In the Name of God

دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

Proceedings of the 9^{th} Seminar on Harmonic Analysis and Applications

27-28 January 2022

Amirkabir University of Technology (Tehran Polytechnic), Iran

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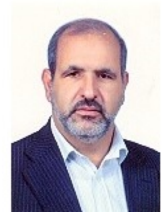

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The organizers of the seminar wish to thank all people of Mathematics and Computer Science Department for their help.

Day 1: Thursday 27 January 2022

Opening Ceremony **08:00-10:00**	
1. Moments with the Holy Quran **2. National Anthem of the Islamic Republic of Iran** **3. A short clip about Amirkabir University of Technology** **4. Speech by the Head of the Seminar** **5. Speech by the Chair of the Department** **6. Speech by the President of the Iranian Mathematical Society (IMS)** **7. Commemoration of the Deceased Professor Mahmood Lashkarizadeh Bami**	
Plenary Talk **10:00-11:00**	
A. Arefijamal	A look at current topics in frame theory
11:00-11:20	
V. Khodakarami	The first Hochschild cohomology groups of Banach algebras with coefficients in special Banach A-bimodules
Z. Hamidi	Fiberization method for identification of Gabor frames
M. Imanfar	Primitive ideals of ultragraph algebras
11:20-11:40	
A. Jabbari	Inner amenability of transformation groups
A. Askarizadeh	On exact and woven g-frames
K. Oustad	On character amenability of weighted convolution algebras
11:40-12:00	
E. Nasrabadi	Characterization of 2-cocycles and 2-coboundaries on direct sum of Banach algebras
H. Ghasemi	Continuous frames and Riesz basis in Hilbert C*-modules
B. Hayati	Approximately module homomorphisms

Launch at Home	
13:40-14:00	
A. Ebadian	Some results on matrix valued group algebras and spaces
J. Cheshmavar	Study on operator representation of frames in Hilbert spaces
E. Tamimi	On l^1-Munn algebras and Connes amenability
14:00-14:20	
H. Lakzian	Biprojectivity and biflatness of bi-amalgamated Banach algebras
F. Azizi	Reconstruction of tomographic images using deep learning and shearlet transform
F. Olyani Nezhad	Orthogonality preserving pairs
Coffee at Home	
Invited Talks	
A. Bagheri Bardi **14:30-15:20**	Operator Valued Measurable Functions
Coffee at Home	
G. Kutyniok **15:30-16:20**	Harmonic Analysis Meets Artificial Intelligence
Coffee at Home	
C. Cabrelli **16:30-17:20**	Dynamical Sampling and Orbits of Operators
Coffee at Home	
I. Todorov **17:30-18:20**	Herz-Schur Multipliers of Dynamical Systems
Coffee at Home	
N. Spronk **18:30-19:20**	Eberlein-de Leeuw-Gicksberg Decompositions For Fourier Stieltjes Algebras
19:20	**Dinner at Home with Your Family**

Day 2: Friday 28 January 2022

09:00-10:00	
M. Rostami	Operator spaces with applications to abstract harmonic analysis
10:00-10:20	
E. Ghaderi	(σ,τ)-Amenability of A×B
H. Keshavarzi	Characterization of forward, vanishing, and reverse Bergman Carleson measures using sparse domination
K. Oustad	Pseudo-amenability of weighted semigroup algebras
10:20-10:40	
A. Alinejad	The notion of quasi-multipliers on topological algebraic Structure
F. Esmaeelzadeh	The irreducible representations on generalized Weyl- Heisenberg groups
A. Hosseini	Diagonal-preserving of C*-algebras
10:40-11:00	
A. Khosravi	Orthogonally additive homogeneous polynomials on the second dual of a Banach algebra
S. S. Jafari	Some remarks on amenable action on the predual of a W*-algebra
M. Jaddi	Fatou's lemma and reverse Fatou's lemma for pseudo-integrals
Coffee at Home	
11:20-11:40	
A.Zivarikazempour	On zero product determined Banach algebras
H. Hosseinnezhad	Representation of the inverse of generalized multipliers in Hilbert C*-modules
A. Dianatifar	Pointwise eventually nonexpansive actions of amenable semigroups in dual Banach spaces and fixed points
11:40-12:00	
M. Ghasemi	Left centralizers on the θ−Lau product of Banach algebras

F. Olyani Nezhad	ε-Orthogonality preseving of Hilbert C*-modules
A. Askarizadeh	Woven and P-woven frames
12:00-12:20	
A. Shirinkalam	Two Banach algebras associated with a locally compact groupoid
S. S. Razavi	Applications on Integral equations in C*-algebra-valued S_b-metric spaces
A. Akrami	A note on (weak) phase retrievable real Hilbert space frames
12:20-12:40	
E. Tamimi	ϕ-Connes module amenability of semigroup algebras
A. Reza	Convolution and convolution type C*-algebras
Launch at Home	
Invited Talks	
M. Hormozi **15:00-15:50**	A Journey Into Real Harmonic Analysis
Coffee at Home	
H. Mousavi **16:00-16:50**	Fixed Scale Improving Inequalities Over Averages Along the Prime Numbers
Coffee at Home	
Y. Choi **17:00-17:50**	Amenable Algebras of Operators on Hilbert Spaces
Coffee at Home	
E. Samei **18:00-18:50**	Spectral Properties of Group Algebras and Their Relations with Amenability
18:50-19:20	**Closing Ceremony**
19:20	**Dinner at Home with Your Family**

سخنرانی‌های عمومی

A Look at Current Topics in Frame Theory

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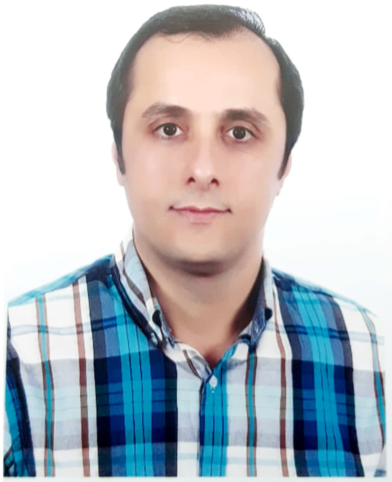
Abstract. In this talk we survey recent results, as well as related results in the theory of frame. Traditionally, dual frames have been used to represent every element of underlying Hilbert space as a linear combination of the frame elements (reconstruction formulas). Finding an optimal dual frame which minimizes the reconstruction errors when erasures occur is a classical problem in frame theory. In some problems in distributed signal processing and wireless sensor networks we need to recover the elements of underlying Hilbert space by the frame coefficients of a pair of frames, no matter which kind of frame coefficients has been made at each sensor. Such frames, are called woven frames, have potential applications in several directions and can replace a pair of dual frames. Phase retrieval is an old problem in signal processing and has been studied for over 100 years by electrical engineers. In the setting of frame theory, the concept of phaseless reconstruction was introduced in 2006 and developed by many researchers.
Keywords: Frames, Optimal dual frames, Weaving of frames, Phase retrieval.
AMS Mathematical Subject Classification [2010]: 42C15, 15A29.



Operator Spaces with Applications to Abstract Harmonic Analysis

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Abstract. In this note wa give a brief introduction to the notion of operator spaces and its basic ideas for which is a refinement of Banach space theory. This is a power tool in the study of quantum mathematics and quantum physics. Some tensor product of operator spaces is investigated. After that some cohomological properties of Banach algebras are described. At the end, we consider some applications in abstract harmonic analysis on locally compact groups.
Keywords: Operator spaces, Amenability, Fourier algebras.
AMS Mathematical Subject Classification [2010]: 46L07, 43A10, 43A30.

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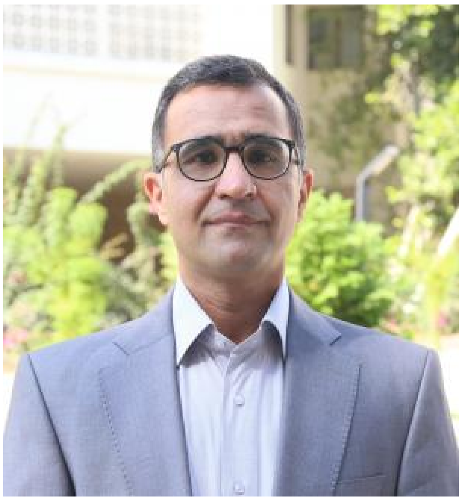

Operator Valued Measurable Functions

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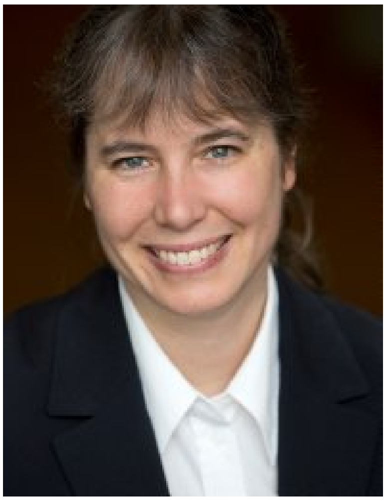

Harmonic Analysis Meets Artificial Intelligence

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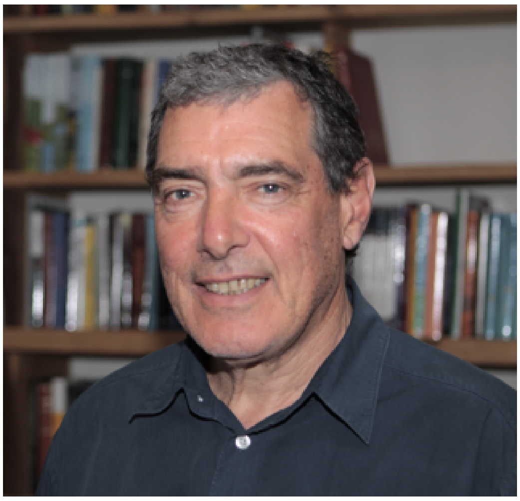

Dynamical Sampling and Orbits of Operators

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Herz-Schur Multipliers of Dynamical Systems

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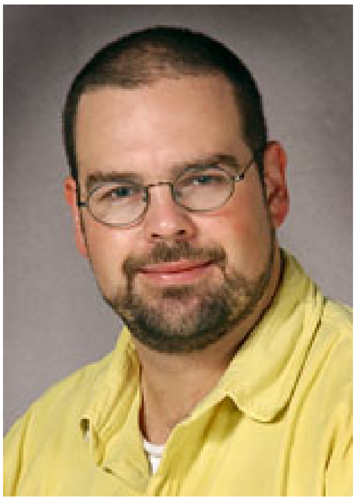

Eberlein-de Leeuw-Glicksberg Decompositions of Fourier Stieltjes Algebras

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A Journey Into Real Harmonic Analysis

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Fixed Scale Improving Inequalities Over Averages Along the Prime Numbers

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Amenable Algebras of Operators on Hilbert Spaces

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Spectral Properties of Group Algebras and Their Relations with Amenability

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مجموعه مقاله‌های انگلیسی

نهمین سمینار آنالیز هارمونیک و کاربردها
۷ و ۸ بهمن ۱۴۰۰
دانشگاه صنعتی امیرکبیر (پلی‌تکنیک تهران)

9th Seminar on Harmonic Analysis and Applications
27 - 28 January 2022
Amirkabir University of Technology

The first Hochschild cohomology groups of Banach algebras with coefficients in special Banach A-bimodules

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Abstract. Let A be a Banach algebra and ϕ, ψ be characters on A. In this paper we consider the class ${}_{\phi}\mathcal{M}_{\psi}^{A}$ of Banach A-bimodules X for which the module actions of A on X is given by $a \cdot x = \phi(a)x$, $x \cdot a = \psi(a)x$ $(a \in A, x \in X)$ and we study the first continuous Hochschild cohomology groups of A with coefficients in $X \in_{\phi} \mathcal{M}_{\psi}^{A}$. We obtain to a characterization of the vanishing of $H^1(A, X)$ for every $X \in_{\phi} \mathcal{M}_{\psi}^{A}$ by $H^1(A, \mathbb{C})$ for $\mathbb{C} \in_{\phi} \mathcal{M}_{\psi}^{A}$.
Keywords: first cohomology group, Banach algebra, bimodule, character on Banach algebra.
AMS Mathematical Subject Classification [2010]: 16E49, 46M20, 46H99.

1. Introduction

Let A be a Banach algebra, X be a Banach A-bimodule, and $H^n(A, X)$ $(n \geq 1)$ be the n^{th} continuous Hochschild cohomology group of A with coefficients in X. It is interesting to study the structure of $H^n(A, X)$ $(n \geq 1)$, where coefficients are in different Banach A-bimodules X, and its vanishing conditions.

Let A be a Banach algebra and $\phi \in \Delta(A) \bigcup \{0\}$ ($\Delta(A)$ is the set of all nonzero characters of A). We denote by $\mathcal{SM}_{\phi}^{A}$ the class of all Banach A-bimodules X with the module actions

$$x \cdot a = a \cdot x = \phi(a)x, \qquad (a \in A, x \in X).$$

In [3], we study the first and second Hochschild cohomology groups of a Banach algebra A with coefficients in X, where $X \in \mathcal{SM}_{\phi}^{A}$ and $\phi \in \Delta(A) \bigcup \{0\}$ is fixed. Also, we check the property that $H^1(A, X) = 0$ for every $X \in \mathcal{SM}_{\phi}^{A}$, and we reach to a characterization of the property that $H^1(A, X) = 0$ for every $X \in \mathcal{SM}_{\phi}^{A}$ by point derivations. We also obtain sufficient conditions under which the first or second cohomology group of A with coefficients in $X \in \mathcal{SM}_{\phi}^{A}$ vanishes or the second group is Hausdorff.

In this paper, as an extension of [3], we remove the symmetry property of Banach A-bimodule $X \in \mathcal{SM}_{\phi}^{A}$ by the module actions

$$a \cdot x = \phi(a)x \quad x \cdot a = \psi(a)x, \qquad ,(a \in A, x \in X, \phi \neq \psi \in \Delta(A))$$

and study $H^1(A, X) = 0$ for every $X \in_{\phi} \mathcal{M}_{\psi}^{A}$.

2. Main results

From this point up to the last part A is a Banach algebra and $\phi \neq \psi \in \Delta(A) \bigcup \{0\}$ are fixed, unless indicated otherwise. In this section we provide some conditions on A under which $H^1(A, X) = 0$ for every $X \in_{\phi} \mathcal{M}_{\psi}^{A}$. The basic properties of A with the property that $H^1(A, X) = 0$ for every $X \in_{\phi} \mathcal{M}_{\psi}^{A}$ are studied.

A derivation d at $\phi, \psi \in \Delta(A) \bigcup \{0\}$ on A is a linear functional satisfying

$$d(ab) = \phi(a)d(b) + \psi(b)d(a) \quad (a, b \in A).$$

That is, d is a derivation into the bimodule $\mathbb{C} \in_{\phi} \mathcal{M}_{\psi}^{A}$, and we call this derivation ϕ, ψ-point derivation. In the next theorem, we obtain to a characterization of the vanishing of $H^1(A, X)$ for every $X \in_{\phi} \mathcal{M}_{\psi}^{A}$ by ϕ, ψ-point derivations.

*speaker

Theorem 2.1. *The following are equivalent:*

(i) $H^1(A,X)=\{0\}$ *for every* $X\in_\phi \mathcal{M}^A_\psi$;
(ii) $H^1(A,X)=\{0\}$ *for some* $\{0\}\neq X\in_\phi \mathcal{M}^A_\psi$;
(iii) $H^1(A,\mathbb{C})=\{0\}$, *where* $\mathbb{C}\in_\phi \mathcal{M}^A_\psi$.

Proof. $(i)\Rightarrow(ii)$ is clear.

$(ii)\Rightarrow(iii)$: Let there be a $\{0\}\neq X\in_\phi \mathcal{M}^A_\psi$ such that $H^1(A,X)=\{0\}$. Assume that $d:A\to\mathbb{C}$ is a continuous derivation, where $\mathbb{C}\in_\phi \mathcal{M}^A_\psi$. Fix $0\neq x\in X$, and define the continuous linear map $D:A\to X$ by $D(a)=d(a)x$. Let $a,b\in A$ then

$$\begin{aligned} D(ab) &= d(ab)x \\ &= (a\cdot d(b)+d(a)\cdot b)x \\ &= (\phi(a)d(b)+\psi(b)d(a))x \\ &= \phi(a)d(b)x+\psi(b)d(a)x \\ &= a\cdot D(b)+D(a)\cdot b. \end{aligned}$$

So, D is a derivation and by the hypothesis, $D=0$. Thus, for every $a\in A$, $D(a)=d(a)x=0$, where $x\neq 0$ and $d(a)\in\mathbb{C}$. Hence, for every $a\in A$, $d(a)=0$. So, $H^1(A,\mathbb{C})=\{0\}$, where $\mathbb{C}\in_\phi \mathcal{M}^A_\psi$.

$(iii)\Rightarrow(i)$: Let $X\in_\phi \mathcal{M}^A_\psi$ and $D:A\to X$ be a continuous derivation. Let $f\in X^*$ and consider the continuous linear map $f\circ D:A\to\mathbb{C}$. Let $a,b\in A$ then

$$\begin{aligned} f\circ D(ab) &= f(D(ab)) \\ &= f(a\cdot D(b)+D(a)\cdot b) \\ &= f(\phi(a)D(b)+\psi(b)D(a)) \\ &= \phi(a)f(D(b))+\psi(b)f(D(a)) \\ &= a\cdot f\circ D(b)+f\circ D(a)\cdot b. \end{aligned}$$

By the hypothesis, $f\circ D=0$. Since $f\in X^*$ is arbitrary, it follows that $D=0$. □

As a collarally of this theorem, we consider the following proposition:

Proposition 2.2. *Let* A *be a unital Banach algebra and* $X\in_\phi \mathcal{M}^A_\psi$.

(a) *If* $\phi=0$ *or* $\psi=0$, *then* $H^n(A,X)=\{0\}$.
(b) *If* A *is commutative and* $\phi,\psi\neq 0$, *then* $H^1(A,X)=H^2(A,X)=\{0\}$.

Proof.

(a) See [1, 2.8.23].
(b) This follows as [1, 2.8.24] and the last theorem. □

References

1. H. G. Dales, *Banach algebra and automatic continuity*, London Math. Soc. Monogr. Ser. Clarendon Press, 2000.
2. E. Feizi, and A. Pourabbas, *Second Hochschlild cohomology group of convolution algebras*, Bull. Iran. Math. Soc. 36(1), 167–173 (2010).
3. E. Feizi, H. Ghahramani and V. Khodakarami, *The first and second Hochschild cohomology groups of Banach algebras with coefficients in special symetric bimodules* , Complex Analysis and Operator Theory. 14, 68 (2020).

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Fiberization method for identification of Gabor frames

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Abstract. In this paper we reformulate the idea of integer oversampling on closed subgroups of a locally compact abelian group. Then, by applying the generalized Zak transform and the fiberization technique, we obtain some characterizations of continuous Gabor frames for $L^2(G)$.

Keywords: Gabor frame, fiberization technique, Zak transform.
AMS Mathematical Subject Classification [2010]: 43A60, 43A22 .

1. Introduction and preliminaries

The main purpose of this work, is to extend and reformulate the idea of integer oversampling for uniform lattices in [3]. However, our formulation rely on the assumption that both translation and modulation groups are only closed subgroups and remove some other limited conditions. Moreover, we discuss the existence conditions of such generalization. Finally, by using the generalized Zak transform and fiberization method, we provide some characterizations of continuous Gabor frames for $L^2(G)$ in term of a family of frames in $l^2(\widehat{H^\perp})$ for a closed co-compact subgroup H of G.

In what follows, let G be a second countable locally compact ablian (LCA) group and it is known that such a group always carries a translation invariant regular Borel measure called Haar measure, denoted by μ_G, and is unique up to a positive constant.

The mathematical theory for Gabor analysis in $L^2(G)$ is based on two classes of operators on $L^2(G)$. The translation by $\lambda \in G$, denoted by T_λ and is defined as $T_\lambda f(x) = f(x-\lambda)$, for all $x \in G$. Also, the modulation by $\gamma \in \widehat{G}$, denoted by E_γ, and defined by $E_\gamma f(x) = \gamma(x) f(x)$, for all $x \in G$.

In the following we give the basic definition and notations of continuous frames.

Definition 1.1. Let $\mathcal{H}$ be a complex Hilbert space, and let $(M, \sum_M, \mu_M)$ be a measure space, where $\sum_M$ denotes the σ-algebra and μ_M the non-negative measure. A family of vectors $\{f_k\}_{k\in M}$ is called a frame for $\mathcal{H}$ with respect to $(M, \sum_M, \mu_M)$ if

(a) the mapping $M \longrightarrow \mathbb{C}$, $k \longmapsto \langle f, f_k\rangle$ is measurable for all $f \in \mathcal{H}$, and

(b) there exist constants $A, B > 0$ such that

$$A \parallel f \parallel^2 \leq \int_M |\langle f, f_k\rangle|^2 d\mu_M(k) \leq B \parallel f \parallel^2, \quad (f \in \mathcal{H}). \tag{1}$$

The constants A and B in (1) are called frame bounds. If $\{f_k\}_{k\in M}$ is weakly measurable and the upper bound in inequality (1) holds, then $\{f_k\}_{k\in M}$ is said to be a Bessel family with bound B. A frame $\{f_k\}_{k\in M}$ is said to be tight if we can choose $A = B$; if furthermore $A = B = 1$, then $\{f_k\}_{k\in M}$ is called a Parseval frame.

Let P be a countable or an uncountable index set, $g_p \in L^2(G)$ for all $p \in P$ and H be a closed co-compact subgroup of G. The translation invariant system generated by $\{g_p\}_{p\in P}$ with translation along the closed co-compact subgroup H is denoted as $\{T_h g_p\}_{h\in H, p\in P}$. Also, let for

*speaker

a topological space T, the Borel algebra of T is denoted by B_T. Then, consider the following standing assumptions of [6, 7];
(I) $(P, \sum_P, \mu_P)$ is a σ-finite measure space,
(II) the mapping $p \longmapsto g_p, (P, \sum_P) \longrightarrow (L^2(G), B_{L^2(G)})$ is measurable,
(III) the mapping $(p, x) \longmapsto g_p(x), (P \times G, \sum_P \bigotimes B_G) \longrightarrow (\mathbb{C}, B_{\mathbb{C}})$ is measurable.
The family $\{g_p\}_{p\in P}$ is called admissible or, when g_p is clear from the context, simply it is said that the measure space P is admissible. Nature of these assumptions was presented in [7]. Every closed subgroup P_j of G with the Haar measure is admissible if $p \longmapsto g_p$ is continuous.

2. Main results

In this section, we present our main results, first we consider the Zak transform associated to a closed subgroup of an LCA group [3].

Definition 2.1. Let Λ be a closed subgroup of G. The Zak transform of a function $f \in L^2(G)$ with respect to Λ is the mapping $Z_\Lambda f$, defined on $G \times \widehat{G}$ as

$$Z_\Lambda f(x, \xi) = \int_\Lambda f(x + \lambda)\xi(\lambda) dm_\Lambda(\lambda).$$

Now, let G be an LCA group, $\Lambda \leq G$ and $\Gamma \leq \widehat{G}$ be closed subgroups. Also, let there exists a closed subgroup $H \leq \Lambda$ so that $H^\perp \leq \Gamma$, and let the quotients $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^\perp}$ to be countable. So, in this case we can choose $\lambda_i \in \Lambda$ so that $\Lambda = \cup_{i=1}^{\infty} (\lambda_i + H)$ and each coset of $\frac{\Lambda}{H}$ contains only one λ_i, and similarly there exist $\gamma_j \in \Gamma$, so that each coset of $\frac{\Gamma}{H^\perp}$ contains only one γ_j. Then the frame operator of the Gabor system $\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$, for a well-fitted window function g on G, can be written as follows

$$\begin{aligned} Sf &= \int_\Gamma \int_\Lambda \langle f, E_\gamma T_\lambda g\rangle E_\gamma T_\lambda g d\mu_\Lambda(\lambda) d\mu_\Gamma(\gamma) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_{H^\perp} \int_H \langle f, E_\omega T_h g_{ij}\rangle E_\omega T_h g_{ij} d\mu_H(h) d\mu_{H^\perp}(\omega) \end{aligned}$$

for all $f \in L^2(G)$ where

$$g_{ij} = T_{\lambda_i} E_{\gamma_j} g.$$

Thus, we can see that

$$Z_H S f = \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mid Z_H g_{ij} \mid^2 \right) Z_H f.$$

The forthcoming theorem, which collects the above computations, shows that the Zak transform on H diagonalize the Gabor frame operator of $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$. In particular, the spectrum of the Gabor frame operator equals the range of $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mid Z_H g_{ij} \mid^2$.

Theorem 2.2. *Let g, Λ, Γ and S be as the above and there exists a closed subgroup H of G so that*

$$H \leq \Lambda, \ \text{and } H^\perp \leq \Gamma. \tag{2}$$

Moreover, assume that $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^\perp}$ are countable. Then, we obtain $Z_H S Z_H^{-1} F = \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mid Z_H g_{ij} \mid^2\right) F$, for all $F \in L^2(M_H)$, where $M_H = \frac{G}{H} \times \frac{\widehat{G}}{H^\perp}$.

As a special case of Theorem 2.2 we record the following corollary.

Corollary 2.3. *Let G be an LCA group, $g \in L^2(G)$, $H, \Lambda \leq G$ and $\Gamma \leq \widehat{G}$ be closed subgroups. Then,*

(i) $Z_H (E_\gamma T_\lambda g)(x, \omega) = \gamma(\lambda) E_{\lambda,\gamma}(x, \omega) Z_H g(x, \gamma + \omega)$, *for all $\lambda \in \Lambda$, $\gamma \in \Gamma$ and a.e. $(x, \omega) \in G \times \widehat{G}$.*

(ii) *If the closed subgroup H of G satisfies (2), then* $Z_H\left(E_\gamma T_\lambda g\right) = E_{\lambda,\gamma} Z_H g$*, for all* $\lambda \in \Gamma^\perp$ *and* $\gamma \in \Lambda^\perp$*.*

The following Proposition, generalize Theorem 11.31 [5] from $\mathbb{R}$ to an LCA group G.

PROPOSITION 2.4. *Let $g \in L^2(G)$, and H be a closed subgroup of G. Then, the following statements hold:*

(i) $\{E_\gamma T_\lambda g\}_{\lambda\in H,\gamma\in H^\perp}$ *is a complete system in $L^2(G)$ if and only if $Z_H g \neq 0$, a.e..*
(ii) *If H is a uniform lattice, then* $\{E_\gamma T_\lambda g\}_{\lambda\in H,\gamma\in H^\perp}$ *is a minimal system in $L^2(G)$ if and only if* $\dfrac{1}{\overline{Z_H g}} \in L^2(M_H)$.

It is worth noticing that for closed subgroups Λ,Γ and H which satisfy (2), we have $\Gamma^\perp \times \Lambda^\perp \subseteq H \times H^\perp \subseteq \Lambda \times \Gamma$. So, if $Z_H \neq 0$ a.e. then the Gabor system $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is complete in $L^2(G)$. However, as the following examples show, the Gabor system $\{E_\gamma T_\lambda g\}_{\lambda\in\Gamma^\perp,\gamma\in\Lambda^\perp}$ is not necessarily complete and in case $\dfrac{1}{\overline{Z_H g}} \in L^2(M_H)$, the Gabor system $\{E_\gamma T_\lambda \phi\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is not minimal, in general.

EXAMPLE 2.5. Fix $0 < \alpha < 1$, set $g(x) = |x|^\alpha$ for $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. It is known that the system $\{T_n E_m g\}_{n,m\in\mathbb{Z}}$ is a Schauder basis for $L^2(\mathbb{R})$ (but not Riesz basis for $L^2(\mathbb{R})$), [2]. So this system is minimal and complete. Take $\Lambda = \frac{1}{2}\mathbb{Z}$, $\Gamma = \frac{1}{4}\mathbb{Z}$ and $H = \mathbb{Z}$, then the closed subgroups H, Λ and Γ satisfy (2). In addition, the Gabor system $\{E_\gamma T_\lambda g\}_{\lambda\in H,\gamma\in H^\perp}$ is complete and minimal by Proposition 2.4. However, the Gabor system $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is not minimal and the Gabor system $\{E_\gamma T_\lambda g\}_{\gamma\in\Lambda^\perp,\lambda\in\Gamma^\perp}$ is not complete in $L^2(\mathbb{R})$.

In the sequel, we show that, $\Gamma^\perp \leq \Lambda$ is not a sufficient condition for the existence of desired H with countable quotient groups, in general.

EXAMPLE 2.6. Let $G = \mathbb{R}^n$, $n > 1$, and $\Lambda = \Gamma = \mathbb{R} \times \mathbb{Z}^{n-1}$. Then $\Gamma^\perp \leq \Lambda$ and for every closed subgroup H so that $\Gamma^\perp \leq H \leq \Lambda$ we can write $H = H_1 \times H_2$ where $H_1 \leq \mathbb{R}$ and $H_2 \leq \mathbb{Z}^{n-1}$. If $H_1 \neq \mathbb{R}$, then $H_1 = \alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$. Thus both $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^\perp}$ are uncountable. Moreover, if $H_1 = \mathbb{R}$ i.e,. $H = \mathbb{R} \times \mathbb{Z}^{n-1}$, then $\frac{\Gamma}{H^\perp}$ is uncountable.

In the next result, we investigate some sufficient conditions for the existence of subgroup H satisfies (2) so that $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^\perp}$ are finite or countable.

THEOREM 2.7. *Let Λ and Γ be subgroups of G and $\widehat{G}$, respectively so that $\Gamma^\perp \leq \Lambda$. Then the following assertions hold;*

(i) *If Λ and Γ are discrete subgroups, then for every subgroup H such that $\Gamma^\perp \leq H \leq \Lambda$, the quotient groups $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^\perp}$ are finite.*
(ii) *If Λ and Γ are open subgroups, then there exists a closed subgroup H satisfies (2) so that either $\frac{\Lambda}{H}$ or $\frac{\Gamma}{H^\perp}$ is countable.*
(iii) *If G is totally-disconnected and Λ, Γ are open subgroups, then there exists a compact subgroup H satisfies (2) so that both $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^\perp}$ are countable.*

The fiberization technique is closely related to Zak transform methods in Gabor analysis. Let H be a closed and co-compact subgroup of G and $\Omega \subset \widehat{G}$ be a Borel section of $H^\perp$ in $\widehat{G}$, we consider the fiberization mapping introduced in [1], $\mathcal{T} : L^2(G) \longrightarrow L^2(\Omega, l^2(H^\perp))$, by

$$\mathcal{T}f(\omega) = \{\widehat{f}(\omega + \alpha)\}_{\alpha\in H^\perp}, \qquad (\omega \in \Omega).$$

It is shown in [1] that, the fiberization is an isometric isomorphic operation.

The next result shows that the frame property of a Gabor system in $L^2(G)$ under certain assumptions is equivalent with the frame property of a family of associated Zak transforms in $l^2(\widehat{H^\perp})$.

THEOREM 2.8. *Let $g \in L^2(G)$, Λ and Γ be closed subgroups of G and $\widehat{G}$ respectively and let H be a closed, co-compact subgroup of G satisfies (2). Then there exists a family $\{g_{ku}\}_{k \in \frac{\Lambda}{H}, u \in \frac{\Gamma}{\Lambda^\perp}}$ in $L^2(G)$ such that following assertions are equivalent.*

(i) $\{E_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ *is a frame for* $L^2(G)$ *with bounds* A *and* B.

(ii) $\{Z_{H^\perp} \widehat{g_{ku}}(\omega, .)\}_{k \in \frac{\Lambda}{H}, u \in \frac{\Gamma}{\Lambda^\perp}}$ *is a frame for* $l^2(\widehat{H^\perp})$ *with bounds* A *and* B*, for a.e.* $\omega \in \Omega$*, where* Ω *is a Borel section of* $H^\perp$ *in* $\widehat{G}$.

Finally, we exploit some connections to the results obtained in [6].

COROLLARY 2.9. *Let $g \in L^2(G)$, Λ be a closed co-compact subgroup of G and Γ be a closed subgroup of $\widehat{G}$ so that $\Gamma^\perp \leq \Lambda$. Then following assertions are equivalent.*

(i) $\{E_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ *is a frame for* $L^2(G)$ *with bounds* A *and* B.

(ii) $\{\widehat{g}(\alpha + \gamma)\}_{\gamma \in \Gamma}$ *is a frame for* $l^2(\Lambda^\perp)$ *with bounds* A *and* B*, for a.e.* $\alpha \in \mathcal{A}$*, where* $\mathcal{A}$ *is a Borel section of* $\Lambda^\perp$ *in* $\widehat{G}$.

(iii) $A \leq \int_{\mathcal{K}} |Z_{\Lambda^\perp} \widehat{g}(\alpha + k, x)|^2 \, d\mu_{\mathcal{K}}(k) \leq B$*, for a.e.* $\alpha \in \mathcal{A}$ *and* $x \in \widehat{\Lambda^\perp}$*, where* $\mathcal{A}$ *is a Borel section of* $\Lambda^\perp$ *in* $\widehat{G}$*,* $\mathcal{K} \subset \Gamma$ *is a Borel section of* $\Lambda^\perp$ *in* Γ.

References

1. M. Bownik and K. Ross, *The structure of translation-invariant space on locally compact abelian group,* J. Fourier Anal. Appl. 21 (2015), 849-884.
2. B. Deng and C. Heil, *Density of Gabor Schauder bases,* Wavelet Application in Signal and Image processing VIII. 4119 (2000), 153-164.
3. K. Grochenig, *Aspect of Gabor analysis on locally compact abelian groups. In Gabor analysis and algorithems,* App. Numer. Harmon. Anal, (1998), 211-231.
4. Z. Hamidi, F. Arabyani-Neyshaburi, R. A. Kamyabi-Gol and M. H. Sattari, *More about continuous Gabor frames on locally compact abelian groups,* Arxiv.org, perprint.
5. C. Heil, *A basis theory primer,* App. Numer. Harmon. Anal, Springer, New York, expanded edition, 2011.
6. M. S. Jakobsen and J. Lemvig, *Co-compact Gabor systems on locally compact abelian groups*, J. Fourier Anal. Appl, 22 (2016), 3670.
7. M. S. Jakobsen and J. Lemvig, *Reproducing formulas for generalized translation invariant systems on locally compact abelian groups,* Trans. Amer. Math, 368 (2016), 8447-8480.

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Primitive ideals of ultragraph algebras

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Abstract. The prime ideals as well as the primitive ideals of ultragrph algebras are completely characterized in terms of downward directed sets. We also give necessary and sufficient conditions on the ultragraph $\mathcal{G}$ under which every prime ideal of $L_K(\mathcal{G})$ is primitive.
Keywords: Ultragraph, Leavitt path algebra, Ultragraph C^*-algebra, Primitive ideal.
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1. Introduction

To study both graph C^*-algebras and Exel-Laca algebras under one theory, Tomforde [6] introduced the notion of an ultragraph and its associated C^*-algebra. Briefly, an ultragraph is a directed graph which allows the range of each edge to be a nonempty set of vertices rather than a singleton vertex. The structure of ultragraph C^*-algebra $C^*(\mathcal{G})$ is more complicated, because in ultragraph $\mathcal{G}$ the range of each edge is allowed to be a nonempty set of vertices rather than a single vertex.

Ultragraph Leavitt path algebra $L_R(\mathcal{G})$ was introduced in [3] as the algebraic version of ultragraph C^*-algebra $C^*(\mathcal{G})$. The algebras $L_R(\mathcal{G})$ are generalizations of the Leavitt path algebras $L_R(E)$ [1, 2]. The class of ultragraph Leavitt path algebras is strictly larger than the class of Leavitt path algebras of directed graphs. Also, every Leavitt path algebra of a directed graph can be embedded as a subalgebra in a unital ultragraph Leavitt path algebra.

The aim of this paper is to give a complete description of the prime ideals as well as the primitive ideals of $L_K(\mathcal{G})$. Similar results obtained in ultragraph C^*-algebra $C^*(\mathcal{G})$ are also presented [4]. We start by recalling the definition of the quotient ultragraph $\mathcal{G}/(H,S)$ and it's C^*-algebra $C^*\big(\mathcal{G}/(H,S)\big)$ and Leavitt path algebra $L_K\big(\mathcal{G}/(H,S)\big)$. We characterize the graded prime ideals in terms of the downward directed sets. To describe the structure of non-graded prime ideals, we investigate the structure of the closed ideals of $L_K\big(\mathcal{G}/(H,S)\big)$ which contain no nonzero set idempotents, and we give a complete description of primitive ideals. We show that a graded prime ideal $I_{(H,B_H)}$ is primitive if and only if the quotient ultragraph $\mathcal{G}/(H,B_H)$ satisfies Condition (L). Finally, we prove that every non-graded prime ideal in $L_K(\mathcal{G})$ is primitive.

1.1. Preliminaries. We begin by reviewing some background material on ultragraph, quotient ultragraph and their algebras. For more details see [6, 5].

An *ultragraph* $\mathcal{G} = (G^0, \mathcal{G}^1, r_{\mathcal{G}}, s_{\mathcal{G}})$ consists of countable sets G^0 of vertices and $\mathcal{G}^1$ of edges, the source map $s_{\mathcal{G}} : \mathcal{G}^1 \to G^0$ and the range map $r_{\mathcal{G}} : \mathcal{G}^1 \to \mathcal{P}(G^0) \setminus \{\emptyset\}$, where $\mathcal{P}(G^0)$ is the collection of all subsets of G^0.

For a set X, a subcollection of $\mathcal{P}(X)$ is called an *algebra* if it is closed under the set operations $\cup$, $\cap$ and $\setminus$. If $\mathcal{G}$ is an ultragraph, we write $\mathcal{G}^0$ for the smallest algebra in $\mathcal{P}(G^0)$ containing $\big\{\{v\}, r_{\mathcal{G}}(e) : v \in G^0 \text{ and } e \in \mathcal{G}^1\big\}$.

Definition 1.1. Let $\mathcal{G}$ be an ultragraph. A subcollection $H \subseteq \mathcal{G}^0$ is *hereditary* if

(1) $\{s_{\mathcal{G}}(e)\} \in H$ implies $r_{\mathcal{G}}(e) \in H$ for all $e \in \mathcal{G}^1$.
(2) $A \cup B \in H$ for all $A, B \in H$.
(3) $A \in H$, $B \in \mathcal{G}^0$ and $B \subseteq A$, imply $B \in H$.

*speaker

The hereditary subcollection $H \subseteq \mathcal{G}^0$ is *saturated* if for every $v \in G^0$ with $0 < |s_{\mathcal{G}}^{-1}(v)| < \infty$ we have

$$\big\{r_{\mathcal{G}}(e) : e \in \mathcal{G}^1 \text{ and } s_{\mathcal{G}}(e) = v\big\} \subseteq H \quad \text{implies} \quad \{v\} \in H.$$

For a saturated hereditary subcollection $H \subseteq \mathcal{G}^0$, the *breaking vertices* of H is denoted by

$$B_H := \Big\{v \in G^0 : \big|s_{\mathcal{G}}^{-1}(v)\big| = \infty \text{ but } 0 < \big|s_{\mathcal{G}}^{-1}(v) \cap \{e : r_{\mathcal{G}}(e) \notin H\}\big| < \infty\Big\}.$$

An *admissible pair in* $\mathcal{G}$ is a pair (H, S) of a saturated hereditary subcollection $H \subseteq \mathcal{G}^0$ and some $S \subseteq B_H$.

In order to define the quotient of ultragraphs we need to recall and introduce some notation. Let $\mathcal{G} = (G^0, \mathcal{G}^1, r_{\mathcal{G}}, s_{\mathcal{G}})$ be an ultragraph and let (H, S) be an admissible pair in $\mathcal{G}$. Given $A \in \mathcal{P}(G^0)$, denote by $\overline{A} := A \cup \{w' : w \in A \cap (B_H \setminus S)\}$. Also, we write $\overline{\mathcal{G}^0}$ for the algebra in $\mathcal{P}(\overline{G^0})$ generated by the sets $\{v\}$, $\{w'\}$ and $\{\overline{r_{\mathcal{G}}(e)}\}$, where $v \in G^0$, $w \in B_H \setminus S$ and $e \in \mathcal{G}^1$.

Let $\sim$ be a relation on $\overline{\mathcal{G}^0}$ defined by $A \sim B$ if and only if there exists $V \in H$ such that $A \cup V = B \cup V$. Then, by [5, Lemma 3.5], $\sim$ is an equivalent relation on $\overline{\mathcal{G}^0}$ and the operations

$$[A] \cup [B] := [A \cup B],\ [A] \cap [B] := [A \cap B] \text{ and } [A] \setminus [B] := [A \setminus B]$$

are well-defined on the equivalent classes $\{[A] : A \in \overline{\mathcal{G}^0}\}$. One can see that $[A] = [B]$ if and only if both $A \setminus B$ and $B \setminus A$ belong to H.

Definition 1.2. Let $\mathcal{G} = (G^0, \mathcal{G}^1, r_{\mathcal{G}}, s_{\mathcal{G}})$ be an ultragraph and let (H, S) be an admissible pair in $\mathcal{G}$. The *quotient ultragraph of* $\mathcal{G}$ *by* (H, S) is the quadruple $\mathcal{G}/(H, S) := (\Phi(G^0), \Phi(\mathcal{G}^1), r, s)$, where

$$\Phi(G^0) := \big\{[\{v\}] : v \in G^0 \setminus H\big\} \cup \big\{[\{w'\}] : w \in B_H \setminus S\big\},$$
$$\Phi(\mathcal{G}^1) := \big\{e \in \mathcal{G}^1 : r_{\mathcal{G}}(e) \notin H\big\},$$

and $s : \Phi(\mathcal{G}^1) \to \Phi(G^0)$ and $r : \Phi(\mathcal{G}^1) \to \{[A] : A \in \overline{\mathcal{G}^0}\}$ are the maps defined by $s(e) := [\{s_{\mathcal{G}}(e)\}]$ and $r(e) := [\overline{r_{\mathcal{G}}(e)}]$ for every $e \in \Phi(\mathcal{G}^1)$, respectively.

For the sake of simplicity, we will write $[v]$ instead of $[\{v\}]$ for every vertex $v \in G^0 \setminus H$. For $A, B \in \overline{\mathcal{G}^0}$, we write $[A] \subseteq [B]$ whenever $[A] \cap [B] = [A]$. The smallest algebra in $\{[A] : A \in \overline{\mathcal{G}^0}\}$ containing

$$\Big\{[v], [w'] : v \in G^0 \setminus H, w \in B_H \setminus S\Big\} \cup \Big\{r(e) : e \in \Phi(\mathcal{G}^1)\Big\}$$

is denoted by $\Phi(\mathcal{G}^0)$. It can be show that $\Phi(\mathcal{G}^0) = \big\{[A] : A \in \overline{\mathcal{G}^0}\big\}$.

A vertex $[v] \in \Phi(G^0)$ is called a *sink* if $|s^{-1}([v])| = \emptyset$ and is called an *infinite emitter* if $|s^{-1}([v])| = \infty$. A *singular vertex* is a vertex that is either a sink or an infinite emitter. The set of singular vertices is denoted by $\Phi_{\mathrm{sg}}(G^0)$.

Definition 1.3. Let $\mathcal{G}/(H, S)$ be a quotient ultragraph. A *Cuntz-Krieger* $\mathcal{G}/(H, S)$*-family* consists of projections $\{q_{[A]} : [A] \in \Phi(\mathcal{G}^0)\}$ and partial isometries $\{t_e : e \in \Phi(\mathcal{G}^1)\}$ with mutually orthogonal ranges such that

(1) $q_{[\emptyset]} = 0$, $q_{[A]} q_{[B]} = q_{[A] \cap [B]}$ and $q_{[A] \cup [B]} = q_{[A]} + q_{[B]} - q_{[A] \cap [B]}$;
(2) $t_e^* t_e = q_{r(e)}$;
(3) $t_e t_e^* \le q_{s(e)}$;
(4) $q_{[v]} = \sum_{s(e)=[v]} t_e t_{e^*}$ whenever $0 < |s^{-1}([v])| < \infty$.

The C^*-algebra $C^*(\mathcal{G}/(H, S))$ is the universal C^*-algebra generated by a Cuntz-Krieger $\mathcal{G}/(H, S)$-family.

Definition 1.4. Let $\mathcal{G}/(H, S)$ be a quotient ultragraph and let K be a field. A *Leavitt* $\mathcal{G}/(H, S)$*-family* in a K-algebra X is a set $\{q_{[A]}, t_e, t_{e^*} : [A] \in \Phi(\mathcal{G}^0) \text{ and } e \in \Phi(\mathcal{G}^1)\}$ of elements in X such that

(1) $q_{[\emptyset]} = 0$, $q_{[A]} q_{[B]} = q_{[A] \cap [B]}$ and $q_{[A] \cup [B]} = q_{[A]} + q_{[B]} - q_{[A] \cap [B]}$;
(2) $q_{s(e)} t_e = t_e q_{r(e)} = t_e$ and $q_{r(e)} t_{e^*} = t_{e^*} q_{s(e)} = t_{e^*}$;

(3) $t_{e^*}t_f = \delta_{e,f}q_{r(e)}$;
(4) $q_{[v]} = \sum_{s(e)=[v]} t_e t_{e^*}$ whenever $[v] \in \Phi(G^0) \setminus \Phi_{\rm sg}(G^0)$.

The *Leavitt path algebra of* $\mathcal{G}/(H,S)$, denoted by $L_K\big(\mathcal{G}/(H,S)\big)$, is defined to be the K-algebra generated by a universal Leavitt $\mathcal{G}/(H,S)$-family.

Due to the fact that in the quotient ultragraph $\mathcal{G}/(\emptyset,\emptyset)$, we have $[A]=\{A\}$ For every $A \in \mathcal{G}^0$, we can consider the ultragraph $\mathcal{G}$ as the quotient ultragraph $\mathcal{G}/(\emptyset,\emptyset)$. So, the definition of ultragraph C^*-algebras [6, Defintion 2.7] (ultragraph Leavitt path algebras [3, Defintion 2.1]) is an special case of the Definition 1.3 (Definition 1.4).

A *path* in $\mathcal{G}/(H,S)$ is a finite sequence $\alpha = e_1e_2\cdots e_n$ of edges with $s(e_{i+1}) \subseteq r(e_i)$ for $1 \le i \le n-1$. We consider the elements of $\Phi(\mathcal{G}^0)$ as the paths of length zero. We let $\mathrm{Path}\big(\mathcal{G}/(H,S)\big)$ denotes the set of all paths in $\mathcal{G}/(H,S)$. We define $[A]^* := [A]$ and $\alpha^* := e_n^*e_{n-1}^*\cdots e_1^*$, for every $[A] \in \Phi(\mathcal{G}^0)$ and $\alpha = e_1e_2\cdots e_n \in \mathrm{Path}\big(\mathcal{G}/(H,S)\big)$. The maps r,s extend to $\mathrm{Path}\big(\mathcal{G}/(H,S)\big)$ in an obvious way.

we have

$$C^*(\mathcal{G}/(H,S)) = \overline{\mathrm{span}}\big\{t_\alpha q_{[A]} t_\beta^* : [A] \in \Phi(\mathcal{G}^0) \text{ and } \alpha,\beta \in (\mathcal{G}/(H,S))^*\big\},$$

$$L_K\big(\mathcal{G}/(H,S)\big) = \mathrm{span}_K\big\{t_\alpha q_{[A]} t_{\beta^*} : \alpha,\beta \in \mathrm{Path}\big(\mathcal{G}/(H,S)\big) \text{ and } r(\alpha)\cap[A]\cap r(\beta) \ne [\emptyset]\big\}.$$

2. Main results

Let $\mathcal{G}$ be an ultragraph. Define a relation on $\mathcal{G}^0$ by setting $A \ge B$ if either $B \subseteq A$ or there is a path α of positive length such that $s_{\mathcal{G}}(\alpha) \in A$ and $B \subseteq r_{\mathcal{G}}(\alpha)$. A subcollection $M \subseteq \mathcal{G}^0$ is called *downward directed* if for every $A,B \in M$ there exists $\emptyset \ne C \in M$ such that $A,B \ge C$.

Lemma 2.1. *Let I be an ideal of $C^*(\mathcal{G})$ or $L_K(\mathcal{G})$. Consider $H_I := \{A \in \mathcal{G}^0 : p_A \in I\}$. If I is prime, then $\mathcal{G}^0 \setminus H_I$ is downward directed.*

Let (H,S) be an admissible pair in $\mathcal{G}$. For any $w \in B_H$, set

$$p_w^H := p_w - \sum_{s_{\mathcal{G}}(e)=w,\ r_{\mathcal{G}}(e)\notin H} s_e s_e^*,$$

and we define $I_{(H,S)}$ as the (two-sided) ideal of $L_K(\mathcal{G})$ generated by the idempotents $\{p_A : A \in H\} \cup \{p_w^H : w \in S\}$. By [3, Theorem 4.4], $L_K\big(\mathcal{G}/(H,S)\big) \cong L_K(\mathcal{G})/I_{(H,S)}$ and the correspondence $(H,S) \mapsto I_{(H,S)}$ is a bijection from the set of all admissible pairs of $\mathcal{G}$ to the set of all graded ideals of $L_K(\mathcal{G})$.

Theorem 2.2. *Let $\mathcal{G}$ be an ultragraph. Set*

$$X_1 = \{I_{(H,B_H)} : \mathcal{G}^0 \setminus H \text{ is downward directed}\}$$

and

$$X_2 = \{I_{(H,B_H\setminus\{w\})} : w \in B_H \text{ and } A \ge w \text{ for all } A \in \mathcal{G}^0 \setminus H\}.$$

Then $X_1 \cup X_2$ is the set of all graded prime ideals of $L_K(\mathcal{G})$.

Definition 2.3. A *loop* in $\mathcal{G}/(H,S)$ is a path α with $|\alpha| \ge 1$ and $s(\alpha) \subseteq r(\alpha)$. A loop $\alpha = e_1\cdots e_n$ has an *exit* if either $r(e_i) \ne s(e_{i+1})$ for some $1 \le i \le n$ or there exists an edge $f \in \Phi(\mathcal{G}^1)$ and an index i such that $s(f) \subseteq r(e_i)$ but $f \ne e_{i+1}$. The quotient ultragraph $\mathcal{G}/(H,S)$ satisfies *Condition (L)* if every loop in $\mathcal{G}/(H,S)$ has an exit.

Lemma 2.4. *Let $\mathcal{G}/(H,S)$ be a quotient ultragraph. If $\gamma = e_1e_2\cdots e_n$ is a loop in $\mathcal{G}/(H,S)$ without exits, then I_{γ^0} and $K[x,x^{-1}]$ are Morita equivalent as rings, where I_{γ^0} is an ideal of $L_K\big(\mathcal{G}/(H,S)\big)$ generated by $\{q_{s(e_i)} : 1 \le i \le n\}$.*

The set of vertices in the loops without exits of $\mathcal{G}/(H,S)$ is denoted by $P_c(\mathcal{G}/(H,S))$. Also, we denote by $I_{P_c(\mathcal{G}/(H,S))}$ the ideal of $C^*\big(\mathcal{G}/(H,S)\big)$ or $L_K\big(\mathcal{G}/(H,S)\big)$ generated by the idempotents associated to the vertices in $P_c(\mathcal{G}/(H,S))$.

LEMMA 2.5. *Let $\mathcal{G}/(H,S)$ be a quotient ultragraph. If I is an ideal of $C^*\big(\mathcal{G}/(H,S)\big)$ or $L_K\big(\mathcal{G}/(H,S)\big)$ with $\{[A]\neq[\emptyset]: q_{[A]}\in I\}=\emptyset$, then $I\subseteq I_{P_c(\mathcal{G}/(H,S))}$.*

Let H be a saturated hereditary subcollection of $\mathcal{G}^0$ and let $\mathcal{G}^0\setminus H$ contains a unique (up to permutation) loop α without exits in $\mathcal{G}^0\setminus H$. For $t\in\mathbb{T}$, the ideal of $C^*\big(\mathcal{G}/(H,B_H)\big)$ generated by

$$\{p_A, p_w^H : A\in H, w\in B_H\}\cup\{tp_{s_{\mathcal{G}}(\alpha)}-s_\alpha\}$$

is denoted by $I_{\langle H,B_H,t\rangle}$.

THEOREM 2.6. [4, Theorem 4.4] *Let $\mathcal{G}$ be an ultragraph and I be a non gauge-invariant ideal of $C^*(\mathcal{G})$. Denote $H:=H_I$. Then I is a primitive (prime) ideal if and only if $\mathcal{G}^0\setminus H$ is downward directed, $\mathcal{G}^0\setminus H$ contains a (unique) loop α without exits in $\mathcal{G}^0\setminus H$ and there exists $t\in\mathbb{T}$ such that $I=I_{\langle H,B_H,t\rangle}$.*

Let $\mathcal{G}/(H,S)$ be a quotient ultragraph, $x\in L_K\big(\mathcal{G}/(H,S)$ and let the ideal of $L_K\big(\mathcal{G}/(H,S)$ generated by $I_{(H,S)}\cup\{x\}$ is denoted by $I_{\langle H,S,x\rangle}$. Suppose that γ is a loop in $\mathcal{G}/(H,S)$ without exits and $f(x)$ is a polynomial in $K[x,x^{-1}]$. It can be shown that $I_{\langle f(t_\gamma)\rangle}=I_{\langle f(t_{\gamma'})\rangle}$, where γ' is a permutation of γ.

THEOREM 2.7. *Let $\mathcal{G}$ be an ultragraph and let I be an ideal of $L_K(\mathcal{G})$. Denote $H:=H_I$. Then I is a non-graded prime ideal if and only if*

(1) *$\mathcal{G}^0\setminus H$ is downward directed,*
(2) *$\mathcal{G}^0\setminus H$ contains a loop γ without exits in $\mathcal{G}^0\setminus H$ and*
(3) *$I=I_{\langle H,B_H,f(s_\gamma)\rangle}$, where $f(x)$ is an irreducible polynomial in $K[x,x^{-1}]$.*

THEOREM 2.8. *Let $\mathcal{G}/(H,S)$ be a quotient ultragraph. Then $L_K\big(\mathcal{G}/(H,S)\big)$ is primitive if and only if one of the following holds:*

(i) *$S=B_H$, $\mathcal{G}/(H,S)$ satisfies Condition (L) and $\mathcal{G}^0\setminus H$ is downward directed.*
(ii) *$S=B_H\setminus\{w\}$ for some $w\in B_H$ and $A\geq w$ for all $A\in\mathcal{G}^0\setminus H$.*

COROLLARY 2.9. *Let $\mathcal{G}$ be an ultragraph. A graded ideal $I_{(H,S)}$ of $L_K(\mathcal{G})$ is primitive if and only if one of the following holds:*

(i) *$S=B_H$, $\mathcal{G}^0\setminus H$ is downward directed and every loop in $\mathcal{G}^0\setminus H$ has an exit in $\mathcal{G}^0\setminus H$.*
(ii) *$S=B_H\setminus\{w\}$ for some $w\in B_H$ and $A\geq w$ for all $A\in\mathcal{G}^0\setminus H$.*

THEOREM 2.10. *Let $\mathcal{G}$ be an ultragraph. A non-graded ideal of $L_K(\mathcal{G})$ is prime if and only if it is primitive.*

References

1. G. Abrams and G. Aranda Pino, *The Leavitt path algebra of a graph*, J. Algebra, **293**(2005), 319-334.
2. G. Abrams and G. Aranda Pino, *The Leavitt path algebras of arbitrary graphs*, Houston J. Math., **34**(2008), 423-442.
3. M. Imanfar, A. Pourabbas and H. Larki, *The leavitt path algebras of ultragraphs*, Kyungpook Math. J., **60(1)**(2020), 21-43.
4. M. Imanfar, A. Pourabbas and H. Larki, *Primitive Ideal Space of Ultragraph C^*-algebras*, Sahand Communications in Mathematical Analysis, **15(1)**(2019), 147-158.
5. H. Larki, *Primitive ideals and pure infiniteness of ultragraph C^*-algebras*, J. Korean Math. Soc., **56**(2019), 1-23.
6. M. Tomforde, *A unified approach to Exel-Laca algebras and C^*-algebras associated to graphs*, *J. Operator Theory* **50**(2003), 345-368.

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Inner amenability of transformation groups

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Abstract. We introduce the inner amenability of transformation groups and characterize this property. As an interesting result, we show that every inner amenable transformation group has property (W).

1. Introduction

Let G be a locally compact group and X be a locally compact space. Then X is called a transformation group if it is a left G-space, i.e., $(x, s) \longmapsto s \cdot x$ is a continuous left action from $X \times G \longrightarrow X$. Note that X is a unitary G-space, i.e., for any $x \in X$, we have $e_G \cdot x = x$. Following [1], if $g \in C_C(X \times G)$, then g^x will be the map $t \longmapsto g^x(t) = g(x, t)$, and $g(t)$ will be the map $x \longmapsto g(t)(x) = g(x, t)$. The transformation group (X, G) (or the G-action on X, or the G-space X) is amenable if there is a net $(m_i)_{i\in I}$ of continuous maps $x \longmapsto m_i^x$ from X into the space Prob(G) (the set of probability measures on G, equipped with the weak*-topology) such that $\lim_i \|sm_i^x - m_i^{sx}\|_1 = 0$, uniformly on compact subsets of $X \times G$. Such a net $(m_i)_{i\in I}$ will be called an approximate invariant continuous mean (a.i.c.m. for short). The amenability of transformation groups and semigroups are investigated in many literatures that we refer to [1, 2, 6, 11, 12], for more details. In [1, 2], Anantharaman-Delaroche by characterizing amenability of transformation groups gave some applications related to amenability of C^*-dynamical systems, nuclearity of the corresponding crossed products and operator algebras. A locally compact group G is called inner amenable if there is a linear functional m on $L^\infty(G)$ such that $m({}_{s^{-1}}f_s) = m(f)$, for all $f \in L^\infty(G)$ and $s \in G$. The class of inner amenable locally compact groups includes all amenable and [IN]-groups. See [3, 4, 5, 7, 8, 13, 14, 15, 16], for inner amenable locally compact groups and its applications. In this section, by a transformation group (X, G), we mean the both left and right actions, i.e., $(x, s) \longmapsto s \cdot x$ and $(x, s) \longmapsto x \cdot s$ are continuous.

2. Main results

Definition 2.1. Let (X, G) be a transformation group. We say that (X, G) is inner amenable if there is a net $(m_\alpha)_{\alpha\in I}$ of continuous maps $x \longmapsto m_\alpha^x$ from X into the space *Prob*(G) such that

$$\lim_\alpha \|m_\alpha^{sx} s^{-1} - s^{-1} m_\alpha^{xs}\|_1 = 0, \tag{1}$$

uniformly on compact subsets of $X \times G$.

Proposition 2.2. *The following conditions are equivalent:*

(i) *(X, G) is an inner amenable transformation group.*
(ii) *There is a net $(f_\alpha)_{\alpha\in I}$ of nonnegative continuous functions on $X \times G$ such that*
  (a) *for all $\alpha \in I$ and $x \in X$, $\int_G f_\alpha^x(t)\, d\lambda_G(t) = 1$;*
  (b) *$\lim_\alpha \int_G \left|f_\alpha^{sx}(s^{-1}t) - f_\alpha^{xs}(ts^{-1})\right|\, d\lambda_G(t) = 0$ uniformly on compact subsets of $X \times G$.*
(iii) *There is a net $(f_\alpha)_{\alpha\in I}$ in $C_C(X \times G)^+$ such that*
  (a) *$\lim_\alpha \int_G f_\alpha^x(t)\, d\lambda_G(t) = 1$ uniformly on compact subsets of X;*
  (b) *$\lim_\alpha \int_G \left|f_\alpha^{sx}(s^{-1}t) - f_\alpha^{xs}(ts^{-1})\right|\, d\lambda_G(t) = 0$ uniformly on compact subsets of $X \times G$.*

*speaker

Proof. (i)$\longrightarrow$(ii) Let $(m_\alpha)_{\alpha\in I}$ be a net of continuous maps $x\longmapsto m_i^x$ from X into the space Prob(G) such that

$$\lim_\alpha \|m_\alpha^{sx}s^{-1}-s^{-1}m_\alpha^{xs}\|_1=0 \tag{2}$$

uniformly on compact subsets of $X\times G$. Pick $f\in C_C(G)^+$ such that $\int_G f(s)\,d\lambda_G(s)=1$. Define

$$f_\alpha(x,s)=\int_G f(t^{-1}s)\,dm_\alpha^x(t),$$

for every $(x,s)\in(X,G)$. Then, for all $x\in X$ and $\alpha\in I$,

$$\int_G f_\alpha^x(s)\,d\lambda_G(s) = \int_G\int_G f(t^{-1}s)\,dm_\alpha^x(t)\,d\lambda_G(s)=\int_G\int_G f(t^{-1}s)\,d\lambda_G(s)\,dm_\alpha^x(t)=1.$$

Thus (3) implies (a). For any $s\in G$ and $x\in X$,

$$\begin{aligned}\int_G\left|f_\alpha^{sx}(s^{-1}t)-f_\alpha^{xs}(ts^{-1})\right|\,d\lambda_G(t)&=\int_G\left|\int_G f(u^{-1}s^{-1}t)\,dm_\alpha^{sx}(u)-\int_G f(u^{-1}ts^{-1})\,dm_\alpha^{xs}(u)\right|\,d\lambda_G(t)\\&\le\int_G\int_G f(u^{-1}s^{-1}ts^{-1})\,d\left|m_\alpha^{sx}s^{-1}-s^{-1}m_\alpha^{xs}\right|(u)\,d\lambda_G(t)\\&\le\|m_\alpha^{sx}s^{-1}-s^{-1}m_\alpha^{xs}\|_1.\end{aligned}\tag{3}$$

Then by (2) and (3) we obtain (b).

(ii)$\longrightarrow$(i) Let $(f_\alpha)_{i\in I}$ be as (ii). Define $m_\alpha^x=f_\alpha^x\lambda_G$, for all $\alpha\in I$ and $x\in X$. Then $dm_\alpha^x(t)=f_\alpha^x(t)\,d\lambda_G(t)$, for all $\alpha\in I$, $t\in G$ and $x\in X$. Applying (a) shows that m_α^x is a probability measure on G, for all $\alpha\in I$ and $x\in X$. Moreover, (b) implies that $\lim_\alpha\|m_\alpha^{sx}s^{-1}-s^{-1}m_\alpha^{xs}\|_1=0$, uniformly on compact subsets of $X\times G$.

(ii)$\longrightarrow$(iii) Since compactly supported functions are dense in continuous functions, the proof is clear.

(iii)$\longrightarrow$(ii) Define $f_{\alpha,n}=\frac{f_\alpha(x,s)+\frac{1}{n}f(s)}{\int_G f_\alpha(x,t)\,d\lambda_G(t)+\frac{1}{n}}$, where $(f_\alpha)_{\alpha\in I}$ is a net that satisfies (iii) and $f\in C_C(G)^+$. □

Remark 2.3. (1) Let (X,G) be an inner amenable transformation group such that X is a point. Proposition 2.2 implies that there is a net $(f_\alpha)_{\alpha\in I}$ such that $\int_G f_\alpha(t)\,d\lambda_G(t)=1$ and $\int_G|f_\alpha(st)-f_\alpha(ts)|\,d\lambda_G(t)\longrightarrow0$ uniformly on compact subsets of $X\times G$. This implies that

$$\int_G\left|f_\alpha(s^{-1}t)-f_\alpha(ts^{-1})\right|\,d\lambda_G(t) = \int_G\left|l_sf_\alpha(t)-r_sf_\alpha(t)\right|\,d\lambda_G(t)=\|\delta_s*f_\alpha-f_\alpha*\delta_s\|_1.$$

This shows that $\|\delta_s*f_\alpha-f_\alpha*\delta_s\|_1$ tends to 0 uniformly on compact subsets of G. Thus G is inner amenable [8, Proposition 1].

(2) Let G be an inner amenable locally compact group and X be a locally compact G-space. Inner amenability of G implies that there is a net $(f_\alpha)_{\alpha\in I}$ of probability measures such that $\|\delta_s*f_\alpha-f_\alpha*\delta_s\|_1\longrightarrow0$ for every $s\in G$. Now, we define m_α on $X\times G$ such that its value on X is a constant value and on G is equal to $f_\alpha\,d\lambda_G$. Thus, (m_α) satisfies in (1). This means that every transformation group (X,G) is inner amenable whenever G is inner amenable.

Example 2.4. (1) We give an example of transformation groups which show that the converse of 2.3(2), in general, is not true. Let $\mathbb{F}_2$ be the free group with two generators a and b and $\partial\mathbb{F}_2$ be the boundary of $\mathbb{F}_2$ that is the set of all infinite reduced words $\omega=a_1a_2\cdots a_n\cdots$ in the alphabet $S=\{a,a^{-1},b,b^{-1}\}$. Suppose that $X=\partial\mathbb{F}_2$, then $(X,\mathbb{F}_2)$ becomes a transformation group. By [1, Example 2.7(4)], $(X,\mathbb{F}_2)$ is inner amenable. But $\mathbb{F}_2$ is not inner amenable [9, Proposition 22.38].

(2) Let $N \in \mathbb{N}$ such that $N \geq 2$. The generalized Thompson group $F(N)$ is the set of piecewise linear homeomorphisms from the closed unit interval $[0,1]$ to itself that are differentiable except at finitely many N-adic rationals and such that at intervals of differentiability the derivatives are powers of N. This group is inner amenable [10, Corollary 3.7]. Thus, by Remark 2.3(2), for any $F(N)$-space X, the transformation group $(X, F(N))$ is inner amenable.

Corollary 2.5. *Let G be a unimodular locally compact group. Then the transformation group (X,G) is inner amenable if and only if there is a net $(\xi_\alpha)_{\alpha\in I}$ in $C_C(X\times G)^+$ such that the following statements hold:*

(i) $\lim_\alpha \int_G |\xi_\alpha(x,t)|^2 \, d\lambda_G(t) = 1$ *uniformly on compact subsets of X;*

(ii) $\lim_\alpha \int_G \left|\xi_\alpha(sx, s^{-1}t) - \xi_\alpha(xs, ts^{-1})\right|^2 d\lambda_G(t) = 0$ *uniformly on compact subsets of $X \times G$.*

Proof. Suppose that (X,G) is inner amenable. Then by Proposition 2.2(iii) there is a net $(f_\alpha)_{\alpha\in I}$ such that satisfies the implications (a) and (b). Set $\xi_\alpha = \sqrt{f_\alpha}$, for any $\alpha \in I$ and define

$$h_\alpha(x,s) = \int_G \overline{\xi_\alpha(sx, s^{-1}t)}\xi_\alpha(xs, ts^{-1})d\lambda_G(t),$$

for all $(x,s) \in X \times G$. Then, for any $x \in X$ and $s \in G$, we have,

$$h_\alpha(sx, e_G) = \int_G \overline{\xi_\alpha(sx,t)}\xi_\alpha(sx,t)d\lambda_G(t) = \int_G \overline{\xi_\alpha(sx,s^{-1}t)}\xi_\alpha(sx,s^{-1}t)d\lambda_G(t)$$

and

$$h_\alpha(xs, e_G) = \int_G \overline{\xi_\alpha(xs,t)}\xi_\alpha(xs,t)d\lambda_G(t) = \int_G \overline{\xi_\alpha(xs,ts^{-1})}\xi_\alpha(xs,ts^{-1})d\lambda_G(t).$$

Then

$$h_\alpha(sx, e_G) + h_\alpha(xs, e_G) - 2h_\alpha(x,s) = \int_G \left|\xi_\alpha(sx, s^{-1}t) - \xi_\alpha(xs, xs^{-1})\right|^2 d\lambda_G(t).$$

By the above equality we get

$$\begin{aligned}\int_G \left|\xi_\alpha(sx, s^{-1}t) - \xi_\alpha(xs, xs^{-1})\right|^2 d\lambda_G(t) &\leq \int_G \left|\xi_\alpha(sx, s^{-1}t)^2 - \xi_\alpha(xs, xs^{-1})^2\right| d\lambda_G(t) \\ &= \int_G \left|f_\alpha^{sx}(s^{-1}t) - f_\alpha^{xs}(xs^{-1})\right| d\lambda_G(t).\end{aligned}$$

Then by the phrase (b) of Proposition 2.2(iii), the statement (ii) holds. We, also for every $x \in X$, have

$$h_\alpha(x, e_G) = \int_G \overline{\xi_\alpha(x,t)}\xi_\alpha(x,t)d\lambda_G(t) = \int_G |\xi_\alpha(x,t)|^2 d\lambda_G(t) = \int_G f_\alpha(x,t)d\lambda_G(t) = \int_G f_\alpha^x(t)d\lambda_G(t).$$

By the phrase (a) of Proposition 2.2(iii), the statement (i) holds.

Conversely, suppose that there is a net $(\xi_\alpha)_{\alpha\in I}$ in $C_C(X\times G)$ such that satisfies (i) and (ii). Set $f_\alpha = |\xi_\alpha|^2$ for any $\alpha \in I$. Then, the statement (a) of Proposition 2.2 holds. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_G \left|f_\alpha^{sx}(s^{-1}t) - f_\alpha^{xs}(xs^{-1})\right| d\lambda_G(t) &= \int_G \left|f_\alpha(sx, s^{-1}t) - f_\alpha(xs, xs^{-1})\right| d\lambda_G(t) \\ &\leq \left(\int_G \left(\left|\xi_\alpha(sx, s^{-1}t)\right| + \left|\xi_\alpha(xs, xs^{-1})\right|\right)^2 d\lambda_G(t)\right)^{\frac{1}{2}} \\ &\quad \left(\int_G \left|\xi_\alpha(sx, s^{-1}t) - \xi_\alpha(xs, xs^{-1})\right|^2 d\lambda_G(t)\right)^{\frac{1}{2}}.\end{aligned}$$

Then by the statement (ii), on compact subsets, we have,

$$\lim_\alpha \int_G \left|f_\alpha^{sx}(s^{-1}t) - f_\alpha^{xs}(xs^{-1})\right| d\lambda_G(t) = 0.$$

Thus, the statement (iii) of Proposition 2.2 holds. Hence, (X,G) is inner amenable. $\square$

Following [1], we call a complex-valued function h defined on the transformation group $X \times G$ a positive type function if, for every $x \in X$, $n \in \mathbb{N}$, $t_1, \ldots, t_n \in G$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, we have

$$\sum_{i,j} h(t_i^{-1}x, t_i^{-1}t_j)\overline{\alpha_i}\alpha_j \geq 0.$$

Let (X,G) be a transformation group. Then with the following action the product space $X \times X$ becomes a $G \times G$-space: $(s,t)\cdot(x,y) = (sx, ty)$, for every $(x,y) \in X \times X$ and $(s,t) \in G \times G$. Then, according to the definition of positive type function on $X \times G$, a complex-valued function h defined on $X \times X \times G \times G$ is called positive type function if, for every $(x,y) \in X \times X$, $n \in \mathbb{N}$ and $(s_1,t_1), \ldots, (s_n,t_n) \in G \times G$, the matrix $\left[h(s_i^{-1}x, t_i^{-1}y, s_i^{-1}s_j, t_i^{-1}t_j)\right]$ is positive. A closed subset B of $(X \times G) \times (X \times G)$ is called proper with respect to projections or π-proper if for every compact subset K of $x \times G$, the sets $[K \times (X \times G)] \cap B$ and $[(X \times G) \times K] \cap B$ are compact [1]. A continuous function h on $(X \times G) \times (X \times G)$ is called properly supported if its support is π-proper. The property (W) is introduced in [1, Definition 4.3]; a transformation group (X,G) has property (W) if, for every compact subset K of $X \times G$ and every $\varepsilon > 0$ there is a continuous bounded positive type, properly supported, function h on $(X \times G) \times (X \times G)$ such that $|h(x,t,x,t) - 1| \leq \varepsilon$, for all $(x,t) \in K$.

Corollary 2.6. *Every inner amenable transformation group has property* (W).

