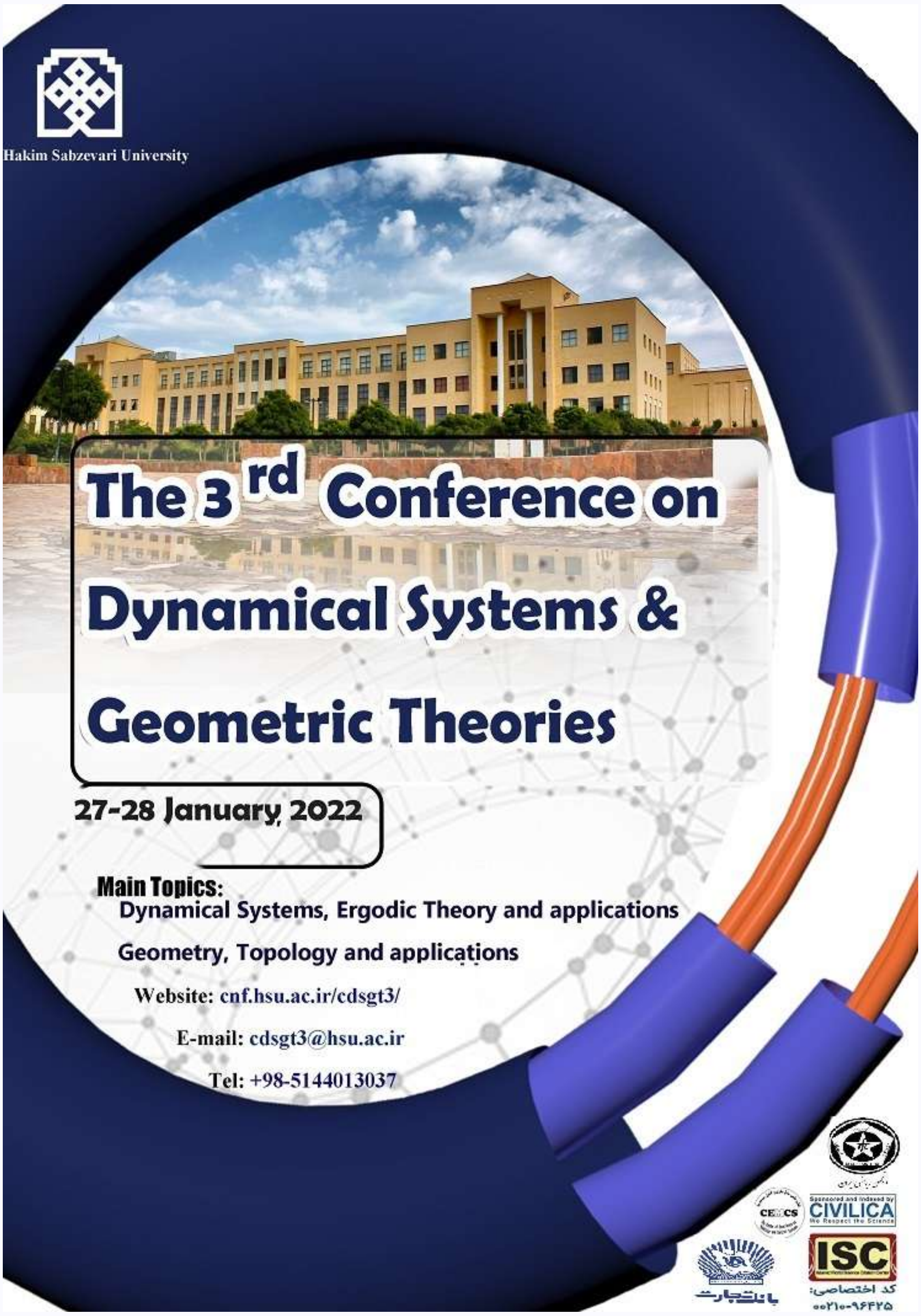
Book of Programs and Extended Abstracts

The 3^{rd} Conference on Dynamical Systems and Geometric Theories

English Papers




In The Name Of God

The 3^{rd} Conference on Dynamical Systems and Geometric Theories
Department of Mathematics, Hakim Sabzevari University, 27-28 January, 2022, Sabzevar, Iran

This book includes the list of names of the members of the Scientific and Executive Committee, the seminar programs and the collection of english articles that have been presented as an extended abstract in the form of lectures and posters in this seminar. It should be noted that the selected articles of the seminar which corresponds to the scope and goals of the Journal of Finsler Geometry and its Applications introduced for publication and only the abstract of these articles has been included in this book.

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Hosseini, Maryam (IPM, Iran)	

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2. Preface

It is our pleasure to welcome you to the 2022 3^{rd} Conference on Dynamical Systems And Geometric Theories (CDSGT2022) in Hakim Sabzevari University, Iran. A major goal and feature of it is to bring academic scientists, mathematical researchers together to exchange and share their experiences and research results about dynamical systems and geometric theories, and discuss about them.
The program consists of invited sessions, workshops and discussions with eminent speakers covering a wide range of Dynamical systems and Geometric theories. This rich program provides all attendees with the opportunities to meet and interact with one another. We hope your experience with CDSG2022 is a fruitful and long lasting one. With your support and participation, the conference will continue its success for a long time.
We would like to thank the organization staff, the members of the program committees and reviewers. They have worked very hard in reviewing papers and making valuable suggestions for the authors to improve their work. We look forward to seeing all of you next years at the conference.

Ali Barzanouni
Head of the Conference

3. English Papers

4. An upper bound for the measure-theoretic pressure of endomorphisms

Maryam Razi1*, Pouya Mehdipour, Sanaz Lamei
1University of Guilan-Iran. .razi264@gmail.com 2Federal University of Viosa-MG, Brazil. Pouya@ufv.br 3University of Guilan-Iran. lamei@guilan.ac.ir

A measure-theoretic pressure was defined by [L. He, J. Lv and L. Zhou, Definition of measure-theoretic pressure using spanning sets, Acta Math. Sinica(English Series), 20 (2004), 709-718] based on Katok entropy formula. In this article, we investigate an upper bound for the measure theoretic pressure of a C^2 endomorphism on a closed s-dimensional Riemannian manifold (compact and boundaryless) preserving a hyperbolic Borel probability measure.

Keywords: Thermodynamic formalism, Periodic points, Measure Pressure, Non-uniform hyperbolicity.

AMS Mathematics Subject Classification [2020]: 18A32, 18F20, 05C65
Code: cdsgt3-00560020

Introduction

Topological and measure theoretical pressures, as generalizations of the topological and measure theoretical entropy, are significant quantities in Ergodic Theory and Statistical Mechanics. They provide a determinant to measure the local complexity of dynamics defined in compact spaces. In this paper we deal with measure theoretical pressure for endomorphisms which are C^2 local diffeomorphism cascades on closed s-dimensional Riemannian manifolds (compact and boundaryless). It is considered the endomorphisms with fixed index, i.e., the dimension of the local unstable manifolds is fixed. Note that when studying endomorphisms, different local dynamics are involved and this can cause a variety of unstable indices at different points. Our aim is to extend the results of [2] which uses the Katok entropy formula [4] to establish an upper bound for measure theoretic pressure of diffeomorphisms in terms of their periodic points. We try to formulate such upper bound for endomorphisms. The difficulty in achieving this result for the case of endomorphism was due to the lack of a version of Katok Closing Lemma, for endomorphisms. In [5] a version of Closing Lemma under the priory mentioned conditions for endomorphisms is obtained. These results together with Theorem 2.1 of [2] on measure-theoretic pressures, defined on spanning sets [1], are the crucial implements used for the proof of the following Theorem.

Multiplicative Ergodic Theorem for Natural Extension. We devote this subsection to introduce the relevant Multiplicative Ergodic Theorem for Natural Extension ($\widetilde{MET}$) and the Katok Closing Lemma for C^2 local diffeomorphisms. These results play a crucial role for our purpose.

4.1. Theorem ([6]). *Let μ be an $f-$invariant Borel probability measure on M. We denote by $\tilde{\mu}$ the $\tilde{f}-$invariant Borel probability measure on M^f such that $\pi_*\tilde{\mu} = \mu$. There exists a full measure* *subset $\tilde{\mathcal{R}}$ called set of regular points such that for all $\tilde{x} = (x_n) \in \tilde{\mathcal{R}}$ and $n \in \mathbb{Z}$ the tangent space $T_{x_n}M$ splits into a direct sum*

$$T_{x_n}M = E_1(\tilde{x}, n) \oplus \cdots \oplus E_{r(x_0)}(\tilde{x}, n)$$

and there exists $-\infty < \lambda_1(\tilde{x}) < \cdots < \lambda_{r(\tilde{x})} < \infty$ and $m_i(\tilde{x})$ $(i = 0, 1, ..., r(\tilde{x}))$ such that:

(1) $\dim E_i(\tilde{x}, n) = m_i(\tilde{x})$;

(2) $D_{x_n}f(E_i(\tilde{x}, n)) = E_i(\tilde{x}, n+1)$, *and* $D_{x_n}f|_{E_i(\tilde{x},n)} : E_i(\tilde{x}, n) \to E_i(\tilde{x}, n+1)$ *is an isomorphism. For* $v \in E_i(\tilde{x}, n)\backslash\{0\}$,

$$\begin{cases} \lim\limits_{m\to\infty} \frac{1}{m} \log \|D_{x_n}f^m(v)\| = \lambda_i(\tilde{x}); \\ \lim\limits_{m\to\infty} -\frac{1}{m} \log \|(D_{x_{n-m}}f^m|_{E_i(\tilde{x},n-m)})^{-1}(v)\| = \lambda_i(\tilde{x}); \end{cases}$$

(3) if $i \neq j$ then

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \sin \angle(E_i(\tilde{x}, n), E_j(\tilde{x}, n)) = 0,$$

where $\angle(V, W)$ denotes the angle between sub-spaces V and W of $T_{x_n}M$.

(4) $r(.)$, $\lambda_i(.)$ and $m_i(.)$ are measurable and $\tilde{f}-$invariant. Moreover $r(\tilde{x}) = r(x_0)$, $\lambda_i(\tilde{x}) = \lambda_i(x_0)$ and $m_i(\tilde{x}) = m_i(x_0)$ for all $i = 1, 2, ..., r(\tilde{x})$.

Let μ be an $f-$invariant Borel probability measure on M. By $\widetilde{MET}$, there exists a full measure subset $\tilde{\mathcal{R}}$ called *"Lyapunov regular set"*. The assumption that the measure *mu* is ergodic and hyperbolic imposes that Lyapunov exponents are constant almost everywhere. The set of Lyapunov regular points without zero Lyapunov exponents, contains a non-uniformly hyperbolic set of full $\tilde{\mu}-$measure with

$$\lambda = \lambda^\mu, \ \ \theta = \theta^\mu, \ \ C(\tilde{x}) = C(\tilde{x}, \epsilon), \ \ K(\tilde{x}) = K(\tilde{x}, \epsilon)$$

where $\lambda = \lambda^\mu$ (resp $\theta = \theta^\mu$) is the least in modulus positive (resp. negative) Lyapunov exponent. Suppose that μ has k positive Lyapunov exponents, then the index of f is considered k. Let (f, μ) be a measure dynamics with μ a non-atomic hyperbolic ergodic measure. Without loss of generality from now on, we set $\tilde{R}$ as the non-uniformly hyperbolic subset of the Lyapunov regular points, with full $\mu-$measure. We denote its projection on M by $\mathcal{R} = \pi(\tilde{R})$.

4.2. Definition (**Pesin Blocks**). Fix $0 < \epsilon \ll 1$. For any $l > 1$, we define a Pesin block $\tilde{\Delta}_l$ of M^f consisting of $\tilde{x} = (x_n) \in M^f$ for which there exists a sequence of splittings $T_{x_n}M = E^s(\tilde{x}, n) \oplus E^u(\tilde{x}, n)$, $n \in \mathbb{Z}$, satisfying:

- $\dim E^s(\tilde{x}, n) = k$;
- $D_{x_n}f(E^s(\tilde{x}, n)) = E^s(\tilde{x}, n+1)$, $D_{x_n}f(E^u(\tilde{x}, n)) = E^u(\tilde{x}, n+1)$;
- for $m \geq 0$, $v \in E^s(\tilde{x}, n)$ and $w \in E^u(\tilde{x}, n)$;

$$\begin{cases} \|D_{x_n}f^m(v)\| \leq e^l e^{-(\theta-\epsilon)m} e^{(\epsilon|n|)} \|v\|, \forall n \in \mathbb{Z}, n \geq 1 \\ \|(D_{x_{n-m}}f^m|_{E^u(\tilde{x},n-m)})^{-1}(w)\| \leq e^l e^{-(\lambda-\epsilon)m} e^{(\epsilon|n-m|)} \|w\|, \forall n \in \mathbb{Z}, n \geq 1; \end{cases}$$

- $\sin \angle(E^s(\tilde{x}, n), E^u(\tilde{x}, n)) \geq e^{-l} e^{-\epsilon|n|}$.

The notation $\tilde{\Delta}_l^k$ represents a Pesin Block with index k being the dimension of the local unstable manifold.

Note that, Pesin blocks are compact subsets of M^f. The sub-spaces $E^s(\tilde{x}, n)$ and $E^u(\tilde{x}, n)$ of $T_{x_n}M$ depend continuously on $\tilde{x}$ and $\tilde{f}^{\pm}(\tilde{\Delta}_l) \subset \tilde{\Delta}_{l+1}$.
In a non-uniformly hyperbolic setting, for C^2 endomorphisms with fixed index, we have the following version of Katok Closing Lemma.

4.3. Lemma (**Katok Closing Lemma** [5]). *Let f be a C^2-endomorphism of a compact Riemannian $s-$dimensional manifold M. Then, for positive numbers χ, l, δ and $0 < k \leq s$, there exists a number $\varrho = \varrho(\chi, l, \delta, k) > 0$ such that, if for some point $\tilde{x} \in \tilde{\Delta}_l^k$ and some integer m, one has*

(1) $$\tilde{f}^m(\tilde{x}) \in \tilde{\Delta}_l^k \quad \textit{and} \quad \tilde{d}(\tilde{x}, \tilde{f}^m(\tilde{x})) < \varrho,$$

then, there exists a point $z \in M$ *and* $\bar{z} \in M^f$*, such that* $z = \pi(\bar{z})$), *and*

- $f^m(z) = z$ *and* $\tilde{f}^m(\bar{z}) = \bar{z}$;
- $\tilde{d}(\tilde{x}, \tilde{z}) < \delta$ *and so* $d(x, z) < \delta$;
- *the point* z *is a hyperbolic periodic point for* f *and its* $W^s_{loc}(x)$ *and* $W^u_{loc}(\bar{z})$ *manifolds are admissible manifolds near the point* x,.

Measure theoretic pressure

Let M be a compact n-dimensional Riemannian manifold and $f : M \to M$ be a C^2 local diffeomorphism preserving an ergodic hyperbolic measure μ (non-atomic). For any $\tilde{x} \in M^f$ with $\pi(\tilde{x}) = x$, let $r(x)$ denote the number of Lyapunov exponents of f at point $x \in M$, $\lambda_i(x)$ $(1 \leq i \leq r(x) \leq s)$ the $i-$th Lyapunov exponent and $k_i(x)$ its multiplicity. Due to Theorem 4.1, and the ergodicity of the measure, the $r^\mu, \lambda_i^\mu, k_i^\mu$ are constant.

Let $f : M \to M$ be a C^2 map and μ an f-invariant Bore1 probability measure on M with $\tilde{\mu}$ the corresponded measure on M^f. Then, the following is a metric on M. This metric is called the d_n metric.

$$d_n(x, y) = \max_i \{d(f^i(x), f^i(y));\ 0 < i \leq n,\ x, y \in M\}.$$

For $\phi \in C(M)$ and $\mu \in \mathcal{M}_f(M)$ the measure theoretic pressure with respect to the measure entropy $h_\mu(f)$ for a definition) is defined as following.

$$P_\mu(f, \phi) = h_\mu(f) + \int \phi \, d\mu. \tag{2}$$

In [2], the authors provide a definition for measure theoretical pressure using measurable sets. The definition is stated as follows. Denote by $B_n(x, \epsilon)$ the ϵ-ball centered on x in d_n-metric. For $\epsilon > 0$ a set $E \subset M$ is said to be an (n, ϵ)-spanning set, if $M \subset \bigcup_{x \in E} B_n(x, \epsilon)$. For $\rho > 0$ one can define the $\mu-$ (n, ϵ, ρ)-spanning set if $\mu(\bigcup_{x \in E} B_n(x, \epsilon)) > 1 - \rho$.

A set $F \subset M$ is said to be an (n, ϵ)-separated set, if for $x \neq y \in F$ there exists some $0 \leq i < n$ such that $d(f^i(x), f^i(y)) \geq \epsilon$. For $\rho > 0$, the $\mu-$ (n, ϵ, ρ)-separated sets are defined similarly. Note that by definition, any (n, ϵ)-separating set is an (n, ϵ)-spanning set.

In what follows the notations $\underline{\lim}$ and $\overline{\lim}$ are used to represent respectively $\liminf_{n \to \infty}$ and $\limsup_{n \to \infty}$. Let $C(M)$ be the space of all continuous real valued functions on M. For $\phi \in C(M)$, one define $S_n\phi(x) = \sum_{i=0}^{n-1} \phi(f^i(x))$ and

$$P(f, \phi, n, \epsilon) = \inf\{\sum_{x \in E} \exp\, S_n(\phi) |\, E \ \ is\ (n, \epsilon)\text{- } spanning\ set\},$$

and for $\rho > 0$;

$$P^*(f, \phi, n, \epsilon, \rho) = \inf\{\sum_{x \in E} \exp\, S_n(\phi) |\, E \ \ is\ \mu\text{-}(n, \epsilon, \rho)\text{-}spanning\ set\}.$$

Then, the **Topological Pressure** of f is defined as the map $P(f, .) : C(M) \to \mathbb{R}$, where

$$P(f, \phi) = \lim_{\epsilon \to 0} \overline{\lim}_{n \to \infty} \frac{1}{n} \log P(f, \phi, n, \epsilon).$$

In a similar way the **Measure Theoretic Pressure** of f with respect to μ is defined as,

$$P^*_\mu(f, \phi) = \lim_{\rho \to 0} \lim_{\epsilon \to 0} \overline{\lim}_{n \to \infty} \frac{1}{n} \log P(f, \phi, n, \epsilon, \rho). \tag{3}$$

The following Theorem, establishes the equality of (3 and 2).

4.4. Theorem (Theorem 2.1 of [2]). *Suppose* (X, d) *is a compact metric space and* $f : X \to X$ *be a continuous map. For any* $\mu \in \mathcal{E}(f)$ *and* $\varphi \in C(X)$,

$$Q^*_\mu(f, \varphi) = P^*_\mu(f, \varphi) = P_\mu(f, \varphi) = h_\mu(f) + \int \varphi d\mu,$$

where $Q^*_\mu(f,\phi) = \lim_{\epsilon\to 0} \underline{\lim}_{n\to\infty} \frac{1}{n} \log P(f,\phi,n,\epsilon)$.

The following Variational Principle assigns the relation between the topological and measure theoretical pressure.

4.5. Theorem. [4] *Let* $f : X \to X$ *be a continuous map and* $\varphi \in C(X)$. *Then*

$$P(f,\varphi) = \sup\{P_\mu(f,\varphi) \mid \mu \in \mathcal{M}(f)\} = \sup\{P_\mu(f,\varphi) \mid \mu \in \mathcal{E}(f)\}.$$

In this work the following upper bound for the measure theoretic pressure of a C^2 endomorphism is investigated.

4.6. Theorem. *Let* $f : M \to M$ *be a* C^2 *local diffeomorphism on a compact s-dimensional Riemannian manifold* M *(with fixed index), and* μ *a hyperbolic measure. Then for any* $\phi \in C(M)$,

$$P_\mu(f,\phi) \leq \lim_{n\to\infty} \sup \frac{1}{n} \log \sum_{x\in Fix(f^n)} \exp(s_n(\phi)(x)).$$

Conclusion

In this paper we give an upper bound for the measure theoretic pressure of a C^2 endomorphism on a closed s-dimensional Riemannian manifold (compact and boundaryless) preserving a hyperbolic Borel probability measure with fixed index.

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5. Chaotic Dynamics of Lorenz-type systems

Roya Makrooni[a]

Faculty of Mathematics, Statistics & Computer Sciences, University of Sistan and Baluchestan, Zahedan, Iran

In the present paper, we consider a family of one-dimensional discontinuous monotone dynamical systems of the interval $[0, 1]$ onto itself, with $N = 2$ branches and investigate their chaotic dynamics. Here we follow the definition suggested by Devaney in order to show robust full chaos.

Keywords: chaos, piecewise smooth dynamical systems, expanding

AMS Mathematics Subject Classification [2020]: 37G15

Code: cdsgt3-00710026

[a]Speaker. Email address: r.makrooni@math.usb.ac.ir

Introduction

The study of piecewise smooth systems had a wide expansion in the last decade. This is due to the large number of physical and engineering systems with nonsmooth vector fields. Many applications come from power electronic circuits in electrical engineering, which gave a wide impulse to the study of piecewise defined systems, both continuous and discontinuous. Several kinds of bifurcations of non-smooth systems also appear in forced impact oscillators, in mechanical engineering, in economics and social sciences.

Piecewise smooth (PWS for short) dynamical systems are characterized by the fact that their state space is divided into partitions by borders also denoted as switching manifolds. Within each partition, the system is smooth (that is C^k up to some k) but the rules which govern the dynamic behavior (that is the right hand side of the system function) change at the boundaries ([1]).

Sometimes, simplifying assumptions on systems lead to piecewise linear models and since many years particular studies have been devoted to piecewise linear (PWL for short) systems, both contin- uous and discontinuous.

In the present work, we consider a map of an interval I into itself. In the last decades, the condition of persistence of chaos in the whole interval (robust full chaos) has become very important in engineering applications, especially those related to grazing bifurcations, for security transmissions, as well as in other applied fields. In such applications, the systems are often ultimately described by piecewise smooth maps. In particular, it is known that the three-dimensional ordinary differential equations called Lorenz flows and discontinuity-induced bifurcation can be analyzed by using suitable Poincaré maps, which are often piecewise smooth and discontinuous. An important class of such systems, for which the Poincaré sections are maps with two branches ($N = 2$), leads to a family of discontinuous maps of an interval, with two increasing branches, called Lorenz maps of Class $\mathcal{A}$ and, as we shall recall, already considered by many authors.

Moreover, particularly important is to investigate the conditions of full chaos, and its robustness, in the class of expanding Lorenz maps. These kind of maps are discontinuous and have increasing branches which has been considered in the literature. In fact, a well known sufficient, but not necessary, condition of robust full chaos for an expanding Lorenz map $f(x)$ is $f'(x) > \sqrt{2}$ for any x (seen in [6], [5]), while the necessary and sufficient conditions for the case $1 < f'(x) < \sqrt{2}$ can be considered outlined in [2].

Differently, the case of a piecewise smooth map with $N > 2$ branches, has got less attention up to now. Besides the basic results related to the piecewise linear map with constant slope, expanding piecewise monotone maps with $N > 2$ branches have been considered by Li and Yorke in [4], where some relevant properties of the chaotic sets are determined, but not the characterization of full chaos.

Still less attention has been paid to the robustness of the chaotic regime. In many applications it is relevant to get robust chaos, i.e. structurally stable chaos, or persistent under parameter perturbations. In particular, the occurrence and robustness of full chaos in a Poincaré map which is a *non-expanding Lorenz map*, can be considered as an open problem. Indeed this is a relevant case, also in applications, which motivates the present work. Besides the class of Lorenz maps, we are interested in a particular class of piecewise monotone discontinuous maps with $N > 2$ branches, which is associated with the first return map in Lorenz maps. Their peculiarity is that the internal branches of the first return maps are *onto* the interval, and we call these maps Baker-like. A relevant fact is that even if a Lorenz map is not expanding, its related first return map may be an expanding Lorenz map or Baker-like map, and this allows to get results otherwise difficult to prove.

So, the main objective of this work is to give the necessary and sufficient conditions for a discontinuous piecewise smooth *expanding* map f of an interval into itself, constituted by N pieces with $N = 2$, to be robustly chaotic in the whole interval. As recalled above, for $N = 2$ the map is a Lorenz map of Class $\mathcal{A}$, and this problem has been investigated by other authors, mainly giving sufficient conditions for full chaos.

Main results

First of all, we state some basic notations related to the chaos theory. Many definitions of chaos can be found in the existing literature. We will follow here the definition that was suggested by Devaney in 1986. It has three ingredients defined as follows:

5.1. Definition. (Chaos) *Let (X, d) be a metric space without isolated points. Then a dynamical system $f : X \longrightarrow X$ is said to be chaotic (in the sense of Devaney) if it satisfies the following conditions:*

(1) **transitivity:** *f is topologically transitive in X; that is, for any pair of non-empty open sets U and V of X there exists a natural number n such that $f^n(U) \cap V \neq \varnothing$;*

(2) **density:** *the periodic points of f are dense in X;*

(3) **sensitivity:** *f has sensitive dependence on initial conditions in X; that is, there is a positive constant δ (sensitivity constant) such that for every point x of X and every neighborhood N of x there exists a point y in N and a non negative integer n such that $d(f^n(x), f^n(y)) \geq \delta$.*

For convenience, suppose that I is the interval $[0,\ 1]$.

5.2. Definition. (Robust or Structurally stable choas) *A map $\phi(x; p) := I \longrightarrow I$ depending on a vector of parameters $p \in P$ is said to be robustly chaotic (or structurally stable chaotic) in X at p_0 if the same property holds in a neighborhood $U(p_0) \cap P$ of p_0 (i.e. the maps are topologically conjugate).*

In this work, we consider a one dimensional discontinuous piecewise monotone $\mathcal{C}^{(1)}$ map $f: I \longrightarrow I$ of the interval $[0,1]$ onto itself, with $N \geq 2$ branches. That is, Lorenz maps for $N = 2$ and a family of expanding Baker-like maps for $N > 2$. We distinguish between maps with one discontinuity point, i.e. the number of branches is $N = 2$, which lead to the well known class of Lorenz maps, or more branches, $N \geq 3$, which we call Baker-like maps. This class of maps is relevant because it may represent the first return map of non-expanding Lorenz maps. We determine the necessary and sufficient conditions to have robust chaos, with robust we mean persistence under parameter perturbations and full chaos refers to the whole interval.

5.3. Definition. (Lorenz map) *A Lorenz map* $x \longmapsto f(x)$ *is defined by a function* $f: I \longrightarrow I$, $I = [0,1]$, *with a single discontinuity point* $\xi_1 \in (0,1)$:

$$f(x) = \begin{cases} f_1(x) & \text{if } 0 \leq x < \xi_1 \\ f_2(x) & \text{if } \xi_1 \leq x \leq 1 \end{cases} \tag{4}$$

such that $f_i(x)$ *are strictly increasing* $\mathcal{C}^{(1)}$ *functions in* I_i, $i = 1, 2$, *with* $f_1(\xi_1) = 1$ *and* $f_2(\xi_1) = 0$, $I_1 = [0, \xi_1]$, $I_2 = [\xi_1, 1]$.

An example of map f is shown in Fig.1.

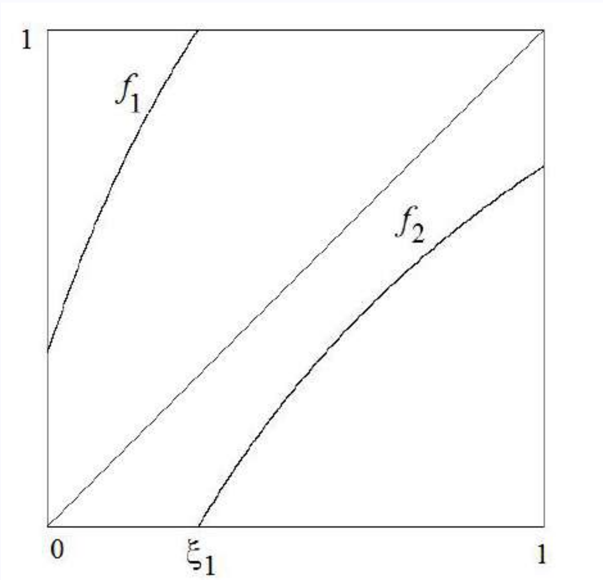

Figure 1. An example of map f and its branches

5.4. Definition. (expanding) *A piecewise* $\mathcal{C}^{(1)}$ *Lorenz map (4) is called expanding if a constant* $\lambda > 1$ *exists such that* $f_i'(x) > \lambda$ *for any* $x \in I_i$ and $1 \leq i \leq N$.

Notice that in our definition of a Lorenz map, it is $f([0,1]) = [0,1)$, but as it is common when dealing with discontinuous maps, we can say the map is chaotic in closed interval $[0,1]$.

In the following, writing a condition on the derivative as $f'(x)$, for any $x \in I$, we mean that for the components f_i the property holds in the closed interval of definiton.

As we have already mentioned in the introduction, we know, from [6], that for an expanding Lorenz map the condition $f'(x) > \sqrt{2}$ is sufficient but not necessary to have chaos in $[0,1]$. In this section we show that the necessary and sufficient conditions, to have full chaos, for $1 < \lambda < \sqrt{2}$ are related to the existing basic cycle RL^{n-1} or $R^{n-1}L$, for some $n > 1$, which must be homoclinic. We emphasize that the existence of such basic cycles is related to a border collision bifurcation, clearly distinguished from the homoclinic bifurcation values.

5.5. Proposition. *([3]). An expanding Lorenz map f is chaotic in $[0, 1]$ iff the existing basic cycle (with symbolic sequence $L^{n-1}R$ or $R^{n-1}L$, for $n > 1$) is homoclinic. Chaos is not robust at the homoclinic bifurcation values detected via the conditions $f^n(0) = x^n_{\max}$ and $f^n(1) = x^n_{\min}$, where $x^n_{\min}$ and $x^n_{\max}$ are the minimum and the maximum of the periodic points of the $n-$cycle.*

The homoclinic bifurcations related to the basic cycles having symbolic sequence RL^{n-1} and $R^{n-1}L$, $n > 1$, have been considered also in [2] for the $\beta-$Transformation $T(x)$. Moreover, similar results hold for a generic expanding Lorenz map (since it is topologically conjugate to a $\beta-$Transformation $T(x)$.

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6. On Einstein Lorentzian Lie groups: type (a1)

Y. AryaNejad[1,a]

[1]Department of Mathematics, Payame noor University, P.O. Box 19395-3697, Tehran, Iran.

The harmonicity vector fields on Einstein Lorentzian Lie groups type (a1) are determined. Left-invariant vector fields defining harmonic maps are also classified.

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[a]Speaker. Email address: y.aryanejad@pnu.ac.ir.

Introduction

A (simply-connected) four-dimensional homogeneous Riemannian manifold is either symmetric or isometric to a Lie group equipped with a left-invariant Riemannian metric. Indeed, the class of n-dimensional simply connected Lorentzian Lie groups (respectively ,Lorentzian Lie algebras) coincides with the class of the Riemannian ones. Using this fact, four-dimensional Einstein Lorentzian Lie groups have been classified [3]. On the other hand, investigating critical points of the energy associated to vector fields is an interesting problem from different points of view. In Riemannian settings, it has been proved that critical points of the energy functional $E : \mathfrak{X}(M) \to R$, restricted to maps defined by vector fields, are parallel vector fields [4, 5]. Moreover, Gil-Medrano [4] studied when V is a harmonic map. So, it is natural to determine the harmonicity properties of vector fields on four-dimensional Lorentzian Einstein Lie groups.

A Riemannian manifold admitting a parallel vector field is locally reducible, and the same is true for a pseudo-Riemannian manifold admitting an either space-like or time-like parallel vector field. This leads us to consider different situations, where some interesting types of non-parallel vector fields can be characterized in terms of harmonicity properties [2].

If $V : (M, g) \longrightarrow (TM, g^s)$ is a critical point for the energy functional, then V is said to define a harmonic map. The Euler-Lagrange equations characterize vector fields V defining harmonic maps as the ones whose tension field $\theta(V) = tr(\nabla^2 V)$ vanishes. Consequently, V defines a harmonic map from (M, g) to (TM, g^s) if and only if

$$tr[R(\nabla_{\cdot} V, V)_{\cdot}] = 0, \quad \nabla^* \nabla V = 0, \tag{5}$$

where with respect to a pseudo-orthonormal local frame $\{e_1, ..., e_n\}$ on (M, g), with $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all indices i, one has

$$\nabla^* \nabla V = \textstyle\sum_i \varepsilon_i (\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V).$$

A smooth vector field V is said to be a harmonic section if it is a critical point of $E^v(V) = (1/2) \int_M ||\nabla V||^2 dv$, where E^v is the vertical energy. The corresponding Euler-Lagrange equations are given by

$$\nabla^*\nabla V = 0. \tag{6}$$

Let $\mathfrak{X}^\rho(M) = \{V \in \mathfrak{X}(M) : ||V||^2 = \rho^2\}$ and $\rho \neq 0$. Then, one can consider vector fields $V \in \mathfrak{X}^\rho(M)$ which are critical points for the energy functional $E|_{\mathfrak{X}^\rho(M)}$, restricted to vector fields of the same constant length. The Euler-Lagrange equations of this variational condition are given by

$$\nabla^*\nabla V \quad \textit{is} \quad \textit{collinear} \quad \textit{to} \quad V. \tag{7}$$

In the non-compact case, the condition (7) is taken as a definition of critical points for the energy functional under the assumption $\rho \neq 0$, that is, if V is not light-like. If $\rho = 0$, then (7) is still a sufficient condition so that V is a critical point for the energy functional $E|_{\mathfrak{X}^0(M)}$, restricted to light-like vector fields ([2], Theorem 26).
Following [3], four-dimensional Einstein Lorentzian Lie groups are classified into four types, denoted by (a1), (a2), (c1) and (c2). In the present paper using a case-by-case argument we shall completely investigat the harmonicity of vector fields on space (a1).

Harmonicity of vector fields

Let (G, g) be a four-dimensional Lorentzian Lie group. Following [3], the Lie algebra $\mathfrak{g}$ of G is a semi-direct product $\mathfrak{r} \ltimes \mathfrak{g}_3$, where $\mathfrak{r} = span\{e_4\}$ acts on $\mathfrak{g}_3 = span\{e_1, e_2, e_3\}$, and the Lorentzian inner product on $\mathfrak{g}$ is described by

$$(a) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (c) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In 2013 Calvaruso and Zaeim [3] obtained the following result:

6.1. THEOREM. *Let G be a four-dimensional simply connected Lie group. If g is a left-invariant Lorentzian Einstein metric on G, then the Lie algebra $\mathfrak{g}$ of G is isometric to $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$, where $\mathfrak{g}_3 = span\{e_1, e_2, e_3\}$ and $\mathfrak{r} = span\{e_4\}$, and one of the following cases occurs.*
(a) *$\{e_i\}_{i=1}^4$ is a pseudo-orthonormal basis, with e_3 time-like. In this case, G is isometric to one of the following semi-direct products $\mathbb{R} \ltimes G_3$:*
(a1) *$\mathbb{R} \ltimes H$, where H is the Heisenberg group and $\mathfrak{g}$ is described by one of the following sets of conditions:*

(1) $[e_1, e_2] = \epsilon A e_1, [e_1, e_3] = A e_1, [e_1, e_4] = \delta A e_1, [e_3, e_4] = -2A\delta(\epsilon e_2 - e_3)$,
(2) $[e_1, e_2] = \frac{\epsilon\sqrt{A^2-B^2}}{2} e_1, [e_1, e_3] = -\frac{\epsilon\delta\sqrt{A^2-B^2}}{2} e_1, [e_1, e_4] = \frac{\delta A+B}{2} e_1, [e_2, e_4] = B(e_2+\delta e_3), [e_3, e_4] = A(e_2 + \delta e_3)$,
(3) $[e_1, e_2] = \frac{\epsilon A\sqrt{A^2-B^2}}{B} e_1, [e_1, e_3] = \epsilon\sqrt{A^2 - B^2} e_1, [e_2, e_4] = B e_2 - A e_3, [e_3, e_4] = A e_2 - \frac{A^2}{B} e_3$,
(4) $[e_1, e_2] = \epsilon\sqrt{A^2 - B^2} e_1 + B e_2, [e_3, e_4] = A e_3$,

(a2) *$\mathbb{R} \ltimes \mathbb{R}^3$, where $\mathfrak{g}$ is described by one of the following sets of conditions:*

(5) $[e_1, e_4] = -(A+B) e_1, [e_2, e_4] = B e_2 - \epsilon\sqrt{A^2 + AB + B^2} e_3, [e_3, e_4] = \epsilon\sqrt{A^2 + AB + B^2} e_2 + A e_3$,
(6) $[e_1, e_4] = -2A e_1, [e_2, e_4] = -5A e_2 + 6\epsilon A e_3, [e_3, e_4] = A e_3$,
(7) $[e_1, e_4] = A e_1, [e_2, e_4] = A e_2 + B e_3, [e_3, e_4] = B e_2 + A e_3$,
(8) $[e_1, e_4] = \epsilon\frac{A+B}{3} e_1, [e_2, e_4] = \epsilon\frac{5B-A}{6} e_2 + B e_3, [e_3, e_4] = A e_2 + \epsilon\frac{5A-B}{6} e_3$,
(9) $[e_1, e_4] = \frac{5A}{2} e_1 + 3\epsilon A e_3, [e_2, e_4] = A e_2, [e_3, e_4] = -\frac{A}{2} e_3$,
(10) $[e_1, e_4] = A e_1 + \epsilon\sqrt{B^2 - A^2 - C^2 - AC} e_2, [e_2, e_4] = \epsilon\sqrt{B^2 - A^2 - C^2 - AC} e_1 - (A + C) e_2 - B e_3, [e_3, e_4] = B e_2 + C e_3$,
(11) $[e_1, e_4] = -\frac{2\epsilon\sqrt{2}A}{3} e_1 + \delta A e_3, [e_2, e_4] = \frac{\epsilon\sqrt{2}A}{3} e_2, [e_3, e_4] = A e_2 - \frac{\epsilon\sqrt{2}A}{6} e_3$,

(c) $\{e_i\}_{i=1}^4$ *is a basis, with the inner product* g *on* $\mathfrak{g}$ *completely determined by* $g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_4) = g(e_4,e_3) = 1$ *and* $g(e_i,e_j) = 0$ *otherwise. In this case,* G *is isometric to one of the following semi-direct products* $\mathbb{R} \ltimes G_3$*:*

(c1) $\mathbb{R} \ltimes H$*,where* $\mathfrak{g}$ *is described by one of the following sets of conditions*

(12) $[e_1,e_2] = \epsilon(A+B)e_3, [e_1,e_4] = Ce_1 + Be_2 + De_3, [e_2,e_4] = Be_1 + Ee_3, [e_3,e_4] = Ce_3,$

(13) $[e_1,e_2] = Be_3, [e_1,e_4] = \frac{(C+D)^2-B^2}{4A}e_1 + De_2 + Fe_3, [e_2,e_4] = Ce_1 + Ae_2 + Ee_3, [e_3,e_4] = \frac{(C+D)^2-B^2+4A^2}{4A}e_3,$

(14) $[e_1,e_2] = \epsilon\sqrt{((A+D)^2+4B^2)}e_3, [e_1,e_4] = -Be_1+De_2+Ee_3, [e_2,e_4] = Ae_1+Be_2+Ce_3,$

(c2) $\mathbb{R} \ltimes \mathbb{R}^3$*, where* $\mathfrak{g}$ *is described by one of the following sets of conditions:*

(15) $[e_1,e_2] = Ae_2 + Be_3, [e_2,e_4] = -Ae_1 + Ce_3,$

(16) $[e_1,e_4] = Ae_1 + Be_2 + Ce_3, [e_2,e_4] = De_1 + Ee_2 + Fe_3, [e_3,e_4] = \frac{(B+D)^2+2(A^2+E^2)}{2(E+A)}e_3$

In all the cases listed above, $\epsilon = \pm 1$*,* $\delta = \pm 1$ *and* A, B, C, D *are real constants.*

All four-dimensional simply connected Einstein Lorentzian Lie groups of type (a1) are symmetric [3] and the study of harmonic invariant vector fields on these spaces would be natural and interesting. The main purpose of this section is to investigat the harmonicity properties of left-invariant vector fields on four-dimensional Lorentzian Lie group of type (a1). The following notation is necessary.

6.2. Remark. Let $\tilde{\mathfrak{X}}^\rho(M)$ denote the set of all vector fields $V \in \mathfrak{X}^\rho(M)$, which are critical points for the energy functional $E|_{\mathfrak{X}^\rho(M)}$, restricted to vector fields of the same constant length. Remember that ρ is not necessarily the same for different cases.

Let (G,g) be a four-dimensional Lorentzian Lie group of type (a1) and $\{e_i\}_{i=1}^4$ a pseudo-orthonormal basis, with e_3 time-like. Under these assumptions, we prove the following result [1].

6.3. Theorem. *Let* $\mathfrak{g}$ *be the Lie algebra of* G *and* $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$ *a left-invariant vector field on* G *for some real constants* a, b, c, d*. For the different cases* $(1)-(4)$ *of type* (a1)*, we have:*

(1) : $V \in \tilde{\mathfrak{X}}^\rho(G)$ *if and only if* $V = c(e_2 - e_3 - e_4)$*, that is,* $b = -c = -d$*. In this case* $\epsilon = 1, \quad \nabla^*\nabla V = 3A^2V.$

(2) : $V \in \tilde{\mathfrak{X}}^\rho(G)$ *if and only if* $V = c(e_2 + e_3 - e_4)$*, that is,* $b = c = -d$*. In this case* $\epsilon = -1$*,* $\delta = 1, \quad \nabla^*\nabla V = -\frac{3}{4}(A+B)^2V.$

(3) : $V \in \tilde{\mathfrak{X}}^\rho(G)$*, in this case,* $\nabla^*\nabla V = -\frac{(A^2-B^2)^2}{B^2}V.$

(4) : $V \in \tilde{\mathfrak{X}}^\rho(G)$ *if and only if* $a = b = 0$*, in this case* $\nabla^*\nabla V = -A^2V$ *or* $c = d = 0$*, in this case* $\nabla^*\nabla V = (B^2 - A^2)V.$

Proof. The above statement is obtained from a case-by-case argument. As an example, we report the details for case (3) here. Let $V \in \mathfrak{g}$ be a critical point for the energy functional. The components of the Levi-Civita connection are the following:

$$\begin{array}{ll} \nabla_{e_1}e_1 = -\frac{\epsilon A\sqrt{A^2-B^2}}{B}e_2 + \epsilon\sqrt{A^2-B^2}e_3, & \nabla_{e_1}e_2 = \frac{\epsilon A\sqrt{A^2-B^2}}{B}e_1 \\ \nabla_{e_1}e_3 = \epsilon\sqrt{A^2-B^2}e_1 \quad \nabla_{e_2}e_2 = -Be_4, & \nabla_{e_2}e_3 = -Ae_4, \\ \nabla_{e_2}e_4 = Be_2 - Ae_3, & \nabla_{e_3}e_2 = -Ae_4, \\ \nabla_{e_3}e_3 = -\frac{A^2}{B}e_4, & \nabla_{e_3}e_4 = Ae_2 - \frac{A^2}{B}e_3, \end{array} \tag{8}$$

while $\nabla_{e_i}e_j = 0$ in the remaining cases. From (8) we obtain

$$\begin{array}{ll} \nabla_{e_1}V = \epsilon\sqrt{A^2-B^2}(\frac{cB+bA}{b}e_1 - \frac{aA}{B}e_2 + ae_3), & \\ \nabla_{e_2}V = dBe_2 - dAe_3 - (cA+bB)e_4, & \nabla_{e_4}V = 0 \\ \nabla_{e_3}V = dAe_2 - \frac{dA^2}{B}e_3 - \frac{A(cA+bB)}{B}e_4. & \end{array} \tag{9}$$

TABLE 1. Equivalent properties for the cases $(1)-(4)$ in Theorem 6.3.

(G,g)	Equivalent properties (denoted by $\equiv$)
(1)	V is geodesic; $\equiv$ $V \in \tilde{\mathfrak{X}}^{\rho}(G)$; $\equiv$ none of these vector fields is harmonic (in particular, defines a harmonic map); $\equiv$ $V = c(e_2 - e_3 - e_4)$,
(2)	V is geodesic; $\equiv$ V is harmonic if and only if $A=-B$; $\equiv$ $V \in \tilde{\mathfrak{X}}^{\rho}(G)$; $\equiv$ V defines harmonic map if and only if $A=-B$; $\equiv$ V is Killing if and only if $A=-B$ and $d=0$; $\equiv$ $V = c(e_2+e_3-e_4)$,
(3)	V is geodesic if and only if $A=\pm B$ and $b=\mp c$; $\equiv$ V is harmonic if and only if $A=\pm B$; $\equiv$ $V \in \tilde{\mathfrak{X}}^{\rho}(G)$; $\equiv$ V defines harmonic map if and only if $A=\pm B$; $\equiv$ V is Killing if and only if $A=\pm B$, $b=\mp c$ and $d=0$,
(4)	V is geodesic if and only if $a=b=c=0$; $\equiv$ $V \in \tilde{\mathfrak{X}}^{\rho}(G)$ if and only if $a=b=0$; $\equiv$ none of these vector fields is harmonic (in particular, defines a harmonic map).

Clearly, there are no parallel vector fields $V \neq 0$ in $\mathfrak{g}$. We can now calculate $\nabla_{e_i}\nabla_{e_i}V$ and $\nabla_{\nabla_{e_i}e_i}V$ for all indices i and we find

$$
\begin{array}{c}
\nabla_{e_1}\nabla_{e_1}V = \frac{-(A^2-B^2)}{B^2}(a(A^2-B^2)e_1 + (cB+bA)(Ae_2-Be_3)),\\
\nabla_{e_2}\nabla_{e_2}V = -(cB+bA)(Be_2-Ae_3) + d(A^2-B^2)e_4,\\
\nabla_{e_3}\nabla_{e_3}V = \frac{-A^2}{B^2}((cB+bA)(Be_2-Ae_3)+d(A^2-B^2)e_4), \quad \nabla_{e_4}\nabla_{e_4}V = 0,\\
\nabla_{\nabla_{e_1}e_1}V = \nabla_{\nabla_{e_3}e_3}V = \nabla_{\nabla_{e_2}e_2}V = \nabla_{\nabla_{e_4}e_4}V = 0.
\end{array} \tag{10}
$$

Thus, we get

$$
\begin{array}{c}
\nabla^*\nabla V = \sum_i \varepsilon_i(\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V) = \frac{-(A^2-B^2)}{B^2}(a(A^2-B^2)e_1 + \\
(cB+bA)(Ae_2-Be_3)) - (cB+bA)(Be_2-Ae_3) + d(A^2-B^2)e_4 - \\
(\frac{-A^2}{B^2}((cB+bA)(Be_2-Ae_3)+d(A^2-B^2)e_4)) = -\frac{(A^2-B^2)^2}{B^2}V.
\end{array}
$$

$\square$

As the definitions already show, V is harmonic if $\nabla^*\nabla V = 0$ and V defines a harmonic map if and only if

$$tr[R(\nabla_{.}V, V)_{.}] = 0, \quad \nabla^*\nabla V = 0.$$

For case (3) in Theorem 6.3, $\nabla^*\nabla V = -\frac{(A^2-B^2)^2}{B^2}V = 0$ if and only if $A = \pm B$, that is, V is harmonic if and only if $A=\pm B$. Let R denote the curvature tensor of (M,g), taken with the sign convention $R(X,Y) = \nabla[X,Y] - [\nabla X, \nabla Y]$. Then, using (9), we find

$$
\begin{array}{c}
R(\nabla_{e_1}V, V)e_1 = \frac{\epsilon(A^2-B^2)^{3/2}}{B^3}((A^2-B^2)a^2 + (bA+cB)^2)(Ae_2-Be_3),\\
\frac{A^2}{B^2}R(\nabla_{e_2}V,V)e_2 = R(\nabla_{e_3}V,V)e_3 = \frac{A^2(A^2-B^2)}{B^3}((A^2-B^2)d^2 - (cA+bB)^2)e_4,\\
R(\nabla_{e_4}V,V)e_4 = 0
\end{array}
$$

and so, when $A=\pm B$ clearly,

$$tr[R(\nabla_{.}V,V)_{.}] = \sum_i \varepsilon_i R(\nabla_{e_i}V,V)e_i = 0.$$

Hence, $tr[R(\nabla_{.}V,V)_{.}] = 0$ if and only if $A = \pm B$. Appling this argument for other cases of type (a1) proves the following classification result [1].

6.4. THEOREM. *Let V be a critical point for the energy functional, described by the conditions* (2) *and* (3) *in Theorem 6.3. Then, for cases* (2) *and* (3)*, V defines a harmonic map if and only if $A=-B$ and $A=\pm B$ respectively.*

A vector field V is geodesic if $\nabla_V V = 0$, and is Killing if $\mathcal{L}_V g = 0$, where $\mathcal{L}$ denotes the Lie derivative. Parallel vector fields are both geodesic and Killing, and vector fields with these special geometric features often have particular harmonicity properties. A straightforward calculation proves the following main classification result [1].

6.5. Corollary. *If g is a left-invariant Lorentzian Einstein metric on G, then for the cases $(1)-(4)$ in Theorem 6.3, the equivalent properties for $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$ are listed in Table 1.*

6.6. Remark. Recall that for a Lorentzian Lie group, a left-invariant vector field V is spatially harmonic if and only if

$$\tilde{X}_V = -\nabla^*\nabla V - \nabla_V\nabla_V V - divV \cdot \nabla_V V + (\nabla V)^t\nabla_V V \quad is \quad collinear \quad to \quad V. \tag{11}$$

Clearly, conditions (7) and (11) coincide for geodesic vector fields. Hence, the results listed in Table 1 show that for cases (1) and (2), V is spatially harmonic and for case (3), V is spatially harmonic if and only if $A = \pm B$ and $b = \mp c$. For case (4), V is spatially harmonic if and only if $a = b = c = 0$.

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7. Topological Asymptotic average Shadowing Property

Seyyed Alireza Ahmadi
Department of Mathematics, Faculty of Mathematics Statistics and Computer Sciences,
University of Sistan and Baluchestan, Zahedan, Iran

We introduce topological definition of asymptotic average shadowing property. We show that this property implies that every two disjoint points in the space are proximal.

