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۱۸-۱۹ اسفند ۱۴۰۰
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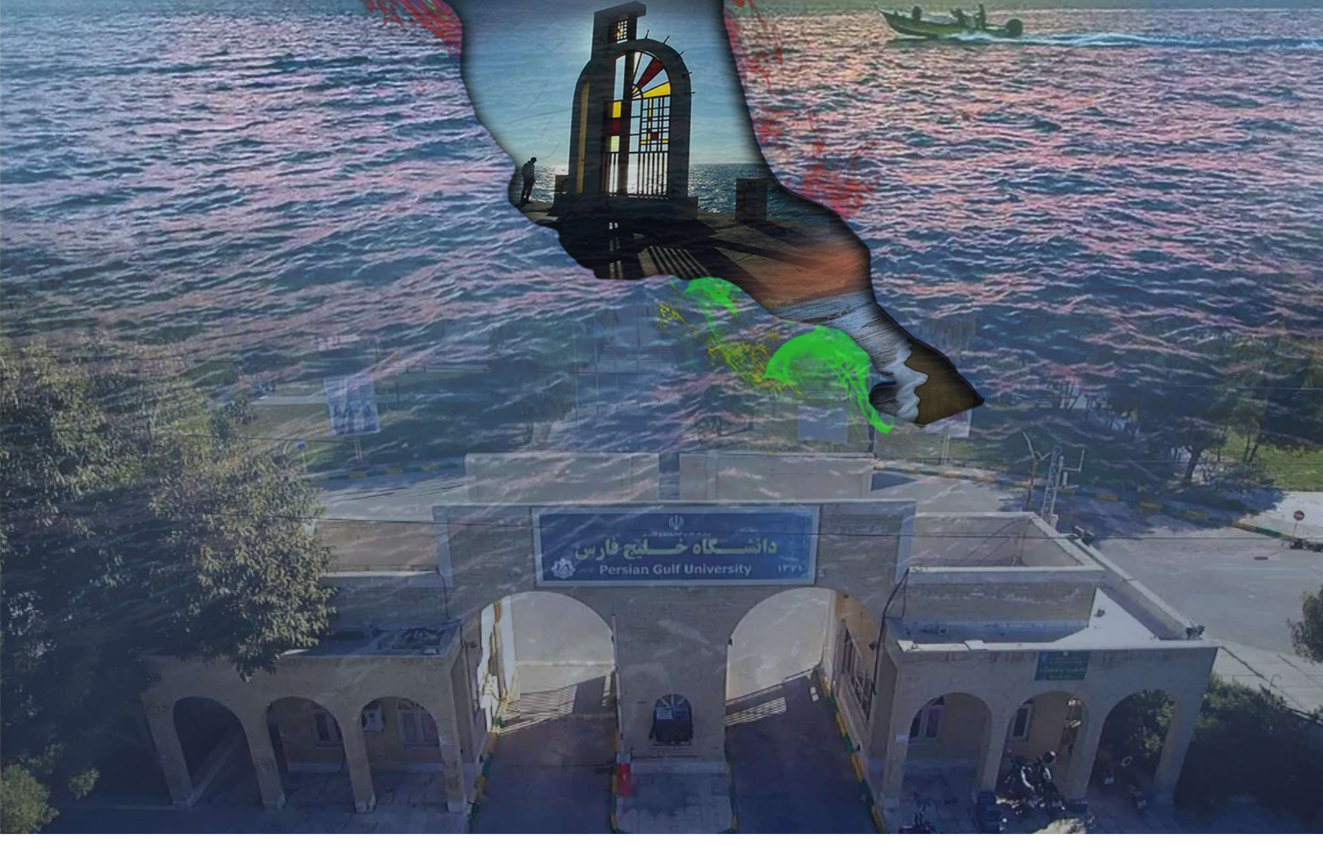
بیست و هفتمین سمینار جبر ایران

Extended Abstracts of

27th Iranian Algebra Seminar

بوشهر دانشگاه خلیج فارس

۱۸ تا ۱۹ اسفند ۱۴۰۰



Extended Abstracts of the

27th Iranian Algebra Seminar

March 9–10, 2022, Bushehr, Iran

March 2022

Persian Gulf University

Editors:
M. Sedaghatjoo
R. Sharafdini
S. Rasouli
N. Dehghani

Organized by the Mathematics Department, Persian Gulf University, Bushehr, Iran

27th Iranian Algebra Seminar (IAS27) - March 9-10, 2022 Timetable

Persian Gulf University, Boushehr, Iran

The program has been set based on the official Iran-Tehran time zone. (GMT + 3: 30)

Seminar timetable at a glance

1st day (2022-03-09)

Opening Ceremony	9-9:50
1st keynote speech	10-11
1st oral presentation session	11-12:20
2nd keynote speech	13-14
2nd oral presentation session	14-15:20
Poster session	15:20-16
3rd oral presentation session	16-17:20
3rd keynote speech	17:30-18:30
Roundtable	19-21

2nd day (2022-03-10)

4th keynote speech	9-10
4th oral presentation session	10-11:20
SAGE workshop	11:30-13
5th keynote speech	13:30-14:30
5th oral presentation session	14:30-15:50
Closing Ceremony	16-16:30

Presentations will be arranged subject classified and color-matched in the following five Halls:

Maryam Mirzakhani Hall	URL: https://vc14.pgu.ac.ir/b/ias-rj1-lkz-6vk
Hall A	URL: https://vc14.pgu.ac.ir/b/ias-c3l-byj-igm
Hall B	URL: https://vc14.pgu.ac.ir/b/ias-64h-srn-pka
Hall C	URL: https://vc14.pgu.ac.ir/b/ias-osb-04x-8i6
Hall D	URL: https://vc14.pgu.ac.ir/b/ias-5xb-kwf-yu7

Click on ➡ to inter the room

Click on the WhatsApp icon to enter the announcement group

Click on pic. to enter the roundtable room

Click on the logo icon to enter the seminar website

1st day (2022-03-09)