Proof. Let (X,G) be an inner amenable transformation group. Given $\varepsilon > 0$ and compact subset K of $X \times G$, in light of Proposition 2.2(iii), there exists $f \in C_C(X \times G)^+$ such that $|f(x,t) - 1| \leq \varepsilon$, for every $(x,t) \in K$. Similar to the proof of Corollary 2.5, define $\xi = \sqrt{f}$. We now define h on $(X \times G) \times (X \times G)$ by $h(x,s,y,t) = \xi(x,s)\xi(y,t)$, for every $(x,s,y,t) \in (X \times G) \times (X \times G)$. Then h is a continuous of positive type function and compactly supported on $(X \times G) \times (X \times G)$ and from Corollary 2.5, we have $|h(x,t,x,t) - 1| = |\xi(x,t)^2 - 1| = |f(x,t) - 1| \leq \varepsilon$, for every $(x,t) \in X \times G$. Thus, (X,G) has property (W). $\square$

References

[1] C. Anantharaman-Delaroche, *Amenability and exactness for dynamical systems and their C^*-algebras*, Trans. Amer. Math. Soc., **364**(10)(2002), 4153-4178.
[2] C. Anantharaman-Delaroche, *Amenability and exactness for groups, group actions and operator algebras*, École thématique. Erwin Schrdinger Institute, Mars 2007, 2007.
[3] M. Choda, *The factors of inner amenable groups*, Math. Japon., **24**(1979), 145-152.
[4] M. Choda, *Inner amenability and fullness*, Proc. Amer. Math. Soc., **86**(1982), 663-666.
[5] E. G. Effros, *Property T and inner amenability*, Proc. Amer. Math. Soc., **47**(1975), 483-486.
[6] A. T-M. Lau, *Topological semigroups with invariant means in the convex hull of multiplicative means*, Trans. Amer. Math. Soc., **148**(1970), 69-84.
[7] A. T-M. Lau and L. T. Paterson, *Inner amenable locally compact groups*, Trans. Amer. Math. Soc., **325**(1)(1991), 155-169.
[8] V. Losert and H. Rindler, *Conjugate invariant means*, Colloq. Math., **15**(1987), 221-225.
[9] J. P. Pier, *Amenable locally compact groups*, Wiley-Interscience, 1984.
[10] G. Picioroaga, *The inner amenability of the generalized Thompson group*, Proc. Amer. Math. Soc., 134(7), 1995-2002.
[11] K. Sakai, *Amenable transformation Groups. II*, Proc. Japan Acad., **49**(1973), 428-431.
[12] K. Sakai, *Amenable transformation groups*, Sci. Report of Kagoshima Univ., **22**(1973), 1-7.
[13] R. Stokke, *Quasi-central bounded approximate identities in group algebras of locally compact groups*, Illinois J. Math., 48 (2004), 151-170.
[14] R. Stokke, *Approximate diagonals and Følner conditions for amenable group and semigroup algebras*, Studia Math., 164(2)(2004), 139-159.
[15] C. K. Yuan, *The existence of inner invariant means on $L^\infty(G)$*, J. Math. Anal. Appl., **130**(1988), 514-524.
[16] C. K. Yuan, *Inner amenable groups and the induced C^*-algebra Properties*, Acta Mathematica Sinica, **5**(2)(1989), 165-169.

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On exact and woven g-frames

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ABSTRACT. Let $\mathcal{U}$ be a Hilbert space. In this paper we consider woven g-frames for $\mathcal{U}$, especially we study the reordered families of a g-frame $\{\Lambda_i\}_{i\in\mathcal{I}}$ with the various woven problems. First, we state some useful results for exact g-frames and excess of g-frames. Then for $\sigma \subset \mathcal{I}$ we consider the families of weavings $\{\Lambda_i\}_{i\in\sigma} \cup \{\Lambda_{\pi(i)}\}_{i\in\sigma^c}$ where π is a permutation function on $\mathcal{I}$ and obtain some conclusions.
Keywords: g-frame, woven g-frame, permutation function, excess of g-frame.
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1. Introduction

The subject which we study in this manuscript is related to woven g-frames [1]. A generalization of frames are g-frames which are defined by Sun in 2005 [3]. Sun in [3] introduced a type of frames that are called g-frames, and he showed that most generalizations of frames can be regarded as special cases of g-frames. For more details about g-frames we refer the reader to [2, 4, 5]. In this section, first we review the definition of a g-frame and other sujects that we need in this paper [3]. Then we study the exact g-frames and provide some requierd content.

For the Hilbert spaces $\mathcal{U}$ and $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$, let $B(\mathcal{U},\mathcal{V}_i)$ be the Banach space of all bounded linear operators from $\mathcal{U}$ in to $\mathcal{V}_i$ and consider $\Lambda_i \in B(\mathcal{U},\mathcal{V}_i)$, $i \in \mathcal{I}$. The sequence $\{\Lambda_i\}_{i\in\mathcal{I}}$ is called a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$ if there exist two positive constants A and B, that are called the lower and upper g-frame bounds, respectively such that:

$$A\|f\|^2 \le \sum_{i\in\mathcal{I}} \|\Lambda_i f\|^2 \le B\|f\|^2, \qquad \forall f \in \mathcal{U}.$$

A g-frame is said to be a tight g-frame if $A = B$, and also it is said Parseval if $A = 1$. When the sequence $\{\mathcal{V}_i : i \in \mathcal{I}\}$ is clear, $\{\Lambda_i\}_{i\in\mathcal{I}}$ is called a g-frame for $\mathcal{U}$. Also it is called $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$ with respect to $\mathcal{V}$ whenever $\mathcal{V}_i = \mathcal{V}$ for each $i \in \mathcal{I}$. A sequence $\{\Lambda_i\}_{i\in\mathcal{I}}$ is called a g-Bessel sequence with bound B if it satisfies in right hand side in the definition of a g-frame. If a g-frame ceases to be a g-frame whenever anyone of its elements is removed, it is called exact g-frame. A sequence $\{\Lambda_i\}_{i\in\mathcal{I}}$ is called g-complete if $\{f \in \mathcal{U} : \Lambda_i f = 0, i \in \mathcal{I}\} = \{0\}$. If $\{\Lambda_i\}_{i\in\mathcal{I}}$ is g-complete and there are $0 < A \le B < \infty$ such that

$$A\sum_{i\in\mathcal{I}_1} \|g_i\|^2 \le \|\sum_{i\in\mathcal{I}_1} \Lambda_i^* g_i\|^2 \le B\sum_{i\in\mathcal{I}_1} \|g_i\|^2,$$

for any finite subsset $\mathcal{I}_1 \subset \mathcal{I}$ and $g_i \in \mathcal{V}_i$, $i \in \mathcal{I}_1$, then $\{\Lambda_i\}_{i\in\mathcal{I}}$ is called a g-Riesz basis for $\mathcal{U}$.

2. Main Results

The space $\left(\sum_{i\in\mathcal{I}} \oplus \mathcal{V}_i\right)_{l_2}$ is defined by

$$\left(\sum_{i\in\mathcal{I}} \oplus \mathcal{V}_i\right)_{l_2} = \left\{\{f_i\}_{i\in\mathcal{I}} : f_i \in \mathcal{V}_i, \quad i \in \mathcal{I} \quad and \quad \sum_{i\in\mathcal{I}} \|f_i\|^2 < \infty\right\},$$

and has the inner product

$$\langle \{f_i\}, \{g_i\}\rangle = \sum_{i\in\mathcal{I}} \langle f_i, g_i\rangle.$$

*speaker

It is clear that $\left(\sum_{i\in\mathcal{I}}\oplus\mathcal{V}_i\right)_{l_2}$ is a Hilbert space. By consider

$$\mathcal{V}'_i = (...,0,0,0,\mathcal{V}_i,0,0,0,...),$$

without lose of generality we can assume that for each $i\in\mathcal{I}$, $\mathcal{V}_i$ is a subspace of $\left(\sum_{i\in\mathcal{I}}\oplus\mathcal{V}_i\right)_{l_2}$.

Here, we give some results that are useful in the rest of this paper. It is well-known that every g-Riesz basis is exact g-frame, but the converse is not true.

PROPOSITION 2.1. *Assume that $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$, $\{\Theta_i\}_{i\in\mathcal{I}}$ is the canonical g-dual frame for $\{\Lambda_i\}_{i\in\mathcal{I}}$, $i_0\in\mathcal{I}$ is arbitrary and* $\dim\mathcal{V}_\mathrm{i}<\infty$ *for each $i\in\mathcal{I}$. The following statements are equivalent:*

- **(1):** $\{\Lambda_i\}_{i\in\mathcal{I}}$ *is an exact g-frame for $\mathcal{U}$.*
- **(2):** $I-\Theta_{i_0}\Lambda^*_{i_0}$ *is not an injective operator on $\mathcal{V}_{i_0}$.*
- **(3):** $I-\Theta_{i_0}\Lambda^*_{i_0}$ *is not a surjective operator on $\mathcal{V}_{i_0}$.*
- **(4):** $\{\Lambda_i : i\in\mathcal{I}, i\neq i_0\}$ *is not g-complete.*

In the next, we define the excess of a g-frame which play a basic role in this paper.

DEFINITION 2.2. Assume that $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$. Consider Ψ as follows:

$$\Psi=\left\{\mathcal{J}\subset\mathcal{I} : \{\Lambda_i\}_{i\in\mathcal{I}\setminus\mathcal{J}}\ \ is\ \ a\ \ g\text{−frame}\ \ \text{for}\ \ \mathcal{U}\right\}.$$

Set $\kappa=\sup\{|\mathcal{J}| : \mathcal{J}\in\Psi\}$, where $|\mathcal{J}|$ is the cardinal number of $\mathcal{J}$. Now for each $\mathcal{J}_0\in\Psi$ with $\kappa=|\mathcal{J}_0|$, we say $\{\Lambda_i\}_{i\in\mathcal{J}_0}$ is excess of $\{\Lambda_i\}_{i\in\mathcal{I}}$.

A result about excess of g-frames and exact g-frames is given in following theorem.

THEOREM 2.3. *Assume that $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$, $\{\Theta_i\}_{i\in\mathcal{I}}$ is the canonical g-dual frame, $\mathcal{J}\subset\mathcal{I}$ and* $\dim\mathcal{V}_\mathrm{i}<\infty$ *for each $i\in\mathcal{I}$. The following statements are equivalent:*

- **(1):** $I-\Theta_i\Lambda^*_i$ *is an invertible operator on $\mathcal{V}_i$ for all $i\in\mathcal{J}$.*
- **(2):** $\{\Lambda_i : i\in\mathcal{I}\setminus\mathcal{J}\}$ *is a g-frame for $\mathcal{U}$.*
- **(3):** $\{\Lambda_i\}_{i\in\mathcal{J}}$ *is a nonempty subset of* excess *of* $\{\Lambda_i\}_{i\in\mathcal{I}}$.

Now, we consider the relation of the families $\{\Lambda_i\}_{i\in\sigma}\cup\{\Lambda_{\pi(i)}\}_{i\in\sigma^c}$ of a g-frame $\{\Lambda_i\}_{i\in\mathcal{I}}$ and permutaion functions on $\mathcal{I}$. At first, we mention some examples that are motivations for us. Then we state and prove a theorem which gives an equivalance condition for the reordered weavings of a g-frame. Throughout the paper, reordered weavings of $\{\Lambda_i\}_{i\in\mathcal{I}}$ are families of the form $\{\Lambda_i\}_{i\in\sigma}\cup\{\Lambda_{\pi(i)}\}_{i\in\sigma^c}$ where $\sigma\subset\mathcal{I}$ and π is a permutation function on $\mathcal{I}$.

EXAMPLE 2.4. Suppose $\{\Lambda_1,\Lambda_2,\dots,\Lambda_M\}$ and $\{\Gamma_1,\Gamma_2,\dots,\Gamma_M\}$ are g-Riesz bases for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i=1}^M$ such that

$$\Gamma_i=\begin{cases}\Lambda_{j_0} & i=i_0\\ \Lambda_{i_0} & i=j_0\\ \Lambda_i & i\neq i_0,j_0.\end{cases}$$

Then $\{\Lambda_1,\Lambda_2,\dots,\Lambda_M\}$ and $\{\Gamma_1,\Gamma_2,\dots,\Gamma_M\}$ are not woven.

In the above example, $\Gamma_i=\Lambda_{\pi(i)}$ where π is a permutation function on $\{1,\dots,M\}$ defined by

$$\pi(i)=\begin{cases}i_0, & i=j_0\\ j_0, & i=i_0\\ i, & i\neq i_0,j_0.\end{cases}$$

EXAMPLE 2.5. Assume that $\{\Lambda_1,\Lambda_2,\Lambda_3,\dots\}$ is a g-Riesz basis for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathbb{N}}$, and consider the family

$$\{\Gamma_1,\Gamma_2,\Gamma_3,\dots\}=\{\Lambda_3,\Lambda_1,\Lambda_5,\Lambda_2,\Lambda_7,\Lambda_4,\Lambda_9,\Lambda_6,\dots\},$$

for $\mathcal{U}$. Then for some $\sigma\subset\mathbb{N}$ the family $\{\Lambda_i\}_{i\in\sigma}\cup\{\Gamma_i\}_{i\in\sigma^c}$ is a g-frame for $\mathcal{U}$, but for each $\sigma\subset\mathbb{N}$ the family $\{\Lambda_i\}_{i\in\sigma}\cup\{\Gamma_i\}_{i\in\sigma^c}$ is not a g-Riesz basis for $\mathcal{U}$.

EXAMPLE 2.6. Let $\{E_1, E_2, \ldots, E_r\}$ be a g-orthonormal basis for $\mathbb{C}^n$ with respect to $\mathbb{C}^m$ and $r > 1$. Define a g-frame $\{\Lambda_i\}_{i\in\mathbb{Z}}$ for $\mathbb{C}^n$ by

$$\Gamma_i = \begin{cases} 2^{-|\frac{k}{2}|}E_1, & i = rk \\ 2^{-|\frac{k}{2}|}E_2, & i = rk+1 \\ \quad\vdots \\ 2^{-|\frac{k}{2}|}E_r, & i = rk+r-1, \end{cases}$$

where $k \in \mathbb{Z}$. Let $S_l(i) = i + l$ be the l-shift operator on $\mathbb{Z}$ for $l \in \mathbb{Z}$. Then $\{\Gamma_i\}_{i\in\mathcal{I}}$ and $\{\Gamma_{S_l(i)}\}_{i\in\mathcal{I}}$ are woven g-frames for $\mathbb{C}^n$ if and only if $l = rq$ for some $q \in \mathbb{Z}$.

The previous examples are motivations for us to study the reordered weavings of a g-frame $\{\Lambda_i\}_{i\in\mathcal{I}}$, becuase in all of them $\Gamma_i = \Lambda_{\pi(i)}$ where π is a permutation function. The next theorem gives a necessary and sufficient condition about this subject. In following, $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$, and $\dim\mathcal{V}_i < \infty$ for each $i \in \mathcal{I}$.

THEOREM 2.7. *Assume that $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$. For $\mathcal{J} \subset \mathcal{I}$, the following statements are equivalent:*

(1): *$\{\Lambda_i\}_{i\in\mathcal{I}}$ is not an exact g-frame for $\mathcal{U}$.*

(2): *There exists a set of permutation functions $\{\pi_j\}_{j\in\mathcal{J}}$ on $\mathcal{I}$ such that for each $j \in \mathcal{J}$, $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Lambda_{\pi_j(i)}\}_{i\in\mathcal{I}}$ are woven g-frames for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$.*

A helpful result is brought in the following corollary.

COROLLARY 2.8. *Assume that $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$. The following statements are equivalent:*

(1): *There exists a permutation function π on $\mathcal{I}$ such that $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Lambda_{\pi(i)}\}_{i\in\mathcal{I}}$ are woven g-frames for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$.*

(2): *The* excess *of $\{\Lambda_i\}_{i\in\mathcal{I}}$ is nonempty.*

(3): *There exists a proper subset $\mathcal{J}$ of $\mathcal{I}$ such that for each $j \in \mathcal{J}$ the operator $I - \Theta_j\Lambda_j^*$ is invertible on $\mathcal{V}_j$, where $\{\Theta_i\}_{i\in\mathcal{I}}$ is the canonical g-dual frame of $\{\Lambda_i\}_{i\in\mathcal{I}}$.*

PROOF. By the use of Theorems 2.3 and 2.7, the proof is easily to seen. □

EXAMPLE 2.9. In Examples 2.4 and 2.5, the g-frame $\{\Lambda_i\}_{i\in\mathcal{I}}$ is exact. Thus by Corollary 2.8, there is no any permutation function π on $\mathcal{I}$ such that $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Lambda_{\pi(i)}\}_{i\in\mathcal{I}}$ can be woven. But the excess of g-frame $\{\Gamma_i\}_{i\in\mathcal{I}}$ in example 2.6 is nonempty and so by Corollary 2.8, there exists a permutation function π on $\mathcal{I}$ such that $\{\Gamma_i\}_{i\in\mathcal{I}}$ and $\{\Gamma_{\pi(i)}\}_{i\in\mathcal{I}}$ are woven.

The following example shows the part (3) in Corollary 2.8 is beneficial.

EXAMPLE 2.10. Let $\{E_i\}_{i=1}^s$ be a g-orthonormal basis for $B(\mathbb{C}^n, \mathbb{C}^m)$. Consider the family $\{\Gamma_i\}_{i=1}^{2s}$ by

$$\{E_1, E_1, E_2, E_2, \ldots, E_s, E_s\}.$$

For each i, the operator $I - \Theta_i\Gamma_i^*$ is invertible on $\mathbb{C}^m$, where $\{\Theta_i\}_{i=1}^{2s}$ is the canonical g-dual frame of $\{\Gamma_i\}_{i=1}^{2s}$. So by Proposition 2.1 and Corollary 2.8, there exists a permutation function π on $\{1, \ldots, 2s\}$ such that $\{\Gamma_i\}_{i=1}^{2s}$ and $\{\Gamma_{\pi(i)}\}_{i=1}^{2s}$ are woven g-frames.

The next theorem, presents conditions on a g-frame $\{\Lambda_i\}_{i\in\mathcal{I}}$ such that the family $\{\Lambda_i\}_{i\in\mathcal{I}\setminus\mathcal{J}}$ satisfies in Corollary 2.8, where $\mathcal{J} \subset \mathcal{I}$.

THEOREM 2.11. *Suppose $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$ and A, B are the lower and upper bounds. If there exists $\mathcal{J} \subset \mathcal{I}$ such that $\{\Lambda_i\}_{i\in\mathcal{I}\setminus\mathcal{J}}$ is a g-frame and $\{\Lambda_i\}_{i\in\mathcal{J}}$ is a g-Bessel sequence with bound $0 < D < A$, then there exists permutation function π on $\mathcal{I}$ such that $\{\Lambda_i\}_{i\in\mathcal{I}\setminus\mathcal{J}}$ and $\{\Lambda_{\pi(i)}\}_{i\in\mathcal{I}\setminus\mathcal{J}}$ are woven g-frames with lower and upper bounds $A - D$ and $2B$ respectively.*

At the end of this section, by using the subsets of a g-Bessel sequence which are g-frames, we give a proposition that tries to furnish the conditions of Corollary 2.8.

PROPOSITION 2.12. *Suppose $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-Bessel sequence with bound B, and for some $\mathcal{J}\subset\mathcal{I}$, $\{\Lambda_i\}_{i\in\mathcal{J}}$ is a g-frame for $\mathcal{U}$ with lower bounds A. There exists a permutation function π on $\mathcal{I}$ such that $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Lambda_{\pi(i)}\}_{i\in\mathcal{I}}$ are woven g-frames with bounds $A, 2B$.*

References

[1] D. Li, J. Leng, T. Hunge, and X. Li, On weaving g-frames for Hilbert spaces, Complex Analysis and Operator Theory, 14(2) (2020), 125.

[2] A. Nejati, and A. Rahimi, G-frames and stability of g-frames in Hilbert spaces, Methods of Functional Analysis and Topology, 14 (3) (2008), 271-286.

[3] W. Sun, g-frames and g-Riesz bases, Journal of Mathematical Analysis and Applications, 322 (2006) 437-452.

[4] W. Sun, Stability of g-frames, Journal of Mathematical Analysis and Applications, 326 (2007) 858-868.

[5] X. Guo, On Redundancy, dilations and canonical duals of g-frames in Hilbert spaces, Banach Journal of Mathematical Analysis, 9 (4) (2015) 81-99.

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On character amenability of weighted convolution algebras

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Abstract. In this work, we study the character amenability of weighted convolution algebras $\ell^1(S,\omega)$, where S is a semigroup, such as inverse semigroup with uniformly locally finite idempotent set, inverse semigroup with a finite number of idempotents, Clifford semigroup and Rees matrix semigroup. We show that for inverse semigroup with a finite number of idempotents and any weight ω, $\ell^1(S,\omega)$ is character amenable if each maximal semigroup of S, is amenable. Then for a commutative semigroup S and $\omega(x) \geq 1$ for all $x \in S$, we also show that character amenability of $\ell^1(S,\omega)$ implies that S is a Clifford semigroup. Finally, we investigate the character amenability of the weighted convolution algebra $\ell^1(S,\omega)$, and its second dual for a Rees matrix semigroup.
Keywords: Character amenability, Rees matrix semigroup, Weighted Rees matrix semigroup algebra, Clifford semigroup.
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1. Introduction and Preliminaries

Let A be a Banach algebra and E be a Banach A-bimodule. We regards the dual space E^* as a Banach A-bimodule with the following module actions:

$$(a.f)(x) = f(x.a)\ ,\ (f.a)(x) = f(a.x)\ (a \in A, f \in E^*, x \in E).$$

The notion of φ- amenability for Banach algebras was introduced by Kaniuth, Lau and Pymin , where $\varphi : A \longrightarrow \mathbb{C}$ is a character. In [6], and also M. S. Monfared in [5] introduced the notion of character amenability for Banach algebras. Let A be a Banach algebra over $\mathbb{C}$ and $\varphi : A \longrightarrow \mathbb{C}$ be a character on A, that is, an algebra homomorphism from A in to $\mathbb{C}$, and let Φ_A denote the character space of A (that is, the set of all character on A).

We recall some standard notion: for more details, see [2]. A continuous linear operator $D : A \longrightarrow B$ is a *derivation* if it satisfies $D(ab) = D(a).b + a.D(b)$ for all $a, b \in A$. Given $x \in E$, the *inner derivation* $ad_x : A \longrightarrow B$ is defined by $ad_x(a) = a.x - x.a$. According to the Johnsons original definition, a Banach algebra A is *amenable* if for every Banach A-bimodule E, every derivation from A into E^* the dual of E, is inner. The concept of amenability introduced by B. E. Johnson. Let A be a Banach algebra, and let X be a Banach A-bimodule, we let $M^A_{\varphi_r}$ denote the class of Banach A- bimodule X for which the right module action of A on X is given by

$$x.a = \varphi(a)x\ (a \in A\ ,\ x \in X\ ,\ \varphi \in \Phi_A),$$

and $M^A_{\varphi_l}$ denote the class of Banach A-bimodule X for which the left module action of A on X is given by

$$a.x = \varphi(a)x\ (a \in A\ ,\ x \in X\ ,\ \varphi \in \Phi_A).$$

It is easy to see that the left module action of A on the dual module X^* is given by

$$a.f = \varphi(a)f\ (a \in A\ ,\ f \in X^*\ ,\ \varphi \in \Phi_A).$$

Thus, we note that $X \in M^A_{\varphi_r}$ (resp .$X \in M^A_{\varphi_l}$) if and only if $X^* \in M^A_{\varphi_l}$ (resp .$X^* \in M^A_{\varphi_r}$).
Let A be a Banach algebra and let $\varphi \in \Phi_A$, we recall from [6] and [5] that
i) A is *left φ-amenable* if every continuous derivation $D : A \longrightarrow X^*$ is inner for every $X \in M^A_{\varphi_r}$;
ii) A is *right φ-amenable* if every continuous derivation $D : A \longrightarrow X^*$ is inner for every $X \in M^A_{\varphi_l}$;

*speaker

iii) A is *left character amenable* if it is left φ-amenable for every $\varphi \in \Phi_A$;
iv) A is *right character amenable* if it is right φ-amenable for every $\varphi \in \Phi_A$;
v) A is *character amenable* if it is both left and right character amenable.
We also recall that a *semigroup* is a non-empty set S with an associative binary operation $(s,t) \longrightarrow st$, $S \times S \longrightarrow S$ $(s,t \in S)$.
Let S be a semigroup, S is said to be *regular* if for all $s \in S$, there is $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. S is an *inverse semigroup* if such s^* exists and is unique for all $s \in S$. An element $p \in S$ is *idempotent* if $p^2 = p$. The set of idempotents in S is denoted by $E(S)$. A semigroup S is *semilattice* if S is commutative and $E(S) = S$.
Let S be a semigroup. The semigroup algebra $\ell^1(S)$ is the completion in the ℓ^1- norm of the algebra $\mathbb{C}S$. It is the Banach algebra generated by the semigroup. For $s \in S$ we write $\delta_s = \chi_{\{s\}}$ for the indicator function of the set $\{s\}$. The convolution product $*$ on $\ell^1(S)$ is uniquely defined by requiring that $\delta_s * \delta_t = \delta_{st}$ $(s,t \in S)$. These Banach algebras have been studied by many authors. There is always on character on the Banach algebra $\ell^1(S)$: this is the augmentation character $\varphi_S : \ \ell^1(S) \longrightarrow \mathbb{C}$ such that $f \mapsto f(s)$ $s \in S$.
Let S be a semigroup. A continuous function $\omega : S \longrightarrow (0,\infty)$ is a *weight* on S if $\omega(st) \leq \omega(s)\omega(t)$, for all $s,t \in$ S and $\Omega(g) := \omega(g)\omega(g^{-1})$. Then

$$\ell^1(S,\omega) = \{f = \Sigma_{s\in S} f(s)\delta_s : ||f||_\omega = \Sigma_{s\in S}|f(s)|\omega(s) < \infty\},$$

with $\|.\|_\omega$ as the norm, is a Banach algebra which is called *weighted convolution algebra.*

2. Main results

In this section, we will consider the character amenability properties of weighted convolution algebras.

PROPOSITION 2.1. *Let S be a semigroup, ω be a weight on S and $\omega \geq 1$, if $\ell^1(S,\omega)$ is character amenable, then S is amenable and regular.*

COROLLARY 2.2. *Let S be a semigroup with $E(S)$ finite, ω be a weight on S and $\omega \geq 1$. If $\ell^1(S,\omega)$ is character amenable, then it has an identity.*

A semigroup S is called *left cancellative* if, for all $a,x,y \in S$, $ax = ay$ implies that $x = y$.

COROLLARY 2.3. *Let S be a left cancellative semigroup. Let ω be a weight on S and $\omega \geq 1$. If $\ell^1(S,\omega)$ is character amenable, then S is a amenable group.*

THEOREM 2.4. *Let S be an inverse semigroup with $E(S)$ finite and ω be a weight on S, if each maximal semigroup of S is amenable, then $\ell^1(S,\omega)$ is character amenable .*

For an inverse semigroup S, and $p \in E(S)$, we set

$$G_p = \{s \in S \ ; \ ss^{-1} = s^{-1}s = p\}.$$

Then G_p is a group with identity p. It is called the maximal subgroup of S at p. We recall that a Clifford semigroup is an inverse semigroup S for which $ss^{-1} = s^{-1}s$ $(s \in S)$. For a Clifford semigroup S, we have $s \in G_{ss^{-1}}$, and so S is a disjoint union of the groups G_p $(p \in E(S))$. See [3] for more details.

COROLLARY 2.5. *Let $S = \cup_{p\in E(S)} G_p$ be a Clifford semigroup such that $E(S)$ is finite and ω be a weight on S. Then $\ell^1(S,\omega)$ is character amenable if G_p is amenable for each $p \in E(S)$.*

The following example shows that finitness of $E(S)$ is necessary.
Example. Let $S = \cup_{e\in E(S)} G_e$ be a Clifford semigroup such that $E(S)$ is uniformly locally finite and each G_e is amenable, ω be a weight on S and $\omega \geq 1$. Then the weighted convolution algebra $\ell^1(S,\omega)$ is not character amenable if $E(S)$ is not finite; If $\ell^1(S,\omega)$ is character amenable, by hypothesis and theorem **??**, $\ell^1(S)$ is character amenable. But since

$$\ell^1(S) \cong \ell^1 - \bigoplus_{e\in E(S)} \ell^1(G_e).$$

(see [7, Theorem 2.16] and [1, Proposition 6.3]). $\ell^1(S)$ is not character amenable, by [1, Proposition 6.3], and this is contradiction.

THEOREM 2.6. *Let S be a commutative semigroup. Let ω be a weight on S and $\omega \geq 1$. If $\ell^1(S,\omega)$ is character amenable, then S is a Clifford semigroup.*

COROLLARY 2.7. *Let S be a commutative semigroup, with $E(S)$ finite. Let ω be a weight on S and $\omega \geq 1$. Then the following statements are equivalent:*
(i) $\ell^1(S,\omega)$ is character amenable;
(ii) S is a Clifford semigroup.

PROPOSITION 2.8. *Let S be a inverse semigroup such that $(E(S),\leq)$ is uniformly locally finite and ω be a weight on S and $\omega \geq 1$. If $\ell^1(S,\omega)$ is character amenable, then each maximal subgroup of S is amenable.*

3. Weighted Rees matrix semigroup algebras.

In this section, we give results on weighted Rees semigroup algebras. Rees semigroups are described in [3] and [2, Chapter 3] . Indeed, let G be a group, $m,n \in N$, and $G^0 = G \cup \{0\}$. Let

$$S = \{(g)_{ij} \ : \ g \in G, 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{0\},$$

where $(g)_{ij}$ denotes the element of $M_{m\times n}(G^0)$ with g in the $(i,j)^{th}$ place and 0 elsewhere and 0 is a matrix with 0 everywhere. Let $P = (p_{ji})$ be an $n \times m$ matrix over G^0. Then the set S with the composition $(g)_{ij} \circ 0 = 0 \circ (g)_{ij} = 0$ and $(g)_{ij} \circ (h)_{lk} = (gp_{jl}h)_{ik}$, $((g)_{ij},(h)_{lk} \in S)$ forms a semigroup which is called a *Rees matrix semigroup* with a zero over G, and it will be denoted by $S = M^0(G,P,m,n)$.The matrix P is called the *sanwich matrix* in each case.
We write $S = M^0(G,P,n)$ for $S = M^0(G,P,n,n)$ in this case where $m = n$.
The above sandwich matrix P is *regular* if every row and column contains at least one entry in G ; the semigroup $S = M^0(G,P,m,n)$ is regular as a semigroup if and only if the sandwich matrix is regular.
In [2], the Rees matrix semigroup algebra $\ell^1(S)$ is described as follows: for $g \in G$, $(g)_{ij}$ is identified with the element of $M_{m\times n}(\ell^1(G))$ which has δ_g in the $(i,j)^{th}$ place and 0 elsewhere, and $\circ$ is identified with δ_0. Furthermore, $P \in M_{n\times m}(G^0)$ is identified with a matrix $P \in M_{n\times m}(\ell^1(G))$ as follows: if the initial matrix P has $g \in G$ in the $(i,j)^{th}$-position, then the new matrix P has the point mass δ_g in the $(i,j)^{th}$-position; if the first matrix P has 0 in the $(i,j)^{th}$-position, then the new matrix P has 0 in the $(i,j)^{th}$-position. Using this identification, it is shown that $\frac{\ell^1(S)}{\mathbb{C}\delta_0}$ is isometrically isomorphic to the Munn algebra $M(\ell^1(G),P,m,n)$, where $\mathbb{C}\delta_0$ is a one-dimensional ideal. $\frac{\ell^1(S)}{\mathbb{C}\delta_0} = M(\ell^1(G),P,m,n)$, is unital. With $m = n$, since $M(\ell^1(G),P,n,n) = M(\ell^1(G),P,n)$, is also unital and so the Munn algebra $M(\ell^1(G),P,n)$, is topologically isometric to $M_n(\ell^1(G))$. For more details, see [2].
Let S be completely 0 - simple with finitely many idempotents, and let ω be a weight on S (not necessary greater than 1). Then there is a maximal subgroup G of S such that

$$S \simeq M^0(G,P,m,n),$$

and

$$\frac{\ell^1(S,\omega)}{\mathbb{C}\delta_0} \simeq M(\ell^1(G,\omega),P,m,n).$$

See [8, Theorem 2.1], for more details. Let G be a group, and let ω be a weight on G. A weight on G is said to be *symmetric* if $\omega(t^{-1}) = \omega(t)$ $(t \in G)$. In the paper that I have mentioned in comment 5, the authors proved more general case than [4] in the Proposition 4.1.

Theorem 3.1. *Let $S = M^0(G,P,I,J)$, ω be a symmetric weight on S. Then the following statements are equivalent:*
i) $\ell^1(S,\omega)$ is character amenable.
ii) $\ell^1(G,\omega)$ is character amenable, $|I| = |J| < \infty$ and P is invertible.
iii) $\ell^1(S)$ is character amenable and Ω is bounded on G.

Corollary 3.2. *Let $S = M^0(G,P,n)$ be a Rees matrix semigroup with a zero over the group G and sandwich matrix P and ω be a weight on S. Then $\ell^1(S,\omega)$ is character amenable if and only if it is amenable.*

Notation Let S be a semigroup, let I be an ideal of S and let ω be a weight on S. For $s,t \in S$, set $s \sim t$ either if $s = t$ or $s,t \in I$. Clearly, $\sim$ is an equivalence relation on S; the equivalence class containing s is denoted by $[s]$. Let $s,t \in S$ and define $[s][t] = [st]$. Evidently, this gives a well-defined semigroup operation on the set of equivalence classes $S/\sim$. So one may form the quotient semigroup S/I with the zero element I. Moreover, the map $S \longrightarrow S/I$, $s \mapsto [s]$ is an epimorphism. See [3] and [8] for further details.
Define $\tilde{\omega} : S/I \longrightarrow \mathbb{C}$, Such that $\tilde{\omega}([s]) = 1$ for all $s \in I$ and $\tilde{\omega}([s]) = \omega(s)$ for all $s \in S - I$. It is easy to see that $\tilde{\omega}$ is a weight on S/I. Now, we need the following Lemma.

Lemma 3.3. *Let S be a semigroup, let I be an ideal of S, and let ω be a weight on S then:*
i) If $\ell^1(S,\omega)$ is character amenable, then $\ell^1(S/I,\tilde{\omega})$ is character amenable.
ii) If both $\ell^1(S/I,\tilde{\omega})$ and $\ell^1_0(I,\omega)$ are character amenable, then $\ell^1(S,\omega)$ is character amenable.
iii) If $\ell^1(S,\omega)$ is character amenable and $\ell^1_0(I,\omega)$ has a bounded approximate identity, then $\ell^1_0(I,\omega)$ is character amenable.

Theorem 3.4. *Let S be a semigroup and let ω be a symmetric weight on S. Then the following statements are equivalent;*
i) $\ell^1(S,\omega)$ is character amenable;
ii) $\ell^1(S)$ is character amenable and Ω is bounded on every maximal subgroup G of S.

Proposition 3.5. *Let $S = M^0(G,P,I,J)$, ω be a weight on S. Then the following statements are equivalent:*
i) $\ell^1(S,\omega)^{**}$ is character amenable.
ii) S is finite, $|I| = |J| = n$ and P is invertible.
iii) $\ell^1(S)$ is character amenable and S is finite.

References

1. H. Pourmahmood- Aghababa, L.Y. Shi and Y. J. Wu, Generalized notions of character amenability, Acta Mathematica Sinica, English series, 29, Issue 7, (2013), 1329-1350.
2. H. G. Dales, A. T. Lau and D.Strauss, *Banach algebras on semigroups and their compactifications*, Mem. Amer. Math.Soc, 205, 2010.
3. J. M. Howie, *Fundamentals of semigroup theory*, London Mathematical Society Monographs, Volume 12, The Clarendon Press Oxford, 1995.
4. O. T. Mewomo and S. M. Maepa, *On character Amenability of Beurling and second dual algebras*, Acta Universitatis Apulensis, No. 38/2014, 67-80.
5. M. S. Monfared, *Character amenability of Banach algebras*, Math. Proc. Cambridge philos. Soc. 144 (2008), 697-706.
6. Z. Hu, M. S. Monfared and T. Traynor, *On character amenable of Banach algebra*, Studia Math. 193 (1) (2009), 53-78.
7. P. Ramsden, *Biflatness of semigroup algebras*, Semigroup Forum 79 (2009) , 515-530.
8. M. Soroshmehr, *Weighted Ress matrix semigroup and their applications*, Arch , Math , 100 (2013) 139-147.

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Characterization of 2-cocycles and 2-coboundaries on Direct Sum of Banach Algebras

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Abstract. Let A and B be Banach algebras. In this paper, we investigate the structure of 2-cocycles and 2-coboundaries on $A \oplus B$, when A and B are unital. Actually, we provide a specific criterion for each 2-cocycle maps and establish a connection between 2-cocycles and 2-coboundaries on $A \oplus B$ and 2-cocycles and 2-coboundaries on $A \oplus B$ on A and B. Finally, our results lead to a connection between $\mathcal{H}^2(A, A^*)$,$\mathcal{H}^2(B, B^*)$ and $\mathcal{H}^2(A \oplus B, A^* \oplus B^*)$.
Keywords: 2-cocycle map, 2-coboundary map, Second Hochschild cohomology group.
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1. Introduction

A derivation from a Banach algebra A to a Banach A-bimodule X is a bounded linear mapping $D : A \to X$ such that $D(a_1a_2) = a_1 \cdot D(a_2) + D(a_1) \cdot a_2 \ \ (a_1, a_2 \in A)$. For each $x \in X$ the mapping $\mathbf{ad}_x : a \mapsto a \cdot x - x \cdot a \ \ (a \in A)$, is a derivation, called the inner derivation implemented by x. The space of all derivations is denoted by $\mathcal{Z}^1(A, X)$ and the space of all ineer derivations is denoted by $\mathcal{B}^1(A, X)$. The first Hochschild cohomology group $\mathcal{H}^1(A, X)$ is defined by the quotient

$$\mathcal{H}^1(A, X) = \frac{\mathcal{Z}^1(A, X)}{\mathcal{B}^1(A, X)}.$$

An 2-cocycle map from a Banach algebra A to a Banach A-bimodule X is a bounded linear mapping $\varphi : A \times A \to X$ such that

$$a_1 \cdot \varphi(a_2, a_3) - \varphi(a_1a_2, a_3) + \varphi(a_1, a_2a_3) - \varphi(a_1, a_2) \cdot a_3 = 0,$$

for all $a_1, a_2, a_3 \in A$. For each bounded linear map $\psi : A \to X$, the mapping $\mathbf{ad}_\psi : A \times A \to X$ defined with

$$\mathbf{ad}_\psi(a_1, a_2) = a_1 \cdot \psi(a_2) - \psi(a_1a_2) + \psi(a_1) \cdot a_2,$$

is an 2-cocycle map, called the 2-coboundary map. The space of all 2-cocycle maps is denoted by $\mathcal{Z}^2(A, X)$ and the space of all 2-coboundary maps is denoted by $\mathcal{B}^2(A, X)$. The second Hochschild cohomology group $\mathcal{H}^2(A, X)$ is defined by the quotient

$$\mathcal{H}^2(A, X) = \frac{\mathcal{Z}^2(A, X)}{\mathcal{B}^2(A, X)}.$$

Let A and B be Banach algebras and M be a Banach A, B-module. Suppose that

$$\mathcal{T} = \left\{ \begin{bmatrix} a & m \\ & b \end{bmatrix} : a \in A, b \in B, m \in M \right\},$$

be the corresponding triangular Banach algebra. Forrest and Marcoux investigated and studied derivations on triangular Banach algebra of $\mathcal{T}$ in [2] and analysed the first Hochschild cohomology group $\mathcal{H}^1(\mathcal{T}, \mathcal{T})$. They in [3] characterize derivation $D : \mathcal{T} \to \mathcal{T}^*$ according to derivations $D_A : A \to A^*$, $D_B : B \to B^*$ and $\gamma_D \in M^*$, as follows

$$D\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right) = \begin{bmatrix} D_A(a) - m\gamma_D & \gamma_D a - b\gamma_D \\ & D_B(b) + \gamma_D m \end{bmatrix}.$$

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They proved using this characterization,

$$\mathcal{H}^1(\mathcal{T},\mathcal{T}^*)\simeq \mathcal{H}^1(A,A^*)\oplus\mathcal{H}^1(B,B^*),$$

of course, if we restrict ourself to the case of unitary A, B and M. In particular, in case of $M=0$, they proved that

$$\mathcal{H}^1(A\oplus B,A^*\oplus B^*)\simeq \mathcal{H}^1(A,A^*)\oplus\mathcal{H}^1(B,B^*).$$

In this paper, somehow we want to provide same characterization for 2-cocycle maps in $\mathcal{Z}^2(A\oplus B,A^*\oplus B^*)$. Indeed, we show that if $\varphi=(\varphi_1,\varphi_2)\in\mathcal{Z}^2(A\oplus B,A^*\oplus B^*)$, then there exist $\varphi_A\in\mathcal{Z}^2(A,A^*)$, $\varphi_B\in\mathcal{Z}^2(B,B^*)$ and $\psi=(\psi_1,\psi_2)\in\mathcal{C}^1(A\oplus B,A^*\oplus B^*)$ such that for every $\omega_1=(a_1,b_1),\omega_2=(a_2,b_2)\in A\oplus B$,

$$\begin{aligned}\varphi_1(\omega_1,\omega_2)&=\varphi_A((a_1,a_2))+a_1\psi_1(\omega_2)-\psi_1(\omega_1\omega_2)+\psi_1(\omega_1)a_2\\ \varphi_2(\omega_1,\omega_2)&=\varphi_B(b_1,b_2)+b_1\psi_2(\omega_2)+\psi_2(\omega_1\omega_2)+\psi_2(\omega_1)b_2,\end{aligned}$$

that leads to $\varphi-\varphi_{AB}=\mathbf{ad}_\psi\in\mathcal{B}^2(A\oplus B,A^*\oplus B^*)$, where $\varphi_{AB}:(A\oplus B)\times(A\oplus B)\to A^*\oplus B^*$ is defined by

$$\varphi_{AB}\left((a_1,b_1),(a_2,b_2)\right)=\left(\varphi_A(a_1,a_2),\varphi_B(b_1,b_2)\right).$$

In the following, as an application of our results, we show that

$$\mathcal{H}^2(A\oplus B,A^*\oplus B^*)\simeq \mathcal{H}^2(A,A^*)\oplus\mathcal{H}^2(B,B^*).$$

2. Main Results

Let A and B be unital Banach algebras with units $\mathbf{1}_A$ and $\mathbf{1}_B$, respectively. ℓ^1-direct sum of Banach algebras $A\oplus B$ with usual multiplication and the norm $\|(a,b)\|=\|a\|+\|b\|$ is a Banach algebra. It is clear that $(A\oplus B)^*\simeq A^*\oplus B^*$. Let $(a,b)\in A\oplus B$ and $(\mu,\theta)\in A^*\oplus B^*$. Then the action of $A^*\oplus B^*$ upon $A\oplus B$ is given by $(\mu,\theta)((a,b))=\mu(a)+\theta(b)$. Also, it is easy to check that module action $A\oplus B$ on $A^*\oplus B^*$ are as follows:

$$(a,b)\cdot(\mu,\theta)=(a\cdot\mu,b\cdot\theta)\quad\text{and}\quad(\mu,\theta)\cdot(a,b)=(\mu\cdot a,\theta\cdot b).\tag{1}$$

Throughout we will remove the dot (the sign " $\cdot$ ") for simplicity.

Remark 2.1. (i) We note that $\varphi\in\mathcal{Z}^2(A\oplus B,A^*\oplus B^*)$ if and only if φ satisfies the 2-cocycle equation

$$\omega_1\varphi(\omega_2,\omega_3)-\varphi(\omega_1\omega_2,\omega_3)+\varphi(\omega_1,\omega_2\omega_3)-\varphi(\omega_1,\omega_2)\omega_3=0\qquad(\omega_1,\omega_2,\omega_3\in A\oplus B).\tag{2}$$

(ii) Let $\psi\in\mathcal{C}^1(A\oplus B,A^*\oplus B^*)$ and $\varphi\in\mathcal{C}^2(A\oplus B,A^*\oplus B^*)$. For $i=1,2$ assume that $\psi_i:=\pi_i\circ\psi$ ($\varphi_i:=\pi_i\circ\varphi$), denote the coordinate functions associated to ψ (φ), when π_i be the projection msps on arrays. That is

$$\psi(\omega)=(\psi_1(\omega),\psi_2(\omega))\quad\text{and}\quad\varphi\left(\omega_1,\omega_2\right)=(\varphi_1(\omega_1,\omega_2),\varphi_2(\omega_1,\omega_2))\quad(\omega_1,\omega_2,\omega_3\in A\oplus B).$$

We will often write $\psi=(\psi_1,\psi_2)$ and $\varphi=(\varphi_1,\varphi_2)$.

Lemma 2.2. *Let $\varphi\in\mathcal{Z}^2(A\oplus B,A^*\oplus B^*)$. Suppose that $\varphi_A:A\times A\to A^*$ and $\varphi_B:B\times B\to B^*$ defined by*

$$\varphi_A(a_1,a_2):=\varphi_1((a_1,0),(a_2,0))\qquad\text{and}\qquad\varphi_B(b_1,b_2):=\varphi_2((0,b_1),(0,b_2)).\tag{3}$$

Then $\varphi_A\in\mathcal{Z}^2(A,A^*)$ and $\varphi_B\in\mathcal{Z}^2(B,B^*)$. Conversely, if $\varphi_A\in\mathcal{Z}^2(A,A^*)$, $\varphi_B\in\mathcal{Z}^2(B,B^*)$ and $\varphi_{AB}:(A\oplus B)\times(A\oplus B)\to A^*\oplus B^*$ is defined by

$$\varphi_{AB}\left((a_1,b_1),(a_2,b_2)\right)=\left(\varphi_A(a_1,a_2),\varphi_B(b_1,b_2)\right),$$

then $\varphi_{AB}\in\mathcal{Z}^2(A\oplus B,A^*\oplus B^*)$. Furthermore, $\varphi_{AB}\in\mathcal{B}^2(A\oplus B,A^*\oplus B^*)$ if and only if $\varphi_A\in\mathcal{B}^2(A,A^*)$ and $\varphi_B\in\mathcal{B}^2(B,B^*)$.

Proof. Left to the reader. □

PROPOSITION 2.3. *Let $\varphi \in \mathcal{Z}^2(A \oplus B, A^* \oplus B^*)$. Then there exists $\psi \in \mathcal{C}^1(A \oplus B, A^* \oplus B^*)$ such that for each $\omega_1 = (a_1, b_1)$ and $\omega_2 = (a_2, b_2)$ in $A \oplus B$ we have*

$$\varphi(\omega_1, \omega_2) = \begin{pmatrix} \varphi_A(a_1, a_2) + a_1\psi_1(\omega_2) & b_1\psi_2(\omega_2) - \psi_2(\omega_1\omega_2) \\ -\psi_1(\omega_1\omega_2) + \psi_1(\omega_1)a_2 & +\psi_2(\omega_1)b_2 + \varphi_B(b_1, b_2) \end{pmatrix}. \tag{4}$$

PROOF. First define

$$\mu : B \to A^* \quad \text{by} \quad \mu(b) := \varphi_1\left((0, b), (\mathbf{1}_A, 0)\right), \tag{5}$$

$$\theta : A \to B^* \quad \text{by} \quad \theta(a) := \varphi_2\left((0, \mathbf{1}_B), (a, 0)\right). \tag{6}$$

We continue the proof in four steps. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$. In each step, we use the actions (1) and 2-cocycle equation (2) for φ with different elements of $\omega_1, \omega_2, \omega_3$ in $A \oplus B$.

step 1. If $\omega_1 = (0, \mathbf{1}_B), \omega_2 = (a_1, 0)$ and $\omega_3 = (a_2, 0)$, then

$$(0, \mathbf{1}_B)\varphi\left((a_1, 0), (a_2, 0)\right) = -\varphi\left((0, \mathbf{1}_B), (a_1a_2, 0)\right) + \varphi\left((0, \mathbf{1}_B), (a_1, 0)\right)(a_2, 0),$$

which according to (3) and (6), we obtain

$$\varphi\left((a_1, 0), (a_2, 0)\right) = \left(\varphi_A(a_1, a_2), -\theta(a_1a_2)\right). \tag{7}$$

step 2. If $\omega_1 = (a_1, 0), \omega_2 = (0, b_2)$ and $\omega_3 = (\mathbf{1}_A, 0)$, then

$$\varphi((a_1, 0), (0, b_2))(\mathbf{1}_A, 0) = (a_1, 0)\varphi((0, b_2), (\mathbf{1}_A, 0)).$$

Also if $\omega_1 = (0, \mathbf{1}_B), \omega_2 = (a_1, 0)$ and $\omega_3 = (0, b_2)$, then

$$(0, \mathbf{1}_B)\varphi((a_1, 0), (0, b_2)) = \varphi((0, \mathbf{1}_B), (a_1, 0))(0, b_2).$$

Using (5), (6) and the two previous equalities, we obtain

$$\varphi((a_1, 0), (0, b_2)) = (a_1\mu(b_2), \theta(a_1)b_2). \tag{8}$$

step 3. If $\omega_1 = (0, b_1), \omega_2 = (0, \mathbf{1}_B)$ and $\omega_3 = (a_2, 0)$, then

$$\varphi((0, b_1), (a_2, 0)) = (0, b_1)\varphi((0, \mathbf{1}_B), (a_2, 0)) - \varphi((0, b_1), (0, \mathbf{1}_B))(a_2, 0),$$

which according to (6) and the previous equality, we obtain

$$\varphi((0, b_1), (a_2, 0)) = (\mu(b_1)a_2, b_1\theta(a_2)). \tag{9}$$

step 4. If $\omega_1 = (0, b_1), \omega_2 = (0, b_2)$ and $\omega_3 = (\mathbf{1}_A, 0)$, then

$$\varphi((0, b_1), (0, b_2))(\mathbf{1}_A, 0) = (0, b_1)\varphi((0, b_2), (\mathbf{1}_A, 0)) - \varphi((0, b_1b_2), (\mathbf{1}_A, 0)).$$

Using (3), (5) and the previous equaliy, we obtain

$$\varphi((0, b_1), (0, b_2)) = (-\mu(b_1b_2), \varphi_B(b_1, b_2)). \tag{10}$$

Now we sum the relationships (7) to (10). We have

$$\varphi(\omega_1, \omega_2) = \begin{pmatrix} \varphi_A(a_1, a_2) + a_1\mu(b_2) & -\theta(a_1a_2) + \theta(a_1)b_2 \\ +\mu(b_1)a_2 - \mu(b_1b_2) & +b_1\theta(a_2) + \varphi_B(b_1, b_2) \end{pmatrix}.$$

Next let us assume that $\psi = (\psi_1, \psi_2) : A \oplus B \longrightarrow A^* \oplus B^*$, to be defined by

$$\psi_1 : A \oplus B \to A^* \quad \text{by} \quad \psi_1((a, b)) := \mu(b),$$
$$\psi_2 : A \oplus B \to B^* \quad \text{by} \quad \psi_2((a, b)) := \theta(a).$$

That ψ is bounded and linear, is given to the reader. Therefore $\psi \in \mathcal{C}^1(A \oplus B, A^* \oplus B^*)$ and also

$$\varphi(\omega_1, \omega_2) = \begin{pmatrix} \varphi_A(a_1, a_2) + a_1\psi_1(\omega_2) & b_1\psi_2(\omega_2) - \psi_2(\omega_1\omega_2) \\ -\psi_1(\omega_1\omega_2) + \psi_1(\omega_1)a_2 & +\psi_2(\omega_1)b_2 + \varphi_B(b_1, b_2) \end{pmatrix}.$$

So (4) is valid and the proof is complete. □

So far, we have established a connection between 2-cocycles on A, B and $A \oplus B$. We want to extend this connection to 2-coboundaries and for this, we need the following Lemma.

Lemma 2.4. *Let* $\varphi \in \mathcal{Z}^2(A \oplus B, A^* \oplus B^*)$. *Then* $\varphi \in \mathcal{B}^2(A \oplus B, A^* \oplus B^*)$ *if and only if* $\varphi_A \in \mathcal{B}^2(A, A^*)$ *and* $\varphi_B \in \mathcal{B}^2(B, B^*)$.

Proof. Let $\varphi \in \mathcal{Z}^2(A \oplus B, A^* \oplus B^*)$. By Proposition 2.3, there exists $\psi \in \mathcal{C}^1(A \oplus B, A^* \oplus B^*)$, so that (4) is valid. For each $\omega_1 = (a_1, b_1), \omega_2 = (a_2, b_2)$ in $A \oplus B$, we have

$$\begin{aligned}
(\varphi - \varphi_{AB})(\omega_1, \omega_2) &= \begin{pmatrix} a_1\psi_1(\omega_2) - \psi_1(\omega_1\omega_2) & & b_1\psi_2(\omega_2) - \psi_2(\omega_1\omega_2) \\ \quad + \psi_1(\omega_1)a_2 & , & \quad + \psi_2(\omega_1)b_2 \end{pmatrix} \\
&= (a_1\psi_1(\omega_2), b_1\psi_2(\omega_2)) - (\psi_1(\omega_1\omega_2), \psi_2(\omega_1\omega_2)) + (\psi_1(\omega_1)a_2, \psi_2(\omega_1)b_2) \\
&= (a_1, b_1)(\psi_1(\omega_2), \psi_2(\omega_2)) - (\psi_1(\omega_1\omega_2), \psi_2(\omega_1\omega_2)) + (\psi_1(\omega_1), \psi_2(\omega_1))(a_2, b_2) \\
&= \omega_1\psi(\omega_2) - \psi(\omega_1\omega_2) + \psi(\omega_1)\omega_2 \\
&= \mathbf{ad}_\psi(\omega_1, \omega_2).
\end{aligned}$$

Hence $\varphi - \varphi_{AB} \in \mathcal{B}^2(A \oplus B, A^* \oplus B^*)$. That this means, φ is 2-coboundary, if and only if φ_{AB} is 2-coboundary, and by Lemma 2.2, if and only if φ_A and φ_B are 2-coboundaries. □

Forrest and Marcoux in [3] have studied the first Hochschild cohomology group of triangular Banach algebra $\mathcal{H}^1(A \oplus B, A^* \oplus B^*)$. They showed that

$$\mathcal{H}^1(A \oplus B, A^* \oplus B^*) \simeq \mathcal{H}^1(A, A^*) \oplus \mathcal{H}^1(B, B^*).$$

In this section we wish to identify the second Hochschild cohomology group $\mathcal{H}^2(A \oplus B, A^* \oplus B^*)$. We get similar results with the results of Forrest and Marcoux in [3], but in second order.

Corollary 2.5. *Let A and B be unital Banach algebras. Then*

$$\mathcal{H}^2(A \oplus B, A^* \oplus B^*) \simeq \mathcal{H}^2(A, A^*) \oplus \mathcal{H}^2(B, B^*).$$

Proof. Let $\varphi \in \mathcal{Z}^2(A \oplus B, A^* \oplus B^*)$. By Lemma 2.2 and Proposition 2.3, there exists $\varphi_A \in \mathcal{Z}^2(A, A^*)$, $\varphi_B \in \mathcal{Z}^2(B, B^*)$ and $\psi \in \mathcal{C}^1(A \oplus B, A^* \oplus B^*)$ such that (4) is valid.

Consider the map

$$\begin{aligned}
\Gamma : \mathcal{Z}^2(A \oplus B, A^* \oplus B^*) &\to \mathcal{H}^2(A, A^*) \oplus \mathcal{H}^2(B, B^*) \\
\varphi \quad &\mapsto (\varphi_A + \mathcal{B}^2(A, A^*),\ \varphi_B + \mathcal{B}^2(B, B^*)).
\end{aligned}$$

Using Lemma 2.2 and Proposition 2.3, it can be easily proved Γ is well-define and surjective and $\ker \Gamma = \mathcal{B}^2(A \oplus B, A^* \oplus B^*)$. This completes the proof. □

Example 2.6. Consider $A = B = \ell^1(S)$, where S is Clifford semigroup. Using Corollary 2.5, [1, Theorem 3.2], [1, Theorem 3.3] and [4, Theorem 4.6], we have

(i) $\mathcal{H}^2(\ell^1(S) \oplus \ell^1(S), \ell^\infty(S) \oplus \ell^\infty(S)) = 0$, for $S = \mathbb{Z}_+$.

(ii) $\mathcal{H}^2(\ell^1(S) \oplus \ell^1(S), \ell^\infty(S) \oplus \ell^\infty(S)) = 0$, for unital semilattice S.

(iii) $\mathcal{H}^2(\ell^1(S) \oplus \ell^1(S), \ell^\infty(S) \oplus \ell^\infty(S))$ is Banach space for unital Clifford semigroup S.

References

1. H. G. Dales and J. Duncan, *Second-order cohomology groups of some semigroup algebras*, In Banach algebras, Walter de gruyter, Berlin, (1998) 101-117.
2. B. E. Forrest and L. W. Marcoux, *Derivation of triangular Banach algrbras*, Indiana, Univ. Math. Soc, 45 (1996), 441-462.
3. B. E. Forrest and L. W. Marcoux, *Weak amenability of triangular Banach algrbras*, Tranc. Amer, Math. Soc, 354(2002), 1435-1452.
4. F. Gordeau, A. R. Pourabbas, and M. C. White, *Simplicial cohomology of some semigroup algebras*, Canadian Mathematical Bulletin, 50(1)(2007), 56-70.

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Continuous frames and Riesz basis in Hilbert C^*-modules

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Abstract. The paper is devoted to continuous frames and Riesz bases in Hilbert C^*-modules. We characterize the operator frame of continuous frames in Hilbert C^*-modules. Then, we define a continuous Riesz basis for Hilbert C^*-modules and give some results about them.
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1. Introduction

In 1952, the concept of discrete frames for Hilbert spaces were introduced by Duffin and Schaeffer [5] to study some problems in nonharmonic Fourier series. Frame theory has been used in many fields such as filter bank theory, image processing, etc. we refer to [4] for an introduction to frame theory in Hilbert spaces and its applications. The concept of a generalization of frames to a family indexed by some locally compact space endowed with a Radon measure was proposed by Ali, Antoine and Gazeau [1]. These frames are known as continuous frames. In 2013, Arefijamaal, Kamyabi Gol, Raisi Tousi and Tavallaei introduced continuous Riesz bases and give some equivalent conditions for a continuous frame to be a continuous Riesz basis [2].

Frank and Larson [6] presented a general approach to the frame theory in Hilbert C^*-modules. Theory of frames have been extended from Hilbert spaces to Hilbert C^*-modules, see [7, 8].

In this paper, we characterize the operator frame of continuous frames in Hilbert C^*-modules. Then, we define a continuous Riesz basis for Hilbert C^*-modules and give some results about them.

First, we recall some definitions and basic properties of Hilbert C^*-modules.

Definition 1.1. A pre-Hilbert module over C^*-algebra $\mathcal{A}$ is a complex vector space U which is also a left $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : U \times U \to \mathcal{A}$ which is $\mathbb{C}$-linear and $\mathcal{A}$-linear in its first variable and satisfies the following conditions:
(i) $\langle f, f\rangle \geq 0$,
(ii) $\langle f, f\rangle = 0$ iff $f = 0$,
(iii) $\langle f, g\rangle^* = \langle g, f\rangle$,
(iv) $\langle af, g\rangle = a\langle f, g\rangle$,
for all $f, g \in U$ and $a \in \mathcal{A}$.

A pre-Hilbert $\mathcal{A}$-module U is called Hilbert $\mathcal{A}$-module if U is complete with respect to the topology determined by the norm $\|f\| = \|\langle f, f\rangle\|^{\frac{1}{2}}$.

2. Main results

In this section, we introduce continuous frames and continuous Riesz bases in Hilbert C^*-modules, and then we give some results for them. We assume that $\mathcal{A}$ is a unital C^*-algebra, U is a Hilbert $\mathcal{A}$-module and (Ω, μ) a measure space.

Let $\mathcal{Y}$ be a Banach space, (Ω, μ) a measure space, and $f : \Omega \to \mathcal{Y}$ a measurable function. Integral of the Banach-valued function f has been defined by Bochner and others. Most properties

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of this integral are similar to those of the integral of real-valued functions. Since every C^*-algebra and Hilbert C^*-module is a Banach space, hence we can use this integral in these spaces.

DEFINITION 2.1. A mapping $F : \Omega \to U$ is called a continuous frame for U if
(i) F is weakly-measurable, i.e, for any $f \in U$, the mapping $\Omega \longmapsto \langle f, F(\Omega)\rangle$ is measurable on Ω.
(ii) There exist $A, B > 0$ such that

$$A\langle f, f\rangle \leq \int_\Omega \langle f, F(\omega)\rangle\langle F(\omega), f\rangle d\mu(\omega) \leq B\langle f, f\rangle, \quad (f \in U). \tag{1}$$

The constant A, B are called lower and upper frame bounds, respectively. The mapping F is called *Bessel* if the right inequality in (1) holds.

A continuous frame $F : \Omega \to U$ is called *exact* if for every measurable subset $\Omega_1 \subseteq \Omega$ with $0 < \mu(\Omega_1) < \infty$, the mapping $F : \Omega\backslash\Omega_1 \to U$ is not a continuous frame for U.

Also we define,

$$L^2(\Omega, A) = \{\varphi : \Omega \to A \quad ; \quad \| \int_\Omega |\varphi(\omega)|^2 d\mu(\omega)\| < \infty\} \tag{2}$$

For any $\varphi, \psi \in L^2(\Omega, A)$, the inner product is defined by $\langle \varphi, \psi\rangle = \int_\Omega \langle \varphi(\omega), \psi(\omega)\rangle d\mu(\omega)$ and the norm is defined by $\|\varphi\| = \|\langle \varphi, \varphi\rangle\|^{\frac{1}{2}}$.

We know that every Hilbert space is a Hilbert C^*-module over $\mathbb{C}$. So we can give an example of continuous Bessel mapping for Hilbert space H as follows.

EXAMPLE 2.2. (see [3])Let H be a Hilbert space. Take an (essentially) unbounded (Lebesgue) measurable function $a : \mathbb{R} \to \mathbb{C}$ such that $a \in L^2(\mathbb{R})\backslash L^\infty(\mathbb{R})$.

Choose a fixed vector $h \in H$, $h \neq 0$. Then the mapping

$$F : \mathbb{R} \to H, \qquad \omega \mapsto F(\omega) := a(\omega).h \tag{3}$$

is weakly (Lebesgue) measurable and a continuous Bessel mapping, since for all $f \in H$,

$$\begin{aligned} \int_\mathbb{R} |\langle f, F(\omega)\rangle|^2 d\mu(\omega) &= \int_\mathbb{R} |a(\omega)|^2 |\langle f, h\rangle|^2 d\mu(\omega) \\ &= |\langle f, h\rangle|^2 \int_\mathbb{R} |a(\omega)|^2 d\mu(\omega) \leq \|h\|^2 \|a\|^2_{L^2(\mathbb{R})} \|f\|^2 \end{aligned}$$

Similar to continuous frames in Hilbert spaces, we introduce a pre-frame operator and frame operator for continuous frames in Hilbert C^*-modules.

THEOREM 2.3. *Let $F : \Omega \to U$ be a Bessel mapping. Then the operator $T_F : L^2(\Omega, A) \to U$ weakly defined by*

$$\langle T_F\varphi, f\rangle = \int_\Omega \varphi(\omega)\langle F(\omega), f\rangle d\mu(\omega), \quad (f \in U). \tag{4}$$

is well defined, bounded, surjective, adjointable A-linear map, and its adjoint $T_F^* : U \to L^2(\Omega, A)$ is defined by

$$(T_F^* f)(\omega) = \langle f, F(\omega)\rangle \quad (\omega \in \Omega). \tag{5}$$

The operator T_F is called a pre-frame operator or synthesis operator and T_F^* is called an analysis operator of F.

DEFINITION 2.4. Let $F : \Omega \to U$ be a continuous frame for Hilbert C^*-module U. Then the frame operator $S_F : U \to U$ is weakly defined by

$$\langle S_F f, f\rangle = \int_\Omega \langle f, F(\omega)\rangle\langle F(\omega), f\rangle d\mu(\omega), \quad (f \in U). \tag{6}$$

COROLLARY 2.5. *Let $F : \Omega \to U$ be a continuous frame for Hilbert C^*-module U. Then $S_F = T_F T_F^*$ is positive, adjointable and invertible.*

Definition 2.6. Let $F : \Omega \to U$ be a Bessel mapping. A Bessel mapping $G : \Omega \to U$ is called a *dual* for F if

$$f = \int_{\Omega} \langle f, G(\omega)\rangle F(\omega) d\mu(\omega), \quad (f \in U). \tag{7}$$

Remark 2.7. For every continuous frame $F : \Omega \to U$ with frame operator S we have,

$$f = SS^{-1}f = \int_{\Omega} \langle f, S^{-1}F(\omega)\rangle F(\omega) d\mu(\omega), \quad (f \in U). \tag{8}$$

Thus $S^{-1}F$, which is a dual for F, so called the *canonical dual*.

Now we define and investigate *Riesz - type* frames and continuous Riesz bases in a Hilbert C^*-module.

Definition 2.8. A continuous frame which has only one dual, is called *Riesz - type* frame.

Theorem 2.9. *Let $F : \Omega \to U$ be a continuous frame for Hilbert C^*-module U. Then F is a Riesz - type frame if and only if T_F^* is surjective.*

Definition 2.10. A mapping $F : \Omega \to U$ is called a continuous Riesz basis for U with respect to (Ω, μ) , if the following two conditions are satisfied:
(i) $\{f \in U \; ; \; \langle f, F(\omega)\rangle = 0 \; a.e \; [\mu]\} = \{0\}$.
(ii) There are two constant $A, B > 0$ such that

$$A\| \int_{\Omega_1} |\varphi(\omega)|^2 d\mu(\omega)\|^{\frac{1}{2}} \leq \| \int_{\Omega_1} \varphi(\omega)F(\omega)d\mu(\omega)\| \leq B\| \int_{\Omega_1} |\varphi(\omega)|^2 d\mu(\omega)\|^{\frac{1}{2}} \tag{9}$$

for every $\varphi \in L^2(\Omega, A)$ and measurable subset Ω_1 of Ω with $\mu(\Omega_1) < +\infty$.

Theorem 2.11. *Let $F : \Omega \to U$ be a continuous frame for Hilbert C^*-module U. Then F is a continuous Riesz basis if and only if F is a Riesz-type continuous frame.*

Corollary 2.12. *A continuous Riesz basis in a Hilbert C^*-module is a continuous exact frame.*

References

1. S. T. Ali, J. P. Antoine, J. P. Gazeau, *Continuous frames in Hilbert spaces*, Annals of Physics 222 (1993), 1-37.
2. A. A. Arefijamaal, R. A. Kamyabi Gol, R. Raisi Tousi and N. Tavallaei, *A new approach to continuous Riesz bases*, J. Sci. Iran, 24(1), 63-69 (2013).
3. P. Balazs, D. Bayer, A. Rahimi, *Multipliers for continuous frames in Hilbert spaces*, J. Phys. A 45 (24) (2012) 244023, 20 pp.
4. O. Christensen, *An introduction to frames and Riesz bases*, Birkhauser, Boston, 2016.
5. R.J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., **72** (1952), 341–366.
6. M. Frank and D. R. Larson, *Frames in Hilbert C^*-modules and C^*-algebras*, J. Operator Theory, **48** (2002), 273–314.
7. S. Kabbaj, H. Labrigui and A. Touri, *Controlled Continuous g-Frames in Hilbert C^*-Modules*, Moroccan J. Pure and Appl. Anal., **6**(2) (2020), 184-197.
8. T.L. Shateri, *$*$-controlled frames in Hilbert C^*-modules*, to appear in Inter. J. Wavelets and Multiresolut. Inf. Process.

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Approximately module homomorphisms

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Abstract. Let A be a Banach algebra and let X, Y be Banach A-modules. We study the conditions under which almost module homomorphisms are near to module homomorphisms in the norm topology on $\mathfrak{L}(X, Y)$, the space of all bounded linear maps from X into Y.
Keywords: Amenability, Banach algebra, Banach module, module homomorphism.
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1. Introduction

Let A be a Banach algebra. A Banach space X which is also a left A-module is called a left Banach A-module if

$$\|a \cdot x\| \leq k_X \|a\| \|x\| \qquad (a \in A, x \in X)$$

for some positve k_X. A right Banach A-module is defined similarly. A Banach space X is a Banach A-module if it is a left and right Banach A-module.

For a Banach algebra A, let X and Y be two Banach A-modules. We note that if X is a Banach A-module, then the actions are compatible; that is,

$$a \cdot (x_1 x_2) = (u \cdot x_1)x_2, \quad a(x_1 \cdot x_2) = (x_1 \cdot a)x_2 \quad (x_1, x_2 \in X, a \in A).$$

The space of all bounded linear maps from X into Y is denoted by $\mathfrak{L}(X, Y)$.

Let A be a Banach algebra, and let X be a Banach A-module. A bounded linear map $D : A \to X$ is called a derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in A).$$

For each $x \in X$, we define $\Delta_x : A \to X$ by $a \mapsto a \cdot x - x \cdot a$. It is easily seen that Δ_x is a derivation. Derivations of this form are called inner derivations. The set of all derivations of A into X is denoted by $\mathfrak{Z}^1(A, X)$, and the set of all inner derivations is denoted by $\mathfrak{B}^1(A, X)$. One can see that $\mathfrak{Z}^1(A, X)$ is a closed subspace of $\mathfrak{L}(A, X)$ and $\mathfrak{B}^1(A, X)$ is a subspace of of $\mathfrak{Z}^1(A, X)$.

Let A be a Banach algebra, and let X be a Banach A-module. Then

$$\mathfrak{H}^1(A, X) := \mathfrak{Z}^1(A, X)/\mathfrak{B}^1(A, X)$$

is the first Hochshild cohomology group of A with coefficients in X, see [2, 5].

For a Banach algebra A and a Banach A-module X, it is easily seen that X^* is a Banach A-module via the following actions:

$$\langle \phi \cdot a, x \rangle = \langle \phi, a \cdot x \rangle, \quad \langle a \cdot \phi, x \rangle = \langle \phi, x \cdot a \rangle \quad (a \in A, x \in X, \phi \in X^*).$$

A Banach algebra A is called amenable if $\mathfrak{H}^1(A, X^*) = \{0\}$ for every Banach A-module X.

In 1986, B. E. Johnson introduced AMNM property for a Banach algebra [3]. A Banach algebra A has AMNM property if any almost multiplicative functional is near to a multiplicative one. Many of the classical commutative Banach algebras were shown to have AMNM property. As an example, the disc algebra $A(\mathbb{D})$ has AMNM property. In [1], K. Jarosz studied AMNM property for general uniform algebras and S. J. Sidney constructed uniform algebras that are not AMNM [6]. In 1988, B. E. Johnson considered AMNM property for a pair of Banach algebras [4]. He studied, for a pair of Banach algebras such as (A, B), when approximately multiplicative maps from A into B are near to multiplicative maps.

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Let A be a Banach algebra and X, Y be Banach A-modules. We say taht a bounded linear map $T : X \to Y$ is almost left (right) module homomorphism if, for some positive δ, we have

$$\|T(a \cdot x) - a \cdot T(x)\| \leq \delta \|a\| \|x\| \;\; (\|T(x \cdot a) - T(x) \cdot a\| \leq \delta \|x\| \|a\|) \qquad (a \in A, x \in X).$$

We say taht a bounded linear map is almost module homomorphism if it is almost left and right module homomorphism. In this paper, we study the conditions under which almost module homomorphisms are near to module homomorphisms in the norm topology on $\mathfrak{L}(X, Y)$.

2. Main results

The space of elements of $\mathfrak{L}(X, Y)$, which are also left (right) A-module homomorphism is denoted by ${}_A\mathfrak{L}(X, Y)(\mathfrak{L}_A(X, Y))$. Also the space of all bounded linear maps from X into Y which are A-module homomorphism is denoted by ${}_A\mathfrak{L}_A(X, Y)$.

