a) = u'(b) = 0, \end{cases} \tag{371}$$

admits an unbounded sequece of weak solutions.

Proof. Put $f(x,t) = h(t)$ and $g(x,t) = q(t)$ for every $x \in [a,b] \times \mathbb{R}$. obviously (A_7) implies (A_1). Moreover, (A_8) that is (A_2) holds and conclusion follows directly from Theorem 3.1 upon observing that $G(x,t) = Q(t)$ for every $x \in [a,b] \times \mathbb{R}$. □

Corollary 4.6. *Let* $h : \mathbb{R} \to \mathbb{R}$ *be a continuous and non-negative function such that:*

$$\liminf_{t\to+\infty} \frac{H(t)}{t^p} < k_2 \limsup_{t\to+\infty} \frac{H(t)}{t^p}.$$

Then, for every

$$\lambda \in \left(\frac{\rho_0}{pk_2(b-a)^{p-1} \limsup_{t\to+\infty} \frac{H(t)}{t^p}}, \frac{\rho_0}{p(b-a)^{p-1} \liminf_{t\to+\infty} \frac{H(t)}{t^p}} \right),$$

for every $q \in \mathbb{C}^0(\mathbb{R})$ *such that for every*

$$tq(t) \geq 0 \tag{372}$$

$$\lim_{|t|\to+\infty} \frac{q(t)}{|t|^{p-1}} = 0 \tag{373}$$

and for every $\mu \geq 0$, *problem (371) admits an unbounded sequence of weak solutions.*

Proof. It follows from Theorem 4.5 on observing that, in view of the non-negativity of h, (A_7) holds and $f'_\infty = \liminf_{t\to+\infty} \frac{H(t)}{t^p}$, and also (372) implies (370). Moreover, by (373) one has

$$0 \leq \limsup_{t\to+\infty} \frac{\max_{|\xi|\leq t} Q(\xi)}{t^p} = \limsup_{t\to+\infty} \frac{\{Q(t), Q(-t)\}}{t^p}.$$

owing to the Hopital rule we have

$$\lim_{t\to+\infty} \frac{Q(t)}{t^p} = \lim_{t\to+\infty} \frac{Q(-t)}{t^p} = \pm \lim_{t\to+\infty} \frac{q(\pm t)}{|t|^{p-1}} = 0.$$

Hence $Q_\infty = 0$ and our conclusion follows. □

Now, put

$$f_0 := \liminf_{\xi\to 0^+} \frac{\int_a^b sup_{|v|<\xi} F(x, v(x))dx}{\xi^p},$$

$$\beta_0 := \limsup_{\xi\to 0^+} \frac{\int_{\frac{a+b}{2}}^b F(x,\xi)dx}{\xi^p},$$

$$\lambda_4 := \frac{\rho_0}{p\beta_0 k_2 (b-a)^{p-1}}$$

and

$$\lambda_5 := \frac{\rho_0}{pf_0 (b-a)^{p-1}}$$

Using Theorem 2.1(c) and arguing as in the proof of Theorem 3.1, we can obtain the following result.

Theorem 4.7. *Assume that* (A_1) *holds and*

(A_9) $\quad f_0 < k_2\beta_0$.

Then, for every $\lambda \in (\lambda_4, \lambda_5)$ *and for every* L^1*-Carathéodory function* g*, such that, there exists* $d > 0$ *such that* $G(x,t) \geq 0$ *for every* $(x,t) \in [a,b] \times [0,d]$,

$$g_0 := \lim_{t \to 0^+} \frac{\int_a^b sup_{|\xi| \leq t} G(x,\xi)dx}{t^p} < +\infty, \tag{374}$$

if we put

$$\mu'_{g,\lambda} := \frac{\rho_0 - \lambda p(b-a)^{p-1} f_0}{p(b-a)^{p-1} g_0}$$

where $\mu'_{g,\lambda} = +\infty$ *when* $g_0 = 0$*, then for every* $\mu \in [0, \mu'_{g,\lambda})$ *problem* (357) *has a sequence of weak solutions, which converges strongly to zero* $\in X$.

Proof. Fix $\overline{\lambda} \in (\lambda_4, \lambda_5)$ and let g be a function that satisfies the condition (374). Since $\overline{\lambda} < \lambda_5$, we obtain

$$\mu'_{g,\overline{\lambda}} := \frac{\rho_0 - \overline{\lambda} p(b-a)^{p-1} f_0}{p(b-a)^{p-1} g_0} > 0.$$

Now fix $\overline{\mu} \in (0, \mu'_{g,\overline{\lambda}})$ and set

$$I_{\overline{\lambda}}(u) := \Phi(u) + \overline{\lambda}\Psi(u)$$

for all $(x,t) \in [a,b] \times \mathbb{R}$. We take Φ, Ψ and $I_{\overline{\lambda}}$ as in the proof of Theorem 3.1. Now, as it has been pointed out before, the functionals Φ and Ψ satisfy the regularity assumptions required in Theorem 2.1. As first step, we will prove that $\overline{\lambda} < \frac{1}{\delta}$. Then, let ξ_n be a sequence of positive numbers such that $\lim_{n \to +\infty} \xi_n = 0$ and

$$\lim_{n \to +\infty} \frac{\int_a^b sup_{|t| < \xi_n} F(x,t)dx}{\xi_n^p} = f_0.$$

By the fact that $inf_X \Phi = 0$ and the defintion of δ, we have $\delta = \liminf_{r \to 0^+} \varphi(r)$.

Putting $r_n := \frac{\rho_0 \xi_n^p}{p(b-a)^{p-1}}$. Then, as in showing (374) in the proof of Theorem 3.1, we can prove that $\delta < +\infty$. From $\overline{\mu} \in (0, \mu'_{g,\overline{\lambda}})$, the following inequalities hold

$$\delta \leq \frac{p(b-a)^{p-1}}{\rho_0}\Big(f_0 + \frac{\overline{\mu}}{\overline{\lambda}} g_0\Big) < \frac{p(b-a)^{p-1}}{\rho_0} f_0 + \frac{\rho_0 - \overline{\lambda} p f_0 (b-a)^{p-1}}{\rho_0 \overline{\lambda}}$$

Therefore, $\overline{\lambda} < \frac{1}{\delta}$. Let $\overline{\lambda}$ be fixed. We claim that the functional $I_{\overline{\lambda}}$ has not a local minimum at zero. Since

$$\frac{1}{\overline{\lambda}} < \frac{p\beta_0 k_2 (b-a)^{p-1}}{\rho_0},$$

there exist a sequence $\{d_n\}$ of positive number and $d > 0$ such that $\lim_{n \to +\infty} d_n = 0^+$ and

$$\frac{1}{\overline{\lambda}} < d < \frac{pk_2(b-a)^{p-1}}{\rho_0} \frac{\int_{\frac{a+b}{2}}^b F(x,d_n)dx}{d_n^p},$$

for each $n \in \mathbb{N}$ large enough. Let $\{w_n\}$ be the sequence in X defined in (363). From (A_1) one has (365) holds. Note that $\overline{\lambda}d > 1$. Then, as in showing (366), we can obtain

$$I_{\overline{\lambda}}(w_n) < \frac{\rho_0 d_n^p}{p(b-a)^{p-1}k_2}(1-\overline{\lambda}d) \leq 0 = \Phi(0) + \overline{\lambda}\Psi(0)$$

for each $n \in \mathbb{N}$ large enough. Then, we see that zero is not a local minimum of $I_{\overline{\lambda}}$. This, together with the fact zero is the only golbal minimum of Φ, we deduce that the energy functional $I_{\overline{\lambda}}$ has not a local minimum at the unique global minimum of Φ.Therefore, by Theorem 2.1, there exists a sequence $\{u_n\}$ of critical points of $I_{\overline{\lambda}}$ which converges weakly to zero. In view of the fact that the embedding $X \hookrightarrow C^0([a,b])$ is copmact, we khnow that the critical points converge strongly to zero, and the proof is compalete. □

Remark 4.8. Under thecondition $f_0 = 0$ and $\beta_0 = +\infty$, Theorem 2.2 ensures that for every $\lambda > 0$ and for each $\mu = [0, \frac{\rho_0}{p(b-a)^{p-1}g_0})$, problem (357) admits a sequence of weak solutions which strongly converges to 0 in X. Moreover, if $g = 0$, the result holds for every $\lambda > 0$ and $\mu \geq 0$.

Remark 4.9. Applying Theorem 2.2, results similar to Theorem 1.3 Corollaries 4.2 and 4.4, can be obtained. We omit the discussions here.

References

[1] G.A. Afrouzi, S. Heidarkhani, *Three solutions for a quasilinear boundary value problem,* Nonl. Anal. TMA 69 (2008) 3330-3336.

[2] R.P. Agarwal, H.B. Thompson, C.C. Tisdell, *On the existence of multiple solutions to boundary value problems for second order ordinary differential equations,* Dynam. Systems Appl. 16 (2007) 595-609.

[3] D. Averna, G. Bonanno, *A mountain pass theorem for a suitable class of functions,* Rocky Mountain J. Math. 39 (2009) 707-727.

[4] D. Averna, G. Bonanno, *A three critical points theorem and its applications to the ordinary Dirichlet problem,* Topol. Meth. Nonl. Anal. 22 (2003) 93-103.

[5] D. Averna, G. Bonanno, *Three solutions for quasilinear two-point boundary value problem involving the one-dimensional p-Laplacian,* Proc. Edinb. Math. Soc. 47 (2004) 257-270.

[6] G. Bonanno, *A critical point theorem via the Ekeland variational principle,* Nonl. Anal. TMA 75 (2012) 2992-3007.

[7] G. Bonanno, S. Heidarkhani, D. O'Regan, *Nontrivial solutions for Sturm-Liouville systems via a local minimum theorem for functionals,* Bull. Aust. Math. Soc. 89 (2014) 8-18.

[8] G. Bonanno, A. Sciammetta, *An existence result of one non-trivial solution for two point boundary value problems,* Bull. Aust. Math. Soc. 84 (2011) 288-299.

[9] G. Bonanno, P. F. Pizzimenti, *Existence results for nonlinear elliptic problems,* Appl. Anal. 92 (2013) 411-423.

[10] G. Bonanno, P. F. Pizzimenti, *Neumann boundary value problems with not coercive potential,* Mediterr. J. Math. 9 (2012) 601-609.

[11] H. Brézis, *Analyse Fonctionnelle: Theorie Et Applications,* Masson, Paris, 1983.

[12] G. D'Aguì, *Multiplicity results for nonlinear mixed boundary value problem,* Bound. Value Probl. 2012, 2012:134.

[13] G. D'Aguì, S. Heidarkhani and G. Molica Bisci, *Multiple solutions for a perturbed mixed boundary value problem involving the one-dimensional p-Laplacian* Electronic Journal. 388 (2013)1-14.

[14] S. Heidarkhani, J. Henderson, *Critical point approaches to quasilinear second order differential equations depending on a parameter,* Topol. Meth. Nonl. Anal. 44, No. 1, (2014) 177-197.

[15] J. Henderson, H.B. Thompson, *Existence of multiple solutions for second order boundary value problems,* J. Differ. Eqs. 166 (2000), no. 2, 443-454.

[16] J. Henderson, H.B. Thompson, *Multiple symmetric positive solutions for a second order boundary value problem,* Proc. Amer. Math. Soc. 128 (2000), no. 8, 2373-2379.

[17] R. Livrea, *Existence of three solutions for a quasilinear two point boundary value problem,* Arch. Math. 79 (2002) 288-298.

[18] P. Pucci and J. Serrin, *A mountain pass theorem,* J. Differ. Eqs. 63 (1985) 142-149.

[19] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations,* CBMS Reg. Conf. Ser. Math. 65, Amer. Mat. Soc., Providence, 1986.

[20] B. Ricceri, *A general variational principle and some of its applications,* J. Comput. Appl. Math. 113 (2000) 401-410.

[21] G. Talenti, *Some inequalities of Sobolev type on two-dimensional spheres,* in: W. Walter (Ed.), General Inequalities, 5, Oberwolfach, 1986, in: Internat. Schriftenreihe Numer. Math., vol. 80, Birkhäuser, Basel, 1987, pp. 401-408.

[22] E. Zeidler, *Nonlinear Functional Analysis and its Applications,* Vol. II, Springer, Berlin-Heidelberg-New Youk 1985.

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Some new refinements of Hermite-Hadamard inequality via a sequence of mappings

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Abstract

In this paper we introduce a new sequence of mappings in connection to Hermite-Hadamard type inequality. Some bounds and refinements of Hermite-Hadamard inequality for convex functions via this sequence are given.

Keywords: Hermite-Hadamard inequality; Jensen inequality; convex functions.
Mathematics Subject Classification [2023]: 26D15, 53C21

1 Introduction

Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a convex function and $a, b \in I, a < b$. We consider the well-known Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{375}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Several refinements and generalizations of the inequality (375) have been found in [1-15] and references therein. In order to provide various refinements of this result, S.S. Dragomir introduced two mappings $H, F : [0, 1] \to \mathbb{R}$, in [5] and [6] respectively as follows and established several results in connection to Hermite-Hadamard inequality;

$$H(t) := \frac{1}{b-a}\int_a^b f\Big(tx + (1-t)\frac{a+b}{2}\Big)dx,$$

$$F(t) := \frac{1}{(b-a)^2}\int_a^b\int_a^b f\Big(tx + (1-t)y\Big)dxdy.$$

Since then numerous articles have appeared in the literature reflecting further applications and properties of these mappings (see [3-11]) and references therein. On the other hand the sequence of mappings $H_n : [0, 1] \to \mathbb{R}$ associated to mapping H defined by;

$$H_n(t) := \frac{1}{(b-a)^n}\int_a^b \ldots \int_a^b f\Big(t\frac{x_1 + \ldots + x_n}{n} + (1-t)\frac{a+b}{2}\Big)dx_1 \ldots dx_n,$$

is introduced by S.S. Dragomir in [9]. We recall some of the main properties of H_n:

Theorem 1.1. *Let $f : I \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then, we have*
(i) H_n is convex on $[0, 1]$.
(ii) One has the following bounds;

$$\inf_{t\in[0,1]} H_n(t) = H_n(0) = f\Big(\frac{a+b}{2}\Big) \tag{376}$$

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and

$$\sup_{t\in[0,1]} H_n(t) = H_n(1). \tag{377}$$

(iii) H_n *increases monotonically on* $[0,1]$.
(iv) For every $n \geq 1$ *and* $t \in [0,1]$ *one has*

$$f\big(\frac{a+b}{2}\big) \leq ... \leq H_{n+1}(t) \leq H_n(t) \leq ... \leq H_1(t) = H(t). \tag{378}$$

(v) If $a, b \in I^\circ$ *with* $a < b$ *then, for every* $n \geq 1$ *and* $t \in [0,1]$ *we have*

$$0 \leq H_n(t) - f\big(\frac{a+b}{2}\big) \leq \frac{t(b-a)M}{2\sqrt{3}\sqrt{n}}, \tag{379}$$

where $M := \sup_{x\in[a,b]} |f'_+(x)|$ *and* $f'_+(x)$ *is the right derivative of* f *at* x. *In particular*

$$\lim_{n\to\infty} H_n(t) = f\big(\frac{a+b}{2}\big). \tag{380}$$

In this paper we introduce a new sequence of mappings associated to the mapping F and establish new inequalities in connection to Hermite-Hadamard inequality.

2 Main results

Motivated by [9] we define the sequence of mappings $F_n : [0,1] \to \mathbb{R}$, associated to mapping F as follows,

$$F_n(t) := \frac{1}{(b-a)^{n+1}} \int_a^b ... \int_a^b f\Big(t\frac{x_1 + ... + x_n}{n} + (1-t)x_{n+1}\Big) dx_1 ... dx_{n+1},$$

where, $f : I \to \mathbb{R}$ is a real valued function, $I \subseteq \mathbb{R}$ is an interval and $a, b \in I$ with $a < b$. Note that for every $n \geq 1$,

$$F_n(1) = H_n(1), \qquad F_n(0) = \frac{1}{b-a}\int_a^b f(x)dx. \tag{381}$$

In this section we study the properties of this sequence and introduce some results in connection to Hermite-Hadamard inequality. We start with the the following theorem.

Theorem 2.1. *Let* $f : I \to \mathbb{R}$ *be a convex function and* $a, b \in I$ *with* $a < b$. *Then;*

(i) The mapping F_n *is convex on* $[0,1]$, *for every* $n \geq 1$.

(ii) For every $n \geq 1$ *and* $t \in [0,1]$ *one has*

$$f\big(\frac{a+b}{2}\big) \leq \int_0^1 F_n(t)dt \leq \frac{2(b-a)H_n(1) + \int_a^b f(x)dx}{2(b-a)}. \tag{382}$$

(iii) If $J_n(t) := \frac{F_n(t)+F_n(1-t)}{2}$ *then, for every* $n \geq 1$, J_n *is convex on* $[0,1]$.

(iv) For every $n \geq 1$ *the following inequalities hold,*

$$\inf_{t\in[0,1]} J_n(t) = J_n(\frac{1}{2}),$$

$$F_n(t) \leq F_n(0) = \frac{1}{b-a}\int_a^b f(x)dx, \quad \text{for all} \quad t \in [0,1]. \tag{383}$$

(*v*) *For every* $n \geq 1$ *and* $t \in [0,1]$ *we have*

$$H_n(t) \leq F_n(t). \tag{384}$$

(*vi*) *For every* $n \geq 1$, $J_n(t)$ *decreases monotonically on* $[0, \frac{1}{2}]$ *and increases monotonically on* $[\frac{1}{2}, 1]$.

Proof. (*i*) It is obvious by definition of F_n and convexity of f.

(*ii*) By simple computation and using Jensen's integral type inequality we have

$$\begin{aligned} & f\Big(\frac{a+b}{2}\Big) \\ &= f\left(\frac{1}{(b-a)^{n+1}}\int_a^b ... \int_a^b \Big(t\frac{x_1+...+x_n}{n} + (1-t)x_{n+1}\Big)dx_1...dx_{n+1}\right) \\ &\leq \frac{1}{(b-a)^{n+1}}\int_a^b ... \int_a^b f\Big(t\frac{x_1+...+x_n}{n} + (1-t)x_{n+1}\Big)dx_1...dx_{n+1} \\ &= F_n(t). \end{aligned} \tag{385}$$

Since F_n is convex, by integrating in (385) and using Hermite-Hadamard inequality we obtain the required result in (382).

(*iii*) It is easy by convexity of F_n.

(*iv*) By convexity of f for every $t \in [0,1]$ we have

$$\begin{aligned} & \frac{1}{2}\left[f\Big(t\frac{x_1+...+x_n}{n} + (1-t)x_{n+1}\Big) + f\Big((1-t)\frac{x_1+...+x_n}{n} + tx_{n+1}\Big)\right] \\ & \geq f\Big(\frac{1}{2}\Big(\frac{x_1+...+x_n}{n} + x_{n+1}\Big)\Big). \end{aligned}$$

Hence by integrating on $[a,b]^{n+1}$ we get

$$\begin{aligned} J_n(t) &= \frac{1}{2}\big(F_n(t) + F_n(1-t)\big) \\ &= \frac{1}{2(b-a)^{n+1}}\left[\int_a^b ... \int_a^b \Big(f\Big(t\frac{x_1+...+x_n}{n} + (1-t)x_{n+1}\Big)\right. \\ &\left. + f\Big((1-t)\frac{x_1+...+x_n}{n} + tx_{n+1}\Big)\Big)dx_1...dx_{n+1}\right] \\ &\geq \frac{1}{(b-a)^{n+1}}\left[\int_a^b ... \int_a^b f\Big(\frac{1}{2}\Big(\frac{x_1+...+x_n}{n} + x_{n+1}\Big)\Big)\right] = F_n(\frac{1}{2}) \\ &= J_n(\frac{1}{2}). \end{aligned}$$

For second inequality in (*iv*) we note that,

$$\begin{aligned} & f\Big(t\frac{x_1+...+x_n}{n} + (1-t)x_{n+1}\Big) \\ & \leq tf\Big(\frac{x_1+...+x_n}{n}\Big) + (1-t)f(x_{n+1}) \\ & \leq t\frac{f(x_1)+...+f(x_n)}{n} + (1-t)f(x_{n+1}), \end{aligned}$$

by discrete Jense's inequality. So integrating on $[a,b]^{n+1}$ implies that

$$\begin{aligned}F_n(t) &\leq \frac{1}{2(b-a)^{n+1}}\\ &\int_a^b ... \int_a^b \Big(t\frac{f(x_1)+...+f(x_n)}{n}+(1-t)f(x_{n+1}))\Big)dx_1...dx_{n+1}\\ &= t\frac{1}{b-a}\int_a^b f(x)dx+(1-t)\frac{1}{b-a}\int_a^b f(x)dx\\ &= \frac{1}{b-a}\int_a^b f(x)dx = F_n(0).\end{aligned}$$

(v) For every $n\geq 1$ and $t\in[0,1)$ applying Jensen's integral type inequality on $[a,b]$ give us

$$\begin{aligned}&\frac{1}{(b-a)}\int_a^b f\Big(t\frac{x_1+...+x_n}{n}+(1-t)x_{n+1}\Big)dx_{n+1}\\ &\geq f\Big[\frac{1}{(b-a)}\int_a^b \Big(t\frac{x_1+...+x_n}{n}+(1-t)x_{n+1}\Big)dx_{n+1}\Big]\\ &= f\Big(t\frac{x_1+...+x_n}{n}+(1-t)\frac{a+b}{2}\Big).\end{aligned}$$

Taking integral on $[a,b]^n$ give us the inequality in (384).

(vi) By statement (iv) for every $t\in[0,1]$, $\quad J_n(t)\geq J_n(\frac{1}{2})$ so, by convexity of J_n, for every $1\geq s>t>\frac{1}{2}$ we have

$$\frac{J_n(s)-J_n(t)}{s-t}\geq\frac{J_n(t)-J_n(\frac{1}{2})}{t-\frac{1}{2}}\geq 0,$$

hence $J_n(s)\geq J_n(t)$. The fact that J_n decreases monotonically on $[0,\frac{1}{2}]$ is similar. □

Now, we give the following result on monotonicity of the sequence F_n which completes the above theorem.

Theorem 2.2. *Let $f:I\to\mathbb{R}$ be a convex function and $a,b\in I$, $a<b$. Then for every $t\in[0,1]$ one has*

$$f\Big(\frac{a+b}{2}\Big)\leq ...\leq F_{n+1}(t)\leq F_n(t)\leq ...\leq F_1(t)=F(t). \tag{386}$$

Proof. If $t=1$ then by (381) the inequality (386) is trivially holds. Suppose that $t\in[0,1)$. Then, for every $x_1,...,x_{n+2}\in[a,b]$ we define the real numbers $y_1,...,y_{n+1}$ as follows

$$\begin{aligned}y_1 :=& t\frac{x_1+...+x_n}{n}+(1-t)x_{n+2},\\ y_2 :=& t\frac{x_2+x_1+...+x_{n-1}}{n}+(1-t)x_{n+2},\\ &.\\ &.\\ &.\\ y_{n+1} :=& t\frac{x_{n+1}+x_1+...+x_{n-1}}{n}+(1-t)x_{n+2}.\end{aligned}$$

Note that

$$\frac{y_1+...+y_{n+1}}{n+1}=t\frac{x_1+...+x_{n+1}}{n+1}+(1-t)x_{n+2}.$$

Hence, by using Jensen's type inequality we get

$$\begin{aligned}
&f\big(t\frac{x_1+...+x_{n+1}}{n+1}+(1-t)x_{n+2}\big)=f\big(\frac{y_1+...+y_{n+1}}{n+1}\big)\\
&\leq\frac{f(y_1)+...+f(y_{n+1})}{n+1}\\
&=\frac{1}{n+1}\Big[f\big(t\frac{x_1+...+x_n}{n}+(1-t)x_{n+2}\big)+...\\
&+f\big(t\frac{x_{n+1}+x_1+...+x_{n-1}}{n}+(1-t)x_{n+2}\big)\Big].
\end{aligned}$$

Taking integral on $[a,b]^{n+2}$ implies that

$$\begin{aligned}
&F_{n+1}(t)\leq\frac{1}{n+1}\Bigg[\frac{1}{(b-a)^{n+2}}\\
&\times\int_a^b...\int_a^b f\big(t\frac{x_1+...+x_n}{n}+(1-t)x_{n+2}\big)dx_1...dx_{n+2}\\
&+...+\frac{1}{(b-a)^{n+2}}\int_a^b...\int_a^b f\big(t\frac{x_{n+1}+x_1+...+x_{n-1}}{n}\\
&+(1-t)x_{n+2}\big)dx_1...dx_{n+2}\Big]=\frac{1}{n+1}\Bigg[(n+1)\frac{b-a}{(b-a)^{n+1}}\\
&\int_a^b...\int_a^b f\big(t\frac{x_1+...+x_n}{n}+(1-t)x_{n+2}\big)dx_1...dx_ndx_{n+2}\Bigg]\\
&=F_n(t).
\end{aligned}$$

This complets the proof. □

Remark 2.3. From (1), (383), (386) for every $n\geq 1$ and $t\in[0,1]$ we have

$$\begin{aligned}
f\big(\frac{a+b}{2}\big)&\leq...\leq F_{n+1}(t)\leq F_n(t)\leq...\leq F_1(t)\\
&\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}.
\end{aligned}\tag{387}$$

Now, it is natural to ask what happens with the difference $\frac{1}{b-a}\int_a^b f(x)dx-F_n(t)$ for all $t\in[0,1)$. The following theorem give us an upper bound for this difference for $t\in[0,1)$.

Theorem 2.4. *Let $f:I\to\mathbb{R}$ be a convex function and $a,b\in I^\circ$, $a<b$. Then for every $t\in[0,1)$ we have the following inequality*

$$\begin{aligned}
0&\leq\frac{1}{b-a}\int_a^b f(x)dx-F_n(t)\\
&\leq\frac{t\sqrt{2}(n+1)^{1/4}}{\sqrt{n}}\left[\int_a^b\left(f'_+(x)\right)^2dx\right]^{1/2}.
\end{aligned}$$

Proof. By convexity of f we have

$$\begin{aligned}
&f\big(t\frac{x_1+...+x_n}{n}+(1-t)x_{n+1}\big)-f(x_{n+1})\\
&\geq tf'_+(x_{n+1})\big(\frac{x_1+...+x_n}{n}-x_{n+1}\big).
\end{aligned}$$

Integrating on $[a,b]^{n+1}$ and using Hölder's inequality deduce that

$$\begin{aligned}
0 &\leq \frac{1}{b-a}\int_a^b f(x)dx - F_n(t)\\
&\leq \frac{t}{(b-a)^{n+1}}\int_a^b ...\int_a^b f'_+(x_{n+1})\big(x_{n+1}-\frac{x_1+...+x_n}{n}\big)dx_1...dx_{n+1}\\
&\leq \frac{t}{(b-a)^{n+1}}\left[\int_a^b ...\int_a^b \big(f'_+(x_{n+1})\big)^2 dx_1...dx_{n+1}\right]^{1/2}\\
&\times\left[\int_a^b ...\int_a^b \Big(x_{n+1}-\frac{x_1+...+x_n}{n}\Big)^2 dx_1...dx_{n+1}\right]^{1/2}\\
&= \frac{t}{(b-a)^{n+2/2}}\left[\int_a^b \big(f'_+(x)\big)^2 dx\right]^{1/2}\\
&\times\left[\int_a^b ...\int_a^b \Big(x_{n+1}-\frac{x_1+...+x_n}{n}\Big)^2 dx_1...dx_{n+1}\right]^{1/2}.
\end{aligned} \tag{388}$$

Let $g(x) := \Big(x_{n+1}-\frac{x_1+...+x_n}{n}\Big)^2 = \frac{1}{n^2}\Big(\sum_{i=1}^n (x_{n+1}-x_i)\Big)^2$ then,

$$\nabla g(x) = \frac{2}{n^2}\sum_{i=1}^n (x_{n+1}-x_i)(-1,...,-1,1).$$

Hence,

$$\begin{aligned}
||\nabla g(x)|| &= \frac{2}{n^2}\left|\sum_{i=1}^n (x_i - x_{n+1})\right|(n+1)^{1/2}\\
&\leq \frac{2(n+1)^{1/2}}{n^2}\sum_{i=1}^n \big|x_i - x_{n+1}\big| \leq \frac{2(n+1)^{1/2}}{n}(b-a).
\end{aligned} \tag{389}$$

By combining (388) and (389) we obtain

$$\begin{aligned}
0 &\leq \frac{1}{b-a}\int_a^b f(x)dx - F_n(t)\\
&\leq \frac{t\sqrt{2}(n+1)^{1/4}}{\sqrt{n}}\left[\int_a^b \Big(f'_+(x)\Big)^2 dx\right]^{1/2},
\end{aligned}$$

and proof is completed. □

The following corollaries are immediate consequence of theorem 2.4.

Corollary 2.5. *Under the assumptions of theorem 2.4 if* $M := \sup_{x\in[a,b]} |f'_+(x)|$, *then for all* $t \in [0,1)$ *and* $n \geq 1$ *we have the inequality*

$$0 \leq \frac{1}{b-a}\int_a^b f(x)dx - F_n(t) \leq \frac{\sqrt{2}(n+1)^{1/4}M\sqrt{b-a}}{\sqrt{n}}.$$

In particular we obtain

$$\lim_{n\to\infty} F_n(t) = \frac{1}{b-a}\int_a^b f(x)dx, \text{ for all } t \in [0,1).$$

Corollary 2.6. *Under the assumptions of theorem 2.4 one has the following inequality*

$$\begin{aligned}
0 &\leq \frac{1}{b-a}\int_a^b f(x)dx - J_n(t)\\
&\leq \frac{(n+1)^{1/4}}{\sqrt{2n}}\left[\int_a^b \Big(f'_+(x)\Big)^2 dx\right]^{1/2}.
\end{aligned}$$

The following result also holds;

Theorem 2.7. *Let $f : I \to \mathbb{R}$ be a convex function and $a, b \in I^\circ$ with $a < b$. Suppose that there exits a constant $K > 0$ such that*

$$\left|f'_+(x) - f'_+(y)\right| \leq K|x - y|, \text{ for all } x, y \in [a, b].$$

Then we have the inequality

$$tF_n(1) + (1-t)F_n(0) - F_n(t) \leq \frac{2t(1-t)(n+1)^{1/2}K}{n}(b-a),$$

for all $t \in [0, 1]$ and $n \geq 1$.

Proof. By convexity of f for every $x_1, ..., x_{n+1} \in [a, b]$ and $t \in [0, 1]$ we have

$$\begin{aligned} &f\big(t\frac{x_1 + ... + x_n}{n} + (1-t)x_{n+1}\big) - f(\frac{x_1 + ... + x_n}{n}) \\ &\geq (1-t)f'_+\big(\frac{x_1 + ... + x_n}{n}\big)\big(x_{n+1} - \frac{x_1 + ... + x_n}{n}\big), \end{aligned} \tag{390}$$

and

$$\begin{aligned} &f\big(t\frac{x_1 + ... + x_n}{n} + (1-t)x_{n+1}\big) - f(x_{n+1}) \\ &\geq -tf'_+\big(x_{n+1}\big)\big(x_{n+1} - \frac{x_1 + ... + x_n}{n}\big). \end{aligned} \tag{391}$$

If we multiply the inequalities (390) and (391) by t and $1 - t$, respectively and added the obtained results we obtain

$$\begin{aligned} &tf(\frac{x_1 + ... + x_n}{n}) + (1-t)f(x_{n+1}) \\ &- f\big(t\frac{x_1 + ... + x_n}{n} + (1-t)x_{n+1}\big) \leq t(1-t) \\ &\Big[f'_+\big(\frac{x_1 + ... + x_n}{n}\big) - f'_+\big(x_{n+1}\big)\Big]\big(\frac{x_1 + ... + x_n}{n} - x_{n+1}\big). \end{aligned}$$

Integrating on $[a, b]^{n+1}$ and using (9) implies that

$$\begin{aligned} &tF_n(1) + (1-t)F_n(0) - F_n(t) \\ &\leq t(1-t)\frac{1}{(b-a)^{n+1}}\int_a^b ... \int_a^b \Big[f'_+\big(\frac{x_1 + ... + x_n}{n}\big) - f'_+\big(x_{n+1}\big)\Big] \\ &\times \big(\frac{x_1 + ... + x_n}{n} - x_{n+1}\big)dx_1...dx_{n+1} \\ &\leq \frac{t(1-t)K}{(b-a)^{n+1}}\int_a^b ... \int_a^b \big(\frac{x_1 + ... + x_n}{n} - x_{n+1}\big)^2 dx_1...dx_{n+1} \\ &\leq \frac{2t(1-t)K(n+1)^{1/2}}{n}(b-a). \end{aligned}$$

This completes the proof. □

Finally an upper bound for the difference $F_n(t) - H_n(t), n \geq 1$, $t \in [0, 1]$, is as follows.

Theorem 2.8. *Let $f : I \to \mathbb{R}$ be a convex function and $a, b \in I^\circ$ with $a < b$. Then, for all $t \in [0,1]$ and $n \geq 1$ we have the inequality*

$$\begin{aligned}
&0 \leq F_n(t) - H_n(t) \\
&\leq \frac{(1-t)(b-a)}{2}\left[\frac{1}{(b-a)^{n+1}}\int_a^b ... \int_a^b \Big(f'_+\big(t\frac{x_1+...+x_n}{n}\right. \\
&\left.+(1-t)x_{n+1}\big)\Big)^2 dx_1...dx_{n+1}\right]^{1/2}.
\end{aligned}$$

Proof. By convexity of f for every $x_1, ..., x_{n+1} \in [a,b]$ and $t \in [0,1]$ we have

$$\begin{aligned}
&f\Big(t\frac{x_1+...+x_n}{n} + (1-t)\frac{a+b}{2}\Big) - f\Big(t\frac{x_1+...+x_n}{n} + (1-t)x_{n+1}\Big) \\
&\geq (1-t)f'_+\Big(t\frac{x_1+...+x_n}{n} + (1-t)x_{n+1}\Big)\Big(\frac{a+b}{2} - x_{n+1}\Big).
\end{aligned}$$

Integrating on $[a,b]^{n+1}$ and using Hölder's inequality implies that

$$\begin{aligned}
&0 \leq F_n(t) - H_n(t) \\
&\leq \frac{1-t}{(b-a)^{n+1}}\int_a^b ... \int_a^b f'_+\Big(t\frac{x_1+...+x_n}{n} + (1-t)x_{n+1}\Big) \\
&\Big(x_{n+1} - \frac{a+b}{2}\Big)dx_1...dx_{n+1} \leq \frac{1-t}{(b-a)^{n+1}}\left[\int_a^b ... \int_a^b\right. \\
&\left.\Big(f'_+\big(t\frac{x_1+...+x_n}{n} + (1-t)x_{n+1}\big)\Big)^2 dx_1...dx_{n+1}\right]^{1/2} \\
&\left[\int_a^b ... \int_a^b \big(x_{n+1} - \frac{a+b}{2}\big)^2 dx_1...dx_{n+1}\right]^{1/2} \\
&= \frac{(1-t)(b-a)}{2}\left[\frac{1}{(b-a)^{n+1}}\int_a^b ... \int_a^b \Big(f'_+\big(t\frac{x_1+...+x_n}{n}\right. \\
&\left.+(1-t)x_{n+1}\big)\Big)^2 dx_1...dx_{n+1}\right]^{1/2}.
\end{aligned}$$

This completes the proof. □

Corollary 2.9. *Under the assumptions as in Theorem (2.8) if $K := \sup_{x\in[a,b]} |f'_+(x)|$ one has*

$$0 \leq F_n(t) - H_n(t) \leq \frac{(1-t)K}{2}(b-a),$$

for every $n \geq 1, \quad t \in [0,1]$.
In particular we have

$$0 \leq F(t) - H(t) \leq \frac{(1-t)K}{2}(b-a).$$

3 Conclusion

In this paper, we have given a sequence of mappings associated to the mapping F. This sequence give us some new refinements and bounds related to well known Hermite-Hadamard inequality.

References

[1] A. Barani, S. Barani, Hermite-Hadamard inequalities for functions when a power of the absolute value of the first derivative is P-convex, Bull. Aust. Math. Soc. 86 (2012), 126-134.

[2] S. Banić, Mapping connected with Hermite-Hadamard inequalities for superquadratic functions, J. Math. Inequal. 3 (2009), 557-589.

[3] A. Barani, Hermite-Hadamard and Ostrowski type inequalities on hemispheres. Mediterr. J. Math. 13, Issue 6 (2016), 4253-4263.

[4] C. Conde, A version of the Hermite-Hadamard inequality in a nonpositive curvature space, Banach J. Math. Anal. 6 (2012), 159-167.

[5] S.S. Dragomir, A mapping in connection to Hadamard's inequalities, An. Öster. Akad. Wiss. Math. Natur. 128 (1991), 17-20.

[6] S.S. Dragomir, Two mappings on connection to Hadamard's inequality, J. Math. Anal. Appl. 167 (1992), 49-56.

[7] S.S. Dragomir, On Hadamard's inequalities for convex functions, Mat. Balkanica. 6 (1992), 215-222.

[8] S.S. Dragomir, C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and applications, (RGMIA Monographs http:// rgmia.vu.edu.au/ monographs/ hermite hadamard.html, Victoria University, 2000.

[9] S.S. Dragomir, A sequence of mappings associated with the Hermite-Hadamard inequalities and applications, Applications of Mathematics. 49 (2004), 123-140.

[10] S.S. Dragomir, I. Gomn, Bounds for two mapping associated to the Hermite-Hadamard inequality, AJMAA. 8 (2011), 1-9.

[11] S.S. Dragomir, I. Gomn, Some new bounds for two mapping related to the Hermite-Hadamard inequality for convex functions, Numer. Algeb. Cont. Optim. 2 (2012), 271-278.

[12] I. İscan New general integral inequalities for quasi geometrically convex functions via fractional integrals, J. Ineq. Appl. 2013 (2013), 491 1-15.

[13] M.S. Moslehian, Matrix Hermite-Hadamard type inequalities, Houst. J. Math. 39 (2013), 177-189.

[14] C.P. Niculescu, The HermiteHadamard inequality for log-convex functions, Nonlinear Analysis. 75 (2012), 662-669.

[15] K.L. Tesing, S.R. Hwang and S.S. Dragomir, New Hermite-Hadamard-type inequalities for convex functions (II), Comput. Math. Appl. 62 (2011), 29-38.

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Module Connes amenability for projective tensor product and Θ-Lau produt of Banach algebras

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Abstract

Let $\mathbb{A}$ and $\mathbb{B}$ be Banach algebras with preduals $\mathbb{A}_*$ and $\mathbb{B}_*$ respectively, and $\Theta : \mathbb{B} \to \mathbb{A}$ be an algebraic homomorphism. In this paper, we develop the notions of module Connes amenability for certain Banach algebras. Indeed, we investigate and give necessary and sufficient conditions for module Connes amenability of projective tensor product $\mathbb{A}\widehat{\otimes}\mathbb{B}$. Moreover, we characterize the (ψ, θ)-module Connes amenability of Θ-Lau product $\mathbb{A} \times_\Theta \mathbb{B}$, which ψ and θ are homomorphisms in $\mathbb{A}_*$ and $\mathbb{B}_*$, respectively.

Keywords: module Connes amenability, projective tensor product, Θ-Lau produt

Mathematics Subject Classification [2023]: 46H25, 46J10, 46L06

1 Introduction and Preliminaries

The concept of amenability of a Banach algebra $\mathbb{A}$ to the case that there exists an additional $\mathbb{U}$-module structure on $\mathbb{A}$, where $\mathbb{U}$ is Banach algebra is extended by Amini. The definition of Connes amenability makes sense for a larger class of Banach algebras (called dual Banach algebras in [5]). Runde in [6], introduced the concept of Connes amenability of dual Banach algebras and this type of Banach algebras is interested by many researchers. The concept of module Connes amenability for dual Banach algebras which are also Banach modules with respect to the compatible action, first defined by Amini. It is shown that there exists a relation between module Connes amenability and existance a normal module virtual diagonal. Let $\mathbb{A}$ and $\mathbb{B}$ be two Banach algebras. The θ-Lau product of $\mathbb{A} \times_\theta \mathbb{B}$ where θ is a nonzero multiplicative functional on $\mathbb{B}$ was introduced by Lau for certain class of Banach algebras and followed by Monfared in [4] for the general case.

Let $\mathbb{E}$ be a Banach $\mathbb{A}$-bimodule. The collection of all elements of $\mathbb{E}$ that module maps from $\mathbb{A}$ onto $\mathbb{E}$ are w^*-weakly continuous, is denoted by $\sigma wc(\mathbb{E})$. A σwc-virtual diagonal for $\mathbb{A}$ is an element $\mu \in \sigma wc((\mathbb{A}\hat{\otimes}\mathbb{A})^*)^*$ such that $a.\mu = \mu.a$ and $a.\Delta_{\sigma wc}\mu = a$ that $\Delta : \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ is the multiplication operator. In [6], it is shown that a Banach algebra $\mathbb{A}$ is Connes amenable if and only if it has a so-called σwc-virtual diagonal. Let $\mathbb{A}$ be a Banach algebra, $\mathbb{A}$-bimodule $\mathbb{E}$ is called normal if it is a dual space such that the module actions are separately w^*-continuous [3].

Recently the authors have introduced the new version, as called φ-Connes amenability of dual Banach algebra $\mathbb{A}$ that $\varphi \in \Delta(\mathbb{A})$, the set of all continuous homomorphisms from $\mathbb{A}$ onto $\mathbb{C}$, and also $\varphi \in \mathbb{A}_*$ [1], as follows:

A dual Banach algebra $\mathbb{A}$ is φ-Connes amenable if, for every normal $\mathbb{A}$-bimodule $\mathbb{E}$, where the left action is of the form

$$a.\mathbf{e} = \varphi(a)\mathbf{e}, \qquad (a \in \mathbb{A}, \mathbf{e} \in \mathbb{E})$$

every bounded w^*-continuous derivation $\mathcal{D} : \mathbb{A} \to \mathbb{E}$ is inner.

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In this paper, we are going to investigate the module Connes amenability and character module Connes amenability for $\mathbb{A}\widehat{\otimes}\mathbb{B}$, Θ-Lau product $\mathbb{A}\times_{\Theta}\mathbb{B}$.

Suppose that $\mathbb{A}$ is a Banach algebra, $\mathcal{F}$ is a subspace of $\mathbb{A}^*$ and $\varphi\in\Delta(\mathbb{A})\cap\mathcal{F}$. A linear functional $\mathbf{m}$ on $\mathcal{F}$ is said to be a mean if $\langle\mathbf{m},\varphi\rangle=1$. A mean $\mathbf{m}$ is φ-invariant mean if $\langle\mathbf{m},a.f\rangle=\varphi(a)\langle\mathbf{m},f\rangle$ for all $a\in\mathbb{A}$ and $f\in\mathcal{F}$. Note that φ-Connes amenability of $\mathbb{A}$ follows from $\mathcal{F}=\mathbb{A}_*$ (see [1, 2]).

The authors showed that a dual Banach algebra $\mathbb{A}$ is φ-Connes amenable if and only if the second dual, $\mathbb{A}^{**}$ has a φ-invariant mean on $\mathbb{A}_*$.

Consider $\mathbb{A}=(\mathbb{A}_*)^*$ is a dual Banach algebra, and $\mathbb{U}$ is a Banach algebra such that $\mathbb{A}$ is a Banach $\mathbb{U}$-bimodule via following actions,

$$v.(ab)=(v.a).b,\quad (vw).a=v.(w.a)\qquad (a,b\in\mathbb{A},v,w\in\mathbb{U}).$$

Let $\mathbb{E}$ be a normal dual Banach $\mathbb{A}$-bimodule. Moreover, if $\mathbb{E}$ is an $\mathbb{U}$-bimodule via

$$v.(a.\mathbf{e})=(v.a).\mathbf{e},\ \ (a.v).\mathbf{e}=a.(v.\mathbf{e}),\ \ (v.\mathbf{e}).a=v.(\mathbf{e}.a),$$

for every $a\in\mathbb{A}$, $v\in\mathbb{U}$ and $\mathbf{e}\in\mathbb{E}$. Then we say that $\mathbb{E}$ is a normal Banach left $\mathbb{A}$-$\mathbb{U}$-bimodule. Similarly, for the right actions. Also, we say that $\mathbb{E}$ is symmetric, if $v.\mathbf{e}=\mathbf{e}.v$ $(v\in\mathbb{U},\mathbf{e}\in\mathbb{E})$.

Let $\mathbb{U}$ and $\mathbb{A}$ be Banach algebras, and let $\mathbb{E}$ be a Banach $\mathbb{A}$-$\mathbb{U}$-bimodule. Then By using [5], $\mathbb{E}^*$ becomes a Banach $\mathbb{U}$-$\mathbb{A}$-bimodule via

$$\langle\mathbf{e},a.f\rangle:=\langle\mathbf{e}.a,f\rangle,\qquad \langle\mathbf{e},f.v\rangle:=\langle v.\mathbf{e},f\rangle,\qquad (v\in\mathbb{U},a\in\mathbb{A},f\in E^*,\mathbf{e}\in\mathbb{E}).$$

Definition 1.1. Let $\mathbb{A}=(\mathbb{A}_*)^*$ be a Banach algebra, $\mathbb{U}$ be a Banach algebra such that $\mathbb{A}$ is a Banach $\mathbb{U}$-bimodule, $\varphi\in\Delta(\mathbb{A})\cap\mathbb{A}_*$ and $\mathbb{E}$ be a Banach $\mathbb{A}$-$\mathbb{U}$-bimodule. A bounded map $\mathcal{D}_{\mathbb{U}}$ from $\mathbb{A}$ to $\mathbb{E}$ is called a *module φ-derivation* if

$$\mathcal{D}_{\mathbb{U}}(v.a\pm b.w)=v.\mathcal{D}_{\mathbb{U}}(a)\pm\mathcal{D}_{\mathbb{U}}(b).w,\qquad \mathcal{D}_{\mathbb{U}}(ab)=\mathcal{D}_{\mathbb{U}}(a).\varphi(b)+\varphi(a).\mathcal{D}_{\mathbb{U}}(b),$$

for all $a,b\in\mathbb{A}$ and $v,w\in\mathbb{U}$.

Definition 1.2. Let $\mathbb{A}$ be a Banach algebra, $\mathbb{U}$ be a Banach algebra such that $\mathbb{A}$ is a Banach $\mathbb{U}$-bimodule and $\varphi\in\Delta(\mathbb{A})\cap\mathbb{A}_*$. We say that $\mathbb{A}$ is *φ-module Connes amenable* if for any symmetric normal Banach $\mathbb{A}$-$\mathbb{U}$-bimodule $\mathbb{E}$, each w^*-continuous module φ-derivation $\mathcal{D}_{\mathbb{U}}:\mathbb{A}\to\mathbb{E}$ is inner.

