MONTHLY PROBLEMS IN MATHEMATICS

December 2023 Problems and November Solutions

1. Find all natural numbers n with property that the set $\{1,2,3,\ldots n\}$ can be partioned into three disjoint subsets, such that the sum of the elements in all three subsets are equal.

2. Find all polynomials P with integer coeffients so that if for natural numbers a and b, $a+b$ is a perfect squre, then so is $P(a)+P(b)$.

[Problems 1 and 2 are from the Farsi language book, *Iranian Mathematical Olympiads and International Results-From the beginning through 2021,* by Ebadollah Mahmoodian.]

3. For any positive integer n let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}$$

4. Prove that a finitie group can not be the union of two of its proper subgroups. Does the statement reman true if "two" is replaced by "three"?

5. Consider the system of inequalities, $|z - v_i| + |z - w_i| < r_i,\ i = 1,2,\ldots n$, where z, v_i, w_i are complex numbers, r_is are positive real numbers with $r_i > |v_i - w_i|$, and $n > 3$.

(a) Prove that the system has a solution if and only if each choice of three inequalities has a common solution. Show that the number "three" cannot be replaced by "two".

(b) If k, $3 < k < n$, inequalities are selected at random and tested by the method in (a) and found that they have a common solution, what is the probability that the entire system will have solution?

(*Proposed by Mahmoud Sayrafiezadeh*)

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November 2023 Problems
Solutions Follow

1. Determine all prime numbers p such that the sum of all the divisors of p^4 is a perfect square.

2. Let $n \geq 2$ be a positive integer and let A be an $n \times n$ matrix with real entries such that $A^2 = -I_n$. If B is an $n \times n$ matrix with real entries and $AB = BA$, prove that $\det B \geq 0$.

3. Denote by G the group $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/36\mathbb{Z}$ and let $f: G \to G$ be thehomomorphism given by $f(g) = 78g$ for all $g \in G$. Find the cardinalities of the kernel and of the image of f.

(*Above problems are proposed by Henry Ricardo*)

4. Find a closed-form expression for $\sum_{k=0}^{n} \sin(k)$

5. Pierre Varignon in a paper published in 1731 proved that the mid points of the sides of an arbitrary quadrilateral form a parallelogram. Prove that if the Varignon parallelogram turns out to be a rectangle, then it will remain a rectangle under any transformation of the quadrilateral that preserves the lengths and the order of its sides.
(*Proposed by Mahmoud Sayrafiezadeh*)

Solutions to November 2023 Problems

Problem 1. Determine all prime numbers p such that the sum of all the divisors of p^4 is a perfect square.

Solution to problem 1 by Henry Ricardo

1. Since p is prime, the only divisors of p^4 are $1, p, p^2, p^3$, and p^4. Therefore we want a solution (p,n) of the Diophantine equation $n^2 = 1 + p + p^2 + p^3 + p^4$, where p is prime and n is a positive integer. We will prove that the only prime p satisfying this condition is **3**.

   First of all, numerical experimentation reveals that $(p,n) = (3,11)$ is a solution: $1 + 3 + 3^2 + 3^3 + 3^4 = 4 + 9 + 27 + 81 = 121 = 11^2$. Next we note that $p = 2$ does not yield a solution. If $p > 3$ is an odd prime, then

$$\left(p^2 + \frac{1}{2}p - \frac{1}{2}\right)^2 \le p^4 + p^3 + p^2 + p + 1 \le \left(p^2 + \frac{1}{2}p + \frac{1}{2}\right)^2.$$

But this says that $p^4 + p^3 + p^2 + p + 1$ lies between two consecutive squares and so cannot be a perfect square. Therefore, $p = 3$ is the only prime that gives a solution.

Problem 2. Let $n \ge 2$ be a positive integer and let A be an $n \times n$ matrix with real entries such that $A^2 = -I_n$. If B is an $n \times n$ matrix with real entries and $AB = BA$, prove that $\det B \ge 0$.

Solution to problem 2

No solution has been received to problem 2.

Problem 3. Denote by G the group $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/36\mathbb{Z}$ and let $f\colon G \to G$ be the homomorphism given by $f(g) = 78g$ for all $g \in G$. Find the cardinalities of the kernel and of the image of f.

Solution 1 to problem 3 by Raymond Thomas

$Ker(f) = \{(a,b,c) \in \mathbb{Z}_5 \times \mathbb{Z}_{10} \times \mathbb{Z}_{36} : (78a, 78b, 78c) = (0,0,0)\}$ thus, we must solve the equation $78x = 0$ in $\mathbb{Z}_5$, $\mathbb{Z}_{10}$, and $\mathbb{Z}_{36}$.