Lemma 2.1. *The space ${}_A\mathfrak{L}_A(X, Y)$ is closed in $\mathfrak{L}(X, Y)$.*

Proof. Let (T_n) be a sequence in ${}_A\mathfrak{L}(X, Y)$ with $T_n \to T$, for some T in $\mathfrak{L}(X, Y)$. Fix $n \in \mathbb{N}$, for each $x \in X$ and $a \in A$, we have

$$\begin{aligned} \|T(a \cdot x) - a \cdot T(x)\| &\leq \|T(a \cdot x) - T_n(a \cdot x)\| + \|a \cdot T_n(x) - a \cdot T(x)\| \\ &\leq k_X \|T_n - T\| \|a\| \|x\| + k_Y \|T_n - T\| \|a\| \|x\|. \end{aligned}$$

By taking limit whenever $n \to \infty$, we see that T is a left A-module homomorphism and so ${}_A\mathfrak{L}(X, Y)$ is closed. Similarly, one can see that $\mathfrak{L}_A(X, Y)$ is closed in $\mathfrak{L}(X, Y)$ and the proof is established. □

Definition 2.2. Let $T \in \mathfrak{L}(X, Y)$, we define two perturbation maps $T^{\lhd} : A \times X \to Y$ and $T^{\rhd} : X \times A \to Y$ by $T^{\lhd}(a, x) = T(a \cdot x) - a \cdot T(x)$ and $T^{\rhd}(x, a) = T(x \cdot a) - T(x) \cdot a$. It is easily seen that $T^{\lhd} \in \mathfrak{L}(A, X; Y)$, the space of all bounded bilinear maps from $A \times X$ into Y, and $T^{\rhd} \in \mathfrak{L}(X, A; Y)$.

One can easily see that

$${}_A\mathfrak{L}_A(X, Y) = \{T \in L(X, Y) : T^{\lhd} = T^{\rhd} = 0\}.$$

For $T \in \mathfrak{L}(X, Y)$, we put

$$d(T) = \operatorname{dist}(T, {}_A\mathfrak{L}_A(X, Y)) = \inf\{\|T - T'\| : \; T' \in \; {}_A\mathfrak{L}_A(X, Y)\}.$$

Since ${}_A\mathfrak{L}_A(X, Y)$ is closed in $\mathfrak{L}(X, Y)$, then $T^{\lhd} = T^{\rhd} = 0$ if and only if $d(T) = 0$.

Lemma 2.3. *Let A be a Banach algebra and let X, Y be Banach A-modules. Then the mappings $\lhd : \mathfrak{L}(X, Y) \to \mathfrak{L}(A, X; Y)$ with $T \mapsto T^{\lhd}$, and $\rhd : \mathfrak{L}(X, Y) \to \mathfrak{L}(X, A; Y)$ with $T \mapsto T^{\rhd}$, are bounded linear maps.*

Proof. Let $T \in \mathfrak{L}(X, Y)$ be fixed. Take $S \in {}_A\mathfrak{L}_A(X, Y)$. Then

$$\begin{aligned} \|T^{\lhd}(a, x)\| &= \|T(a \cdot x) - a \cdot T(x)\| \\ &\leq \|T(a \cdot x) - S(a \cdot x)\| + \|a \cdot S(x) - a \cdot T(x)\| \\ &\leq k_X \|T - S\| \|a\| \|x\| + k_Y \|T - S\| \|a\| \|x\| \end{aligned}$$

for all $a \in A$ and $x \in X$. So $\|T^{\lhd}\| \leq (k_X + k_Y)\|T - S\|$. By taking infimum over all elements in ${}_A\mathfrak{L}_A(X, Y)$, we have

$$\|T^{\lhd}\| \leq (k_X + k_Y) d(T), \qquad (T \in \mathfrak{L}(X, Y)).$$

Similarly, we may see that

$$\|T^{\rhd}\| \leq (k_X + k_Y) d(T), \qquad (T \in \mathfrak{L}(X, Y)).$$

Since $d(T) \leq \|T\|$, the claim is proved. □

By the above lemma, we see that if $d(T)$ is small then so are $\|T^{\lhd}\|$ and $\|T^{\rhd}\|$. We are interested in whether $T^{\lhd}$ and $T^{\rhd}$ being small implies $d(T)$ being small.

Definition 2.4. We say that $(X,Y)_A$ is **ALMHNLMH** (almost left module homomorphisms are near left module homomorphisms)(respect to A) if for each positive ϵ and K there is a positve δ such that if $T \in \mathfrak{L}(X,Y)$ with $\|T\| < K$ and $\|T^{\triangleleft}\|, \|T^{\triangleright}\| < \delta$ then $dist(T,_A \mathfrak{L}(X,Y)) < \epsilon$.

Definition 2.5. We say that $(X,Y)_A$ is **ARMHNRMH** (almost right module homomorphisms are near right module homomorphisms)(respect to A) if for each positve ϵ and K there is a positve δ such that if $T \in \mathfrak{L}(X,Y)$ with $\|T\| < K$ and $\|T^{\triangleleft}\|, \|T^{\triangleright}\| < \delta$ then $dist(T, \mathfrak{L}_A(X,Y)) < \epsilon$.

Definition 2.6. We say that $(X,Y)_A$ is **AMHNMH** (almost module homomorphisms are near module homomorphisms)(respect to A) if for each positve ϵ and K there is a positve δ such that if $T \in \mathfrak{L}(X,Y)$ with $\|T\| < K$ and $\|T^{\triangleleft}\|, \|T^{\triangleright}\| < \delta$ then $d(T) < \epsilon$.

For Banach A-modules X and Y, $\mathfrak{L}(X,Y)$ can be considered as a Banach A-module via the following actions:

$$\langle a \cdot T, x\rangle = a \cdot T(x), \qquad \langle T \cdot a, x\rangle = T(a \cdot x) \qquad (a \in A, x \in X,\ T \in \mathfrak{L}(X,Y)).$$

One can easily see that T is a left A-module homomorphism if and only if $a \cdot T = T \cdot a$ for each $a \in A$.

Theorem 2.7. *Let A be a Banach algebra; X and Y Banach A-modules. If $\mathfrak{H}^1(A, \mathfrak{L}(X,Y))$ is a Banach space, then $(X,Y)_A$ is ALMHNLMH and ARMHNRMH.*

Proof. Consider the mapping

$$\Delta : \mathfrak{L}(X,Y) \to \mathfrak{Z}^1(A, \mathfrak{L}(X,Y))$$
$$T \mapsto \Delta_T$$

where $\Delta_T : A \to \mathfrak{L}(X,Y)$ is defined by $\Delta_T(a) = a \cdot T - T \cdot a$.

It is clear that Δ_T is an inner derivations and so $Im(\Delta) = \mathfrak{B}^1(A, \mathfrak{L}(X,Y))$. Also one can see that $Ker(\Delta) =_A \mathfrak{L}(X,Y)$ and

$$\text{dist}(T, {}_A\mathfrak{L}(X,Y)) = \text{dist}(T, Ker(\Delta)) = \| T + Ker(\Delta) \| .$$

On the other hand, since $Im(\Delta)$ is closed there exists $K > 0$ such that for each $T \in \mathfrak{L}(X,Y)$

$$\| T + Ker(\Delta) \| \leq K \| \Delta_T \| .$$

For each left δ-module homomorphism T, we have

$$\begin{aligned} \| \Delta_T \| &= \sup\{\| \Delta_T(a) \|:\ a \in A,\ \| a \| \leq 1\} \\ &= \sup\{\| a \cdot T - T \cdot a \|:\ a \in A,\ \| a \| \leq 1\} \\ &= \sup\{\| a \cdot T(x) - T(a \cdot x) \|:\ a \in A,\ x \in X,\ \| a \| \leq 1,\ \| x \| \leq 1\} \\ &\leq \delta. \end{aligned}$$

So $\text{dist}(T, {}_A\mathfrak{L}(X,Y)) \leq K\delta$ is true for each left δ-module homomorphism. Choosing $\delta \leq \epsilon/K$, we have $\text{dist}(T, {}_A\mathfrak{L}(X,Y)) \leq \epsilon$. Thus $(X,Y)_A$ is ALMHNLMH.

Similarly, by replacing the module actions on $\mathfrak{L}(X,Y)$ via:

$$\langle a \cdot T, x\rangle = T(x) \cdot a, \qquad \langle T \cdot a, x\rangle = T(x \cdot a) \qquad (a \in A, x \in X,\ T \in \mathfrak{L}(X,Y)),$$

we may see that $(X,Y)_A$ is ARMHRMH. □

Corollary 2.8. *Let A be an amenable Banach algebra; X a Banach A-module and Y a dual Banach A-module. Then $(X,Y)_A$ is ALMHNLMH and ARMHRMH.*

Proof. Since Y is a dual Banach A-modle, Y has a predual such as Y_* with $X\hat{\otimes}Y_*$ as a Banach A-module. On the other hand, $\mathfrak{L}(X,Y) \cong (X\hat{\otimes}Y_*)^*$. Since A is amenable,

$$\mathfrak{H}^1(A, \mathfrak{L}(X,Y)) \cong \mathfrak{H}^1(A, (X\hat{\otimes}Y_*)^*) = \{0\}.$$

So $\mathfrak{H}^1(A, \mathfrak{L}(X,Y))$ is a Banach space; by Theorem 2.7, the proof is established. □

THEOREM 2.9. *Let A be a Banach algebra and let X, Y be finite dimensional Banach A-modules. Then $(X, Y)_A$ is AMHNMH.*

PROOF. Given $\epsilon > 0$ and K, put

$$C = \{T \in \mathfrak{L}(X, Y) : \ \|T\| \leq K, \ d(T) \geq \epsilon\}.$$

Also for $\delta > 0$, put $G_\delta = G_\delta^{\triangleleft} \cap G_\delta^{\triangleright}$, where

$$G_\delta^{\triangleleft} = \{T \in \mathfrak{L}(X, Y) : \ \|T^{\triangleleft}\| > \delta\}$$

and

$$G_\delta^{\triangleright} = \{T \in \mathfrak{L}(X, Y) : \ \|T^{\triangleright}\| > \delta\}.$$

By Lemma 2.3, $G_\delta^{\triangleleft}$ and $G_\delta^{\triangleright}$ are open in $\mathfrak{L}(X, Y)$, so G_δ is open. On the other hand, we have $C \subseteq \mathfrak{L}(X, Y) \setminus {}_A\mathfrak{L}_A(X, Y) \subseteq \cup\{G_\delta : \ \delta > 0\}$. Since C is compact, there exists $\delta > 0$ such that $C \subseteq G_\delta$; that is, $(X, Y)_A$ is AMHNMH. □

References

1. K. Jarosz, *Almost multiplicative functionals*, Studia Math. 124 (1997), 37-58.
2. B. E. Johnson, *Cohomology in Banach algebras*, Memoirs 127 (1972), American. Math. Soc. Providence.
3. B. E. Johnson, *Approximately multiplicative functionals*, J. London Math. Soc. 34 (1986) 489-510.
4. B. E. Johnson, *Approximately multiplicative maps between Banach algebras*, J. London Math. Soc. 37 (1988), 294-316.
5. V. Runde, *Lectures on Amenability*, Lecture Notes in Mathematics 1774, Springer-Verlag, Berlin-Heidelberg-New York, 2002.
6. S. J. Sidney, *Are all uniform algebras AMNM?*, Bull. London Math. Soc. 29 (1997), 327-330.

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Some results on matrix valued group algebras and spaces

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Abstract. For a locally compact group G we consider the Fourier transform of matrix valued functions and representations induced by the representations of G. We show that for a locally compact abelian group G and for any $\phi \in L^\infty(\widehat{G}, M_n)$, $\Lambda_{\widehat{G}}$ defined by $\Lambda_{\widehat{G}}(\phi)(f) = \mathcal{F}^{-1}(\phi\mathcal{F}(f))$, $(f \in L^2(G, M_{n,2}))$ is an isometric $*$-isomorphism from $L^\infty(\widehat{G}, M_n)$ onto $CV_2(G, M_n)$.

1. Introduction

Let G be a locally compact group, m_G be the unique Haar measure on G, $1 < p, q < \infty$ such that $1/p + 1/q = 1$ and M_n be an $n \times n$, $n \in \mathbb{N}$, matrix with entries in $\mathbb{C}$. The trace $\mathrm{Tr} : M_n \longrightarrow \mathbb{C}$ is a positive linear functional of norm n. Suppose that $\mathcal{B}$ is a σ-algebra of Borel sets in G, $\mu : G \longrightarrow M_n$ is countably additive function that we call it an M_n-valued measure on G and denote by an $n \times n$ matrix $\mu = (\mu_{ij})$ of complex valued measures μ_{ij} on G. The variation of μ is $|\mu|$ that is a positive real finite measure on G defined by

$$|\mu|(E) = \sup_{\mathcal{P}} \left\{ \sum_{E_i \in \mathcal{P}} \|\mu(E_i)\| : E \in \mathcal{B} \right\},$$

where $\mathcal{P}$ is a partitions of E into a finite number of pairwise disjoint Borel sets. Define the norm of μ as $\|\mu\| = |\mu|(G)$. Following [1, 2], μ has a polar representation $\mu = \omega \cdot |\mu|$ where $\omega : G \longrightarrow M_n$ is a Bochner integrable function with $\|\omega(\cdot)\| = 1$. A function $f = (f_{ij}) : G \longrightarrow M_n$ is called μ-integrable if each f_{ij} is a Borel function and the integral $\int_G f_{ij} \mathrm{d}\mu_{k\ell}$ exist in which case. For any $E \in \mathcal{B}$, the integral $\int_E f \mathrm{d}\mu$ is an $n \times n$ matrix with ij-th entry

$$\sum_k \int_E f_{ik} \mathrm{d}\mu_{kj}.$$

Then, we have

$$\left\| \int_G f \mathrm{d}\mu \right\| = \left\| \int_G f(x)\omega(x) \mathrm{d}|\mu|(x) \right\| \leq \int_G \|f(x)\| \,\mathrm{d}|\mu|(x). \tag{1}$$

By [1, Lemma 5], $M(G, M_n^*)$ is linearly isomorphic to the dual of $C_0(G, M_n)$, with the following duality formula:

$$\langle \cdot, \cdot \rangle : C_0(G, M_n) \times M(G, M_n^*) \longrightarrow \mathbb{C}$$

$$\langle f, \mu \rangle = \mathrm{Tr}\left(\int_G f \mathrm{d}\mu \right) = \sum_{i,k} \int_G f_{ik} \mathrm{d}\mu_{k,j}, \tag{2}$$

for $f = (f_{ij}) \in C_0(G, M_n)$ and $\mu = (\mu_{ij}) \in M(G, M_n^*)$. By [3, Proposition 2.4], $(M(G, M_n^*), \|\cdot\|_{tr})$ is a Banach algebra with the following convolution product:

$$\langle f, \mu * \nu \rangle = \mathrm{Tr}\left(\int_G \int_G f(xy) \mathrm{d}\mu(x) \mathrm{d}\nu(y) \right), \tag{3}$$

for all $f \in C_0(G, M_n)$ and $\mu, \nu \in M(G, M_n^*)$. Also, $(M(G, M_n), \|\cdot\|)$ becomes a Banach algebra with the convolution product and is algebraically isomorphic to $(M(G, M_n^*), \|\cdot\|_{tr})$. Let $f = (f_{ij})$

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be a Borel M_n-valued function on G and $\mu = (\mu_{ij})$ be a M_n-valued measure on G. An M_n-valued convolution $f * \mu$, if exists at $x \in G$, is defined by

$$(f * \mu)(x) = \int_G f(xy^{-1}) \mathrm{d}\mu(y). \tag{4}$$

The left convolution $\mu *_\ell f$ is the following integral:

$$(\mu *_\ell f)(x) = \int_G \mathrm{d}\mu(y) f(y^{-1}x) \qquad (x \in G). \tag{5}$$

The transposed integral $\int_G \mathrm{d}\mu(x) f(x)$ which is defined to have ij-entry

$$\left(\int_G \mathrm{d}\mu(x) f(x) \right)_{ij} = \sum_k \int_G f_{kj}(x) \mathrm{d}\mu_{ik}(x).$$

Also,

$$\left\| \int_G \mathrm{d}\mu(x) f(x) \right\| \leq \int_G \|f(x)\| \mathrm{d}|\mu|(x). \tag{6}$$

Definition 1.1. [6] Let G be a locally compact group, $1 < p < \infty$ A bounded operator $T : L^p(G, M_n) \longrightarrow L^p(G, M_n)$ is called a matrix valued left p-convolution operator of G if $T({}_af) = {}_a T(f)$ for every $a \in G$ and $f \in L^p(G, M_n)$. We denote the set of all matrix valued p-convolution operators of G by $LCV_p(G, M_n)$. Similarly, we define the right p-convolution operator with entries in M_n, if $T(f_a) = T(f)_a$ for every $a \in G$ and $f \in L^p(G, M_n)$ and we denote the set of all such operators by $RCV_p(G, M_n)$. We denote the space of matrix valued p-convolution operators by $CV_p(G, M_n)$ that is $LCV_p(G, M_n) \cap RCV_p(G, M_n)$.

For more details regarding to these form of Banach algebras and spaces, we refer to [7, 8].

2. Fourier transform and p-convolution operators

In this section, we consider Fourier transform of matrix valued functions, we suppose that G is a locally compact abelian group with the Haar measure m_G and by $\widehat{G}$, we denote the dual of G. Following [2, Section 3. §3], for $\pi \in \widehat{G}$, $\mu \in M(G, M_n)$ and $f \in L^1(G, M_n)$, we define their Fourier transforms by

$$\widehat{\mu}(\pi) = \int_G \pi\left(x^{-1}\right) \mathrm{d}\mu(x), \tag{7}$$

and

$$\widehat{f}(\pi) = \int_G f(x) \pi\left(x^{-1}\right) \mathrm{d}m_G(x). \tag{8}$$

Also,

$$\widehat{\mu *_\ell f} = \widehat{\mu}\widehat{f} \quad \text{and} \quad \widehat{f * \mu} = \widehat{f}\widehat{\mu}, \tag{9}$$

for every $\mu \in M(G, M_n)$ and $f \in L^1(G, M_n)$. We denote by $M_{n,2}$ the vector space M_n equipped with the Hilbert-Schmidt norm and we also consider $L^2(G, M_{n,2})$ as a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. For every $f \in L^1(G, M_{n,2}) \cap L^2(G, M_{n,2})$, by [2, Lemma 3.3.9], $\|\widehat{f}\|_2 = \|f\|_2$ and there is a unique continuous map $\mathcal{F} : L^2(G, M_{n,2}) \longrightarrow L^2(\widehat{G}, M_{n,2})$ such that $\mathcal{F}(f) = \widehat{f}$. Similar to [4, Definition 1.3.2], for $\phi \in L^\infty(\widehat{G}, M_n)$, we define

$$\Lambda_{\widehat{G}}(\phi)(f) = \mathcal{F}^{-1}\left(\phi \mathcal{F}(f)\right), \quad (f \in L^2(G, M_{n,2})). \tag{10}$$

We now give a result similar to [4, Theorem 1.3.2] as follows:

Theorem 2.1. *Let G be a locally compact abelian group. Then $\Lambda_{\widehat{G}}$ is an isometric $*$-isomorphism from $L^\infty(\widehat{G}, M_n)$ onto $CV_2(G, M_n)$.*

Proof. By a similar argument in the proof of [4, Theorem 1.3.2], $\Lambda_{\widehat{G}}$ is an algebraic homomorphism i.e., for every $\phi, \psi \in L^\infty(\widehat{G}, M_n)$ and $f \in L^2(G, M_{n,2})$,

$$\Lambda_{\widehat{G}}(\phi\psi)(f) = \Lambda_{\widehat{G}}(\phi)\left(\Lambda_{\widehat{G}}(\psi)(f)\right). \tag{11}$$

Also, it is injective and for $\phi \in L^\infty(\widehat{G}, M_n)$ and $f, g \in L^2(G, M_{n,2})$,

$$\begin{aligned}\langle g, \Lambda_{\widehat{G}}(\phi^*)(f)\rangle &= \langle g, \mathcal{F}^{-1}(\phi^*\mathcal{F}(f))\rangle = \langle \mathcal{F}(g), \phi^*\mathcal{F}(f)\rangle \\ &= \langle g, \Lambda_{\widehat{G}}(\phi)^*(f)\rangle.\end{aligned}$$

This shows that $\Lambda_{\widehat{G}}$ is a $*$-homomorphism. Let $T \in CV_2(G, M_n)$ and define $S(f) = \mathcal{F}\left(T\left(\mathcal{F}^{-1}(f)\right)\right)$, for every $f \in L^2(\hat{G}, M_{n,2})$. Moreover, for any $g \in L^1(G, M_n)$

On the other hand, every $*$-homomorphism between C^*-algebras is an isometry, thus, $\Lambda_{\widehat{G}}$ is an isometry. Denote by $\imath_G$ the canonical map of G onto $\widehat{\widehat{G}}$. For every $a \in G$, $\pi \in \widehat{G}$ and $f \in L^2(G, M_{n,2})$,

$$\begin{aligned}\mathcal{F}\left({}_af\right)(\pi) &= \int_G f(ax)\pi(x^{-1})\mathrm{d}m_G(x) = \int_G f(x)\pi(x^{-1})\pi(a)\mathrm{d}m_G(x) \\ &= \pi(a)\int_G f(x)\pi(x^{-1})\mathrm{d}m_G(x) = \imath_G(a)(\pi)\mathcal{F}(f)(\pi) \\ &= \left(\imath_G(a)\mathcal{F}(f)\right)(\pi).\end{aligned} \tag{12}$$

One can identify $L^\infty(\widehat{G}, M_n)$ as a C^*-subalgebra of $B(L^2(G, M_{n,2})$. Then $\widehat{\mu} \in L^\infty(\widehat{G}, M_n)$. Then by (9) and (12), we have

$$\begin{aligned}{}_a\left(\Lambda_{\widehat{G}}(\widehat{\mu})(f)\right) &= {}_a\left(\mathcal{F}^{-1}\left(\widehat{\mu}\mathcal{F}(f)\right)\right) = {}_a\left(\mathcal{F}^{-1}\left(\mathcal{F}(\mu *_\ell f)\right)\right) = {}_a\left(\mu *_\ell f\right) \\ &= \mu *_\ell {}_af = \mathcal{F}^{-1}\left(\mathcal{F}(\mu *_\ell {}_af)\right) = \mathcal{F}^{-1}\left(\widehat{\mu}\mathcal{F}({}_af)\right) \\ &= \Lambda_{\widehat{G}}(\widehat{\mu})({}_af),\end{aligned} \tag{13}$$

for every $a \in G$, $f \in L^2(G, M_{n,2})$ and $\widehat{\mu} \in L^\infty(\widehat{G}, M_n)$. Similarly, we have

$$\left(\Lambda_{\widehat{G}}(\widehat{\mu})(f)\right)_a = \Lambda_{\widehat{G}}(\widehat{\mu})(f_a), \tag{14}$$

for every $a \in G$, $f \in L^2(G, M_{n,2})$ and $\widehat{\mu} \in L^\infty(\widehat{G}, M_n)$. Thus $\Lambda_{\widehat{G}}(\widehat{\mu})$ is in $CV_2(G)$, for every $\widehat{\mu} \in L^\infty(\widehat{G}, M_n)$. □

Similar to (10), for every $\phi \in L^\infty(\widehat{G}, M_n)$, we define

$$\Theta_{\widehat{G}}(\phi)(f) = \mathcal{F}^{-1}\left(\mathcal{F}(f)\phi\right), \quad (f \in L^2(G, M_{n,2})). \tag{15}$$

Similar to Theorem 2.1, we have the following result for $\Theta_{\widehat{G}}$.

Theorem 2.2. *Let G be a locally compact abelian group. Then $\Theta_{\widehat{G}}$ is an isometric $*$-anti-isomorphism from $L^\infty(\widehat{G}, M_n)$ onto $CV_2(G, M_n)$.*

Proof. For every $\phi, \psi \in L^\infty(\widehat{G}, M_n)$ and $f \in L^2(G, M_{n,2})$,

$$\begin{aligned}\Theta_{\widehat{G}}(\phi\psi)(f) &= \mathcal{F}^{-1}\left(\mathcal{F}(f)\phi\psi\right) = \mathcal{F}^{-1}\left(\left(\mathcal{F}(f)\phi\right)\psi\right) = \mathcal{F}^{-1}\left(\mathcal{F}\left(\Theta_{\widehat{G}}(\phi)(f)\right)\psi\right) \\ &= \Theta_{\widehat{G}}(\psi)\left(\Theta_{\widehat{G}}(\phi)(f)\right).\end{aligned}$$

Thus, $\Theta_{\widehat{G}}$ is an anti-homomorphism. Also, similar to $\Lambda_{\widehat{G}}$ one can show that $\Theta_{\widehat{G}}$ is a $*$-map and consequently, it is a $*$-anti-homomorphism between C^*-algebras. This implies that it is an isometry. By (9), similar to the relations (13) and (14), we obtain for $\Theta_{\widehat{G}}$. Hence it is in $CV_2(G)$. □

Following [2], for any $f \in L^p(G, M_n)$ and the scalar valued map Δ (the modular function of G), the product $f(x) \otimes \Delta(y)$ is given by

$$f(x) \otimes \Delta(y) = \begin{pmatrix} f_{11}(x)\Delta(y) & \cdots & f_{1n}(x)\Delta(y) \\ \vdots & \ddots & \vdots \\ f_{n1}(x)\Delta(y) & \cdots & f_{nn}(x)\Delta(y) \end{pmatrix} \tag{16}$$

THEOREM 2.3. *Let π be a unitary representation of G. Then, for all $f, g \in L^1(G, M_n)$,*

$$\overline{\pi}(f * g) = \int_G (f * g)(x) \otimes \pi(x) \mathrm{d}m_G(x)$$

satisfies the following

(i) $\overline{\pi}(f * g) = \overline{\pi}(f)\overline{\pi}(g)$.

(ii) $\pi(x) \otimes \overline{\pi}(f) = \pi(\ell_x f)$ *and* $\pi(f) \otimes \pi(x) = \Delta(x^{-1}) \otimes \pi(r_{x^{-1}} f)$.

PROOF. For any $f, g \in L^1(G, M_n)$,

$$\begin{aligned}\overline{\pi}(f * g) &= \int_G \int_G f\left(xy^{-1}\right) g(y) \mathrm{d}m_G(y) \mathrm{d}m_G(x) \\ &= \int_G \int_G f(y) g(x) \otimes \pi(x)\pi(y) \mathrm{d}m_G(y) \mathrm{d}m_G(x) \\ &= \int_G \int_G (f(y) \otimes \pi(x))(g(x) \otimes \pi(y)) \mathrm{d}m_G(y) \mathrm{d}m_G(x) \\ &= \overline{\pi}(f)\overline{\pi}(g).\end{aligned}$$

The case (ii) by a similar calculations holds. □

THEOREM 2.4. *Let π be a unitary representation of G. Then, for all $f, g \in L^1(G, M_n)$,*

$$\overline{\pi}(f *_\ell g) = \int_G (f *_\ell g)(x) \otimes \pi(x) \mathrm{d}m_G(x)$$

satisfies the following equality

$$\overline{\pi}(f *_\ell g) = \overline{\pi}(f)\overline{\pi}(g).$$

References

1. C-H. Chu, *Matrix-valued harmonic functions on groups*, J. Reine Angew. Math., **552**, (2002), 15-52.
2. C-H. Chu, *Matrix convolution operators on groups*, Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 2008.
3. C-H. Chu and A. T-M. Lau, *Jordan structures in harmonic functions and Fourier algebras on homogeneous spaces*, Math. Ann., **336**, (2006), 803–840.
4. A. Derighetti, *Convolution operators on groups*, Lecture Notes of the Union Mathematica Italiana, Springer, 2011.
5. J. Diestel and J. J. Uhl, *Vector measures*, Math. Surv. **15**, Amer. Math. Soc., 1977.
6. A. Ebadian and A. Jabbari, *Matrix valued p-convolution operators*, Oper. Mat., **14**, 1 (2020), 117–128.
7. A. Ebadian and A. Jabbari, *Matrix valued conjugate convolution operators on matrix valued L^p-spaces*, Math. Inequal. Appl., 24(3) (2021), 827844.
8. A. Jabbari, *Positive type and positive definite functions on matrix valued group algebras*, Results Math., **75**, 4 (2020), Art. N. 149 (25 pages), https://doi.org/10.1007/s00025-020-01278-1.

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Study on operator representation of frames in Hilbert spaces

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Abstract. One of the main problems in frame theory is investigation of frame properties of sequences in a Hilbert space $\mathcal{H}$ obtained by iterates of operators on the form $\{T^k\phi\}_{k\in\mathbb{N}_0}$, where the operator $T:\mathcal{H}\to\mathcal{H}$ belongs to certain classes of linear operators and the element ϕ belongs to $\mathcal{H}$. In this note, we discuss the size of the set of such elements.
Keywords: Frames, Operator representation of frames.
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1. Introduction

The system of iterations $\{T^k\phi\}_{k\in\mathbb{N}_0}$, where T is a bounded linear operator on a separable Hilbert space $\mathcal{H}$ and $\phi\in\mathcal{H}$, the so-called dynamical sampling problem is a relatively new research topic in Harmonic analysis. This topic has been studied since the work of Aldroubi and Petrosyan [1] for some new results concerning frames and Bessel systems, see for example Aldroubi et al. [2] to study dynamical sampling problem in finite dimensional spaces. Christensen et al. [5, 6, 7] studied some properties of systems arising via iterated actions of operators and also, frame properties of operator orbits. For more details, we refer to [1, 3, 4]. Let $\{f_k\}_{k\in\mathbb{N}}$ is a frame for $\mathcal{H}$ which spans an infinite dimensional subspace. A natural question to ask is whether there exists a linear operator T such that $f_{k+1}=Tf_k$, for all $n\in\mathbb{N}$? In [7] it was proved that such an operator exists if and only if the frame $\{f_k\}_{k\in\mathbb{N}}$ is linearly independent; also, T is bounded if and only if the kernel of the synthesis operator of $\{f_k\}_{k\in\mathbb{N}}$ is invariant under the right shift operator on $\ell^2(\mathbb{N})$, in the affirmative case, $\{f_k\}_{k\in\mathbb{N}}=\{T^kf_1\}_{k\in\mathbb{N}_0}$. In this note, necessary conditions for the frame $\{f_k\}_{k\in\mathbb{N}}$ to have a representation of the form $\{T^kf_1\}_{k\in\mathbb{N}_0}$ is discussed (Proposition 2.1, below). Note that the problem considered in this note is of some interest from other points of view. Indeed, assuming that the system $\{T^kf_1\}_{k\in\mathbb{N}_0}$ is a frame for $\mathcal{H}$. It is natural to ask for a characterization of $\mathcal{V}(T)$: the set of all $\phi\in\mathcal{H}$ such that $\{T^k\phi\}_{k\in\mathbb{N}_0}$ is a frame for $\mathcal{H}$. In [6], $\mathcal{V}(T)$ is obtained by applying all invertible operators from the set of commutant T' of T to ϕ. In this note we discuss the size of the set $\mathcal{V}(T)$ (Theorem 2.2, below).

Throughout this note, let $\mathcal{H}$ be a separable Hilbert spaces. We denote by $B(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$, the index set is the natural set $\mathbb{N}$ and $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$.

Definition 1.1. [4] A sequence of vectors $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $A,B>0$ so that

$$A\parallel f\parallel^2\leq\sum_{k\in\mathbb{N}}|\langle f,f_k\rangle|^2\leq B\parallel f\parallel^2\ , \tag{1}$$

for all $f\in\mathcal{H}$.

It follows from the definition that if $\{f_k\}_{k\in\mathbb{N}}$ is a frame for $\mathcal{H}$, then

$$\overline{span}\{f_k\}_{k\in\mathbb{N}}=\mathcal{H}\ . \tag{2}$$

The sequence $\{f_k\}_{k\in\mathbb{N}}$ is a *Bessel sequence* if at least the upper condition in (1) holds. If $\{f_k\}_{k\in\mathbb{N}}$ is a Bessel sequence, the *synthesis operator* is defined by

$$U:\ell^2(\mathbb{N})\to\mathcal{H};\ \ U\{c_k\}_{k\in\mathbb{N}_0}=\sum_{k\in\mathbb{N}}c_kf_k\ ;$$

*speaker

it is well known that U is well-defined and bounded. The availability of the representation $\{f_k\}_{k\in\mathbb{N}} = \{T^k f_1\}_{k\in\mathbb{N}_0}$ is characterized in [7]:

PROPOSITION 1.2. *Consider any sequence* $\{f_k\}_{k\in\mathbb{N}}$ *in* $\mathcal{H}$ *for which* $span\{f_k\}_{k\in\mathbb{N}}$ *is infinite-dimensional. Then the following are equivalent:*

(i) $\{f_k\}_{k\in\mathbb{N}}$ *is linearly independent.*
(ii) *There exists a linear operator* $T : span\{f_k\}_{k\in\mathbb{N}} \to \mathcal{H}$ *such that* $\{f_k\}_{k\in\mathbb{N}} = \{T^k f_1\}_{k\in\mathbb{N}_0}$.

2. Main results

For a given operator $T \in B(\mathcal{H})$, let

$$\mathcal{V}(T) := \left\{\phi \in \mathcal{H} : \{T^k\phi\}_{k\in\mathbb{N}_0} \text{ is a frame for } \mathcal{H}\right\}.$$

We state necessary conditions for the frame $\{f_k\}_{k\in\mathbb{N}}$ to have a representation of the form $\{T^k f_1\}_{k\in\mathbb{N}_0}$:

PROPOSITION 2.1. *Let* $\{f_k\}_{k\in\mathbb{N}}$ *be a frame for* $\mathcal{H}$ *of the form* $\{T^k f_1\}_{k\in\mathbb{N}_0}$*, for some linear operator* $T \in B(\mathcal{H})$*. Then the adjoint operator* T^* *is not power bounded, i.e.,* $\sup_{k\in\mathbb{N}_0} \|(T^*)^k\| = \infty$.

PROOF. it is enough to show that, for every $\phi \in \mathcal{H}\backslash\{0\}$ and a sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers:

$$\lim_{k\to\infty} \|(T^*)^{n_k}\phi\| = \infty .$$

Suppose not. That is, there exists $0 \neq \phi \in \mathcal{H}$, $M > 0$ and $\{n_{k_j}\}_{j\in\mathbb{N}}$ subsequence of $\{n_k\}_{k\in\mathbb{N}}$ such that

$$\|(T^*)^{n_{k_j}}\phi\| \leq M, \quad j = 1, 2, \cdots . \tag{3}$$

Therefore, $\{\|(T^*)^{n_{k_j}}\phi\|\}_{j\in\mathbb{N}}$ is bounded and so is $span\{\|(T^*)^{n_{k_j}}\phi\|\}_{j\in\mathbb{N}}$.
Since $\{T^k f_1\}_{k\in\mathbb{N}_0}$ is a frame for $\mathcal{H}$, it is well-known that $\{T^{n_{k_j}} f_1\}_{j\in\mathbb{N}}$ is a frame for closed subspace $V := \overline{span}\{T^{n_{k_j}} f_1\}_{j\in\mathbb{N}} \cup \{\phi\}$ of $\mathcal{H}$. Without loss of generality, we can assume that $\|\varphi\| = 1$. Since $\{T^{n_{k_j}} f_1\}_{j\in\mathbb{N}} \cup \{\phi\}$ is dense in V, thus

$$span\{\langle T^{n_{k_j}} f_1, \phi\rangle\}_{j\in\mathbb{N}} = span\{\langle f_1, (T^*)^{n_{k_j}}\phi\rangle\}_{j\in\mathbb{N}} ,$$

must be dense in $\mathbb{F}$ (the scalar field of the closed subspace V of $\mathcal{H}$); because for any $\alpha \in \mathbb{F}$, we have $\alpha\varphi \in V$, so there exists a sequence of positive numbers $\{m_{k_j}\}_{j\in\mathbb{N}}$ such that $\sum_{j=1}^{\infty} c_j T^{m_{k_j}} f_1 \to \alpha\varphi$. Hence,

$$\left\langle \sum_{j=1}^{\infty} c_j T^{m_{k_j}} f_1, \varphi \right\rangle \to \langle \alpha\varphi, \varphi\rangle, \ or \ \sum_{j=1}^{\infty} c_j \langle T^{m_{k_j}} f_1, \varphi\rangle \to \alpha,$$

that is,

$$\overline{span}\{\langle T^{m_{k_j}} f_1, \varphi\rangle\}_{j\in\mathbb{N}} = \overline{span}\{\langle f_1, (T^*)^{m_{k_j}}\phi\rangle\}_{j\in\mathbb{N}} = \mathbb{F}.$$

Thus given $\epsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that for any $j > N$, we obtain

$$\left| \sum_{j=1}^{m_j} c_j \langle T^{n_{k_j}} f_1, \varphi\rangle - \alpha \right| < \epsilon,$$

and then

$$\begin{aligned} |\alpha| &< \left|\sum_{j=1}^{m_j} c_j \langle f_1, (T^*)^{m_{k_j}} \varphi\rangle\right| + \epsilon \\ &= \left|\langle f_1, \sum_{j=1}^{m_j} \overline{c_j} (T^*)^{m_{k_j}} \varphi\rangle\right| + \epsilon \\ &\leq \|f_1\| \| \sum_{j=1}^{m_j} \overline{c_j} (T^*)^{m_{k_j}} \| + \epsilon. \end{aligned}$$

But this contradicts with the boundedness of $span\{\|(T^*)^{n_{k_j}} \varphi\|\}_{j\in\mathbb{N}}$, so the proof is complete. □

Let $T \in B(\mathcal{H})$ be an operator for which there exists some $f_0 \in \mathcal{H}$ such that $\{T^n f_1\}_{n\in\mathbb{N}_0}$ is a frame for $\mathcal{H}$. A natural question to ask is whether there exist other vectors $\phi \in \mathcal{H}$ for which $\{T^k \phi\}_{k\in\mathbb{N}_0}$ also is a frame for $\mathcal{H}$. The answer of this question is already given in [6]. In the following theorem we consider another viewpoint of this question; we discussed the size of the set of vectors $\phi \in \mathcal{H}$ for which $\{T^n \phi\}_{n\in\mathbb{N}_0}$ is a frame for $\mathcal{H}$.

PROPOSITION 2.2. *Let $T \in B(\mathcal{H})$. Assume that T is invertible. Then*

$$\mathcal{V}(T) = \bigcap_{f\in\mathcal{V}(T), k\in\mathbb{N}} \mathbf{B}(f, k),$$

where,

$$\mathbf{B}(f, k) = \bigcup_{n\in\mathbb{N}_0} \{\phi \in \mathcal{H} : \|T^n \phi - f\| < \frac{1}{k}\}.$$

PROOF. For any $f \in \mathcal{V}(T)$, it is clear that $T \in E(\mathcal{H})$. Now, let $B_{f,k}$'s are open balls centered at $f \in \mathcal{V}(T)$ and with radius $\frac{1}{k}$, $(k \in \mathbb{N})$. Then by continuity, for any $n \in \mathbb{N}_0$,

$$(T^n)^{-1} B_{f,k} = \{\phi \in \mathcal{H} : \|T^n \phi - f\| < \frac{1}{k}\},$$

is open in $\mathcal{H}$. Therefore, for any $k \in \mathbb{N}$,

$$\mathbf{B}(f, k) := \bigcup_{n\in\mathbb{N}_0} (T^n)^{-1} B_{f,k},$$

is open in $\mathcal{H}$.

claim:

$$\mathcal{V}(T) = \bigcap_{f\in\mathcal{V}(T),\ k\in\mathbb{N}} \mathbf{B}(f, k).$$

Let $\phi \in \mathcal{V}(T)$. Then, for any $f \in \mathcal{V}(T)$ and any $k \in \mathbb{N}$, there exists $n \in \mathbb{N}_0$ such that, $T^n \phi \in B_{f,k}$, that is, $\phi \in (T^n)^{-1} B_{f,k}$. Therefore, $\phi \in \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}_0} (T^n)^{-1} B_{f,k}$, that is, $\phi \in \bigcap_{k\in\mathbb{N}} \mathbf{B}(f, k)$, for any $f \in \mathcal{V}(T)$.

Conversely, let $\phi \in \bigcap_{f\in\mathcal{V}(T),k\in\mathbb{N}} \mathbf{B}(f, k)$, then for any $f \in \mathcal{V}(T)$ and any $k \in \mathbb{N}$, $\phi \in \mathbf{B}(f, k)$. That is, for any $f \in \mathcal{V}(T)$ and any $k \in \mathbb{N}$, there exists $m \in \mathbb{N}_0$ such that $\phi \in (T^m)^{-1} B_{f,k}$, or $T^m \phi \in B_{f,k}$. Now it clear that, for $k \in \mathbb{N}$ sufficiently large, $B_{f,k} \subset \mathcal{V}(T)$, therefore, $T^m \phi \in \mathcal{V}(T)$, and then the system $\{T^n(T^m \phi)\}_{n\in\mathbb{N}_0}$ is a frame of $\mathcal{H}$. By adding of a finite elements of points $\{\phi, T\phi, \cdots T^{m-1}\phi\}$ to the system $\{T^n(T^m \phi)\}_{n\in\mathbb{N}_0}$, we have the system $\{T^n \phi\}_{n\in\mathbb{N}_0}$ which is a frame for $\mathcal{H}$, that is, $\phi \in \mathcal{V}(T)$, and this completes the proof. □

Let

$$E(\mathcal{H}) := \left\{ T \in B(\mathcal{H}) : \{T^k \phi\}_{k \in \mathbb{N}_0} \text{ is a frame for } \mathcal{H}, \text{ for some } \phi \in \mathcal{H} \right\}.$$

It is clear that $E(\mathcal{H})$ cannot be dense in $B(\mathcal{H})$ with respect to the norm topology. Indeed, by Prop. 2.2. in [5], every operator $T \in E(\mathcal{H})$ has norm greater or equal than 1. So, the norm topology is not always the most natural topology on $B(\mathcal{H})$. It is often more useful to consider the weakest topology on $B(\mathcal{H})$, so-called the strong operator topology, it is defined by the family of seminorms $\{p_h : h \in \mathcal{H}\}$, where $p_h(T) = \|Th\|$. We conclude this note by raising the following question.

Question. For $T \in B(\mathcal{H})$, pick $c = \|T\| + \alpha$, $(\alpha > 0)$ and replace the norm topology by the strong operator topology. What can we say for the size of the set of all operators in $E(\mathcal{H})$ with the norm of at most c?

References

1. A. Aldroubi and A. Petrosyan, *Dynamical sampling and systems from iterative actions of operators* 2016.
2. A. Aldroubi, C. Cabrelli, U. Molter and S. Tang, *Dynamical sampling*, Appl. Harmon. Anal. Appl. 42 (3), 378-401 (2017)
3. F. Bayart, E. Matheron, *Dynamics of linear operators*, Cambridge University Press (2009).
4. O. Christensen, *An Introduction to Frame and Riesz Bases*, Second expanded, Birkhäuser, Basel (2016).
5. O. Christensen, M. Hasannasab, *Operator Representations of Frames: Boundedness, Duality and stability*, Integr. Equ. Oper. Theory 88, 483-499 (2017).
6. O. Christensen, M. Hasannasab and F. Philipp, *Frames properties of operator orbits*, Math. Nach., 2019
7. O. Christensen, M. Hasannasab, *Frame properties of systems arising via iterated actions of operators*, Appl. Comput. Har. Anal., 46 (3), (2019), 664-673.

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On l^1-Munn Algebras and Connes Amenability

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Abstract. In this paper, we study and investigate Connes amenability for l^1-Munn algebra $\mathcal{LM}(\mathcal{A}; P, I, J)$ where $\mathcal{A}$ is a Banach algebra, I and J are nonempty sets and P is invertible matrix. Then we use the obtained results to semigroup algebras $l^1(S)$ that S is a semigroup. Also, we prove that if S is a weakly cancellative semigroup and $l^1(S)$ be Connes amenable then the idempotents set of S will be finite.

Keywords: Connes amenability, Banach algebras, l^1-Munn algebras, semigroup algebras.

AMS Mathematical Subject Classification [2010]: 65F05, 46L05, 11Y50.

1. Introduction

l^1-Munn algebras are introduced by Eslamzadeh in [2]. Eslamzadeh characterized amenable semigroup algebras by l^1-Munn algebras. In [5], Munn introduced a certain type of mentioned algebras. l^1-Munn algebras has been investigated in some texts. For instance, Blackmore showed the l^1-Munn algebra of the group algebra $l^1(G)$ is weakly amenable. Also, the structure of this algebras studied by Eslamzadeh in [3]. On the other hand, the authors in [1] used the l^1-Munn algebras to study of semigroup algebras of completely simple semigroups. We know that special concept of amenability was called Connes amenability In [6], Runde extended the notion of Connes-amenability to dual Banach algebras. For a locally compact group G, the group algebra $l^1(G)$ and the measure algebra $M(G)$ are two examples of dual Banach algebras. Runde introduced normal, virtual diagonals for a dual Banach algebra and showed that the existence of a normal virtual diagonal for $M(G)$ is equivalent to it being Connes amenable. In particular, $l^1(G)$ is amenable if and only if $l^1(G)$ is Connes amenable. In this paper we investigate the semigroup algebra of a weakly cancellative semigroup in theorem of Runde. Also, the concept of Connes amenability of l^1-Munn algebras is investigated. We apply the l^1-Munn algebras to study of Connes amenability of mentioned semigroup algebras.

2. Main results

Let $\mathcal{A}$ be a dual Banach algebra such that $(\mathcal{A}_*)^* = \mathcal{A}$. A dual Banach $\mathcal{A}$-bimodule X is called normal Banach $\mathcal{A}$-bimodule if for each $x \in X$, the maps $a \to a.x$, $a \to x.a$ are ω^*-continuous $(a \in \mathcal{A})$.

Now, in the following we present some definitions.

Definition 2.1. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and X be a Banach $\mathcal{A}$-bimodule. The linear bounded map $D : \mathcal{A} \to X$ is called derivation if for every $a, b \in \mathcal{A}$ we have $D(ab) = D(a).b + a.D(b)$. We say D is inner derivation if there exists $x \in X$ such that for every $a \in \mathcal{A}$, $D(a) = a.x - x.a$. In this case we denote D by ad_x.

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. $\mathcal{A}$ is called amenable if every derivation $D : \mathcal{A} \to X^*$ is inner.

Definition 2.2. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and X be a normal dual Banach $\mathcal{A}$-bimodule. $\mathcal{A}$ is called Connes amenable if every ω^*-continuous derivation $D : \mathcal{A} \to X$ is inner.

*speaker

REMARK 2.3. Let X be a Banach $\mathcal{A}$-bimodule. An element $x \in X$ is called ω^*-weakly continuous if the module maps $a \to x.a, a \to a.x$ are ω^*-weak continuous $(a \in \mathcal{A})$. The collection of all ω^*-weakly continuous elements of X is denoted by $\sigma wc(X)$. We define the map $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ by $\pi(a \otimes b) = ab$, $(a, b \in \mathcal{A})$. Then we have $\pi^* : \mathcal{A}_* \to \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$. Consequently, π^{**} drops to a homomorphism $\pi_{\sigma wc} : \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^* \to \mathcal{A}$.

DEFINITION 2.4. An element $M \in \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A}^*)^*)$ is called a σwc-virtual diagonal for $\mathcal{A}$, if $M.u = u.M$ for every $u \in \mathcal{A}$ and $u.\pi_{\sigma wc}(M) = u$ for every $u \in \mathcal{A}$.

It is shown that Connes amenability of Banach algebra $\mathcal{A}$ is equal to existence a σwc-virtual diagonal for $\mathcal{A}$.

DEFINITION 2.5. Let $\mathcal{A}$ be a unital Banach algebra, let $I \neq \varnothing$, $J \neq \varnothing$ be arbitrary sets and $P = (p_{ij}) \in M_{J\times I}(\mathcal{A})$ be a matrix such that $\parallel P \parallel_\infty = \sup\{\parallel p_{ji} \parallel : j \in J, i \in I\} \leq 1$. The set $M_{I\times J}(\mathcal{A})$ of all $I \times J$ matrices $a = (a_{ij})$ on $\mathcal{A}$ with l^1-norm and the product $A \bullet B = APB$, $(A, B \in M_{I\times J}(\mathcal{A}))$ is a Banach algebra that is called l^1-Munn algebra on $\mathcal{A}$ with sandwich matrix P. In this case l^1-Munn algebra is denoted by $\mathcal{LM}(\mathcal{A}; P, I, J)$ [2].

The following statements are from [2].
A semigroup S is called regular if for every $a \in S$ there is $b \in S$ such that $a = aba$. S is an inverse semigroup if for every $a \in S$ there is a unique $a^* \in S$ such that $aa^*a = a$ and $a^*aa^* = a^*$. Let G be a group, $I \neq \varnothing$ and $J \neq \varnothing$ be arbitrary sets, and $G^0 = G \cup \{0\}$ be the group with zero arising from G by adjunction of a zero element. An $I \times J$ matrix A over G^0 that has at most one nonzero entry $a = A(i, j)$ is called a Rees $I \times J$ matrix over G^0 and is denoted by $(a)_{ij}$. Let P be a $J \times I$ matrix over G^0. The set $S = G \times I \times J$ with the composition $(a, i, j) \circ (b, l, k) = (aP_{jl}b, i, k)$, $(a, i, j), (b, l, k) \in S$ is a semigroup that we denote by $\mathcal{M}(G, P)$ [4, p. 68]. Similarly if P is a $J \times I$ matrix over G^0, then $S = G \times I \times J \cup \{0\}$ is a semigroup under the following composition operation:

$$(a, i, j) \circ (b, l, k) = \begin{cases} (aP_{jl}b, i, k); & P_{jl} \neq 0 \\ 0; & P_{jl} = 0 \end{cases}$$

$$(a, i, j) \circ 0 = 0 \circ (a, i, j) = 0 \circ 0 = 0$$

Mentioned semigroup which is denoted by $\mathcal{M}^0(G, P)$ also can be described in the following way: The set of all Rees $I \times J$ matrices over G^0 form a semigroup under the binary operation $A \bullet B = APB$, which is called the Rees $I \times J$ matrix semigroup over G^0 with the sandwich matrix P and is isomorphic to $\mathcal{M}^0(G, P)$ [4, pp. 61-63]. An $I \times J$ matrix P over G^0 is called regular (invertible) if every row and every column of P contains at least (exactly) one nonzero entry.

Already in[2], Esslamzadeh shows that if $\mathcal{LM}(\mathcal{A}; P, I, J)$ is amenable then $\mathcal{A}$ is amenable, $I \neq \varnothing$ and $J \neq \varnothing$ are finite and P is invertible. In the following theorem we investigate this subject for Connes amenablility of a Banach algebra.

THEOREM 2.6. *Let $\mathcal{A}$ be a dual Banach algebra and $\mathcal{LM}(\mathcal{A}; P, I, J)$ be Connes amenable. Then $\mathcal{A}$ is Connes amenable, moreover the sets I and J are finite and matrix P is invertible.*

PROOF. Let $\mathcal{LM}(\mathcal{A}; P, I, J)$ be Connes amenable. Thus it has a bounded approximate identity. By applying [2, Lemma 3.7], we implise that I and J are finite sets. So, P must be invertible. For prove Connes amenability of $\mathcal{A}$, it is sufficient that suppose $X = (X_*)^*$ be a normal Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \to X$ be a ω^*-continuous derivation. We show that D is inner. □

If in the above theorem, $\mathcal{A}$ has an identity element then we conclude the following corollary.

COROLLARY 2.7. *Let $\mathcal{A}$ be a Connes amenable unital dual Banach algebra, I and J are finite sets and P is invertible matrix. Then Banach algebra $\mathcal{LM}(\mathcal{A}; P, I, J)$ is Connes amenable.*

PROOF. Without losing the generality let $|I| = |J| = n$. Suppose that X be a normal Banach $\mathcal{LM}(\mathcal{A}; I_n, I, J)\widehat{\otimes}\mathcal{A}$-bimodule and $D : \mathcal{LM}(\mathcal{A}; I_n, I, J)\widehat{\otimes}\mathcal{A} \to X$ be a ω^*-continuous derivation. Now, we define the right and left module actions on $\mathcal{A}$, then we prove that D is inner. This complete the proof. □

Let S be a semigroup. We say that S is cancellative semigroup, if for every $r, p \neq q \in S$ we have $rs \neq rq$ and $pr \neq qr$.

An element $e \in S$ is called an idempotent if $e = e^* = e^2$. For $p \in S$, we define $L_p, R_p : S \to S$ by $L_p(q) = pq, R_p(q) = qp; (q \in S)$. If for each $p \in S$, L_p and R_p are finite-to-one maps, in this case S is named weakly cancellative. Suppose that S be a semigroup. S is called simple semigroup if only ideal in S is itself. S is called completely simple semigroup if it is simple and contain a principle idempotent.

Lemma 2.8. *Let S be a weakly cancellative semigroup and let $l^1(S)$ be Connes amenable unital semigroup algebra. Then the idempotents set S is finite.*

Theorem 2.9. *Let G be a group and S be a weakly cancellative semigroup and let $l^1(S)$ be Connes amenable unital semigroup algebra. Then S is a Rees matrix semigroup of the form $S = \mathcal{M}(G; P, I, J)$ and l^1-Munn algebra $\mathcal{LM}(l^1(G); P, I, J)$ has an identity.*

Proof. By above lemma, S is a simple semigroup with finite idempotents set, thus S is a completely simple semigroup. Therefore, S is a Rees matrix semigroup of the form $S = \mathcal{M}(G; P, I, J)$. This complete the proof. □

Theorem 2.10. *Let S be a weakly cancellative semigroup with finite idempotents set. Let $l^1(S)$ be an unital semigroup algebra. Moreover, suppose that S be a Rees matrix semigroup of the form $S = \mathcal{M}(G; P, I, J)$. If $l^1(S)$ is Connes amenable then $l^1(S)$ is amenable and vice versa.*

Proof. If $l^1(S)$ is Connes amenable then by Theorem 2.6 and [2] the proof is clearly. The converse follows directly from the above theorem. □

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References

1. J. Duncan and A. L. T. Paterson, *Amenability for discrete convolution semigroup algebras*, Math. Scand., 66 (1990), 141-146.
2. G. H. Esslamzadeh, *Banach algebra structure and amenability of a class of matrix algebras with applications*, J. Funct. Anal., 161 (1999), 364-383.
3. G. H. Esslamzadeh, *Representation theory and positive functionals of involutive l^1-Munn algebras*, Semigroup Forum, 69 (2004), 51-62.
4. O. T. Mewomo and O. J. Ogunsola, *On n-weak amenability of semigroup algebras*, J. Nigerian Math. Soc., 32 (2013), 298-301.
5. W. D. Munn, *On semigroup algebras*, Math. Proc. Cambridge Philos. Soc., 51 (1955), 1-15.
6. V. Runde, *Amenability for dual Banach algebras*, Studia Math., 148 (2001), 47-66.

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Biprojectivity and biflatness of bi-amalgamated Banach algebras

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Abstract. In this paper, we investigate biprojectivity and biflatness of a bi-amalgamated Banach algebra $A \boxtimes_\Theta X$ which depends on the Banach algebras A and X, and X is an algebraic Banach A-module and $\Theta : X \times X \to A$ is a bounded bilinear map that is compatible with the A-module operations of X.

Our results extend several results in the literature and provide simple direct proofs for some known results. In particular, we characterize the biprojectivity and biflatness of certain classes of the module extension Banach algebras and generalized matrix Banach algebras.
Keywords: Bi-amalgamated Banach algebras; Biprojective; Biflat.
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1. Introduction

Let A be a Banach algebra and $\Delta_A : A\hat{\otimes}A \to A$ be the diagonal operator induced by the multiplication of A, where $A\hat{\otimes}A$ stands for the projective tensor product of A by itself. Then A is called *biprojective* if $\Delta_A : A\hat{\otimes}A \to A$ has a bounded right inverse $\rho_A : A \to A\hat{\otimes}A$ which is an A-bimodule map. A Banach algebra A is said to be *biflat* if the adjoint $\Delta_A^* : A^* \to (A\hat{\otimes}A)^*$ of Δ_A has a bounded left inverse $\lambda_{A^*} : (A\hat{\otimes}A)^* \to A^*$ which is an A-bimodule map. Taking adjoints implies that every biprojective Banach algebra is biflat. These homological notions initiated and studied by Helemskii [2].

Let A, X be two Banach algebras and let X be a Banach A-module such that:

(i) X is an algebraic Banach A-module, that is,

$$a(xy) = (ax)y, (xy)a = x(ya), x(ay) = (xa)y \quad (a \in A, x, y \in X), \text{ and}$$

(ii) $\Theta : X \times X \to A$ is a bounded bilinear mapping which is compatible with the module operations and the multiplication of X in the sense of the following identities:

$$a\Theta(x, y) = \Theta(ax, y), \Theta(x, y)a = \Theta(x, ya), \Theta(xa, y) = \Theta(x, ay),$$

$$\Theta(xy, z) = \Theta(x, yyz), \Theta(x, y)z = x\Theta(y, z) \quad (a \in A, x, y \in X).$$

Then the ℓ^1-direct sum $A \times X$ equipped with the pointwise vector space operations and the multiplication

$$(a, x)(b, y) = (ab + \Theta(x, y), ay + xb + xy) \quad (a, b \in A, x, y \in X),$$

is a Banach algebra. We call this Banach algebra a *bi-amalgamated Banach algebra* with respect to Θ and we denote it by $A \boxtimes_\Theta X$. This Banach algebra firstly introduced in [4] where many important properties such as n-weak amenability, topological centers and the ideal structure have been studied.

The main examples of bi-amalgamated Banach algebras are (generalized) module extension Banach algebras, Lau product Banach algebras and generalized matrix Banach algebras.

The main results of this paper concerning biprojectivity and biflatness of $A \boxtimes_\Theta X$. We show that there is a close relation between the biprojectivity (resp. biflatness) of $A \boxtimes_\Theta X$ and that of A

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and X. We also apply our result to the certain examples of bi-amalgamated Banach algebras to give a simple direct proof for some older known results.

2. Biprojectivity and biflatness of $A \boxtimes_\Theta X$

In this section we first investigate the relation between biprojectivity of $A \boxtimes_\Theta X$ and that of A and X and then we seek the same for the biflatness of $A \boxtimes_\Theta X$.

2.1. Biprojectivity of $A \boxtimes_\Theta X$. We commence with the following result.

THEOREM 2.1. *Let A and X be biprojective. Let X be unital with $1_X a = a 1_X$ for all $a \in A$, and let there exist bounded linear maps $\nu : X \to X$ and $\mu : X \to A$ and a homomorphism $\eta : A \to A$ satisfying the following conditions:*

(i) $\Theta(1_X\eta(a), 1_X\eta(b)) = 0$;
(ii) $\mu(xy) = \mu(x)\mu(y) + \Theta(\nu(x), \nu(y))$ *and* $\nu(xy) = \nu(x)\nu(y) + \nu(x)\mu(y) + \mu(x)\nu(y)$;
(iii) $a\nu(1_X) = 1_X\eta(a)$, $x\mu(1_X) + x\nu(1_X) = x, \Theta(x, \nu(1_X)) = 0$ *and* $\eta(a) + a\mu(1_X) = a$,

for every $a, b \in A$ and $x, y \in X$. Then for $G = A \boxtimes_\Theta X$ there exists a bounded linear map $\rho_G : G \to G\hat{\otimes}G$ with $\Delta_G \circ \rho_G = id_G$. In adition if

(iv) *There exists a homomorphism $K : A \to A$ such that $\Theta(x, 1_X K(a)) = 0 = \Theta(1_X K(a), x)$, for all $x \in X, a \in A$,*
(v) *The involved A-bimodule map ρ_A satisfies the condition $\rho_A \circ \Theta = 0$, and*
(vi) *The involved X-bimodule map ρ_X satisfies the identity $(U\hat{\otimes}U) \circ \rho_X(1_X)g = g(U\hat{\otimes}U) \circ \rho_X(1_X)$ for all $g \in G$, where $U : X \to G$ is defined by $U(x) = (\mu(x), \nu(x))$ for each $x \in X$,*

then G is biprojective.

The following result studies the converse direction of Theorem 2.1.

THEOREM 2.2. *Suppose that $G = A \boxtimes_\Theta X$ is biprojective. Then,*

(1) *If there are A-bimodule maps $S : X \to A, L : A \to X, K : A \to A$ and $T : A \to A$ such that T is also a homomorphism satisfying*
   (i) $S(x)S(y) = S(xy) + T(\Theta(x, y))$,
   (ii) $S(x)T(a) = S(xa)$ *and* $T(a)S(x) = S(ax)$, *and*
   (iii) $T \circ K + S \circ L = id_A$,
   for every $a, b \in A$ and $x, y \in X$, then A is biprojective.
(2) *If there are bounded linear maps $U : A \to X, V : X \to X, \mu : X \to A$ and $\nu : X \to X$ satisfying*
   (i) *U is a homomorphism, $V(ax) = U(a)V(x), V(xa) = V(x)U(a)$ and $V(x)V(y) = V(xy) + U(\Theta(x, y))$,*
   (ii) $U \circ \mu + V \circ \nu = id_X$,
   (iii) $\Theta(\nu(x), y) = \mu(xy) = \Theta(x, \nu(y)), x\mu(y) + x\nu(y) = \nu(xy) = \mu(x)y + \nu(x)y$, *and*
   (iv) $V(x)y = U(\Theta(x, y)) + V(xy) = xV(y), V(xa) = xU(a), V(ax) = U(a)x$,
   for every $x, y \in X$ and $a \in A$, then X is biprojective.

If we use Theorems 2.1 and 2.2 for the special case $\Theta = 0$, we arrive at the following result for the generalized module extension Banach algebra $A \bowtie X$ which has already proved by Ettefagh [1].

COROLLARY 2.3 ([1, Theorems 3.1, 3.2]). *Suppose that there exists a homomorphism $U : A \to X$ such that $U(a)x = ax, xU(a) = xa$, for all $a \in A, x \in X$, (In particular, $U(a) = 1_X a$ $(a \in A)$, when X is unital with $1_X a = a1_X$ for all $a \in A$). Then $A \bowtie X$ is biprojective if and only if A and X are biprojective.*

EXAMPLE 2.4. Here we presents some special examples illustrating Theorems 2.1 and 2.2. Let A be a Banach algebra equipped with the multiplication π, which is also considered as a Banach A-module under its own multiplication. Then we have:

(i) If A is biprojective and unital, then for $G = A \boxtimes_\pi A$ there exists a G-bimodule map ρ_G such that $\Delta_G \circ \rho_G = id_G$.
(ii) If $A \boxtimes_{2\pi} A$ is biprojective, then so is A.
(iii) If $A \boxtimes_\pi A_0$, (A_0 stands for A with the zero multiplication) is biprojective, then so is A.

2.2. Biflatness of $A \boxtimes_\Theta X$. We now investigate the biflatness of $A \boxtimes_\Theta X$. For this purpose we need the following straightforward lemma.

LEMMA 2.5. *Set $G = A \boxtimes_\Theta X$.*

(1) *Let $T : A \to A$ and $S : A \to X$ be A-bimodule maps which are also homomorphisms and define $\psi : A \to G$ by the role $\psi(a) = (T(a), S(a))$ $(a \in A)$. If $\Theta(X, S(A)) = 0 = \Theta(S(A), X)$, $T(a)x = -S(a)x$ and $xT(a) = -xS(a)$, for all $x \in X$ and $a \in A$. Then*
  (i) *ψ is A-bimodule map and a homomorphism.*
  (ii) $(\psi\hat{\otimes}\psi)^* \circ \Delta_G^* = \Delta_A^* \circ \psi^*$.
  (iii) $(\psi\hat{\otimes}\psi)^*(\xi(a,x)) = (\psi\hat{\otimes}\psi)^*(\xi)a$ *and* $(\psi\hat{\otimes}\psi)^*((a,x)\xi) = a(\psi\hat{\otimes}\psi)^*(\xi)$, *for every* $(a,x) \in G$ *and* $\xi \in (G\hat{\otimes}G)^*$.
(2) *Let $K_X : X \to G$ be an A-bimodule map and a homomorphism and let $V : A \to X$ be a bounded linear map satisfying $V(a)x = ax$ and $xV(a) = xa$, for all $a \in A, x \in X$. Define $\phi : G \to X$ by by the role $\phi(a,x) = x + V(a)$ $(a \in A, x \in X)$. Then*
  (i) $(K_X\hat{\otimes}K_X)^* \circ \Delta_G^* = \Delta_X^* \circ K_X^*$.
  (ii) $(K_X\hat{\otimes}K_X)^*(\xi(a,x)) = (K_X\hat{\otimes}K_X)^*(\xi)\phi((a,x))$ *and*
  $(K_X\hat{\otimes}K_X)^*((a,x)\xi)) = \phi((a,x))(K_X\hat{\otimes}K_X)^*(\xi)$ *for all* $(a,x) \in G, \xi \in (G \otimes G)^*$.

For $G = A \boxtimes_\Theta X$, let $P_A : G \to A$ and $P_X : G \to X$ be the usual projections on A and X, respectively and let $J_A : A \to G$ and $J_X : X \to G$ be the canonical injections.

In the next result we study some conditions under which the biflatness of A and X induce a left inverse for Δ_G^*.

THEOREM 2.6. *Suppose that A and X are biflat. If T, S and K_X are as in Lemmas 2.5 satisfying the additional conditions*

$$T - P_A \circ K_X \circ S = id_A,\ P_X \circ K_X - S \circ P_A \circ K_X = id_X,\ S = P_X \circ K_X \circ S,\ \text{and}\ P_A \circ K_X = T \circ P_A \circ K_X.$$

Then for $G = A \boxtimes_\Theta X$ there exists a bounded linear map $\lambda_{G^*} : (G\hat{\otimes}G)^* \to G^*$ such that $\lambda_{G^*} \circ \Delta_G^* = id_{G^*}$.

Investigating on the reverse direction of Theorem 2.6, we arrive at the next result.

THEOREM 2.7. *Suppose that $G = A \boxtimes_\Theta X$ is biflat. If $q_A : G \to A$, $K_A : A \to G$ are A-bimoule maps and $q_X : G \to X$ and $K_X : X \to G$ are X-bimodule maps such that q_A and q_X are homomorphisms satisfying the identities $q_A \circ K_A = id_A$ and $q_X \circ K_X = id_X$. Then A and X are biflat.*

We apply Theorems 2.6 and 2.7 for the generalized module extension Banach algebra $A \bowtie X$ which has already studied by Ettefagh [1].

COROLLARY 2.8 ([1, Theorems 4.1, 4.2]). *Suppose that there exists a homomorphism $U : A \to X$ such that $U(a)x = ax, xU(a) = xa$, for all $a \in A, x \in X$, (In particular, $U(a) = 1_X a$ $(a \in A)$, when X is unital with $1_X a = a 1_X$ for all $a \in A$). Then $A \bowtie X$ is biflat if and only if A and X are biflat.*

As consequences of Corollaries 2.3 and 2.8 we get the following results.

COROLLARY 2.9. *Let A and B be two Banach algebras. Then*
(i) *the bi-amalgamated Banach algebra $A \bowtie A$ is biflat (resp. biprojective) if and only if A is biflat (resp. biprojective).*
(ii) *the ℓ^1-direct product $A \oplus B$ is biflat (respr. biprojective) if and only if A and B are biflat (resp. biprojective).*

We also obtain the following result concerning the biprojectivity and biflatness of the τ-Lau product Banach algebra $A\ {}_{\tau}\!\times B$ which already proved by Khodami and Ebrahimi Vishki [3].

COROLLARY 2.10 ([3, Theorem 1]). *Let A and B be two Banach algebras such that B is unital and $\tau \in \sigma(A)$. Then $A\ {}_{\tau}\!\times B$ is biflat (resp. biprojective) if and only if A and B are biflat (resp. biprojective).*

COROLLARY 2.11. *Let $G = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ be a trivial generalized matrix Banach algebra such that $\overline{AM} = M = \overline{MB}$ and $\overline{NA} = N = \overline{BN}$. Then G is biflat (rep. biprojective) if and only if A and B are biflat (resp. biprojective), $M = 0$ and $N = 0$.*

As an immediate consequence of Corollary 2.11 we have the following result of Medghalchi and Sattari [5] concerning the triangular Banach algebra.

COROLLARY 2.12 ([5, Theorem 2.2]). *Let $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ be a triangular Banach algebra such that $\overline{AM} = M = \overline{MB}$. Then T is biflat (rep. biprojective) if and only if A and B are biflat (resp. biprojective) and $M = 0$.*

References

[1] M. Ettefagh, *Biprojectivity and biflatness of generalized module extension Banach algebras*, Filomat **32** (2018), 5895–5905.

[2] A.Ya. Helemskii, *Flat Banach modules and amenable algebras*, Trudy Moskov. Math. Obshch. **47** (1985), 199–244.

[3] A.R. Khodami and H.R. Ebrahimi Vishki, *Biflatness and biprojectivity of Lau product of Banach algebras*, Bull. Iranian Math. Soc. **39** (2013), 559–568.

[4] H. Lakzian, H.R. Ebrahimi Vishki and S. Barootkoob, *Itrated duals and weak amenability of bi-amalgamated Banach algebras*, *Preprint.*

[5] A.R. Medghalchi and M.H. Sattari, *Biflatness and biprojectivity of triangular Banach algebras*, Bull. Iranian Math. Soc. **34** (2008), 115–120.

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Orthogonality preserving pairs

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Abstract. In this survey, we would use the categorical approach to Hilbert C^*-modules over a commutative C^*-algebra to investigate the orthogonality preserving pair of maps on Hilbert C^*-modules over a commutative C^*-algebra.

Suppose that Γ and Γ' are two continuous fields of Hilbert spaces $(H_z)_{z\in Z}$ and $(K_z)_{z\in Z}$ on a locally compact Hausdorff space Z. Let (Ψ, Φ) be an orthogonality preserving pair of $C_0(Z)$-module maps from Γ into Γ'. Then there exists a bounded complex-valued function ϕ on Z that is continuous on $Z_\Gamma = \{z \in Z : \langle x, x\rangle(z) \neq 0 \text{ for some } x \in \Gamma\}$ and satisfies

$$\langle \Psi(x), \Phi(y)\rangle = \phi \cdot \langle x, y\rangle$$

on Z, for all $x, y \in \Gamma$.

Keywords: Orthogonality preserving maps, Hilbert C^*-modules, Continuouse field of Hilbert spaces.

1. Introduction

The notion of orhogonality is initially associated with inner product spaces. Let $\big(H, (\cdot, \cdot)\big)$ be an inner product space, two elements $x, y \in H$ are said to be orthogonal if $(x, y) = 0$. For two inner product spaces H and K, a mapping $T : H \longrightarrow K$ is called orthogonality preserving, OP in short, if it preserves orthogonality, that is if

$$\forall x, y \in H : \ x \perp y \quad \Longrightarrow \quad T(x) \perp T(y).$$

By [2], for a pair of linear mappings $T, S : H \longrightarrow K$ between inner product spaces H and K. The following conditions are equivalent, for some $\gamma \in \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$:

$$1.\ \forall x, y \in H : \ x \perp y \ \Rightarrow \ T(x) \perp S(y),$$

$$2.\ \forall x, y \in H : \ \big(T(x), S(y)\big) = \gamma(x, y).$$

Some authors generalized orthogonality preserving property to Hilbert C^*-modules [4, 6, 7]. Frank, Moslehian and Zamani in [3] studied orthogonality preserving pairs of maps on Hilbert A-modules over standard C^*-algebra A. Recall that A is a standard C^*-algebra on a Hilbert space H if $\mathcal{K}(H) \subseteq A \subseteq \mathcal{B}(H)$. In fact, they prove if A is a standard C^*-algebra and $\Psi, \Phi : E \longrightarrow F$ are two A-linear maps between Hilbert A-modules such that (Ψ, Φ) is an orthogonality preserving pair, then there exists an element γ of the center $Z(M(A))$ of the multiplier algebra $M(A)$ of A such that

$$\langle \Psi(x), \Phi(y)\rangle = \gamma \cdot \langle x, y\rangle$$

for all $x, y \in E$.

In this survey, we cosider the above statement for Hilbert C^*-modules over a commutative C^*-algebra. We use the fact that in the case of commutative C^*-algebras $C_0(Z)$, Hilbert C_0-modules are the same as continuous field of Hilbert spaces over a locally compact Hausdorff space Z.

We first recall some basic facts on these structures.