9-9:50		**The opening Ceremony**	
		1st keynote speech (→ to Maryam Mirzakhani Hall)	
	Speaker	**Title**	
10-11	M. R. Darafsheh	On rational irreducible characters of finite groups University of Tehran, Iran; (→ to Homepage)	
		1st oral presentation session	
	Presenter	**title**	**Paper ID**
		Chairman: R. Soleimani (→ to Hall A)	
11-11:20	S. Madadi Moghadam	گروه‌های متناهی-گروه‌های شبه ساده-گروه‌های تقریباً ساده-درجه سرشت‌های تحویل‌ناپذیر- حدس هوپرت	1104
11:20-11:40	F. Karimi	The autocentralizer automorphism of groups.	1092
11:40-12	F. Mirzaei	Bounds for the index of the second center subgroup of a pair of finite groups	1084
12-12:20	R. Soleimani	Automorphism group, nth autoclass-preserving automorphism, absolute	1131
		Chairman: B. Amini (→ to Hall B)	
11-11:20	F. Fattahi	On skew Armendariz ideals of rings	1059
11:20-11:40	R. Ghaseminejad	S-almost prime submodule	1034
11:40-12	A. Hajizamani	A cotorsion theory in the homotopy category of complexes of flat R-modules	1076
12-12:20	B. Amini	On submodules of the set of rational numbers	1124
		Chairman: S. Rasouli (→ to Hall C)	
11-11:20	S. Rasouli	A survey on some subclasses of residuated lattices	1110
11:20-11:40	D. Heydari	A hypergroup for the control flow graph	1070
11:40-12	S. Mirvakili	On EL^2-semihypergroups of order 2	1065
12-12:20	A. delfan	Co-Intersection graph of act	1083
		Chairman: M. R. Oboudi (→ to Hall D)	
11-11:20	M. Safazadeh	Fair domination polynomial of a graph	1129
11:20-11:40	S. Fallahpoor	Semisymmetric cubic graphs whose order has prime factors: a comprehensive review	1118
11:40-12	A. Shukur	Energy of Monad Graphs	1087
12-12:20	M. R. Oboudi	Results on signless Laplacian spectral characterization of broken graphs	1128
		2nd keynote speech (→ to Maryam Mirzakhani Hall)	
13-14	V. Laan	Morita equivalence of finite semigroups University of Tartu, Estonia; (→ to Homepage)	
		2nd oral presentation session	
		Chairman: H. Mousavi (→ to Hall A)	
14-14:20	A. K. Asboei	Sums of Sylow numbers of finite groups	1047
14:20-14:40	S. Barin	IA-central series	1033
14:40-15	M. Ghasemi	Groups which do not have four irreducible characters of degrees divisible by a prime p	1028
15-15:20	A. R. Doustabadi	Relation between the power graph of finite group and commutative elements	1006
		Chairman: N. Dehghani (→ to Hall B)	
14-14:20	M. A. Dehghanzadeh	On the center and automorphisms of crossed modules	1058
14:20-14:40	R. Mahtabi	On complement to a submodule of multiplication modules	1057
14:40-15	GH. Safakish	φ −primary subsemimodules	1060
15-15:20	N. Dehghani	The Schröder-Bernstein Theorem for the class of Baer modules	1048
		Chairman: D. Heydari (→ to Hall C)	
14-14:20	F. K. Haghani	Some results on internal state residuated lattices	1102
14:20-14:40	D. Heydari	GCP-graphs	1051
14:40-15	M. Yaghmaei	The conductor ideal of simplicial affine semigroups	1115
15-15:20	M. Roueentan	A note on mono-covered acts	1089
		Chairman: S. Mirvakili (→ to Hall D))	
14-14:20	F. Kazemnejad	Connected domination number of central trees	1085
14:20-14:40	Z. Barati	Distance spectral of the unitary Cayley graphs of commutative rings	1071
14:40-15	B. Ahmadi	Some relations between the distinguishing and some graph parameters	1068
15-15:20	A. Bahmani	Domination set for bipartite graph $\Gamma(v, k, 3, 2)$	1053
15:20-16		**Poster session (The timetable is located at the end)**	
		3rd oral presentation session	
		Chairman: M. Ghasemi (→ to Hall A)	
16-16:20	F. Karimi	گراف برخی از گروه‌های آبلی بی تاب از رتبه یک	1024
16:20-16:40	M. Zameni	Deficient square graph of finite group	1099
16:40-17	B. Khosravi	Reconstructing normal edge-transitive Cayley graphs of abelian groups	1098
17-17:20	J. Rezaei	Cubic edge-transitive graphs of order $40p$	1113
		Chairman: H. Sharif (→ to Hall B)	
16-16:20	L. Heidarizadeh	Ring morphisms and their orderings	1040
16:20-16:40	A. R. Najafizadeh	The torsion theory of a completely prime radical of a module	1038
16:40-17	M. Ajdani	Conjecture on the k-Cohen-Macaulay simplicial complexes of codimension 3	1019
17-17:20	A. Fathi	On the annihilators of Ext modules	1011
		Chairman: M. Sedaghatjoo (→ to Hall C)	
16-16:20	R. Khosravi	A note on Rees large subacts	1077
16:20-16:40	H. Rasouli	Supplemented acts over monoids and their properties	1075
16:40-17	M. A. Naghipoor	On differential semigroups and radical differential ideals	1064
17-17:20	M. Sedaghatjoo	Principal right congruences over completely 0-simple semigroups	1123
		Chairman: M. Azadi Motlagh (→ to Hall D)	
16-16:20	M. Beigi	One-sided repeated-root two-dimensional constacyclic codes	1063
16:20-16:40	F. Farhang	Lee weight for $(u, u+v)$-construction of codes over $Z4$	1005
16:40-17	A. Eisapoor	بررسی کدهای MRD روی حلقه‌های متناهی ایده‌آل اصلی با استفاده از گراف‌های ماتریسی	1020
17-17:20	R. M. Hessari	$ZqZq(Zq + uZq)$-Additive skew cyclic codes	1000
		3rd keynote speech (→ to Maryam Mirzakhani Hall)	
17:30-18:30	S. T. Rizvi	Continuous module hulls The Ohio State University, USA; (→ to Homepage)	
		Roundtable (→ to Maryam Mirzakhani Hall)	
19-21		Algebra; future, opportunities and challenges	

2nd day (2022-03-10)