Remark 1.3. In [2], it is defined a certain version of φ-module Connes amenability where $\varphi:\mathbb{A}\to\mathbb{A}$ is a module homomorphism, as follows:

Let $\mathbb{A}$ be a Banach algebra, $\mathbb{U}$ be a Banach algebra such that $\mathbb{A}$ is a Banach $\mathbb{U}$-bimodule and $\varphi:\mathbb{A}\to\mathbb{A}$ is a map that satisfied

$$\varphi(v.a+b.w)=v.\varphi(a)+\varphi(b).w,\quad \varphi(ab)=\varphi(a)\varphi(b),$$

for every $a,b\in\mathbb{A},v,w\in\mathbb{U}$.

In this case, $\mathbb{A}$ is called φ-module Connes amenable if for any symmetric normal Banach $\mathbb{A}$-$\mathbb{U}$-bimodule $\mathbb{E}$, each w^*-continuous module φ-derivation from $\mathbb{A}$ to $\mathbb{E}$ is inner [2].

2 $\varphi \otimes \psi$-module Connes Amenability of projective tensor product of Banach algebras and $\varphi \otimes \psi$-invariant mean

Our aim in this section is to study of $\varphi \otimes \psi$-module Connes Amenability of projective tensor product of Banach algebras. In the sequel, in special case and by using Remark 1.3, we conclude the following result.

Proposition 2.1. *Suppose that $\mathbb{A}$ and $\mathbb{B}$ are two Banach algebras. Let $\varphi : \mathbb{A} \to \mathbb{A}$ and $\psi : \mathbb{B} \to \mathbb{B}$ be two maps on Banach algebras. Suppose that $\mathbb{A}$ is φ-module Connes amenable and $\mathbb{B}$ is ψ-module Connes amenable. Then $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is $\eta \circ (\varphi \otimes \psi)$-module Connes amenable for any map $\eta : \mathbb{A}\widehat{\otimes}\mathbb{B} \to \mathbb{A}\widehat{\otimes}\mathbb{B}$.*

Theorem 2.2. *With above notations, if $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is a Banach algebra. Then $\mathbb{A}$ is φ-module Connes amenable and $\mathbb{B}$ is ψ-module Connes amenable if and only if $\sigma wc(\mathbb{A}\widehat{\otimes}\mathbb{B})$ is $\varphi \otimes \psi$-module Connes amenable.*

Proof. Let $\mathbb{A}$ be φ-module Connes amenable and $\mathbb{B}$ be ψ-module Connes amenable. Let $\mathbb{U}$ be a Banach algebra. Take $\mathbb{E}$ as a symmetric normal Banach $\mathbb{A}\widehat{\otimes}\mathbb{B}$-$\mathbb{U}$-bimodule and let $\mathcal{D}_{\mathbb{U}} : \mathbb{A}\widehat{\otimes}\mathbb{B} \to \mathbb{E}$ be a bounded w^*-continuous module $\varphi \otimes \psi$-derivation defined by

$$\mathcal{D}_{\mathbb{U}}\big((a \otimes b)(a' \otimes b')\big) = \mathcal{D}_{\mathbb{U}}(a \otimes b).(a' \otimes b') + (a \otimes b).\mathcal{D}_{\mathbb{U}}(a' \otimes b'),$$

for all $a, a' \in \mathbb{A}$ and $b, b' \in \mathbb{B}$. Hence, we obtain

$$\begin{aligned}\mathcal{D}_{\mathbb{U}}(a \otimes b) &= \varphi \otimes \psi(a \otimes b).\mathbf{y} - \mathbf{y}.(a \otimes b)\\ &= (a \otimes b).\mathbf{y} - \mathbf{y}.(a \otimes b) = (ad_{\mathbb{U}})_{\mathbf{y}}(a \otimes b),\end{aligned}$$

for all $a \in \mathbb{A}$ and $b \in \mathbb{B}$, as required. □

Example 2.3. Set $\mathbb{A}=\begin{pmatrix} 0 & 0 \\ \mathbb{C} & \mathbb{C} \end{pmatrix}$. Consider usual matrix multiplication and l^1-norm, $\mathbb{A}$ is a dual Banach algebra. We consider $\varphi : \mathbb{A} \to \mathbb{C}$; $\varphi \begin{pmatrix} 0 & 0 \\ z_1 & z_2 \end{pmatrix} = z_2$, for all $z_1, z_2 \in \mathbb{C}$. Note that $\varphi \in \Delta(\mathbb{A}) \cap \mathbb{A}_*$ is norm continuous and w^*-continuous. Let $u = \begin{pmatrix} 0 & 0 \\ 1 & -i \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ i & i \end{pmatrix} \in \mathbb{A}\widehat{\otimes}\mathbb{A}$. We can see that $u \in \sigma wc((\mathbb{A}\widehat{\otimes}\mathbb{A})^*)^*$ and therefore u has the property of $\varphi - \sigma wc-$virtual diagonal for Banach algebra $\mathbb{A}$. Consequently, there exists a $\varphi \otimes \varphi$-invariant mean on $(\mathbb{A}\widehat{\otimes}\mathbb{A})^*$ and so, on $(\mathbb{A}\widehat{\otimes}\mathbb{A})_*$ (for more details see [1]).

Corollary 2.4. *Let $\mathbb{A}$ and $\mathbb{B}$ be Banach algebras, $\mathcal{F}$ be a subspace of $\mathbb{A}^*$ and $\mathcal{H}$ be a subspace of $\mathbb{B}^*$. Let $\varphi \in \Delta(\mathbb{A}) \cap \mathcal{F}$, $\psi \in \Delta(\mathbb{B}) \cap \mathcal{H}$, $\mathcal{F}$ be φ-module Connes amenable and $\mathcal{H}$ be ψ-module Connes amenable. Then $\mathcal{F}\widehat{\otimes}\mathcal{H}$ is $\varphi \otimes \psi$-module Connes amenable.*

Theorem 2.5. *Let $\mathbb{A} = (\mathbb{A}_*)^*$, $\mathbb{B} = (\mathbb{B}_*)^*$ and $\mathbb{A}\widehat{\otimes}\mathbb{B}$ be Banach algebras, and let $\varphi \in \Delta_{\omega^*}(\mathbb{A}) \cap \mathbb{A}_*$ and $\psi \in \Delta_{\omega^*}(\mathbb{B}) \cap \mathbb{B}_*$. Let $\mathbb{I}$, $\mathbb{J}$ be Banach algebras and closed two-sided ideals of $\mathbb{A}$, $\mathbb{B}$, respectively. If $\varphi\mid_I \neq 0$, $\psi\mid_{\mathbb{J}} \neq 0$ and $\mathbb{I}\widehat{\otimes}\mathbb{J}$ is a Banach algebra that is $\varphi\otimes\psi\mid_{\widehat{\mathbb{I}\otimes J}}$-module Connes amenable, then $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is $\varphi\otimes\psi$-module Connes amenable.*

In the sequel, under condition existence of bounded approximate identities for above ideals, $\mathbb{I}$ and $\mathbb{J}$, we show that the converse of Theorem 2.5 is hold.

Theorem 2.6. *Let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{A}\widehat{\otimes}\mathbb{B}$, $\mathbb{I}$, $\mathbb{J}$, φ and ψ be as above. Let $\mathbb{I}$, $\mathbb{J}$ be with bounded approximate identities such that $\varphi\mid_{\mathbb{I}}$ and $\psi\mid_{\mathbb{J}}$ are non-zero and $\mathbb{I}\widehat{\otimes}\mathbb{J}$ be a Banach algebra. If $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is $\varphi\otimes\psi$-module Connes amenable, then $\mathbb{I}\widehat{\otimes}\mathbb{J}$ is $\varphi\otimes\psi\mid_{\widehat{\mathbb{I}\otimes\mathbb{J}}}$-module Connes amenable.*

3 (ψ,θ)-module Connes amenability of Θ-Lau product of Banach algebras

Suppose that $\mathbb{A}$ is an unital dual Banach algebra with predual $\mathbb{A}_*$, the identity $e_{\mathbb{A}}$ and $\mathbb{B}$ is a dual Banach algebra with predual $\mathbb{B}_*$. Suppose that $\psi \in \Delta(\mathbb{A}) \cap \mathbb{A}_*$, $\theta \in \Delta(\mathbb{B}) \cap \mathbb{B}_*$ and $\Theta : \mathbb{B} \to \mathbb{A}$ is an algebraic homomorphism, which $\Theta(b) = \theta(b)e_{\mathbb{A}}$. The Θ-Lau product $\mathbb{A} \times_\Theta \mathbb{B}$ is defined with

$$(a, b).(a', b') = (a.a' + \Theta(b').a + \Theta(b).a', bb')$$

and the norm $\|(a, b)\|_{\mathbb{A}\times_\Theta\mathbb{B}} = \|a\|_{\mathbb{A}} + \|b\|_{\mathbb{B}}$ for all $a, a' \in \mathbb{A}$ and $b, b' \in \mathbb{B}$. This definition is a certain case of the product that is presented in [4]. In this section, we investigate the notions of (ψ,θ)-module Connes amenability and $(0,\theta)$-module Connes amenability. Since $\theta \in \Delta(\mathbb{B}) \cap \mathbb{B}_*$, then $\mathbb{A} \times_\Theta \mathbb{B}$ is a dual Banach algebra with predual $\mathbb{A}_* \times \mathbb{B}_*$.

Lemma 3.1. *Let $\mathbb{A}$ be an unital dual Banach algebra with predual $\mathbb{A}_*$ and let $\mathbb{B}$ be a dual Banach algebra with predual $\mathcal{B}_*$. Then the following two statements are equevalent,*

1. *$\mathbb{A}$ and $\mathbb{B}$ are id-module Connes amenable.*
2. *$\mathbb{A} \times_\Theta \mathbb{B}$ is $id \otimes id$-module Connes amenable.*

Theorem 3.2. *Let $\mathbb{A} = (\mathbb{A}_*)^*$ be an unital dual Banach algebra that is Arens regular and $\mathbb{B} = (\mathbb{B}_*)^*$ be a dual Banach algebra. Let $\psi \in \Delta(\mathbb{A}) \cap \mathbb{A}_*$ and $\theta \in \Delta(\mathbb{B}) \cap \mathbb{B}_*$. Then the following statements are hold:*

1. *$\mathbb{A} \times_\Theta \mathbb{B}$ is (ψ,θ)-module Connes amenable if and only if $\mathbb{A}$ is ψ-module Connes amenable.*
2. *$\mathbb{A} \times_\Theta \mathbb{B}$ is $(0,\theta)$-module Connes amenable if and only if $\mathbb{B}$ is θ-module Connes amenable.*

4 Conclusion

We give necessary and sufficient conditions for module Connes amenability of $\mathbb{A}\widehat{\otimes}\mathbb{B}$ that $\mathbb{A}$ and $\mathbb{B}$ are two Banach algebras. Moreover, we characterize the (ψ,θ)-module Connes amenability of Θ-Lau product $\mathbb{A}\times_\Theta\mathbb{B}$, which $\Theta : \mathbb{B} \to \mathbb{A}$ be an algebraic homomorphism and ψ and θ are homomorphisms in $\mathbb{A}_*$ and $\mathbb{B}_*$, respectively.

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References

[1] A. Ghaffari and S. Javadi, *φ-Connes amenability of dual Banach algebras*, Bull. Iranian Math. Soc., 43 (2017), 25-39.

[2] A. Ghaffari, S. Javadi and E. Tamimi, *φ-Connes moddule amenability of dual Banach algebras*, Journal of Algebraic Systems, 8(1) (2020), 69-82.

[3] B. E. Johnson, R. V. Kadison and J. Ringrose, *Cohomology of operator algebras III*, Bull. Soc. Math. France, 100 (1972), 73-79.

[4] M. S. Monfared, *On certain products of Banach algebras with applications to harmonic analysis*, Studia Math., 178 (2007), 277-294.

[5] V. Runde, *Amenability for dual Banach algebras*, Studia Math., 148(1) (2001), 47-66.

[6] Runde4 V. Runde, *Dual Banach algebras: Connes amenability, normal, virtual diagonals, and injectivity of the predual bimodule*, Math. Scand., 95(1) (2004), 124-144.

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Two families derivative-free with memory methods for solving nonlinear equations

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Abstract

In this work, we have proposed two families of two-step methods with sixth-order memory and three-step methods of twelfth-order with 50% convergence improvement. These families have an accelerator parameter that is approximated using Newton's interpolation polynomials. And their efficiency index is 1.81712 and 1.86120, respectively. We also present numerical results to compare our proposal with some previously developed ones for solving some nonlinear equations.

Keywords: Nonlinear equations, R-order convergence, Self accelerating parameter.

Mathematics Subject Classification [2023]: 65H05

1 Introduction

In this work, we have proposed iterative methods with-memory higher efficiency index than methods without memory. Nonlinear equations by using with-memory methods have been solved. Based on these methods is the existence of a parameter or parameters of the accelerator itself. For a detailed discussion of these methods, consider the following memory method proposed by Traub [6]:

$$\begin{cases} \gamma_k = -\frac{1}{f(x_k)-f(x_{k-1})}, \ k = 1, 2, 3, \ldots, \\ w_k = x_k + \gamma_k f(x_k), \ x_{k+1} = x_k - \frac{f(x_k)}{f[x_k, w_k]}, \ k = 0, 1, 2, \ldots. \end{cases} \tag{392}$$

The error equation of this method is as follows $e_{k+1} = (1 + \gamma f'(\alpha))c_2 e_k^2 + O(e_k^3)$. As can be seen, there is an expression $(1 + \gamma f'(\alpha))$ in the error equation. Here we call the variable γ parameter accelerator. The γ parameter can be approximated in various ways using the information given current. Traub approximated the parameter γ to the quasi-secant method as relation (392) and constructed a memory method with a convergence order of $1 + \sqrt{2}$. Also, Dzunic and Petkovic [1], by approximating the self-accelerator parameter using a Newton interpolator polynomial, in three points x_k, x_{k-1}, w_{k-1}, constructed a method with a convergence order of 3 as follows.

$$\begin{cases} \gamma_0 \ is \ given, \ \gamma_k = -\frac{1}{N_2'(x_k)}, \ k = 1, 2, 3, \ldots, \\ w_k = x_k + \gamma_k f(x_k), \ x_{k+1} = x_k - \frac{f(x_k)}{f[x_k, w_k]}, \ k = 0, 1, 2, \ldots. \end{cases} \tag{393}$$

Torkashvand and Kazemi also proposed a family of multi-step with-memory methods that approximated the accelerator parameters by Newton's interpolation polynomial [7]. Petkovic et al. [3] used the inverse interpolation method to compute the free parameter. Wang and his colleague Zhang [8] proposed the two-step method with memory and approximated the self-accelerator parameter using a Hermit interpolator

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polynomial. We also developed memory methods using the input of maximum accelerator parameters and convergence order improvement [7]. In these methods, we have used the adaptive technique.
In this work, we present an iterative procedure for constructing a memory method. In Section 2, we consider two families methods that serve as the base of multipoint iterative schemes Solaiman et al.'s method. Construction of two and three-point methods with memory relied on the Newton interpolation is displayed in Section 2. The presented multipoint methods are compared with existing methods with and without memory in Section 3. Finally, the conclusions are provided in Section 4.

2 Main results

Solaiman et al. [5] obtained two families of derivative-free optimal methods. Their methods are as follows. Two-step methods

$$\begin{cases} w_k = x_k + f(x_k),\ y_k = x_k - \frac{f(x_k)}{f[x_k,w_k]}, \\ g(x_k) = f[w_k,x_k] + 2(w_k - x_k)f[w_k,x_k,y_k] - f[y_k,w_k] + f[x_k,y_k], \\ x_{k+1} = y_k - \frac{f(y_k)}{g(x_k)}\frac{f(x_k+\beta f(y_k)}{f(x_k+(\beta-2)f(y_k)},\ k = 0,1,2,\ldots. \end{cases} \tag{394}$$

And, three-step method

$$\begin{cases} w_k = x_k + f(x_k),\ y_k = x_k - \frac{f(x_k)}{f[x_k,w_k]},\ k = 0,1,2,\ldots, \\ g(x_k) = f[w_k,x_k] + 2(w_k - x_k)f[w_k,x_k,y_k] - f[y_k,w_k] + f[x_k,y_k], \\ z_k = y_k - \frac{f(y_k)}{g(x_k)}\frac{f(x_k+\beta f(y_k)}{f(x_k+(\beta-2)f(y_k)},\ m_1 = f(y_k)(f(z_k)(z_k - y_k), \\ m_2 = f(w_k)(f(z_k)(w_k - z_k),\ m_3 = f(y_k)(f(w_k)(y_k - w_k), \\ x_{k+1} = x_k - \frac{f(x_k)(m_1+m_2+m_3)}{m_1 f[w_k,x_k]+m_2 f[y_k,x_k]+m_3 f[z_k,x_k]}. \end{cases} \tag{395}$$

By entering a self-accelerator parameter, these two methods can be rewritten as follows:

$$\begin{cases} w_k = x_k + \gamma f(x_k),\ y_k = x_k - \frac{f(x_k)}{f[x_k,w_k]}, \\ g(x_k) = f[w_k,x_k] + 2(w_k - x_k)f[w_k,x_k,y_k] - f[y_k,w_k] + f[x_k,y_k], \\ x_{k+1} = y_k - \frac{f(y_k)}{g(x_k)}\frac{f(x_k)+\beta f(y_k)}{f(x_k)+(\beta-2)f(y_k)},\ k = 0,1,2,\ldots. \end{cases} \tag{396}$$

And

$$\begin{cases} w_k = x_k + \gamma f(x_k),\ y_k = x_k - \frac{f(x_k)}{f[x_k,w_k]},\ k = 0,1,2,\ldots, \\ g(x_k) = f[w_k,x_k] + 2(w_k - x_k)f[w_k,x_k,y_k] - f[y_k,w_k] + f[x_k,y_k], \\ z_k = y_k - \frac{f(y_k)}{g(x_k)}\frac{f(x_k)+\beta f(y_k)}{f(x_k)+(\beta-2)f(y_k)},\ m_1 = f(y_k)(f(z_k)(z_k - y_k), \\ m_2 = f(w_k)(f(z_k)(w_k - z_k),\ m_3 = f(y_k)(f(w_k)(y_k - w_k), \\ x_{k+1} = x_k - \frac{f(x_k)(m_1+m_2+m_3)}{m_1 f[w_k,x_k]+m_2 f[y_k,x_k]+m_3 f[z_k,x_k]}. \end{cases} \tag{397}$$

Theorem 2.1. *Assume $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be an adequately smooth function and has a simple root α in D. Then the two-step and three-step without memory schemes denoted by (396) and (397) are of local order of convergence four and eight and satisfies in the following error equations, respectively.*

$$e_{k+1} = (1+\gamma f'(\alpha))^2 c_2((1+2\beta+2f'(\alpha)(-1+\beta)\gamma)c_2^2 - c_3)e_k^4 + O(e_k^5), \tag{398}$$

$$e_{k+1} = (1+\gamma f'(\alpha))^4 c_2^2((1+2\beta+2(-1+\beta)\gamma f'(\alpha))c_2^2 - c_3)(c_2^3 - 2c_2c_3 + c_4)e_k^8 + O(e_k^9), \tag{399}$$

where $c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}$ for $k = 2,3,\ldots$.
Now, we are going to construct new iterative methods with memory from (396), and (397) using self-

accelerating parameter.

$$\begin{cases} \gamma_k = -\frac{1}{N_3'(x_k)},\ k = 1, 2, 3, \ldots, \\ w_k = x_k + \gamma_k f(x_k),\ y_k = x_k - \frac{f(x_k)}{f[x_k, w_k]}, \\ g(x_k) = f[w_k, x_k] + 2(w_k - x_k) f[w_k, x_k, y_k] - f[y_k, w_k] + f[x_k, y_k], \\ x_{k+1} = y_k - \frac{f(y_k)}{g(x_k)} \frac{f(x_k)+\beta f(y_k)}{f(x_k)+(\beta-2)f(y_k)},\ k = 0, 1, 2, \ldots. \end{cases} \tag{400}$$

And

$$\begin{cases} \gamma_k = -\frac{1}{N_4'(x_k)},\ k = 1, 2, 3, \ldots, \\ w_k = x_k + \gamma f(x_k),\ y_k = x_k - \frac{f(x_k)}{f[x_k, w_k]},\ k = 0, 1, 2, \ldots, \\ g(x_k) = f[w_k, x_k] + 2(w_k - x_k) f[w_k, x_k, y_k] - f[y_k, w_k] + f[x_k, y_k], \\ z_k = y_k - \frac{f(y_k)}{g(x_k)} \frac{f(x_k)+\beta f(y_k)}{f(x_k)+(\beta-2)f(y_k)},\ m_1 = f(y_k)(f(z_k)(z_k - y_k), \\ m_2 = f(w_k)(f(z_k)(w_k - z_k),\ m_3 = f(y_k)(f(w_k)(y_k - w_k), \\ x_{k+1} = x_k - \frac{f(x_k)(m_1+m_2+m_3)}{m_1 f[w_k,x_k]+m_2 f[y_k,x_k]+m_3 f[z_k,x_k]}. \end{cases} \tag{401}$$

Theorem 2.2. *Let that x_0 is an approximation to a simple zero α of $f(x) = 0$, then the convergence order of the two-step and three-step methods with memory (400) and (401) is at 6 and 12, respectively.*

3 Numerical results

In this section, the family of the with memory methods (400) and (401) are experimented using four examples of nonlinear equations. The errors $|x_k - \alpha|$ of guesses to the sought zeros, are given in Table 1 where $x(-y)$ presents $x \times 10^{-y}$. Computational order of convergence COC, in Table 1 calculated by following expression [4]:

$$COC = \frac{\log|f(x_n)/f(x_{n-1})|}{\log|f(x_{n-1})/f(x_{n-2})|} \tag{402}$$

In this table symbol, EI is the abbreviation from the efficiency Index. We present some numerical test results with the following functions:

$$\begin{cases} f_1(x) = \sin(x) - x^2 + 1,\ \alpha \approx 1.409624004002596,\ x_0 = 1, \\ f_2(x) = -3x - e^x + x^2 + 2,\ \alpha \approx 0.257530285439861,\ x_0 = 0.8, \end{cases}$$

Table 14: Numerical results

$f_1(x)$					
Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	COC	EI
Method (392)	0.97546(−1)	0.17489(−2)	0.99575(−7)	2.41734	1.55478
Method (393)	0.12300(0)	0.45563(−3)	0.29068(−10)	3.00000	1.73205
Method (394)	0.15703(−1)	0.92894(−7)	0.22872(−15)	4.00000	1.58740
Method (395)	0.43850(−4)	0.22872(−15)	0.22872(−15)	8.00000	1.68179
Method (396)	0.15703(−1)	0.92894(−7)	0.22872(−15)	4.00000	1.58740
Method (397)	0.43850(−4)	0.22872(−15)	0.22872(−15)	8.00000	1.68179
Method (400)	0.15703(−1)	0.21921(−11)	0.22872(−15)	6.00000	1.81712
Method (401)	0.14106(−3)	0.22872(−15)	0.22872(−15)	12.00000	1.86121
$f_2(x)$					
Method (392)	0.86287(−1)	0.19441(−4)	0.27131(−12)	2.46169	1.56898
Method (393)	0.64658(−2)	0.52085(−7)	0.24595(−15)	3.00000	1.73205
Method (394)	0.10382(−2)	0.25577(−13)	0.24595(−15)	4.00000	1.58740
Method (395)	0.14427(−5)	0.24595(−15)	0.24595(−15)	8.00000	1.68179
Method (396)	0.10382(−2)	0.25577(−13)	0.24595(−15)	4.00000	1.58740
Method (397)	0.14427(−5)	0.24595(−15)	0.24595(−15)	8.00000	1.68179
Method (400)	0.10382(−2)	0.24594(−15)	0.24594(−15)	6.00000	1.81712
Method (401)	0.65767(−5)	0.24595(−15)	0.24595(−15)	12.00000	1.86121

4 Conclusion

In this work, we proposed the Ostrowski-Type methods with memory for solving nonlinear equations. In Table 1, we have examined some methods with different kinds of convergence order. The last column of table shows the efficiency index. Proposed methods (400), and (401) do not need any derivatives and can be used even for non-smooth functions. The efficiency index of the proposed families with memory are 1.81712 and 1.86121, which is much better than optimal methods without memory. Table 1 shows that all of the methods work in concordance with theoretical results.

References

[1] J. Dzunic, M. Petkovic, *A cubically convergent Steffensen-like method for solving nonlinear equations*, Applied Mathematics Letters, 25 (2012) 1881–1886.

[2] J. Herzberger, *Uber Matrixdarstellungen fur Iterationverfahren bei nichtlinearen Gleichungen*, Computing, 12 (1974) 215–222.

[3] M.S. Petkovic, J. Dzunic, B. Neta, *Interpolatory multipoint methods with memory for solving nonlinear equations*, Applied Mathematics and Computation, 218 (2011) 2533–2541.

[4] M.S. Petkovic, B. Neta, L.D. Petkovic, J. Dzunic, *Multipoint methods for solving nonlinear equations*, Elsevier, Amsterdam, (2013).

[5] O. Said Solaiman, S.A.A. Karim, I. Hashim, Optimal fourth-and eighth-order of convergence derivative-free modifications of King's method, Journal of King Saud University-Science, 31 (4) (2019) 1499–1504.

[6] J.F. Traub, *Iterative Methods for the Solution of Equations*, Prentice Hall, New York, USA, (1964).

[7] V. Torkashvand, T. Lotfi, M.A. Fariborzi Araghi, *A new family of adaptive methods with memory for solving nonlinear equations*, Mathematical Sciences, 13 (1) (2019) 1–20.

[8] X. Wang, T. Zhang, *A new family of Newton-type iterative methods with and without memory for solving nonlinear equations,* Calcolo, 51 (14) (2014) 1–15.

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A new Ky Fan theorem for weakly irreducible nonnegative tensors

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Abstract

In this paper, by using the ratio of the smallest and largest values of a Perron vector and Brualdi-type eigenvalue inclusion sets, we propose a new Ky Fan-type theorem for weakly irreducible nonnegative tensors. Finally, a numerical example is given to verify the theoretical result.

Keywords: Ky Fan theorem, Brualdi-type eigenvalue inclusion sets, weakly irreducible nonnegative tensors

Mathematics Subject Classification [2023]: 15A18, 15A69 15A42

1 Introduction and preliminaries

A high order tensor is a multi-way array whose entries are addressed via multiple indices in the following form [1, 2, 3]

$$\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R},\ i_j \in [n],\ j = 1, ..., m.$$

Tensor $\mathcal{A}$ is called nonnegative if $a_{i_1 i_2 \ldots i_m} \geq 0$.

Tensor eigenvalue problems have attracted much attention of researchers in recent decades due to their wide applications in data analysis, automatical control and etc.

Definition 1.1. Let $\mathcal{A}$ be a tensor of order m dimension n.

(i) Let $\lambda \in \mathbb{C},\ x \in \mathbb{C}^n \backslash \{0\}$, then (λ, x) is called an eigenpair of tensor $\mathcal{A}$ if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}.$$

where $(\mathcal{A}x^{m-1})_i = \sum_{i_2, i_3, \ldots, i_m = 1}^{n} a_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m}$.

(ii) We call $\sigma(\mathcal{A})$ as the set of all eigenvalues of $\mathcal{A}$ and the spectral radius of $\mathcal{A}$ is denoted by

$$\rho(\mathcal{A}) = \max \{|\lambda| \ :\ \lambda \in \sigma(A)\}.$$

(iii) We call a tensor $\mathcal{A}$ reducible if there exists a nonempty proper index subset $I \subset \langle n \rangle := \{1, 2, ..., n\}$ such that

$$a_{i_1 i_2 \ldots i_m} = 0, \quad \forall\, i_1 \in I, \quad i_2, ..., i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.

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(iv) We call a tensor $\mathcal{A}$ nonnegative weakly irreducible, if for any nonempty proper index subset $I \subset \langle n \rangle$, there is at least an entry $a_{i_1 i_2 \ldots i_m} > 0$, where $i_1 \in I$, and at least an $i_j \notin I, \quad j = 2, ..., m$.

(v) We denote by $\delta_{i_1 i_2 \ldots i_m}$, the Kronecker symbol for the case of m indices, that is,

$$\delta_{i_1 i_2 \ldots i_m} = \begin{cases} 1 & i_1 = i_2 = ... = i_m \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.2. If $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ is a real tensor of order m dimension n and D is a diagonal matrix, then the product of $\mathcal{A}$ and D is defined as follows:

$$\mathcal{A}_D = \mathcal{A}.D^{-(m-1)}.\overbrace{D....D}^{m-1}$$

where

$$(\mathcal{A}_D)_{i_1 i_2 \ldots i_m} = a_{i_1 \ldots i_m} D_{i_1, i_1}^{-(m-1)} D_{i_2, i_2} ... D_{i_m, i_m}.$$

If D is a diagonal nonsingular matrix, then $\mathcal{A}_D$ and $\mathcal{A}$ have the same eigenvalues.

2 Main result

In this section, we obtain an improvement of Ky Fan theorem. For that, we need the following lemma.

Lemma 2.1. *[3] Let $\mathcal{A}$ be a weakly irreducible nonnegative tensor of order m dimension n with maximal eigenvector x. Then*

$$\tau(\mathcal{A}) = \max_{1 \le k \le m-1} \tau^{(k)}(\mathcal{A}) \le \frac{x_{min}}{x_{max}}, \tag{403}$$

where

$$\tau^{(k)}(\mathcal{A}) = \max_{\substack{\{j_1,...,j_k\} \subseteq \{2,...,m\} \\ \{j'_1,...,j'_k\} \subseteq \{2,...,m\}}} \quad \min_{\substack{1 \le i_1, i'_1 \le n \\ 1 \le i_{j_t} = i'_{j_t} \le n \\ 1 \le t \le k}} \left(\frac{\sum\limits_{\underbrace{i_2, ..., i_m}_{except\ i_{j_1}, ..., i_{j_k}} = 1}^{n} a_{i_1, ..., i_m}}{\sum\limits_{\underbrace{i'_2, ..., i'_m}_{except\ i'_{j_1}, ..., i'_{j_k}} = 1}^{n} a_{i'_1, ..., i'_m}} \right)^{\frac{1}{k}}. \tag{404}$$

Theorem 2.2. *Let $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{[m,n]}$. If $\mathcal{B}$ is irreducible and $\mathcal{B} \ge |\mathcal{A}|$, then for any eigenvalue λ of $\mathcal{A}$, there exist $i \ne j$ such that*

$$\begin{aligned} &|\lambda - \mathcal{A}_{i,\cdots,i}|^{m-1} |\lambda - \mathcal{A}_{j,\cdots,j}| \\ \le &\left[\rho(\mathcal{B}) - \mathcal{B}_{i,\cdots,i} - (\tau(\mathcal{B}))^{m-1} r_i(\mathcal{E})\right]^{m-1} \left[\rho(\mathcal{B}) - \mathcal{B}_{j,\cdots,j} - (\tau(\mathcal{B}))^{m-1} r_j(\mathcal{E})\right] \end{aligned}$$

where $\mathcal{E} = \mathcal{B} - |\mathcal{A}|, r_i(\mathcal{E}) = \sum_{i_2,\cdots,i_m=1}^{n} |\mathcal{E}_{i,i_2,\cdots,i_m}| - \mathcal{E}_{i,\cdots,i}$ and $\tau(\mathcal{B})$ is given in (403).

Proof. Since $\mathcal{B}$ is nonnegative and irreducible, by Theorem 2.2 of [2], there exists a positive vector $\mathbf{y}$ such that

$$\rho(\mathcal{B})y_i^{m-1} = \sum_{i_2,\cdots,i_m=1}^{n} \mathcal{B}_{i,i_2,\cdots,i_m} y_{i_2}\cdots y_{i_m}$$

Let $D = \operatorname{diag}(y_1,\cdots,y_m)$ and define $\mathcal{A}_D$ and $\mathcal{B}_D$ as in (1.2). Then we get

$$\rho(\mathcal{B}) = \sum_{i_2,\cdots,i_m=1}^{n} (\mathcal{B}_D)_{i,i_2,\cdots,i_m} \qquad and \qquad \lambda(\mathcal{A}) = \lambda(\mathcal{A}_D). \tag{405}$$

Let $\mathbf{x}$ be an eigenvector of $\mathcal{A}_D$ corresponding to λ, and $\mathcal{E} = \mathcal{B} - |\mathcal{A}|$. Hence $\mathcal{E} \geq 0$, and for any $1 \leq i \leq n$

$$\lambda x_i^{m-1} = \sum_{i_2,\cdots,i_m=1}^{n} (\mathcal{A}_D)_{i,i_2,\cdots,i_m} x_{i_2}\cdots x_{i_m} \tag{406}$$

Let $|x_l| = \max_{1\leq i\leq n}|x_i|, \Delta_i^{(k)}$ be a set of all sequences $(i_2,\cdots,i_m)$, each of which exactly consists of k ' i 's, and $\bar{\Delta}_i^{(k)}$ be the complement of $\Delta_i^{(k)}$. Therefore it follows from (405) and (406) that

$$\begin{aligned}
\left|\lambda - (\mathcal{A}_D)_{l,\cdots,l}\right| &\leq \sum_{(i_2,\cdots,i_m)\in\bar{\Delta}_l^{(m-1)}} \left|(\mathcal{A}_D)_{l,i_2,\cdots,i_m}\right| \frac{|x_{i_2}|\cdots|x_{i_m}|}{|x_l|^{m-1}} \\
&\leq \sum_{(i_2,\cdots,i_m)\in\bar{\Delta}_l^{(m-1)}} \left|(\mathcal{A}_D)_{l,i_2,\cdots,i_m}\right| \frac{|x_t|}{|x_l|} \\
&= \left[\sum_{(i_2,\cdots,i_m)\in\bar{\Delta}_l^{(m-1)}} (\mathcal{B}_D)_{l,i_2,\cdots,i_m} - \sum_{(i_2,\cdots,i_m)\in\bar{\Delta}_l^{(m-1)}} (\mathcal{E}_D)_{l,i_2,\cdots,i_m}\right] \frac{|x_t|}{|x_l|} \\
&= \left[\rho(\mathcal{B}) - \mathcal{B}_{l,\cdots,l} - \sum_{(i_2,\cdots,i_m)\in\bar{\Delta}_l^{(m-1)}} \mathcal{E}_{l,i_2,\cdots,i_m} \frac{y_{i_2}\cdots y_{i_m}}{y_l^{m-1}}\right] \frac{|x_t|}{|x_l|} \\
&\leq \left[\rho(\mathcal{B}) - \mathcal{B}_{l,\cdots,l} - r_l(\mathcal{E})\left(\frac{y_{\min}}{y_{\max}}\right)^{m-1}\right] \frac{|x_t|}{|x_l|} \\
&\leq \left[\rho(\mathcal{B}) - \mathcal{B}_{l,\cdots,l} - (\tau(\mathcal{B}))^{m-1} r_l(\mathcal{E})\right] \frac{|x_t|}{|x_l|},
\end{aligned} \tag{407}$$

where $|x_t| = \max_{1\leq i\leq n}|x_i|\,(i\neq l), y_{\min} = \min_{1\leq i\leq n} y_i, y_{\max} = \max_{1\leq i\leq n} y_i$ and $\tau(\mathcal{B})$ is a lower bound for $\frac{y_{\min}}{y_{\max}}$. Similarly, by (405) and (406) again, for index t we have

$$\begin{aligned}
\left|\lambda - (\mathcal{A}_D)_{t,\cdots,t}\right| |x_t|^{m-1} &\leq \sum_{(i_2,\cdots,i_m)\in\bar{\Delta}_t^{(m-1)}} \left|(\mathcal{A}_D)_{t,i_2,\cdots,i_m}\right| |x_{i_2}|\cdots|x_{i_m}| \\
&\leq \left[\rho(\mathcal{B}) - \mathcal{B}_{t,\cdots,t} - (\tau(\mathcal{B}))^{m-1} r_t(\mathcal{E})\right] |x_l|^{m-1}
\end{aligned} \tag{408}$$

Combining (407) and (408), we get

$$\begin{aligned}
&|\lambda - \mathcal{A}_{l,\cdots,l}|^{m-1} |\lambda - \mathcal{A}_{t,\cdots,t}| \\
\leq &\left[\rho(\mathcal{B}) - \mathcal{B}_{l,\cdots,l} - (\tau(\mathcal{B}))^{m-1} r_l(\mathcal{E})\right]^{m-1} \left[\rho(\mathcal{B}) - \mathcal{B}_{t,\cdots,t} - (\tau(\mathcal{B}))^{m-1} r_t(\mathcal{E})\right],
\end{aligned}$$

which proves the desired inequality. □

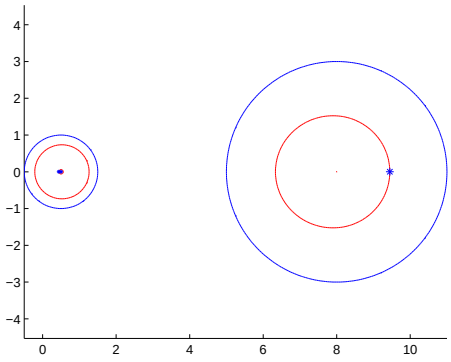


Figure 7: Comparisons of Ky Fan theorem, Theorem 3.3 of [2] and Theorem 2.2. The "$*$" symbols show the location of the eigenvalues of $\mathcal{A}$

Example 2.3. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ be a tensor with entries defined as follows:

$$a_{1111} = \frac{1}{2},\ a_{1222} = a_{2122} = a_{2212} = a_{2221} = -1,\ a_{2222} = 8 \ \ \textit{and other}\ a_{ijkl} = 0.$$

Let $\mathcal{B} = (b_{ijkl}) \in \mathbb{R}^{[4,2]}$ be a tensor, where $b_{ijkl} = 8$ for nay $1 \leq i, j, k, l \leq 2$. We get that all the H-eigenvalues of $\mathcal{A}$ are $0.4376,\ 0.5000$ *and* 9.4452. By [2, Lemma 5.1], we have $\rho(\mathcal{B}) = 64$. From Theorem 3.3 in [2], we obtain

$$\sigma(\mathcal{A}) \subseteq \left\{z : \left|z - \frac{1}{2}\right| \leq 64 - 8 - 1^3 \times 55 = 1\right\} \cup \left\{z : |z - 8| \leq 64 - 8 - 1^3 \times 53 = 3\right\}.$$

But from Theorem 2.2, we obtain

$$\begin{aligned}\sigma(\mathcal{A}) \subseteq &\left\{z : \left|z - \frac{1}{2}\right|^3 |z - 8| \leq [64 - 8 - (1^3 \times 55)]^3 [64 - 8 - (1^3 \times 53)] = 3\right\} \\ &\cup \left\{z : |z - 8|^3 \left|z - \frac{1}{2}\right| \leq [64 - 8 - (1^3 \times 53)]^3 [64 - 8 - (1^3 \times 55)] = 27\right\}.\end{aligned}$$

3 Conclusion

In this paper, we proposed a new improvement of Ky Fan theorem for tensors. Based on the ratio of the smallest and largest entries of the Perron vector, we obtain Theorem 2.2, which improves the bounds in [2], and show by an example, that this bound is sharper than the one given in [2].

References

[1] Bu C, Jin X, Li H, Deng C (2017) Brauer-type eigenvalue inclusion sets and the spectral radius of tensors. Linear Algebra and its Applications 512: 234-240.

[2] Li, W., Ng, M. K. (2015). Some bounds for the spectral radius of nonnegative tensors. Numerische Mathematik, 130(2), 315-335.

[3] Tourang, M. and Zangiabadi, M. (2021). Some improvements on the Ky Fan theorem for tensors. Computational and Applied Mathematics, 40(2), 1-20.

[4] Yang Y, Yang Q (2010) Further results for Perron-Frobenius theorem for nonnegative tensors. SIAM Journal on Matrix Analysis and Applications 31(5): 2517-2530.

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The comparative study of uncertainty measures in evidence theory

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Abstract

In this paper, a comparative study of various types of uncertainty measurements was done. The results of this study showed that the most consistent and complete measure compared to previous researches was defined by Zhang et al. But this uncertainty measure does not have a subadditivity property. Therefore, the introduction of a comprehensive measure is recommended by researchers.

Keywords: uncertainty measures, evidence theory, entropy

Mathematics Subject Classification [2023]: 18A32, 18F20, 05C65

1 Introduction

In recent years, with the development of sensor technology, equipping complex systems with several sensors has become the subject of research. In this regard, multi-sensor monitoring that can effectively combine multi-sensor information and have higher reliability is a key issue.Many techniques are proposed to solve the issue, such as the Dempster-Shafer theory (DST), Kalman filtering , fuzzy theory, Bayesian reasoning method ,neural network, and so on.Among them, DST is an effective framework to deal with uncertain information. This theory was first proposed by Dempster and further developed by Shafer. It is widely used in decision-making, risk assessment, and so on. Many studies in recent years have focused on conflict resolution , evidence revision,combination rules , and information volume. Many methods about uncertainty quantification have also been proposed. However, the existing methods have some limitations, and the uncertainty measure of basic probability assignment (BPA) is still an open issue in DST. The concept of entropy was first proposed by the German physicist Clausius in 1865. In thermodynamics, entropy is a measure of the chaos of a system. In information theory, entropy, also known as Shannon entropy, represents a measure of the uncertainty of a random variable. Parker and Jeynes also showed from a MaxEnt standpoint that the (stupendously gigantic) entropy of the supermassive black hole at the center of the Milky Way can account for the geometrical stability of the galaxy. Among them, the Shannon entropy, verified in the past few decades, is an effective way to measure uncertainty in probability theory (PT), but its direct application in DST is inappropriate. That is because that PT describes the probability of the occurrence of singletons, while the evidence theory is based on the theory of non-additive probabilities, which can represent the possibility of the occurrence of propositions with multiple elements. Based on the above analysis, many scholars have proposed different entropy-like measures to quantify the uncertainty of a body of evidences (BOEs) in DST.

The purpose of current research is the meta-analysis of different entropies (measures of uncertainty) by several researchers. After introducing the uncertainty measures, the comparative study of uncertainty measures was carried out. The remainder of this paper is organized as follows. Some preliminaries are introduced in Section 2. In Section 3, we illustrate the uncertainty measures introduced by authors in detail. Section 4 is a conclusion of the paper.

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2 Preliminaries

The Dempster-Shafer theory, proposed by Dempster and expanded by Shafer, is a mathematical theory of multi-source information. This method can effectively cope with uncertainty. It is widely used in target identification, fusion decision, and so on. Some definitions about this theory are as follows [5, 6].

Definition 2.1. If $\mathbf{\Theta} = \{\theta_1, \theta_2, \cdots \theta_r\}$ is a finite complete set of r mutually exclusive elements, it is called a Frame of Discernment (FOD). Set $\mathbf{\Theta}$ is a FOD, and its power set constitutes a set of propositions. If a function $m : 2^{\mathbf{\Theta}} \to [0, 1]$ satisfies the following formula:

$$\begin{cases} m(\varnothing) = 0 \\ \sum_S m(S) = 1 \end{cases}$$

the mass function m is a Basic Probability Assignment (BPA). In this definition, $m(S)$ is the basic probability number of proposition S, and indicates the belief assigned to S.

Definition 2.2. Let m be a BPA on FOD $\mathbf{\Theta}$, if $Bel : 2^{\mathbf{\Theta}} \to [0, 1]$ statisfies; $Bel(S) = \sum_{B \subseteq S} m(B), S \in 2^{\mathbf{\Theta}}$ then, $Bel(S)$ is called the belief measure of proposition S.

$Pl(S)$ is the plausibility function that is defined as follows:

$$Pl(S) = 1 - Bel(\bar{S}) = \sum_{B \cap S \neq \varnothing} m(B)$$

$Pl(S)$ is the degree to which you do not disagree with proposition S.

The commonality function Q_m corresponding to BPA m is defined as follows:

$$Q_m(S) = \sum_{S \subseteq B} m(B), S \in 2^{\mathbf{\Theta}}$$

Definition 2.3. Let X be a sample space with possible values $\{x_1, x_2, \cdots, x_n\}$, then, the Shannon entropy is defined as:

$$H_s = \sum_{x_i \in X} p(x_i) \log_2 \frac{1}{p(x_i)}$$

where $p(x_i)$ is the probability of x_i. It also satisfies $\sum_{x_i \in X} p(x_i) = 1$. It is maximal only for the uniform probability distribution and then $H_S = \log n$, where $n = |X|$ (cardinality of X). It is minimal and is zero for x such that$p(x) = 1$.

Definition 2.4. In set theory the Hartley measure is defined by $H(A) = \log |A|$. Since logarithmic function is strictly increasing, it is monotonic with respect to set inclusion.

Definition 2.5. If a subset a satisfies $a \in 2^X$ and $m(a) > 0$, then a is called the focal element of m. If a BPA m contains a focal element X with belief 1, then the BPA m is a vacuous BPA.

3 Main results

In evidence theory, there are two types of uncertainty, namely, the discord and the non-specificity. The discord measure and non-specificity measure are also aggregated in different forms to measure the uncertainty of belief structures. The measures of discord are defined to describe the randomness (or discord or conflict) in a BPA. Shannon entropy is the classical measure of discord in probability theory. Non-specificity means two or more alternatives are left unspecified and represents an imprecision degree. In classical set theory, the only measurable kind of uncertainty is the non-specificity, which is related to its cardinality. The Hartley measure is regarded as the standard measure of non-specificity for classical sets. Generally in science and engineering, Entropies have been widely used, but their have essential restrictions of investigation the uncertainty in DST. So, many authors proposed methods of uncertainty measurements. Let X be a FOD, and existing uncertainty measures in DST are listed in Table15.

Table 15: Comparison of uncertainty measures [3], [2], [5], [1], [4], [6]

Definition	Year	Cons. DST	Non-neg	Max. ent	Monot.	Prob. cons.	Addit.
Hhle entropy	1982	yes	no	no	no	yes	yes
Smets	1983	yes	no	no	no	no	yes
Yagers entropy	1983	yes	no	no	no	yes	yes
Weighted Hartley entropy	1986	yes	no	yes	yes	no	yes
Nguyens entropy	1987	yes	no	no	no	yes	yes
Lamata and Moral	1987	yes	yes	no	yes	yes	yes
Klir-Ramer Eq.	1990	yes	yes	no	yes	yes	yes
Inuiguchi, et al.	1991	yes	yes	no	yes	yes	yes
Klir-Parviz	1992	yes	yes	no	yes	yes	yes
Pal et al.	1993	yes	yes	no	yes	yes	yes
Maeda and Ichihashi	1993	no	yes	yes	yes	yes	yes
Aggregated measure	1994	no	yes	no	yes	yes	yes
Harmanec and Klir	1994	no	yes	no	yes	yes	yes
Abelln and Moral	1999	no	yes	yes	yes	yes	yes
Jousselme et al.	2006	no	yes	no	yes	yes	yes
Pouly et al.	2013	no	yes	no	yes	yes	yes
TUI measure	2016	yes	yes	no	no	yes	no
Deng entropy	2016	yes	yes	no	no	yes	no
Jirouek and Shenoy	2017	yes	yes	yes	yes	yes	yes
$TUEI$ measure	2018	Yes	yes	yes	yes	yes	yes
Qin et al. measure	2020	Yes	yes	yes	yes	yes	yes
Zhang et al. measure	2021	Yes	yes	yes	yes	yes	yes

4 Conclusion

In this article, there was a review of the uncertainty measures introduced by researchers in the years 1982 to 2021. The results presented in Table (1) show that in previous years, the defined uncertainty measures did not have all the properties; But in recent years and from 2018 onwards, the uncertainty measures had all the properties. The best uncertainty measures was introduced by Zhang et al. [6]. Examining numerical examples by the measure of Zhang et al. was more consistent. On the other hand, another important property of uncertainty measures is the subset property, unfortunately, the measure defined by Zhang et al. does not have this property, and the researchers were unable to prove the subadditivity property of this measure. Therefore, researchers in the future are suggested to introduce the uncertainty measure with all the properties, especially the subadditivity property, based on the changes in the measure of Zhang et al.