The notion of an inner product (respectively Hilbert) C^*-module is a generalization of a complex inner product (respectively Hilbert) space in which the inner product takes its values in

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a C^*-algebra rather than in field of complex numbers. Hilbert C^*-modules are useful tools in theory of operator algebra, operator K-theory and theory of operator spaces. A (left) Hilbert C^*-module E over a C^*-algebra A is a left A-module equipped with an "A-valued inner product" ${}_A\langle .,.\rangle : E \times E \longrightarrow A$ such that the following conditions hold for all $x, y, z \in E$, $a \in A$ and $\alpha, \beta \in \mathbb{C}$:

$$\begin{aligned} (i) &\quad {}_A\langle \alpha x + \beta y, z\rangle = \alpha \ {}_A\langle x, z\rangle + \beta \ {}_A\langle y, z\rangle, \\ (ii) &\quad {}_A\langle ax, y\rangle = a \ {}_A\langle x, y\rangle, \\ (iii) &\quad {}_A\langle x, y\rangle^* = \ {}_A\langle y, x\rangle, \\ (iv) &\quad {}_A\langle x, x\rangle \geq 0, \ and \ {}_A\langle x, x\rangle = 0 \ if \ and \ only \ if \ x = 0. \end{aligned}$$

If E is complete with respect to the induced norm by the A-valued inner product, $\|x\| = \|{}_A\langle x, x\rangle\|^{\frac{1}{2}}$, $x \in E$, then E is called a (left) Hilbert C^*-module over A or, simply a Hilbert A-module (in the sequel, we will omit the subscripts). Any C^*-algebra A is a Hilbert C^*-module over itself via $\langle a, b\rangle = ab^*$ $(a, b \in A)$. Note that the theory of inner product C^*-modules is quite defferent from that of inner product spaces. As example, not any closed submodule of an inner product C^*-modules is complemented. For a good introduction to Hilbert C^*-modules, we refer the reader to book [5].

Let A be a C^*-algebra. Two elements x, y in an inner product A-module $(E, \langle .,.\rangle)$ are said to be orthogonal if $\langle x, y\rangle = 0$. Recall that a map $\Psi : E \longrightarrow F$, where E and F are Hilbert C^*-modules, is said to be orthogonality preserving if $\langle \Psi(x), \Psi(y)\rangle = 0, \ whenever \ \langle x, y\rangle = 0, \ x, y \in E$. Also two maps $\Psi, \Phi : E \longrightarrow F$ between Hilbert A-modules are said to be orthogonality preserving if $\langle \Psi(x), \Phi(y)\rangle = 0, \ whenever \ \langle x, y\rangle = 0, \ x, y \in E$.

Recall that a complex linear map $\Psi : E \longrightarrow F$ between Hilbert A-module, is called an A-module map if $\Psi(ax) = a\Psi(x)$ for all $a \in A$ and $x \in E$.

In this survey, we would use the categorical approach to Hilbert C^*-modules over a commutative C^*-algebra to investigate the orthogonality preserving pair of maps on Hilbert C^*-modules over a commutative C^*-algebra. The categorical approach says that the category of (left) Hilbert C^*-modules over a commutative C^*-algebra $A = C_0(Z)$ is equivalent to the category of continuous fields of Hilbert spaces over a locally compact Hausdorff space Z [8].

2. Main results

Definition 2.1. Let Z be a locally compact Hausdorff space. Consider $((H_z)_{z \in Z}, \Gamma)$, where $(H_z)_{z \in Z}$ is a family of Hilbert spaces and Γ is a subset of $\prod_{z \in Z} H_z$. Also, we set

$$C_0 - \prod_{z \in Z} H_z = \{x \in \prod_{z \in Z} H_z : \ [z \mapsto \|x(z)\|] \in C_0(Z)\}.$$

The pair $((H_z)_{z \in Z}, \Gamma)$ satisfying the following properties is said to be a *continuous field of Hilbert spaces*.

1) Γ is a linear subspace of $C_0 - \prod_{z \in Z} H_z$.
2) The set $\{x(z) : x \in \Gamma\}$ equals to H_z, for every $z \in Z$.
3) If $x \in C_0 - \prod_{z \in Z} H_z$ and for every $z \in Z$ and every $\epsilon > 0$ there is a $x' \in \Gamma$ such that $\|x(s) - x'(s)\| < \epsilon$ in some neighbourhood of z, then $x \in \Gamma$.

If there is no confusion, we denote a continuous field of Hilbert spaces $((Hz)_{z \in Z}, \Gamma)$ by Γ.

If Γ is a continuous field of Hilbert spaces, then the function $z \mapsto \langle x(z), y(z)\rangle$ is an element of $C_0(Z)$, for every $x, y \in \Gamma$. In fact, Γ is a (left) Hilbert $C_0(Z)$-module equipped with the following pointwise multiplication and $C_0(Z)$-valued inner product

$$(f \cdot x)(z) = f(z)x(z) \qquad \& \qquad \langle x, y\rangle(z) = \langle x(z), y(z)\rangle,$$

for all $f \in C_0(Z)$, $x, y \in \Gamma$ and $z \in Z$. Conversely, it is well known that every Hilbert $C_0(Z)$-module is isomorphic to one of this form [8]. In this survey, we set $Z_\Gamma = \{z \in Z : \langle x, x\rangle(z) \neq 0$ for some $x \in \Gamma\}$.

The following theorem is the main theorem, and we prove it after some lemmas.

THEOREM 2.2. [1] *Let Z be a locally compact Hausdorff space, and Γ and Γ' be continuous fields of Hilbert spaces $(H_z)_{z\in Z}$ and $(K_z)_{z\in Z}$ over Z. Let $\Psi, \Phi : \Gamma \longrightarrow \Gamma'$ be two nonzero $C_0(Z)$-module maps that preserve orthogonality. Then there exists a bounded complex function ϕ on Z, continuous on Z_Γ, such that for all $x, y \in \Gamma$,*

$$\langle \Psi(x), \Phi(y)\rangle = \phi \ \cdot \langle x, y\rangle.$$

Moreover, if $Im(\Psi) \subseteq Im(\Phi)$ (or $Im(\Phi) \subseteq Im(\Psi)$) then Ψ (or Φ) is bounded. In particular, if $Im(\Psi) = Im(\Phi)$ then Ψ and Φ are both bounded.

Throughout this work Z is a locally compact Hausdorff space, and Γ and Γ' are two continuous fields of Hilbert spaces $(H_z)_{z\in Z}$ and $(K_z)_{z\in Z}$ on Z, respectively.

The following lemma determines the structure of $C_0(Z)$-module maps between continuous field of Hilbert spaces.

LEMMA 2.3. [1] *Let $\Psi, \Phi : \Gamma \longrightarrow \Gamma'$ be two nonzero $C_0(Z)$-module maps. For every $z \in Z$, two maps $\Psi_z, \Phi_z : H_z \longrightarrow K_z$ defined by $\Psi_z(x(z)) = (\Psi(x))(z)$ and $\Phi_z(x(z)) = (\Phi(x))(z)$ are well-defined and linear. Moreover, the $C_0(Z)$-module maps Ψ and Φ are bounded if and only if Ψ_z and Φ_z are bounded for all $z \in Z$ and $sup_{z\in Z}\|\Psi_z\| < \infty$ and also $sup_{z\in Z}\|\Phi_z\| < \infty$. Indeed, $sup_{z\in Z}\|\Psi_z\| = \|\Psi\|$ and also $sup_{z\in Z}\|\Phi_z\| = \|\Phi\|$.*

To achieve the main result, we first show that for every $z \in Z$, two nonzero linear maps $\Psi_z, \Phi_z : H_z \longrightarrow K_z$ are orthogonality preserving between Hilbert spaces H_z and K_z. Therefore, some results that hold in the setting of Hilbert spaces can be generalized to Hilbert $C_0(Z)$-modules.

LEMMA 2.4. [1] *For every $z \in Z$, the linear maps $\Psi_z, \Phi_z : H_z \longrightarrow K_z$ are orthogonality preserving.*

Proof of the main theorem:

Now we have two linear maps $\Psi_z, \Phi_z : H_z \longrightarrow K_z$ are orthogonality preserving, then by [2, Theorem 3.2], there exists a complex number γ_z such that for every $h, h' \in H_z$, $\langle \Psi_z(h), \Phi_z(h')\rangle = \gamma_z\langle h, h'\rangle$.

Now, we define $\phi : Z \longrightarrow \mathbb{C}$ by

$$\phi(z) = \begin{cases} 0 & \mathrm{H}_z = \{0\} \\ \gamma_z & \mathrm{H}_z \neq \{0\} \end{cases}$$

Obviously, for every $x, y \in \Gamma$ and $z \in Z$, we have

$$\big\langle \Psi(x)(z), \Phi(y)(z)\big\rangle = \phi(z)\langle x(z), y(z)\rangle.$$

The authors in [1] proved that the nonnegative complex-valued function ϕ is bounded on Z, and continuos on $Z_E = \{z \in Z : \langle x, x\rangle(z) \neq 0 \text{ for some } x \in E\}$. Hence, $\phi \in C_b(Z_\Gamma)$.

The following example show that a pair of orthogonality preserving maps need not be bounded.

EXAMPLE 2.5. [1] Let H be a Hilbert space and $T : H \to H$ be a linear operator that is not bounded. We define linear operators $\Phi, \Psi : H \to H \oplus H$ by $\Phi(h) = (T(h), 0)$ and $\Psi(h) = (0, T(h))$. Clearly (Ψ, Φ) is an orthogonality preserving pair of linear operators, but Φ, and also Ψ, are not bounded.

The following result is Corollary 3.7 in [7], and indeed it show that a $C_0(Z)$-module map $\Psi : E \longrightarrow F$ between Hilbert $C_0(Z)$-modules, under conditions, is a $C_b(Z)^+$-scalar multiple of a unitary.

COROLLARY 2.6. *Suppose that E and F are two Hilbert $C_0(Z)$-modules and $\Psi : E \longrightarrow F$ is an orthogonality preserving $C_0(Z)$-module map. Then Ψ is bounded and there exists a bounded nonnegative function ϕ on Z that is continuous on Z_E and satisfies*

$$\langle \Psi(x), \Psi(y)\rangle = \phi \cdot \langle x, y\rangle,$$

for all $x, y \in E$. Moreover,

(i) if E is full, then $Z_E = Z$ and so $\phi \in C_b(Z)^+$,

(ii) if F is full and Ψ is surjective, then ϕ is a strictly positive element of $C_b(Z)^+$ and so $\phi^{\frac{1}{2}}\Psi$ is unitary, i. e., Ψ is a $C_b(Z)^+$-scalar multiple of a unitary.

References

1. M. B. Asadi, F. Olyaninezhad, Orthogonality preserving pairs of operators on Hilbert $C_0(Z)$-modules. https://doi.org/10.1080/03081087.2020.182561.
2. J. Chmieliński, Orthogonality equation with two unknown functions, Aequationes Math. 90 (2016) 11-23.
3. M. Frank, M. S. Moslehian and A. Zamani, Orthogonality preserving property for pairs of operators on Hilbert C^*-modules, arXiv:1711.04724v2 [Math.OA] 2019.
4. D. Ilievíc and A. Turnek, Approximately orthogonality preserving mappings on C*-modules. 341 (2008) 298-308.
5. E.C. Lance, *Hilbert C^*-modules*, London Math. Soc. Lecture Note Ser., vol. 210, Cambridge Univ. Press, Cambridge 1995.
6. C.-W. Leung, C.-K. Ng and N.-C. Wong, Linear orthogonality preservers of Hilbert C^*-modules, J. Operator Theory 71 (2014) 571-584.
7. C.-W. Leung, C.-K. Ng and N.-C. Wong, Linear orthogonality preservers of Hilbert bundles, J. Aust. Math. Soc. 89 (2010), 245-254.
8. Heunen C, Reyes ML. Frobenius structures over Hilbert C-modules. Commun Math Phys. 2018;361:787824.

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$(\sigma,\tau)-$amenability of $A \rtimes B$

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Abstract. For Banach algebra B and Banach B-bimodule A, the amalgamated of B along a Banach B-bimodule A, i.e. $A \rtimes B$, was introduce by Javanshiri and Nemati [1]. In this paper, for continuous endomorphisms σ,σ' and τ,τ' on A and B respectively, we investigate the $((\sigma,\tau),(\sigma',\tau'))-$amenability of $A\rtimes B$, and also we study the relations between of $((\sigma,\tau),(\sigma',\tau'))-$amenability of $A \rtimes B$ with similar concept on A and B.

Keywords: Amenability, Banach algebras, $(\sigma,\tau)-$amenability.
AMS Mathematical Subject Classification [2010]: 46H05, 46H25.

1. Introduction

Let A be a Banach algebra and X be a Banach $A-$bimodule. Then X^*, the dual space of X, by thy following module actions is a Banach $A-$bimodule.

$$\langle x^* \cdot a, x\rangle = \langle x^*, a \cdot x\rangle, \qquad \langle a \cdot x^*, x\rangle = \langle x^*, x \cdot a\rangle,$$

for all $a \in A$, $x \in X$ and $x^* \in X^*$. The linear map $d : A \longrightarrow X$ is derivation if $d(aa') = a \cdot d(a') + d(a) \cdot a'$ for all $a, a' \in A$. Also, for $x \in X$ the derivation $\mathrm{ad}_x : A \longrightarrow X$ defined by $\mathrm{ad}_x(a) = a \cdot x - x \cdot a$ called inner derivation and if for derivation $d : A \longrightarrow X$ there exists a net $(x_\alpha) \subseteq X$ such that $d(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a)$ for all $a \in A$, then d is an approximate inner derivation. The set of all continuous derivations and inner derivations from A to Banach $A-$bimodule X are denoted by $Z^1(A, X)$ and $N^1(A, X)$, respectively. Also, the quotient space $H^1(A, X) = Z^1(A, X)/N^1(A, X)$ is first cohomology group of A with coefficients in X. The Banach algebra A is called amenable if $H^1(A, X^*) = \{0\}$ for all Banach $A-$bimodule X. The concept of amenability of Banach algebra was first introduced by Johnson [2] in 1972.

We recall that from [5], for any two bounded continuous endomorphisms σ and τ on Banach algebra A, the linear map $d : A \longrightarrow X$ is called $(\sigma,\tau)-$derivation if

$$d(aa') = \sigma(a) \cdot d(a') + d(a) \cdot \tau(a'),$$

for all $a, a' \in A$. For any $x \in X$, the derivation $\mathrm{ad}_x : A \longrightarrow X$ defined via $\mathrm{ad}_x(a) = \sigma(a)\cdot x - x\cdot\tau(a)$ is called $(\sigma,\tau)-$inner derivation for all $a \in A$. Similarly, the set of all continuous $(\sigma,\tau)-$derivations and $(\sigma,\tau)-$ inner derivations from A to X denoted by $Z^1_{(\sigma,\tau)}(A, X)$ and $N^1_{(\sigma,\tau)}(A, X)$, respectively, and $H^1_{(\sigma,\tau)}(A, X) = Z^1_{(\sigma,\tau)}(A, X)/N^1_{(\sigma,\tau)}(A, X)$ is the first $(\sigma,\tau)-$cohomology group of A with coefficients in X. The Banach algebra A is $(\sigma,\tau)-$amenable if any $(\sigma,\tau)-$derivation $d : A \longrightarrow X^*$ be $(\sigma,\tau)-$inner, for all Banach $A-$bimodule X. The notion of $(\sigma,\tau)-$amenability of Banach algebra introduced and studied by Moslehian [4].

2. Main results

Let A and B be Banach algebras such that A is a Banach $B-$bimodule. Then the left and right actions of B on A are compatible, if

$$b \cdot (aa') = (b \cdot a)a', \ (aa') \cdot b = a(a' \cdot b), \ a(b \cdot a') = (a \cdot b)a', \ (a, a' \in A, \ b \in B).$$

*speaker

So, the Cartesian product $A \rtimes B$ with the multiplication

$$(a,b)(a',b') = (aa' + a\cdot b' + b\cdot a', bb'),$$

and the norm

$$\|(a,b)\| = \|a\| + \|b\|,$$

is amalgamated duplication of B along A and denoted by $A \rtimes B$. It is easy to see that A and B are identify with $A \times \{0\}$ and $\{0\} \times B$, respectively. Also, A is a closed ideal and B is a closed subalgebra of $A \rtimes B$. Moreover $\frac{(A \rtimes B)}{A}$ is isometric isomorphism with to B. $A \rtimes B$ is commutative if and only if A and B are commutative Banach algebra.

THEOREM 2.1. *Suppose that A and B are two Banach algebras, $e \in A$ is unit and σ and τ are two homomorphisms on A and B, respectively. Then $(\sigma,\tau) : A \rtimes B \longrightarrow A \rtimes B$ defined by $\langle(\sigma,\tau),(a,b)\rangle = (\sigma(a),\tau(b))$ is a homomorphism if and only if $\sigma(e\cdot b) = e\cdot\tau(b)$ for all $a \in A$ and $b \in B$.*

Proof. First suppose that (σ,τ) is a homomorphism. Then

$$\langle(\sigma,\tau),(a,b)(a',b')\rangle = (\sigma(aa' + ab' + ba'), \tau(bb')),$$

for all $a, a' \in A$ and $b, b' \in B$. Set $a = e$ and $a' = 0$. Thus, we have that $\sigma(e\cdot b) = e\cdot\tau(b)$ for all $b \in B$. Conversely, let $\sigma(e\cdot b) = e\cdot\tau(b)$ for all $b \in B$. Then we have that

$$\begin{aligned}
\sigma(aa' + ab' + ba') &= \sigma(aa') + \sigma(ab') + \sigma(ba') \\
&= \sigma(a)\sigma(a') + \sigma(ae\cdot b') + \sigma(e\cdot ba') \\
&= \sigma(a)\sigma(a') + \sigma(a)\sigma(e\cdot b') + \sigma(e\cdot b)\sigma(a') \\
&= \sigma(a)\sigma(a') + \sigma(a)e\cdot\tau(b') + e\cdot\tau(b)\sigma(a') \\
&= \sigma(a)\sigma(a') + \sigma(a)\cdot\tau(b') + \cdot\tau(b)\sigma(a').
\end{aligned}$$

These relations show that $((\sigma,\tau),()\sigma',\tau')$ is a continuous homomorphisms on $A \rtimes B$. □

PROPOSITION 2.2. *Let A and B are two Banach algebras, $e \in A$ is nuit and σ, σ' and τ, τ' are two homomorphisms on A and B, respectively such that $\sigma(e\cdot b) = e\cdot\tau(b)$ and $\sigma'(e\cdot b) = e\cdot\tau'(b)$ for all $b \in B$. Then $((\sigma,\tau),(\sigma',\tau'))-$amenability of $A \rtimes B$ imply the $(\sigma,\sigma')-$amenability of A and $(\tau,\tau')-$amenability of B.*

Proof. First, if X be Banach $A-$bimodule and $d_1 : A \longrightarrow X^*$ be $(\sigma,\sigma')-$derivation, then with the module actions defined by

$$y * (a,b) = y\cdot a + y\cdot(e\cdot b), \qquad (a,b) * y = a\cdot y + (e\cdot b)\cdot y$$

X is a Banach $(A \rtimes B)-$bimodule, for all $a \in A$, $b \in B$ and $y \in X$. Also, the map $D_1 : A \rtimes B \longrightarrow X^*$ via $D_1(a,b) = d_1(a + e\cdot b)$ is $((\sigma,\tau),(\sigma',\tau'))-$derivation. Hence, there exists $f \in X^*$ such that $D_1(a,b) = \langle(\sigma,\tau),(a,b)\rangle * f - f * \langle(\sigma',\tau'),(a,b)\rangle$ for all $a, a' \in A$ and $b, b' \in B$. Therefore, if set $b = 0$, then we have that

$$\begin{aligned}
\langle d_1(a), y\rangle &= \langle D_1(a,0), y\rangle \\
&= \langle f, y * (\sigma(a),0)\rangle - \langle f, (\sigma'(a),0) * y\rangle \\
&= \langle \sigma(a)\cdot f - f\cdot\sigma'(a), y\rangle
\end{aligned}$$

for all $a \in A$ and $y \in X$. So, d_1 is $(\sigma,\sigma')-$inner derivation. Thus, A is $(\sigma,\sigma')-$amenable. For the $(\tau,\tau')-$amenability of B, let X be Banach $B-$bimodule and $d_2 : B \longrightarrow X^*$ be $(\tau,\tau')-$derivation. Then by the following module actions X is a Banach $(A \rtimes B)-$bimodule and $D_2 : A \rtimes B \longrightarrow X^*$ defined via $D_2(a,b) = d_2(b)$ is a $((\sigma,\tau),(\sigma',\tau'))-$derivation for all $a \in A$ and $b \in B$. So, there exists $g \in X^*$ such that $D_2(a,b) = \langle(\sigma,\tau),(a,b)\rangle * g - g * \langle(\sigma,\tau),(a,b)\rangle$, for all $a \in A$ and $b \in B$. Thus, it is routine that $d_2(b) = \tau(b)\cdot g - g\cdot\tau'(b)$ for all $b \in B$. □

THEOREM 2.3. *Let A and B be two Banach algebras, $e \in A$ is nuit and σ, σ' and τ, τ' are two homomorphisms on A and B, respectively such that $\sigma(e \cdot b) = e \cdot \tau(b)$ and $\sigma'(e \cdot b) = e \cdot \tau'(b)$ for all $b \in B$. Then $A \bowtie B$ is $((\sigma,\tau),(\sigma',\tau'))-$amenable if A be $(\sigma,\sigma')-$amenable and B be $(\tau,\tau')-$amenable.*

Proof. Suppose that X is Banach $(A \bowtie B)-$bimodule and $D : A \bowtie B \longrightarrow X^*$ is $((\sigma,\tau),(\sigma',\tau'))-$derivation. Then by module actions

$$a \cdot y = (a,0) \cdot y, \qquad y \cdot a = y \cdot (a,0),$$

X is $A-$bimodule and $D_1 : A \longrightarrow A^*$ by $D_1(a) = D(a,0)$ is $(\sigma,\sigma')-$derivation, for all $a \in A$ and $y \in A$. Thus, there exists $f \in X^*$ such that $D_1(a) = \sigma(a) \cdot f - f \cdot \sigma'(a)$ for all $a \in A$. It is clear that $(D - D_1)|_A = 0$. So, by replacing D with $D - D_1$, $D(a) = 0$ for all $a \in A$. Therefore, we have that $\langle D((a,b)(a',0)), y)\rangle = \langle D(a,b), (\sigma'(a'),0) \cdot y\rangle = 0$ and $\langle D((a',0)(a,b)), y\rangle = \langle D(a,b), y \cdot (\sigma(a'),0)\rangle = 0$ for all $a, a' \in A$ and $y \in X$. If $X_A = \overline{\langle \sigma'(A) \cdot X \bigcup X \cdot \sigma(A)\rangle}$, then X_A is a closed submodule of X. So, $D(A)|_{X_A} = 0$. Hence $D(A) \subseteq X_A^{\perp} = (X/X_A)^*$. Therefore, by module actions

$$b \cdot (y + X_A) = (0,b) \cdot y + X_A, \qquad (y + X_A) \cdot b = y \cdot (0,b) + X_A,$$

X/X_A is Banach $B-$bimodule, for all $b \in B$ and $y \in X$. It is clear that the map $\tilde{D} : B \longrightarrow (X/X_A)^*$ defined via

$$\langle \tilde{D}(b), y + X_A\rangle = \langle D(0,b), y\rangle,$$

is $(\tau,\tau')-$derivation and there exists $g \in X_A^{\perp}$ such that $\tilde{D} = \mathrm{ad}_g$, for all $b \in B$ and $y \in X$. Thus we have that $D = \mathrm{ad}_{f+g}$, as required. □

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References

1. H. Javanshiri, and M. Nemati, *Amalgamated duplication of the Banach algebra B along a $B-$bimodule A,* Preprint.
2. B. E. Johnson, *Cohomology in Banach Algebras*, Mem. Amer. Math. Soc., **127** (1972).
3. M. Momeni, T. Yazdanpanah, and M. R. Mardanbeigi, *Sigma-ideal amenability of Banach algebras*, J. Math. Computer Sci., **8** (2014) 319-325.
4. M. Sal Moslehian, *Approximate $(\sigma-\tau)-$contractibility*, Nonlinear Funct. Anal. Appl., **11**(no. 5,) (2006) 805-813.
5. M. Sal Moslehian, and A. Niazi Motlagh, *Some notes on $(\sigma,\tau)-$amenability of Banach algebras,* Studia Univ. "Babes-Bolyai", Mathematica, **3** (2008) 57-68.

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Characterization of forward and reverse Bergman Carleson measures using sparse domination

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Abstract. In this paper, using a new technique from harmonic analysis called sparse domination, we characterize the positive Borel measures including forward and reverse Bergman Carleson measures. The main novelty of this paper is determining the reverse Bergman Carleson measures which have remained open from the work of Luecking [Am. J. Math. 107 (1985) 85111]. Moreover, in the case of forward and vanishing measures, our results extend the results of [J. Funct. Anal. 280 (2021), no. 6, 108897, 26 pp] from $1 \leq p \leq q < 2p$ to all $0 < p \leq q < \infty$. Although we consider the weighted Bergman spaces induced by two-side doubling weights, the results are new even on classical weighted Bergman spaces.
Keywords: Bergman Carleson measures, Sparse domination, Radial difference operators.
AMS Mathematical Subject Classification [2010]: 30H20, 47B39, 42B99.

1. Introduction

Throughout the paper, n is a fixed positive integer. Let $\mathbb{C}$ be the complex plane, $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$, and $\mathbb{S}$ be the boundary of $\mathbb{B}$. Let $H(\mathbb{B})$ be the space of all analytic functions on $\mathbb{B}$. A radial weight on $\mathbb{B}$ is a positive and integrable function ω on $\mathbb{B}$ such that $\omega(z) = \omega(|z|)$. If $0 < p < \infty$, the weighted Bergman space is denoted by $A^p_\omega(\mathbb{B})$ and defined as $A^p_\omega = A^p_\omega(\mathbb{B}) = H(\mathbb{B}) \bigcap L^p(\mathbb{B}, \omega dV)$, where V is the normalized Lebesgue measure on $\mathbb{B}$. Throughout the paper, we use the notations $dV_\omega = \omega dV$ and $\|.\|_\omega = \|.\|_{\omega,2}$. For every $\alpha > -1$, the classical weighted Bergman space, denoted by A^p_α, is the space $A^p_{(1-|z|^2)^\alpha}$.

Let $\widehat{\mathcal{D}}$ be the class of all radial weights ω on $\mathbb{B}$ where $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s)ds$ satisfies the doubling condition $\widehat{\omega}(r) \lesssim \widehat{\omega}(\frac{1+r}{2})$. We say that the weight $\omega \in \widehat{\mathcal{D}}$ is a two-sides doubling weight, denoted by $\omega \in \mathcal{D}$, if there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that $\widehat{\omega}(r) \geq C\widehat{\omega}(1 - \frac{1-r}{K})$, for all $0 \leq r < 1$. J. A. Pelez [5] introduced the classes $\widehat{\mathcal{D}}$ and $\mathcal{D}$ and studied the weighted Bergman spaces $A^q_\omega(\mathbb{D})$ induced by these weights.

A. Lerner [3], with an alternative proof of the A_2 theorem, showed that the norm of the CalderónZygmund operators can be controlled from above by a very special dyadic type of operators called sparse operators. For a survey of the advances on the topic, see [2, 3] and the references therein. In Theorem (3.1), we show that the modulus of the k-th radial derivative of functions in $H(\mathbb{B})$ will be controlled by the sparse operators from above.

Suppose that μ is a positive Borel measure on the domain $D \subset \mathbb{C}^n$. Let A be a Banach space of holomorphic functions on D where A is contained in $L^q(D, \mu)$ for some $q > 0$. Then, μ is called a (q, A)-Carleson measure if the inclusion map $i : A \to L^q(D, \mu)$ is bounded. A (q, A)-Carleson measure is called a reverse (q, A)-Carleson measure if $i : A \to L^q(\mu)$ is bounded below. If $A = A^p_\omega$, then a (q, A^p_ω)-Carleson measure is called a (p, q, ω)-Bergman Carleson measure and if $p = q$, it is called a ω-Bergman Carleson measure.

In 1962, Carleson introduced and characterized the Carleson measures to study the structure of the Hardy spaces and solved the corona problem on the disc. B. Hu et al. [1], using sparse domination, gave a new characterization for the boundedness of weighted composition operators from A^p_α to A^q_α for $\alpha > -1$ and $1 \leq p \leq q < 2p$ on the upper half-plane and the unit ball. Checking

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the validity of the results of this paper for the classical $(p, q, (1-|.|)^\alpha)$-Bergman Carleson measures is easy. In this project, we will extend the results of [1] from $1 \le p \le q < 2p$ to all $0 < p \le q < \infty$. Moreover, in a more general case, we determine the positive Borel measures μ on $\mathbb{B}$ so that the radial differential operator $R^{(k)} : A^q_\omega \to L^q(\mu)$ is bounded for $\omega \in \mathcal{D}$ and $0 < p \le q < \infty$.

Luecking [4] has conducted a valuable study on the reverse (q, A^q_α)-Carleson measures. In Theorem (3.5), using the sparse domination technique, we give a complete characterization for the reverse ω-Bergman Carleson measures when $\omega \in \mathcal{D}$. Furthermore, we present new versions of [4, Theorems 4.2 and 4.3].

2. Preliminaries

The radial derivative of a holomorphic function $f : \mathbb{B} \to \mathbb{C}$ is defined as $Rf(z) = \sum_{i=1}^n z_i \frac{\partial f}{\partial z_i}(z)$. Moreover, the k-th radial differentiation operator on $H(\mathbb{B})$ is denoted by $R^{(k)}$ and defined as usual.

Let $\beta : \mathbb{B} \times \mathbb{B} \to [0, \infty)$ be the Bergman metric. If $r > 0$ and $z \in \mathbb{B}$, the Bergman ball centered at z with radius r is defined as $B(z, r) = \{\zeta \in \mathbb{B} : \ \beta(z, \zeta) < r\}$.

For $a \in \mathbb{B}$, let Pa be the radial projection of a onto $\mathbb{S}$. The pseudometric on $\mathbb{S}$ is defined as $\rho(z, \zeta) = |1 - \langle z, \zeta\rangle|$. As usual, $D(z, r) := \{\zeta \in \mathbb{S} : \rho(z, \zeta) < r\}$. For $z \in \mathbb{B} \setminus \{0\}$, let $Q_z = D(Pz, 1 - |z|)$. Then, the Carleson square $S(z)$ will be defined as

$$S(z) = \{\zeta \in \mathbb{B},\ P\zeta \in Q_z,\ |z| < |\zeta| < 1\}, \quad S(0) = \mathbb{B}.$$

Let μ and B be a Borel measure and a Borel subset of $\mathbb{B}$, respectively. Then, we set $\mu(B) = \int_B d\mu$. For the function $g \in L^p(\mathbb{B}, \mu)$, the average of g based on μ is defined as $\langle g\rangle_{\mu,B} := \frac{1}{\mu(B)} \int_B g(z) d\mu(z)$, For convenience, we put $\langle h\rangle_B = \langle h\rangle_{V,B}$ for $h \in L^p(\mathbb{B}, V)$.

2.1. Dyadic structure on $\mathbb{B}$. Let $S_r = \partial B(0, r)$. Fix $\theta, \lambda > 0$. For $N \in \mathbb{N}$, we can find a sequence of points $E_N = \{w_j^N\}_{j=1}^{J_N}$, and a corresponding sequence of Borel subsets $\{X_j^N\}_{j=1}^{J_N}$ of $S_{N\theta}$, that satisfy: $S_{N\theta} = \bigcup_{j=1}^{J_N} X_j^N$, $X_j^N \bigcap X_i^N = \emptyset$ for all $i \ne j$, and

$$S_{N\theta} \bigcap B(w_j^N, \lambda) \subset X_j^N \subset S_{N\theta} \bigcap B(w_j^N, C\lambda).$$

For $z \in \mathbb{B}$, let $P_r z$ denote the radial projection of z onto the sphere S_r. We now define the set K_j^N as follows: $K_1^0 := \{z \in \mathbb{B} : \ \beta(0, z) < \theta\}$ and

$$K_j^N \ := \ \{z \in \mathbb{B} : \ N\theta \le \beta(0, z) < (N + 1)\theta \textit{ and } P_{N\theta} z \in X_j^N\},\ N \ge 1,\ j \ge 1.$$

We will refer to the subset K_j^N of $\mathbb{B}$ as a unit cube centered at $c_j^N = P_{(N+\frac{1}{2})\theta} w_j^N$, while K_1^0 is centered at 0. We say that c_i^{N+1} is a child of c_j^N if $P_{N\theta} c_i^{N+1} \in X_j^N$.

A tree structure is a set $\mathcal{T} := \{c_j^N\}$ which contains the centers of the cubes. We will denote the elements of the tree by α and β, while K_α will be the cube with center α. There is no problem to abuse the notation and use α to denote both an element of the tree $\mathcal{T}$ and the center of the corresponding cube or any element of the cube. The notation $\beta \ge \alpha$ means that β is a descendant of α. A *dyadic tent* under K_α is defined as: $\widehat{K_\alpha} := \bigcup_{\beta \in \mathcal{T} : \beta \ge \alpha} K_\beta$.

A *dyadic grid* on $\mathbb{S}$ with calibre δ is a collection of Borel subsets $\mathfrak{D} := \{Q_i^k\}_{i,k \in \mathbb{Z}}$ and points $\{z_i^k\}_{i,k \in \mathbb{Z}}$ in $\mathbb{S}$ that satisfy:

(i) There are $c_1, C_2 > 0$ such that $D(z_i^k, c_1\delta^k) \subset Q_i^k \subset D(z_i^k, C_2\delta^k)$ for every $k, i \in \mathbb{Z}$.
(ii) For all $k \in \mathbb{Z}$, there holds $\mathbb{S} = \bigcup_{i \in \mathbb{Z}} Q_i^k$ and the sets are disjoint.
(iii) If $Q, R \in \mathfrak{D}$ and $Q \bigcap R \ne \emptyset$, then either $Q \subset R$ or $R \subset Q$.

Consider $\mathfrak{D}$ as the dyadic grid on $\mathbb{S}$ with calibre δ as mentioned above. Then, this dyadic system creates a Bergman tree with suitable parameters θ and λ which depend on δ. Indeed, we can construct a tree structure as: $X_j^N := P_{N\theta} Q_j^N$, and $c_j^N := P_{(N+\frac{1}{2})\theta} z_j^N$, there are $\lambda_2 > \lambda_1 > 0$ so that

$$S_{N\theta} \bigcap B(P_{N\theta} z_j^N, \lambda_1) \subset X_j^N \subset S_{N\theta} \bigcap B(P_{N\theta} z_j^N, \lambda_2)$$

Thus, if $\mathfrak{D} := \{Q_i^k\}_{i,k\in\mathbb{Z}}$ and $\mathcal{T} := \{c_j^N\}_{j,N\in\mathbb{Z}}$ are constructed as above, then the function $c : \mathfrak{D} \to \mathcal{T}$, defined as $c(Q_j^N) = c_j^N$, is one-to-one and onto.

Lemma 2.1. *Let $\omega \in \widehat{\mathcal{D}}$. For every $\delta > 0$, there is a finite collection of dyadic grids $\{\mathfrak{D}^i\}_{i=1}^M$ with the following property: For all $z \in \mathbb{B}$, there are $\zeta \in \mathbb{B}$ and $1 \le i \le M$ such that there is a dyadic cube $Q \in \mathfrak{D}^i$ where $S(z) \subseteq \widehat{K_{c(Q)}} \subseteq S(\zeta)$ and $V_\omega(\widehat{K_{c(Q)}}) \simeq V_\omega(S(z)) \simeq V_\omega(S(\zeta))$.*

Notation. From now on, for every calibre $\delta > 0$, we assume that $\mathfrak{D}_\delta = \{\mathfrak{D}_\delta^i\}_{i=1}^M$ is the finite collection of *dyadic grids* obtained in Lemma (2.1).

3. Main results

3.1. Sparse domination. In this subsection, we show that the modulus of the k-th radial derivative of functions in $H(\mathbb{B})$ will be controlled by sparse operators from above.

Theorem 3.1. *Let $0 < p, \delta < \infty$ and $k \in \mathbb{N}\bigcup\{0\}$. Then, there are positive constants r_δ, C_δ, and D_δ which depend only on δ such that*

$$|R^{(k)}f(z)|^p \le D_\delta^{k+1} C_\delta^{kp} \sum_{i=1}^{M} \sum_{Q\in\mathfrak{D}_\delta^i} \frac{\mathbf{1}_{K_{c(Q)}}(z)}{(1-|c(Q)|)^{kp}} \langle |f|^p \rangle_{B(c(Q),2^{k+1}r_\delta)}, \tag{1}$$

for all $f \in H(\mathbb{B})$ and $z \in \mathbb{B}$.

Notation. From now on, for $k \in \mathbb{N}\bigcup\{0\}$, we fix $R_\delta = 2^{k+1} r_\delta$ and $B_{c(Q)} = B(c(Q), R_\delta)$ for Q in $\mathfrak{D}_\delta$.

3.2. Forward Bergman Carleson measures. Let ν be a weight on $\mathbb{B}$, $k \in \mathbb{N}\bigcup\{0\}$ and δ, s, and t be positive numbers. We say that a positive Borel measure μ is in $\mathcal{C}_{\nu,\delta}^{t,ks}$ if

$$[\mu]_{\nu,\delta}^{t,ks} := \sup_{Q\in\mathfrak{D}_\delta} \frac{\mu(B_{c(Q)})}{V_\nu(B_{c(Q)})^t (1-|c(Q)|^2)^{ks}} < \infty.$$

Theorem 3.2. *Let $\omega \in \mathcal{D}$, $\nu(r) = \widehat{\omega}(r)/(1-r)$, $0 < p \le q < \infty$, and $k \in \mathbb{N}\bigcup\{0\}$. If μ is a positive Borel measure, then the following statements are equivalent.*

(i) *$R^{(k)} : A_\omega^p \to L_\mu^q$ is bounded,*

(ii) *$\mu \in \mathcal{C}_{\nu,\delta}^{\frac{q}{p},kq}$ for some (every) calibre $\delta > 0$.*

(iii) *For some (every) calibre $\delta > 0$ and $m \in \mathbb{N}$, there are positive numbers C_δ and D_δ which depend only on δ so that for all $f \in A_\omega^p$ we have*

$$\|R^{(k)}f\|_{\mu,q}^q \le D_\delta^{m(k+1)} C_\delta^{kq} [\mu]_{\nu,\delta}^{\frac{q}{p},kq} \inf_{r_j>0,\ r_1+\ldots+r_m=q} \Big(\sum_{i=1}^{M} \sum_{Q\in\mathfrak{D}_\delta^i} V_\nu(B_{c(Q)})^{\frac{q}{p}} \cdot \prod_{j=1}^{m} \langle |f|^{r_j} \rangle_{B_{c(Q)}} \Big),$$

3.3. Reverse Bergman Carleson measures. First, we give two results from [4] which are the best obtained results on this topic.

Theorem 3.3. [4, Theorem 4.2] *Let $\alpha > -1$, $0 < \varepsilon, C, q < \infty$, and μ be a (q, A_α^q)-Carleson measure. Then, there is an $r > 0$ such that if $G = \{z \in \mathbb{D};\ \frac{\mu(B(z,r))}{(1-|z|)^{\alpha+2}} > \varepsilon\}$ is a C-dominated set for $A_\alpha^q(\mathbb{D})$, then μ is a reverse (q, A_α^q)-Carleson measure.*

Let C be a positive constant and G be a Borel subset of $\mathbb{B}$. Then, we say that G is a C-dominated set for A_ω^q, if

$$\int_G |f|^q dV_\omega \ge C \int_{\mathbb{B}} |f|^q dV_\omega, \quad f \in A_\omega^q.$$

Sometimes when the quantity of C is not important, we say that G is a dominated set for A_ω^q.

Theorem 3.4. [4, Theorem 4.3] *Let $\alpha > -1$ and $0 < q < \infty$. If μ is a reverse (q, A_α^q)-Carleson measure, then there are $r, \varepsilon > 0$ such that $\frac{\mu(B(z,r))}{(1-|z|)^{\alpha+2}} > \varepsilon$ for all $z \in \mathbb{D}$.*

Consider $\delta > 0$. Define the sets

$$H_\varepsilon(\mu;\omega;\delta) := \{Q \in \mathfrak{D}_\delta;\ \frac{\mu(B_{c(Q)})}{V_\omega(B_{c(Q)})} > \varepsilon\},\quad G_\varepsilon(\mu;\omega;\delta) := \bigcup_{Q \in H_\varepsilon(\mu;\omega;\delta)} B_{c(Q)},$$

and for $f \in A^q_\omega$,

$$H_\varepsilon(f;\mu;\delta;r) := \Big\{Q \in \mathfrak{D}_\delta;\ \langle |f|^r \rangle_{\mu,B_{c(Q)}} > \varepsilon \langle |f|^r \rangle_{B_{c(Q)}}\Big\}.$$

Now we present the main result of the paper.

THEOREM 3.5. *Let $\omega \in \mathcal{D}$, $\nu(r) = \widehat{\omega}(r)/(1-r)$, and $0 < q < \infty$. If μ is ω-Bergman Carleson measure, then the following statements are equivalent.*

(i) *μ is a reverse ω-Bergman Carleson measure.*

(ii) *For some (every) $m \in \mathbb{N}$ and calibre δ, there is an $\varepsilon > 0$ such that*

$$\|f\|^q_{\nu,q} \lesssim \inf_{r_1+\ldots+r_m=q} \Big(\sum_{Q \in H_\varepsilon(\mu;\nu;\delta)} V_\nu(B_{c(Q)}).\langle |f|^{r_m} \rangle_{\mu,B_{c(Q)}}.\prod_{j=1}^{m-1} \langle |f|^{r_j} \rangle_{B_{c(Q)}}\Big),\quad f \in A^q_\omega.$$

(iii) *For some (every) $m \in \mathbb{N}$ and calibre δ, there is an $\varepsilon > 0$ such that*

$$\|f\|^q_{\nu,q} \lesssim \inf_{r_1+\ldots+r_m=q} \Big(\sum_{Q \in H_\varepsilon(\mu;\nu;\delta)} V_\omega(B_{c(Q)}).\prod_{j=1}^{m} \langle |f|^{r_j} \rangle_{B_{c(Q)}}.\mathbf{1}_{H_\varepsilon(f;\mu;\delta;r_m)}(Q)\Big),\quad f \in A^q_\omega.$$

REMARK 3.6. Note that we can choose the power r_m in Theorem (3.5) (iii, iv) as small as we like.

REMARK 3.7. The gap between Theorems (3.3) and (3.4) is that the r obtained in Theorem (3.3) is small, while the r in Theorem (3.4) is large. To fill this gap, we used the sets $H_\varepsilon(\mu;\omega;\delta)$ and $G_\varepsilon(\mu;\omega;\delta)$ in Theorem (3.5) instead of set G in Theorem (3.3).

Now, we will try to give the new versions of Theorems (3.3) and (3.4) using the sets $G_\varepsilon(\mu;\omega;\delta)$. The following theorem is the new version of Theorem (3.3). However, again again in this new version, the calibre (or equivalently the radius of the Bergman balls) must be small enough.

THEOREM 3.8. *Let $\omega \in \mathcal{D}$, $\nu(r) = \widehat{\omega}(r)/(1-r)$, $0 < \varepsilon, C, q < \infty$, and μ be ω-Carleson measure. Then, there is a calibre δ such that if $G_\varepsilon(\mu;\nu;\delta)$ is a C-dominated set for A^q_ω, then μ is a reverse ω-Carleson measure.*

The following Theorem is a new version of Theorem (3.4). Note that in part (i) of this new version, the radius of the Bergman balls does not need to be large.

THEOREM 3.9. *Let $\omega \in \mathcal{D}$, $\nu(r) = \widehat{\omega}(r)/(1-r)$ and $0 < q < \infty$. If μ is a reverse ω-Bergman Carleson measure, then the following statements hold.*

(i) *For every calibre δ, there is an $\varepsilon > 0$ such that $G_\varepsilon(\mu;\nu;\delta)$ is a dominated set for $A^q_\omega(\mathbb{B})$.*

(ii) *There are $\delta, \varepsilon > 0$ such that $H_\varepsilon(\mu;\nu;\delta) = \mathfrak{D}_\delta$.*

References

1. B. Hu, S. Li, Y. Shi, B. Wick, Sparse domination of weighted composition operators on weighted Bergman spaces, J. Funct. Anal. 280 (2021), no. 6, 108897, 26 pp.
2. T. Hytönen, The sharp weighted bound for general CalderónZygmund operators, Ann. of Math. (2) 175 (2012), no. 3, 14731506.
3. A.K. Lerner, A simple proof of the A_2 conjecture, Int. Math. Res. Not. IMRN, 14 (2013), 31593170.
4. D. H. Luecking, Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives, Am. J. Math. 107 (1985) 85111.
5. J. A. Pelez, Small weighted Bergman spaces. In: Proceedings of the Summer School in Complex and Harmonic Analysis, and Related Topics (2016).

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Pseudo-amenability of weighted semigroup algebras

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Abstract. Amenability and pseudo-amenability of $\ell^1(S,\omega)$ is characterized, where S is a left (right) zero semigroup or it is a rectangular band semigroup. The equivalence conditions to amenability of $\ell^1(S,\omega)$ are provided, where S is a band semigroup. The equivalence properties of amenability of $\ell^1(S,\omega)^{**}$ are described, where S is an inverse semigroup. For a locally compact group G, pseudo-amenability of $\ell^1(G,\omega)$ is also discussed.
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1. Introduction

For a Banach algebra $\mathfrak{A}$ the projective tensor product $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ is a Banach $\mathfrak{A}$-bimodule in a natural manner and the multiplication map $\pi : \mathfrak{A}\widehat{\otimes}\mathfrak{A} \longrightarrow \mathfrak{A}$ defined by $\pi(a \otimes b) = ab$ for $a, b \in \mathfrak{A}$ is a Banach $\mathfrak{A}$-bimodule homomorphism.

Amenability for Banach algebras introduced by B. E. Johnson [6]. Let $\mathfrak{A}$ be a Banach algebra and E be a Banach $\mathfrak{A}$-bimodule. A continuous linear operator $D : \mathfrak{A} \longrightarrow E$ is a *derivation* if it satisfies $D(ab) = D(a) \,.\, b + a \,.\, D(b)$ for all $a, b \in \mathfrak{A}$. Given $x \in E$, the *inner* derivation $ad_x : \mathfrak{A} \longrightarrow E$ is defined by $ad_x(a) = a \,.\, x - x \,.\, a$. A Banach algebra $\mathfrak{A}$ is *amenable* if for every Banach $\mathfrak{A}$-bimodule E, every derivation from $\mathfrak{A}$ into E^*, the dual of E, is inner.

An *approximate diagonal* for a Banach algebra $\mathfrak{A}$ is a net $(m_i)_i$ in $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that $a \,.\, m_i - m_i \,.\, a \longrightarrow 0$ and $a\pi(m_i) \longrightarrow a$, for each $a \in \mathfrak{A}$. The concept of pseudo-amenability introduced by F. Ghahramani and Y. Zhang in [4]. A Banach algebra $\mathfrak{A}$ is *pseudo-amenable* if it has an approximate diagonal. It is well-known that amenability of $\mathfrak{A}$ is equivalent to the existence of a *bounded* approximate diagonal.

The notions of biprojectivity and biflatness of Banach algebras introduced by Helemskiĭ. A Banach algebra $\mathfrak{A}$ is *biprojective* if there is a bounded $\mathfrak{A}$-bimodule homomorphism $\rho : \mathfrak{A} \longrightarrow \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that $\pi o \rho = I_{\mathfrak{A}}$, where $I_{\mathfrak{A}}$ is the identity map on $\mathfrak{A}$. We say that $\mathfrak{A}$ is *biflat* if there is a bounded $\mathfrak{A}$-bimodule homomorphism $\rho : \mathfrak{A} \longrightarrow (\mathfrak{A}\widehat{\otimes}\mathfrak{A})^{**}$ such that $\pi^{**} o \rho = k_{\mathfrak{A}}$, where $k_{\mathfrak{A}} : \mathfrak{A} \longrightarrow \mathfrak{A}^{**}$ is the natural embedding of $\mathfrak{A}$ into its second dual.

Let S be a semigroup. A continuous function $\omega : S \longrightarrow (0, \infty)$ is a *weight* on S if $\omega(st) \leq \omega(s)\omega(t)$, for all $s, t \in S$. Then it is standard that

$$\ell^1(S,\omega) = \left\{ f = \sum_{s\in S} f(s)\delta_s : \|f\|_\omega = \sum_{s\in S} |f(s)|\omega(s) < \infty \right\}$$

is a Banach algebra with the convolution product $\delta_s * \delta_t = \delta_{st}$. These algebras are called *Beurling algebras.*

In this note, we study the earlier mentioned properties of Banach algebras for Beurling algebras. Firstly in section 2, we characterize amenability and pseudo-amenability of $\ell^1(S,\omega)$, for some certain class of semigroups. We prove that pseudo-amenability of $\ell^1(S,\omega)$, for a left or right zero semigroup S, is equivalent to it's amenability and these equivalent conditions imply that S is singleton. We show that the same result holds for $\ell^1(S,\omega)$, whenever S is a rectangular band

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semigroup and ω is separable. Further, we investigate biprojectivity of $\ell^1(S,\omega)$ whenever S is either left (right) zero semigroup or a rectangular band semigroup. For a band semigroup S, we show that amenability of $\ell^1(S,\omega)$ is equivalent to that of $\ell^1(S)$ and these are equivalent to S being a finite semilattice. We find necessary and sufficient conditions for $\ell^1(S,\omega)^{**}$ to be amenable, where S is an inverse semigroup.

Finally in section 3, we investigate pseudo-amenability of $L^1(G,\omega)$ where G is a locally compact group and ω is a weight on G. We prove that pseudo-amenability of $L^1(G,\omega)$ implies amenability of G, and under a certain condition it implies diagonally boundedness of ω. Next, if $L^1(G,\omega)$ is pseudo-amenable we may obtain a character φ on G for which $\varphi \leq \omega$.

2. Amenability and pseudo-amenability of $\ell^1(S,\omega)$

A semigroup S is a *left zero semigroup* if $st = s$, and it is a *right zero semigroup* if $st = t$ for each $s,t \in S$. Then for $f,g \in \ell^1(S,\omega)$, it is obvious that $f * g = \varphi_S(f)g$ if S is a right zero semigroup, and $f*g = \varphi_S(g)f$ if S is a left zero semigroup, where φ_S is the *augmentation character* on $\ell^1(S,\omega)$.

We extend the results for $\ell^1(S)$ in [**1, 2**] to the weighted case $\ell^1(S,\omega)$.

Proposition 2.1. *Suppose that S is a right (left) zero semigroup and ω be a weight on S. Then $\ell^1(S,\omega)$ is biprojective.*

Proof. We only give the proof in the case S is a right zero semigroup. Define $\rho : \ell^1(S,\omega) \longrightarrow \ell^1(S,\omega)\widehat{\otimes}\ell^1(S,\omega)$ by $\rho(f) = \delta_{t_\circ} \otimes f,$, where t_0 is an arbitrary element S. Then for each $f,g \in \ell^1(S,\omega)$ we have

$$\rho(f*g) = \delta_{t_\circ} \otimes (f*g) = \varphi_S(f)(\delta_{t_\circ} \otimes g) = (f*\delta_{t_0}) \otimes g = f \,.\, (\delta_{t_0} \otimes g) = f \,.\, \rho(g)$$

and similarly $\rho(f*g) = \rho(f) \,.\, g$. Further, $\pi\rho$ is the identity map on $\ell^1(S,\omega)$, as required. □

Remark 2.2. It is known that every biprojective Banach algebra is biflat. Hence Proposition 2.1 shows that for every right or left zero semigroup S, $\ell^1(S,\omega)$ is biflat.

Given two semigroups S_1 and S_2, we say that a weight ω on $S := S_1 \times S_2$ is *separable* if there exist two weights ω_1 and ω_2 on S_1 and S_2, respectively such that $\omega = \omega_1 \otimes \omega_2$. It is easy to verify that $\ell^1(S,\omega) \cong \ell^1(S_1,\omega_1)\widehat{\otimes}\ell^1(S_2,\omega_2)$.

Let S be a semigroup and let $E(S) = \{p \in S : p^2 = p\}$. We say that S is a *band semigroup* if $S = E(S)$. A band semigroup S satisfying $sts = s$, for each $s,t \in S$ is called a *rectangular band semigroup*. For a rectangular band semigroup S, it is known that $S \simeq L \times R$, where L and R are left and right zero semigroups, respectively.

Proposition 2.3. *Let S be a rectangular band semigroup and ω be a separable weight on S. Then $\ell^1(S,\omega)$ is biprojective, and so it is biflat.*

Proof. In view of earlier argument, it follows From Proposition 2.1, and then from [**7**, Proposition 2.4].

Theorem 2.4. *Let S be a rectangular band semigroup and ω be a weight on S. Then $\ell^1(S,\omega)$ is amenable if and only if S singleton.*

Proof. From [**8**, Theorem 3.6], $\ell^1(S)$ is amenable. Then it is immediate by [**1**, Theorem 3.3]. □

For a semigroup S, we denote by S^{op} the semigroup whose underlying space is S but whose multiplication is the multiplication in S reversed.

Proposition 2.5. *Let S be a right (left) zero semigroup and ω be a weight on S. Then $\ell^1(S,\omega)$ is amenable if and only if S is singleton.*

Let $\mathfrak{A}$ be Banach algebra, $\mathcal{I}$ be a *semilattice* (i.e., $\mathcal{I}$ is a commutative band semigroup) and $\{\mathfrak{A}_\alpha : \alpha \in \mathcal{I}\}$ be a collection of closed subalgebras of $\mathfrak{A}$. Then $\mathfrak{A}$ is ℓ^1-*graded* of $\mathfrak{A}_\alpha$'s over the semilattice $\mathcal{I}$, denoted by $\mathfrak{A} = \bigoplus_{\alpha\in\mathcal{I}}^{\ell^1} \mathfrak{A}_\alpha$, if it is ℓ^1-directsum of $\mathfrak{A}_\alpha$'s as Banach space such that $\mathfrak{A}_\alpha\mathfrak{A}_\beta \subseteq \mathfrak{A}_{\alpha\beta}$, for each $\alpha, \beta \in \mathcal{I}$.

Suppose that S^1 is the unitization of a semigroup S. An equivalence relation τ on S is defined by $s\tau t \iff S^1sS^1 = S^1tS^1$, for all $s, t \in S$. If S is a band semigroup, then by [?, Theorem 4.4.1], $S = \bigcup_{\alpha\in\mathcal{I}} S_\alpha$ is a semilattice of rectangular band semigroups, where $\mathcal{I} = \frac{S}{\tau}$ and for each $\alpha = [s] \in \mathcal{I}$, $S_\alpha = [s]$.

Theorem 2.6. *Let S be a band semigroup and ω be a weight on S. Then the following are equivalent:*
(i) $\ell^1(S,\omega)$ is amenable.
(ii) S is finite and each $\tau-$class is singleton.
(iii) $\ell^1(S)$ is amenable.
(iv) S is a finite semilattice.

Proof: The implications (ii) to (iv) are equivalent [1, Theorem 3.5]. We establish $(i) \longrightarrow (ii)$ and $(iv) \longrightarrow (i)$.

$(i) \longrightarrow (ii)$ If $\ell^1(S,\omega)$ is amenable, then $E(S) = S$ is finite and so $\mathcal{I} = \frac{S}{\tau}$ is a finite semilattice. Hence $\ell^1(S,\omega) \cong \bigoplus_{\alpha\in\mathcal{I}}^{\ell^1} \ell^1(S_\alpha,\omega_\alpha)$, where $\omega_\alpha = \omega|_{S_\alpha}$. Then by [5, Proposition 3.1], each $\ell^1(S_\alpha,\omega_\alpha)$ is amenable. Now by Theorem 2.4, S_α is singleton for each $\alpha \in \mathcal{I}$, as required.

$(iv) \longrightarrow (i)$ In this case $\ell^1(S,\omega) \cong \ell^1(S)$, and $\ell^1(S)$ is amenable. □

Theorem 2.7. *Let S be a rectangular band semigroup, and let ω be a separable weight on S. Then $\ell^1(S,\omega)$ is pseudo-amenable if and only if S is singleton.*

Proof. There is a left zero semigroup L and a right zero semigroup R, and there are weights ω_L and ω_R on L and R, respectively such that $S \cong L\times R$ and $\omega = \omega_L \otimes \omega_R$. We have $\ell^1(S,\omega) \cong \ell^1(L,\omega_L)\widehat{\otimes}\ell^1(R,\omega_R)$. Hence the map $\theta : \ell^1(S,\omega) \longrightarrow \ell^1(L,\omega_L)$ defined by $\theta(f\otimes g) = \varphi_R(g)f$ for $f \in \ell^1(L,\omega_L)$ and $g \in \ell^1(R,\omega_R)$, is an epimorphism of Banach algebras, whereas φ_R is the *augmentation* character on $\ell^1(R,\omega_R)$. Whence $\ell^1(L,\omega_L)$ has left and right approximate identity. Therefore L is singleton, because it is left zero semigroup. Similarly R is singleton, so is S. □

Corollary 2.8. *Let S be a right (left) zero semigroup and ω be a weight on S. Then the following are equivalent:*
(i) $\ell^1(S,\omega)$ is pseudo-amenable.
(ii) S is singleton.
(iii) $\ell^1(S,\omega)$ is amenable.

The following is a combination of Theorems 2.4 and 2.7. Notice that in Theorem 2.4, we need not ω to be separable.

Corollary 2.9. *Let S be a rectangular band semigroup, and let ω be a separable weight on S. Then the following are equivalent:*
(i) $\ell^1(S,\omega)$ is pseudo-amenable.
(ii) S is singleton.
(iii) $\ell^1(S,\omega)$ is amenable.

For the left cancellative semigroups we have the following.

Theorem 2.10. *Suppose that S is a left cancellative semigroup and ω is a weight on S. If $\ell^1(S,\omega)$ is pseudo-amenable, then S is a group.*

proof: This is a more or less verbatim of the proof of [2, Theorem 3.6 $(i) \longrightarrow (ii)$]. □

Let $(P,\leq)$ is a partially ordered set. Then $(P,\leq)$ is *locally finite* if $(x] = \{y \in S : y \leq x\}$ is finite for every $x \in S$, and it is *uniformly locally finite* if $\sup\{|(x]| : s \in S\} < \infty$.

We recall that a semigroup S is an *inverse semigroup* if for each $s \in S$ there exists a unique element $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s^*$. The maximal subgroup of S at $p \in E(S)$ is denoted by G_p. It is known that $G_p = \{s \in S : ss^* = s^*s = p\}$.

Theorem 2.11. *Let S be an inverse semigroup, let ω be a weight on S, and let $\ell^1(S,\omega)$ has a bounded approximate identity. Then the following are equivalent:*
(i) $\ell^1(S,\omega)^{**}$ is amenable;
(ii) $\ell^1(S)$ is biprojective and S is finite;
(iii) $\ell^1(S,\omega)^{**}$ is biprojective.

Proof.$(i) \to (ii)$ Suppose that $\ell^1(S,\omega)^{**}$ is amenable. Hence $\ell^1(S)$ is amenable and S is finite [8, Theorem 3.7]. Then by [7, Thorem 3.7 (i)], S is uniformly locally finite and for each $p \in E(S)$, G_p is an amenable group. Finiteness of S implies that G_p is finite and then [7, Theorem 3.7 (ii)] shows that $\ell^1(S)$ is biprojective.

$(ii) \to (iii)$ Since S is finite, $\ell^1(S) \cong \ell^1(S,\omega)$, and $\ell^1(S)$ is finite-dimensional. Therefore $\ell^1(S)^{**} \cong \ell^1(S)$, and so $\ell^1(S,\omega)^{**}$ is biprojective.

$(iii) \to (i)$ Biprojectivity of $\ell^1(S,\omega)^{**}$ implies its biflatness. Thus, since $\ell^1(S,\omega)^{**}$ has a bounded approximate identity, $\ell^1(S,\omega)^{**}$ is amenable. □

3. Pseudo-amenability of $L^1(G,\omega)$

Throughout G is a locally compact group and ω is a weight on G. The weight ω is *diagonally bounded* if $\sup_{g\in G}\omega(g)\omega(g^{-1}) < \infty$. It seems to be a *right* conjecture that $L^1(G,\omega)$ will fail to be pseudo-amenable whenever ω is not diagonally bounded. Although we are not able to prove (or disprove) the conjecture, we have the following.

Theorem 3.1. *Suppose that there exists an approximate diagonal $(m_i)_i$ for $L^1(G,\omega)$ such that $m_i - \delta_g \,.\, m_i \,.\, \delta_{g^{-1}} \longrightarrow 0$ uniformly on G. Then ω is diagonally bounded.*

Theorem 3.2. *Suppose that $L^1(G,\omega)$ is pseudo-amenable, and that ω is bounded away from 0. Then G is amenable.*

We conclude by the following which is an analogue of [3, Proposition 8.9].

Proposition 3.3. *Let $L^1(G,\omega)$ be pseudo-amenable. Then there is a continuous positive character φ on G such that $\varphi \leq \omega$.*

References

1. M. Essmaili and A. Medghalchi, *Biflatness of certain semigroup algebra*, *Bull. Iran. Math. Soc.*, **39** (2013), 959-969.
2. M. Essmaili and M. Rostami and A. R. Medghalchi, *Pseudo-contractibility and Pseudo-amenability of semigroup algebras*, *Arch. Math*, **97** (2011), 167-177.
3. F. Ghahramani, R. J. Loy and Y. Zhang, *Generalized notions of amenability II*, *J. Funct. Anal.*, **254** (2008), 1776-1810.
4. F. Ghahramani and Y. Zhang, *Pseudo-amenable and Pseudo-contractible Banach algebras*, *Math. Proc. Cambridge Phil. Soc.*, **142** (2007), 111-123.
5. M. Ghandahari, H. Hatami and N. Spronk, *Amenability constant for semilattice algebras*, *Semigroup Forum* **79**, no. 2 (2009), 279-297.
6. B. E. Johnson, *Cohomology in Banach algebras*, *Mem. Amer. Math. Soc.* **127** (1972).
7. P. Ramsden, *Biflatness of semigroup algebras*, *Semigroup Forum* **79**, no. 3 (2009), 515-530.
8. M. Soroshmehr, *Weighted Ress matrix semigroups and their applications*, *Arch. Math.*, **100**, no. 2 (2013), 139-147.

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The notion of quasi-multipliers on topological algebraic structure

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Abstract. We introduce and study the notion of quasi-multipliers on a semi-topological semigroup S. The set of all quasi-multipliers on S is denoted by $\mathfrak{QM}(S)$. First, we study the problem of extension of quasi-multipliers on topological semigroups to its Stone-Čech compactification. Indeed, we prove if S is a topological semigroup such that $S \times S$ is pseudocompact, then $\mathfrak{QM}(S)$ can be regarded as a subset of $\mathfrak{QM}(\beta S)$. Moreover, with an extra condition we describe $\mathfrak{QM}(S)$ as a quotient subsemigroup of βS.

Keywords: Quasi-multiplier, topological semigroup, Stone-Čech compactification.
AMS Mathematical Subject Classification [2010]: 22A15, 22A20, 54C20.

1. Introduction

The main theme of this paper is motivated by the problem of whether the Stone-Čech compactification βS of a topological semigroup S is a compact right (left) topological semigroup. Precisely, to find conditions on a topological semigroup S guaranteeing that the semigroup operation of S extends to a continuous semigroup operation on the Stone-Čech compactification βS of S. In [1], Baker and Butcher showed that for a certain class of locally compact semi-topological semigroup S, a necessary and sufficient condition to the above problem is discreteness or countably compactness of S. Moreover, Filali and Vedenjuoksu showed in [6, Proposition 3.4] that for a countably compact semi-topological semigroup S, the Stone-Čech compactification βS is a semi-topological semigroup. Also, Reznichenko in [7] proved that the semigroup operation on a pseudocompact topological semigroup S can be extended to a separately continuous semigroup operation on βS. Furthermore, Banakh and Dimitrova in [4] obtained some conditions on topological semigroup S for which the semigroup operation of S extends to a jointly continuous semigroup operation on its Stone-Čech compactification.

Bhatt et al. in [2] introduced and studied the notion of multipliers of weighted discrete abelian semigroups. For an discrete abelian semigroup S, the multiplier semigroup $M(S)$ consists of all $T\colon S \longrightarrow S$ such that

$$T(st) = sT(t) = T(s)t \quad (s, t \in S).$$

Also for a weighted discrete abelian semigroup (S, ω), they defined the set $M_w(S)$ consists of all $T \in M(S)$ such that there exists a constant $K > 0$ with the following property:

$$\omega(T(s)) \leq K\omega(s) \quad (s \in S).$$

Recently in [3], the authors also considered multipliers on the weighted abelian semigroup (S, ω) and its relation with the multiplier algebra of the associated Beurling algebra $\ell^1(S, \omega)$. Multipliers on a weighted discrete abelian semigroup have been studied by many authors, for example see [2], [3].

*speaker

The above discussion motivates us to generalize the notion of multipliers on semigroups to what we shall call quasi-multipliers. As the name suggests, the motivation for this grouping comes from the quasi-multipliers on Banach algebras. Actually, we can regard a semigroup operation as a quasi-multiplier on topological semigroups. In the theory of semigroups, the Stone-Čech compactification of S plays the same role of second dual of Banach algebra in some aspects such as Arens products and topological centers. Precisely, βS can be regarded as a subsemigroup of $M(\beta S) = \ell^1(S)''$ equipped with the first Arens product. By considering these facts, it would be interesting to study quasi-multiplier maps on Stone-Čech compactification and to discuss the relationship between multipliers and quasi-multipliers on semigroups and their Stone-Čech compactification. This paper is organized as follows:

We concentrate on finding conditions on a topological semigroup S such that a quasi-multiplier on S extends to the Stone-Čech compactification βS as a quasi-multiplier. Indeed, we show that if S is a topological semigroup such that $S \times S$ is pseudocompact, then any element $\mathfrak{m} \in \mathfrak{QM}(S)$ can be extended to an element in $\mathfrak{QM}(\beta S)$. Moreover, we describe a characterization of $\mathfrak{QM}(S)$ in the sense of quotient semigroups of βS.

2. Main results

We start the section by defining the new notion of quasi-multipliers on semi-topological semigroups which can be regarded as a generalization of multipliers on semigroups. The standard references on topological semigroups and their Stone-Čech compactification are [5].

Definition 2.1. Let S be a semi-topological semigroup. A jointly continuous map $\mathfrak{m} : S \times S \longrightarrow S$ is called a *quasi-multiplier* if for all $s, t, u, v \in S$, we have

$$\mathfrak{m}(st, uv) = s\mathfrak{m}(t, u)v. \tag{1}$$

The set of all such mappings is denoted by $\mathfrak{QM}(S)$. Moreover, the set of all separately continuous maps $\mathfrak{m} : S \times S \longrightarrow S$ satisfying the equation (1) is denoted by $\mathfrak{QM}_w(S)$. It is clear that $\mathfrak{QM}(S) \subseteq \mathfrak{QM}_w(S)$.

Example 2.2. (i) Suppose that S is a topological semigroup and consider the semigroup multiplication map $\mathfrak{m} : S \times S \longrightarrow S$ which is defined by

$$\mathfrak{m}(s, t) = st \quad (s, t \in S).$$

It is easy to see that $\mathfrak{m}$ is a quasi-multiplier and in this case $\mathfrak{QM}(S)$ is always non-empty.

(ii) Suppose that S is a left (right) zero semigroup, that is for each $s, t \in S$ we have $st = s (st = t)$. Then it is straightforward to check that $\mathfrak{QM}(S) = \mathfrak{QM}_w(S)$ is a singleton.

(iii) Consider a semi-topological semigroup S which is not a topological semigroup, see [5, Example 3.3(h)]. Thus the semigroup multiplication map $\mathfrak{m} : S \times S \longrightarrow S$ is not a quasi-multiplier but $\mathfrak{m} \in \mathfrak{QM}_w(S)$.

Recall that a topological space X is *pseudocompact* if every real-valued continuous function on X is bounded. In [7], E. Reznichenko has shown that the semigroup operation on a pseudocompact topological semigroup S can be extended to a separately continuous semigroup operation on βS. Moreover, T. Banakh and S. Dimitrova in [4] obtained some conditions on topological semigroup S for which the semigroup operation of S extends to a jointly continuous semigroup operation on its Stone-Čech compactification.

In the sequel, we consider this view of fact for extension of quasi-multipliers on a topological semigroup to its Stone-Čech compactification. Indeed, the following theorem can be regarded as a generalization of [4, Theorem 1.3].

Theorem 2.3. *Let S be a topological semigroup such that $S \times S$ is pseudocompact. Then*

(i) *$(\beta S, \Box)$ is a topological semigroup, where $\Box : \beta S \times \beta S \longrightarrow \beta S$ is the extension of the semigroup operation of S.*
(ii) *any element $\mathfrak{m} \in \mathfrak{QM}(S)$ can be extended to an element in $\mathfrak{QM}(\beta S)$.*

As a consequence, we give the following prompt result. Recall that the inverse semigroup S is *primitive* if all non-zero idempotents of S is primitive.

REMARK 2.4. For a topological group G, it is well-known that pseudocompactness of G implies that $G \times G$ is pseudocompact. We note that this is not true when S is a topological semigroup. For example, we can consider a pseudocompact topological space X such that $X \times X$ is not pseudocompact. Now, we can regard X as a topological semigroup with the semigroup operation defined by

$$x * y = x \qquad (x, y \in X).$$

COROLLARY 2.5. *If S is a primitive Hausdorff pseudocompact topological inverse semigroup, then any element $\mathfrak{m} \in \mathfrak{QM}(S)$ can be extended to an element $\mathfrak{QM}(\beta S)$.*

Now, our aim is to obtain conditions that give a similar version of Theorem 2.3 for separately continuous quasi-multipliers on topological semigroups. Recall that a topological space X is *countably compact* if every countable open cover has a finite subcover. It is worthwhile to mention that Filali and Vedenjuoksu in [6] studied this problem for an extension of the semigroup operation of S to βS, where S is a countably compact semi-topological semigroup. The following theorem is an improvement of [6, Proposition 3.4] to separately continuous quasi-multipliers.

THEOREM 2.6. *Let S be a countably compact topological semigroup. Then*

(i) *$(\beta S, \Box)$ is a semi-topological semigroup.*
(ii) *for each $y \in S$, the map $\Psi : \beta S \longrightarrow \beta S$ defined by $\boldsymbol{x} \mapsto y \Box \boldsymbol{x}$ is continuous.*
(iii) *for every element $\mathfrak{m} \in \mathfrak{QM}_w(S)$, we have an extension $\widetilde{\mathfrak{m}} \in \mathfrak{QM}_w(\beta S)$.*

Now, for a certain class of topological semigroup S, we describe $\mathfrak{QM}(S)$ as a quotient of a subsemigroup of its Stone-Čech compactification. First, we need some definitions.

DEFINITION 2.7. A topological semigroup S is called *faithful* if the map $\Phi : S \longrightarrow \mathfrak{QM}(S)$ defined by

$$\Phi(s)(x, y) = xsy \qquad (s, x, y \in S),$$

is one-to-one.

DEFINITION 2.8. Suppose that S is a topological semigroup. We say a net $\{\mathfrak{m}_\beta\} \subseteq \mathfrak{QM}(S)$ converges to some $\mathfrak{m} \in \mathfrak{QM}(S)$ in the $q(S)$-topology and denote by $\mathfrak{m}_\beta \xrightarrow{q(S)} \mathfrak{m}$ if $\mathfrak{m}_\beta(s, t) \longrightarrow \mathfrak{m}(s, t)$, for all $s, t \in S$.

Let S be a topological semigroup with an approximate identity, that is there is a net (e_α) in S such that for each $s \in S$, $e_\alpha s \longrightarrow s$ and $s e_\alpha \longrightarrow s$. Therefore, S is faithful and for each $\mathfrak{m} \in \mathfrak{QM}(S)$ we have,

$$\begin{aligned}
\lim_\alpha \left[\Phi\left(\mathfrak{m}(e_\alpha, e_\alpha)\right)\right](s, t) &= \lim_\alpha s\mathfrak{m}(e_\alpha, e_\alpha)t \\
&= \lim_\alpha \mathfrak{m}(s e_\alpha, e_\alpha t) \\
&= \mathfrak{m}(s, t).
\end{aligned}$$

This shows that $\Phi\left(\mathfrak{m}(e_\alpha, e_\alpha)\right) \xrightarrow{q(S)} \mathfrak{m}$ and so S can be considered as a dense subset of $\mathfrak{QM}(S)$ equipped with the $q(S)$-topology.