	Presenter	title	Paper ID
		4th keynote speech (→ to Maryam Mirzakhani Hall)	
9-10	R.K. Nath	Commuting and non-commuting graphs of finite groups and their extensions Tezpur University, India; (→ **to** Homepage)	
		4th oral presentation session	
		Chairman: R. Orfi (→ to Hall A)	
10-10:20	F. Abedi	Some results on 15-Valent 2-arc-transitive graphs	1049
10:20-10:40	H. Moshtagh	Some remarks on regular association schemes of order pqr	1106
10:40-11	M. Kowkabi	When is a local homeomorphism a full subsemicovering?	1119
11-11:20	M. Sadeghlou	On the isoclinism of a pair of Hom-Lie algebras	1111
		Chairman: M. Jahangiri (→ **to Hall B**)	
10-10:20	A. Fathi	On the annihilator of local cohomology	1012
10:20-10:40	S. Bandari	Polymatroidal ideals and linear resolution	1026
10:40-11	F. Vahdanipoor	Cofiniteness and Artinianness of generalized local cohomology modules	1027
11-11:20	M. Jahangiri	The rate of graded modules over some graded algebras	1052
		Chairman: M. Mahmmoodi (→ **to Hall C**)	
10-10:20	Z. Khaki	Weakly Right Noetherian Semigroups	1078
10:20-10:40	F. Jahanian	شبه ت-کمینگی برای جبرهای استون مضاعف	1080
10:40-11	H. Barzegar	When a quotient of a distributive lattice is a Boolean algebra	1016
11-11:20	A. A. Gholipoor	S-acts with finitely generated universal right congruence	1029
		Chairman: M. Shahriari (→ to Hall D)	
10-10:20	B. Sadeghi	ارایه الگوریتم جدید برای پایه های گربنر	1062
10:20-10:40	A. Javan	Precrossed modules in Lie algebras	1025
10:40-11	A. Javan	Tensor product of crossed modules in Lie algebras	1023
11-11:20	A. Shamsaki	On the triple tensor product of some class of nilpotent Lie algebra	1042
		SAGE workshop (→ to Maryam Mirzakhani Hall)	
11:30-13	A. R. Ashrafi		
		5th keynote speech (→ to Maryam Mirzakhani Hall)	
13:30-14:30	I. Ponomarenko	The 3-closure of a solvable permutation group is solvable Petersburg department of V. A. Steklov institute of mathematics St.Petersburg, Russia; (→ **to** Homepage)	
		5th oral presentation session	
		Chairman: B. Edalatzadeh (→ to Hall A)	
14:30-14:50	F. Johari	Capability of nilpotent Lie algebras with one-dimensional center	1091
14:50-15:10	K. Ghasemi	n-Jordan $*$-Derivations in Fréchet locally $C*$-algebras	1055
15:10-15:30	Z. A. Rostami	Bogomolov multiplier and the Lazard correspondence	1043
15:30-15:50	B. Edalatzadeh	Some inequalities for the dimension of the second homology of nilpotent Leibniz algebras	1021
		Chairman: M. Jahangiri (→ **to Hall B**)	
14:30-14:50	F. Vahdanipour	مدول‌های کوهمولوژی موضعی غیرآرتینی از بعد صفر	1045
14:50-15:10	F. Rashedi	W-neat rings	1114
15:10-15:30	P. Abbaspoor	Definable Monotone Functions in Type Complete Ordered Fields	1088
15:30-15:50	H. Pasbani	Annihilator graph of modules over commutative rings	1044
		Chairman: M. A. Naghipoor (→ **to Hall C**)	
14:30-14:50	H. Moghbeli	Z-Scott topology and Z-refinement property	1101
14:50-15:10	R. Rasouli	Anti-fuzzy BCK-subalgebras and ideals under S-norms	1013
15:10-15:30	S. Borhaninejad	On hyper CI-algebras: as a generalization of hyper BE-algebras	1097
15:30-15:50	P. Khamehchi	New results on Condition (P') and (PF'')-cover	1116
16-16:30		**Closing Ceremony** (→ to Maryam Mirzakhani Hall)	

The Poster session timetable

15:20-16	Presenter	title		Poster ID
15:20-15:30	H. Shalal	The Solvablitiy Degree of the Alternation Group	→ to Hall A	1135
	V. Hashemi	On eccentric adjacency index of graphs and trees	→ to Hall B	1130
	M. Noori	The first Zagreb indices of Scalar Product Graph of some Modules	→ to Hall C	1125
	S. Fallahpoor	A note on automorphism groups of cubic semisymmetric graphs of special order	→ to Hall D	1117
15:30-15:40	H. Shalal	Relation between Sylowility degree and Sylow Graph	→ to Hall A	1134
	SH. Heidarian	Commutativity degree of crossed modules	→ to Hall B	1095
	V. Rahmani	از چه مرتبه‌هایی دقیقا چهار یا پنج گروه وجود دارند؟	→ to Hall C	1094
	K. Ghadimi	On characters of polygroups	→ to Hall D	1093
15:40-15:50	S. S. Gholami	The dimensions of certain cartesian symmetry classes	→ to Hall A	1108
	S. Rahnama	مشتق BL جبر و شبه BL- جبرها	→ to Hall B	1082
	GH. Aghababaei	گراف‌های نامتباین گروه‌های دو دوری	→ to Hall C	1074
	M. Shafiee	مواردی از مدول‌های کوهمولوژی موضعی تعمیم‌یافته‌ی مدرج آرتینی	→ to Hall D	1073
15:50-16	M. Shafiee	مدول کوهمولوژی موضعی تعمیم‌یافته‌ی آرتینی با چند جمله‌ای هیلبرت-کیربی	→ to Hall A	1072
	H. Moghbeli	Information systems and algebraic complete semi-lattices	→ to Hall B	1069
	R. Fallah	Crossed product condition and skew linear groups	→ to Hall C	1041
	P. Karimi	A Subgraph of the strongly annihilating submodule graph	→ to Hall D	1039