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Introduction

The pseudo-orbit tracing property is one of the most important notions in dynamical systems, which is closely related to stability and chaos of systems . This concept is motivated by computer simulations. More precisely, let X be a set and $f : X \to X$ be a map. Then in the computation of f with initial value $x_0 \in X$, computer approximates $f(x_0)$ by some point x_1. To continue the process, it computes the value x_2 as an approximation of $f(x_1)$ and so on. For formulating this concept we have to use the 'distance' between points to control approximation errors. In a metric space (X, d) one can approximates points using metric d and define a pseudo-orbit with error δ as a sequence $x_0, x_1, \dots$ with $d(x_{j+1}, f(x_j)) < \delta$ for all $j \geq 0$. However, for general topological spaces such a distance cannot be found unless we have somewhat more structure than what the topology itself provides. This issue will be solved if we consider a completely regular topological spaces which is equipped with an structure, called uniformity, enabling us to control the distance between points in these spaces. Using this structure, Das et al. generalized the usual definitions of shadowing, and chain recurrence for homeomorphisms to topological spaces. Then, we [1] proved that a dynamical system with ergodic shadowing is topologically chain transitive. Wu [2] introduced the topological concepts of weak uniformity, uniform rigidity, and multi-sensitivity and obtained some equivalent characterizations of uniform rigidity. Then, we [3] proved that a point transitive dynamical system in a Hausdorff uniform space is either almost (Banach) mean equicontinuous or (Banach) mean sensitive. Recently, we [?] generalized concepts of entropy points, expansivity and shadowing property for dynamical systems to uniform spaces and obtained a relation between topological shadowing property and positive uniform entropy. Good and Macías [4] obtained some equivalent characterizations and iteration invariance of various definitions of shadowing in the compact uniform spaces.

Nevertheless when calculating approximate trajectories, it makes sense to consider errors small on average, since controlling them in each iteration may be impossible. The notion of average pseudo-orbit introduced by Blank in 1988. In a metric space (X, d) an average pseudo-orbit with error δ is a sequence $x_0, x_1, \dots$ for which there is $N \in \mathbb{N}$ such that $\frac{1}{n}\sum_{j=0}^{n-1} d(x_{j+k+1}, f(x_{j+k})) < \delta$ for any $n \geq N$ and $k \in \mathbb{N}$. The average shadowing property is related to finding an averagely close real orbit for any average pseudo-orbit. But in a general topological space we need some method to control the average of errors in a pseudo-orbit . Motivated by mentioned ideas, We introduced average shadowing property on uniform spaces.[5]

Here we show that asymptotic average shadowing property can be defined in a natural way on uniform spaces. In order to do this, we control the average of errors of a pseudo-orbit in a non-metrizable topological space via infinite sequences of neighborhoods of diagonal. Then we prove that asymptotic average shadowing property implies topological chain transitivity.

Topological asymptotic average shadowing property

Denote by $\Sigma_{\mathcal{U}}$ the family of all sequences $\mathcal{E} = \{E_i\}_{i=0}^{\infty}$ of entourages in $\mathcal{U}$ with $E_0 = X \times X$, such that $E_{i+1} \subset E_i$ for all $i \in \mathbb{N}_0$. For a sequence $\mathcal{E} = \{E_i\}_{i=0}^{\infty} \in \Sigma_{\mathcal{U}}$, a map $f : X \to X$ and a sequence $\xi = \{x_0, x_1, \dots\}$ in X we define

$$\begin{aligned}\mathcal{A}_n(\xi, f, \mathcal{E}) = \mathcal{A}_n(\xi, f, \{E_i\}_{i=0}^{\infty}) &= \inf\{\sum_{j=0}^{n} \frac{1}{2^{\sigma(j)}} | \quad (x_{j+1}, f(x_j)) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\}; \quad n \in \mathbb{N} \\ &= \sum_{j=0}^{n} \inf\{\frac{1}{2^{\sigma(j)}} | \quad (x_{j+1}, f(x_j)) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^j\}; \quad n \in \mathbb{N},\end{aligned}$$

and

$$\mathcal{A}_n(\xi, z, f, \mathcal{E}) = \mathcal{A}_n(\xi, z, f, \{E_i\}_{i=0}^{\infty}) = \inf\{\sum_{j=0}^{n} \frac{1}{2^{\sigma(j)}} | \quad (x_j, f^j(z)) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\},$$

where $\mathbb{N}_0^n$ is the set of all maps from $\{0, 1, \dots, n\}$ to $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

7.1. Remark. Let $(X,\mathcal{U})$ be a compact uniform space and $f : X \to X$ be a continuous map. If $\mathcal{E} = \{E_i\}_{i=0}^{\infty} \in \Sigma_{\mathcal{U}}$ and $\xi = \{x_i\}_{i=0}^{\infty}$ be a sequence in X, then for any $m, n, k \in \mathbb{N}$ we have

(1) $0 \leq \mathcal{A}_n(\xi, f, \mathcal{E}) \leq n$;
(2) $\mathcal{A}_n(\xi, f, \mathcal{E}) \leq \mathcal{A}_{n+1}(\xi, f, \mathcal{E}) \leq \mathcal{A}_n(\xi, f, \mathcal{E}) + 1$;
(3) $\mathcal{A}_n(\xi, f, \{E_i\}_{i=0}^{\infty}) = \frac{1}{2^k}\mathcal{A}_n(\xi, f, \{E'_i\}_{i=0}^{\infty})$, where $E'_i = E_{i+k}$ for $i \geq 1$ and $E'_0 = X \times X$.
(4) $\mathcal{A}_n(\xi, f, \{E'_i\}_{i=0}^{\infty}) \leq \mathcal{A}_n(\xi, f, \{E_i\}_{i=0}^{\infty})$, where $E'_i = E_{ik}$;
(5) $\mathcal{A}_{m+n}(\xi, f, \mathcal{E}) = \mathcal{A}_m(\xi, f, \mathcal{E}) + \mathcal{A}_n(T^m(\xi), f, \mathcal{E})$, where $T : X^{\mathbb{N}_0} \to X^{\mathbb{N}_0}$ is the shift map.

7.2. Definition. [5] For $\mathcal{D} \in \Sigma_{\mathcal{U}}$, a *topological average $\mathcal{D}$-pseudo-orbit* of f is a sequence $\{x_i\}$ in X such that $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\xi, f, \mathcal{D}) = 0$. Let $\mathcal{E} = \{E_i\} \in \Sigma_{\mathcal{U}}$. We say that the sequence $\{x_i\}$ is *$\mathcal{E}$-shadowed on average by some point* $z \in X$, if $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\xi, z, f, \mathcal{E}) = 0$. We say that the map f has the *topological average shadowing property* **TASP**, if for every $\mathcal{E} = \{E_i\} \in \Sigma_{\mathcal{U}}$, there exists $\mathcal{D} = \{D_i\}_{i=0}^{\infty} \in \Sigma_{\mathcal{U}}$ such that every topological average $\mathcal{D}$-pseudo-orbit is $\mathcal{E}$-shadowed on average by some point of X.

If (X, d) is a compact metric space, then for any neighborhood U of Δ_X, we can find $\delta > 0$ such that $V_\delta^d \subset U$. On the other hand, every V_δ^d is a neighborhood of Δ_X. Moreover if $\{x_i\}$ is a topological average $\{V_{\frac{\delta}{2^i}}^d\}$-pseudo-orbit, then $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\xi, f, \{V_{\frac{\delta}{2^i}}^d\}) = 0$ and there exists $N \in \mathbb{N}$ such that $\frac{1}{n}\mathcal{A}_n(\xi, f, \{V_{\frac{\delta}{2^i}}^d\}) < 1$ for $n \geq N$. One can easily check that $\frac{1}{n}\sum_{i=0}^{n-1} d(x_{i+k+1}, f(x_{k+i})) < \delta$ for all $n \geq N$ and $k \in \mathbb{N}$. This shows that for any $\delta > 0$, we can find a $\mathcal{D} = \{D_i\}_{i=0}^\infty \in \Sigma_\mathcal{U}$ such that every topological average $\mathcal{D}$-pseudo orbit is an average δ-pseudo orbit. But the converse is not true, for example, if we consider the identity map on S^1 with the usual topology, then for $\mathcal{D} = \{V_{\frac{1}{i}}\}_{i=0}^\infty \in \Sigma_{\mathcal{U}_{S^1}}$ and any $\delta > 0$, the sequence $x_{j+1} = x_j + \delta/2$ with $x_0 = 0$ is an average δ-pseudo orbit which is not a topological average $\mathcal{D}$-pseudo orbit. That is, this remark does not implies that the topological average shadowing property is equivalent to the usual average shadowing property when the uniform structure is came from a metric d.

7.3. Definition. An asymptotic-average-pseudo-orbit is a sequence $\{x_i\}$ in X such that for each $\mathcal{D} \in \Sigma_\mathcal{U}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \inf\{\sum \frac{1}{2^{\sigma(j)}} : \quad (fx_{j-1}, x_j) \in D_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\} = 0$$

We say that the sequence $\{x_i\}$ is asymptotically shadowed on average by some point $y \in X$, if for each $\mathcal{E} = \{E_i\} \in \Sigma_\mathcal{U}$ we obtain

$$\lim_{n\to\infty} \frac{1}{n} \inf\{\sum \frac{1}{2^{\sigma(j)}} : \quad (f^j(x), x_j) \in D_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\} = 0$$

7.4. Definition. A map f is said to have the *asymptotic average shadowing property*, if every asymptotic-average-pseudo-orbit of f can be asymptotically shadowed on average by some point in X.

7.5. Definition. A map f is said to have the *weak asymptotic average shadowing property*, if if for every $\mathcal{E} = \{E_i\} \in \Sigma_\mathcal{U}$ and any asymptotic pseudo-orbit $\xi = \{x_i\}$, there exists $z \in X$ such that $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\xi, z, f, \mathcal{E}) = 0$.

The following example shows that **AASP** $\not\Rightarrow$ **TASP**.

7.6. Example. Let $X = \{a, b, c\}$. Consider following subsets of $X \times X$

$U_0 = \Delta_X \cup \{(a, b), (b, a)\}$
$U_1 = \Delta_X \cup \{(a, b), (b, a), (a, c)\}$
$U_2 = \Delta_X \cup \{(a, b), (b, a), (a, c), (c, a)\}$
$U_3 = \Delta_X \cup \{(a, b), (b, a), (a, c), (b, c)\}$
$U_4 = \Delta_X \cup \{(a, b), (b, a), (a, c), (c, b)\}$
$U_5 = \Delta_X \cup \{(a, b), (b, a), (c, a)\}$
$U_6 = \Delta_X \cup \{(a, b), (b, a), (c, a), (b, c)\}$
$U_7 = \Delta_X \cup \{(a, b), (b, a), (c, a), (c, b)\}$
$U_8 = \Delta_X \cup \{(a, b), (b, a), (b, c)\}$
$U_9 = \Delta_X \cup \{(a, b), (b, a), (c, b)\}$
$U_{10} = \Delta_X \cup \{(a, b), (b, a), (b, c), (c, b)\}$
$U_{11} = \Delta_X \cup \{(a, b), (b, a), (b, c), (c, b), (a, c).\}$
$U_{12} = \Delta_X \cup \{(a, b), (b, a), (b, c), (c, b), (c, a)\}$
$U_{13} = \Delta_X \cup \{(a, b), (b, a), (b, c), (a, c).(c, a)\}$
$U_{14} = \Delta_X \cup \{(a, b), (b, a), (c, b), (a, c).(c, a)\}$

Then $\mathcal{U} = \{U_0, U_1, U_2, \ldots, U_{14}, X \times X\}$ is a uniformity on X for which $\tau_\mathcal{U} = \{\emptyset, \{a, b\}, \{c\}, X\}$. Let $f : X \to X$ be a permutation defined by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then f is uniformly continuous. We show that f does not have the average shadowing property. Let $E_0 = X \times X$ and $E_i = U_0$ for all $i \geq 1$. Then $\mathcal{E} = \{E_i\} \in \Sigma_\mathcal{U}$. Put $x_0 = a$ and for each $i \in \mathbb{N}$ and $k \geq 0$ define

$$x_i = \begin{cases} f^i(a) & \text{if } 2^{2k} \leq i < 2^{2k+1}, \\ c & \text{if } 2^{2k+1} \leq i < 2^{2k+2} \end{cases}.$$

In other word

$$x_0, x_1, x_2, \cdots = \underbrace{a, b}_{2}, \underbrace{c, c}_{2}, \underbrace{a, b, a, b}_{4}, \underbrace{c, c, \ldots, c}_{8}, \underbrace{a, b, a, b, \ldots, b}_{16}, \underbrace{c, c, \ldots, c, c}_{32}, \ldots.$$

Let $\xi = \{x_i\}$. Since any element of $\mathcal{U}$ contains (a, b) and (b, a), we obtain

$$\begin{aligned}\frac{1}{n}\inf\{\sum_{j=1}^{n}\frac{1}{2^{\sigma(j)}} : \quad (f(x_{j-1}), x_j) \in D_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\} &\leq \frac{2(k+1)}{n} \\ &< \frac{2(k+1)}{2^k},\end{aligned}$$

for $2^k \leq n < 2^{k+1}$ and arbitrary $\mathcal{D} = \{D_i\} \in \Sigma_{\mathcal{U}}$. That is ξ is an average $\mathcal{D}$-pseudo-orbit. For $2^k \leq n < 2^{k+1}$, we obtain

$$\begin{aligned}&\frac{1}{n}\mathcal{A}_n(\xi, a, f, \mathcal{E}) = \frac{1}{n}\mathcal{A}_n(\xi, b, f, \mathcal{E}) \geq \frac{\sum_{i=0}^{k} 2^{2i-1}}{\sum_{i=0}^{2k} 2^i}. \\ &\frac{1}{n}\mathcal{A}_n(\xi, c, f, \mathcal{E}) \geq \frac{\sum_{i=0}^{k} 2^{2i}}{\sum_{i=0}^{2k} 2^i}.\end{aligned}$$

Hence

$$\begin{aligned}&\lim n \to \infty \frac{1}{n}\mathcal{A}_n(\xi, a, f, \mathcal{E}) = \lim n \to \infty \frac{1}{n}\mathcal{A}_n(\xi, b, f, \mathcal{E}) \geq \frac{1}{3} \\ &\lim n \to \infty \frac{1}{n}\mathcal{A}_n(\xi, c, f, \mathcal{E}) \geq \frac{2}{3}\end{aligned}$$

Therefore ξ could not be $\mathcal{E}$-shadowed in average by any point in X. This implies that f does not have the topological average shadowing property. It is easy to show that f has the topological shadowing property and topological asymptotic average shadowing

7.7. Proposition. *Let (X, f) be a dynamical system with uniform space. Then f has the asymptotically average shadowing property if f^k has for every $k \in \mathbb{N}$.*

Let $f : X \longrightarrow X$ be a homeomorphism of uniform compact Hausdorff space. Points $x, y \in X$ are called *proximal* if closure $\overline{\mathcal{O}((X, Y))}$ of the orbit of (x, y) under $f \times f$ intersects the diagonal $\Delta = \{(z, z) \in X \times X : z \in X\}$. Denote by $PR(X, f)$ the set of all pairs (x, y) where x and y are proximal.

7.8. Theorem. *Let $(X, \mathcal{U})$ be a uniform space and $f : X \to X$ be continuous function. If f has the asymptotically average shadowing property, then $(x, y) \in PR \circ PR(X, f)$.*

7.9. Corollary. *Let $(X, \mathcal{U})$ be a compact uniform space and f be a continuous map from X onto itself. If f has asymptotic average shadowing property, then every point $x \in X$ is topological chain recurrent point.*

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8. Mean Ergodic Shadowing: relation with other shadowing

Ali Darabi[a]

Department of mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran

In this paper, we study mean ergodic shadowing property and obtain some results in relation with other partial shadowing. We show that mean ergodic shadowing implies $\overline{d}$-shadowing property, and any minimal system with mean ergodic shadowing does not have $\mathcal{F}_s$-shadowing property. In addition, by giving some examples, we show that the shadowing property and mean ergodic shadowing are different.

Keywords: Shadowing, mean ergodic shadowing, $\overline{d}$-shadowing.

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[a]Speaker. Email address: adarabi@scu.ac.ir,

Introduction

A pair (X, f), where (X, d) is a metric space and $f : X \to X$ is a continuous map is called a *topological dynamical system*. The shadowing theory is an important part of the global and stability theory of dynamical systems [9]. The shadowing property means that near a pseudo-orbit (numerically computed orbit) there exists an exact orbit. In other words, numerical computations reflect the real dynamical behavior of f. Throughout the study (X, d) is a compact metric space. We use $\mathbb{Z}^+$ for the set of non-negative integers.

Given $\delta > 0$ a sequence $\xi = \{x_i\}_{i\in\mathbb{Z}^+} \subset X$ with the property

$$d(f(x_i), x_{i+1}) < \delta, \quad \forall i \in \mathbb{Z}^+$$

is called a δ-*pseudo-orbit* for f, and if

$$\underline{d}(\{i \geq 0 : d(f(x_i), x_{i+1}) < \delta\}) = 1,$$

where $\underline{d}(A)$ is the *lower density* of the set $A \subset \mathbb{Z}^+$ defined by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap \{0, 1, \cdots, n-1\}|}{n},$$

is called a δ-*ergodic pseudo-orbit* of f [3]. If we replace lim inf with lim sup in the above formula we get $\overline{d}(A)$, the *upper density* of A. We say the set A has density zero if $\overline{d}(A) = 0$. A sequence $\xi = \{x_i\}_{i=0}^{\infty}$ is said to be ϵ-*shadowed* by a point $z \in X$ if

$$d(f^i(z), x_i) < \epsilon, \quad \forall i \geq 0.$$

A map $f : X \to X$ is said to has *shadowing property* (POTP, for short) if for any $\epsilon > 0$ there exists $\delta > 0$ in which every δ-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ can be ϵ-shadowed by some point in X.

8.1. Definition. A δ-ergodic pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}^+}$ is said to be *ϵ-ergodic shadowed* by some point z in X if

$$\underline{d}(\{i \geq 0 : d(f^i(z), x_i) < \epsilon\}) = 1.$$

A dynamical system (X, f) has *ergodic shadowing property* if for any given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ in which that any δ-ergodic pseudo-orbit of f can be ϵ-ergodic shadowed by some point in X [3].

A set $A \subset \mathbb{Z}^+$ is called *syndetic* if it has bounded gaps, i.e., there is $k > 0$ such that $A \cap \{i, i+1, \cdots, i+k-1\} \neq \emptyset$ for each $i \geq 0$. The family of syndetic sets is denoted by $\mathcal{F}_s$.

8.2. Definition. If for any $\epsilon > 0$ there exists $\delta > 0$ such that for every δ-pseudo-orbit $\{x_i\}_{i\geq 0}$ there is a point $z \in X$ in which

$$\{i : d(f^i(z), x_i) < \epsilon\} \in \mathcal{F}_s,$$

then we say f has *$\mathcal{F}_s$-shadowing property.*

Analogously, we denote $\mathcal{F}_{\underline{d}}$ for the family of subsets of $\mathbb{Z}^+$ with positive lower density, and define $\mathcal{F}_{\underline{d}}$-shadowing property similarly.

8.3. Definition. (X, f) has *$\underline{d}$-shadowing property* if for each $\epsilon > 0$ there exists $\delta > 0$ such that for every δ-ergodic pseudo-orbit $\{x_i\}_{i\geq 0}$ there is a point $z \in X$ such that

$$\underline{d}(\{i : d(f^i(z), x_i) < \epsilon\}) > 0.$$

Similarly, if $\overline{d}(\{i : d(f^i(z), x_i) < \epsilon\}) > \frac{1}{2}$ it is said *$\overline{d}$-shadowing property* (see [1, Definition 2.1]).

Recently, Das et al. [2] introduced a new type of average shadowing named mean ergodic shadowing and studied some aspects of it. We bring the following definition from [2].

8.4. Definition. [2] A map f has *mean ergodic shadowing property* if for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ in which that any δ-ergodic pseudo-orbit of f can be ϵ-shadowed in average by some point in X. In other words, there exists $z \in X$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon.$$

It is well known that mean ergodic shadowing is a weaker form of ergodic shadowing property[2, Proposition 4.2].

8.1. Topological dynamical systems. A finite δ-pseudo-orbit $\{x_i\}_{i=0}^{b}$ is called a *δ-chain* from x_0 to x_b. A dynamical system (X, f) is called *chain transitive* if for any two points $x, y \in X$ and any $\delta > 0$ there exists a δ-chain from x to y. If for any $\delta > 0$ and any $x, y \in X$ there is $N > 0$ so that for every $n > N$ there is a δ-chain from x to y of length n, we say f is *chain mixing*. It is known that f is chain mixing if and only if f^n is chain transitive for every $n > 0$ [10].

For two nonempty open sets $U, V \subset X$, define $N(U, V) := \{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\}$. In this regard, f is called *topological transitive* if for any two nonempty open sets $U, V \subset X$, $N(U, V) \neq \emptyset$. Moreover, if $\mathbb{N} \setminus N(U, V)$ is finite set f is called *topological mixing.*

Main results

First of all, it is easy to see that mean ergodic shadowing is independent of choosing an equivalent meter. In fact, suppose that d_1 and d_2 are two equivalent meters, and $\epsilon > 0$ is arbitrary. Take $0 < \epsilon_1 < \epsilon$ so that for each $x \in X$

$$B_{d_1}(x, \epsilon_1) \subset B_{d_2}(x, \epsilon_).$$

Note that this holds due to compactness of (X, d). Let $\delta_1 > 0$ be corresponding to ϵ_1 for f in definition of mean ergodic shadowing. Again choose $\delta_2 > 0$ in which

$$B_{d_2}(x, \delta_2) \subset B_{d_1}(x, \delta_1)$$

holds for every $x \in X$. Now, let $\{x_i\}_{i\geq 0}$ be a δ_2-ergodic pseudo-orbit for f with respect to d_2. So,

$$\overline{d}(\{i : \ d_2(f(x_i), x_{i+1}) \geq \delta_2\}) = 0.$$

By choosing δ_1 and δ_2 we also have

$$\overline{d}(\{i : \ d_1(f(x_i), x_{i+1}) \geq \delta_1\}) = 0.$$

Note that the latter set is subset of the former set. Hence, the sequence $\{x_i\}_{i\geq 0}$ is a δ_1-ergodic pseudo-orbit for f with respect to d_1. It follows there exists $z \in X$ such that $\overline{d}(E_1) < \epsilon_1$, where

$$E_1 = \{i : \ d_1(f^i(z), x_i) \geq \epsilon_1\}.$$

Eesily, by putting

$$E_2 = \{i : \ d_2(f^i(z), x_i) \geq \epsilon\},$$

we obtain $\overline{d}(E_2) < \epsilon$, because $E_2 \subset E_1$.

In the following, we show that mean ergodic shadowing is invariant of conjugacy.

8.5. Theorem. *Let (X, d_X) and (Y, d_Y) be two compact metric spaces, and let f and g be two continuous maps on (X, d_X) and (Y, d_Y), respectively. If $h : X \to Y$ is a homeomorphism (conjugacy) between f and g, then $g = hofoh^{-1}$ has mean ergodic shadowing if and only if f has mean ergodic shadowing property.*

Proof. Suppose that f has mean ergodic shadowing, and $\epsilon > 0$ be given. Take $0 < \epsilon' < \epsilon$ by uniform continuity of h, i.e., $d_X(x, y) < \epsilon'$ implies $d_Y(h(x), h(y)) < \epsilon$. Suppose $\delta > 0$ is given for ϵ' by mean ergodic shadowing for f and δ' be given for δ by uniform continuity of h^{-1}, i.e. $d_Y(x, y) < \delta'$ implies $d_X(h^{-1}(x), h^{-1}(y)) < \delta$. Let $\{y_i\}_{i\in\mathbb{N}}$ be a δ'-ergodic pseudo-orbit for $g = hofoh^{-1}$, i.e. $\overline{d}(E) = 0$, where $E = \{i \in \mathbb{N} \mid d_Y(hofoh^{-1}(y_i), y_{i+1}) \geq \delta'\} \supset \{i \in \mathbb{N} \mid d_X(foh^{-1}(y_i), h^{-1}(y_{i+1})) \geq \delta\}$. This shows that $\{h^{-1}(y_i)\}_{i\in\mathbb{N}}$ is a δ-ergodic pseudo-orbit for f. So there is $z \in X$ by mean ergodic shadowing such that $\overline{d}(E') < \epsilon'$, where

$$\begin{aligned} E' =& \{i \in \mathbb{N} \mid d_X(f^i(z), h^{-1}(y_i)) \geq \epsilon'\} \supset \{i \in \mathbb{N} \mid d_Y(hof^i(z), y_i) \geq \epsilon\} \\ &= \{i \in \mathbb{N} \mid d_Y(g^i oh(z), y_i) \geq \epsilon\}. \end{aligned}$$

That is, $h(z)$ ϵ-shadowed $\{y_i\}_{i=0}^{\infty}$ in average, so g has mean ergodic shadowing. The remaining part is similar. □

In [2, Theorem 4.1] it is proved that in the presence of shadowing property the followings are equivalent for surjective dynamical system (X, f):

(1) f is totally transitive,
(2) f has almost average shadowing,
(3) f has mean ergodic shadowing,
(4) f has $\underline{d}$-shadowing.

We can also equalize specification property and topologically mixing with others, because by the above hypothesis totally transitivity implies chain mixing...

8.6. Corollary. *Under the hypothesis of* [2, Theorem 4.1] *the followings are equivalent:*

(1) f is totally transitive,
(2) f is topological mixing,
(3) f has almost average shadowing,
(4) f has mean ergodic shadowing,
(5) f has $\underline{d}$-shadowing,
(6) f has specification property.

We go on with the relation between $\mathcal{F}_s$-shadowing and mean ergodic shadowing property. We prove the following result:

8.7. Corollary. *The $\mathcal{F}_s$-shadowing property does not imply mean ergodic shadowing.*

PROOF. Because every Morse-Smale diffeomorphism has shadowing property so clearly has $\mathcal{F}_s$-shadowing property, but they are not chain mixing and therefore by Corollary 8.6 do not have mean ergodic shadowing property. Refer to Example 8.14 as another example. $\square$

8.8. THEOREM. *Mean ergodic shadowing implies $\mathcal{F}_{\underline{d}}$-shadowing property.*

PROOF. It is well known that $\underline{d}$-shadowing implies $\mathcal{F}_{\underline{d}}$-shadowing property. So, [2, Proposition 4.4] completes the proof. Alternatively, without loss of generality, let $0 < \epsilon < 1$ be given and $\delta > 0$ corresponds to ϵ in mean ergodic shadowing. Suppose $\{x_i\}_{i\geq 0}$ is a δ-psuedo-orbit for f, obviously, it is also a δ-ergodic psuedo-orbit for f. There exists $z \in X$ so that $\overline{d}(E) < \epsilon$, where $E = \{i \in \mathbb{N} \mid d(f^i(z), x_i) \geq \epsilon\}$. By the equality $\underline{d}(E^c) = 1 - \overline{d}(E)$ we obtain $\underline{d}(E^c) > 1 - \epsilon > 0$. That is f has $\mathcal{F}_{\underline{d}}$-shadowing property. $\square$

In the following, we also prove that mean ergodic shadowing follows $\overline{d}$-shadowing property.

8.9. PROPOSITION. *If f has mean ergodic shadowing, then it has $\overline{d}$-shadowing property.*

PROOF. By inspiring of the proof of [8, Theorem 5], suppose $\epsilon > 0$ be given, and f has mean ergodic shadowing. Let $\delta > 0$ be correspond to $\frac{\epsilon}{2}$ in the definition of mean ergodic shadowing property for f. Take any δ-ergodic pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ so there exists $z \in X$ such that $\frac{\epsilon}{2}$-shadowed $\{x_i\}_{i=0}^{\infty}$ in average. Let $A = \{i : d(f^i(z), x_i < \epsilon)\}$ then we have

$$\begin{aligned}\frac{\epsilon}{2} > \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) &\geq \limsup_{n\to\infty} \frac{\epsilon}{n}(n - \#A \cap \{0, \cdots, n-1\})\\ &\geq \epsilon - \epsilon\overline{d}(A),\end{aligned}$$

so $\overline{d}(A) > \frac{1}{2}$. $\square$

8.10. COROLLARY. *Every minimal dynamical system with mean ergodic shadowing does not have $\mathcal{F}_s$-shadowing property.*

PROOF. It is well known that mean ergodic shadowing implies chain mixing. By [8, Theorem 16] every chain mixing minimal dynamical system with at least two points, does not have $\mathcal{F}_s$-shadowing property. So, every minimal dynamical system with mean ergodic shadowing does not have $\mathcal{F}_s$-shadowing property. $\square$

Question. Is there any minimal system with mean ergodic shadowing property?

8.11. COROLLARY. *The asymptotic average shadowing property (AASP, for short) implies mean ergodic shadowing, but not vice versa.*

PROOF. By [11, Theorem 4.3] AASP implies almost average shadowing property, and by [2, Proposition 4.3] almost average shadowing property in turn implies mean ergodic shadowing. However, the next example shows the converse does not hold. $\square$

8.12. EXAMPLE. By the statement of Example 5.4 from [5] we construct the map φ on interval $[0, 1]$ such that $\varphi(x) > x$ if and only if $x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ It is easy to see that φ has mean ergodic shadowing, but by [6, Theorem 3.1, Remark 1] it does not have AASP.

Easily it can be shown that the analogous of [1, Theorem 4.2] holds in case of mean ergodic shadowing.

8.13. COROLLARY. *Any transitive sofic subshift σ with mean ergodic shadowing is mixing.*

PROOF. $\square$

Examples

In this section, we bring two examples show that the shadowing property does not imply the mean ergodic shadowing property and vice versa. Also, we bring a class of maps without mean ergodic shadowing property.

8.14. Example. The permutation of two points does not have the mean ergodic shadowing property, but has shadowing property.

Proof. Let $X = \{a, b\}, f(a) = b, f(b) = a$. Without loss of generality suppose that $0 < \epsilon < \frac{d(a,b)}{4}$ is given and let $\delta > 0$ be arbitrary. If $\{m_i\}_{i=0}^{\infty}$ is a sequence of natural numbers defined by $m_i = 2^i$, it is obvious that $\{m_i\}$ has density zero. Now, let $\{x_i\}_{i=0}^{\infty} = \{a; a, b; b, a; a, b, a, b; b, a, \cdots\}$ be such that $x_{m_i} = x_{m_i+1}$ for each $i \geq 0$ and $\{x_{m_i+1}, \cdots, x_{m_{i+1}}\}$ is a finite δ-chain for each i. We show that the sequence $\{x_i\}$ cann't be ϵ-shadowed in average. Indeed, for $z = a$ we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{2^{2n}} \sum_{i=0}^{2^{2n}-1} d(f^i(z), x_i) &\geq \limsup_{n\to\infty} \frac{1}{2^{2n}} \sum_{i=2^{2n-2}+1}^{2^{2n-1}} d(f^i(z), x_i) \\ &= \limsup_{n\to\infty} \frac{2^{2n-2}}{2^{2n}}\, d(a, b) = \frac{1}{4}\, d(a, b) > \epsilon.\end{aligned}$$

Similarly, for $z = b$ we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{2^{2n}} \sum_{i=0}^{2^{2n}-1} d(f^i(z), x_i) &\geq \limsup_{n\to\infty} \frac{1}{2^{2n}} \sum_{i=2^{2n-1}+1}^{2^{2n}-1} d(f^i(z), x_i) \\ &= \limsup_{n\to\infty} \frac{2^{2n-1}}{2^{2n}}\, d(a, b) = \frac{1}{2}\, d(a, b) > \epsilon.\end{aligned}$$

Therefore, it doesn't have the mean ergodic shadowing. However, it is an easy exercise to check that the permutation of two points has the shadowing property. □

The following provides an example indicating mean ergodic shadowing does not imply the shadowing property.

8.15. Example. [7, Example 3.13] $f : \mathbb{S}^1 \ni e^{2\pi i x} \mapsto e^{2\pi i x^2} \in \mathbb{S}^1$ (where $x \in [0, 1)$ and $\mathbb{S}^1$ is the unit circle) has the AASP, and hence mean ergodic shadowing property by [4] and [2, Proposition 4.3], but obviously does not have the shadowing property.

8.16. Example. The circle rotations do not have mean ergodic shadowing.

Proof. □

Conclusion

In this paper we show that mean ergodic shadowing is a dynamical property, and study its relation with other type of partial shadowing.

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9. The shadowing and ergodic shadowing properties of semigroup actions on non-compact metric spaces

Zahra Shabani[1,a], Ali Barzanouni [2]

[1]Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

[2]Department of Mathematics, Hakim Sabzevari University, Sabzevar, Iran

In this talk, we introduce the notions of shadowing and ergodic shadowing properties of finitely generated semigroup actions on non-compact metric spaces which are dynamical properties and equivalent to the classical definitions in case of compact metric spaces.

Keywords: ergodic shadowing property, shadowing property, semigroup actions

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[a]Speaker. Email address: zshabani@math.usb.ac.ir,

Introduction and Preliminaries

The concept of shadowing was emanated from the Anosov closing lemma and because of its rich consequences, it has a significant role in the study of dynamical systems. It is considerably developed in recent years and many authors have studied several kinds of shadowing including ergodic shadowing [2], $\underline{d}$-shadowing, and average shadowing, which have the common motivation of studying the behavior of a dynamical system by using the closeness of approximate orbits and true orbit. The shadowing and average shadowing properties of IFSs(iterated function systems) were introduced by Bahabadi in [1]. He obtained average shadowing property of an iterated function system implies chain transitivity. Ergodic shadowing property of semigroup actions with finitely many generators were introduced in [4], and the author showed that if a semigroup G has the shadowing property then the ergodic shaowing property is equivalent to some kind of specification which is called pseudo-orbital specification.

However, the definitions of shadowing and ergodic shadowing properties for a continuous map f and also for the finitely generated semigroup actions on a compact metric space depend on the metrics on non-compact metric spaces. In other words, the map f (or semigroup G) has the shadowing (ergodic shadowing) property with respect to one metric, may does not have the shadowing (ergodic shadowing) property with respect to another metric inducing the same topology (see [3, Example 2.2] and Example 9.2). Lee et al. [3] introduced the notions of ε-chain and shadowing property for homeomorphisms on non-compact metric spaces, which are dynamical properties and equivalent to the classical definitions in case of compact metric spaces. Here, we extend the notions of shadowing and ergodic shadowing properties to case of finitely generated semigroup actions on non-compact metric spaces and we show that the definitions of shadowing in the new notions are dynamical properties.

9.1. Definition. [5] Let X and Y be two metric spaces. We say that two semigroups F and G with generating sets $\{id, f_1, \ldots, f_m\}$ and $\{id, g_1, \ldots, g_m\}$ on X and Y, receptively, are *(topologically) conjugate* if there is a homeomorphism $h : X \to Y$ such that $h \circ f_i = g_i \circ h$ for all $i = 1, \ldots, m$. The homeomorphism h is called a *conjugacy* between F and G.

A property P is called a *dynamical property* if a semigroup G has the property P, then any other semigroup F which is conjugate to G also has the property P. Note that shadowing and ergodic shadowing properties of semigroup action on compact metric spaces are independent of metric and they are dynamical properties. However, they depend on the metrics on non-compact metric spaces, as we see in the following example.

9.2. Example. Let $T : \mathbb{R} \to \mathbb{S}^1 \setminus \{(0,1)\}$ be a map given by

$$T(t) = \left(\frac{2t}{1+t^2}, \frac{t^2-1}{t^2+1}\right), \quad \text{for all } t \in \mathbb{R},$$

and let $X = T(\mathbb{Z})$. Let d be the metric on X induced by the Riemannian metric on S^1, and let d' be a discrete metric on X. It is clear that d and d' induce the same topology on X. Let $g_1 : X \to X$ be a homeomorphism defined by $g_1(a_i) = a_{i+1}$. Denote by G the finitely generated semigroup action associated with $\{id, g_1, g_2\}$, where g_2 is any homeomorphism on X. Since the metric d' is discrete, it is easy to see that G has the ergodic shadowing property with respect to d'. We show that g_1 does not have the shadowing property with respect to d. Therefore the semigroup G does not have the shadowing property. By contradiction, let $g_1 : X \to X$ have the shadowing property. For $\varepsilon = \frac{1}{4}$, let $\delta > 0$ be an ε-modulus of the shadowing property of the mapping g_1. Choose $N_0 \in \mathbb{N}$ such that $d'(a_{N_0}, a_{-N_0}) < \frac{\delta}{2}$. For any $i \geq 0$, put $j := i \bmod 2N_0$. Then, the sequence $\{x_i\}_{i\geq 0}$ given by

$$x_i = \begin{cases} a_j, & j \in \{0, 1, 2, \ldots, N_0 - 1\}, \\ a_{j-2N_0}, & j \in \{N_0, N_0 + 1, \ldots, 2N_0 - 1\}. \end{cases}$$

is a δ-pseudo orbit for g_1. So, there is a point $z \in X$ such that $d(g_1^i(z), x_i) < \varepsilon$, for any $i \geq 0$. Since for any $z \in X$, $g_1^i(z)$ attract to $(0,1)$, so we can find an integer $i \in \mathbb{N}$ such that $d(g_1^i(z), x_i) \geq \varepsilon$, which is a contradiction. So, G does not have the shadowing and ergodic shadowing properties with respect to d.

The shadowing and ergodic shadowing on non-compact metric spaces

In this section, we define the notions of shadowing and ergodic shadowing properties for the finitely generated semigroup actions on non-compact metric spaces, which are independent of metrics.

Let $\mathcal{C}(X)$ be the collection of all continuous functions from X to $(0, \infty)$. Let X be a metrizable space and let G be a finitely generated semigroup action with the set of generators $\{id, g_1, \ldots, g_m\}$. For a sequence $\xi = \{x_i\}_{i\geq 0} \subset X$, $\delta \in \mathcal{C}(X)$, and $\omega = \omega_0\omega_1\omega_2 \ldots \in \Sigma^m$, put

$$Np(\xi, G, \omega, \delta) = \{i \in \mathbb{Z}^+ : d(g_{\omega_i}(x_i), x_{i+1}) \geq \delta(g_{\omega_i}(x_i))\},$$

$$Np^c(\xi, G, \omega, \delta) = \mathbb{Z}^+ \setminus Np(\xi, G, \omega, \delta),$$

and

$$Np_n(\xi, G, \omega, \delta) = Np(\xi, G, \omega, \delta) \cap \{0, \ldots, n-1\}.$$

Given a sequence $\xi = \{x_i\}_{i\geq 0}$ and a point $z \in X$, consider

$$Ns(\xi, G, \omega, z, \delta) = \{i \in \mathbb{Z}^+ : d(g^i_\omega(z), x_i) \geq \delta(g^i_\omega(z))\},$$

$$Ns^c(\xi, G, \omega, z, \delta) = \mathbb{Z}^+ \setminus Ns(\xi, G, \omega, z, \delta),$$

and

$$Ns_n(\xi, G, \omega, z, \delta) = Ns(\xi, G, \omega, z, \delta) \cap \{0, \ldots, n-1\}.$$

9.3. Definition. Let X be a metrizable space, let G be a finitely generated semigroup action with the set of generators $\{id, g_1, \dots, g_m\}$, and let $\delta \in \mathcal{C}(X)$.

(1) [5] For $w \in \mathcal{A}^m$ and $x, y \in X$, a (δ, w)-chain of semigroup G from x to y is a finite sequence $x_0 = x, x_1, \dots, x_n = y$ such that $d(g_{w_i}(x_i), x_{i+1}) < \varepsilon(g_{w_i}(x_i))$, for all $i = 1, \dots, n-1$.
(2) We say that $\{x_i\}_{i \geq 0} \subset X$ is a (δ, ω)-pseudo orbit of G for some $\omega = \omega_0\omega_1 \ldots \in \Sigma^m$, if for any $i \in \mathbb{Z}^+$, $d(g_{\omega_i}(x_i), x_{i+1}) < \delta(g_{\omega_i}(x_i))$.
(3) We say that $\{x_i\}_{i \geq 0} \subset X$ is a (δ, ω)-*ergodic pseudo orbit* of G for some $\omega = \omega_0\omega_1 \ldots \in \Sigma^m$ provided that the set $Np(\xi, G, \omega, \delta)$ has zero density, that is,

$$\lim_{n \to \infty} \frac{|Np_n(\xi, G, \omega, \delta)|}{n} = 0.$$

9.4. Definition. Let X be a metrizable space, let G be a finitely generated semigroup action with the set of generators $\{id, g_1, \dots, g_m\}$. We say that

(1) G has the shadowing property, if for every $\epsilon \in \mathcal{C}(X)$, there is $\delta \in \mathcal{C}(X)$ such that for every (δ, ω)-pseudo orbit $\{x_i\}_{i \geq 0}$ of G, for some $\omega \in \Sigma^m$, there is a point $z \in X$ satisfying $d(g^i_\omega(x), x_i) < \epsilon(g^i_\omega(x))$ for all $i \geq 0$.
(2) G has the ergodic shadowing property if for each $\epsilon \in \mathcal{C}(X)$, there exists $\delta \in \mathcal{C}(X)$ such that every (δ, ω)-ergodic pseudo orbit ξ of G can be ϵ-ergodic shadowed by some point z in X, that is, there exists $\varphi \in \Sigma^m$ with $\varphi_i = \omega_i$ for $i \in Np^c(\{x_i\}_{i \geq 0}, G, \omega, \delta)$, such that

$$\lim_{n \to \infty} \frac{|Ns_n(\{x_i\}_{i \geq 0}, G, \varphi, z, \epsilon)|}{n} = 0.$$

In the following, we show that Definition 9.4 for the semigroup G on the non-compact metric space X can be preserved by conjugacy. Hence, they do not depend on the choices of metrics on X. For this, we need two lemmas.

9.5. Lemma. [3, Lemmas 2.7 and 2.8] *Let (X, d) and (Y, d') be two metric spaces.*

(1) A function f from X to Y is continuous if and only if, for any $\varepsilon \in \mathcal{C}(Y)$, there exists $\delta \in \mathcal{C}(X)$ such that if $d(x, y) < \delta(x)$ $(x, y \in X)$, then $d'(f(x), f(y)) < \varepsilon(f(x))$.
(2) For every $\alpha \in \mathcal{C}(X)$, there exists $\gamma \in \mathcal{C}(X)$ such that

$$\gamma(x) \leq \inf \{\alpha(z) : z \in B(x, \gamma(x))\}. \tag{12}$$

The next lemma is cited from [5] and is an immediate result of Lemma 9.5.

9.6. Lemma. [5] *Let (X, d) be a metric space and let $f_i : X \to X$ $(i = 1, \dots, m)$ be continuous maps. Then, for every $\varepsilon \in \mathcal{C}(X)$, there exists $\delta \in \mathcal{C}(X)$ such that if $d(x, y) < \delta(x)$, then $d(f_i(x), f_i(y)) < \varepsilon(f_i(x))$ for all $i = 1, \dots, m$.*

9.7. Proposition. *Let X be a metric space and let G be a semigroup action with generating set $\{id, g_1, \dots, g_m\}$. Then the shadowing and ergodic shadowing properties of G introduced in Definition 9.4, are dynamical properties.*

Proof. Let G and F be two semigroups generated by $G_1 = \{id, g_1, \dots, g_m\}$ and $F_1 = \{id, f_1, \dots, f_m\}$ on the metric space (X, d) and (Y, d'), respectively. Suppose that G and F are topologically conjugate with conjugacy $h : X \to Y$. We show that the ergodic shadowing property preserves by topological conjugacy. Assume that G has the ergodic shadowing property. For every $\varepsilon' \in \mathcal{C}(Y)$, there exists $\varepsilon \in \mathcal{C}(X)$ such that if $d(x, y) < \varepsilon(x)$, then $d'(h(x), h(y)) < \varepsilon'(h(x))$. Take $\delta \in \mathcal{C}(X)$ as an ε-modulus of ergodic shadowing property of G, and let $\delta' \in \mathcal{C}(Y)$ be such that if $d'(x, y) < \delta'(x)$, then $d(h^{-1}(x), h^{-1}(y)) < \delta(h^{-1}(x))$. Let $\{x_i\}_{i \geq 0} \subseteq Y$ be a (δ', ω)-ergodic pseudo orbit of F for $\omega = \omega_0\omega_1 \ldots \in \Sigma^m$.

We show that $\{h^{-1}(x_i)\}_{i \geq 0}$ is a (δ, ω)-ergodic pseudo orbit of G. Indeed for any $i \in Np^c(\{x_i\}_{i \geq 0}, F, \omega, \delta')$, we have $d'(f_{\omega_i}(x_i), x_{i+1}) < \delta'(f_{\omega_i}(x_i))$ implies that

$$d(g_{\omega_i}(h^{-1}(x_i)), h^{-1}(x_{i+1})) = d(h^{-1}(f_{\omega_i}(x_i)), h^{-1}(x_{i+1})) < \delta(g_{\omega_i}(h^{-1}(x_i))).$$

It yields that $Np^c(\{x_i\}_{i\geq 0}, F, \omega, \delta') \subset Np^c(\{h^{-1}(x_i)\}_{i\geq 0}, G, \omega, \delta)$ and so $\{h^{-1}(x_i)\}_{i\geq 0}$ is a δ-ergodic pseudo orbit of G. Since the G has the ergodic shadowing property, there exist $z \in X$ and $\varphi \in \Sigma^m$ with $\varphi_i = \omega_i$ for $i \in Np^c(\{h^{-1}(x_i)\}_{i\geq 0}, G, \omega, \delta)$, such that $Ns(\{h^{-1}(x_i)\}_{i\geq 0}, G, \varphi, z, \epsilon)$ has zero density. Since for any $i \in Ns^c(\{h^{-1}(x_i)\}_{i\geq 0}, G, \varphi, z, \epsilon)$, $d(g^i_\varphi(z), h^{-1}(x_i)) < \epsilon(g^i_\varphi(z))$, we have

$$d'(h(g^i_\varphi(z)), x_i) = d'(f^i_\varphi(h(z)), x_i) < \epsilon'(f^i_\varphi(h(z))).$$

This means that $Ns^c(\{h^{-1}(x_i)\}_{i\geq 0}, G, \varphi, z, \epsilon) \subset Ns^c(\{x_i\}_{i\geq 0}, F, \varphi, h(z), \epsilon')$, which implies that $h(z)$, ϵ'-ergodic shadows $\{x_i\}_{i\geq 0}$. Thus F the has the ergodic shadowing property. $\square$

The proof of the next lemma for a semigroup G on a compact metric space X was appeared in [4]. In the following theorem, shows that it holds for semigroup G on non-compact metric spaces with the new notions for δ-chain and shadowing property introduced in Definitions 9.3 and 9.4.

9.8. Theorem. *Let (X, d) be a metric space and G be a semigroup associated with finite family $\{id, g_1, \ldots, g_m\}$ of continuous maps on X. If the semigroup G has the shadowing property, then G is topologically mixing if and only if it is chain mixing.*

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10. Various shadowing properties for time varying maps

Javad Nazarian Sarkooh
Department of Mathematics, Ferdowsi University of Mashhad, Mashhad, IRAN

In this paper we study various notions of shadowing of dynamical systems so-called time varying maps. We define and study the h-shadowing, limit shadowing, slimit shadowing and exponential limit shadowing properties of these dynamical systems. We investigate the relationships between these notions of shadowing and examine the role that expansivity plays in shadowing properties of such dynamical systems. Specially, we prove some results linking s-limit shadowing property to limit shadowing property, and h-shadowing property to s-limit shadowing and limit shadowing properties.

Keywords: Time varying map, h-shadowing, Limit shadowing, s-limit shadowing, Exponential limit shadowing.

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Introduction

The time varying maps (so-called non-autonomous or time-dependent dynamical systems), describe situations where the dynamics can vary with time and yield very flexible models than autonomous cases for the study and description of real world processes. They may be used to describe the evolution of a wider class of phenomena, including systems which are forced or driven. In the recent past, lots of studies have been done regarding dynamical properties in such systems, but a global theory is still out of reach. In general time varying maps can be rather complicated. Thus, we are inclined to look at approximations of orbits, also called pseudo orbits. Systems for which pseudo orbits can be approximated by true orbits are said to satisfy the shadowing property. The shadowing property plays a key role in the study of the stability of dynamical systems. This property is found in hyperbolic dynamics, and it was used to prove their stability. In this literature, some remarkable results were further obtained through works of several authors, see e.g. [**1**, **2**, **3**]. Since the approximation by true orbits can be expressed in various ways, different notions of shadowing have been introduced. In this paper, what we want to study on time varying maps is shadowing, h-shadowing, limit shadowing, s-limit shadowing and exponential limit shadowing properties.

This article is organized as follows. In Section 8.1, we present an overview of the main concepts and introduce notations. Next, we give our main results in Section 8.1.

Preliminaries

Throughout this paper we consider (X, d) to be a metric space, $f_n : X \to X$, $n \in \mathbb{N}$, to be a sequence of continuous maps and $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ to be a time varying map on X that its time evolution is defined by composing the maps f_n in the following way

$$\mathcal{F}_n := f_n \circ f_{n-1} \circ \cdots \circ f_1, \text{ for } n \geq 1, \text{ and } \mathcal{F}_0 := Id_X.$$

For time varying map $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ defined on X, we set $\mathcal{F}_{[i,j]} := f_j \circ f_{j-1} \circ \cdots \circ f_{i+1} \circ f_i$ for $1 \leq i \leq j$, and $\mathcal{F}_{[i,j]} := Id_X$ for $i > j$. Also, for any $k > 0$, we define a time varying map (k^{th}-iterate of $\mathcal{F}$) $\mathcal{F}^k = \{g_n\}_{n\in\mathbb{N}}$ on X, where

$$g_n = f_{nk} \circ f_{(n-1)k+k-1} \circ \ldots \circ f_{(n-1)k+2} \circ f_{(n-1)k+1} \text{ for } n \geq 1.$$

Thus $\mathcal{F}^k = \{\mathcal{F}_{[(n-1)k+1,nk]}\}_{n\in\mathbb{N}}$.

Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a metric space (X, d). For a point $x_0 \in X$, put $x_n := \mathcal{F}_n(x_0)$ for all $n \geq 0$. Then the sequence $\{x_n\}_{n\geq 0}$, denoted by $\mathcal{O}(x_0)$, is said to be the *orbit* of x_0 under time varying map $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$. Moreover, a subset Y of X is said to be *invariant* under $\mathcal{F}$ if $f_n(Y) = Y$ for all $n \geq 1$, equivalently $\mathcal{F}_n(Y) = Y$ for all $n \geq 0$.

10.1. Definition (Conjugacy). Let (X, d_1) and (Y, d_2) be two metric spaces. Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ and $\mathcal{G} = \{g_n\}_{n\in\mathbb{N}}$ be time varying maps on X and Y, respectively. If there exists a homeomorphism $h : X \to Y$ such that $h \circ f_n = g_n \circ h$, for all $n \in \mathbb{N}$, then $\mathcal{F}$ and $\mathcal{G}$ are said to be *conjugate* (with respect to the map h) or *h-conjugate*. In particular, if $h : X \to Y$ is a uniform homeomorphism, then $\mathcal{F}$ and $\mathcal{G}$ are said to be *uniformly conjugate* or *uniformly h-conjugate*. (Recall that homeomorphism $h : X \to Y$, such that h and h^{-1} are uniformly continuous, is called a uniform homeomorphism.)