References

[1] X. Deng, *Analyzing the monotonicity of belief interval based uncertainty measures in belief function theory*. Int. J. Intell. Syst, 33, (2018), 17751985.

[2] D. Dubois, Prade, H. *Properties of measures of information in evidence and possibility theories*. Fuzzy Sets Syst, 24, (1987), 161182.

[3] R. Jirouek, P.P. Shenoy, *A new definition of entropy of belief functions in the DempsterShafer theory*. Int. J.Approx. Reason, (2018), 92, 4965.

[4] M. Qin, T. Yongchuan, W. Junhao, *An improved total uncertainty measure in the evidence theory and its application in decision making*.Entropy 22, 4 (2020), 487.

[5] Y. Yang, D. Han, *A New Distance-Based Total Uncertainty Measure in the Theory of Belief Functions*. Know. Based Syst. 94, (2016), 114123.

[6] Y. Zhang, F. huang, X. Deng, W. Jiang. *A New Total Uncertainty Measure from A Perspective of Maximum Entropy Requirement* . Entropy, 23(2021), 1-20.

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Generalized (Jordan) derivations on C^*-algebras through zero products

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Abstract

Let A be a unital C^*-algebra and X be a unitary Banach A-bimodule. We characterize continuous generalized derivations and generalized Jordan derivations $D : A \longrightarrow X$ through the action on zero product.

Keywords: Generalized derivation, Generalized Jordan derivation, Bilinear maps.

Mathematics Subject Classification [2023]: 47A07, 47B47

1 Introduction

Let A be a unital Banach algebra with unit e_A and X be a unitary (Banach) A-bimodule. A linear map $D : A \longrightarrow X$ is said to be a *derivation* [*generalized derivation*] if for all $a, b \in A$,

$$D(ab) = D(a)b + aD(b), \qquad [D(ab) = D(a)b + aD(b) - aD(e_A)b],$$

and it is called a *Jordan derivation* [*generalized Jordan derivation*] if

$$D(a \circ b) = D(a) \bullet b + a \bullet D(b), \quad D(a \circ b) = D(a) \bullet b + a \bullet D(b) - aD(e_A)b - bD(e_A)a,$$

for all $a, b \in A$. Here and subsequently, $'\circ'$ denotes the Jordan product $a \circ b = ab + ba$ on A and $'\bullet'$ denotes the Jordan product on X:

$$a \bullet x = x \bullet a = ax + xa, \qquad a \in A, \quad x \in X.$$

Clearly, each (generalized) derivation is a (generalized) Jordan derivation, but the converse is fails in general. It is proved by Johnson in [6, Theorem 6.3] that every continuous Jordan derivation from C^*-algebra A into any A-bimodule X is a derivation.

We recall that a C^*-algebra A is called a W^*-algebra if it is a dual space as a Banach space [5]. Note that every W^*-algebra is unital.

Let A be a Banach algebra and X be a arbitrary Banach space. Then the continuous bilinear mapping $\phi : A \times A \longrightarrow X$ preserves zero products if

$$ab = 0 \quad \Longrightarrow \quad \phi(a, b) = 0, \qquad a, b \in A. \tag{409}$$

The study of zero products preserving bilinear maps was initiated in [1] for a very special setting and then it was studied in [2] for the general case. Motivated by (409) the following concept was introduced in [2].

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Definition 1.1. A Banach algebra A *has the property* $(\mathbb{B})$ if for every continuous bilinear mapping $\phi : A \times A \longrightarrow X$, where X is an arbitrary Banach space, the condition (409) implies that $\phi(ab, c) = \phi(a, bc)$, for all $a, b, c \in A$.

It follows from [2, Theorem 2.11] that C^*-algebras, group algebras and Banach algebras that generated by idempotents have the property $(\mathbb{B})$.

It should be pointed out that the property $(\mathbb{B})$ is a powerful tool for characterizing homomorphisms, derivations and Jordan derivations on that class of algebras through the action on zero products. We refer the reader to [1, 2, 3] for a full account of the topic and a list of references.

The following result, which closely related to the property $(\mathbb{B})$, plays a key role in this sequel.

Theorem 1.2. *[3, Theorem 2.2] Let A be a C^*-algebra and X be a Banach space and let $\phi : A \times A \longrightarrow X$ be a continuous bilinear mapping such that*

$$ab = ba = 0 \implies \phi(a, b) = 0, \qquad a, b \in A.$$

Then

$$\phi(xa, yb) + \phi(bx, ay) = \phi(x, ayb) + \phi(bxa, y),$$

for all $a, b, x, y \in A$.

In this paper we consider the subsequent conditions on a linear map D from a Banach algebra A into a Banach A-bimodule X;

$(\mathbb{G}1)$ $a, b \in A,\ ab = 0 \implies D(a)b + aD(b) = 0,$

$(\mathbb{G}2)$ $a, b, c \in A,\ ab = bc = 0 \implies aD(b)c = 0,$

$(\mathbb{J}1)$ $a, b \in A,\ ab = ba = 0 \implies D(a)b + aD(b) = 0,$

$(\mathbb{J}2)$ $a, b \in A,\ ab = ba = 0 \implies D(a) \bullet b + a \bullet D(b) = 0,$

$(\mathbb{J}3)$ $a, b \in A,\ ab = ba = 0 \implies aD(b) + bD(a) = 0.$

Our purpose is to investigate whether those conditions characterizes generalized derivations and generalized Jordan derivations.

2 Generalized derivations on C^*-algebras

In this section, we characterizes generalized derivations from unital C^*-algebra A into unitary Banach A-bimodule X, that satisfy the condition $(\mathbb{G}1)$ or $(\mathbb{G}2)$, through their action on zero products.

Theorem 2.1. *Let A be a unital C^*-algebra. If $D : A \longrightarrow X$ is a continuous linear map satisfying $(\mathbb{G}1)$, then D is a generalized derivation.*

Note that if D is a generalized derivation, then the linear map $\delta : A \longrightarrow X$ defined by $\delta(a) = D(a) - D(e_A)a$ is a derivation, thus we get the following result.

Corollary 2.2. *[1, Corollary 4.2]. Let A be a unital C^*-algebra. If $D : A \longrightarrow X$ is a continuous linear map satisfying $(\mathbb{G}1)$, then $aD(e_A) = D(e_A)a$ for all $a \in A$ and there is a derivation $\delta : A \longrightarrow X$ such that $D(a) = D(e_A)a + \delta(a)$.*

If D is a continuous linear map from unital C^*-algebra A into unitary A-bimodule X such that $D(b)a + bD(a) = 0$ for all $a, b \in A$ with $ab = 0$, then D is a generalized Jordan derivation. Indeed, by a similar proof of preceding theorem we can obtain

$$D(ab) = D(b)a + bD(a) - D(e_A)ab, \quad a, b \in A.$$

Taking $a = b$, we get $D(a^2) = D(a) \bullet a - aD(e_A)a$ for all $a \in A$. Moreover, if $D(e_A) = 0$, then D is a Jordan derivation and hence it is a derivation by [6, Theorem 6.3].

Example 2.3. *Let*

$$A = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} : \quad a_{11}, a_{12}, a_{22} \in \mathbb{C} \right\}.$$

We make $X = \mathbb{C}$ an A-bimodule by defining

$$a\lambda = a_{22}\lambda, \quad \lambda a = \lambda a_{11}, \quad \lambda \in \mathbb{C},\ a \in A.$$

Consider the linear map $D : A \longrightarrow X$ defined by $D(a) = a_{12}$. Then $D(ab) = D(b)a + bD(a)$ for all $a, b \in A$, and $D(e_A) = 0$. In particular, D is a generalized Jordan derivation, but it is not a generalized derivation. Therefore the Johnson's result is not valid for unital Banach algebras instead of C^*-algebras, in general.

In the next result, we characterize generalized derivations by using condition ($\mathbb{G}2$).

Theorem 2.4. *Suppose that A is a unital C^*-algebra and $D : A \longrightarrow X$ is a continuous linear map such that the condition ($\mathbb{G}2$) holds. Then D is a generalized derivation.*

3 Generalized Jordan derivations on C^*-algebras

In this section, we prove that each linear mapping D from unital C^*-algebra A into unitary Banach A-bimodule X which satisfies the condition ($\mathbb{J}1$), ($\mathbb{J}2$) or ($\mathbb{J}3$) is a generalized Jordan derivation.

Our first main theorem is the following.

Theorem 3.1. *Let A be a W^*-algebra and let $D : A \longrightarrow X$ be a continuous linear map satisfying ($\mathbb{J}1$). Then D is a generalized Jordan derivation.*

It is well-known that on the second dual space A^{**} of a Banach algebra A there are two multiplications, called the first and second Arens products which make A^{**} into a Banach algebra [5]. If these products coincide on A^{**}, then A is said to be Arens regular.

It is shown [5] that every C^*-algebra A is Arens regular. Also the second dual of each C^*-algebra is a W^*-algebra.

There exists another related concept of generalized derivation that first appeared in [4]. Let A be a Banach algebra and X be an A-bimodule. A linear operator $D : A \longrightarrow X$ is said to be a *generalized derivation* if there exist $\xi \in X^{**}$ such that

$$D(ab) = D(a)b + aD(b) - a\xi b, \quad a, b \in A. \tag{410}$$

It should be note that if A is unital and D is a generalized derivation according to (410), then by [2, Proposition 4.2],

$$D(ab) = D(a)b + aD(b) - aD(e_A)b,$$

for all $a, b \in A$ and hence D is a generalized derivation in the usual sense.

Motivated by (410) we introduce the concept of generalized Jordan derivation. A linear operator $D : A \longrightarrow X$ is said to be a *generalized Jordan derivation* if there exist $\xi \in X^{**}$ such that

$$D(a \circ b) = D(a) \bullet b + a \bullet D(b) - a\xi b - b\xi a, \tag{411}$$

for all $a, b \in A$. Similarly, if A is unital and D satisfies in (411) for some $\xi \in X^{**}$, then

$$D(ab) = D(a) \bullet b + a \bullet D(b) - aD(e_A)b - bD(e_A)a, \qquad a, b \in A.$$

Now we prove Theorem 3.1 to the C^*-algebra case. First note that the linear span of projections are dense in a unital C^*-algebra of real rank zero, hence the conclusion of Theorem 3.1 also holds for such C^*-algebras.

By applying the techniques in [3, Theorem 4.1], we have the following result.

Theorem 3.2. *Let A be a unital C^*-algebra and let $D : A \longrightarrow X$ be a continuous linear map satisfying* ($\mathbb{J}1$). *Then D is a generalized Jordan derivation.*

Corollary 3.3. *Let A be a unital C^*-algebra. If $D : A \longrightarrow X$ is a continuous linear map satisfying* ($\mathbb{J}1$), *then D is a derivation if and only if $D(e_A) = 0$.*

The following result characterize generalized Jordan derivation by using condition ($\mathbb{J}2$).

Theorem 3.4. *Let A be a W^*-algebra and let $D : A \longrightarrow X$ be a continuous linear map satisfying* ($\mathbb{J}2$). *Then D is a generalized Jordan derivation.*

Similar to the proof of Theorem 3.2, we can obtain the following result.

Corollary 3.5. *Let A be a unital C^*-algebra and $D : A \longrightarrow X$ be a continuous linear map satisfying* ($\mathbb{J}2$). *Then D is a generalized Jordan derivation.*

Theorem 3.6. *Let A be a W^*-algebra and $D : A \longrightarrow X$ be a continuous linear map such that the condition* ($\mathbb{J}3$) *holds. Then $D(a) = aD(e_A)$ for all $a \in A$. In particular, D is a generalized derivation.*

As a consequence of Theorem 3.6 we conclude the following result.

Corollary 3.7. *Let A be a unital C^*-algebra. If $D : A \longrightarrow X$ is a continuous linear map satisfying* ($\mathbb{J}3$), *then $D(a) = aD(e_A)$ for all $a \in A$. Moreover, if $D(e_A) = 0$, then $D \equiv 0$.*

References

[1] J. Alaminos, M. Bresar, J. Extremera and A. R. Villena, *Characterizing homomorphisms and derivations on C^*-algebras*, Proc. Roy. Soc. Edinb., A 137 (2007), 1-7.

[2] J. Alaminos, M. Bresar, J. Extremera and A. R. Villena, *Maps preserving zero products*, Studia Math., 193 (2009), 131–159.

[3] J. Alaminos, M. Bresar, J. Extremera and A. R. Villena, *Characterizing Jordan maps on C^*-algebras through zero products*, Proc. Roy. Soc. Edinb. Sect., 53 (2010), 543–555.

[4] M. Bresar, *On the distance of the compositions of two derivations to the generalized derivations*, Glasgow Math. J., 33 (1991), 89–93.

[5] H. G. Dales, *Banach Algebras and Automatic Continuity*, Clarenden Press, Oxford, 2000.

[6] B. E. Johnson, *Symmetric amenability and the nonexistence of Lie and Jordan derivations*, Math. Proc. Camb. Phil. Soc., 120 (1996), 455–473.

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The existence and uniqueness of a solution for the nonlinear Fredholm integral equation

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Abstract

The aim of is finding a solution for the nonlinear Fredholm integral equation in the setting of w-distance. It is a new and distinct solution to the open problem of Rhoades. For this purpose, first, we introduce generalized weak two-sided convex contraction mapping f, which is k-continuous or S-orbitally lower semi-continuous instead of continuity at the fixed point. We prove some new fixed point theorems on a complete metric space with a w-distance. Some examples are illustrated our main results.

Keywords: Fixed point, w-distance, generalized contractive, k-continuous mapping.

Mathematics Subject Classification [2023]: 47H10,47H09

1 Introduction

In 1981, Istrătescu generalized the Banach–Picard–Caccioppoli (BCP) contraction principle and proved that each convex contraction has a unique fixed point on a complete metric space. The concept of convex contractions was introduced by Istrătescu.

Definition 1.1. A continuous mapping $f : M \to M$ is said to be a convex contraction of order 2 if there exist $a, b \in (0, 1)$, $a + b < 1$ such that for all $x, y \in M$,

$$d(f^2(x), f^2(y)) \leq ad(f(x), f(y)) + bd(x, y)$$

In 2017, Pant introduced the following notion of k-continuity for the metric space.

Definition 1.2. Suppose that (M, d) is a metric space and S is a self-mapping. S is said to be k-continuous, (k = 1, 2, 3, $\cdots$) if for every sequence $\{x_n\}$ in X such that $S^{k-1}x_n \to z$ implies that $S^k x_n \to Sz$.

The following theorem was proved by Istrătescu that the assumption of continuity condition can be replaced by a relatively weaker condition of k-continuity.

Theorem 1.3. *Let S is k-continuous ($k \in \mathbb{N}$) of a complete metric space (M, d). Then S has a unique fixed point if for each distinct $x, y \in M$;*

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(i) $0 \leq p_0 + p_1 + q_1 + q_2 + r_1 + r_2 < 1$;

(ii) $d(S^2x, S^2y) \leq p_0 d(x,y) + p_1 d(Sx,Sy) + q_1 d(x,Sx) + q_2 d(Sx,S^2x) + r_1 d(y,Sy) + r_2 d(Sy,S^2y)$.

The concept of a w-distance on a metric space was introduced by Kada, Suzuki and Takahashi. They generalized Caristi's fixed point theorem.

Definition 1.4. Let M be a metric space endowed with a metric d. A function $\rho : M \times M \longrightarrow [0,\infty)$ is called a w-distance on M if it satisfies the following properties:

(1) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for any $x, y, z \in M$;

(2) ρ is lower semi-continuous in its second variable; i.e., if $x \in M$ and $y_n \to y$ in M, then $\rho(x,y) \leq \liminf_n \rho(x,y_n)$;

(3) For each $\epsilon > 0$, there exists a $\delta > 0$ such that $\rho(z,x) \leq \delta$ and $\rho(z,y) \leq \delta$ imply $d(x,y) \leq \epsilon$.

Definition 1.5. Let (M,d) be a metric space. For $x \in M$, $S : M \to M$, $O(x;\infty) = \{x, Sx, S^2x, ...\}$ is called the S-orbit of x. Mapping S is orbitally continuous at x if for any sequence $\{x_n\}$ in $O(x;\infty)$ which is convergent to z, then Sx_n is convergent to Sz, as $n \to \infty$. A function $T : M \to \mathbb{R}^+$ is called S-orbitally lower semi-continuous at x if $\{x_n\}$ is a sequence in $O(x;\infty)$ and $x_n \to z$ implies $T(z) \leq \liminf_{n\to\infty} T(x_n)$.

Definition 1.6. Mapping $T : M \to M$ on a metric space (M,d) obeys the condition $(C;k)$ if there is a constant $k \geq 0$ such that for every sequence $x_n \in M$,

$$x_n \to x_0 \in M \quad \Rightarrow \quad \mathfrak{D}(x_0) \leq k \cdot \limsup \mathfrak{D}(x_n) \quad s.t. \quad \mathfrak{D}(x) = d(x,Tx), x \in M.$$

Currently, an active field of research is the study of contractive mappings without condition of continuity at the fixed point. Rhoades question influences researches in applications. The purpose of this paper is to establish the new fixed point theorem on a complete metric space with w-distance that is generalization of [1] and [2]. We show that the assumption of continuity can be replaced by a relatively weaker condition of k-continuity. We propose the existence of a solution for a nonlinear Fredholm integral equation, that is a new distinct solution to the open problem of Rhoades [5]. we give examples to confirm our results.

2 Main results

Definition 2.1. Let (M,d) be a metric space with w-distance ρ and $S : M \to M$ be a given self-mapping. We say that S is a generalized weak two-sided convex contractive mapping if there exist $\alpha, \beta, \theta, \eta, \gamma, \zeta \in [0,1)$ with $\alpha + \beta + \theta + \eta + \gamma + \zeta < 1$ such that for each $x, y \in M$, we have

$$\rho(S^2x, S^2y) \leq \alpha\rho(x,y) + \beta\rho(Sx,Sy) + \theta\rho(x,Sx) + \eta\rho(Sx,S^2x) + \gamma\rho(y,Sy) + \zeta\rho(Sy,S^2y).$$

Theorem 2.2. *Let ρ be a w-distance on a complete metric space (M,d). Suppose that $S : M \to M$ is a generalized weak two-sided convex contractive mapping. Assume that we have one of the following conditions:*

(i) S is a k-continuous mapping or S^k is a continuous mapping;

(ii) $\inf\{\rho(x,y)+\rho(x,Sx) : x \in M\} > 0$ *for every* $y \in M$ *with* $y \neq Sy$;

(iii) $\mathfrak{D}(x) = d(x,Sx)$ *is S-orbitally lower semi-continuous.*

(iv) S obeys the condition $(C;k)$.

Then S has a unique fixed point. Moreover, for this fixed point u, we have $\rho(u,u)=0$.

Example 2.3. Let $M=[-1,1]$ and let $d: M\times M \to \mathbb{R}^+$ be defined by $d(x,y)=|x-y|$. Also, define the function ρ as $\rho: M\times M \to [0,+\infty)$ such that for every $x,y\in M$, $\rho(x,y)=|y|$. Then ρ is a w-distance on M. Define $S: M\to M$ by

$$S(x)=x+(0.2)signx=\begin{cases} x-0.2, & x<0 \\ 0, & x=0 \\ x+0.2 & x>0 \end{cases}$$

Then S satisfies all the conditions of Theorem 2.2. So S has a unique fixed point $x=0$, and S is a discontinuous mapping at the fixed point. Also S is not a generalized two-sided convex contractive mapping by d.

Remark 2.4. In 1977, in a classical paper on contractive definitions, Rhoades [4] compared 250 contractive definitions. Later, in 1988, Rhoades [5] studied in detail the continuity of a large number of contractive mappings at their fixed points. Those contractive mappings were not necessarily continuous in their domain of definition, but they were continuous at their fixed points. In the above mentioned paper Rhoades set the question without continuity condition at the fixed point. In 1999, Pant [3] got the first result for contractive discontinuous mappings at the fixed point. In this paper, by Example 2.3 and Theorem 2.2, we provide a new and distinct solution to the open problem of Rhoades [5].

3 Application

Theorem 3.1. *Consider the nonlinear Fredholm integral equation*

$$x(t)=f(t)+\int_a^b K(t,s,x(s))ds, \tag{412}$$

where $a,b\in\mathbb{R}$ with $a<b$, and $f:[a,b]\to\mathbb{R}$ and $K:[a,b]^2\times\mathbb{R}\to\mathbb{R}$ are given continuous mappings. Let $M=C[a,b]$ be the set of all real continuous functions on $[a,b]$, with the metric $d: M\times M\to[0,+\infty)$ given by $d(x,y)=\sup_{t\in[a,b]}|x(t)-y(t)|$ $(\forall x,y\in M)$.

Suppose that the following conditions hold:

(i) the mapping $S : C[a,b] \to C[a,b]$ defined by

$$(Sx)(t) = f(t) + \int_a^b K(t,u,x(u))du, \quad \text{for all } x \in C[a,b] \text{ and } t \in [a,b]$$

is a continuous mapping;

(ii) Let S be k-continuous, and there exist $\alpha, \beta, \theta, \eta, \gamma, \zeta \in (0,1)$ with $\lambda = \alpha + \beta + \theta + \eta + \gamma + \zeta < 1$ such that for each $x, y \in X$ with $x \neq y$ and $t, u \in [a,b]$, we have

$$\begin{aligned} |K(t,u,Sx(u)) - K(t,u,Sy(u))| \quad &\leq \frac{1}{b-a}(\alpha(|x(u)| + |y(u)|) + \beta(|Sx(u)| + |Sy(u)|) \\ &\quad +\theta(|x(u)| + |Sx(u)|) + \eta(|Sx(u)| + |S^2x(u)|) \\ &\quad +\gamma(|y(u)| + |Sy(u)|) + \zeta(|Sy(u)| + |S^2y(u)|) - 2|f(t)|). \end{aligned} \tag{413}$$

Then the nonlinear integral equation (412) has a unique solution. Moreover, for each $x \in C[a,b]$, the Picard iteration (x_n) defined by $x_n(t) = f(t) + \int_a^b K(t,u,x_{n-1}(u))ds$ for all $n \in \mathbb{N}$ converges to a unique solution of the nonlinear integral equation (412).

Now, we give an example to illustrate the usefulness of our result.

Example 3.2. Let $M = C[0,1]$ be a set of all real continuous functions on $[0,1]$ equipped with the usual metric d, such that for every $x, y \in M$ define $d : M \times M \to [0, +\infty)$ such that $d(x,y) = \max_{t\in[0,1]} |x(t) - y(t)|$. Let the function $p : M \times M \to [0, +\infty)$ by $p(x,y) = \max_{t\in[0,1]} |x(t)| + \max_{t\in[0,1]} |y(t)|$ be a w-distance on M. Also assume the mapping $S : M \to M$ is defined by

$$(Sx)(t) = f(t) + \int_0^1 K(t,u,x(u))du, \quad \text{for all } x \in M \text{ and } t \in [0,1] \tag{414}$$

is a k-continuous mapping and $Sx \in M$, where $x \in M$, $f : [0,1] \to \mathbb{R}$ and $K : [0,1]^2 \times \mathbb{R} \to \mathbb{R}$ are given continuous mappings. Thus

$$S^2x(t) = f(t) + \int_0^1 K\Big(t,u,f(u) + \int_0^1 K(u,u,x(u))du\Big)du. \tag{415}$$

Let $f(t) = \frac{8}{15}t^2$ and $K(t,u,x(u)) = \frac{1}{4}t^2(1+u)(x^2(u)+1)$ such that $\sup_{t\in[0,1]} \frac{1}{4}t^2(1+u) \leq \frac{1}{2}$ for every $(t,u) \in [0,1] \times [0,1]$. Therefore from (414), we have

$$(Sx)(t) = \frac{8}{15}t^2 + \int_0^1 \frac{1}{4}t^2(1+u)(x^2(u)+1)du. \tag{416}$$

We have $|Sx - Sy| \leq |x - y|$, for every $x, y \in M$ and so the integral operator S is not a contractive mapping by d. In addition, we get

$$\begin{aligned} p(S^2x, S^2y) = |S^2x| + |S^2y| &= \max_{t\in[0,1]} \int_0^1 \int_0^1 \Big|\Big[\frac{1}{4}t^2(1+u)\Big]^2 (x^2(u) + y^2(u))\Big| dudu. \\ &= \frac{1}{4}(|x^2| + |y^2|) \leq \frac{1}{2}(|x| + |y|). \end{aligned}$$

Thus, all the conditions of Theorem (3.1) are satisfied and so the integral Equation (414) has a unique solution. It follows that $x(t) = t^2$ is the exact solution of the Equation (414). From (416), we obtain

$$x_{n+1} = Sx_n(t) = \frac{8}{15}t^2 + \frac{1}{4}\int_0^1 [t^2(1+u)(x_n^2(u)+1)]du. \tag{417}$$

We choose $x_0(t) = 0$ that it is an initial solution. Putting $n = 0, 1, 2, ...,$ in (417), we get

$$x_1(t) = 0.90833333t^2, \quad x_2(t) = 0.98396470t^2, \quad x_3(t) = 0.99708377t^2, \quad x_4(t) = 0.99946614t^2,$$

$$x_5(t) = 0.99990215t^2, \quad x_6(t) = 0.99998206t^2, \quad x_7(t) = 0.99999671t^2, \quad x_8(t) = 0.99999940t^2,$$

$$, \qquad x_9(t) = 0.99999985t^2, \quad x_{10}(t) = 0.99999998t^2, \quad x_{11}(t) = t^2.$$

So $x(t) = t^2$ is the unique solution.

References

[1] R. K. Bisht, N. Ozgur, *Discontinuous convex contraction and their applications in neural networks*, Computational and applied mathematics, (2021), 1-11.

[2] R. K. Bisht, N. K. Singh, V. Rakočević, B. Fisher *On discotinuity at fixed point via power quasi-contraction*, Publ. de L'Institut Math., 108(122), (2020), 5-11.

[3] A. Pant, R. P. Pant, *Fixed points and continuity of contractive maps*, Filomat 31(11), (2017), 3501-3506.

[4] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Am. Math. Soc. 226, (1977), 257-290.

[5] B. E. Rhoades, *Contractive definitions and continuity*, Contemp. Math. 72, (1988), 233-245.

[6] M. Zakeri, A. Sahleh, H. Lakzian and V. Rakočević *Weacly Istrătescu convex Contractions*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. 117, 16 (2022), 1-23.

POSTER

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Solution of the linear Schrodinger equation via piecewise potential: Sinc-Galerkin methods

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Abstract

The Sinc-Galerkin methods are presented to obtain the eigenvalues and eigenfunctions of linear Schrodinger equation via piecewise potential. Some properties of the Sinc methods required for our subsequent development are given. Numerical example is included to applicability of the presented techniques.

Keywords: Galerkin methods, Piecewise potential, Schrdinger equation, Sinc methods

Mathematics Subject Classification [2023]: 65L15, 65L60, 65Z05

1 Introduction

Sinc numerical methods have been extensively used to solve differential equations because of their exponential convergence rate[5]. The aim of this paper is to extend the Sinc techniques to calculate the eigenvalues and eigenfunctions of Schrodinger equation on interval (a, b), within homogeneous boundary conditions $Y(a)$=$Y(b)$=0 as

$$cY''(x) + V(x)Y(x) = EY(x), \qquad \int_a^b Y^2(x)dx = 1 \tag{418}$$

where unknown value E and function $Y(x)$ are eigenvalues and corresponding eigenfunctions, respectively. We call $V(x)$ as potential profile. There are various methods to obtain the exact solution of the equation (418), i.e. eigenvalues and corresponding eigenfunctions. But for some potentials, the Eq (418) has not exact solution and it must be solved with the numerical methods such as finite difference, Shooting method [4] sinc-collocation, sinc-Galerkin[2], Sinc-Integral[1] etc. In this paper we will use the sinc-Galekin method(SGM) for solve the equation Eq (418) via piecewise potential.

2 preliminaries

Sinc function properties are thoroughly discussed in [5]. The Sinc function is defined on the whole real line as

$$Sinc(x) = \begin{cases} \frac{sin(\pi x)}{\pi x} & x \neq 0; \\ 1 & x = 0 \end{cases} \tag{419}$$

For $h > 0$, the translated Sinc functions with evenly spaced nodes are given as

$$S(k, h)(x) = Sinc\left(\frac{x - kh}{h}\right) \quad k = 0, \pm 1, \pm 2, ... \tag{420}$$

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The basis functions on (a,b) are then taken to be the composite translated Sinc functions as

$$S_k(x) = S(k,h)o\phi(x) = Sinc\left(\frac{\phi(x) - kh}{h}\right) \quad k = 0, \pm 1, \pm 2, ... \tag{421}$$

where $\phi : (a,b) \longrightarrow (-\infty, \infty)$ is a conformal map. Suppose the infinite strip D_d as

$$D_d = \{z \in c : |Im(z)| < d\}.$$

Then the range of ϕ^{-1} is as

$$\Gamma = \left\{\phi^{-1}(t) \in D_d : t \in (-\infty, \infty)\right\}.$$

Definition 2.1. Let D be a simply connected domain which satisfies $(a,b) \subset D$ and α and c be positive constants. Then $L_\alpha(D)$ denotes the family of all functions $u \in Hol(D)$ which satisfies

$$|u(z)| \leq c|Q(z)|^\alpha$$

for all $z \in D$ where $Q(z) = (z-a)(b-z)$.

The following results will be useful to obtain the discretize our system[3]. In the following, D be a simply connected domain, $\phi : D \to D_s$ be a conformal one-to-one map, a, b are bounded of D and $\phi(a) = -\infty$, $\phi(b) = \infty$ and $x_i = \phi^{-1}(ih)$.

Corollary 2.2.

$$\delta_{ji}^{(0)} = S_j(x_i) = \begin{cases} 0 & i \neq j, \\ \\ 1 & i = j. \end{cases} \tag{422}$$

$$\delta_{ji}^{(1)} = hS_j'(x_i) = \begin{cases} \frac{(-1)^{(i-j)}}{(i-j)} & i \neq j, \\ \\ 0 & i = j. \end{cases} \tag{423}$$

$$\delta_{ji}^{(2)} = h^2 S_j''(x_i) = \begin{cases} \frac{-2(-1)^{(i-j)}}{(i-j)^2} & i \neq j, \\ \\ \frac{-\pi^2}{3} & i = j. \end{cases} \tag{424}$$

Proof. See [6]. □

Corollary 2.3. *For $F \in L_\alpha(D_d)$, positive integer N and $h = \sqrt{\frac{\pi d}{\alpha N}}$ we have:*

$$\int \lim_{\Gamma} F(x)dx \approx h \sum \lim_{k=-N}^{N} \frac{F(x_k)}{\phi'(x_k)}$$

$$F(x) \approx \sum \lim_{k=-N}^{N} F(x_k)S_k(x)$$

Corollary 2.4. *Assume that there exist positive constants α, β and K such that*

$$|F(x)| \leq K \begin{cases} e^{-\alpha|\phi(x)|} & x \in (-\infty, 0), \\ \\ e^{-\beta|\phi(x)|} & x \in [0, \infty). \end{cases} \tag{425}$$

where $F(x) = YW(x)$, $Y\phi' W(x)$ ore $Y(\frac{\phi''}{\phi'}W + 2W')(x)$. Make the selections $N = [|\frac{\alpha}{\beta}M + 1|]$ and $h = \sqrt{\frac{\pi d}{\alpha M}}$. Let $Y(S_j W)'' \in B(D)$. Then

$$\int_a^b Y''(x)W(x)S_j dx \approx h \sum_{k=-M}^{M} Y\left(\delta_{jk}^{(2)}(\phi' W) + \delta_{jk}^{(1)}\left(\frac{\phi''}{\phi'}W + 2W'\right)\right)(x_k) - h\left(\frac{W''Y}{\phi'}\right)(x_j)$$

Proof. See [6]. □

3 Formalism

By the Galerkin method with the Sinc functions we have

$$c\langle Y'', S_j\rangle + \langle VY, S_j\rangle = E\langle Y, S_j\rangle \quad for j = 0, \pm 1, \pm 2, ... \tag{426}$$

Define the inner product by

$$\langle f, g\rangle = \int_a^b f(x)g(x)W(x)dx \tag{427}$$

where W is a weight function. By corollary 2.4 we have

$$\langle Y'', S_j\rangle \approx h \sum_{k=-N}^{N} Y_k\left(\delta_{jk}^{(2)}(\phi' W) + \delta_{jk}^{(1)}\left(\frac{\phi'' W}{\phi'} + 2W'\right)\right)(x_k) + hY_j\left(\frac{W''}{\phi'}\right)(x_j) \tag{428}$$

Also by corollary 2.3 we have

$$\langle Y, S_j\rangle \approx h \sum_{k=-N}^{N} Y_k\left(\frac{S_j W}{\phi'}\right)(x_k) = hY_j\left(\frac{w}{\phi'}\right)(x_j). \tag{429}$$

Now we approximate $\langle YV, S_j\rangle$ where the potential V is piecewise continuous as follow

$$V(x) = \begin{cases} V_1(x) & a_1 \leq x < b_1 \\ V_2(x) & a_2 \leq x < b_2 \\ \vdots & \\ V_q(x) & a_q < x \leq b_q \end{cases} \tag{430}$$

Then

$$\langle YV, S_j\rangle \approx \sum_{k=-N}^{N} \sum_{m=1}^{q} Y_k\left(\int_{a_m}^{b_m} V_m(x)W(x)S_k S_j dx\right). \tag{431}$$

To calculate the integrals in relation 431,for m from 1..q, let ϕ_m such that $\phi_m(a_m) = -\infty$, $\phi_m(b_m) = \infty$. Also let $x_{mi} = \phi_m^{-1}(ih)$. Then we have

$$\langle YV, S_j\rangle \approx h \sum_{k=-N}^{N} \sum_{i=-N}^{N} \sum_{m=1}^{q} Y_k\left(\frac{V_m W S_k S_j}{\phi_m'}(x_{mi})\right). \tag{432}$$

Now, by substituting 428, 429, 432 in Eq. 426, and evaluating the results at x_j, we obtain

$$\sum_{k=-N}^{N} cY_k\left(\delta_{jk}^{(2)}(\phi' W)(x_k)+\delta_{jk}^{(1)}\left(\frac{\phi''}{\phi'}W+2W'\right)(x_k)+\frac{1}{c}\sum_{i=-N}^{N}\sum_{m=1}^{q}\frac{V_m W S_k S_j}{\phi'_m}(x_{mi})\right) +Y_j\left(\frac{cW''}{\phi'}\right)(x_j)=EY_j\left(\frac{W}{\phi'}\right)(x_j)$$

Then

$$\frac{\phi'(x_j)}{W(x_j)}\sum_{k=-N}^{N} cY_k\left(\delta_{jk}^{(2)}(\phi' W)(x_k)+\delta_{jk}^{(1)}\left(\frac{\phi'' W}{\phi'}+2W'\right)(x_k)+\frac{1}{c}\sum_{i=-N}^{N}\sum_{m=1}^{q}\frac{V_m W S_k S_j}{\phi'_m}(x_{mi})\right) +Y_j\left(\frac{cW''}{W}\right)=EY_j \tag{433}$$

Equations 418 give 2N + 2 linear algebraic equations and we can transform them to a matrix eigenvalue problem. So we can obtain the electronic spectrum and non-normal wave function of Eq. 418 by diagonalizing this matrix. Finally, we normalize the eigenfunction Y to obtain the normalized one Y_N as

$$Y_N(x)=\frac{Y(x)}{\sqrt{\int_a^b Y^2(t)dt}}\approx\sum_{k=-n}^{n}\frac{Y_k S_k(x)}{\sqrt{h\sum_{m=-n}^{n}\frac{Y_m^2}{\phi'(x_m)}}} \tag{434}$$

4 Numerical results

Example 4.1. Consider Eq. 418 on (0, 2) with c=−57.25 and quantum well potential as:

$$V(x)=\begin{cases}0.228 & 0\le x<100\\ 0 & 100<x<200\\ 0.228 & 200\le x\le 300\\ 0 & 300<x<500\\ 0.228 & 500\le x\le 600\end{cases} \tag{435}$$

We transfer the interval $(0, 600)$ to $(-\infty, +\infty)$ with $\phi(x) = Ln(\frac{x}{600-x})$. Select $N = 60$, $h = \sqrt{\frac{\pi}{2N}}$ and $W = \frac{1}{\sqrt{\phi'(t)}}$. Table 16 represents the eigenvalues obtained from SGM. Figure 8 shows the eigenfunctions obtain by SGM methods. Also, figure 9 shows the convergent of first eigenfunction.

References

[1] S.M.A. Aleomraninjad, P. Abbasi, *Sinc-Integral method to solve the linear Schrodinger equation* Mathematics and Computational Sciences 1(1) (2020), 43-50.

[2] S.M.A. Aleomraninejada, M. Solaimani, *Electronic spectrum of linear Schrodinger equations with some potentials by Sinc-Galerkin and Sinc-Collocation methods*, 16(3) (2022), 251-260.

[3] T. Okayama, T. Matsuo, M. Sugihara, *Error estimates with explicit constants for Sinc approximation, Sinc quadrature and Sinc indefinite integration*, Mathematical Engineering Technical Reports 2009-01, The University of Tokyo, 2009.

Table 16: Comparison of electronic spectrum of example 4.1 obtained by SGM method.

Eigenvalue	SGM
E_1	0.010520
E_2	0.032254
E_3	0.041807
E_4	0.092752
CPU time(s)	61.4

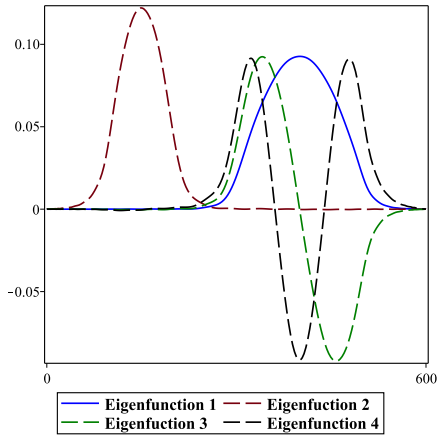


Figure 8: Eigenfunctions of example 4.1 obtained by SGM method

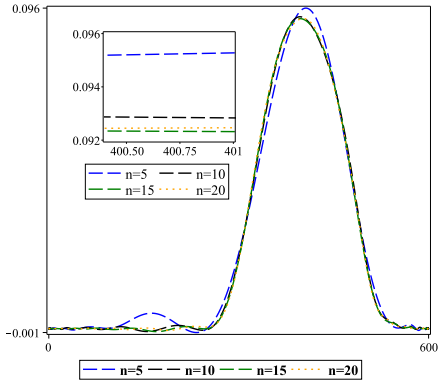


Figure 9: Convergence of first eigenfunction of example 4.1 obtained by SGM method

[4] M. Solaimani, S.M.A. Aleomraninejad, *Optical Properties of Energy-Dependent Effective Mass GaAs/Ga x In1 x As and GaAs/Al x Ga1 x As Quantum Well Systems: A Shooting Method Study* Journal of Electronic Materials 48(2) (2019),942-950.

[5] F. Stenger, *Numerical Methods Based on Sinc and Analytic Functions*, Springer, Berlin, New York, 1993.

[6] F. Stenger, *Approximations via Whittaker's Cardinal Function*, J. Approx. Theory, 17 (1976) 222-240.

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On the inverse eigenvalue problem for matrices whose graph is a double starlike tree

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Abstract

In this paper, we study a partially described inverse eigenvalue problem of matrices whose graph is a double starlike tree, and prove some spectral properties of such matrices. The problem involves the reconstruction of the matrix by the minimal eigenvalues of all leading principal submatrices and eigenpair $(\lambda_2^{(n)}, x)$ such that $\lambda_2^{(n)}$ is maximal eigenvalue of the required matrix. To solve this problem, the recurrence relations between leading principal submatrices are used. We investigate conditions for the solvability of the problem.

Keywords: Eigenvalue, eigenpair, leading principal submatrices, inverse eigenvalue problem.

Mathematics Subject Classification [2023]: 05C50, 65F18.

1 Introduction

The problem of reconstructing a matrix with a specific structure from a partial or total spectral data is known as inverse eigenvalue problem which arises in a variety of applications. In [2] detailed characterization of inverse eigenvalue problems is mentioned. The inverse eigenvalue problem of a graph is to determine the possible spectra among real symmetric matrices whose pattern of nonzero offdiagonal entries is described by a graph. The problem in this paper involves the construction of a specific symmetric matrix. This is carried out through the minimal eigenvalue of each of its leading principal submatrices and an eigenpair of the matrix. The symmetric matrix will be of the following form:

$$A_n = \begin{pmatrix} a_1 & b_1 & \dots & b_{p+1} & & & & & & & \\ b_1 & a_2 & & & & & & & & & \\ \vdots & & \ddots & & & & & & & & \\ b_{p+1} & & & a_{p+1} & b_{p+2} & & & & & & \\ & & & b_{p+2} & a_{p+2} & b_{p+3} & & & & & \\ & & & & b_{p+3} & \ddots & \ddots & & & & \\ & & & & & \ddots & \ddots & b_{p+m} & & & \\ & & & & & & b_{p+m} & a_{p+m} & b_{p+m+1} & \dots & b_{n-1} \\ & & & & & & & b_{p+m+1} & a_{p+m+1} & & \\ & & & & & & & \vdots & & \ddots & \\ & & & & & & & b_{n-1} & & & a_n \end{pmatrix} \tag{436}$$

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where $b_j > 0$ for $j = 1, \ldots, n-1$ and $n = p + m + q$.
In the current paper, we attempt to solve the following IEP, denoted by IEP1, consisting of constructing a matrix of form (436) with partial eigendata.
IEP1: The list of real numbers $\lambda_2^{(n)}$, $\lambda_1^{(j)}$, $j = 1, \ldots, n$, and real vector $x = (x_1, x_2, \ldots, x_n)$ are given. Find a matrix A_n of the form (436) such that $\lambda_2^{(n)}$ is the maximal eigenvalue of A_n, $\lambda_1^{(j)}$ is the minimal eigenvalue of the leading principal submatrix A_j of A_n and $(\lambda_2^{(n)}, x)$ is an eigenpair of A_n.

2 Main results

Let $\{\varphi_j(\lambda) = \det(\lambda I_j - A_j)\}_{j=1}^n$ be the sequence of characteristic polynomials of A_n. We obtain the recurrence relation $\varphi_j(\lambda)$ of the matrix A_n in Lemma 2.1.

Lemma 2.1. *Let A_n be an $n \times n$ matrix of the form (436). Then the sequence of $\{\varphi_j(\lambda)\}_{j=1}^n$ satisfies the following recurrence relations:*

i. $\varphi_1(\lambda) = (\lambda - a_1)$,

ii. $\varphi_j(\lambda) = (\lambda - a_j)\varphi_{j-1}(\lambda) - b_{j-1}^2 \prod_{k=2}^{j-1}(\lambda - a_k), \quad j = 2, \ldots, p+2$,

iii. $\varphi_j(\lambda) = (\lambda - a_j)\varphi_{j-1}(\lambda) - b_{j-1}^2 \varphi_{j-2}(\lambda), \quad j = p+3, \ldots, p+m+1$,

iv. $\varphi_j(\lambda) = (\lambda - a_j)\varphi_{j-1}(\lambda) - b_{j-1}^2 \varphi_{p+m-1}(\lambda) \prod_{k=p+m+1}^{j-1}(\lambda - a_k)$,
$j = p+m+2, \ldots, n$,

where $\varphi_0(\lambda) = 1$.

Proof. The relations can be verified by expanding the determinant. □

Lemma 2.2. *[1] Let A_n be an $n \times n$ matrix of the form (436) and $\lambda_1^{(j)}$, $\lambda_2^{(j)}$ be the minimal and maximal eigenvalue of the leading principal submatrix A_j of A_n, $j = 1, 2, \ldots, n$. Then:*

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_2^{(n)} \tag{437}$$

and

$$\lambda_1^{(j)} < a_k < \lambda_2^{(j)}, \quad k = 1, 2, \ldots, j; \quad j = 2, \ldots, n. \tag{438}$$

The Theorem 2.3 presents the solution to the IEP1 and the conditions under which the problem is solvable.

Theorem 2.3. *There are solutions to IPE1 if the following conditions are satisfied:*

a: There is a solution $\alpha > 0$ of the equation

$$\alpha^2 \prod_{i=2}^{j-1}(\lambda_1^{(j)} - a_i) - \frac{\alpha x_1 \varphi_{j-1}(\lambda_1^{(j)})}{x_j} + (\lambda_2^{(n)} - \lambda_1^{(j)})\varphi_{j-1}(\lambda_1^{(j)}) = 0,$$

for $j = 2, \ldots, p+1$.

b: There is a solution $\beta > 0$ *of the equation*

$$\beta^2 \varphi_{p+m-1}(\lambda_1^{(j)}) \prod_{i=p+m+1}^{j-1} (\lambda_1^{(j)} - a_i) - \frac{\beta x_1 \varphi_{j-1}(\lambda_1^{(j)})}{x_j} + (\lambda_2^{(n)} - \lambda_1^{(j)}) \varphi_{j-1}(\lambda_1^{(j)}) = 0,$$

for $j = p + m + 1, \ldots, n$.

3 Conclusions

In the current paper, a partially described inverse eigenvalue problem was considered for construction of specific symmetric matrix. The problem involves the construction of this matrix by one eigenpair of the required matrix and minimal eigenvalue of all leading principal submatrices. The significance of the IEP1 stems in the fact that it partially describes inverse eigenvalue problem while it constructs matrices from partial eigendata. Such partially described problems may be encountered in computations in which obtaining the entire spectrum is difficult.

References

[1] M. Babaei ,S.A. Shahzadeh Fazeli and S.M. Karbassi , *Inverse eigenvalue problem for constructing a kind of acyclic matrices with two eigenpairs*, Appl Math, 65(2020) ,89-103.

[2] M. T. Chu and H.Golub, Inverse Eigenvalue Problems: Theory, Algorithms, and Applications, 406. Oxford University Press (2005).

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On existence the solution of the group of elliptic differential equations

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Abstract

In this paper, we prove the existence of a non trivial solution for a nonlinear equation via minimization on the Nehari manifold.

Keywords: solution, equation, nonlinear, Nehari manifold.

Mathematics Subject Classification [2023]: 35J25.