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بیست و هفتمین سمینار جبر ایران
27th Iranian ALGEBRA Seminar
سخنرانان مدعو
10:00am March 9 (GMT + 3: 30)
University of Tehran, Iran
Mohammad Reza Darafsheh
13:30pm March 10 (GMT + 3: 30)
PETERSBURG DEPARTMENT of V. A. STEKLOV INSTITUTE of MATHEMATICS, St. Petersburg, RUSSIA
Ilia Ponomarenko
13:00pm March 9 (GMT + 3: 30)
University of Tartu Estonia
Valdis Laan
17:30pm March 9 (GMT + 3: 30)
Ohio State University, USA
Syed Tariq Rizvi
9:00am March 10 (GMT + 3: 30)
Tezpur University India
Rajat Kanti Nath
۱۸ – ۱۹ اسفند ماه ۱۴۰۰
9-10 March 2022
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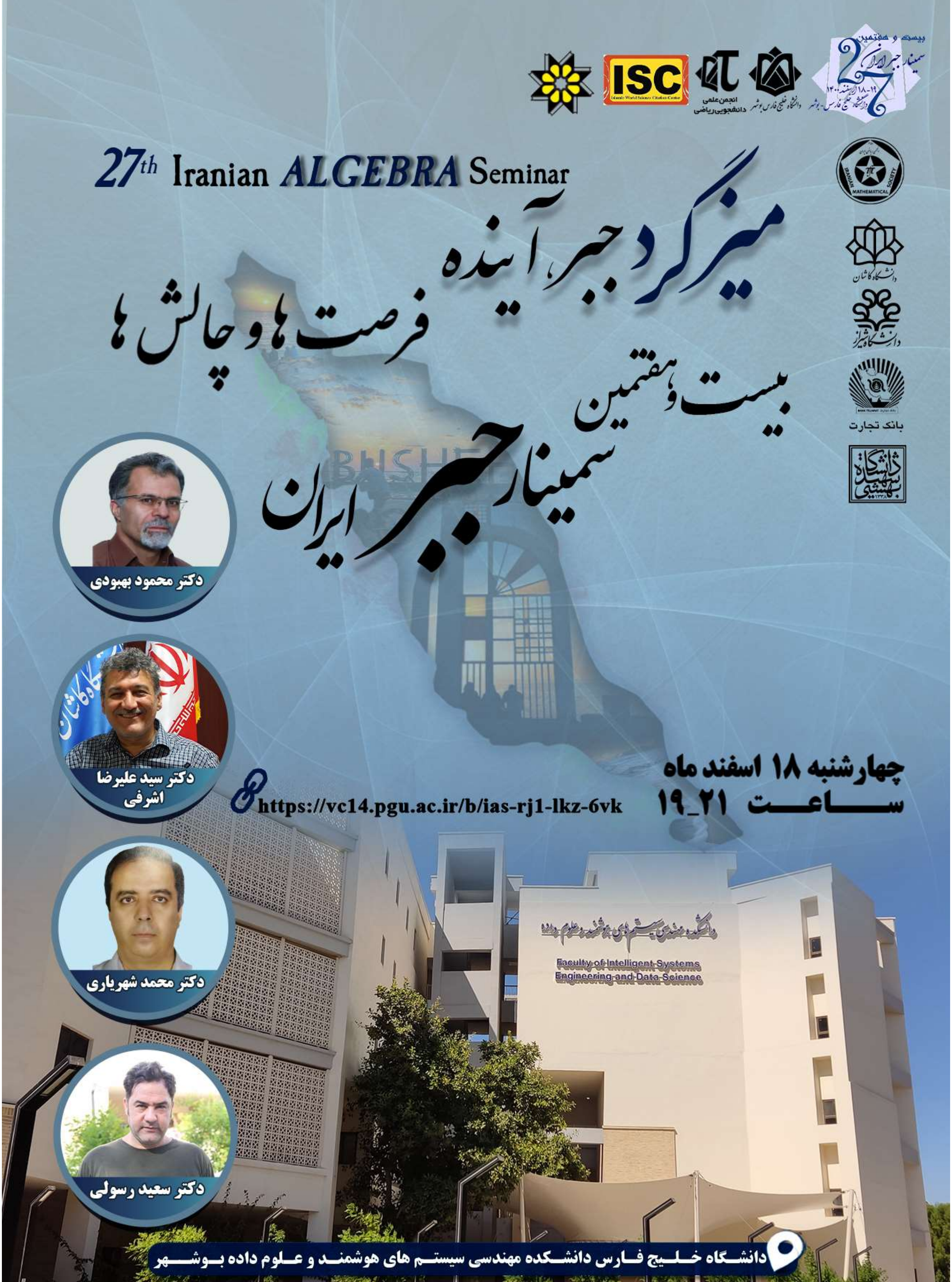
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27th Iranian ALGEBRA Seminar

دانشگاه شیراز

دانشگاه شهید بهشتی ۱۳۳۸

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بانک تجارت

Contributed Talks

Contents

27^{th} Iranian Algebra Seminar
Persian Gulf University, Bushehr, Iran
18–19 Esfand 1400 (March 9–10, 2022)

$\mathbb{Z}_q\mathbb{Z}_q(\mathbb{Z}_q + u\mathbb{Z}_q)$-Additive skew cyclic codes

ROGHAYE MOHAMMADI HESARI*

Abstract

In this paper, we study the algebraic structure additive skew cyclic codes over $\mathbb{Z}_q\mathbb{Z}_qR$, where $q = p^m$ is a prime power and $R = \mathbb{Z}_q + u\mathbb{Z}_q$ with $u^2 = 0$. Also, we describe the generator polynomials of these codes. We classify that there are eight different types of explicit generators of $\mathbb{Z}_q\mathbb{Z}_qR$-additive skew cyclic codes.

Keywords and phrases: Skew cyclic codes, Additive codes, Generator polynomials.
2010 *Mathematics subject classification:* 94B15,94B05.

1. Introduction

Aydogdu et al. presented the structure of cyclic and constacyclic codes and their duals in [1]. Wu et al. have been studided $\mathbb{Z}_2\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes in [5]. One of the most applicable type of cyclic codes is skew cyclic codes which were introduced by Boucher et al. in [2]. Jitman et al. extended the results of skew cyclic codes to skew constacyclic codes over finite chain rings. They have obtained Euclidean and Hermitian dual of these codes in [3].

The class of skew cyclic codes plays a very significant role in the theory of error-correcting codes. Since there are much more additive skew cyclic codes, this class of codes allows to systematically search for codes with good properties and improve the previously best known linear codes.

In this paper, we generalize the approach used by Melakhessou et al. in [4] to determine the structure of $\mathbb{Z}_q\mathbb{Z}_q(\mathbb{Z}_q + u\mathbb{Z}_q)$-additive skew cyclic codes.