10.2. Definition (Shadowing property). Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a metric space (X, d) and Y be a subset of X. Then,

(1) for $\delta > 0$, a sequence $\{x_n\}_{n\geq 0}$ in X is said to be a *δ-pseudo orbit* if
$$d(f_{n+1}(x_n), x_{n+1}) < \delta \text{ for all } n \geq 0;$$
(2) for given $\varepsilon > 0$, a δ-pseudo orbit $\{x_n\}_{n\geq 0}$ is said to be *ε-shadowed* by $x \in X$ if $d(\mathcal{F}_n(x), x_n) < \varepsilon$ for all $n \geq 0$;
(3) the time varying map $\mathcal{F}$ is said to have *shadowing property* on Y if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that every δ-pseudo orbit in Y is ε-shadowed by some point of X. If this property holds on $Y = X$, we simply say that $\mathcal{F}$ has *shadowing property.*

10.3. Definition (h-shadowing property). Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a metric space (X, d) and Y be a subset of X. We say that $\mathcal{F}$ has *h-shadowing property* on Y if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite δ-pseudo orbit $\{x_0, x_1, \ldots, x_m\}$ in Y there is $x \in X$ such that $d(\mathcal{F}_n(x), x_n) < \varepsilon$ for every $0 \leq n < m$ and $\mathcal{F}_m(x) = x_m$. If this property holds on $Y = X$, we simply say that $\mathcal{F}$ has *h-shadowing property.*

10.4. Definition (Limit shadowing property). Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a metric space (X, d) and Y be a subset of X. Then,

(1) a sequence $\{x_n\}_{n\geq 0}$ in X is called a *limit pseudo orbit* if $d(f_{n+1}(x_n), x_{n+1}) \to 0$ as $n \to +\infty$;
(2) a sequence $\{x_n\}_{n\geq 0}$ in X is said to be *limit shadowed* if there is $x \in X$ such that $d(\mathcal{F}_n(x), x_n) \to 0$, as $n \to +\infty$;
(3) the time varying map $\mathcal{F}$ has the *limit shadowing property* on Y whenever every limit pseudo orbit in Y is limit shadowed by some point of X. If this property holds on $Y = X$, we simply say that $\mathcal{F}$ has *limit shadowing property.*

The notion of limit shadowing property was extended to a notion so called s-limit shadowing property, to account the fact that many systems have limit shadowing property but not shadowing property.

10.5. Definition (s-Limit shadowing property). Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a metric space (X, d) and Y be a subset of X. We say that $\mathcal{F}$ has *s-limit shadowing property* on Y if for every $\varepsilon > 0$ there is $\delta > 0$ such that

(1) for every δ-pseudo orbit $\{x_n\}_{n\geq 0}$ in Y, there exists $x \in X$ satisfying $d(\mathcal{F}_n(x), x_n) < \varepsilon$ for all $n \geq 0$, and,
(2) if in addition, $\{x_n\}_{n\geq 0}$ is a limit pseudo orbit in Y then $d(\mathcal{F}_n(x), x_n) \to 0$ as $n \to +\infty$.

If this property holds on $Y = X$, we simply say that $\mathcal{F}$ has *s-limit shadowing property.*

We say that a sequence $\{a_n\}_{n\geq 0}$ of real numbers converges to zero with rate $\theta \in (0,1)$ and write $a_n \xrightarrow{\theta} 0$ as $n \to +\infty$, if there exists a constant $L > 0$ such that $|a_n| \leq L\theta^n$ for all $n \geq 0$.

10.6. Definition (Exponential limit shadowing property). Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a metric space (X, d) and Y be a subset of X. Then,

(1) for $\theta \in (0,1)$, a sequence $\{x_n\}_{n\geq 0}$ in X is called a *θ-exponentially limit pseudo orbit* of $\mathcal{F}$ if $d(f_{n+1}(x_n), x_{n+1}) \xrightarrow{\theta} 0$ as $n \to +\infty$;
(2) the time varying map $\mathcal{F}$ has the *exponential limit shadowing property with exponent* ξ on Y if there exists $\theta_0 \in (0,1)$ so that for any θ-exponentially limit pseudo orbit $\{x_n\}_{n\geq 0} \subseteq Y$ with $\theta \in (\theta_0, 1)$, there is $x \in X$ such that $d(\mathcal{F}_n(x), x_n) \xrightarrow{\theta^\xi} 0$, as $n \to +\infty$. In the case $\xi = 1$ we say that F has the *exponential limit shadowing property* on Y. If this property holds on $Y = X$, we simply say that $\mathcal{F}$ has *exponential limit shadowing property.*

Main results

In this section, we mention our main results.

10.7. Theorem. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ and $\mathcal{G} = \{g_n\}_{n\in\mathbb{N}}$ be time varying maps on metric spaces (X, d_1) and (Y, d_2), respectively, such that $\mathcal{F}$ is uniformly conjugate to $\mathcal{G}$. Then, the following statements hold:*

(a) *If $\mathcal{F}$ has the h-shadowing property, then so does $\mathcal{G}$.*
(b) *If $\mathcal{F}$ has the limit shadowing property, then so does $\mathcal{G}$.*
(c) *If $\mathcal{F}$ has the s-limit shadowing property, then so does $\mathcal{G}$.*

10.8. Theorem. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ and $\mathcal{G} = \{g_n\}_{n\in\mathbb{N}}$ be time varying maps on metric spaces (X, d_1) and (Y, d_2), respectively. Define metric d on $X \times Y$ by*

$$d((x_1, y_1), (x_2, y_2)) := \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} \quad \textit{for any } (x_1, y_1), (x_2, y_2) \in X \times Y.$$

Then,

(a) *$\mathcal{F}$ and $\mathcal{G}$ have the h-shadowing property if and only if so does $\mathcal{F} \times \mathcal{G} := \{f_n \times g_n\}_{n\in\mathbb{N}}$.*
(b) *$\mathcal{F}$ and $\mathcal{G}$ have the limit shadowing property if and only if so does $\mathcal{F} \times \mathcal{G}$.*
(c) *$\mathcal{F}$ and $\mathcal{G}$ have the exponential limit shadowing property if and only if so does $\mathcal{F} \times \mathcal{G}$.*
(d) *$\mathcal{F}$ and $\mathcal{G}$ have the s-limit shadowing property if and only if so does $\mathcal{F} \times \mathcal{G}$.*

10.9. Theorem. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on metric space (X, d) and $k \in \mathbb{N}$. Then, the following statements hold:*

(a) *If $\mathcal{F}$ has the limit shadowing property, then so does $\mathcal{F}^k$.*
(b) *If $\mathcal{F}$ has the exponential limit shadowing property, then so does $\mathcal{F}^k$.*
(c) *If $\mathcal{F}$ has the s-limit shadowing property, then so does $\mathcal{F}^k$.*

10.10. Definition (Equicontinuity). Time varying map $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ on a metric space (X, d) is said to be *equicontinuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(\mathcal{F}_{[i,j]}(x), \mathcal{F}_{[i,j]}(y)) < \varepsilon$ for all $1 \leq i \leq j$.

10.11. Theorem. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be an equicontinuous time varying map on a compact metric space (X, d) and Y be an invariant subset of X. Then, the following conditions are equivalent:*

(a) *$\mathcal{F}$ has the h-shadowing property on Y.*
(b) *$\mathcal{F}^k$ has the h-shadowing property on Y for some $k \in \mathbb{N}$.*
(c) *$\mathcal{F}^k$ has the h-shadowing property on Y for all $k \in \mathbb{N}$.*

10.12. Lemma. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a metric space (X, d) and Y be a subset of X. If $Y \subseteq f_n(Y)$ for every $n \in \mathbb{N}$ and $\mathcal{F}$ has s-limit shadowing property on Y then $\mathcal{F}$ also has limit shadowing property on Y. In particular, if $\mathcal{F}$ is a time varying map of surjective maps and has s-limit shadowing property then $\mathcal{F}$ also has limit shadowing property.*

10.13. Theorem. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a compact metric space (X, d) and Y be a closed subset of X. Then, the following statements hold:*

(a) *If there is an open set U such that $Y \subseteq U$ and $\mathcal{F}$ has h-shadowing property on U, then $\mathcal{F}$ has s-limit shadowing property on Y. If in addition, $Y \subseteq f_n(Y)$ for every $n \in \mathbb{N}$ then $\mathcal{F}$ has limit shadowing property on Y.*

(b) *If Y is invariant and $\mathcal{F}|_Y$ has h-shadowing property then $\mathcal{F}|_Y$ has s-limit shadowing property and limit shadowing property.*

(c) *If $\mathcal{F}$ has h-shadowing property then $\mathcal{F}$ has s-limit shadowing property. If in addition, $\mathcal{F}$ is a time varying map of surjective maps then $\mathcal{F}$ has limit shadowing property.*

10.14. Definition (Expansivity). An time varying map $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ on a metric space (X, d) is called *strongly expansive* if there exists $\gamma > 0$ (called expansivity constant) such that for any two distinct points $x, y \in X$ and every $N \in \mathbb{N}$, $d(\mathcal{F}_{[N,n]}(x), \mathcal{F}_{[N,n]}(y)) > \gamma$ for some $n \geq N$. Equivalently, if for $x, y \in X$ and some $N \in \mathbb{N}$, $d(\mathcal{F}_{[N,n]}(x), \mathcal{F}_{[N,n]}(y)) \leq \gamma$ for all $n \geq N$, then $x = y$.

10.15. Corollary. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a compact metric space (X, d).*

(a) *If $\mathcal{F}$ is strongly expansive then $\mathcal{F}$ has the shadowing property if and only if $\mathcal{F}$ has the h-shadowing property.*

(b) *If $\mathcal{F}$ is strongly expansive and has the shadowing property then $\mathcal{F}$ has the h-shadowing and s-limit shadowing properties. If in addition, $\mathcal{F}$ is a time varying map of surjective maps then $\mathcal{F}$ has the limit shadowing property.*

10.16. Definition (Uniformly contracting and uniformly expanding time varying map). Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a metric space (X, d). Then,

(1) the time varying map $\mathcal{F}$ is *uniformly contracting* if its contracting ratio which denoted by α exists and is less than one, where

$$\alpha := \sup_{n\in\mathbb{N}} \sup_{\substack{x,y\in X \\ x\neq y}} \frac{d(f_n(x), f_n(y))}{d(x, y)};$$

(2) the time varying map $\mathcal{F}$ is *uniformly expanding* if its expanding ratio which denoted by β exists and is greater than one, where

$$\beta := \inf_{n\in\mathbb{N}} \inf_{\substack{x,y\in X \\ x\neq y}} \frac{d(f_n(x), f_n(y))}{d(x, y)}.$$

10.17. Theorem. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a uniformly contracting time varying map on a metric space (X, d). Then, $\mathcal{F}$ has the shadowing, limit shadowing, s-limit shadowing and exponential limit shadowing properties.*

10.18. Theorem. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a uniformly expanding time varying map of surjective maps on a complete metric space (X, d). Then, $\mathcal{F}$ has the shadowing, limit shadowing, s-limit shadowing and exponential limit shadowing properties.*

10.19. Definition. Let $f : X \to X$ be a linear homeomorphism on a Banach space X. Then, f is said to be *hyperbolic* if there exist Banach subspaces $X_s, X_u \subset X$, called stable and unstable subspaces, respectively, and a norm $\|.\|$ on X compatible with the original Banach structure such that

$$X = X_s \oplus X_u, \quad f(X_s) = X_s, \quad f(X_u) = X_u, \quad \|f|_{X_s}\| < 1 \text{ and } \|f^{-1}|_{X_u}\| < 1.$$

10.20. THEOREM. *Let X be a Banach space, and let $\mathcal{A}$ be a finite set of hyperbolic linear homeomorphisms with the same stable and unstable subspaces. Then, any time varying map $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ with $f_n \in \mathcal{A}$ has the shadowing, limit shadowing, s-limit shadowing and exponential limit shadowing properties.*

11. Conclusion

Time varying maps which are a natural generalization of autonomous dynamical systems, are more flexible tools for the description and study of real world processes. Hence, their study is important.

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12. A Stabilized diagonal-preservin of C*-algebras

Arezoo Hosseini[1 a],

[1]Faculty of Mathematics, College of Science, Farhangian University, Tehran, Iran

We give a stabilized version of any *-isomorphism $O_X \to O_Y$ which maps $C(X)$ onto $C(Y)$ is in fact diagonal-preserving under mild conditions on X and Y.

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[a]Speaker. Email address: a.hosseini@cfu.ac.ir,

Introduction

Let X and Y be one-sided shift spaces. A *-isomorphism $\Psi : O_X \to O_Y$ is *diagonal-preserving* if $\Psi(D_X) = D_Y$. In this paper we prove that a *-isomorphism $\Psi : O_X \to O_Y$ satisfying $\Psi(C(X)) = C(Y)$ is diagonal-preserving is stabilized. First we need some preliminary results. Everyone can read more in [1, 2].

Main Section

12.1. Lemma. *Let X be a one-sided shift space. Then*

$$C^*(Iso(\mathcal{G}_X)^\circ) = D'_X \subseteq C(X)'.$$

If X contains a dense set of aperiodic points, then $D'_X = C(X)'$.

Proof. Let $\iota \in C_c(\mathcal{G}_X)$. The condition that $\iota \star g = g \star \iota$ for all $g \in D_X$ means that ι is supported on elements $\gamma \in \mathcal{G}_X$ with $s(\gamma) = r(\gamma)$. It follows that $C^*(Iso(\mathcal{G}_X)^\circ) = D'_X$. The inclusion $D'_X \subseteq C(X)'$ follows from the inclusion $C(X) \subseteq D_X$. □

Consider the equivalence relation $\sim$ on the space $\tilde{X} \times \mathbb{T}$ given by $(\tilde{x}, \iota) \sim (\tilde{y}, \theta)$ if and only if $\tilde{x} = \tilde{y}$ and $\iota^p = \theta^p$ for all $p \in Stab(\tilde{x})$. Then the quotient $\tilde{X} \times \mathbb{T}/\sim$ is compact and Hausdorff and as we shall see (homeomorphic to) the spectrum of $C^*(Iso(\mathcal{G}_X)^\circ)$. We read more in groupoid [3].

12.2. Lemma. *Let $\sim$ be the equivalence relation on $\tilde{X} \times \mathbb{T}$ defined above. There is a *-isomorphism*

$$\Omega : C^*(Iso(\mathcal{G}_X)^\circ) \to C(\tilde{X} \times \mathbb{T}/\sim),$$

given by

$$\Omega(f)([\tilde{x}, \iota]) = \sum_{p \in Stab(\tilde{x})} f(\tilde{x}, p, \tilde{x})\iota^n \tag{13}$$

for $f \in C_c(Iso(\mathcal{G}_X)^\circ)$ and $[\tilde{x}, \iota] \in \tilde{X} \times \mathbb{T}$.

Main Section

12.3. Theorem. *Let X and Y be one-sided shift spaces with dense sets of aperiodic points and let $\Psi : O_X \to O_Y$ be a *-isomorphism satisfying $\Psi(C(X)) = C(Y)$. Then $\Psi(D_X) = D_Y$.*

Proof. If $\Psi : O_X \to O_Y$ is a *-isomorphism satisfying $\Psi(C(X)) = C(Y)$, then $\Psi(C(X)') = C(Y)'$. By Lemmas 116 and 113, there is a homeomorphism

$$h : \tilde{X} \times \mathbb{T}/\sim \to \tilde{Y} \times \mathbb{T}/\sim,$$

such that $\Psi(f) = f \circ h^{-1}$ for $f \in C(\tilde{X} \times \mathbb{T}/\sim)$.
Define the map $q_X : \tilde{X} \times \mathbb{T}/\sim \to \tilde{X}$ by $q_X([\tilde{x}, z]) = \tilde{x}$. This is well-defined, continuous and surjective. Furthermore, q_X induces the inclusion $D_X \subseteq C(X)'$. Let $\tilde{x} \in \tilde{X}$ and put $\tilde{y}_{\tilde{x}} = q_Y(h([\tilde{x}, 1])) \in \tilde{Y}$. The connected component of any $[\tilde{x}, z]$ is the set $[\tilde{x}, w] \mid w \in \mathbb{T}$, so since any homeomorphism will preserve connected components, we have

$$h(q_X^{-1}(\tilde{x})) = q_Y^{-1}(h([\tilde{x}, 1])).$$

We may now define a map $\tilde{h} : \tilde{X} \to Y$ by

$$\tilde{h}(\tilde{x}) = \tilde{y}_{\tilde{x}} = q_Y(h([\tilde{x}, 1]))$$

for $\tilde{x} \in \tilde{X}$, which is well-defined, continuous and surjective. The above considerations show that h is also injective. As both $\tilde{X}$ and $\tilde{Y}$ are compact and Hausdorff, $\tilde{h}$ is a homeomorphism. The relation $\tilde{h} \circ q_X = q_Y \circ h$ ensures that that $\Psi(D_X) = D_Y$ as wanted. $\square$

12.4. Corollary. *Let X and Y be one-sided shift spaces and let $\Psi : O_X \to O_Y$ be a *-isomorphism satisfying $\Psi(C(X)) = C(Y)$ and $\Psi \circ \gamma^X = \gamma^Y \circ \Psi$. Then $\Psi(D_X) = D_Y$.*

Proof. This follows from the observation that $D_X = C(X)' \cap \mathcal{F}_X$ and $D_Y = C(Y)' \cap \mathcal{F}_Y$. $\square$

12.5. Remark. Let X be any strictly sofic one-sided shift and let $Y = \tilde{X}$ be its cover. Then Y is (conjugate to) a shift of finite type so $D_Y = C(Y)$ but $D_X = C(Y) \ncong C(X)$. The identity map is a *-isomorphism $O_X \to O_Y$ with sends $D_X$onto $D_Y = C(Y)$, but there is no *-isomorphism $\Psi : O_X \to O_Y$ which satisfies $\Psi(C(X)) = C(Y)$.

Below, we give a stabilized version of Theorem 12.3. Consider the product $\tilde{X} \times \mathbb{N} \times \mathbb{T}$ equipped with the equivalence relation $\approx$ defined by $(\tilde{x}, m_1, z) \approx (\tilde{y}, m_2, w)$ if and only if $\tilde{x} = \tilde{y}$ and $m_1 = m_2$ and $z^n = w^n$ for all $n \in Iso(\tilde{x})$. The spaces $\tilde{X} \times \mathbb{N} \times \mathbb{T}/\approx$ and $(\tilde{X} \times \mathbb{T}/\sim) \times \mathbb{N}$ are now homeomorphic. An argument similar to the above then yields the following result.

12.6. Corollary. *Let X and Y be one-sided shift spaces with dense sets of aperiodic points and let $\Psi : O_X \otimes \mathbb{K} \to O_Y \otimes \mathbb{K}$ be a *-isomorphism satisfying $\Psi(C(X) \otimes c_0) = C(Y) \otimes c_0$. Then $\Psi(D_X \otimes c_0) = D_Y \otimes c_0$.*

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13. A note on causal conditions fail along a null geodesic

Mehdi Vatandoost[1] and Neda Ebrahimi[a2]

[1]Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran.
[2]Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman

It is known for some of the causality conditions that they can't fail at a single isolated point. Recetlely, it is shown that if causal continuity or stable causality fail at a point p then there is a null geodesic segment containing p at every point of which the condition fails. In this paper, we show that if causal simplicity fails at a point p of a reflecting spacetime M then there exists a future or past inextendible maximal null geodesics with endpoint p at every point of which causal simplicity fails

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[a]Speaker. Email address:n_ebrahimiuk.ac.ir

Introduction

In the theory of General Relativity, a space-time (M, g) is a connected C^∞ Hausdorff manifold of dimension two or greater which has a countable basis, a Lorentzian metric g of signature $(-, +, ..., +)$ and a time orientation. The metric g determines the causal structure of the space-time based on which causality conditions are defined. Causality conditions are important in determining how physical a space-time is and proving mathematical theorems about its global structure.

We say that a vector $v \in T_pM$ is timelike if $g_p(v, v) < 0$, causal if $g_p(v, v) \leq 0$, null if $g_p(v, v) = 0$ and spacelike if $g_p(v, v) > 0$. A smooth curve is timelike (future pointing) if its tangent vector is everywhere timelike (future pointing). When speaking about future pointing curves, we usually omit future pointing and simply write causal or timelike curve. Causal and null, future or past pointing and space-like curves are defined similarly. Suppose $p, q \in M$. q is in the chronological future of p, written $q \in I^+(p)$ or $p \ll q$, if there is a timelike future pointing curve $\gamma : [0, 1] \to M$ with $\gamma(0) = p$, and $\gamma(1) = q$; similarly, q is in the causal future of p, written $q \in J^+(p)$ or $p \prec q$, if there is a future pointing causal curve from p to q. For any point, p, $I^\pm(p)$ is open; but $J^\pm(p)$ need not, in general, be closed. $J^\pm(p)$ is, however, always a subset of the closure of $I^\pm(p)$. The set of all the Lorentzian metrics on M is denoted by $Lor(M)$. The fine C^0 topology on $Lor(M)$ is defined using a fixed locally finite countable covering $B = \{B_i\}$ of M by coordinate neighborhoods with the property that the closure of each B_i lies in a coordinate chart of M. Let $\delta : M \longrightarrow (0, \infty)$ be a continuous function. Then $g_1, g_2 \in Lor(M)$ are said to be σ close in the C^0 topology, if for each

$p \in M$ all of the corresponding coefficients of the two metrics are $\sigma(p)$ close at p when calculated in the fixed coordinates of all $B_i \in B$ which contain p. For all $g, h \in Lor(M)$, $h > g$ iff $p \prec q$ in space-time (M, g) implies $p \ll q$ in space-time (M, h).

To be more careful, it is useful to remind the following conditions.[2, 3] A space-time M is:

- Causal if there is no non-degenerate causal curve which starts and ends at the same point. If M is causal at all points it's simply called causal.
- Strongly causal at p if p has arbitrarily small neighborhoods which every causal curve intersects in a single component.
- Stably causal if there is a fine C^0 neighborhood $U(g)$ of g in Lor(M) such that each $h \in U(g)$ is causal (equivalently there exists some causal $h \in Lor(M)$ with $g < h$).
- Causally continuous at a point p if for each compact set K in the exterior of $I^+(p)$ there exists some neighborhood $U(p)$ of p such that for each $q \in U(p)$, K is in the exterior of $I^+(q)$.
- Causally simple at p if it is strongly causal and $J^{\pm}(p)$ is closed.
- Globally hyperbolic if it is strongly causal and $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$.

Main results

Some causality conditions are defined pointwisely i.e. they either hold or not *at individual points*. A natural question is whether the failure of a pointwise causality condition at a point of space-time, implies the failure of the condition at some other points.

As a result of Proposition 4.29 and Theorem 3.31 in [4], one can deduce that the failure of future or past distinction and strong causality conditions at some point $p \in M$ imply the failure of them at all points of a null geodesic segment containing p. It is also not hard to show that chornologicality and causality have this property too. In Ref. [1], it is shown that causal continuity and an equivalent pointwise defition of stable causality also have this property i.e. if they fail at a point p there is a null geodesic segment containing p along which the conditions fail.

13.1. Theorem. [1] *If causal continuity fails at a point p then there is a future endless null geodesic γ with past point p at every point of which causal continuity fails.*

13.2. Theorem. [1] *Let S^c be the set of points atwhich stable causality fails. If stable causality fails at p then at least one of the following holds:*
a) $p \in int(S^c)$;
b) p is a non-endpoint point on a future endless null geodesic on ∂S^c;
c) p is a non-endpoint point on a past endless null geodesic on ∂S^c;
d) p is the endpoint of a future endless and a past endless null geodesic on ∂S^c;
e) p lies on an endless null geodesic on ∂S^c at every point of which stable causality fails.

So, the above theorem shows that when stable causality fails at a point p, there is a null geodesic segment containing p along which the condition fails. But, the case of causal simplicity is a challenging problem and remains a conjecture [1]. Now, by the following theorem, we prove it.

13.3. Theorem. *If causal simplicity fails at a point p of a reflecting spacetime M then there exists a future or past inextendible maximal null geodesics with endpoint p at every point of which causal simplicity fails.*

Proof. Let M not be causally simple. Therefore, there exists a point p that $J^+(p)$ or $J^-(p)$ is not closed. We show that if $J^+(p)$ $(J^-(p))$ is not closed then there exists a null geodesic segment with past (future) endpoint p at every point of which causal simplicity fails. Since $J^+(p)$ is not closed, there is a point $r \in \partial J^+(p) \setminus J^+(p)$ and a sequence of points $r_n \in I^+(p)$ which $r_n \longrightarrow r$ and also there is a sequence of causal curves γ_n from p to r_n . Let $U(p)$ be a strictly convex normal neighborhood of p such that $\partial U(p)$ is compact. So, γ_n intersects $\partial U(p)$ in p'_n. Assume p'_n converge to $p' \in \partial U(p)$. So, there exists a causal geodesic pp'. There exists two possibilities:

Case 1: The geodesic pp' is timelike.
In this case, $p \in I^-(p')$ and any future null geodesic with past endpoint p has a segment in $I^-(p')$ such that for every point s of this segment, $r \in \partial J^+(s) \setminus J^+(s)$ and $J^+(s)$ is not closed.
Case 2: The geodesic pp' is null.
In this case, we show that causal simplicity fails at every points of the null geodesic pp'. By the reflectivity of M, $r \in \partial J^+(p') \setminus J^+(p')$ and we conclude that $J^+(p')$ is not closed. Now, for every point q on the null geodesic pp' we have $r \in \partial J^+(p') \subseteq \overline{J^+(q)}$ and $r \notin J^+(q) \subseteq J^+(p)$ and therefore, $J^+(q)$ is not closed.

$\square$

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14. Contact Equivalence Problem for the general form of Burgers' equations

Mostafa Hesamiarshad[1, a]

[1] Department of Mathematics, Tuyserkan Branch, Islamic Azad University, Tuyserkan, Iran

The moving coframe method is applied to solve the local equivalence problem for the equation of the form $u_{xx} = u_t + Q(u)u_x$ in two independent variables under an action of the pseudo-group of contact transformations. The structure equations, invariants and equivalent condition of this equations are found.

Keywords: Equivalence problem, general form of Burgers' equation, invariants, Moving coframe

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[a]Speaker. Email address: mostaf.hesami@gmail.com,

Introduction

In this article we consider a local equivalence problem for the class of equations

$$u_{xx} = u_t + Q(u)u_x \tag{14}$$

under a contact transformation pseudo-group. Two equations are said to be equivalent if there exists a contact transformation mapping one equation to the other.We use Elie Cartan's method of equivalence, [1], in its form developed by Fels and Olver, [2, 3], to compute the Maurer - Cartan forms, the structure equations, the basic invariants, and the invariant derivatives for symmetry groups of equations from the class . All differential invariants are functions of the basic invariants and their invariant derivatives. Cartan's solution to the equivalence problem states that two equations are (locally) equivalent if and only if Cartan test's satisfied.

Equivalence problem of differential equations

In this section we describe the local equivalence problem for differentials equations under the action of the pseudo group of contact transformations. Two equations are said to be equivalent if there exists a contact transformation which maps the equations to each other. We apply Elie Cartan's structure theory of Lie pseudo-groups to obtain necessary and sufficient conditions under which equivalence mappings can be found. This theory describes a Lie pseudo-group in terms of a set of invariant differential 1-forms called Maurer-Cartan forms. Expressions of exterior differentials of Maurer-Cartan forms in terms of the forms themselves yield Cartan structure equations for the pseudo-group. The Maurer-Cartan forms contain all information about the pseudo-group, in particular, they give basic invariants and operators of invariant differentiation and allow one to

solve equivalence problems for submanifolds under the action of the pseudo-group.
As is shown in [4], the following differential 1-forms,

$$\begin{aligned}
\Theta^\alpha &= a^\alpha_\beta(du^\beta - u^\beta_{x^j}dx^j),\\
\Xi^i &= b^i_j dx^j + c^i_\beta \Theta^\beta,\\
\Sigma^\alpha_i &= a^\alpha_\beta B^i_j du^\beta_{x^j} + f^\alpha_{i\beta}\Theta^\beta + g^\alpha_{ij}\Xi^j
\end{aligned}$$

are Maurer-Cartan forms of $Cont(J^1(\pi))$. They are defined on $J^1(\pi)\times\mathcal{H}$, where $\mathcal{H} = (a^\alpha_\beta, b^i_j, c^i_\beta, f^\alpha_{i\beta}, g^\alpha_{ij}) \mid \alpha, \beta \in \{1,\dots,q\}, i, j \in \{1,\dots,n\}, det(a^\alpha_\beta).det(b^i_j) \neq 0, g^\alpha_{ij} = g^\alpha_{ji}, (B^i_j)$ is the inverse matrix for (b^i_j). They satisfy the structure equations

$$\begin{aligned}
d\Theta^\alpha &= \Phi^\alpha_\beta \wedge \Theta^\beta + \Xi^k \wedge \Sigma^\alpha_k,\\
d\Xi^i &= \Psi^i_k \wedge \Xi^k + \Pi^i_\gamma \wedge \Theta^\gamma,\\
d\Sigma^\alpha_i &= \Phi^\alpha_\gamma \wedge \Sigma^\gamma_i - \Psi^k_i \wedge \Sigma^\alpha_k + \Lambda^\alpha_{i\beta} \wedge \Theta^\beta + \Omega^\alpha_{ij} \wedge \Xi^j.
\end{aligned}$$

where the forms$\Phi^\alpha_\beta, \Psi^i_j, \Pi^i_\beta, \Lambda^\alpha_{i\beta}$ and Ω^α_{ij} depend on differentials of the coordinates of $\mathcal{H}$. Differential equations defines a submanifold $\mathcal{R} \subset J^1(\pi)$. The Maurer-Cartan forms for its symmetry pseudo-group Cont($\mathcal{R}$) can be found from restrictions $\theta^\alpha = \imath^*\Theta^\alpha, \xi^i = \imath^*\Xi^i$and $\sigma^\alpha_i = \imath^*\Sigma^\alpha_i$. where $\imath = \imath_0 \times id : \mathcal{R}\times\mathcal{H} \longrightarrow J^1(\pi)\times\mathcal{H}$ with $\imath_0 : \mathcal{R} \longrightarrow J^1(\pi)$ defined by our differential equations. In order to compute the Maurer.Cartan forms for the symmetry pseudo-group, we implement Cartan's equivalence method. Firstly, the forms $\theta^\alpha, \xi^i, \sigma^\alpha_i$ are linearly dependent, i.e. there exists a nontrivial set of functions $U_\alpha, V_i, W^i_\alpha$ on $\mathcal{R}\times\mathcal{H}$ such that $U_\alpha\theta^\alpha + V_i\xi^i + W^i_\alpha\sigma^\alpha_i \equiv 0$. Setting these functions equal to some appropriate constants allows one to express a part of the coordinates of $\mathcal{H}$ as functions of the other coordinates of $\mathcal{R}\times\mathcal{H}$. Secondly, we substitute the obtained values into the forms $\phi^\alpha_\beta = \imath^*\Phi^\alpha_\beta$ and $\psi^i_k = \imath^*\psi^i_k$ coefficients of semi-basic forms ϕ^α_β at σ^γ_j, ξ^j, and the coefficients of semi-basic forms ψ^i_j at σ^γ_j are lifted invariants of $Cont(\mathcal{R})$. We set them equal to appropriate constants and get expressions for the next part of the coordinates of $\mathcal{H}$, as functions of the other coordinates of $\mathcal{R}\times\mathcal{H}$. Thirdly, we analyze the reduced structure equations. If the essential torsion coefficients dependent on the group parameters appear, then we should normalize them to constants and find some new part of the group parameters, which, on being substituted into the reduced modified Maurer-Cartan forms, allows us to repeat the procedure of normalization. There are two possible results of this process. The first result, when the reduced lifted coframe appears to be involutive, outputs the desired set of invariant 1-forms which characterize the pseudo-group $Lie(\mathcal{R})$. In the second result, when the coframe is not involutive, we should apply the procedure of prolongation [[5]].

Structure of symmetry groups for general form of Burgers' equations

We apply the method described in the previous section to the class of equations (14).we take the equivalent system of first order

$$u_x = v, \qquad u_t = v_x + Q(u)v. \tag{15}$$

Denoting, $x = x_1, t = x_2, v = u_1, u = u_2, v_x = p^1_1, v_t = p^1_2, u_t = p^2_2, u_x = p^2_1$. We consider this system as a sub-bundle of the bundle $J^1(\varepsilon), \varepsilon = \mathbb{R}^2\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, with local coordinates $\{x_1, x_2, u_1, u_2, p^1_1, p^1_2, p^2_1, p^2_2\}$, where the embedding ι is defined by the equalities:

$$p^2_1 = u_1, \qquad p^1_1 = p^2_2 - Q(u_2)u_1. \tag{16}$$

The forms $\theta^\alpha = \iota^*\Theta^\alpha, \alpha \in \{1,2\}, \xi^i = \iota^*\Xi^i, i \in \{1,2\}$, are linearly dependent. The group parameters a^α_β, b^i_j must satisfy the conditions $det(a^\alpha_\beta) \neq 0, det(b^i_j) \neq 0$.
linear dependence between the forms σ^α_i are

$$\sigma^2_1 = 0, \qquad \sigma^1_1 = \sigma^2_2 \tag{17}$$

Computing the linear dependence conditions (17) and The analysis of the semi-basic modified Maurer-Cartan forms and the structure equations at the obtained values of the group parameters gives the following group parameters as a functions of other group parameters and the local coordinates $\{x_1, x_2, u_1, u_2, p_2^1, p_2^2\}$ of $\mathcal{R}$.

$$\begin{aligned} &a_1^2 = 0, \qquad b_1^2 = 0, \qquad b_2^2 = \frac{b_1^1 a_2^2}{a_1^1} \\ &a_2^2 = a_1^1 b_1^1, \qquad c_1^2 = 0, \qquad c_2^2 = c_1^1 \\ &c_2^1 = 0, \qquad c_1^1 = 0, \qquad a_2^1 = \frac{1}{2}\frac{a_1^1(Qb_1^1 - b_2^1)}{b_1^1}. \end{aligned} \tag{18}$$

Regarding the appearance of different derivatives of $Q(u)$ in the essential torsion coefficients and with respect to vanishing or non-vanishing of these derivatives and their effects on normalizations process, we have to impose some restrictions on the function $Q(u_2)$. As a result of these restrictions, the following cases arise.

Case-1:

After normalization (31.16), if Q is a constant then we have the following structure equations

$$\begin{aligned} d\theta^1 &= \phi_1^1 \wedge \theta^1 - \frac{1}{2}\psi_2^1 \wedge \theta^2 + \xi^1 \wedge \sigma_2^2 + \xi^2 \wedge \sigma_2^1, \\ d\theta^2 &= \phi_1^1 \wedge \theta^2 + \psi_1^1 \wedge \theta^2 - \theta^1 \wedge \xi^1 + \xi^2 \wedge \sigma_2^2, \\ d\xi^1 &= \psi_1^1 \wedge \xi^1 + \psi_2^1 \wedge \xi^2, \\ d\xi^2 &= 2\psi_1^1 \wedge \xi^2, \\ d\sigma_2^1 &= \phi_1^1 \wedge \sigma_2^1 - 2\psi_1^1 \wedge \sigma_2^1 - \frac{3}{2}\psi_2^1 \wedge \sigma_2^2 + \lambda_{21}^1 \wedge \theta^1 + \omega_{12}^1 \wedge \xi^1 + \omega_{22}^1 \wedge \xi^2, \\ d\sigma_2^2 &= \phi_1^1 \wedge \sigma_2^2 - \psi_1^1 \wedge \sigma_2^2 - \psi_2^1 \wedge \theta^1 + \frac{1}{3}\lambda_{21}^1 \wedge \theta^2 + \omega_{12}^1 \wedge \xi^2 + \xi^1 \wedge \sigma_2^1. \end{aligned} \tag{19}$$

The structure equations (52) do not contain any torsion coefficient depending on the group parameters. The first reduced character is $s_1' = 5$, and the degree of indeterminancy is 2. The Cartan involutivity test is not satisfied. Therefore we should use the procedure of prolongation, which gives us the following structure equations.

$$\begin{aligned} d\theta^1 &= \eta^1 \wedge \theta^1 - \frac{1}{2}\eta^3 \wedge \theta^2 + \xi^1 \wedge \sigma_2^2 + \xi^2 \wedge \sigma_2^1, \\ d\theta^2 &= \eta^1 \wedge \theta^2 + \eta^2 \wedge \theta^2 - \theta^1 \wedge \xi^1 + \xi^2 \wedge \sigma_2^2, \\ d\xi^1 &= \eta^2 \wedge \xi^1 + \eta^3 \wedge \xi^2, \\ d\xi^2 &= 2\eta^2 \wedge \xi^2, \\ d\sigma_2^1 &= \eta^1 \wedge \sigma_2^1 - 2\eta^2 \wedge \sigma_2^1 - \frac{3}{2}\eta^3 \wedge \sigma_2^2 + \eta^4 \wedge \theta^1 + \eta^5 \wedge \xi^1 + \eta^6 \wedge \xi^2, \\ d\sigma_2^2 &= \eta^1 \wedge \sigma_2^2 - \eta^2 \wedge \sigma_2^2 - \eta^3 \wedge \theta^1 + \frac{1}{3}\eta^4 \wedge \theta^2 + \eta^5 \wedge \xi^2 + \xi^1 \wedge \sigma_2^1. \\ d\eta_1 &= \frac{1}{2}\eta^3 \wedge \xi^1 + \eta^4 \wedge \xi^2, \\ d\eta_2 &= \frac{2}{3}\eta^4 \wedge \xi^2, \\ d\eta_3 &= \frac{2}{3}\eta^4 \wedge \xi^1 - \eta^2 \wedge \eta^3, \\ d\eta_4 &= -2\eta^2 \wedge \eta^4, \\ d\eta_5 &= -\pi_1 \wedge \xi^2 - \eta^6 \wedge \xi^1 + 2\eta^3 \wedge \sigma_2^1 - 2\eta^4 \wedge \sigma_2^2 + \eta^1 \wedge \eta^5 - 3\eta^2 \wedge \eta^5, \\ d\eta_6 &= -\pi_1 \wedge \xi^1 - \pi_2 \wedge \xi^2 - \frac{10}{3}\eta^4 \wedge \sigma_2^1 + \eta^1 \wedge \eta^6 - 4\eta^2 \wedge \eta^6 - \frac{5}{2}\eta^3 \wedge \eta^5. \end{aligned} \tag{20}$$

The forms $\eta_1, ..., \eta_6$ depend on differentials of the parameters of $\mathcal{H}$, while the forms π_1, π_2 depend on differentials of the prolongation variables.
In structure equations (54), the degree of indeterminancy is 2 and the reduced characters of the coframe are $s'_1 = 2, s'_2 = ... = s'_{12} = 0$. Since the Cartan involutivity test for the lifted coframe $\{\theta^1, \theta^2, \xi^1, \xi^2, \sigma_2^1, \sigma_2^2, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6\}$ is satisfied, then the coframe is involutive. Also all the essential torsion coefficients in the structure equations (54) are constants, then from the Theorem 11.8 of [5], we have:

14.1. THEOREM. *The equation $u_t = \kappa u_x + u_{xx}$ is equivalent to the $u_t = u_{xx}$ under a contact transformation.*

Case-2:
Suppose $Q = \kappa u_2 + \lambda$, is a linear function ($\kappa \neq 0$). In this case the analysis of the structure equations gives the following extra normalizations to (31.16).

$$
\begin{aligned}
& b_2^1 = \frac{1}{2}(\kappa u_2 + \lambda)(4\kappa^2 u_1 u_2 + 4\kappa\lambda u_1 - 4\kappa p_2^2)^{\frac{1}{3}}, \\
& a_1^1 = 4\kappa(4\kappa^2 u_1 u_2 + 4\kappa\lambda u_1 - 4\kappa p_2^2)^{-\frac{2}{3}}, \quad b_1^1 = \frac{1}{2}\kappa(4\kappa^2 u_1 u_2 + 4\kappa\lambda u_1 - 4\kappa p_2^2)^{\frac{1}{3}}, \\
& f_{21}^1 = -8\kappa u_1(4\kappa^2 u_1 u_2 + 4\kappa\lambda u_1 - 4\kappa p_2^2)^{-\frac{2}{3}}, \quad g_{12}^1 = 6\sqrt[3]{2}\kappa u_1(\kappa^2 u_1 u_2 + \kappa\lambda u_1 - \kappa p_2^2)^{-\frac{2}{3}}.
\end{aligned} \tag{21}
$$

The expression for g_{22}^1 are too long to be written out in full here. Now, all the group parameters are expressed as functions of the local coordinates $\{x_1, x_2, u_1, u_2, , p_2^1, p_2^2\}$.
After normalization (21) the structure equations of coframe $\{\theta^1, \theta^2, \xi^1, \xi^2, \sigma_2^1, \sigma_2^2\}$, is

$$
\begin{aligned}
d\theta^1 &= -\frac{2I}{3}\theta^1 \wedge \xi^1 - \frac{1}{3}\theta^1 \wedge \sigma_2^2 + \theta^2 \wedge \xi^2 + \xi^1 \wedge \sigma_2^2 + \xi^2 \wedge \sigma_2^1, \\
d\theta^2 &= -\theta^1 \wedge \xi^1 - \frac{I}{3}\theta^2 \wedge \xi^1 - \frac{1}{6}\theta^2 \wedge \sigma_2^2 + \xi^2 \wedge \sigma_2^2, \\
d\xi^1 &= \theta^2 \wedge \xi^2 + \frac{1}{6}\xi^1 \wedge \sigma_2^2, \\
d\xi^2 &= -\frac{2I}{3}\xi^1 \wedge \xi^2 + \frac{1}{3}\xi^2 \wedge \sigma_2^2, \\
d\sigma_2^1 &= \theta^1 \wedge \xi^1 - 8I\theta^1 \wedge \xi^2 - I\theta^2 \wedge \xi^2 - \frac{1}{2}\theta^2 \wedge \sigma_2^2 + 40I\xi^1 \wedge \xi^2 + \\
&\quad \frac{4I}{3}\xi^1 \wedge \sigma_2^1 - 13\xi^2 \wedge \sigma_2^2 - \frac{2}{3}\sigma_2^1 \wedge \sigma_2^2, \\
d\sigma_2^2 &= 6\theta^1 \wedge \xi^2 - \theta^2 \wedge \xi^1 - 2\theta^2 \wedge \xi^2 + \xi^1 \wedge \sigma_2^1 + I\xi^1 \wedge \sigma_2^2,
\end{aligned} \tag{22}
$$

where

$$
I = \frac{\sqrt[3]{2}(\kappa^2 u^2 u_x + 2\kappa\lambda u u_x - \kappa u u_t - \kappa(u_x)^2 + \lambda^2 u_x - \lambda u_t + u_{tx})\kappa}{\sqrt[3]{(\kappa^2 u u_x + \kappa\lambda u_x - \kappa u_t)^4}}
$$

is the only invariant of the symmetry group o equations of the from **Case-2**.
Note that, the exterior differential of I is

$$
dI = \frac{1}{2}\theta^2 + \frac{4I^2}{3}\xi^1 + 6\xi^2 + \frac{1}{2}\sigma_2^1 + \frac{2I}{3}\sigma_2^2.
$$

All derived invariants of the group are expressed as functions of I. Therefore, the rank of the coframe, is 1 and our manifold is 6-dimensional and by theorem 8.22 from [5], we deduce the following theorem.

14.2. THEOREM. *The equation $u_t = (\kappa u + \lambda)u_x + u_{xx}$, ($\kappa \neq 0$) admits a contact transformation symmetry group of dimension 5.*

If Q is not a linear function, the analysis of the structure equations gives the following normalizations in addition to (31.16).

$$a_1^1 = \frac{4\left(\frac{d^2Q}{du^2}\right)^2}{\left(\frac{dQ}{du}\right)^3}, \quad b_1^1 = -\frac{1}{2}\frac{4\left(\frac{dQ}{du}\right)^2}{\left(\frac{d^2Q}{du^2}\right)}, \quad b_2^1 = -\frac{1}{2}\frac{\left(\frac{dQ}{du}\right)\left(-2\left(\frac{d^2Q}{du^2}u_x\right)+\left(\frac{dQ}{du}\right)Q\right)}{\left(\frac{d^2Q}{du^2}\right)}.$$

The expression for $f_{21}^1, g_{22}^1, g_{12}^1$ are too long to be written out in full here. Now, all the group parameters are expressed as functions of the local coordinate.

Case-3:

$Q(u)$ is a quadratic polynomial or

$$\frac{2(\frac{d^2Q}{du^2})^2 - (\frac{d^3Q}{du^3})(\frac{dQ}{du})}{(\frac{d^2Q}{du^2})^2} = Constant. \tag{23}$$

If (23) satisfied then the structure equations of the coframe are

$$\begin{aligned}
d\theta^1 &= J_3\theta^1\wedge\xi^2 - 8J_1\theta^2\wedge\xi^1 + J_2\theta^2\wedge\xi^2 + \xi^1\wedge\sigma_2^2 + \xi^2\wedge\sigma_2^1,\\
d\theta^2 &= -\theta^1\wedge\xi^1 + \frac{J_3}{3}\theta^2\wedge\xi^2 + \xi^2\wedge\sigma_2^2,\\
d\xi^1 &= \theta^1\wedge\xi^2 + \theta^2\wedge\xi^2 - (16J_1 + \frac{J_3}{2})\xi^1\wedge\xi^2,\\
d\xi^2 &= 0,\\
d\sigma_2^2 &= -16J_1\theta^1\wedge\xi^1 + 2(20J_1 - J_2)\theta^1\wedge\xi^2 - J_2\theta^2\wedge\xi^1 - \theta^2\wedge\sigma_2^2 -\\
&\quad (64{J_1}^2 + 4J_1J_3 - 8J_1 + J_2)\theta^2\wedge\xi^2 + \xi^1\wedge\sigma_2^1 - \xi^2\wedge\sigma_2^1 - \frac{J_3}{2}\xi^2\wedge\sigma_2^2.
\end{aligned} \tag{24}$$

The expression for $d\sigma_2^2$ is too long to be written out in full here.

If $Q(u)$ is a quadratic polynomial then the structure equations of coframe is different from (24), and expressed only by $\{J_1, J_2, J_3\}$, where

$$\begin{aligned}
J_1 &= \frac{\left(\frac{d^2Q}{du^2}\right)^3\left(\left(\frac{d^2Q}{du^2}\right){u_x}^2 - Q\left(\frac{dQ}{du}\right)u_x + \left(\frac{dQ}{du}\right)u_t\right)}{\left(\frac{dQ}{du}\right)^6},\\
J_2 &= -\frac{4\left(\frac{d^2Q}{du^2}\right)^3}{\left(\frac{dQ}{du}\right)^9}\left(\left(\frac{dQ}{du}\right)^4 Qu_x + \left(\frac{d^2Q}{du^2}\right)\left(\frac{dQ}{du}\right)^3{u_x}^2 -\right.\\
&\quad 2\left(\frac{d^2Q}{du^2}\right)\left(\frac{dQ}{du}\right)^2 Q^2u_x + 6\left(\frac{d^2Q}{du^2}\right)^2\left(\frac{dQ}{du}\right)Q{u_x}^2 -\\
&\quad 4\left(\frac{d^2Q}{du^2}\right)^3{u_x}^3 + 2\left(\frac{d^2Q}{du^2}\right)\left(\frac{dQ}{du}\right)^2 Qu_t - 6\left(\frac{d^2Q}{du^2}\right)^2\left(\frac{dQ}{du}\right)u_xu_t\\
&\quad \left.-2\left(\frac{d^2Q}{du^2}\right)\left(\frac{dQ}{du}\right)^2 u_{tx} - \left(\frac{dQ}{du}\right)^4 u_t\right),\\
J_3 &= -\frac{8\left(\frac{d^2Q}{du^2}\right)^2}{\left(\frac{dQ}{du}\right)^6}\left(\left(\frac{dQ}{du}\right)^3 u_x + 2\left(\frac{d^2Q}{du^2}\right)^3{u_x}^2 - 2\left(\frac{d^2Q}{du^2}\right)\left(\frac{dQ}{du}\right)Qu_x\right.\\
&\quad \left.+2\left(\frac{d^2Q}{du^2}\right)\left(\frac{dQ}{du}\right)u_t\right),
\end{aligned} \tag{25}$$

are invariants of the symmetry group of an equation from **Case-3**.
All derived invariants of the group are expressed as functions of $\{J_1, J_2, J_3\}$. Therefore the rank of the coframe, is 3. Again by theorem 8.22 from [5], we have

14.3. THEOREM. *If $Q(u)$ is a quadratic polynomial or $\frac{2(\frac{d^2Q}{du^2})^2-(\frac{d^3Q}{du^3})(\frac{dQ}{du})}{(\frac{d^2Q}{du^2})^2}$ be a constant and $\frac{d^3Q}{du_x{}^3} \neq 0$ then, the equation $u_t = Q(u)u_x + u_{xx}$, admits a contact transformation symmetry group of dimension 3.*

Case-4:
We will make the following assumption for (14):

$$\frac{2(\frac{d^2Q}{du^2})^2 - (\frac{d^3Q}{du^3})(\frac{dQ}{du})}{(\frac{d^2Q}{du^2})^2} \neq constant, \qquad \frac{d^3Q}{du^3} \neq 0. \tag{26}$$

The structure equations of the coframe, in this case, is

$$\begin{aligned}
d\theta^1 &= -J_4\theta^1 \wedge \theta^2 + (4J_1J_3J_4 + J_3 + 32{J_1}^2J_4 + \frac{1}{8}{J_3}^2J_4 + 8J_1J_4)\theta^1 \wedge \xi^2 - \\
&\quad (16{J_1}^2J_4 + 2J_1J_3J_4 + \frac{1}{16}{J_3}^2J_4 - 8J_1)\theta^2 \wedge \xi^1 + \frac{J_4(16J_1 + J_3)}{2}\theta^1 \wedge \xi^1 + \\
&\quad (64{J_1}^3J_4 + \frac{1}{64}{J_3}^3J_4 + 16{J_1}^2J_4 + 12{J_1}^2J_3J_4 + \frac{3}{4}J_1{J_3}^2J_4 + J_1J_3J_4 + \\
&\quad J_2)\theta^2 \wedge \xi^2 + \xi^1 \wedge \sigma_2^2 + \xi^2 \wedge \sigma_2^1, \\
d\theta^2 &= -\theta^1 \wedge \xi^1 + \frac{J_4(16J_1 + J_3)}{4}\theta^2 \wedge \xi^1 + (2J_1J_3J_4 + \frac{1}{2}J_3 + 16{J_1}^2J_4 + \\
&\quad \frac{1}{16}{J_3}^2J_4 + 4J_1J_4)\theta^2 \wedge \xi^2 + \xi^2 \wedge \sigma_2^2, \\
d\xi^1 &= \theta^1 \wedge \xi^2 - \frac{J_4}{2}\theta^2 \wedge \xi^1 + (1 - 4J_1J_4 - \frac{1}{4}J_3J_4)\theta^2 \wedge \xi^2 + \\
&\quad (2J_1J_3J_4 - 16J_1 - \frac{1}{2}J_3 + 16{J_1}^2J_4 + \frac{1}{16}{J_3}^2J_4 - 4J_1J_4))\xi^1 \wedge \xi^2, \\
d\xi^2 &= -J_4\theta^2 \wedge \xi^2 + \frac{J_4(16J_1 + J_3)}{2}\xi^1 \wedge \xi^2.
\end{aligned} \tag{27}$$

The expression for $d\sigma_2^1, d\sigma_2^2$ are too long to be written out in full here.
There is an invariant extra to (25), for the symmetry group of equations from **case4**, which is

$$J_4 = \frac{2(\frac{d^2Q}{du^2})^2 - (\frac{d^3Q}{du^3})(\frac{dQ}{du})}{(\frac{d^2Q}{du^2})^2}.$$

All derived invariants of the group are functionally expressed as functions of
$\{J_1, J_2, J_3, J_4\}$. The rank of the coframe, is 4, therefore we have:

14.4. THEOREM. *If $Q(u)$ satisfy, $\frac{2(\frac{d^2Q}{du^2})^2-(\frac{d^3Q}{du^3})(\frac{dQ}{du})}{(\frac{d^2Q}{du^2})^2} \neq constant$ and $(\frac{d^3Q}{du_1^3} \neq 0)$ then, the equation $u_t = Q(u)u_x + u_{xx}$, admits a contact transformation symmetry group of dimension 2.*

15. Conclusion

In this paper, the moving coframe method of [4] is applied to the local equivalence problem for a class of systems of the general form of Burgers' equations under the action of a pseudo-group of contact transformations. We have found four subclasses and showed that every type of the general form of Burgers' equations belongs to a system from one of these subclasses. The equivalence condition of first subclass, structure equations and invariants for all subclasses are found.