In $\mathbb{Z}_5$, $78x = 0 \Leftrightarrow 3x = 0 \Leftrightarrow x \in \{0\}$

In $\mathbb{Z}_{10}$, $78x = 0 \Leftrightarrow 8x = 0 \Leftrightarrow x \in \{0, 5\}$

In $\mathbb{Z}_{36}$, $78x = 0 \Leftrightarrow 6x = 0 \Leftrightarrow x \in \{0,6,12,18,24,30\}$

Hence $Ker(f) = \{0\} \times \{0,5\} \times \{0,6,12,18,24,30\}$, and the cardinality of $Ker(f)$ is therefore 1(2)(6) = **12.** Since $Im(f)$ is isomorphic to $G/Ker(f)$, we have that the cardinality of $Im(f) = \frac{card(G)}{card(Ker(f))} = \frac{5(10)(36)}{12} = \mathbf{150}$

Solution 2 to problem 3 by Henry Ricardo

Let $g=(g_1,g_2,g_3)$ be an element of G with $g_1\in\mathbb{Z}/5\mathbb{Z}$, $g_2\in\mathbb{Z}/10\mathbb{Z}$, and $g_3\in\mathbb{Z}/36\mathbb{Z}$.

The element g is in the kernel of f if and only $f(g)=78(g_1,g_2,g_3)=(3g_1,-2g_2,6g_3)=(0,0,0)$. In other words, $f(g)=(0,0,0)$ if and only if $g_1\equiv 0 \pmod 5$, $g_2\equiv 0 \pmod 5$, and $g_3\equiv 0 \pmod 6$. Therefore, we conclude that $\mathrm{Ker}(f)$ has $1\cdot 2\cdot 6=12$ elements. This implies that $\mathrm{Im}(f)\,;\ G/\mathrm{Ker}(f)$ has $|G|/|\mathrm{Ker}(f)|=5\cdot 10\cdot 36/12=150$ elements.

Problem 4. Find a closed-form expression for $\sum_{k=0}^{n}\sin(k)$

Solution 1 to problem 4 by Henry Ricardo

We invoke the *product-to-sum formula* $2\sin\theta\sin\varphi = \cos(\theta-\varphi) - \cos(\theta+\varphi)$ to prove a general result:

$$\sum_{k=0}^{n}\sin(kx) = \sum_{k=1}^{n}\sin(kx)$$

$$= \sum_{k=1}^{n}\frac{2\sin\left(\frac{x}{2}\right)\cdot\sin(kx)}{2\sin\left(\frac{x}{2}\right)}$$

$$= \frac{\sum_{k=1}^{n}\left[\cos\left(kx-\frac{x}{2}\right) - \cos\left(kx+\frac{x}{2}\right)\right]}{2\sin\left(\frac{x}{2}\right)}$$

$$= \frac{\cos\left(\frac{x}{2}\right) - \cos\left(n+\frac{1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)},\quad x\neq 2m\pi,\ m\in\mathbb{Z}.$$

Taking $x=1$, we see that

$$\sum_{k=0}^{n}\sin(k) = \frac{\cos\left(\frac{1}{2}\right) - \cos\left(n+\frac{1}{2}\right)}{2\,\sin\left(\frac{1}{2}\right)} = \frac{\sin\left(\frac{n}{2}\right)\sin\left(\frac{n+1}{2}\right)}{\sin\left(\frac{1}{2}\right)}.$$

Solution 2 to problem 4 by Raymond Thomas

$$\sum_{k=0}^{n}\sin(k) = \mathrm{Im}\,(\sum_{k=0}^{n}e^{ik}) = \mathrm{Im}\left(\frac{e^{i(n+1)}-1}{e^{i}-1}\right) = \mathrm{Im}\left[\left(\frac{e^{i(n+1)}-1}{e^{i}-1}\right)\cdot\frac{e^{-i}-1}{e^{-i}-1}\right]$$