DEFINITION 2.9. An approximate identity (e_α) in a topological semigroup S is called an *ultra-approximate identity* if for each $s \in S$ and $\mathfrak{m} \in \mathfrak{QM}(S)$ the nets $\{\mathfrak{m}(s, e_\alpha)\}$ and $\{\mathfrak{m}(e_\alpha, s)\}$ are convergent.

In the following, we will be able to consider $\mathfrak{QM}(S)$ as a right topological semigroup when S admits an ultra-approximate identity.

Lemma 2.10. *Let S be a topological semigroup and (e_α) is an ultra-approximate identity. For each $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathfrak{QM}(S)$ define*

$$(\mathfrak{m}_1 \odot \mathfrak{m}_2)(s,t) = \mathfrak{m}_1\left(s, \lim_\alpha \mathfrak{m}_2(e_\alpha, t)\right) \quad (s,t \in S).$$

Then the following statements hold.

(i) *$(\mathfrak{QM}(S), \odot)$ is a right topological semigroup with the $q(S)$-topology.*
(ii) *S is an ideal in $(\mathfrak{QM}(S), \odot)$.*
(iii) *$\mathcal{Z}_q(S) = \{\mathfrak{m} \in \mathfrak{QM}(S) : L_\mathfrak{m}$ is continuous$\}$ is a semi-topological subsemigroup of $(\mathfrak{QM}(S), \odot)$ such that contains S, where $L_\mathfrak{m} : \mathfrak{QM}(S) \longrightarrow \mathfrak{QM}(S)$ is the left translation map.*

Suppose that $(S, \cdot)$ is a topological semigroup such that $(\beta S, \bullet)$ is a topological semigroup, where $\bullet$ is an extension of the semigroup operation of S. Define

$$\beta S_\sigma = \{F \in \beta S : \ x \bullet F \bullet y \in S \text{ for all } x, y \in S\}.$$

It is easy to see that βS_σ is a subsemigroup of βS which contains S. Also, the relation

$$\sigma(F)(x,y) = x \bullet F \bullet y \qquad (x, y \in S, \ F \in \beta S_\sigma),$$

defines a map σ from βS_σ into $\mathfrak{QM}(S)$.

As a main result, we now are able to present a characterization of $\mathfrak{QM}(S)$ in the sense of quotient semigroups as follows.

Theorem 2.11. *Let S be a topological semigroup with an ultra-approximate identity. If $S \times S$ is pseudocompact, then*

(i) *the map $\sigma : (\beta S_\sigma, \Box) \longrightarrow (\mathfrak{QM}(S), \odot)$ is a continuous epimorphism.*
(ii) *$\mathfrak{QM}(S) \cong \frac{\beta S_\sigma}{J}$, where $J = \{(F,G) \in \beta S_\sigma \times \beta S_\sigma : \ \sigma(F) = \sigma(G)\}$.*

References

1. J. W. Baker, R. J. Butcher, *The Stone-Čech compactification of a topological semigroup*, Math. Proc. Camb. Philos. Soc. **80** (1976), 103–107.
2. S. J. Bhatt, P .A. Dabhi, and H. V. Dedania, *Multipliers of weighted semigroups and associated Beurling Banach algebras*, Proc. Indian Acad. Sci. (Math. Sci.), **121** (4) (2011), 417–433.
3. S. J. Bhatt, P. A. Dabhi and H. V. Dedania, *The multiplier algebra of a Beurling algebra*, Bull. Aust. Math. Soc. **90** (2014), 113–120.
4. T. Banakh, S. Dimitrova, *Openly factorizable spaces and compact extensions of topological semigroups*, Comment. Math. Univ. Carolin. **51**(1) (2010), 113–131.
5. J. F. Berglund, H. D. Junghenn and P. Milnes, Analysis on Semigroups: Function Spaces, Compactifications, Representations, Wiley, New York (1989).
6. M. Filali and T. Vedenjuoksu, *The extension of the operation in the Stone-Čech compactification of semigroups*, Semigroup Forum, **79** (2009), 400–412.
7. E. A. Reznichenko, *Extensions of functions defined on products of pseudocompact spaces and continuity of the inverse in pseudocompact groups*, Topology Appl. **59** (1994), 233–244.

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The irreducible representations on generalized Weyl- Heisenberg groups

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Abstract. For a locally compact group H and a locally compact abelian group K, consider the semi-direct product H and K with respect to the continuous homomorphism $\theta : H \to Aut(K)$. We introduce a representation of generalized Weyl- Heisenberg group associate with the semi-direct product group G_θ. We show that this representation is irreducible if H is compact.

Keywords: Generalized Weyl-Heisenberg group, irreducible representation, semi-direct product group.
AMS Mathematical Subject Classification [2010]: (Primary 43A65).

1. Introduction

Let H and K be two locally compact groups with the identity elements e_H and e_K, respectively and let $\tau : H \to Aut(K)$ be a homomorphism such that the map $(h, k) \mapsto \tau_h(k)$ is continuous from $H\times K$ onto K, where $H\times K$ equips with the product topology. The semi- direct product topological group $G_\tau = H \times_\tau K$ is the locally compact topological space $H \times K$ under the product topology, with the group operations:

$$(h_1, k_1) \times_\tau (h_2, k_2) = (h_1h_2, k_1\tau_{h_1}(k_2)),$$
$$(h, k)^{-1} = (h^{-1}, \tau_{h^{-1}}(k^{-1})).$$

It is worth to note that $K_1 = \{(e_H, k); k \in K\}$ is a closed normal subgroup and $H_1 = \{(h, e_K); h \in H\}$ is a closed subgroup of G_τ such that $G_\tau = HK$. Moreover, the left Haar measure of the locally compact group G_τ is

$$d\mu_{G_\tau}(h, k) = \delta_H(h)d\mu_H(h)d\mu_K(k),$$

in which $d\mu_H, d\mu_K$ are the left Haar measures on H and K, respectively and $\delta_H : H \to (0,\infty)$ is a positive continuous homomorphism that satisfies

$$d\mu_K(k) = \delta_H(h)d\mu(\tau_h(k)),$$

for $h \in H, k \in K$. Moreover, the modular function Δ_{G_τ} is

$$\Delta_{G_\tau} = \delta_H(h)\Delta_H(h)\Delta_K(k),$$

where Δ_H, Δ_K are the modular functions of H, K, respectively.
When K is also abelian, one can define $\hat{\tau} : H \to Aut(\hat{K})$ via $h \mapsto \hat{\tau_h}$ where

$$\hat{\tau_h}(\omega) = \omega \circ \tau_{h^{-1}},$$

for all $\omega \in \hat{K}$. We usually denote $\omega \circ \tau_{h^{-1}}$ by ω_h. With this notation, it is easy to see

$$\omega_{h_1h_2} = (\omega_{h_2})_{h_1},$$

where $h_1, h_2 \in H$ and $\omega \in \hat{K}$. The semi-direct product $G_{\hat{\tau}} = H \times_{\hat{\tau}} \hat{K}$ is a locally compact group with the left Haar measure

$$d\mu_{\hat{G}}(h, \omega) = \delta_H(h)^{-1}d\mu_H(h)d\mu_{\hat{K}}(\omega),$$

where $d\mu_{\hat{K}}$ is the Haar measure on $\hat{K}$. Also, for all $h \in H$,

$$d\mu_{\hat{K}}(\omega_h) = \delta_H(h)d\mu_{\hat{K}}(\omega),$$

*speaker

for $\omega \in \hat{K}$, (see more details in [4, 1, 3].)
Let $G_\tau = H \times_\tau K$, and define $\theta : G_\tau \to Aut(\hat{K} \times \mathbb{T})$ via

$$(h,k) \mapsto \theta_{(h,k)}(\omega, z) = (\hat{\tau_h}(\omega), \hat{\tau_h}(\omega)(k)z) = (\omega_h, \omega_h(k)z),$$

for all $(h,k) \in H \times_\tau K$ and $(\omega, z) \in \hat{K} \times \mathbb{T}$. The mapping θ is a continuous homomorphism. Thus the semi-direct prodoct

$$G_\tau \times_\theta (\hat{K} \times \mathbb{T}) = (H \times_\tau K) \times_\theta (\hat{K} \times \mathbb{T}),$$

is a locally compact group and it is called the generalized Weyl Heisenberg group associated with the semi direct product group $G_\tau = H \times_\tau K$, and denoted by $\mathbb{H}(G_\tau)$. It is easy to see that the group operations of $\mathbb{H}(G_\tau)$ are

$$(h_1, k_1, \omega_1, z_1).(h_2, k_2, \omega_2, z_2) = (h_1 h_2, k_1 \tau_{h_1}(k_2), \omega_1 \omega_{2h_1}, \omega_{2h_1}(k) z_1 z_2),$$
$$(h_1, k_1, \omega_1, z_1)^{-1} = (h_1^{-1}, \tau_{h_1}^{-1}(k^{-1}), \bar{\omega}_{h_1^{-1}}, \bar{\omega}_{h_1^{-1}}(\tau_{h_1}^{-1}(k^{-1}))z^{-1}),$$

for $(h_1, k_1, \omega_1, z_1), (h_2, k_2, \omega_2, z_2) \in \mathbb{H}(G_\tau)$ (see [4]) and the left Haar measure of $\mathbb{H}(G_\tau)$ is:

$$d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z) = d\mu_H(h) d\mu_K(k) d\mu_{\hat{K}}(\omega) d\mu_{\mathbb{T}}(z).$$

2. Main results

Throughout this section, we assume that H and K are locally compact topological groups and that K is abelian, too. We denote the left Haar measures of H and K by $d\mu_H, d\mu_K$, respectively. Suppose that $h \mapsto \tau_h$ from H to $Aut(K)$ is a homomorphism such that $(h,k) \mapsto \tau_h(k)$ from $H \times K$ into K is continuous. $G_\tau = H \times_\tau K$ is the semi-direct product of H and K that is a locally compact topology group with the left Haar measure $d\mu_{G_\tau}(h,k) = \delta_H(h) d\mu_H((h) d\mu_K(k)$, where $\delta_H : H \mapsto (0, \infty)$ is a continuous homomorphism. Consider the homomorphism $\theta : G_\tau \to Aut(\hat{K} \times \mathbb{T})$ is defined by

$$((h,k),(\omega,z)) \mapsto \theta_{(h,k)}(\omega, z),$$

where $\theta_{(h,k)}(\omega, z) = (\omega \circ \tau_{h^{-1}}, \omega \circ \tau_{h^{-1}}(k).z)$. This makes $\mathbb{H}(G_\tau) = G_\tau \times_\theta (\hat{K} \times \mathbb{T})$ a locally compact topological group where $\mathbb{H}(G_\tau)$ is equipped with the product topology and the group operations as

$$(h_1, k_1, \omega_1, z_1).(h_2, k_2, \omega_2, z_2) = (h_1 h_2, k_1 \tau_{h_1}(k_2), \omega_1 \omega_{2h_1}, \omega_{2h_1}(k) z_1 z_2),$$
$$(h_1, k_1, \omega_1, z_1)^{-1} = (h_1^{-1}, \tau_{h_1}^{-1}(k^{-1}), \bar{\omega}_{h_1^{-1}}, \bar{\omega}_{h_1^{-1}}(\tau_{h_1}^{-1}(k^{-1}))z^{-1}),$$

for $(h_1, k_1, \omega_1, z_1), (h_2, k_2, \omega_2, z_2) \in \mathbb{H}(G_\tau)$. The left Haar measure of $\mathbb{H}(G_\tau)$ is

$$d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z) = d\mu_H(h) d\mu_K(k) d\mu_{\hat{K}}(\omega) d\mu_{\mathbb{T}}(z).$$

Now, we are going to define a irreducible representation on $\mathbb{H}(G_\tau)$. With the above notations define $\pi : \mathbb{H}(G_\tau) \to U(L^2(\hat{K}))$ by

$$\pi(h,k,\omega,z) f(\xi) = \delta_H^{-1/2}(h) z \xi(k) \overline{\omega(k)} f((\xi \overline{\omega})_{h^{-1}}), \tag{1}$$

then π is a homomorphism. Indeed,

$$\pi\left((h_1, k_1, \omega_1, z_1)(h_2, k_2, \omega_2, z_2)\right) f(\xi) = \pi(h_1 h_2, k_1 \tau_{h_1}(k_2), \omega_1 (\omega_2)_{h_1}, (\omega_2)_{h_1}(k_1) z_1 z_2) f(\xi)$$
$$= \delta_H^{-1/2}(h_1 h_2)(\omega_2)_{h_1}(k_1) z_1 z_2 \xi(k_1 \tau_{h_1}(k_2)) \overline{\omega_1 (\omega_2)_{h_1}}(k_1 \tau_{h_1}(k_2)) f((\xi \overline{\omega_1 (\omega_2)_{h_1}})_{(h_1 h_2)^{-1}} \quad .$$
$$= \delta_H^{-1/2}(h_1 h_2)(\omega_2)_{h_1}(k_1) z_1 z_2 \xi(k_1) \xi_{h_1^{-1}}(k_2) \overline{\omega_1(k_1)(\omega_1)_{h_1^{-1}}(k_2)\omega_2(k_2)} f(\xi_{h_2^{-1} h_1^{-1}} \overline{(\omega_1)}_{h_2^{-1} h_1^{-1}} \overline{(\omega_2)}_{h_2^{-1}})$$

Also,

$$\pi(h_1, k_1, \omega_1, z_1) \pi(h_2, k_2, \omega_2, z_2) f(\xi)$$
$$= \delta_H^{-1/2}(h_1) z_1 \xi(k_1) \overline{\omega_1}(k_1) \pi(h_2, k_2, \omega_2, z_2) f((\xi \overline{\omega_1})_{h_1^{-1}}$$
$$= \delta_H^{-1/2}(h_1) \delta_H^{-1/2}(h_2) z_1 z_2 \xi(k_1) \overline{\omega_1}(k_1) \overline{\omega_2}(k_2) (\xi \overline{\omega_1})_{h_1^{-1}}(k_2) f((\xi \overline{\omega_1})_{h_1^{-1}} (\overline{\omega_2})_{h_2^{-1}}) \quad .$$
$$= \delta_H^{-1/2}(h_1 h_2) z_1 z_2 \xi(k_1) \xi_{h_1^{-1}}(k_2) \overline{\omega_1(k_1)(\omega_1)_{h_1^{-1}}(k_2)\omega_2(k_2)} f(\xi_{h_2^{-1} h_1^{-1}} \overline{(\omega_1)}_{h_2^{-1} h_1^{-1}} \overline{(\omega_2)}_{h_2^{-1}}).$$

Moreover, π is unitary. In fact we have,

$$\begin{array}{lll} \|\pi(h,k,\omega,z)f\|_2^2 & = & \int_{\hat{K}} |\pi(h,k,\omega,z)f(\xi)|^2 d\mu_{\hat{K}}(\xi) \\ & = & \int_{\hat{K}} \delta_H^{-1}(h)|f((\xi\overline{\omega})_{h^{-1}}|^2 d\mu_{\hat{K}}(\xi) \\ & = & \int_{\hat{K}} \delta_H^{-1}(h)|f((\xi)_{h^{-1}}|^2 d\mu_{\hat{K}}(\xi) \\ & = & \int_{\hat{K}} f((\xi)|^2 d\mu_{\hat{K}}(\xi) \\ & = & \|f\|_2^2. \end{array}$$

And it is easy to check that π is continuous and onto. So, π is a continuous unitary representation of group $\mathbb{H}(G_\tau)$ to the Hilbert space $L^2(\hat{K})$. In the sequel, we show that π is irreducible when H is compact. Note that when H is a compact group, we normalize the Haar measure μ_H such that $\mu_H(H)=1$.

Theorem 2.1. *Let $\mathbb{H}(G_\tau)=(H\times_\tau K)\times_\theta(\hat{K}\times\mathbb{T})$ where H is a locally compact group and K is a locally compact abelian group. Then for φ,ψ in $L^2(\hat{K})$,*

$$\int_{\mathbb{H}(G_\tau)} |\prec\varphi,\pi(h,k,\omega,z)\psi\succ|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)=\|\varphi\|_2^2\|\psi\|_2^2. \tag{2}$$

if and only if H is compact.

Proof. For φ,ψ in $L^2(\hat{K})$ we first consider the following observations:

$\int_{\mathbb{H}(G_\tau)} |\prec\varphi,\pi(h,k,\omega,z)\psi\succ|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)$
$= \int_{\mathbb{H}(G_\tau)} |\int_{\hat{K}} \varphi(\xi)\overline{\pi(h,k,\omega,z)\psi(\xi)} d\mu_{\hat{K}}(\xi)|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)$
$= \int_{\mathbb{H}(G_\tau)} |\int_{\hat{K}} \varphi(\xi)\delta_H^{-1/2}(h)\overline{z}\overline{\xi(k)}\omega(k)\overline{\psi(\xi\overline{\omega})_{h^{-1}}} d\mu_{\hat{K}}(\xi)|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)$
$= \int_{\mathbb{H}(G_\tau)} |\int_{\hat{K}} \varphi(\xi\omega)\delta_H^{-1/2}(h)\overline{z}\overline{\xi(k)\psi(\xi)_{h^{-1}}} d\mu_{\hat{K}}(\xi)|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)$
$= \int_{\mathbb{H}(G_\tau)} |\int_{\hat{K}} R_\omega\varphi(\xi)\delta_H^{-1/2}(h)\overline{z}\overline{\xi(k)\psi(\xi\circ\tau_h)} d\mu_{\hat{K}}(\xi)|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)$
$= \int_{\mathbb{H}(G_\tau)} |\int_{\hat{K}} R_\omega\varphi(\xi\circ\tau_{h^{-1}})\delta_H^{-1/2}(h)\overline{z}\overline{\xi\circ\tau_{h^{-1}}(k)\psi(\xi)} d\mu_{\hat{K}}(\xi_h)|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)$
$= \int_{\mathbb{H}(G_\tau)} |\int_{\hat{K}} R_\omega\varphi(\xi\circ\tau_{h^{-1}})\delta_H^{1/2}(h)\overline{z}\overline{\xi(\tau_{h^{-1}}(k))\psi(\xi)} d\mu_{\hat{K}}(\xi)|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)$
$= \int_{\mathbb{H}(G_\tau)} \delta_H(h)| \int_{\hat{K}}(R_\omega\varphi(.\circ\tau_{h^{-1}}).\overline{\psi})(\xi)\overline{\xi(\tau_{h^{-1}}(k))} d\mu_{\hat{K}}(\xi)|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)$.
$= \int_{\mathbb{H}(G_\tau)} \delta_H(h)|(R_\omega\widehat{\varphi(.\circ\tau_{h^{-1}}).\overline{\psi}})(\tau_{h^{-1}}(k))|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)$
$= \int_H \delta_H(h)\int_{\hat{K}}\int_K |(R_\omega\widehat{\varphi(.\circ\tau_{h^{-1}}).\overline{\psi}})(\tau_{h^{-1}}(k))|^2 d\mu_K(k)d\mu_{\hat{K}}(\omega)d\mu_H(h)$
$= \int_H\int_{\hat{K}}\int_K |(R_\omega\widehat{\varphi(.\circ\tau_{h^{-1}}).\overline{\psi}})(k)|^2 d\mu_K(k)d\mu_{\hat{K}}(\omega)d\mu_H(h)$
$= \int_H\int_{\hat{K}}\int_{\hat{K}} |(R_\omega\varphi(.\circ\tau_{h^{-1}}).\overline{\psi})(\xi)|^2 d\mu_{\hat{K}}(\xi)d\mu_{\hat{K}}(\omega)d\mu_H(h)$
$= \int_H\int_{\hat{K}}\int_{\hat{K}} |R_\omega\varphi(\xi\circ\tau_{h^{-1}}).\overline{\psi}(\xi)|^2 d\mu_{\hat{K}}(\xi)d\mu_{\hat{K}}(\omega)d\mu_H(h)$
$= \int_H\int_{\hat{K}}\int_{\hat{K}} \delta_H(h)|R_\omega\varphi(\xi).\overline{\psi}(\xi\circ\tau_h)|^2 d\mu_{\hat{K}}(\xi)d\mu_{\hat{K}}(\omega)d\mu_H(h)$
$= \int_H\int_{\hat{K}} \|\varphi\|_2^2\delta_H(h)|\overline{\psi}(\xi\circ\tau_h)|^2 d\mu_{\hat{K}}(\xi)d\mu_H(h)$
$= \|\varphi\|_2^2\|\psi\|_2^2\mu_H(H)$

Now, if H is compact, then $\mu_H(H)=1$. So, (2) holds. Conversely, if (2) holds,the above observation implies that $\mu_H(H)=1$. So, we can conclude that H is compact. □

Corollary 2.2. *With notation as above, the representation π of $\mathbb{H}(G_\tau)$ on $L^2(\hat{K})$ is irreducible if H is compact.*

Proof. If H is compact, then (2) in Theorem 3.1 holds. Now, suppose that M is a closed subspace of the Hilbert space $L^2(\hat{K})$ that is invariant under π. Then for any $\varphi\in M$ we have,

$$\{\pi(h,k,\omega,z)\varphi;\quad (h,k,\omega,z)\in\mathbb{H}(G_\tau)\}\subseteq M.$$

Let $\psi \in L^2(\hat{K})$ be orthogonal to M, that is $\prec \psi, \pi(h,k,\omega,z)\varphi \succ = 0$, for all $(h,k,\omega,z) \in \mathbb{H}(G_\tau)$. Thus by (2), $\|\varphi\|_2\|\psi\|_2 = 0$, and hence $\psi = 0$. So, $M^\perp = \{0\}$, that is, $M = L^2(\hat{K})$. Namely, π is irreducible. □

Example 2.3. Let K be an abelian locally compact group and $H = \{e\}$ (the trivial group). In this case the generalized weyl Heisenberg group $\mathbb{H}(G_\tau)$ coincides with the standard weyl Heisenberg group $G := K \times_\theta (\hat{K} \times \mathbb{T})$. In this case the irreducible representation of $G = K \times_\theta (\hat{K} \times \mathbb{T})$ on $L^2(\hat{K})$ is as follows:

$$\pi(k,\omega,z)f(\xi) = z\xi(k)\overline{\omega(k)}f(\xi\overline{\omega}). \tag{3}$$

References

1. A.A. Arefijamaal, R.A. Kamyabi Gol, *A Characterization of Square Integrable Representations Associated with CWT*, J. Sci. Islam. Repub. Iran., Vol 18, No. 2, 159-166, 2007.
2. R. Fabec, G. lafsson *The Continuous Wavelet Transform and Symmetric Spaces*, Acta. Appl. Math., Vol. 77, No. 1, 4169, 2003.
3. H. Fuehr, M. Mayer, *Continuous wavelet transforms from semidirect products: Cyclic representations and Plancherel measure*, J. Fourier Anal. Appl., Vol. 8, 375-398, 2002.
4. A. Ghaani Farashahi, *Generalized Weyl-Heisenberg group*, Anal. Math. Phys., Vol.4, 187-197, 2014.

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Diagonal-preserving of C*-algebras

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Abstract. We show that any *-isomorphism $O_X \to O_Y$ which maps $C(X)$ onto $C(Y)$ is in fact diagonal-preserving under mild conditions on X and Y.
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1. Introduction

Let X and Y be one-sided shift spaces. A *-isomorphism $\Psi : O_X \to O_Y$ is *diagonal-preserving* if $\Psi(D_X) = D_Y$. In this paper we prove that a *-isomorphism $\Psi : O_X \to O_Y$ satisfying $\Psi(C(X)) = C(Y)$ is diagonal-preserving. First we need some preliminary results. Everyone can read more in [1, 2].

Lemma 1.1. *Let X be a one-sided shift space. Then*

$$C^*(Iso(\mathcal{G}_X)^\circ) = D'_X \subseteq C(X)'.$$

If X contains a dense set of aperiodic points, then $D'_X = C(X)'$.

Consider the equivalence relation $\sim$ on the space $\tilde{X} \times \mathbb{T}$ given by $(\tilde{x}, \iota) \sim (\tilde{y}, \theta)$ if and only if $\tilde{x} = \tilde{y}$ and $\iota^p = \theta^p$ for all $p \in Stab(\tilde{x})$. Then the quotient $\tilde{X} \times \mathbb{T}/\sim$ is compact and Hausdorff and as we shall see (homeomorphic to) the spectrum of $C^*(Iso(\mathcal{G}_X)^\circ)$. We read more in groupoid [3].

Lemma 1.2. *Let $\sim$ be the equivalence relation on $\tilde{X}\times\mathbb{T}$ defined above. There is a *-isomorphism*

$$\Omega : C^*(Iso(\mathcal{G}_X)^\circ) \to C(\tilde{X} \times \mathbb{T}/\sim),$$

given by

$$\Omega(f)([\tilde{x}, \iota]) = \sum_{p \in Stab(\tilde{x})} f(\tilde{x}, p, \tilde{x})\iota^n \tag{1}$$

for $f \in C_c(Iso(\mathcal{G}_X)^\circ)$ and $[\tilde{x}, \iota] \in \tilde{X} \times \mathbb{T}$.

Proof. The map $\Omega : C^*(Iso(\mathcal{G}_X)^\circ) \to C(\tilde{X}\times\mathbb{T}/\sim)$ given in 1 is well-defined by the definition of $\sim$ and linear. If $f, g \in C_c(Iso(\mathcal{G}_X)^\circ)$ and $[\tilde{x}, z] \in \tilde{X} \times \mathbb{T}$, then

$$\Omega(f)([\tilde{x}, \iota])\Omega(g)([\tilde{x}, \iota]) = \sum_{k,l \in Stab(\tilde{x})} f(\tilde{x}, k, \tilde{x})g(\tilde{x}, l, \tilde{x})\iota^{k+1}$$

$$= \sum_{m,n \in Stab(\tilde{x})} f(\tilde{x}, n-m, \tilde{x})g(\tilde{x}, m, \tilde{x})\iota^n$$

$$= \Omega(f \star g)([\tilde{x}, \iota]),$$

so Ω is multiplicative.
In order to see that Ω is injective, let $f, g \in C_c(Iso(\mathcal{G}_X)^\circ)$ such that both $supp(f)$ and $supp(g)$ are bisections and suppose that $\Omega(f) = \Omega(g)$. Suppose $f(\tilde{x}, p, \tilde{x}) \neq 0$. The $p \in Iso(\tilde{x})$ is necessarily unique because $supp(f)$ is a bisection. The assumption implies the existence of a unique $q \in Iso(\tilde{x})$ such that $g(\tilde{x}, q, \tilde{x})$, and then

$$0 \neq f(\tilde{x}, p, \tilde{x}) = \Omega(f)([\tilde{x}, 1]) = \Omega(g)([\tilde{x}, 1]) = g(\tilde{x}, q, \tilde{x}).$$

Similarly,

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$$0 \neq f(\tilde{x}, p, \tilde{x})\iota^p = \Omega(f)([\tilde{x}, \iota]) = \Omega(g)([\tilde{x}, \iota]) = g(\tilde{x}, q, \tilde{x})\iota^q.$$

for all $1 \neq \iota \in \mathbb{T}$. It follows that $p = q$. Since $C_c(Iso(\mathcal{G}_X)^\circ)$ is spanned by functions whose support is a bisection, we conclude that ρ is injective. We show that Ω separates points. First, if$[\tilde{x}, \iota] \neq [\tilde{x}, \theta]$, then there is $p \in Iso(\tilde{x})$ such that $\iota^p \neq \theta^p$. Choose a compact open bisection $U \subseteq \mathcal{G}_X$ satisfying $U \cap Iso(\tilde{x}) = (\tilde{x}, p, \tilde{x})$ and observe that

$$\Omega(1_U)([\tilde{x}, \iota]) = \iota^p$$

and $\rho(1_U)([\tilde{x}, \theta]) = \theta^p$. Second, if $\tilde{x} \neq \tilde{y}$ in $\tilde{X}$ then we choose a compact open bisection U satisfying $(\tilde{x}, 0, \tilde{x}) \in U$ and $Iso(\tilde{y}) \cap U = \emptyset$. Then $\Omega(1_U)([\tilde{x}, \iota]) = 1$ while $\Omega(1_U)([\tilde{y}, \theta]) = 0$. By the StoneWeierstrass theorem, the image of Ω is dense in $C(\tilde{X} \times \mathbb{T}/\sim)$ and Ω thus extends to a *-isomorphism as wanted. □

2. Main Theorem

THEOREM 2.1. *Let X and Y be one-sided shift spaces with dense sets of aperiodic points and let $\Psi : O_X \to O_Y$ be a *-isomorphism satisfying $\Psi(C(X)) = C(Y)$. Then $\Psi(D_X) = D_Y$.*

PROOF. If $\Psi : O_X \to O_Y$ is a *-isomorphism satisfying $\Psi(C(X)) = C(Y)$, then $\Psi(C(X)') = C(Y)'$. By Lemmas 1.1 and 1.2, there is a homeomorphism

$$h : \tilde{X} \times \mathbb{T}/\sim \to \tilde{Y} \times \mathbb{T}/\sim,$$

such that $\Psi(f) = f \circ h^{-1}$ for $f \in C(\tilde{X} \times \mathbb{T}/\sim)$.
Define the map $q_X : \tilde{X} \times \mathbb{T}/\sim \to \tilde{X}$ by $q_X([\tilde{x}, z]) = \tilde{x}$. This is well-defined, continuous and surjective. Furthermore, q_X induces the inclusion $D_X \subseteq C(X)'$. Let $\tilde{x} \in \tilde{X}$ and put $\tilde{y}_{\tilde{x}} = q_Y(h([\tilde{x}, 1])) \in \tilde{Y}$. The connected component of any $[\tilde{x}, z]$ is the set $[\tilde{x}, w] \mid w \in \mathbb{T}$, so since any homeomorphism will preserve connected components, we have

$$h(q_X^{-1}(\tilde{x})) = q_Y^{-1}(h([\tilde{x}, 1])).$$

We may now define a map $\tilde{h} : \tilde{X} \to Y$ by

$$\tilde{h}(\tilde{x}) = \tilde{y}_{\tilde{x}} = q_Y(h([\tilde{x}, 1]))$$

for $\tilde{x} \in \tilde{X}$, which is well-defined, continuous and surjective. The above considerations show that h is also injective. As both $\tilde{X}$ and $\tilde{Y}$ are compact and Hausdorff, $\tilde{h}$ is a homeomorphism. The relation $\tilde{h} \circ q_X = q_Y \circ h$ ensures that that $\Psi(D_X) = D_Y$ as wanted. □

References

1. T.M. Carlsen, Operator algebraic applications in symbolic dynamics, PhD-thesis (2004) University of Copenhagen.
2. V. Deaconu, Groupoids associated with endomorphisms, Trans. Amer. Math. Soc. 347 (1995), 17791786.
3. J. Renault, Cartan subalgebras in C*-algebras, Irish Math. Soc. Bull. 61 (2008), 2963.

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Orthogonally additive homogeneous polynomials on the second dual of a Banach algebra

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Abstract. In this talk, we present some conditions under which orthogonally additive homogeneous polynomials on the second dual of a Banach algebra can be presented in its standard form.
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1. Introduction

For Banach space $\mathcal{A}$, the space of all bounded k-linear forms on $\mathcal{A}$, denoted by $L(\mathcal{A}^{(k)}, \mathbb{C})$, is a normed space under the norm

$$\|\Phi\| = \sup\{|\Phi(x_1, \cdots, x_k)| : \|x_i\| \le 1, i = 1, \cdots, k\}.$$

A form Φ is called symmetric if $\Phi(x_1, \cdots, x_k) = \Phi(x_{\sigma(1)}, \cdots, x_{\sigma(k)})$ for each $\sigma \in S_k$ (=the symmetric group on k letters). Every k-linear form corresponds to a symmetric k-linear form Φ_s defined by

$$\Phi_s(x_1, \cdots, x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \Phi(x_{\sigma(1)}, \cdots, x_{\sigma(k)}).$$

A map $P : \mathcal{A} \to \mathbb{C}$ is called a k-homogeneous polynomial if there exists a symmetric k-linear form $\mathcal{A} \in L(\mathcal{A}^{(k)}, \mathbb{C})$ such that $P(x) = \Phi(x, \cdots, x)$, for every $x \in \mathcal{A}$.

Two elements x, y in a Banach algebra $\mathcal{A}$ are called orthogonal if $xy = yx = 0$. A k-homogeneous polynomial P is called orthogonally additive if $P(x + y) = P(x) + P(y)$ whenever x and y are orthogonal.

An example of a k-homogeneous orthogonally additive polynomial is the polynomial P defined by $P(x) = \phi(x^k)$ for some $\phi \in X^*$. This is called an standard k-homogeneous orthogonally additive polynomial. In this case the symmetric k-linear form associated with P is:

$$\Phi(x_1, \cdots, x_k) = \langle \phi, \frac{1}{k!} \sum_{\sigma \in S_k} x_{\sigma(1)} \cdots x_{\sigma(k)} \rangle,$$

where we write $\langle \cdot, \cdot \rangle$ for the dual pairing of $\mathcal{A}$ and $\mathcal{A}^*$. In this setting, we say that a Banach algebra $\mathcal{A}$ is standard if, for each $k \in \mathbb{N}$, all k-homogeneous orthogonally additive polynomials on $\mathcal{A}$ are standard. A well-known example of a standard Banach algebra is $C(\Omega)$, for some compact Hausdorff space Ω. Indeed, Pérez-García and Villanueva [8] proved that for each orthogonally additive k-homogeneous polynomial $P : C(\Omega) \to \mathbb{C}$, there exists a regular Borel measure μ on Ω such that $P(f) = \int_\Omega f^k d\mu$, for each $f \in C(\Omega)$. Therefore all commutative unital C*-algebras are standard. Later, Palazuelos, Peralta and Villanueva [7] showed that this result holds for all C*-algebras. Besides C*- algebras, the Fourier algebra $A(\mathbb{T})$ of the unit circle in the complex plan is also standard. Indeed, in [3], it is shown that every completely bounded and orthogonally additive k-homogeneous polynomial on the Fourier algebra $A(G)$ of a locally compact group G is standard. There are also non-standard Banach algebras. In [2], beside proving that the convolution algebra

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$L^1(G)$ is standard for a compact group G, it is also shown that the convolution algebras $C(\mathbb{T})$ and $L^p(\mathbb{T})$ for $1 < p \leq \infty$ are not standard.

It seems natural to ask whether the orthogonally additive property of a polynomial is preserved by extending a polynomial to the second dual; and that, under what conditions being standard of a Banach algebra implies being standard for the second dual and vice versa. Before any thing we need to clarify what we mean by extending a polynomial to the second dual.

Definition 1.1 (Extension to the second dual [1]). Let P be a k-homogeneous polynomial on a Banach algebra $\mathcal{A}$ and let Φ be its corresponding k-linear form. Then P can be extended to the second dual through:

$$\overline{P}(m) = \overline{\Phi}(\underbrace{m, \cdots, m}_{k}) = \text{w}^*\text{-}\lim_{\alpha_1} \cdots \lim_{\alpha_k} \Phi(x_{\alpha_1}, \cdots, x_{\alpha_k}),$$

where $(x_\alpha)_{\alpha \in I}$ is a net in $\mathcal{A}$, w*-converging to $m \in \mathcal{A}^{**}$.

Davie and Gamelin [4] showed that $\|P\| = \|\overline{P}\|$. When $\mathcal{A}$ is a dual space, there exists another way for extending a polynomial to the second dual which is discussed in the next section.

2. Results

If P is an orthogonal additive k-homogeneous polynomial, then as the following example demonstrates, the extension $\overline{P}$ of P to the second dual need not be orthogonally additive, in general.

Example 2.1. let $\mathcal{A}$ be a non-commutative Banach algebra which is not Arens regular, and let $P : \mathcal{A} \to \mathbb{C}$ be defined by $P(x) = \phi(x^2)$, for some $\phi \in \mathcal{A}^*$. Then P is obviously orthogonally additive. It is known that $\Phi(x, y) = \frac{(xy+yx)}{2}$ is its associated bilinear form. It follows that

$$\overline{\Phi}(x^{**}, y^{**}) = \langle \frac{1}{2}(x^{**} \Box y^{**} + y^{**} \Diamond x^{**}), f \rangle.$$

So $\overline{P}(m) = \langle \frac{1}{2}(m \Box m + m \Diamond m), f \rangle$, $m \in \mathcal{A}^{**}$. Now let $m, n \in \mathcal{A}^{**}$ and $m \Box n = n \Box m = 0$ Then:

$$\overline{P}(m + n) = \overline{P}(m) + \overline{P}(n) + \langle \frac{1}{2}(m \Diamond n), \phi \rangle + \langle \frac{1}{2}(n \Diamond m), f \rangle.$$

In the sequel we investigate some conditions under which the orthogonality of P implies that of $\overline{P}$.

Theorem 2.2. *Let $\mathcal{A}$ be an Arens regular Banach algebra. If $P : \mathcal{A} \to C$ is a standard k-homogeneous orthogonal additive polynomial, then so is its extension $\overline{P} : \mathcal{A}^{**} \to C$.*

In the following, another way for extending a k-homogeneous polynomial to the second dual is stated:

Let $\mathcal{A} = F^*$ where F is a complex Banach space. Then $\mathcal{A}^{**} = \mathcal{A} \oplus F^\perp$. So each $m \in \mathcal{A}$ can be represented as $x + x^\perp$. Let $\pi : \mathcal{A}^{**} \to \mathcal{A}$ be the projection onto $\mathcal{A}$. If P is a polynomial from $\mathcal{A}$ into $\mathbb{C}$ then $P \circ \pi$ is a polynomial from $\mathcal{A}^{**}$ into $\mathbb{C}$ that extends P. In general $\overline{P}$ is not necessarily equal to $P \circ \pi$. In [6], the polynomials P whose Aron-Berner extensions are equal to $P \circ \pi$ are characterized.

Note that $\pi : (\mathcal{A}^{**}, \Box) \to (\mathcal{A}, \cdot)$ is a homomorphism and $F^\perp$ is a closed ideal in $(\mathcal{A}^{**}, \Box)$. see [5, Theorem 2.2].

Lemma 2.3. *Let $\mathcal{A}$ be an Arens regular Banach algebra and $\mathcal{A} = F^*$. If P is orthogonally additive, then so is $P \circ \pi$ with each of the Arens products.*

Lemma 2.4. *Let $\mathcal{A}$ be a non-reflexive dual Banach algebra with the predual F and let $\gamma \in \mathcal{A}^*$.*

- *If $\mathcal{A}$ is either Arens regular or commutative, then $P_\gamma \circ \pi = \overline{P_\gamma} + Q$, where Q is a standard k-homogeneous polynomial on $\mathcal{A}^{**}$.*
- *If $\gamma \in F$, then $P_\gamma \circ \pi = \overline{P_\gamma}$. The converse also holds in the case where $\mathcal{A}$ is unital.*

Theorem 2.5. *Let $\mathcal{A}$ be an Arens regular Banach space and $\mathcal{A} = F^*$. If $\mathcal{A}^{**}$ is standard, then so is $\mathcal{A}$.*

Theorem 2.6. *Let $\mathcal{A}$ be a semisimple Banach algebra with a predual F. Furthermore suppose $a.F^{\perp} = F^{\perp}.a = 0$ for each $a \in \mathcal{A}$. Then $\mathcal{A}$ being standard implies that $\mathcal{A}^{**}$ is standard.*

Example 2.7. $(\ell^{\infty})^*$ is Standard. Indeed ℓ^1 is a semisimple Banach algebra whose predual is c_0. It is not hard to see that $a.c_0^{\perp} = c_0^{\perp}.a = 0$ for each $a \in \ell^1$. Furthermore, ℓ^1, together with pointwise product, is standard.

The following theorem presents conditions under which the second dual of a Banach algebra is not standard.

Theorem 2.8. *let $\mathcal{A}$ be a non-commutative, non-Arens regular Banach algebra with a bounded approximate identity. Assume that there exists $m_0 \in \mathcal{A}^{**}$ such that $m_0 \Box m_0 \neq m_0 \Diamond m_0$. Furthermore, let $m \Box n = n \Box m = 0$ implies $m \Diamond n = n \Diamond m = 0$ for each $m, n \in \mathcal{A}^{**}$. Then $\mathcal{A}^{**}$ is not standard.*

References

[1] R. Aron and P. Berner, *A Hahn-Banach extension theorem for analytic mappings*, Bull. Soc. Math. France, **106** (1978), 3-24.

[2] J. Alaminos, J. Extremera, M. L. C. Godoy and A. R. Villena, *Orthogonally additive polynomials on convolution algebras associated with a compact group*, Journal of Mathematical Analysis and Applications, **472**(1)(2019), 285-302.

[3] J. Alaminos, J. Extremera and A. R. Villena, *Orthogonally additive polynomials on Fourier algebras*, J. Math.Anal.Appl., **422** (2015), 72-83.

[4] A. M. Davie and T. W. Gamelin, *A theorem on polynomial-star approximation*, Proc. Amer. Math. Soc., **106** (1989), 351-356.

[5] F. Ghahramani and J. Laali, *Amenability and topological centres of the second duals of Banach algebras*, Bull. Australian Math. Soc., **65** (2002), 191-197.

[6] J. G. Llavona and L. A. Moraes, *The Aron-Berner Extension for Polynomials Defined in the Dual of a Banach Space*, Publ. RIMS, Kyoto Univ., **40** (2004), 221-230.

[7] C. Palazuelos, A.M. Peralta and A. Villanueva, *Orthogonally additive polynomials on C^*-algebras*, Quart. J. Math., **59** (2008), 363-374.

[8] D. Pérez-García and I. Villanueva, *Orthogonally additive polynomials on spaces of continuous functions*, J. Math. Anal. Appl., **306** (2005), 97-105.

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Some remarks on amenable action on the predual of a W*-algebra

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Abstract. In this note, we present a characterization for locally compact groups that act amenably on the predual of a W*-algebra.
Keywords: Amenability, bounded linear operator, locally compact group, module, W*-algebra..
AMS Mathematical Subject Classification [2010]: 22D10, 46H25, 43A07.

1. Introduction

Throughout G is a locally compact group with the unit e, a fixed left Haar-measure. Let X be a left Banach G-module. Then X^* is a right G-module, not necessarily Banach. Also, X^* is a right Banach $L^1(G)$-module, where $L^1(G)$ denote the group algebra of G with the convolution product as defined in [4]. Recall that the convolution product of two complex-valued functions f and g on G is defined as follows.

$$f * g(s) = \int_G f(t)g(t^{-1}s)\,dt,$$

when the integral makes sense.

Also, $L^1(G)$ is a left G-module with the action $s \cdot \phi = l_{s^{-1}}\phi$ for all $s \in G$ and $\phi \in L^1(G)$. It's known that $L^\infty(G)$ can be identified by the first dual space of $L^1(G)$ under the pairing

$$\langle x, \phi\rangle = \int_G x(s)\phi(s)\,ds \qquad (x \in L^\infty(G),\, \phi \in L^1(G)).$$

A locally compact group G is called amenable if there is a mean on $L^\infty(G)$, which is left translation invariant.

One of the most important objects of study in the category of harmonic analysis is the von-Neumann algebra. Many of the central objects of study in abstract harmonic analysis, such as amenable theory, are in this class. Amenability, which is a very distinctive property for locally compact groups, was defined firstly for discrete groups by von-Neumann [9]. The definition of amenability for arbitrary locally compact groups was later given by Day [2]. A general notion of an amenable action on the predual of a von-Neumann algebra developed by Stokke [8] in 2004.

One of the motivations of this paper is the existence of a vast body of results on equivalents of locally compact groups equipped with the amenable property.

2. The results

Let $\mathcal{M}$ be a W*-algebra with predual $\mathcal{M}_*$ and the identity element $e_{\mathcal{M}}$. A bounded linear functional M on $\mathcal{M}$ is called a state if it satisfies $\|M\| = M(e_{\mathcal{M}}) = 1$. Also, $(\mathcal{M}_*)_1^+$ is the collection of normal states on $\mathcal{M}$; i.e., the collection of all w*-continuous states on $\mathcal{M}_*$.

Using the idea $\mathcal{M} = L^\infty(G)$, $\mathcal{M}_* = L^1(G)$ with $s \cdot \phi = l_{s^{-1}}\phi$ for all $s \in G$ and $\phi \in L^1(G)$, Stokke in [8] was introduced a unified approach under which the standard techniques used to develop the basic theory of amenable groups as follows.

Definition 2.1. A locally compact group G will be said to have positive action on $\mathcal{M}_*$ if $\mathcal{M}_*$ is a left Banach G-module such that

*speaker

(a) $\|s \cdot \phi\| \leq \|\phi\|$ for all $s \in G$ and $\phi \in \mathcal{M}_*$,
(b) $s \cdot \phi \in (\mathcal{M}_*)_1^+$ for all $s \in G$ and $\phi \in (\mathcal{M}_*)_1^+$.

The reader can see several known examples of the above notion in [8, Example 1.2].

Throughout this paper, G is a locally compact group, and $\mathcal{M}$ is a W*-algebra such that G has positive action on the predual $\mathcal{M}_*$. An element $x \in \mathcal{M}$ is called uniformly continuous if the mapping $G \longrightarrow (\mathcal{M}, \|.\|)$ that given by $s \mapsto x \cdot s$ is continuous. The collection of all such elements denotes $UC(\mathcal{M})$. According to [6], $x \in UC(\mathcal{M})$ if and only if the mapping $s \mapsto x \cdot s$ from G into $\mathcal{M}$ is weakly continuous. We always have $UC(\mathcal{M}) = \mathcal{M} \cdot L^1(G)$; see [8]. Note that $UC(\mathcal{M})$ is weak*-dense in $\mathcal{M}$. Also, one can easily verify that $x \in UC(\mathcal{M})$ if and only if $x \cdot e_i \longrightarrow x$, if and only if $x \cdot e_i \longrightarrow x$ with respect to the weak topology of $\mathcal{M}$, where (e_i) is the bounded approximate identity of $L^1(G)$.

Definition 2.2. By the above assumpption,

(a) a state M on $\mathcal{M}$ is called an invariant mean if $\langle M, x \cdot s\rangle = \langle M, x\rangle$ for all $x \in \mathcal{M}$ and $s \in G$.
(b) a state M on $\mathcal{M}$ is called a toplogical invariant mean if $\langle M, x \cdot f\rangle = \langle M, x\rangle$ for all $x \in \mathcal{M}$ and $f \in L^1(G)_1^+$, where

$$L^1(G)_1^+ = \{f \in L^1(G) \,|\, f \geq 0, \int_G f = 1\}.$$

(c) G acts amenably on $\mathcal{M}_*$ if there exists an invariant mean on $\mathcal{M}$.

The above notion, as a unified approach to developing the basic theory of amenable groups, was presented and studied by Stokke [8] in which he proved that the following statements are equivalent.

(a) there is a topological invariant mean on $\mathcal{M}$,
(b) there is an invariant mean on $\mathcal{M}$,
(c) there is an invariant mean on $UC(\mathcal{M})$,
(d) there is a topological invariant mean on $UC(\mathcal{M})$.

Now, we show that G is amenable if and only if every statement in the above holds for all W*-algebras that G has a positive action on its predual.

Theorem 2.3. *A locally compact group G is amenable if and only if G acts amenably on $\mathcal{M}_*$ for every W*-algebra $\mathcal{M}$ such that G has positive action on the predual $\mathcal{M}_*$.*

Proof. Suppose that G is amenable and $\phi \in (\mathcal{M}_*)_1^+$. For each $x \in \mathcal{M}$, we define the complex-valued function ψ_x on G by $\psi_x(s) = \langle x, s \cdot \phi\rangle$ for all $s \in G$. Clearly, ψ_x is bounded, and since $\mathcal{M}_*$ is a left Banach G-module, ψ_x is continuous. Therefore, $\psi_x \in L^\infty(G)$. Define now $M : \mathcal{M} \longrightarrow \mathbb{C}$ by $M(x) = m(\psi_x)$ for all $x \in \mathcal{M}$, where m is a left invariant mean on $L^\infty(G)$. It is readily checked that M is an invariant mean on $\mathcal{M}$. For the converse, we regard the W*-algebra $\mathcal{M} = L^\infty(G)$ with $\mathcal{M}_* = L^1(G)$. □

The above theorem has also been proved in [8, Corollary 1.11] by Day's fixed theorem [3, Theorem 3.3.5].

The following lemma is well-known when $\mathcal{M} = L^\infty(G)$. The proof is also similar way.

Lemma 2.4. *Let $\mathcal{M}$ is a W*-algebra such that G has a positive action on the predual $\mathcal{M}_*$. Then a bounded linear functional M on $\mathcal{M}$ (or $UC(\mathcal{M})$) is a state if and only if M satisfies in any pair of the following conditions:*

(a) *M is positive; i.e., $M(x) \geq 0$ for all positive elements $x \in \mathcal{M}$,*
(b) *$M(e_{\mathcal{M}}) = 1$,*
(c) *$\|M\| = 1$.*

For each M in the dual of $UC(\mathcal{M})$, we define the bounded linear mapping ω_M from $\mathcal{M}$ into $L^\infty(G)$ as follows

$$\langle \omega_M(x),\, f\rangle = \langle M,\, x\cdot f\rangle \quad (x\in\mathcal{M},\, f\in L^1(G)).$$

THEOREM 2.5. *Let $\mathcal{M}$ is a W*-algebra such that G has a positive action on the predual $\mathcal{M}_*$. Then the following statements are equivalent.*

(a) *G acts amenably on $\mathcal{M}_*$,*
(b) *There exists a weakly compact positive operator $\omega : \mathcal{M} \longrightarrow L^\infty(G)$ of norm 1 such that $\omega(x\cdot s) = l_s\omega(x)$ for all $x\in\mathcal{M}$ and $s\in G$.*

PROOF. (a) $\Longrightarrow$ (b). Suppose that G acts amenably on $\mathcal{M}_*$. Then there exists a toplogical invariant mean M on $UC(\mathcal{M})$. Therefore, $\omega_M(x) = M(x)1$, and so $m\circ\omega_M = m(1)M$ for all $m\in L^1(G)^{**}$. It follows effortlessly that

$$\omega_M^{**}(x^{**}) = x^{**}(M)1 \in L^\infty(G) \quad (x^{**}\in\mathcal{M}^{**}).$$

So, ω_M is weakly compact by Theorem 3.5.8 of [7]. Now, we show that $\|\omega_M\| = 1$. It is clear that $\|\omega_M\| \le \|M\|$. To prove the reverse inequality, let (e_i) be an approximate identity of $L^1(G)$ bounded to 1. Then for each $x\in UC(\mathcal{M})$, we have

$$\|x\cdot e_i - x\| \longrightarrow 0,$$

and so

$$|\omega_M(x)(e_i)| \longrightarrow |\langle M,\, x\rangle|.$$

Furthermore, for each i, we have

$$\|\omega_M(x)\|_\infty \ge |\omega_M(x)(e_i)|.$$

Consequently, $\|\omega_M\| \ge \|M\|$. So, $\|\omega_M\| = 1$. Note that since M is a state, ω_M is positive by Lemma 2.4. Also, one can readily checking that $\omega_M(x\cdot s) = l_x\omega_M(s)$ for all $x\in\mathcal{M}$ and $s\in G$.

(b) $\Longrightarrow$ (a). By assumption, for each $x\in\mathcal{M}$

$$\{l_s\omega(x)|\, s\in G\} = \{\omega(x\cdot s)|\, s\in G\}$$

is relatively weakly compact in the weak topology of $L^\infty(G)$. It follows that $\omega(x)\in WAP(G)$, where $WAP(G)$ denotes the space of all weakly compact periodic functions on G. It's known that there exists a unique left invariant mean on $WAP(G)$, see for example [3]. We define $\langle M,\, x\rangle = \langle m,\, \omega(x)\rangle$ for all $x\in\mathcal{M}$. Then M is the invariant mean on $\mathcal{M}$. □

We end the work by the following result that is an immediate consequence of Theorem 2.3 and Theorem 2.5.

COROLLARY 2.6. *A locally compact group G is amenable if and only if either, and hence all, of the following statements hold.*

(a) *For every W*-algebra $\mathcal{M}$ such that G has positive action on $\mathcal{M}_*$, there exists a weakly compact positive operator ω of norm 1 from $\mathcal{M}$ into $L^\infty(G)$ such that $\omega(x\cdot s) = l_x\omega(x)$ for all $x\in\mathcal{M}$ and $s\in G$.*
(b) *There exists a weakly compact positive operator ω of norm 1 on $L^\infty(G)$ such that $l_s\omega(x) = \omega(l_sx)$ for all $x\in L^\infty(G)$ and $s\in G$.*

References

1. M.E.B. Bekka, *Amenable unitary representations of locally compact groups.* Invent. Math. **100** (1990), 383-401.
2. M. M. Day, *Means on semigroups and groups.* Bull. Amer. Math. Soc. **55** (1949), 1054-1055.
3. F. P. Greenleaf, *Invariant means on topological groups and their applications.* Van Nostrand Mathematical Studies, **16**, 1969.
4. E. Hewitt and K. Ross, *Abstract Harmonic Analysis I.* Springer, New York, 1970.
5. A. T. Lau, *Operators which commute with convolutions on subspace of* $L^{\infty}(G)$. Colloquium Math. **39** (1978), 357-359.
6. B. E. Johnson, *Cohomology in Banach algebras.* Memoirs of the American Mathematical Society, **127** American Mathematical Society, Providence, RI, 1972.
7. R. E. Megginson, *An Introduction to Banach Space Theory.* Springer-Verlag, New York, 1998.
8. R. Stokke, *Quasi-central bounded approximate identities in group algebras.* Illinois Journal of Mathematics 48 (2004), 151-170.
9. J. Von Neumann, *Zur allgemeinen Theorie des Mafies.* Fund. Math. **13** (1929), 73-116.

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Fatou's lemma and Reverse Fatou's lemma for pseudo-integrals

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ABSTRACT. Fatou's lemma is a useful result in the theory of integration with applications in probability and measure theory. The Lebesgue Dominated Convergence Theorem is an important result in Real Analysis that the Fatou's lemma is used during its proof. Pseudo-integrals form an important subclass of nonlinear integrals. Using the pseudo-operations given by strictly monotone and continuous function g, we have presented a new style the Fatou's lemma and Reverse Fatou's lemma for pseudo-integrals.

Keywords: Pseudo-addition, Pseudo-multiplication; Pseudo-integral; Non-additive measure; Fatou's lemma.
AMS Mathematical Subject Classification [2010]: 26D15, 28A25, 28E15, 39B62.

1. Introduction

Pseudo-analysis has been raised as a generalization of classical analysis where a real interval $[a, b] \subseteq [-\infty, \infty]$ equipped with two operations $\oplus$ (called pseudo-addition) and $\otimes$ (called pseudo-multiplication) acts a similar role as the field of real numbers $\Re$ for real analysis. Based on this structure there were created the concepts of pseudo-additive measure ($\oplus - measure$), pseudo-division, pseudo-integral, pseudo-scalar product, pseudo-analytic exponential, pseudo-logarithm, etc. The important advantage of the pseudo-analysis is that there are covered with one theory, and so with unified methods, problems (usually nonlinear and under involved uncertainty) from many different fields.

Many important integral inequalities have been proved in the context of pseudo-integral. Pap and Štrboja generalized the Jensen integral inequality for the pseudo-integral. Abbaszadeh et al. proved Hölder's type integral inequality and Hadamard inequality for the pseudo-integral in the above two cases of the real semirings. Agahi et al. proved Chebyshev type inequalities and a generalization of the Hölder's and Minkowski's integral inequalities for the pseudo-integral.

We know that the classical Fatou's Lemma [6] is as follows :

$$\int_X \lim_{n\to\infty} inf f_n d\mu \leq \lim_{n\to\infty} inf \int_X f_n d\mu,$$

where $\{f_n\}$ is a sequence of non-negative measurable functions defined on a the classical measure space $(X, \mathcal{E}, \mu)$ such that $\int_X \liminf_{n\to\infty} f_n d\mu < \infty$.

Agahi et al. [2] proved the following Fatou's type lemma for Sugeno integral:

$$⨍_X \lim_{n\to\infty} inf f_n d\mu \leq \lim_{n\to\infty} inf ⨍_X f_n d\mu,$$

where $\{f_n\}$ is a non-decreasing sequence of non-negative and measurable functions defined on the non-additive measure space $(X, \mathcal{E}, \mu)$ with $\mu(X) < \infty$.

We also know that the classical reverse Fatou's Lemma [7] is as follows :

$$\int_X \lim_{n\to\infty} sup f_n d\mu \geq \lim_{n\to\infty} sup \int_X f_n d\mu,$$

where $\{f_n\}$ is a sequence of non-negative measurable functions defined on a the classical measure space $(X, \mathcal{E}, \mu)$ such that for any n, $f_n \leq g$, for some non-negative measurable function g with $\int g d\mu < \infty$.

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In this paper we show that the Fatou's lemma and the reverse Fatou's lemma are also true for pseudo-integrals.

Assume that $[a,b] \subseteq [-\infty,\infty]$ is a closed interval and $\preceq$ be a total order on $[a,b]$. Pseudo-addition $\oplus$ and Pseudo-multiplication $\otimes$ are defined as follows.

Definition 1.1. [3] A binary operation $\oplus$ on $[a,b]$ is pseudo-addition if for all $x,y,z \in [a,b]$,

(1) $x \oplus y = y \oplus x$ (commutativity);
(2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ (associativity);
(3) If $x \preceq y$, then $x \oplus z \preceq y \oplus z$ (monotonicity);
(4) $\mathbf{0}_\oplus \oplus x = x$, where $\mathbf{0}_\oplus \in [a,b]$ is a neutral (zero) element (boundary condition).

Definition 1.2. [3] Let $\oplus$ be a given pseudo-addition on $[a,b]$. A binary operation $\otimes$ on $[a,b]$ is pseudo-multiplication if for all $x,y,z \in [a,b]$ and $w \in [a,b]_+$,

(1) $x \otimes y = y \otimes x$ (commutativity);
(2) $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ (associativity);
(3) If $x \preceq y$, then $x \otimes w \preceq y \otimes w$ (monotonicity);
(4) $(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$ (left distributivity);
(5) $\mathbf{1}_\otimes \otimes x = x$, where $\mathbf{1}_\otimes \in [a,b]$ is a neutral (unit) element (boundary condition).

It is easy to see that the structure $([a,b],\oplus,\otimes)$ is a semiring, see [4].

If both operations $\oplus$ and $\otimes$ are not idempotent, then The pseudo-operations are generated by a strictly monotone and continuous function g. In this case we will consider only strict pseudo-addition $\oplus$.

By Aczel's representation theorem [1] for each strict pseudo-addition $\oplus$ there exists a strictly monotone and continuous surjective function g (generator for $\oplus$), $g : [a,b] \to [0,+\infty]$ such that $g(\mathbf{0}_\oplus) = 0$ and

$$x \oplus y := g^{-1}(g(x) + g(y)).$$

Using a generator g of a strict pseudo-addition $\oplus$ we can define a pseudo-multiplication $\otimes$ by

$$x \otimes y := g^{-1}(g(x)g(y)).$$

with the convention $0 \times (+\infty) := 0$. This is the only way to define a pseudo-multiplication $\otimes$, which is distributive with respect to a given pseudo-addition $\oplus$ generated g. If the zero element for the pseudo-addition is a, we will consider increasing generators. Then $g(a) = 0$ and $g(b) = +\infty$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = +\infty$.

Definition 1.3. [5] Let X be a non-empty set and $\mathcal{E}$ be a σ-algebra of the subsets of X. The set function $m : \mathcal{E} \to [a,b]_+$ is a σ-$\oplus$-measure if

(1) $m(\emptyset) = \mathbf{0}_\oplus$;
(2) For any sequence $(E_i)_{i\in\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{E}$,

$$m(\overset{+\infty}{\underset{i=1}{\cup}} E_i) = \overset{+\infty}{\underset{i=1}{\oplus}} m(E_i) := \lim_{n\to+\infty} \overset{n}{\underset{i=1}{\oplus}} m(E_i)$$

If pseudo-addition $\oplus$ is idempotent, then condition (1) and pairwise disjointedness of sets can be left out.

Definition 1.4. [5] Suppose that X is a non-empty set, $\mathcal{E}$ is a σ-algebra of the subsets of X and $m : \mathcal{E} \to [a,b]_+$ is a σ-$\oplus$-measure. The pseudo-integral of a bounded measurable function $f : X \to [a,b]$, where the pseudo-operations are defined by a monotone and continuous function $g : [a,b] \to [0,\infty]$, is defined by

$$\int_X^{\oplus} f(x) \otimes \mathrm{d}m := g^{-1}\left(\int_X (g \circ f)\ \mathrm{d}(g \circ m)\right).$$

If $X \subseteq [-\infty, +\infty]$ is a closed (semiclosed) interval, $\mathcal{E} = \mathcal{B}_X$ is σ-algebra of Borel subsets of X and $m = g^{-1} \circ \mu$ where μ is the standard Lebesgue measure on X, then the pseudo-integral for the function f has the form

$$\int_X^{\oplus} f(x) \otimes \mathrm{d}m = g^{-1}\left(\int_X g\big(f(x)\big)\mathrm{d}\mu\right),$$

where the integral applied on the right side is the standard Lebesgue integral and is called the g-integral of the function f. If we consider the semiring $([a, b], \sup, \otimes)$, where $\otimes$ is a pseudo-multiplication defined by means of a generator $g : [a, b] \to [0, +\infty]$ and g is increasing bijection, the pseudo-integral of a function $f : X \to [a, b]$ has the form

$$\int_X^{\oplus} f \otimes \mathrm{d}m := \sup_{x \in X} \big(f(x) \otimes \psi(x)\big),$$

where the function ψ defines sup-measure m. In fact $\psi : X \to [a, b]$ is a density function given by $\psi(x) = m(\{x\})$. In this case, we will use the notation $\int_X^{\sup} f \otimes \mathrm{d}m$ instead of $\int_X^{\oplus} f \otimes \mathrm{d}m$.

Theorem 1.5. [5] Let $([0, \infty], \sup, \otimes)$ be a semiring, when $\otimes$ is generated by the continuous and increasing function g. Let m be sup-measure on $\big([0, \infty], \mathcal{B}_{[0,+\infty]}\big)$, where $\mathcal{B}_{[0,+\infty]}$ is σ-algebra of Borel subsets of the interval $[0, \infty]$, $m(A) = \sup\big\{a | \mu(\{x | x \in A,\ x > a\}) > 0\big\}$ and $\psi : [0, \infty] \to [0, \infty]$ be a continuous density. Then, there exists a family m_λ of $\oplus_\lambda$-measure where $\oplus_\lambda$ is generated by g^λ, $\lambda \in (0, \infty)$ such that for every continuous function $f : [0, \infty] \to [0, \infty]$,

$$\begin{aligned}\int^{\sup} f \otimes \mathrm{d}m &= \lim_{\lambda \to +\infty} \int^{\oplus_\lambda} f \otimes \mathrm{d}m_\lambda \\ &= \lim_{\lambda \to +\infty} \big(g^\lambda\big)^{-1}\left(\int g^\lambda\big(f(x)\big)\mathrm{d}x\right).\end{aligned}$$

2. Main results

Theorem 2.1. *(***Fatou's lemma for the pseudo-integrals***) If $\{f_n\}$ is a sequence of non-negative measurable functions defined on a the measurable space $(X, \mathcal{E})$, and let a generator $g : [0, \infty] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be a strictly monotone increasing and continuous function such that $g(0) = 0$, then for any σ-$\oplus$-measure m, the inequality:*

$$\int_X^{\oplus} \lim_{n \to \infty} inf f_n \otimes dm \le \lim_{n \to \infty} inf \int_X^{\oplus} f_n \otimes dm,$$

holds.

Lemma 2.2. *(***Reverse Fatou's lemma for the pseudo-integrals***) If f_n is a sequence of non-negative measurable functions defined on a the measurable space $(X, \mathcal{E})$, and let a generator $g : [0, \infty] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be a strictly monotone increasing and continuous function such that $g(0) = 0$ and for any n, $g(f_n) \le h$, for some non-negative measurable function h with $\int_X h(x) d(g \circ m) < \infty$, then for any σ-$\oplus$-measure m, the inequality:*

$$\int_X^{\oplus} \lim_{n \to \infty} sup f_n \otimes dm \ge \lim_{n \to \infty} sup \int_X^{\oplus} f_n \otimes dm,$$

holds.

Theorem 2.3. *(***Another version of the Fatou's lemma***) If $\{f_n\}$ is a sequence of non-negative measurable functions defined on a the measurable space $(X, \mathcal{E})$, and let the generator $g : [0, \infty] \to [0, \infty]$ of the pseudo-multiplication $\odot$ be a strictly monotone increasing and continuous function, then for complete sup-measure m, the inequality:*

$$\int_X^{\sup} \lim_{n \to \infty} inf f_n \otimes dm \le \lim_{n \to \infty} inf \int_X^{\sup} f_n \otimes dm,$$

holds.

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References

1. J. Aczél, *Lectures on Functional Equations and their Applications*, Academic Press, New York, 1966.
2. H. Agahi, A. Flores-Franulic, S. Mansour Vaezpour, *Fatou's lemma for Sugeno integral*, Applied Mathematics and Computation 217 (2011) 6092-6096.
3. P. Benvenuti, R. Mesiar and D. Vivona, *Monotone set functions-based integrals*, in Handbook of Measure Theory, eds. E. Pap (Elsevier, Amsterdam, 2002), pp. 13291379.
4. W. Kuich, Semirings, *Automata*, Languages, Springer-Verlag, Berlin, 1986.
5. R. Mesiar and E. Pap, *Idempotent integral as limit of g-integrals*, Fuzzy Sets and Systems, 102 (1999), 385-392.
6. W. Rudin, *Real and Complex Analysis*, third ed., McGraw-Hill Book Company, New York, 1986.
7. D. Williams, Probability with Martingales, *Cambridge University Press*, New York, 1991.

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On zero product determined Banach algebras

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Abstract. Let $\mathcal{A}$ be a Banach algebra with a left approximate identity. If

(i) For every continuous bilinear mapping ϕ from $\mathcal{A}\times\mathcal{A}$ into $\mathcal{X}$, where $\mathcal{X}$ is a Banach space, there exists $k>0$ such that $\|\phi(a,b)\|\leq k\|ab\|$, for all $a,b\in\mathcal{A}$, or

(ii) $\mathcal{A}$ is generated by idempotents,

that $\mathcal{A}$ is zero product determined.

Keywords: Banach algebra, Zero product determined, approximate identity.
AMS Mathematical Subject Classification [2010]: 47B47.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a arbitrary Banach space. Then the continuous bilinear mapping $\phi:\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{X}$ preserves zero products if

$$ab=0 \quad\Longrightarrow\quad \phi(a,b)=0, \qquad a,b\in\mathcal{A}. \tag{1}$$

A Banach algebra $\mathcal{A}$ is said to be *zero product determined* if every continuous zero product preserving bilinear mapping $\phi:\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{X}$ is implemented by a linear map $T:\mathcal{A}\longrightarrow\mathcal{X}$, i.e., $\phi(a,b)=T(ab)$ for all $a,b\in\mathcal{A}$.

Characterizing homomorphisms, derivations and multipliers on Banach algebras, matrix algebras and C^*-algebras through the action on zero products have been studied by many authors, see for example [1, 3, 4, 6, 7] and the references therein.

In [4], Bresar *et. al.* proved that the matrix algebra $M_n(\mathcal{A})$ of $n\times n$ matrices over a unital algebra $\mathcal{A}$ is zero product determined. In [7], Ghahramani studied the (centeralizers) multipliers on Banach algebras through identity products by consideration of bilinear mapping satisfying a related condition.

Motivated by (1) the following concept was introduced in [1].

Definition 1.1. [1] A Banach algebra $\mathcal{A}$ *has the property* $(\mathbb{B})$ if for every continuous bilinear mapping $\phi:\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{X}$, where $\mathcal{X}$ is an arbitrary Banach space, the condition (1) implies that $\phi(ab,c)=\phi(a,bc)$, for all $a,b,c\in\mathcal{A}$.

Under mild assumptions, the condition $\phi(ab,c)=\phi(a,bc)$ for every $a,b,c\in\mathcal{A}$, implies that $\mathcal{A}$ is zero product determined. For example, if $\mathcal{A}$ is unital, then just take $c=e_{\mathcal{A}}$ and note that T can be defined according to $T(a)=\phi(a,e_{\mathcal{A}})$.

However, throughout the paper we focus for nonunital Banach algebras, but we will assume the existence of a left approximate identity.

Recall that a left (right) approximate identity for $\mathcal{A}$ is a net $\{e_\lambda\}_{\lambda\in I}$ in $\mathcal{A}$ such that $e_\lambda a\longrightarrow a$ ($ae_\lambda\longrightarrow a$) for all $a\in\mathcal{A}$. For example, it is known that the group algebra $L^1(G)$ for locally compact group G and C^*-algebras have an approximate identity bounded by one [5].

In this paper, we introduce the property $(\mathbb{P})$, which closely related to the property $(\mathbb{B})$, and prove that the property $(\mathbb{B})$ follows from the property $(\mathbb{P})$. We show that every Banach algebra $\mathcal{A}$ with a left approximate identity is zero product determined if either $\mathcal{A}$ has the property $(\mathbb{P})$ or it is generated by idempotents.

*speaker

2. The property $(\mathbb{P})$

We commence with the next concept which is closely related to the property $(\mathbb{B})$.

Definition 2.1. A Banach algebra $\mathcal{A}$ *has the property* $(\mathbb{P})$ if for every continuous bilinear mapping ϕ from $\mathcal{A}\times\mathcal{A}$ into $\mathcal{X}$, where $\mathcal{X}$ is an arbitrary Banach space, there exists $k>0$ such that

$$\|\phi(a,b)\| \leq k\|ab\|, \tag{2}$$

for all $a,b\in\mathcal{A}$.

The following theorem is the main result of this section.

Theorem 2.2. *Let $\mathcal{A}$ be a Banach algebra with a left approximate identity $\{e_\lambda\}$. If $\mathcal{A}$ has the property $(\mathbb{P})$, then $\mathcal{A}$ has the property $(\mathbb{B})$.*

From Theorem 2.2 we get the following results.

Corollary 2.3. *Let $\mathcal{A}$ be a Banach algebra with a left approximate identity $\{e_\lambda\}$. If $\mathcal{A}$ has the property $(\mathbb{P})$, then $\mathcal{A}$ is zero product determined.*

Corollary 2.4. *Let $\mathcal{A}$ be a Banach algebra with an approximate identity $\{e_\lambda\}$. If $\mathcal{A}$ has the property $(\mathbb{P})$, then $\mathcal{A}$ is commutative.*

Corollary 2.5. *Let $\mathcal{A}$ be a Banach algebra with a left approximate identity $\{e_\lambda\}$. If $\mathcal{A}$ has the property $(\mathbb{P})$, then every bilinear mapping $\phi:\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{X}$ is symmetric, that is, $\phi(a,b)=\phi(b,a)$ for all $a,b\in\mathcal{A}$.*

Corollary 2.6. *Suppose that $\mathcal{A}$ is a Banach algebra with a left approximate identity, and let $f:\mathcal{A}\longrightarrow\mathcal{A}$ be a continuous linear mapping. If $\mathcal{A}$ has the property $(\mathbb{P})$, then f is a left multiplier.*

Theorem 2.7. [2, Lemma 1.1] *Let G be a locally compact group, and $\phi: L^1(G)\times L^1(G)\longrightarrow\mathcal{X}$ be a continuous linear map, where $\mathcal{X}$ is a Banach space. If*

$$f,g\in L^1(G),\quad f*g=0 \implies \phi(f,g)=0,$$

then

$$\phi(f*g,h)=\phi(f,g*h),$$

for all $f,g,h\in L^1(G)$.

The following example shows that the converse of Theorem 2.2 is false, in general.