This paper has been organized as follows. Section 2 contains some basic definitions, some notations and previous results related to our work. Also, we specify the $\mathbb{Z}_q\mathbb{Z}_q(\mathbb{Z}_q + u\mathbb{Z}_q)$-additive skew cyclic codes, where $u^2 = 0$.

2. Additive skew cyclic codes of length (α, β, γ) over $\mathbb{Z}_q\mathbb{Z}_qR$

In this section we determine the algebraic structure of all additive skew cyclic codes of length (α, β, γ) over $\mathbb{Z}_q\mathbb{Z}_qR$.

* speaker

Recall that $R = \mathbb{Z}_q + u\mathbb{Z}_q = \{a + ub : a, b \in \mathbb{Z}_q\}$ is a finite ring of nilpotency index 2 and characteristic p. The ring R is not a chain ring, whereas it is a local ring, and the only maximal ideal is $\langle u, p \rangle$. Also, the ring R is isomorphic to $\mathbb{Z}_q^2$. It is known that the ring $\mathbb{Z}_q$ is a subring of the ring R.

Definition 2.1. *[4] An automorphism θ of R is defined as $\theta(a+ub) = a + \sigma(u)b$, where $\sigma(u) = c + ud$ such that c is a non-unit in $\mathbb{Z}_q$, $c^2 \equiv 0 \mod q$ and $2cd \equiv 0 \mod q$. Therefore,*

$$\theta(a+ub) = a + \sigma(u)b = (a+cb) + ubd.$$

Definition 2.2. *Let $R[x;\theta]$ the set of all (skew) polynomials*

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n,$$

where $a_i \in R$, x is an indeterminate and $n \in \mathbb{N}_0$. Equality and addition of these polynomials is defined in the standard manner while multiplication is defined by the basic rule $xa = \theta(a)x$ ($a \in R$). The multiplication is extended to all elements in $R[x;\theta]$ by associativity and distributivity. The set $R[x;\theta]$ with the above operations forms a ring called the skew polynomial ring *over R, and every element in $R[x;\theta]$ is called the* skew polynomial*. It is easily seen that the ring $R[x;\theta]$ is non-commutative unless θ is the identity automorphism on R.*

Proposition 2.3. *[3,* Proposition 2.3*] Let $h(x), g(x) \in R[x;\theta]$. If $h(x)g(x)$ is a monic central skew polynomial, then $h(x)g(x) = g(x)h(x)$.*

The ring $R[x;\theta]$ is neither left nor right Euclidean. However, left and right divisions can be defined for some suitable elements. Let $f(x), g(x)$ be skew polynomials in $R[x;\theta]$, with $f(x) \neq 0$. Then there exist $q(x), r(x) \in R[x,\theta]$ such that $g(x) = q(x)f(x) + r(x)$, where $r(x) = 0$ or $\deg(r(x)) < \deg(f(x))$. Note that $q(x)$ and $r(x)$ are unique.

We say that $f(x)$ is a right divisor of $g(x)$ in $R[x,\theta]$ and we write $f(x) \mid_r g(x)$ if there exists a skew polynomial $h(x)$ such that $g(x) = h(x)f(x)$.

Let (α, β, γ) denote $n = \alpha + \beta + \gamma$, where α, β are positive integers and γ is a positive integer coprime to characteristic of R.

Throughout this paper, we use the following symbols for simplicity:

$$\mathcal{R}_\alpha = \frac{\mathbb{Z}_q[x]}{\langle x^\alpha - 1 \rangle}, \mathcal{R}_\beta = \frac{\mathbb{Z}_q[x]}{\langle x^\beta - 1 \rangle}, \mathcal{R}_\gamma = \frac{R[x;\theta]}{\langle x^\gamma - 1 \rangle}, \mathcal{R} = \mathcal{R}_\alpha \times \mathcal{R}_\beta \times \mathcal{R}_\gamma,$$

and by [3, Proposition 2.2], we assume that $o(\theta) \mid \gamma$, where $o(\theta)$ is the order of θ. Since $x^\gamma - 1$ is a monic central skew polynomial, therefore by Proposition 2.3, a right divisor of $x^\gamma - 1$ is a two-sided divisor.

Definition 2.4. *A code C over R is called skew θ-cyclic, if C is closed under θ-cyclic shift $\rho_\theta : R^\gamma \longrightarrow R^\gamma$ which is defined by*

$$\rho_\theta\big((a_0, a_1, \ldots, a_{\gamma-1})\big) = \big(\theta(a_{\gamma-1}), \theta(a_0), \ldots, \theta(a_{\gamma-2})\big).$$

When there is no ambiguity, we say "skew cyclic" instead of "skew θ-cyclic".

Let $\mu : R \longrightarrow \mathbb{Z}_q$ be defined by $\mu(a+ub) = b$, for any $a+ub \in R$. Then μ is ring homomorphism. Consider the set

$$\mathbb{Z}_q\mathbb{Z}_q R = \{(a|b|c) : \quad a,b \in \mathbb{Z}_q, c \in R\}.$$

By the following scalar multiplication, $\mathbb{Z}_q\mathbb{Z}_q R$ is a left R-module,

$$\begin{aligned} . : R \times \mathbb{Z}_q\mathbb{Z}_q R &\longrightarrow \mathbb{Z}_q\mathbb{Z}_q R, \\ r.(a|b|c) &= (\mu(r)a|\mu(r)b|c). \end{aligned}$$

This multiplication can be generalized over the set $\mathbb{Z}_q^\alpha\mathbb{Z}_q^\beta R^\gamma$ in the following way. For any $r \in R$ and $(a_0,...,a_{\alpha-1}|b_0,...,b_{\beta-1}|c_0,...,c_{\gamma-1}) \in \mathbb{Z}_q^\alpha\mathbb{Z}_q^\beta R^\gamma$ define $r.(a_0,...,a_{\alpha-1}|b_0,...,b_{\beta-1}|c_0,...,c_{\gamma-1}) = (\mu(r)a_0,...,\mu(r)a_{\alpha-1}|\mu(r)b_0,...,\mu(r)b_{\beta-1}|rc_0,...,rc_{\gamma-1})$.