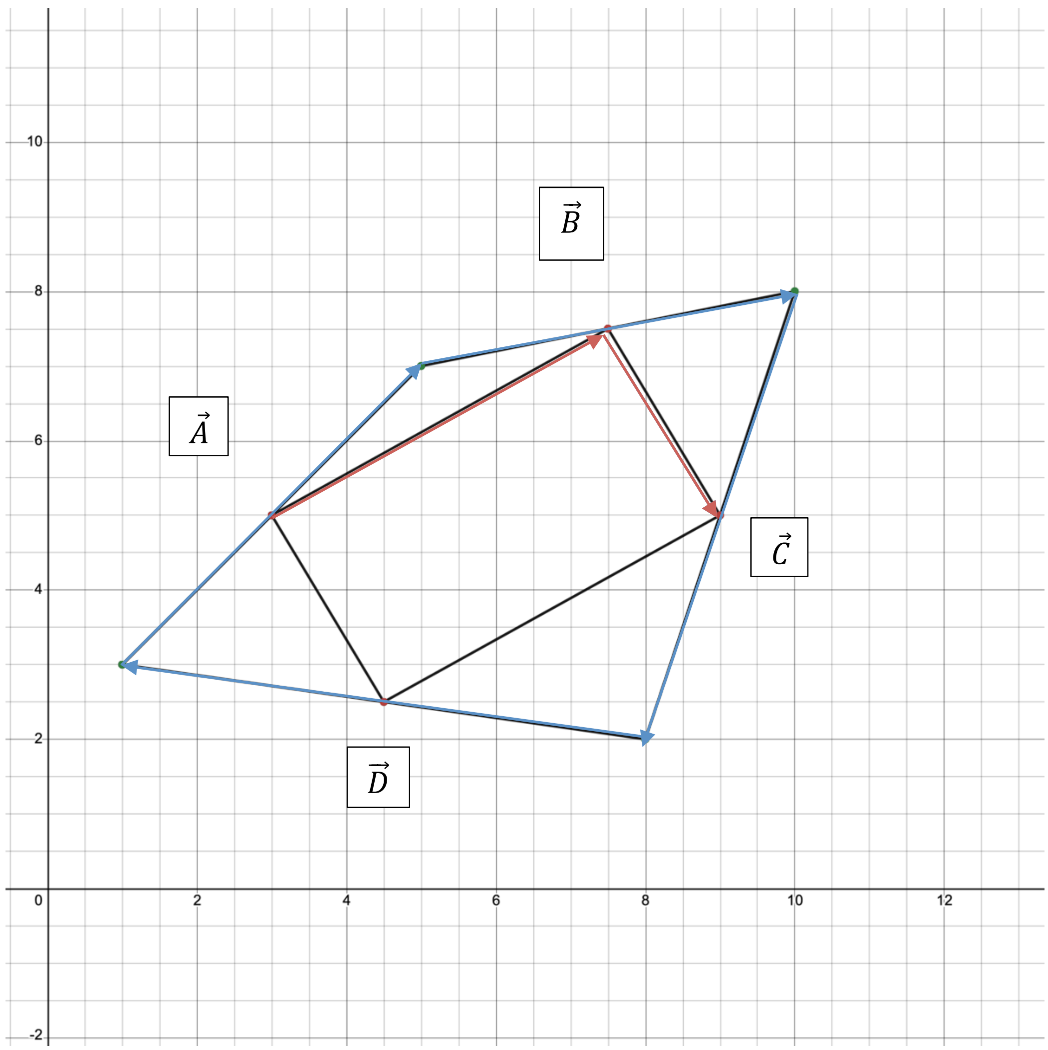
$$= \mathrm{Im}\left[\frac{e^{in}-e^{i(n+1)}-e^{-i}+1}{1-e^{i}-e^{-i}+1}\right] = \mathrm{Im}\left[\frac{\cos(n)+i\sin(n)-\cos(n+1)-i\sin(n+1)-\cos(1)+i\sin(1)+1}{1-\cos(1)-i\sin(1)-\cos(1)+i\sin(1)+1}\right]$$

$$=\frac{\sin(n)-\sin(n+1)+\sin(1)}{2-2\cos(1)}$$

Problem 5. Pierre Varignon in a paper published in 1731 proved that the mid points of the sides of an arbitrary quadrilateral form a parallelogram. Prove that if the Varignon parallelogram turns out to be a rectangle, then it will remain a rectangle under any transformation of the quadrilateral that preserves the lengths and the order of its sides.

Solution 1 to problem 5 by Raymond Thomas

The following picture is meant only as an aid in following the proof:



Let the sides of the quadrilateral be the vectors $\vec{A}, \vec{B}, \vec{C},$ and $\vec{D}$, then we have

$\vec{A}+\vec{B}+\vec{C}+\vec{D}=\vec{0}$, the zero vector. Furthermore, the two non-parallel sides

of the Varignon parallelogram are given by $\frac{1}{2}\vec{A}+\frac{1}{2}\vec{B}$ and $\frac{1}{2}\vec{B}+\frac{1}{2}\vec{C}$ so they are

parallel to the diagonals of the quadrilateral: $\vec{A}+\vec{B}$, and $\vec{B}+\vec{C}$. We immediately deduce that the Varignon parallelogram is a rectangle if and only if the diagonals of the quadrilateral are perpendicular i.e. iff $(\vec{A}+\vec{B})\cdot(\vec{B}+\vec{C})=0$ where the dot is the usual vector dot product.