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16. Study of qualitative behavior of a new coronavirus disease model

Mehran Namjoo[1,a], Mehran Aminian[2], Mehdi Karami[3] and
Mohammad Javad Mohammad Taghizadeh[4]

[1,2,3]Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran.
[4]Atherosclerosis Research Center, Ahvaz Jundishapur University of Medical Sciences, Ahvaz, Iran.

namjoo@vru.ac.ir, mehran.aminian@vru.ac.ir, m.karami@vru.ac.ir, dr.taghizadeh87@gmail.com
The aim of this manuscript is to discuss the dynamics of a coronavirus disease 2019 (COVID–19) model. We first prove the positivity and boundedness of the solution of the proposed COVID–19 model. Thence, we determine the equilibrium points and discuss the stability analysis of the model. In continuation, we show that the equilibrium points are locally asymptotically stable. We apply the nonstandard finite difference (NSFD) scheme to study the dynamic behaviors COVID–19 model. In order to the efficiency and accuracy of the proposed NSFD, some numerical results are presented.

Keywords: Coronavirus model, positivity, boundedness, stability analysis, nonstandard finite difference scheme.

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[a]Speaker. Email address: namjoo@vru.ac.ir

Introduction

Over the years, mathematical modelling is proved its ability to obtain more understanding dynamics of disease models in the community. These models can help the researchers to understand more about the spread process of a virus that may turn into a pandemic situation and may predict the conditions that will show the continuation or end of these infections. In March 2020, the COVID–19 disease begins to spread throughout the world which is originated from Wuhan in China causing a global fear and devastating effect which conclude the governments and scientists to find a suitable cure [1, 2]. This virus can mainly transmit through the droplets of an infected person that can spread when the person coughs, sneezes, or even while talking. These small droplets from an infected person can be transmitted to an uninfected person who may breathe them and causing the person to be infected with this virus. These viruses are heavy and they will not be hanging in air and eventually, they should land on any other surface or the floor. Another way of getting infected is that when an infected person is caught or sneezes in his hand and touches some surfaces and then an infected person touches these surfaces and then touches his eye or nose and then he becomes infected. Due to the above reasons, scientists have been working extensively overall the last two years [3, 4]. In many cases, mathematical modelling of disease can be described by a nonlinear autonomous initial value problem. Since analytical solution a few numbers of these equations can be obtained, hence, various numerical methods were constructed to solve such equations. In this research, in order to approximate the solution of the proposed COVID–19 model, we are going to construct an efficient NSFD scheme. Our model takes the following form:

$$\begin{cases} s'(t) = \gamma r(t) - \alpha s(t)i(t) - \mu_s s(t) + \mu^*, \\ i'(t) = \alpha s(t)i(t) - \beta i(t) - \mu_i i(t), \\ r'(t) = \beta i(t) - \gamma r(t) - \mu_r r(t), \\ s(0) = s_0,\ i(0) = i_0,\ r(0) = r_0. \end{cases} \tag{28}$$

In this model, the total population individuals at each time t is divided into three groups. Here, $s(t)$ is the number of susceptible group at time t, $i(t)$ is the number of infected group at time t and $r(t)$ denotes the recovered group at time t. Also, moving from the susceptible group to the infected group occurs at a rate α and infected groups are supposed to recover at a constant rate β. Moreover, the recovered individuals can again return to the susceptible group at a constant rate of γ. According to the [6] in diseases COVID–19 because of two exposures over a small time period a single contact produces infection at the rate αsi. Here the parameters μ_s, μ_i and μ_r denote death rates of the susceptible, infected and recovered groups, respectively and the parameter μ^* presents rate of birth the susceptible group. The organization of the manuscript is as follows. In Section 2, we prove positivity and boundedness of the solution model of (28). Section 3 deals with stability analysis of COVID–19 model. In Section 4, we construct an efficient NSFD scheme for the COVID–19 model (28). The numerical results are obtained by the NSFD scheme, show the efficiency of the NSFD scheme.

Positivity and boundedness of solutions

In this part, we are going to show that the state variables are nonegative and bounded that describes the COVID–19 model meaningful. First, we want to show that the solutions $s(t)$, $i(t)$ and $r(t)$ of the model (28), when they exist, are positive for all $t \geq 0$ with nonegative initial conditions.

16.1. Theorem. *Consider the initial conditions as given in* (28). *Then the solutions* (s, i, r) *are positive for all time* $t \geq 0$.

Proof. Since the sr-coordinate plane is invariant under the flows of system, this implies that $i(t) > 0$ for all $t \geq 0$. Let $A = \{t \geq 0 | r(t) < 0\}$, we will show that $A = \emptyset$. Suppose that $A \neq \emptyset$ and let $t_0 = \inf(A)$. Since $r(0) > 0$, so $t_0 > 0$. Now the continuity of r implies that $r(t_0) = 0$ and by the third equation of system (28), $r'(t_0) = \beta i(t_0) > 0$. Hence, there is $\varepsilon > 0$ such that $r(t) > 0$ for all $t \in (t_0 - \varepsilon,\ t_0 + \varepsilon)$. Consequently, $r(t) \geq r(t_0) > 0$, for all time $t \in (t_0,\ t_0 + \varepsilon)$ which contradicts $t_0 = \inf(A)$. By a similar argument, we can show that $s(t) \geq 0$, for all $t \geq 0$. □

In order to prove boundedness of the solutions of the system (28), we first state the following proposition.

16.2. Proposition. *Let* $K(t) : [0, +\infty) \longrightarrow \mathbb{R}$ *be a derivative function such that* $K(t) \geq 0$ *for all* $t \geq 0$. *If* $\alpha > 0$, $\beta \in \mathbb{R}$, *such that* $K'(t) + \alpha K(t) \leq \beta$, *for all* $t \geq 0$, *then* $K(t) \leq K(0) + \frac{\beta}{\alpha}$.

16.3. Lemma. *All the solutions* $(s(t), i(t), r(t))$ *of the system* (28) *are bounded.*

Proof. Set $K(t) = s(t) + i(t) + r(t)$ and suppose that $m = \min\{\mu_s,\ \mu_i,\ \mu_r\}$. Hence $K(t) + mK'(t) \leq \mu^*$. It follows from proposition 16.2 that $s(t) + i(t) + r(t) \leq s(0) + i(0) + r(0) + \frac{\mu^*}{m}$. This shows that the solutions s, i, r of model (28) are bounded. □

Stability analysis for the COVID–19 model

The equilibrium points of the COVID–19 model (28) are given by $E_1 = (\frac{\mu^*}{\mu_s}, 0, 0)$ and $E_2 = (\frac{\mu_i + \beta}{\alpha}, i^*, \frac{\beta i^*}{\gamma + \mu_r})$, where $i^* = \frac{\mu_s(\frac{\mu_i+\beta}{\alpha}) - \mu^*}{\frac{\gamma\beta}{\gamma+\mu_r} - \mu_i - \beta}$.

16.4. Theorem. *The system* (28) *is*

(i) locally asymptotically stable at the equilibrium point E_1 *if* $\alpha\frac{\mu^*}{\mu_s} - \beta - \mu_i < 0$.
(ii) locally asymptotically stable at the equilibrium point E_2 *if* $i^* > 0$.

Proof. The Jacobian matrix of system (28) corresponding to any equilbrium point (s_1, i_1, r_1) can be written as

$$J(s_1,i_1,r_1) = \begin{bmatrix} -\alpha i_1 - \mu_s & -\alpha s_1 & \gamma \\ -\alpha i_1 & \alpha s_1 - \beta - \mu_i & 0 \\ 0 & \beta & -\gamma - \mu_r \end{bmatrix}.$$

The Jacobian matrix of (28) at the equilibrium point E_1 is obtained as given below

$$J(E_1) = \begin{bmatrix} -\alpha\dfrac{\mu^*}{\mu_s} - \mu_s & -\alpha\dfrac{\mu^*}{\mu_s} & \gamma \\ 0 & \alpha\dfrac{\mu^*}{\mu_s} - \beta - \mu_i & 0 \\ 0 & \beta & -\gamma - \mu_r \end{bmatrix}.$$

The corresponding eigenvalues are $\lambda_1 = -\alpha\frac{\mu^*}{\mu_s} - \mu_s$, $\lambda_2 = \alpha\frac{\mu^*}{\mu_s} - \beta - \mu_i$ and $\lambda_3 = -\gamma - \mu_r$. Therefore the equilibrium point E_1 is locally asymptotically stable if $\alpha\frac{\mu^*}{\mu_s} - \beta - \mu_i < 0$. At the equilibrium point E_2 the Jacobian matrix is given

$$J(E_2) = \begin{bmatrix} -\alpha i^* - \mu_s & -\mu_i - \beta & \gamma \\ \alpha i^* & 0 & 0 \\ 0 & \beta & -\gamma - \mu_r \end{bmatrix}.$$

Hence, we obtain that the characteristic equation can be presented in the following form

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \tag{29}$$

where

$$a_1 = \alpha i^* + \mu_s + \gamma + \mu_r,\ a_2 = (\alpha i^* + \mu_s)(\gamma + \mu_r) + \alpha i^*(\mu_i + \beta),\ a_3 = \alpha i^*(\mu_i + \beta)(\gamma + \mu_r) - \gamma\alpha i^*\beta.$$

Using the Routh-Hurwitz criteria, all roots of Eq. (29) have negative real parts if

$$a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ a_1a_2 - a_3 > 0. \tag{30}$$

clearly $a_1 > 0$, $a_2 > 0$, and $a_3 > 0$, and to see whether $a_1a_2 - a_3$ is greater than zero, we check the following working

$$a_1a_2 - a_3 = (\alpha i^* + \mu_s)^2(\gamma + \mu_r) + \alpha i^*(\mu_i + \beta)(\alpha i^* + \mu_s) + (\gamma + \mu_r)^2(\alpha i^* + \mu_s) + \gamma\alpha_i^*\beta > 0.$$

Hence, the equilibrium point E_2 is locally asymptotically stable if $i^* > 0$. □

A NSFD scheme for the COVID–19 model

In this section, we are going to develop an explicit numerical scheme using NSFD scheme which were firstly proposed by Mickens for an initial value problem. Many applications are available in literature using NSFD scheme [5]. In order to introduce the general aspect of a NSFD scheme consider the following autonomous initial value problem

$$X'(t) = f(X(t)), \quad X(0) = X_0, \quad t \in [0, t_f]. \tag{31}$$

Suppose that a discretization $t_k = kh$ is given. A NSFD scheme for the problem (31) is constructed by the following two steps.

(i) The first order deviation in the problem (31) at the k-th time step can be replaced by a discrete form $X'(t_k) \approx \frac{X_{k+1} - X_k}{\phi(h)}$, where X_k is an approximation of the exact solution $X(t_k)$ and moreover the denominator function $\phi(h)$ has to satisfy the condition $\phi(h) = h + O(h^2)$ with $0 < \phi(h) < 1$.

(ii) The nonlinear and linear terms in the right–hand–side equation have to replace by non-local discrete approximations. According to the Mickens rules, a NSFD scheme for the proposed COVID–19 model (28) can be written as

$$\begin{cases} \dfrac{s_{k+1}-s_k}{\phi_1} = \gamma r_k - \alpha s_{k+1} i_k - \mu_s s_{k+1} + \mu^*, \\ \dfrac{i_{k+1}-i_k}{\phi_2} = \alpha s_{k+1} i_k - \beta i_{k+1} - \mu_i i_{k+1}, \\ \dfrac{r_{k+1}-r_k}{\phi_3} = \beta i_{k+1} - \gamma r_{k+1} - \mu_r r_{k+1}, \end{cases} \tag{32}$$

where the denominator functions are defined as

$$\phi_1(h) = \frac{e^{\mu_s h}-1}{\mu_s}, \qquad \phi_2(h) = \frac{e^{(\beta+\mu_i)h}-1}{\beta+\mu_i}, \qquad \phi_3(h) = \frac{e^{(\gamma+\mu_r)h}-1}{\gamma+\mu_r}.$$

The explicit form of (32) can be written as

$$\begin{cases} s_{k+1} = \dfrac{s_k + \phi_1 \gamma r_k + \phi_1 \mu^*}{1+\alpha\phi_1 i_k + \phi_1 \mu_s}, \\ i_{k+1} = \dfrac{(1+\alpha\phi_2 s_{k+1}) i_k}{1+(\beta+\mu_i)\phi_2}, \\ r_{k+1} = \dfrac{\beta\phi_3 i_{k+1} + r_k}{1+(\gamma+\mu_r)\phi_3}. \end{cases} \tag{33}$$

16.5. Proposition. *If $s_0 > 0$, $i_0 > 0$ and $r_0 > 0$, then for all stepsize h, the numerical solutions are obtained from (33) are always positive.*

17. Numerical analysis

This section is devoted to numerical interpretation of COVID–19 model using the proposed NSFD scheme simulated with the help of Matlab software. In order to investigate the numerical solutions of the proposed NSFD we consider two cases. At the first simulation, we choose the parameter values $\mu^* = 0.02$, $\mu_s = 0.2$, $\alpha = 0.6$, $\beta = 0.1$, $\gamma = 0.001$ and $\mu_r = \mu_i = 0.02$ with the initial condition $s_0 = 30$, $i_0 = 25$ and $r_0 = 20$ for simulating time 1000 and the stepsize $h = 0.4$. Figure 1 confirms that the NSFD scheme (32) converges to the equilibrium point $E_1 = (0.1, 0, 0)$. In Figure 2, we plot the behaviour of the NSFD scheme (32) for the parameter values $\beta = 0.1$, $\alpha = 0.05$, $\mu_s = 0.2$, $\mu_i = \mu_r = 0.02$, $\gamma = 0.001$ and $\mu^* = 0.5$ with choosing stepsize $h = 2$ and initial condition $s_0 = 30$, $I_0 = 25$ and $r_0 = 20$. The Figure 2, shows that (s_k, i_k, r_k) approaches to the equilibrium point $E_2 = (2.4, 0.1735, 0.8261)$.

Conclusion

In this article, the dynamics of the new COVID–19 model is investigated. The positivity and boundedness of the model is proved. The stability analysis for both equilibrium points is obtained proving that the model is locally asymptotically stable for both equilibrium points. The proposed COVID–19 model is solved using a NSFD scheme. The simulation results show the effective of the NSFD scheme, even for choosing the large stepsize h. As a future research work, we can focus on the fractional–order COVID–19 model and obtain an efficient NSFD scheme which preserves the positivity and stability properties of the fractional order COVID–19 model.

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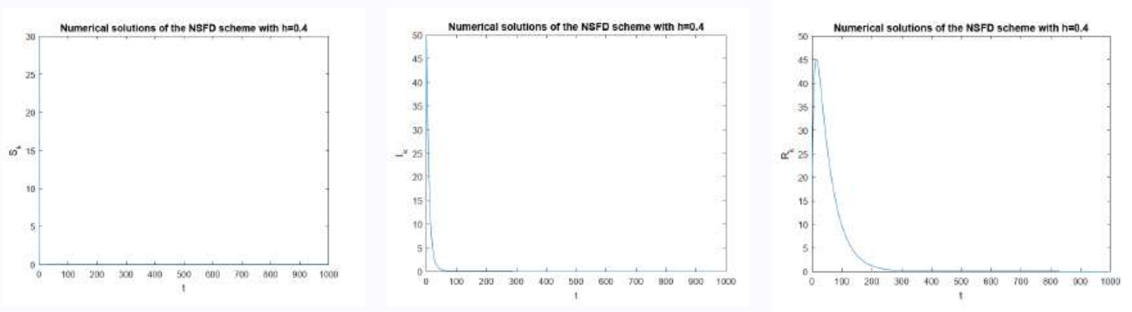

FIGURE 2. Numerical simulation with $h = 0.4$ for the NSFD scheme (32)

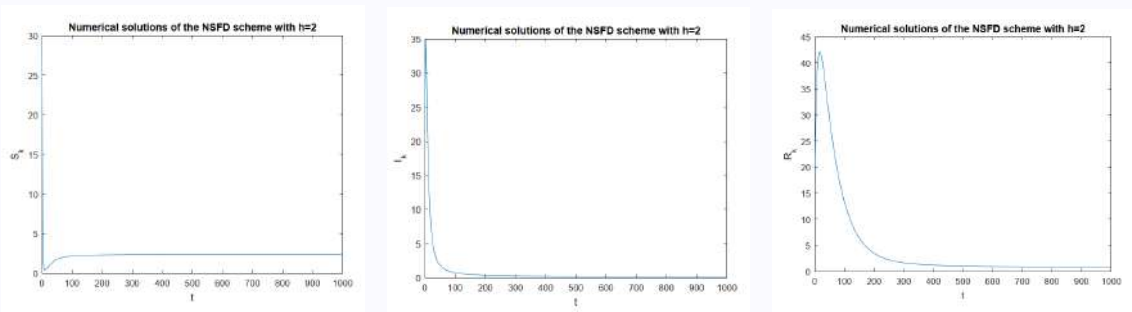

FIGURE 3. Numerical simulation with $h = 2$ for the NSFD scheme (32)

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18. Analysis of a nonlinear election model in fractional order

Z. Dadi [1], and N. Nazari [2] [a],

[1]Department of Mathematics, University of Bojnord , Bojnord, Iran
[2]Department of Mathematics, University of Hormozgan, Bandarabbas, Iran

This article is devoted to studying a fractional model of the election with three parties. For this aim, we used the fractional derivative in Caputo sense. The stability of some equlibria in this model is investigated by the fractional RouthHurwitz stability criterion. Also, our numerical results of the proposed model show the simplicity and the efficiency of the proposed criterion.

Keywords: Fractional-order nonlinear election model , Population dynamics, Stability.
AMS Mathematics Subject Classification [2020]: 91F10, 92D25, 34D20
Code: cdsgt3-00990043

[a]Speaker. Email address: nazarinargesmath@gmail.com

19. Introduction

The 2000 and 2016 United States presidential elections show us political parties shape public opinion, but their influence is limited. Indeed, we should be noted that although many countries have dual political parties which play a main role in their elections, there is a special parameter which is people's opinions. Some researches show sometimes people change the mind of these political parties or maybe build a new political party. Indeed, a new party arises which a growing number of voters move into it. The 2000 and 2016 U.S. presidential elections and 2005 Iran presidential election are some samples of this idea.

According to the importance of elections in the democracy of countries, our attention is attracted to study the movement of voters between political parties and the population dynamics amongst each group. For this aim, a nonlinear mathematical model in fractional order with a constant population assumption is considered.

In most countries, it is important that political independence is retained by the people and exercised directly by citizens. The usual mechanism in these countries is a decision-making process by which citizens who have necessary conditions choose an individual to hold formal office. In this approach, the influence of political parties on people's views is treated as a disease that person is affected. Therefore, it can be assumed that members move from one political party to another one when they are exposed to the ideology of other parties. Some modeling studies have been conducted regarding the growth of political parties and voters [3, 1, 2, 3, 4]. In the modeling process, we assume that the total population is constant, N. This total population has been divided into four classes:

(1) V: Population of eligible voters
(2) A: Population of Political Party A

(3) B: Population of Political Party B
(4) C: Population of Political Party C.

Bauelos et al. [3] introduced the following model after simplification

$$\frac{dX}{dt} = X(a(-X-Y-Z+1)-\mu-\psi Y+\Omega Z), \tag{34}$$
$$\frac{dY}{dt} = Y(b(-X-Y-Z+1)-\mu+\psi(X-Z)), \tag{35}$$
$$\frac{dZ}{dt} = Z(c(-X-Y-Z+1)-\mu-X\Omega+\psi Y), \tag{36}$$

which

- X : Proportion of Political Party A
- Y : Proportion of Political Party B
- Z : Proportion of Political Party C
- a : per capita recruitment rate of Party A from V
- b : per capita recruitment rate of Party B from V
- c : per capita recruitment rate of Party$C$ from V
- μ : rate at which individuals enter and leave voting system
- Ω : Net Shift between Party A and Party C
- ψ : Net Shift between Party A and Party B and Net Shift between Party B and Party C.

In the next Section, we consider the fractional order form of this model as follows

$$D^\alpha X(t) = X(a(-X-Y-Z+1)-\mu-\psi Y+\Omega Z), \tag{37}$$
$$D^\alpha Y(t) = Y(b(-X-Y-Z+1)-\mu+\psi(X-Z)), \tag{38}$$
$$D^\alpha Z(t) = Z(c(-X-Y-Z+1)-\mu-X\Omega+\psi Y), \tag{39}$$

to investigate its dynamics by determining equilibrium points of this system analytically and discuss their stability.

20. Main results

The equilibrium points of our model are denoted by (X^*, Y^*, Z^*), where $X^* = \frac{A^*}{N}$, $Y^* = \frac{B^*}{N}$, $Z^* = \frac{C^*}{N}$, and $V^* = N-(A^*+B^*+C^*)$. These equilibria will be in the one of four forms

- the party-free equilibria
- the single-party equilibria
- the dual-party equilibria
- the interior equilibria

which is obtained in [3] as it is summarized in Table 2 On the other hand, it is not hard to compute

Table 2. Equilibrium Points

case	equilibrium point	value	status
1	p1	$X=0, Y=0, Z=1$	—
2	p2	$X=0, Y=1, Z=0$	—
3	p3	$X=0, Y=-1, Z=1$	not acceptable
4	p4	$X=\frac{0.99}{\Omega}, Y=0, Z=\frac{-0.99}{\Omega}$	not acceptable
5	p5	$X=\frac{0.99}{(2+\Omega)}, Y=\frac{0.99\Omega}{(2+\Omega)}, Z=\frac{0.99}{(2+\Omega)}$	—
6	p6	$X=0., Y=0., Z=0.$	—
7	p7	$X=1, Y=0., Z=0.$	—
8	p8	$X=-1, Y=1, Z=0.$	not acceptable

the characteristic equation of the equilibria as the following polynomial:

$$\phi(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 \tag{40}$$

Now, expressing the discriminant of $\phi(\lambda)$ as

$$D(\phi) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3a_1^2 - 4a_2^2 - 27a_3^2 \tag{41}$$

and using the result of Ahmed et al. [1], following fractional RouthHurwitz conditions associated with are observed:

(1) If $D(\phi) > 0$, then the necessary and sufficient condition for the equilibrium point to be locally asymptotically stable is $a_1 > 0, a_3 > 0, a_1a_2 > a_3$;
(2) If $D(\phi) < 0, a_1 \geq 0, a_2 \geq 0, a_3 > 0$, then the equilibrium point is locally asymptotically stable for $\alpha < \frac{2}{3}$,
(3) If $D(\phi) < 0, a_1 < 0, a_2 < 0, \alpha > \frac{2}{3}$, then all roots of Eq. 40 satisfy the condition $|arg(\lambda_i)| < \alpha\frac{\pi}{2}, i = 1, 2, 3$

To obtain the equilibria of system (4)-(6), we consider the following parameter values which is used in [3]

$$\begin{aligned} a &= b = c = \psi = 1, \\ \mu &= 0.01. \end{aligned}$$

Therefore, if $\Omega = \psi$ we have the results in Table3

By applying fractional RouthHurwitz stability criterion, the results is summarized in Table 4

Table 3. Characteristic Equation

case	equilibrium point	value	characteristic equation
1	p1	$X = 0, Y = 0, Z = 1$	$\lambda^3 + 1.03\lambda^2 - 0.9797\lambda - 1.0099$
2	p2	$X = 0, Y = 1, Z = 0$	$\lambda^3 + 0.03\lambda^2 - 0.9997\lambda - 0.009999$
3	p3	$X = 0, Y = -1, Z = 1$	not acceptable
4	p4	$X = \frac{0.99}{\Omega}, Y = 0, Z = \frac{-0.99}{\Omega}$	not acceptable
5	p5	$X = \frac{0.99}{(2+\Omega)}, Y = \frac{0.99\Omega}{(2+\Omega)}, Z = \frac{0.99}{(2+\Omega)}$	$\lambda^3 + 0.99\lambda^2 + 0.3267\lambda + 0.035937$
6	p6	$X = 0., Y = 0., Z = 0.$	$\lambda^3 - 2.97\lambda^2 + 2.9403\lambda - 0.970299$
7	p7	$X = 1, Y = 0., Z = 0.$	$\lambda^3 + 0.99\lambda^2 - 0.9801\lambda - 0.970299$
8	p8	$X = -1, Y = 1, Z = 0.$	not acceptable

Now, we consider p_6. By Table 4 and 3, it is an unstable point, see Figure 4-6. The result for

Table 4. Fractional RouthHurwitz Stability Criterion

case	equilibrium point	$D(\phi)$
1	p1	-7.73824
2	p2	-3.97242
3	p3	not acceptable
4	p4	not acceptable
5	p5	-0.407442
6	p6	201.692
7	p7	-7.56213
8	p8	not acceptable

other points is similar.

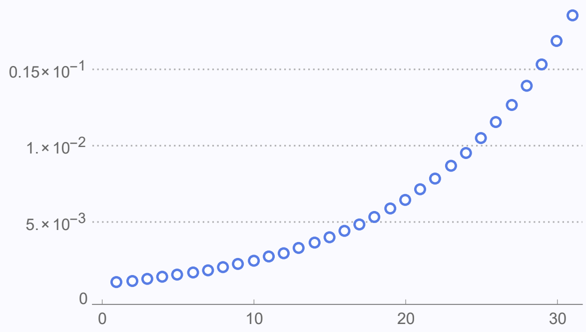

FIGURE 4. Behavior of the numerical solution X(t) using GABMM , $\alpha = 0.95$.

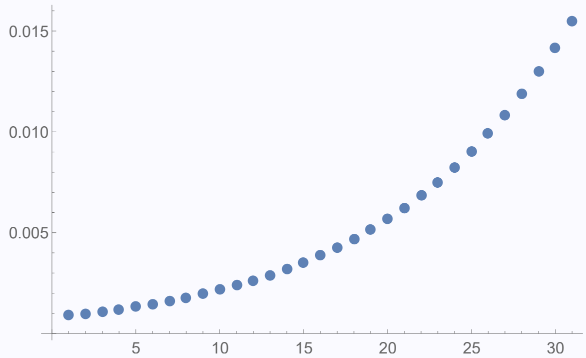

FIGURE 5. Behavior of the numerical solution Y(t) using GABMM , $\alpha = 0.95$.

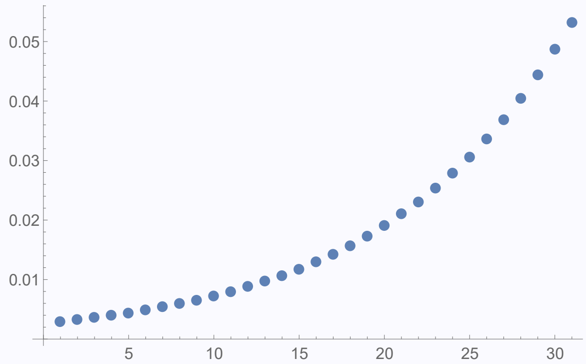

FIGURE 6. Behavior of the numerical solution Z(t) using GABMM ,$\alpha = 0.95$.

21. Conclusion

This paper is devoted to implement the analytical and numerical method for studying a nonlinear election model in fractional-order. In this work, we interested to discuss and study the discriminant of characteristic equation of our model to investigate the stability of equilibria. Also, some numerical results is presented for one of equlibria, origin. Our analytical and numerical results showed that it was unstable.

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22. Synchronized systems and entropy minimality

Somayyeh Jangjooye Shaldehi[1, a], Dawoud Ahmadi Dastjerdi[2]

[1]Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, Iran
[2]Department of Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran

Let X be an entropy minimal synchronized system and $\varphi : X \to Y$ a factor code. We show that Y is synchronized whenever φ is entropy preserving. With this property, entropy preserving is equivalent to having a degree. Moreover, entropy minimality is equivalent to X being intrinsically ergodic of full support and in this situation, the entropy of X is identical with the synchronized entropy of X.

Keywords: shift of finite type, sofic, coded, entropy, degree

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Code: cdsgt3-00570013

[a]Speaker. Email address: s.jangjoo@alzahra.ac.ir.

Introduction

Entropy minimality was introduced by Coven and Smtal [2] as a property of dynamical systems which is stronger than topological transitivity and weaker than minimality. There has been some recent work which describes some conditions which are equivalent to entropy minimality for shifts of finite type. Our goal of this note is to look for entropy minimality among the synchronized systems which are a well-known subclass of coded systems.

As any other topological dynamical system, the study of possible measures preserved by the shift map is of interest in coded systems. In particular, the investigation for the existence and uniqueness of a measure of maximal entropy has a long history and those systems with this unique invariant measure are called *intrinsically ergodic*. This measure, if exists, is the most natural measure on subshifts and is the main tool for studying their statistical properties.

Parry [5] established intrinsically ergodic for topologically transitive shifts of finite type and all their subshift factors (sofic shifts) and Bowen [1] proved for shifts with specification property. We extend their results and will show that a synchronized system (X, σ) with positive entropy is intrinsically ergodic of full support if and only if it is entropy minimal.

In Theorem 22.2, we will prove that entropy minimality is a property invariant by entropy preserving factor codes. Also, we will show that if X is an entropy minimal synchronized system, then any factor of X by an entropy preserving factor code will be synchronized (Theorem 22.3). Furthermore, for entropy minimal synchronized systems, entropy preserving is equivalent to having a degree (Theorem 22.5).

Finally, for a synchronized system X with the underlying graph G for its Fischer cover, we show that $h(G) = h_{\text{syn}}(X)$ and if X is entropy minimal or equivalently if X is intrinsically ergodic of full support, then $h(X) = h_{\text{syn}}(X)$ [Theorem 22.8].

Background and Notations

The notations has been borrowed from [4] and a brief reminder of the main definitions of symbolic dynamics has been brought here. Let $\mathcal{A}$ be a finite set of alphabet. A *full $\mathcal{A}$-shift* is defined as $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ where the *shift map* σ is defined by $\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$. Any closed invariant set X of $\mathcal{A}^{\mathbb{Z}}$ is called a *subshift* or a *shift space*. Let $\mathcal{B}_n(X)$ denote the set of all admissible n words, i.e. words of length n and set $\mathcal{B}(X) := \cup_{n\in\mathbb{N}}\mathcal{B}_n(X)$ to be the language of X.

Recall the definition of an $(m+n+1)$-*block map* from [4, §1.5]; however, without loss of generality we will use only one-block maps ($m = n = 0$) which induces a map φ called *code*. Thus if φ is a code from X into another shift space, then $\varphi(\cdots x_{-1}x_0x_1\cdots) = (\cdots\Phi(x_{-1})\Phi(x_0)\Phi(x_1)\cdots)$ where Φ is the $(m + n + 1)$-block map.

A *factor code* is an onto code. A code $\varphi : X \to Y$ is finite-to-one if there is an integer M such that $|\varphi^{-1}(y)| \leq M$ for every $y \in Y$. A point x in a shift space X is doubly transitive if every word in X appears in x infinitely many often to the left and to the right. We denote by $D(X)$ the set of doubly transitive points of X.

A shift space X is *irreducible* if for every pair of words $u, v \in \mathcal{B}(X)$ there is a word $w \in \mathcal{B}(X)$ so that $uwv \in \mathcal{B}(X)$. A word $v \in \mathcal{B}(X)$ is *synchronizing* if whenever $uv, vw \in \mathcal{B}(X)$, then $uvw \in \mathcal{B}(X)$.

Let G be a graph with edge set $\mathcal{E}$. The *edge shift* X_G is the shift space specified by $X_G = \{\xi = (\xi_i)_{i\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : t(\xi_i) = i(\xi_{i+1}) \text{ for all } i \in \mathbb{Z}\}$. A *labeled graph* $\mathcal{G}$ is a pair $(G, \mathcal{L})$ where G is a graph with the labeling $\mathcal{L} : \mathcal{E} \to \mathcal{A}$. Let $\mathcal{L}_\infty(\xi)$ be the label of a bi-infinite path $\xi \in X_G$. Set $X_{\mathcal{G}} := \{\mathcal{L}_\infty(\xi) : \xi \in X_G\}$ which will be denoted by $\mathcal{L}_\infty(X_G)$ as well. If there is a graph G such that $X = \overline{X_{\mathcal{G}}}$, then we say $\mathcal{G}$ is a *presentation* (or *cover*) of $X_{\mathcal{G}}$.

A shift space is *sofic* if there is a finite graph G such that $X = X_{\mathcal{G}}$. Equivalently X is sofic, if it is a factor of an SFT, or shift of finite type (shifts characterized by a set of finite forbidden words). A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is *right-resolving* if for any vertex I of G and any symbol $a \in \mathcal{A}$, there is at most one edge labeled by a and going out of I. A *minimal right-resolving cover* (or *Fischer cover*) of a sofic shift X is a right-resolving cover of X having the fewest vertices among all right-resolving covers of X.

The entropy of a subshift X is defined by $h(X) = \lim_{n\to\infty}(1/n)\log|\mathcal{B}_n(X)|$. There are some other entropies which will be used in this note. One is the entropy related to graphs given by Gurevich and that is defined as follows. Let G be a connected oriented graph. Then for any vertices I, J

$$h(G) = \lim_{n\to\infty}\frac{1}{n}\log B_{IJ}(n) \tag{42}$$

where $B_{IJ}(n)$ is the number of paths of length n which starts at I and ends at J.

Main results

22.1. Definition. A topological dynamical system (X, T) is said to be entropy minimal if all closed T-invariant subsets of X have entropy strictly less than (X, T).

First, we investigate some dynamical properties of a subshift satisfying entropy minimality. The next two theorems deals with the application of entropy preserving factor codes in such systems. Also see Theorem 22.5.

22.2. Theorem. *Suppose $\varphi : X \to Y$ is an entropy preserving factor code and assume that X is entropy minimal. Then, Y is entropy minimal as well.*

Proof. Let Z be a proper subshift of Y and assume that $h(Z) = h(Y)$. By surjectivity, $\varphi^{-1}(Z)$ is a proper subshift of X and since X is entropy minimal, $h(\varphi^{-1}(Z)) < h(X)$. Now Z is a factor of $\varphi^{-1}(Z)$ and so $h(Z) \leq h(\varphi^{-1}(Z)) < h(X) = h(Y)$ violating our assumption. □

22.3. Theorem. *Suppose X is an irreducible entropy minimal shift space and let $\varphi : X \to Y$ be an entropy preserving factor code. Then, $\varphi^{-1}(D(Y)) = D(X)$. In particular, if X is synchronized, then φ has a degree and Y is synchronized as well.*

Proof. For the first part a similar result holds for irreducible sofic shifts [4, Lemma 9.1.13]. The main ingredients for the proof of that result is to have X compact, φ entropy preserving and the fact that entropy minimality holds for irreducible sofics [4, Corollary 4.4.9]. All of them are provided here.

The second part is a direct application of [3, Theorem 3.3] and [3, Theorem 4.2]. $\square$

22.4. Theorem. *If X is an entropy minimal synchronized system with Fischer cover $\mathcal{G} = (G, \mathcal{L})$, then $h(X) = h(G)$.*

Proof. One has $h(X) = \max\{h(G), h(\partial X)\}$ [6, Theorem 6.16]. Also, ∂X is a proper subsystem of X, and so $h(X) = h(G)$. $\square$

22.5. Theorem. *Let X be an entropy minimal synchronized system and $\varphi : X \to Y$ a factor code. Then, φ is entropy preserving if and only if it has a degree.*

Proof. By Theorem 22.3 necessity is at hand, so we prove sufficiency. Assume φ has a degree. Since X is entropy minimal, $h(X) = h(G_X)$ where G_X is the underlying graph of Fischer cover of X (Theorem 22.4). So, $h(X) = h(G_X) = h(G_Y) = h(Y)$. $\square$

22.6. Theorem. *Suppose X is a subshift with positive topological entropy. Then, any invariant measure on X with maximal entropy is of full support if and only if X is entropy minimal.*

Proof. First let μ_X be the invariant measure on X with maximal entropy of full support; so $h(X) = h_{\mu_X}$. Suppose Y is a proper subsystem of X and $h(Y) = h(X)$. By the variational principle

$$h(Y) = \sup\{h_\nu : \nu \in \mathcal{M}(Y, \sigma)\} \tag{43}$$

where $\mathcal{M}(Y, \sigma)$ is the set of all invariant measures. The shift map σ is expansive and so there is a measure ν_Y with $h(Y) = h_{\nu_Y}$. Set $\nu_X(A) = \nu_Y(A \cap Y)$ for $A \in \mathcal{M}(X)$ and notice that ν_X is an invariant measure on X vanishing at the open set $X \setminus Y$. Now by a direct verification, $h_{\nu_X} = h_{\nu_Y} = h(Y) = h(X)$ which is absurd by the hypothesis.

For the converse assume that X is entropy minimal and let μ be the measure with maximal entropy which is not of full support. Then there has to be a cylinder $[u] \subset X$ with $\mu([u]) = 0$. Now $Y = X \setminus \cup_{i=-\infty}^{\infty} \sigma^{-i}([u])$ is a closed invariant subset of X and in fact a proper subsystem of X with $\mu(Y) = \mu(X)$. Restrict μ to Y and call it μ_Y. Then, $h_{\mu_Y}(Y) = h_\mu(X) = h(X)$ and this in turn by applying (43) implies that $h(Y) = h(X)$ which violates our assumption. $\square$

22.7. Theorem. *Let X be a subshift with positive topological entropy. If X is intrinsically ergodic of full support, then X is entropy minimal. The converse is true whenever X is synchronized.*

Proof. Apply Theorem 22.6 and the fact that a maximal measure of full support for synchronized systems is unique [6]. $\square$

Let X be synchronized and fix a synchronizing word $\alpha \in \mathcal{B}(X)$. Let $C_n(\alpha)$ be the set of words $v \in \mathcal{B}_n(X)$ such that $\alpha v \alpha \in \mathcal{B}(X)$. Then the *synchronized entropy* $h_{\mathrm{syn}}(X)$ is defined by

$$h_{\mathrm{syn}}(X) = \limsup_{n\to\infty} \frac{1}{n} \log |C_n(\alpha)|.$$

This value is independent of α and $h(X) \geq h_{\mathrm{syn}}(X)$. In general, $h(X) \neq h_{\mathrm{syn}}(X)$; however, Thomsen showed that for irreducible sofic shifts $h(X) = h_{\mathrm{syn}}(X)$. Later Jung extended this result to SVGL shifts.

22.8. Theorem. *Let X be a synchronized system with Fischer cover $\mathcal{G} = (G, \mathcal{L})$. Then $h(G) = h_{\mathrm{syn}}(X)$. In particular, when X is entropy minimal or equivalently if X is intrinsically ergodic of full support, then $h(X) = h_{\mathrm{syn}}(X)$.*

Proof. First we show that $h(G) \leq h_{\text{syn}}(X)$. Let $\alpha \in \mathcal{B}(X)$ be a synchronizing word. Then all elements of $\mathcal{L}^{-1}(\alpha)$ have the same terminal vertex, say I and suppose L_n denote the set of cycles in X_G of length n starting and terminating at I. Set $\pi \in L_n(I)$ and $\mathcal{L}(\pi) = v$. Also, let $\min\{|w| : \alpha w \alpha \in \mathcal{B}(X)\} = k$ and $w' \in C_k(\alpha)$. Then $vw' \in C_{n+k}(\alpha)$. Since $\mathcal{L}_\infty$ is right-resolving,

$$\mathcal{L} : L_n(I) \longrightarrow C_{n+k}(\alpha)$$

is injective. So,

$$\limsup_{n\to\infty} \frac{1}{n} \log L_n(I) \leq \limsup_{n\to\infty} \frac{1}{n} \log C_{n+k}(\alpha) = h_{\text{syn}}(X).$$

By (42), $\limsup_{n\to\infty} \frac{1}{n} \log L_n(I) = h(G)$. So $h(G) \leq h_{\text{syn}}(X)$.

The converse is quite similar! Let $|\alpha| = l$ and $v \in C_n(\alpha)$ with $\pi \in \mathcal{L}^{-1}(v)$. Since all elements of $\mathcal{L}^{-1}(\alpha)$ have the same terminal vertex I, then for some $\pi' \in \mathcal{L}^{-1}(\alpha)$, $\pi\pi'$ is a cycle in G starting and terminating at I. So

$$h_{\text{syn}}(X) = \limsup_{n\to\infty} \frac{1}{n} \log C_n(\alpha) \leq \limsup_{n\to\infty} \frac{1}{n} \log L_{n+l}(I).$$

Now suppose X is entropy minimal. Then by Theorem 22.4, $h(X) = h(G)$ and hence by above result, $h(X) = h_{\text{syn}}(X)$.

$\square$

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23. On Sannon entropy bounds

Yamin Sayyari

Entropy, has many applications in thermodynamics, code theory, physics, statistics and information theory. In this paper, we present some new and interesting results related to the bounds of the Shannon entropy.

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Introduction

Entropy plays an important role in many areas of mathematics, probability and physics. Shannon's entropy, as metric entropy, is in general difficult to calculate and even to estimate. See [1] for other methods to estimate the Shannon entropy and [2, 4, 5, 6] for a review on entropy estimation. In [9, 11], the authors presented some bounds for the classical Shannon's entropy. The results of this paper improve the results in [3, 7, 8, 10, 11].

Basic notions

Let $p_1, ..., p_n$ be a positive weight sequence with $\sum_{i=1}^n p_i = 1$, and let $\overline{x} = \{x_1, ..., x_n\} \subseteq I := [a, b]$ be a sequence. The well-known Jensen's inequality states that: If f is convex on I, then $\sum_{i=1}^n p_i f(x_i) - f(\sum_{i=1}^n p_i x_i) \geq 0$. The sum $\sum_{i=1}^n p_i x_i$ is called the convex combination of x_i.

23.1. Lemma. [2] *Let f be a differentiable convex mapping. Then*

$$0 \leq \sum_{i=1}^n p_i f(x_i) - f(\sum_{i=1}^n p_i x_i) \leq \frac{1}{4}(b-a)(f'(b) - f'(a)) := D_f(a, b). \tag{44}$$

Dragomir's result (44), implies $0 \leq \log n - H(X) \leq \frac{(\nu-\mu)^2}{4\mu\nu} := D(\mu, \nu)$.

23.2. Proposition. [10] *For $\mu := \min_{1\leq i\leq n}\{p_i\}$ and $\nu := \max_{1\leq i\leq n}\{p_i\}$, have*

$$\begin{aligned} m(\mu, \nu) := \mu \log(\frac{2\mu}{\mu+\nu}) + \nu \log(\frac{2\nu}{\mu+\nu}) &\leq \log n - H(X) \\ &\leq \log(\frac{(\mu+\nu)^2}{4\mu\nu}) := M(\mu, \nu). \end{aligned} \tag{45}$$

23.3. Proposition. [10] *Under the notation of Proposition 23.2, have*

$$m(\mu, \nu) \leq \log n - H(X) \leq n m(\mu, \nu). \tag{46}$$

23.4. PROPOSITION. [7] *Under the notation of Proposition 23.2, have*

$$\tilde{m}(\mu,\nu) \leq \log n - H(X) \leq \tilde{M}(\mu,\nu), \tag{47}$$

where $\tilde{m}(\mu,\nu) := m(\mu,\nu) + \frac{\mu^2(2-n\mu-n\nu)^2}{2(\mu+\nu)(1-\mu-\nu)}$, *and*

$$\tilde{M}(\mu,\nu) := M(\mu,\nu) - \frac{(\mu+\nu-2n\mu\nu)^2 + 2\mu\nu(1-\mu n)(\nu n-1)}{4\nu^2}.$$

23.5. PROPOSITION. [7] *Let* $\mu := \min_{1\leq i\leq n}\{p_i\}$ *and* $\nu := \max_{1\leq i\leq n}\{p_i\}$. *Then*

$$\overline{m}(\mu,\nu) \leq \log n - H(X) \leq \overline{M}(\mu,\nu), \tag{48}$$

where $\overline{m}(\mu,\nu) := m(\mu,\nu) + \frac{(2-n\mu-n\nu)^2}{4\nu n(n-2)}$ *and*

$$\overline{M}(\mu,\nu) := nm(\mu,\nu) - \frac{(2-n\mu-n\nu)^2 + 2(n\nu-1)(1-n\mu)}{4\nu n}.$$

23.6. THEOREM. [11] *If* $X = \{p_i\}_{i=1}^n$ *is a positive probability distribution, then*

$$H(X) \leq \log n - \max_{1\leq\mu_1<\ldots<\mu_{n-1}\leq n}\{log([(\frac{n-1}{\sum_{i=1}^{n-1} p_{\mu_k}})^{\sum_{k=1}^{n-1} p_{\mu_k}}][\prod_{k=1}^{n-1} p_{\mu_k}^{p_{\mu_k}}])\}.$$

23.7. THEOREM. [3] *Let* $X = \{p_i\}_{i=1}^n$ *be a positive probability distribution and* $\mu = (\mu_1, .., \mu_n)$. *Then*

$$H(X) \leq \log(n) - \frac{1}{n}\sum_{i=1}^n (e^{1-np_i} - 1) - \max_{1\leq\mu_1<\ldots<\mu_{n-1}\leq n}\{F(\mu) + G(\mu)\},$$

where

$$F(\mu) = log([(\frac{n-1}{\sum_{i=1}^{n-1} p_{\mu_k}})^{\sum_{k=1}^{n-1} p_{\mu_k}}][\prod_{k=1}^{n-1} p_{\mu_k}^{p_{\mu_k}}]),$$

$$G(\mu) = \frac{n-1}{n}(e^{1-\frac{n}{n-1}\sum_{i=1}^{n-1} p_{\mu_i}} - 1) - \frac{1}{n}\sum_{i=1}^{n-1}(e^{1-np_{\mu_i}} - 1).$$

Main results

In this section we obtain new upper bounds for Shannons entropy of a positive probability distribution.

23.8. THEOREM. *Let* $X = \{p_1, ..., p_n\}$ *be a positive probability distribution and* $\mu_0 := \min_{1\leq i\leq n}\{p_i\}$, *then*

$$H(X) \leq$$

$$\log n - \max_{2\leq i\leq n-1}\{\max_{1\leq\mu_1<\ldots<\mu_i\leq n}\{F_i(\mu)\exp(\frac{\mu_0^2(n\sum_{k=1}^i p_{\mu_k} - i)^2}{2(1-\sum_{k=1}^i p_{\mu_k})(\sum_{k=1}^i p_{\mu_k})})\}\}.$$

where

$$F_i(\mu) := \log([(\frac{i}{\sum_{k=1}^i p_{\mu_k}})^{\sum_{k=1}^i p_{\mu_k}}][\prod_{k=1}^i p_{\mu_k}^{p_{\mu_k}}].$$

$2 \leq i \leq n-1$ *and* $\mu = (\mu_1, .., \mu_n)$.

23.9. Corollary. *Let $X = \{p_1, ..., p_n\}$ be a positive probability distribution and $\mu_0 := \min_{1\leq i\leq n}\{p_i\}$, then*

$$H(X) \leq \log n - \max_{1\leq \mu_1<\ldots<\mu_{n-1}\leq n}\{F(\mu)\exp(\frac{\mu_0^2(n\sum_{k=1}^{n-1}p_{\mu_k} - n + 1)^2}{2(1-\sum_{k=1}^{n-1}p_{\mu_k})(\sum_{k=1}^{n-1}p_{\mu_k})})\}.$$

where

$$F(\mu) := \log([(\frac{n-1}{\sum_{k=1}^{n-1}p_{\mu_k}})^{\sum_{k=1}^{n-1}p_{\mu_k}}][\prod_{k=1}^{n-1}p_{\mu_k}^{p_{\mu_k}}]$$

and $\mu = (\mu_1, .., \mu_n)$.

Note that, the estimation in Corollary 23.9 is better than the estimation in Theorem 23.6.

23.10. Theorem. *Let $X = \{p_1, ..., p_n\}$ be a positive probability distribution. Let $\mu := \min_{1\leq i\leq n}\{p_i\} = p_\alpha$, $\mu_1 := \min_i\{p_i : i \neq \alpha\}$, $\nu := \max_{1\leq i\leq n}\{p_i\} = p_\beta$ and $\nu_1 := \max\{p_i : i \neq \beta\}$. Then*

$$0 \leq \log n - H(X) \leq \tilde{M}_1(\mu, \nu, \mu_1, \nu_1), \tag{49}$$

where $\tilde{M}_1(\mu, \nu, \mu_1, \nu_1) := \tilde{M}(\mu, \nu) - \frac{\mu^3(\nu_1-\mu_1)^2}{(1-\mu-\nu)2\nu_1^2\mu_1}$.

23.11. Remark. Since $\tilde{M}_1(\mu, \nu, \mu_1, \nu_1) \leq \tilde{M}(\mu, \nu) \leq M(\mu, \nu) \leq D(\mu, \nu)$, the estimation (49) is better than (47) and (45).

23.12. Theorem. *Let $X = \{p_1, ..., p_n\}$ be a positive probability distribution. Let $\mu := \min_{1\leq i\leq n}\{p_i\} = p_\alpha$, $\mu_1 := \min_i\{p_i : i \neq \alpha\}$, $\nu := \max_{1\leq i\leq n}\{p_i\} = p_\beta$ and $\nu_1 := \max\{p_i : i \neq \beta\}$. Then*

$$0 \leq \log n - H(X) \leq \overline{M}_1(\mu, \nu, \mu_1, \nu_1), \tag{50}$$

where $\overline{M}_1(\mu, \nu, \mu_1, \nu_1) := \overline{M}(\mu, \nu) - \frac{\mu\mu_1(\nu_1-\mu_1)^2}{2\nu(1-\mu-\nu)}$.

23.13. Remark. Since $\overline{M}_1(\mu, \nu, \mu_1, \nu_1) \leq \overline{M}(\mu, \nu) \leq nm(\mu, \nu)$, the estimation (50) is better than (48) and (46).