1 Introduction

We are concerned with existence of solution for boundary value problem with Dirichlet boundary conditions. For these types of problems the techniques used for example the functionals associated to these problems are generally unbounded from below and they present a lack of coercivity properties that the arguments of existence of solution for these to class of problems.
We present approach to the search of solution to (439), still based on constrained minimization. It does not require the nonlinearity [4] to be homogeneous [5], and can thus be applied to a wider class of problems.

The spaces $H^1(\Omega)$ are reflexive Banach spaces when endowed with the norm $\|u\|^2_{H^1(\Omega)} = |u|^2_{L^2(\Omega)} + \int_\Omega |\nabla u|^2 \, dx$.
Let $\Omega = \mathbb{R}^N$, with $N \geq 3$, be open and bounded, and let $1 < p < N$ and $1 < r < 2 < p < 2^*$. Consider the value boundary problem

$$\begin{cases} -\Delta u = |u|^{p-2}u - |u|^{r-2}\, u & in\, \Omega, \\ u = 0 & on\, \partial\Omega. \end{cases} \tag{439}$$

Our paper is the first time of use of the Nehari manifold method to study solutions for the elliptic equations (439).
For these types of problems the techniques used for example the functionals associated to these problems are generally unbounded from below and they present a lack of coercivity properties that the arguments of existence of solution for these to class of problems.
The energy functional and the Nehari manifold define for this problem (439) and prove the existence of a nontrival solution via minimization on the Nehari manifold. We recall that the functional associated to (439) it is

$$I(u) = \frac{1}{2}\int\limits_\Omega |\nabla u|^2 dx - \frac{1}{p}\int\limits_\Omega |u|^p \, dx + \frac{1}{r}\int\limits_\Omega |u|^r dx,$$

defined and differentiable on $H^1_0(\Omega)$ [3]. Recall that I is unbounded below, one again we restrict I to a suitable set in order to get red of this problem. Now we use the set

$$\mathcal{N} = \{u \in H^1_0(\Omega) \, : \, u \neq 0 \, , \, I'(u)u = 0\} =$$

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$$\{u \in H_0^1(\Omega) : u \neq 0\,,\ \|u\|^2_{H_0^1(\Omega)} = \int\limits_\Omega |u|^p dx - \int\limits_\Omega |u|^r dx\}.$$

Notice that on $\mathcal{N}$ the functional I reads

$$I(u) = (\frac{1}{2} - \frac{1}{p})\,\|u\|^2_{H_0^1(\Omega)} + (\frac{1}{r} - \frac{1}{p}) \int\limits_\Omega |u|^r dx.$$

This shows that I is coercive on $\mathcal{N}$, in the sense that if $\{u_k\}_k \subset \mathcal{N}$ satisfies $\|u_k\| \to \infty$, then $I(u_k) \to \infty$. We define

$$m = \inf_{u \in \mathcal{N}} I(u),$$

and we show, through a series of lemmas, that m is attained by some $u \in \mathcal{N}$ which is a critical point of I considered on the whole space $H_0^1(\Omega)$, and therefore a solution to (439).

Semilinear elliptic equations with concave-convex nonlinearities in bounded domainsare widely studied. For example, Ambrosetti et al. [9] considered the following equation:

$$\begin{cases} -\Delta u = \left|u^{q-2}\right| u + \xi \left|u^{q-2}\right| u & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{440}$$

where $\xi > 0$, $1 < q < 2 < p \leq 2^*$ $(2^* = \frac{2N}{N-2})$, with $N \geq 3$.
In [6], C.O. Alves and A. El Hamidi existence and multiplicity results to the following nonlinear elliptic equation via minimization Nehri manifold. Giovany M. Figueiredo and Fernando Bruno M. Nunes has been extended quasilinear elliptic problems[7]. In this paper our nonlinear elliptic equation is similar to semilinear elliptic equation (440), that we search of solution to (439) by used Nehari manifold.

1.1 Main results

Lemma 1.1. The Nehari manifold is not empty.

Proof. For every not identically zero $u \in H_0^1(\Omega)$, one sees immediately that $tu \in \mathcal{N}$ for some $t > 0$. Indeed, $tu \in \mathcal{N}$ is equivalent to

$$tu \in \mathcal{N} \Leftrightarrow \|tu\|^2_{H_0^1(\Omega)} = \int\limits_\Omega |tu|^p dx - \int\limits_\Omega |tu|^r dx,$$

$$t^2\,\|u\|^2_{H_0^1(\Omega)} = \int\limits_\Omega t^p |u|^p dx - \int\limits_\Omega t^r |u|^r dx,$$

$$\|u\|^2_{H_0^1(\Omega)} = \int\limits_\Omega t^{p-2} |u|^r dx - \int\limits_\Omega t^{r-2} |u|^r dx.$$

We proved, by the mean value theorem

$$\exists t > 0 \ \ s.t\ f(t) = 0.$$

Thus, if $f(t) = At^{p-2} + Bt^{r-2} - C$, then,

$$\lim_{t \to 0^+} f(t) = -C,$$

$$\lim_{t\to+\infty} f(t) = \lim_{t\to+\infty} t^{p-2} = +\infty,$$

Thus,

$$\exists t > 0 \;\; s.t \; f(t) = 0,$$

thus Nehari manifold is not empty. □

Lemma 1.2. We have $m = \inf_{u\in\mathcal{N}} I(u) > 0$.

Proof.

$$\begin{aligned} m &= \inf_{u\in\mathcal{N}} I(u) = \inf_{u\in\mathcal{N}} (\frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega |u|^p\, dx + \frac{1}{r}\int_\Omega |u|^r dx) \\ &= (\frac{1}{2}-\frac{1}{p})\int_\Omega |u|^p dx + (\frac{1}{r}-\frac{1}{2})\int_\Omega |u|^r dx > 0. \end{aligned}$$

□

Lemma 1.3. The level m is attained by a nonnegative function, namely there exists $u \in \mathcal{N}$, $u(x) \geq 0$ a.e, such that $I(u) = m$.

Proof. Let $\{u_k\}_k \subset \mathcal{N}$ be a minimizing sequence for I, namely such that $I(u_k) \to m$. Clearly $|u_k| \in N$ and $I(|u_k|) = I(u_k)$, so that $\{|u_k|\}_k$ is another minimizing sequence, for this reason we assume straight away that $u_k(x) \geq 0$ are in Ω for all k, we have obseved that I is coercive on $\mathcal{N}$, this implies that the sequence $\{u_k\}_k$ is bounded in $H_0^1(\Omega)$, and as usual this means that, up to subsequence,

1. $u_k \to u \; in \, H_0^1(\Omega)$,
2. $u_k(x) \to u(x) \; a.e \; in \; \Omega$,
3. $u_k \to u \; in \, L^p(\Omega)$.

Then we have $u \geq 0$ a.e and by weak lower semicontinuity,

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_\Omega |u|^p\, dx + \frac{1}{r}\int_\Omega |u|^r dx \\ &\leq \liminf_{k\to\infty} (\frac{1}{2}\|u_k\|^2 - \frac{1}{p}\int_\Omega |u_k|^p\, dx + \frac{1}{r}\int_\Omega |u_k|^r dx) \end{aligned}$$

$$= \liminf_{k\to\infty} \; I(u_k) = m, \tag{441}$$

$$I(u) \leq m.$$

And,

$$\|u\|^2 \leq \liminf_{k\to\infty} \|u_k\|^2 = \liminf_{k\to\infty}(\int_\Omega |u_k|^p dx - \int_\Omega |u_k|^r dx).$$

Since $u_k \in \mathcal{N}$, we have $\|u\|^2 = \int_\Omega |u_k|^p dx - \int_\Omega |u_k|^r dx$. By it cannot be $\|u_k\| \to 0$, and therefore $\int_\Omega |u_k|^p dx - \int_\Omega |u_k|^r dx$ cannot tend to zero, thus, by strong convergence, $\int_\Omega |u_k|^p dx - \int_\Omega |u_k|^r dx \neq 0$, which shows that $u \neq 0$. Passing to the limit we obtain

$$\|u\|^2 \leq \int_\Omega |u_k|^p dx - \int_\Omega |u_k|^r dx. \tag{442}$$

If

$$\|u\|^2 = \int_\Omega |u_k|^p dx - \int_\Omega |u_k|^r dx, \tag{443}$$

then $u \in \mathcal{N}$ and shows that u is the required minimizer. Since holds, we only have to treat the case where $\|u\|^p < \int_\Omega |u_k|^r dx - \int_\Omega |u_k|^q dx$. We now show that if this happens, we reach a contradiction. Indeed, take $t > 0$ such that $tu \in \mathcal{N}$, namely. Since we are assuming , we deduce that $0 < t < 1$. But $tu \in \mathcal{N}$, so that

$$\begin{aligned} 0 < m < I(tu) &= \frac{1}{2}t^2\|u\|^2 - \frac{1}{p}t^p \int_\Omega |u|^p \, dx + \frac{1}{r}t^r \int_\Omega |u|^r dx \\ &\leq \liminf_{k\to\infty}(\frac{1}{2}\|u_k\|^2 - \frac{1}{p}\int_\Omega |u_k|^p \, dx + \frac{1}{r}\int_\Omega |u_k|^r dx) \\ &= \liminf_{k\to\infty} I(u_k) = m. \end{aligned}$$

This is impossible, and the proof is complete. □

References

[1] A. Ambrosetti, A. Malchiodi, *Nonlinear Analysis and Semilinear Elliptic Problems*. Cambridge Studies in Advanced Mathematics, vol. 104 (Cambridge University Press,Cambridge 2007).

[2] M.Willem, *Minimax Theorems.Progress in Progress in Nonlinear Differential Equations and Their Applications*, vol. 24 (Birkhauser, Boston, 1996).

[3] R. Adams, *Sobolev spaces*, Academic Press, New York (1975).

[4] K. Deimling, *Nonlinear Functional Analysis* (Springer,Berlin,1985).

[5] L. C. Evans, *Partial Differential Equations*. Graduate Studies in Mathematics, vol. 19 (American Mathematical Society, Providence, 1998).

[6] C. O. Alves, A. El Hamidi, *Nehari manifold and existence of positive solutionsto a class of quasilinear problems*, Nonlinear Analysis 60 (2005) 611-624.

[7] Giovany M. Figueiredo, Fernando Bruno M. Nunes,*Existence of positive solutions for a class of quasilinear elliptic problems with exponential growth via the Nehari manifold method*, Jan 2019 in Revista Matematica Complutense, Volume 32 issue1, 1-18.

[8] Tsing-San Hsu, Huei-Li Lin, *Multiple Positive Solutions for Semilinear Elliptic Equations in Involving Concave-Convex Nonlinearities and Sign-Changing Weight Functions*, Applied Analysis volume 2010, 1-21.

[9] A. Ambrosetti, H. Brezis, G. Cerami, *Combined Effects of Concave and Convex Nonlinearities in Some Elliptic Problems*, Journal of Functional Analysis, volume 122 issue 2, 519 -543.

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Investigating some methods for solving nonlinear equations

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Abstract

Nonlinear equations appear frequently in many branch of science and so they are of special importance. In this paper, we investigate the interval version of two methods for enclosing the roots of nonlinear equations with a single independent variable. Some test problems are presented for comparing the investigated methods.

Keywords: Nonlinear equations, Newton method, Solution bounds

Mathematics Subject Classification [2023]: 65G30; 34G20

1 Introduction

There are many real problems which can be modeled by mathematical forms and they lead to nonlinear equations. Nonlinear equations appear often in algebra, calculus, and physics (for example, when solving problems that involve gravity or acceleration). Mostly, it is difficult to calculate the exact solutions for these equations. Therefore, many numerical methods have been proposed in order to be able to find the approximate solutions for these equations. But solving the problems in floating-point arithmetic is inevitably associated with round-off errors, and so the obtained solution to the problem is accompanied by some errors. Interval analysis is a tool for bounding the errors and providing rigorous bounds on the solution to the problems. Here we want to use interval arithmetic to provide secure bounds for the roots of these equations.

Consider the nonlinear equation

$$f(x) = 0, \tag{444}$$

where $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a real-valued nonlinear function on the open interval D. Nonlinear problems are of interest to engineers, physicists and many other scientists because most systems are inherently nonlinear in nature. Up to now, many modified methods for solving nonlinear equations have been developed to improve the local order of convergence of some classical methods such as Newton, Chebyshev, Potra-Pták and Ostrowski methods, see [1, 3, 4].

Many modified methods for solving nonlinear equation (444) with high-order of convergence are based on the well-known Newton method. Newton method is one of the best iterative methods for solving nonlinear equations by using

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{445}$$

which converges quadratically, see [1, 4].

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The Ostrowski method given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \frac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)} \frac{f(y_n)}{f'(x_n)}, , \end{cases} \tag{446}$$

is an improvement of Newton method (445) with the order of convergence equal to 4. A detailed description of this method is given in [4].

As we said previously, we want to apply interval version of the above methods for enclosing the solution of nonlinear equations. We use the interval extension of the Newton method with quadratic convergence [2] and the interval extension of the Ostrowski method [2].

Here, we use boldface letters to denote intervals. The set of real intervals is denoted by $\mathbb{IR} = \{\mathbf{x} = [\underline{x}, \overline{x}] : \underline{x} \leq \overline{x}\}$. The midpoint and width of an interval number $\mathbf{x} = [\underline{x}, \overline{x}]$ are defined by $\mathrm{m}(\mathbf{x}) = \frac{\overline{x}+\underline{x}}{2}$ and $\mathrm{w}(\mathbf{x}) = \overline{x} - \underline{x}$, respectively. An interval number $\mathbf{x}$ is said to be degenerate if $\mathrm{w}(\mathbf{x}) = 0$. The absolute value of $\mathbf{x}$ is $|\mathbf{x}| = \max\{|x| : x \in \mathbf{x}\}$. The interval extension of real-valued function f is denoted by its corresponding uppercase and bold letter $\mathbf{F}$.

2 Investigation of the methods

Definition 2.1. We say that F is an interval extension of f, if for degenerate interval arguments, F agrees with f, i.e.,

$$F([x, x]) = f(x).$$

The idea of the interval Newton method for the first time was discussed in [2]. Suppose the real differentiable function f in (444) has the inclusion monotonic interval extension $\mathbf{F}'(\mathbf{x})$ of its derivative $f'(x)$ and $\mathbf{x}_0$ is an initial point, then the interval Newton method is

$$\mathbf{x}_{n+1} = \Big\{\mathrm{m}(\mathbf{x}_n) - \frac{f(\mathrm{m}(\mathbf{x}_n))}{\mathbf{F}'(\mathbf{x}_n)}\Big\} \cap \mathbf{x}_n, \quad n = 0, 1, \ldots. \tag{447}$$

Recursive relation (447) produces a sequence $\{\mathbf{x}_n\}$ of interval numbers. If the initial interval $\mathbf{x}_0$ contains a zero x^* of $f(x)$ and $0 \notin \mathbf{F}'(\mathbf{x}_0)$, then all iterates contain x^* and the method converges to x^*.

Interval extension of the classic Ostrowski method for computing the enclosure solutions of nonlinear equations is given by the following recursive formula

$$\begin{cases} \mathbf{y}_n = \left\{m(\mathbf{x}_n) - \frac{f(m(\mathbf{x}_n))}{F'(\mathbf{x}_n)}\right\} \cap \mathbf{x}_n, \\ \mathbf{x}_{n+1} = \left\{m(\mathbf{y}_n) - \frac{f(m(\mathbf{x}_n))f(m(\mathbf{y}_n))}{(f(m(\mathbf{x}_n))-2f(m(\mathbf{y}_n)))F'(\mathbf{x}_n)}\right\} \cap \mathbf{x}_n. \end{cases} \tag{448}$$

Now, we give some numerical examples to compare the performance of the above interval methods in the sense of quality of the obtained enclosures. In all examples, the procedures are stopped when $\mathrm{w}(\mathbf{x}_k) < 10^{-16}$. We utilize INTLAB [5] to compute the verified results on the computer. We study the following examples:

$$f_1(x) = \cos(x^3 - 2x + 4) - \sin(x) + 1, \quad x_1^* \approx 0.152,$$

$$f_2(x) = \log(3x^2 - 6x + 7) - 3x + 1, \quad x_2^* \approx 0.804.$$

The graphs of the functions f_1 and f_2 have been drawn in Figure 10 over the intervals $[0, 1]$ and $[-1, 1]$, respectively.

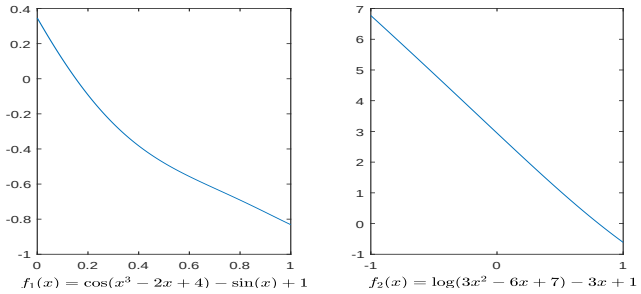


Figure 10: Graphs of the functions f_1, f_2

In tables below, the results obtained by implementing the interval Newton method and the interval Ostrowski method have been displayed. The third column of tables show the tolerance parameter $e_n = \max_{x \in \mathbf{x}_n} |f(x)|$.

Table 17: Results of interval Newton method for $f_1(x)$

n	$\mathbf{x}_n$	e_n
1	[0.00000000000000, 0.18310310736628]	3.46×10^{-1}
2	[0.14252686098096, 0.15989136856245]	1.94×10^{-2}
3	[0.15209635713953, 0.15212764187109]	3.23×10^{-5}
4	[0.15211243161051, 0.15211243163758]	2.75×10^{-11}
5	[0.15211243162421, 0.15211243162422]	0

Table 18: Results of interval Ostrowski method for $f_1(x)$

n	$\mathbf{x}_n$	e_n
1	[0.11130347322823, 0.17996206770790]	8.51×10^{-2}
2	[0.15211228880029, 0.15211255170808]	2.87×10^{-7}
3	[0.15211243162421, 0.15211243162422]	0

Table 19: Results of interval Newton method for $f_2(x)$

n	$\mathbf{x}_n$	e_n
1	[0.80441506257318, 0.80508630385581]	1.33×10^{-3}
2	[0.80482157639978, 0.80482160839011]	5.60×10^{-8}
3	[0.80482159345911, 0.80482159345912]	10^{-15}
4	[0.80482159345911, 0.80482159345912]	0

The presented results in the above tables show that for $f_1(x)$, the interval Ostrowski method successes in fewer number of iterations with respect to the interval Newton method which confirms that the order of convergence of interval Ostrowski method is bigger than the interval Newton method. However, for $f_2(x)$ the interval Ostrowski method fails. Also, for two functions f_1 and f_2 sharp bounds have been obtained by two interval methods. As the number of repetitions increases, the quality of the obtained results increases, too.

3 Conclusion

In this work, we study the interval version of two methods for enclosing the roots of nonlinear equations, namely interval Newton method and interval Ostrowski method. Similar to their non-interval case, the order of convergence of interval Newton and interval Ostrowski methods are 2 and 4, respectively. We also observed this fact when some test problems were implemented. These two interval methods provide sharp bounds for the roots of nonlinear equations.

Acknowledgement

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References

[1] A. Cordero, C. Jordan, J.R. Torregrosa, *One-point Newton-type iterative methods: A unified point of view*, Journal of Computational and Applied Mathematics 275 (2015) 366-374.

[2] T. Eftekhari, *A new proof of interval extension of the classic Ostrowskis method and its modified method for computing the enclosure solutions of nonlinear equations*, Numerical Algorithms, Numerical Algorithms 69(1) (2015) 157–165.

[3] M.A. Noor, M. Waseem, K.I. Noor, *New iterative technique for solving a system of nonlinear equations*, Applied Mathematics and Computation 271 (2015) 446-466.

[4] A.M. Ostrowski, *Solution of Equations in Euclidean and Banach Spaces*, 3rd edn. Academic Press, New York, 1973.

[5] S.M. Rump, *INTLAB-INTerval LABoratory*, in: Tibor Csendes (Ed.), Developments in Reliable Computing, Kluwer, Dordrecht, Netherlands, (1999) 77-105.

[6] T. Sunaga, *Theory of an Interval Algebra and its Applications to Numerical Analysis*, in RAAG Memoirs II, Gakujutsu, Bunken Fukyu-Kai, 1958.

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Existence result for a critical p-laplacian problem

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Abstract

Using minimization arguments we establish the existence solution for a boundry value p-Laplacian problem. here, we use Lagrange multiplier method to study existence non trivial solution

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The rule of divisibility of any number by another number using the sum-of-digits

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Abstract

Abstract Using the sum of digits a number , interesting features are obtained ,some features were already obtained , for example, if the sum of digits a number is divisible by 3 or 9, that number must be divisible by 3 or 9 or if the rightmost digit of a number is 0 or 5, that number is divisible by 5, or any even number is divisible by 2 , or the remainder of division of any number by 9 equal the sum of its digits, now new information has been obtained about the remainder of division, this paper, other properties were investigated and each was written in the form of equations , for example , the rule of divisibility by 11,13, in other words, finding rules that obtain the remainder of the division of any number by another number. when we have a number with many very long digits, the mentioned methods can create a simple algorithm to find the remainder of that number over another number.

Keywords: Divisibility of any number, Sum of Digits, Remainder of division

Mathematics Subject Classification [2023]: 05C50, 65F18.

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Non-linear maps preserving the ϵ-pseudo spectrum of the Jordan triple $*$-product

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Abstract

Let $B(H)$ be the algebra of all bounded linear operators on infinite-dimensional complex Hilbert space H. Fix $\epsilon > 0$ and $A \in B(H)$, let $\sigma_\epsilon(A)$ denote the ϵ-pseudo spectrum of A. In this note, we show that if bijective map $\varphi : B(H) \to B(H)$ satisfies

$$\sigma_\epsilon(A \bullet B \bullet C) = \sigma_\epsilon(\varphi(A) \bullet \varphi(B) \bullet \varphi(C)), \quad (A, B, C \in B(H)),$$

then there exist a unitary operator $U \in B(H)$ and a scalar $\lambda \in \{-1, 1\}$ such that either $\varphi(A) = \lambda UAU^*$ for every $A \in B(H)$ or $\varphi(A) = \lambda UA^tU^*$ for every $A \in B(H)$, where A^t denotes the transpose of A relative to an arbitrary but fixed orthonormal basis of H and $A \bullet B = AB + BA^*$ is the Jordan $*$-product of A and B.

Keywords: Nonlinear preserve, ϵ-Pseudo Spectrum, Spectrum

Mathematics Subject Classification [2023]: 47B49, 47A10, 47B48

1 Introduction

Throughout this paper, $B(H)$ stands for the algebra of all bounded linear operators acting on an infinite dimensional complex Hilbert space $(H, \langle , \rangle)$ and its unit will be denoted by I. Let $B_s(H)$ and $Z(B(H))$ be the real linear space of all self-adjoint operators in $B(H)$ and center of $B(H)$, respectively. For an operator $A \in B(H)$, the adjoint and the spectrum of A are denoted by A^* and $\sigma(A)$, respectively. For $\epsilon > 0$, the ϵ-pseudo spectrum of A, $\sigma_\epsilon(A)$, is defined by $\sigma_\epsilon(A) = \cup\{\sigma(A + S) : S \in B(H), \|S\| \leq \epsilon\}$ and coincides with the set

$$\{\lambda \in \mathbb{C} : \|(\lambda I - A)^{-1}\| \geq \epsilon^{-1}\}$$

with the convention that $\|(\lambda I - A)^{-1}\| = \infty$ if $\lambda I - A$ is not invertible. It is a compact subset of $\mathbb{C}$ and contains $\sigma(A)$, the spectrum of A. Unlike the spectrum, which is a purely algebraic concept, the ϵ-pseudo spectrum depends on the norm. The ϵ-pseudo spectral radius of A, $r_\epsilon(A)$, is given by

$$r_\epsilon(A) = sup\{|\lambda| : \lambda \in \sigma_\epsilon(A)\}.$$

For more information about these notions, one can see [5]. Recently, there has been interest in studying maps φ on matrices or operators satisfying $F(\varphi(A) \circ \varphi(B)) = F(A \circ B)$, here, $F()$ is a spectral function or a spectral set such as the spectrum, the numerical range, the ϵ-pseudo spectral radius, the ϵ-pseudo spectrum and the local spectrum. While $A \circ B$ stands for different kinds of products. Linear preserver problems,

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in the most general setting, demands the characterization of maps between algebras that leave a certain property, a particular relation, or even a subset invariant. In all cases that have been studied by now, the maps are either supposed to be linear, or proved to be so. This subject is very old and goes back well over a century to the so-called first linear preserver problem, due to Frobenius [4], who characterized linear maps that preserve the determinant of matrices. The study of nonlinear pseudo spectrum preserver problems attracted the attention of a number of authors. Cui et al. [2, Theorem 3.3] characterized maps on $M_n(\mathbb{C})$ that preserve the ϵ-pseudo spectrum of the usual product of matrices. They proved that a map φ on $M_n(\mathbb{C})$ satisfies

$$\sigma_\epsilon(\varphi(A)\varphi(B)) = \sigma_\epsilon(AB) \quad (A, B \in M_n(\mathbb{C}))$$

if and only if there exist a scalar $c = \pm 1$ and a unitary matrix $U \in M_n(\mathbb{C})$ such that $\varphi(A) = cUAU^*$ for all $A \in M_n(\mathbb{C})$. This result was extended to the infinite dimensional case by Cui et al. [3, Theorem 4.1], where the authors showed that a surjective map φ on $B(H)$ preserves the ϵ-pseudo spectrum of the product of operators if and only if it is a unitary similarity transform up to a scalar $c = \pm 1$. The Jordan $*$-product of two elements $A, B \in B(H)$ is defined by $A \bullet B = AB + BA^*$. The aim of this note is to characterize mappings on $B(H)$ that preserve the ϵ-pseudo spectral of the Jordan $*$-product $A \bullet B = AB + BA^*$ of operators. For two nonzero vectors x and y in H, let $x \otimes y$ stands for the operator of rank at most one defined by

$$(x \otimes y)z = \langle z, y\rangle\, x, \qquad \forall\ z \in H.$$

Note that every rank one operator in $B(H)$ can be written in this form, and that every finite rank operator $A \in B(H)$ can be written as a finite sum of rank one operators; i.e., $A = \sum_{i=1}^{n} x_i \otimes y_i$ for some $x_i, y_i \in H$ and $i = 1, 2, ..., n$. We denote by $F(H)$ the set of all finite rank operators in $B(H)$ and $F_n(H)$ the set of all operators of rank at most n, n is a positive integer.
In the following proposition, we collects some known properties of the ϵ-pseudo spectrum which are needed in the proof of the main result.
Let $\epsilon > 0$ be arbitrary and $D(0, \epsilon) = \{\mu \in \mathbb{C} : |\mu - a| < \epsilon\}$, where $a \in \mathbb{C}$.

Proposition 1.1. *(See [3, 5].) Let $\alpha > 0$ and $A \in B(H)$. Then*

(1) $\sigma(A) + D(0, \epsilon) \subseteq \sigma_\epsilon(A)$.

(2) If A is normal, then $\sigma_\epsilon(A) = \sigma(A) + D(0, \epsilon)$.

(3) For any $\alpha \in \mathbb{C}, \sigma_\epsilon(A + \alpha I) = \alpha + \sigma_\epsilon(A)$.

(4) For any nonzero $\alpha \in \mathbb{C}, \sigma_\epsilon(\alpha A) = \alpha\sigma_{\frac{\epsilon}{|\alpha|}}(A)$.

(5) For any $\alpha \in \mathbb{C}$, we have $\sigma_\epsilon(A) = D(\alpha, \epsilon)$ if and only if $A = \alpha I$.

2 Main results

The following lemma is a key tool for the proof of main result and describes the spectrum of the Jordan product $A(x \otimes x) + (x \otimes x)A$ for any nonzero vector $x \in H$ and operator $A \in B(H)$.

Lemma 2.1. *(See [1, Corollary 2.1].) For any nonzero vector $x \in H$ and $A \in B(H)$, we have*

$$\sigma(A(x \otimes x) + (x \otimes x)A) = \{0, \langle Ax, x\rangle \pm \sqrt{\langle A^2x, x\rangle}\ \}.$$

The following theorem will be useful in the proof of the main result.

Theorem 2.2. *(See [3, Theorem 5.1].) Let $\epsilon > 0$. Then a surjective map $\varphi : B_s(H) \to B_s(H)$ satisfies*

$$\sigma_\epsilon(AB + BA) = \sigma_\epsilon(\varphi(A)\varphi(B) + \varphi(B)\varphi(A))$$

if and only if there exists a unitary operator $U \in B(H)$ such that φ has the form

$$A \to \mu UAU^* \text{ or } \quad A \to \mu UA^tU^*,$$

where $\mu \in \{-1, 1\}$ and A^t denotes the transpose of A relative to an arbitrary but fixed orthonormal basis of H.

The followig theorem is the main result of this paper.

Theorem 2.3. *Suppose that a bijective map $\varphi : B(H) \to B(H)$ satisfies*

$$\sigma_\epsilon(A \bullet B \bullet C) = \sigma_\epsilon(\varphi(A) \bullet \varphi(B) \bullet \varphi(C)), \quad (A, B, C \in B(H)).$$

Then there exist a unitary operator $U \in B(H)$ and a scalar $\lambda \in \{-1, 1\}$ such that either $\varphi(A) = \lambda UAU^*$ for every $A \in B(H)$ or $\varphi(A) = \lambda UA^tU^*$ for every $A \in B(H)$, where A^t denotes the transpose of A relative to an arbitrary but fixed orthonormal basis of H.

Proof. The proof of it will be completed after checking several claims.

Claim 1. $\varphi(0) = 0$.

Claim 2. $\varphi(iI)^* = -\varphi(iI) \in Z(B(H))$.

Claim 3. φ preserves anti-self adjoint operators in both directions.

Claim 3. There exist a unitary operator $U : H \to H$ and a scalar $\lambda \in \{-1, 1\}$ such that either $\varphi(A) = \lambda UAU^*$ for every $A \in B_s(H)$ or $\varphi(A) = \lambda UA^tU^*$ for every $A \in B_s(H)$, where A^t denotes the transpose of A relative to an arbitrary but fixed orthonormal basis of H.

Claim 4. The result in the theorem holds.

□

3 Conclusion

In this paper, we have followed the same path of studies by considering general pseudo spectrum preservers, and characterized the form of all bijective maps $\varphi : B(H) \to B(H)$ that preserve the ϵ-pseudo spectral of the Jordan triple $*$-product $A \bullet B \bullet C$ of operators.

References

[1] E. Alzedani and M. Mabrouk, *Maps preserving the spectrum of skew lie oroduct of operators*, Kragujevac Journal of Mathematics., 64(4) (2022), 525–532.

[2] J. Cui, V. Forstall, C.K. Li and V. Yannello, *Properties and preservers of the pseudospectrum*, Linear Algebra Appl., 436 (2012), 316–325.

[3] J. Cui, C.K. Li and Y.T. Poon, *Pseudospectra of special operators and pseudospectrum preservers*, J. Math. Anal. Appl., 419 (2014), 1261–1273.

[4] G. Frobenius, *Ueber die Darstellung der endlichen Gruppen durch lineare Substitutionen*, Berl Ber. Appl., 203 (1897), 994–1015.

[5] L.N. Trefethen and M. Embree, *Spectra and pseudospectra*, Princeton University Press, Princeton, NJ, 2005.

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Existence solution for quasi-equilibrium problems under a weaker equilibrium conditions in topological vector space

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Abstract

The first we extend the Kakutani's fixed point theorem in topological vector spaces, then establish existence results for quasi-equilibrium problems with this, when the Previous equilibrium condition is replaced by a weaker one.

Mathematics Subject Classification [2023]: 18A32, 18F20, 05C65 (at least 1 and at most 3)

1 Introduction

The extended abstract should have at most 4 pages. Papers prepared more than 4 pages or out of the style of the meeting, will be returned.

Here you should state the introduction, preliminaries and your notations. Authors are required to state clearly the contribution of the extended abstract and its significance in the introduction. There could be some short survey of relevant literature.

1.1 Instructions for speakers

In scientific contexts the term equilibrium has been widely used at least in Physics, Chemistry, Engineering and Economics within different frameworks, relying on different mathematical models such as optimization, variational inequalities and noncooperative games among others. In turn, these mathematical models share an underlying common structure which allows to cque format. Theronveniently formulate them in a unique format. Therefore, theoretical studies developed for one of these models can be generally modified to cope with the others through the common format in unifying language. This format reads

topological vector space X, subset C of X, bifunction $T : C \times C \to R$ and set-valued map $k : C \to 2^C$ are assumed. The quasi-equilibrium problem (QEP) is the find of $x \in k(x)$ such that for each $y \in k(x)$,

$$T(x, y) \geq 0$$

So far, many conditions for the existence of a quasi-equilibrium solution to the problem have been explored by mathematicians, including the conditions for the existence of an answer in the uniform state provided by Aussel and Cotrina [1, 2]. Cotrina and Zuniya later presented the conditions for the existence of an answer without assuming a generalized uniformity in topological spaces with infinite dimensions, and their applications of the quasi-optimization problem were examined. Cotrina and Zuniya proved the existence of an answer to the question of quasi-equilibrium using the Kakutani's theorem in locally convex spaces [3].

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In this dissertation, we intend to prove the case of Kakutani used by Cotrina and Zuniga with weaker conditions in topological vector spaces, and to use the obtained result to have the answer to the solutions of quasi-equilibrium.

Let a real topological vcector space X, a subset C of X, a bifunction $T : C \times C \to \mathbb{R}$ and a set-valued map $K : C \to 2^C$, the quasi-equilibrium problem (QEP) consists in finding

$$x \in K(x) \ \ \textit{such that} \ \ T(x,z) \geqslant 0, \textit{for all} \ \ z \in K(x)$$

We denote its set solution by $\mathrm{QEP}(T,K)$ and define the following set-valued map $S : C \to 2^C$ by

$$S(x) = \{y \in K(x) : \ T(y,z) \geq 0 \ \ \textit{for all} \ \ z \in K(x)\}.$$

If for every $x \in C$ we put $K(x) = C$ then quasi-equilibrium problem becomes equilibrium problem. Therefore, the issue of equilibrium problem is a special case of quasi-equilibrium problem.

Definition 1.1. Let X and Y be two topological spaces, and let $T : X \to 2^Y$ be a set-valued function. T is said to be upper semicontinuous if for each open subset G of Y , the set $\{x \in X : T(x) \subset G\}$ is open in X. It is well known that if Y is a compact Hausdorff space and $T(x)$ is closed for each $x \in X$, then T is upper semicontinuous if and only if the graph $\{(x,y) \in X \times Y : y \in T(x)\}$ of T is closed in $X \times Y$.

Definition 1.2. Given two sets X and Y, let F be a multivalued map from X and Y. Equivalently, $F : X \to 2^Y$ is a function from X to the power set of Y. A function $T : X \to Y$ is said to be a selection of F, if for all $x \in X$, $T(x) \in F(x)$.

2 existence results

Our existence result will be obtained as a consequence of a new version of Kakutani's fixed point theorem in a topological vector space which is stated in the next result.

Theorem 2.1. *Let C be a nonempty compact convex subset of a Hausdorf topological space X; and let $T : C \to 2^C$ be a set-valued map such that for all $x \in C$, $f(x)$ is convex and non-empty. If for all $y \in C$, $T^{-1}(y) = \{x \in C : y \in T(x)\}$is open in C then T admits a fixed point.*

One can consider this theorem as a generalized version of Kakutani's fixed point theorem.

Theorem 2.2. *Let $F : C \times C \to \mathbb{R}$ be a function and $K : C \to 2^C$ be a set-valued mapping, where C is a non-empty compact and convex subset of a linear Hausdorff topological vector space of X. Assume that the following conditions are satisfied:*

(a) *for all $x \in C$, the set $\{u \in K(x) : F(u,z) \geq 0, \forall z \in K(x)\}$ is nonempty and convex;*

(b) *For each $x \in C$, the $\{u \in K(x) : F(u,z) < 0\}$ is open;*

(c) *for all $x \in C$, $K(x)$ is nonempty, closed and convex set and $F(x,x) \geq 0$.*

Then the solution set of quasi equilibrium problem is non-empty.

References

[1] D. Aussel, J. Cotrina, Quasi monotone quasi variational inequalities: existence results and applications, J. optim. Theory Appl, 158 (2013) 6337-652.

[2] D. Aussel, J. Cotrina and A. Iusem, An existence result for quasi-equilibrium problem, J. Convex Anal, 24(1)(2017) 55-66.

[3] J. Cotrina, J. Zunniya, A note on quasi equilibrium problems, Operations Research letters, 46 (2018) 138-140.

[4] H. Ben. El. Mechaiekh, Fixed points for compact set-valued maps, Q and A in General Topology, Vol. 10 (1992).

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Some extensions of the Kakutani-Glicksberg-Fan theorem

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Abstract

In this paper, we provide an extension of the Kakutani-Glicksberg-Fan theorem via the measure of noncompactness. Also, we present a few generalizations of the Kakutani-Glicksberg-Fan theorem by the concept of measure of noncompactness in Banach spaces. In addition, we prove some coupled fixed point theorems by the measure of noncompactness in Banach spaces.

Keywords: Measure of noncompactness, Kakutani-Glicksberg-Fan theorem, Set valued mapping.

Mathematics Subject Classification [2023]: 47H09, 39B62, 34A12

1 Introduction

Assume that E is a given Banach space with the norm $\| \cdot \|$. Denote by B(x, r) the closed ball in E centered at x and with radius r. If X is a subset of E, then the symbols $\bar{X}$ and $ConvX$ stand for the closure and the convex hull of X, respectively. Moreover, we denote by $\mathcal{M}_E$ the family of all nonempty bounded subsets of E and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact sets. In 1930, Kuratowski [4] introduced the first measure of noncompactness in the following way:

$$\alpha(X) := \inf\{\delta > 0 | X = \bigcup_{i=1}^{n} X_i \text{ for some } X_i \text{ with } diam(X_i) < \delta \text{ for } 1 \leq i \leq n < \infty\}$$

for all $X \in \mathcal{M}_E$ and

$$diam(X) := \sup\{d(x, y) | x, y \in X\}.$$

Now, we provide some definitions and auxiliary facts which will be needed in the sequel.

Definition 1.1. A mapping $\mu : \mathcal{M}_E \to \mathbb{R}_+ = [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1. The family $ker\mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathcal{N}_E$.
2. $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$.
3. $\mu(X) = \mu(\bar{X})$.
4. $\mu(ConvX) = \mu(X)$.

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5. $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.

6. If $\{X_n\}$ is a nested sequence of closed sets from $\mathcal{M}_E$ such that $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Definition 1.2. A set-valued mapping $F : E \to 2^E$ is upper hemicontinuous if for every open set $W \subset E$, the set $\{x|\, F(x) \subset W\}$ is open in X.

Definition 1.3. Let $F : E \to 2^X$ be an upper hemicontinuous mapping and E be a convex subset of X, then F is termed a Kakutani mapping if the set $F(x)$ is non-empty, compact, and convex for all $x \in E$.

Kakutani's fixed-point theorem was extended to infinite-dimensional locally convex topological vector spaces by Irving Glicksberg [6] and Ky Fan [5] as follows.

Theorem 1.4 (Kakutani-Glicksberg-Fan theorem)**.** *Let S be a non-empty, compact, and convex subset of a Banach space E. Suppose that $F : S \to 2^S$ be a Kakutani map. Then F has a fixed point.*

In 2013, Aghajani et al. [2] proved two following generalizations of the Darbo fixed point theorem.

Theorem 1.5. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and $F : C \longrightarrow C$ be a continuous operator such that*

$$\mu(FX) \leq \varphi(\mu(X))$$

for any nonempty subset X of C, where μ is an arbitrary measure of noncompactness and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is given mapping such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \geq 0$. Then F has at least one fixed point in C.

Theorem 1.6. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $F : C \longrightarrow C$ be a continuous operator such that*

$$\psi(\mu(FX)) \leq \psi(\mu(X)) - \varphi(\mu(X))$$

for any nonempty subset X of C, where μ is an arbitrary measure of noncompactness and $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ are given mappings such that φ is lower semicontinuous and ψ is continuous on $\mathbb{R}_+$. Moreover, $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$. Then F has at least one fixed point in C.

Definition 1.7. [1] Let ϖ denote the class of functions $\beta : \mathbb{R}_+ \to [0,1)$ which satisfies the condition $\beta(t_n) \to 1$ implies $t_n \to 0$.

In 2014, Aghajani et al. [1] proved the following fixed point theorem.

Theorem 1.8. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and $F : C \longrightarrow C$ be a continuous operator satisfying*

$$\mu(FX) \leq \beta(\mu(X))\mu(X) \tag{449}$$

for any subset X of C, where μ is an arbitrary measure of noncompactness and $\beta \in \varpi$. Then F has at least one fixed point.

In a normed space X, an element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to 2^X$ if $x \in F(x, y)$ and $y \in F(y, x)$.

Theorem 1.9. *[1] Let C be a nonempty, bounded, closed, and convex subset of a Banach space E, μ be an arbitrary measure of noncompactness, and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing and upper semicontinuous mapping such that $\varphi(t) < t$ for $t > 0$. Suppose that $F : C \times C \longrightarrow C$ be a continuous operator satisfying*

$$\mu(F(X_1 \times X_2)) \leq \varphi(\frac{\mu(X_1) + \mu(X_2)}{2})$$

for all $X_1, X_2 \subset C$. Then F has at least a coupled fixed point.

Theorem 1.10. *[3] Let C be a nonempty, bounded, closed, and convex subset of Banach space E. Let $F : C \times C \longrightarrow C$ be a continuous operator satisfying*

$$\mu(F(X_1 \times X_2)) \leq k \max\{\mu(X_1), \mu(X_2)\}$$

for all $X_1, X_2 \subset C$, where $k \in [0, 1)$. Then F has at least a coupled fixed point.

2 Main results

In this section, we present new fixed point theorems for the particular class of Kakutani set valued mappings by the concept of measure of noncompactness.

Lemma 2.1. *Let C be a nonempty, closed, and convex subset of a Banach space E. Suppose that $F : C \longrightarrow 2^C$ be an upper hemicontinuous mapping, C_1 be a closed subset of C and $F : C_1 \to 2^{C_1}$ be a well-defined mapping. Then $F : C_1 \to 2^{C_1}$ is upper hemicontinuous.*

First, we extend Theorem 1.4 as follows.

Theorem 2.2. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E. Suppose that $F : C \longrightarrow 2^C$ be an upper hemicontinuous mapping, Fx be a nonempty, closed, and convex for all $x \in C$ and*

$$\mu(FX) \leq k\mu(X)$$

for any subset X of C, where $k \in [0, 1)$ and $FX = \bigcup_{x \in X} Fx$. Then F has a coupled fixed point.

Then, we extend Theorem 1.10 for the Kakutani mappings.

Theorem 2.3. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E. Let $F : C \times C \longrightarrow 2^C$ be an upper hemicontinuous mapping, $F(x, y)$ be a nonempty, closed and convex set for all $(x, y) \in C \times C$ and*

$$\mu(F(X_1 \times X_2)) \leq k \max\{\mu(X_1), \mu(X_2)\}$$

for all $X_1, X_2 \subset C$, *where* $k \in [0,1)$ *and* $F(X \times Y) = \bigcup_{(x,y)\in X\times Y} F(x,y)$. *Then* F *has a coupled fixed point.*

After that, an extension of Theorem 1.5 for the Kakutani mappings is showed.

Theorem 2.4. *Let* C *be a nonempty, bounded, closed, and convex subset of a Banach space* E *and* $F : C \longrightarrow 2^C$ *be an upper hemicontinuous mapping such that*

$$\mu(FX) \leq \varphi(\mu(X))$$

for any nonempty subset X *of* C, *where* μ *is an arbitrary measure of noncompactness,* $FX = \bigcup_{x\in X} Fx$ *and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is given mapping such that* $\lim_{n\to\infty} \varphi^n(t) = 0$ *for each* $t \geq 0$. *Then* F *has a fixed point.*

Next, we present an extension of Theorem 1.6 for the Kakutani mappings as follows.

Theorem 2.5. *Let* C *be a nonempty, bounded, closed, and convex subset of a Banach space* E *and* $F : C \longrightarrow 2^C$ *be a upper hemicontinuous mapping such that*

$$\psi(\mu(FX)) \leq \psi(\mu(X)) - \varphi(\mu(X))$$

for any nonempty subset X *of* C, *where* μ *is an arbitrary measure of noncompactness,* $FX = \bigcup_{x\in X} Fx$ *and* $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ *are given mappings such that* φ *is lower semicontinuous and* ψ *is continuous on* $\mathbb{R}_+$. *Moreover,* $\varphi(0) = 0$ *and* $\varphi(t) > 0$ *for* $t > 0$. *Then* F *has a fixed point.*

Here, we have an extension of Theorem 1.8 for the Kakutani mappings.

Theorem 2.6. *Let* C *be a nonempty, bounded, closed, and convex subset of a Banach space* E *and* $F : C \longrightarrow 2^C$ *be an upper hemicontinuous mapping satisfying*

$$\mu(FX) \leq \beta(\mu(X))\mu(X) \tag{450}$$

for any subset X *of* C, *where* μ *is an arbitrary measure of noncompactness,* $FX = \bigcup_{x\in X} Fx$ *and* $\beta \in \varpi$. *Then* F *has a fixed point.*

Finally, an extension of Theorem 1.9 for the Kakutani mappings is presented.

Theorem 2.7. *Let* C *be a nonempty, bounded, closed, and convex subset of a Banach space* E, μ *be an arbitrary measure of noncompactness,* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a nondecreasing and upper semicontinuous mapping such that* $\varphi(t) < t$ *for* $t > 0$. *Suppose that*
$F : C \times C \longrightarrow 2^C$ *be an upper hemicontinuous mapping,* $F(x,y)$ *is a nonempty, closed, and convex set for all* $(x,y) \in C \times C$ *and*

$$\mu(F(X_1 \times X_2)) \leq \varphi(\frac{\mu(X_1) + \mu(X_2)}{2})$$

for all $X_1 \times X_2 \subset C$ *where* $F(X \times Y) = \bigcup_{(x,y)\in X\times Y} F(x,y)$. *Then* F *has a coupled fixed point.*

The presented generalizations of the Kakutani-Glicksberg-Fan theorem have many applications in game theory, for example in the Nash equilibrium.

References

[1] A. Aghajani, R. Allahyari, M. Mursaleen, *A generalization of Darbos theorem with application to the solvability of systems of integral equations*, J. Comput. Appl. Math. Appl. Math 260 (2014), 6877.

[2] A. Aghajani, J. Banas, N. Sabzali, *Some generalizations of Darbo fixed point theorem and applications*, Bull. Belg. Math. Soc. Simon Stevin 20 (2013), 345358.

[3] A. Aghajani, N. Sabzali, *Existence of coupled fixed point via measure of noncmpactness and applications*, J. Nonlinear Convex Anal., Volume 15, Number 5, (2014), 941-952.

