$$I=\int_{\frac{1}{2}}^{2}\frac{\tan^{-1}(x)}{x^2-x+1}dx \quad , \quad \tan^{-1}(x)+\tan^{-1}\left(\frac{1}{x}\right)=\frac{\pi}{2} \quad (x>0)$$

Solution:

$$\begin{cases} x=\frac{1}{t} \Rightarrow dx=-\frac{1}{t^2}dt \\ x=2 \Rightarrow t=\frac{1}{2} \text{ , } x=\frac{1}{2} \Rightarrow t=2 \end{cases}$$

$$I=\int_{2}^{\frac{1}{2}}\frac{\tan^{-1}\left(\frac{1}{t}\right)}{\frac{1}{t^2}-\frac{1}{t}+1}\cdot\left(-\frac{1}{t^2}dt\right)=\int_{\frac{1}{2}}^{2}\frac{\tan^{-1}\left(\frac{1}{t}\right)}{t^2-t+1}dt$$

$$\begin{cases} I=\int_{\frac{1}{2}}^{2}\frac{\tan^{-1}(x)}{x^2-x+1}dx \\ I=\int_{\frac{1}{2}}^{2}\frac{\tan^{-1}\left(\frac{1}{x}\right)}{x^2-x+1}dx \end{cases} \overset{+}{\Rightarrow} 2I=\int_{\frac{1}{2}}^{2}\frac{\tan^{-1}(x)+\tan^{-1}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

$$2I=\frac{\pi}{2}\int_{0}^{\frac{3}{2}}\frac{1}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}du \Rightarrow I=\frac{\pi}{4}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\left[\tan^{-1}\left(\frac{x}{\frac{\sqrt{3}}{2}}\right)\right]_{0}^{\frac{3}{2}}$$

$$I=\frac{\pi}{2\sqrt{3}}\tan^{-1}\left(\frac{3}{\sqrt{3}}\right)=\frac{\pi}{2\sqrt{3}}\tan^{-1}(\sqrt{3})=\frac{\pi}{2\sqrt{3}}\left(\frac{\pi}{3}\right)=\frac{\pi^2}{6\sqrt{3}}$$

$$I=\int_{\frac{1}{2}}^{2}\frac{\tan^{-1}(x)}{x^2-x+1}dx=\frac{\pi^2}{6\sqrt{3}}$$

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