23.14. Example. Let $n = 10^k, \mu = 10^{-k-1}$, $\nu = 10^{-k+1}(k > 2)$ and

$$X = \{10^{-k-1}, 10^{-k-1}, x_3, x_4, ..., x_{10^k-2}, 10^{-k+1}, 10^{-k+1}\}.$$

Then $M(\mu, \nu) \simeq 1.406$, $\tilde{M}(\mu, \nu) \simeq 1.202058$. Since,

$$\begin{aligned}\tilde{M}_1(\mu, \nu, \mu_1, \nu_1) &= \tilde{M}(\mu, \nu) - \frac{\mu^3(\nu_1-\mu_1)^2}{(1-\mu-\nu)2\nu_1^2\mu_1}\\ &= 1.202058 - \frac{10^{-3k-3}(10^{-k+1} - 10^{-k-1})^2}{(1 - 10^{-k-1} - 10^{-k+1})2 \times 10^{-2k+2}10^{-k-1}}\\ &= 1.202058 - \frac{9.99^2}{2\times 10^4} \times \frac{10^{-2k}}{1 - 10^{-k+1} - 10^{-k-1}}\\ &\leq 1.202058 - \frac{9.99^2}{2\times 10^4} \times 3 \times 10^{-2k}\\ &= 1.202058 - 0/0149 \times 10^{-2k},\end{aligned}$$

$$0 \leq \log n - H(X) \leq 1.202058 - 14/9 \times 10^{-2k-3}.$$

23.15. Example. Let $n = 100^k, \mu = 100^{-k-1}$, $\nu = 100^{-k+1}(k > 2)$ and

$$X = \{100^{-k-1}, 100^{-k-1}, x_3, x_4, ..., x_{100^k-2}, 100^{-k+1}, 100^{-k+1}\}.$$

Then

$$nm(\mu, \nu) - \overline{M}(\mu, \nu) \simeq 24.5049.$$

Also,

$$\begin{aligned}
\overline{M}_1(\mu,\nu,\mu_1,\nu_1) &= \overline{M}(\mu,\nu) - \frac{\mu\mu_1(\nu_1-\mu_1)^2}{2\nu(1-\mu-\nu)} \\
&= \overline{M}(\mu,\nu) - \frac{100^{-k-1}\times 100^{-k-1}(100^{-k+1}-100^{-k-1})^2}{2100^{-k+1}(1-100^{-k-1}-100^{-k+1})} \\
&= \overline{M}(\mu,\nu) - \frac{99.99^2}{2\times 100^3}\times\frac{100^{-2k}}{1-100^{-k+1}-100^{-k-1}} \\
&\leq \overline{M}(\mu,\nu) - \frac{99.99^2}{2\times 100^3}\times 3\times 100^{-2k} \\
&= \overline{M}(\mu,\nu) - 149/9\times 100^{-2k-2}.
\end{aligned}$$

So, $0 \leq \log n - H(X) \leq \overline{M}(\mu,\nu) - 149/9\times 100^{-2k-2}$.

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24. The entropy of relative dynamical systems having countably many atoms

Uosef Mohammadi[a] ,
Department of Mathematics and Faculty of Science, University of Jiroft, Jiroft, Iran

In this paper, in order to develop a mathematical model underlying uncertainty and fuzziness in a dynamical system, which is called relative mathematical modeling, we are going to apply the notion of observer. First, by using a mathematical model of a one dimensional observer, the notion of relative entropy for a relative dynamical system having countably many atoms is considered. Also, some ergodic properties of relative dynamical systems are investigated. At the end, a new version of Kolmogorov-Sinai theorem for a relative dynamical system having countably many atoms is given.

Keywords: Entropy, observer, relative mathematical modeling, relative dynamical system
AMS Mathematics Subject Classification [2020]: 37A35, 28D05, 28E05
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[a]Uosef Mohammadi. Email address: u.mohamadi@ujiroft.ac.ir

Introduction

Entropy is applicable and useful in studying the behavior of stochastic processes since it represents the ambiguity and disorder of the processes without being restricted to the forms of the theoretical probability distributions. Different entropy measures have been studied and presented including Shannon entropy, Renyi entropy, Tsallis entropy, Sample entropy, Permutation entropy, Approximate entropy, and Transfer entropy. Since in mathematical modeling of physical systems the role of observer is important, so a method is needed to measure the entropy of a system from the point of view of an observer. Any mathematical model according to the view point of an observer is called a relative model [6, 7]. The notion of a relative dynamical system as a generalization of a fuzzy dynamical system has been defined in [7]. Also, the concept of entropy of a relative dynamical system has been introduced in [6, 7]. This article is an attempt to present a new approach to the entropy of relative dynamical systems having countably many atoms.

Basic Notions

This section is devoted to provide some basic notions of relative structures. A modeling for an observer of a set X is a fuzzy set $\Theta : X \to [0,1]$ [6]. In fact this kinds of fuzzy sets are called " one dimentional observes". The idea is based on the relation between "experiance " and "information" from the view point of an observer. Let Θ be an observer on X, then we say $\lambda \subseteq \Theta$ if $\lambda(x) \leq \Theta(x)$ for all $x \in X$. Moreover, if $\lambda_1, \lambda_2 \subseteq \Theta$ then $\lambda_1 \vee \lambda_2$ and $\lambda_1 \wedge \lambda_2$ are subsets of Θ, and defined by

$$(\lambda_1 \vee \lambda_2)(x) = \sup\{\lambda_1(x), \lambda_2(x)\},$$

and

$$(\lambda_1 \wedge \lambda_2)(x) = \inf\{\lambda_1(x), \lambda_2(x)\},$$

where $x \in X$.

24.1. DEFINITION. A collection F_Θ of subsets of Θ is said to be a σ_Θ-algebra in Θ if F_Θ satisfies the following conditions [6],

(i) $\Theta \in F_\Theta$,
(ii) $\lambda \in F_\Theta$ then $\lambda' = \Theta - \lambda \in F_\Theta$. λ' is the complement of λ with respect to Θ,
(iii) if $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence in F_Θ then $\vee_{i=1}^{\infty}\lambda_i = \sup_i \lambda_i \in F_\Theta$,
(iv) $\dfrac{\Theta}{2}$ doesn't belong to F_Θ.

If P_1 and P_2 are σ_Θ-algebras on X then $P_1 \vee P_2$ is the smallest σ_Θ-algebra that contains $P_1 \cup P_2$, denoted by $[P_1 \cup P_2]$.

24.2. DEFINITION. A positive $\Theta-$measure m_Θ over F_Θ is a function $m_\Theta : F_\Theta \to I$ which is countably additive. This means that if λ_i is a disjoint countable collection of members of F_Θ, (i.e. $\lambda_i \subseteq \lambda_j' = \Theta - \lambda_j$ whenever $i \neq j$) then

$$m_\Theta(\vee_{i=1}^{\infty}\lambda_i) = \sum_{i=1}^{\infty} m_\Theta(\lambda_i).$$

The $\Theta-$measure m_Θ has the following properties [6],

(i) $m_\Theta(\chi_\emptyset) = 0$,
(ii) $m_\Theta(\lambda' \vee \lambda) = m_\Theta(\Theta)$ and $m_\Theta(\lambda') = m_\Theta(\Theta) - m_\Theta(\lambda)$ for all $\lambda \in F_\Theta$,
(iii) $m_\Theta(\lambda \vee \mu) + m_\Theta(\lambda \wedge \mu) = m_\Theta(\lambda) + m_\Theta(\mu)$ for each $\lambda, \mu \in F_\Theta$,
(iv) m_Θ is a nondecreasing function i.e. if $\lambda, \eta \in F_\Theta$ and $\lambda \subseteq \Theta$, then $m_\Theta(\lambda) \leq m_\Theta(\eta)$.

The triple (X, F_Θ, m_Θ) is called a $\Theta-$ measure space and the elements of F_Θ are called relative measurable sets. The $\Theta-$ measure space, (X, F_Θ, m_Θ), is called a relative probability $\Theta-$measure space if $m_\Theta(\Theta) = 1$ [6].

24.3. EXAMPLE. Let (X, β, p) be a classical probability measure space and $\Theta = \chi_X$. Then $F_\Theta = \{\chi_A : A \in \beta\}$ is a σ_Θ-algebra on X. Define $m_\Theta(\chi_A) = p(A)$, $A \in \beta$. Then (X, F_Θ, m_Θ) is a relative probability $\Theta-$ measure space.

24.4. DEFINITION. Let (X, F_Θ, m) be a $\Theta-$measure space, the elements μ,λ of F_Θ are called m_Θ-disjoint if $m_\Theta(\lambda \wedge \mu) = 0$.

A $\Theta-$relation '=(mod m_Θ)' on F_Θ is defined as bellow

$$\lambda = \mu (\mathrm{mod}\, m_\Theta) \quad \text{iff} \quad m_\Theta(\lambda) = m_\Theta(\mu) = m_\Theta(\lambda \wedge \mu),$$

for each $\lambda, \mu \in F_\Theta$.
$\Theta-$relation '=(mod m_Θ)' is an equivalence relation. $\tilde{F_\Theta}$ denotes the set of all equivalence classes induced by this relation, and $\tilde{\mu}$ is the equivalence class determined by μ. For $\lambda, \mu \in F_\Theta$, $\lambda \wedge \mu = 0$ (mod m_Θ) iff λ, μ are m_Θ-disjoint. We shall identify $\tilde{\mu}$ with μ.

24.5. DEFINITION. Let (X, F_Θ, m_Θ) be a $\Theta-$measure space, and P be a sub-σ_Θ-algebra of F_Θ. Then an element $\tilde{\lambda} \in \tilde{P}$ is an atom of P if

(i) $m_\Theta(\lambda) > 0$,
(ii) for each $\tilde{\mu} \in \tilde{P}$such that $m_\Theta(\lambda \wedge \mu) = m_\Theta(\mu) \neq m_\Theta(\lambda)$ then $m_\Theta(\mu) = 0$.

24.6. THEOREM. *Let (X, F_Θ, m_Θ) be a $\Theta-$measure space, and P be a sub-σ_Θ-algebra of F_Θ. If $\tilde{\lambda_1}$,$\tilde{\lambda_2}$ are disjoint atoms of P then they are m_Θ-disjoint.*

Entropy of a sub-σ_Θ-algebra with countable atoms

In this section we introduce the notion of entropy of a sub-σ_Θ-algebra with countable atoms. At the following, the set of all sub-σ_Θ-algebra of F_Θ with countable atoms is denoted by $R^*(F_\Theta)$. Assume that F_Θ is a σ_Θ-algebra and $P_1, P_2 \in R^*(F_\Theta)$, and $\{\lambda_i; i \in \mathbb{N}\}$ and $\{\mu_j; j \in \mathbb{N}\}$ denote the atoms of P_1 and P_2 respectively, then the atoms of $P_1 \vee P_2$ are $\lambda_i \wedge \mu_j$ which $m_\Theta(\lambda_i \wedge \mu_j) > 0$ for each $i, j \in \mathbb{N}$.
If $\gamma \in \bar{F_\Theta}$ we set

$$P_1 \vee \gamma = \{\lambda_i \wedge \gamma; m_\Theta(\lambda_i \wedge \gamma) > 0, i \in \mathbb{N}\}.$$

24.7. Theorem. *Let $\{\lambda_i; i \in \mathbb{N}\}$ be a m_Θ-disjoint collection of relative measurable sets of relative probability Θ-measure space (X, F_Θ, m_Θ), then,*

$$m_\Theta(\vee_{i=1}^{\infty}(\lambda_i)) = \sum_{i=1}^{\infty} m_\Theta(\lambda_i).$$

24.8. Definition. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space and $P_1, P_2 \in R^*(F_\Theta)$. We say that P_2 is an m_Θ-refinement of P_1, denoted by $P_1 \leq_{m_\Theta} P_2$, if for each $\mu \in \bar{P_2}$ there exists $\lambda \in \bar{P_1}$ such that,

$$m_\Theta(\lambda \wedge \mu) = m_\Theta(\mu).$$

24.9. Theorem. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space and $P_1, P_2, P_3 \in R^*(F_\Theta)$ if $P_1 \leq_{m_\Theta} P_2$ then,*

$$P_1 \vee P_3 \leq_{m_\Theta} P_2 \vee P_3.$$

24.10. Definition. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and P be a sub σ_Θ-algebra of F_Θ which $P \in R^*(F_\Theta)$, the entropy of P is defined as

$$H_\Theta(P) = -\log \sup_{i \in \mathbb{N}} m_\Theta(\mu_i),$$

where $\{\mu_i; i \in \mathbb{N}\}$ are atoms of P.

24.11. Definition. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space and $P \in R^*(F_\Theta)$. The conditional entropy of P given $\gamma \in \bar{F_\Theta}$ is defined by

$$H_\Theta(P|\gamma) = -\log \sup_{i \in \mathbb{N}} m_\Theta(\mu_i|\gamma),$$

where,

$$m_\Theta(\mu_i|\gamma) = \frac{m_\Theta(\mu_i \wedge \gamma)}{m_\Theta(\gamma)} \qquad (m_\Theta(\gamma) \neq 0).$$

24.12. Theorem. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2 \in R^*(F_\Theta)$ which $\bar{P_1} = \{\lambda_i; i \in \mathbb{N}\}$ and $\bar{P_2} = \{\mu_j; j \in \mathbb{N}\}$. Then,*

(i) $P_1 \leq_{m_\Theta} P_2 \Rightarrow H_\Theta(P_1) \leq H_\Theta(P_2)$,
(ii) $P_1 \leq_{m_\Theta} P_2 \Rightarrow H_\Theta(P_1|\gamma) \leq H_\Theta(P_2|\gamma)$.

24.13. Definition. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space and $P_1, P_2 \in R^*(F_\Theta)$. We say that P_1 and P_2 are m_Θ-equivalent, denoted by $P_1 \approx_{m_\Theta} P_2$, if the following axioms are satisfied:

(i) If $\lambda \in \bar{P_1}$ then $m_\Theta(\lambda \wedge (\vee\{\mu; \mu \in \bar{P_2}\})) = m_\Theta(\lambda)$.
(ii) If $\mu \in \bar{P_2}$ then $m_\Theta(\mu \wedge (\vee\{\lambda; \lambda \in \bar{P_1}\})) = m_\Theta(\mu)$.

24.14. Theorem. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2 \in R^*(F_\Theta)$. If $P_1 \approx_{m_\Theta} P_2$ then,*

$$P_1 \approx_{m_\Theta} P_1 \vee P_2.$$

24.15. Theorem. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2 \in R^*(F_\Theta)$. If $P_1 \approx_{m_\Theta} P_2$ then,*

$$H_\Theta(P_1) \leq H_\Theta(P_1 \vee P_2).$$

24.16. Definition. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space and $P \in R^*(F_\Theta)$. The diameter of P is defined as follows

$$\mathrm{diam}P = \sup_{\lambda_i \in \bar{P}} m_\Theta(\lambda_i).$$

24.17. Definition. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space,and $P_1, P_2 \in R^*(F_\Theta)$, which $\bar{P_1} = \{\lambda_i; i \in \mathbb{N}\}$, $\bar{P_2} = \{\gamma_k; k \in \mathbb{N}\}$. The conditional entropy of P_1 given P_2 is defined as

$$\begin{aligned} H_\Theta(P_1|P_2) &= -\log\sup_{i\in\mathbb{N}} \frac{\mathrm{diam}(\lambda_i \vee P_2)}{\mathrm{diam}P_2} \\ &= -\log\sup_{j\in\mathbb{N}} \frac{\mathrm{diam}(P_1 \vee \mu_j)}{\mathrm{diam}P_2}. \end{aligned}$$

24.18. Theorem. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2, P_3 \in R^*(F_\Theta)$. Then,*

(i) $P_2 \leq_{m_\Theta} P_3 \Rightarrow H_\Theta(P_1|P_2) \leq H_\Theta(P_1 \vee P_3)$,
(ii) $H_\Theta(P_1|P_2) \leq H_\Theta(P_1 \vee P_2)$.

24.19. Theorem. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2, P_3 \in R^*(F_\Theta)$. If $P_1 \leq_{m_\Theta} P_2$ then,*

$$H_\Theta(P_1|P_3) \leq H_\Theta(P_2|P_3).$$

Entropy of a relative dynamical system having countably many atoms

24.20. Definition. Suppose (X, F_Θ, m_Θ) be a $\Theta-$measure space and Θ be a constant observer on X. A transformation $\varphi : (X, F_\Theta, m_\Theta) \to (X, F_\Theta, n_\Theta)$, is said to be a $\Theta-$measure preserving if $m_\Theta(\varphi^{-1}(\mu)) = n_\Theta(\mu) \quad$ for all $\mu \in \bar{F_\Theta}$.

24.21. Theorem. *Suppose $\varphi : (X, F_\Theta, m_\Theta) \to (X, F_\Theta, n_\Theta)$, be a $\Theta-$measure preserving transformation. Then for each $P \in R^*(F_\Theta)$ we have,*

$$H_\Theta(P) = H_\Theta(\varphi^{-1}(P)).$$

24.22. Definition. Suppose $\varphi : (X, F_\Theta, m_\Theta) \to (X, F_\Theta, n_\Theta)$, be a $\Theta-$measure preserving transformation. If $P \in R^*(F_\Theta)$, we define the entropy of φ with respect to P as:

$$h_\Theta(\varphi, P) = \lim_{n\to\infty} \frac{1}{n} H_\Theta(\vee_{i=0}^{n-1} \varphi^{-i}(P)).$$

24.23. Theorem. *Let $\varphi : (X, F_\Theta, m_\Theta) \to (X, F_\Theta, n_\Theta)$, be a $\Theta-$measure preserving transformation and $P \in R^*(F_\Theta)$. Then,*

(i) $h_\Theta(\varphi, \varphi^{-1}(P)) = h_\Theta(\varphi, P)$,
(ii) $h_\Theta(\varphi, \vee_{i=0}^{r-1} \varphi^{-i}(P)) = h_\Theta(\varphi, P)$ *for every* $r \geq 1$.

24.24. Theorem. *Let $\varphi : (X, F_\Theta, m_\Theta) \to (X, F_\Theta, n_\Theta)$, be a $\Theta-$measure preserving transformation and $P_1, P_2 \in R^*(F_\Theta)$. Then,*

(i) $P_1 \leq_{m_\Theta} P_2 \Rightarrow h_\Theta(\varphi, P_1) \leq h_\Theta(\varphi, P_2)$,
(ii) if $P_1, P_2 \in R^*(F_\Theta)$ *such that* $P_1 \approx_{m_\Theta} P_2$ *then,*

$$\varphi^{-1}(P_1) \approx_{m_\Theta} \varphi^{-1}(P_2).$$

24.25. Definition. The entropy of the relative dynamical system $(X, F_\Theta, m_\Theta, \varphi)$ is the number $h_\Theta(\varphi)$ defined by,

$$h_\Theta(\varphi) = \sup_P h_\Theta(\varphi, P),$$

where the supremum is taken over all sub-σ_Θ-algebras of F_Θ which $P \in R^*(F_\Theta)$.

24.26. Definition. $P \in R^*(F_\Theta)$ is said to be a m_Θ-generator of the relative dynamical system $(X, F_\Theta, m_\Theta, \varphi)$ if there exists an integer $r > 0$ such that,

$$Q \leq_{m_\Theta} \vee_{i=0}^{r} \varphi^{-i} P,$$

for each $Q \in R^*(F_\Theta)$.

24.27. Theorem. *If P is a m_Θ-generator of the relative dynamical system $(X, F_\Theta, m_\Theta, \varphi)$ then,*

$$h_\Theta(\varphi, Q) \leq h_\Theta(\varphi, P),$$

for each $Q \in R^*(F_\Theta)$.

Now we can deduce the following version of Kolmogorov-Sinai theorem for relative dynamical systems having countably many atoms.

24.28. Theorem. *If P is a m_Θ-generator of relative dynamical system $(X, F_\Theta, m_\Theta, \varphi)$ then,*

$$h_\Theta(\varphi) = h_\Theta(\varphi, P).$$

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25. Hypernormed Entropy

Khatereh Ghasemi[1, a], Javad Jamalzadeh [2]
[1]University of Sistan and Baluchestan, Zahedan, Iran
[2]Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

In this paper, we introduce the notion of topological entropy on topological hypernormed hypergroup and provide some interesting examples. So, we obtain the fundamental properies of this entropy such as invariance under conjugation; Invariance under inversion;logarithmic law; Monotonicily for subflows; Continuity for direct limits.

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[a]Speaker. Email address:khatere.ghasemi@pgs.usb.ac.ir,

Introduction

Entropy is a tool to measure the amount of uncertainty in random events.The entropy has been applied in the information theory, physics, computer sciences, statistics, chemistry, biology, sociology, general systems theory and many other fields. The classical approach in the information theory was based on Shannon entropy. Shannon entropy of a probability distribution was studied in. Kolmogorov and Sinai used the Shannon entropy to define the entropy of measurable partitions and then they defined the entropy of dynamical systems. Kolmogorov-Sinai entropy is a useful tool in studying the isomorphism of dynamical systems. Adler, Konheim and McAndrew defined the topological entropy of a continuous self-map of a compact space. So, Bowen extended this notion to uniformly continuous self-maps of metric spaces. The notion of algebraic entropy was studied later by Weiss and Peters. Topological and Algebraic entropy were deeply studied[**1**, **2**, **3**, **4**, **5**]. Recently, Mehrpooya, Sayyari, Molaei proposed other definitions of Algebraic and Shannon entropies on commutative hypergroups.

In this paper, we introduce the notion of entropy on topological hypernormed hypergroup and prove Some interesting examples. So, we obtain the fundamental properties of this entropy such as Invariance under conjugation, Logarithmic Law, Monotonicity for subflows; Continuity for direct limits.

Preliminaries

The notion of hyperstructure, as a generalization of algebraic structure, was introduced by F. Marty at the 8th congress of Scandinavian Mathematicians in 1934 . One of the most important instances of hyperstructures is hypergroupoid. Let H be a nonempty set and $\mathcal{P}^{\star}(H)$ be the set of all non-empty subsets of H. A hyperoperation on H is a mapping $\circ : H \times H \to \mathcal{P}^{\star}(H)$. The pair $(H, \circ)$ is called a hypergroupoid. In the above definition, if A and$B$ are two non-empty subsets of

H, then we define $A \circ B = \bigcup_{a\in A, b\in B} a \circ b; a \in A, b \in B$. A semihypergroup is a hypergroupoid $(H, \circ)$ such that:

$$\forall (a, b, c) \in H^3; a \circ (b \circ c) = (a \circ b) \circ c$$

A hypergroup is a semihypergroup $(H, \circ)$ such that: $\forall a \in H, a \circ H = H \circ a$. This condition is called the reproduction axiom.

A mapping $\varphi : (H, \circ) \to (H, \circ)$ is called good homomorphism if for every $x, y \in H$;

$$f(x \circ y) = f(x) \circ f(y).$$

We denote by $End(H)$ the set of good homomorphism of H.

Algebraic structures which also have a topology are useful in mathematics. In the same direction, some of mathematicians have studied the properties of hypergroupoids endowed with a topology. Ameri and Hoskova have defined and studied τ_u-topological hypergroup. Now we introduce a topology on $\mathcal{P}^\star(H)$.

25.1. Lemma. *Let (H, τ) be a topological space. Then the family is consisting of all sets $V = \{U \in \mathcal{P}^\star(H) : U \subseteq V; V \in \tau\}$, is a basis for a topology τ_u on $\mathcal{P}^\star(H)$.*

25.2. Definition. Let $(H, \circ)$ be a hypergroup and (H, τ) be a topological space. Then, the hyperoperation $\circ$ is said to be continuous if $\circ : H \times H \to \mathcal{P}^\star(H)$ is continuous. Here $H \times H$ and $\mathcal{P}^\star(H)$ are equipped with the product topology and τ_u , respectively.

Let $(H, \circ, \tau, \tau_u)$ be a topological hypergroup. We denote $End_c(H)$ the set of continuous good homomorphism.

Hypernormed entropy on topological hyper normed hypergroup

In this section, Let $(H, \circ, \tau, \tau_u)$ be a topological hypergroup and $\varphi \in End_c(H)$.

A mapping $\nu : \mathcal{P}^\star(H) \to \mathbb{R}^{\geq 0}$ is called hypernormed on H. For any $x \in H$; we define $\nu(x) = \nu(\{x\})$. If ν is a continuous on $(H, \circ, \tau, \tau_u)$, then $(H, \circ, \tau, \tau_u, \nu)$ is called topological hypernormed hypergroup.

25.3. Definition. A mapping $\varphi : (H, \circ, \tau, \tau_u, \nu) \to (H', \circ', \tau', \tau'_u, \nu')$ is called contractive if

$$\nu'(\varphi(A)) \leq \nu(A), \tag{51}$$

for every $A \in \mathcal{P}^\star(H)$.

25.4. Definition. A hypernormed ν on $\mathcal{P}^*(H)$ is called :

(a) subadditive, if $\nu(A \circ B) \leq \nu(A) + \nu(B)$ for every $A, B \in P^*(H)$.
(b) arithmetic, if for $x \in H$ there exists $c_x \in \mathbb{R}$ such that $\nu(x^n) = c_x logn$ for every $n \in \mathbb{N}$.
(c) ν is incresing respect to hyoeroperation such that $\nu(x), \nu(y) \leq \nu(x \circ y)$ for every $x, y \in H$.

Let $(H, \circ, \tau, \tau_u, \nu)$ be a topolpgical hypernormed hypergroup and $\varphi \in End_c(H)$. consider the n-th φ-trajectory of $x \in H$

$$H_n(\varphi, x) = x \circ \varphi(x) \circ \cdots \circ \varphi^{n-1}(x)$$

and let

$$h_n(\varphi, x) = \nu(H_n(\varphi, x)).$$

now, we define

$$h(\varphi, x) = \limsup_n \frac{h_n(\varphi, x)}{n}.$$

25.5. Lemma. *Let $(H, \circ, \tau, \tau_u, \nu)$ be a topological hypernormed hypergroup. Then $h(\varphi, x)$ is finite for every $x \in H$.*

Now, we define hypernormed entropy of $\varphi \in End_c(H)$, where $(H, \circ, \tau, \tau_u, \nu)$ is a topological hypernormed hypergroup,

$$h(\varphi) = \sup\{h(\varphi, x), x \in H\}.$$

In the following, we provide some partical examples of hypernormed entropy.

25.6. Example. Let$(H, \circ, \tau, \tau_u)$ be a topological hypergroup and hypernormed $\nu : \mathcal{P}^*(H) \to \mathbb{R}^{\geq 0}$ be a continuous and arthmetic.Then $h(I_H) = 0$ where $I : H \to H$ is idetitical map.

25.7. Example. Let $(\mathbb{R}, \circ, \tau, \tau_u)$ be a topological hypergroup with the standard topology, where $x \circ y = \{x, y\}$. Consider continuous hypernormed $\nu = ||.||$ and $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) = x + 1$ for every $x \in \mathbb{R}$.

$$H_n(\varphi, x) = \{x, x + 1, \cdots, x + n - 1\}$$

Then for every $x \in \mathbb{R}$ and so $h_n(\varphi, x) = n$, thus

$$h(\varphi, x) = \lim_n \sup \frac{n}{n} = 1$$

therefore $h(\varphi) = 1$.

Fundamental properties of hypernormed entropy

In this section, we define some of fundamental properties of entropy such as Invariance under conjugation, Logarithmic Law, Monotonicily for subflows; Continuity for direct limits.

25.8. Theorem. (Existence of limit) *Let $(H, \circ, \tau, \tau_u, \nu)$ be a Topological hypernormed hypergroup. If ν is a subadditive hypernormed, then for every $x \in H$, $\lim_n \sup \frac{h_n(\varphi,x)}{n}$ exists.*

25.9. Theorem. (monotonicily for factors) *Let $(H, \circ, \tau, \tau_u, \nu)$ and $(H', \circ', \tau', \tau'_u, \nu')$ be topological hypernormed hypergroup. And $\varphi \in End_c(H)$, $\psi \in End_c(H')$. If $\alpha : H \to H'$ is continuous contractive good epimorphism.*

Such that $\alpha \circ \psi = \psi \circ \alpha$. Then

$$h(\psi) \leq h(\varphi).$$

The following corollary is a direct consequence of Theorem 2.

25.10. Corollary. (Invariance under conjugation) *Let $(H, \circ, \tau, \tau_u, \nu)$ be a topological hypergroup and $\varphi \in End_c(H)$. If $(H', \circ', \tau', \tau'_u, \nu')$ is another topological hypernormed hypergroup and there exist a topological good isomorphism $\alpha : H \to H'$ such that $\nu(a) = \nu'(\alpha(a))$ for every $a \in H$. Then*

$$h(\varphi) = h(\alpha \circ \varphi \circ \alpha^{-1}).$$

25.11. Lemma. *(Monotonicily for subflow)*

Let $(H, \circ, \tau, \tau_u, \nu)$ be a topological hypernormed hypergroup and $\varphi \in End_c(H)$. If G is a $\varphi-$invariant ($\varphi(G) = G$) subhypergroup. Then

$$h(\varphi|_G) \leq h(\varphi)$$

One of the important properties of entropy is the law of logarithm. The defined entropy in lemma 3 has this property.

25.12. Theorem. *Let $(H, \circ, \tau, \tau_u, \nu)$ be a topological hypernormed hypergroup and $\varphi \in End_c(H)$. If ν is increasing respect to hyperoperation $\circ$, then*

$$h(\varphi^k) = kh(\varphi),$$

for every $k \in \mathbb{N}$.

PROOF. Fix $k \in \mathbb{N}$.For every $x \in H$, we put

$$y = x \circ \varphi(x) \circ \cdots \circ \varphi^{k-1}(x)$$

we have

$$\begin{aligned} h_n(\varphi^k, y) &= \nu(y \circ \varphi^k(y) \circ \cdots \circ \varphi^{n-1}(\varphi^k(x))) \\ &= h_{nk}(\varphi, x) \end{aligned}$$

for every $n \in \mathbb{N}$. So

$$\begin{aligned} h(\varphi^k) &\geq h(\varphi^k, y) \\ &= \limsup_n \frac{h_n(\varphi^k, y)}{n} \\ &= k \limsup_n \frac{h_{nk}(\varphi, x)}{kn} \\ &= kh(\varphi, x) \end{aligned}$$

consequently, $h(\varphi^k) \geq kh(\varphi)$.

Now,we prove the converse inequality. Since ν is increasing respect to $\circ$, then for every $n \in \mathbb{N}, x \in H$, we finde that deduce

$$\begin{aligned} h_{nk}(\varphi, x) &= v(x \circ \varphi(x) \circ \cdots \circ \varphi^{nk-1}(x)) \\ &\geq V(x \circ \varphi^k(x) \circ \cdots \circ (\varphi^k)^{n-1}(x)) \\ &= V(H_n(\varphi^k, x)) \\ &= h_n(\varphi^k, x) \end{aligned}$$

Thus

$$\begin{aligned} h(\varphi, x) &= \limsup_n \frac{h_{nk}(\varphi, x)}{nk} \\ &\geq \frac{1}{k} \limsup_n \frac{h_n(\varphi^k, x)}{n} \\ &= \frac{h(\varphi^k, x)}{k}. \end{aligned}$$

There fore

$$kh(\varphi) \geq h(\varphi^k),$$

this complate the proof.

$\square$

25.13. THEOREM. *(ciontinuity for direct limits) Let* $(H, \circ, \tau, \tau_u, \nu)$ *be a direct limit of* $\varphi -$ *invariant subhypergroups* $\{H_i; i \in I\}$. *Then*

$$h(\varphi) = \sup\{h(\varphi|_{H_i}); i \in I\}.$$

Conclusion

In this paper, we introduce the notion of entropy on topological hypernormed hypergroup and prove Some interesting examples. So, we obtain the fundamental properties of this entropy such as Invariance under conjugation, Logarithmic Law, Monotonicity for subflows; Continuity for direct limits.

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26. Gradient Ricci harmonic-Bourguignon solitons on multiply warped products

Sakineh Hajiaghasi[1 a] and Shahroud Azami [2]

[1]Department of pure mathematics, Imam Khomeini international university, Qazvin, Iran
[2]Department of pure mathematics, Imam Khomeini international university, Qazvin, Iran

In this paper, we study certain conditions that a multiply warped product could be a gradient Ricci harmonic-Bourguignon soliton. Also, we obtain some conditions that potential function be constant and consequently multiply warped product be a harmonic-Einstein manifold.

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[a]Speaker. Email address: s4004196001@edu.ikiu.ac.ir,

Introduction

Let (M, g) and (N, h) be complete Riemannian manifolds and $\varphi : M \longrightarrow N$ be a critical point of the energy integral $E(\varphi) = \int_M |\nabla\varphi|^2 dv_g$, where N is isometrically embeded in $\mathbb{R}^d$, $d \geq n$. By a one parameter family of Riemannian metrics $(g(x,t), \varphi(x,t))$, $t \in [0,T)$ and a family of smooth functions $\varphi(x,t)$, a Ricci harmonic-Bourguignon flow on manifold M is defined as

$$\begin{aligned} \frac{\partial}{\partial t} g(x,t) &= -2\mathrm{Ric}(x,t) + 2\rho R(x,t) + 2\alpha\nabla\varphi(x,t) \otimes \nabla\varphi(x,t), \\ \frac{\partial}{\partial t} g(x,t) &= \tau_g \varphi(x,t). \end{aligned}$$

Here α and ρ are positive constants, Ric is the Ricci tensor of M, R is the scalar curvature, and $\tau_g\varphi$ is the intrinsic Laplacian of φ which denotes the tension field of map φ [2]. The system $(M, g, X, \lambda, \rho, \varphi)$ is said to define a Ricci harmonic-Bourguignon soliton (RHBS for short) when it satisfies in the following coupled equation

$$\begin{aligned} \mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g + \rho R g + \alpha\nabla\varphi \otimes \nabla\varphi, \\ \tau_g\varphi - \mathcal{L}_X \nabla\varphi = 0, \end{aligned}$$

where λ, α and ρ are constants, R is scalar curvature and φ is a smooth function $\varphi : (M, g) \to (N, h)$ where M and N are static Riemannian manifolds. In definition of RHBS if $X = \nabla f$, which f is a smooth function on M, then we say M is a gradient Ricci-harmonic-Bourguignon soliton (GRHBS for short). In this case we have

$$\begin{aligned} \mathrm{Ric} + \mathrm{Hess} f - \rho R g - \alpha\nabla\varphi \otimes \nabla\varphi &= \lambda g \\ \tau_g\varphi - < \nabla\varphi, \nabla f > &= 0. \end{aligned} \tag{52}$$

The function f is called the potential. The GRHBS is steady, expanding or shrinking if $\lambda = 0$, $\lambda < 0$ or $\lambda > 0$ respectively. Actually gradient Ricci solitons are a particularly interesting family of Ricci solitons. These arise as self similar solutions of the Ricci flow under certain conditions. If in (52), $\alpha = 0$ or φ is a constant function, then it defines gradient Ricci Bourguignon soliton and if $\rho = 0$, then it defines gradient Ricci-harmonic soliton. For more study about these kind of solitons see [1, 3].
Let (B^r, g_B) and $(F_s^{m_s}, g_{F_s})$ be semi-Riemannian manifolds for $1 \leq s \leq l$ and $M = B \times F_1 \times F_2 \times ... \times F_l$ be an n-dimentional semi-Riemannian manifold. Let $b_s : B \longrightarrow (0, \infty)$ be positive smooth functions for $1 \leq s \leq l$. The multiply warped product manifold is the product manifold $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_l} F_l$ endowed with the metric tensor $g = \pi^*(g_B) \oplus (b_1 \circ \pi)^2 \sigma_1^*(g_{F_1}) \oplus ... \oplus (b_l \circ \pi)^2 \sigma_l^*(g_{F_l})$, where π and σ are the natural projections on B and F_i, respectively[5]. In [4], Fatma Karaca studied about the necessary conditions for a multiply warped product to be a gradient Ricci-harmonic soliton. Till now so many different results have been found about the sufficient conditions for a multiply warped product and also doubly warped product to be different kinds of Ricci soliton such as gradient Ricci solitons, gradient harmonic solitons and gradient Yamabe solitons. Motivated by those work we studied some conditions that a multiply warped product could be a GRHBS.

Main results

We shall denote ∇, ∇_B and ∇_{F_s}; Ric, Ric_B and Ric_{F_s}; Δ, Δ_B and Δ_{F_s}; the Levi-civita connections, the Ricci tensors and the Laplacians of M, B and F respectively. Here is our main results. First of all we want to characterize the harmonic map φ by means of the potential function f. For this aim we obtain:

26.1. Proposition. *Let $(M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_l} F_l, g, f, \varphi, \lambda, \rho)$ be a GRHBS on a multiply warped product with non-constant harmonic function φ, then for a neighborhood V around $(p, q_1, ..., q_l)$, it can be shown like $\varphi = \varphi_B \circ \pi$ or $\varphi = \varphi_{F_s} \circ \sigma_s$ for $1 \leq s \leq l$ iff $h = h_B \circ \pi$.*

Now we want to know the structure of Ric_B and Ric_{F_s} for a multiply warped product which could be a GRHBS.

26.2. Theorem. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_l} F_l$ be a multiply warped product manifold. M is a GRHBS iff*
1) For $\varphi = \varphi_B \circ \pi$ we have

$$\begin{cases} \mathrm{Ric}_B - \sum_{s=1}^{l} \dfrac{m_s}{b_s} \mathrm{Hess}_B(b_s) + \mathrm{Hess}_B h_B - \rho R g_B - \alpha \nabla_B \varphi_B \otimes \nabla_B \varphi_B = \lambda g_B, \\ \Delta_w \varphi_B = 0 \quad in\ B. \end{cases} \tag{53}$$

Here $\Delta_w = \Delta - < \nabla, \nabla w >$, $w = h - \sum_{s=1}^{l} m_s log(b_s)$. F_s is Einstein manifold for all $1 \leq s \leq l$ with $\mathrm{Ric}_{F_s} = \mu_s g_{F_s}$, where

$$\begin{aligned} \mu_s &= \lambda b_s^2 + b_s(\Delta_B b_s) + (m_s - 1)\|\nabla_B b_s\|^2 + b_s \nabla_B h_B(b_s) \\ &+ \sum_{k=1, k\neq s}^{l} \frac{m_k}{b_k} g_B(\nabla_B b_s, \nabla_B b_k) b_s + \rho R b_s^2. \end{aligned}$$

2) For $\varphi = \varphi_{F_s} \circ \sigma_s$, we have

$$\mathrm{Ric}_B - \sum_{s=1}^{l} \frac{m_s}{b_s} \mathrm{Hess}_B(b_s) + \mathrm{Hess}_B h_B - \rho R g_B = \lambda g_B, \tag{54}$$

and F_s are harmonic-Einstein manifolds so that

$$\begin{cases} \mathrm{Ric}_{F_s} - \alpha \nabla_{F_s} \varphi_{F_s} \otimes \nabla_{F_s} \varphi_{F_s} = \mu_s g_{F_s} \\ \Delta_{F_s} \varphi_{F_s} = 0 \end{cases} \tag{55}$$

which for all $1 \leq s \leq l$, *we have*

$$\begin{aligned}\mu_s =&\lambda b_s^2 + \rho R b_s^2 + b_s(\Delta_b b_s) + (m_s-1)\|\nabla_B b_s\|^2 + b_s \nabla_B h_B(b_s)\\ &+ \sum_{k=1,k\neq s}^{l} \frac{m_k}{b_k} g_B(\nabla_B b_s, \nabla_B b_k) b_s.\end{aligned}$$

Now, we give some results for the potential function and harmonic function with use of maximum principle and give some conditions that cause the multiply warped product M to be a harmonic-Einstein manifold.

26.3. Theorem. *Suppose that* $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_l} F_l$ *is a GRHBS on multiply warped product with non-constant harmonic map* φ.
1) For $1 \leq s \leq l$, *either* $\varphi = \varphi_B \circ \pi$ *or* $\varphi = \varphi_{F_s} \circ \sigma_s$, *it is a constant function and* M *is a gradient Ricci-Bourguignon soliton if* φ_B *or* φ_{F_s} *has the maximum or minimum in* B *and* F_s.
2) For $\lambda, \rho, R \geq 0$, h_B *reaches the maximum or minimum in* B *and* $h = h_B \circ \pi$ *is a constant map. Therefore* M *is a harmonic-Einstein manifold.*

We consider a GRHBS on a multiply warped product with harmonic map $\varphi = \varphi_B \circ \pi$ when the base manifold is conformal to an n-dimentional semi-Euclidean space, invariant under the action of an $(r-1)$-dimentional tranlation group. Let $M = (\mathbb{R}^r, \phi^{-2} g_{\mathbb{R}}) \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_l} F_l$ be a multiply warped product endowed with the metric tensor

$$g = \frac{1}{\phi^2} g_{\mathbb{R}} + b_1^2 g_{F_1} + ... + b_l^2 g_{F_l}, \tag{56}$$

which here $g_{\mathbb{R}}$ is the canonical semi-Riemannian metric and φ is the conformal factor. Actually, we have the semi-Riemannian metric $(g_{\mathbb{R}})_{i,j} = \epsilon_i \delta_{i,j}$ in the coordinates $x = (x_1, ..., x_r)$ of $\mathbb{R}^r$, $\epsilon_i = \pm 1$. We consider the function $\xi(x_1, ..., x_r) = \sum_{i=1}^r \beta_i x_i$, where $\beta_i \in \mathbb{R}$. We take $\varphi = \varphi_B \circ \pi$ for the next theorem.

26.4. Theorem. *Let* $M = \mathbb{R}^r \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_l} F_l$ *be a multiply warped product with non-constant harmonic map* φ *and* $b_s = b_s \circ \xi$, $h = h \circ \xi$, $\varphi = \varphi \circ \xi$, $\phi = \phi \circ \xi$ *defined in* $(\mathbb{R}^r, \phi^{-2} g_{\mathbb{R}})$ *endowed with the metric (56), then* M *is a GRHBS iff the functions* b_s, h, φ *and* ϕ *satisfy in the following equations:*

$$(r-2)\frac{\phi''}{\phi} - \sum_{s=1}^{l} m_s \frac{b_s''}{b_s} - 2\sum_{s=1}^{1} m_s \frac{b_s'}{b_s}\frac{\phi'}{\phi} + h'' + 2\frac{\phi'}{\phi}h' - \alpha(\varphi)^2 = 0, \tag{57}$$

$$\left(\frac{\phi''}{\phi} - (r-1)\left(\frac{\phi'}{\phi}\right)^2 + \sum_{s=1}^{l} m_s \frac{b_s'}{b_s}\frac{\phi'}{\phi} - \frac{\phi'}{\phi}h'\right)\|\beta\|^2 = \frac{\lambda + \rho R}{\phi^2}, \tag{58}$$

$$\begin{aligned}&\left(\frac{b_s''}{b_s} - (r-2)\frac{\phi'}{\phi}\frac{b_s'}{b_s} + (m_s-1)\left(\frac{b_s'}{b_s}\right)^2 + \sum_{k=1,k\neq s}^{l}\left(m_k \frac{b_s'}{b_s}\frac{b_k'}{b_k}\right) + \frac{b_s'}{b_s}h'\right)\|\beta\|^2\\ &= \frac{\mu_s}{b_s^2\phi^2} - \frac{\lambda+\rho R}{\phi^2},\end{aligned} \tag{59}$$

$$\left(\varphi'' - (r-2)\frac{\phi'}{\phi}\varphi' + \sum_{s=1}^{1} m_s \frac{b_s'}{b_s}\varphi' - \varphi' h'\right)\|\beta\|^2 = 0. \tag{60}$$

Now, we consider a GRHBS on a multiply warped product with harmonic map $\varphi = \varphi_{F_s} \circ \sigma_s$ when the base manifold and fibers are conformal to r-dimentional and m_i-dimentional semi-Euclidean spaces, invarient under the action or $(r-1)$-dimentional and (m_i-1)-dimentional translation groups for $1 \leq i \leq l$, respectively. Let $M = (\mathbb{R}^r, \phi^{-2} g_{\mathbb{R}}) \times_{b_1} (\mathbb{R}^{m_1}, \tau_1^{-2} g_{\mathbb{R}}) \times_{b_2} (\mathbb{R}^{m_2}, \tau_2^{-2} g_{\mathbb{R}}) \times \dots \times_{b_l} (\mathbb{R}^{m_l}, \tau_l^{-2} g_{\mathbb{R}})$ be a multiply warped product endowed with the metric tensor

$$g = \frac{1}{\phi^2} g_{\mathbb{R}} + b_1^2 \frac{1}{\tau_1^2} g_{\mathbb{R}} + \dots + b_l^2 \frac{1}{\tau_l^2} g_{\mathbb{R}}, \tag{61}$$

here ϕ and τ_i for $1 \leq i \leq l$ are the conformal factors of base and fibers, respectively. We define function ζ_s for nonzero arbitrary vectors $a = (a_{r+1}, \dots, a_{r+m_s})$ and $y = (x_{r+1}, \dots, x_{r+m_s})$ as follows

$$\zeta_s(x_{r+1}, \dots, x_{r+m_s}) = a_{r+1} x_{r+1}, \dots, a_{r+m_s} x_{r+m_s}.$$

26.5. Theorem. *Let $M = \mathbb{R}^r \times_{b_1} \mathbb{R}^{m_1} \times_{b_2} \mathbb{R}^{m_2} \times \dots \times_{b_l} \mathbb{R}^{m_l}$ be a multiply warped product with non-constant harmonic map $\varphi = \varphi_{F_s \circ \sigma_s}$ and $b_s = b_s \circ \xi$, $h = h \circ \xi$, $\phi = \phi \circ \xi$, $\varphi = \varphi \circ \zeta_s$ defined in $(\mathbb{R}^r, \phi^{-2} g_{\mathbb{R}})$ and $(\mathbb{R}^{m_s}, \tau_s^{-2} g_{\mathbb{R}})$ for $1 \leq s \leq l$ with the metric tensor (61), then M is a GRHBS iff the functions b_s, h, ϕ, φ satisfy*

$$(r-2)\frac{\phi''}{\phi} - \sum_{s=1}^{l} m_s \frac{b_s''}{b_s} - 2\sum_{s=1}^{l} m_s \frac{b_s'}{b_s}\frac{\phi'}{\phi} + h'' + 2\frac{\phi'}{\phi} h' = 0, \tag{62}$$

$$\left(\frac{\phi''}{\phi} - (r-1)(\frac{\phi'}{\phi})^2 + \sum_{s=1}^{l} m_s \frac{b_s'}{b_s}\frac{\phi'}{\phi} - \frac{\phi'}{\phi} h'\right)\|\beta\|^2 = \frac{\lambda + \rho R}{\phi^2}, \tag{63}$$

$$\begin{aligned} &\left(b_s b_s'' \phi^2 - (r-2)\phi\phi' b_s b_s' + (m_s-1)\phi^2 (b_s')^2 + \sum_{k=1, k\neq s}^{l} \left(m_k \phi^2 \frac{b_k'}{b_k} b_s b_s'\right) + b_s b_s' \phi^2 h'\right)\|\beta\|^2 \\ &+(\lambda + \rho R)(b_s)^2 = [\tau_s \tau_s'' - (m_s-1)(\tau_s')^2]\|a\|^2, \end{aligned} \tag{64}$$

$$(m_s - 2)\frac{\tau_s''}{\tau_s} - \alpha(\varphi')^2 = 0, \tag{65}$$

$$(\varphi'' \tau_s^2 - (m_s-2)\tau_s \tau_s' \varphi')\|a\|^2 = 0. \tag{66}$$

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27. Mathematical Modeling for Variation Factors of Persian Gazelles Population in a Wildlife Environment

Mohammad Hossein Rahmani Doust, Mohammad Nasser Modoodi, Arash Mowdoudi
Department of mathematics, faculty of Basic Sciences, University of Neyshabur , Neyshabur, Iran
Department of horticulture scince and engineering, Torbat-e Jam University, Torbat-e Jam, Iran
BSc student in Informatics, Universita della Svizzera Italiana, Loganu, switzerland

Evaluating Persian Gazelles population can improve our understanding about the population fluctuations of large mammals in eastern Iran. The present article showed that the most important threatening factors for the population reduction of Persian Gazelle are natural and human-wise parameters. In the present research work, we consider an animal population concluding some parameters in that, such as illegal hunts, preferential migration and roads collision, which have negative impacts on animals population, all of them are considered as disturbing factors. By constructing some hypothesis, we model a population model of single species. After analyzing the obtained model, we can study the Gazelles population on the respective hunting prohibited region.

Keywords: Gazelle population, Differential Equations, Equilibrium Point.
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Introduction

One of the most symbolic species of Irans wildlife is the Persian Gazelle (Gazella subgutturosa) which its population significantly declined during the last decades due to several factors such as illegal hunts and habitat destruction, so that currently, the mentioned animal has placed among the protected species of Irans Environmental Protection Agency and has inserted at the Vulnerable Class (VU) of IUCN Red List (Ashouri et al., 2017). Numerous studies have shown that several factors influence its population dynamic changes and even the herd's physiological parameters and its biological reactions may stimulate the populations fluctuation in a distinctive area (Malekian et al.,2020).

The study district enjoys a semiarid climate with an area of 108,000 hectare of hunting prohibited region located on the east of Khorasan Razavi province and on the border of Iran and Afghanistan. The region consists of two mountainous and plain lands. The predominant vegetation species of the area include Salsola spp, Scariola orientalis, siberi and Euphorbia spp. Lacking the systematic census and only based on the direct observations of rangers, the number of Persian Gazelle in the area is estimated between 350 to 400 animals which despite the supportive strategies, it seems to partially decline due to the shortage or undesirable forage, reduction of drinking water supplies, human interventions and unavoidable migrations. Periodic visits were conducted in accompany with knowledgeable experts and environmental NGOS, in addition to library studies and questionnaires distributed among the road drivers and local people.

In general, the purpose of this study is to investigate the causes of Gazelles population changes and modeling the relationship of some of these parameters with the population decline of this species. In an overview, the threatening factors affecting this animals population included two categories of natural-wise and human-wise reasons.

The main natural reasons are: unfavorable habitat, the non-sustainable climate conditions such as the continuation of drought periods, the presence of large carnivores such as leopards and wolves, age and sex structure of the species, reproductive failure rate, physiological weakness leading to natural mortality and preferential migration of herds to neighboring areas such as Afghanistan;

Considering the human-wise reasons, the main causes were: illegal hunts (leading to gender ratio change), destruction of natural habitats (such as expansion of human settlements, roads troubles, innovations and agricultural lands, and intensification of livestock grinder pressure on pastures leading to destructive competition), stray dog attacks especially for young Gazelles near to rural settlements and agricultural fields, road casualties and other human-centered causes such as deer dispersion, falling into a water pools, trapping in enclosed spaces, swallowing deadly waste, etc.