[4] R.R. Akmerov, M.I. Kamenski, A.S. Potapov, A.E. Rodkina, B.N. Sadovskii, *Measures of Noncompactness and Condensing Operators*, Birkhauser-Verlag, Basel, 1992.

[5] k.y. Fan, *Fixed-point and Minimax Theorems in Locally Convex Topological Linear Spaces*, Proc. Natl. Acad. Sci. U S A, 38 (2), (1952) 121126.

[6] I.L. Glicksberg, *A Further Generalization of the Kakutani Fixed Point Theorem, with Application to Nash Equilibrium*, Proc. Amer. Math. Soc, 3 (1) (1952) 170174.

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New algorithm for computing squar root

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Abstract

In this paper, we present an algorithm for approximate calculating the square root of arithmetic numbers until n decimal places by using the square expansion. This algorithm is accurate and simple. It's more accurate than other methods such as Newton-Raphson method.

Keywords: Square root, square expansion, New algorithm

Mathematics Subject Classification [2023]: 18A32, 18F20, 05C65 (at least 1 and at most 3)

1 Introduction

The Babylonians devised a remarkable iterative algorithm [1] for calculation of square roots in about 1500 BC. This algorithm is easy to understand and remarkably fast, yielding about 26 decimal digits accuracy in only five iterations. The algorithm has appeared in introductory computer programming texts as well as precalculus, calculus and numerical analysis texts. [2]
In this paper, we present an algorithm for approximate calculating the square root of arithmetic numbers until n decimal places by using the square expansion.

2 Process of computing

Let c be an arithmetic number.

Step 1. Let

$$\sqrt{c} = a + e_1 \times 10^{-1} + E, \quad a, e_1 \in \mathbb{N} \cup \{0\}$$

In fact $a + e_1 \times 10^{-1} + E$ is the largest arithmetic number satisfy

$$(a + e_1 \times 10^{-1})^2 \leq c.$$

So

$$a^2 + 10^{-2}e_1^2 + 2(10^{-1}ae_1) \leq c. \tag{451}$$

Now we put $c_1 = c - a^2$ and a is the largest arithmetic number so that $a^2 \leq c$ then,

$$e_1(2a \times 10 + e_1) \leq 10^2 c_1.$$

To calculate e_1, we find the largest arithmetic number that satisfy (451).

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Step 2. Now we calcute the root of c until secound decimal.

$$\sqrt{c} = a + e_1 \times 10^{-1} + e_2 \times 10^{-2} + E \quad a, e_1, e_2 \in \mathbb{N} \cup \{0\}$$

In fact $a + e_1 \times 10^{-1} + e_2 \times 10^{-2} + E$ is the largest arithmetic number that

$$a + e_1 \times 10^{-1} + e_2 \times 10^{-2})^2 \leq c.$$

Hence

$$\begin{aligned} & a^2 + e_1^2 \times 10^{-2} + e_2^2 \times 10^{-4} \\ & \quad + 2ae_1 \times 10^{-1} + 2ae_2 \times 10^{-2} + 2e_1e_2 \times 10^{-3} \\ & \leq c. \end{aligned} \tag{452}$$

Now suppose that a is the largest arithmetic number so that $a^2 \leq c$ and $c_1 = c - a^2$. Then by using (452),

$$\begin{aligned} e_1(2a \times 10 + e_1) + e_2^2 \times 10^{-2} \\ + 2ae_2 + 2e_1e_2 \times 10^{-1} \leq 10^2 c_1. \end{aligned} \tag{453}$$

To compute e_1, we find the largest arithmetic number that satisfy

$$e_1(2a \times 10 + e_1) \leq 10^2 c_1.$$

By replacing

$$c_2 = 10^2 c_1 - e_1(2a \times 10 + e_1),$$

in (453),

$$e_2\Big(2(a \times 10 + e_1) \times 10^1 + e_2\Big) \leq 10^2 c_2. \tag{454}$$

Similarly, by using induction, we can compute square root the arithmetic numbers until n decimal pleces.

3 Application

Example 3.1. We want to calculate $\sqrt{17}$ until secound decimal.

$$\sqrt{17} = a + e_1 \times 10^{-1} + e_2 \times 10^{-2} + E = a/e_1e_2 + E$$

$$a^2 \leq 17 \to a = 4$$

$$c_1 = 17 - 4^2 = 1$$

$$e_1(2a \times 10 + e_1) \leq 100c_1,$$

$$8\square \times \square \leq 100$$

we have $e_1 = 1$

To compute e_2

$$c_2 = 10^2 c_1 - e_1(2a \times 10 + e_1) = 19.$$

$$e_2\Big(2(a \times 10 + e_1) \times 10^1 + e_2\Big) \leq 10^2 c_2.$$

Therefore

$$e_2\Big(2(4\times 10+1)\times 10^1+e_2\Big)\leq 1900.$$

$$82\square\times\square\leq 1900$$

So $e_2=2$

$$\sqrt{17}=4/12$$

Example 3.2. We calculate $\sqrt{5}$ until fourth decimal.

$$a/\overline{e_1e_2e_3e_4}$$

$$a=2.$$

$$4\square\times\square\leq 100\rightarrow e_1=2$$

$$44\square\times\square\leq 1600\rightarrow e_2=3$$

$$446\square\times\square\leq 27100\rightarrow e_3=6$$

$$4472\square\times\square\leq 30400\rightarrow e_4=0$$

$$\sqrt{5}=2/2360$$

References

[1] D. M. Burton, *History of Mathematics: An Introduction*, McGraw Hill, (1991), 79.

[2] H. Anton, *Brief Edition*, Wiley, (6 th ed), 1999.

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Three nontrivial solutions for Kirchhoff type problems with nonlinear Neumann boundary conditions

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Abstract

In this article, we prove existence of at least three solutions for Kirchhoff type problem with nonlinear Neumann boundary conditions on a bounded domain. The proof is based on the variational methods and critical points theory.

Keywords: Three solutions; Kirchhoff type problems; Neumann boundary condition; Variational method.

Mathematics Subject Classification [2023]: 18A32, 18F20, 05C65

1 Introduction

We are interested to study the existence of at least three weak solutions for the following elliptic problem

$$\begin{cases} -\left[M\left(\int_\Omega |\nabla u|^p\,dx\right)\right]^{p-1}\Delta_p u = \lambda f(x,u), & \text{in } \Omega,\\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda k(u) + \mu g(u), & \text{on } \partial\Omega, \end{cases} \tag{455}$$

where $\Delta_p u := div\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian operator, $M(t) := a + bt$, $a > 0, b \geq 0$, $p > N$, Ω is a nonempty bounded open subset of $\mathbb{R}^N$ with a boundary of class C^1, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative, λ, μ are positive real parameters, the functions f, k, g satisfy hypothesis stated as follows:

(H_1) $k \in C(\mathbb{R}, \mathbb{R})$ and there exist two positive constants ρ, ρ^* with $\rho < p$ such that

$$|K(t)| \leq \rho^*(1 + |t|^\rho)$$

for all $t \in \mathbb{R}$, where $K(t) = \int_0^t k(s)\,ds$.

(H_2) $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ and there exist two positive constants a_1, a_2 such that

$$a_1|u|^p \leq -F(x,u) \leq a_2|u|^p$$

for all $(x,u) \in \Omega \times \mathbb{R}$, where $F(x,u) = \int_0^u f(x,s)\,ds$.

(H_3) $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ and there exist two positive constants ρ, ρ^* such that $|F(x,u)| \leq \rho^*(|u|^\rho + 1)$ for all $(x,u) \in \mathbb{R}$.

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2 Preliminaries

Theorem 2.1. *([1, Theorem 3.3]) Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a convex, coercive, and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*. Let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact. Suppose that*

(*i*) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

(*ii*) *for any $\lambda > 0$ and for every u_1 and u_2 that are local minima for the functional $\Phi - \lambda\Psi$ such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, the following inequality be true:*

$$\inf_{s\in[0,1]} \Psi\left(su_1 + (1-s)u_2\right) \geq 0.$$

Assume that there are two positive constants r_1 and r_2 and $\bar{v} \in X$ with $2r_1 < \Phi(\bar{v}) < \frac{r_2}{2}$ such that

(b_1) $\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(\bar{v})}{\Phi(\bar{v})}$,

(b_2) $\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(\bar{v})}{\Phi(\bar{v})}$.

Then, for each

$$\lambda \in \left]\frac{3}{2}\frac{\Phi(\bar{v})}{\Psi(\bar{v})},\ \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)},\ \frac{r_2/2}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}\right\}\right[,$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points lying in $\Phi^{-1}(]-\infty, r_2[)$.

Let $W^{1,p}(\Omega)$ be the usual Sobolev space endowed with the norm

$$\|u\| := \left(\int_\Omega \left(|\nabla u|^p + |u|^p\right) dx\right)^{\frac{1}{p}}.$$

Let

$$\kappa := \sup_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\max_{x\in\bar{\Omega}} |u(x)|}{\|u\|}. \tag{456}$$

Since $p > N$, the embedding $W^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$ is compact, and thus $0 < \kappa < \infty$,

$$|u(x)| \leq \kappa\|u\|, \quad \text{for } u \in W^{1,p}(\Omega). \tag{457}$$

Fix $x^0 \in \Omega$, pick $D > 0$ such that

$$S(x^0, \frac{D}{2}) \subset S(x^0, D) \subseteq \Omega,$$

where $S(x^0, D)$ denotes the ball with center x^0 and radius D.

Set

$$\alpha := \int_{S(x^0,D)\setminus S(x^0,\frac{D}{2})} \left(D - |x - x^0|\right)^p dx,$$

and

$$h_1 := \kappa\left[(\frac{2}{D})^p(1+\alpha) + \frac{(\frac{D}{2})^N \pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}\right]^{\frac{1}{p}}.$$

Definition 2.2. A function $u \in X := W^{1,p}(\Omega)$ said weak solution of the problem (455) if u satisfies

$$\left[M\left(\int_\Omega |\nabla u|^p\,dx\right)\right]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx - \lambda\int_\Omega f(x,u)v\,dx - \int_{\partial\Omega}(\lambda k(u)+\mu g(u))\,v\,d\sigma = 0$$

for every $v \in W^{1,p}(\Omega)$.

Moreover, if $I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ for every $u \in X$, by the standard arguments, one infers that the critical points of the functional I_λ are the weak solutions of the problem (455).

We need the following Proposition in the proof of our main results.

Proposition 2.3. ([2, Proposition 2.3]) Let $\Upsilon : X \to X^*$ be the operator defined by

$$\Upsilon(u)(v) = \left[M\left(\int_\Omega |\nabla u|^p\,dx\right)\right]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx$$

for any $u, v \in X$. Then Υ admits a continuous inverse on X^*.

3 Main results

In this section, we establish the main abstract results of this paper. Before introducing our result we put $|\Omega| = \int_\Omega dx$, $|\partial\Omega| = \int_{\partial\Omega} d\sigma$.

Define the functionals on $W^{1,p}(\Omega)$ by

$$\Gamma(u) = \frac{1}{p}\int_0^{\int_\Omega |\nabla u|^p\,dx} M^{p-1}(s)\,ds = \begin{cases} \dfrac{1}{bp^2}\left[\left(a+b\displaystyle\int_\Omega |\nabla u|^p\,dx\right)^p - a^p\right], & b>0,\\ \dfrac{a^{p-1}}{p}\displaystyle\int_\Omega |\nabla u|^p\,dx, & b=0,\end{cases}$$

$$\Phi(u) = \Gamma(u), \quad \Psi(u) = \int_\Omega F(x,u)\,dx + \int_{\partial\Omega}\left[K(u)+\frac{\mu}{\lambda}G(u)\right]d\sigma, \quad I_\lambda := \Phi - \lambda\Psi, \tag{458}$$

where $G(t) := \int_0^t g(s)\,ds$ for each $t \in \mathbb{R}$.

Conditions $(H_1), (H_2), (H_3)$ and $p > N$ guarantee that the functionals Φ and Ψ are well defined and of class C^1.

Theorem 3.1. *Assume that conditions* $(H_1)-(H_3)$ *hold and there exist two positive constants* θ, τ *with* $\theta < \tau$ *and* $h_1\tau > \theta$, *such that*

(α_1) $f(x,t) \geq 0, \quad \forall\ (x,t) \in \Omega \times [-\theta, \tau]$;

(α_2) $\dfrac{K^\theta|\partial\Omega| + \int_\Omega \sup_{|t|\leq\theta} F(x,t)\,dx}{m_2} < \dfrac{K_\tau|\partial\Omega| + \int_{S(x^0,\frac{s}{2})} F(x,\tau)\,dx}{m_1}$;

(α_3) *there exist, a positive function* $\alpha \in L^\infty$ *and positive constant* $\gamma < p$ *such that*

$$F(x,t) \leq \alpha(x)(1+|t|^\gamma)$$

for almost all $x \in \Omega$ *and for all* $t \in \mathbb{R}$.

Then, for each

$$\lambda \in \left]\frac{m_1}{K_\tau|\partial\Omega| + \int_{S(x^0,\frac{s}{2})} F(x,\tau)\,dx},\ \frac{m_2}{K^\theta|\partial\Omega| + \int_\Omega \sup_{|t|\le\theta} F(x,t)\,dx}\right[,$$

and for every continuous function $g:\mathbb{R}\to\mathbb{R}$ such that

(α_4) $\limsup_{|t|\to+\infty}\frac{G(t)}{t^p} < +\infty$, *where* $G(t)=\int_0^t g(s)\,ds$, *for aech* $t\in\mathbb{R}$,

there exists $\bar\delta_{\lambda,g,k}>0$ *such that, for each* $\mu\in[0,\bar\delta_{\lambda,g,k}[$, *the problem* (455) *admits at least three weak solutions in* X, *where*

$$m_1 := \frac{(2)^{p-1}\tau^p h_2}{p}\left(a^{p-1}+b^{p-1}\tau^{p(p-1)}h_2^{p-1}\right),\quad h_2 := \left(\frac{\tau h_1}{\kappa}\right)^p$$

and

$$m_2 := \frac{a^{p-1}\theta^p}{p\kappa^p},\quad K_\tau := \inf_{|\xi|\le\tau} K(\xi),\quad K^\theta := \sup_{|\xi|\le\theta} K(\xi)$$

and $G_\tau := \inf_{|\xi|\le\tau} G(\xi)$, $G^\theta := \sup_{|\xi|\le\theta} G(\xi)$. *Clearly,* $K^\theta, G^\theta \ge 0$ *and* $K_\tau, G_\tau \le 0$.

Remark 3.2. We observe that, if $f(x,0)\neq 0$, then, by Theorem 3.1, we obtain the existence of at least three non-zero weak solutions.

Theorem 3.3. *Assume that conditions* $(H_1)-(H_3)$ *hold and there exist three positive constants* θ_1,θ_2,τ *with*

$$\theta_1 < h_1\tau < \frac{\theta_2}{2\theta_3},$$

where $\theta_3 := \sqrt[p]{\frac{2\left[a^{p-1}+b^{p-1}\left(\frac{h_1\tau}{\kappa}\right)^{p(p-1)}\right]}{a^{p-1}}}$, *such that*

(α_5) $f(x,t)\ge 0$; *for all* $x\in\Omega$ *and* $t\in\mathbb{R}$;

(α_6) $\dfrac{\int_\Omega \sup_{|t|\le\theta_1} F(x,t)\,dx+|\partial\Omega|K^{\theta_1}}{m_5} < \dfrac{2}{3}\dfrac{\int_{S(x^0,\frac{D}{2})} F(x,\tau)\,dx+|\partial\Omega|K_\tau}{m_1}$;

(α_7) $\dfrac{\int_\Omega \sup_{|t|\le\theta_2} F(x,t)\,dx+|\partial\Omega|K^{\theta_2}}{m_6} < \dfrac{1}{3}\dfrac{\int_{S(x^0,\frac{D}{2})} F(x,\tau)\,dx+|\partial\Omega|K_\tau}{m_1}$;

(α_8) $K(t)\ge 0,\ \forall t\in\mathbb{R}$.

Then, for every $\lambda\in\Lambda :=]\lambda_1,\lambda_2[$, *where*

$$\lambda_1 := \frac{3}{2}\frac{m_1}{\int_{S(x^0,\frac{D}{2})} F(x,\tau)\,dx + |\partial\Omega|K_\tau},$$

$$\lambda_2 := \min\left\{\frac{m_5}{\int_\Omega \sup_{|t|\le\theta_1} F(x,t)\,dx + |\partial\Omega|K^{\theta_1}}, \frac{\frac{m_6}{2}}{\int_\Omega \sup_{|t|\le\theta_2} F(x,t)\,dx + |\partial\Omega|K^{\theta_2}}\right\},$$

with m_1 *given by Theorem* 3.1, $m_5 := \frac{a^{p-1}}{2p}(\frac{\theta_1}{\delta^*_{\lambda,g,k}\kappa})^p$, $m_6 := \frac{a^{p-1}}{2p}(\frac{\theta_2}{\kappa})^p$, and for every positive continuous function $g:\mathbb{R}\to\mathbb{R}$, there exists $\delta^*_{\lambda,g,k}>0$ such that, for each $\mu\in]0,\delta^*_{\lambda,g,k}[$, the problem (455) has at least three classical solutions u_i, $i=1,2,3$, such that

$$0 < u_i(x) < \frac{\theta_2}{\theta_3},\quad \forall x\in\Omega,\ i=1,2,3.$$

References

[1] G. Bonanno, P. Candito, *Nondifferentiable functionals and applications to elliptic problems with discontinuous nonlinearities,* J. Different. Equat. 244 (2008), 3031-3059.

[2] G. Bonanno, S.A. Marano, *On the structure of the critical set of non-differentiable functions with a weak compactness condition,* Appl. Anal. 89 (2010), 1-10.

[3] G. Bonanno, A. Chinnì, *Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent,* J. Math. Anal. Appl. 418 (2014), 812-827.

[4] W-B. Wang, W. Tang, *Infinitely many solutions for Kirchhoff type problems with nonlinear Neumann boundary conditions,* Electron. J. Diff. Eq. Vol. 2016 (2016), no. 188, 1-9.

مراجع

[1] Johnson BE. An Introduction to the Theory of Centralizers. Proc London Math Soc 1964;3:299–320.

[2] Busby RC. Double Centralizers and Extensions of C*-algebras. Trans Am Math Soc 1968;132:79–99.

[3] Fontenot RA. The Double Centralizer Algebra as a Linear Space. Proc Am Math Soc 1975;53:99–103.

[4] Taylor DC. The Strict Topology for Double Centralizer Algebras. Trans Am Math Soc 1970:633–43.

[5] Malviya BD, Tomiuk BJ. Multiplier Operators on B*-Algebras. Proc Am Math Soc 1972;31:505–10.

[6] Phillips NC. Inverse Limits of C*-algebras. J Oper Theory 1988;19:159–95.

[7] Van Daele A. Multiplier Hopf Algebras. Trans Am Math Soc 1994;342:917–32.

[8] Riazi A, Adib M.$\phi-$ Multipliers on Banach Algebras without Order. J Math Anal 2009;3:121–32.

[9] Rudin W. Functional analysis 1973.

[10] Mallios A. Topological algebras: Selected topics. vol. 124. Elsevier; 2011.

[11] Adib M, Riazi A. Double ϕ- Multipliers and Some of their Properties. Int. Math. Forum, vol. 5, 2010, p. 2497–504.

[12] Bonsall FF, Duncan J. Complete Normed Algebras. vol. 80. Springer Science & Business Media; 2012.

[13] Tomiuk BJ. Arens regularity and the algebra of double multipliers. Proc Am Math Soc 1981:293–8.

[14] Máté L. Embedding Multiplier Operators of a Banach Algebra B into its Second Conjugate Space B^{**}. Bull L Acad Pol Des Sci Des Sci Math Astron Phys 1965;13:809.

پارامتر A نشان دهنده‌ی یک نیم‌گروه μ شرکت‌پذیر[۴۱] بدون فضای خالی [۴۲] است.

فرض کنیم $M_d(A)$ نشانگر دسته‌ای ضربگرهای دوگانه بر روی A باشد. نخستین سیستم حساب از نظریه عمومی ضربگرهای دوگانه بر روی جبرهای باناخ و جبرهای سی استار [۴۳] که در پژوهش بوسبای [۲] فونتنات [۳] تیلور [۴]، تومیوک [۵] و دیگران آورده شده است.

هدف دیگر این پژوهش بررسی برخی از خصوصیات جبری ضربگرهای دوگانه بر روی جبرهای توپولوژیکی است. فرض کنیم $M_d(A)$ یک جبر ضربگر دوگانه بر روی جبر توپولوژیکی A باشد و u و s به ترتیب توپولوژی‌های عملگری یکنواخت [۴۴] و قوی [۴۵] بر روی $M_d(A)$ باشند. نشان داده شده است برخی از فرضیات اضافی بر روی A شامل موارد زیر را می‌توان در نظر گرفت:

$M_d(A)$ کامل u و s باشد.

A ایده‌آل بسته u در $M_d(A)$ باشد.

s و u مجموعه‌های کران‌دار یکسانی داشته باشند.

هر هم‌ریخت پیوسته پوشای $\varphi : A \to B$ یک توسیع[۴۶] یکنواخت[۴۷] از $\tilde{\phi} : (M_d(A), s_A) \to (M_d(B), s_B)$ دارا است.

با در نظر گرفتن هر یک از فرضیات گفته شده، خصوصیات ضربگرهای دوگانه را می‌توان بررسی کرد و مورد مطالعه قرار داد.

تعریف ۱.۲. تعریف: زوج a, b از عناصر A یک هم‌مولد[۴۸] نامیده می‌شود اگر برای تمام $x \in A$ $a \neq b$ و $ax = bx$ اگر A هیچ جفتی از عناصر هم‌مولد چپ را نداشته باشد می‌گوییم A وفادار از طرف چپ نامیده می‌شود. اگر A هم از طرف چپ و هم از طرف راست وفادار باشد آن‌گاه می‌گوییم A وفادار است.

قضیه ۲.۲. *منظم چپ بیان‌کننده‌ی هم‌ریختی از A بر روی $M_l(A)$ است. آن‌گاه یک هم‌ریخت نامیده می‌شود اگر و تنها اگر A وفادار چپ و در صورتی پوشا است که اگر و تنها اگر A عنصر همانی چپ داشته باشد. اگر A دارای خاصیت جابه‌جایی باشد آن‌گاه $M_l(A) = M_r(A)$ و اگر A وفادار و دارای خاصیت جابه‌جایی باشد آن‌گاه $M_l(A)$ نیز دارای خاصیت جابه‌جایی است. اگر H دارای خاصیت جابه‌جایی باشد و قانون لغو[۴۹] داشته باشد آن‌گاه $M_l(A)$ یک زیر- نیم‌گروه[۵۰] از توسیع گویای H است و به طور خاص دارای قانون لغو است.*

اثبات. اثبات: فرض کنیم A وفادار و دارای خاصیت جابه‌جایی باشد. همچنین فرض کنیم $S, T \in M_l(A)$ و $x, y \in A$. آن‌گاه $S(Tx)y = SyTx = TxSy = T(x, Sy) = T(Sxy) = T(Sx)y$. چون این رابطه برای تمام yهای در H برقرار است، رابطه‌ی $STx = TSx$ برای تمام xهای در A نیز درست است که این بدین معنی است که $ST = TS$. فرض کنیم A و B دو جبر توپولوژیکی سره هاسدورف باشند و $\varphi : A \to B$ یک هم‌ریخت پوشا باشد. با استفاده از قضیه (۱-۵) (که در بخش‌های بعدی آورده خواهد شد) به ازای $T \in M_d(A)$ و $a \in A$ یک نگاشت هم‌ریخت (توسیع) یکتای $\tilde{\varphi} : M_d(A) \to M_d(B)$ وجود دارد که توسط روابط زیر ارائه می‌شود:

$$[\tilde{\varphi}(T)]'(\tilde{\varphi}(a)) = \varphi([T]'(a)), \qquad [\tilde{\varphi}(T)]''(\tilde{\varphi}(a)) = \varphi([T]''(a))$$

□

قضیه ۳.۲. *فرض کنید A, B و $\tilde{\varphi}$ مانند توضیحات پاراگراف قبل تعریف شوند و s_A و s_B به ترتیب بیان‌کننده‌ی توپولوژی عملگرهای قوی بر روی $M_d(A)$ و $M_d(B)$ باشند. اگر $\tilde{\varphi} : A \to$ پیوسته باشد، آن‌گاه نگاشت $\tilde{\varphi} : (M_d(A), s_A) \to (M_d(B), s_B)$ وجود دارد.*

اثبات. فرض کنیم $\varphi : A \to B$ پیوسته باشد و $\{T_a\}$ یک دنباله با $T_a \xrightarrow{S_a} T \in M_d(A)$ در $M_d(A)$ باشد. آن‌گاه اگر $b \in B$ و $b = \varphi(a)$ باشد، برای $a \in A$:

$$[\tilde{\varphi}(T)]'(b)) = \varphi([T_a]'(a)) \to \varphi([T]'(a)) = [\tilde{\varphi}(T)]'(b)$$

و به طور مشابه، در B داریم: $[\tilde{\varphi}(T_a)]''(b) \to [\tilde{\varphi}(T)]''(b)$. که نشان دهنده‌ی پیوسته بودن $\tilde{\varphi}$ است. □

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[۴۱] Associative
[۴۲] Non-void
[۴۳] algebras -C^*
[۴۴] Uniform
[۴۵] Strong
[۴۶] Extension
[۴۷] extension Unique
[۴۸] Isoproductive
[۴۹] law Cancellation
[۵۰] Sub-Semigroup

چکیده مبسوط بیست و پنجمین سمینار آنالیز ریاضی و کاربردهای آن
۱۳ تا ۱۵ اردیبهشت ماه ۱۴۰۲، دانشگاه لرستان

ضربگرهای دوگانه

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چکیده

نظریه ضربگرهای دوگانه (یا مرکزی‌سازهای دوگانه[۲۷]) برای نخستین بار توسط جانسون[۲۸] [۱] برای جبرهای توپولوژیکی توسعه پیدا کرد و بعداً برای جبرهای باناخ و جبرهای سی استار توسط بوسبای[۲۹] [۲]، فونتنات[۳۰] [۳]، تیلور[۳۱] [۴] و تومیک[۳۲] [۵] و دیگران بررسی شد. در صورتی که A یک جبر سی استار با خاصیت جابه‌جایی باشد و $A = C_0(X)^-$ جبر تمام توابع پیوسته با مقادیر مختلط باشد که بر روی محدوده‌ی فضای هاسدورف X^- در بی‌نهایت به سمت صفر میل می‌کند، آنگاه $M_d(A)$ جبر تمام ضربگرهای دوگانه از A ، جبر $C_b(X)^-$ از تمام توابع پیوسته کران‌دار با مقادیر مختلط بر روی X است. تعمیم ناجابه‌جایی[۳۳] از روابط بین $C_0(X)$ و $C(X)$ در کار بوسبای، تیلور و دیگران آورده شده است.

واژه‌های کلیدی: جبر تولوژیکی، ضربگر های دوگانه،عملگرهای نرم دار

رده بندی موضوعی [۲۰۱۰]: ۴۲A۴۵

۱. مقدمه

از دیدگاه کاربردهای ضربگرهای دوگانه جبرهای توپولوژیکی غیرنرم‌دار[۳۴] در زمینه‌های دیگر مانند مکانیک کوانتومی و استاتیک کوانتومی و پیشرفت‌های اخیر در نظریه‌ی جبرهای توپولوژیکی در نظر گرفتن این عملگرها بر روی دسته‌ی عمومی‌تر از جبرهای توپولوژیکی بسیار مهم است. در سال ۱۹۸۸، فیلیپس[۳۵] [۶] مشتقات درونی و تقریباً درونی بر روی جبرهای موافق سی استار (حدهای وارون جبرهای سی استار که جبرهای موضعاً محدب جبری[۳۶] یا جبرهای LMC^* نیز گفته می‌شوند) با استفاده از ضربگرها را مطالعه کرده است، ون دائلی[۳۷] [۷] ضربگرها را برای جبرهای هاپف[۳۸] در نظر گرفته است که یک چارچوب طبیعی را برای مطالعه‌ی گروه‌های کوانتومی فراهم می‌کند. بنابراین، توسعه‌ی نظریه‌ی ضربگرها بر روی جبرهای توپولوژیکی به ویژه جبرهای توپولوژیکی متریک‌پذیر[۳۹] امری مهم است.

۲. مفهوم ضربگرهای (مرکزی‌سازهای) دوگانه و کاربرد آن‌ها بر روی شبه‌گروه‌ها

ضربگر دوگانه یا مرکزی‌ساز دوگانه[۴۰] یک تعریف عمومی از مفهوم ضربگر است که اولین بار توسط جانسون [۱] به صورتی که در ادامه توضیح داده می‌شود تعریف شد. یک زوج (S,T) از نگاشت $S,T : A \to A$ یک ضربگر دوگانه (مرکزی‌ساز دوگانه) بر روی A نامیده می‌شود اگر به ازای تمام $a, y \in A$:

$$x(Sy) = (Tx)y$$

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[۲۷] centralizers Double
[۲۸] Johnson
[۲۹] Busby
[۳۰] Fontenot
[۳۱] Taylor
[۳۲] Tomiuk
[۳۳] Noncommutative
[۳۴] Nonnormed
[۳۵] Phillips
[۳۶] LMC^*-Algebras
[۳۷] Daele Van
[۳۸] Algebras Hopf
[۳۹] Metrizable
[۴۰] Centralizer Double

نتیجه می‌شود که برای هر n:

$$۰٫۳ < \|(-۴, x_۱^n, x_۲^n, \cdots)\|$$

برای هر انتخاب از $x_۱^n, x_۲^n, \cdots$ خواهیم داشت:

$$N(x_n - (۴, ۰, ۰, \cdots), ۰٫۳) = N((-۴, x_۱^n, x_۲^n, \cdots), ۰٫۳) = ۰$$

در نتیجه برای هر n:

$$\lim_{n\to\infty} N(x_n - x, ۰٫۳) = ۰$$

پس برای هر دنباله‌ی $\{x_n\}$ در $Range(P)$، $t = ۰٫۳ > ۰$ وجود دارد به قسمی که

$$\lim_{n\to\infty} N(x_n - x, ۰٫۳) = ۰$$

در نتیجه x به بستار $Range(P)$ تعلق ندارد. (اگر x به بستار تعلق می‌داشت باید بنا به تعریف:

$$\lim_{n\to\infty} N(x_n - x, ۰٫۳) = ۱$$

در حالی که مساوی صفر شده است. پس x به بستار $Range(P)$ تعلق ندارد. درنتیجه به خود مجموعه $Range(P)$ هم تعلق نخواهد داشت زیرا هر مجموعه، زیرمجموعه‌ی بستارش می‌باشد، بنابراین $Range(P)$ در $\ell_۲(\mathbb{C})$ چگال نیست).

بنابراین:

$$\lambda = ۰ \in \sigma_r(P)$$

۱. نتیجه‌گیری

با مقدمات فراهم شده از تعاریف و قضایا در طیف و با اثبات خاصیت انتقال کرانداری از عملگر P به عملگرطیفی در حالت فازی شهودی ضمن یک مثال با تعریف یک نرم در فضای نرم‌دار فازی $\ell_۲(\mathbb{C})$ ثابت شده است که ۰ عضو طیف مانده‌ای P می‌باشد.

مراجع

[1] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst., 20 (1986), pp. 87–96.

[2] T. Bag and S.K.Samanta, "Finite Dimensional Fuzzy Normed Linear Spaces", J. Fuzzy Math., Vol.11, No.3, 2003, pp. 687-705.

[3] S. Cheng and J. Mordeson, "Fuzzy Linear Operators and Fuzzy Normed Linear Spaces", Bull. Cal. Math. Soc., Vol. 86, 1994, pp. 429-436.

[4] E. Kreyszig, Introductory functional analysis with applications, John Wiley & Sons, New York-London-Sydney, 1978.

[5] A. Samee A., A. Jameel K. and F. Ali C., Spectral theory in fuzzy normed spaces, Al-Nahrain Journal of Science, 14(2), (2011), pp.178–185

۲. طیف پیوسته $\sigma_c(P)$ مجموعه‌ای است به قسمی که $R_\lambda(P)$ وجود دارد و در شرط (۳) تعریف ۶ صدق نمی‌کند(یعنی $R_\lambda(P)$ عملگر خطی کراندار فازی روی $Range(P_\lambda)$ است.)

۳. طیف $\sigma_r(P)$ (مانده‌ای) مجموعه‌ای است. به قسمی که $R_\lambda(P)$ وجود داشته باشد(و ممکن است کراندار فازی باشد یا نباشد) اما در شرط (۳) در تعریف ۶ صدق نمی‌کند.(یعنی $R_\lambda(P)$ تعریف شده روی یک مجموعه که چگال در X است می‌باشد) (که در اینجا چگال نمی‌باشد).

لم ۹.۰ *فرض کنید $(X, \|\cdot\|)$ یک فضای نرم‌دار کامل روی میدان $\mathbb{C}$ باشد، به قسمی که $X \neq \{0\}$ و $P : X \to X$ یک عملگر خطی باشد، N نرم فازی تعریف شده بصورت ذیل باشد*

برای هر $x \in X$ و $t \in \mathbb{R}$

$$N(x,t) = \begin{cases} 0 & t \leqslant \|x\| \\ 1 & t > \|x\| \end{cases}$$

و فرض کنید $\lambda \in \rho(P)$ نسبت به (X, N) باشد، اگر P روی X کراندار باشد آنگاه $R_\lambda(P)$ روی X کراندار فازی است.برهان ارجاع به [۵،۷-۲]

بستار فازی

در فضای نرم‌دار فازی (X, N)، $\overline{U}$بستار U می‌باشد در صورتی که به ازای هر $x \in U$ یک دنباله‌ی $\{x_n\}$ در $\overline{U}$ وجود داشته باشد به قسمی که برای هر $t > 0$ داشته باشیم

$$\lim_{n\to\infty} N(x_n - x, t) = 1$$

مثال ۱۰.۰ فرض کنید $(X, \|\cdot\|)$ یک فضای نرم‌دار باشد و فرض کنید N نرم فازی تعریف شده به صورت زیر باشد:

برای هر $x \in X$ و $t \in \mathbb{R}$،

$$N(x,t) = \begin{cases} 0 & t \leqslant \|x\| \\ 1 & t > \|x\| \end{cases}$$

و فرض کنید $X = \ell_2(\mathbb{C})$ که در آن

$$\ell_2(\mathbb{C}) = \{x = (x_1, x_2, \cdots); \Sigma_{i=1}^{\infty} \mid x_i \mid^2 < \infty, x_i \in \mathbb{C}\}$$

تعریف می‌کنیم:

$$X = \ell_2(\mathbb{C}) \ , \ \|x\| = < x, x >^{\frac{1}{2}} = \left(\Sigma_{i=1}^{\infty} \mid x_i \mid^2\right)^{\frac{1}{2}}$$

فرض کنید $N : \ell_2(\mathbb{C}) \times \mathbb{R} \to [0, 1]$ نرم فازی تعریف شده در فوق باشد.

فرض کنید $P(x) = P(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$ به قسمی که $x \in \ell_2(\mathbb{C})$ نشان می‌دهیم که

$$0 \in \sigma_r(P)$$

برای این منظور واضح است که P عملگر خطی کراندار نسبت به $\|\cdot\|$ است(در حالت کلاسیک قبلا ثابت کرده‌ایم که $\|\cdot\|$ تعریف شده در این مثال نرم اقلیدسی است و در تعریف هم آمده که $\Sigma_{i=1}^{\infty} \mid x_i \mid^2 < \infty$ پس کراندار است.)

حال با استفاده از لم قبل می‌توان داشت: $R_\lambda(P)$ روی $\ell_2(\mathbb{C})$ کراندار فازی است و با توجه به ضابطه تابع P صرف نظر از مولفه‌ی ۰ در برد، می‌توان تابع P را همانی دانست پس $R_\lambda(P) = P_\lambda^{-1} = P^{-1}$ و $P : \ell_2(\mathbb{C}) \to \ell_2(\mathbb{C})$ یک به یک است. درنتیجه $P^{-1} : Range(P) \to \ell_2(\mathbb{C})$ وجود دارد. $\lambda = 0 \Rightarrow P_\lambda = P - \lambda I = P - 0I = P$

نشان می‌دهیم که $Range(P) = \{x \in \ell_2(\mathbb{C}) \mid x = (0, x_1, x_2, \cdots)\}$ در $\ell_2(\mathbb{C})$ چگال نیست.

برای این منظور فرض کنید $x \in \ell_2(\mathbb{C})$ به قسمی که $x = (4, 0, 0, \cdots)$ و فرض کنید $t = 0/3 > 0$ و $\{x_n\}$ دنباله‌ای در $Range(P)$ باشد به طوری که $x_n = (0, x_1^n, x_2^n, \cdots)$ از اینکه:

$$\|(0, x_1^n, x_2^n, \cdots) - (4, 0, 0, \cdots)\| = \|(-4, x_1^n, x_2^n, \cdots)\|$$
$$= (16 + |x_1^n|^2 + |x_2^n|^2 + \cdots)^{\frac{1}{2}}$$

۲. $R_\lambda(P)$عملگر خطی کراندار فازی روی $Range(P_\lambda)$ باشد.

۳. $R_\lambda(P)$تعریف شده روی یک مجموعه که چگال در X است؛ باشد.

که در آن $R_\lambda(P) = P_\lambda^{-1} = (P-\lambda I)^{-1}$ **عملگر حلّال P** و $Range(P_\lambda)$ یا **برد** $\mathbf{P}_\lambda$ نامیده می‌شوند.

تعریف ۵.۰. فرض کنید (X,N) یک فضای نرم‌دار فازی روی میدان $\mathbb{C}$ به طوری که $X \neq \{0\}$ و $P : X \to X$ یک عمگر خطی باشد مجموعه‌ی حلّال P که با $\rho(P)$ نشان داده می‌شود مجموعه‌ی تمام مقادیر عادی λ از P می‌باشد. [۳]

تعریف ۶.۰. عملگر خطی $P : (X,N) \to (X,N)$ روی X کراندار فازی نامیده می‌شود، در صورتی که $M > 0$ وجود داشته باشد به قسمی که برای هر $x \in X$ و $t \in \mathbb{R}$،

$$N(P(x),t) \geqslant N\left(x, \frac{t}{M}\right)$$

گزاره ۷.۰. *فرض کنید $(X, \|\cdot\|)$ یک فضای نرم‌دارکامل و N نرم فازی تعریف شده به صورت ذیل برای هر $x \in X$،*

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|} & t > 0 \\ 0 & t \leqslant 0 \end{cases}$$

و فرض کنید λ عضو $\rho(P)$ نسبت به (X,N) اگر P روی X کراندار فازی باشد، آنگاه $R_\lambda(P)$ روی (X,N) کراندار فازی است.

برهان: *اگر P روی X کراندار فازی باشد، آنگاه بنا به تعریف $M > 0$ وجود دارد به قسمی‌که برای هر $x \in X$ و $t > 0$:*

$$N(P_{(x)},t) \geqslant N\left(x, \frac{t}{M}\right)$$
$$\frac{t}{t+\|P(x)\|} \geqslant \frac{t/M}{t/M+\|x\|} \Rightarrow \frac{t}{t+\|P(x)\|} \geqslant \frac{t}{t+\|Mx\|}$$

در نتیجه برای هر $x \in X$،

$$\|P(x)\| \leqslant M\|x\|$$

*بنابراین P **عملگر خطی کراندار** است.*

علاوه بر این چون λ عضو $\rho(P)$ نسبت به (X,N) می‌باشد، پس $R_\lambda(P)$ وجود دارد، طبق تعریف۶ قسمت (۲) $R_\lambda(P)$، روی $Range(P_\lambda)$ که کراندار است و $Range(P_\lambda)$ در $(X, \|\cdot\|)$ چگال است.

(چون P_λ^{-1} وجود دارد) پس λ به مجموعه‌ی حلّال عملگر خطی کراندار P تعلق دارد و (مشابه قسمت اول اثبات به جای $P(x)$ مقدار $R_\lambda(P(X))$ قرار می‌دهیم؛ در نتیجه $R_\lambda(P)$ روی $R_\lambda(P(X))$ عملگر خطی کراندار است). پس $M > 0$ وجود دارد به قسمی‌که برای هر $x \in X$:

$$\|R_\lambda(P)(x)\| \leqslant M\|x\|$$

پس برای هر t بزرگتر از صفر و هر x عضو X، با جمع طرفین با t خواهیم داشت:

$$t + \|R_\lambda(P)(x)\| \leqslant t + M\|x\|$$
$$\frac{t}{t+\|R_\lambda(P)(x)\|} \geqslant \frac{t}{t+M\|x\|} \Rightarrow N(R_\lambda(P)(x),t) \geqslant N(x, \frac{t}{M})$$

درنتیجه بنا به تعریف۶.۰ عملگر خطی کراندار فازی روی (X,N) است.

تعریف ۸.۰. فرض کنید (X,N) یک فضای نرم‌دار فازی روی میدان $\mathbb{C}$ باشد به طوری که $X \neq \{0\}$ و $P : X \to X$ یک عملگر خطی باشد طیف P به سه مجموعه الحاقی زیر تقسیم می‌شود:

۱. طیف نقطه‌ای $\sigma_p(P)$ مجموعه‌ای است به قسمی که $R_\lambda(P)$ یعنی
$R_\lambda(P) = P_\lambda^{-1} = (P-\lambda I)^{-1}$وجود نداشته باشد.
هر $\lambda \in \sigma_p(P)$ یک مقدار ویژه P نامیده می‌شود.

چکیده مبسوط بیست و پنجمین سمینار آنالیز ریاضی و کاربردهای آن
۱۳ تا ۱۵ اردیبهشت ماه ۱۴۰۲، دانشگاه لرستان

کاوشی در نظریه طیفی عملگرهای خطی کراندار روی فضاهای نرم دار فازی شهودی

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چکیده

در این مقاله نوعی طیف از یک عملگر خطی در فضاهای نرم دار فازی شهودی مورد بررسی قرار گرفته و با استفاده از خاصیتی از کرانداری فازی شهودی، با مثالی در یک فضای نرم دار فازی شهودی نشان داده شده است که صفر عضو طیف مانده ای عملگر موردنظر می باشد.

واژه‌های کلیدی: ساختارهای تجزیه‌ای،عملگرهای بستاری،دستگاه معادلات دیفرانسیل خارجی

تعریف ۱.۰. فرض کنیم X و Y دو فضای برداری روی میدان K باشند. تابع $P : X \to Y$ را یک عملگر خطی از X به Y گوئیم هرگاه به ازای هر x و y عضو X و هر c عضو K داشته باشیم:

الف) $P(x+y) = P(x) + P(y)$

ب) $P(cx) = cP(x)$

تعریف ۲.۰. طیف P با $\sigma(P)$ نمایش داده می‌شود و به صورت زیر تعریف می‌شود

$$\sigma(P) = \{\lambda \in \mathbb{C}; (P - \lambda I) \text{معکوس‌پذیر نباشد}\}$$

$\lambda \in \sigma(P)$ مقدار طیفی از P نامیده می‌شود.

تعریف ۳.۰ (فضای نرم‌دار فازی). فرض کنید X یک فضای برداری روی میدان $\mathbb{C}$ باشد، یک تابع N از $X \times [0, \infty)$ یک **نرم فازی** روی X نامیده می‌شود اگر در خواص زیر صدق کند:

برای هر $x, y \in X$ و $c \in \mathbb{C}$

(N_1) $N(x,t) = 0$ برای هر $t \leqslant 0$

(N_2) $N(x,t) = 1 \Leftrightarrow x = 0$ برای هر $t > 0$

(N_3) $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ برای هر $c \in \mathbb{C}$

(N_4) $N(x+y, s+t) \geqslant \min(N(x,s), N(y,t))$ برای هر $s, t \in \mathbb{R}$

(N_5) $\lim_{t\to\infty} N(x,t) = 1$

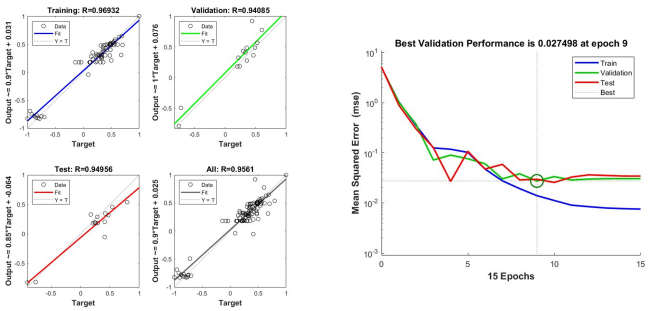
در این حالت، زوج (X, N) یک فضای نرم‌دار فازی نامیده می‌شود (همان‌گونه که ملاحظه می‌شود، در این تابع N، برد در فاصله $[0, 1]$ قرار می‌گیرد)

تعریف ۴.۰. فرض کنید (X, N) یک فضای نرم‌دار فازی روی میدان $\mathbb{C}$ به طوری که $X \neq \{0\}$ و $P : X \to X$ یک عمگر خطی باشد.

یک مقدار عادی از P یک عدد λ است به طوری که:

۱. $R_\lambda(P) = P_\lambda^{-1} = (P - \lambda I)^{-1}$ وجود داشته باشد.

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شکل ۱۰: نمودار عملکرد یادگیری شبکه و رگرسیون خطی داده‌ها.

۵. درونیابی چندجمله‌ای

در این پژوهش برای درونیابی، روش درونیابی چندجمله‌ای انتخاب شده است. درونیابی چند جمله‌ای را می‌توان با شیوه‌های متفاوتی مانند درونیابی لاگرانژ و نیوتن انجام داد. اما از آنجا که درونیابی چندجمله‌ای یکتا است پس فرقی نمیکند که از درونیاب لاگرانژ استفاده شود یا درونیاب نیوتن. در این پژوهش از روش درونیاب چندجمله‌ای لاگرانژ برای محاسبه درونیابی با روش‌های عددی استفاده می‌شود. برای بررسی عملکرد این روش از ۱۰ داده‌ای که در تست شبکه‌عصبی بکار رفته، استفاده شده است. مقادیر R و MSE حاصل از درونیابی لاگرانژ به ترتیب برابر هستند با $۸٫۷۸۹۸ \times ۱۰^{۳۶}$ و $۲٫۶۷۲۹ \times ۱۰^{۳۳}$.

۶. نتیجه‌گیری

نتایج حاصل از درونیابی به دو روش عددی و شبکه عصبی نشان می‌دهد که درونیابی با شبکه عصبی عملکرد بهتری نسبت به روش عددی چند جمله‌ای لاگرانژ دارد. این در حالی است که درون یابی چندجمله‌ای حتی خروجی نزدیک به داده‌های واقعی را نیز ارائه نداده است. علاوه بر دقت بالاتر شبکه‌های عصبی در درونیابی قیمت می‌توان از همان شبکه برای پیشبینی قیمت نیز استفاده کرد اما برای پیشبینی قیمت به روش‌های عددی نمی‌توان از درونیابی لاگرانژ استفاده کرد.