Example 2.8. *Let G be a nonabelian locally compact group and $\mathcal{A}=L^1(G)$. Then $\mathcal{A}$ has the property $(\mathbb{B})$ by Theorem 2.7. Since $\mathcal{A}$ is not commutative, it fails to satisfy the property $(\mathbb{P})$ by Corollary 2.4.*

3. Subalgebras generated by idempotents

By the subalgebra of an algebra $\mathcal{A}$ generated by a subset E of $\mathcal{A}$ we mean the linear subspace of $\mathcal{A}$ spanned by the set of all finite products of elements in E.

Theorem 3.1. *If the Banach algebra $\mathcal{A}$ is generated by idempotents, then $\mathcal{A}$ has the property $(\mathbb{B})$.*

Corollary 3.2. *Let $\mathcal{A}$ be a Banach algebra with a bounded left approximate identity. If $\mathcal{A}$ is generated by idempotents, then it is zero product determined.*

Combining Theorem 3.1 and [8, Theorem 1] we deduce the next result.

Corollary 3.3. *Let $\mathcal{A}$ be a Banach algebra which is generated by k of its elements and by the identity element. Then, for all $n\geq k+2$, the algebra $M_n(\mathcal{A})$ of all $n\times n$ matrices with entries in $\mathcal{A}$, has the property $(\mathbb{B})$.*

Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear mapping between Banach algebras. Then we say that f preserves zero products if

$$ab = 0 \quad \Longrightarrow \quad f(a)f(b) = 0, \quad (a, b \in \mathcal{A}).$$

It is obvious that homomorphisms from $\mathcal{A}$ into $\mathcal{B}$ preserve zero products.

Corollary 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, and let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a continuous linear mapping preserving zero products. If $\mathcal{A}$ is generated by idempotents, then*

$$f(ab)f(c) = f(a)f(bc).$$

Moreover, if $\mathcal{A}$ and $\mathcal{B}$ are unital and $f(e_{\mathcal{A}}) = e_{\mathcal{B}}$, then f is a homomorphism.

Let p be a nontrivial idempotent ($p \neq 0$ and $p \neq 1$) in $\mathcal{A}$. Then p cannot be contained in the centre of $\mathcal{A}$. This implies that the subalgebra M of $\mathcal{A}$ generated by idempotents contains a nonzero ideal of $\mathcal{A}$ by [3, Lemma 2.1], from which it follows that M is dense in $\mathcal{A}$. Thus,

Corollary 3.5. *Suppose that $\mathcal{A}$ is a topologically simple Banach algebra containing a nontrivial idempotent. Then $\mathcal{A}$ has the property ($\mathbb{B}$).*

Next we prove that the Banach algebra $C_0(X)$ is generated by idempotents if and only if X is totally disconnected.

Theorem 3.6. *The Banach algebra $C_0(X)$, for a locally compact Hausdorff space X, is generated by idempotents if and only if X is totally disconnected.*

It follows from [1, Theorem 2.11] that every C^*-algebra $\mathcal{A}$ is zero product determined. Now as an upcoming consequence of Theorem 3.6 and Corollary 3.2 we have the following result.

Corollary 3.7. *The Banach algebra $C([0, 1])$ is zero product determined.*

Example 3.8. *Let $\mathcal{A} = C([0, 1])$. Then by Corollary 3.7, $\mathcal{A}$ is zero product determined. Thus, for every continuous bilinear mapping $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}$ satisfying (1), there exist a linear mapping $T : \mathcal{A} \longrightarrow \mathcal{X}$ such that $\phi(a, b) = T(ab)$ for every $a, b \in \mathcal{A}$. Since T is continuous, there exist $k > 0$ such that*

$$\|\phi(a, b)\| = \|T(ab)\| \leq k\|ab\|, \qquad a, b \in \mathcal{A}.$$

Hence $\mathcal{A}$ has the property ($\mathbb{P}$), for each continuous bilinear mapping ϕ satisfying (1). Moreover,

$$\phi(a, b) = T(ab) = T(ba) = \phi(b, a), \qquad a, b \in \mathcal{A},$$

and hence ϕ is symmetric.

Example 3.9. *Let K denote the cross $[-1, 1] \cup i[-1, 1]$ and let $C^{2\times2}(K)$ denote the algebra of all continuous 2×2 matrix functions on a compact set K. Then by [8, Theorem 4], the Banach algebra $\mathcal{A} = C^{2\times2}(K)$ is generated by two idempotents and by the identity function. Therefore it has the property ($\mathbb{B}$) by Theorem 3.1, howover it fails to satisfy the property ($\mathbb{P}$).*

References

1. J. Alaminos, M. Bresar, J. Extremera and A. R. Villena, *Maps preserving zero products*, Studia Math., 193 (2009), 131-159.
2. J. Alaminos, J. Extremera and A. R. Villena, *Orthogonality preserving linear maps on group algebras*, Math., Proc. Camb. Philos. Soc., 158 (2015), 493-504.
3. M. Bresar, *Characterizing homomorphisms, multipliers and derivations in rings with idempotents*, Proc. Roy. Soc. Edinburgh. Sect. A. 137 (2007), 9-21.
4. M. Bresar, M. Grasic and J. S. Ortega, *Zero product determined matrix algebras*, Linear Algebra Appl., 430 (2009), 1486-1498.
5. H. G. Dales, *Banach algebras and automatic continuity*, London Math. Soc. Monogr. Ser. Clarendon Press, 2000.
6. H. Ghahramani, *On derivations and Jordan derivations through zero products*, Oper. Matrices, 8 (2014), 759-771.
7. H. Ghahramani, *On centeralizers of Banach algebras*, Bull. Malaya. Math. Sci. Soc., 38(1) (2015), 155-164.
8. N. Krupnik, S. Roch and B. Silbermann, *On C^*-algebras generated by idempotents*, J. Func. Anal., 137 (1996), 303-319.

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Representation of the inverse of generalized multipliers in Hilbert C^*-modules

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Abstract. This note includes a general version of Bessel multipliers in Hilbert C^*-modules. In fact, by combining analysis, an operator on the standard Hilbert C^*-module $\ell^2(A, \mathbb{I})$ and synthesis, we reach so called generalized Bessel multipliers. Because of their importance for applications, we are interested to determine cases when generalized multipliers are invertible. We investigate some necessary or sufficient conditions for the invertibility of such operators. Specially, our attention is on how to express the inverse of an invertible generalized multiplier as a multiplier. In fact, we show that for all frames, the inverse of any invertible frame multiplier with an invertible symbol can always be represented as a multiplier with an invertible symbol and appropriate dual frames of the given ones.
Keywords: Bessel multiplier, Modular Riesz basis, Standard frame..
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1. Introduction

Bessel multipliers in Hilbert spaces were introduced by Balazs in [2]. Bessel multipliers are operators that are defined by a fixed multiplication pattern which is inserted between the analysis and synthesis operators. This class of operators is important for applications in modern life, for example in acoustics, psychoacoustics and denoising. Recently, M. Mirzaee Azandaryani and A. Khosravi generalized multipliers to Hilbert C^*-modules [5]. As we know, the invertibility of the operators related to frames has great importance in frame theory mostly because of the reconstruction of signals. In this respect, it is important to find the inverse of a multiplier if it exists. Moreover, a question is how to express the inverse of an invertible frame multiplier as a multiplier. In [6], the authors have provided some necessary and/or sufficient conditions for invertibility of multipliers in Hilbert spaces. In this note, first, we introduce the concept of generalized multipliers for Hilbert C^*-modules. Subsequently, special attention is devoted to the study of invertible generalized multipliers.

Throughout, A is a unital C^*-algebra, E and F are finitely or countably generated Hilbert A-modules and $\mathbb{I}$ is an at most countable index set. Moreover, for a C^*-algebra A, the standard Hilbert module $\ell^2(A, \mathbb{I})$ is defined as:

$$\ell^2(A, \mathbb{I}) := \left\{ \{a_i\}_{i\in\mathbb{I}} \subseteq A : \sum_{i\in\mathbb{I}} a_i a_i^* \quad \text{converges in norm in } A \right\}.$$

First, we recall the concept of frame in Hilbert C^*-modules.

Definition 1.1. [3] A sequence $\{x_i\}_{i\in\mathbb{I}} \subset E$ is said to be a frame if there exist two constant $C, D > 0$ such that

$$C\langle x, x\rangle \leq \sum_{i\in\mathbb{I}} \langle x, x_i\rangle \langle x_i, x\rangle \leq D\langle x, x\rangle, \tag{1}$$

for every $x \in E$. If the sum in (1) converges in norm, the frame is called a *standard frame*.

*speaker

The sequence $\{x_i\}_{i\in\mathbb{I}}$ is called a Bessel sequence with bound D if the upper inequality in (1) holds for every $x \in E$.

Suppose that $X = \{x_i\}_{i\in\mathbb{I}}$ is a Bessel sequence in Hilbert A-module E with bound D. The operator $T_X : \ell^2(A, \mathbb{I}) \to E$ defined by

$$T_X \{a_i\}_{i\in\mathbb{I}} = \sum_{i\in\mathbb{I}} a_i x_i$$

is called the synthesis operator. The adjoint operator $T_X^* : E \to \ell^2(A, \mathbb{I})$ which is given by

$$T_X^* x = \{\langle x, x_i\rangle\}_{i\in\mathbb{I}}$$

is called the analysis operator. Composing T_X and T_X^*, we obtain the frame operator $S_X : E \to E$ as

$$S_X x = T_X T_X^* x = \sum_{i\in\mathbb{I}} \langle x, x_i\rangle\, x_i.$$

Now, let us take a brief review of the definition of Bessel multipliers in Hilbert C^*-modules.

Definition 1.2. [5] Let $X = \{x_i\}_{i\in\mathbb{I}} \subseteq E$ and $Y = \{y_i\}_{i\in\mathbb{I}} \subseteq F$ be standard Bessel sequences. Moreover, let $m = \{m_i\}_{i\in\mathbb{I}} \in \ell^\infty(A, \mathbb{I})$ be such that $m_i \in Z(A)$ (Z(A) is the center of a Banach algebra A which is defined as $Z(A) = \{a \in A; ab = ba, \forall b \in A\}$), for each $i \in \mathbb{I}$, and $\mathcal{M}_m$ defined on $\ell^2(A, \mathbb{I})$ as $\mathcal{M}_m \{a_i\}_{i\in\mathbb{I}} = \{m_i a_i\}_{i\in\mathbb{I}}$.

The operator $\mathbf{M}_{m,Y,X} : E \to F$ which is defined by $\mathbf{M}_{m,Y,X} = T_Y \mathcal{M}_m T_X^*$ is called the Bessel multiplier for the Bessel sequences $\{x_i\}_{i\in\mathbb{I}}$ and $\{y_i\}_{i\in\mathbb{I}}$. It is easy to see that

$$\mathbf{M}_{m,Y,X}(x) = \sum_{i\in\mathbb{I}} m_i \langle x, x_i\rangle\, y_i.$$

In the sequel, first, we introduce the concept of Generalized Bessel multipliers for countably generated Hilbert C^*-modules and then, we will discuss about invertibility of such operators.

Definition 1.3. [1] Let E and F be two Hilbert C^*-modules over a unital C^*-algebra A and $X = \{x_i\}_{i\in\mathbb{I}} \subset E$ and $Y = \{y_i\}_{i\in\mathbb{I}} \subset F$ be standard Bessel sequences. Also, let $U \in L\left(\ell^2(A, \mathbb{I})\right)$ be an arbitrary non-zero operator. The operator $\mathbf{M}_{U,Y,X} : E \to F$ which is defined as

$$\mathbf{M}_{U,Y,X}(x) = T_Y U T_X^*(x), \qquad (x \in E), \tag{2}$$

is called the Generalized Bessel multiplier associated with X and Y with symbol U.

2. Invertibility of Generalized Bessel multipliers

In the following, we investigate sufficient conditions for being standard frames by properties of the generalized Bessel multipliers.

Proposition 2.1. *Assume that $X = \{x_i\}_{i\in\mathbb{I}} \subset E$ and $Y = \{y_i\}_{i\in\mathbb{I}} \subset F$ are Bessel sequences.*

1. *If $\mathbf{M}_{U,Y,X}$ has a left inverse, then X is a standard frame for E.*
2. *If $\mathbf{M}_{U,Y,X}$ has a right inverse, then Y is a standard frame for F.*

The following propositions contain some sufficient conditions for the invertibility of frame multipliers.

Proposition 2.2. *Let $Y = \{y_i\}_{i\in\mathbb{I}}$ be a standard frame for Hilbert A-module E with bounds C, D and $W : E \to E$ be an adjointable and bijective operator such that $x_i = W y_i$ for each $i \in \mathbb{I}$. Moreover, let U be a bounded operator on $\ell^2(A, \mathbb{I})$ such that $\|U - I\| < \dfrac{C}{D}$. Then the following statements hold:*

1. *$X = \{x_i\}_{i\in\mathbb{I}}$ is a standard frame for E.*
2. *$\mathbf{M}_{U,Y,X}$ (resp. $\mathbf{M}_{U,X,Y}$) is invertible and $\mathbf{M}^{-1}_{U,Y,X} = (W^{-1})^*\, \mathbf{M}^{-1}_{U,Y,Y}$ (resp. $\mathbf{M}^{-1}_{U,X,Y} = \mathbf{M}^{-1}_{U,Y,Y}(W^{-1})$).*

Proposition 2.3. *Let $X = \{x_i\}_{i\in\mathbb{I}}$ be a standard frame for Hilbert A-module E with upper bound D and $X^d = \{x_i^d\}_{i\in\mathbb{I}}$ be a dual frame of X with upper bound D'. Also, let U be a bounded operator on $\ell^2(A,\mathbb{I})$ such that $\|U - I\| < \frac{1}{\sqrt{DD'}}$. Then the multiplier $\mathbf{M}_{U,X,X^d}$ (resp. $\mathbf{M}_{U,X^d,X}$) is invertible.*

Proposition 2.4. *Let $Y = \{y_i\}_{i\in I}$ be a standard frame for Hilbert A-module E with bounds C and D and $\tilde{Y} = \{\tilde{y}_i\}_{i\in I}$ be its canonical dual frame.*

(1) *If $X = \{x_i\}_{i\in\mathbb{I}}$ is a standard Bessel sequence such that*

$$\sum_{i\in\mathbb{I}} \|x_i - \tilde{y}_i\|^2 < \frac{1}{4D}, \tag{3}$$

then $\mathbf{M}_{I,Y,X}$ is invertible.

(2) *Let $X = \{x_i\}_{i\in\mathbb{I}}$ be a standard Bessel sequence and (3) holds. Also, let U be a bounded operator on $\ell^2(A,\mathbb{I})$ with $\|U\| < 1$ and $\|U - I\| < \frac{\sqrt{C}}{2\sqrt{D}}$. Then $\mathbf{M}_{U,Y,X}$ is invertible.*

The next proposition give a necessary and sufficient condition for invertibility of generalized multipliers. First, we recall the following definition.

Definition 2.5. [4] A sequence $\{x_i\}_{i\in\mathbb{I}}$ is a modular Riesz basis for E if there exists an adjointable and invertible operator $U : \ell^2(A,\mathbb{I}) \to E$ such that $U\delta_i = x_i$ for each $i \in I$, where $\{\delta_i\}_{i\in\mathbb{I}}$ is the standard orthonormal basis of $\ell^2(A,\mathbb{I})$.

Proposition 2.6. *Let U be a bounded linear operator on $\ell^2(A,\mathbb{I})$ and $X = \{x_i\}_{i\in\mathbb{I}}$ and $Y = \{y_i\}_{i\in\mathbb{I}}$ be two modular Riesz bases for Hilbert A-module E. Then U is invertible if and only if the generalized multiplier $\mathbf{M}_{U,Y,X}$ is invertible.*

3. Representation of the inverse of a multiplier

As we have seen in Proposition 2.6, for modular Riesz bases $X = \{x_i\}_{i\in\mathbb{I}}$ and $Y = \{y_i\}_{i\in\mathbb{I}}$, if $U \in L\left(\ell^2(A,\mathbb{I})\right)$ is invertible then the generalized multiplier $\mathbf{M}_{U,Y,X}$ is automatically invertible and vise versa. Moreover,

$$\mathbf{M}^{-1}_{U,Y,X} = \mathbf{M}_{U^{-1},\tilde{X},\tilde{Y}}.$$

This result motivates us to generalize this idea for frames and even non-Bessel sequences. In more details, we will show that there are other invertible frame multipliers $\mathbf{M}_{U,Y,X}$ whose inverses can be represented as multipliers using the inverted symbol and suitable dual frames of X and Y.

Proposition 3.1. *Let $X = \{x_i\}_{i\in\mathbb{I}}$ and $Y = \{y_i\}_{i\in\mathbb{I}}$ be two standard frames for Hilbert A-module E and U be an invertible operator on $\ell^2(A,\mathbb{I})$. Assume that $\mathbf{M}_{U,Y,X}$ is invertible. Then the following hold.*

(1) *There exists a dual frame $Y^\dagger$ of Y such that for any dual frame X^d of X we have*

$$\mathbf{M}^{-1}_{U,Y,X} = \mathbf{M}_{U^{-1},X^d,Y^\dagger}.$$

(2) *There exists a dual frame $X^\dagger$ of X such that for any dual frame Y^d of Y we have*

$$\mathbf{M}^{-1}_{U,Y,X} = \mathbf{M}_{U^{-1},X^\dagger,Y^d}.$$

(3) *If $F = \{f_i\}_{i\in\mathbb{I}}$ is a Bessel sequence in E such that $\mathbf{M}^{-1}_{U,Y,X} = \mathbf{M}_{U^{-1},X^\dagger,F}$ (resp. $\mathbf{M}^{-1}_{U,Y,X} = \mathbf{M}_{U^{-1},F,Y^\dagger}$), then F must be a dual of Y (resp. X).*

Remark 3.2. It is worth mentioning that in Proposition 3.1, if $E = \ell^2(A,\mathbb{I})$, one can show that $X^\dagger$ and $Y^\dagger$ are unique.

The next proposition determines a class of multipliers which are invertible and whose inverses can be written as a multiplier. While in Proposition 3.1, it is assumed that the frame multiplier is invertible, in the following we investigate a sufficient condition for invertibility of frame multipliers.

Proposition 3.3. *Let $X = \{x_i\}_{i\in\mathbb{I}}$ and $Y = \{y_i\}_{i\in\mathbb{I}}$ be standard frames for Hilbert A-module E and $U \in L(\ell^2(A,\mathbb{I}))$ be invertible. if X is equivalent to $\{T_Y U\delta_i\}_{i\in\mathbb{I}}$ (resp. Y is equivalent to $\{T_X U^*\delta_i\}_{i\in\mathbb{I}}$), then $\mathbf{M}_{U,Y,X}$ is invertible and $\mathbf{M}^{-1}_{U,Y,X} = \mathbf{M}_{U^{-1},\tilde{X},Y^d}$ (resp.$\mathbf{M}^{-1}_{U,Y,X} = \mathbf{M}_{U^{-1},X^d,\tilde{Y}}$), for all dual frame Y^d(resp. X^d) of Y(resp. X). Furthermore, the proposition also holds in the opposite side if $E = \ell^2(A,\mathbb{I})$.*

Now, we extend the results of Proposition 3.1 to non-Bessel sequences. First, consider the following definition.

Definition 3.4. Let $X = \{x_i\}_{i\in\mathbb{I}}$ be a standard frame for Hilbert A-module E. The sequence $Y = \{y_i\}_{i\in\mathbb{I}}$ with elements from E is called

(i) an *analysis pseudo-dual* (in short, *a-pseudo-dual*) of X, if for every $x \in E$,

$$x = \sum_{i\in\mathbb{I}} \langle x, y_i\rangle x_i. \tag{4}$$

(ii) a *synthesis pseudo-dual* (in short, *s-pseudo-dual*) of X, if for every $x \in E$,

$$x = \sum_{i\in\mathbb{I}} \langle x, x_i\rangle y_i. \tag{5}$$

Example 3.5. Let $\{e_i\}_{i\in\mathbb{I}}$ be an orthonormal basis of E. Consider the standard frame $Y = \{e_1, e_1, e_1, e_2, e_2, e_2, e_3, e_3, e_3, \ldots\}$ for E and the sequence $X = \{e_1, e_1, -e_1, e_2, e_1, -e_1, e_3, e_1, -e_1, \ldots\}$ which is not a Bessel sequence of E. Then, it is easy to check that (4) holds, but (5) does not hold.

The next propositions determine how to represent the inverse of an invertible generalized multiplier for non-Bessel sequences and invertible symbol. In following, the synthesis operators are assumed to be closed operators. Moreover, $\{\delta_i\}_{i\in\mathbb{I}}$ is the standard orthonormal basis of $\ell^2(A,\mathbb{I})$.

Proposition 3.6. *Let $X = \{x_i\}_{i\in\mathbb{I}}$ be a standard frame for Hilbert A-module E, $Y = \{y_i\}_{i\in\mathbb{I}}$ be a sequence with elements from E and $U \in L(\ell^2(A,\mathbb{I}))$ be invertible such that $U\delta_i \in Dom(T_Y)$, for every $i \in \mathbb{I}$. Moreover, assume that $\mathbf{M}_{U,Y,X}$ is invertible. Then, there exists a dual frame $X^\dagger$ of X such that for any a-pseudo-duals Y^{ad} of Y, $\mathbf{M}^{-1}_{U,Y,X} = \mathbf{M}_{U^{-1},X^\dagger,Y^{ad}}$.*

Proposition 3.7. *Let $Y = \{y_i\}_{i\in\mathbb{I}}$ be a standard frame for Hilbert A-module E, $X = \{x_i\}_{i\in\mathbb{I}}$ be a sequence with elements from E and $U \in L(\ell^2(A,\mathbb{I}))$ be invertible such that $U^*\delta_i \in Dom(T_X)$, for every $i \in \mathbb{I}$. Moreover, assume that $\mathbf{M}_{U,Y,X}$ is invertible. Then, there exists a dual frame $Y^\dagger$ of Y such that for any s-pseudo-duals X^{sd} of X, $\mathbf{M}^{-1}_{U,Y,X} = \mathbf{M}_{U^{-1},X^{sd},Y^\dagger}$.*

References

1. Gh. Abbaspour Tabadkan and H. Hosseinnezhad, *Invertibility of Generalized Bessel multipliers in Hilbert C^*-modules*, Bull. Korean Math. Soc. **58** (2021), no. 2, 461–479.
2. P. Balazs, *Basic definition and properties of Bessel multipliers*, J. Math. Anal. Appl. **325** (2007), no. 1, 571–585.
3. M. Frank and D. Larson, *Frames in Hilbert C^*-modules and C^*-algebras*, J. Operator Theory. **48** (2002), 273–314.
4. A. Khosravi and B. Khosravi, *G-frames and modular Riesz bases*, Int. J. Wavelets Multiresolut. Inf. Process. **10** (2012), no. 02, p.1250013.
5. A. Khosravi and M. Mirzaee Azandaryani, *Bessel multipliers in Hilbert C^*-modules*, Banach J. Math. Anal. **9** (2015), no. 3, 153–163.
6. D. Stoeva and P. Balazs, *Invertibility of multipliers*, Appl. Comput. Harmon. Anal. **33** (2012), no. 2, 292–299.

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Pointwise eventually nonexpansive actions of amenable semigroups in dual Banach spaces and fixed points

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Abstract. A long-standing problem posed by Lau in a conference in Halifax in 1976 to characterize semitopological semigroups which have common fixed point property when acting on a nonempty weak* compact convex subset of a dual Banach space as weak* continuous and norm nonexpansive mappings. According to this question we prepared a fixed point property for pointwise eventually nonexpasnive actions which introduced by Kirk and Xu on weak* compact convex subsets.

Keywords: Pointwise eventually non-expansive action, weak* normal structure, Right reversible semi-topological semigroup, Left invariant mean.
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1. Introduction

Let S be a *semi-topological semigroup*, i.e., S is a semigroup equipped with a topology such that for every $a \in S$, the mappings $s \mapsto as$ and $s \mapsto sa$ from S into itself, are continuous. A semi-topological semigroup S is called right (left) reversible if any two closed left (right) ideals of S have non-void intersection. An *action* of S on a subset K of a Hausdorff topological space E is a mapping $\varphi : S \times K \to K$, denoted by $\varphi(s, x) = s \cdot x$ for $s \in S$ and $x \in K$, such that $(st) \cdot x = s(t \cdot x)$ for all $s, t \in S$, $x \in K$. A point $x \in K$ is called a *common fixed point* for φ if $s \cdot x = x$ for all $s \in S$. The action is said to be *separately continuous* if it is continuous in each of the variables when the other is fixed. We say that an action is *jointly continuous* if the mapping $(s, x) \mapsto s \cdot x$ from $S \times K$ to K is continuous, when $S \times K$ has the product topology. Let S be a semi-topological semigroup. Denote by $C(S)$ the C*-algebra consisting of all bounded continuous complex-valued functions on S with respect to the supremum norm and F be a closed subspace of $C(S)$. We say that F is left translation invariant if,

$$L_s F \subseteq F$$

for all $s \in S$, where for each $s \in S$, we consider the left translation operator $L_s : C(S) \to C(S)$ defined by

$$(L_s f)(t) = f(st)$$

for all $f \in C(S)$ and $t \in S$. Let F be a left translation invariant C*-subalgebra of $C(S)$ containing the constant functions. A linear functional μ on F is called a mean if $\|\mu\| = \mu(1) = 1$; a mean is called multiplicative if $\mu(fg) = \mu(f)\mu(g)$ for all $f, g \in F$. Recall that a mean μ on F is left invariant if $\mu(L_s f) = \mu(f)$ for all $s \in S$ and $f \in F$.

Let $LUC(S)$ be the subspace of $C(S)$ consisting of left uniformly continuous functions in $C(S)$; all functions $f \in C(S)$ for which the mapping $s \to L_s f$ from S into $C(S)$ is continuous when $C(S)$ has the supremum norm topology. A semigroup S is called left amenable if $LUC(S)$ has a left invariant mean. Left amenable semitopological semigroups include all commutative semigroups, all compact groups and all solvable groups. Let $LMC(S)$ be the C*-subalgebra of $C(S)$ consisting

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of all functions $f \in C(S)$ such that the map $s \to L_s f$ from S into $C(S)$ is $\sigma(C(S), \beta S)$-continuous, where $\beta S := \Delta(C(S))$ is the Stone-Čech compactification of S, i.e, the set of all multiplicative means on $C(S)$. In general, $LUC(S) \subseteq LMC(S) \subseteq C(S)$ We refer the reader to [3, 15], for more details.

Let E be a Banach space with dual E^* and let K be a nonempty weak* closed convex subset of E^*. A self-mapping $T : K \to K$ is called *non-expansive* if $\|T(x) - T(y)\| \leq \|x - y\|$, for all $x, y \in K$. We say that K has weak* *normal structure*, if every nontrivial weak* compact convex subset C of K with more than one point contains a nondiametral point; i.e., there exists a point $x \in C$ such that $sup\{\|x - y\| : y \in C\} < sup\{\|u - v\| : u, v \in C\} = diam(C)$. With a slight modification of the proof of Kirk [6], it is shown that every non-expansive mapping on a weak* compact convex set of a dual Banach space with weak* normal structure has a fixed point. Note that in general weak* normal structure assumption cannot be dropped. In fact, Lim [14] has prepared a fixed point free affine isometry on a nonvoid weak* compact convex set in l^1 (considered as the dual of c_0). A result of Karlovitz [5] shows that every non-expansive mapping on a weak* compact convex subset K of l^1 has a fixed point. These results can be further generalized for some semigroups of non-expansive self-mappings on K. In this connection consider the following fixed point property:

(F_*): Let S be a semi-topological semigroup. Whenever $\varphi : S \times K \to K$ is a non-expansive action on a nonempty weak* compact convex subset K of a dual Banach space E and the action is jointly continuous, where K is equipped with the weak* topology of E, then there is a common fixed point for S in K.

It is obvious to see that the property (F_*) implies that $LUC(S)$ has a left invariant mean [10]. Whether the converse is true is a long-standing open problem posed by A. T. -M. Lau during a conference in Halifax in 1976 [8]. The answer of this question is affirmative for commutative semigroups [9]. For noncommutative semigroups, it has been established that if S is left reversible or $LUC(S)$ has a left invariant mean, then S has fixed point property obtained from (F_*) by requiring K to be separable in the norm topology [11] or K has normal structure [12].

Kirk and Xu [7] introduced the concept of pointwise eventually non-expansive mappings as following:

Definition 1.1. Let K be a subset of a Banach space. A mapping $T : K \to K$ is said to be *pointwise eventually nonexpansive* if for each $x \in K$ there exists $N(x) \in \mathbb{N}$, such that for $n \geqslant N(x)$,

$$\|T^n(x) - T^n(y)\| \leq \|x - y\| \text{ for all } y \in K.$$

The following question raised by Kirk and Xu:

Question. Does a Banach space E have the fixed point property for pointwise eventually nonexpansive mappings if E has the fixed point property for nonexpansive mappings?

The first result in this direction was given by Butsan et al. [2, 4]. They proved a fixed point property for a pointwise eventually non-expansive mapping in a nearly uniformly convex Banach space. On the other hand, inspired by the above definition, the notion of pointwise eventually non-expansive action as a generalize of non-expansive action of semigroups introduced by Amini et al. in [1] as follows: Let K be a subset of a Banach space. The action is said to be pointwise eventually non-expansive if for each $x \in K$ there is a left ideal $J \subseteq S$ such that $\|s \cdot x - s \cdot y\| \leqslant \|x - y\|$ for all $s \in J$, $y \in K$.

Motivated by this question, in this paper we are interested to provide an affirmative answer to the question given by Kirk and Xu in dual Banach spaces. To do this we extend and improve some fixed point theorems of A.T.-M. Lau and Y. Zhang from nonexpansive actions to pointwise eventually nonexpansive actions. Indeed, we established fixed point properties for pointwise eventually nonexpansive actions of a right reversible semitopological semigroup on a weak* compact convex subset of a dual Banach space with weak* normal structure, assuming that certain subspaces of $LUS(S)$ or $LMC(S)$ has a left invariant mean.

2. Main results

We begin with a general fixed point theorem for pointwise eventually nonexpansive mappings in dual Banach spaces which gives a positive answer to the open problem raised by Kirk and Xu.

THEOREM 2.1. *Let K be a non-empty weak* compact convex subset of a dual Banach space such that K possesses weak* normal structure. Suppose that a right reversible semi-topological semigroup S acts on K such that the action is weak* separately continuous (i.e., separately continuous when K is equipped with the weak* topology) and pointwise eventually non-expansive. Let X be a closed linear subspace of $C(S)$ containing constants and invariant under translations. If X has a left invariant mean and each $f \in C(K)$ and $y \in K$, defines an element $\psi_y f \in X$ such that for all $s \in S$,*

$$\psi_y f(s) = f(s \cdot y).$$

Then there exists a common fixed point of S in K.

PROOF. It is easy to show that by Zorn's lemma, there exists a subset A_0 of K which is minimal with respect to being non-empty, weak* compact, convex and satisfying:
(P) There exists a collection ξ of weak* closed subsets of K such that $A_0 = \cap\xi$ and, for each $x \in A_0$, $L \in \xi$, there exists a left ideal $J \subseteq S$ such that $J \cdot x \subseteq L$.

We next show that A_0 contains a non-empty weak* closed S-invariant subset. For this end, let $x \in A_0$ be fixed and $\mathcal{C}'$ be the collection of all finite intersection of sets in ξ. For each $\alpha \in \mathcal{C}'$, $\alpha = B_1 \cap B_2 \cap ... \cap B_n$, where $B_i \in \xi$. For each i, choose a left ideal J_i such that $J_i(x) \subseteq B_i$ and let $a_\alpha \in \cap\left\{\overline{J_i} : i = 1, 2, ..., n\right\}$. In fact, this last intersection is non-empty by the right reversibility of S. Thus, we have $Sa_\alpha(x) \subseteq \alpha$. If z is a cluster point of net $\{a_\alpha : \alpha \in \mathcal{C}'\}$ where $\mathcal{C}'$ is directed by inclusion, then $z \in A_0$ and $\overline{S\,(z)}$ is a weak* closed S-invariant subset of A_0. Now a second application of Zorn'z lemma shows that there is a subset $M \subseteq \overline{S\,(z)}$ which is minimal with respect to being non-empty, weak* closed and S-invariant. We claim that M is S-preserved, i.e, $M = s \cdot M$ for all $s \in S$.

We follow an idea of Lau and Takahashi in Lemma 5.1 of [11] to prove that M would be S-preserved. Fix $y \in M$ and let m be a left invariant mean on X. Define a positive functional ϕ on $C(M)$ by $\phi(f) = m(\psi_y f)$ for all $f \in C(M)$. Since $\psi_y f \in X$ and $\|\phi\| = 1$, ϕ is well defined. For every $s \in S$ and $f \in C(M)$ we define

$${}_sf(x) = f(s \cdot x)$$

for every $x \in M$. Then ${}_sf : M \to \mathbb{C}$ is weak* continuous since the action is weak* separately continuous. Since m is a left invariant mean, it follows that

$$\phi(f) = \phi({}_sf).$$

According to Riesz representation theorem, we can assume that μ is the probability measure on M corresponding to ϕ. Thus $\mu(B) = \mu(s^{-1} \cdot B)$ for all $s \in S$ and for every Borel subset B of M (endowed with the weak* topology), where $s^{-1} \cdot B = \{x \in M : s \cdot x \in B\}$. The support of probability measure μ on M is defined by

$$supp(\mu) := \cap\{H;\ H \text{ is weak* closed subset of } M \text{ such that } \mu(H) = 1\ \}.$$

Put $K_0 = supp(\mu)$ and notice that $\mu(s^{-1} \cdot K_0) = \mu(K_0) = 1$. Therefore, $K_0 \subseteq s^{-1} \cdot K_0$ since $s^{-1} \cdot K_0$ is weak* closed. Similarly, $\mu(s \cdot K_0) = \mu(s^{-1}(s \cdot K_0)) = \mu(K_0) = 1$. Hence $K_0 \subseteq s \cdot K_0$. So, there exists a weak* compact subset K_0 of M such that $s \cdot K_0 = K_0$ for every $s \in S$ and by minimality of M, $K_0 = M$. Thus M would be S-preserved.
Let $F = \overline{co}^{w*}(M)$ and suppose that $r = \text{diam(F)} > 0$. Since K has weak* normal structure and F is a weak* compact, convex subset of K, there exists $u \in F$ such that

$$r_0 = \sup\{\|u - x\| : x \in F\} < r.$$

For each $\varepsilon > 0, B \in \xi$ if we put

$$K_{\varepsilon,B} = B \cap (\cap_{m\in M} B\,[m, r_0 + \varepsilon]).$$

Clearly, $K_{\varepsilon,B}$ is a non-empty, weak* compact and convex set. Define

$$K_0 = \cap \{K_{\varepsilon,B} : \varepsilon > 0, B \in \xi\}.$$

We show that K_0 has property (P). Take, $x \in K_0, \varepsilon > 0$ and $B \in \xi$. By property (P) there exists a left ideal $I \subseteq S$ such that $I \cdot x \subseteq B$. Since, the action is pointwise eventually nonexpansive, there exists $c \in S$ with $\|(tc) \cdot m - (tc) \cdot x\| \leq \|m - x\|$ for all $t \in S$. Take $t_0 \in \overline{I} \cap \overline{Sc}$. Put $J := St_0$. It is easily seen that, for each $s \in S$, $(st_0) \cdot x \in B$, and by weak* lower semicontinuity of norm $\|(st_0) \cdot m - (st_0) \cdot x\| \leqslant \|m - x\| \leqslant r_0 + \varepsilon$. Therefore, $st_0 \cdot x \in B\,[(st_0) \cdot m, r_0 + \varepsilon]$ for each $s \in S$. But $(st_0) \cdot M = M$ for any $s \in S$, hence for the left ideal $J = St_0$ we see that $J \cdot x \subseteq K_{\varepsilon,B}$ and (P) holds, contradicting the minimality of A_0. This completes the proof. □

The following example shows that, in the above theorem, the existence of a left invariant mean, can not be entirely dropped:

EXAMPLE 2.2. Let S be a finite left zero semigroup; i.e., a finite semigroup S with the operation $st = s$ for all $s, t \in S$. Then S with the discrete topology is a right reversible semi-topological semigroup. Set $X = LUC(S)$. Consider the action of S on the non-empty compact convex subset $K := \Delta(LUC(S))$, the set of all multiplicative means on $LUC(S)$, of the Banach space $LUC(S)$ given by $s.\mu := {L_s}^*\mu$ for all $s \in S$ and $\mu \in \Delta(LUC(S))$, where ${L_s}^*\mu(f) = \mu(L_S f)$ for all $f \in LUC(S)$. Then, it is easy to verify that the action of S on K is even non-expansive, however does not have common fixed point.

According to Theorem 2.1, we obtain common fixed point theorems for semi-topological semigroup S acting on a weak* compact convex set of a dual Banach space, when X being $LUC(S)$ and $LMC(S)$.

THEOREM 2.3. *Let S be a right reversible semi-topological semigroup. If $LUC(S)$ has a left invariant mean then the following fixed point property holds.*

(L∗): Whenever S acts on a non-empty weak compact convex subset K of the dual space such that K possesses weak* normal structure and the action is (jointly) weak* continuous and pointwise eventually nonexpansive, then there exists a common fixed point for S in K.*

PROOF. If the action is jointly continuous when K is endowed with the weak* topology by applying [**11**, Lemma 5.1], $\psi_y f \in LUC(S)$ for all $f \in C(K)$ and $y \in K$. Since the action is non-expansive and $LUC(S)$ has a left invariant mean, the result follows from Theorem 2.1. □

REMARK 2.4. We point out that, the above theorem is related to the open problem raised by Lau [**8**]. In [**12**, Proposition 3.4] Lau partially answered to this question with an additional assumption on K. Indeed, the above theorem extends this answer for pointwise eventually non-expansive actions.

In the following we prepare a fixed point theorem for dual Banach space in which $LMC(S)$ has a left invariant mean, although there has not been discovered any fixed point results in dual Banach spaces when $LMC(S)$ has a left invariant mean.

THEOREM 2.5. *Let S be a right reversible semi-topological semigroup. If $LMC(S)$ has a left invariant mean then the following fixed point property holds.*

(M∗): Whenever S acts on a non-empty weak compact convex subset K of the dual space such that K possesses weak* normal structure and the action is separately continuous and pointwise eventually non-expansive, then there exists a common fixed point for S in K.*

Proof. By assumption and [15, Theorem 3], $\psi_y f \in LMC(S)$ for all $f \in C(K)$ and $y \in K$. Therefore, if $LMC(S)$ has a left invariant mean, we can apply Theorem 2.1 with $X = LMC(S)$ to obtain a common fixed point of S in K. □

References

1. A. Amini, A. Medghalchi, F. Naderi, *Pointwise eventually non-expansiveaction of semi-topological semigroups and fixed points*, J. Math. Anal. Appl., 437 (2016), 1176–1183.
2. A. Aminpour, A. Dianatifar, R. Nasr-Isfahani, *Asymptotically Non-expansive Actions of Strongly Amenable Semigroups and Fixed Points*, J. Math. Anal. Appl., 461 (2018), 364–377.
3. J. F. Berglund, H. D. Junghenn, P. Milnes, *Analysis on Semigroups*, John Wiley and Sons, New York, 1988.
4. T. Bustan, S. Dhompongsa, W. Takahashi, *A fixed point theorem for pointwise eventually nonexpansive mappings in nearly uniformly convex Banach spaces*, Nonlinear Anal., 74 (2011), 16941701.
5. L. A. Karlovitz, *On nonexpansive mappings*, Proc. Am. Math. Soc., 55 (1976), 321–325.
6. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Math. Monthly., 72 (1965), 1004–1006.
7. W. A. Kirk, H. K. Xu, *Asymptotic powintwise contraction*, Nonlinear Anal., 69 (2008), 4706–4712.
8. A. T. -M. Lau, *Amenability and fixed point property for semigroup of nonexpansive mappings*, in: M.A. Thera, J.B. Baillon (Eds.), Fixed Point Theory and Applications, in: Pitman Res. Notes Math. Ser., vol. 252, Longman Sci. Tech., Harlow, 303–313 (1991).
9. A. T. -M. Lau, *Normal structure and common fixed point properties for semigroups of nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl., (2010), Art. ID 580956, 14 pp.
10. A.T. -M. Lau, W. Takahashi, *Fixed point properties for semigroup of nonexpansive mapping on Frechet spaces*, Nonlinear Anal., 70 (1995), 3837–3841.
11. A. T. -M. Lau, W. Takahashi, *Invariant means and fixed point properties for non-expansive representations of topological semigroups*, Topol. Methods Nonlinear Anal., 5 (1995), 39–57.
12. A. T. -M. Lau, Y. Zhang, *Fixed point properties of semigroups of nonlinear mappings and amenability*, J. Funct. Anal. ,263 (2012), 2949–2977.
13. A. T. -M. Lau, Y. Zhang, *Fixed point properties of semigroups of non-expansive mappings*, J. Funct. Anal., 254 (2008), 2534–2554.
14. T. C. Lim, *Asymptotic centers and nonexpansive mappings in conjugate Banach spaces*, Pacific J. Math., 90 (1980), 135–143.
15. T. Mitchell, *Topological semigroups and fixed points*, Illinois J. Math., 14 (1970), 630–641.

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Left centralizers on the $\theta-$Lau product of Banach algebras

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Abstract. Let A and B be Banach algebras and $\sigma(B) \neq \emptyset$. Let $\theta, \phi \in \sigma(B)$ and $\mathrm{LM}(A \times_\theta^\phi B)$ be the set of all linear mappings $T : A \times R \to A \times B$ satisfying $T((a, b) \cdot_\theta (x, y)) = T(a, b) \cdot_\phi (x, y)$ for all $a, x \in A$ and $b, y \in B$. In this paper, we prove that if A is a Banach algebra without identity and $T \in \mathrm{LM}(A \times_\theta^\phi B) \neq 0$, then $\theta = \phi$. We then investigate the concept of skew commuting for elements of $\mathrm{LM}(A \times_\theta^\phi B)$. We also apply some results to group algebras.

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1. Introduction

Let A and B be Banach algebras and $\theta \in \sigma(B)$, the spectrum of B. Let us recall from [4] that the $\theta-$Lau product A and B is the direct product $A \times B$ together with the component wise addition and the multiplication

$$(a, b) \cdot_\theta (x, y) = (ax + \theta(y)a + \theta(b)x, by).$$

Note that if we permit $\theta = 0$, the $\theta-$Lau product $A \times_\theta B$ is the usual direct product of Banach algebras. Hence we disregard the possibility that $\theta = 0$.

The $\theta-$Lau products $A \times_\theta B$ were first introduced by Lau [2], for Banach algebras that are pre-duals of von Neumann algebras, and for which the identity of the dual is a multiplicative linear functional. Sanjani Monfared [4] extended this product to arbitrary Banach algebras A and B. In fact, he introduced a strongly splitting Banach algebra extension of B by A which present many properties that are not shared by arbitrary strongly splitting extension. He also gave characterizations of bounded approximate identity, spectrum, topological center and minimal idempotents of these products.

Let $T : A \to A$ be a linear map. Then T is called *skew commutative* if for every $a, x \in A$

$$\langle T(a), a\rangle = 0,$$

where for each $a, x \in A$

$$\langle a, x\rangle = ax + xa.$$

Also, $T : A \to A$ is called *Left centralizer* if for every $a, x \in A$

$$T(ax) = T(a)x.$$

For results concerning left centralizer on rings and algebras we refer the reader to [5, 6, 7, 8].

In this paper, let $\theta, \phi \in \sigma(B)$ and $\mathrm{LM}(A \times_\theta^\phi B)$ be the set of all linear mappings $T : A \times B \to A \times B$ satisfying

$$T((a, b) \cdot_\theta (x, y)) = T(a, b) \cdot_\phi (x, y)$$

for all $a, x \in A$ and $b, y \in B$. We show that if A is a Banach algebra without identity and $T \in \mathrm{LM}(A \times_\theta^\phi B) \neq 0$, then $\theta = \phi$. We also give some related results. Finally, we investigate the concept of skew commuting for elements of $\mathrm{LM}(A \times_\theta^\phi R)$.

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2. Main results

In the following, let A be a Banach algebra with a right identity and the center of A

$$Z(A) = \{a \in A : ax = xa \quad \text{for all} \quad x \in A\}$$

and B be arbitrary Banach algebra. Let also θ and ϕ be elements of the spectrum of B.

Let G denote a locally compact group with a fixed left Haar measure λ. Let $L^\infty(G)$ denote the usual Lebesgue space as defined in [1] equipped with the essential supremum norm $\|.\|_\infty$. We denote by $L_0^\infty(G)$ the subspace of $L^\infty(G)$ consisting of all functions $f \in L^\infty(G)$ vanishing at infinity: that is, for every positive number ε, there is a compact subset K of G for which $\|f\chi_{G\setminus K}\|_\infty < \varepsilon$, where $\chi_{G\setminus K}$ denotes the characteristic function of $G \setminus K$ on G.

It is well-known from [3] that the dual of $L_0^\infty(G)$, represented by $L_0^\infty(G)^*$, is a Banach algebra with the first Arens product "·" defined by

$$\langle F \cdot H, f\rangle = \langle F, Hf\rangle,$$

where

$$\langle Hf, \phi\rangle = \langle H, f\phi\rangle \quad \text{and} \quad \langle f\phi, \psi\rangle = \langle f, \phi * \psi\rangle$$

for all $F, H \in L_0^\infty(G)^*$, $f \in L_0^\infty(G)$ and $\phi, \psi \in L^1(G)$.

Theorem 2.1. *Let A be a Banach algebra without identity and $T \in LM(A \times_\theta^\phi B) \neq 0$. Then $\theta = \phi$.*

Corollary 2.2. *Let $\theta \neq \phi$ and $LM(L_0^\infty(G)^* \times_\theta^\phi B) \neq 0$. Then G is discrete.*

Theorem 2.3. *Let $\eta \in \sigma(B)$ and A be a Banach without identity. If T is an element of $LM(A \times_\theta^\phi B)$ satisfying $T((a, b) \cdot_\theta (x, y)) - (a, b) \cdot_\eta (x, y) \in Z(A) \times Z(B)$ for all $a, x \in A$ and $b, y \in B$, then $\theta = \phi = \eta$.*

Let $\eta_1, \eta_2 \in \sigma(B)$. We define for every $a, x \in A$ and $b, y \in B$

$$[(a, b), (x, y)]_{\eta_1,\eta_2} := (a, b) \cdot_{\eta_1} (x, y) - (x, y) \cdot_{\eta_2} (a, b).$$

Theorem 2.4. *Let $\eta_i, \rho_i \in \sigma(B)$ for $i = 1, 2$ and A be a Banach algebra without identity. If T is an element of $LM(A \times_\theta^\phi B)$ satisfying $T([(a, b), (x, y)]_{\eta_1,\eta_2}) = [(a, b), (x, y)]_{\rho_1,\rho_2}$ for all $a, x \in A$ and $b, y \in B$, then $\theta = \phi$.*

Corollary 2.5. *Let $A = L_0^\infty(G)^*$ in Theorem 2.4. If $\theta \neq \phi$, then G is discrete*

Theorem 2.6. *Let $\eta \in \sigma(R)$ and $T \in LM(A \times_\theta^\phi B)$. If for every $a, x \in A$ and $b, y \in B$*

$$T((a, b) \cdot_\theta (x, y)) = \pm(a, b) \cdot_\eta (x, y),$$

then $\theta = \phi = \eta$.

Corollary 2.7. *Let $\eta \in \sigma(B)$ and T be an element of $LM(L_0^\infty(G)^* \times_\theta^\phi B)$ satisfying*

$$T((a, b) \cdot_\theta (x, y)) = \pm(a, b) \cdot_\eta (x, y),$$

for all $a, x \in L_0^\infty(G)^*$ and $b, y \in B$. If $\theta \neq \phi$, then G is discrete

Theorem 2.8. *Let $\eta \in \sigma(B)$. If T is an element of $LM(A \times_\theta^\phi B)$ satisfying*

$$T((a, b) \cdot_\theta (x, y)) = (x, y) \cdot_\eta (a, b)$$

for all $a, x \in A$ and $b, y \in B$, then $\theta = \phi = \eta$ and A is unital.

Corollary 2.9. *Let $\eta \in \sigma(B)$ and T be an element of $LM(L_0^\infty(G)^* \times_\theta^\phi B)$ satisfying*

$$T((a, b) \cdot_\theta (x, y)) = (x, y) \cdot_\eta (a, b)$$

for all $a, x \in L_0^\infty(G)^*$ and $b, y \in B$. If $\theta \neq \phi$, then G is discrete

A mapping $T : A \times R \to A \times B$ is called $(\eta_1, \eta_2)-$*skew commuting* if for every $a \in A$ and $b \in B$

$$\langle T(a, b), (a, b)\rangle_{\eta_1,\eta_2} := T(a, b) \cdot_{\eta_1} (a, b) + (a, b) \cdot_{\eta_2} T(a, b) = 0.$$

Theorem 2.10. *Let* $\eta_1, \eta_2 \in \sigma(B)$, $T \in LM(A \times_\theta^\phi B)$ *and* A *be an algebra without identity. If* T *is an* $(\eta_1, \eta_2)-$*skew commuting, then* $\eta_1 T(b) = \eta_2 T(b)$ *for all* $b \in B$, .

References

1. E. Hewitt and K. Ross, *Abstract Harmonic Analysis I*, Springer-Verlag, New-York, 1970.
2. A. T. Lau, *Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups*, Fund. Math., **118** (1983), 161–175.
3. A. T. Lau and J. Pym, *Concerning the second dual of the group algebra of a locally compact group*, J. London Math. Soc., (41) (1990), 445-460.
4. M. Sangani Monfared, On certain products of Banach algebras with applications to harmonic analysis, Studia Math., **178** (3) (2007), 277–294.
5. J. Vukman, *An identity related to centralizer in semiprime rings*, Comment. Math. Univ. Carolin., **40** (1999), 447–456.
6. J. Vukman, *Centralizer on semiprime rings*, Comment. Math. Univ. Carolin., **42** (2001), 237–245.
7. J. Vukman and I. Kosi-Ulbl, *Centralizers on rings and algebras*, Bull. Austral. Mth. Soc., **71** (2005), 225–234.
8. B. Zalar, *On centralizer of semiprime rings*, Comment. Math. Univ. Carolin., **32** (1991), 609–614.

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ε-Orthogonality preserving of Hilbert C^*-modules

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Abstract. Let $\mathcal{A}$ be a standard C^*-algebra. We will study the continuity of ε-orthogonality preserving mappings between Hilbert $\mathcal{A}$-modules. Moreover, we will show that a local mapping between Hilbert $\mathcal{A}$-modules is $\mathcal{A}$-linear. Furthermore, we will prove that for two nonzero $\mathcal{A}$-linear mappings $T, S : E \longrightarrow F$, between Hilbert $\mathcal{A}$-modules, satisfying ε-orthogonality preserving property, there exists $\gamma \in \mathbb{C}$,

$$\|\langle T(x), S(y)\rangle - \gamma\langle x, y\rangle\| \leq \varepsilon\|T\|\|S\|\|x\|\|y\|, \quad x, y \in E.$$

Our results generalize the known ones in the context of Hilbert spaces.
Keywords: ε-orthogonality preserving mappings, local mappings, Hilbert C^*-modules.
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1. Introduction

Let $\big(H, (.,.)\big)$ be an inner product space, two elements $x, y \in H$ are said to be orthogonal, and is denoted by $x \perp y$, if $(x, y) = 0$. For two inner product spaces H and K, a mapping $T : H \longrightarrow K$ is called orthogonality preserving, OP in short, if $x \perp y \;\Rightarrow\; T(x) \perp T(y),\ (x, y \in H)$. By [2], a mapping T is orthogonality preserving if and only if it is scalar multiple of an isometry, that is $T = \gamma U$, where $\gamma \geq 0$ and U is an isometry. Two mappings $T, S : H \longrightarrow K$ are called orthogonality preserving, if $x \perp y \;\Rightarrow\; T(x) \perp S(y),\ (x, y \in H)$, this notion has been charactrized in [3].

A generalization of the orthogonality notion, namely approximately orthogonality preserving or ε-orthogonality preserving mappings, for a given $\varepsilon \in [0, 1)$, between inner product spaces was considered in [2]. Recall that for $\varepsilon \in [0, 1)$ two vectors $x, y \in H$ are approximately orthogonal or ε-orthogonal, denoted by $x \perp^{\varepsilon} y$, if $|(x, y)| \leq \varepsilon\|x\|\|y\|$.

For a given $\varepsilon \in [0, 1)$, a mapping $T : H \longrightarrow K$ is called ε-orthogonality preserving, ε-OP in short, if

$$x \perp y \;\Rightarrow\; T(x) \perp^{\varepsilon} T(y), \quad (x, y \in H).$$

Two linear mappings $T, S : H \longrightarrow K$ between inner product spaces are called ε-orthogonality preserving if

$$x \perp y \Rightarrow T(x) \perp^{\varepsilon} S(y), \quad (x, y \in H).$$

ε-orthogonality preserving property for two mappings T, S between inner product spaces has been charactrized in [4]. In this servey, in fact, we generalize this charactrization.

Now, we start with some prerequisites.

Recall that a (left) Hilbert C^*-module E over a C^*-algebra A is a left A-module equipped with an A-vallued inner product " ${}_A\langle .,. \rangle : E \times E \longrightarrow A$ such that the following conditions hold for all $x, y \in E$ and all $a \in A$ and $\alpha, \beta \in \mathbb{C}$:

$$\begin{aligned}
&(i) \quad {}_A\langle \alpha x + \beta y, z\rangle = \alpha\ {}_A\langle x, z\rangle + \beta\ {}_A\langle y, z\rangle,\\
&(ii) \quad {}_A\langle ax, y\rangle = a\ {}_A\langle x, y\rangle,\\
&(iii) \quad {}_A\langle x, y\rangle^* = {}_A\langle y, x\rangle,\\
&(iv) \quad {}_A\langle x, x\rangle \geq 0,\ and\ {}_A\langle x, x\rangle = 0\ if\ and\ only\ if\ x = 0.
\end{aligned}$$

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Moreover, E is a Banach space equipped with the norm $\|x\| = \|_A\langle x, x\rangle\|^{\frac{1}{2}}$. In this case we call E a Hilbet A-module. A comlex linear mapping $T : E \longrightarrow F$ between two Hilbert A-modules is called an A-linear if $T(ax) = aT(x)$ for all $a \in A$ and $x \in E$.

Hilbert C^*-modules are used as powerful tool in operator algebra theory. They serve as a major class of examples in operator C^*-module theory. See [7] for a general intoduction to the theory of Hilbet C^*-module.

Let A be a C^*-algebra. Two elements x, y in an inner product A-module $(E, \langle ., .\rangle)$ are said to be orthogonal if $\langle x, y\rangle = 0$ and, for a given $\varepsilon \in [0, 1)$, they are approximately orthogonal or ε-orthogonal if $\|\langle x, y\rangle\| \leq \varepsilon\|x\|\|y\|$. A mapping $T : E \longrightarrow F$, where E and F are inner product A-modules, is called ε-orthogonality preserving if $\langle x, y\rangle = 0$ (where $x, y \in E$) implies $\|\langle Tx, Ty\rangle\| \leq \varepsilon\|Tx\|\|Ty\|$.

Throughout, $\mathcal{K}(H)$ and $\mathcal{B}(H)$ denote the C^*-algebras of all compact operators and all bounded operators on a Hilbert spaces H, respectively. Recall that $\mathcal{A}$ is a standard C^*-algebra on a Hilbert space H if $\mathcal{K}(H) \subseteq \mathcal{A} \subseteq \mathcal{B}(H)$.

It is natural to explore the ε-orthogonality preserving mappings between inner product C^*-modules, for $\varepsilon \in [0, 1)$. D. Iliević and A. Turnek in [6] consider ε-orthogonality preserving property for a "single" mapping. In this survey, we consider ε-orthogonality preserving property for "two mappings" in the framework of Hilbert $\mathcal{A}$-modules.

In [5], Frank, Moslehian and Zamani proved that if two nonzero local mappings $T, S : E \longrightarrow F$ between Hilbert $\mathcal{A}$-modules are orthogonal preserving, then there exists $\gamma \in \mathbb{C}$ such that

$$\langle T(x), S(y)\rangle = \gamma\langle x, y\rangle, \quad x, y \in E.$$

It is interesting to ask whether it is possible to consider ε-orthogonality preserving property for two these mappings. In this servey, we study ε-orthogonality preserving property for a pair of mappings in the setting of Hilbert C^*-modules over standard C^*-algebra $\mathcal{A}$. Then we give the estimate of $\|\langle T(x), S(y)\rangle - \gamma\langle x, y\rangle\|$ for two nonzero local ε-orthogonality preserving mappings $T, S : E \longrightarrow F$ when E and F are Hilbert $\mathcal{A}$-modules, where $\gamma \in \mathbb{C}$.

We recall that, for a C^*-algebra A, a complex linear mapping $T : E \longrightarrow F$ between inner product A-modules E and F, is called local if

$$aT(x) = 0 \quad \text{whenever} \quad ax = 0, \quad a \in A; x \in E.$$

As example, linear differential mappings are local mapping. Note that every A-linear mapping is local, but the converse is not true, in general.

Suppose that E and F are Hilbert A-modules. Let $\mathcal{L}(E, F)$ to be the set of all mappings $T : E \longrightarrow F$ for which there is a mapping $T^* : F \longrightarrow E$ such that for all $x \in E$ and $y \in F$,

$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

By [7], $\mathcal{L}(E, F)$ is called the set of all adjointable mappings from E to F. Every element of $\mathcal{L}(E, F)$ is a bounded A-linear, and in general, a bounded A-linear mapping may fail to possess an adjoint. But each bounded $\mathcal{K}(H)$-linear mapping on $\mathcal{K}(H)$-modules is essentially adjointable [1].

In the following we give some preliminaries about minimal projections in C^*-algebras and their role in our work.

Let $\xi, \eta \in H$ be elements of a Hilbert space $\big(H, (., .)\big)$, the rank one operator defined by $(\xi \otimes \eta)\zeta = (\zeta, \eta)\xi$, where $\zeta \in H$. Observe that $\xi \otimes \xi$ is a rank one projection by the unit vector ξ. Let T be an arbitrary bounded operator on $\big(H, (., .)\big)$, then

$$(\xi \otimes \xi)T(\xi \otimes \xi) = (T\xi, \xi)\xi \otimes \xi.$$

Recall that a projection (i.e., a self-adjoint idempotent.) e in $\mathcal{A}$ is called minimal if $e\mathcal{A}e = \mathbb{C}e$. Hence, $\xi \otimes \xi$ is a minimal projection.

By citeIT, let $(E, \langle ., .\rangle)$ be an inner product(respectively Hilbert) $\mathcal{A}$-module, and for a unit vector $\xi \in H$, let $e = \xi \otimes \xi$ be any minimal projection. Then

$$E_e = \{ex : x \in E\},$$

is a complex inner product (respectively Hilbert) space contained in E with respect to the inner product $(x, y) = tr(\langle x, y\rangle)$, $x, y \in E_e$.

Let $x = eu, y = ev$ such that $u, v \in E$,

$$\langle x, y\rangle = e\langle u, v\rangle e = (\xi \otimes \xi)\langle u, v\rangle(\xi \otimes \xi) = \big(\langle u, v\rangle\xi, \xi\big)(\xi \otimes \xi),$$

by $\mathrm{tr}(\langle x, y\rangle) = \big(\langle u, v\rangle\xi, \xi\big)$, thus

$$\langle x, y\rangle = (x, y)e.$$

Authors, in [6], also showed that:

1) two elements $x, y \in E_e$ are orthogonal in $\big(E_e, (\cdot, \cdot)\big)$ if and only if they are orthogonal in $\big(E, \langle ., .\rangle\big)$,

2) if $x \in E_e$, then $\|x\|_{E_e} = \|x\|_E$, where the norm $\|.\|_{E_e}$ comes from the inner product $(\cdot, \cdot)$,

3) if $T : E \to F$ between Hilbert $\mathcal{A}$-modules E and F is an $\mathcal{A}$-linear OP (respectively ε-OP) mapping, then $T_e = T\mid_{E_e}: E_e \to F_e$ is a linear OP (respectively ε-OP) mapping.

The following lemme is important, which is used to prove the main theorem.

Lemma 1.1. *Let $L \in \mathcal{B}(H)$, then*

$$\|L\| = sup\left\{\|eLf\| \ : \ e, f \ \textit{are rank one projections}\right\}.$$

2. Main results

As mentioned in previous section, $\mathcal{A}$ is a standard C^*-algebra on a Hilbert space H if $\mathcal{K}(H) \subseteq \mathcal{A} \subseteq \mathcal{B}(H)$. To achieve our main result, Theorem 2.6, we give some results. First we prove the continuity of ε-orthogonality-preserving nonzero pair of $\mathcal{A}$-linear mappings between Hilbert $\mathcal{A}$-modules.

Lemma 2.1. [4, Lemma 3.1]

For a given $\varepsilon \in [0, 1)$, ε-orthogonality preserving property for two nonzero linear mappings f and g between inner product spaces X and Y, with the same inner product $(., .)$, is equivalent to

$$\left|(f(x), g(y)) - \frac{(f(y), g(x))}{\|y\|^2}(x, y)\right| \leq \varepsilon\left\|f(x) - \frac{(x, y)}{\|y\|^2}f(y)\right\|\|g(y)\|,$$

for $x, y \in X, y \neq 0$.

As an immediate generalization, we give the next result in setting of inner product $\mathcal{A}$-modules. Let $\varepsilon \in [0, 1)$, and let $T, S : E \longrightarrow F$ be pair of nonzero $\mathcal{A}$-linear ε-orthogonality preserving mappings between inner product $\mathcal{A}$-modules E and F. Then $T_e, S_e : E_e \longrightarrow F_e$ are pair of two nonzero linear ε-orthogonality preserving mappings between inner product spaces E_e and F_e. Where e is a minimal projection in $\mathcal{A}$.

Proposition 2.2. *Let $\varepsilon \in [0, 1)$, and let $T, S : E \longrightarrow F$ be pair of nonzero $\mathcal{A}$-linear ε-orthogonality preserving mappings between inner product $\mathcal{A}$-modules E and F. Then for every minimal projection $e \in \mathcal{A}$, and for all $x, y \in E_e$,*

$$\|\langle y, y\rangle\langle T(x), S(y)\rangle - \langle x, y\rangle\langle T(y), S(y)\rangle\| \leq \varepsilon\|\langle y, y\rangle T(x) - \langle x, y\rangle T(y)\|\|S(y)\|.$$

Consequently, T_e, S_e are a pair of linear ε-orthogonality preserving mappings.

The following proposition proves the continuity of two $\mathcal{A}$-linear mappings $T, S : E \longrightarrow F$ between Hilbert $\mathcal{A}$-modules.

Proposition 2.3. *Let $\varepsilon \in [0, 1)$, and let $\mathcal{A}$ has an approximate unit, which contains finite combinations of minimal projections in $\mathcal{A}$, and let E and F be Hilbert $\mathcal{A}$-modules, and e be an arbitrary minimal projection in $\mathcal{A}$. Suppose that $T, S : E \longrightarrow F$ are two nonzero surjective $\mathcal{A}$-linear ε-orthogonality preserving mappings. Then T and S are continuous.*

In the following, we give a stability result in this context. Note that, in Proposition 2.4 and Theorem 2.6, H is a complex Hilbert space.

PROPOSITION 2.4. *Let $\varepsilon \in [0,1)$. Let $\mathcal{A} = \mathcal{K}(H)$, and let E, F be Hilbert $\mathcal{A}$-modules. Suppose that $T, S : E \longrightarrow F$ are two nonzero surjective $\mathcal{A}$-linear ε-orthogonality preserving mappings. Then there exists $\gamma \in \mathbb{C}$ such that*

$$\|\langle T(x), S(y)\rangle - \gamma\langle x, y\rangle\| \leq \varepsilon\|T\|\|S\|\|x\|\|y\|, \quad x, y \in E.$$

To prove, we use this fact that each bounded $\mathcal{K}(H)$-linear mapping on $\mathcal{K}(H)$-modules is adjointable.

As mentioned in previous section, in general, for any C^*-algebra A, a local mapping on Hilbert A-modules is not A-linear. In the following, for standard C^*-algebra $\mathcal{A}$, we will show that a local mapping between Hilbert $\mathcal{A}$-modules is $\mathcal{A}$-linear.

PROPOSITION 2.5. *Let $T : E \longrightarrow F$ be a local mapping between Hilbert $\mathcal{A}$-modules E and F, then $T : E \longrightarrow F$ is an $\mathcal{A}$-linear mapping.*

In Proposition 2.3, the boundedness of two nonzero $\mathcal{A}$-linear mappings $T, S : E \longrightarrow F$ over Hilbert $\mathcal{A}$-modules is proved. Now, we are in a position to give the main result.

THEOREM 2.6. *Let $\varepsilon \in [0,1)$, and let E and F be Hilbert $\mathcal{A}$-modules. Let $T, S : E \longrightarrow F$ be two nonzero surjective local ε-orthogonality preserving mappings. Then there exists $\gamma \in \mathbb{C}$ such that*

$$\|\langle T(x), S(y)\rangle - \gamma\langle x, y\rangle\| \leq \varepsilon\|T\|\|S\|\|x\|\|y\|, \quad x, y \in E.$$

References

1. D. Baki, B. Gulja, Hilbert C*-modules over C*-algebras of compact operators, Acta Sci. Math. (Szeged) 68 (2002) 249269.
2. J. Chmieliński, Linear mappings approximately preserving orthogonality, J. Math. Annal. Appl. 304 (2005) 158-169.
3. J. Chmieliński, Orthogonality equation with two unknown functions, Aequationes Math. 90 (2016) 11-23.
4. J. Chmieliński, R. ukasik and P. Wjcik, On the stability of the orthogonality equation and the orthogonality-preserving property with two unknown functions, Banach, J. Math Anal. 10 (2016) 828-847.
5. M. Frank, M. S. Moslehian and A. Zamani, Orthogonality preserving property for pairs of operators on Hilbert C^*-modules, arXiv:1711.04724v2 [Math.OA] 2019.
6. D. Iliević and A. Turnek, Approximately orthogonality preserving mappings on C*-modules. 341 (2008) 298-308.
7. E.C. Lance, *Hilbert C^*-modules*, London Math. Soc. Lecture Note Ser., vol. 210, Cambridge Univ. Press, Cambridge 1995.
8. G. J. Murphy, C^*-algebras and operator theory, Academic Press Inc. Boston, MA, 1990.

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Woven and P-woven frames

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Abstract. In this manuscript, for a Hilbert space H we study woven frames and P-woven frames and give some results by the effect of reordering of the elements of weavings. In particular, we consider full spark frames and perturbation concepts.
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1. Introduction

The motivation of this paper is a problem in woven frames. In this paper, we consider a frame $\{\phi_i\}_{i\in\mathcal{I}}$, the permutation functions π on $\mathcal{I}$ and the famalies $\{\phi_i\}_{i\in\sigma} \cup \{\phi_{\pi(i)}\}_{i\in\sigma^c}$ for $\sigma \subset \mathcal{I}$. In fact, we study the conditions which the frames $\{\phi_i\}_{i\in\mathcal{I}}$ and $\{\phi_{\pi(i)}\}_{i\in\mathcal{I}}$ are woven. In this section at first, we mention the definition of frames and some basic properties of them in a separable Hilbert space H. After that we review the definition of woven frames. For more details, the reader is referred to [1, 2, 3, 4]. Throughout the paper, H is a separable Hilbert space and $\mathcal{I}$ is a countable index set.

A family $\{\phi_i\}_{i\in\mathcal{I}}$ in H is a frame for H if there exists constants $0 < A \leq B < \infty$ such that for all $x \in H$,

$$A\|x\|^2 \leq \sum_{i\in\mathcal{I}} |\langle x, \phi_i\rangle|^2 \leq B\|x\|^2,$$

where A and B are called lower and upper frame bounds, respectively. If only B is assumed to exist, then $\{\phi_i\}_{i\in\mathcal{I}}$ is called a Bessel sequence. If $A = B$, then $\{\phi_i\}_{i\in\mathcal{I}}$ is called a tight frame, also it is called Parseval if $A = B = 1$. A frame $\{\phi_i\}_{i\in\mathcal{I}}$ is called exact when it ceases to be a frame when an arbitrary element is removed. Corresponding to each Bessel sequence $\{\phi_i\}_{i\in\mathcal{I}}$ in H, one can consider some important operators. The synthesis operator $T : l^2(\mathcal{I}) \to H$, which is defined by $T\{c_i\}_{i\in\mathcal{I}} = \sum_{i\in\mathcal{I}} c_i\phi_i$. The analysis operator $T^* : H \to l^2(\mathcal{I})$, which is really the adjoin of the synthesis operator T, is given by $T^*x = \{\langle x, \phi_i\rangle\}_{i\in\mathcal{I}}$. In the case that $\{\phi_i\}_{i\in\mathcal{I}}$ is a frame for H, the frame operator $S : H \to H$ is defined by

$$Sx := TT^*x = \sum_{i\in\mathcal{I}} \{\langle x, \phi_i\rangle\}\phi_i,$$

for each $x \in H$. It is well known that, S is bounded, positive, self-adjoint and invertible. A frame which is a Schauder basis is called a Riesz basis. A frame which is not a Riesz basis is said to be overcomplete. Moreover, a frame $\{\phi_i\}_{i\in\mathcal{I}}$ is called near-Riesz basis if it consists of a Riesz basis and a finite number of extra elements. The excess of a frame is equal to the number of elements which have to removed in order to obtain a Riesz basis. Two frames $\{\phi_i\}_{i\in\mathcal{I}}$ and $\{\psi_i\}_{i\in\mathcal{I}}$ for H is called woven if for each subset σ of $\mathcal{I}$, the families $\{\phi_i\}_{i\in\sigma} \cup \{\phi_i\}_{i\in\sigma^c}$ are frames for H with the same frame bounds. Each family $\{\phi_i\}_{i\in\sigma} \cup \{\phi_i\}_{i\in\sigma^c}$ for $\sigma \subset \mathcal{I}$ is called a weaving of $\{\phi_i\}_{i\in\mathcal{I}}$ and $\{\psi_i\}_{i\in\mathcal{I}}$.

2. Main Results

The next example shows that if $\{\phi_i\}_{i\in\mathcal{I}}$ is a basis for H such as Riesz basis, orthonormal basis, Schauder basis, exact frame and so on, then for each nontrivial permutation function (The trivial

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permutation function on $\mathcal{I}$ is the identity operator on $\mathcal{I}$ and is denoted by I_d) π on $\mathcal{I}$, the families $\{\phi_i\}_{i\in\sigma} \cup \{\phi_{\pi(i)}\}_{i\in\sigma^c}$ are not frames for each $\sigma \subset \mathcal{I}$.

EXAMPLE 2.1. Assume that $\{\phi_i\}_{i\in\mathcal{I}}$ is a basis for H. Then for every non trivial permutation function π on $\mathcal{I}$ there exists a subset $\sigma \subset \mathcal{I}$ such that the weavings $\{\phi_i\}_{i\in\sigma} \cup \{\phi_{\pi(i)}\}_{i\in\sigma^c}$ are not frames for H.

PROOF. The proof is similar to the proof of the [2, Theorem 2.7]. □

One can consider shift operators on the set of integer numbers $\mathbb{Z}$ as permutation functions which are given in the next example.

EXAMPLE 2.2. Consider the frame $\{\phi_i\}_{i\in\mathbb{Z}}$ for the Hilbert space $\mathbb{C}^2$ by $\phi_{2i} = e_1$ and $\phi_{2i+1} = e_2$, where $\{e_1, e_2\}$ is the orthonormal basis for $\mathbb{C}^2$. For $m \in \mathbb{Z}$, consider the shift operators π on $\mathbb{Z}$ by $\pi(i) = i + m$, which are in fact, permutation functions on $\mathbb{Z}$. Then $\{\phi_i\}_{i\in\mathbb{Z}}$ and $\{\phi_{\pi(i)}\}_{i\in\mathbb{Z}}$ are woven if m is even and are not woven if m is odd.