Definition 2.5. *A non-empty subset C of $\mathbb{Z}_q^\alpha\mathbb{Z}_q^\beta R^\gamma$ is called a $\mathbb{Z}_q\mathbb{Z}_q R$-additive skew cyclic code if*
1) C is a subgroup of $\mathbb{Z}_q^\alpha\mathbb{Z}_q^\beta R^\gamma$, and
2) For any codeword $(a_0,...,a_{\alpha-1}|b_0,...,b_{\beta-1}|c_0,...,c_{\gamma-1}) \in C$, its θ-cyclic shift $(a_{\alpha-1},a_0,...,a_{\alpha-2}|b_{\beta-1},b_0,...,b_{\beta-2}|\theta(c_{\gamma-1}),\theta(c_0),...,\theta(c_{\gamma-2}))$ is also in C.

There is a bijection map between $\mathbb{Z}_q^\alpha\mathbb{Z}_q^\beta R^\gamma$ and $\mathcal{R} = \mathcal{R}_\alpha \times \mathcal{R}_\beta \times \mathcal{R}_\gamma$ given by

$$(a_0,...,a_{\alpha-1}|b_0,...,b_{\beta-1}|c_0,...,c_{\gamma-1}) \longmapsto (a(x)|b(x)|c(x)).$$

Suppose $(f(x)|g(x)|h(x)) \in \mathcal{R}$ and $r(x) \in R[x;\theta]$, we have

$$\begin{aligned} . : R[x;\theta] \times \mathcal{R} &\longrightarrow \mathcal{R}, \\ r(x).(f(x)|g(x)|h(x)) &= (\mu(r(x))f(x)|\mu(r(x))g(x)|r(x)h(x)), \end{aligned}$$

where $\mu(r(x)) = \mu(\sum_j r_j x^j) = \sum_j \mu(r_j)x^j$ and $r_j \in R$.

Lemma 2.6. *A code C is a $\mathbb{Z}_q\mathbb{Z}_q R$-additive skew cyclic code of length (α,β,γ) if and only if C is a left $R[x;\theta]$-submodule of $\mathcal{R}$.*

Definition 2.7. *We define a* Gray map

$$\begin{aligned} \phi : R &\longrightarrow \mathbb{Z}_q, \\ \phi(a+ub) &= (b, a+b), \end{aligned}$$

and we can generalize this Gray map for all $(x_0,...,x_{\alpha-1}) \in Z_q^\alpha$, $(y_0,...,y_{\beta-1}) \in Z_q^\beta$ and $(z_0,...,z_{\gamma-1}) \in R^\gamma$ as follows:

$$\begin{aligned} \psi : \mathbb{Z}_q^\alpha\mathbb{Z}_q^\beta R^\gamma &\longrightarrow \mathbb{Z}_q^{\alpha+\beta+2\gamma}, \\ \psi(x_0,...,x_{\alpha-1}|y_0,...,y_{\beta-1}|z_0,...,z_{\gamma-1}) &= (x_0,...,x_{\alpha-1}|y_0,...,y_{\beta-1}|\phi(z_0),...,\phi(z_{\gamma-1})). \end{aligned}$$

Therefore, $\mathcal{C} = \psi(C)$ is a cyclic code of length $\alpha+\beta+2\gamma$ over $\mathbb{Z}_q$.

Theorem 2.8. *Every left $R[x;\theta]$-submodule of $\mathcal{R}$ is of the form*

$$\langle (a(x)|0|0), (0|b(x)|0), (\ell_1(x)|\ell_2(x)|c(x)+ug(x))\rangle,$$

where

$$a(x), \ell_1(x) \in \mathcal{R}_\alpha, \quad a(x)|x^\alpha - 1, \quad b(x), \ell_2(x) \in \mathcal{R}_\beta,$$

$$b(x)|x^\beta - 1, \quad \deg(\ell_1(x)) < \deg(a(x)), \quad \deg(\ell_2(x)) < \deg(b(x)),$$

and $g(x)|_r c(x)|x^\gamma - 1$. Moreover, $g(x)$ with the above condition is unique.

Now, we can list all $\mathbb{Z}_q\mathbb{Z}_qR$-additive skew cyclic code of length (α, β, γ) as follows:

Theorem 2.9. *$\mathbb{Z}_q\mathbb{Z}_qR$-Additive skew cyclic code of length (α, β, γ) are of the following types:*

•*Type* $1:0$, $\mathcal{R}$.

•*Type* $2: \langle (a(x)|0|0)\rangle$, *where* $a(x) \in \mathcal{R}_\alpha$, $a(x)|x^\alpha - 1$ *and* $0 \leq \deg(a(x)) \leq \alpha - 1$.

•*Type* $3: \langle (0|b(x)|0)\rangle$, *where* $b(x) \in \mathcal{R}_\beta$, $b(x)|x^\beta - 1$ *and* $0 \leq \deg(b(x)) \leq \beta - 1$.

•*Type* $4: \langle (\ell_1(x)|\ell_2(x)|c(x)+ug(x))\rangle$, *where*

$$\ell_1(x) \in \mathcal{R}_\alpha, \quad \ell_2(x) \in \mathcal{R}_\beta, \quad g(x)|_r c(x)|x^\gamma - 1,$$

and $0 \leq \deg(c(x)) \leq \gamma - 1$. *Moreover, $g(x)$ with the above condition is unique.*

• *Type* $5: \langle (a(x)|0|0), (0|b(x)|0)\rangle$, *where* $a(x) \in \mathcal{R}_\alpha$,

$$a(x)|x^\alpha - 1, 0 \leq \deg(a(x)) \leq \alpha - 1, b(x) \in \mathcal{R}_\beta, b(x)|x^\beta - 1,$$

and $0 \leq \deg(b(x)) \leq \beta - 1$.