مراجع

[۱] ک.ا.اتکینسن، *آشنایی با آنالیز عددی*، ترجمه ع.دانایی، چاپ دوم، نشر دانشگاهی،۱۳۹۴.

[۲] ط.رشید، *شبکه عصبی‌تان را بسازید*، ترجمه غ.صابری تبریزی، ۱۳۹۹.

[۳] ت.هاگان، *طراحی شبکه‌های عصبی*، ترجمه م.کیا، چاپ هفتم، انتشارات دانشگاهی کیان، ۱۴۰۱.

[4] M. Aghashahi, Sh. Bamdad, *Analysis of different artificial neural networks for Bitcoin price prediction*, International Journal of Management Science and Engineering Management,2022, DOI: 10.1080/17509653.2022.2032442

[5] P. Yu, X. Yan, *Stock price prediction based on deep neural networks*, Neural Comput and Applic 32, 1609–1628, 2020, DOI: 10.1007/s00521-019-04212-x

[6] H.T. Ünal, F. Başçiftçi, *Evolutionary design of neural network architectures: a review of three decades of research*, Artificial Intelligence Review 55, 1723 - 1802, 2022, DOI: 10.1007/s10462-021-10049-5

۳. آماده سازی داده‌ها

برای انجام مقایسه کارآمدی درونیابی با شبکه‌های عصبی و روش‌های عددی بر روی داده‌های تاریخی قیمت، از قیمت پایانی ارز دیجیتال بیت‌کوین در بازه زمانی ۱۵ دقیقه استفاده شده است. این داده‌ها از زمان یک بامداد ۲۰ فوریه تا یک بامداد ۲۱ فوریه سال جاری میلادی استخراج شده که شامل ۹۶ داده است.

به کمک درونیابی مقادیر مابین داده‌ها تخمین زده می‌شوند باتوجه به آنکه از داده‌های قیمتی در بازه زمانی ۱۵ دقیقه استفاد شده است پس درونیابی، قیمت بین دو بازه ۱۵ دقیقه می‌باشد. به عبارتی قیمت در بازه زمانی پایین‌تر با درونیابی تخمین زده می‌شود. داده‌های تست به‌گونه‌ای مرتب و برچسب گذاری شده‌اند تا درونیابی قیمت پایانی در بازه ۱۵ دقیقه، قیمت پایانی در زمان ۵ دقیقه در همان بازه زمانی شود. برای محاسبه خطای درونیابی از این قیمت به‌عنوان داده واقعی استفاده می‌شود. برای عملکرد بهتر شبکه‌های عصبی در ابتدا به مقیاس بندی داده‌ها پرداخته شده است. شبکه عصبی با استفاده از تابع فعالساز اجازه ورود داده‌ها به شبکه و لایه‌های بعدی آن را می‌دهد. تابع فعالساز استفاده شده، تابع Tan-Sigmoid است. شیب این تابع در بازه $[-1, 1]$ تغییر محسوسی دارد و در خارج از این بازه شیب تابع تقریبا تغییری نمی‌کند. برای همین داده‌های ورودی خارج از این بازه از فرآیند یادگیری کنار گذاشته می‌شوند. از این رو داده‌ها با مقیاس بندی داخل این بازه قرار داده می‌شوند تا در فرآیند یادگیری شرکت داده شوند.

۴. درونیابی با شبکه‌های عصبی

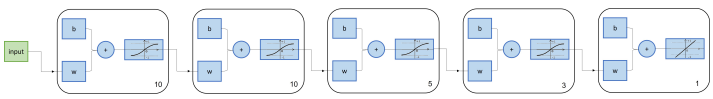
شبکه‌عصبی مدل‌ها و معماری‌های مختلفی دارد که می‌توان هر مدل را با الگوریتم‌های متفاوتی برای کاربردهای مختلف آموزش داد. [۶] باتوجه به کاربردهای شبکه‌عصبی دو مدل شبکه‌عصبی المان و پس‌انتشار مورد بحث قرار گرفته‌اند. شبکه‌عصبی المان یک مدل بازگشتی است که با پسخورد وزن‌ها را به‌روز می‌کند. این پسخورد با ایجاد حافظه کوتاه مدت وابستگی به زمان را برای شبکه ممکن می‌کند. شبکه‌عصبی پس‌انتشار نیز یک شبکه پرسپترون چند لایه با تابع انتقال غیرخطی می‌باشد. این نوع شبکه در تخمین یک تابع و پیشبینی قیمت کاربرد دارد. [۳] [۴] [۵]

درونیابی، پیدا کردن یک تقریب برای تابع است از این رو شبکه پس‌انتشار انتخاب شده است از طرفی چون داده‌ها وابسته به زمان هستند، شبکه‌عصبی المان نیز بکار گرفته شده است. در ابتدا شبکه‌عصبی پس‌انتشار و المان مورد مقایسه قرار می‌گیرند، تا مناسب‌ترین شبکه برای درونیابی با روش‌های‌عددی مقایسه شود. در طراحی این دو شبکه مقادیر اولیه وزن‌ها و سوگیری [۲۶]به صورت تصادفی انتخاب شده‌اند. این شبکه‌ها علاوه بر یک لایه ورودی شامل ۴ لایه پنهان و یک لایه خروجی می‌باشند. چهار لایه پنهان به ترتیب دارای ۱۰، ۱۰، ۵ و ۳ گره هستند و تابع فعالساز Tan-Sigmoid است. لایه خروجی نیز شامل ۱ گره و تابع فعالساز خطی Purelin می‌باشد. در جدول ۵ مقایسه بهترین نتایج دو شبکه نشان داده شده است.

جدول ۵: مقایسه شبکه‌های عصبی.

نوع شبکه	الگوریتم یادگیری	میانگین مربع تفاضلات (MSE)	R
BP	LM	۰.۰۲۴۷	۰.۷۲۰۱
Elman	LM	۰.۰۵۴۰	۰.۶۷۸۲

باتوجه به مقایسه انجام شده در جدول بالا شبکه‌عصبی پس‌انتشار (BP) با الگوریتم یادگیری Levenberg-Marquardt(LM) بهترین نتیجه را داده است. شکل‌های ۹ و ۱۰ به‌ترتیب معماری این شبکه و نمودار عملکرد اعتبارسنجی یادگیری شبکه و رگرسیون خطی داده‌ها را نمایش می‌دهند.



شکل ۹: معماری شبکه BP.

[۲۶]Bios

با ادامه روند فوق و با فرض آن‌که n تعداد گره‌های لایه i و m تعداد گره های لایه بعدی آن باشد. داریم

$$O_{i+1} = FA((W_i \times O_i) + B_{i+1}) = \begin{bmatrix} FA((\sum_{a=1}^{n} W_i^{a1} \times O_i^a) + B_{i+1}) \\ FA((\sum_{a=1}^{n} W_i^{a2} \times O_i^a) + B_{i+1}) \\ . \\ . \\ FA((\sum_{a=1}^{n} W_i^{am} \times O_i^a) + B_{i+1}) \end{bmatrix}, i = 0, 1, 2, 3, 4. \quad (۳)$$

خروجی شبکه‌عصبی المان[۲۵] مشابه شبکه BP است با این تفاوت که خروجی هر لایه پنهان به ورودی آن برگشت داده می‌شود.
برای بروزرسانی وزن‌ها و حداقل سازی خطای شبکه از گرادیان کاهشی استفاده شده است. تابع $E = (t - O_5)^2$ تابع خطای خروجی نهایی شبکه فرض می‌شود که در آن t مقدار واقعی خروجی است. گرادیان کاهشی با حرکت در خلاف جهت شیب تابع خطا وزن‌ها را اصلاح می‌کند. از این رو به شیب تابع خطا نیاز است. ابتدا شیب تابع خطای لایه خروجی که دارای تابع خطی با شیب ۱ است را حساب می‌کنیم.

$$\begin{aligned} \frac{\partial E}{\partial W_4} &= \frac{\partial (error_5)}{\partial O_5} \times \frac{\partial O_5}{\partial W_4} \\ &= -2(t - O_5) \times \frac{\partial L((W_4 \times O_4) + B_5)}{\partial W_4} \\ &= -2(t - O_5) \times O_4 \, . \end{aligned} \quad (۴) \; [۲]$$

حال شیب تابع خطای لایه‌های پنهان بررسی می‌شود که دارای تابع فعالساز TS هستند. یعنی داریم

$$\frac{\partial E}{\partial W_i} = \frac{\partial (error_{i+1})}{\partial O_{i+1}} \times \frac{\partial O_{i+1}}{\partial W_i} = -(error_{i+1}) \times [1 - \frac{(e^{w_i O_i} - e^{-w_i O_i})^2}{(e^{w_i O_i} + e^{-w_i O_i})^2}] \times O_i \, , \quad (۵)$$

که در آن $error_i$ خطای لایه i است که به صورت زیر بدست می‌آید.

$$error_i = W_i^T \times error_{i+1} \, . \quad (۶) \; [۲]$$

پس از مشخص شدن شیب تابع خطا در تمامی لایه‌ها نوبت به بروزرسانی وزن‌ها می‌رسد. وزن‌ها با توجه به ضریبی از شیب تابع خطا بروز می‌شوند. این ضریب که نرخ یادگیری نام دارد با α نشان داده شده است.

$$W_i' = W_i - \alpha \times \frac{\partial E}{\partial W_i} \, . \quad (۷) \; [۲]$$

تمامی این فرآیند برای یادگیری شبکه با اولین داده‌ی ورودی است. این فرآیند به تعداد داده‌ها تکرار می‌شود و وزن جدید (W_i') به عنوان وزن اولیه مرحله بعد آموزش انتخاب می‌شود.

قضیه ۱.۲. *فرض کنید x_i ها، $i = 1, 2, ..., n$، داده‌های ورودی متمایز شبکه‌های عصبی مفروض و $f(x_i)$ خروجی‌های واقعی آنها باشند، آنگاه یک و تنها یک چندجمله‌ای $p(x_i)$ از درجه حداکثر n وجود دارد، به‌طوری‌که*

$$p(x_i) = f(x_i) \, . \quad (۸) \; [۱]$$

چند جمله‌ای درونیاب لاگرانژ $p(x_i)$ به شکل زیر است.

$$p(x_i) = \sum_{i=0}^{n} L_i(x) f(x_i) \, , \quad (۹) \; [۱]$$

که در آن $L_i(x)$ برابر است با

$$L_i(X) = \prod_{j=0, j \neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)}, \; j = 1, 2, ..., n \, . \quad (۱۰) \; [۱]$$

[۲۵] Elman Neural Network

مقایسه درونیابی داده‌های قیمتی با شبکه‌های عصبی و روش‌های عددی

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[۱,۲,۳] دانشکده علوم ریاضی، دانشگاه صنعتی شاهرود، شاهرود، ایران

چکیده

استفاده از روش‌های عددی برای درونیابی امری متداول است. اما با پیشرفت روش‌های نوین مانند شبکه‌های عصبی وقت آن رسیده است که روش‌های بهینه‌تری جایگزین روش‌های عددی شوند. در این مقاله دو شبکه‌عصبی المان و پس‌انتشار مورد مقایسه قرار می‌گیرند. شبکه عصبی پس انتشار با توجه به داده‌های قیمتی که از بازار ارزهای دیجیتال گرفته شده، نتایج بهتری نسبت به شبکه عصبی المان ارائه کرده است. در ادامه این شبکه با چندجمله‌ای درونیاب لاگرانژ مقایسه می‌شود. نتایج حاصل از درونیابی برتری شبکه‌عصبی پس‌انتشار را برای درونیابی داده‌های قیمتی نشان می‌دهد.

واژه‌های کلیدی: درونیابی، شبکه‌عصبی، جواب تقریبی

رده بندی موضوعی [۲۰۱۰]: ۹۲B۲۰ ۹۷N۵۰،

۱. مقدمه

باتوجه به آنکه بازارهای مالی دارای چرخه‌ای از روندها هستند. تحلیل سابقه قیمت از اهمیت ویژه‌ای برخوردار است به همین دلیل بسیاری از افراد به تحلیل داده‌های قیمتی روی آورده‌اند که به آنان تحلیلگران تکنیکال [۲۳] می‌گویند. این تحلیلگران، برای تحلیل نیاز به نموداری پیوسته از داده‌های قیمتی دارند.

اگر در بازه‌ای از زمان داده‌های قیمتی به‌طور کامل وجود نداشته باشند. دراینصورت نبود سابقه‌ی کامل از قیمت‌ها باعث ایجاد ناپیوستگی در نمودار و به دنبال آن باعث ایجاد خطا در تحلیل تحلیلگران می‌شود. برای رفع این مشکل می‌توان با انجام درونیابی داده‌ها، نقاط قیمتی مابین آنها را که در اختیار نیستند تخمین زده و با آن یک نمودار پیوسته رسم کرد. با اهمیت یافتن درونیابی برای سابقه تاریخی قیمت، نیاز به پیدا کردن بهترین روش درونیابی برای این مدل از داده‌ها تشدید شده است. به طور کلی درونیابی با روش‌های عددی مانند درونیابی چندجمله‌ای ، اسپلاین‌ها و... انجام می‌شود اما با پیشرفت و گسترش هوش مصنوعی و شبکه‌های عصبی این درونیابی‌ها با شبکه‌های عصبی انجام می‌شوند و نتیجه آن با روش‌های عددی مورد مقایسه قرار می‌گیرد.

۲. مبانی نظری

$W_{i,i+1}^{a,b}$، نماد نشانگر ماتریس وزن‌های ارتباطی لایه‌ها است که در آن i، i=0,1,2,3,4، نشان دهنده شماره لایه ، a شماره گره لایه i و b شماره گره لایه i+1 می‌باشند که برای ساده نویسی آن را به صورت $W_i^{a,b}$ نشان می‌دهیم. O_i^a، خروجی گره a از لایه i می‌باشد که ورودی گره‌های لایه بعدی آن نیز به حساب می‌آید. B_i، مقدار سوگیری لایه i است. برای راحتی کار تابع Tan-Sigmoid را با نماد TS، تابع خطی Purelin را با L و به طور کلی تابع فعالساز را با FA نمایش می‌دهیم.

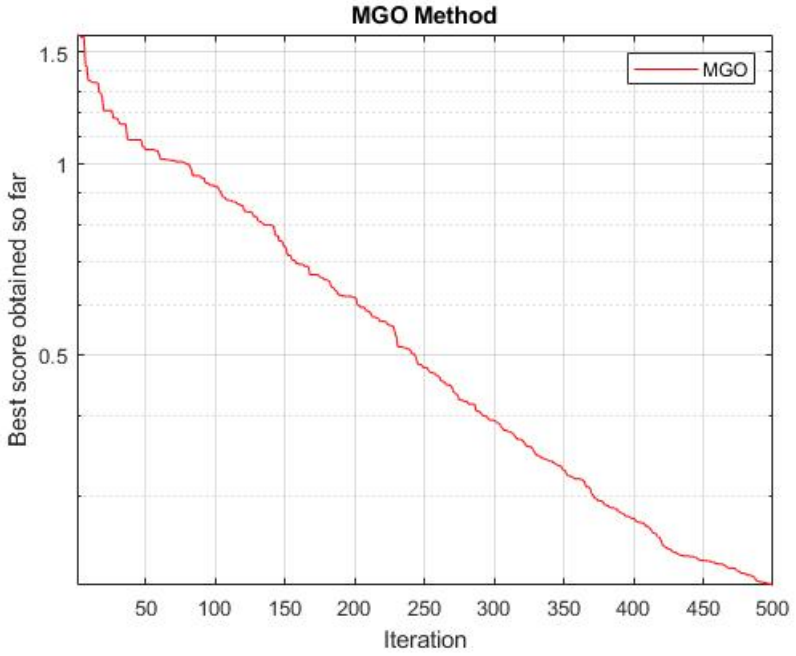
خروجی شبکه در گام اول آموزش برای شبکه پس‌انتشار[۲۴] به صورت زیر است.

$$O_1 = TS((W_0 \times O_0) + B_1), \tag{۱۴}$$

$$O_2 = TS((W_1 \times O_1) + B_2), \tag{۱۵}$$

*ارائه دهنده. آدرس ایمیل: mh.seifi79@gmail.com

[۲۳] Technical analysts
[۲۴] Backpropagation(BP)



شکل ۸: روند همگرایی الگوریتم برای دستگاه خطی با ماتریس ضرایب اسپارس $bcsstk۰۶$

[2] K.M.Dunham, *The social organization of mountain gazelles Gazella gazella in a population reintroduced to central Arabia*, Journal of Arid Environments, 43(3) (1999) 251-266.

[3] G.A.Grau and F.R.Walther, *Mountain gazelle agonistic behaviour*, Animal Behaviour, 24(3) (1976) 626-636.

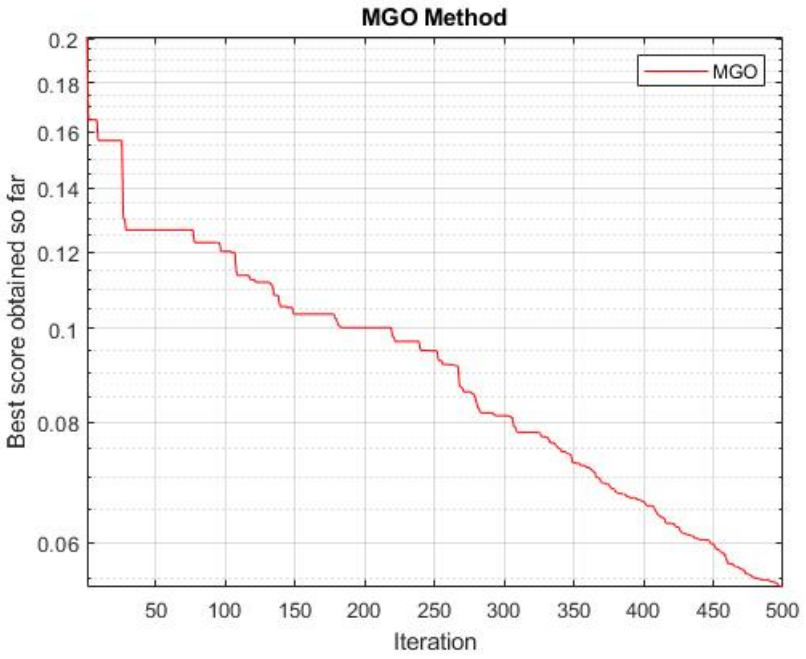
[4] E. Mezura-Montes and C.A.C. Coello, *An empirical study about the usefulness of evolution strategies to solve constrained optimization problems*, Int. J. Gen. Syst., 37(4) (2008) 443–473.

۳. آزمایشات عددی

الگوریتم MGO برای یافتن جواب بهینه دستگاه‌های خطی که در بالا بدان اشاره گردیده، با نرم افزار $MATLAB$ پیاده سازی شده است. در ادامه از دستگاه‌های خطی متعددی برای بررسی روند همگرایی این الگوریتم استفاده شده است. در جدول ۴ خلاصه مشخصات ماتریس ضرایب دو دستگاه خطی نشان داده شده است که در مثال‌ها، ماتریس‌های A و بردارهای b به‌طور تصادفی و به‌گونه‌ای انتخاب شده‌اند تا دارای جواب باشند. شکل ۷ مربوط به یک دستگاه خطی با ماتریس ضرایب A ۱۰۰ × ۱۰۰ و شکل ۸ مربوط به دستگاه خطی با ماتریس ضرایب اسپارس به نام bcsstk۰۶ می‌باشد که ماتریسی ۴۲۰ × ۴۲۰ است و هر یک روند همگرایی جواب بهینه حاصل از الگوریتم را برای رسیدن به جواب دستگاه $Ax = b$ نشان می‌دهند.

جدول ۴: مشخصات ماتریس ها

ماتریس	اندازه
A	۱۰۰ × ۱۰۰
$bcsstk۰۶$	۴۲۰ × ۴۲۰



شکل ۷: روند همگرایی الگوریتم برای دستگاه خطی با ماتریس ضرایب A

۴. نتیجه گیری

در این مقاله الگوریتم MGO برای یافتن جواب بهینه دستگاه خطی ارائه شده است. این روش برخلاف روش‌های تکراری دیگر هیچ محدودیتی ندارد. الگوریتم MGO روشی ساده برای به‌دست آوردن جواب بهینه است. همانطور که در مثال ها نشان داده شده است، روند همگرایی حاکی از کارآیی الگوریتم می‌باشد. همچنین این الگوریتم برای دستگاه‌های خطی با ماتریس ضرایب غیر مربعی قابل استفاده است و بعلاوه کارآیی قابل توجه آن برای ماتریس ضرایب اسپارس نیز مشاهده می‌شود.

مراجع

[1] B.Abdollahzadeh, F.Soleimanian Gharehchopogh, N.Khodadadi and S.A.Mirjalili, *Mountain Gazelle Optimizer: A new Nature-inspired Metaheuristic Algorithm for Global Optimization Problems*, Advances in Engineering Software, 174 (2022): 103282.

- گله‌های نر مجرد[۱۹]
- گله‌های زایمان[۲۰]
- نرهای منفرد منطقه[۲۱]
- مهاجرین برای جستجوی غذا[۲۲]

در الگوریتم MGO، هر غزال طی عملیات بهینه‌سازی می‌تواند عضو یکی از گله‌های نر مجرد، نرهای منفرد منطقه‌ای و یا گله‌های زایمان شود. یک غزال جدید هم از یکی از این سه گله متولد می‌شود و غزال نر بالغ در قلمرو گله را بهترین جواب سراسری MGO در نظر می‌گیریم [۳]. در تخمینی که زده می‌شود، حدود یک سوم از جمعیت جستجو در گله‌های نر مجرد، جوان هستند که آنچنان از قدرت و توانایی لازم برخوردار نیستند که قادر به غلبه بر غزال ماده باشند یا اینکه بتوانند آنها را تحت کنترل خود درآورند. بر همین اساس می‌توان گفت این غزال‌های جوان، در مقایسه با سایر گله‌های دیگر دارای کمترین هزینه هستند. از جمله جواب‌های دیگر برای کل جمعیت می‌توان به غزال‌های گله زایمان نیز اشاره کرد. در هر تکرار غزال‌های قوی با جواب‌های با کیفیت را نگه می‌داریم. نوع دیگری از جواب‌ها نیز غزال‌های قدیمی و بیمار بوده که به کل جمعیت اضافه می‌شوند و دارای هزینه بسیار کمتری هستند. لازم به ذکر است این گروه از جواب‌ها از کل جمعیت حذف می‌شوند [۱]. در ادامه، به فرآیند عملکرد الگوریتم MGO اشاره شده است.

الگوریتم MGO برای یافتن جواب بهینه دستگاه‌های خطی

۱. ورودی‌ها: ماتریس A، بردار b، اندازه جمعیت Nو حداکثر تعداد تکرار T.
خروجی‌ها: موقعیت بهترین غزال و جواب دستگاه خطی.

۲. مقداردهی اولیه:

ایجاد جمعیت اولیه
X_i ، $i = \{۱, ۲, ..., N\}$.

۳. مراحل زیر را تا رسیدن به شرط توقف انجام دهید:

(آ) برای هر غزال موجود در جمعیت مراحل زیر را انجام دهید:
۱. گله‌های نر منفرد منطقه را طبق معادله (۱) [۱] محاسبه کنید.
۲. گله‌های زایمان را طبق معادله (۶) [۱] محاسبه کنید.
۳. گله‌های نر مجرد را طبق معادله (۷) [۱] محاسبه کنید.
۴. مهاجرین برای جستجوی غذا را طبق معادله (۹) [۱] محاسبه کنید.
۵. مقدار تابع برازش را برای TSM، MH، BMH و MSF محاسبه و آنها را به زیستگاه اضافه کنید.

(ب) کل جمعیت را به ترتیب صعودی مرتب کنید.

(ج) موقعیت بهترین غزال را به‌روز رسانی کنید.

(د) N غزال برتر را ذخیره کنید.

۴. با پایان الگوریتم بهترین مقدار تابع برازش و جواب دستگاه خطی را برگردانید.

[۱۹] Bachelor Male Herds (BMH)
[۲۰] Maternity Herds (MH)
[۲۱] Territorial Solitary Males (TSM)
[۲۲] Migration to Search for Food (MSF)

چکیده مبسوط بیست و پنجمین سمینار آنالیز ریاضی و کاربردهای آن
۱۳ تا ۱۵ اردیبهشت ماه ۱۴۰۲، دانشگاه لرستان

یافتن جواب بهینه دستگاه‌های خطی با استفاده از روش فراابتکاری غزال کوهستان

الهه اسلامی[۱*] ، سیدابوالفضل شاهزاده فاضلی[۲]، اسراء موسوی [۳]

[۱] کارشناسی ارشد علوم کامپیوتر، دانشگاه یزد

[۲]عضو هیات علمی، بخش علوم کامپیوتر، دانشکده علوم ریاضی، دانشگاه یزد

[۳] دانشجوی دکتری علوم کامپیوتر، دانشگاه یزد

چکیده

روش‌های گوناگونی به منظور حل دستگاه‌های خطی ارائه شده است. با این وجود از میان همه روش‌های مستقیم و تکراری، در انتخاب صحیح روش، همچنان با مشکلات عدیده‌ای روبرو هستیم و انتخاب نادرست در این زمینه می‌تواند منجر به زمان بیشتر و همینطور خطاهای زیادی شود که دقت لازم در این راستا را می‌طلبد. روش‌های بهینه‌سازی می‌کوشند تا بهترین جواب ممکن برای مسئله‌ای خاص را بیابند؛ بر همین اساس روش‌های ابتکاری و فراابتکاری بسیاری به منظور یافتن جواب بهینه مسائل متعدد و مختلف ارائه شده‌اند. در این مقاله سعی بر این است تا با استفاده از روش فراابتکاری غزال کوهستان، برای دستگاه‌های خطی، بهترین جواب ممکن را به دست آوریم. این بهینه‌ساز با الهام از زندگی اجتماعی غزال‌های کوهستانی، زندگی سلسله مراتبی آن‌ها را به‌صورت ریاضی فرمول بندی نموده و به منظور توسعه یک الگوریتم بهینه‌سازی مورد استفاده قرار می‌گیرد. در ادامه این مقاله به استفاده از روش بهینه سازی غزال کوهستان پرداخته می‌شود که به بهبود کارایی در حل دستگاه خطی کمک چشمگیری داشته است. در این راستا به منظور بررسی کارایی روش مذکور، این روش پیاده سازی شده است که نتایج آزمایشات صورت گرفته حاکی از کارایی موثر این روش در یافتن جواب بهینه دستگاه‌های خطی است.

واژه‌های کلیدی: الگوریتم غزال کوهستان، بهینه‌سازی، دستگاه خطی، روش‌های فراابتکاری

رده بندی موضوعی [۲۰۱۰]: ۶۸T۲۰ ۶۵F۱۰ ۷۸M۵۰

۱. مقدمه

بسیاری از مسائل در دنیای واقعی به دلیل محدودیت‌ها و متغیرهای مختلف، ابعاد گسترده‌ای دارند که به عنوان مسائل $NP-hard$ طبقه بندی می شوند. با افزایش پیچیدگی مسائل طی دهه‌های اخیر، وجود روش‌های بهینه‌سازی جدید بیش از گذشته ضرورت دارد؛ روش‌هایی که بطور تصادفی به ارائه بهترین جواب ممکن برای مسائل مختلف، می‌پردازند [۴]. الگوریتم‌های بهینه‌سازی مختلفی برای حل چنین مسائل چالش برانگیزی، از جمله موارد دقیق و تقریبی، توسعه یافته‌اند. الگوریتم های تقریبی هم به عنوان یک تکنیک جدید برای حل مسائل پیچیده با ابعاد بالا و چند حالته به دلیل در دسترس نبودن روش‌های بهینه سازی دقیق ارائه شده‌اند. الگوریتم‌های ابتکاری و فراابتکاری دو نوع الگوریتم تقریبی هستند که راه حل های قابل قبولی را در یک زمان معقول ارائه می دهند. در نتیجه، تحقیقات طی دو دهه گذشته تلاش‌های متعددی را برای مسائل مختلف با استفاده از الگوریتم‌های فراابتکاری مبتنی بر رویکرد تقریبی و الگوی رویدادهای طبیعی و رفتار اجتماعی حیوانات و انسان‌ها نشان می‌دهد. یکی از روش‌هایی فراابتکاری الهام گرفته از طبیعت، بهینه‌ساز غزال کوهستان [۱۸] است که در این مقاله سعی بر یافتن جواب بهینه برای دستگاه خطی دارد. این الگوریتم که به اختصار MGO نامیده می‌شود مبتنی بر زندگی اجتماعی و گروهی گوزن‌های کوهی یا غزال‌های کوهستانی در طبیعت است و از چهار مکانیزم مختلف برای انجام بهینه‌سازی استفاده می‌کند [۱] که در بخش بعدی به شرح آن پرداخته شده است.

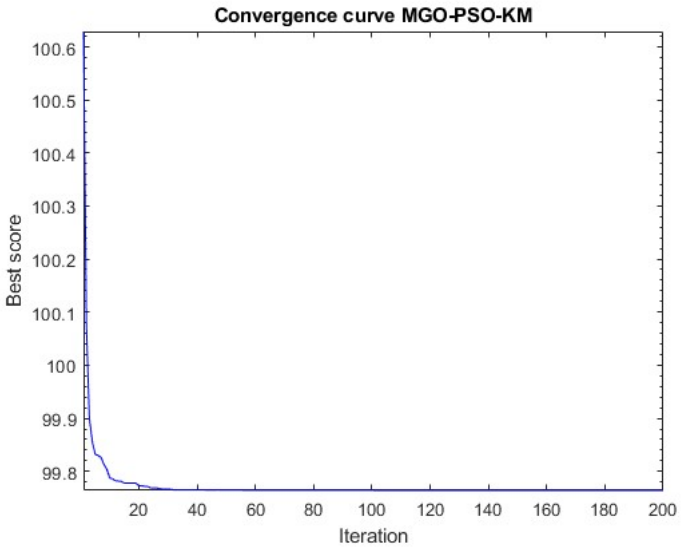
۲. روش بهینه‌سازی غزال کوهستان

الگوریتم غزال کوهستان، یک الگوی فراابتکاری مبتنی بر رفتارهای اجتماعی و زندگی غزال‌های کوهستان است که در سال ۲۰۲۲ توسط عبدالله‌زاده و همکاران ارائه شد [۱]. این بهینه‌ساز با استفاده از چهار مکانیزم و عامل اصلی در زندگی غزال‌های کوهستان، عملیات بهینه‌سازی را انجام می‌دهد [۲] که عبارتند از:

*ارائه دهنده. آدرس ایمیل: elahe.eslami@ymail.com

[۱۸] Mountain Gazelle Optimizer (MGO)

پوستر



شکل ۶: ارزیابی عملکرد الگوریتم MGO_PSO_KM با داده‌های $Iris$

جدول ۳: مقدار تابع هدف الگوریتم MGO_PSO_KM

داده	MGO_PSO_KM
$Iris$	۹۹٫۷۶۴۴
$Football$	۳۱۳٫۲۷۳۳

(2022): 103282.

[2] R. Eberhart and J. Kennedy, *Particle swarm optimization*, In Proceedings of the IEEE international conference on neural networks., 4 (1995) 1942-1948.

[3] G. Gan, C . Ma and J. Wu, *Data clustering: theory, algorithms and applications*, Society for Industrial and Applied Mathematics, 2020.

الگوریتم ۱ الگوریتم پیشنهادی MGO_PSO_KM

۱. شروع

۲. پارامترهای ورودی را مقداردهی کنید.

۳. جمعیت اولیه خوشه‌ها که برابر با K مرکز خوشه است، را به صورت تصادفی بسازید.

۴. تازمانی که به معیار توقف نرسیده‌اید گام‌های زیر را انجام دهید:

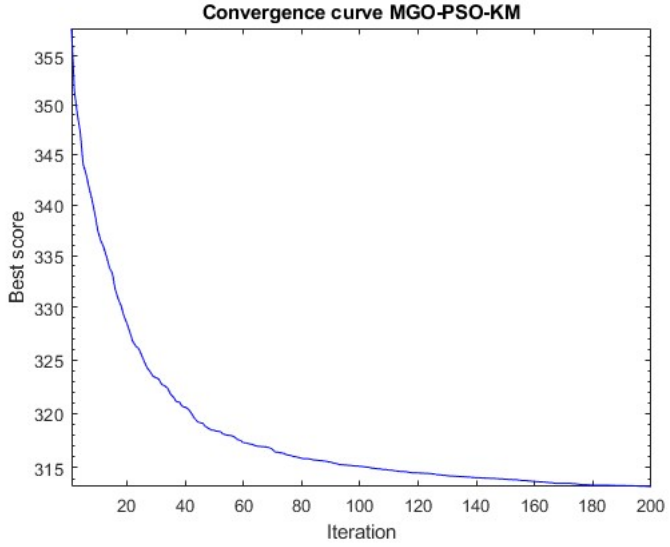
(آ) الگوریتم PSO را با داده‌ها اجرا کنید و مقدار تابع هدف و مرکز خوشه‌ها را بدست آورید.

(ب) الگوریتم MGO را با داده‌های بروزشده اجرا کنید و مقدار تابع هدف و مرکز خوشه‌ها را بدست آورید.

(ج) با توجه به مقدار تابع هدف دو الگوریتم مراکز را بروز و روند را تکرار کنید.

۵. مراکز خوشه و داده‌های خوشه‌بندی شده را به عنوان خروجی برگردانید.

۶. پایان



شکل ۵: ارزیابی عملکرد الگوریتم MGO_PSO_KM با داده‌های $Football$

۶. نتیجه‌گیری

الگوریتم‌های ترکیبی به عنوان یک روش قدرتمند در بهبود کارایی الگوریتم‌های بهینه‌سازی مورد استفاده قرار می‌گیرند. در این روش، الگوریتم‌های مختلفی با هدف بهبود عملکرد هم‌دیگر، ترکیب می‌شوند. الگوریتم‌های مختلف به گونه‌ای ترکیب می‌شوند که نقاط ضعف یکدیگر را پوشش می‌دهند. نتایج حاصل از اجرای الگوریتم بهینه‌سازی خوشه‌بندی توسط الگوریتم ترکیبی MGO_PSO_KM کارآیی آن را نشان می‌دهد. در واقع، استفاده از الگوریتم MGO در کنار الگوریتم PSO به خوبی می‌تواند ضعف آن را در خوشه‌بندی رفع نماید.

با استفاده از روش ارائه شده، می‌توان به نتایج بهتری در مقایسه با استفاده از الگوریتم PSO دست یافت، به عبارتی بهبود عملکرد و دقت خوشه‌بندی است که با استفاده از الگوریتم MGO_PSO_KM میسر شده است.

مراجع

[1] B. Abdollahzadeh, F. S. Gharehchopogh, N. Khodadadi, and S. Mirjalili , *Mountain gazelle optimizer: a new nature-inspired metaheuristic algorithm for global optimization problems*, Advances in Engineering Software. 174

حرکت را پیدا می‌کند و این روند به صورت مکرر ادامه می‌یابد تا بهترین جواب ممکن برای مسئله بهینه‌سازی پیدا شود. برای استفاده از الگوریتم PSO، باید مسئله به شکل یک تابع هدف تعریف و فضای جستجو برای حل مسئله مشخص شود. سپس، یک گروه از ذرات با موقعیت و سرعت تصادفی در فضای جستجو ایجاد می‌شود. در هر مرحله، هر ذره با توجه به موقعیت و سرعت خود، بهترین موقعیت خود را جستجو می‌کند و این اطلاعات را با همسایگان خود به اشتراک می‌گذارد تا به بهترین موقعیت دست یابند. الگوریتم PSO با استفاده از ایده‌های اجتماعی و تعاملی، به دنبال یافتن بهینه‌ترین نقطه، فضای مسئله را جستجو می‌کند [۲].

۳. الگوریتم بهینه‌سازی غزال کوهستان

الگوریتم بهینه‌سازی غزال کوهستان، یک الگوی فراابتکاری نوین مبتنی بر رفتارهای اجتماعی و زندگی غزال‌های کوهستان است که در سال ۲۰۲۲ توسط عبداللهزاده و همکاران ارائه گردید [۱]. در این الگوریتم با استفاده از چهار عامل اصلی زندگی غزال‌های کوهستان، شامل گله‌های نر مجرد، گله‌های زایمان، نرهای منفرد منطقه و مهاجرین برای جستجوی غذا، عملیات بهینه‌سازی انجام می‌شود. این عوامل در ادامه توضیح داده شده‌اند:

۱. گله‌های نر مجرد: شامل غزال‌های جوان و نر بدون همنوع هستند که در تلاش برای پیدا کردن همنوع برای تشکیل گله نر هستند. غزال‌های جوان در این گله‌ها تجربه کمتری دارند و به دنبال یادگیری از غزال‌های بالغ هستند.

۲. گله‌های زایمان: شامل غزال‌های نر بالغ و غزال‌های ماده هستند که در حال تربیت فرزندان خود هستند. غزال‌های بالغ در این گله‌ها تجربه بیشتری دارند و به دنبال حفظ منطقه خود هستند.

۳. نرهای منفرد منطقه: شامل غزال‌های بالغ نر هستند که قلمرو خود را در اختیار دارند و به دنبال حفظ منطقه خود هستند.

۴. مهاجرین برای جستجوی غذا: شامل غزال‌هایی هستند که برای جستجوی غذا از منطقه خود به سمت مناطق جدید حرکت می‌کنند. این غزال‌ها توانایی یافتن منابع غذایی جدید را دارند و به دنبال تأمین غذا برای خود و گله خود هستند.

فرزندان تولید شده توسط یک غزال می‌توانند از ویژگی‌های مختلف و از گله‌های مختلف الگوریتم ترکیب شوند. این ویژگی باعث می‌شود که MGO از تنوع بیشتری در جستجوی بهینه برخوردار باشد. بهترین جواب سراسری MGO، غزال نر بالغ در قلمرو گله است، به این معنا که بهترین جوابی که در فضای جستجو وجود دارد، شامل یک غزال نر بالغ در یکی از گله‌های نر مجرد، گله‌های زایمان یا نرهای منفرد منطقه‌ای است. در پایان هر تکرار از الگوریتم، غزال‌های قوی با جواب‌های با کیفیت حفظ می‌شوند و غزال‌های قدیمی و بیمار از کل جمعیت حذف می‌شوند.

۴. الگوریتم پیشنهادی MGO_PSO_KM

خوشه‌بندی یک روش مهم در داده‌کاوی است و همان‌طور که در بخش‌های قبل بیان شد انتخاب مراکز اولیه و گرفتار شدن در بهینه محلی از مشکلاتی است که خوشه‌بندی $k-$میانگین با آن مواجه می‌شود. یکی از روش‌های بهبود خوشه‌بندی، استفاده از الگوریتم‌های بهینه‌سازی است. این الگوریتم‌ها با به دست آوردن بهترین مراکز خوشه، فاصله درون خوشه‌ها را کمینه و فاصله بین خوشه‌ها را بیشینه می‌کنند. این رویکرد باعث می‌شود که خوشه‌بندی داده‌ها به صورت بهتری انجام شود. الگوریتم‌های بهینه‌سازی بر اساس یک تابع هدف، به دنبال یافتن بهترین پاسخ برای آن تابع هستند. در خوشه‌بندی، تابع هدف می‌تواند فاصله داده‌ها تا مراکز خوشه‌ها باشد. بر اساس این تابع هدف، الگوریتم بهینه‌سازی، مراکز خوشه را بهینه می‌کند تا هدف خوشه‌بندی $k-$میانگین که همان کمینه‌سازی فاصله درون خوشه‌ای و بیشینه‌سازی فاصله بین خوشه‌ای است را برآورده کند. در این راستا، الگوریتم پیشنهاد شده در این مقاله جهت بهبود خوشه‌بندی استفاده از الگوریتم ترکیبی بهینه‌سازی PSO و MGO است. در این الگوریتم، در ابتدا الگوریتم PSO برای بهبود خوشه‌بندی اجرا می‌شود و سپس با بهترین نتایج حاصله، الگوریتم MGO اجرا می‌شود. با بررسی نتایج حاصل از هر دو الگوریتم و در نظر گرفتن بهترین مقدار تابع هدف، خروجی این مرحله به عنوان ورودی مرحله بعد در نظر گرفته می‌شود. این عمل به تعداد مشخص شده تکرار می‌شود. خروجی نهایی، مراکز خوشه حاصل از اخرین تکرار خواهند بود. با استفاده از این الگوریتم، بهبود قابل توجهی در خوشه‌بندی داده‌ها حاصل می‌شود و می‌توان از آن به‌عنوان یک روش موثر و کارا برای خوشه‌بندی استفاده کرد.

۵. آزمایشات عددی

در این بخش به نتایج حاصل از پیاده‌سازی الگوریتم MGO_PSO_KM که با نرم افزار $MATLAB$ صورت گرفته شده است، می‌پردازیم. داده‌های مورد استفاده $Iris$ و $Football$ و نتایج مورد بررسی در جدول ۴ نشان داده شده است. شکل‌های ۹ و ۱۰ روند همگرایی با استفاده از الگوریتم MGO_PSO_KM برای رسیدن به جواب بهینه را نشان می‌دهد.

چکیده مبسوط بیست و پنجمین سمینار آنالیز ریاضی و کاربردهای آن
۱۳ تا ۱۵ اردیبهشت ماه ۱۴۰۲، دانشگاه لرستان

بهبود خوشه‌بندی میانگین-k با به کارگیری الگوریتم ترکیبی بهینه‌سازی ازدحام ذرات و غزال کوهستان

اسراء موسوی[۱،*] ، سید ابوالفضل شاهزاده فاضلی[۲]، الهام عباسی هرفته[۲]، فاطمه کاوه یزدی[۳]

[۱،۲] بخش علوم کامپیوتر، دانشکده علوم ریاضی، دانشگاه یزد

[۳] دانشکده مهندسی کامپیوتر، دانشگاه یزد

چکیده

خوشه‌بندی یکی از مهمترین مسائل در داده کاوی و یادگیری ماشین است که به دسته‌بندی داده‌ها بر اساس شباهت و ویژگی‌های مشابه آن‌ها می‌پردازد. هدف اصلی در خوشه‌بندی، یافتن گروه‌های همگن و متمایز از داده‌ها است تا بتوان اطلاعات مفیدی از داده‌ها استخراج نمود. از جمله مشکلاتی که این الگوریتم‌ها با آن مواجه می‌شوند، گرفتار شدن در بهینه محلی است. برای حل این مشکل می‌توان از الگوریتم‌های فراابتکاری استفاده نمود. در این مقاله جهت بهبود خوشه‌بندی ،میانگین-k از الگوریتم ترکیبی بهینه‌سازی ازدحام ذرات و غزال کوهستان استفاده شده است و نشان داده می‌شود نسبت به حالتی که تنها از الگوریتم ازدحام ذرات استفاده شده است بهتر عمل می‌کند.

واژه‌های کلیدی: الگوریتم بهینه‌سازی ازدحام ذرات، الگوریتم بهینه‌سازی غزال کوهستان، خوشه‌بندی میانگین-k

رده بندی موضوعی [۲۰۱۰]: ۹۱C۲۰ ۷۸M۵۰ ۶۸T۲۰

۱. مقدمه

با گسترش فناوری و افزایش حجم داده‌ها در دنیای امروز، خوشه‌بندی به یکی از ابزارهای مهم در داده‌کاوی تبدیل شده است. با توجه به حجم بالای داده‌های امروزی، الگوریتم‌های خوشه‌بندی موجود، نیازمند بهینه‌سازی و پیشرفت بیشتری هستند. با استفاده از خوشه‌بندی و تجزیه‌ی داده‌های بزرگ به بخش‌های کوچک‌تر، می‌توان اطلاعات مفید و دانش جدیدی را از داده‌ها استخراج کرد و این موضوع به عنوان یکی از مزایای مهم خوشه‌بندی، در داده‌کاوی محسوب می‌شود. یکی از الگوریتم‌های خوشه‌بندی، الگوریتم $k-$میانگین است. در این روش هر خوشه دارای یک مرکز خوشه است و این مراکز طوری انتخاب می‌شوند که فاصله درون خوشه‌ها را کمینه و فاصله بین خوشه‌ها را بیشینه کند [۳]. یکی از مشکلات رایج در الگوریتم‌های خوشه‌بندی، انتخاب مراکز اولیه است، زیرا مراکز اولیه به‌صورت تصادفی برای خوشه‌بندی انتخاب می‌شوند. مشکل دیگری که الگوریتم‌های خوشه‌بندی با آن مواجه هستند مساله گرفتار شدن در بهینه محلی است. برای حل این مشکل، از الگوریتم‌های فراابتکاری استفاده می‌شود.
الگوریتم بهینه‌سازی ازدحام ذرات [۱۶] یکی از الگوریتم‌های هوشمند و محبوب برای حل مسائل بهینه‌سازی است که در سال ۱۹۹۵ توسط ابِرهارت و کندی معرفی شد [۲]. هدف این الگوریتم، بهینه‌سازی یک تابع هدف با استفاده از تعدادی ذره در فضای جستجو است. الگوریتم بهینه‌سازی جدیدی بنام الگوریتم بهینه‌سازی غزال کوهستان[۱۷] برای مسائل بهینه‌سازی محاسباتی ارائه شده است که از رفتار غزال کوهستان الهام گرفته شده است [۱]. در این مقاله سعی بر این است تا با ترکیب دو الگوریتم PSO و MGO، مراکز اولیه مناسب برای خوشه‌بندی $k-$میانگین را به دست آوریم و از این طریق به خوشه‌بندی مناسب‌تری دست یابیم. در ادامه توضیح مختصری در مورد این دو الگوریتم ارائه می‌شود و سپس به الگوریتم پیشنهادی و نتایج حاصل از آن می‌پردازیم.

۲. الگوریتم بهینه‌سازی ازدحام ذرات

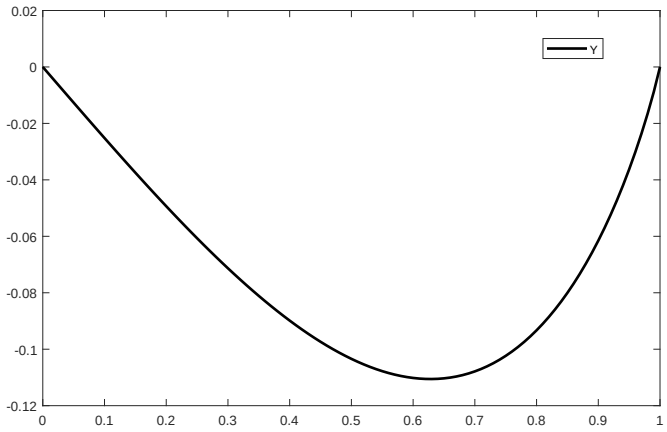
الگوریتم بهینه‌سازی ازدحام ذرات یک روش بهینه‌سازی مبتنی بر جمعیت است که برای حل مسائل بهینه‌سازی مورد استفاده قرار می‌گیرد. در این الگوریتم، یک جمعیت از ذرات در فضای جستجو حرکت می‌کنند و با توجه به تجربیات قبلی خود، به دنبال یافتن بهترین جواب ممکن برای مسئله بهینه‌سازی هستند. هر ذره در این الگوریتم دارای موقعیتی در فضای مسئله است و با تغییر سرعت و جابجایی در فضای مسئله، بهترین موقعیت خود را پیدا می‌کند. الگوریتم PSO در هر مرحله بهترین موقعیت هر ذره و کل ذرات موجود در جمعیت را با هم به اشتراک می‌گذارد و جهت مناسب برای

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[۱۶] Particle Swarm Optimization (PSO)
[۱۷] Mountain Gazelle Optimizer (MGO)

جدول ۲: تقریبهای تنها مقدار ویژه مسئله

λ	n
-۰,۸۷۷۲۱۸۰۳۴۰۹۹	۱۵
-۰,۸۷۴۵۱۵۰۰۴۵۴۴	۲۳
-۰,۸۷۴۱۰۰۵۱۶۹۶۴	۲۵



شکل ۴: منحنی توابع ویژه متناظر با مقدار ویژه.