Modeling and Discussion

In the following, we consider a single-species model for deer population(Rahmani Doust et al., 2021, 2020 and 2015). In this model, we enter some parameters such as illegal hunts, preferential migration of herds to neighboring habitats and collisions with road vehicles that have a negative impact on animal mortality. It is assumed that deers, in the absence of above factors, have a logistic growth rate, about the two factors of illegal hunts and collisions with road vehicles, although both of them cause animal deaths, the decrease in deer population has no effect on increasing the number of poachers and vehicles.In fact, these two factors appear in the role of disturbing factors. Now, we consider some following hypothesis:

The independent variable t and dependent variable x represent time and number of Gazelle population, respectively.
Parameter a shows the exponential growth rate of deer population.
Parameter b illustrates the logistic growth rate of deer population.
Parameter c_1 shows the impact factor of road casualties on deer population.
Parameter c_2 indicates the coefficient of impact of migration on deer population.
Parameter c_3 demonstrates the impact factor of illegal hunts on deer population
Parameter K indicates the maintenance capacity of the environment for the deer population.

By considering the above assumptions, the following model may be obtained:

$$x' = x(a - \frac{bx}{K} - c_1 - c_2 - c_3) \tag{67}$$

Since the parameters c_1, c_2 and c_3 are considered constant, after simplifying, model (67) may be written as follow:

$$x' = x(a - \frac{bx}{K} - c) \tag{68}$$

The above equation has trivial equilibrium point which is origin. Moreover, model (68) has non-trivial equilibrium point which is $x = \frac{K}{b}(a - c)$.

By analyzing the equilibrium points of equation (68) we are able to study the deer population. The analysis of results shows that the situation of Persian Gazelle herds in the study area, like other herds in the protected areas of Khorasan Razavi province, is in a defensive model (WT) and urgent management strategies and supportive mechanisms should be applied and implemented to maintain the population reductive factors at the lowest level of threat (Modoodi et al., 2016).

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28. Transitivity in IFS over arbitrary shift spaces

Mahdi Aghaee[1,a], Dawoud Ahmadi Dastjerdi [2]

[1]Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran
[2]Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran

The orbit of a point $x \in X$ in a classical iterated function system (IFS) is defined as $\{f_u(x) = f_{u_n} \circ \cdots \circ f_{u_1}(x) : u = u_1 \cdots u_n$ is a word of a full shift on finite symbols$\}$. In other words, an IFS is parameterized over the full shift. Here, we parameterize our IFS over an arbitrary shift space Σ. In particular, we associate to $\sigma \in \Sigma$ a non-autonomous system (X, f_σ) where trajectory of $x \in X$ is defined as $x, f_{\sigma_1}(x), f_{\sigma_1\sigma_2}(x), \ldots$. We show that for a transitive IFS and a sofic Σ, there is a transitive $t \in \Sigma$ such that the non-autonomous system (X, f_t) is transitive. This is not true for the case where Σ is non-sofic.

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[a]Speaker. Email address: mahdi.aghaei66@email.com,

Introduction

In a classical dynamical system, here called *conventional dynamical system*, we have a phase space and a unique map where the trajectories of points are obtained by iterating this map. However, in various problems, including applied ones, one may have some finite sequence of maps in place of a single map acting on the same phase space. For instance, in Physics by two or more maps have appeared in [1, 11], Economy in [13] and Biology in [4]. In Mathematics, this has been studied either by non-autonomous systems in many literature such as [9] or as iterated function system (IFS) for constructing and studying some fractals in [5, 8] or for investigating dynamical properties in many places such as [2, 3, 6, 7].

In a "classical" IFS, a compact metric space X and a set of some k finite continuous functions $\{f_0, \cdots, f_{k-1}\}$ on X are assumed and the trajectory of a point $x \in X$ is considered to be the action on x of the sequence of freely combination of those maps, or action on x of combination of those maps over the words of a full shift: just write

$$f_u = f_{u_1} \circ \cdots \circ f_{u_m} \tag{69}$$

where $u = u_1 \cdots u_m$ is a word of the full shift over k symbols. However, in some physical problems, such freely action is not possible. In other words, there are some words that one cannot perform (69). This is the case where a subshift instead of the full shift must be considered and it is of our interest.

Preliminaries

Iterated function systems. Throughout the paper, X will be a compact metric space. The *classical* iterated function system (IFS) consists of finitely many continuous self maps $\mathcal{F} = \{f_0, \dots, f_{k-1}\}$ on X. The *forward orbit* of a point $x \in X$, denoted by $\mathcal{O}^+(x)$, is the set of all values of finite possible combinations of f_i's at x. We need the following equivalent statement: Let Σ_F be the full shift on k symbols and let $\mathcal{L}(\Sigma_F)$ called the *language of* Σ_F be the set of words or blocks. Define $f_u(x)$ as in (69) and set $\mathcal{O}^+(x) := \{f_u(x) : \ u \in \mathcal{L}(\Sigma_F)\}$. Such iterated function systems, here called *classical IFS*, have been the subject of study for quite a long time.

Here we define an IFS to be

$$\mathfrak{I} = (X, \mathcal{F} = \{f_0, \dots, f_{k-1}\}, \Sigma). \tag{70}$$

where f_i is continuous and Σ is an arbitrary subshift on k symbols, not necessarily the full shift Σ_F as in the classical IFS. Hence we use a good deal of symbolic dynamics, see for instance [10] for this subject. By this setting, Σ_F above will be replaced with Σ and thus $\mathcal{O}^+(x) = \{f_u(x) : \ u \in \mathcal{L}(\Sigma)\}$ is the forward orbit of x. In particular, $f_u(f_v(x)) = f_{vu}(x)$ whenever vu is admissible or equivalently $vu \in \mathcal{L}(\Sigma)$. Let $u = u_1 \cdots u_n \in \mathcal{L}(\Sigma)$ and set $u^{-1} := u_n \cdots u_1$. Then for $A \subseteq X$,

$$\begin{aligned}(f_u)^{-1}(A) &= (f_{u_n} \circ \cdots \circ f_{u_1})^{-1}(A)\\ &= f_{u_1}^{-1} \circ \cdots \circ f_{u_n}^{-1}(A)\\ &= f_{u^{-1}}^{-1}(A),\end{aligned}$$

where for the last equality, we used (69). Also

$$\begin{aligned}f_{u^{-1}}^{-1}(f_{v^{-1}}^{-1}(A)) &= f_{v^{-1}u^{-1}}^{-1}(A) = f_{(uv)^{-1}}^{-1}(A)\\ &= (f_{uv})^{-1}(A).\end{aligned}$$

Thus the backward orbit and the (full) orbit of a point $x \in X$ are $\mathcal{O}_-(x) = \{f_{u^{-1}}^{-1}(x) : \ u \in \mathcal{L}(\Sigma)\}$ and $\mathcal{O}(x) = \mathcal{O}_-^+(x) = \mathcal{O}^+(x) \cup \mathcal{O}_-(x)$ respectively.

When all f_i's are homeomorphisms, the backward, forward and full trajectory of x is defined.

We say $\mathcal{F} = \{f_0, \dots, f_{k-1}\}$ in $(X, \mathcal{F} = \{f_0, \dots, f_{k-1}\}, \Sigma)$ is *surjective*, if all f_i's are surjective.

When $k = 1$ and $\Sigma = \{0^\infty\}$, we simply have the classical dynamical system, here called *conventional dynamical system* denoted either by the pair (X, f) or $\mathfrak{I} = (X, \{f_0\}, \{0^\infty\})$. By the above setting, the following definition looks natural.

28.1. Definition. Consider $\mathfrak{I}$ as in (70) and let U and V be arbitrary open sets in X. Then $\mathfrak{I}$ is *forward transitive*, if there is $x \in X$ such that $\{f_u(x) : \ u \in \mathcal{L}(\Sigma)\}$ is dense in X. *Backward transitivity* and *transitivity* is likewise defined.

28.2. Definition. Let $\mathfrak{I}$ be an IFS. Then $\mathfrak{I}$ is called *(forward) transitive along an orbit* $\sigma \in \Sigma$, if the non-autonomous system (X, f_σ) is (forward) transitive.

Transitivity in IFS vs transitivity in the subshift

Transitivity in dynamical systems is a sort of richness in dynamics. Thus when an IFS has transitivity along an orbit, we in fact have a non-autonomous transitive system. Recall that transitivity along an orbit defined in Definition 28.2 implies the transitivity of the system defined in Definition 28.1 and the converse is not necessarily true. Having these in mind, we like to address the following questions.

1. Is there any sufficient condition on Σ such that two notions of transitivity in Definition 28.1 and Definition 28.2 coincides?
2. In which situation there is a transitive $t \in \Sigma$ such that for some $x \in X$, $\overline{O_t^+(x)} = X$?
3. How large is the set

$$S = S(\mathfrak{I}) := \{\sigma \in \Sigma : \ \exists x \in X \text{ s.t. } \overline{\mathcal{O}_\sigma^+(x)} = X\}? \tag{71}$$

The following relatively similar propositions are stated due to the different types of transitivity given in Definitions 28.1 and 28.2.

28.3. Proposition. *Let* $(X, \mathcal{F} = \{f_0, \ldots, f_{k-1}\}, \Sigma)$ *be an IFS with* $\mathcal{F}$ *surjective. Then the following are equivalent.*

(1) *For some* $x \in X$, $\overline{\mathcal{O}^+(x)} = X$.
(2) *Whenever* E *is a closed subset of* X *and for any* $i \in \mathcal{A}$, $E \subseteq f_i^{-1}E$, *then either* $E = X$ *or* E *is nowhere dense. (Equivalently, whenever* U *is an open subset of* X *and for any* $i \in \mathcal{A}$, $f_i^{-1}U \subseteq U$, *then* $U = \emptyset$ *or* U *is dense.)*
(3) *Whenever* W, V *are non-empty open sets, then there is* $u \in \mathcal{L}(\Sigma)$ *such that* $(f_u)^{-1}V \cap W \neq \emptyset$.
(4) *The set* $\{x \in X | \overline{\mathcal{O}^+(x)} = X\}$ *is residual in* X.

Although the following proposition, a very classical result in conventional dynamical systems, has been declared for the case when subshift is over an alphabet with finite characters, the proof (not presented here) is valid for a subshift over infinite characters and thus for a general non-autonomous system as well. For non-autonomous systems, Yan, Zeng and Zang in [12] have a similar result for the special case where their non-autonomous system $\{f_{\sigma_i}\}_{i=1}^{\infty}$ on a compact metric space (X, d) is uniformly convergent to a map f and besides $d(f_{\sigma_n \cdots \sigma_{2n-1}}, f^n) \to 0$ as $n \to \infty$.

28.4. Proposition. *Let* $(X, \mathcal{F} = \{f_0, \ldots, f_{k-1}\}, \Sigma)$ *be an IFS with* $\mathcal{F}$ *surjective and* $\sigma = \sigma_1\sigma_2 \cdots \in \Sigma$. *Then the following are equivalent.*

(1) *For some* $x \in X$, $\overline{O_\sigma^+(x)} = X$.
(2) *Assume* E *is a closed subset of* X *and for some* j, $f^{-1}_{\sigma_1 \cdots \sigma_j}E \subset f^{-1}_{\sigma_1 \cdots \sigma_j u}E$ *whenever* $\sigma_1 \cdots \sigma_j u \subset \sigma$, *then either* $E = X$ *or* E *is nowhere dense. (Equivalently, suppose* U *is an open subset and for some* j, $f^{-1}_{\sigma_1 \cdots \sigma_j u}U \subset f^{-1}_{\sigma_1 \cdots \sigma_j}U$ *whenever* $\sigma_1 \cdots \sigma_j u \subset \sigma$, *then* $U = \emptyset$ *or* U *is dense.)*
(3) *Whenever* W, V *are non-empty open sets, then there is* $n \in \mathbb{N}$ *such that*

$$(f_{\sigma_1 \cdots \sigma_n})^{-1}W \cap V \neq \emptyset. \tag{72}$$

(4) *The set* $\{x \in X | \overline{\mathcal{O}_\sigma^+(x)} = X\}$ *is residual in* X.

In our next proposition we will show that when Σ is an irreducible sofic and the respective IFS is transitive, then for some $x \in X$, there is a transitive $\sigma \in \Sigma$ such that the transitivity of the IFS occurs along this transitive σ. This will give an answer to questions 1 and 2 on the beginning of this section for special cases where Σ is an irreducible sofic. First we recall a classical result.

28.5. Proposition. *Let* $\mathfrak{I} = (X, \mathcal{F}, \Sigma)$ *be an IFS,* $\mathcal{F}$ *surjective and* Σ *an irreducible sofic. Then,* $\mathfrak{I}$ *is forward transitive iff there is a forward transitive* $t \in \Sigma$ *and some* $x \in X$ *such that* $\overline{\mathcal{O}_t^+(x)} = X$.

The structure of $S(\mathfrak{I})$. Let $S = S(\mathfrak{I}) \subseteq \Sigma$ be the set given in (71). In general, except in few cases, a definite structure cannot be given for S, though its largeness can be understood in some cases. In fact in the sequel, we give sufficient conditions where S is dense in Σ. First a weaker version of specification property for subshifts:

28.6. Definition. A subshift Σ is called a *subshift of variable gap length* or SVGL, if there exists $M \in \mathbb{N}$ such that for u and v in $\mathcal{L}(\Sigma)$, there is w with $|w| \leq M$ and $uwv \in \mathcal{L}(\Sigma)$.

When Σ is mixing and SVGL, then Σ has *specification property* and in this situation, there exists $M \in \mathbb{N}$ such that for u and v in $\mathcal{L}(\Sigma)$ there is w with $|w| = M$ and $uwv \in \mathcal{L}(\Sigma)$. Clearly an SVGL is irreducible. Moreover, all sofics are SVGL; however, there are SVGL's which are not sofic [10].

28.7. Proposition. *Let* $\mathfrak{I} = (X, \mathcal{F}, \Sigma)$ *be transitive along some* $\sigma \in \Sigma$, $\mathcal{F}$ *surjective and* Σ *an SVGL. Then,* S *defined in* (71) *is dense in* Σ. *If* $S \neq \Sigma$, *then* $\Sigma \setminus S$ *is also dense in* Σ.

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29. When every shift spaces are flow equivalent?

Arezoo Hosseini[1 a],

[1]Faculty of Mathematics, College of Science, Farhangian University, Tehran, Iran

This paper will show that all shift spaces are flow equivalent to shifts of arbitrarily small entropy.

Keywords: Shift space, Small entropy, Flow equivalent.

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Code: cdsgt3-00580018

[a]Speaker. Email address: a.hosseini@cfu.ac.ir,

Introduction

Let $\mathcal{A}$ be an alphabet. A word over $\mathcal{A}$ is a finite sequence over $\mathcal{A}$, written $w = w_1...w_n$ for $w_i \in \mathcal{A}$. We say that w has length $|w| = n$ and let the empty word of length 0 be denoted ϵ. If $w = w_1....w_n$ and $v = v_1...v_m$ are words over $\mathcal{A}$ then $wv = w_1...w_n v_1...v_m$ denotes their concatenation, and for $k \in \mathbb{N}$, w^k is the concatenation of k copies of w [2, 5].
For a point $x = (x_i)_{i \in \mathbb{Z}}$ of the full shift $\mathcal{A}^{\mathbb{Z}}$, we let $x_{[i;j]}$; $i \leq j$, signify the word $w = x_i...x_j$ and say that w *occurs* in x. Similarly, for a word $w = w_1...w_n$ over $\mathcal{A}$ and $i; j \in \mathbb{N}$ with $1 \leq i \leq j \leq n$, we denote by $w_{[i;j]}$ the subword $u = w_i...w_j$ of w and say that u is a factor of w.
The canonical way of defining a shift space combinatorially is by the words that do not occur in any of its points. Let $\mathcal{A}$ be a finite alphabet, $\mathcal{F}$ a set of words over $\mathcal{A}$, and $X_{\mathcal{F}}$ the set of points $x \in \mathcal{A}^{\mathbb{Z}}$ such that no word of $\mathcal{F}$ occurs in x. The set $\mathcal{F}$ is called a set of *forbidden words* for $X_{\mathcal{F}}$.

29.1. Definition. Let $\mathcal{A}$ be an alphabet. A set $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a *shift space* if there is a set of forbidden words $\mathcal{F}$ over $\mathcal{A}$ such that $X = X_{\mathcal{F}}$. If an arbitrary shift space X is given, we denote its alphabet by $\mathcal{A}(X)$, and the shift map restricted to X by σ_X.

We recall that for some alphabet $\mathcal{A}$, a shift space X over $\mathcal{A}$ is a compact, shiftinvariant subset of $\mathcal{A}^{\mathbb{Z}}$.

29.2. Definition. [1, 2] Let X be a shift space and define an equivalence relation $\sim$ on $X \times \mathbb{R}$ generated by $(x; t+1) \sim (\sigma_X(x); t)$. Giving $X \times \mathbb{R}$ the product topology we let the *suspension flow* of X be given by the quotient space

$$SX = X \times \mathbb{R}/\sim \tag{73}$$

We denote by $[x; t]$ the equivalence class in SX of $(x; t) \in X \times \mathbb{R}$.
A flow equivalence is a homeomorphism between the suspension flows of two shift spaces that preserves direction in $\mathbb{R}$.

29.3. Definition. [6] Let X and Y be shift spaces and SX and SY their suspension flows. A homeomorphism $\Phi : SX \to SY$ is a *flow equivalence* if for each $[x; t] \in SX$ there is a monotonically increasing function $\phi_{[x;t]} : \mathbb{R} \to \mathbb{R}$ such that $\Phi([x;t]) = [y;t']$ implies $\Phi([x;t+r]) = [y;t' + \phi_{[x;t]}(r)]$. If such a homeomorphism exists we say that X and Y are flow equivalent and write $X \sim_{FE} Y$.

Entropy describes the information density or complexity of a shift space by the asymptotic number of words of a given length.

29.4. Definition. Let X be a shift space. Then the entropy of X is given by

$$h(X) = lim_{n\to\infty}\frac{1}{n}log|B_n(X)|, \tag{74}$$

where *log* is the base 2 logarithm.

The limit exists (see for instance Lind and Marcus [[4], Prop. 4.1.8]), so the entropy is always well-defined. Entropy can be said to describe the information density of a shift space in the sense that if $h(X) = t > 0$ for some shift space X, then there are roughly 2^{tn} words of length n in X. A very intuitive example of entropy is that of the full shift.

29.5. Example. Let $X = X_{[r]}$ be the full r-shift. Then $|Bn(X)| = r^n$, so

$$h(X) = lim_{n\to\infty}\frac{1}{n}log|B_n(X)| = lim_{n\to\infty}\frac{n}{n}logr.$$

Flow equivalence as symbol expansions

To understand flow equivalence in combinatorial terms we need the concept of a symbol expansion. A symbol expansion of a shift space X takes a symbol $a \in \mathcal{A}(X)$ and appends to every occurrence of a in every point of X a symbol $\lozenge \notin \mathcal{A}(X)$ such that a is replaced by $a\lozenge$ everywhere in X.

Let X be a shift space, $a \in \mathcal{A}(X)$, and $\lozenge \notin \mathcal{A}(X)$. Let for $b \in \mathcal{A}(X)$,

$$\tau(b) = \begin{cases} a\lozenge, & b = a \\ b, & b \neq a \end{cases}$$

and define a function $\mathcal{T}$ on the points of X by $\mathcal{T}(x) = \cdots\tau(x_{-1})\tau(x_0)\tau(x_1)\cdots$.
The shift space $X^{a\mapsto a\lozenge} = \mathcal{T}(X) \cup \sigma(\mathcal{T}(X))$ is said to be obtained by a symbol expansion of X. If B is a set of words over an alphabet containing a, we write

$$B^{a\mapsto a\lozenge} = \{\tau(w_1)\cdots\tau(w_n)|w_1\cdots w_n \in B\}.$$

29.6. Remark. Adding the set $\mathcal{T}(\sigma(X))$ in the definition of the symbol expansion of a shift space is necessary for $X^{a\mapsto a\lozenge}$ to be closed under the shift map. We will be rather liberal with the notation $X^{a\mapsto b}$, which will simply mean replacing every occurrence of a in X by b and taking the closure under the shift map.

We now make sure that symbol expansion does in fact yield a shift space.

29.7. Proposition. *(Johansen [3]) Let X be a shift space, $a \in \mathcal{A}(X)$, and $\lozenge \notin \mathcal{A}(X)$ be a symbol. Then $X^{a\mapsto a\lozenge}$ is a shift space.*

Proof. Let $\mathcal{F}$ be a set of forbidden words for X and $\mathcal{B} = \mathcal{A}(X) \cup \{\lozenge\}$ be an alphabet. The set

$$F' = F^{a\mapsto a\lozenge} \cup \{b\lozenge\}|b \in \mathcal{A}(X)\{a\}\} \cup \{\lozenge\lozenge\}$$

is a set of forbidden words for $X^{a\mapsto a\lozenge} \subseteq \mathcal{B}^{\mathbb{Z}}$. □

A result by Parry and Sullivan makes the concept of flow equivalence available to our combinatorial interpretation of the theory of shift spaces through symbol expansion. They showed that flow equivalence is the coarsest equivalence relation which is closed under both conjugacy and symbol expansion.

29.8. THEOREM. *(Parry and Sullivan [5]) Let X, Y be shift spaces. Then $X \sim_{FE} Y$ if and only if there exists a sequence of shift spaces $X_0 = X, X_1, X_2, \cdots, X_n = Y$ such that for each $0 \leq i < n$ one of the following conditions hold.*

- *X_i is obtained by a symbol expansion of X_{i+1},*
- *X_{i+1} is obtained by a symbol expansion of X_i,*
- *X_i and X_{i+1} are conjugate.*

Main Section

This section will show that all shift spaces are flow equivalent to shifts of arbitrarily small entropy. First of all, we prove the already known result that having entropy zero is an invariant under flow equivalence.

29.9. THEOREM. *Let X and Y be shift spaces with $X \sim_{FE} Y$. Then $h(X) = 0$ if and only if $h(Y) = 0$.*

PROOF. Let X be a shift space. Since entropy is invariant under conjugacy and flow equivalence is generated by conjugacy and symbol expansion, we only need to show that for some shift space X and some shift space $Y = X^{a \mapsto a\Diamond}$ obtained by a symbol expansion of X, we have $h(X) = 0$ if and only if $h(Y) = 0$.
First, for $u_1; u_2 \in B_n(Y)$, it holds that $u_1^{\Diamond \mapsto \epsilon}$ is a prefix of $u_2^{\Diamond \mapsto \epsilon}$ if and only if u_2 can be achieved by adding and removing $\Diamond$ at the ends of u_1. This can be done in maximally of two different ways (e.g. if $u_1 = \Diamond u_1'$, where u_1' does not end in a $\Diamond$, then $u_2 = u_1$ or $u_2 = u_1'\Diamond$ are the two possibilities), so for every $w \in B_n(X)$ there is at most two words $u \in B_n(Y)$ such that $u^{\Diamond \mapsto \epsilon}$ is a prefix of w, and for every $u \in B_n(Y)$, $u^{\Diamond \mapsto \epsilon}$ is a prefix of some $w \in B_n(X)$. Thus, $2|B_n(X)| \geq |B_n(Y)|$, which yields

$$h(X) = lim_{n \to \infty} \frac{1}{n} log 2|B_n(X)| \geq lim_{n \to \infty} \frac{1}{n} log|B_n(Y)| = h(Y). \tag{75}$$

Second, for two different word $w_1; w_2 \in B_n(X)$, none of the words $w_1^{a \mapsto a\Diamond}$ and $w_2^{a \mapsto a\Diamond}$ can be a prefix of the other, and for every $w \in B_n(X)$ there is at least one word $u \in B_{2n}(Y)$, which has $w^{a \mapsto a\Diamond}$ as a prefix. So $B_{2n}(Y) \geq B_n(X)$, and we can make the estimate

$$h(Y) = lim_{n \to \infty} \frac{1}{2n} log|B_{2n}(Y)| \geq lim_{n \to \infty} \frac{1}{2n} log|B_n(X)| = \frac{1}{2} h(X). \tag{76}$$

Thus, $h(X) \geq h(Y) \geq \frac{1}{2} h(X)$ and the result follows. $\square$

29.10. PROPOSITION. *(Johansen [3]) Let X be a shift space and $a, b \in \mathcal{A}(X)$ with $a \neq b$. Then $X \sim_{FE} X^{a \mapsto ab}$.*

Moving on to shift spaces with non-zero entropy, we need a procedure that given a shift space can produce shift spaces flow equivalent to it of arbitrarily small entropy.

29.11. THEOREM. *Let X be a shift space and $n \in \mathbb{N}$. Then there exists $Y \sim_{FE} X$ with $h(Y) = \frac{1}{n} h(X)$.*

PROOF. The case $n = 1$ is trivially true, so assume that $n > 1$. Let $A = e_1, e_2, ..., e_m$ be the alphabet of X, $\Diamond \notin \mathcal{A}$, and $w = \Diamond^{n-1}$. Further, set $X_0 = X$ and consider the series of symbol expansions

$$X_i = X_{i-1}^{e_i \mapsto e_i w}, 1 \leq i \leq m.$$

Now,$X \sim_{FE} X_m$ by repeated use of Proposition 116, and for every $s \in \mathbb{N}$ the words of X_m of length ns can be described by

$$B_{ns}(X_m) = \{\Diamond^k f_1 w f_2 w ... w f_s \Diamond^{n-1-k} \mid 0 \leq k \leq n-1 \text{ and } f_1 f_2 ... f_s \in B_s(X)\}.$$

So, noting that $|B_{ns}(X_m)| = n|B_s(X)|$, we find that

$$\frac{1}{n} h(X) = \frac{1}{n} lim_{s \to \infty} \frac{1}{s} log|B_s(X)| = lim_{s \to \infty} \frac{1}{ns} log \frac{1}{n} |B_{ns}(X_m)| = h(X_m).$$

□

The main result of the section now follows easily.

29.12. Corollary. *Any shift space X is flow equivalent to shifts of arbitrarily small entropy.*

Proof. Follows directly from Theorem 88 □

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30. On causality conditions along limit curves in space-time

Rahimeh Pourkhandani[a]

Department of Mathematics and Computer Science, Hakim Sabzevari University, Sabzevar, Iran

Limit curve theorem is the strongest tools for the study of causal structure of space- time. This tries to formulate the treatments of causal curve families and their limit. In this psper, we focus on some causal conditions along any elements of causal curve families and study the satisfacation of these conditions along the limit curve.

Keywords: Causality conditions, Limit curve, Limit curve theorem

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Code: cdsgt3-00400041

[a]Speaker. Email address: r.pourkhandanie@hsu.ac.ir

Introduction

motivations. One of the first steps to Lorentzian causality theory is the publication of the papers " Conformal Treatment of Null Infinity" (1964) and " Gravitational Collapse and Space-time Singularities" (1965) by Roger Penrose. In this theory, global methods from differential geometry are employed to predict the formation of singularities of space- time. The fact that any two points have a causal relation is equivalent to the existence of , at least, one causal curve connecting these points to each other and so, the study of the treatments of causal curves in space-time is important. In this way, it seems that limit curve theorem is the strongest tools available in causality theory [3]. There one can find a discussion of the history of limit curve theorem results in Lorentzian geometry. In fact, many authors contributed to their formulation, e.g. Hawking and Ellis (1973), Penrose (1972), Beem et al. (1996), Galloway (1986), Eschenburg and Galloway (1992) and Minguzzi (2008) (see for more infomation[1, 2]). The classical references (Hawking and Ellis 1973; Beem et al. 1996) contain versions that are weaker in several respects, most notably they might impose global causality conditions, such as strong causality, or the deduced convergence might be weak, e.g. in the C^0 topology on curves. The main difference between the different versions of this theorem is the difference in their definitions of limit curve. Furthermore, the domains of curves might be different by domain of each other or be different by the domain of the limit. Also, the theorem can be generlized to curves by restriced domain and we consider this version of definition and theorem [2].

In general, a sequence of curves may have no limit curve or may have many limit curves. Here, we study some case in limit situation.

preliminaries and notations. By a space-time, M, we will mean a C^∞ Hausdorff manifold with a C^∞ Lorentz metric of signature $(+,-,\ldots,-)$ defined on it. Otherwise stated, M will be four-dimensional Let the tangent bundle of M is TM, with fibre T_pM at p. We say that a vector

$v \in T_pM$ is timelike if $g_p(v,v) > 0$, causal if $g_p(v,v) \geq 0$, null if $g_p(v,v) = 0$ and spacelike if $g_p(v,v) < 0$. Also, M is said to be time-orientable if there exists a continuous timelike vector field t on M; we will always assume that M is time-orientable. We state these basic definitions here, only for convination about contracts relating to the temporality and location of events being studied in space- time; other common definitions in the Lorentzian geometry literature which have used in what follows, can be found in [1]. There are different forms of convergence for a sequence of nonspacelike curves $\{\gamma_n\}$ in Lorentzian geometry and general relativity. For example, the limit curve convergence, the C^0 convergence, and the uniform convergence. For arbitrary space-times, each of the limit curve convergence and the C^0 convergence is not stronger than the other. But in strongly causal space-times, these two types of convergence are almost equivalent for sequences of causal curves (see [1, Proposition 3.34]). Recently, Minguzzi introduced a discussion of the history of limit curve theorems results in Lorentzian geometry and proved a strong version of limit curve theorems by a generalized version of uniform convergence [2].

30.1. Definition. [2, Definition 2.1] (In this definition a_n, b_n, a, and b may take an infinite value.) Let h be a Riemannian metric on M and let d_0 be the associated Riemannian distance. The sequence of curves $\gamma_n : [a_n, b_n] \to M$ converges h-uniformly to $\gamma : [a,b] \to M$ if $a_n \to a, b_n \to b$, and for every $\epsilon > 0$ there is $N > 0$, such that for $n > N$, and for every $t \in [a,b] \cap [a_n, b_n]$, $d_0(\gamma(t), \gamma_n(t)) < \epsilon$.

The sequence of curves $\gamma_n : [a_n, b_n] \to M$ converges h-uniformly on compact subsets to $\gamma : [a,b] \to M$ if for every compact interval $[a', b'] \subseteq [a,b]$, there is a choice of sequences $a'_n, b'_n \in [a_n, b_n]$, $a'_n < b'_n$, such that $a'_n \to a'$, $b'_n \to b'$, and for any such choice $\gamma_n|_{[a'_n, b'_n]}$ converges h-uniformly to $\gamma|_{[a',b']}$. Also, Minguzzi proved that the h-uniform convergence implies the C^0 convergence and on compact subsets, it is independent of the Riemannian metric h chosen (see [2, Theorem 2.4]). In this paper, by the limit curve or the limit geodesic segment, we mean that the h-uniform convergence on compact subsets is applied.

Main results

30.2. Proposition. [4] *Any limit curve of a sequence of maximal null geodesics is a maximal null geodesic.*

Proof. For this, let γ be a future directed causal curve from p to q as a limit curve of a sequence of future directed maximal null geodesics γ_n from p_n to q_n such that $p_k \to p$ and $q_k \to q$. By the maximality of γ_n, the Lorentzian arc length of γ_n is equal to the Lorentzian distance from p_n to q_n namely, $L(\gamma_n) = d(p_n, q_n)$ (see [1, Definitions 4.1 and 4.10]. Now, By using [1, Lemma 4.4], we have $L(\gamma) \leq d(p,q) \leq lim\, inf\, d(p_n, q_n) = 0$. Hence $L(\gamma) = d(p,q) = 0$ and γ may be reparametrized to a maximal null geodesic segment from p to q by [1, Theorem 4.13]. Because every null geodesic is locally maximal, then it immediately implies that any limit curve of a sequence of null geodesics is a null geodesic. For this, we can cover γ with a finite number of strictly convex normal neighborhoods $U_1, ..., U_m$ such that $p \in U_1$, $q \in U_m$, and $\gamma_n|_{U_i}$ is a maximal null geodesic segment for all i. So, $\gamma|_{U_i}$ is a maximal null geodesic segment for all i. □

30.3. Proposition. *Let M be a past [resp. future] reflecting spacetime and $\{p_n\}$ and $\{q_n\}$ are sequences in M converging to p and q respectively and there are causal curves γ_n from p_n to q_n for all value of n. Then $q \in \overline{J^+(p)}$ [resp. $p \in \overline{J^-(q)}$]. Especially, if M is causally simple then $q \in J^+(p)$.*

Proof. For each point $x \in I^+(q)$, the open set $I^-(x))$ contains all but a finite number of $\{q_n\}$. So, there is $N > 0$ such that $p_n \in I^-(x)$ for all value $n \geq N$. It implies that $p \in \overline{J^-(x)}$. By the past reflectivity of M, $x \in \overline{J^-(p)}$ and we have $q \in \overline{J^+(p)}$. □

30.4. Remark. We note that if $\tilde{p}$ and $\tilde{q}$ are two distinct limit points of the sequence γ_n in Lemma 30.3 and $U(\tilde{p})$ and $U(\tilde{q})$ are two distinct strictly convex normal neighborhoods that each γ_n lefts $U(\tilde{p})$ and enter to $U(\tilde{q})$ then $\tilde{p} \in \overline{J^+(\tilde{q})}$.

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31. A note on MV-pseudo norm

F. Rajabisotudeh[1,a], N. Kouhestani[2]
[1] Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran
[2] Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

In this paper, we define the notions of MV-pseudo norm and MV-pseudo metric on MV-algebras and study some of their algebraic properties. The notion of uniform MV-algebra is also introduced and its relationship to MV-pseudo metrics is studied.

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[a]Speaker. Email address: Rajabisotudeh@gmail.com, Kohestani@math.usb.ac.ir.

Introduction

MV-algebras, which were introduced by Chang in [2] in 1958, prove the completeness theorem for $\aleph_0$-valued Lukasiewicz logic. Our aim in this article is to introduce and study *MV-pseudo metrics* on MV-algebras. To this end, we first define MV-pseudo norms on MV-algebras, and study their algebraic properties.

The article is organized as follows: in Section 2 we present some definitions and results of the MV-algebra theory and uniform spaces which will be used later in the paper.

In Section 3 we define the concept of MV-pseudo norm, and discuss its algebraic properties and its relation to filters and ideals. Also, the relationship between MV-pseudo norm on MV-algebras and qoutient MV-algebras will be examined in this section. Finally, we show that if $f : A_1 \to A_2$ is an isomorphism between MV-algebras, and N_{A_1} is an MV-pseudo norm on A_1, then $N_{A_2} = N_{A_1} \circ f^{-1}$ is an MV-pseudo norm on A_2.

In Section 4, we define MV-pseudo metrics and examine their relations to MV-pseudo norms. There are also a few theorems about the relationship between MV-pseudo metrics and uniform MV-algebras. Theorem 31.14 in particular provides an efficient way to construct an MV-pseudo metric on MV-algebras.

MV-algebras. An *MV-algebra* is an algebra $(A, \oplus, *, 0)$ of type $(2, 1, 0)$ such that for every $x, y \in A$,
$(M1)$ $(A, \oplus, 0)$ is a commutative monoid,
$(M2)$ $x \oplus 0^* = 0^*$,
$(M3)$ $(x^*)^* = x$, and
$(M4)$ $(x^* \oplus y)^* \oplus y = (x \oplus y^*)^* \oplus x$.[4]
In an MV-algebra A, for every $x, y \in A$, define
$(M5)$ $1 := 0^*$;
$(M6)$ $x \odot y := (x^* \oplus y^*)^*$;

$(M7)$ $x \ominus y := x \odot y^*$;
$(M8)$ $x \to y := (x \odot y^*)^*$;
$(M9)$ $x \rightsquigarrow y := (x \oplus y^*)^*$.

In an MV-algebra A, for every $x, y \in A$, we write $x \le y$ if and only if $x^* \oplus y = 1$. It is well-know that $\le$ is a partial order on A, which gives A the structure of a distributive lattice, where the join and meet are defined by $x \wedge y = y \odot (y^* \oplus x)$ and $x \vee y = x \oplus (y \ominus x)$, respectively, 0 is the least element and 1 is the greatest element. By $(M6)$ and $(M7)$, for every $x, y \in A, x \le y \iff x \ominus y = 0$.

31.1. Definition. Let A be an MV-algebra.
(1) A non-empty subset I of A is called an *ideal* if it satisfies the following conditions.
(I1) For every $x, y \in I, x \oplus y \in I$.
(I2) If $x \in I$ and $y \le x$, then $y \in I$. [2]
(2) A non-empty subset F of A is called a *filter* if it satisfies the following conditions.
(F1) For every $x, y \in F, x \odot y \in F$.
(F2) If $x \in F$ and $x \le y$, then $y \in F$. [4]

31.2. Proposition. [4] *Let F be a filter and I be an ideal of an MV-algebra A. Then the following are congruence relations on A.*

$$x \overset{F}{\equiv} y \iff x \to y \in F \text{ and } y \to x \in F.$$
$$x \overset{I}{\equiv} y \iff x \ominus y \in I \text{ and } y \ominus x \in I.$$

Moreover, if $x/F = \{y \in A : x \overset{F}{\equiv} y\}$, $A/F = \{x/F : x \in A\}$, $x/I = \{y \in A : x \overset{I}{\equiv} y\}$ and $A/I = \{x/I : x \in A\}$, then both A/F and A/I are quotient MV-algebras with the operations

$$x/F \odot y/F = (x \odot y)/F,\ x/I \oplus y/I = (x \oplus y)/I,\ (x/F)^* = x^*/F \text{ and } (x/I)^* = x^*/I.$$

MV-pseudo norms on MV-algebras

31.3. Definition. Let A be an MV-algebra. Then, we say that a map $N : A \longrightarrow \mathbb{R}$ is an *MV-pseudo norm* on A if the following hold.
$(N1)$ $N(x \oplus y) \le N(x) + N(y)$.
$(N2)$ $N(x^*) \le N(1) - N(x)$.
An MV-pseudo norm is an *MV-norm* if $N(x) = 0 \Leftrightarrow x = 0$.

31.4. Example. Let X be a finite set and $(P(X), \cup, *, \varnothing, X)$ be the MV-algebra in which for each $B \in P(X)$, B^* is the complement of B in X. The map $N : P(X) \longrightarrow \mathbb{R}$ by $N(B) = cardB$ is a MV-pseudo norm.

31.5. Theorem. *Let N_1 and N_2 be MV-pseudo norms on A and $\alpha \ge 0$, then*
(i) the function $N : A \longrightarrow \mathbb{R}$, defined by $N(x) = \alpha N_1(x) + N_2(x)$, is an MV-pseudo norm. Moreover, N is an MV-norm, if N_1 and N_2 are MV-norms.
(ii) the map $N(x) = inf\{N_1(z) : z \in \frac{x}{I}\}$ is an MV-pseudo norm, where I is an ideal in A.

31.6. Theorem. *Let I be an ideal in an MV-algebra A, and N be an MV-pseudo norm on it. Then,*
(i) the map $n : \frac{A}{I} \longrightarrow \mathbb{R}$ defined by $n(\frac{x}{I}) = \inf\{N(z) : z \in \frac{x}{I}\}$ is an MV-pseudo norm on $\frac{A}{I}$ morovere if for every $x \in A$, $\min \frac{x}{I}$ exists and N is an MV-norm on A, then $n(\frac{x}{I})$ is also an MV-norm on $\frac{A}{I}$.

If F is filter, similar to the Theorem 31.6, $n(\frac{x}{F})$ is also an MV-pseudo norm on $\frac{A}{F}$.

31.7. Theorem. *Let I be an ideal in an MV-algebra A. Then,*
(i) the set $I_N = \{x \in A : N(x) = 0\}$ is an ideal in A if N is an MV-pseudo norm on A; moreover if n is an MV-pseudo norm on $\frac{A}{I}$, then $N(x) = n(\frac{x}{I})$ is an MV-pseudo norm on A. Moreover, n is an MV-norm on $\frac{A}{I}$ if and only if $I = I_N$.

31.8. Theorem. *Let f be an isomorphism from an MV-algebra $(A_1, \oplus, 0)$ to an MV-algebra $(A_2, \oplus, 0)$. If N_{A_1} is an MV-pseudo norm on A_1, then $N_{A_2} : A_2 \longrightarrow \mathbb{R}$, defined by $N_{A_2}(y) = N_{A_1} \circ f^{-1}(y)$ for every $y \in A_2$, is an MV-pseudo norm on A_2, and $N_{A_2}(f(x)) = N_{A_1}(x)$.*

31.9. Theorem. *Let A_1 and A_2 be MV-algebras, and N_{A_1} be an MV-pseudo norm on A_1. If $f : A_1 \longrightarrow A_2$ is an epimorphism, then $N_{A_2} : A_2 \longrightarrow \mathbb{R}$ defined by $y \longmapsto \inf\{N_{A_1}(z) : f(z) = y\}$ is an MV-pseudo norm on A_2, and $N_{A_2}(f(x)) \leq N_{A_1}(x)$.*

MV-pseudo metrics on MV-algebras

31.10. Definition. A pseudo metric d on an MV-algebra A is called an *MV-pseudo metric* if for every $x, y, a, b \in A$,
$(D5)$ $d(x \oplus y, a \oplus b) \leq d(x, a) + d(y, b)$, and
$(D6)$ $d(x^*, y^*) \leq d(x, y)$.
An *MV-metric* on A is an MV-pseudo metric that satisfies $d(x, y) = 0 \Longleftrightarrow x = y$.

31.11. Theorem. *If N is an MV-pseudo norm on an MV-algebra A, then $d_N(x, y) = N(x \ominus y) + N(y \ominus x)$ is an MV-pseudo metric on A.*

31.12. Corollary. *MV-pseudo metric d_N of Theorem 31.11, satisfies the following properties. (i) For every x, $d_N(0, x) + d_N(1, x) = N(1)$, (ii) The mapping d_N is an MV-metric if and only if N is an MV-norm, (iii) For every x, $d_N(x, x^*) \leq N(1)$.*

Remark. From now on, if N is an MV-pseudo norm on an MV-algebra, then d_N is the MV-pseudo metric induced by N in Theorem 31.11.
Let A be an MV-algebra and $\mathcal{U}$ be a uniformity on A. By Definition uniformly continuous,
(i) the operation $\oplus : (A \times A, \mathcal{U} \times \mathcal{U}) \to (A, \mathcal{U})$ is *uniformly continuous* if for every $W \in \mathcal{U}$, there exist $U, V \in \mathcal{U}$ such that $U \oplus V \subseteq W$ or equivalently, for every $(x, x') \in U$ and $(y, y') \in V$, $(x \oplus y, x' \oplus y') \in W$;
(ii) the map $* : (A, \mathcal{U}) \to (A, \mathcal{U})$ is *uniformly continuous* if for every $W \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that if $(x, y) \in V$, then $(*(x), *(y)) \in W$.
The pair $(A, \mathcal{U})$ is called a *uniform MV-algebra* if $\oplus$ and $*$ are uniformly continuous.

Let d be an MV-pseudo metric on an MV-algebra A. Then, it is easy to prove that the set $\mathcal{B} = \{U_\epsilon : \varepsilon > 0\}$ is a base for a uniformity $\mathcal{U}_d$ on A, where $U_\epsilon = \{(x, y) : d(x, y) < \epsilon\}$. Thus, by Definition uniformly continuos and $(D5)$ and $(D6)$, the operations $\oplus$ and $*$ are uniformly continuous.
A subset S of an MV-algebra A is said to be *convex* if for any $x, y, z \in A$, $x \leq z \leq y$, and $x, y \in S$ imply that $z \in S$.

31.13. Proposition. *Let A be an MV-algebra, $S \subseteq A$ and $\widehat{S} = \{x \in A : \exists\ y \in S \text{ such that } x \leq y\}$. Then,*
(i) if $0 \in S$, then S is convex if and only if for any $x, y \in A$, if $x \leq y$ and $y \in S$, then $x \in S$;
(ii) $0 \in \widehat{S}$ and $\widehat{S}$ is the smallest convex set of A containing S;
(iii) if $S \subseteq T$, then $\widehat{S} \subseteq \widehat{T}$;
(iv) $\widehat{S} \oplus \widehat{T} \subseteq \widehat{S \oplus T}$.

Remark. Let d be a pseudo metric on MV-algebra A. We denote the set $\{x : d(x, 0) < r\}$ by $B(r)$ i.e $B(r) = \{x : d(x, 0) < r\}$. Also, we recall that the first part of the proof of the following theorem is from [1].

31.14. Theorem. *Let $\{U_n\}_{n \geqslant 0}$ be a family of subsets of an MV-algebra A such that $0 \in U_n$ and $U_{n+1} \oplus U_{n+1} \subseteq U_n$ for any $n \geq 0$. Then there is an MV-pseudo metric d on A such that the operations $\oplus$ and $*$ are uniformly continuous on $(A, \mathcal{U}_d)$ and for any $n \geq 0$,*

$$\{x : d(x, 0) < 1/2^n\} \subseteq \widehat{U_n} \subseteq \{x : d(x, 0) < 2/2^n\}.$$

Moreover, d is an MV-metric if and only if $\bigcap_{n \geq 0} \widehat{U}_n = 0$.

PROOF. Let $V(1) = U_0$, $n \geq 0$ and assume that $V(\frac{m}{2^n})$ are defined for each $m = 1, 2, 3, ..., 2^n$ such that $0 \in V(\frac{m}{2^n})$. Put then $V(\frac{1}{2^{n+1}}) = U_{n+1}$, $V(\frac{2m}{2^{n+1}}) = V(\frac{m}{2^n})$ for $m = 1, 2, 3, ..., 2^n$ and for each $m = 1, 2, 3, ..., 2^n - 1$, $V(\frac{2m+1}{2^{n+1}}) = V(\frac{m}{2^n}) \oplus U_{n+1} = V(\frac{m}{2^n}) \oplus V(\frac{1}{2^{n+1}})$. We also define $V(\frac{m}{2^n}) = A$, when $m > 2^n$. By induction on n we prove that for any $m > 0$ and $n \geq 0$,

$$(*) \quad V(\frac{m}{2^n}) \oplus V(\frac{1}{2^n}) \subseteq V(\frac{m+1}{2^n}).$$

First notice that if $m + 1 > 2^n$, then $(*)$ is obviously true. Let $m < 2^n$. If $n = 1$, then m is also 1, so $V(\frac{1}{2}) \oplus V(\frac{1}{2}) = U_1 \oplus U_1 \subseteq U_0 = V(1)$. Asume that $(*)$ holds for some n. We verify it for $n+1$. If $m = 2k$, then by the definition of $V(\frac{2m+1}{2^{n+1}})$, $V(\frac{m}{2^{n+1}}) \oplus V(\frac{1}{2^{n+1}}) = V(\frac{2k}{2^{n+1}}) \oplus V(\frac{1}{2^{n+1}}) = V(\frac{k}{2^n}) \oplus V(\frac{1}{2^{n+1}}) = V(\frac{2k+1}{2^n})$. Suppose now that $m = 2k+1 < 2^{n+1}$ for some $n \geq 0$. Then $V(\frac{m}{2^{n+1}}) \oplus V(\frac{1}{2^{n+1}}) = V(\frac{2k+1}{2^{n+1}}) \oplus U_{n+1} = V(\frac{k}{2^n}) \oplus U_{n+1} \oplus U_{n+1} \subseteq V(\frac{k}{2^n}) \oplus U_n = V(\frac{k}{2^n}) \oplus V(\frac{1}{2^n})$. But by the inductive assumption, $V(\frac{m}{2^{n+1}}) \oplus V(\frac{1}{2^{n+1}}) \subseteq V(\frac{k+1}{2^n}) = V(\frac{m+1}{2^{n+1}})$. By Proposition 31.13, for any $r \geq 0$, $\widehat{V(r)}$ is a convex set containing 0, it is easy to derive that the map $f : A \longrightarrow \mathbb{R}$ defined by $f(x) = inf\{r : x \in \widehat{V(r)}\}$ is increasing bounded function. Define the map $N : A \longrightarrow \mathbb{R}$ by $N(x) = sup\{f(x \oplus z) - f(z) : z \in A\}$. The function N is obviously well defined and increasing. In a similar method with the proof of Theorem 31.11, we can show that $d_N(x, y) = N(x \ominus y) + N(y \ominus x)$ is an MV-pseudo metric. By $(D5)$ and $(D6)$, we can prove that the operations $\oplus$ and $*$ are uniformly continuous on $(A, \mathcal{U}_{d_N})$. Let us prove that d_N satisfies

$$\{x : d_N(x, 0) < \frac{1}{2^n}\} \subseteq \widehat{U_n} \subseteq \{x : d_N(x, 0) \leq \frac{2}{2^n}\}.$$

Notice that $f(0) = 0$, hence if $d_N(x, 0) < \frac{1}{2^n}$, then $f(x) = f(x \oplus 0) - f(0) \leq N(x) = d_N(x, 0) < \frac{1}{2^n}$. Hence for some $0 \leq r < \frac{1}{2^n}$, $x \in \widehat{V(r)}$. Since $V(r) \subseteq V(\frac{1}{2^n}) = U_n$, $x \in \widehat{V(r)} \subseteq \widehat{V(\frac{1}{2^n})} = \widehat{U_n}$. Now let $x \in \widehat{U_n}$. Then there is a $x' \in U_n$ such that $x \leq x'$. Clearly for any $z \in A$, there exists a $k \geq 0$ such that $\frac{k-1}{2^n} \leq f(z) \leq \frac{k}{2^n}$. Since $z \in \widehat{V(\frac{k}{2^n})}$, there is a $z' \in V(\frac{k}{2^n})$ such that $z \leq z'$. From condition $(*)$ it follows that $z' \oplus x' \in V(\frac{k}{2^n}) \oplus V(\frac{1}{2^n}) \subseteq V(\frac{k+1}{2^n})$ and from $z \oplus x \leq z' \oplus x'$ deduces that $z \oplus x \in \widehat{V(\frac{k+1}{2^n})}$. Hence $f(x \oplus z) - f(z) \leq \frac{k+1}{2^n} - \frac{k-1}{2^n} = \frac{2}{2^n}$.

In the end of proof, let us prove that d_N is an MV-metric if and only if $\bigcap_{n \geq 0} \widehat{U_n} = 0$. Let $\bigcap_{n \geq 0} \widehat{U_n} = 0$ and $d_N(x, y) = 0$. Then $N(x \ominus y) = N(y \ominus x) = 0$. Hence for any $n \geq 0$, $x \ominus y$ and $y \ominus x$ are in $\widehat{U_n}$. This concludes that $x \ominus y = y \ominus x = 0$ and so $x = y$. Therefore d_N is metric. Conversely let d_N be metric and $x \in \bigcap_{n \geq 0} \widehat{U_n}$. Since $\widehat{U_n} \subseteq \{x : d_N(x, 0) \leq \frac{2}{2^n}\}$ for every $n \geq 0$, we derive that $d_N(x, 0) = 0$. This implies that $x = 0$. □

31.15. THEOREM. *Let A be a MV-algebra. Then, there is an MV-pseudo metric d on A such that $(A, \mathcal{U}_d)$ is a uniform MV-algebra if and only if there is a topology τ on A such that (A, τ) is a topological MV-algebra and τ hsa a countable local base at 0. Moreover, d is continuous in (A, τ).*

31.16. PROPOSITION. *Let $S = \{N_i : i \in I\}$ be a chain of MV-pseudo norms on an MV-algebra A. Then, there exists a uniformity $\mathcal{U}$ on A such that $(A, \mathcal{U})$ is a uniform MV-algebra.*

31.17. PROPOSITION. *Suppose A is an MV-algebra, I is an ideal and $q : A \longrightarrow \frac{A}{I}$, given by $q(x) = \frac{x}{I}$, is the quotient map. Then there are uniformities η_I and ε_I on A and $\frac{A}{I}$ such that (A, η_I) and $(\frac{A}{I}, \varepsilon_I)$ are uniform MV-algebras and $q : (A, \eta_I) \to (\frac{A}{I}, \varepsilon_I)$ is uniformly continuous.*

31.18. PROPOSITION. *Let N be an MV-pseudo norm and I be an ideal in an MV-algebra A. Then there exists an MV-pseudo metric D_n on $\frac{A}{I}$ such that $(\frac{A}{I}, \mathcal{U}_{D_n})$ is a uniform MV-algebra and the quotient map $q : (A, \mathcal{U}_{d_N}) \longrightarrow (\frac{A}{I}, \mathcal{U}_{D_n})$, given by $q(x) = \frac{x}{I}$, is uniformly continuous.*

Conclusion

In this article is to introduce MV-pseudo norms, MV-pseudo metric and MV-metric and its relation to uniform continuity are discussed.