PROOF. At first, assume that m is an even integer number. Let $\sigma \subset \mathbb{Z}$ be arbitrary. In this case, $\sigma \cup \pi(\sigma^c)$ consists of even numbers and of odd numbers. Thus, $\{\phi_i\}_{i\in\sigma} \cup \{\phi_{\pi(i)}\}_{i\in\sigma^c}$ is a frame for $\mathbb{C}^2$ and so $\{\phi_i\}_{i\in\mathbb{Z}}$ and $\{\phi_{\pi(i)}\}_{i\in\mathbb{Z}}$ are woven. But, when m is an odd integer number, let $\sigma \subset \mathbb{Z}$ be the set of all even integers. Then $\sigma \cup \pi(\sigma^c)$ consists only of all even numbers, and in this case $\{\phi_i\}_{i\in\sigma} \cup \{\phi_{\pi(i)}\}_{i\in\sigma^c}$ is not a frame for $\mathbb{C}^2$. Thus, in this case $\{\phi_i\}_{i\in\mathbb{Z}}$ and $\{\phi_{\pi(i)}\}_{i\in\mathbb{Z}}$ are not woven. □

The next proposition is a useful result, and its proof is an algorithm to build the new woven frames of a given frame $\{\phi_i\}_{i\in\mathcal{I}}$.

PROPOSITION 2.3. *A sequence $\{\phi_i\}_{i\in\mathcal{I}}$ is an overcomplete frame for H if and only if there exists a non trivial permutation function π on $\mathcal{I}$ such that for each $\sigma \subset \mathcal{I}$, the families $\{\phi_i\}_{i\in\sigma} \cup \{\phi_{\pi(i)}\}_{i\in\sigma^c}$ are frames for H.*

The following example is a consequence of the above proposition.

EXAMPLE 2.4. Let $H = \mathbb{R}^3$ and let $\{e_i\}_{i=1}^3$ be an orthonormal basis for H. Consider the frame $\{\phi_i\}_{i=1}^M$ for H as

$$\varphi_i = \begin{cases} e_i & i = 1, 2, 3 \\ \sum_{i=1}^N e_i & i = 4 \\ 0 & i = 5, \dots, M, \end{cases}$$

where $M > 3$. Define the permutation function π on $\mathcal{I}$ as follows:

$$\pi(i) = \begin{cases} i & i = 2, \dots, 3, 5, \dots, M \\ 1 & i = 4 \\ \quad 4 & i = 1. \end{cases}$$

Now, by using the proof of Proposition 2.3, $\{\phi_i\}_{i=1}^M$ and $\{\phi_{\pi(i)}\}_{i=1}^M$ are woven.

Two frames $\{\phi_i\}_{i\in\mathcal{I}}$ and $\{\psi_i\}_{i\in\mathcal{I}}$ for H are called P-woven if there exists a non trivial subset σ of $\mathcal{I}$ such that the family $\{\phi_i\}_{i\in\sigma} \cup \{\phi_i\}_{i\in\sigma^c}$ is a frame for H. The following result is interesting.

PROPOSITION 2.5. *Suppose that $\{\phi_i\}_{i\in\mathcal{I}}$ is a frame for H. The following statements hold.*

- **(i):** *For each permutation function π on $\mathcal{I}$, $\{\phi_i\}_{i\in\mathcal{I}}$ and $\{\phi_{\pi(i)}\}_{i\mathcal{I}}$ are P-woven.*
- **(ii):** *For each $\sigma \subset \mathcal{I}$ there exists a non trivial permutation function π on $\mathcal{I}$ such that $\{\phi_i\}_{i\in\sigma} \cup \{\phi_{\pi(i)}\}_{i\in\sigma^c}$ is a frame for H.*

In finite dimensional case a set of vectors $\{\phi_i\}_{i=1}^M$ in an N-dimensional Hilbert space H is a full spark frame if either they are independent or if $M \geq N + 1$, then they have spark $N + 1$. A family $\{\phi_i\}_{i\in\mathcal{I}}$ for a separable Hilbert space H is called full spark, whenever for each $\sigma \subset \mathcal{I}$ with $|\sigma|$=dim$H$, $\{\phi_i\}_{i\in\sigma}$ is a generator for H.

Proposition 2.6. *Let H be an infinite dimensional separable Hilbert space. If $\{\phi_i\}_{i\in\mathcal{I}}$ is a full spark family for H and π is a permutation function on $\mathcal{I}$, then the family $\{\phi_i\}_{i\in\sigma}\cup\{\phi_{\pi(i)}\}_{i\in\sigma^c}$ spans H for each $\sigma\subset\mathcal{I}$.*

Proof. Suppose that the Hilbert space H is of infinite dimention and $\{\phi_i\}_{i\in\mathcal{I}}$ is a full spark family for H. If π is a permutation function on $\mathcal{I}$ and $\sigma\subset\mathcal{I}$, then at least one of the subfamilies $\{\phi_i\}_{i\in\sigma}$ or $\{\phi_{\pi(i)}\}_{i\in\sigma^c}$ of $\{\phi_i\}_{i\in\mathcal{I}}$ has infinite elements. Now this complete the proof, becuase $\{\phi_i\}_{i\in\mathcal{I}}$ is a full spark family for H. □

Proposition 2.7. *Assume that $\{\phi_i\}_{i=1}^M$ is a frame for N-dimentional Hilbert space H and π is a permutation function on $\mathcal{I}$. Then for each $\sigma\subset\mathcal{I}$ the weavings $\{\phi_i\}_{i\in\sigma}\cup\{\phi_{\pi(i)}\}_{i\in\sigma^c}$ are full spark frames if and only if $\{\phi_i\}_{i=1}^M$ is full spark and $\pi=I_d$.*

Proof. The proof of the sufficiency part is trivial. For the proof of the necessity part, assume if possible that $\{\phi_i\}_{i=1}^M$ is not full spark or $\pi\neq I_d$. If $\{\phi_i\}_{i=1}^M$ is not full spark, then for $\sigma=\mathcal{I}$ the weaving $\{\phi_i\}_{i\in\sigma}\cup\{\phi_{\pi(i)}\}_{i\in\sigma^c}$ for each permutation function π is not a full spark frame which is a cotradiction. On the other hand assume if possible that $\pi\neq I_d$. So there exist $i_0,j_0\in\mathcal{I}$ with $i_0\neq j_0$ such that $\pi(i_0)=j_0$. □

The following example, shows that there are frames which are woven with each reordering of itself.

Example 2.8. Take the frame $\{\psi_i\}_{i=1}^M$ for the N-dimentional Hilbert space $H=\mathbb{C}^N$ as

$$\psi_i=\begin{cases} e_i & i=1,2,\ldots,N\\ \sum_{i=1}^N e_i & i=N+1,\ldots,M,\end{cases}$$

where $M\geq 2N$, and $\{e_i\}_{i=1}^N$ is the standard orthonormal basis for H. Then for each permutation function π on $\mathcal{I}$, $\{\psi_i\}_{i=1}^M$ and $\{\psi_{\pi(i)}\}_{i=1}^M$ are woven.

In the following theorem, we give some optimal conditions for the perturbation of reordered weavings of a frame $\{\phi_i\}_{i\in\mathcal{I}}$.

Theorem 2.9. *Suppose that $\{\phi_i\}_{i\in\mathcal{I}}$ is a frame for H with bounds $0<A\leq B<\infty$, and π is a permutation function on $\mathcal{I}$. Suppose that there esists a positive number λ, such that $\lambda<\frac{A}{2}$ and*

$$\sum_{i\in\mathcal{I}}|\langle x,\phi_i-\phi_{\pi(i)}\rangle|^2\leq\lambda\|x\|^2, \tag{1}$$

for each $x\in H_0$, where $H_0\subset H$. Then $\{\phi_i\}_{i\in\mathcal{I}}$ and $\{\phi_{\pi(i)}\}_{i\in\mathcal{I}}$ are P-woven.

The next proposition is an interesting result in the concept of pertorbation of reordered weavings and operators.

Proposition 2.10. *Assume that $\{\phi_i\}_{i\in\mathcal{I}}$ is a frame for H with lower and upper frame bounds A and B respectively. Also, assume that $\pi:\mathcal{I}\to\mathcal{I}$ is a permutation function such that $\{\phi_i\}_{i\in\mathcal{I}}$ and $\{\phi_{\pi(i)}\}_{i\in\mathcal{I}}$ are P-woven. If T is a bounded and onto operator, then $\{\phi_i\}_{i\in\mathcal{I}}$ and $\{T\phi_{\pi(i)}\}_{i\in\mathcal{I}}$ are P-woven.*

References

[1] T. Bemrose, P. G. Casazza, K. Gröchenig, M. C. Lammers and R. G. Lynch, Weaving frames, Operators and Matrices, 10 , 4 (2016), 1093-1116.

[2] P. G. Casazza, and R.G. Lynch, Weaving properties of Hilbert space frames, 2015 International Conference on Sampling Theory and Applications (SampTA), doi:10.1109/sampta.2015.7148861

[3] O. Christensen, An Introduction to Frames and Riesz Bases, 2nd edn, Birkhauser, Boston, (2015).

[4] O. Christensen, Frames and Bases, Birkhauser, Boston, (2008).

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Two Banach algebras associated with a locally compact groupoid

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ABSTRACT. Let G be a locally compact groupoid with a fixed Haar system λ and a quasi invariant measure μ. We introduce the notion of λ-measurability and we put a new norm on $C_c(G)$ the space of continuous functions on G with compact support to make it a Banach algebra denoting by $L^1(G, \lambda, \mu)$ and we show that it is a two sided ideal of the algebra $M(G)$ of complex Radon measures on G.
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1. The algebra $L^1(G, \lambda, \mu)$

For a locally compact group G with a Haar measure λ, the Banach algebra $L^1(G, \lambda)$ plays a central role in harmonic analysis on G. This motivated us to define a similar notion in the case where G is a locally compact groupoid. Following [1], in this article we give a new norm on $C_c(G)$ and we denote by $L^1(G, \lambda, \mu)$ its completion with respect to this norm. We show that the Banach algebra $L^1(G, \lambda, \mu)$ plays a similar role to the group algebra when we replace groups with groupoids. For the rest of the paper, G is a locally compact, Hausdorff, second countable groupoid which admits a left Haar system $\lambda = \{\lambda^u\}$. First we have a definition.

A Borel measurable set $E \subseteq G$ is called λ-measurable if for each $u \in G^0$, $E \cap G^u$ belongs to the σ-algebra $\mathfrak{M}_{\lambda^u}$. A function $f : G \to \mathbb{C}$ is λ-measurable if for every $u \in G^0$ and every open set $O \subseteq \mathbb{C}, f^{-1}(O) \cap G^u \in \mathfrak{M}_{\lambda^u}$. For each $f : G \to \mathbb{C}$, λ-measurability of f is equivalent to ν-measurability of f.

Here is our main definition.

DEFINITION 1.1. Suppose μ is a quasi-invariant probability measure on G^0 and ν is Radon measure induced by μ. We define

$$L^1(G, \nu) = L^1(G, \lambda, \mu) = \left\{ f : G \to \mathbb{C} : f \text{ is } \lambda\text{-measurable}, \ \|f\|_1 = \int_G |f(x)| d\nu(x) < \infty \right\},$$

with the product given by $(f * g)(x) = \int_{G^{r(x)}} f(y)g(y^{-1}x)d\nu(y)$.

If $f, g \in L^1(G, \lambda, \mu)$, then

$$\begin{aligned}
\|f * g\|_1 &= \int_G |f * g(x)| d\nu(x) \le \int_G \int_{G^{r(x)}} |f(y)||g(y^{-1}x)| d\nu(y) d\nu(x) \\
&\le \int_G \int_G |f(y)||g(y^{-1}x)| d\nu(y) d\nu(x) = \int_G \int_G |f(y)||g(x)| d\nu(y) d\nu(yx) \\
&= \int_{G^0} \int_G |g(x)| \int_{G^0} \int_G |f(y)| d\lambda^u(y) d\mu(u) d\lambda^u(yx) d\mu(u) = \int_G |g(x)| (\int_G |f(y)| d\nu(y)) d\nu(x) \\
&= \|f\|_1 \|g\|_1.
\end{aligned}$$

Also the measurability of $f * g$ follows from λ-measurability of f, g.

Next we define an involution on $L^1(G, \lambda, \mu)$. We say that the assertion $P(x)$ holds for λ-a.e.x if for $E = \{x : \neg P(x)\}$, $\mu\{u : \lambda_u(E) > 0\} = 0$. Clearly an assertion holds λ-almost everywhere if and only if it holds ν-almost everywhere.

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Lemma 1.2. *Suppose* $D_u : G^u \to \mathbb{R}^+$ *with* $D_u(x) = \frac{d\lambda^u(x)}{d\lambda_u(x)}$ $(x \in G)$. *Then* $D = D_u$ *on* G^u *(a.e.).*

The map $* : L^1(G,\lambda,\mu) \to L^1(G,\lambda,\mu);\ f \longmapsto f^*$, where $f^*(x) = \bar{f}(x^{-1})D(x^{-1})$, is an isometric involution on $L^1(G,\lambda,\mu)$. Note that from [2, page 9] we have

$$\|f\|_{L^1(G,\lambda,\mu)} = \|f^*\|_{L^1(G,\lambda,\mu)} = \|f\|_{L^1(G,\nu_0)} = \|f^*\|_{L^1(G,\nu_0)} \leq \|f\|_{II,\mu} = \|f^*\|_{II,\mu} \leq \|f\|_{I,\mu} = \|f^*\|_{I,\mu}.$$

Hence the space of $L^1(G,\lambda,\mu)$ is in general bigger than $I(G,\nu,\mu)$ and $II_\mu(G,\nu,\mu)$ with respect to I-norm and II-norm, indeed $I(G,\nu,\mu) \subseteq II_\mu(G) \subseteq L^1(G,\lambda,\mu)$.

By [2, page 15] the space of continuous functions with compact support $C_c(G)$ has a two-sided bounded approximate identity. Since $C_c(G)$ is dense in $L^1(G,\lambda,\mu)$, thus $L^1(G,\lambda,\mu)$ has a two-sided bounded approximate identity.

For each $f \in L^1(G,\lambda,\mu)$ define

$$L_x f(y) = f(x^{-1}y), \quad R_x f(y) = f(yx),$$

when the multiplications on the right hand sides are defined. It is easy to check that the maps L_x, R_x are homomorphisms.

Proposition 1.3. *Let I be a closed subspace of $L^1(G,\lambda,\mu)$. Then I is a left ideal if and only if it is closed under left translation, and I is a right ideal if and only if it is closed under right translation.*

Proof. Note that since $f * g = \int_{G^{r(y)}} f(y) L_y g \, d\nu(y)$,

$$\begin{aligned} L_x(f*g) &= \int_{G^{r(y)}} f(y) L_x L_y g \, d\nu(y) = \int_{G^{r(y)}} f(y) L_{xy} g \, d\nu(y) \\ &= \int_{G^{r(y)}} f(x^{-1}y) L_y g \, d\nu(y) = \int_{G^{r(y)}} L_x f(y) L_y g \, d\nu(y) = (L_x f) * g. \end{aligned}$$

Now suppose $(e_n)_n$ is a bounded approximate identity for $L^1(G,\lambda,\mu)$. For the first assertion, if $f \in L^1(G,\lambda,\mu)$ and $g \in I$ and I is a left ideal, then we have

$$L_x(e_n) * f = L_x(e_n * f) \longrightarrow L_x f.$$

Conversely, if I is closed under left translation and $f \in L^1(G,\lambda,\mu)$ and $g \in I$,

$$f * g = \int_{G^{r(y)}} f(y) L_y g \, d\nu(y)$$

is in the closed linear span of the functions $L_y g$ and hence in I. The other assertion is proved similarly. □

2. The involutive Banach algebra $M(G)$

In this section we show that $L^1(G,\lambda,\mu)$ is a closed ideal in the algebra of complex Radon measures on G. Let $M(G)$ be the space of complex Radon measures on G. If $\eta, \theta \in M(G)$, then the map $\psi \mapsto I(\psi)$ on $C_0(G)$ defined by $I(\psi) = \int_G \int_{G^{r(y)}} \psi(xy) d\eta(x) d\theta(y)$ is a linear functional on $C_0(G)$ satisfying $|I(\psi)| \leq \|\psi\|_{sup} \|\eta\| \|\theta\|$, so by Riesz representation theorem, it is given by a measure shown as $\eta * \theta$ called the convolution of η, θ with $\|\eta * \theta\| \leq \|\eta\| \|\theta\|$. If we define $\eta^*(E) = \overline{\eta(E^{-1})}$ then $\eta \mapsto \eta^*$ is an involution on $M(G)$, and $M(G)$ is a Banach $*$-algebra. In this section we show that the space $L^1(G,\lambda,\mu)$ is a closed two-sided ideal of $M(G)$.

Proposition 2.1. *The map $L^1(G,\lambda,\mu) \hookrightarrow M(G); f \mapsto \nu_f$ defined by $\nu_f(E) = \int_E f \chi_E \, d\nu \quad (E \subseteq G)$ is an isometric embedding.*

Proof. If $f \in L^1(G,\lambda,\mu)$, then f is λ-measurable so the integral exists and it is easy to check that ν_f is a measure on G. We show that ν_f is Radon. If $f = u + iv$ then $\nu_f = \nu_u + i\nu_v$, so ν_f is Radon if and only if ν_u and ν_v are Radon. Since G is locally compact Hausdorff and second countable, we have $\nu_u(K) = \int_K u \, d\nu \leq \int_G |u| d\nu = \|u\|_1 < \infty$, for each compact set K, thus ν_u is Radon. Similarly ν_v is Radon, and so is ν_f.

By definition, $\|\nu_f\| = \sup\{\sum_1^n |\nu_f(E_i)| : n \in N, G = \bigsqcup_1^n E_i\}$, so for each $\epsilon > 0$ there exists a partition $\{E_i\}_1^n$ of G such that

$$\|\nu_f\| - \epsilon < \sum_1^n |\nu_f(E_i)| = \sum_1^n |\int_G f\chi_{E_i} d\nu| \leq \int_G (|f| \sum_1^n \chi_{E_i}) d\nu = \|f\|_1.$$

Thus $\|\nu_f\| \leq \|f\|_1$. Conversely, suppose $f \geq 0$ then $\nu_f \geq 0$ and for every partition $\{E_i\}_1^n$ of G we have,

$$\|\nu_f\| \geq \sum_1^n \nu_f(E_i) = \sum_1^n \int_G f\chi_{E_i}\, d\nu = \int_G f d\nu = \|f\|_1.$$

If $f = u + iv = (f_1 - f_2) + i(f_3 - f_4)$, where $f_i \geq 0$ then $\|\nu_f\| = \|\nu_{f_1}\| + \|\nu_{f_2}\| + \|\nu_{f_3}\| + \|\nu_{f_4}\| \geq \|f_1\|_1 + \|f_2\|_1 + \|f_3\|_1 + \|f_4\|_1 \geq \|f\|_1$. Hence $\|\nu_f\| \geq \|f\|_1$ and equality holds. □

The above proposition shows that $L^1(G, \lambda, \mu)$ is a closed subspace of $M(G)$. Next we show that it is indeed an ideal.

Lemma 2.2. *If $f, g \in L^1(G, \lambda, \mu)$, then $\nu_{(f*g)} = \nu_f * \nu_g$.*

Proof. For each compact set K we have

$$\begin{aligned}\nu_f * \nu_g(K) &= \int_G \chi_K(x) d(\nu_f * \nu_g)(x) = \int_G \int_{G^{r(x)}} \chi_K(yx) d\nu_f(y) d\nu_g(x)\\ &= \int_G f(y) \int_{G^{s(y)}} \chi_K(yx) d\nu_g(x) d\nu(y) = \int_G \int_{G^{s(y)}} f(y) g(y^{-1}x) \chi_K(x) d\nu(y) d\nu(x)\\ &= \int_G \chi_K(x) \int_{G^{r(x)}} f(y) g(y^{-1}x) d\nu(y) d\nu(x) = \int_G (f * g)(x) \chi_K(x)\, d\nu(x) = \nu_{f*g}(K).\end{aligned}$$

Since ν_{f*g} and $\nu_f * \nu_g$ are regular measures, the equality holds for each open set and then for each measurable set. □

If $f \in L^1(G, \lambda, \mu)$ and $\eta \in M(G)$, we shall define $\eta * f$ in such a way that $\nu_{\eta * f} = \eta * \nu_f$. Suppose $\varphi \in C_0(G)$ and put

$$\nu(\eta * f)(\varphi) = \int_G \varphi(x)\, d\nu_{\eta*f}(x) = \int_G \varphi(x)(\eta * f)(x)\, d\nu(x).$$

On the other hand,

$$\begin{aligned}\eta * \nu_f(\varphi) &= \int_G \int_{G^{r(x)}} \varphi(yx) d\eta(y) d\nu_f(x) = \int_G \int_{G^{r(x)}} \varphi(yx) d\eta(y) f(x) d\nu(x)\\ &= \int_G \int_{G^{s(y)}} f(y^{-1}x) \varphi(x) d\eta(y) d\nu(y^{-1}x) = \int_G \varphi(x) \int_{G^{s(y)}} f(y^{-1}x) d\eta(y) d\nu(x).\end{aligned}$$

Comparing these equalities implies that

$$(\eta * f)(x) = \int_{G^{r(x)}} f(y^{-1}x) d\eta(y).$$

If $f \in L^1(G, \lambda, \mu)$, then it is easy to check that

$$\int_{G^u} R_y f(x) d\lambda^u(x) = \int_{G^u} f(x) d\lambda^u(xy^{-1}) = D(y^{-1}) \int_{G^u} f(x) d\lambda^u(x).$$

Thus

$$\int_G R_y f(x) d\nu(x) = D(y^{-1}) \int_G f(x) d\nu(x).$$

Similarly, we want to define $f * \eta$ in such a way that the equality $\nu_{(f*\eta)} = \nu_f * \eta$ holds. Again suppose $\varphi \in C_0(G)$. We have

$$\nu_{(f*\eta)}(\varphi) = \int_G \varphi(x)\, d\nu_{f*\eta}(x) = \int_G \varphi(x)(f*\eta)(x)\, d\nu(x) = \int_{G^0}\int_G \varphi(x)(f*\eta)(x)\, d\lambda^u(x) d\mu(u)$$
$$= \int_{G^0}\int_{G^u} \varphi(x)(f*\eta)(x)\, d\lambda^u(x) d\mu(u) = \int_{G^u} \varphi(x)(f*\eta)(x)\, d\nu(x).$$

On the other hand,

$$(\nu_f * \eta)(\varphi) = \int_G \int_{G^{r(y)}} \varphi(xy) d\nu_f(x) d\eta(y) = \int_G \int_{G^{r(y)}} \varphi(xy) f(x) d\nu(x) d\eta(y)$$
$$= \int_G \int_{G^{r(y)}} \varphi(x) f(xy^{-1}) d\nu(xy^{-1}) d\eta(y) = \int_G \int_{G^{r(y)}} \varphi(x) f(xy^{-1}) D(y^{-1}) d\nu(x) d\eta(y)$$
$$= \int_{G^{s(x)}} \varphi(x) \int_G f(xy^{-1}) D(y^{-1}) d\eta(y) d\nu(x).$$

Comparing the above equalities, we have

$$(f*\eta)(x) = \int_G f(xy^{-1}) D(y^{-1}) d\eta(y).$$

Corollary 2.3. *$L^1(G, \lambda, \mu)$ is a two sided closed ideal of $M(G)$.*

Proof. Suppose $f \in L^1(G, \lambda, \mu)$ and $\eta \in M(G)$. Then we have

$$\|\eta * f\|_1 = \int_G |\eta * f(x)| d\nu(x) \le \int_G \int_{G^{r(x)}} |f(y^{-1}x)|\, d|\eta|(y) d\nu(x)$$
$$= \int_G \int_{G^{r(x)}} |f(x)|\, d|\eta|(y) d\nu(yx) = \int_G |f(x)| \int_{G^{r(x)}} d|\eta|(y) d\nu(x)$$
$$\le \|\eta\| \|f\|_1 < \infty.$$

Thus $\eta * f \in L^1(G, \lambda, \mu)$. Also

$$\|f*\eta\|_1 = \int_G |(f*\eta)(x)| d\nu(x) \le \int_G \int_G |f(xy^{-1}) D(y^{-1})| d|\eta|(y) d\nu(x)$$
$$= \int_G \int_G |f(x)| D(y^{-1}) D(y) d\nu(x) d|\eta|(y)$$
$$= \int_G \int_G |f| d\nu d|\eta| = \|f\|_1 \|\eta\| < \infty.$$

Hence $f * \eta \in L^1(G, \lambda, \mu)$. □

References

1. M. Amini, A. Medghalchi and A. Shirinkalam, L^1*-Algebra of a locally compact groupoid*, ISRN Algebra, 2011, doi:10.5402/2011/856709.
2. M. R. Buneci, The equality of the reduced and the full C*-algebras and the amenability of a topological groupoid, *Oper. Theory Adv. Appl.* **153** (2005), 61–78.

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Applications on integral equations in C*-algebra-valued S_b-metric spaces

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Abstract. Existence and uniqueness results for one type of integral equations in C*-algebra-valued S_b-metric spaces is discussed by using some common fixed point results in these spaces.

Keywords: Integral equation, C*-algebra, S_b-metric space, Common fixed point.
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1. Preliminaries

At first, we review some facts of the theory of C*-algebras which need in this paper, the references [1]–[6] are useful.

Throughout this paper, suppose that $\mathcal{A}$ is an unital C*-algebra with the unit I. Set $\mathcal{A}_h = \{t \in \mathcal{A} \ : \ t = t^*\}$. We say $t \in \mathcal{A}$ a positive element, showed it by $t \succeq 0_{\mathcal{A}}$ if $t = t^*$ and $\sigma(t) \subseteq [0, \infty)$, in which $0_{\mathcal{A}}$ in $\mathcal{A}$ is the zero element and the spectrum of t is $\sigma(t)$.
On $\mathcal{A}_h$ we can find a natural partial ordering given by $u \preceq v$ iff $v - u \succeq 0_{\mathcal{A}}$. From now on, we will denote $\mathcal{A}_+$ and $\mathcal{A}'$ for the set $\{t \in \mathcal{A} \ : \ t \succeq 0_{\mathcal{A}}\}$ and the set $\{t \in \mathcal{A} \ : tk = kt\,, \ \forall k \in \mathcal{A}\}$, respectively.

Definition 1.1. Assume that $\mathcal{X}$ is a nonempty set, and $\mathcal{A}$ is a C*-algebra. A function $S : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ is called a C*-algebra-valued S-metric on $\mathcal{X}$ if for every $u, v, t, a \in \mathcal{X}$:

(1) $S(u, v, t) \succeq 0_{\mathcal{A}}$
(2) $S(u, v, t) = 0$ iff $u = v = t$
(3) $S(u, v, t) \preceq S(u, u, a) + S(v, v, a) + S(t, t, a)$.

Then $(\mathcal{X}, \mathcal{A}, S)$ is a C*-algebra-valued S-metric space (in short C*-AV-SM space).

Definition 1.2. Assume that $\mathcal{X}$ is a nonempty set and $b \in \mathcal{A}'$ such that $||b|| \geq 1$. A function $S_b : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ is called a C*-algebra-valued S_b-metric on $\mathcal{X}$ if for every $u, v, t, a \in \mathcal{X}$:

(1) $S_b(u, v, t) \succeq 0_{\mathcal{A}}$
(2) $S_b(u, v, t) = 0$ iff $u = v = t$
(3) $S_b(u, v, t) \preceq b[S_b(u, u, a) + S_b(v, v, a) + S_b(t, t, a)]$.

Then $(\mathcal{X}, \mathcal{A}, S_b)$ is called C*-algebra-valued S_b-metric space (in short C*-AV-S_bM space with coefficient b.)

Definition 1.3. A C*-AV-S_bM S_b is said to be symmetric if

$$S_b(u, u, v) = S_b(v, v, u), \ \ \forall u, v \in \mathcal{X}.$$

By the above definitions, we give an example in C*-AV-S_bM space:

Example 1.4. Let $\mathcal{X} = \mathbb{R}$ and $\mathcal{A} = M_2(\mathbb{R})$ be all 2×2-matrices with the usual operations of addition, scalar multiplication and matrix multiplication. It is clear that

$$||A|| = (\sum_{i,j=1}^{2} |a_{ij}|^2)^{\frac{1}{2}}$$

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defines a norm on $\mathcal{A}$ where $A = (a_{ij}) \in \mathcal{A}$. $* : \mathcal{A} \to \mathcal{A}$ defines an involution on $\mathcal{A}$ where $\mathcal{A}^* = \mathcal{A}$. Then $\mathcal{A}$ is a C^*-algebra. For $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathcal{A}$, a partial order on $\mathcal{A}$ can be given as follows:

$$A \leq B \Leftrightarrow (a_{ij} - b_{ij}) \leq 0 \ \forall i, j = 1, 2$$

Let $(\mathcal{X}, d)$ be a b-metric space with $||b|| \geq 1$ and $S_b : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to M_2(\mathbb{R})$ be defined by

$$S_b(u, v, t) = \begin{bmatrix} d(u,v) + d(v,t) + d(u,t) & 0 \\ 0 & d(u,v) + d(v,t) + d(u,t) \end{bmatrix}$$

then it is a C*-AV-S_bM space. Now, we check the condition (3) of Definition 1.2:

$$\begin{aligned} S_b(u, v, t) &= \begin{bmatrix} d(u,v) + d(v,t) + d(u,t) & 0 \\ 0 & d(u,v) + d(v,t) + d(u,t) \end{bmatrix} \\ &\leq [2 \begin{bmatrix} d(u,a) & 0 \\ 0 & d(u,a) \end{bmatrix} + 2 \begin{bmatrix} d(v,a) & 0 \\ 0 & d(v,a) \end{bmatrix} + 2 \begin{bmatrix} d(t,a) & 0 \\ 0 & d(t,a) \end{bmatrix}] \\ &= b[S_b(u,u,a) + S_b(v,v,a) + S_b(t,t,a)] \end{aligned}$$

For all $u, v, t, a \in \mathcal{X}$. So $(\mathcal{X}, \mathcal{A}, S_b)$ is a C*-AV-S_bM space.

2. Main results

In this section, we define some concepts in C*-AV-S_bM space and present some Lemmas which will be needed in the sequel:

Definition 2.1. Let $(\mathcal{X}, \mathcal{A}, S_b)$ be a C*-AV-S_bM space and $\{u_n\}$ be a sequence in $\mathcal{X}$:

(1) If $||S_b(u_n, u_n, u)|| \to 0$, $(n \to \infty)$, then it is said that $\{u_n\}$ converges to u, and we denote it by $\lim_{n\to\infty} u_n = u$.
(2) If for any $p \in \mathbb{N}$, $||S_b(u_{n+p}, u_{n+p}, u_n)|| \to 0$, $(n \to \infty)$, then $\{u_n\}$ is called a Cauchy sequence in $\mathcal{X}$.
(3) If every Cauchy sequence is convergent in $\mathcal{X}$, then $(\mathcal{X}, \mathcal{A}, S_b)$ is called a complete C*-AV-S_bM space.

Definition 2.2. Let $(\mathcal{X}, \mathcal{A}, S_b)$ and $(\mathcal{X}_1, \mathcal{A}_1, S_{b_1})$ be C*-AV-S_bM spaces, and let $f : (\mathcal{X}, \mathcal{A}, S_b) \to (\mathcal{X}_1, \mathcal{A}_1, S_{b_1})$ be a function, then f is said to be continuous at a point $u \in \mathcal{X}$ iff for every sequence $\{u_n\}$ in $\mathcal{X}$, $S_b(u_n, u_n, u) \to 0_{\mathcal{A}}$, $(n \to \infty)$ implies $S_{b_1}(f(u_n), f(u_n), f(u)) \to 0_{\mathcal{A}}$, $(n \to \infty)$. A function f is continuous at $\mathcal{X}$ iff it is continuous at all $u \in \mathcal{X}$.

Lemma 2.3. *Let $(\mathcal{X}, \mathcal{A}, S_b)$ be a symmetric C*-AV-S_bM space and $\{u_n\}$ be a sequence in $\mathcal{X}$. If $\{u_n\}$ converges to u and v, respectively, then $u = v$.*

Definition 2.4. Let $(\mathcal{X}, \mathcal{A}, S_b)$ be a C*-AV-S_bM space. A pair $\{\psi, \varphi\}$ is said to be compatible iff $S_b(\psi\varphi u_n, \psi\varphi u_n, \varphi\psi u_n) \to 0_{\mathcal{A}}$, whenever $\{u_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} \psi u_n = \lim_{n\to\infty} \varphi u_n = u$, for some $u \in \mathcal{X}$.

Definition 2.5. A point $u \in \mathcal{X}$ is called a coincidence point of ψ and φ iff $\psi u = \varphi u$. In this case, $t = \psi u = \varphi u$ is called a point of coincidence of ψ and φ. If ψ and φ commute at all of their coincidence points, then they are weakly compatible but the converse is not true.

Theorem 2.6. *If the mapping ψ and φ on the C*-AV-S_bM space $(\mathcal{X}, \mathcal{A}, S_b)$ are compatible, then they are weakly compatible.*

Lemma 2.7. [1] *Let ψ and φ be a weakly compatible mappings of a set $\mathcal{X}$. If ψ and φ have a unique point of coincidence, then it is the unique common fixed point (in short FP) of ψ and φ.*

Theorem 2.8. *Assume that $(\mathcal{X}, \mathcal{A}, S_b)$ is a complete symmetric C*-AV-S_bM space and suppose that $\psi, \varphi : \mathcal{X} \to \mathcal{X}$ satisfy*

$$S_b(\psi u, \psi u, \varphi v) \preceq a^* S_b(u, u, v) a, \tag{1}$$

for every $u, v \in \mathcal{X}$, in which $a \in \mathcal{A}$ with $||a|| < 1$. Then ψ and φ have a unique common FP in $\mathcal{X}$.

Corollary 2.9. *Assume that $(\mathcal{X}, \mathcal{A}, S_b)$ is a complete symmetric C*-AV-S_bM space and suppose that $\psi, \varphi : \mathcal{X} \to \mathcal{X}$ be two mappings such that*

$$||S_b(\psi u, \psi u, \varphi v) \preceq ||a||||S_b(u, u, v)||,$$

for every $u, v \in \mathcal{X}$ in which $a \in \mathcal{A}$ with $||a|| < 1$. Then ψ and φ have a unique common FP in $\mathcal{X}$.

2.1. Existence and uniqueness. Consider the next equations:

$$x(m) = \int_{\mathcal{E}} (T_1(m, n, x(n))dn + J(m), \ m \in \mathcal{E}$$

$$x(m) = \int_{\mathcal{E}} (T_2(m, n, x(n))dn + J(m), \ m \in \mathcal{E} \tag{2}$$

where $\mathcal{E}$ is a Lebesgue measurable set and $m(\mathcal{E}) < \infty$.

In what follows, we always let $\mathcal{X} = L^\infty(\mathcal{E})$ denote class of essentially bounded measurable functions on $\mathcal{E}$, where $\mathcal{E}$ is a Lebesgue measurable set such that $m(\mathcal{E}) < \infty$.

Now, we consider the functions T_1, T_2, α, β fulfill the following assumptions:

(i) $T_1, T_2 : \mathcal{E} \times \mathcal{E} \times \mathbb{R} \to \mathbb{R}$ are integrable. Also, an integrable function α is from $\mathcal{E} \times \mathcal{E}$ to $\mathbb{R}^{\geq 0}$, and $J \in L^\infty(\mathcal{E})$;

(ii) there exists $\ell \in (0, 1)$ such that

$$|T_1(m, n, x) - T_2(m, n, y)| \leq \ell|\alpha(m, n)||x - y|,$$

for $m, n \in \mathcal{E}$ and $x, y \in \mathbb{R}$;

(iii) $\sup_{m\in\mathcal{E}} \int_{\mathcal{E}} |\alpha(m, n)|dn \leq 1$.

Theorem 2.10. *Suppose that assumptions (i)-(iii) hold. Then the integral equation 2 has a unique common solution in $L^\infty(\mathcal{E})$.*

Proof. Let $\mathcal{X} = L^\infty(\mathcal{E})$ and $B(L^2(\mathcal{E}))$ be the set of bounded linear operators on a Hilbert space $L^2(\mathcal{E})$. We endow $\mathcal{X}$ with the S_b-metric $S_b : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to B(L^2(\mathcal{E}))$ defined by

$$S_b(\alpha, \beta, \gamma) = M_{(|\alpha-\gamma|+|\beta-\gamma|)^p}$$

where $M_{(|\alpha-\gamma|+|\beta-\gamma|)^p}$ is the multiplication operator on $L^2(\mathcal{E})$ defined by

$$M_h(\alpha) = h.\alpha\,; \ \alpha \in L^2(\mathcal{E})$$

Hence $(\mathcal{X}, B(L^2(\mathcal{E})), S_b)$ is a complete C*-AV-S_bM space. Define the self-mappings $\Psi, \Phi : \mathcal{X} \to \mathcal{X}$ by

$$\Psi x(m) = \int_{\mathcal{E}} T_1(m, n, x(n))dn + J(m),$$

$$\Phi x(m) = \int_{\mathcal{E}} T_2(m, n, x(n))dn + J(m),$$

for all $m \in \mathcal{E}$. Now, we have

$$S_b(\Psi x, \Psi x, \Phi y) = M_{(|\Psi x-\Phi y|+|\Psi x-\Phi y|)^p}.$$

we can obtain,

$$\begin{aligned}
||S_b(\Psi x,\Psi x,\Phi y)|| &= \sup_{||h||=1}\langle M_{(|\Psi x-\Phi y|+|\Psi x-\Phi y|)^p}h,h\rangle \\
&= \sup_{||h||=1}\langle M_{(2|\Psi x-\Phi y|)^p}h,h\rangle \\
&= \sup_{||h||=1}\langle 2^p M_{|\Psi x-\Phi y|^p}h,h\rangle \\
&= \sup_{||h||=1}\int_{\mathcal{E}}(2^p|\Psi x-\Phi y|^p)h(t)\overline{h(t)}dt \\
&\preceq 2^p\sup_{||h||=1}\int_{\mathcal{E}}[\int_{\mathcal{E}}|T_1(m,n,x(n))-T_2(m,n,y(n))|]^p|h(t)|^2dt \\
&\preceq 2^p\sup_{||h||=1}\int_{\mathcal{E}}[\int_{\mathcal{E}}\ell|\alpha(m,n)(x(n)-y(n))|dn]^p|h(t)|^2dt \\
&\preceq 2^p\ell^p\sup_{||h||=1}\int_{\mathcal{E}}[\int_{\mathcal{E}}|\alpha(m,n)|dn]^p|h(t)|^2dt.||x-y||_\infty^p \\
&\preceq \ell\sup_{m\in\mathcal{E}}\int_{\mathcal{E}}|\alpha(m,n)|dn.\sup_{||h||=1}\int_{\mathcal{E}}|h(t)|^2dt2^p||x-y||_\infty^p \\
&\preceq 2^p\ell||x-y||_\infty^p \\
&= \ell||2(x-y)||_\infty^p \\
&= \ell||M_{(|x-y|+|x-y|)^p}|| \\
&= ||a||||S_b(x,x,y)||
\end{aligned}$$

Set $a=\ell 1_{B(L^2(\mathcal{E}))}$, then $a\in B(L^2(\mathcal{E}))$ and $||a||=\ell<1$. Hence applying Corollary 2.9, we get the desired result. □

References

1. M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl., **341**(2008), 416-420.
2. M.E. Ege and C. Alaca, *C^*-algebra-valued S-metric spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **66 (2)**(2018), 165-177.
3. Z.H. Ma, L.N. Jiang, H.K. Sun, *C*-algebra-valued metric spaces and related fixed point theorems*, Fixed Point Theory Appl. 2014, **206**(2014).
4. G.J. Murphy, *C*-Algebras and Operator Theory*, Academic Press, London, (1990).
5. S.S. Razavi, H.P. Masiha, *Common fixed point theorems in C*-algebra valued b-metric spaces with applications to integral equations*, Fixed Point Theory, **20**(2019), No. 2, 649-662.
6. S.S. Razavi, H.P. Masiha, *C*-algebra-valued S_b-metric spaces and applications to integral equations*, In Progress.

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A note on (weak) phase retrievable Real Hilbert space frames

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Abstract. In this manuscript, we answer to some of longstanding open problems on weak phase retrieval including: (1) A complete classification of the vectors $\{x_i\}_{i=1}^2$ in $\mathbb{R}^2$ that do weak phase retrieval; (2) Weak phase retrieval is preserved by orthogonal projection operators; (3) We show that frames doing weak phase retrieval in $\mathbb{R}^n$ must span $\mathbb{R}^n$.
Keywords: Real Hilbert frames, Phase retrieval, Full spark, Weak phase retrieval.
AMS Mathematical Subject Classification [2010]: 43A60, 43A22.

Frames have the redundancy property that make them more applicable than bases. Phase retrieval is one of the most applied and studied areas of research today. Phase retrieval for Hilbert space frames was introduced in [2] and quickly became an industry. The concept of weak phase retrieval weakened of notion of phase retrieval and it has just been defined for vectors ([4] and [3]). We first give the background material needed for the paper. The natural numbers and real numbers are denoted by "$\mathbb{N}$" and "$\mathbb{R}$", respectively. We use $[m]$ instead of the set $\{1, 2, 3, \ldots, m\}$. Also I is a finite or countable subset of $\mathbb{N}$. We denote by $\mathbb{R}^n$ a n dimensional Real Hilbert space.
We start with the definition of a Real Hilbert space frame.

Definition 0.1. A family of vectors $\{x_i\}_{i\in I}$ in a separable Real Hilbert space $\mathbb{R}^n$ is a **frame** if there are constants $0 < A \leq B < \infty$ so that $A\|x\|^2 \leq \sum_{i\in I} |\langle x, x_i\rangle|^2 \leq B\|x\|^2$ for all $x \in \mathbb{R}^n$.

Definition 0.2. A family of vectors $\{x_k\}_{k=1}^m$ in a finite dimensional Hilbert space $\mathbb{R}^n$ has the **complement property** if for any subset $I \subset [m]$,

$$\textit{either} \quad \overline{span}\{x_k\}_{k\in I} = \mathbb{R}^n \quad \textit{or} \quad \overline{span}\{x_k\}_{k\in I^c} = \mathbb{R}^n.$$

Theorem 0.3. *A family of vectors $\{x_i\}_{i\in I}$ does phase retrieval if and only if it has the complement property.*

Definition 0.4. A family of vectors $\{x_i\}_{i\in I}$ in a Real Hilbert space $\mathbb{R}^n$ does **phase retrieval** if whenever $x, y \in \mathbb{R}^n$, satisfy $|\langle x, x_i\rangle| = |\langle y, x_i\rangle|$ for all $i \in I$, then $x = \pm y$.

Now we define "spark" and "full spark".

Definition 0.5. A family of vectors $\{x_i\}_{i=1}^m$ in $\mathbb{R}^n$ $(m \geq n)$ has **spark k** if for every $I \subset [m]$ with $|I| = k-1$, $\{x_i\}_{i\in I}$ is linearly independent. It is full spark if $k = n+1$ and hence every n-element subset spans $\mathbb{R}^n$.

Corollary 0.6. *If $\{x_i\}_{i=1}^m$ does phase retrieval in $\mathbb{R}^n$, then $m \geq 2n-1$. If $m = 2n-1$, the frame does phase retrieval if and only if it is full spark.*

The notion of "Weak phase retrieval by vectors" in $\mathbb{R}^n$ was introduced in [4] and was developed further in [3]. For $x \in \mathbb{R}^n$, $sgn(x) = 1$ if $x > 0$ and $sgn(x) = -1$ if $x < 0$.

Definition 0.7. Two vectors $x = (a_1, a_2, \ldots, a_n)$ and $y = (b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^n$ **weakly have the same phase** if there is a $|\theta| = 1$ so that $phase(a_i) = \theta phase(b_i)$ for all $i \in [n]$, for which $a_i \neq 0 \neq b_i$.
If $\theta = 1$, we say x and y weakly have the same signs and if $\theta = -1$, they weakly have the opposite signs.

*speaker

Definition 0.8. A family of vectors $\{\phi_i\}_{i=1}^m$ does **weak phase retrieval** in $\mathbb{R}^n$ if for any $x = (a_1, a_2, \ldots, a_n)$ and $y = (b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^n$ with $|\langle x, \phi_i\rangle| = |\langle y, \phi_i\rangle|$ for all $i \in [m]$, then x and y weakly have the same phase.

Proposition 0.9. [4] *Let $x = (a_1, a_2, \ldots, a_n)$ and $y = (b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^n$. The following are equivalent:*
(1) We have $sgn(a_i a_j) = sgn(b_i b_j)$, for all $1 \leq i \neq j \leq n$
(2) Either x, y have weakly the same sign or they have the opposite signs.

It is clear that if $\{x_i\}_{i=1}^m$ does phase retrieval (respectively, weak phase retrieval) in $\mathbb{R}^n$ then $\{c_i x_i\}_{i=1}^m$ does phase retrieval (respectively, weak phase retrieval) as long as $c_i > 0$ for all $i = 1, 2, \ldots, m$.
The following appears in [3]

Theorem 0.10. *If $X = \{x_i\}_{i=1}^{2n-2}$ does weak phase retrieval in $\mathbb{R}^n$ then X is full spark.*

The converse to this theorem fails.

Example 0.11. Now we give an example of a full spark not weak phase retrievable frame in $\mathbb{R}^3$ with 4 vectors. The frame $\{x_i\}_{i=1}^4$ given by

$$x_1 = (1, 0, 0), x_2 = (0, 1, 0), x_3 = (0, 0, 1), x_4 = (1, 1, -3)$$

is full spark but it cannot do weak phase retrieval for $\mathbb{R}^3$. Take $x = (4, 3, 1)$ and $y = (4, -3, -1)$, then $|\langle x, x_i\rangle| = |\langle y, x_i\rangle|$ for $i \in [4]$, but x and y do not weakly have the same sign.

Theorem 0.12. [4] *If $\{x_i\}_{i=1}^m$ does weak phase retrieval in $\mathbb{R}^n$ then $m \geq 2n - 2$.*

By above Theorem the set of vectors doing weak phase not phase retrieval, must have $2n - 2$ vectors.

Theorem 0.13. *A set of vectors $\{x_i\}_{i=1}^2$ does weak phase retrieval in $\mathbb{R}^2$ if and only if after normalization the two vectors are of the form $(1, 1),\ (1, -1)$ or $(1, b),\ (1, -b)$.*

We will prove the theorem in a series of 3 lemmas.

Lemma 0.14. *Sets $\{(a, 0),\ (b, c)\}$ and $\{(0, b),\ (c, d)\}$ with $a, b, c, d \neq 0$ fail weak phase retrieval but they are full spark.*

Lemma 0.15. *If $0 < ab, cd$ or $ab, cd < 0$ then $\{(a, b), (c, d)\}$ fail weak phase retrieval.*

Lemma 0.16. *If $ab > 0$ and $cd < 0$ then $\{(a, b), (c, d)\}$ do weak phase retrieval if and only if $a = b,\ c = -d$ or $a = c,\ b = -d$.*

Theorem 0.17. *If $\{x_i\}_{i=1}^2$ does weak phase retrieval in $\mathbb{R}^2$ then $\{x_1^\perp, x_2^\perp\}$ do weak phase retrieval in $\mathbb{R}^2$.*

Proposition 0.18. *If the frame $\{x_i\}_{i=1}^m$ does weak phase retrieval in $\mathbb{R}^n$ then $\{Px_i\}_{i=1}^m$ yields weak phase retrieval for all orthogonal projections P on $\mathbb{R}^n$.*

It is known in literature that every phase retrievable set is a spanning set in $\mathbb{R}^n$. We now solve a longstanding similar open problem in the field about weak phase retrievable sets in $\mathbb{R}^n$.

Theorem 0.19. *If $\{x_i\}_{i=1}^m$ is a weak phase retrievable vectors in $\mathbb{R}^n$, then*

$$span\{x_i\}_{i=1}^m = \mathbb{R}^n.$$

References

1. F. Akrami, P. G. Casazza, A. Rahimi, M. A. Hasankhani Fard, B. Daraby, *A note on phase (norm) retrievable real Hilbert space (fusion) frames*, arXiv:2107.09606, 2021.
2. R. Balan, P. G. Casazza, D. Edidin, *On signal reconstruction without phase*, Appl. Comput. Harmonic Anal. 20, 3, 2006, 345-356.
3. S. Botelho-Andrade, Peter G. Casazza, D. Ghoreishi, S. Jose, J. C. Tremain, *Weak phase retrieval and phaseless reconstruction*, arXiv:1612.08018, 2016.
4. S. Botelho-Andrade, Peter G. Casazza, H. V. Nguyen, And J. C. Tremain, *Phase retrieval versus phaseless reconstruction*, Journal of Mathematical Analysis and Applications, Volume 436, Issue 1, 2015.

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χ-Connes Module Amenability of Semigroup Algebras

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ABSTRACT. In this paper, we define χ-Connes module amenability of a semigroup algebra $l^1(S)$, where χ is a bounded module homomorphism from $l^1(S)$ to $l^1(S)$ that is ω^*-continuous and S is an inverse weakly cancellative semigroup with subsemigroup E of idempotents. We are mainly concerned with the study of χ-module normal, virtual diagonals. We show that if $l^1(S)$ as a Banach module over $l^1(E)$ is χ-Connes module amenable, then it has a χ-module normal, virtual diagonal. In the case $\chi = id$, the converse also holds.
Keywords: χ-Connes module amenable, χ-module normal virtual diagonal, Inverse semigroup algebra, Module φ-derivation, Weakly cancellative semigroup.
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1. Introduction

In [1], Amini introduced the concept of module amenability for Banach algebras, and proved that when S is an inverse semigroup with subsemigroup E of idempotents, then $l^1(S)$ as a Banach module over $U = l^1(E)$ is module amenable if and only if S is amenable. We may refer the reader e.g. to [1, 2, 6], for extensive treatments of various notions of module amenability. All of these concepts generalized the earlier concept of amenability for Banach algebras introduced by Johnson [4]. In this paper, we introduce the concept of χ-Connes module amenability for semigroup algebra $l^1(S)$ and give a characterization of χ-Connes module amenability in terms of χ-modul normal virtual diagonals. In particular, we show that if χ is a bounded module homomorphism from $l^1(S)$ to $l^1(S)$ that is ω^*-continuous and $l^1(S)$ as a Banach module over $l^1(E)$ is χ-Connes module amenable, then it has a χ-module normal virtual diagonal. In the case $\chi = id$, the converse also holds.

2. Main results

Let S be a semigroup. Then S is named cancellative semigroup, if for every $r, s \neq t \in S$ we have $rs \neq rt$ and $sr \neq tr$.

A discrete semigroup S is called an inverse semigroup if for each $x \in S$ there is a unique element $x^* \in S$ such that $xx^*x = x$ and $x^*xx^* = x^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotent elements of S is denoted by E. For $s \in S$, we define $L_s, R_s : S \to S$ by $L_s(t) = st, R_s(t) = ts; (t \in S)$. If for each $s \in S$, L_s and R_s are finite-to-one maps, then we say that S is weakly cancellative.

Before turning our result, we note that if S is a weakly cancellative semigroup, then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$[3].

Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, and $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-bimodule via,

$$\alpha.(ab) = (\alpha.a).b, \quad (\alpha\beta).a = \alpha.(\beta.a) \qquad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

Let I be the closed ideal of $\mathcal{A}\widehat{\otimes}\mathcal{A}$ generated by elements of the form $\alpha.(a \otimes b) - (a \otimes b).\alpha$, for $a, b \in \mathcal{A}$ and $\alpha \in \mathcal{U}$. $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{A}$ is defined to be the quitient Banach space $\frac{\mathcal{A}\widehat{\otimes}\mathcal{A}}{I}$.

Let J be the closed ideal of $\mathcal{A}$ generated by elements of the form $(\alpha.a).b - a.(b.\alpha)$. Since J is ω^*-closed, then the quotient algebra $\frac{\mathcal{A}}{J}$ is again dual with predual ${}^{\perp}J = \{\phi \in \mathcal{A}_* : \langle \phi, a\rangle = 0 \ for\ all\ \ a \in J\}$. Also we have $J^{\perp} = \{\phi^* \in \mathcal{A}^* : \langle \phi, a\rangle = 0\ for\ all\ \ \phi \in J\}$.

*speaker

Definition 2.1. Let $\mathcal{A}$ be a dual Banach algebra. A module homomorphism from $\mathcal{A}$ to $\mathcal{A}$ is a map $\varphi : \mathcal{A} \to \mathcal{A}$ with

$$\varphi(\alpha.a + b.\beta) = \alpha.\varphi(a) + \varphi(b).\beta, \quad \varphi(ab) = \varphi(a)\varphi(b) \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

In this paper we let that $\mathcal{L}^2_{\omega^*}(\frac{\mathcal{A}}{J}, \mathbb{C})$ denote the separately ω^*-continuous two-linear maps from $\frac{\mathcal{A}}{J} \times \frac{\mathcal{A}}{J}$ to $\mathbb{C}$, $\tilde{\omega} : \mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{A} \to \frac{\mathcal{A}}{J}$ be the multiplication operator with $\tilde{\omega}(a \otimes b + I) = ab + J$ and $\tilde{\varphi} : \frac{\mathcal{A}}{J} \to \frac{\mathcal{A}}{J}$ be the map that is defined by $\tilde{\varphi}(a + J) = \varphi(a) + J,\ a \in \mathcal{A}$.

Definition 2.2. Let $\mathcal{A}$ be a dual Banach algebra and $\varphi : \mathcal{A} \to \mathcal{A}$ be a bounded ω^*-continuous module homomorphism. An element $M \in \mathcal{L}^2_{\omega^*}(\frac{\mathcal{A}}{J}, \mathbb{C})^*$ is called a φ-module normal virtual diagonal for $\mathcal{A}$ if $\tilde{\omega}^{**}(M)$ is an identity for $\frac{\varphi(\mathcal{A})}{J}$ and

$$M.\tilde{\varphi}(c + J) = \tilde{\varphi}(c + J).M \qquad (c \in \mathcal{A}).$$

Let X be a dual Banach $\mathcal{A}$-bimodule. X is called normal if for each $x \in X$, the maps

$$\mathcal{A} \to X; \qquad a \to a.x, \quad a \to x.a$$

are ω^*-continuous. If moreover X is a $\mathcal{U}$-bimodule such that for $a \in \mathcal{A}, \alpha \in \mathcal{U}$ and $x \in X$

$$\alpha.(a.x) = (\alpha.a).x, \quad (a.\alpha).x = a.(\alpha.x), \quad (\alpha.x).a = \alpha.(x.a),$$

then X is called a normal Banach left $\mathcal{A}$-$\mathcal{U}$-module. Similarly for the right and two sided actions. Also, X is called symmetric, if $\alpha.x = x.\alpha \qquad (\alpha \in \mathcal{U}, x \in X)$.

Throughout this paper $\mathcal{H}_{\omega^*}(\mathcal{A})$ will denotes the space of all bounded module homomorphisms from $\mathcal{A}$ to $\mathcal{A}$ that are ω^*-continuous.

Definition 2.3. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, $\varphi \in \mathcal{H}_{\omega^*}(\mathcal{A})$ and let that X be a dual Banach $\mathcal{A}$-bimodule. A bounded map $D_{\mathcal{U}} : \mathcal{A} \to X$ is called a module φ-derivation if for every $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathcal{U}$, we have

$$D_{\mathcal{U}}(\alpha.a \pm b.\beta) = \alpha.D_{\mathcal{U}}(a) \pm D_{\mathcal{U}}(b).\beta, \quad D_{\mathcal{U}}(ab) = D_{\mathcal{U}}(a).\varphi(b) + \varphi(a).D_{\mathcal{U}}(b).$$

When X is symmetric, each $x \in X$ defines a module φ-derivation

$$(D_{\mathcal{U}})_x(a) = \varphi(a).x - x.\varphi(a) \qquad (a \in \mathcal{A}).$$

Derivations of this form are called inner module φ-derivation.

Definition 2.4. Let $\mathcal{A}$ be a dual Banach algebra, $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-module and $\varphi \in \mathcal{H}_{\omega^*}(\mathcal{A})$. $\mathcal{A}$ is called φ-Connes module amenable if for any symmetric normal Banach $\mathcal{A}$-$\mathcal{U}$-module X, each ω^*-continuous module φ-derivation $D_{\mathcal{U}} : \mathcal{A} \to X$ is inner.

Theorem 2.5. *Let $\mathcal{A}$ and $\mathcal{U}$ be dual Banach algebras, let $\mathcal{A}$ be a unital dual Banach $\mathcal{U}$-module and let $\mathcal{A}$ has an id-module normal virtual diagonal. Then $\mathcal{A}$ is id-Connes module amenable.*

Proof. Let X be a symmetric normal Banach $\mathcal{A}$-$\mathcal{U}$-module. We first note that $\mathcal{A}$ has an identity. It is therefore sufficient for $\mathcal{A}$ to be *id*-Connes module amenable that we suppose that X is unital. Let $D_{\mathcal{U}} : \mathcal{A} \to X$ be a module derivation that is ω^*-continuous. It is straightforward to see that E is a normal Banach $\frac{\mathcal{A}}{J}$-$\mathcal{U}$-module. Let $X = (X_*)^*$. Since X is symmetric, then $D_{\mathcal{U}}|_J = 0$. We define $\tilde{D_{\mathcal{U}}} : \frac{\mathcal{A}}{J} \to X$; $\tilde{D_{\mathcal{U}}}(a + J) := D_{\mathcal{U}}(a)\ (a \in \mathcal{A})$. To each $x \in X_*$, there corresponds $V_x : \frac{\mathcal{A}}{J} \times \frac{\mathcal{A}}{J} \to \mathbb{C}$ via $V_x(a + J, b + J) = \langle x, (a + J)\tilde{D_{\mathcal{U}}}(b + J)\rangle(a, b \in \mathcal{A})$. It is clearly that $V_x \in \mathcal{L}^2_{\omega^*}(\frac{\mathcal{A}}{J}, \mathbb{C})$. For each $a, b \in \mathcal{A}$ and $a_* \in \mathcal{A}_*$ we have

$$\langle \int ab + JdM, a_* + J^{\perp}\rangle = \langle M, \tilde{\omega}^*(a_* + J^{\perp})\rangle = \langle \tilde{\omega}^{**}(M), a_* + J^{\perp}\rangle.$$

Now, put $f(x) = \langle M, \nu_x\rangle(x \in X_*)$. Let $c \in \mathcal{A}$. After a little calculation, we obtain

$$\langle (c + J).f - f.(c + J)\rangle = \int \langle (ab + J)\tilde{D_{\mathcal{U}}}(c + J), x\rangle dM = \langle \tilde{\omega}^{**}(M).\tilde{D_{\mathcal{U}}}(c + J), x\rangle.$$

All in all, $D_{\mathcal{U}}(c) = c.f - f.c$ holds. $\square$

In Theorem 2.5 it is shown that if a unital Banach algebra $\mathcal{A}$ has an id-module normal virtual diagonal, then $\mathcal{A}$ is id-Connes module amenable. Let S be a semigroup, it would be interesting to know that the converse holds for inverse semigroup algebra $l^1(S)$. Thus for an inverse semigroup S, we consider an equivalence relation on S where $s \sim t$ if and only if there is $e \in E$ such that $se = te$. The quotient semigroup $S_G = \frac{S}{\sim}$ is a group [5]. Also, E is a symmetric subsemigroup of S. Therefore, $l^1(S)$ is a Banach $l^1(E)$-module with compatible canonical actions. Let $l^1(E)$ acts on $l^1(S)$ via

$$\delta_e.\delta_s = \delta_s, \qquad \delta_s.\delta_e = \delta_{se} = \delta_s * \delta_e \qquad (s \in S, e \in E).$$

With above notation, $l^1(S_G)$ is a quotient of $l^1(S)$ and so the above action of $l^1(E)$ on $l^1(S)$ lifts to an action of $l^1(E)$ on $l^1(S_G)$, making it a Banach $l^1(E)$-module [1].

The following theorem is the main result of the present paper.

Theorem 2.6. *Let S be a weakly cancellative semigroup. Let S be an inverse semigroup with idempotents E, let $l^1(S)$ be a Banach $l^1(E)$- module and let $\chi \in \mathcal{H}_{\omega^*}(l^1(S))$. If $l^1(S)$ is χ-Connes module amenable, then $l^1(S)$ has a χ-module normal virtual diagonal.*

Proof. Let $\pi : S \to S_G$ be the quotient map. By [1, Lemma 3.2], we define a bimodule action of $l^1(S)$ on $l^\infty(S_G)$ by

$$\delta_s.x = \delta_{\pi(s)} * x, \qquad x.\delta_s = x * \delta_{\pi(s)} \qquad (s \in S, x \in l^\infty(S_G)).$$

Since $c_0(S_G)$ is an introverted subspace of $l^1(S_G)$ then $l^1(S_G)^*$ is a normal Banach $l^1(S)$-$l^1(E)$-module. Choose $n \in l^1(S_G)^*$ such that $\langle n, 1\rangle = 1$, and define $D : l^1(S) \to l^1(S_G)^*$ by $D(\delta_s) = \chi(\delta_s).n - n.\chi(\delta_s)$. Moreover, D attains its values in the ω^*-closed submodule $(\frac{l^\infty(S_G)}{\mathbb{C}})^*$. Since $l^1(S)$ is χ-Connes module amenable, then D is inner. Consequently, there exists $\tilde{n} \in (\frac{l^\infty(S_G)}{\mathbb{C}})^*$ such that $D(\delta_s) = ad_{\tilde{n}}$, so

$$\tilde{\chi}(\delta_{\pi(s)}).n - n.\tilde{\chi}(\delta_{\pi(s)}) = \tilde{\chi}(\delta_{\pi(s)}).\tilde{n} - \tilde{n}.\tilde{\chi}(\delta_{\pi(s)})$$

Then we may define

$$\langle M, f\rangle = \lim_\alpha \int f\big(\tilde{\chi}(\delta_{\pi(x^*)}), \tilde{\chi}(\delta_{\pi(x)})\big) f_\alpha(x)dx.$$

Also for each s we obtain

$$\tilde{\omega}^{**}(M).\tilde{\chi}(\delta_{\pi(s)}) = \langle M, \tilde{\omega}^*(\tilde{\chi}(\delta_{\pi(s)}))\rangle = \lim_\alpha \int (\omega^*(\tilde{\chi}(\delta_{\pi(s)})))(\tilde{\chi}(\delta_{\pi(x^*)})), \tilde{\chi}(\delta_{\pi(x)}))f_\alpha(x)dx$$
$$= \lim_\alpha \tilde{\chi}(\delta_{\pi(s)}) \int f_\alpha(x)dx = \tilde{\chi}(\delta_{\pi(x)}).$$

Consequently, M is a χ-normal module virtual diagonal for $l^1(S)$. □

Theorem 2.7. *Let S be a weakly cancellative semigroup with idempotents E, let $l^1(S)$ be a unital dual Banach $l^1(E)$-module and let $l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S)$ be a dual Banach $l^1(E)$-module and $\chi \in \mathcal{H}_{\omega^*}(l^1(S))$. If $l^1(S)$ is χ-Connes module amenable, then $l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S)$ is $\chi\otimes_{l^1(E)}\chi$-Connes module amenable.*

Corollary 2.8. *Let S be a weakly cancellative semigroup, let S be an inverse semigroup with idempotents E and let $l^1(S)$ be a Banach $l^1(E)$-module. Then $l^1(S)$ is Connes module amenable if and only if $l^1(S)$ has a module normal virtual diagonal.*

Proof. This follows immediately from Theorem 2.5 and Theorem 2.6. □

Example 2.9. Let $(\mathbb{N}, \vee)$ be the semigroup of positive integers with maximum operation. Since $\mathbb{N}$ is weakly cancellative, then $l^1(\mathbb{N})$ is a dual Banach algebra with predual $c_0(\mathbb{N})$. By [3, Theorem 5.13], $l^1(\mathbb{N})$ is not Connes amenable. Moreover $l^1(\mathbb{N})$ is module amenable on $l^1(E_\mathbb{N})$, so it is Connes module amenable.

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References

1. M. Amini, *Module amenability for semigroup algebras*, Semigroup Forum, 69 (2004), 243-254.
2. M. Amini, A. Bodaghi, D. Bagha and D. Ebrahimi, *Module amenability of the second dual and module topological center of semigroup algebras*, Semigroup Forum, 80 (2010), 302-312.
3. M. Daws, *Connes amenability of bidual and weighted semigroup algebras*, Math. Scand., 99 (2006), 217-246.
4. B. E. Johnson, *Cohomology in Banach Algebras*, Mem. Amer. Math. Soc., 127 (1972).
5. W. D. Munn, *A class of irreducible matrix representations of an arbitrary inverse semigroup*, Proc. Glasgow Math. Assoc., 5 (1961), 41-48.
6. R. Rezavand, M. Amini, M. H. Sattari and D. Ebrahimi Bagha, *Module Arens regularity for semigroup algebras*, Semigroup Forum, 77 (2008), 300-305.

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Convolution and convolution type C*-algebras

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ABSTRACT. Product of two functions, Dirichlet product of two arithmetic functions are examples of convolution. Convolutions play important roles in functional analysis, harmonic analysis and measure theory. Veldsman defined a more general convolution under the name convolution type in order to introduce convolution rings in a very general algebraic setting. Although convolution types are independent of any algebraic considerations, but they can imposed on rings so that many important rings constructions can be covered through this approach. Any semigroup define a convolution type, while there are some convolution types that can not define by a semigroup. We define a kind of convolution type as the definition of Veldsman by a little modification of the original definition of convolution type. The modification is useful to define convolution type C*-algebra. For example, we define a notion of involution type. We use the notion of involution type to our new definition of convolution type. We use C*- algebras instead of rings and define the notion of convolution type C*- algebra. There are some familiar C*- algebras which can be considered as a convolution type C*- algebra. For example, discrete crossed product of C*- algebras, matrix C*-algebras, finite dimensional C*- algebras, are some examples of convolution type C*-algebra. We show that any convolution type C*- algebra can be written as a tensor product of C*- algebras. The notion of weighted convolution type C*-algebra is also introduced. This notion is a generalization of convolution type C*-algebra.
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1. Introduction

The term convolution is used to many fields of mathematics. For example, the Dirichlet product of arithmetic functions f, g defined by $f * g(n) = \sum_{d|n} f(d)g(\frac{n}{d})$ is called convolution of f and g. The Cauchy product of arithmetic functions f, g defined by $f * g(n) = \sum_{n=i+j} f(i)g(j)$ is also a convolution of f and g. Dirichlet product and cauchy product are useful for Dirichlet series and power series. Let M be a semigroup and let $\ell_1(M)$ be the set of all complex functions $f : M \to \mathbb{C}$ with a countable support(the set $\{x : f(x) \neq 0\}$is at most countable)such that $\sum_{x\in M} |f(x)|$ is finite. Then $\ell_1(M)$ is a Banach *-algebra with the usual involution and addition. The multiplicatin is defined by $(f \star g)(x) = \sum_{st=x} f(s)g(t)$. An involutive semigroup is a semigroup M together with a function$* : M \to M$ such that $x^{**} = x$ and $(xy)^* = y^*x^*$ for all x, y in M. Any multiplicative semigroup of a *-algebra is an involutive semigroup. An inverse semigroup is a semigroup M together with a function $* : M \to M$ such that $xx^*x = x$ and $x^*xx^* = x^*$. It is shown that any inverse semigroup is an involutive semigroup with involution $*$. Any group G is an involutive semigroup with the involution $x^* = x^{-1}$ for any $x \in G$. Suppose that M is an involutive semigroup and A is a C*-algebra. Consider the set $\ell_1(M, A)$ as before (replace $\mathbb{C}$ by A) with involution $f^*(x) = (f(x^*))^*$. The enveloping C*-algebra of $\ell_1(M, A)$ is a C*-algebra. We are going to generalize this notion. We generalize M to the convolution type C*-algebras. Convolution type is defined by Veldsman [5]. He defined the notion of convolution type $\mathcal{T}$ and by using a ring R he defined a new ring $\mathcal{T}(R)$. We use the definition of Veldsman (by a little modification)and replace R by a C*-algebra A. We define a C*-algebra denoted by $\mathcal{T} \star A$ that can be considered as a generalization of discrete crossed product of C*-algebras and C*-semigroups. Most of the results of this paper are in the reference [2] with a little generalization. Any convolution type define a

*speaker

functor on the category of C*-algebras. We show that $\mathcal{T} \star A \cong (\mathcal{T} \star \mathbb{C}) \otimes A$. We define the notion of weighted convolution type C*-algebras as a generalization of convolution type C*-algebras.

2. Main results

The following definition is from [2].

Definition 2.1. A convolution type $\mathcal{T}$ is a quadruple $\mathcal{T} = (X, \gamma, \beta, T)$ satisfy the following conditions

(i) X is a non-empty set and $T \subseteq X$ is non-empty
(ii) For any x in X there is a non-empty subset $\gamma(x)$ of $X \times X$ such that $\gamma(x) \cap \gamma(y) = \varnothing$ for $x \neq y$,
(iii) For any x in X if $(s,t) \in \gamma(x)$ and $(p,q) \in \gamma(s)$ there is a unique v in X such that $(p,v) \in \gamma(x)$ and $(q,t) \in \gamma(v)$. If $(s,t) \in \gamma(x)$ and $(p',q') \in \gamma(t)$ there is a unique u in X such that $(u,q') \in \gamma(x)$ and $(s,p') \in \gamma(u)$,
(iv) $(t,t) \in \gamma(t)$ for any t in T,
(v) For any x of X there exists a unique (t,t') of $T \times T$ with $(t,x) \in \gamma(x)$ and $(x,t') \in \gamma(x)$,
(vi) If $(s,t) \in \gamma(x)$ and $s \in T$ then $t = x$ and if $(s,t) \in \gamma(x)$ and $t \in T$ then $s = x$,
(vii) An involution type β is a function $\beta : X \to X$ such that $\beta^2 = id_X$ and $(s,t) \in \gamma(x)$ if and only if $(\beta(t), \beta(s)) \in \gamma(\beta(x))$.

Note that we add the conditions (ii) and (iv) to the original definition of convolution type. In the following we define three kinds of convolution types.

Definition 2.2. Let $\mathcal{T} = (X, \gamma, \beta, T)$ be a convolution type.$\mathcal{T}$ is called

(i) Simple if $T = \{t\}$ and $\gamma(x) = \{(t,t)\}$,
(ii) Full if for any elements s,t of X there is an element x of X such that $(s,t) \in \gamma(x)$
(iii) Symmetric if $\beta(x) = x$ for any element x of X.

It is easy to say that a convolution type $\mathcal{T}$ is symmetric if and only if

$$(s,t) \in \gamma(x) \iff (t,s) \in \gamma(x)$$

and this shows that the term symmetric is suitable.

In the following there are some examples of convolution types.

Example 2.3.

(i) The convolution type defined by $\mathcal{T}_P = (X, \gamma, \beta = id_X, T = X)$,and $\gamma(x) = \{(x,x) : x \in X\}$, is called product convolution type. The convolution type $\mathcal{T}_P$ is neither full nor simple,
(ii) The convolution type defined by $\mathcal{T}_C = (X = \{0,1,2,3,...\}, \gamma, \beta = id_X, T = \{0\})$ and $\gamma(n) = \{(i,j) : i + j = n\}$, is called Cauchy product convolution type. The convolution type $\mathcal{T}_C$ is both full and simple,
(iii) The convolution type defined by $\mathcal{T}_C = (X = \{1,2,3,...\}, \gamma, \beta = id_X, T = \{1\})$ and $\gamma(n) = \{(i,j) : ij = n\}$, is called Dirichlet product convolution type. $\mathcal{T}_C$ is both full and simple,
(iv) Let M be an involutive semigroup (monoid) and $\mathcal{T}_M = (M, \gamma, \beta, T = \{e\})$ and $\gamma(x) = \{(s,t) : st = x\}$ and $\beta(x) = x^*$. $\mathcal{T}_M$ is a full convolution type,
(v) Let S be a set of k prime numbers and let $X = \{p^m : p \in S, m \geq 0\}$. Define $\mathcal{T}_S = (X, \gamma, \beta = id_X, T = \{1\})$ and $\gamma(n) = \{(i,j) : ij = n\}$. Then $\mathcal{T}_S$ is a simple and full convolution type. $\mathcal{T}_S$ is called prime power product convolution type,
(vi) Let $\mathcal{T}_U = (X, \gamma, \beta = id_X, T = \{1\})$, where $X = \{1,2,3,...\}$ and $\gamma(n) = \{(i,j) : ij = n, (i,j) = 1\}$, where (i,j) is the greatest common divisor of i,j.$\mathcal{T}_U$ is called the unitery convolution type which is simple but not full,

(vii) $\mathcal{T}_n = (X, \gamma, \beta = id_X, T)$ where $X = \{(i,j) : i,j = 1,2,3,..,n\}$, $T = \{(i,i) : i = 1,2,3,...,n\}$ and $\gamma(i,j) = \{(i,k),(k,j)) : k = 1,2,3,...,n\}, \beta(i,j) = (j,i)$ is the matrix convolution type.