مراجع

[1] Q. M. Al-Mdallal, *An efficient method for solving fractional Sturm-Liouville problem*, Chaos, Solitons and Fractals, 40 (2009), 183–189.

[2] A. El-Ajou, *A modification to the conformable fractional calculus with some applications*, Alexandria Eng. J. 59 (2020), no. 4, 2239–2249.

[3] A. ERCAN, *Adomian decomposition method for solving nonlinear fractional sturm-liouville problems*, Cumhuriyet Sci. J. 41 (2020), no. 1, 169–175.

[4] R. Khalil, M. Horani, A. Al Yousef and M. Sababheh, *A new definition of fractional derivative*, J. Comp. Appl. Math. 264 (2014), 65–70.

[5] H. Mortazaasl, A.A. Jodayree Akbarfam, *Two classes of conformable fractional Sturm-Liouville problems: Theory and applications*, Math. Meth. Appl. Sci. 44 (2021), no. 1 166–195.

نشان می دهیم، که درآن A_n چند جمله ایهای آدومیان اند و

$$A_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\mu^n}\left[\Lambda\left(\sum_{i=0}^{\infty}\mu^i y_i\right)\right]_{\mu=0},$$

از (۱۰) و (۱۱) داریم

$$\sum_{n=0}^{\infty} y_n(x) = y(a) + p(b)\mathcal{D}_a^{\alpha} y(b)\int_a^x \frac{1}{p(t)}\mathrm{d}_\alpha(t,a) + \sum_{n=0}^{\infty}\Phi^{-1}A_n(x). \tag{۱۲}$$

ویژگی خطی بودن عملگر Φ^{-1} در بالا استفاده شده است. پس، روابط بازگشتی زیر را از معادله (۱۲) معرفی کنیم

$$y_0(x) = y(a) + p(b)\mathcal{D}_a^{\alpha} y(b)\int_a^x \frac{1}{p(t)}\mathrm{d}_\alpha(t,a), \quad y_{n+1}(x) = \Phi^{-1}A_n(x), \quad n \geqslant 0, \tag{۱۳}$$

که در آن $A_n(x) = \big(\lambda w_\alpha(x) - q(x)\big)y_n(x)$. سری جواب، (سری $y(x)$) مستقیماً نتیجه می شود و دقت جواب قطعاً به تعداد جملات محاسبه شده بستگی دارد. در نتایج عددی حداکثر تا ۲۵ جمله انتخاب می کنیم.

۳. نتایج عددی

در این بخش، مسئله مقدار ویژه کسری تطبیق پذیر منظم زیر را با استفاده از روشی که در بالا توضیح داده شد، حل میکنیم.

مثال ۱.۳. مسئله مقدار ویژه کسری تطبیق پذیر منظم $^{1/2}\mathcal{D}_1 y'(x) + \lambda y(x) = 0,\ x \in (0,1)$ با شرایط مرزی $y(0) = y(1) = 0$ را در نظر بگیرید، معادله را می توان با عملگر به فرم $\Phi y(x) = -\lambda y(x)$ نوشت. با اعمال Φ^{-1} و استفاده از شرایط اولیه در $x = 0$ خواهیم داشت:

$$y(x) = y(0) + p(1)y'(1)\int_0^x \mathrm{d}t - \lambda\int_0^x {}^{1/2}I_1 y(t)\mathrm{d}_1(x,0), \quad \alpha = 1, \beta = 1/2,$$

در نتیجه

$$y(x) = Bx - \lambda\int_0^x\int_1^t y(s)(1-s)^{-1/2}\mathrm{d}s\mathrm{d}t, \quad B = p(1)y'(1).$$

طبق فرآیند گفته شده جملات دنباله عبارتند از:

$$\begin{aligned}
y_0(x) &= Bx,\\
y_1(x) &= -\frac{4B\lambda}{15}\big(x(1-x)^{3/2} + (1-x)^{3/2} - 1\big),\\
y_2(x) &= \frac{B\lambda^2}{45}\big(64 - 26x + 24x^2 - 6x^3 - x^4 - 64(1-x)^{3/2}\big),\\
y_3(x) &= -\frac{B\lambda^3}{31185}\big(43840 - 50820(1-x)^{3/2} - 412(1-x)^{9/2} - 7392(x-1)^3 - 28(1-x)^{9/2}\big),\\
&\vdots
\end{aligned}$$

و تا $y_{25}(x)$ محاسبه می کنیم. این تقریب $y(x)$ را به صورت سری توسط $y(x) \cong \mathbf{y}(x) = \sum_{n=0}^{25}\tilde{y}_n(x)$ بدست می دهد. مقدار ویژه با تنها ریشه تابع مشخصه $\Delta(\lambda) = y(1) = 0$ منطبق هست. با توجه به تعداد جملات سری $y(x)$ ، تقریب مقدار ویژه در جدول ۵ ارائه شده است. تابع ویژه متناظر با مقدار ویژه بالا در شکل ۹ نشان داده شده است.

۴. نتیجه‌گیری

در این مقاله، حل عددی رده ای از مسائل اشتورم -لیوویل کسری تطبیق پذیر را پیشنهاد کرده‌ایم. با بررسی نتایج عددی روش تجزیه آدومیان برای محاسبه مقادیر و توابع ویژه بسیار ساده و کارآمد بوده و نتایج خوبی را بدست آورد. در این تحقیق، روزنه ای جدید برای حل این گونه مسائل باز شده است که در مطالعات بعدی، بررسی سایر روشهای عددی می تواند مفید باشد.

تعریف ۲.۱. انتگرال کسری تطبیق پذیر چپ تابع $f : [s, \infty) \to \mathbb{R}$ شروع شده از $s \in \mathbb{R}$، از مرتبه α عبارت است از

$$I_s^\alpha f(t) = \frac{1}{(m-1)!} \int_s^t (t-x)^{m-1} f(x) \mathrm{d}_\alpha(x,s), \quad \mathrm{d}_\alpha(x,s) = (x-s)^{\alpha-m} \mathrm{d}x, \quad t > s.$$

انتگرال کسری تطبیق پذیر راست تابع $f : (-\infty, s] \to \mathbb{R}$ ختم شده به $s \in \mathbb{R}$، از مرتبه α عبارت است از

$${}^\alpha I_s f(t) = \frac{1}{(m-1)!} \int_t^s (x-t)^{m-1} f(x) {}_\alpha\mathrm{d}(s,x), \quad {}_\alpha\mathrm{d}(s,x) = (s-x)^{\alpha-m} \mathrm{d}x, \quad t < s.$$

قضیه ۳.۱. فرض کنید تابع $f : [s, \infty) \to \mathbb{R}$، $(m-1)$ -بار دیفرانسیل پذیر باشد، آنگاه

$$I_s^\alpha T_s^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(s)}{k!} (t-s)^k, \quad T_s^\alpha I_s^\alpha f(t) = f(t), \quad \forall t, t > s.$$

قضیه ۴.۱. فرض کنید تابع $f : (-\infty, s] \to \mathbb{R}$، $(m-1)$ -بار دیفرانسیل پذیر باشد، آنگاه

$${}^\alpha I_s {}^\alpha T_s f(t) = f(t) - \sum_{k=0}^{m-1} (-1)^k \frac{f^{(k)}(s)}{k!} (s-t)^k, \quad {}^\alpha T_s {}^\alpha I_s f(t) = f(t), \quad \forall t, t < s.$$

روش حل مورد استفاده برای حل دسته معادلات (۷) بر اساس یک تکنیک عددی کارآمد به نام روش تجزیه آدومیان که توسط بسیاری از نویسندگان به منظور حل چندین نوع معادله دیفرانسیل استفاده شده است [۳،۱]. در ادامه مقاله، اجرای روش تجزیه آدومیان روی مسئله (۷) با استفاده از حساب کسری تطبیق پذیر می باشد که قبلا در مورد مشتقات کسری ریمان-لیوویل و کاپوتو مورد بحث قرار گرفته است. نتایج عددی در بخش آخر ارائه خواهد شد.

۲. تجزیه و تحلیل روش تجزیه آدومیان

در این بخش، بحث مختصری از الگوریتم برای حل عملگر $\mathcal{L}_{\alpha,\beta}$ با دو مقدار مرزی را ارائه می دهیم، مسئله مقدار ویژه ارائه شده توسط (۷) را به صورت زیر بازنویسی می کنیم

$$\Phi y(x) = \Lambda, \tag{۹}$$

که در آن $\Phi = \Phi(x, \mathcal{D}_a^\alpha, {}^\beta\mathcal{D}_b)$ به عنوان یک عملگر معکوس است و $\Lambda = \Lambda(\lambda, x, y, \mathcal{D}_a^\alpha)$ یک عملگر خطی شامل سایر عبارات است. به وضوح، عملگر Φ به فرم $\Phi(x, \mathcal{D}_a^\alpha, {}^\beta\mathcal{D}_b) = {}^\beta\mathcal{D}_b p \mathcal{D}_a^\alpha$ انتخاب می گردد. معکوس عملگر Φ را به شکل

$$\Phi^{-1} = \int_a^x \frac{1}{p(t)} {}^\beta I_b(\cdot) \mathrm{d}_\alpha(t,a)$$

معرفی کرده، با بکارگیری قضیه ۱.۲، و اعمال Φ^{-1} در سمت چپ معادله (۹) داریم

$$(\Phi^{-1}\Phi)(y(x)) = y(x) - y(a) - p(b)\mathcal{D}_a^\alpha y(b) \int_a^x \frac{1}{p(t)} \mathrm{d}_\alpha(t,a)$$

، که در آن شرایط اولیه $y(a)$ و $\mathcal{D}_a^\alpha y(b)$ باید معلوم باشند. بنابراین، از معادله (۹) نتیجه خواهد شد

$$y(x) = y(a) + p(b)\mathcal{D}_a^\alpha y(b) \int_a^x \frac{1}{p(t)} \mathrm{d}_\alpha(t,a) + \int_a^x \frac{1}{p(t)} {}^\beta I_b \Lambda(\lambda, t, y(t), \mathcal{D}_a^\alpha y(t)) \mathrm{d}_\alpha(t,a). \tag{۱۰}$$

در روش تجزیه آدومیان، فرض می کنیم که جواب $y(x)$ را می توان با یک سری نامتناهی به شکل $y(x) = \sum_{n=0}^\infty y_n(x)$ نمایش داد، و عبارت Λ را با یک سری نامتناهی از چندجمله ایها

$$\Lambda = \sum_{n=0}^\infty A_n, \tag{۱۱}$$

یک روش کارآمد برای حل مسائل اشتورم-لیوویل کسری تطبیق پذیر

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چکیده

در این مقاله، یک روش کارآمد برای حل مسائل اشتورم -لیوویل کسری تطبیق پذیر را در نظر می گیریم. به کمک روش تجزیه آدومیان مقادیر و توابع ویژه رده ای از معادلات دیفرانسیل کسری تطبیق پذیر را تقریب خواهیم زد. نتایج عددی ارائه در جدول و نمودار نشان دهنده کارایی و سادگی روش بر روی معادلات مورد نظر می باشد.

واژه‌های کلیدی: مسائل اشتورم-لیوویل کسری تطبیق پذیر، مقادیر و توابع ویژه، روش تجزیه آدومیان

رده بندی موضوعی [۲۰۱۰]: ۳۴B۲۴ ،۳۴L۱۰ ،۲۶A۳۳،

۱. مقدمه

اخیراً، در [۴] تعریف جدید از مشتق کسری موضعی ارائه شده است که اکثر خواص مشتقات معمولی را برآورده می کند. این تعریف به مشتق کسری تطبیق پذیر[۱۵] معروف است و یک تکنیک قدرتمند برای حل معادلات دیفرانسیل حاصل از مدلهای فیزیکی مانند معادله انتشار و موج کسری به کمک آن ارائه می شود [۵]. البته مشتق کسری تطبیق پذیر دارای یک سری معایب هم هست که خوشبختانه بسیاری از آنها در [۲] بهبود یافته است. در این مقاله، عملگر کسری زیر را تعریف می کنیم

$$\mathcal{L}_{\alpha,\beta} := {}^{\beta}\mathcal{D}_b\left[p(x)\mathcal{D}_a^{\alpha}\right] + q(x), \tag{۶}$$

حال، مسئله مقدار ویژه به فرم

$$\mathcal{L}_{\alpha,\beta}y(x) - \lambda w_\alpha(x)y(x) = 0, \quad x \in [a,b], \tag{۷}$$

را در نظر می گیریم، که $\mathcal{D}_a^{\alpha}$ و ${}^{\beta}\mathcal{D}_b$ به ترتیب، مشتق کسری تطبیق پذیر چپ، راست از مرتبه $0 < \alpha, \beta \leq 1$ بوده و $\forall x \in [a,b], p(x) \neq 0$ و $w_\alpha(x)$ تابع وزن نامنفی و $q(x)$ تابع پتانسیل می باشد. همچنین توابع w_α، p و q در بازه $[a,b]$ حقیقی مقدار پیوسته اند، و شرایط مرزی عبارتند از:

$$c_1y(a) + c_2\mathcal{D}_a^{\alpha}y(a) = 0, \;\; d_1y(b) + d_2\mathcal{D}_a^{\alpha}y(b) = 0, \tag{۸}$$

با $c_1^2 + c_2^2 \neq 0$ و $d_1^2 + d_2^2 \neq 0$.

فرض کنید $\alpha \in (m-1, m]$، $m \in \mathbb{N}$، در اینصورت تعاریف و قضایای زیر داریم:

تعریف ۱.۱. مشتق کسری تطبیق پذیر چپ تابع $f : [s,\infty) \to \mathbb{R}$ شروع شده از $s \in \mathbb{R}$، از مرتبه α عبارت است از

$$T_s^{\alpha}f(t) = \lim_{\epsilon \to 0} \frac{f^{(m-1)}\left(t + \epsilon(t-s)^{m-\alpha}\right) - f^{(m-1)}(t)}{\epsilon}, \; T_s^{\alpha}f(s) = \lim_{t \to s^+} T_s^{\alpha}f(t), \quad t > s,$$

و $T_s^{\alpha}f(s) = \lim_{t\to s^+} T_s^{\alpha}f(t)$ به شرط وجود حد و $(m-1)$ -بار تابع $f(t)$ در $t > s$ مشتق پذیر باشد. همچنین، مشتق کسری تطبیق پذیرراست تابع $f : (-\infty, s] \to \mathbb{R}$ ختم شده به $s \in \mathbb{R}$، از مرتبه α عبارت است از

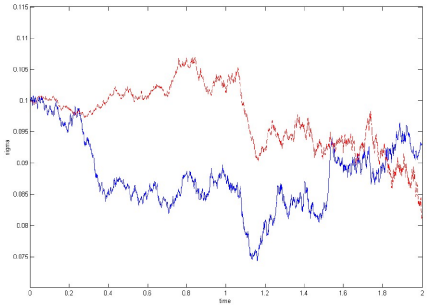
$${}^{\alpha}T_sf(t) = (-1)^m \lim_{\epsilon \to 0} \frac{f^{(m-1)}\left(t + \epsilon(s-t)^{m-\alpha}\right) - f^{(m-1)}(t)}{\epsilon}, \quad t < s,$$

و ${}^{\alpha}T_sf(s) = \lim_{t\to s^-} {}^{\alpha}T_sf(t)$ به شرط وجود حد و $(m-1)$ -بار تابع $f(t)$ در $t < s$ مشتق پذیر باشد.

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[۱۵]Conformable fractional derivative

نمودار زیر شبیه‌سازی یک مسیر از مدل کسری را در مقایسه با مدل غیر کسری ارائه می دهد. مسیر نمونه فرایند تلاطم مبتنی بر OU کسری (قرمز) روند ماندگارتری را نشان می‌دهد، یعنی روند آن در مقایسه با فرایند نوسان مبتنی بر OU معمولی هموارتر است که مطابق با انتظار ماست.



شکل ۳:

۴. ایده‌ای برای توسیع مدل

یک حرکت براونی کسری مختلط [۱۴] ترکیبی خطی از یک حرکت براونی و یک حرکت براونی کسری با شاخص هرست H است، یعنی

$$M_t^H(\alpha,\beta) = \alpha B_t + \beta B_t^H, \quad \circ \le t < \infty,$$

که در آن B_t حرکت براونی استاندارد و B_t^H یک حرکت براونی کسری استاندارد مستقل با شاخص هرست H است و α, β ثابت‌های حقیقی هستند که هر دو صفر نیستند. فرایند $M_t^H(\alpha,\beta)$ یک فرایند گاوسی با $M_\circ^H = \circ$ است. تابع اتوکوواریانس آن عبارت است از

$$\operatorname{cov}\left(M_t^H, M_s^H\right) = \alpha^۲ \min(t,s) + \frac{\beta^۲}{۲}\left(t^{۲H} + s^{۲H} - |t-s|^{۲H}\right).$$

همچنین نموهای این فرایند ایستا و خودمتشابه هستند و بنابراین جایگزین مناسبی برای نویز تصادفی در مدل‌های مالی است که دارای ویژگی‌های حرکت براونی و حرکت براونی کسری می‌باشد.

۵. نتیجه‌گیری

با انگیزه عدم توانایی در نشان دادن ویژگی وابستگی بلند مدت در فرایند تلاطم سنتی، امکان استفاده از حرکت براونی کسری در مدل‌سازی مالی مطرح شد. با شروع از تعریف کلی، fBm را می‌توان به عنوان بسط حرکت براونی معمولی با تابع اتوکوواریانسی در نظر گرفت که به هر دو شاخص زمان بستگی دارد و نه فقط به مینیم آن‌ها. با مقادیر مختلف شاخص هرست، می‌توان حرکت براونی کسری را در سه حالت مختلف بیان کرد: $H < \frac{۱}{۲}, H = \frac{۱}{۲}, H > \frac{۱}{۲}$. از آنجایی که فقط حالت $H > \frac{۱}{۲}$ وابستگی بلند مدت را نشان می‌دهد در مدل‌سازی مالی دارای اهمیت بیشتری است.

مراجع

[1] A. N. Kolmogorov, *Wienersche Spiralen und einige andere interessante Kurvenim Hilbertschen Raum*, C.R. (Doklady) Acad. URSS (N.S), 26 (1940), 115–118.

[2] B.B. Mandelbrot, and J.W. Van Ness, *Fractional Brownian Motions, Fractional Noises and Applications*, SIAM Review, 10 (1968), 422–437.

[3] J. Hull, and A. White, *The Pricing of Options on Assets with Stochastic Volatilities*, Rev. Finan. Studies, 3 (1987), 281–300.

[4] Kijima, and C.M. Tam,*Fractional Brownian motions in financial models and their Monte Carlo simulation.* Theory and Applications of Monte Carlo Simulations, Intech, 2013, 53–85.

[5] F. Comte, and E. Renault, *Long Memory in Continuous-Time Stochastic Volatility Models*, Mathematical Finance, 8 (1998), 291–323.

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[۱۴] mixed fractional Brownian motion

۲. حرکت براونی کسری در حوزه مالی

ابتدا چندین مثال را که از حرکات براونی کسری در حوزه مدل‌سازی مالی استفاده می‌کنند، بیان می‌کنیم.

۱.۲ قیمت‌گذاری دارایی پایه

از حرکت براونی کسری در مدل‌های قیمت‌گذاری دارایی پایه استفاده می‌شود. هر چند که در حالتی که $H \neq \frac{۱}{۲}$، حرکت براونی کسری یک نیم‌مارتینگل پیوسته نخواهد بود و لذا شرط عدم وجود آربیتراژ در حالت کلی نقض می‌شود، اما ثابت می‌شود که اگر زمان بین معاملات صفر نباشد در این صورت کلاس استراتژی‌های سرمایه‌گذاری بدون آربیتراژ می‌شود. با توجه به محدودیت فیزیکی و هزینه غیرصفر معاملات، چنین فرضی در عمل منطقی است.

۲.۲ مدل تلاطم تصادفی

از حرکت براونی کسری همچنین در مدل‌سازی ماندگاری تلاطم نیز استفاده می‌شود که به یک نمونه از این مدل‌ها اشاره می‌کنیم. کمت و رنو[۱۳] [۵] یک مدل تلاطم تصادفی به صورت

$$\begin{cases} \frac{dS(t)}{S(t)} = rdt + \sigma(t)dB(t), \\ \sigma(t) = \sigma_{\circ} e^{x(t)}, \\ dx(t) = -kx(t)dt + vd\hat{B}^H(t), \end{cases}$$

معرفی کردند که در آن دارایی پایه یک حرکت براونی هندسی با تلاطم ماندگار است، زیرا لگاریتم نوسان از یک فرایند اورنشتین-اولنبک (OU) تحت یک حرکت براونی کسری قطع شده پیروی می‌کند. در بخش بعد شبیه‌سازی عددی این مدل تلاطم انجام شده است.

۳. نتایج عددی

در نهایت، در این بخش یک مثال عددی از شبیه‌سازی یک مدل تلاطم fBm در مقایسه با یک مدل تلاطم غیر کسری ارائه می‌شود. این مثال مدل نوسانات تصادفی بررسی شده توسط کمت و رنو را تشریح می کند. فرایند تلاطم تصادفی

$$\begin{cases} \sigma(t) = \sigma_{\circ} e^{x(t)}, \\ dx(t) = -kx(t)dt + vd\hat{B}^H(t), \end{cases}$$

را در نظر بگیرید که در آن v ثابت تلاطم در فرایند لگاریتم نوسان است و $\frac{۱}{۲} < H < ۱$، $k > ۰$، $x(۰) = ۰$. فرایند نوسان فرم نمایی از یک فرایند OU حاصل از یک حرکت براونی کسری است، بنابراین با حل معادله فوق داریم:

$$x(t) = \int_{\circ}^{t} ve^{-k(t-s)} d\hat{B}^H(s).$$

به کمک حسابان تصادفی کسری محاسبه می‌شود:

$$x(t) = \int_{\circ}^{t} a(t-s)dB(s),$$

که در آن $B(t)$ یک حرکت براونی است و

$$a(\theta) = \frac{v}{\Gamma\left(H+\frac{۱}{۲}\right)} \frac{d}{dx} \int_{\circ}^{\theta} e^{-ku}(\theta-u)^{H-\frac{۱}{۲}} du = \frac{v}{\Gamma\left(H+\frac{۱}{۲}\right)} \left(\theta^{\alpha} - ke^{-\theta} \int_{\circ}^{\theta} e^{ku} u^{\alpha} du \right).$$

با گسسته‌سازی معادله فوق نسبت به زمان داریم:

$$\tilde{x}(t) = \sum_{j=۱}^{t_N=t} a\left(t_N - t_{j-۱}\right) \left(B\left(t_j\right) - B\left(t_{j-۱}\right) \right).$$

[۱۳] Comet and Renault

چنین مدل‌هایی در ثبت تقارن و چولگی لبخند نوسان ضمنی با در نظرگرفتن روابط بین حرکات براونی در (۴) موفق هستند. با توجه به اثر مشهود انباشتگی زمانی در (۷)، اثر لبخند با نزدیک شدن به زمان انقضا بدتر می‌شود زیرا انباشتگی زمانی به تدریج اثر لبخند را خنثی می‌کند. در مدل استاندارد نوسانات تصادفی، این تنزل خاص بسیار سریعتر از آنچه در داده‌های بازار مشاهده می‌شود رخ می‌دهد. پدیده تنزل آهسته لبخند نوسان به عنوان ماندگاری نوسان (وابستگی بلند مدت فرایند نوسان) شناخته می‌شود. از طرف دیگر چنین وابستگی در داده‌های مالی به وفور مشاهده شده است. برای جبران این نارسایی، فرایند حافظه بلند مدت جایگزین مناسب‌تری برای نوفه تصادفی معمولی تلقی می‌شود. از این رو، حرکت براونی کسری با توجه به شباهت آن با حرکت براونی معمولی، نامزد اصلی در میان تمام فرایندها با حافظه بلند مدت است: حرکت براونی کسری و حرکت براونی معمولی خود متشابه با ساختار گاوسی مشابه هستند.

تعریف ۱.۱. فرض کنید $H \in (۰, ۱)$. یک حرکت براونی کسری با اندیس هرست H یک فرایند گاوسی پیوسته با تابع کوواریانس زیر است [۴]

$$\mathbb{E}\left[B^H(t)B^H(s)\right] = \frac{۱}{۲}\left(t^{۲H} + s^{۲H} - |t-s|^{۲H}\right).$$

در حالت خاص برای $H = \frac{۱}{۲}$، این فرایند یک حرکت براونی معمولی است و

$$\mathbb{E}\left[B^H(t)B^H(s)\right] = \min(t,s).$$

حرکت براونی کسری به سه گروه تقسیم می‌شود: $H < \frac{۱}{۲}, H = \frac{۱}{۲}, H > \frac{۱}{۲}$، که در ادامه به بیان ویژگی آن می‌پردازیم.

تعریف ۲.۱. یک دنباله ایستا دارای حافظه بلند مدت است اگر تابع اتوکواریانس $\gamma(n) := \operatorname{cov}(X_k, X_{k+n})$ در رابطه زیر به ازای ثابت‌های $c, \alpha \in (۰, ۱)$ صدق کند:

$$\lim_{n\to\infty} \frac{\gamma(n)}{cn^{-\alpha}} = ۱,$$

که در این حالت وابستگی بین X_k, X_{k+n} با افزایش n کاهش می‌یابد و داریم:

$$\lim_{n\to\infty} \frac{\gamma(n)}{cn^{-\alpha}} = ۱.$$

از آنجا که حرکت براونی کسری دارای نموهای ایستا است با تعریف

$$X_k = B^H(k) - B^H(k-۱),$$
$$X_{k+n} = B^H(k+n) - B^H(k+n-۱),$$

داریم:

$$\gamma_H(n) = \frac{۱}{۲}\left[(n+۱)^{۲H} + (n-۱)^{۲H} - ۲n^{۲H}\right],$$

و

$$\lim_{n\to\infty} \frac{\gamma_H(n)}{H(۲H-۱)n^{۲H-۲}} = ۱.$$

بنابراین برای $H > \frac{۱}{۲}$، حرکات براونی کسری وابستگی بلند مدت را نشان می دهد. از آنجایی که ما علاقه‌مند به ثبت پدیده حافظه بلند مدت مشاهده شده در بازارهای مالی هستیم، $H > \frac{۱}{۲}$ را در نظر می‌گیریم.

۲.۱ نمایش به صورت انتگرال تصادفی

در مقاله اصلی [۲]، مندلبروت و ون نس fBm را به صورت انتگرال تصادفی نسبت به حرکت براونی معمولی نشان می‌دهند.

$$B^H(t) = \frac{۱}{\Gamma\left(H+\frac{۱}{۲}\right)}\left(\int_{-\infty}^{۰}\left[(t-s)^{H-\frac{۱}{۲}} - (-s)^{H-\frac{۱}{۲}}\right]dB(s) + \int_{۰}^{t}(t-s)^{H-\frac{۱}{۲}}dB(s)\right),$$

که در آن $\Gamma(\cdot)$ تابع گاما است. معمولا از نمایش قطع شده فوق به صورت زیر استفاده می شود:

$$\hat{B}^H(t) = \int_{۰}^{t}\frac{(t-s)^{H-\frac{۱}{۲}}}{\Gamma\left(H+\frac{۱}{۲}\right)}dB(s).$$

چکیده مبسوط بیست و پنجمین سمینار آنالیز ریاضی و کاربردهای آن
۱۳ تا ۱۵ اردیبهشت ماه ۱۴۰۲، دانشگاه لرستان

حرکت براونی کسری در مدل‌های مالی

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چکیده

امروزه معادلات دیفرانسیل تصادفی کسری با ورود به بازارهای مالی نقش مهمی در توسعه مدل‌ها دارند. در این نوشتار با استفاده از حرکت براونی کسری به عنوان یک نوفه تصادفی که مارکوف نیست و بنابراین می‌تواند دارای حافظه بلند مدت باشد، مدل‌های مالی دینامیک دارایی پایه و تلاطم (نوسان) تصادفی را بررسی می‌کنیم.

واژه‌های کلیدی: حرکت براونی کسری، تلاطم تصادفی، بازار مالی

رده بندی موضوعی [۲۰۱۰]: ۳۴A۰۸ ۶۰G۲۲، ۳۴A۰۸

۱. مقدمه

حرکت براونی کسری[۱۰] (fBm) برای اولین بار توسط کولموگروف [۱] در چهارچوب فضای هیلبرت معرفی شد و پس از مطالعات بیشتر در سال ۱۹۶۸ نام «حرکت براونی کسری» را توسط مندلبروت و ون نس[۱۱] [۲] به خود گرفت. این فرایند تصادفی امروزه به طور گسترده‌ای در زمینه‌های مختلف علمی استفاده می‌شود و اولین بار در حوزه آب شناسی مورد مطالعه قرار گرفت. در سال‌های اخیر، این فرایند در حوزه مالی نیز مورد توجه بسیاری قرار گرفته است، زیرا به نظر می‌رسد مدل تلاطم تصادفی سنتی که توسط حرکت براونی معمولی تعریف می‌شود، دربردارنده نوعی تنزل زمانی هندسی در مدل لبخند نوسان است که بسیار سریع‌تر از تنزل هذلولوی که در داده‌های واقعی بازار رخ می‌دهد، می‌باشد. این ویژگی، ماندگاری نوسان نامیده می‌شود و نشان‌دهنده حافظه بلند مدت فرایند است. fBm به دلیل شباهت آن به حرکت براونی معمولی و ویژگی‌های گاوسی آن انتخاب مناسبی برای مطالعه فرایندهای با حافظه بلند مدت در مدل‌های قیمت‌گذاری است.

در این مقاله، ما حرکت براونی کسری با $H > ۱/۲$ را در چند مدل مالی به کار می‌گیریم، که در آن شاخص هرست، پارامتری است که در ادبیات رایج برای تعمیم حرکت براونی به حرکت براونی کسری مورد استفاده قرار می‌گیرد و اولین بار توسط مندلبروت معرفی شد. ابتدا حرکت براونی کسری را در بخش ۱ معرفی کرده و انگیزه استفاده از آن را در مدل دینامیک دارایی پایه و تلاطم تصادفی در بخش ۲ بیان می‌کنیم. در بخش ۳، کاربرد حرکت براونی کسری در مدل تلاطم تصادفی و شبیه سازی آن مطرح شده است. همچنین، در بخش ۴ مدل توسط نوفه براونی مختلط توسیع داده شده است.

۱.۱ مقدمات و انگیزه‌های مالی

مطالعات تجربی متعددی نشان داده‌اند که در بازارهای اختیار معاملات، تلاطم ضمنی حاصل از معادله بلک–شولز، به اثر لبخند و چولگی تلاطم و فرم U شکل رویه نوسان ضمنی اشاره دارد. برای این منظور هال و وایت[۱۲] [۳] مدل تلاطم تصادفی زیر را معرفی کردند:

$$\begin{cases} \frac{dS(t)}{S(t)} = \mu\,(t, S_t)\,dt + \sigma(t) dB_۱(t), \\ d(\ln \sigma(t)) = k\,(\theta - \ln(\sigma(t))dt + v dB_۲(t))\,. \end{cases} \tag{۴}$$

در اینجا $S(t)$ قیمت دارایی، $\sigma(t)$ نوسان لحظه‌ای در زمان t است و $\{B_۱(t), B_۲(t)\}$ حرکت براونی استاندارد هستند. هال و وایت نشان داده‌اند که قیمت اختیار اروپایی در زمان t و زمان انقضای T، مقدار مورد انتظار شرطی از قیمت بلک–شولز است که در آن نوسان ثابت در فرمول اصلی با میانگین درجه دوم آن در دوره $[t, T]$جایگزین می‌شود:

$$\sigma^۲_{\mathrm{t,T}} = \frac{۱}{T-t}\int_t^T \sigma^۲(u) d(u). \tag{۵}$$

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[۱۰] fractional Brownian motion

[۱۱] Mandelbrot and Van Ness

[۱۲] Hull and White

اثبات. مجموعه‌های $\mathfrak{A}$ و $\mathfrak{B}$ را مانند قضیه ۱.۳ درنظربگیرید، با استفاده از لم ۳.۲ و رابطه (۳) نتیجه می‌شود

$$\begin{aligned} d(A,B) \leq d(\mathfrak{A},\mathfrak{B}) \leq & \lim_{n\to\infty} d(T^{۲n}x, T^{۲n+۱}x) \\ \leq & k_x \lim_{n\to\infty} \max\left\{d(T^{۲n-۱}x, T^{۲n}x), d(T^{۲n}x, T^{۲n+۱}x)\right\} \\ & + (۱-k_x)d(A,B) \\ = & k_x d(A,B) + (۱-k_x)d(A,B) = d(A,B). \end{aligned}$$

پس برای هر $a \in \mathfrak{A}$ و $b \in \mathfrak{B}$ رابطه زیر برقرار است

$$d(Tb,Ta) \leq k_x \max\left\{d(b,a), d(b,Tb), d(a,Ta)\right\} + (۱-k_x)d(\mathfrak{A},\mathfrak{B}).$$

از طرفی از نتیجه ۳.۳ از [۶] دنباله $\{T^{۲n}x\}$ کشی است و چون A کامل است همگرا به نقطه‌ای مانند $x^* \in A$ می‌باشد، پس با استفاده مجدد از لم ۳.۲ و رابطه (۳) می‌توان نتیجه گرفت

$$\begin{aligned} d(x^*,Tx^*) = & \lim_{n\to\infty} d(T^{۲n}x, Tx^*) \\ \leq & \lim_{n\to\infty} k_x \max\left\{d(T^{۲n-۱}x, x^*), d(T^{۲n-۱}x, T^{۲n}x), d(x^*,Tx^*)\right\} \\ & + (۱-k_x)d(A,B) \\ = & k_x\, d(x^*,Tx^*) + (۱-k_x)d(A,B). \end{aligned}$$

پس $d(x^*,Tx^*) = d(A,B)$. برای اثبات یکتایی، فرض کنید $\overline{x}$ بهترین نقطه تقریب دیگری از T در A باشد، ثابت شد x^* یک بهترین نقطه تقریبی T است، از طرفی $\lim_{n\to\infty} d(x^*, T^{۲n+۱}x) = d(A,B)$، درنتیجه چون دوتایی (B,A) دارای خاصیت UC است، می‌توان نتیجه گرفت که دنباله $\{T^{۲n+۱}x\}$ همگرا به Tx^* است. بدون کاستن از کلیت فرض کنید $d(Tx^*,\overline{x}) \leq d(x^*,T\overline{x})$، پس روابط زیر برقرار هستند

$$\begin{aligned} d(x^*,T\overline{x}) = & \lim_{n\to\infty} d(T^{۲n}x, T\overline{x}) \\ \leq & \lim_{n\to\infty} k_x \max\left\{d(T^{۲n-۱}x, \overline{x}), d(T^{۲n-۱}x, T^{۲n}x), d(\overline{x},T\overline{x})\right\} \\ & + (۱-k_x)d(A,B) \\ = & k_x\, d(Tx^*,\overline{x}) + (۱-k_x)d(A,B) \\ \leq & k_x\, d(x^*,T\overline{x}) + (۱-k_x)d(A,B). \end{aligned}$$

درنتیجه $d(x^*,T\overline{x}) = d(A,B)$، از طرفی $d(\overline{x},T\overline{x}) = d(A,B)$ و (A,B) دارای خاصیت UC است پس $x^* = \overline{x}$ است. □

مراجع

[1] CM. Chen, *Fixed point theorems of generalized cyclic orbital Meir-Keeler contractions*, Fixed Point Theory Appl., 91 (2013)., 10 pages. https://doi.org/10.1186/1687-1812-2013-91

[2] E. Karapinar, S. Romaguera and K. Taş, *Fixed points for cyclic orbital generalized contractions on complete metric spaces*, Cent. Eur. J. Math., 11 (2013), 552–560.

[3] S. Karpagam and S., Agrawal, *Best proximity point theorems for cyclic orbital Meir-Keeler contraction maps*, Nonlinear Anal., 74 (2011), 1040–1046.

[4] S. Karpagam and B. Zlatanov, *Best proximity points of p-cyclic orbital Meir–Keeler contraction maps*, Nonlinear Anal., 21(6) (2016), 790–806.

[5] A. Safari-Hafshejani, *Generalizations of Banach's contraction principle and Kannan and Chatterjea's theorems for cyclic and noncyclic mappings*, J. Mahani Math. Res., 12(2) (2023), 349–362.

[6] A. Safari-Hafshejani, A. Amini-Harandi and M. Fakhar, *Best proximity points and fixed points results for non-cyclic and cyclic Fisher quasi-contractions*, Numer. Funct. Anal. Optim., 40(5) (2019), 603–619.

[7] T. Suzuki, M. Kikkawa and C. Vetro, *The existence of best proximity points in metric spaces with the property UC*. Nonlinear Anal., 71 (2009), 2918–2926.

اکنون از قضیه ۲.۲ دنباله $\{T^n x\}$ کشی است و بنابراین $x^* \in A \cup B$ وجود دارد به گونه ای که دنباله $\{T^n x\}$ به آن همگرا است. اما زیر دنباله $\{T^{۲n} x\}$ از $\{T^n x\}$ دنباله‌ای در A و زیردنباله $\{T^{۲n+۱} x\}$ از $\{T^n x\}$ دنباله‌ای در B هستند که هر دو به x^* همگرا هستند. از طرفی A و B هر دو بسته هستند پس $x^* \in A \cap B$. برای اتمام برهان کافی است ثابت شود که x^* نقطه ثابت یکتایی از T است. از رابطه (۲) و قضیه ۲.۲ روابط زیر را می‌توان نتیجه گرفت

$$\begin{aligned} d(x^*, Tx^*) &= \lim_{n\to\infty} d(T^{۲n}x, Tx^*) \\ &\le \lim_{n\to\infty} k_x \max\left\{d(T^{۲n-۱}x, x^*), d(T^{۲n-۱}x, T^{۲n}x), d(x^*, Tx^*)\right\} \\ &= k_x\, d(x^*, Tx^*). \end{aligned}$$

پس $Tx^* = x^*$. برای اثبات یکتایی، فرض کنید $\overline{x}$ نقطه ثابت دیگری از T در $A \cup B$ باشد، دراین‌صورت

$$\begin{aligned} d(x^*, \overline{x}) &= \lim_{n\to\infty} d(T^{۲n}x, T\overline{x}) \\ &\le \lim_{n\to\infty} k_x \max\left\{d(T^{۲n-۱}x, \overline{x}), d(T^{۲n-۱}x, T^{۲n}x), d(\overline{x}, T\overline{x})\right\} \\ &= k_x\, d(x^*, \overline{x}), \end{aligned}$$

که ایجاب می‌کند $\overline{x} = x^*$. □

همانگونه که در اثبات ملاحظه کردید قضیه قبل در واقع نتیجه‌ای از قضیه ۲.۲ می‌باشد. به عنوان نتیجه‌ای از قضیه قبل می توان به تعمیمی از قضیه ۳.۲ از [۳] به شکل زیر اشاره کرد.

نتیجه ۲.۳. *فرض کنید برای خودنگاشت T روی فضای متریک کامل (X, d) عنصر $x \in X$ و عدد ثابت $k_x \in (۰, ۱)$ موجود باشند به گونه ای که برای هر $n \in \mathbb{N}$ و $y \in X$ نابرابری زیر برقرار باشد*

$$d(T^{۲n}x, Ty) \le k_x \max\left\{d(T^{۲n-۱}x, y), d(T^{۲n-۱}x, T^{۲n}x), d(y, Ty)\right\},$$

دراین‌صورت T دارای یک نقطه ثابت یکتا است.

با یک مثال نشان داده خواهد شد که قضیه ۱.۳ قوی‌تر از قضیه ۲.۲ از [۳] است.

مثال ۳.۳. فضای $\mathbb{R}^۲$ مجهز به متر معمولی و زیرمجموعه‌های

$$A = \{(۰, ۰), (۲٫۲۵, ۰), (۲٫۵, ۰), (۴, ۳)\}, \quad B = \{(۰, ۰), (۱, ۰), (۴, ۰)\},$$

از آن‌را درنظربگیرید. چنان‌چه نگاشت دوری $T : A \cup B \to A \cup B$ به صورت زیر تعریف شود

$$\begin{aligned} &T(۴, ۳) = (۴, ۰), T(۲٫۲۵, ۰) = (۱, ۰), T(۲٫۵, ۰) = (۰, ۰), \\ &T(۰, ۰) = (۰, ۰), T(۴, ۰) = (۲٫۲۵, ۰), T(۱, ۰) = (۰, ۰). \end{aligned}$$

با در نظر گرفتن $x = (۴, ۳)$ و $k_x = ۰٫۹۵$، می‌توان چک کرد که رابطه (۲) برقرار است و می‌توان دید که $x^* = (۰, ۰)$ تنها نقطه ثابت T در $A \cap B$ است. همچنین می‌توان بررسی کرد که رابطه (۱) در این مثال برقرار نیست و درنتیجه قضیه ۲.۲ برای این مثال استفاده‌ای نخواهد داشت.

در ادامه این بخش مفهوم یک انقباض مداری را به حالت کلی‌تری که لزوماً دو مجموعه A و B دارای اشتراکی نباشند، تعمیم می‌دهیم و شرایط وجود بهترین نقطه تقریب را برای این کلاس جدید از انقباض‌های دوری بررسی می‌کنیم.

قضیه ۴.۳. *فرض کنید A و B زیرمجموعه‌هایی ناتهی و بسته از فضای متریک کامل (X, d) باشند که (A, B) دارای خاصیت UC است. فرض کنید برای نگاشت دوری $T : A \cup B \to A \cup B$ عنصر $x \in A$ و عدد ثابت $k_x \in (۰, ۱)$ موجود باشند به گونه ای که برای هر $n \in \mathbb{N}$ و $y \in A$ نابرابری زیر برقرار باشد*

$$\begin{aligned} d(T^{۲n}x, Ty) \le k_x \max&\left\{d(T^{۲n-۱}x, y), d(T^{۲n-۱}x, T^{۲n}x), d(y, Ty)\right\} \\ &+ (۱ - k_x) d(A, B). \end{aligned} \tag{۳}$$

دراین‌صورت T دارای حداقل یک بهترین نقطه تقریبی در A است که در صورتی که دوتایی (B, A) نیز دارای خاصیت UC باشد یکتا می‌باشد.

تعریف ۱.۲. [۷] فرض کنید A و B زیرمجموعه‌هایی ناتهی از فضای متریک (X,d) باشند. دراین‌صورت دوتایی (A,B) دارای خاصیت UC است در صورتی که اگر $\{x_n\}$ و $\{x'_n\}$ دنباله‌هایی در A و $\{y_n\}$ دنباله‌ای در B باشند به طوری که

$$\lim_{n\to\infty} d(x_n,y_n)=\lim_{n\to\infty} d(x'_n,y_n)=d(A,B),$$

آن‌گاه

$$\lim_{n\to\infty} d(x_n,x'_n)=0.$$

آن‌ها نشان دادند اگر A و B زیرمجموعه‌هایی ناتهی از فضای متریک (X,d) باشند به گونه ای که $d(A,B)=0$، آن‌گاه دوتایی (A,B) دارای خاصیت UC است. این بخش را با قضیه و لم کاربردی زیر به پایان می بریم.

قضیه ۲.۲. *[۵] فرض کنید A و B زیرمجموعه‌هایی ناتهی و بسته از فضای متریک کامل (X,d) باشند. فرض کنید T یک نگاشت دوری روی $A\cup B$ باشد چنان‌که برای هر $x\in A$ و $y\in B$ نابرابری*

$$d(Tx,Ty)\le c\max\big\{d(x,y),d(x,Tx),d(y,Ty)\big\},$$

که در آن $c\in[0,1)$ عددی ثابت است، برقرار باشد. دراین‌صورت T دارای یک نقطه ثابت یکتا مانند $x^*\in A\cap B$ است که دنباله $\{T^nx_0\}$ به ازای هر $x_0\in A\cup B$ همگرا به آن است.

لم ۳.۲. *[۶] فرض کنید A و B زیرمجموعه‌هایی ناتهی از فضای متریک (X,d) بوده و T یک نگاشت دوری روی $A\cup B$ باشد چنان‌که برای هر $x\in A$ و $y\in B$ نابرابری*

$$d(Tx,Ty)\le c\max\big\{d(x,y),d(x,Tx),d(y,Ty)\big\}+(1-c)d(A,B),$$

که در آن $c\in[0,1)$ عددی ثابت است، برقرار باشد. با فرض $x_0\in A$، برای هر $n\ge 0$ قرار دهید $x_{n+1}:=Tx_n$. دراین‌صورت داریم

$$\lim_{m,n\to\infty} d(x_{2n},x_{2m+1})=d(A,B).$$

۳. تعمیمی از انقباض‌های مداری

این بخش را با ارائه تعمیمی از قضیه ۲.۲ از [۳] آغاز می‌کنیم.

قضیه ۱.۳. *فرض کنید A و B زیرمجموعه‌هایی ناتهی و بسته از فضای متریک کامل (X,d) باشند. فرض کنید برای نگاشت دوری $T:A\cup B\to A\cup B$ عنصر $x\in A$ و عدد ثابت $k_x\in(0,1)$ موجود باشند به گونه ای که برای هر $n\in\mathbb{N}$ و $y\in A$ نابرابری زیر برقرار باشد*

$$d(T^{2n}x,Ty)\le k_x\max\big\{d(T^{2n-1}x,y),d(T^{2n-1}x,T^{2n}x),d(y,Ty)\big\},\qquad(2)$$

دراین‌صورت $A\cap B$ ناتهی است و T دارای یک نقطه ثابت یکتا در $A\cup B$ می‌باشد.