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32. n-Jordan $*$-homomorphisms in Fréchet locally C^*-algebras

Shahram Ghaffary[1,a], Javad Jamalzadeh [2]

[1]University of Sistan and Baluchestan, Zahedan, Iran

[2]Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

Using the fixed point method, we prove the Hyers-Ulam stability and the superstability of n-Jordan $*$-homomorphisms in Fréchet locally C^*-algebras for the following generalized Jensen-type functional equation

$$rf\left(\frac{a+b}{r}\right)+rf\left(\frac{a-b}{r}\right)=2f(a).$$

where r is a fixed real number with $r>1$.

Keywords: n-Jordan $*$-homomorphism, Fréchet locally C^*-algebra, Fréchet algebra, fixed point method, Hyers-Ulam stability

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Code: cdsgt3-00870029

[a]Speaker. Email address:shahram.ghaffary@pgs.usb.ac.ir,

Introduction

The stability of functional equations was first introduced by Ulam in 1940. Hyers gave a partial solution of *Ulam's* problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. Aoki generalized the Hyers' theorem for approximately additive mappings. In 1978, Th.M. Rassias generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. The paper of Th.M. Rassias has provided a lot of influence in the development of what we call Hyers-Ulam-Rassias stability of functional equations.

32.1. Theorem. *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(a+b)-f(a)-f(b)\| \leq \epsilon(\|a\|^p+\|b\|^p) \tag{77}$$

for all $a, b \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(a)-T(a)\| \leq \frac{2\epsilon}{2-2^p}\|a\|^p \tag{78}$$

for all $a \in E$. If $p < 0$ then inequality (77) holds for all $a, b \neq 0$, and (78) holds for $a \neq 0$. Also, if the function $t \to f(ta)$ from R into E' is continuous for each fixed $a \in X$, then T is linear.

The result of the Th.M. Rassias theorem was generalized by Forti and Gavruta who permitted the Cauchy difference to become arbitrary unbounded. Some results on the stability of functional equations in single variable and nonlinear iterative equations can be found in . G. Isac and Th.M.

Rassias were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. The concept of n-Jordan homomorphisms in complex algebras was introduced by Eshaghi Gordji *et al.* J. Jamalzadeh *et al.* introduced the Hyers-Ulam stability and the superstability of n-Jordan $*$-derivations in Fréchet locally C^*-algebras.

During the last decades several stability problems of functional equations have been investigated by many mathematicians (see [**1**, **2**, **3**, **4**, **5**]).

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

32.2. Theorem. *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In this paper, assume that n is an integer greater than 1.

32.3. Definition. Let A, B be complex algebras. A C-linear mapping $h : A \to B$ is called an n-Jordan homomorphism if

$$h(a^n) = h(a)^n$$

for all $a \in A$.

32.4. Definition. Let A, B be C^*-algebras. An n-Jordan homomorphism $h : A \to B$ is called an n-Jordan $*$-homomorphism if

$$h(a^*) = h(a)^*$$

for all $a \in A$.

32.5. Definition. A topological vector space X is a Fréchet space if it satisfies the following three properties:

(1) it is complete as a uniform space,

(2) it is locally convex,

(3) its topology can be induced by a translation invariant metric, i.e., a metric $d : X \times X \to R$ such that $d(x, y) = d(x + a, y + a)$ for all $a, x, y \in X$.

For more detailed definitions of such terminologies, we can refer to . Note that a ternary algebra is called a ternary Fréchet algebra if it is a Fréchet space with a metric d.

Fréchet algebras, named after Maurice Fréchet, are special topological algebras as follows.

Note that the topology on A can be induced by a translation invariant metric, i.e. a metric $d : X \times X \to R$ such that $d(x, y) = d(x + a, y + a)$ for all $a, x, y \in X$.

Trivially, every Banach algebra is a Fréchet algebra as the norm induces a translation invariant metric and the space is complete with respect to this metric.

A locally C^*-algebra is a complete Hausdorff complex $*$-algebra A whose topology is determined by its continuous C^*-seminorms in the sense that a net $\{a_i\}_{i \in I}$ converges to 0 if and if the net $\{p(a_i)\}_{i \in I}$ converges to 0 for each continuous C^*-seminorm p on A. The set of all continuous C^*-seminorms on A is denoted by $S(A)$. A Fréchet locally C^*-algebra is a locally C^*-algebra whose topology is determined by a countable family of C^*-seminorms. Clearly, any C^*-algebra is a Fréchet locally C^*-algebra.

For given two locally C^*-algebras A and B, a morphism of locally C^*-algebras from A to B is a continuous $*$-morphism φ from A to B. An isomorphism of locally C^*-algebras from A to B is a bijective mapping $\varphi : A \to B$ such that φ and φ^{-1} are morphisms of locally C^*-algebras.

Hilbert modules over locally C^*-algebras are generalization of Hilbert C^*-modules by allowing the inner product to take values in a locally C^*-algebra rather than in a C^*-algebra.

In this paper, using the fixed point method, we prove the Hyers-Ulam stability and the superstability of n-Jordan $*$-homomorphisms in Fréchet locally C^*-algebras for the the following generalized Jensen-type functional equation

$$rf\left(\frac{a+b}{r}\right) + rf\left(\frac{a-b}{r}\right) = 2f(a).$$

Stability of n-Jordan $*$-homomorphisms

32.6. Lemma. *Let A, B be linear spaces, and $f : A \to B$ be an additive mapping such that $f(\mu a) = \mu f(a)$ for all $a \in A$ and all $\mu \in T^1 := \{\lambda \in C : |\lambda| = 1\}$. Then the mapping $f : A \to B$ is C-linear.*

32.7. Theorem. *Let A, B be Fréchet locally C^*-algebras, and $f : A \to B$ be a mapping for which there exists a function $\varphi : A \times A \to [0, \infty)$ such that*

$$r\mu f\left(\frac{a+b}{r}\right) + r\mu f\left(\frac{a-b}{r}\right) - 2f(\mu a) \leq \varphi(a,b), \tag{79}$$

$$\|f(a^n) - f(a)^n\| \leq \varphi(a,b), \tag{80}$$

$$\|f(a^*) - f(a)^*\| \leq \varphi(a,b) \tag{81}$$

for all $\mu \in T^1$ and all $a, b \in A$. If there exists an $L < 1$ such that $\varphi(a,b) \leq rL\varphi(\frac{a}{r}, \frac{b}{r})$ for all $a, b \in A$, then there exists a unique n-Jordan $*$-homomorphism $h : A \to B$ such that

$$\|f(a) - h(a)\| \leq \frac{L}{1-L}\varphi(a,0) \tag{82}$$

for all $a \in A$.

Now, we prove the Hyers-Ulam stability problem for n-Jordan $*$-homomorphisms in Fréchet locally C^*- algebras.

32.8. Corollary. *Let $p \in (0,1)$ and $\theta \in [0, \infty)$ be real numbers. Suppose $f : A \to B$ satisfies*

$$\|r\mu f\left(\frac{a+b}{r}\right) + r\mu f\left(\frac{a-b}{r}\right) - 2f(\mu a)\| \leq \theta(\|a\|^p + \|b\|^p),$$

$$\|f(a^n) - f(a)^n\| \leq 2\theta\|a\|^p,$$

$$\|f(a^*) - f(a)^*\| \leq 2\theta\|a\|^p$$

for all $\mu \in T$ and $a, b \in A$. Then there existes a unique n-Jordan $*$-homomorphism $h : A \to B$ such that

$$\|f(a) - h(a)\| \leq \frac{2^p\theta}{2-2^p}$$

for all $a \in A$.

Superstability of n-Jordan $*$-homomorphisms

In this section, we prove the superstability of n-Jordan $*$-homomorphisms on Fréchet locally C^*-algebras for the generalized Jensen-type functional equation. we need the following lemma in our main results.

32.9. LEMMA. *Let A, B be Fréchet locally C^*-algebras, Let $\theta \geq 0$, p and q be real numbers with $q > 0$ and $p + q \neq 1$. Suppose $f : A \to B$ satisfies $f(0) = 0$ and*

$$\left\| r\mu f\left(\frac{a+b}{r}\right) + r\mu f\left(\frac{a-b}{r}\right) - 2f(\mu a)\right\| \leq \theta \|a\|^p \|b\|^q \tag{83}$$

for all $\mu \in T$ and all $a, b \in A$. Then f is C-linear.

Now, we prove the superstability problem for n-Jordan $*$-homomorphisms in Fréchet locally C^*-algebras.

32.10. COROLLARY. *Let $p, s \in R$ and $\theta, q \in (0, \infty)$ with $p + q \neq 1, s \neq 2$. Let A, B be Fréchet locally C^*-algebras. Suppose $f : A \to B$ satisfies $f(0) = 0$ and*

$$\left\| r\mu f\left(\frac{a+b}{r}\right) + r\mu f\left(\frac{a-b}{r}\right) - 2f(\mu a)\right\| \leq \theta \|a\|^p \|b\|^q$$

$$\|f(a^n) - f(a)^n\| \leq \theta \|a\|^s$$

for all $\mu \in T$ and all $a, b \in A$. Then f is an n-Jordan $*$-homomorphism.

32.11. COROLLARY. *Let $p \in R$ and $\theta, q \in (0, 1)$ with $p + q \neq 1, 2$. Suppose A, B are Fréchet locally C^*-algebras, $f : A \to B$ satisfies $f(0) = 0$, and*

$$\max\{\|f(a^*) - (f(a))^*\|, \|f(a^n) - (f(a))^n\|,$$
$$\left\| r\mu f\left(\frac{a+b}{r}\right) + r\mu f\left(\frac{a-b}{r}\right) - 2f(\mu a)\right\|\} \leq \theta \|a\|^p \|b\|^q$$

for all $\mu \in T$ and all $a, b \in A$. Then f is an n-Jordan $*$-homomorphism.

Conclusion

In this paper, using the fixed point method, we prove the Hyers-Ulam stability and the superstability of n-Jordan $*$-homomorphisms in Fréchet locally C^*-algebras

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33. Coexistence of Periods in Majority Parallel Dynamical Systems over Directed Graphs

Leila Musavizadeh Jazaeri[a], Leila Sharifan

Hakim Sabzevari University, Sabzevar, Iran

In this talk, we solve the problem of the coexistence of periodic orbits in homogeneous parallel Boolean dynamical systems which are induced by majority function, with a directed dependency graph. In particular, we show that periodic orbits of any period can coexist. This result contrasts with the properties of their counterparts over simple graphs with the same evolution operator, where only fixed points and 2−periodic points can exist and coexist.

Keywords: Parallel dynamical systems, Boolean network , Fixed points, Periodic points, Dependency graph, Majority function

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[a]Speaker. Email address: leilamusavi.math@gmail.com,

Introduction

Many real-world phenomena are modeled as (finite) dynamical systems over large complex networks. For example we can list them as the interactions of gene regulatory networks, the virus spreading thorough a computer network, the spread of a disease through a social network, etc. Such coherent utilizations of (finite) dynamical systems in social network, science and engineering make the research on this topic an interesting and important subject. We devote this paper to study a class of finite dynamical system that we call it majority-PDDS.

Given a finite (non-empty) set of elements X and a function $F : X \to X$, the pair (X, F), or simply F, is named a finite dynamical system. Throughout this work, X is called the state space and F is named the evolution operator of the system.

Let (X, F) be a finite dynamical system, a point $\mathbf{x} \in X$ is called a periodic point of F of period $t > 0$ whenever $F^t(\mathbf{x}) = \mathbf{x}$ and $F^s(\mathbf{x}) \neq \mathbf{x}$ for each $0 < s < t$. We denote by $Per_t(F)$ the set of periodic points of F of period t. In particular, if $t = 1$ then $\mathbf{x}$ is called a fixed point of F, and we denote by $Fix(F)$ the set of fixed points of F. Note that (X, F) is called a fixed point system when all the periodic points are fixed points.

A *Boolean finite dynamical system* is a finite dynamical system where the state space and the evolution operator are Boolean. More precisely, in a Boolean finite dynamical system (X, F), $X = \{0, 1\}^n$ for some natural n and

$$F : \{0,1\}^n \to \{0,1\}^n, \quad F(x_1, \dots, x_n) = (F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n))$$

where each F_i is a Boolean function. Corresponding to this system, we consider the underlying graph $G = (V, E)$ on the vertex set $\{1, \ldots, n\}$ whose edge/arc set is

$$E = \{(i, j)\ ;\ \text{ the variable } x_i \text{ is involved in the component function } F_j\}.$$

Throughout this work, we assume that for each $1 \leq i \leq n$ the variable x_i appears in the component function F_i, but to simplify, we remove all self-loops of G. The graph G defined in this way is called *dependency graph* of F. The Boolean evolution operator F of a Boolean finite dynamical system can update the (states of) variables in a synchronous or in an asynchronous manner. In the first case, the system is called parallel or synchronous, while in the second case it is named sequential or asynchronous. In the literature, when the dependency graph $G = (V, E)$ is simple (resp. directed), these systems are denoted by PDS and SDS (resp. PDDS and SDDS), respectively. If the evolution operator F is induced by a function

$$f : \{0, 1\}^n \to \{0, 1\}$$

such that each F_j is computed by the restriction f_j of f to the state of the entry j and the entries i such that $(i, j) \in E$, then the system is called homogeneous. In this setting we simply say F is an $f-$PDS, $f-$SDS (resp. $f-$ PDDS and $f-$SDDS). In this paper, we are going to study $f-$PDDS in the case that f is a majority function.

Following the notations of [3], let

$$sum_n : \{0, 1\}^n \to \mathbb{N}, \quad sum_n(a_1, \ldots, a_n) = a_1 + \cdots + a_n$$

and assume that the evolution operator F induced by

$$majority_n : \{0, 1\}^n \to \{0, 1\}, \quad majority_n(a_1, \ldots, a_n) = \begin{cases} 1, & sum_n(a_1, \ldots, a_n) \geq \lceil \frac{n}{2} \rceil \\ 0, & \text{otherwise} \end{cases}$$

In this situation, we simply say F is a majority-PDS or majority-PDDS depending on G is a simple or directed graph.

Goles and Olivos proved that every periodic point of a majority-PDS has period 2 or 1 (see [2]). Poljak and Turzik showed that for any arbitrary point $\mathbf{x}$, $\mathbf{x}^t$ is a periodic point when $t \geq O(n^2)$ (see [5]). Moreover, Kaaser, Mallmann-Trenn, and Natale ([1]) proved that given an integer t and some graph G, it is NP-hard to decide whether there exists an initial point $\mathbf{x}$ for which $\mathbf{x}^t$ is a periodic point.

Many papers were devoted to the study of majority-PDS while majority-PDDS has net been study well. The main contribution of this paper is to show that periodic orbits of any period can exist and coexist together in majority-PDDS despite the fact that in majority-PDS only fixed points and 2-periodic points can exist and coexist together.

Main results

Let F be a majority-PDS over a simple graph G, then, as we mentioned before, $Per_t(F)$ can be a non-empty set only in the case that $t = 1, 2$. As $(1, \ldots, 1)$ and $(0, \ldots, 0)$ are always fixed points of F, it is clear that $Per_1(F) = Fix(F) \neq \emptyset$. It is worth remarking that in many situations F is a fixed-point system. For example, if G is a tree or a complete graph or a cycle of odd length, then F is a fixed point system, while fixed points and 2-periodic points present simultaneously if G is a cycle of even length (see [4]). In this Section, we show that majority-PDDS behaves completely different from majority-PDS and periodic points of any period may happens for a majority-PDDS.

33.1. Theorem. *Given $\{n_1, \ldots, n_r\} \subset \mathbb{N}, r \geq 2$, there exists a majority-PDDS which presents periodic orbits of periods $n_1, \ldots, n_r$ simultaneously.*

Proof. In order to prove the result, we introduce a specific majority-PDDS F and we find all t that $Per_t(F) \neq \emptyset$. Consider the majority-PDDS F over the directed graph $G = (V, E)$ where

$$V = \{1, \ldots, m, m + 1, \ldots, 2m\}$$

and

$E = \{(i, i+1), (i, i+m+1) \mid 1 \le i \le m-1\} \cup \{(i, i+1), (i, i-m+1) \mid m \le i \le 2m-1\} \cup \{(2m, 1), (2m, m+1)\}$

So, $F : \{0,1\}^{2m} \to \{0,1\}^{2m}$ is given by $(F_1, \dots, F_m, F_{m+1}, \dots, F_{2m})$ where

$$F_1(x_1, \dots, x_{2m}) = majority_3(x_1, x_m, x_{2m}),$$
$$F_{m+1}(x_1, \dots, x_{2m}) = majority_3(x_{m+1}, x_m, x_{2m}),$$
$$\forall 2 \le i \le m \quad F_i(x_1, \dots, x_{2m}) = majority_3(x_i, x_{i-1}, x_{m+i-1})$$

and

$$\forall m+2 \le i \le 2m \quad F_i(x_1, \dots, x_{2m}) = majority_3(x_i, x_{i-1}, x_{i-m-1}).$$

Let $\mathbf{a} = (a_1, \dots, a_m)$ be an arbitrary point of $\{0,1\}^m$ and define

$$\mathbf{b_a} = (a_1, \dots, a_m, a_1, \dots, a_m) \in \{0,1\}^{2m}$$

It is straightforward to see that

$$F(\mathbf{b_a}) = (a_m, a_1, \dots, a_{m-1}, a_m, a_1, \dots, a_{m-1}).$$

Now consider the finite Boolean dynamical system $R : \{0,1\}^m \to \{0,1\}^m$ where $R(x_1, \dots, x_m) = (x_m, x_1, \dots, x_{m-1})$. It is clear that for each positive integer i,

$$F^i(\mathbf{b_a}) = (R^i(\mathbf{a}), R^i(\mathbf{a})).$$

This shows that for each $\mathbf{a} \in \{0,1\}^m$, the orbit of $\mathbf{b_a}$ in F is in one-to-one correspondence with the orbit of $\mathbf{a}$ in R. On the other hand, all orbits of R are periodic orbits and $Per_t(R) \neq \emptyset$ if and only if $t|m$. So, we conclude that for each $\mathbf{a} \in \{0,1\}^m$, the orbit of $\mathbf{b_a}$ in F is a periodic orbit and in particular $Per_t(F) \neq \emptyset$ for each t dividing m.

Now suppose that $\mathbf{b} = (b_1, \dots, b_m, b_{m+1}, \dots, b_{2m}) \in \{0,1\}^{2m}$ be such that for each $1 \le i \le m$, $b_i \neq b_{m+i}$, then one can easily see that $\mathbf{b}$ is a fixed point of F.

Finally, if $\mathbf{b} = (b_1, \dots, b_m, b_{m+1}, \dots, b_{2m}) \in \{0,1\}^{2m}$ is such that $b_i = b_{m+i}$ for some $1 \le i \le m$, then one can easily check that there exists a positive integer n and $\mathbf{a} \in \{0,1\}^m$ such that $F^n(\mathbf{b}) = \mathbf{b_a}$ and so the orbit of $\mathbf{b}$ in F converges to the periodic orbit of $\mathbf{b_a}$ in F.

In few words, we have shown that $Per_t(F) \neq \emptyset$ if and only if t divides m. Now to prove the result, for a given $\{n_1, \dots, n_r\} \subset \mathbb{N}, r \ge 2$, let m be the least common multiple of $n_1, \dots, n_r$ and F as defined in previous paragraphs. As discussed before $Per_t(F) \neq \emptyset$ for each t dividing m. So, F presents periodic orbits of periods $n_1, \dots, n_r$ simultaneously and the conclusion follows □

Conclusion

Let F be a majority-PDS over a simple graph then, it is well-known that $Per_t(F)$ can be a non-empty set only in the case that $t = 1, 2$. In this paper by a careful study of periodic structure of a specific majority-PDDS, we conclude that for a given $\{n_1, \dots, n_r\} \subset \mathbb{N}, r \ge 2$, one can find a majority-PDDS which presents periodic orbits of periods $n_1, \dots, n_r$ simultaneously. This shows that majority-PDDS behave completely different from majority-PDS and studying their periodic structure is more difficult than the case that the dependency graph is a simple graph.

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34. Conjugate of dynamical systems in locales

Ali Akbar Estaji[1,a], Maryam Robat Sarpoushi[2]

[1,2]Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran

For a dynamical system (X, f), the concept of "conjugate" is studied by many authors. Our goal is to introduce and study this concept in pointfree topology to give a description of categorical properties of conjugate in localic dynalical systems. We give the relation between the category **TDS** of dynamical systems and the category **LDS** of localic dynamical sysytems and continuous maps between them.

Keywords: dynamical system, locale, functor, adjunction

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[a]Speaker. Email address: aaestaji@hsu.ac.ir; aaestaji@gmail.com,

Introduction

In this section we briefly touch category theory and give the definitions and results needed in the next section. For more details, the reader can see [5] and [4].

34.1. Definition. Let $F, G : \mathbb{C} \to \mathbb{D}$ be functors. A *natural transformation* $\tau : F \to G$ is a function $\tau : Obj\mathbb{C} \to \mathrm{Mor}\mathbb{D}$ assigning to each object A in $\mathbb{C}$ a morphism $\tau_A : FA \to GA$ in $\mathbb{D}$ such that for every morphism $f : A \to B$ in $\mathbb{C}$, $(Gf)(\tau_A) = \tau_A(Ff)$. In this case we denote τ by $(\tau_A)_{A\in\mathbb{C}}$, and call each τ_A a component of τ.

Let $G : \mathbb{D} \to \mathbb{C}$ be a functor and $A \in Obj\mathbb{C}$. A *universal arrow* from A to G is an object B of D together with a morphism $u : A \to GB$ in $\mathbb{D}$ such that for each $D \in Obj\mathbb{D}$ and each morphism $f : A \to GD$ in $\mathbb{C}$ there is a unique morphism $\bar{f} : B \to D$ in $\mathbb{D}$ with $(G\bar{f}) \circ u = f$. Dualizing this, we get the notion of a couniversal arrow from A to G.

34.2. Definition. Let $G : \mathbb{D} \to \mathbb{C}$ be a functor. A *left adjoint* to G is a functor $F : \mathbb{C} \to \mathbb{D}$ such that for each $A \in Obj\mathbb{C}$, $B \in Obj\mathbb{D}$ there is an isomorphism $\alpha_{A,B} : Hom_{\mathbb{D}}(FA, B) \to Hom_{\mathbb{C}}(A, GB)$ which is natural in A and B (that is, $(\alpha_{A,-})_{A\in Obj\mathbb{C}} : Hom(FA, -) \to Hom(A, G-)$ and $(\alpha_{-,B})_{B\in Obj\mathbb{D}} : Hom(F-, B) \to Hom(-, GB)$ are natural isomorphisms). In this case we write $F \dashv G$, and say (F, G, α) is an *adjunction*, or also G is a *right adjoint* to F.

34.3. Theorem. *For functors $F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{C}$, the following are equivalent:*

(a) F is a left adjoint to G.
(b) There exists a natural transformation $\eta : I_{\mathbb{C}} \to G \circ F$, called unit or front adjunction, such that for each $A \in Obj\mathbb{C}, \eta_A : A \to GFA$ is a universal arrow from A to G.
(c) There exists a natural transformation $\epsilon : F \circ G \to I_{\mathbb{D}}$, called counit or back adjunction, such that for each $B \in Obj\mathbb{D}$, $\epsilon_B : FGB \to B$ is a couniversal arrow from B to F.

Our references for frames and locales are [4] and [5].

A *frame* (or *locale*) is a complete lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \colon s \in S\}$$

holds for all $a \in L$ and $S \subseteq L$. We denote by 0 and 1, respectively, the bottom and top elements of L. A *frame homomorphism* (or *frame map*) is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element. An element $p \in L$ is said to be prime if $p < 1$ and $x \wedge y \leq p$ implies $a \leq p$ or $b \leq p$. Recall the contravariant functor Σ from **Frm** to the category **Top** of topological spaces which assigns to each frame L its *spectrum* ΣL of prime elements with $\Sigma_a = \{p \in \Sigma L \colon a \not\leq p\}$ $(a \in L)$ as its open sets. Also, we have $\Sigma L = \{f \colon L \to \mathbf{2} \colon f \text{ is a frame map }\}$, where $\mathbf{2} = \{0, 1\}$. The frame of all open sets of a topological space X is denoted by $\mathfrak{O}(X)$. The right adjoint of a frame homomorphism $h \colon L \to M$ is denoted by h_*. We shall use the terms "frame" and "locale" interchangeably, but when we wish to consider frame homomorphisms we will rather use "frame".

A *localic map* $f \colon L \to M$ is a meet-preserving mapping between locales the left adjoint of which preserves binary meets.

For the category **Loc**, we have the functor $\mathrm{Lc} \colon \mathbf{Top} \to \mathbf{Loc}$ by setting $\mathrm{Lc}(X) = \mathfrak{O}(X)$ and $\mathrm{Lc}(f) = \mathfrak{O}(f)_*$ where $\mathfrak{O}(f) = f^{-1}$.

A discrete-time dynamical system (X, f) is a continuous map f on a nonempty topological space X, i.e. $f : X \to X$. The dynamics is obtained by iterating the map. The reader can see [2, 1] and [3] for more informations on dynamical systems.

34.4. Definition. A homomorhism of dynamical sysytems from (X, f) to (Y, g) is a map $\varphi : X \to Y$ such that $\varphi \circ f = g \circ \varphi$. In this case, we say that two dynamical systems are conjugated.

Adapting by this, we call (L, f) a locallic dynamical system if L is a locale and $f : L \to L$ is a locallic map.

Main results

Recall that for dynamical systems (X, f) and (Y, g), we call continuous function $\varphi : X \to Y$ a *morphism* if the following diagram commuts.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "f"'] & Y \end{tikzcd}$$

We denote the category of all topological dynamical systems and their morphisms by **TDS**. Now, we give the next definition.

34.5. Definition. A *localic dynamical system* (L, f) is a localic map f on a locale L, that is, $f \colon L \to L$.

34.6. Definition. Let (L, f) and (M, g) be localic dynamical systems. We call continuous function $h : L \to M$ a *morphism* if the following diagram commuts.

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "h"'] & L \arrow[d, "h"] \\ M \arrow[r, "f"'] & M \end{tikzcd}$$

We denote the category of all localic dynamical systems and their morphisms by **LDS**. Here, we are going to give a relation between **TDS** and **LDS**.

Foe this, we define $\Sigma : \mathbf{LDS} \longrightarrow \mathbf{TDS}$ with $(L, f) \longmapsto (\Sigma L, \Sigma f^*)$. It is clear that if $h : L \to M$ be a localic map such that $h \in Hom\big((L, f), (M, g)\big)$, then $g \circ h = h \circ f$. This implies that $\Sigma h^* \circ \Sigma g^* = \Sigma f^* \circ \Sigma h^*$ which means that

$$\begin{tikzcd} \Sigma M \arrow[r, "\Sigma g^*"] \arrow[d, "\Sigma h^*"'] & \Sigma M \arrow[d, "\Sigma h^*"] \\ \Sigma L \arrow[r, "\Sigma f^*"'] & \Sigma L \end{tikzcd}$$

and therefore $\Sigma h^* \in Hom\big((\Sigma M, \Sigma g^*), (\Sigma L, \Sigma f^*)\big)$. Threfore we have:

34.7. Proposition. *The functor* Σ *is a countravariant functor from* ***LDS*** *to* ***TDS***.

Now, we define $\mathrm{Lc} : \mathbf{TDS} \longrightarrow \mathbf{LDS}$ with $(X, f) \longmapsto (\mathrm{Lc}(X), \mathrm{Lc}(f))$. It is clear that if $h : X \to Y$ be a morphism such that $h \in Hom\big((X, f), (Y, g)\big)$, then $g \circ h = h \circ f$. This implies $\mathrm{Lc}(h) \circ \mathrm{Lc}(f) = \mathrm{Lc}(g) \circ \mathrm{Lc}(h)$, which means that the next diagram commuts

$$\begin{tikzcd} \mathrm{Lc}(X) \arrow[r, "\mathrm{Lc}(f)"] \arrow[d, "\mathrm{Lc}(h)"'] & \mathrm{Lc}(X) \arrow[d, "\mathrm{Lc}(h)"] \\ \mathrm{Lc}(Y) \arrow[r, "\mathrm{Lc}(g)"'] & \mathrm{Lc}(Y) \end{tikzcd}$$

and therefore $\mathrm{Lc}(h) \in Hom\big((\mathrm{Lc}(X), \mathrm{Lc}(f)), (\mathrm{Lc}(Y), \mathrm{Lc}(g))\big)$. Thus we have:

34.8. Proposition. *The functor* Lc *is a covariant functor from* ***TDS*** *to* ***TDS***.

Now, Put $\lambda : I_{\mathbf{TDS}} \to \Sigma\mathrm{Lc}$ such that, for every X, $\lambda_X : X \longrightarrow \Sigma Lc(X)$ is gicen by $x \longmapsto \{U \in \mathfrak{O}(X) : x \in U\}$. Now, let $h : (X, f) \to (Y, g)$ be a morphism in **TDS**, then $g \circ h = h \circ f$. This implies $\Sigma\mathrm{Lc}(h) \circ \lambda_X = \lambda_Y \circ h$ and so the following diagram is commuted:

$$\begin{tikzcd} X \arrow[r, "\lambda_X"] \arrow[d, "h"'] & \Sigma\mathrm{Lc}(X) \arrow[d, "\Sigma\mathrm{Lc}(h)"] \\ Y \arrow[r, "\lambda_Y"'] & \Sigma\mathrm{Lc}(Y) \end{tikzcd}$$

We put $\varphi : L \longrightarrow \mathrm{Lc}\Sigma L$ with $a \longmapsto \{F : a \in F\ ,\ F$ is a completely prime filter on $L\}$. It is clear that φ is a frame map. Now, we define $\sigma : \mathrm{Lc}\Sigma \to I_{\mathbf{LDS}}$ and set $\sigma_L = (\varphi_L)_*$. Now, let $f : X \to X$ be a continuous function, then, for a completely prime filter F, we have $(\Sigma\mathrm{Lc}(f))(F) = (\mathrm{Lc}(f)^*)^{-1}(F)$ This implies that, for $h : X \to Y$, we have $\Sigma\mathrm{Lc}(h)\lambda = \lambda_Y h$.

34.9. Proposition. *The function* λ_X *is a natural transformation.*

Now, let $f : L \to M$ be a localic map. We have

$$\begin{tikzcd} (\mathrm{Lc}\Sigma)(L) \arrow[r, "\sigma_L"] \arrow[d, "(\mathrm{Lc}\Sigma)(f)"'] & L \arrow[d, "f"] \\ (\mathrm{Lc}\Sigma)(M) \arrow[r, "\sigma_M"'] & M \end{tikzcd}$$

and so $f \circ \sigma_L = \sigma_M \circ (\mathrm{Lc}\Sigma)(f)$ which implies that $\sigma_L^* \circ f^* = (\mathrm{Lc}\Sigma)(f)^* \circ \sigma_M^*$.

Let $\lambda : I_{\mathbf{TDS}} \to \Sigma\mathrm{Lc}$. For every $(X, f) \in Obj(\mathbf{TDS})$, we have $\lambda_{(X,f)} : (X, f) \to \Sigma\mathrm{Lc}(X, f)$ where $\lambda_{(X,f)} = \lambda_X : X \to \Sigma\mathrm{Lc}(X)$ is a continuous fuction. It is easy to see that, for every $x \in X$, we have $\Sigma(\mathrm{Lc}(f))\lambda_{(X,f)}(x) = \lambda_{(X,f)} f(x)$ which shows that the following diagram is commuted:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\lambda_{(X,f)}"'] & X \arrow[d, "\lambda_{(X,f)}"] \\ \Sigma\mathrm{Lc}(X) \arrow[r, "\Sigma(\mathrm{Lc}(f))"'] & \Sigma\mathrm{Lc}(X) \end{tikzcd}$$

Now, let $h : (X, f) \to (Y, g)$ be a morphism in **TDS**. Thus, for every $x \in X$, we have $\Sigma(\mathrm{Lc}(h)) \circ \lambda_{(X,f)}(x) = \lambda_{(Y,g)} \circ h(x)$ which shows that the following diagram commutes.

$$\begin{tikzcd}
(X, f) \arrow[r, "\lambda_{(X,f)}"] \arrow[d, "h"'] & (\Sigma(\mathrm{Lc}(X)), \Sigma(\mathrm{Lc}(f))) \arrow[d, "\lambda_{(X,f)}"] \\
(Y, g) \arrow[r, "\lambda_{(Y,g)}"'] & (\Sigma(\mathrm{Lc}(Y)), \Sigma(\mathrm{Lc}(g)))
\end{tikzcd}$$

For every (L, f), we define $\sigma_{(L,f)} : (\mathrm{Lc}\Sigma(L), \mathrm{Lc}\Sigma(f)) \to (L, f)$ where $\sigma_{(L,f)} = (\varphi)_* : \mathrm{Lc}\Sigma L \to L$ is a localic map. One can see that $\sigma_{(L,f)} \circ \mathrm{Lc}\Sigma f = f \circ \sigma_{(L,f)}$ which shows that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Lc}\Sigma L \arrow[r, "\mathrm{Lc}\Sigma f"] \arrow[d, "\sigma_{(L,f)}"'] & \mathrm{Lc}\Sigma L \arrow[d, "\sigma_{(L,f)}"] \\
L \arrow[r, "f"'] & L
\end{tikzcd}$$

Now, let $h : (L, f) \to (M, g)$ be a morphism in **LDS**. We have $\sigma_{(M,g)} \circ \mathrm{Lc}\Sigma h = h \circ \sigma_{(L,f)}$ which shows that the following diagram commutes.

$$\begin{tikzcd}
(\mathrm{Lc}\Sigma L, \mathrm{Lc}\Sigma f) \arrow[r, "\sigma_{(L,f)}"] \arrow[d, "\mathrm{Lc}\Sigma L"'] & (L, f) \arrow[d, "h"] \\
(\mathrm{Lc}\Sigma M, \mathrm{Lc}\Sigma g) \arrow[r, "\sigma_{(M,g)}"'] & (M, g)
\end{tikzcd}$$

34.10. Proposition. *The function σ is a natural transformation.*

Conclusion

34.11. Corollary. *Let Σ : **LDS** $\to$ **TDS** and* Lc : ***TDS*** $\to$ ***LDS*** *be as the same in the previous section. Then* $\mathrm{Lc} \dashv \Sigma$.

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35. On the dynamics of lattice networks

Ghazaleh Malekbala[a], Leila Sharifan

Hakim Sabzevari University, Sabzevar, Iran

Let $(L, \leq)$ be a finite lattice and $f : L^n \to L^n$ be a lattice network over dependency graph $G = (V, E)$. We prove that f is a fixed-point system. Also we analyse fixed points of f is some special cases.

Keywords: Lattice network , Fixed points, Periodic points, Dependency graph.

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[a]Speaker. Email address: gmalekbala@gmail.com,

Introduction

Finite dynamical systems play a crucial role in studying problems of several different contexts. In particular, these mathematical models, which naturally arise in computer processes, are profusely used in other sciences as biology, mathematics, physics, chemistry, or even sociology. Some relevant examples of finite dynamical systems are (finite) cellular automata and, more generally, deterministic Boolean networks, also called Boolean finite dynamical systems, diffusion models and recently semilattice networks (see [5, 4]. It is worth remarking that semilattice networks are generalization of conjunctive Boolean networks in [3] and some diffusion models that studied in [2] and the results of [5] recovers and extends some main theorems of those researches.

The benefit of finite dynamical systems is that they can be easily simulated on a computer in most cases and the difficulty is that few analytical devices beyond simulation are available. A common combinatorial device to study dynamics of models is the *dependency graph* that extensively has been used in Boolean networks, diffusion models and recently in the study of semilattice networks. Actually, [5] provides a general mathematical (algebraic and combinatorial) method to study semilattice networks.

A partially ordered set $(L, \leq)$ is called a join-semilattice if each two-element subset $\{a, b\} \subseteq L$ has a join (i.e. least upper bound, denoted by $a \vee b$), and is called a meet-semilattice if each two-element subset has a meet (i.e. greatest lower bound, denoted by $a \wedge b$). $(L, \leq)$ is called a lattice if it is both a join- and a meet-semilattice.

Let $(L, \leq)$ be a finite meet-semilattice (or join-semilattice). A finite dynamical system

$$f = (f_1, \dots, f_n) : L^n \to L^n,$$

is called a semilattice network when $f_j = \bigwedge_{x_i \in I_j} x_i$ for all $j = 1, \dots, n$ where I_j is the set of variables that influence the variable x_j (or $f_j = \bigvee_{x_i \in I_j} x_i$ for all $j = 1, \dots, n$). In this paper, we introduce and study the concept of lattice network. Let $(L, \leq)$ be a finite lattice. We say that a finite dynamical system

$$f = (f_1, \dots, f_n) : L^n \to L^n,$$

is a lattice network when for each $j = 1, \dots, n$, $f_j = \bigwedge_{x_i \in I_j} x_i$ or $f_j = \bigvee_{x_i \in I_j} x_i$. Note that the notion of lattice network is a generalization of the notion of AND-OR network where $L = \{0, 1\}$, $\wedge = AND$, and $\vee = OR$. Also note that AND-OR networks have been studied well in the literature. In the following, we assume that f is a lattice network. The dynamics of f is characterized by its *phase space* which is the directed graph with vertex set L^n and each $(\mathbf{u}, \mathbf{v})$ is a directed edge if $f(\mathbf{u}) = \mathbf{v}$. A *limit cycle* of length t is a set with t elements $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_t\}$ such that $f(\mathbf{u}_i) = \mathbf{u}_{i+1}$ for $i = 1, \dots, t-1$ and $f(\mathbf{u}_t) = \mathbf{u}_1$. All elements of that limit cycle are called *periodic points* of period t. If $t = 1$ then $\mathbf{u}_1$ is called a fixed point and if all periodic points of f are fixed points then f is called a fixed-point network (system). Note that since L is a finite set, each $\mathbf{u} \in L^n$ converges to a periodic point (there exists $k \geq 0$ such that $f^k(\mathbf{u})$ is a periodic point). Dependency graph associated to f is a directed graph $G = (V, E)$ on the vertex set $V = \{1, 2, \dots, n\}$ with the edge set

$$E = \{(i, j) \mid \text{ the function } f_j \text{ depends on } x_i\}$$

which is a powerful combinatorial tool that is used for detecting cycle structure of the network without direct computation of the phase space. In this paper, we assume that for each $1 \leq i \leq n$, $(i, i) \in E$ which means that each function f_i depends on x_i (Note that in many research papers this assumption has a crucial role in the study of the network), and we show that f is a fixed-point network.

Main results

Let $(L, \leq)$ be a finite lattice and $f : L^n \to L^n$ be a lattice network. In this section we show that f is a fixed-point system. Before presenting the main result let us give an example of lattice network.

35.1. Example. Let $L = \{1, 2, 3, 6\}$. Define $\leq$ in L by

$$a \leq b \quad \text{if and only if} \quad a \quad \text{divides} \quad b.$$

So, by $a \wedge b$ we mean the greatest common divisor of a and b, and by $a \vee b$ we mean the least common multiple of a and b.

Now define $f : L^3 \to L^3$ by $f(x_1, x_2, x_3) = (x_1 \vee x_2, x_2 \wedge x_3, x_3 \vee x_1)$. Then the dependency graph of f is $G = (V, E)$ where $V = \{1, 2, 3\}$ and $E = \{(1, 1), (2, 1), (2, 2), (3, 2), (3, 3), (1, 3)\}$. One can check that f is a fixed-point system and

$$fix(f) = \{(a, b, a) \in L^3 \mid b \mid a \text{ and } a \mid c\}$$

Now we are ready to present our main result.

35.2. Theorem. *Let $(L, \leq)$ be a finite lattice and $f : L^n \to L^n$ be a lattice network. Then f is a fixed point system.*

Proof. Let $G = (V, E)$ be the dependency graph of f. Since f is a lattice network, there exists $I, J \subseteq \{1, \dots, n\}$ ($I \cup J = \{1, \dots, n\}$, $I \cap J = \emptyset$) such that for each $j \in I$, $f_j = x_j \wedge \bigwedge_{(s,j) \in E} x_s$ and for each $j \in J$, $f_j = x_j \vee \bigvee_{(s,j) \in E} x_s$. Let $\mathbf{a} = (a_1^0, \dots, a_n^0)$ be a t-periodic point of f. We show that $\mathbf{a}$ is a fixed point of f. For each $1 \leq r \leq t$ let $f^r(\mathbf{a}) = (a_1^r, \dots, a_n^r)$ and let $L_1 := \{a_i^r \mid 1 \leq i \leq n, 0 \leq r \leq t\}$. For each $b \in L_1$ define $A_{r,b}$ ($0 \leq r \leq t$) as

$$A_{r,b} = \{j \in I \mid a_j^r = b\}$$

Since $\mathbf{a}$ is a $t-$periodic point, it is clear that $A_{0,b} = A_{t,b}$ for all $b \in L_1$.

Suppose that b is a minimal element of L_1. Let $0 \leq r \leq t-1$ be such that $j \in A_{r,b}$, then by $f_j = x_j \wedge \bigwedge_{(s,j) \in E} x_s$, we have $a_j^{r+1} = a_j^r \wedge \bigwedge_{(s,j) \in E} a_s^r$. So $a_j^{r+1} \leq a_j^r = b$ and by the minimality of b in L_1, We get $a_j^{r+1} = b$ which shows that $j \in A_{r+1,b}$ and so for each $0 \leq r \leq t-1$, $A_{r,b} \subseteq A_{r+1,b}$.mThus

$$A_{0,b} \subseteq A_{1,b} \subseteq \cdots \subseteq A_{r,b} = A_{0,b}$$

In particular,

$$A_{0,b} = A_{1,b}$$

Now suppose that $b \in L_1$ is a cover of some minimal element of L_1. Let $0 \leq r \leq t-1$ be such that $j \in A_{r,b}$. As previous paragraph, we can show that $a_j^{r+1} \leq a_j^r = b$. If $a_j^{r+1} < a_j^r$, then a_j^{r+1} is a minimal element of L_1 and so $A_{r,a_j^{r+1}} = A_{r+1,a_j^{r+1}}$ and it shows that $a_j^r = a_j^{r+1}$ which is a contradiction. So $a_j^{r+1} = a_j^r$ and again we conclude that for each $0 \leq r \leq t-1$, $A_{r,b} \subseteq A_{r+1,b}$. So,

$$A_{0,b} \subseteq A_{1,b} \subseteq \cdots \subseteq A_{r,b} = A_{0,b},$$

and we have

$$A_{0,b} = A_{1,b}.$$

Continuing this process, we finally conclude that

$$\forall b \in L_1, \ A_{0,b} = A_{1,b} \tag{84}$$

Now for each $b \in L_1$ define $B_{r,b}$ $(0 \leq r \leq t)$ as

$$B_{r,b} = \{j \in J \mid a_j^r = b\}$$

Since $\mathbf{a}$ is a $t-$periodic point, it is clear that $B_{0,b} = B_{t,b}$ for all $b \in L_1$. Suppose that b is a maximal element of L_1. Let $0 \leq r \leq t-1$ be such that $j \in B_{r,b}$, then by $f_j = x_j \vee \bigvee_{(s,j)\in E} x_s$, we have $a_j^{r+1} = a_j^r \vee \bigvee_{(s,j)\in E} a_s^r$. So $b = a_j^r \leq a_j^{r+1}$ and by the maximality of b in L_1, We get $a_j^{r+1} = b$ which shows that $j \in A_{r+1,b}$ and as above this yields to the fact that

$$A_{0,b} = A_{1,b}$$

Next we do the same method for an arbitrary element of L_1 which is covered by a maximal element of L_1 and continuing in this way we get that

$$\forall b \in L_1, \ A_{0,b} = A_{1,b} \tag{85}$$

Now by equations (1) and (2), $\mathbf{a}$ is a fixed point of f. □

We remark that Theorem 2.2 is an extension of [**?**, Theorem 1] where f is finite Boolean dynamical system and $\wedge, \vee$ are AND, OR respectively. Next we study some special cases.

35.3. Corollary. *Let $(L, \leq)$ be a finite lattice and $f : L^n \to L^n$ be a lattice network with dependency graph $G = (V, E)$. Then the following holds.*

(i) If for each $1 \leq j \leq n$, $f_j = \bigwedge_{(s,j)\in E} x_s$, then an arbitrary point $\mathbf{a} = (a_1, \ldots, a_n)$ converges to the fixed point $\mathbf{b_a} = (b_1, \ldots, b_n)$ where for each $1 \leq i \leq n$,

$$b_i = \bigwedge_{\textit{there is a directed path from } s \textit{ to } i} a_s$$

(ii) If for each $1 \leq j \leq n$, $f_j = \bigvee_{(s,j)\in E} x_s$, then an arbitrary point $\mathbf{a} = (a_1, \ldots, a_n)$ converges to the fixed point $\mathbf{b_a} = (b_1, \ldots, b_n)$ where for each $1 \leq i \leq n$,

$$b_i = \bigvee_{\textit{there is a directed path from } s \textit{ to } i} a_s$$

(iii) If f is as (i) or (ii), and G is strongly connected then an arbitrary point $\mathbf{a} = (a_1, \ldots, a_n)$ converges to the fixed point $\mathbf{b_a} = (b_1, \ldots, b_n)$ where for each $1 \leq i \leq n$, $b_i = \bigvee_{1\leq s\leq n} a_s$ and in particular $\mathbf{a} = (a_1, \ldots, a_n)$ is a fixed point of f if and only if $a_1 = \cdots = a_n$.

Conclusion

Let f be a lattice network over a dependency graph $G = (V, E)$. We prove that f is a fixed-point system provided that all vertices of G have a self-loop.

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36. A remark on the definition of Ricci flow in Finsler geometry

Mohamad Yar Ahmadi[a], Sina Hedayatian

Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran

The Ricci flow in Finsler geometry is defined as a natural generalization of the Hamilton's Ricci flow on Riemannian maniflods. It seems that, this definition is completely suitable for Ricci flow in Finsler geometry. Here, it's stated the reasons for the appropriateness of this definition, similar to the Hamilton's work.

Keywords: Ricci flow, Finsler space, gradient flow, Einstein-Hilbert functiona.
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[a]Speaker. Email address: m.yarahmadi@scu.ac.ir,

Introduction

Ricci flow is a branch of general geometric flows, which is an evolution equation for a Riemannian metric in the set of all Riemannian metrics defined on a manifold. Geometric flow can be used to deform an arbitrary metric into an informative metric, from which one can determine the topology of the underlying manifold and hence innovate numerous progress in the proof of some geometric conjectures. In 1982 Hamilton introduced the notion of Ricci flow on Riemannian manifolds by the evolution equation

$$\frac{\partial}{\partial t} g_{ij} = -2Ric_{ij}, \quad g(t=0) := g_0. \tag{86}$$

The Ricci flow, which evolves a Riemannian metric by its Ricci curvature is a natural analogue of the heat equation for metrics. In Hamilton's celebrated paper [4], it is shown that there is a unique solution to the Ricci flow for an arbitrary smooth Riemannian metric on a closed manifold over a sufficiently short time.

Let (M, g) be a closed Riemannian manifold. One of the most natural functionals one can construct on $\mathcal{M}$ is the so-called Einstein-Hilbert functional $E : \mathcal{M} \longrightarrow \mathbb{R}$, which is the integral of the scalar curvature: $E(g) = \int_M R_g d\mu$, where R_g is the scalar curvature related to g. Hamilton has computed the variation of E at g, in direction $\frac{\partial}{\partial s} g_{ij} = v_{ij}$, and he has concluded that

$$\frac{d}{ds} E(g) = \int_M < v, \frac{1}{2} Rg - Rc > d\mu.$$

Note that (twice) the gradient flow of E is

$$\frac{\partial}{\partial s} g_{ij} = 2(\nabla E(g))_{ij} = Rg_{ij} - 2R_{ij}.$$

We note that this equation looks similar to the Ricci flow, but the extra term means that this equation is not parabolic and as such, short time existence is not expected to hold. Dropping the Rg term on the RHS of last formula yields the Ricci flow.
The concept of Ricci flow on Finsler manifolds is defined first by D. Bao, cf., [3], choosing the Ricci tensor introduced by H. Akbar-Zadeh, [2]. It seems to the present authors that, this choice of D. Bao for definition of Ricci tensor, is completely suitable for definition of Ricci flow in Finsler geometry. In fact, in order to define the concept of Ricci tensor, Akbar-Zadeh has used Einstein-Hilbert's functional in general relativity, although it has some computation negligence and introduced definition of Einstein-Finsler spaces as critical points of this functional, similar to the Hamilton's work.