We can now rephrase the problem at hand as follows:

Given vectors $\vec{A},\vec{B},\vec{C}$, and $\vec{D}$ such that $\vec{A}+\vec{B}+\vec{C}+\vec{D}=\vec{O}$ and $(\vec{A}+\vec{B})\cdot(\vec{B}+\vec{C})=0$,

Show that if $\vec{a},\vec{b},\vec{c}$, and $\vec{d}$ are such that $|\vec{a}|=|\vec{A}|,|\vec{b}|=|\vec{B}|,|\vec{c}|=|\vec{C}|,|\vec{d}|=|\vec{D}|$ and $\vec{a}+\vec{b}+\vec{c}+\vec{d}=\vec{O}$, then $(\vec{a}+\vec{b})\cdot(\vec{b}+\vec{c})=0$

Proof: Since $|\vec{d}|=|\vec{D}|$ we have $|\vec{a}+\vec{b}+\vec{c}|^2=|\vec{A}+\vec{B}+\vec{C}|^2$

or $(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=(\vec{A}+\vec{B}+\vec{C})\cdot(\vec{A}+\vec{B}+\vec{C})$

or $|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\vec{a}\cdot\vec{b}+2\vec{a}\cdot\vec{c}+2\vec{b}\cdot\vec{c}=|\vec{A}|^2+|\vec{B}|^2+|\vec{C}|^2+2\vec{A}\cdot\vec{B}+2\vec{A}\cdot\vec{C}+2\vec{B}\cdot\vec{C}$

Now noting that $|\vec{a}|^2=|\vec{A}|^2,|\vec{b}|^2=|\vec{B}|^2$, and $|\vec{c}|^2=|\vec{C}|^2$, we conclude that

$$\boxed{2\vec{a}\cdot\vec{b}+2\vec{a}\cdot\vec{c}+2\vec{b}\cdot\vec{c}=2\vec{A}\cdot\vec{B}+2\vec{A}\cdot\vec{C}+2\vec{B}\cdot\vec{C}} \qquad (*)$$

We now compute $(\vec{a}+\vec{b})\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+|\vec{b}|^2+\vec{b}\cdot\vec{c}$

$=\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}+|\vec{B}|^2+\vec{B}\cdot\vec{C}$ by (*) and b/c $|\vec{b}|^2=|\vec{B}|^2$

$=(\vec{A}+\vec{B})\cdot(\vec{B}+\vec{C})$

$=0$ by hypothesis.

This completes the proof.

Solution 2 to problem 5 by Henry Ricardo

First we see that

$$(\boldsymbol{u}+\boldsymbol{v})\bullet(\boldsymbol{u}+\boldsymbol{x})=\boldsymbol{u}\bullet\boldsymbol{u}+\boldsymbol{u}\bullet\boldsymbol{x}+\boldsymbol{v}\bullet\boldsymbol{u}+\boldsymbol{v}\bullet\boldsymbol{x}. \qquad (*)$$

Now suppose that $\|\boldsymbol{u}\|^2+\|\boldsymbol{w}\|^2=\|\boldsymbol{v}\|^2+\|\boldsymbol{x}\|^2$, or $\|\boldsymbol{w}\|^2=-\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2+\|\boldsymbol{x}\|^2$. Since $\boldsymbol{w}=-(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{x})$, we have

$$
\begin{aligned}
\|\boldsymbol{w}\|^2 &= \boldsymbol{w} \bullet \boldsymbol{w} = (\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{x}) \bullet (\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{x}) \\
&= \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 + \|\boldsymbol{x}\|^2 + 2(\boldsymbol{u}\bullet\boldsymbol{v} + \boldsymbol{u}\bullet\boldsymbol{x} + \boldsymbol{x}\bullet\boldsymbol{v}) \\
&= -\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 + \|\boldsymbol{x}\|^2 + 2(\boldsymbol{u}\bullet\boldsymbol{u} + \boldsymbol{u}\bullet\boldsymbol{v} + \boldsymbol{u}\bullet\boldsymbol{x} + \boldsymbol{x}\bullet\boldsymbol{v}).
\end{aligned}
$$

Thus $\boldsymbol{u}\bullet\boldsymbol{u} + \boldsymbol{u}\bullet\boldsymbol{v} + \boldsymbol{u}\bullet\boldsymbol{x} + \boldsymbol{x}\bullet\boldsymbol{v} = 0$—that is, looking at **(*)**, $\boldsymbol{u}+\boldsymbol{v}$ is orthogonal to $\boldsymbol{u}+\boldsymbol{x}$.

On the other hand, if $(\boldsymbol{u}+\boldsymbol{v}) \bullet (\boldsymbol{u}+\boldsymbol{x}) = 0$, then

So $\|\boldsymbol{u}\|^2 + \|\boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + \|\boldsymbol{x}\|^2$.

[This appeared as part of the solution to “Quickie” 630 in *Mathematics Magazine* (Vol **48** (1975), pp. 295 and 303—a problem about a skew quadrilateral proposed by Murray Klamkin and Mahmoud Sayrafiezadeh—and appears as an exercise in my text *A Modern Introduction to Linear Algebra*.]