اثبات. فرض کنید $\mathfrak{A}:=\{x,T^2x,T^4x,\cdots\}$ و $\mathfrak{B}:=\{Tx,T^3x,T^5x,\cdots\}$. بدیهی است که $T(\mathfrak{A})\subseteq\mathfrak{B}$ و $T(\mathfrak{B})\subseteq\mathfrak{A}$. با توجه به رابطه (۲) برای هر $n\in\mathbb{N}$ و $a\in\mathfrak{A}$ می‌توان نتیجه گرفت

$$d(T(T^{2n-1}x),Ta)\le k_x\ \max\big\{d(T^{2n-1}x,a),d(T^{2n-1}x,T^{2n}x),d(a,Ta)\big\},$$

بنابراین

$$d(Tb,Ta)\le k_x\max\big\{d(b,a),d(b,Tb),d(a,Ta)\big\},\quad\forall b\in\mathfrak{B},\ \forall a\in\mathfrak{A}.$$

وجود نقاط ثابت و بهترین نقاط تقریب برای کلاس جدیدی از انقباض‌های مداری در فضاهای متریک کامل

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چکیده

در این مقاله ضمن معرفی کلاس جدیدی از انقباض های مداری به عنوان تعمیمی از انقباض مداری تعریف شده توسط کارپاگام و آگراوال به بررسی شرایط وجود نقاط ثابت و بهترین نقاط تقریب برای آنها خواهیم پرداخت. نشان خواهیم داد این نتایج به همراه قضیه کارپاگام و آگراوال حالت خاصی از نتایج اصلی موجود در برخی مقالات مربوط به انقباض های دوری فیشر می باشند.

واژه‌های کلیدی: انقباض مداری، نقطه ثابت، بهترین نقطه تقریب، فضای متریک کامل

رده بندی موضوعی [۲۰۱۰]: ۵۴H۲۵ ۴۷H۱۰،

۱. مقدمه

فرض کنید A و B زیرمجموعه‌هایی از فضای متریک (X,d) باشند. چنان‌چه برای نگاشت دوری $T : A\cup B \to A\cup B$ عنصر $x \in A$ و عدد ثابت $k_x \in (۰,۱)$ موجود باشند به گونه ای که برای هر $n \in \mathbb{N}$ و $y \in A$ نابرابری

$$d(T^{۲n}x, Ty) \leq k_x\, d(T^{۲n-۱}x, y), \qquad (۱)$$

برقرار باشد، نگاشت دوری T یک انقباض مداری[۶] نامیده می‌شود. در سال ۲۰۱۱ [۳]، کارپاگام[۷] و آگراوال[۸] ثابت کردند که اگر A و B زیرمجموعه‌هایی ناتهی و بسته از فضای متریک کامل (X,d) باشند و نگاشت دوری $T : A\cup B \to A\cup B$ یک انقباض مداری باشد، آنگاه $A\cap B$ ناتهی است و T دارای یک نقطه ثابت یکتا در $A\cap B$ است. آن‌ها علاوه بر این ثابت کردند که چنان‌چه برای خودنگاشت T روی فضای متریک کامل (X,d) عنصر $x \in X$ و عدد ثابت $k_x \in (۰,۱)$ موجود باشند به گونه ای که برای هر $n \in \mathbb{N}$ و $y \in X$ نابرابری زیر برقرار باشد

$$d(T^{۲n}x, Ty) \leq k_x d(T^{۲n-۱}x, y),$$

آنگاه T دارای یک نقطه ثابت یکتا در X است. برای مطالعه بیشتر انواعی از انقباض‌های مداری می توانید به [۱–۴] مراجعه کنید.

در این مقاله ضمن معرفی کلاس جدیدی از انقباض های مداری به عنوان تعمیمی از انقباض مداری تعریف شده توسط کارپاگام و آگراوال به بررسی شرایط وجود نقاط ثابت و بهترین نقاط تقریب برای آنها خواهیم پرداخت. نشان خواهیم داد این نتایج به همراه قضیه کارپاگام و آگراوال حالت خاصی از نتایج اصلی موجود در برخی مقالات مربوط به انقباض های دوری فیشر می باشند.

۲. تعاریف و پیش‌نیازها

در سال ۲۰۰۸ سوزوکی[۹] و همکارانش [۷] خاصیت UC را معرفی کردند.

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[۶] *Orbital contraction*
[۷] *Karpagam*
[۸] *Agrawal*
[۹] *Suzuki*

قضیه ۲.۲. *un- توپولوژی روی مشبکه باناخ X مترپذیر است اگر و تنها اگر X یک نقطه شبه درونی داشته باشد.*

اثبات. فرض کنید $e \in X^+$ یک نقطه شبه درونی باشد. برای هر $x, y \in X$ قرار می‌دهیم $d(x,y) = \| \mid x-y \mid \wedge e\|$. برای هر $x, y \in X$، $d(x,y) \geq ۰$ و $d(x,y) = d(y,x)$. اگر $d(x,y) = ۰$، آنگاه $\mid x-y \mid \wedge e = ۰$. چون e یک واحد ضعیف است، بنابراین $\mid x-y \mid = ۰$. لذا $x = y$. همچنین برای هر $x, y, z \in X$ داریم

$$d(x,z) = \| \mid x-z \mid \wedge e\| \leq \| \mid x-y \mid \wedge e\| + \| \mid y-z \mid \wedge e\| = d(x,y) + d(y,z).$$

بنابراین d یک متر روی X تعریف می‌کند. حال برای هر تور (x_α) در X داریم

$$x_\alpha \xrightarrow{un} x \iff \| \mid x_\alpha - x \mid \wedge e\| \to ۰ \iff d(x_\alpha, x) \to ۰.$$

لذا un- توپولوژی روی X مترپذیر است. برعکس، فرض کنید un- توپولوژی مترپذیر باشد و d یک متر سازگار با این توپولوژی باشد. برای هر n، فرض کنید

$$B_{\frac{۱}{n}} = \{x \in X \ : \ d(x, ۰) \leq \frac{۱}{n}\}.$$

چون $B_{\frac{۱}{n}}$ یک همسایگی صفر تحت un- توپولوژی است، لذا $\varepsilon_n > ۰$ و $u_n > ۰$ موجودند که $V_{\varepsilon_n, u_n} \subseteq B_{\frac{۱}{n}}$. فرض کنید $M_n = ۲^n \|u_n\| + ۱$. در این‌صورت $\sum_{n=۱}^{\infty} \frac{u_n}{M_n}$ یک سری همگرا در X است. قرار می‌دهیم $e = \sum_{n=۱}^{\infty} \frac{u_n}{M_n}$. برای هر n، $M_n > ۱$ و $u_n \leq M_n e$. ادعا می‌کنیم که e یک نقطه شبه درونی است. برای این منظور کافی است حکم (ب) قضیه ۱.۲ را ثابت کنیم. فرض کنید برای توری مانند (x_α) در X^+، $x_\alpha \wedge e \xrightarrow{\|.\|} ۰$. n را ثابت در نظر می‌گیریم. با توجه به این‌که

$$x_\alpha \wedge u_n \leq (M_n x_\alpha) \wedge (M_n e) = M_n(x_\alpha \wedge e) \xrightarrow{\|.\|} ۰,$$

نتیجه می‌شود که $x_\alpha \wedge u_n \xrightarrow{\|.\|} ۰$. لذا $\alpha_۰$ موجود است که برای هر $\alpha \geq \alpha_۰$، $\|x_\alpha \wedge u_n\| < \varepsilon_n$. بنابراین $x_\alpha \in V_{\varepsilon_n, u_n}$. لذا $x_\alpha \in B_{\frac{۱}{n}}$. پس $x_\alpha \to ۰$ تحت متر d. لذا طبق فرض $x_\alpha \xrightarrow{un} ۰$. □

نتیجه ۳.۲. *un- توپولوژی روی مشبکه باناخ X یک پایه موضعی شمارا دارد اگر و تنها اگر X یک نقطه شبه درونی داشته باشد.*

اثبات. با توجه به‌اینکه یک فضای برداری توپولوژیکی هاسدورف مترپذیر است اگر و تنها اگر یک پایه موضعی شمارا داشته باشد ([۵، فصل ۱])، حکم از قضیه ۲.۲ به‌دست می‌آید. □

مراجع

[1] Y. Abramovich, C.D. Aliprantis, *An Invitation to Operator Theory*, Vol 50, American Mathematical Society, Providence, RI, 2002.

[2] C. Aliprantis, O. Burkinshaw, *Positive Operators*, Springer, The Netherlands (2006).

[3] Y. Deng, M. O'Brien, V.G. Troitsky, *Unbounded norm convergence in Banach lattices*, Positivity, 21 (2017) 963-974.

[4] K. Kandic, M.A.A. Marabeh, V.G. Troitsky, *unbounded norm topology in Banach lattices*, J. Math. Anal. Appl. 451 (2017) 259-279.

[5] W. Rudin, *Functional Analysis*, International second Editions 1991.

ج) e یک یکانی ضعیف گفته می‌شود هرگاه $B_e = X$؛ بطور معادل برای هر $x \in X^+$، $x \wedge ne \uparrow x$.

طبق [۲، قضیه ۳,۴۶]، B_e بسته نرمی است. لذا $I_e \subseteq \bar{I}_e \subseteq B_e$. بنابراین اگر e یک یکانی قوی باشد، آنگاه یک نقطه شبه درونی است. لذا یک یکانی ضعیف نیز می‌باشد. یادآوری می‌شود که اگر $e \in X^+$، آنگاه I_e تحت نرم

$$\|x\|_e = \inf\{\lambda > 0 \ : \ |x| \leq \lambda e\}$$

با $C(K)$ ایزومتری مشبکه‌ای است که در آن K یک فضای هاسدورف فشرده است. در این حالت e با تابع ثابت 1 متناظر است (به [۱، قضایای ۳,۴ و ۳,۶] مراجعه شود). اگر e یک یکانی قوی در X باشد، آنگاه $I_e = X$؛ به آسانی دیده می‌شود که $\|.\|_e$ با نرم اصلی معادل است؛ این نشان می‌دهد که X با $C(K)$ ایزومورف نرمی و مشبکه‌ای است.

۲. نتایج اصلی

یادآوری می‌شود که برای تور (x_α) در مشبکه برداری X با یکانی ضعیف e، $x_\alpha \xrightarrow{uo} x$ اگر و تنها اگر $|x_\alpha - x| \wedge e \xrightarrow{o} 0$. ما در این‌جا یک حکم مشابه برای همگرایی نرمی بی‌کران و نقاط شبه درونی بررسی می‌کنیم([۴]).

قضیه ۱.۲. *فرض کنید $e \in X^+$. در این‌صورت احکام زیر معادلند:*

الف) e یک نقطه شبه درونی است.

ب) برای هر تور (x_α) در X^+، اگر $x_\alpha \wedge e \xrightarrow{\|.\|} 0$، آنگاه $x_\alpha \xrightarrow{un} 0$.

ج) برای هر دنباله (x_n) در X^+، اگر $x_n \wedge e \xrightarrow{\|.\|} 0$، آنگاه $x_n \xrightarrow{un} 0$.

اثبات. (ب) $\Rightarrow$ (الف) از [۳، لم ۲,۱۱] به‌دست می‌آید. (ج) $\Rightarrow$ (ب) بدیهی است. (الف) $\Rightarrow$ (ج) فرض کنید (ج) برقرار باشد. فرض کنید $x \in X^+$ دلخواه باشد. نشان می‌دهیم که $x \wedge ne \xrightarrow{\|.\|} x$. طبق [۲، قضیه ۱,۷]، کافی است نشان دهیم که $(x - ne)^+ \xrightarrow{\|.\|} 0$. قرار می‌دهیم $u = x \vee e$. در این‌صورت u یک یکانی قوی در X^+ است. لذا فضای فشرده‌ای مانند K موجود است که I_u به عنوان یک مشبکه برداری با $C(K)$ ایزومورف مشبکه‌ای است که در آن u با 1 متناظر است. چون $x, e \in I_u$، لذا $x, e \in C(K)$. با توجه به اینکه $x \vee e = 1$، نتیجه می‌شود که x و e به‌طور همزمان صفر نمی‌شوند. برای هر n تعریف می‌کنیم

$$F_n = \{t \in K \ : \ x(t) \geq ne(t)\}, O_n = \{t \in K \ : \ x(t) > ne(t)\}.$$

در این‌صورت $O_n \subseteq F_n$، O_n باز است و F_n بسته است.
ادعا ۱: $F_{n+1} \subseteq O_n$. در حقیقت، فرض کنید $t \in F_{n+1}$. در این‌صورت $x(t) \geq (n+1)e(t)$. اگر $e(t) > 0$، آنگاه $x(t) > ne(t)$. بنابراین $t \in O_n$. اگر $e(t) = 0$، آنگاه $x(t) > 0$. لذا $t \in O_n$.
طبق لم اوریزون، $z_n \in C(K)$ موجود است به‌طوری که $z_n \leq x$، $0 \leq z_n$ روی F_{n+1}، با x یکی است و خارج از O_n صفر می‌شود. همچنین مشاهده می‌شود که z_n عضوی از X است.
ادعا ۲: $n(z_n \wedge e) \leq x$. فرض کنید $t \in K$. اگر $t \in O_n$، آنگاه $n(z_n \wedge e)(t) \leq ne(t) < x(t)$. اگر $t \notin O_n$، آنگاه $z_n(t) = 0$ و لذا ادعا ثابت می‌شود.
ادعا ۳: $(x - (n+1)e)^+ \leq z_n$. فرض کنید $t \in K$. اگر $t \in F_{n+1}$، آنگاه

$$(x - (n+1)e)^+ \leq x(t) = z_n(t).$$

اگر $t \notin F_{n+1}$، آنگاه $x(t) < (n+1)e(t)$. لذا $(x - (n+1)e)^+(t) = 0$. بنابراین ادعا برقرار است.
از ادعای ۲ نتیجه می‌شود که $0 \leq z_n \wedge e \leq \frac{1}{n}x \xrightarrow{\|.\|} 0$. بنابراین $z_n \wedge e \xrightarrow{\|.\|} 0$. طبق فرض $z_n \xrightarrow{un} 0$. چون برای هر n، $0 \leq (z_n) \leq x$، لذا (z_n) یک دنباله کران‌دار ترتیبی است. پس طبق [۳، گزاره ۲,۴]، $z_n \xrightarrow{\|.\|} 0$. از ادعای ۳ نتیجه می‌شود که $(x - (n+1)e)^+ \xrightarrow{\|.\|} 0$. لذا (الف) برقرار است. □

چکیده مبسوط بیست و پنجمین سمینار آنالیز ریاضی و کاربردهای آن
۱۳ تا ۱۵ اردیبهشت ماه ۱۴۰۲، دانشگاه لرستان

مترپذیری توپولوژی نرمی بی‌کران روی یک مشبکه باناخ

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چکیده

در چند سال اخیر همگرایی نرمی بی‌کران روی یک مشبکه باناخ تعریف و مطالعه شده است و توپولوژی نرمی بی‌کران به‌عنوان یک توپولوژی برداری هاسدورف معرفی شده است. در این مقاله مترپذیری توپولوژی نرمی بی‌کران روی یک مشبکه باناخ را بررسی می‌کنیم.

واژه‌های کلیدی: توپولوژی نرمی بی‌کران، نقطه شبه درونی، مترپذیری

رده بندی موضوعی [۲۰۱۰]: ۴۶A۴۰ ،۴۶B۴۲

۱. مقدمه

در مقالات اخیر همگرایی نرمی بی‌کران روی یک مشبکه باناخ تعریف شده است. این همگرایی را تروستکی، اوبراین و دنگ در [۳] مطالعه کرده و نشان داده اند که این همگرایی به‌عنوان تعمیمی از همگرایی در اندازه روی مشبکه‌های باناخ قابل مشاهده است.
یک مشبکه برداری، فضای برداری به صورت جزیی مرتب X است که برای هر جفت از بردارهای $x, y \in X$ سوپریمم و اینفیمم $\{x, y\}$ هر دو در X موجود باشند. بردار x در مشبکه برداری X یک بردار مثبت گفته می‌شود هرگاه $x \geq 0$. مجموعه تمام بردارهای مثبت X با X^+ نشان داده می‌شود، یعنی $X^+ = \{x \in X \ : \ x \geq 0\}$. یادآوری می‌شود که نرم $\|.\|$ روی یک مشبکه برداری نرم مشبکه‌ای گفته می‌شود هرگاه $|x| \leq |y|$ نتیجه دهد که $\|x\| \leq \|y\|$. مشبکه برداری‌ای که به یک نرم مشبکه‌ای مجهز باشد، مشبکه برداری نرم دار گفته می‌شود. اگر مشبکه برداری نرم دار X تحت این نرم کامل باشد، آنگاه X یک مشبکه باناخ نامیده می‌شود. تور (x_α) در مشبکه برداری نرم دار X همگرای نرمی بی‌کران(un- همگرا) به $x \in X$ گفته می‌شود و با نماد $x_\alpha \xrightarrow{un} x$ نشان داده می‌شود هرگاه برای هر $u \in X^+$، $|x_\alpha - x| \wedge u \xrightarrow{\|.\|} 0$. فرض کنید X یک مشبکه باناخ باشد. برای هر $\varepsilon > 0$ و هر $u \in X^+ \setminus \{0\}$، قرار می‌دهیم

$$V_{\varepsilon,u} = \{x \in X \ : \ \| |x| \wedge u\| < \varepsilon\}.$$

گردایه تمام مجموعه‌هایی به‌شکل فوق یک پایه همسایگی صفر برای یک توپولوژی است که همگرایی تحت این توپولوژی با un- همگرایی معادل است. به این توپولوژی، توپولوژی نرمی بی‌کران (un- توپولوژی) گفته می‌شود. برای جزئیات به [۳] مراجعه شود. یادآوری می‌شود که ایده‌آل اصلی تولید شده توسط بردار x در مشبکه برداری X عبارت است از

$$I_x = \{y \in X \ : \ \exists \lambda > 0 \ ; \ |y| \leq \lambda |x|\}.$$

همچنین باند اصلی تولید شده توسط بردار x در مشبکه برداری X عبارت است از

$$B_x = \{y \in X \ : \ |y| \wedge n|x| \uparrow |y|\}.$$

فرض کنید X یک مشبکه باناخ باشد و $e \in X^+$. یادآوری می‌شود که

الف) e یک یکانی قوی است هرگاه $I_e = X$؛ بطور معادل برای هر $x \in X^+$، $\lambda > 0$ ای موجود باشد به‌طوری که $x \leq \lambda e$.

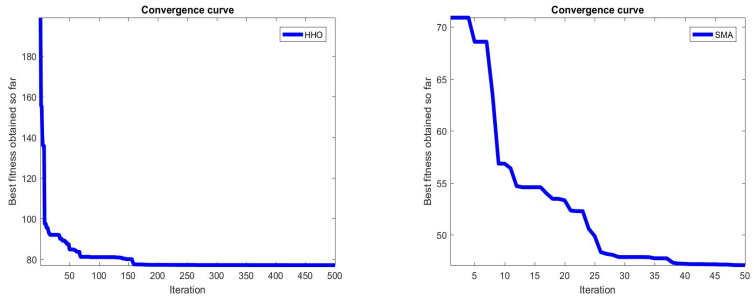
ب) e یک نقطه شبه درونی است هرگاه $\bar{I}_e = X$؛ بطور معادل برای هر $x \in X^+$، $x \wedge ne \xrightarrow{\|.\|} x$.

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جدول ۱: مقایسه عملکرد تابع برازندگی

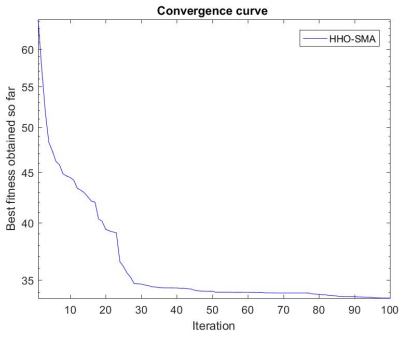
الگوریتم	میانگین تابع برازندگی	تعداد تکرار
SMA	۴۳٫۲۵	۱۰
HHO	۷۵٫۰۱	۱۰
$HHO-SMA$	۳۵٫۶۵	۱۰

شکل‌های ۹ و ۲، عملکرد این سه الگوریتم را نشان می‌دهد که حاکی از بهبود عملکرد الگوریتم پیشنهادی است.



(ب) (الف)

شکل ۱: الف: ارزیابی عملکرد الگوریتم SMA ب: ارزیابی عملکرد الگوریتم HHO



شکل ۲: ارزیابی عملکرد الگوریتم پیشنهادی $HHO-SMA$

۵. نتیجه‌گیری

یکی از چالش‌های اساسی در بحث خوشه‌بندی، یافتن مراکز بهینه داده‌ها می‌باشد تا از گرفتار شدن در بهینه‌های محلی جلوگیری شود. در این مقاله سعی شد با توجه به عملکرد مؤثر روش‌های فراابتکاری در این مورد، روش ترکیبی از دو الگوریتم فراابتکاری HHO و SMA جهت پیداکردن مراکز بهینه جهت خوشه‌بندی داده‌ها ارائه شود که با توجه به انجام آزمایشات و بررسی‌ها، نتایج حاکی از عملکرد مناسب روش پیشنهادی می‌باشد.

مراجع

[1] A.A. Heidari, S. Mirjalili, H. Faris, I. Aljarah, M. Mafarja, and H. Chen, *Harris hawks optimization: Algorithm and applications*, Future generation computer systems, 97 (2019) 849-872.

[2] T. Singh, N. Saxena, M. Khurana, D. Singh, V. Abdalla, and H. Alshazly, *Data clustering using moth-flame optimization algorithm*, *Sensors*, 21(7) (2021).

[3] Li. Shimin, H. Chen, M. Wang, A.A. Heidari and S. Mirjalili. *Slime mould algorithm: A new method for stochastic optimization*, Future Generation Computer Systems, 111 (2020) 300-323.

بهینه‌ساز شاهین هریس HHO

ورودی: جمعیت اولیه و حداکثر تعداد تکرار
خروجی: بهترین موقعیت و مقدار تابع مقصد

۱. مقداردهی اولیه: انرژی خرگوش (E) را بی‌نهایت و موقعیت آن را صفر قرار دهید و موقعیت شاهین‌ها را به طور تصادفی مقداردهی کنید.

۲. مراحل زیر را تا رسیدن به حداکثر تکرار، انجام دهید:

(آ) در صورتی‌که جوابی خارج از فضای جستجو باشد آن را در محدوده فضای جستجو قرار دهید.
(ب) تابع برازندگی موقعیت هر شاهین را محاسبه کنید.
(ج) موقعیت و انرژی خرگوش را در صورت پیدا کردن تابع برازندگی بهتر، به روزرسانی کنید.
(د) اگر $|E| \geq ۱$ مرحله‌ی اکتشاف را اجرا کنید و موقعیت شاهین‌ها را بر اساس مقدار تصادفی متغیر q به‌روزرسانی کنید (معادله (۱) [۱]):
i. اگر $q < ۰/۵$ مکان شاهین‌ها را بر اساس موقعیت سایر شاهین ها و خرگوش به‌روز کنید.
ii. اگر $q \geq ۰/۵$ مکان شاهین‌ها را بر روی درختان بلند بصورت تصادفی به‌روز کنید.
(ه) اگر $|E| < ۱$ مرحله‌ی بهره‌برداری را با توجه به E و مقدار تصادفی متغیرr را اجرا کنید:
i. اگر $r \geq ۰/۵$ و $|E| \geq ۰/۵$ محاصره نرم را اجرا کنید (معادله (۴) [۱]).
ii. اگر $r \geq ۰/۵$ و $|E| < ۰/۵$ محاصره سخت را اجرا کنید (معادله (۶) [۱]).
iii. اگر $r > ۰/۵$ و $|E| \geq ۰/۵$ محاصره نرم با شیرجه‌های پیش‌رونده را اجرا کنید (معادله (۱۰) [۱]).
iv. اگر $r > ۰/۵$ و $|E| < ۰/۵$ محاصره سخت با شیرجه‌های پیش‌رونده را اجرا کنید (معادله (۱۱) [۱]).

۳. شرط پایان آزمایش: اگر شرط پایان برآورده شود، الگوریتم خاتمه می‌یابد. در غیر این صورت، به مرحله ۲ بروید.

پیشنهادی $HHO-SMA$

ورودی: جمعیت اولیه، ابعاد داده‌ها، K تعداد خوشه و حداکثر تعداد تکرار
خروجی: K مرکز بهینه با بعد D

۱. مقداردهی اولیه: مقدار تابع برازندگی و تابع مقصد را بی‌نهایت و بهترین موقعیت‌ها را صفر قرار دهید و K مرکز خوشه برای هر کپک مخاطی را به طور تصادفی اختصاص دهید.

۲. تا زمانی‌که به معیار توقف نرسیده‌اید، گام‌های زیر را انجام دهید:

(آ) روش خوشه‌بندی مبتنی بر SMA را اجرا کنید.
(ب) در صورت یافتن مقادیر بهتر برای تابع برازندگی و بهترین موقعیت‌ها، مقادیر آن‌ها را روزرسانی کنید.
(ج) روش خوشه‌بندی مبتنی بر HHO را اجرا کنید.
(د) در صورت یافتن مقادیر بهتر برای تابع برازندگی و بهترین موقعیت‌ها، مقادیر آن‌ها را به روزرسانی کنید.

۳. مقدار بهترین موقعیت‌ها که همان مراکز بهینه برای خوشه‌بندی است را بازگردانید.

بهینه‌ساز کپک مخاطی SMA
ورودی: جمعیت اولیه و حداکثر تعداد تکرار
خروجی: بهترین موقعیت و مقدار تابع مقصد

۱. مقداردهی اولیه: مقدار تابع برازندگی را بی‌نهایت و بهترین موقعیت‌ها و تابع مقصد را صفر قرار دهید و موقعیت کپک M را به طور تصادفی مقداردهی کنید.

۲. مراحل زیر را تا رسیدن به حداکثر تکرار، انجام دهید:

(آ) در صورتی‌که جوابی خارج از فضای جستجو باشد آن را در محدوده فضای جستجو قرار دهید.

(ب) تابع برازندگی موقعیت هر کپک و وزن آن را محاسبه و آن‌ها را مرتب‌سازی کنید.

(ج) در صورتی‌که بهترین مقدار تابع برازندگی کمتر از مقدار تابع مقصد باشد، مقدار تابع مقصد را برابر با بهترین مقدار تابع برازندگی قرار دهید.

(د) مکان کپک‌ها را با استفاده از یک عدد تصادفی برای جلوگیری از گرفتار شدن در بهینه محلی، به‌روز کنید.

۳. شرط پایان آزمایش: اگر شرط پایان برآورده شود، الگوریتم خاتمه می‌یابد. در غیر این صورت، به مرحله ۲ بروید.

۲.۲ روش بهینه‌سازی شاهین هریس

بهینه‌ساز شاهین هریس، یک الگوریتم بهینه‌سازی مبتنی بر جمعیت است که با الهام از سبک تعقیب پرنده شکارچی، در سال ۲۰۱۹ پیشنهاد شده است. این شکارچی، توانایی‌های مبتکرانه‌ای را در تعقیب در ردیابی، محاصره کردن، بیرون ریختن و در نهایت حمله به طعمه احتمالی را از خود نشان می‌دهد. در این روش، چندین شاهین سعی می کنند به طور مشترک از جهات مختلف به طعمه حمله کنند و به طور همزمان روی یک خرگوش در حال فرار شناسایی شده در خارج از پوشش، همگرا شوند [۱]. در ادامه، مراحل الگوریتم HHO در الگوریتم ۲.۲ شرح داده شده است.

۳. یافتن مراکز بهینه با استفاده از روش پیشنهادی

این روش از خوشه‌بندی مبتنی بر SMA و HHO مشتق شده است. در الگوریتم پیشنهادی موارد زیر در نظر گرفته شده است:

۱. برای هر داده O_i فاصله اقلیدسی آن داده را تا هر مرکز خوشه K_j محاسبه کنید.

۲. در هر مرحله داده O_i را به نزدیک‌ترین خوشه S_k ، $k = \{۱, ۲, ..., K\}$ انتساب دهید.

۳. مقدار تابع برازندگی را به‌صورت زیر محاسبه کنید:

$$F(O_i, K_j) = \sum_{i=۱}^{N} \sum_{j=۱}^{K} w_{ij} \|O_i - K_j\|^۲$$

مراحل الگوریتم پیشنهادی به صورت الگوریتم ۳ می‌باشد:

۴. نتایج عددی

تمام آزمایشات بر روی گل زنبق با ۴ ویژگی انجام گرفته است. در این آزمایشات، از الگوریتم‌های کپک مخاطی و شاهین هریس جهت خوشه‌بندی استفاده شده است. میانگین تابع برازندگی این سه الگوریتم در ده تکرار در یافتن مراکز بهینه در جدول ۴ ارزیابی شده است که نشان از رسیدن به جواب مطلوب‌تر در الگوریتم پیشنهادی است.

چکیده مبسوط بیست و پنجمین سمینار آنالیز ریاضی و کاربردهای آن
۱۳ تا ۱۵ اردیبهشت ماه ۱۴۰۲، دانشگاه لرستان

بهبود خوشه‌بندی با استفاده از ترکیب الگوریتم‌های شاهین هریس و کپک مخاطی

سعیده برخورداری فیروزآبادی[۱،*] ، سید ابوالفضل شاهزاده فاضلی[۲]، جمال زارع‌پور احمدآبادی[۲]، سید مهدی کرباسی[۳]

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[۳] بخش ریاضی کاربردی، دانشکده علوم ریاضی، دانشگاه یزد

چکیده

امروزه الگوریتم‌های مختلفی برای حل مساله خوشه‌بندی، مورد استفاده قرار می‌گیرند. بعضی از این الگوریتم‌ها در مقابل داده‌های بزرگ با احتمال زیاد در بهینه‌های محلی قرار می‌گیرند که اگر به طور مناسب به این مشکل پرداخته نشود، روند خوشه‌بندی با اشکال مواجه می‌شود. به منظور حل مشکل فوق، می‌توان از الگوریتم‌های فراابتکاری برای یافتن مراکز بهینه برای خوشه‌بندی استفاده نمود. در این مقاله جهت دستیابی به جواب بهتر، یک رویکرد ترکیبی با استفاده از دو الگوریتم فراابتکاری شاهین هریس و الگوریتم کپک مخاطی جهت یافتن مراکز بهینه برای خوشه‌بندی داده‌ها، پیشنهاد شده است. نتایج نشان می‌دهد روش پیشنهادی می‌تواند به شکل مطلوبی مراکز بهینه برای خوشه‌بندی را ارائه دهد.

واژه‌های کلیدی: خوشه‌بندی، الگوریتم‌های فراابتکاری، الگوریتم شاهین هریس، الگوریتم کپک مخاطی

رده بندی موضوعی [۲۰۱۰]: ۶۸T۲۰ ۹۱C۲۰ ۷۸M۵۰

۱. مقدمه

خوشه‌بندی یکی از روش‌های مهم کشف دانش در پایگاه داده است [۳]. با توجه به داده‌ها و شرایط مساله، الگوریتم‌های مختلفی برای خوشه‌بندی به کار گرفته می‌شود. یافتن مراکز بهینه در الگوریتم‌های خوشه‌بندی یکی از راه‌کارهای مؤثری است که استفاده از آن به جای مراکز اولیه تصادفی، سبب تسریع در همگرایی و یافتن جواب‌های دقیق‌تر می‌شود. براساس مطالعات، الگوریتم‌های فراابتکاری در کاهش مشکلات فوق، موفق عمل کرده‌اند و استفاده از نقاط قوت آن‌ها می‌تواند بسیار کمک‌کننده باشد که در ادامه به این الگوریتم‌ها می‌پردازیم.

۲. الگوریتم‌های فراابتکاری

الگوریتم فراابتکاری یا متاهیوریستیک [۱] که به آن‌ها الگوریتم های فراتکاملی نیز گفته می‌شود، دسته‌ای از الگوریتم‌هایی هستند که برای حل مسائل بهینه‌سازی به کار برده می‌شوند. الگوریتم های فراابتکاری، با بهره‌گیری از دو مکانیزم مهم و کلیدی بنام اکتشاف [۲] و بهره برداری [۳] توانایی فرار از بهینه محلی و رسیدن به بهینه سراسری را دارند. الگوریتم بهینه‌سازی کپک مخاطی[۴] و الگوریتم شاهین هریس [۵] دو نمونه از این الگوریتم‌ها هستند که در سال‌های اخیر معرفی شده‌اند.

۱.۲ روش بهینه‌سازی کپک مخاطی

بهینه‌ساز کپک مخاطی، از روش‌های فراابتکاری و یک بهینه‌ساز قدرتمند مبتنی بر جمعیت است که با الهام از رفتار هوشمندانه یک موجود تک‌سلولی به نام کپک مخاطی، در سال ۲۰۲۰ معرفی شده است. این روش به دلیل ساختار ساده و همگرایی قابل قبول، مورد توجه بسیاری از محققان قرار گرفته‌است. موجودات تک سلولی در این کپک، با وجودی‌که از قدرت تفکر محروم هستند، اما می توانند به طور هوشمندانه‌ای، در مسیرهای پر پیچ و خم راه خود را پیدا کنند [۳]. در ادامه، مراحل الگوریتم SMA در الگوریتم ۱.۲ شرح داده شده است.

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[۱] Meta Heuristic
[۲] Exploration
[۳] Exploitation
[۴] Slim Mould Algorithm (SMA)
[۵] Harris Hawks Optimization Algorithm (HHO)

مراجع

[1] C.D. Aliprantis, O. Burkinshaw *Locally Solid Riesz Spaces with Applications to Economics*, second ed., American Mathematical Society, Providence, RI, 2003.

[2] Y. Deng, M. O'Brien, V.G. Troitsky *Unbounded norm convergence in Banach lattices*, Positivity (2017) 963-974.

۲. نتایج اصلی

لم ۱.۲. *فرض کنید Ω فضای توپولوژیکی هاسدورف بطورکامل منتظم باشد و $A \subseteq C(\Omega)^+$. در این‌صورت احکام زیر معادلند:*

(الف) $\inf A = ۰$

(ب) برای هر مجموعه‌ی غیر خالی و باز U و هر $\varepsilon \geq ۰$، $t \in U$ و $h \in A$ موجودند به‌طوری‌که $h(t) < \varepsilon$.

(ج) برای هر مجموعه‌ی غیر خالی و باز U و هر $\varepsilon \geq ۰$، مجموعه باز و غیر خالی $V \subseteq U$ و $h \in A$ موجودند به‌طوری‌که h روی V کوچکتر از ε می‌باشد.

اثبات. (ب) $\Rightarrow$ (الف) فرض کنید $inf A = ۰$ و همچنین فرض کنید(ب) برقرار نباشد. در این‌صورت مجموعه‌ی باز و غیرخالی U و $\varepsilon > ۰$ موجود است به‌طوری‌که برای هر $t \in U$ و هر $h \in A$، $h(t) \geq \varepsilon$. چون Ω بطور کامل منتظم است، پس $g \in C(\Omega)^+$ موجود است که در آن $g \leq \varepsilon 1$ و به‌ازای هر $t \in U^c$، $g(t) = ۰$. بنابراین $g \leq A$. با توجه به تعریف اینفیمم داریم $g \leq inf A$. از طرفی چون $\inf A = ۰$، پس $g \leq ۰$ که این تناقض می‌باشد.
(الف) $\Rightarrow$ (ب) فرض کنید $\inf A \neq ۰$. در این‌صورت $g \in C(\Omega)$ موجود است که $۰ < g \leq A$. چون $۰ < g$، لذا به‌ازای هر $a \in \Omega$، $۰ < g(a)$. بنابراین $\varepsilon > ۰$ ای موجود است به‌طوری‌که به‌ازای هر $a \in \Omega$، $\varepsilon < g(a)$. لذا $g(a) \in (\varepsilon, \infty)$. چون g پیوسته است، پس $g^{-۱}(\varepsilon, \infty)$ باز است لذا به‌ازای هر $a \in \Omega$ مجموعه‌ی باز U موجود است به‌طوری‌که f روی V بزرگتر از ε می‌باشد. پس به‌ازای هر $h \in A$، h روی V بزرگتر از ε می‌باشد که این یک تناقض می‌باشد.
(ج) $\Leftrightarrow$ (ب) بدیهی است. □

قضیه ۲.۲. *فرض کنید Ω یک فضای توپولوژیکی هاسدورف بطورکامل منتظم باشد و $(f_\alpha)_{\alpha \in I}$ توری در $C(\Omega)^+$ باشد. در این‌صورت $f_\alpha \xrightarrow{uo} ۰$ اگروتنهااگر برای هر مجموعه‌ی باز و غیر خالی U و هر $\varepsilon > ۰$، مجموعه‌ی باز و غیر خالی $V \subseteq U$ و اندیس $\alpha_۰$ موجود باشد به‌طوری‌که برای هر $\alpha \geq \alpha_۰$ داشته باشیم:*

$$\forall\, t \in V, \quad f_\alpha(t) \leq \varepsilon.$$

اثبات. فرض کنید $f_\alpha \xrightarrow{uo} ۰$. در این‌صورت چون 1 یک یکانی ضعیف برای $C(\Omega)$ است، لذا طبق [۲]، $f_\alpha \wedge 1 \xrightarrow{o} o$. طبق لم ۱.۱، $A \subseteq C(\Omega)^+$ موجود است به‌طوری‌که $\inf A = ۰$ و به‌ازای هر $h \in A$، $\alpha_۰$ ای موجود است به‌طوری‌که به‌ازای هر $\alpha \geq \alpha_۰$، $f_\alpha \wedge 1 \leq h$.
فرض کنید U مجموعه‌ی باز و غیر خالی دلخواهی باشد و همچنین $\varepsilon \in (۰, ۱)$ دلخواه باشد. بنابه لم ۱.۲، مجموعه‌ی باز و غیرخالی $V \subseteq U$ و $h \in A$ موجود است به‌طوری‌که به‌ازای هر $t \in V$، $h(t) < \varepsilon$. چون هر عضو از A بر زیرتوری از $f_\alpha \wedge 1$ تسلط دارد پس $\alpha_۰$ ای وجود دارد که به‌ازای هر $\alpha \geq \alpha_۰$، $f_\alpha \wedge 1 \leq h$. به‌ازای هر $s \in V$، $f_\alpha(s) \wedge 1 \leq h(s) < \varepsilon$ بنابراین به‌ازای هر $s \in V$ و هر $\alpha \geq \alpha_۰$، $f_\alpha(s) < \varepsilon$.

برعکس، فرض کنید برای هر مجموعه‌ی باز و غیرخالی U و هر $\varepsilon > ۰$ مجموعه‌ی باز و غیرخالی $V \subseteq U$ و اندیس $\alpha_۰$ موجود باشد به‌طوری‌که به‌ازای هر $\alpha \geq \alpha_۰$، f_α روی V کوچکتر از ε باشد. چون 1 واحد ضعیف $C(\Omega)$ می‌باشد لذا کافی است نشان دهیم $f_\alpha \wedge 1 \xrightarrow{o} ۰$. فرض کنید مجموعه‌ی باز و غیرخالی U و $\varepsilon > ۰$ دلخواه باشد. با استفاده از فرض، $V \subseteq U$ باز و غیرخالی و همچنین اندیس $\alpha_۰$ موجود است به‌طوری‌که برای هر $\alpha \geq \alpha_۰$ و هر $t \in V$، $f_\alpha(t) < \varepsilon$. فرض کنید $t \in V$ دلخواه و بعد از این ثابت باشد. چون Ω بطور کامل منتظم است پس $g \in C(\Omega)^+$ موجود است به‌طوری‌که $g(t) = ۰$ و g خارج از V برابر 1 می‌باشد. قرار می‌دهیم $h = g \vee \varepsilon 1$. در این‌صورت $h(t) = \varepsilon$. حال ادعا می‌کنیم برای هر $\alpha \geq \alpha_۰$، $f_\alpha \wedge 1 \leq h$. دو حالت در نظر می‌گیریم.
حالت اول: اگر $s \in V$، آنگاه به‌ازای هر $\alpha \geq \alpha_۰$، $(f_\alpha \wedge 1)(s) \leq f_\alpha(s) < \varepsilon \leq h(s)$.
حالت دوم: اگر $s \notin V$، آنگاه به‌ازای هر $\alpha \geq \alpha_۰$، $(f_\alpha \wedge 1)(s) \leq 1(s) = ۱ = g(s) \leq h(s)$.
روند بالا را برای هر مجموعه باز و غیرخالی U و هر $\varepsilon > ۰$ تکرار می‌کنیم. فرض کنید A مجموعه‌ی تمام توابع h ای باشد که با تکرار روند بالا بدست آمده باشد. در نتیجه به‌ازای هر $h \in A$، $\alpha_۰$ ای وجود دارد به‌طوری‌که به‌ازای هر $\alpha \geq \alpha_۰$، $f_\alpha \wedge 1 \leq h$. با توجه به لم ۱.۲، نتیجه می‌شود $\inf A = ۰$. بنابراین با در نظر گرفتن لم ۱.۱، داریم $f_\alpha \wedge 1 \xrightarrow{o} ۰$.

□

بررسی همگرایی ترتیبی بی‌کران در یک مشبکه باناخ

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چکیده

فرض کنید $C(\Omega)$ مشبکه باناخ تمام تابعک‌های خطی و پیوسته از Ω به $\mathbb{R}$ باشد، که در آن Ω یک فضای توپولوژیکی هاسدورف بطور کامل منتظم می‌باشد. در این مقاله، همگرایی ترتیبی بی‌کران در مشبکه باناخ $C(\Omega)$ را مطالعه می‌کنیم.

واژه‌های کلیدی: همگرای ترتیبی بی‌کران، تابعک‌های خطی و پیوسته، بطور کامل منتظم

رده بندی موضوعی [۲۰۱۰]: ۴۶A۴۰ ۴۶B۴۲،

۱. مقدمه

همگرایی ترتیبی بی‌کران در چندین مقاله اخیر توسط گائو به‌کار برده شده است و همچنین این همگرایی به عنوان تعمیمی از همگرایی تقریباً همه‌جا در مشبکه‌های برداری بررسی شده است.
مشبکه برداری (فضای ریس) X یک فضای برداری به صورت جزیی مرتب می‌باشد که نسبت به سوپریمم و اینفیمم هر دو عضو دلخواه از X بسته می باشد، یعنی

$$\forall\, x,y\in X,\quad x\vee y\ \&\ x\wedge y\in X.$$

بردار x در مشبکه برداری X مثبت گفته می‌شود هرگاه $x\geq 0$. همچنین مجموعه‌ی تمام بردارهای مثبت X را با نماد X^+ نشان می‌دهند. به عبارت دیگر

$$X^+=\{x\in X\ :\ x\geq 0\}.$$

تور $(x_\alpha)_{\alpha\in I}$ در مشبکه برداری X همگرای ترتیبی به $x\in X$ گفته می‌شود و با نماد $x_\alpha \xrightarrow{o} x$ نمایش داده می‌شود هرگاه تور دیگری مانند $(y_\gamma)_{\gamma\in\beta}$ موجود باشد به‌طوری‌که $y_\gamma\downarrow 0$ و برای هر γ، α_0 ای موجود باشد به‌طوری‌که برای هر $\alpha\geq\alpha_0$ داشته باشیم $|x_\alpha-x|\leq y_\gamma$.
تور $(x_\alpha)_{\alpha\in I}$ در مشبکه برداری X همگرای ترتیبی بی‌کران به $x\in X$ گفته می‌شود و با نماد $x_\alpha \xrightarrow{uo} x$ نمایش می‌دهند هرگاه برای هر $u\in X^+$، $|x_\alpha-x|\wedge u\xrightarrow{o} 0$. برای جزئیات بیشتر در مورد همگرایی ترتیبی و همگرای ترتیبی بی‌کران به [۱] مراجعه شود.
فضای توپولوژیکی X بطور کامل منتظم گفته می‌شود هرگاه برای هر مجموعه‌ی بسته $C\subseteq X$ و هر $x\in C^c$، تابع پیوسته $f:X\longmapsto[0,1]$ وجود داشته باشد به‌طوری‌که $f(x)=0$ و $f(C)=\{1\}$.

لم ۱.۱. *فرض کنید X یک مشبکه برداری باشد و $(x_\alpha)_{\alpha\in I}$ تور دلخواهی در X باشد. در این‌صورت $x_\alpha\xrightarrow{o}0$ اگروتنهااگر مجموعه‌ی $A\subseteq X^+$ موجود باشد به‌طوری‌که $\inf A=0$ و برای هر $h\in A$، α_0 ای موجود باشد به‌طوری‌که برای هر $\alpha\geq\alpha_0$، $|x_\alpha|\leq g$.*

فرض کنید X یک مشبکه برداری باشد. $e\in X^+$ یکانی ضعیف گفته می‌شود هرگاه باند تولید شده توسط e برابر X باشد. یعنی $B_e=X$، بطور معادل برای هر $x\in X^+$، $x\wedge ne\uparrow x$.

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سخنرانان عمومی

تهدیدها در جنگهای فرسایشی

دکتر مجید اسحاقی

دانشگاه سمنان

چکیده

جنگهای فرسایشی که به عنوان یکی از بازهای با استراتژی پیوسته می باشند، توسط مینارد اسمیت برای مدلسازی رقابت حیوانات جهت انتخاب جفت یا تصاحب قلمرو بیان شده اند. از طرفی تهدیدها موضوعی جداگانه در اقتصاد و سیاست هستند که توسط توماس شیلینگ معرفی شده اند. در دنیای واقعی بیشتر جنگهای فرسایشی در معرض تهدیدهای خاص قرار دارند. در این مقاله انواع تهدیدها در جنگهای فرسایشی مطرح می شوند. تهدید برای بازیکنان، تهدید برای منبع و تهدید برای هردوی بازیکنان و منبع به عنوان سه دسته اصلی تهدیدها در جنگهای فرسایشی مطرح می شوند. تهدیدها برای بازیکنان را مدلسازی کرده و با مثال‌های متنوع انواع این تهدیدها را از هم متمایز می کنیم.

واژه‌های کلیدی: جنگهای فرسایشی، تهدیدها، برانگیختگی رقابتی، نظریه بازیها

رده بندی موضوعی [۲۰۱۰]: ۹۱AXX، ۹۲XX

مراجع:

1. Smith, J Maynard, The Theory of Games and the Evolution of Animal Conflicts, Journal of Theoretical Biology, 1974, 47 (1), 209 221.

2.Topkis, Donald M, Equilibrium Points in Nonzero-Sum n-Person Submodular Games, Siam Journal of Control and Optimization, 1979, 17 (6), 773 787.

3.Wang, Zhongmin, (Mixed) Strategy in Oligopoly Pricing: Evidence from Gasoline Price Cycles Before and Under a Timing Regulation, Journal of Political Economy, 2009, 117 (6), 987 10

4. Schelling, Thomas C, The Strategy of Conflict, Harvard University Press, 1960.

5. Slantchev, Branislav L, The Power to Hurt: Costly Conflict with Completely Informed States, American Political Science Review, 2003, 97 (1), 123 133.

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$$\left\|\left|A^{\frac{\alpha+\beta}{2}}XB^{1-\frac{\alpha+\beta}{2}}+A^{1-\frac{\alpha+\beta}{2}}XB^{\frac{\alpha+\beta}{2}}\right|\right\|$$

$$\leq |\beta-\alpha|^{-1}\left\|\left|\int_\alpha^\beta (A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu)dv\right|\right\|$$

$$\leq \frac{1}{2}\left\|\left|A^\alpha XB^{1-\alpha}+A^{1-\alpha}XB^\alpha+A^\beta XB^{1-\beta}+A^{1-\beta}XB^\beta\right|\right\|$$

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