• *Type* $6: \langle (a(x)|0|0), (\ell_1(x)|\ell_2(x)|c(x)+ug(x))\rangle$, *where*

$$a(x) \in \mathcal{R}_\alpha, \quad a(x)|x^\alpha - 1, \quad 0 \leq \deg(a(x)) \leq \alpha - 1, \quad \ell_1(x) \in \mathcal{R}_\alpha,$$

$$\ell_2(x) \in \mathcal{R}_\beta, \quad g(x)|_r c(x)|x^\gamma - 1, \quad 0 \leq \deg(c(x)) \leq \gamma - 1,$$

and $\deg(\ell_1(x)) < \deg(a(x))$. *Moreover, $g(x)$ with the above condition is unique.*

• *Type* $7: \langle (0|b(x)|0), (\ell_1(x)|\ell_2(x)|c(x)+ug(x))\rangle$, *where*

$$b(x) \in \mathcal{R}_\beta, \quad b(x)|x^\beta - 1, \quad 0 \leq \deg(b(x)) \leq \beta - 1,$$

$$g(x)|_r c(x)|x^\gamma - 1, \quad 0 \leq \deg(c(x)) \leq \gamma - 1,$$

and $\deg(\ell_2(x)) < \deg(b(x))$. *Moreover,* $g(x)$ *with the above condition is unique.*

- *Type* 8 : $\langle (a(x)|0|0), (0|b(x)|0), (\ell_1(x)|\ell_2(x)|c(x) + ug(x))\rangle$, *where*

$$a(x) \in \mathcal{R}_\alpha, \quad a(x)|x^\alpha - 1, \quad 0 \leq \deg(a(x)) \leq \alpha - 1,$$

$$b(x) \in \mathcal{R}_\beta, \quad b(x)|x^\beta - 1, \quad 0 \leq \deg(b(x)) \leq \beta - 1,$$

$$\ell_1(x) \in \mathcal{R}_\alpha, \quad \ell_2(x) \in \mathcal{R}_\beta, \quad g(x)|_r c(x)|x^\gamma - 1,$$

$$0 \leq \deg(c(x)) \leq \gamma - 1, \;\; \deg(\ell_1(x)) < \deg(a(x)), \;\; \deg(\ell_2(x)) < \deg(b(x)).$$

Moreover, $g(x)$ *with the above condition is unique.*

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Lee Weight for $(u, u+v)$-construction of codes over Z_4

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Abstract

For a linear code C of length n over Z_4, the Lee support weight of C, denoted by $wt_L(C)$, is the sum of Lee weights of all columns of $A(C)$, $A(C)$ is $|C| \times n$ array of all codewords in C. For $1 \leq r \leq rank(C)$, the r-th generalized Lee weight with respect to rank (GLWR) for C, denoted by $d_r^L(C)$, is defined as

$$d_r^L(C) = \min\{wt_L(D); D \text{ is a } Z_4 - \text{submodule of C}, rank(D) = r\}.$$

Let $C_i, i = 1,2$ be codes over Z_4 and C denote $(u, u+v)$-construction of them. In this paper, we obtained $d_1^L(C)$ in terms of $d_1^L(C_1), d_1^L(C_2)$ and we generally obtained an upper bound for $d_r^L(C)$ for all r, $1 \leq r \leq rank(C)$. We found a relationship between $wt_L x$, $wt_L y$ and $wt_L(x+y)$, for any $x, y \in Z_4^n$ and we showed that Lee support weight is invariant under multiplication by 3.

Keywords and phrases: Linear code, Hamming Weight, Lee Weight, Generalized Lee Weight, $(u, u+v)$- construction of Codes.

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1. Introduction

Consider Z_m as code alphabet. The Lee Weight of an integer i, for $0 \leq i \leq m$ is defined as $wt_L(i) = \min\{i, m-i\}$. For $m = 4$, namely in Z_4, we have $wt_L(0) = 0, wt_L(1) = wt_L(3) = 1, wt_L(2) = 2$. The Lee metric on Z_m^n is defined by

$$wt_L(a) = \sum_{i=1}^{n} wt_L(a_i),$$

where the sum is defined in N_0. We define Lee distance by $d_L(x,y) = wt_L(x-y)$. For more information, see [5]. Generalized Lee Weight (GLW) for codes over Z_4 introduced by B. Hove in [4] for the first time. He showed that there is a relationship between Generalized Hamming Weight (GHW) and GLW. After him, several authors studied this concept, see [1] and [7]. The concept of GHW introduced by V. K. Wei in [6]. After Wei, several authors worked on this topic, see [2] and [3].

* speaker

A code of length n over Z_4 is a subset of the free module Z_4^n and it is called linear if it is a Z_4- submodule of Z_4^n.

Let C be a linear code of length n over Z_4 and Let $A(C)$ be the $|C| \times n$ array of all code words in C. Each arbitrary column of $A(C)$, say c, corresponds to the following three cases:
i) c contains only 0
ii) c contains 0 and 2 equally often
iii) c contains all elements of Z_4 equally often,

We define the Lee support weight of these columns as 0, 2 and 1, respectively. Also, we define the Lee support weight of code C, denoted by $wt_L(C)$, as the sum of the Lee support weights of all columns of $A(C)$. As an example, let $C = \{(0,0,0),(2,1,2),(0,3,2),(0,2,0),(2,3,2),(2,0,2),(0,1,0),(2,2,0)\}$. Hence we have

$$A(C) = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 2 & 0 \\ 2 & 3 & 2 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 2 & 0 \end{pmatrix}.$$

If c_i be the i-th column of C, then we have $wt_L(c_1) = 2$, $wt_L(c_2) = 1$ and $wt_L(c_3) = 2$. Hence we obtain that $wt_L(C) = 2+1+2 = 5$. For code C with one generator, say x, we have $wt_L(C) = wt_L(x)$.

Let C be a code of length n over ring Z_4. The rank of C, denoted by $rank(C)$, is defined as the minimum number of generators of C, see [1]. For $1 \leq r \leq rank(C)$, the r-th generalized Lee weight with respect to *rank* (GLWR) for C, denoted by $d_r^L(C)$, is defined as follows

$$d_r^L(C) = \min\{wt_L(D) \mid D \text{ is a } Z_4 - \text{submodule of C with rank(D)} = r\}.$$

In this paper, we denote by $C = [n,k]$, the linear code C of length n and $rank = k$.

2. Main Results

Theorem 2.1. *Let C_i be an $[n,k_i]$ linear code over Z_4, for $i = 1,2$. Then the $(u,u+v)$-construction of C_1 and C_2 defined by*

$$C = \{(c_1, c_1 + c_2) : c_1 \in C_1, c_2 \in C_2\},$$

is a $[2n, k_1 + k_2]$ linear code over Z_4.

Theorem 2.2. *[1] Let C_1 and C_2 be $[n;k_1,k_2]$ codes over Z_4. Then we have*

$$wt_L(C) = \frac{4}{|C|}\sum_{x\in C}(wt_L(x) - wt(x)).$$

Note that $wt(x)$ is the Hamming weight for vector x.