Main results

The concept of Ricci flow on Finsler manifolds is defined first by D. Bao, cf., [3], choosing the Ricci tensor introduced by H. Akbar-Zadeh, [2]. Let M be a compact Finslerian manifold and Ric_{jk} the symmetric tensor defined by $\frac{1}{2}\frac{\partial^2(F^2\mathcal{R}ic)}{\partial y^j y^k}$. Let λ be a differentiable function on M. We consider the scalar function on $S(M)$ defined by $\hat{H} = \tilde{H} - \lambda(x)\mathcal{R}ic$, where $\tilde{H} = g^{jk}Ric_{jk}$. Akbar-Zadeh consider the functional $E(g) = \int_{S(M)} \hat{H} d\mu$ as energy-functional similar to Hamilton's approach in Finsler space, cf. [1]. Here, we look for the gradient flow of the energy functional introduced by Akbar-Zadeh. It is computed that the variation of E at g, in direction $\frac{\partial}{\partial s}g_{ij} = v_{ij}$ is

$$\frac{d}{ds}E(g) = -< A, v >= -\int_{S(M)} A^{jk}v_{jk}\, d\mu,$$

where $<,>$ denotes the global scalar product and A is defined by

$$A^{jk} = Ric^{jk} - \lambda\mathcal{R}ic l^j l^k - (n\tau - \phi)l^j l^k - \hat{H}(g^{jk} - \frac{1}{2}nu^j u^k).$$

The (twin) gradient flow of energy functional introduced by Akbar-Zadeh is $\frac{\partial}{\partial s}g_{jk} = -2A_{jk}$, that is

$$\frac{\partial}{\partial s}g_{jk} = -2(Ric_{jk} - \lambda\mathcal{R}ic l_j l_k - (n\tau - \phi)l_j l_k - \hat{H}(g_{jk} - \frac{1}{2}nl_j l_k)).$$

By multiplying the two sides by l^j and l^k successively we get

$$\frac{d}{ds}\ln(F(t)) = -(1 - \frac{1}{2}n\lambda)Ric + (n\tau - \phi) - (1 - \frac{1}{2}n)\tilde{H}.$$

Case $\lambda = 0$. In this case the integral $E(g)$ (the energy functional introduced by Akbar-Zadeh) is reduced to $E_1(g) = \int_{SM} \tilde{H} d\mu$. The derivative of $E_1(g)$ is $E_1'(g) = -< \tilde{A}, v >$, where

$$\tilde{A}_{ij} = Ric_{ij} - n\tau l_i l_j - \tilde{H}(g_{ij} - \frac{1}{2}nl_i l_j).$$

The (twin) gradient flow of energy functional $E_1(g)$ is

$$\frac{\partial}{\partial s}g_{ij} = -2\tilde{A}_{ij}.$$

By multiplying the two sides by u^j and u^k successively we get

$$\frac{d}{ds}\ln(F(t)) = -H(u, u) + n\tau - (1 - \frac{1}{2}n)\tilde{H}.$$

Similar to Hamilton's approach for definition of Ricci flow, we note that this equation looks similar to the Ricci flow, but the extra term means that this equation is not parabolic and as such, short time existence is not expected to hold. Dropping the $n\tau - (1 - \frac{1}{2}n)\tilde{H}$ term on the RHS of last formula yields the Finsler Ricci flow $\frac{d}{ds}\ln(F(t)) = -H(u, u)$. Therefore the Finsler Ricci flow introduced by David Bao is completely suitable for definition of Ricci flow in Finsler geometry.

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37. Mean curvature of semi-symmetric metric connections

Elham Zangiabadi1;, Zohreh Nazari2
1Department of mathematics, Vali-e-Asr university, Rafsanjan, Iran
2Department of mathematics, Vali-e-Asr university, Rafsanjan, Iran

In this paper, we first introduce semi-symmetric metric and semi-symmetric non-metric connections on a (n+p)-dimensional semi-Riemannian manifold (M; g). Then we obtain relations between mean curvatures of these connections and the Levi-Civita connection.

Keywords: Mean curvature, Semi-symmetric metric connection, semi-symmetric non metric connection.

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Introduction

The notion of semi-symmetric connections on a differentiable manifold is introduced by Friedman and Schouten in 1924 [2]. The study of it was further developed by some researcher such as Yano [5].
These mentioned connections have applications in Physics. There are various physical problems involving them.
In this work, we find relations between mean curvatures of these connections and the Levi-Civita connection.
In the following, we provide basic information used in the paper.
A linear connection ∇ on a semi-Riemannian manifold (M, g) is said to be semi-symmetric if the torsion tensor T of the connection ∇ satisfies

$$T(X,Y) = \omega(Y)X - \omega(X)Y, \tag{87}$$

for any vector fields X, Y on M and ω is a 1-form given by $\omega(X) = g(X, W)$, where W is the vector field associated with the 1-form ω.
If $\nabla g = 0$, then the connection ∇ is said to be a metric connection; otherwise, it is called non-metric [3].
Now let (M, g) be an $(n+p)-$dimensional semi-Riemannian manifold endowed with an $n-$distribution $\mathcal{D}$. The real vector space of all symmetric bilinear mappings $g_x : \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{R}$, is denoted by $L^2_s(\mathcal{D}_x, \mathbb{R})$. Then we consider the vector bundle

$$L^2_s(\mathcal{D}, \mathbb{R}) = \bigcup_{x \in M} L^2_s(\mathcal{D}_x, \mathbb{R})$$

over M. The metric tensor g induces a global section of $L^2_s(\mathcal{D}, \mathbb{R})$ which is denoted by the same symbol g. If g is non-degenerate, the pair $(\mathcal{D}, g)$ is a semi-Riemannian distribution.

Also the vector bundle $D^{\perp}$ is considered as follows

$$\mathcal{D}^{\perp} = \bigcup_{x\in M} \mathcal{D}_x^{\perp},$$

where $\mathcal{D}_x^{\perp}$ is the complementary orthogonal subspace to $\mathcal{D}_x$ in (T_xM, g_x). The metric tensor g induces a semi-Riemannian metric on $\mathcal{D}^{\perp}$, and here again we denote it by g. Therefore $(\mathcal{D}^{\perp}, g)$ is a semi-Riemannian distribution. Thus we have

$$TM = \mathcal{D} \oplus \mathcal{D}^{\perp}. \tag{88}$$

Also the mappings $\mathcal{P}$ and $\mathcal{Q}$ are the projection morphisms of TM on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively.

$$a)\ \tilde{H}(X, \mathcal{P}Y) = \mathcal{Q}\tilde{\nabla}_X \mathcal{P}Y, \quad b)\ \tilde{H}^{\perp}(X, \mathcal{Q}Y) = \mathcal{P}\tilde{\nabla}_X \mathcal{Q}Y, \tag{89}$$

where $\tilde{H}$ and $\tilde{H}^{\perp}$ are the second fundamental forms of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ with respect to $\tilde{\nabla}$, respectively [1].

Main Results

We now suppose that the semi-Riemannian manifold (M, g) admits a semi-symmetric metric connection given by

$$\nabla_X Y = \tilde{\nabla}_X Y + \omega(Y)X - g(X, Y)W, \tag{90}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on (M, g), ω is a 1-form and W is the vector field defined by

$$g(W, X) = \omega(X), \tag{91}$$

for any vector field X of M (see [4], [5]).

$$a)\ H(X, \mathcal{P}Y) = \mathcal{Q}\nabla_X \mathcal{P}Y, \quad b)\ H^{\perp}(X, \mathcal{Q}Y) = \mathcal{P}\nabla_X \mathcal{Q}Y. \tag{92}$$

We call H (resp. $H^{\perp}$) the second fundamental forms of $\mathcal{D}$ (resp. $\mathcal{D}^{\perp}$) with respect to ∇. Since ∇ is metric by using (92) we obtain that

$$\begin{aligned} g(H(X, \mathcal{P}Y), \mathcal{Q}Z) &= g(\nabla_X \mathcal{P}Y, \mathcal{Q}Z) \\ &= -g(\mathcal{P}Y, \nabla_X \mathcal{Q}Z) \\ &= -g(H^{\perp}(X, \mathcal{Q}Z), \mathcal{P}Y). \end{aligned} \tag{93}$$

By defination of the semi-symmetric metric connection ∇ and by (89a), (89b), (92a) and (92b) we deduce that

$$H(X, \mathcal{P}Y) = \tilde{H}(X, \mathcal{P}Y) + \omega(\mathcal{P}Y)\mathcal{Q}X - g(X, \mathcal{P}Y)\mathcal{Q}W,$$

and

$$H^{\perp}(X, \mathcal{Q}Y) = \tilde{H}^{\perp}(X, \mathcal{Q}Y) + \omega(\mathcal{Q}Y)\mathcal{P}X - g(X, \mathcal{Q}Y)\mathcal{P}W.$$

Therefore

$$H(\mathcal{P}X, \mathcal{P}Y) = \tilde{H}(\mathcal{P}X, \mathcal{P}Y) - g(\mathcal{P}X, \mathcal{P}Y)\mathcal{Q}W, \tag{94}$$

$$H^{\perp}(\mathcal{Q}X, \mathcal{Q}Y) = \tilde{H}^{\perp}(\mathcal{Q}X, \mathcal{Q}Y) - g(\mathcal{Q}X, \mathcal{Q}Y)\mathcal{P}W. \tag{95}$$

The second fundamental forms and the shap operators of the distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are related by [1]

$$g(\tilde{H}(\mathcal{P}X, \mathcal{P}Y), \mathcal{Q}Z) = g(\tilde{A}_{\mathcal{Q}Z}\mathcal{P}X, \mathcal{P}Y), \tag{96}$$

and

$$g(\tilde{H}^{\perp}(\mathcal{Q}X, \mathcal{Q}Y), \mathcal{P}Z) = g(\tilde{A}^{\perp}_{\mathcal{P}Z}\mathcal{Q}X, \mathcal{Q}Y). \tag{97}$$

Let $E_1, ..., E_n$ be an orthonormal basis for $\mathcal{D}_p$ of signature $\varepsilon_1, ..., \varepsilon_n$, then

$$\tilde{\text{II}} = \frac{1}{n}\Sigma_{i=1}^{n}\varepsilon_i \tilde{H}(E_i, E_i)$$

is the mean curvature vector field of $(\mathcal{D}, g)$ with respect to $\tilde{\nabla}$ and

$$\text{II} = \frac{1}{n}\Sigma_{i=1}^{n}\varepsilon_i H(E_i, E_i)$$

is the mean curvature of $(\mathcal{D}, g)$ with respect to ∇. We obtain from (94)

$$\begin{aligned}
\text{II} &= \frac{1}{n}\Sigma_{i=1}^{n}\varepsilon_i H(E_i, E_i) && (98)\\
&= \frac{1}{n}\Sigma_{i=1}^{n}\varepsilon_i\{\tilde{H}(E_i, E_i) - g(E_i, E_i)\mathcal{Q}W\} && (99)\\
&= \frac{1}{n}\Sigma_{i=1}^{n}\varepsilon_i\{\tilde{H}(E_i, E_i)\} - \frac{1}{n}\Sigma_{i=1}^{n}\varepsilon_i g(E_i, E_i)\mathcal{Q}W\}\\
&= \tilde{\text{II}} - \mathcal{Q}W.
\end{aligned}$$

We can state the following.

37.1. Theorem. *The mean curvature of $(\mathcal{D}, g)$ with respect to $\tilde{\nabla}$ concides with that of $(\mathcal{D}, g)$ with respect to ∇, if a vector W lies in $\mathcal{D}$.*

If $\tilde{H}$ vanishes, then $(\mathcal{D}, g)$ is totally geodesic with respect to $\tilde{\nabla}$, and if $\tilde{H}(\mathcal{P}X, \mathcal{P}Y) = g(\mathcal{P}X, \mathcal{P}Y)\tilde{\text{II}}$, then $(\mathcal{D}, g)$ is totally umblical with respect to $\tilde{\nabla}$. Similarly, If H vanishes, then $(\mathcal{D}, g)$ is totally geodesic with respect to ∇, and if $H(\mathcal{P}X, \mathcal{P}Y) = g(\mathcal{P}X, \mathcal{P}Y)\text{II}$, then $(\mathcal{D}, g)$ is totally umblical with respect to ∇.
From (94) and (98), we have the following Proposition:

37.2. Proposition. *The semi-symmetric distribution $(\mathcal{D}, g)$ is totally umblical with respect to $\tilde{\nabla}$ if and only if it is totally umblical with respect to the semi-symmetric metric connection ∇.*

If $\tilde{\text{II}} = 0$ $(resp.\text{II} = 0)$, then $(\mathcal{D}, g)$ is called minimal with respect to $\tilde{\nabla}$ $(resp.\nabla)$ from equation (98) we have the following result:

37.3. Theorem. *$(\mathcal{D}, g)$ is minimal with respect to the semi-symmetric metric connection ∇ if and only if it is minimal with respect to the Levi-Civita connection, when a vector field W lies in $\mathcal{D}$.*

A linear connection $\breve{\nabla}$ on a semi-Riemannian manifold (M, g) defined by

$$\breve{\nabla}_X Y = \tilde{\nabla}_X Y + \omega(Y)X \tag{100}$$

is a semi-symmetric non-metric connection, where $\tilde{\nabla}$ is the Levi-Civita connection of (M, g) and ω is a 1-form.

$$a)\ \breve{H}(X, \mathcal{P}Y) = \mathcal{Q}\breve{\nabla}_X\mathcal{P}Y, \quad b)\ \breve{H}^{\perp}(X, \mathcal{Q}Y) = \mathcal{P}\breve{\nabla}_X\mathcal{Q}Y. \tag{101}$$

$\breve{H}$ and $\breve{H}^{\perp}$ are the second fundamental forms of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively. By (100) and (101)

$$\breve{H}(X, \mathcal{P}Y) = \tilde{H}(X, \mathcal{P}Y) + \omega(\mathcal{P}Y)\mathcal{Q}X, \tag{102}$$

$$\breve{H}^{\perp}(X, \mathcal{Q}Y) = \tilde{H}^{\perp}(X, \mathcal{Q}Y) + \omega(\mathcal{Q}Y)\mathcal{P}X. \tag{103}$$

Thus we obtain that

$$a)\ \breve{H}(\mathcal{P}X, \mathcal{P}Y) = \tilde{H}(\mathcal{P}X, \mathcal{P}Y) \quad b)\ \breve{H}^{\perp}(\mathcal{Q}X, \mathcal{Q}Y) = \tilde{H}^{\perp}(\mathcal{Q}X, \mathcal{Q}Y). \tag{104}$$

$$\breve{\text{II}} = \frac{1}{n}\Sigma_{i=1}^{n}\varepsilon_i \breve{H}(E_i, E_i) \tag{105}$$

is the mean curvature of $(\mathcal{D}, g)$ with respect to the semi-symmetric non-metric connection $\breve{\nabla}$, where $E_1, ..., E_n$ is an orthonormal basis for $\mathcal{D}_p$ of signature $\varepsilon_1, ..., \varepsilon_n$.
Hence by 102 we may state the following results.

37.4. Theorem. *The mean curvature of $(\mathcal{D}, g)$ with respect to $\tilde{\nabla}$ concides with that of $(\mathcal{D}, g)$ with respect to the semi-symmetric non-metric connection $\breve{\nabla}$.*

If $\breve{\mathrm{II}} = 0$, then $(\mathcal{D}, g)$ is called minimal with respect to $\breve{\nabla}$.

37.5. Theorem. *$(\mathcal{D}, g)$ is minimal with respect to $\breve{\nabla}$ if and only if it is minimal with respect to $\tilde{\nabla}$.*

If $\breve{H}$ vanishes, then $(\mathcal{D}, g)$ is totally geodesic with respect to $\breve{\nabla}$ and if $\breve{H}(\mathcal{P}X, \mathcal{P}Y) = g(\mathcal{P}X, \mathcal{P}Y)\breve{\mathrm{II}}$, then $(\mathcal{D}, g)$ is said to be totally umblical with respect to $\breve{\nabla}$.

37.6. Theorem. *(see[6]) The semi-Riemannian distribution $(\mathcal{D}, g)$ is totally geodesic (totally umblical) with respect to $\tilde{\nabla}$ if and only if it is totally geodesic (totally umblical) with respect to $\breve{\nabla}$.*

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38. Black-Scholes pricing model and Tepix of Iran

R. Fallah-Moghaddam1
, 1Department and Computer Science, University of Garmsar, Garmsar, Iran

Many classic finance, such as the Black-Scholes option pricing model, has its origins equation:

$$\frac{1}{P}dP = \mu dt + \sigma dW$$

In this article, we try to review this model on the Iran Stock Exchange index in 1399. This led to finding relation to model the future forecast of the Tepix of Iran.

Keywords: Financial markets, Black-Scholes pricing model, Tepix of Iran..
AMS Mathematics Subject Classification [2020]: 37C05, 37H99
Code: cdsgt3-00860030

Introduction

Block-chain technology enables a large number of traders to conduct electronic transactions. In fact, a new set of currencies called cryptocurrencies, in recent years, has attracted the attention of many traders. The price of Bitcoin focused the spotlight of public attention on cryptocurrencies that evolved into a new asset class. Following the pattern of other nascent assets, speculators dominated trading and pushed prices toward a bubble. Directly without intermediaries, and in recent years has led to a new form of payment. Financial markets in the world today, whether forex or cryptocurrencies like Bitcoin or in general, all financial markets have very complex fluctuations. But with all these fluctuations that are practically a kind of chaotic property for such markets, by considering these markets as a category of dynamic systems in order to model and formulate such markets can be done.

Black and Scholes attempted to apply the formula to the markets, but incurred financial losses, due to a lack of risk management in their trades. In 1970, they decided to return to the academic environment. Scholes received the 1997 Nobel Memorial Prize in Economic Sciences for his work, the committee citing their discovery of the risk neutral dynamic revision as a breakthrough that separates the option from the risk of the underlying security. Many classic finance, such as the Black-Scholes option pricing model, has its origins equation:

$$\frac{1}{P}dP = \mu dt + \sigma dW,$$

for the change in the relative price $P^{-1}dP$ in terms of the expected return, μ, the standard deviation of the return, σ, and independent increments of Brownian motion, dW. The SDE can solved this equation analytically and the solution has the form:

$$P(t) = P(0)exp([\mu - \frac{\sigma^2}{2}]t + \sigma W(t)),$$

where $(W(t))_{t\geq 0}$ is a Brownian motion.

In fact, Black-Scholes is a pricing model used to determine the fair price or theoretical value of a buy or sell option based on six variables such as fluctuations, type of option, stock price, time, strike price and risk-free rate. Quantum is more speculation about stock market derivatives, and therefore proper pricing of options eliminates the possibility of any arbitrage. There are two important models for option pricing, the binomial model and the Black Scholes model. This model is used to determine the price of a European purchase option, which simply means that this option is only valid on the expiration date. For more information in this regard, we refer dear readers to references [1], [2], [3],[4], [5] and [6].

Main results

In this article, we intend to examine the effects of the Black-Scholes model on the total index of Iran Stock Exchange. Table (1) contains the monthly information of the total index of Iran Stock Exchange based on the closing number of the monthly candlestick. We get data through the Tse Clint software.

TABLE 5. Your table's caption

Month	Monthly Tepix of Iran	Returns
1	741960	
2	986759	0.329935576
3	1419453	0.438500181
4	1901147	0.339351849
5	1718783	-0.095923145
6	1611582	-0.062370293
7	1288330	-0.200580548
8	1367248	0.061256045
9	1447915	0.058999538
10	1183978	-0.182287634
11	1205832	0.018458113
12	1294521	0.073550047

Therefore, we have $\mu = 0.070808157$ and $\sigma = 299957.6543$.

Notice that

$$dP = \mu P dt + \sigma P dW.$$

So, in the interval $[a, b]$, we have

$$\int_a^b dP = \int_a^b \mu P dt + \int_a^b \sigma P dW.$$

Therefore, considering the numerical approximation $\int_a^b f(t) = f(a)(b-a)$, we have $P(b) = (1+\mu)P(a) + \sigma P(a)(W(b) - W(a))$. We know that $W(t)$ is a Brownian motion. Therefore,

$$W(b) - W(a) \sim N(0, b-a).$$

Thus,

$$P(b) = (1+\mu)P(a) + \sigma P(a)N(0, b-a)).$$

Hence, if we use Table 1, we can get the following relation beyond the total index of Iran Stock Exchange.

$$P(1) = (1+\mu)P(0) + \sigma P(a)N(0,1)) = P(0)((1.070808157) + 299957.6543N(0,1)).$$

When $P(1)$ is the Tepix of Iran for one year later. This relation can be used as a approximation method using numerical methods and can practically be considered as one of the available approximate methods to predict the future trend of the Iranian stock market.

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39. Schouten and Vranceanu connections on metallic manifold

Zohreh Nazari1;, Elham Zangiabadi2
1Department of mathematics, Vali-e-Asr university, Rafsanjan, Iran
2Department of mathematics, Vali-e-Asr university, Rafsanjan, Iran

In this paper, we introduce two linear connections, that are called Schouten and Vranceanu connections, on a metallic Riemannian manifold and study the notion of parallelism for distributions derived frome metallic structure with respect to the Schouten and Vranceanu connections.

Keywords: metallic structure, Schouten connection, Vranceanu connection, parallelism, half parallesim.
AMS Mathematics Subject Classification [2020]: 53C05, 53C15
Code: cdsgt3-00640046

Introduction

In 1999, the concept of the metallic ratio was introduced by Vera W. de Spinadel [4]. Recently, many researchers studied metallic structures [2, 3].
Now, we recall some necessary notions.
Let M be a C^∞ manifold, A tensor J of type $(1,1)$ is said to be a metallic structure if

$$J^2 = pJ + qI,$$

where p and q are two positive real numbers.
A C^∞ manifold M equipped with a metallic structure J is called a metallic manifold and denoted by (M,J)[2]. The solutions of the equation $x^2 - px - q = 0$ are named the metallic means family and denoted by

$$\sigma_+ = \frac{p+\sqrt{p^2+4q}}{2}, \; \sigma_- = \frac{p-\sqrt{p^2+4q}}{2} \tag{106}$$

The projections operator with respect to metallic structure J are as follows:

$$\mathcal{P} = \frac{-1}{\sqrt{p^2+4q}} J + \frac{\sigma_+}{\sqrt{p^2+4q}} I,$$

$$\mathcal{P}' = \frac{1}{\sqrt{p^2+4q}} J - \frac{\sigma_-}{\sqrt{p^2+4q}} I. \tag{107}$$

These maps satisfy in the following relations:

$$\mathcal{P}^2 = \mathcal{P}, \; \mathcal{P}' = \mathcal{P}', \; \mathcal{P} + \mathcal{P}' = I.$$

Also, we have $J = \sigma_+\mathcal{P} + \sigma_-\mathcal{P}'$, where J is a metallic structure.

The corresponding distributions with respect to $\mathcal{P}$ and $\mathcal{P}'$ are as follows:

$$D_p = \{X_p \in T_pM,\ JX_p = \sigma_+ X_p\},\ D = \bigcup_p D_p,$$

$$D'_p = \{X_p \in T_pM,\ JX_p = \sigma_- X_p\},\ D' = \bigcup_p D'_p, \tag{108}$$

so $TM = D \oplus D'$.

Main Results

In this section, we investigate the notion of parallelism for the distributions D and D', defined in (108), with respect to the Schouten and Vranceanu connections.

39.1. Definition. The Schouten and Vranceanu connections with respect to metallic structure are defined as follows [1]:

$$\nabla^S_X Y = \mathcal{P}\nabla_X \mathcal{P}Y + \mathcal{P}'\nabla_X \mathcal{P}'Y,$$
$$\nabla^V_X Y = \mathcal{P}\nabla_{\mathcal{P}X}\mathcal{P}Y + \mathcal{P}'\nabla_{\mathcal{P}'X}\mathcal{P}'Y + \mathcal{P}[\mathcal{P}'X, \mathcal{P}Y] + \mathcal{P}'[\mathcal{P}X, \mathcal{P}'Y].$$

39.2. Definition. Let D be a distribution on M. D is parallel with respect to linear connection ∇ if the vector $\nabla_X Y \in \Gamma(D)$.

39.3. Proposition. *Let (M, J) be a metallic Riemannian manifold and D and D' be the distributions with respect to $\mathcal{P}$ and $\mathcal{P}'$ defined in (107) and (108). The following statements are valid.*

1) *Both of the distributions D and D' are parallel with respect to the Schouten and Vranceanu connection.*
2) $\nabla^S_X \mathcal{P}Y = \mathcal{P}\nabla_X \mathcal{P}Y.$

for all $X, Y \in TM$.

Proof. 1) We must show that $\nabla^S_X Y, \nabla^V_X Y \in \Gamma(D)$. For this purpose we have if $Y \in \Gamma(D)$ then $\mathcal{P}'(Y) = 0$, so

$$\nabla^S_X Y = \mathcal{P}\nabla_X \mathcal{P}Y + \mathcal{P}'\nabla_X \mathcal{P}'Y = \mathcal{P}\nabla_X \mathcal{P}Y \in \Gamma(D).$$

and

$$\begin{aligned}\nabla^V_X Y &= \mathcal{P}\nabla_{\mathcal{P}X}\mathcal{P}Y + \mathcal{P}'\nabla_{\mathcal{P}'X}\mathcal{P}'Y + \mathcal{P}[\mathcal{P}'X, \mathcal{P}Y] + \mathcal{P}'[\mathcal{P}X, \mathcal{P}'Y] \\ &= \mathcal{P}\nabla_{\mathcal{P}X}\mathcal{P}Y + \mathcal{P}[\mathcal{P}'X, \mathcal{P}Y] \in \Gamma(D).\end{aligned}$$

Similarly, we can prove this for D'.

2) By a straightforward calculation we get

$$\begin{aligned}\nabla^S_X \mathcal{P}Y &= \mathcal{P}\nabla_X \mathcal{P}^2 Y + \mathcal{P}'\nabla_X \mathcal{P}'\mathcal{P}Y \\ &= \mathcal{P}\nabla_X \mathcal{P}Y.\end{aligned} \tag{109}$$

□

39.4. Proposition. *Let (M, J) be a metallic Riemannian manifold. Then $\nabla^S_X \mathcal{P} = 0$, where $\mathcal{P}$ is the projection map from TM to D.*

Proof. we have

$$\begin{aligned}\nabla^S_X \mathcal{P}Y = \mathcal{P}(\nabla^S_X Y) + (\nabla^S_X \mathcal{P})Y &= \mathcal{P}(\mathcal{P}\nabla_X \mathcal{P}Y + \mathcal{P}'\nabla_X \mathcal{P}'Y) + (\nabla^S_X \mathcal{P})Y \\ &= \mathcal{P}\nabla_X \mathcal{P}Y + (\nabla^S_X \mathcal{P})Y,\end{aligned} \tag{110}$$

According to Eqs. (109) and (110), we obtained $(\nabla^S_X \mathcal{P})Y = 0$, for all $Y \in TM$, so $\nabla^S_X \mathcal{P} = 0$ □

39.5. Proposition. *Let (M, J) be a metallic Riemannian manifold. Then the metallic structure J is parallel with respect to Schouten and Vranceanu connections.*

PROOF. we have $\nabla^S_X JY = \mathcal{P}\nabla_X \mathcal{P}JY + \mathcal{P}'\nabla_X \mathcal{P}'JY$. Since

$$\mathcal{P}J = J\mathcal{P} = \sigma_+\mathcal{P},$$

and

$$\mathcal{P}'J = J\mathcal{P}' = \sigma_-\mathcal{P},$$

so we have

$$\begin{aligned} \nabla^S_X JY &= \mathcal{P}\nabla_X \sigma_+\mathcal{P}Y + \mathcal{P}'\nabla_X \sigma_-\mathcal{P}Y \\ &= \sigma_+\mathcal{P}\nabla_X \mathcal{P}Y + \sigma_-\mathcal{P}'\nabla_X \mathcal{P}Y. \end{aligned} \tag{111}$$

on the other hand

$$\begin{aligned} \nabla^S_X JY &= (\nabla^S_X J)Y + J(\nabla^S_X Y) \\ &= (\nabla^S_X J)Y + J(\mathcal{P}\nabla_X \mathcal{P}Y + \mathcal{P}'\nabla_X \mathcal{P}'Y) \\ &= (\nabla^S_X J)Y + \sigma_+\mathcal{P}\nabla_X \mathcal{P}Y + \sigma_-\mathcal{P}'\nabla_X \mathcal{P}Y). \end{aligned} \tag{112}$$

By Eqs. (111) and (112), we have $(\nabla^S_X J)Y = 0$, for all $Y \in TM$. So $\nabla^S_X J = 0$.
Similarly, we can show that $\nabla^V_X J = 0$. □

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40. McGinley dynamic indicator and Tepix of Iran

R. Fallah-Moghaddam1
, 1Department and Computer Science, University of Garmsar, Garmsar, Iran

This paper examines the impact of McGinley dynamic indicator on the total index of Iran Stock Exchange. The trend of the total index of Iran Stock Exchange is considered as a dynamic system. Two points are important here. The first point is that this indicator, using recursive relations, presents a very close and formulable approximation of the trend chart of the total index of Iran Stock Exchange. The next point is that the chaotic behaviors in the chart of the total index of the Iranian Stock Exchange have been leveled by using this indicator.

Keywords: Financial markets, McGinley dynamic indicator, Tepix of Iran.
AMS Mathematics Subject Classification [2020]: 37C05, 37H99
Code: cdsgt3-00860028

Introduction

The foundation of stochastic finance is return independence, which is the key assumption in the random walk model. Real stock prices exhibit higher-order and nonlinear correlations, thus according to the ARCH and GARCH models, the classical approach to deal with this problem is to model the volatility parameter in the random walk model as a random process. This means that the price returns are in general not independent. The classical models are non-linear stochastic equations. Also, they are descriptive in nature and they could not provide quantitative links between return independence and trader actions. One of the best topics in this field has been considered by researchers is that how the returns generated by our price dynamical model are changing from positively correlated to uncorrelated and then to negatively correlated as the model parameters change.

Indicators in financial markets are in fact a kind of observer of financial markets as a dynamic system. The highly complex and volatile behavior of global financial markets increases the need for these indicators every day. Very important indicators such as RSI, MACD, MA, SMA, ... are some of these indicators that predict the future trend of a financial market as a dynamic system. The McGinley Dynamic is a little-known yet highly reliable indicator invented by John R. McGinley, a chartered market technician and former editor of the Market Technicians Association's Journal of Technical Analysis. Working within the context of moving averages throughout the 1990s, McGinley sought to invent a responsive indicator that would automatically adjust itself in relation to the speed of the market. His eponymous dynamic, first published in the Journal of Technical Analysis in 1997, is a 10-day simple and exponential moving average with a filter that smooths the data to avoid whipsaws. The McGinley Dynamic indicator is a type of moving average that was designed to track the market better than existing moving average indicators. It is a technical indicator that improves upon moving average lines by adjusting for shifts in market speed. This indicator solves the issue of varying market speeds by incorporating an automatic adjustment factor into its formula, which speeds (or slows) the indicator in trending, or ranging, markets. The McGinley Dynamic indicator improves upon conventional moving averages by minimizing price separations and volatile whipsaws so that price action is more accurately reflected. McGinley Dynamic Formula is:

$$MD_i = MD_{i-1} + \frac{Close - MD_{i-1}}{k \times N \times (\frac{Close}{MD_{i-1}})^4}.$$

Where:

(1) MD_i=Current McGinley Dynamic;
(2) Close=Closing price ;
(3) MD_{i-1}=Previous McGinley Dynamic;
(4) k=0.6 (Constant 60 percent of selected period N;
(5) N=Moving average period.

In fact, McGinley found that the moving averages were too often applied incorrectly. The period of the moving averages should be adjusted to the speed of the market changes. Another problem McGinley saw in the moving averages was that they are often too far separated from the prices. They should follow the price to give the right signals to open a position. To read more about the structure of indicators and how they work, we refer dear readers to references [1], [2], [3], [4] and [5].

Main results

Global financial markets today are known as highly complex dynamic systems. These markets have volatile movements and with these volatile movements they cause losses to traders. In fact, it is these fluctuations that cause bubbles in financial markets. These bubbles cause the real price of a financial market to never happen easily. Positive and negative price bubbles actually cause drastic changes in the price movements of a financial market. In this article, we intend to examine the trend of the total index of the Iranian Stock Exchange in the past two years by using McGinley dynamic indicator. Last year, the Iranian stock market experienced one of its biggest historical declines. But was this fall unpredictable? In fact, the indicators, that are based on the trend of the financial market, which in practice is a complex dynamic system, can predict the future of this financial market by putting together past information as a random process. The information that is examined in this article is the information of the total index of Iran Stock Exchange, the source of which is the site of the Iran Stock Exchange Organization. First, in Figure (1), we see the behavior of the indicator on the total index of Iran Stock Exchange. Sometimes an indicator on the chart of a financial trend can be very telling and useful. As can be seen in Figure (1), McGinley dynamic indicator, Expresses important points from the Iranian stock market process.

It can be seen that this indicator can practically act as a simulator of the Iranian stock market trend in the last two years. Also, this indicator has formalized the kind of chaotic movements of the Iranian stock market by eliminating minor oscillating movements. In a dynamic system, the existence of a formulation is very important. From the trading point of view, this indicator has both support and resistance properties. Whenever this indicator is located above the chart of the total index of the Iranian Stock Exchange, it has acted as a resistance of the trend. This can be clearly seen in Figure (1). Also, whenever this indicator is below the chart of the Iranian Stock Exchange index, it has acted as a supporter of the trend. In other words, whenever this indicator is at the bottom of the chart of the total index of Iran Stock Exchange, the trend of the stock market index is upward, and whenever this indicator is at the top of the chart of the total stock index of Iran, the trend of the stock market index is downward. But, what is important here in terms of discussing dynamic systems are two points. The first point is that this chart, using recursive relations, presents a very close and formulable approximation of the trend chart of the total index of Iran Stock Exchange. The next point is that the chaotic behaviors in the chart of the total index of the Iranian Stock Exchange have been leveled using this indicator.

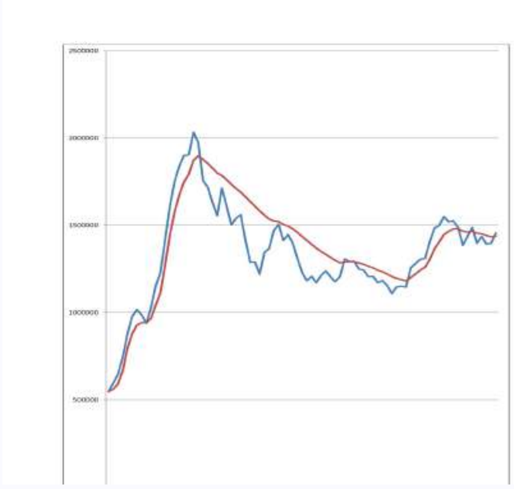

Figure 7. McGinley Dynamic Indicator and Tepix of Iran.

Acknowledgement

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41. A Study on Epidemic Models; Stability and Basic Reproduction Number

M.H. Rahmanidoust1,, A. Farahmandfard
Department of mathematics, University of Neyshbur, Neyshbur, Iran

Over the past century, mathematical modeling has made the connection between important public health questions and the basic parameters of infection for a proper understanding of the spread of disease has been used. Nowdays, every scientist and researcher knows the importance and appreciation of dynamical systems and differential equations in ecology, biology, medicine, , epidemiology and etc. The major topic in epidemiology is when time a disease is epidemic, endemic or pandemic. This is usually done by finding the basic reproduction number, R0. In this paper, we study SIR and SEIR models. In continuation after finding equilibrium point, we prove three theorems which analyzes locally and globally asymptotically stability and backward bifurcation.

Keywords: Stability, Basic reproduction number, Backward bifurcation.
AMS Mathematics Subject Classification [2020]: 93D05, 92D30, 34C23
Code: cdsgt3- 00930058

Introduction

We first give a brief literature to the modeling of epidemics; more thorough descriptions may be found in [1, 5]. One of the early triumphs of mathematical epidemiology was the formulation of a simple model by Kermack and McKendrick (1927) whose predictions are very similar to this behavior, observed in countless epidemics. The Kermack-Mendrick model is a compartmental model based on relatively simple assumptions on the rates of flow between different classes of members of the population and there is a threshold quantity which is called the basic reproduction number and denoted by R_0 which determines whether there is an epidemic [3].

Modeling

The special case of the model proposed by Kermack and McKendrick in 1927 which is the starting point for our study of epidemic models is as follows:

$$
\begin{aligned}
S^{'} &= -\beta SI, \\
I^{'} &= \beta SI - \alpha I, \\
R^{'} &= \alpha I.
\end{aligned}
\tag{113}
$$

In this model, we assume that $S(t)$ denotes the number of individuals who are susceptible to the disease, that is, who are not infected at time t. $I(t)$ denotes the number of infected individuals, assumed infectious and able to spread the disease by contact with susceptibles. $R(t)$ denotes the number of individuals who have been infected and then removed from the possibility of being

infected again or of spreading infection. In this model infected neighbors recover at rate α and infected neighbors transmit infection at rate β and it is based on the following assumptions:
(i) An average member of the population makes contact sufficient to transmit infection with βN others per unit time, where N represents total population size (mass action incidence).
(ii) Infectives leave the infective class at rate I per unit time.
(iii) There is no entry into or departure from the population, except possibly through death from the disease.
(iv) There are no any disease deaths, and the total population size is a constant N.
In many infectious diseases there is an exposed period after the transmission of infection from susceptibles to potentially infective members but before these potential infectives develop symptoms and can transmit infection. Cosider the $SEIR$ model with some infectivity in the exposed period, to incorporate an exposed period with mean exposed period $\frac{1}{\kappa}$, we add an exposed class E and use compartmentsS, E, I, R and total population size $N = S + E + I + R$ to give a generalization of the epidemic model (1) as follows:

$$\begin{aligned} S^{'} &= -\beta S I, \\ E^{'} &= \beta S I - \kappa E, \\ I^{'} &= \kappa E - \alpha I. \end{aligned} \tag{114}$$

42. Main Results

Now, we consider the $SEIR$ model infectivity in the exposed stage,

$$\begin{aligned} S^{'} &= -\beta S(I + \epsilon E), \\ E^{'} &= \beta S(I + \epsilon E) - \kappa E, \\ I^{'} &= \kappa E - \alpha I, \\ R^{'} &= \alpha I. \end{aligned} \tag{115}$$

The analysis of this model is the same as the analysis of (1), but with I replaced by $E + I$. That is, instead of using the number of infectives as one of the variables, we use the total number of infected members, whether or not they are capable of transmitting infection. In some diseases there is some infectivity during the exposed period. This may be modeled by assuming infectivity reduced by a factor ε during the exposed period. Here, the disease states are E and I, and hence the Jacobin matrix is as follows:

$$J = \begin{bmatrix} \epsilon\beta N - \kappa & \beta N \\ \kappa & -\alpha \end{bmatrix}$$

So the next generation matrix obtanied by the following matrices

$$F = \begin{bmatrix} \epsilon\beta N & \beta N \\ 0 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} \kappa & 0 \\ -\kappa & \alpha \end{bmatrix}$$

the matrix $K = FV^{-1}$ is referred to as the next generation matrix for the system at the disease-free equilibrium. Since FV^{-1} has rank 1, it has only one nonzero eigenvalue, and since the trace of the matrix is equal to the sum of the eigenvalues, we see that

$$R_0 = \frac{\varepsilon\beta N}{\kappa} + \frac{\beta N}{\alpha},$$

the element in the first row and first column FV^{-1}. If all of new infections are in a single compartment, as the case here, the basic reproduction number is the trace of the matrix FV^{-1}. There are some situations in $R_0 < 1$ in which it is possible to show that the asymptotic stability of the disease-free equilibrium is global, that is, all solutions approach the disease-free equilibrium, only those with initial values sufficiently close to this equilibrium.

System (3) has a continuum of disease-free equilibria (DFE), given by: $E_0 = (N, 0, 0)$ and the next generation operator method can be used to analyse the asymptotic stability property of the DFE.

42.1. Theorem. *Assume that the disease transmission model is given by*

$$
\begin{aligned}
x_i^{'} &= f_i(x, y) - v_i(x, y) \qquad && i = 1, ..., n \\
y_j^{'} &= g_j(x, y) && j = 1, ..., m
\end{aligned} \tag{116}
$$

The diseasefree equilibrium of (3.1) is locally asymptotically stable if $R_0 < 1$, *but unstable if* $R_0 > 1$.

Proof. Let F and V be as defined as above, and let J_{21} and J_{22} be the matrices of partial derivatives of g with respect to x and y evaluated at the disease-free equilibrium. The Jacobian matrix for the linearization of the system about the disease-free equilibrium has the block structure

$$
J = \begin{bmatrix} F - V & 0 \\ J_{21} & J_{22} \end{bmatrix}
$$

The disease-free equilibrium is locally asymptotically stable if the eigenvalues of the Jacobian matrix all have negative real parts. Since the eigenvalues of J are those of $(F - V)$ andJ_{22}, and the latter all have negative real parts by assumption, the diseasefree equilibrium is locally asymptotically stable if all eigenvalues of $(F - V)$ have negative real parts.
By the assumptions on F and V , F is nonnegative and V is a nonsingular M-matrix. Hence, all eigenvalues of $(F - V)$ have negative real parts if and only if $\rho(FV^{-1}) < 1$. It follows that the disease-free equilibrium is locally asymptotically stable if $R_0 = \rho(FV^{-1}) < 1$.
Instability for $R_0 > 1$ can be established by a continuity argument. If $R_0 \leq 1$, then for any $\varepsilon \geq 0$, $((1 + \varepsilon)I - FV^{-1})$ is a nonsingular M-matrix and by Lemma 3.1, $((1 + \varepsilon)I - FV^{-1})^{-1} \geq 0$.
By Lemma 3.2, all eigenvalues of $((1 + \varepsilon)V - F)$ have positive real parts. Since $\varepsilon > 0$ is arbitrary, and eigenvalues are continuous functions of the entries of the matrix, it follows that all eigenvalues of $(V - F)$ have nonnegative real parts. To reverse the argument, suppose all the eigenvalues of $(V - F)$ have nonnegative real parts. For any positive ε, $(V + \varepsilon I - F)$ is a nonsingular M-matrix, and by Lemma 3.2, $\rho(F(V + \varepsilon I)^{-1}) < 1$.
Again, since $\varepsilon > 0$ is arbitrary, it follows that $\rho(FV^{-1}) \leq 1$. Thus, $(F - V)$ has at least one eigenvalue with positive real part if and only if $\rho(FV^{-1}) > 1$, and the disease-free equilibrium is unstable whenever $R_0 > 1$.

□

For globally asymptotically stable theorem, we will say that a vector is nonnegative if each of its components is nonnegative, and that a matrix is if each of its entries is non-negative. We rewrite the system (4) as

$$
\begin{aligned}
x^{'} &= -Ax - \widehat{f}(x, y) \\
y_j^{'} &= g_j(x, y) \qquad && j = 1, ..., m.
\end{aligned} \tag{117}
$$

42.2. Theorem. *If -A is a nonsingular M-matrix and* $\widehat{f} \geq 0$, *if the assumptions on the model (4) are satisfied, and if* $R_0 < 1$, *then the disease-free equilibrium of (5) is globally asymptotically stable.*

PROOF. The variation of constants formula for the first equation of (4) gives

$$x(t) = e^{-tA}x(0) - \int_0^t e^{-(t-s)}\widehat{f}(x(s), y(s))ds.$$

It can be shown that $e^{-tA} \geq 0$ if $-A$ is an M-matrix. Because we have $-A = B - sI$ with $B \geq 0$,

$$e^{-tA} = e^{tB}e^{-stI} = e^{tB}e^{-st}I = e^{tB}e^{-st}$$

and $e^{tB} \geq 0$, since $B \geq 0$. This, together with the assumption that $f \geq 0$, implies that $0 \leq x(t) \leq e^{-tA}x(0)$ and since $e^{tA}x(0) \to 0$ as $t \to \infty$ follows that $t \to \infty$.
On other hand, there are another examples to show that the disease-free equilibrium may not be globally asymptotically stable if the condition $\widehat{f} \geq 0$ is not satisfied.

□

42.3. THEOREM. *Consider model (3). A backward bifurcation occurs at $R_0 = 1$.*

PROOF. the Jacobin matrix for system (3) at $E_0 = (N, 0, 0)$ is as follows:

$$J = \begin{bmatrix} \epsilon\beta_1 N - \kappa & \beta_1 N \\ \kappa & -\alpha \end{bmatrix}$$

Choosing β_1 as the bifuraction parameter, then $R_0 = 1$ and $\beta_1 = \dfrac{\kappa\alpha}{N(\epsilon\alpha + \kappa)}$

□

Conclusion

We have established that the simple Kermack-McKendrick epidemic model (3) has some basic properties:
(i) There is a basic reproduction number R_0 such that if $R_0 < 1$, the disease dies out while if $R_0 > 1$, there is an epidemic.
(ii) There is a relationship between the reproduction number and the final size of the epidemic, which is an equality if there are no disease deaths. And also, in epidemic models the disease-free equilibrium is asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.
(iii) In models for which endemic equilibria exist near the disease-free equilibrium for $R_0 < 1$ the bifurcation is called a backward bifurcation.

Acknowledgement

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Selected articles of The 3^{rd} Conference on Dynamical Systems and Geometric Theories which corresponds to the scope and goals of the Journal of Finsler Geometry and its Applications introduced for publication

43. On Ricci curvature of Finsler warped product metrics

Mehran Gabrani[1,a], Bahman Rezaei[2] and Esra Sengelen Sevim[3]

[1,2] Department of Mathematics, Faculty of Science, Urmia University, Urmia, Iran
[3]Department of Mathematics, Istanbul Bilgi University, 34060, Eski Silahtaraga Elektrik Santrali, Kazim Karabekir Cad. No: 2/13 Eyupsultan, Istanbul, Turkey

In this paper, we study a rich and important class of Finsler metrics called Finsler warped product metrics. We find an equation that characterizes locally projectively flat warped product metrics. Further, we study Einstein Finsler warped product metrics.

Keywords: Finsler warped product metrics, locally projectively flat, Einstein metrics
AMS Mathematics Subject Classification [2020]: 53B40, 53C60
Code: cdsgt3-01040056

[a]Speaker. Email address: m.gabrani@urmia.ac.ir.

44. On generalized symmetric Finsler spaces with Matsumoto metrics

Milad Zeinali Laki[1,a], Dariush Latifi [2]

[1]Department of Mathematics, University of Mohaghegh Ardabili, Ardabil, Iran
[2]Department of Mathematics, University of Mohaghegh Ardabili, Ardabil, Iran

In this paper, we study generalized symmetric $(\alpha, \beta)-$spaces. We prove that generalized symmetric $(\alpha, \beta)-$spaces with Matsumoto metric are Riemannian.

Keywords: $(\alpha, \beta)-$metric, generalized symmetric space, Matsumoto metric.
AMS Mathematics Subject Classification [2020]: 18A32, 18F20, 05C65
Code: cdsgt3-01090059

[a]Speaker. Email address: miladzeinali@gmail.com,

45. Characterization of a special case of hom-Lie superalgebra

Mohammad Reza Farhangdoost [a] and Ahmad Reza Attari Polsangi

Department of Mathematics, College of Sciences,
Shiraz University, P.O. Box 71457- 44776, Shiraz, Iran

In this paper, we introduce the notion of sympathetic hom-Lie superalgebras. We prove some results on sympathetic multiplicative hom-Lie superalgebras with surjective α. In particular, we find some equivalence condition in which a sympathetic graded hom-ideal is direct factor of multiplicative hom-Lie superalgebra.

Keywords: hom-Lie superalgebra, Sympathetic hom-Lie superalgebra, multiplicative hom-Lie superalgebra

AMS Mathematics Subject Classification [2020]: 17B65, 17B70, 17B99
Code: cdsgt3-00430027

[a]Speaker. Email address: farhang@shirazu.ac.ir,

46. On pseudoconvexity conditions and static spacetimes

Mehdi Vatandoost[1] and Rahimeh Pourkhandani[2a]

[1,2]Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran.

Recently, the relationship between (geodesics) convexity, connectedness, and completeness properties in Riemannian manifolds $(\Sigma; h)$ and the causal properties in Lorentzian static spacetimes $(M; g) = (\mathcal{R} \times \Sigma; -dt^2 + h)$ is studied. In this paper, some sufficient conditions are introduced to $(\Sigma; h)$ be geodesically convex.

Keywords: Spacetimes, Causal structure, Pseudoconvexity.

AMS Mathematics Subject Classification [2020]: 83Cxx, 53C50, 53C22
Code: cdsgt3-00400049

[a]Speaker. Email address:r.pourkhandani@hsu.ac.ir

47. Controllability on the infinite-dimensional group of orientation-preserving diffeomorphisms of the unit circle

Mahdi Khajeh Salehani[1,2,a],

[1]School of Mathematics, Statistics and Computer Science, College of Science
University of Tehran, P.O. Box: 14155-6455, Tehran, Iran
[2]School of Mathematics, Institute for Research in Fundamental Sciences (IPM) P.O. Box: 19395-5746, Tehran, Iran

In this paper, we give a generalization of Chow–Rashevsky's theorem for control systems in regular connected manifolds modeled on convenient locally convex vector spaces which are not necessarily normable. To indicate an application of our approach to the infinite-dimensional geometric control problems, we conclude with a novel controllability result on the group of orientation-preserving diffeomorphisms of the unit circle, which has applications in, e.g., conformal field theory as well as string theory and statistical mechanics.

Keywords: Controllability, Infinite-dimensional manifolds, Geometric control, Orientation-preserving diffeomorphisms

AMS Mathematics Subject Classification [2020]: 93B05, 93B27
Code: cdsgt3-00310025

[a]Speaker. Email address: salehani@ut.ac.ir

48. Hypergroups and Lie hypergroups

T. Waezizadeh[1,] and N. Ebrahimi [2]

[1]Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman

[2]Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman

Using the action of a Lie group on a hypergroup, the notion of Lie hypergroup is defined. It is proved that tangent space of a Lie hypergroup is a hypergroup and that a differentiable map between two Lie hypergroup is good homomorphism if and only if its differential map is a good homomorphism.

Keywords: Hypergroup; Action, Quotient Space, Sub Hypergroup, Hypergroup Bundle.

AMS Mathematics Subject Classification [2020]: 20N20, 58A32, 22F05

Code: cdsgt3-00620053

49. Geometric Analysis of the Lie Algebra of Killing Vector Fields for a Significant Cosmological Model of Rotating Fluids

Fatemeh Ahangari[1,a]

[1]Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, P.O.Box 1993893973, Iran.

The investigation of rotating fluids in the context of general relativity received remarkable consideration principally after Godel proposed relativistic model of a rotating dust universe. In this paper, a comprehensive analysis regarding the structure of the Lie algebra of Killing vector fields for a specific solution of field equations describing the behaviour of rotating fluid models is presented. Killing vector fields can be undoubtedly reckoned as one of the most substantial types of symmetries and are denoted by the smooth vector fields which preserve the metric tensor. Additionally, the flow corresponding to a Killing vector field generates a symmetry in a way that if each point moves on an object at the same distance in the direction of the Killing vector field then distances on the object will not distorted at all. Therefore, Killing vector fields are inherently expected to be of significant application in the study of geodesic motion. When one investigates the Lagrangian explaining the motion of a particle, one can realize that Killing vectors are the symmetries of the system and lead to conserved canonical momenta analogous to cyclic coordinates in classical mechanics. Taking into account the outstanding properties declared above, in this paper, we specifically concentrate on detailed investigation of the Killing vector fields by reexpressing the analyzed cosmological solution in the orthogonal frame. Significantly, for the resulted Lie algebra of Killing vector fields, the associated basis for the original Lie algebra is determined in which the Lie algebra will be appropriately decomposed into an internal direct sum of subalgebras, where each summand is indecomposable.

Keywords: Killing vector fields, five dimensional spacetime, rotating fluids.

AMS Mathematics Subject Classification [2020]: 53Z05, 83C20

Code: cdsgt3-00920034

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[a]Speaker. Email address: f.ahangari@alzahra.ac.ir, fa.ahangari@gmail.com

50. Index Name

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51. Thanksgivings

The organizer of the seminar would like to express his gratitude to all the dignitaries who helped us in this seminar, all the esteemed participants, guest speakers, specialized speakers and the scientific and executive committee of the seminar, and wish good health to all from God Almighty. Hoping to see you dear ones in another scientific congres.

Ali Barzanouni *Head of the Conference*

Rahimeh Pourkhandani *Head of the Executive Committee*

Mehdi Vatandoost *Head of the Scientific Committee*