Note that the convolution types (vi) and (vii) can not define by a semigroup so the convolution type is a generalization of a semigroup.

Denote by $\ell_1(\mathcal{T}, A)$ the set of all functions $f : X \to A$ such that $||f||_1 = \sum_{x\in X} \|f(x)\| < \infty$. Define on $\ell_1(\mathcal{T}, A)$ addition ,multiplication and involution by

(1) $(f+g)(x) = f(x) + g(x)$
(2) $(f \star g)(x) = \sum_{(s,t)\in\gamma(x)} f(s)g(t)$
(3) $f^*(x) = (f(\beta(x))^*$

for any x, y in X. part (2) shows that any convolution type define a convolution between two functions f and g. The normed space $\ell_1(\mathcal{T}, A)$ is a Banach *-algebra and it has a non-trivial representation provided that $\mathcal{T}$ is full or simple.We denote by $\mathcal{T} \star A$ the C*-envelope of $\ell_1(\mathcal{T}, A)$ as the convolution type C*-algebra of A by $\mathcal{T}$. In [2] we show that if $\mathcal{T}$ is a simple or full convolution type, then $\ell_1(\mathcal{T}, A)$ has a non-trivial representation. Therefore the C*-envelope of $\ell_1(\mathcal{T}, A)$ is not trivial for any non-trivial C*-algebra A. Suppose that T is a finite set in the convolution type $\mathcal{T} = (X, \gamma, \beta, T)$. Then the map $L : A \to \ell_1(\mathcal{T}, A)$ defined by

$$L(a)(x) = L_a(x) = \begin{cases} a & \text{if } x \in T \\ 0 & \text{if} x \notin T \end{cases}$$

is an embedding of A in $\ell_1(\mathcal{T}, A)$ as a Banach $*$-subalgebra. Therefore if A has an identity 1 then L_1 is the identity of $\ell_1(\mathcal{T}, A)$. Suppose that $\mathcal{T}$ is a convolution type. $\mathcal{T}$ define a covariant functor $- * \mathcal{T}$. Therefore if $A \cong B$ for two C*-algebras A and B, then $\mathcal{T} * A \cong \mathcal{T} * B$.

The following results can be found in [2]

Example 2.4.

(i) $\mathcal{T}_C \star A \cong C([0,1], A)$,
(ii) $\mathcal{T}_S \star A \cong C([0,1]^k, A)$,
(iii) $\mathcal{T}_D \star A \cong C([0,1]^{\mathbb{N}}, A)$,
(iv) $\mathcal{T}_n \star A \cong M_n(A)$,
(v) $\mathcal{T}_G \star A \cong G \times_t A$,where G is a discrete group and $G \times_t A$ is the trivial crossed product,
(vi) For any finit dimensional C*-algebra A there is a convolution type $\mathcal{T}$ such that $A \cong \mathcal{T} \star \mathbb{C}$.

We denote $\ell_1(\mathcal{T}, \mathbb{C})$ and $\mathcal{T} * \mathbb{C}$ by $\ell_1(\mathcal{T})$ and $C^*(\mathcal{T})$, respectivly. The notation $\ell_1(\mathcal{M})$ is used instead of $\ell_1(\mathcal{T}_M)$ provided that M is an involutive semigroup. We have

$$\mathcal{T}_M \star \mathbb{C} = C^*(M)$$

. In particular $C^*(M)$ is the full semigroup C*-algebra, where M is an inverse semigrop. We see from the above example the following equivalences

(i) $\mathcal{T}_C \star A \cong C[0,1] \otimes A$,
(ii) $\mathcal{T}_S \star A \cong C([0,1]^k) \otimes A$,
(iii) $\mathcal{T}_D \star A \cong C([0,1]^{\mathbb{N}}) \otimes A$,

which lead to the following theorem

Theorem 2.5. *Let $\mathcal{T} = (X, \gamma, \beta, T)$ be a convolution type and A be a C*-algebra. Then*

$$\mathcal{T} \star A \cong C^*(\mathcal{T}) \otimes A$$

Proof. To prove the theorem we need the statement[1, Theoem 2]:
Let (Y, μ) be a measure space, where Y is a locally compact hausdorff space. Let A be a Banach space and let $\ell_1(Y, A)$ be the set of all measurable functions $f : Y \to A$ such that $\int_Y ||f||d\mu < \infty$. Then

$$\ell_1(Y, A) \cong \ell_1(Y)\hat{\otimes}A$$

, where the tensor product $\hat{\otimes}$ is the projective tensor product of Banach algebras [3]. Therefore we have the following equivalence

$$\ell_1(\mathcal{T}, A) \cong \ell_1(\mathcal{T})\hat{\otimes}A$$

. Hence [3, Theorem 3]

$$C^*(\ell_1(\mathcal{T}, A)) = C^*(\ell_1(\mathcal{T})\hat{\otimes}A) = C^*(\mathcal{T}) \otimes A$$

, where the tensor product $\otimes$ is the maximal tensor product of C*-algebras. Therefore

$$\mathcal{T} \star A = C^*(\mathcal{T}) \otimes A$$

□

We are interested in a generalization of the convolution type which may not satisfies(or satisfies) the above Theorem. Weighted semigroup is used for many papers as in [4]. We generealize this notion to convolution types. The following generalization may be useful. Let $\mathcal{T} = (X, \gamma, \beta, T)$ be a convolution type and let A be a C*-algebra. A weight ω is a function

$$\omega : X \to (0, +\infty)$$

such that

$$\omega(x) \leq \omega(s)\omega(t) \qquad ((s,t) \in \gamma(x))$$

for all $x \in X$.Let $\ell_1(\mathcal{T}, A, \omega)$ be the set of all functions $f : X \to A$ such that

$$||f||_\omega = \sum_{x\in X} \omega(x)||f(x)|| < \infty$$

. Let $\omega \geq 1$, then $\ell_1(\mathcal{T}, A, \omega)$ is a Banach *-subalgebra of $\ell_1(\mathcal{T}, A)$. Denote by

$$\mathcal{T} \star_\omega A$$

the C*-envelope of $\ell_1(\mathcal{T}, A, \omega)$ as a weighted convolution type C*-algebra. For example, let M be an involutive semigrop and let $\omega(m_1m_2) \leq \omega(m_1)\omega(m_2)$ be a weight on M. Then $\ell_1(M, \omega)$ is a Banach *-algebra and $C^\star(M, \omega)$ is the C*-envelope of $\ell_1(M, \omega)$.

Example 2.6.

(i) Let $\mathcal{T}_C = (X, \gamma, \beta, T)$ be the Cauchy product convolution type. Define $\omega(n = i + j) = e^{\sqrt{i+j}} \leq e^{\sqrt{i}}e^{\sqrt{j}} = \omega(i)\omega(j)$

(ii) Let $\mathcal{T}_D = (X, \gamma, \beta, T)$ be the Dirichlet product convolution type. Define $\omega(n) = n = ij = \omega(i)\omega(j)$

(ii) Let $\mathcal{T}_n$ be the matrix convolution type. Define $\omega(i, j) = ij$, then

$$ij = \omega(i, j) \leq ijk^2 = \omega(i, k)\omega(k, j)$$

I ask two important questions about the weighted convolution type C*-algebras:

Question 2.7. Is there a relation between $\mathcal{T} \star_\omega A$ and $\mathcal{T} \star A$?

Question 2.8. Is $\mathcal{T} \star_\omega A$ equivalent to a tensor product as the Theorem?

References

1. A.Grothendieck, *Produits tensoriels topologiqueset et espaces nucleaires.*, Mem.Amer.Math.soc.no 16 (1955).
2. K.Nourouzi,A.Reza, *Convolution type C*-algebras*, BIMS Vol 46 issue 3 (2020)
3. T.Okayasu, *On the tensor products of C*-algebras*,Tohoku Math.J(2)18(1966)
4. M. Soroshmehr, *Weighted Ress matrix semigroup and their applications*,Arch.Math.100(2013)
5. S.Veldsman,*Convolution rings*,Algebra Colloq.13 ,no 2 (2006)

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مجموعه مقاله‌های فارسی

ارائه روشی برای بازسازی تصاویر توموگرافی با استفاده از یادگیری عمیق و تبدیل قیچک

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چکیده. امروزه روش های تصویربرداری، پردازش و بازسازی تصاویر در زمینه پزشکی دچار رشد چشمگیری شده است.اما همچنان روش های به کار رفته برای بازسازی تصاویر، به طورخاص بازسازی تصاویر توموگرافی از دقت بسیار بالایی برخوردار نمی باشد. ایده اصلی موجود در این مقاله، استفاده ازاطلاعات شاخه ای از ریاضیات به نام آنالیز ریزموضعی است که در رابطه با موقعیت و جهت انتشار تکینگی های یک جسم پس از اثر یک تبدیل بر آن مانند تبدیل ری به دست می آید.در واقع می توانیم با استفاده از تکینگی ها یا همان لبه های موجود در یک تصویر و اطلاعات به دست آمده از انالیز ریزموضعی، لبه ها و تکینگی های موجود در تصویر بازسازی شده را تشخیص دهیم و در نتیجه دقت را هنگام بازسازی تصاویر توموگرافی افزایش دهیم. برای تشخیص جهت و موقعیت این تکینگی ها می توانیم از یک تبدیل جهت دار در آنالیزهارمونیک به نام تبدیل قیچک و خواصی جهتی آن بهره ببریم و با استفاده از ضرایب دیجیال تبدیل قیچک یک شبکه عصبی عمیق را برای تشخیص این تکینگی ها و بازسازی با کیفیت تصاویر توموگرافی آموزش دهیم.

واژه‌های کلیدی: منظم سازی، مسائل معکوس، جبهه موج، آنالیزهارمونیک،تبدیل چند مقیاس جهت دار.
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۱. پیش‌گفتار

امروزه شاهد پیشرفت چشمگیری در بسیاری از زمینه های صنعتی، پزشکی، کشاورزی، صنایع دفاعی و امنیتی هستیم. این رشد و پیشرفت تنها به دلیل مطالعه و بررسی عمیق و مفهومی علومی مانند مدلسازی، شبیه سازی، بهینه سازی و ریاضیات محض می باشد. ماشین های خودران، ربات های جراحی از راه دور، مراقبت های بهداشتی، سیستم های امنیتی مبتنی بر پردازش تصویر و بسیاری از فناوری های دیگر که زندگی انسان امروز را دچار تغییرات فراوانی کرده است همه حاصل چنین مطالعاتی می باشند. اما مدل سازی های مبتنی بر نظریات کلاسیک و محاسبات نمادین به تنهایی در پس این تغییرات نیستند در واقع دانشمندان با استفاده ازهوش مصنوعی و روش های یادگیری ماشین توانسته اند بسیاری از پدیده های طبیعی را پیش بینی و شبیه سازی کنند. یکی از مفاهیمی که در زمینه ی پردازش تصاویر پزشکی مانند تصاویر حاصل از CT و MRI وجود دارد مفهومی به نام حل مسائل معکوس می باشد. به دلیل اهمیت و کاربرد این مفهوم، امروزه در زمینه های محاسبات کامپیوتری و روش های عددی قابل اطمینان و پیچیده در دهه های اخیر به طور گسترده ای توسعه یافته است. تئوری اساسی برای حل مسائل معکوس، منظم سازی می باشد و شامل حل یک معادله عملگری به صورت $Kx = y$ از طریق روش های منظم سازی است. که $K : X \to Y$ یک عملگر خطی یا غیرخطی فشرده بین دو فضای هیلبرت X, Y می باشد. دو معیار منحصربه فرد بودن جواب و پایداری آن در مسائل معکوس از اهمیت خاصی برخوردار هستند. اگر پاسخ مسئله منحصر به فرد نباشد، آنگاه بایستی ثبات روش های حل مسئله مورد بررسی قرار بگیرد به عبارت دیگر مشخص شود که شرایط اولیه و یا تنوع داده ها تا چه اندازه بر روی پاسخ نهایی موثر است. مسائلی را که پاسخ منحصر به فرد نداشته و روش های حل آن ناپایدار است را یک "مسئله بد وضعیت" می نامیم. اطلاعاتی را که با استفاده از روابط یا مشاهدات پیرامون مسئله معکوس برای حل آن و کمک به پایداری پاسخ مسئله معکوس استفاده می شود را "اطلاعات پیشین" می نامیم و این روش حل مسائل معکوس را منظم سازی می نامیم.

همانطورکه پیش تر بیان شد بازسازی تصاویر مربوط به سی تی اسکن یک مسئله معکوس در زمینه پردازش تصاویر است که تا کنون تکنیک های متفاوتی برای بازسازی این تصاویر، مانند روش منظم سازی تیخونوف با استفاده از ضرایب موجک و قیچک یا FBP ارائه شده است. ایده اصلی ما در ادامه مقاله برای حل این مسئله معکوس، ایجاد یک تعریف جدید برای تبدیل قیچک و تلفیق آن با مفاهیم موجود در آنالیز ریز موضعی برای بازسازی تصاویر توموگرافی می باشد،فرض کنید یک پزشک به دنبال یافتن تومور در بدن بیمار می باشد، اغلب شکل و مکان قرار گرفتن آن تومور برای گرفتن تصمیمات لازم جهت درمان بیمار کافی است. محل و شکل تومور را می توان به راحتی از قسمت منفرد تصویر تعیین کرد. به همین دلیل، کاربردهای فراوانی از آنالیز ریزموضعی در تصویربرداری توموگرافی وجود دارد. در این کاربردها، تبدیلی که اجسام را به داده تبدیل می کند تبدیل ری است که می تواند به عنوان یک عملگر انتگرال فوریه تفسیر شود. بدین ترتیب، می توان رابطه بین مجموعه تکینگی ها وجهت تابع (جسم مورد مطالعه) که در ادامه از آن به نام جبهه موج

* سخنران

تابع یاد می شود و نسخه تبدیل شده آن (داده های توموگرافی) را به صراحت توصیف کرد. از این گونه روابط به عنوان روابط متعارف (ریزموضعی) یاد می شود. در ادامه استخراج جبهه های موج تصویر یک شبکه عصبی عمیق را با اطلاعات لبه های موجود در تصاویر و یا از دیدگاه ریاضی، همان تکینگی های موجود در $L^2(\mathbb{R}^2)$ و جهت آن ها، برای بازسازی تصاویر آموزش می دهیم. در پایان خواهیم دید که شبکه عصبی ایجاد شده نسبت به سایر روش های بازسازی تصاویر توموگرافی عملکرد مناسب تری دارد.

۲. هندسه تصاویر و تقریب لبه ها با قیچک

برای این که بتوانیم یک تقریب مناسب از لبه های موجود در تصاویر داشته باشیم تبدیل های مختلفی مانند موجک،خمک و لبک [۵] توسط ریاضیدانان ارائه و بررسی شده است که از میان آن ها تبدیل قیچک به دلیل وجود دو پارامتر اتساع ناهمسانگرد و جهت برای تقریب بهینه این لبه ها بسیارمناسب است. تبدیل قیچک، مبتنی بر سه عمل انتقال، اتساع نایکروند و قیچی، روی تابع مولد است.دلیل این نام گذاری، کنترل خواص جهتی تصویر، از طریق عملگر قیچی است. برای اتساع وقیچی یک تابع ، سه ماتریس زیر را تعریف می کنیم

$$A_a := \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix} \quad \tilde{A}_a := \begin{pmatrix} \sqrt{a} & 0 \\ 0 & a \end{pmatrix}, \quad S_s := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad a > 0, s \in \mathbb{R}$$

سپس در نظر می گیریم $(a, s, t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^2, \quad \psi \in L^2(\mathbb{R}^2), \quad x \in \mathbb{R}^2$ و عملگرهای زیر را تعریف می کنیم

$$\psi_{a,s,t,1}(x) := a^{\frac{-3}{4}} \psi(A_a^{-1} S_s^{-1}(x - t)), \qquad \psi_{a,s,t,-1}(x) := a^{\frac{-3}{4}} \tilde{\psi}(\tilde{A}_a^{-1} S_s^{-T}(x - t)) \tag{1}$$

که در آن $\tilde{\psi}(x_1, x_2) := \psi(x_2, x_1)$ برای هر $x = (x_1, x_2) \in \mathbb{R}^2$.

تعریف ۱.۲. فرض کنید $\psi \in L^2(\mathbb{R}^2)$.تابع $\psi_{a,s,t,\iota} : \mathbb{R}^2 \to \mathbb{R}$ که در آن $(a, s, t, \iota) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^2 \times \{-1, 1\}$. و در ۱ تعریف شد، یک سیستم قیچک نامیده می شود. قیچک پیوسته متناظر با سیستم قیچک فوق به صورت زیر تعریف می شود

$$\mathcal{SH}_\psi : L^2(\mathbb{R}^2) \to L^\infty(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2 \times \{-1, 1\})$$
$$\mathcal{SH}_\psi(f)(a, s, t, \iota) := \langle f, \psi_{a,s,t,\iota} \rangle.$$

تبدیل قیچک هیچ اطلاعات اضافی در باره هندسه مجموعه تکینگی های تابع نمی دهد. در صورتی که برای تشخیص لبه ها نه تنها مکان تکینگی بلکه شکل هندسی تکینگی نیز شدیدا حائز اهمیت است در ادامه نشان می دهیم تبدیل قیچک، چگونه مکان و هندسه مجموعه تکینگی ها را به دقت آشکارسازی می کند. یک نماد گذاری مناسب برای نمایش مکان تکینگی ها و جهت آن ها برای تابعی مانند f، $WF(f)$ است که در ادامه آن را تعریف می کنیم[۲].

تعریف ۲.۲. فرض کنید $f \in L^2(\mathbb{R}^2)$ و $k \in \mathbb{N}$. نقطه $(x, \lambda) \in \mathbb{R}^2 \times \mathbb{S}^1$ را یک $k-$ امین نقطه منظم جهت دار f می نامیم اگر همسایگی های U_x از x،V_λ از λ و تابع هموار $\phi \in C^\infty(\mathbb{R}^2)$ که $\phi(x) = 1, supp\phi \subset U_x$ وجود داشته باشند، به طوریکه

$$|\widehat{(\phi f)}(\xi)| \leq C_k(1 + |\xi|)^{-k} \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\} \quad s.t \quad \frac{\xi}{|\xi|} \in V_\lambda$$

به ازای $C_k > 0$ برقرار باشد.

$k-$ امین مجموعه جبهه موج تابع f را با نماد $WF_k(f)$ نمایش می دهیم و متمم مجموعه ی تمام $k-$ امین نقاط منظم جهت دار است. مجموعه جبهه موج تابع f را بانماد $WF(f)$ نمایش می دهیم و به صورت زیر تعریف می شود:

$$WF(f) := \cup_{k \in \mathbb{N}} WF_k(f)$$

تعریف و ایده اصلی مجموعه جبهه موج، مبنی بر توصیف میزان همواربودن یک تابع از نظر تباهیدگی تبدیل فوریه ی آن تابع است. در واقع هرچه میزان تباهیدگی یک تابع بیشتر شود آن تابع در آن نقطه هموارتر است. در [۵] نشان داده شده است که اگر تابع مولد ψ دارای جهت محوشدگی باشد، رفتار مجانبی متغیر a وقتی $a \to 0$ تبدیل قیچک پیوسته تابع $f \in L^2(\mathbb{R}^2)$ مجموعه جبهه موج آن را مشخص می کند.به عبارت دیگر داریم:

$$WF(f)^c = \{(t_0, s_0) \in \mathbb{R}^2 \times [-1, 1] : \forall (t, s) \in U_{(t_0, s_0)} :$$
$$|\mathcal{SH}_\psi f(a, s, t)| = \mathcal{O}(a^k), \quad a \to 0, \forall k \in \mathbb{N}\}$$

۳. تبدیل رادون

یک اسکنر CT با نمونه برداری از تبدیلی به نام تبدیل رادون، عمل می کند. تبدیل رادون یک تابع مانند $f \in L^{2}(\mathbb{R}^{2})$ روی خط L به صورت زیر تعریف می شود.

$$\mathcal{R}f = \int_{L(s,\theta)} f(x)dS(x)$$
$$L(s,\theta) = \{x \in \mathbb{R}^{2} : x_1 \cos(\phi) + x_2 \sin(\phi) = s\}, \phi \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}), s \in \mathbb{R}$$

وظیفه بازسازی CT بازیابی تابع f از نمونه های مخدوش شده سینوگرام آن است. این کار یک مثال کلاسیک از یک مسئله معکوس بد وضعیت است [۶]. تبدیل رادون به صورت گسترده‌ای در توموگرافی یا برش‌نگاری کاربرد دارد. در این نوع از مسائل تصویر برداری، اشعه همواره در فاصله $[-\frac{\pi}{2}, \frac{\pi}{2}]$ تابیده می شود و هنگام بازسازی تصاویر ایجاد شده مشکلی ایجاد نمی شود، اما چالش اصلی زمانی ایجاد می شود که پرتو ها، به هر دلیلی در بازه محدود تری به شی مورد مطالعه تابیده شود یعنی هنگامی که $[-\phi, \phi] \subset [-\frac{\pi}{2}, \frac{\pi}{2})$. در ادامه یک روش داده محور براساس شبکه های عصبی عمیق که براساس ضرایب قیچک آموزش می بینند را معرفی میکنیم. فرض کنید یک استخراج کننده جبهه موج دیجیتال که در[۲] معرفی شده است داریم، لذا می توانیم الگوریتم مربوطه را بر روی اطلاعات نمونه گیری شده از سینو گرامf، که آن را g می نامیم اعمال کنیم. رابطه بین جبهه موج یک تابع و جبهه موج آن تابع تحت تبدیل عملگر انتگرال فوریه با نام رابطه میکروکانونی [1] شناخته می شود. عملگر تبدیل رادون $\mathcal{R}$ و عملگر الحاقی آن $\mathcal{R}^*$ مثال هایی از عملگرهای انتگرال فوریه هستند . برای توصیف رابطه یاد شده قضیه ی زیر رابیان می کنیم [۶] .

قضیه ۱.۳ (رابطه میکروکانونی تبدیل رادون). فرض کنید f یک تابع با تکیه گاه فشرده باشد و $x_0 \in L(s_0, \varphi_0)$، $\theta_0 = \theta(\varphi_0)$، $\lambda_0 = \frac{(1, -x_0 . \theta_0^{\perp})}{(1+(x_0 . \theta_0^{\perp})^2)}^{\frac{1}{2}}$ و $k \in \mathbb{N}_0$ باشد.رابطه میکروکانونی همان تناظر جبهه های موج است یعنی

$$(x_0; \theta_0) \in WF_k(f) \Longleftrightarrow ((s_0, \varphi_0); \lambda_0) \in WF_{k+\frac{1}{2}}(\mathcal{R}f), \tag{۲}$$

و در حالت خاص با تعمیم مفهوم موج به صورت $WF(f) = \cup_{k\in\mathbb{N}_0} WF_k(f)$ نیز روابط بالا برقرار است وداریم

$$(x_0; \theta_0) \in WF(f) \Longleftrightarrow ((s_0, \varphi_0); \lambda_0) \in WF(\mathcal{R}f).$$

از آنجا که قضیه ۱.۳ یک تناظر یک به یک بین اعضای مجموعه $WF(f)$ و اعضای مجموعه $WF(\mathcal{R}f)$، بیان می کند می توانیم یک نگاشت به نام **can**، را برای هر تابع f با تکیه گاه فشرده به صورت زیر تعریف کنیم

$$\mathbf{can} : \mathbb{R}^2 \times \mathcal{P}(\mathbb{S}^1) \to \mathbb{R}^2 \times \mathcal{P}(\mathbb{S}^1)$$
$$\mathbf{can}(WF(f)) = WF(\mathcal{R}f)$$

اکنون که موفق به محاسبه جبهه موج مربوط به $\mathcal{R}f$ شده ایم می توانیم آن ها را به عنوان اطلاعات پیشین در نظر گرفته و برای منظم سازی مسائل معکوس بهره ببریم، به عبارت دیگر هدف ما مینیم سازی مقادیر زیر است

$$\|\mathcal{R}f - g\|_2^2, \quad d(WF(f), \mathbf{can}^{-1}(WF(g)) \text{ ویا } d(\mathbf{can}(WF(f)), WF(g)),$$

که در آن d یک متر مناسب به جهت سنجش فاصله بین دو جبهه موج یک تابع مانند متر هاسدورف است.

۴. یادگیری عمیق وبازسازی تصاویر

در رویکرد ارائه شده در این مقاله برای طبقه بندی و تشخیص تکینگی ها، از ضرایب قیچک برای آموزش شبکه عصبی استفاده می کنیم به این طریق که شبکه عصبی ضرایب قیچک دیجیتال یک تصویر [۳] را دریافت و سپس زوج مرتبی از موقعیت و جهت را به عنوان جبهه موج تصاویر ورودی معرفی می کند. سپس با استفاده از جبهه موج تصاویر به دست آموده و اعمال روابط کانونی معرفی شده در قسمت قبل به بازسازی تصاویر توموگرافی می پردازیم. اندازه این دسته ها به ابعاد ۲۱ × ۲۱ × ۴۹ است. معماری شبکه شامل چهار لایه کانولوشن است ، با حداکثر ۲ × ۲ پیچش ، تابع فعالسازی آن تابع فعال ساز ReLU و نرمال سازی آن نرمالسازی دسته ای می باشد ،پس از آن یک لایه کاملاً متصل با ۱۰۲۴ نورون وتابع فعال سازی softmax قرار دارد [۱].

[1] microcanonical relation

۵. مقایسه ونتیجه گیری

استخراج مجموعه جبهه موج داده ها و سپس اعمال روابط متعارف کانونی این مزیت محاسباتی را دارد که دیگر نیازی به حل مسائل معکوس نداریم. برای اثبات این ادعا آزمایش زیر را انجام می دهیم. با استفاده از سه روش منظم سازی مسائل معکوس با نام های تصویرمعکوس فیلتر شده، تیخونوف منظم شده و وارونگی منظم تغییرات کلی، ابتدا یک معکوس تقریبی از یک تصویر توموگرافی را محاسبه می کنیم و سپس مجموعه جبهه موج مرتبط را با استفاده از شبکه ایجاد شده محاسبه می کنیم[۴]. سپس میانگین مربعات خطای مجموعه جبهه موج به دست آمده با جبهه موج اصلی تصاویر را محاسبه می کنیم. سپس اعداد به دست آمده را با خطای حاصل از روش محاسبه مجموعه جبهه موج تصویر سینوگرام و تبدیل آن به مجموعه جبهه موج تصویر شی مورد مطالعه، از طریق روابط متعارف مقایسه می کنیم. نتایج جدول زیر به وضوح مزیت استخراج مجموعه جبهه موج سینوگرام و سپس اعمال روابط متعارف را نسبت به رویکردی که ابتدا معکوس سازی و سپس استخراج جبهه موج را انجام می دهد، بیان می کند.

جدول ۱: عنوان جدول

تکنیک های معکوس سازی	**میانگین مربع خطا**
Tikhonov	۴۴۳٫۰
variation Total	۳۸۰٫۹
backprojection Filter	۵۰۴٫۳
relation Canonical	۱۶۸٫۱

مراجع

1. A.K. Dombrowski, C.J. Anders, K.R. Müller,P. Kessel, “Towards robust explanations for deep neural networks”, Pattern Recognition 121 (2022) 108–194
2. A. Loarca, G. Kutyniok, O. Oktem, and P.C. Petersen, “Extraction of digital wavefront sets using applied harmonic analysis and deep neural networks”, SIAM Journal on Imaging Sciences, 12(2019), pp.1936–1966.
3. G. Kutyniok, W.-Q. Lim, and X. Zhuang, Digital Shearlet Transforms, in Shearlets, Springer, New York, 2012, pp. 239–282
4. G. Kutyniok, Shearlets: From Theory to Deep Learning. Handbook of Mathematical Models and Algorithms in Computer Vision and Imaging: Mathematical Imaging and Vision,2021, pp.1-38.
5. P. Grohs, “Continuous shearlet frames and resolution of the wavefront set”, Monatsh. Math., 164 (2011), pp. 393–426
6. E. T. Quinto, An Introduction to X-Ray Tomography and Radon Transforms, Proc. Sympos. Appl.Math., 63, AMS, Providence, RI, 2006

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Investigation of some problems on the generalized frames

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Abstract. Frame theory is an interesting topic that has been studied in both abstract and applied aspects. In recent years, we have several articles on generalized frames, because they sometimes are more flexible than ordinary frames. This paper investigates some of the problems with the conditions that make a sequence a frame, as a generalized form of frames.
Keywords: Frame, generalized frame, K-frame, Controlled frame.
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1. Introduction

Every element in a Hilbert space can be represented as a linear combination of the elements of a frame, but this representation is not necessarily unique. This fact is essential in the application. Moreover, K-frames were introduced as special generalizations of frames [5]. Furthermore, controlled frames also have been introduced and investigated [3]. These generalized frames may provide more flexibility in some approaches. Frame theory also plays a foundational role in signal processing, image processing, data compression, sampling theory, and more; It is also productive for researching abstract mathematics.

In this paper, an exercise and a proposition of [4] are examined and then extended to K frames and controlled frames due to their attractive and flexible features.

Before proceeding the main results, some important definitions are provided in the following.

A sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ is called a Bessel sequence if there exists $B>0$ such that

$$\sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\,\|f\|^2 \qquad (f\in\mathcal{H}).$$

By [4, Theorem 3.2.3], $\{f_n\}$ is a Bessel sequence if and only if the mapping, $T:\ell^2\to\mathcal{H},\ \ T(\{c_n\}_{n=1}^{\infty}) := \sum_{n=1}^{\infty} c_n f_n$ is a well-defined bounded linear operator with $\|T\|\leq\sqrt{B}$. Then T is called synthesis operator, related to $\{f_n\}_{n=1}^{\infty}$. Also, the adjoint operator of T is given by $T^*:\mathcal{H}\longrightarrow\ell^2,\quad T^*(f)=\{\langle f, f_n\rangle\}_{n=1}^{\infty}$ and is called analysis operator for $\{f_n\}_{n=1}^{\infty}$. Moreover, a sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ is called a frame if there exist constants $A,B>0$ such that

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2 \qquad (f\in\mathcal{H}).$$

The constants A and B are called lower and upper frame bounds, which are not unique. The optimal lower frame bound (resp. the optimal upper frame bound) is the supremum over all lower frame bounds, (resp. the infimum over all upper frame bounds).

The frame operator is defined as $S:\mathcal{H}\longrightarrow\mathcal{H},\quad S(f)=TT^*(f)=\sum_{n=1}^{\infty}\langle f, f_n\rangle f_n,$

Now suppose that $K\in\mathcal{B}(\mathcal{H})$. The range of K is denoted by $R(K)$. A sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ is called a K-frame for $\mathcal{H}$, if there exist constants $A,B>0$ such that

$$A\|K^*f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2 \qquad (f\in\mathcal{H}).$$

The constants A and B are called lower and upper K-frame bounds of $\{f_n\}_{n=1}^{\infty}$, respectively. The synthesis operator, analysis operator and frame operator of K-frames are similarly defined to frames. Note that the frame operator S of K-frames is not necessarily invertible. However, S is invertible on $R(K)$, whenever K has closed range.

Moreover, let $GL(\mathcal{H})$ denotes the set of all bounded linear operators which have bounded inverses. Following [3], a countable family of vectors $\Psi = \{\psi_n\}_{n=1}^{\infty}$ is controlled by the operator $C \in GL(H)$, or is called a C-controlled frame if there exist two constants $0 < m_{C\Psi} \leq M_{C\Psi} < \infty$, such that

$$m_{C\Psi}\|f\|^2 \leq \sum_{n=1}^{\infty}\langle f, \psi_n\rangle\langle C\psi_n, f\rangle \leq M_{C\Psi}\|f\|^2, \quad (f \in \mathcal{H}).$$

The controlled frame operator for frame Ψ is defined as $L_{C\Psi}f = \sum_{n=1}^{\infty}\langle f, \psi_n\rangle C\psi_n$, $(f \in \mathcal{H})$. For more details about generalized frames we refer to [[1], [2], [3], [6].] Finally, we finish this part by [lemma3.2.6, [4]], which says if $\{f_k\}_{k=1}^{\infty}$ is a sequence of elements in Hilbert space $\mathcal{H}$ and there exists constant B ¿ 0 such that the condition of Bessel sequence there holds for all f in a dense subset V of $\mathcal{H}$. Then $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence for $\mathcal{H}$ with the same bound.

2. Main results

In [Lemma 5.1.9 [4]], author shows that it is enough to check the frame condition on a dense set, i.e., if $\{f_k\}_{k=1}^{\infty}$ is a sequence of elements in Hilbert space $\mathcal{H}$ and there exist constants A, B¿0 such that the condition of frame there holds for all f in a dense subset V of $\mathcal{H}$. Then $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with the same bounds.

Now there is a generalization of this lemma to K-frames.

Proposition 2.1. Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of elements in $\mathcal{H}$ and there exists constants $A, B > 0$ such that $A\|K^*f\|^2 \leq \sum_{n=1}^{\infty}|\langle f, f_k\rangle|^2 \leq B\|f\|^2$ for all f in a dense subset V of $\mathcal{H}$. Then $\{f_k\}_{k=1}^{\infty}$ is a K-frame for $\mathcal{H}$ with bounds A, B.

Proof. By [lemma 3.2.6, [4]], $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence. So, the right hand of K-frame inequality holds for all elements in $\mathcal{H}$. Now let g $\in (H = \overline{V}) - V$ such that

$$\|T^*g\|^2 < A\|K^*g\|^2.$$

By density of V in $\mathcal{H}$ there exist a sequence $\{f_n\}_{n=1}^{\infty} \subseteq V$ such that $\lim_{n\to\infty} f_n = g$ Then, Since $T, K \in B(H)$, We have $\lim_{n\to\infty}\|T^*f_n\|^2 < A\lim_{n\to\infty}\|K^*f_n\|^2$. Thus, there exists $N \in \mathbb{N}$ such that $\|T^*f_N\|^2 < A\|K^*f_N\|^2$. This is contradiction. □

Now we want to extend this proposition to controlled frames.

Proposition 2.2. Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of elements in $\mathcal{H}$ and there exists constants $A, B > 0$ such that for which $C \in GL(H)$

$$A\|f\|^2 \leq \sum_{n=1}^{k}\langle f, f_n\rangle\langle Cf_n, f\rangle \leq B\|f\|^2 \tag{1}$$

for all $n \in \mathbb{N}$ and f in a dense subset V of $\mathcal{H}$. Then $\{f_n\}_{n=1}^{\infty}$ is C-controlled frame for H.

Proof. Let $g \in H = \overline{V}(-V)$ such that $A\|g\|^2 > \sum_{n=1}^{k}\langle g, f_n\rangle\langle Cf_n, g\rangle$, Thus, by density of V in $\mathcal{H}$ there exists $h \in V$ such that $\sum_{n=1}^{k}\langle h, f_n\rangle\langle Cf_n, h\rangle < A\|h\|^2$. This is contradiction and shows that the left hand of (1) holds for all elements in $\mathcal{H}$. The right hand is proved as the same way. Hence, for all $f \in \mathcal{H}$ and $n \in \mathbb{N}$ we have the (1), that shows the sequence of partial Sum below

$$\left\{ \sum_{n=1}^{k} \langle f, f_n \rangle \langle C f_n, f \rangle \right\}_{k \in \mathbb{N}},$$

is a positive, bounded sequence. Therefore, $A\|f\|^2 \leq \sum_{n=1}^{\infty} \langle f, f_n \rangle \langle C f_n, f \rangle \leq B\|f\|^2$, which means the proof is complete. □

In the following [Lemma 5.1.10 [4]], that is in fact Exercise 5.4, is proved.

PROPOSITION 2.3. *if $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for Hilbert space $\mathcal{H}$ and $\{f_k\}_{k=1}^{\infty}$ is a sequence in this space, then we define*

$$S := \left\{ \{c_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}) : \text{ is a finite sequence } \{c_n\}_{n=1}^{\infty} \text{ and} \sum_{n=1}^{\infty} |c_n|^2 = 1 \right\}.$$

If there exists A and $B > 0$ such that for all $\{c_n\}_{n=1}^{\infty} \in S$ we have $A \leq \sum_{n=1}^{\infty} \left| \sum_{j=1}^{\infty} c_j \langle e_j, f_n \rangle \right|^2 \leq B$, Then $\{f_n\}_{n=1}^{\infty}$ is a frame for $\mathcal{H}$ with bounds A and B.

PROOF. Let $\Omega := \{f \in \mathcal{H} : \ \|f\| = 1\}$, and $D = \left\{ \sum_{j=1}^{\infty} c_j e_j : \ \{c_j\}_{j=1}^{\infty} \in S \right\}$. By the definition of S it is clear that $D \subseteq \Omega$. Now we show that D is dense in Ω. Based on [Theorem 4.18, [7]], for every $f \in \Omega$ we have $f = \sum_{j=1}^{\infty} \langle f, e_j \rangle e_j$ and $\sum_{j=1}^{\infty} |\langle f, e_j \rangle|^2 = \|f\| = 1$. If we let $g_n := \dfrac{\sum_{j=1}^{n} \langle f, e_j \rangle e_j}{\left(\sum_{j=1}^{n} |\langle f, e_j \rangle|^2 \right)^{\frac{1}{2}}}$, for every $n \in \mathbb{N}$. Then by simple calculation we can show $\{g_n\}_{n=1}^{\infty} \subseteq D$. Moreover, because of tendency of $\{g_n\}_{n=1}^{\infty}$ to f, we have density of D in Ω. Now we show that inequality of frame holds for D. for every, $f = \sum_{j=1}^{\infty} c_j e_j$ in D, we have $A\|f\|^2 = A \leq \sum_{n=1}^{\infty} \left| \sum_{j=1}^{\infty} c_j \langle e_j, f_n \rangle \right|^2 \leq B = B\|f\|^2$. Since $\sum_{n=1}^{\infty} \left| \sum_{j=1}^{\infty} c_j \langle e_j, f_n \rangle \right|^2 = \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2$, then, for every $f \in D$

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq B\|f\|^2. \tag{2}$$

Thus, (2) holds for every $f \in \Omega$ Now let $f \in \mathcal{H}$ is not zero. Hence $\dfrac{f}{\|f\|} \in \Omega$ and by (2),

$$A \left\| \frac{f}{\|f\|} \right\|^2 \leq \sum_{n=1}^{\infty} |\langle \frac{f}{\|f\|}, f_n \rangle|^2 \leq B \left\| \frac{f}{\|f\|} \right\|^2.$$

Thus, for all $f \in \mathcal{H}$ we have $A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq B\|f\|^2$. □

PROPOSITION 2.4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in a Hilbert space $\mathcal{H}$, and $\{e_n\}_{n=1}^{\infty}$ be orthonormal basis for H. additionally there exist constants $A, B > 0$, $K \in B(H)$ and

$$S := \left\{ \{c_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}) | \{c_n\}_{n=1}^{\infty} \ \ is \ \ finite \ \ and \ \ \sum_{n=1}^{\infty} |c_k|^2 = 1 \right\},$$

Then If $A\|K^*\|^2 \leq \sum_{n=1}^{\infty} \left| \sum_{j=1}^{\infty} c_j \langle e_j, f_n \rangle \right|^2 \leq B$ for all $\{c_n\}_{n=1}^{\infty} \in S$, $\{f_n\}_{n=1}^{\infty}$ is a K-frame for H.

Proof. By proposition 2.3 $\{f_n\}_{n=1}^{\infty}$ is a frame for $\mathcal{H}$ by bounds $A\|K\|^2$ and B. Therefore, for every $f \in \mathcal{H}$ we have, $A\|K\|^2\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2$. Since $\|K^*f\|^2 \leq \|K\|^2\|f\|^2$, then $A\|K^*f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2$. □

Now we extend this proposition to controlled-frames.

Proposition 2.5. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in a Hilbert space H, and $\{e_n\}_{n=1}^{\infty}$ be orthonormal basis for H. Let

$$S := \left\{ \{c_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}) | \ \{c_n\}_{n=1}^{\infty} \quad is \ \ finite \ \ and \ \ \sum_{n=1}^{\infty} |c_k|^2 = 1 \right\}.$$

Additionally if there exist constants $A, B > 0$, $C \in GL(H)$ such that

$$A \leq \sum_{n=1}^{\infty} \sum_{i,j=1}^{\infty} c_i \overline{c_j} \langle e_i, f_n\rangle \langle Cf_n, e_j\rangle \leq B$$

for all $\{c_n\}_{n=1}^{\infty} \in S$, Then $\{f_n\}_{n=1}^{\infty}$ is a C-controlled frame for H.

Proof. This proposition is proved by the same way as proposition 2.3. Let $\Omega := \{f \in \mathcal{H} : \quad \|f\| = 1\}$ and $D = \left\{ \sum_{j=1}^{\infty} c_j e_j : \quad \{c_j\}_{j=1}^{\infty} \in S \right\}$. Hence by the proof of proposition 2.3, D is dense in Ω. Now we show that the controlled frame inequality holds for D. By some simple calculation we can see for every $f = \sum_{j=1}^{\infty} c_j e_j$ in D we have

$$\sum_{n=1}^{\infty} \sum_{i,j=1}^{\infty} c_i \overline{c_j} \langle e_i, f_n\rangle \langle Cf_n, e_j\rangle = \sum_{n=1}^{\infty} \langle f, f_n\rangle \langle Cf_n, f\rangle.$$

By the assumption, for every $f \in D$ we have

$$A = A\|f\|^2 \leq \sum_{n=1}^{\infty} \langle f, f_n\rangle \langle Cf_n, f\rangle \leq B = B\|f\|^2. \tag{3}$$

Since D is dense in Ω, then (3) is valid for all $f \in \Omega$. Now let $f \in \mathcal{H}$ be nonzero, hence, $\frac{f}{\|f\|} \in \Omega$ and by (3), $A \leq \sum_{n=1}^{\infty} \langle \frac{f}{\|f\|}, f_n\rangle \langle Cf_n, \frac{f}{\|f\|}\rangle \leq B$. Therefore, easily we can see the proof is complete. □

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References

1. F. Abtahi, Z. Kamali and Z. Keyshams, *On the Schur-Horn problem*, Rocky Mountain J. Math., 51 (4) 1157 - 1170, August 2021.
2. F. Abtahi, Z. Kamali and Z. Keyshams, *On the sum of generalized frames in Hilbert spaces*, Mediterr. J. Math., 18, 178 (2021).
3. P. Balasz, J-P. Antonie and A. Graybos, *Weighted and controlled frames: mutual Relationship and first numerical properties*, Int. J. Wavelets Multiresolut. Inf. Process., **8**, (2010), 109-132.
4. O. Christensen, An Introduction to Frames and Riesz Bases, Springer, Birkhauser (2016).
5. L. Găvruta, *Frames for operators*, Appl. Comput. Harmon. Anal., **32(1)**, (2012), 139-144.
6. M. Nouri, A. Rahimi and SH. Najafzadeh, *Some results on controlled K-frames in Hilbert spaces*, Int. J. Analysis and applications., **16** (2018), 62-74.
7. W. Rudin, Real and Complex Analysis, Third edition, McGraw-Hill Book Co., New York., 1987.

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A vector approach of the best proximity pair

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Abstract. In this paper, the vector version of the best proximity pair problem is discussed in a cone metric. the conditions for the existence and uniqueness of this type version Is expressed. I will define a kind of cone cyclic contraction map and studied the problem.
Keywords: The best proximity pair, Cone metric Riesz space, cyclic contraction map..
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1. Introduction

The problem of the best proximity pair is one of the significant issues that has called a lot of attention in recent years. In all relevant papers, the research done on metric space (X, d) has made use of metric function $d : X \times X \to \mathbb{R}$. As examples, Eldred and Veeramani [6] discussed the best proximity pair problem for cyclic contraction maps on uniformly convex Banach spaces. This problem was examined for relatively nonexpansive maps [14] and pointwise contraction maps in [2] and [3]. The best approximation problem in Banach lattices is connected to monotonicity in [5], [7], [8], [10], [11] and [12]. Afterwards, we will review some basic definitions in Riesz space X. If X is a parially ordered vector space, then X is called a Riesz space (or a vector lattice space) if $x \vee y = \sup\{x, y\}$, and $x \wedge y = \inf\{x, y\}$, both exist in X, for any $x, y \in X$. For any vector x in Riesz space X, define $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x \vee (-x)$. The set $X^+ = \{x \in X : x \geq 0\}$ is called the positive cone of X. Riesz space X is called Dedekind complete whenever every nonempty bounded above subset has a supremum (or equivalently, whenever every nonempty bounded below subset has an infimum). Also X is said Archimedean if $x = 0$ holds whenever, $0 \leq nx \leq y \in X^+$ for all $n \in \mathbb{N}$. More details about Riesz spaces could be find in [1], [4] , [13] and [15].

Definition 1.1. The mapping $d : X \times X \to X^+$ is said to be a cone metric on X if it satisfies:

(a) $d(x, y) = 0$ if and only if $x = y$.
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

In this way, we recognize (X, d) as a cone metric Riesz space. We define $d(x, y) = |x - y| \in X^+$ for any $x, y \in X$, so $(X, |\cdot|)$ is a cone metric Riesz space. Recall that $f_n \downarrow f$ it means the sequence $\{f_n\} \subseteq X$ is decreasing and $f = \inf f_n$ in X.

In the continuation of the article, it is supposed that E and F are two nonempty subsets of Riesz space X, and $\tau : E \to F$ is an arbitrary map, and $Dist(E, F) = \bigwedge |E - F| = \inf\{|x - y| : x \in E, y \in F\}$ exists in the set $|E - F|$. It means, there exist $a \in E$ and $b \in F$ such that $Dist(E, F) = |a - b|$. For instance, if X is Dedekind complete and $|E - F|$ is order closed, then there exist $a \in E$ and $b \in F$ such that $Dist(E, F) = |E - F|$ ($|E - F|$ is a bounded below subset of X). Let $x \in E$. If $|x - \tau x| = Dist(E, F)$, we say $(x, \tau X)$ is a cone best proximity point for τ. We show the set of all such points by $P^c_\tau(E, F)$, i.e., $P^c_\tau(E, F) = \{x \in E : |x - \tau x| = Dist(E, F)\}$.

Definition 1.2. Let $(X, |\cdot|)$ be a cone metric Riesz space and $u, u_n \in X$ $(n = 1, 2, \ldots)$.

(a) The sequence $\{u_n\}$ is order convergence to u if there exists a sequence $f_n \downarrow 0$ such that $|u - u_n| \leq f_n$ holds for any $n \in \mathbb{N}$. (In symbols $u_n \xrightarrow{o} u$). Also the subset $E \subseteq X$ is order closed whenever for the all sequences $\{x_n\} \subseteq E$ such that $x_n \xrightarrow{o} x$, imply $x \in E$.
(b) The sequence $\{u_n\}$ is order Cauchy if there exists a sequence $f_n \downarrow 0$ which $|u_n - u_m| \leq f_n$ for all $n \geq m \geq 1$. Clearly, order convergence sequences are order Cauchy.

(c) The cone metric Riesz space $(X, |\cdot|)$ is order complete if any order Cauchy sequence is order convergence.
(d) Let (E, F) be a pair of non-empty subsets of a metric space. A cyclic mapping $\tau : E \cup F \to E \cup F$ is said to be an cone absolute proximal cyclic contraction if the following conditions hold:
(a) There exists a non-negative real number $\alpha < 1$ and a real number $\delta > 0$ such that

$$\left.\begin{array}{r} d(u, \tau x) \leq \beta d(E, F) \\ d(v, \tau y) \leq \beta d(E, F) \\ 1 \leq \beta \leq 1 + \delta \end{array}\right\} \Rightarrow d(\tau u, \tau v) \leq (\alpha + \beta - 1) d(\tau x, \tau y) + [1 - (\alpha + \beta - 1)] d(E, F)$$

for all $u, x \in E$ and $v, y \in F$.
(b) T is relatively continuous.

Some definitions and notation:

$$E_0 = \{a \in E : d(a, b) = d(E, F) \;\; , \; for \; some \; b \in F\}$$

$$F_0 = \{b \in F : d(a, b) = d(E, F) \;\; , \; for \; some \; a \in E\}$$

$$E_0^* = \{a \in E : d(a, b_n) \to d(E, F) \;\; , \; for \; some \; sequence \; \{b_n\} \subset F_0\}$$

$$F_0^* = \{b \in F : d(a_n, b) \to d(E, F) \;\; , \; for \; some \; sequence \; \{a_n\} \subset E_0\}$$

$$E_0^+ = \{a \in E : d(a, b_n) \to d(E, F) \;\; , \; for \; some \; sequence \; \{b_n\} \subset F\}$$

$$F_0^+ = \{b \in F : d(a_n, b) \to d(E, F) \;\; , \; for \; some \; sequence \; \{a_n\} \subset E\}$$

It should be mentioned that the order convergence in a Riesz space X does not necessarily correspond to a topology on X.

Remark 1.3. Let $C(K)$ be the set of all real continuous functions on K by ordering $f_1 \leq f_2$ if $f_1(x) \leq f_2(x)$ for any $x \in K$. We know $f_1 \vee f_2 = \frac{1}{2}(f_1 + f_2) + \frac{1}{2}|f_1 - f_2| \in C(K)$ and $f_1 \wedge f_2 = \frac{1}{2}(f_1 + f_2) - \frac{1}{2}|f_1 - f_2| \in C(K)$ for any $f_1, f_2 \in C(K)$. Therefore $C(K)$ is a Riesz space. Also, if K is a Hausdorff topological space, compact and extremally disconnected (i.e., the closure of any open set is open) then $C(K)$ is order complete and Dedekind complete [9].

Definition 1.4. Let $(X, |\cdot|)$ be a cone metric Riesz space.
(a) A sequence $\{x_n\} \subseteq E$ is said to be a cone τ-minimizing sequence in E whenever $|x_n - \tau x_n| \xrightarrow{o} Dist(E, F)$.
(b) The subset $E \subseteq X$ is a τ-absolutely direct set if for any $x, y \in E$, there exists $z \in E$ such that

$$|z - \tau x| \leq |x - \tau x| \wedge |y - \tau x| \quad \text{and} \quad |z - \tau y| \leq |x - \tau y| \wedge |y - \tau y|$$

Example 1.5. Suppose $E \subseteq X$ is a sublattice, it means $x \vee y$ and $x \wedge y$ both exist in E for any $x, y \in E$, and also $E \geq F$ (or $F \geq E$). Then E is a τ-absolutely direct set.
The notation $E \geq F$ means that $a \geq b$ for any $a \in E$ and $b \in F$.

Cone best proximity pair problem is τ-solvable (τ-uniquely solvable) if $P_\tau^c(E, F) \neq \emptyset$ ($card\ P_\tau^c(E, F) = 1$).

2. Main results

In this part, we aim to provide conditions to investigate the existence and uniqueness of cone best proximity pair problem.

Theorem 2.1. *Let $(X, |\cdot|)$ be a cone metric Riesz space and $E \subseteq X$ be a convex τ-absolutely direct set. Then card $P_\tau^c(E, F) \leq 1$*

Example 2.2. Suppose $X = \mathbb{R}^2$ with coordinatewise ordering (i.e., $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$). Put $E = \{(1, x) : x \in \mathbb{R}\}$, $F = \{(2, x) : x \in \mathbb{R}\}$ and $\tau : E \to F$ defined by

$$\tau(1, x) = \begin{cases} (2, 1), & x \in \mathbb{Q} \\ (2, \sqrt{2}), & x \notin \mathbb{Q} \end{cases}$$

W can see that $Dist(E, F) = (1, 0)$ and E is a convex set, but E is not a τ-absolutely direct set. It is easy to see that $P^c_\tau(E, F) = \{(1, 1), (1, \sqrt{2})\}$.

Theorem 2.3. *Let $(X, |\cdot|)$ be order complete, $E \subseteq X$ be an order closed and a convex τ-absolutely direct set. Then any cone τ-minimizing sequence in E, is order convergence.*

It is necessary to mention that $f : E \to F$ is a σ-order continuse map if $f(x_n) \xrightarrow{o} f(x)$ for all sequences $\{x_n\} \subseteq E$ such that $x_n \xrightarrow{o} x$.

Corollary 2.4. *Let $(X, |\cdot|)$ be order complete and $\tau : E \to F$ be a σ-order continuous map. Let E be order closed, and a convex τ-absolutely direct set. If E has a cone τ-minimizing sequence then cone best proximity pair problem is τ-uniquely solvable.*

Theorem 2.5. *Let $(X, |\cdot|)$ be an Archimedean cone metric Riesz space and $\tau : E \cup F \to E \cup F$ be a cone absolute proximal cyclic contraction. If $x_0 \in E$ and $x_{n+1} = \tau x_n = \tau^{n+1} x_0$ $(n = 0, 1, 2, \ldots)$, then the sequence $\{x_{2n}\} \subseteq E$ is a cone τ-minimizing sequence in E.*

Theorem 2.6. *Let $(X, |\cdot|)$ be Archimedean and order complete. Let $E \subseteq X$ be an order closed sublattice and $E \geq F$ (or $F \geq E$). If $\tau : E \cup F \to E \cup F$ is a cone cyclic contraction map then cone best proximity pair problem is uniquely solvable.*

Theorem 2.7. *Let (E, F) be a pair of non-empty subsets of a metric space such that E_0 and F_0 are non-void, and (E, F) is cyclically complete. Let $\tau : E \cup F \to E \cup F$ be a cyclic mapping satisfying the following conditions:*
(a) τ is an absolute proximal cyclic contraction.
(b) $\tau(E_0) \subset F_0^*$ and $\tau(F_0) \subset E_0^*$.
Then, the mapping τ has a best proximity point in E or F. Moreover, if x^* and y^* are best proximity points of τ in E and F, respectively, then

$$d(\tau x^*, \tau y^*) = d(E, F)$$

References

1. C. D. Aliprantis and O. Burkinshaw, *Positive operators*, Springer, Dordrecht, 2006, Reprint of the 1985 original.
2. J. Anuradha and P. Veeramani, *Proximal pointwise contraction*, Topology Appl. 156 (2009), no. 18, 2942–2948.
3. S. Sadiq Basha, *Best proximity point theorems for some special proximal contractions*, Numer. Funct. Anal. Optim. 40 (2019), no. 10, 1182–1193.
4. G. Birkhoff, *Lattice theory*, Corrected reprint of the 1967 third edition. American Mathematical Society Colloquium Publications, 25. American Mathematical Society, Providence, R.I., 1979.
5. Chen, Shu Tao; He, Xin; Hudzik, H. *Monotonicity and best approximation in Banach lattices*, Acta Math. Sin. (Engl. Ser.) 25 (2009), no. 5, 785–794.
6. A. A. Eldred and P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl. 323 (2006), no. 2, 1001–1006.
7. P. Foralewski, H. Hudzik, W. Kowalewski and M. Wisa *Monotonicity properties of Banach lattices and their applications a survey*, Ordered structures and applications, 203–232, Trends Math., Birkhuser/Springer, Cham, 2016.
8. H. Hudzik and W. Kurc, *Monotonicity properties of Musielak-Orlicz spaces and dominated best approximation in Banach lattices*, J. Approx. Theory, 95 (1998), no. 3, 353–368.
9. A. F. Kalton, J. Nigel, *Topics in Banach space theory*, Graduate Texts in Mathematics, 233. Springer, New York, 2006.
10. H. R. Khademzadeh H. Mazaheri, *Monotonicity and the dominated farthest points problem in Banach lattice*, Abstr. Appl. Anal. 2014, 7 pp.

11. N. S. Kukushkin, *Increasing selections from increasing multifunctions*, Order 30 (2013), no. 2, 541–555.
12. W. Kurc, *Strictly and uniformly monotone Musielak-Orlicz spaces and applications to best approximation*, J. Approx. Theory, 69 (1992), no. 2, 173–187.
13. P. Meyer-Nieberg, *Banach lattices*, Universitext, Springer-Verlag, Berlin, 1991.
14. V. Sankar Raj and P. Veeramani, *Best proximity pair theorems for relatively nonexpansive mappings*, Appl. Gen. Topol. 10 (2009), no. 1, 21–28.
15. A.C. Zaanen, *Introduction to operator theory in Riesz spaces*, Springer-Verlag, Berlin, 1997.

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پایداری K-G قاب های پیوسته

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چکیده. در این مقاله ابتدا گزاره ایی در مورد پایایی k-g قاب های پیوسته تحت برخی از عملگرهای جبری می اوریم و سپس با اعمال شرایط لازم و به کمک خواص عملگرها،ترکیباتی از k-g قاب ها را می سازیم به گونه ایی که مجددا k-g قاب پیوسته به دست اید.واضح است که می توان ترکیبات متعدد و جالبی از این دسته از قاب ها ایجاد نمود.در قسمت اخر به نتیجه ایی در مورد قاب های پیوسته چسبان که شکل خاصی از قاب های پیوسته هستند،اشاره شده است.
واژه‌های کلیدی: g-k قاب چسبان ، g-k قاب پیوسته ، قاب پیوسته.
طبقه‌بندی موضوعی [۲۰۱۰]: 06D22 , 00A69.

۱. پیش‌گفتار

نظریه قاب ها (قاب های گسسته) برای اولین بار توسط دافین و شیفر در سال ۱۹۵۲ در زمینه مسائل سری های فوریه غیر هارمونیک مطرح گردید [۴].دابشی،گراسمن و میر به طور جداگانه نظریه قاب ها را در سال ۱۹۸۶ معرفی کردند [۷].خواص متنوع و کاربردی قاب ها آن را به ابزاری قدرتمند در فضاهای تابعی، کدنویسی و مخابرات، رشته های مهندسی، کوانتوم و فیزیک تبدیل کرده است.مفهوم قاب های پیوسته در [۱] به صورت خانواده ایی اندیس شده در فضای موضعا فشرده به همراه یک اندازه رادون عنوان شد.همچنین به طور جداگانه توسط کیزر و گازو نیز معرفی شدند[۶]. گاباردو و هان [۵] این قاب ها را قاب های متناظر با فضای اندازه نام گذاری کرده اند و اقایان عسکری همت،رجبعلی پور و دهقان آنها را قاب های توسعه یافته نامیده اند [۳]. در تمام این نوشته H ، (Ω, μ) و $\{H_\omega\}_{\omega\in\Omega}$ به ترتیب یک فضای هیلبرت جدایی پذیر، یک فضای اندازه و خانواده ایی از فضاهای هیلبرت را تداعی می کنند.در ابتدا برای اشنایی بیشتر خواننده با مفاهیم مورد بحث چند تعریف را می آوریم.

تعریف ۱.۱. فرض کنید $K \in B(H)$ عملگری خطی و کراندار روی H باشد.دنباله $\{f_n\}_{n=1}^{\infty}$ یک K - قاب برای H نامیده می شود اگر ثابتهای $A, B > 0$ موجود باشند به طوری که

$$A\|K^*f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \quad (f \in H). \tag{۱}$$

ثابتهای A و B کرانهای پایینی و بالایی K - قاب $\{f_n\}$ نامیده می شود.اگر فقط قسمت راست نامساوی فوق برقرار باشد ان را یک دنباله بسل می نامیم و در حالتی که $K = I$ یک قاب معمولی خواهیم داشت.

تعریف ۲.۱. فرض کنید $K \in B(H)$ و $\Lambda = \{\Lambda_i \in B(H, H_i) : i \in I\}$. Λ یک g - K قاب برای H متناظر با $\{H_i\}_{i\in I}$ یا یک g - K قاب برای H نامیده می شود اگر ثابت های A و Bموجود باشند به طوری که

$$A\|K^*f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad (f \in H). \tag{۲}$$

ثابت های A و B به ترتیب کرانهای پایینی و بالایی g - K قاب هستند.

تعریف ۳.۱. [۲] فرض کنید $\prod_{\omega\in\Omega} H_\omega = \{f : \Omega \longrightarrow \bigcup_{\omega\in\Omega} H_\omega : f(\omega) \in H_\omega\}$. نگاشت $F \in \prod_{\omega\in\Omega} H_\omega$ اندازه پذیر قوی است اگر F نگاشتی اندازه پذیراز Ω به$\bigoplus_{\omega\in\Omega} H_\omega$ باشد.

تعریف ۴.۱. [۲] خانواده $\Omega = \{\Omega_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ یک g - قاب پیوسته متناظر با $\{H_\omega\}_{\omega\in\Omega}$ است اگر

(الف) به ازای هر بردار $f \in H$ ، خانواده $\{\Lambda_\omega f\}_{\omega\in\Omega}$ اندازه پذیر قوی باشد.

(ب) ثابتهای A و B موجود باشند به طوری که

$$A\|f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2, \quad (f \in H). \tag{۳}$$

تعریف ۵.۱. اگر (Ω, μ) فضای اندازه با اندازه ایی مثبت μ و $K \in B(H)$ باشد خانواده $\Lambda = \{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ که در آن $\{H_\omega\}_{\omega\in\Omega}$ خانواده ایی از فضاهای هیلبرت هستند، یک g-K قاب پیوسته یا یک g-K-c قاب برای H متناظر با $\{H_\omega\}_{\omega\in\Omega}$ است اگر
الف) به ازای هر بردار $f \in H$ ، $\{\Lambda_\omega f\}_{\omega\in\Omega}$ به طور قوی اندازه پذیر باشد.
ب) ثابتهای $0 \leq A, B < \infty$ موجود باشند به طوری که

$$A\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2, \qquad (f \in H). \tag{۴}$$

ثابتهای A و B به ترتیب کرانهای پایینی و بالایی g-K-c قاب هستند.اگر $A = B$ ، $\{\Lambda_\omega\}_{\omega\in\Omega}$ یک g - K - c قاب چسبان و اگر $A = B = 1$ یک g - K - c قاب پارسوال است.در صورتی که سمت راست نامساوی (۴) برقرار باشد یک خانواده g - بسل خواهیم داشت. می توان دید که هر g-K-c قاب یک g - c دنباله بسل متناظر با $\{H_\omega\}_{\omega\in\Omega}$ می باشد.

۲. پایداری k - قاب های پیوسته تحت برخی اعمال جبری

فرض کنید $\{\Lambda_\omega\}_{\omega\in\Omega} \in B(H, H_\omega)$ که در آن $\{H_\omega\}_{\omega\in\Omega}$ خانواده ایی از فضاهای هیلبرت است را در اختیار داریم.

گزاره ۱.۲. فضای هیلبرت H ، $L_1, L_2 \in B(H)$ و $\alpha \in C$ مفروضند.اگر $\{\Lambda_\omega\}_{\omega\in\Omega}$ یک L_1 - g قاب پیوسته و L_2 - g قاب پیوسته باشد، در این صورت $\{\Lambda_\omega\}_{\omega\in\Omega}$ یک αL_1 - g قاب پیوسته ، $L_1 L_2$ - g قاب پیوسته و $L_1 + L_2$ - g قاب پیوسته است.

اثبات. اعداد مثبت A_1 ، A_2 و B موجود هستند به طوری که به ازای هر عضو f در H ،

$$A_1\|L_1{}^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2 \tag{۵}$$

و

$$A_2\|L_2{}^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2. \tag{۶}$$

با استفاده از نامساوی (۵)داریم

$$\tfrac{A_1}{|\alpha|^2}\|(\alpha L_1)^*f\|^2 \leq \textstyle\int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2$$

و نتیجه می گیریم که $\{\Lambda_\omega\}_{\omega\in\Omega}$ یک αL_1 - g قاب پیوسته است. از طرفی

$$\begin{aligned}\|(L_1L_2)^*f\|^2 &= \|(L_2{}^*L_1{}^*)f\|^2 \\ &\leq \|L_2\|^2\|L_1{}^*f\|^2 \\ &\leq \frac{\|L_2\|^2}{A_1}\int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega)\end{aligned}$$

و نامساوی زیر به دست می آید

$$\tfrac{A_1}{\|L_2\|^2}\|(L_1L_2)^*f\|^2 \leq \textstyle\int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2$$

و لذا $\{\Lambda_\omega\}_{\omega\in\Omega}$ یک L_1L_2 - g قاب پیوسته است. اگر نامساوی های (۵) و (۶) و نامساوی کشی شوارتز را به کار ببریم داریم

$$\begin{aligned}\|(L_1 + L_2)^*f\|^2 &= \|L_1{}^*f + L_2{}^*f\|^2 \\ &\leq \|L_1{}^*f\|^2 + \|L_2{}^*f\|^2 + 2|\langle L_1{}^*f, L_2{}^*f\rangle| \\ &\leq \|L_1{}^*f\|^2 + \|L_2{}^*f\|^2 + 2\|L_1{}^*f\|\|L_2{}^*f\| \\ &\leq (\frac{1}{A_1} + \frac{1}{A_2} + \frac{2}{\sqrt{A_1A_2}})\int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega)\end{aligned}$$

لذا خواهیم داشت

$$\|(L_1 + L_2)^*f\|^2 \leq (\tfrac{1}{A_1} + \tfrac{1}{A_2} + \tfrac{2}{\sqrt{A_1A_2}})\textstyle\int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega).$$

پس $\{\Lambda_\omega f\}_{\omega\in\Omega}$یک $(L_1 + L_2)$ - g قاب پیوسته است . □

اگر از استقرای ریاضی استفاده کنیم می توان گزاره قبل را برای هر تعداد متناهی عملگرهای $L_1, L_2, ..., L_n \in B(H)$ تعمیم داد.

قضیه ۲.۲. فرض کنید $\{\Lambda_\omega f\}_{\omega\in\Omega}$ یک K - g قاب پیوسته برای H متناظر با $\{H_\omega\}_{\omega\in\Omega}$ و $T \in B(H)$ به گونه ایی انتخاب شود که T^*KK^* عملگر مثبت باشد. آنگاه $\{\Lambda_\omega + \Lambda_\omega T\}_{\omega\in\Omega}$ یک K - g قاب پیوسته برای H متناظر با $\{H_\omega\}_{\omega\in\Omega}$ خواهد بود.

اثبات. چون $\{\Lambda_\omega f\}_{\omega\in\Omega}$ یک K - g قاب پیوسته برای H است پس ثابت های مثبت A و B موجودند به طوری که به ازای هر $f \in H$ ،

$$A\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2$$

به ازای هر $f \in H$ ،

$$\begin{aligned} A\|K^*(I+T)f\|^2 &\leq \int_\Omega \|\Lambda_\omega (I+T)f\|^2 d\mu(\omega) \\ &\leq B\|(I+T)f\|^2 \\ &\leq B\|I+T\|^2\|f\|^2 \end{aligned}$$

با توجه به اینکه T^*KK^* عملگری مثبت می باشد می توان نوشت

$$\begin{aligned} A\|K^*(I+T)f\|^2 &= A(\|K^*f\|^2 + 2Re\langle K^*f, K^*Tf\rangle + \|K^*Tf\|^2) \\ &= A(\|K^*f\|^2 + 2Re\langle T^*KK^*f, f\rangle + \|K^*Tf\|^2) \\ &\geq A(\|K^*f\|^2 + \|K^*Tf\|^2) \\ &\geq A\|K^*f\|^2 \end{aligned}$$

درنتیجه خواهیم داشت

$$\begin{aligned} A\|K^*f\|^2 &\leq \int_\Omega \|\Lambda_\omega (I+T)f\|^2 d\mu(\omega) \\ &\leq B\|I+T\|^2\|f\|^2 \end{aligned}$$

و لذا $\{\Lambda_\omega + \Lambda_\omega T\}_{\omega\in\Omega}$ یک K - g قاب پیوسته برای H متناظر با $\{H_\omega\}_{\omega\in\Omega}$ خواهد بود.

□

۳. نتیجه ایی در مورد k - g قابهای چسبان

در این قسمت به یک نتیجه در مورد قاب های پیوسته چسبان اشاره میکنیم .

قضیه ۱.۳. فرض کنید $K_1 \in B(H_1)$ و $\{\Lambda_\omega f\}_{\omega\in\Omega}$ یک K_1 - g قاب پیوسته چسبان برای H_1 باشد.اگر $K_2 \in B(H_2)$ عملگری یک به یک و پوشا و $T \in B(H_1, H_2)$ به گونه ایی انتخاب شود که تساوی $TK_1 = K_2T$ برقرار باشد،در این صورت $\{\Lambda_\omega T^* f\}_{\omega\in\Omega}$ یک K_2 - g قاب پیوسته برای H_2 خواهد بود اگر و تنها اگر T عملگری پوشا باشد.

اثبات. ابتدا فرض میکنیم T عملگری پوشا باشد.در این صورت با توجه به نتیجه ۶-۳ [۸] ، $\{\Lambda_\omega T^* f\}_{\omega\in\Omega}$ یک K_2- g قاب پیوسته برای فضای H_2 خواهد بود.برعکس فرض کنیم $\{\Lambda_\omega T^* f\}_{\omega\in\Omega}$ یک K_2 - g قاب پیوسته برای H_2 با کران های A_1 و B_1 باشد، به ازای هر بردار $h \in H_2$ داریم

$$A_1\|K_2{}^*h\|^2 \leq \int_\Omega \|\Lambda_\omega T^*h\|^2 d\mu(\omega) \leq B_1\|h\|^2. \quad (۷)$$

اگر $\{\Lambda_\omega f\}_{\omega\in\Omega}$ یک K_1 - g قاب چسبان در فضای H_1 با کرانهای A و B باشد،به ازای $f \in H_1$

$$A\|K_1{}^*f\|^2 = \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega).$$

از طرفی چون $TK_1 = K_2T$ پس داریم $K_1{}^*T^* = T^*K_2{}^*$. حال به ازای هر $h \in H_2$ ،

$$A\|T^*K_2{}^*h\|^2 = A\|K_1{}^*T^*h\|^2 = \int_\Omega \|\Lambda_\omega T^*h\|^2 d\mu(\omega).$$

با استفاده از (۷)

$$A\|T^*{K_۲}^*h\|^۲ \geq A_۱\|{K_۲}^*h\|^۲$$

و لذا

$$\|T^*{K_۲}^*h\|^۲ \geq \frac{A_۱}{A}\|{K_۲}^*h\|^۲, \qquad (\forall h \in H_۲).$$

حال ثابت می‌کنیم T پوشاست. با توجه به لم ۳-۲ در [۸] کافی است ثابت کنیم ${K_۲}^*$ پوشاست و چون $K_۲ \in B(H_۲)$ یک به یک با برد بسته فرض شده است،این مورد محقق میشود . □

مراجع

۱. S. Ali، T.، Antoine، J.P.، Gazeau، J.P.: Continuous frames in Hilbert spaces. Ann. Phys. ۲۲۲، ۱-۳۷ (۱۹۹۳).
۲. M. Abdollahpour. R.، Faroughi، M.H.: Continuous G-frames in Hilbert spaces. Southeast Asian Bull. Math. ۳۲، ۱-۱۹(۲۰۰۸).
۳. A. Askari-Hemmat، A. M. Dehghan، M. Radjabalipour، Generalized frames and their redundancy، Proc. Amer. Math. Soc. ۱۲۹(۲۰۰۱)، no. ۴، ۱۱۴۳-۱۱۴۷.
۴. Duffin، R.J.، Scheaffer، A.C.: A class of nonharmonic Fourier series. Trans. Am. Math. Soc. ۳۴۱،۷۲-۳۶۶(۱۹۵۲).
۵. Gabardo، J.P.، Han، D.:Frames associated with measurable space. Adv. Comp. Math. ۱۸۹۳)، ۱۲۷-۱۴۷(۲۰۰۳).
۶. G. Kaiser، A Friendly Guide to Wavelets، Birkha"user، Boston، ۱۹۹۴.
۷. I. Daubechies، A. Grossmann، Y. Meyer، Painless nonorthogonal expansions، J. Math. Phys، ۲۷(۱۹۸۶) ۱۲۷۱-۱۲۸۳.
۸. Rahmani، M.؛ On some properties of c-frames. J. Math. Res. Appl. ۳۷(۴)، ۴۶۶-۴۷۶(۲۰۱۷).

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