Lemma 2.3. ***(Main Result)** For any $x,y \in Z_4^n$, we have*

$$wt_L(x) + wt_L(y) \geq wt_L(x+y).$$

PROOF. It is enough to show that for $1 \leq i \leq n$,

$$wt_L(x_i) + wt_L(y_i) \geq wt_L(x_i + y_i).$$

Considering all cases for x_i and y_i, the proof is completed. □

Lemma 2.4. ***(Main Result)** Let $x = (x_1,x_2,...,x_n) \in Z_4^n$, so we have*

$$wt_L(x) = wt_L(3x).$$

PROOF. It is enough to show that $wt_L(x_i) = wt_L(3x_i)$, for any $i, 1 \leqslant i \leqslant n$. Considering all cases for x_i, the proof is completed. This means that the Lee weight of any coordinate is not changed after multiplication by 3. It is desired. □

Theorem 2.5. ***(Main result)** Let C_1 and C_2 be linear codes over Z_4. Let $C = \{(c_1,c_1+c_2) : c_1 \in C_1, c_2 \in C_2\}$. Then we have*

$$d_1^L(C) = \min\{2d_1^L(C_1), d_1^L(C_2)\}.$$

PROOF. Let $d_1^L(C_1) = wt_L(D_1), D_1 =< x >$ for x in C_1 and

$$d_1^L(C_2) = wt_L(D_2), \quad D_2 =< y >,$$

for y in C_2. We have $d_1^L(C_1) = wt_L(x)$ and $d_1^L(C_2) = wt_L(y)$. Note that $(x,x) \in C$. Let $D =< (x,x) >$. Hence $rank(D) = 1$. We have $wt_L(D) = wt_L(x,x) = 2wt_L(x) = 2d_1^L(C_1)$. Also $(0,y) \in C$. Now let $D' =< (0,y) >$, so we have $wt_L(D') = wt_L(0,y) = wt_L(y) = d_1^L(C_2)$. Note that D and D' are satisfying $\{wt_L(H); H \leqslant C, rank(H) = 1\}$, and

$$\min\{wt_L(H); H \leqslant C, rank(H) = 1\} = d_1^L(C_1).$$

So we have

$$d_1^L(C) \leqslant wt_L(D) = 2d_1^L(C_1),$$
$$d_1^L(C) \leqslant wt_L(D') = d_1^L(C_2).$$

Therefore we have

$$d_1^L(C) \leqslant \min\{2d_1^L(C_1), d_1^L(C_2)\}. \tag{1}$$

On the other hand, let $d_1^L(C) = wt_L(H)$. So, $rank(H) = 1$ and $H =< (x, x+y) >$ for $x \in C_1$ and $y \in C_2$. We have

$$wt_L(H) = wt_L(x, x+y) = wt_L(x) + wt_L(x+y).$$

We have the following three cases:
i) If $x = 0, y \neq 0$ then $wt_L(H) = wt_L(y) \geq d_1^L(C_2)$.
ii) If $x \neq 0, y = 0$ then $wt_L(H) = 2wt_L(x) \geq 2d_1^L(C_1)$.
iii) If $x \neq 0, y \neq 0$ then by using Lemmas 2.3 and 2.3, we have

$$wt_L(H) = wt_L(3x) + wt_L(x+y) \geq wt_L(4x+y) = wt_L(y) \geq d_1^L(C_2).$$

Finally, we have

$$d_1^L(C) \geq \min\{2d_1^L(C_1), d_1^L(C_2)\}. \tag{2}$$

By using Eqs (1) and (2), the proof is completed. □

Theorem 2.6. ***(Main result)*** *Let C_1 and C_2 be linear codes over Z_4 and let $C = \{(c_1, c_1 + c_2) : c_1 \in C_1, c_2 \in C_2\}$. Then we have*

$$d_r^L(C) \leq \min\{2d_r^L(C_1), d_r^L(C_2)\}.$$

PROOF. Suppose that $d_r^L(C_1) = wt_L(D_1), D_1 =< x_1, x_2, \cdots, x_r >$ and $d_r^L(C_2) = wt_L(D_2), D_2 =< y_1, y_2, \cdots, y_r >$. Let $D_1' =< (x_1, x_1), (x_2, x_2), \cdots, (x_r, x_r) >$. By Theorem 2.2, we have

$$\begin{aligned} wt_L(D_1') &= \frac{4}{|D_1'|} \sum_{\alpha_1, \cdots, \alpha_r \in Z_4} [wt_L(\alpha_1(x_1, x_1) + \cdots + \alpha_r(x_r, x_r)) \\ &\qquad - wt(\alpha_1(x_1, x_1) + \cdots + \alpha_r(x_r, x_r))] \\ &= \frac{4}{|D_1'|} \sum 2wt_L(\alpha_1 x_1 + \cdots + \alpha_r x_r) \\ &\qquad - 2wt(\alpha_1 x_1 + \cdots + \alpha_r x_r) \\ &= \frac{2 \times 4}{|D_1|} \sum_{t \in D_1} wt_L(t) - wt(t) = 2wt_L(D_1) = 2d_r^L(C_1), \end{aligned}$$

implying that $wt_L(D_1') = 2d_r^L(C_1)$. By using the above method for

$$D_2' =< (0, y_1), (0, y_2), \cdots, (0, y_r) >,$$

we have $wt_L(D_2') = wt_L(D_2) = d_r^L(C_2)$. Since D_1' and D_2' are submodule of C of rank r, satisfying $\{wt_L(H); H \leq C, rank(H) = r\}$. Moreover, we have $\min\{wt_L(H); H \leq C, rank(H) = r\} = d_r^L(C)$. so we have

$$d_r^L(C) \leq wt_L(D_1') = 2d_r^L(C_1), \quad d_r^L(C) \leq wt_L(D_2') = d_r^L(C_2)$$

Finally, we obtain

$$d_r^L(C) \leq \min\{2d_r^L(C_1), d_r^L(C_2)\}.$$

□

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Relation between the power graph of finite group and commutative elements

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Abstract

Let G be a finite group. The *power graph* of a group G, with notation $\mathcal{P}(G)$ is a graph such that it's vertex set is the group G and two distinct elements x, y are adjacent if and only if $x = y^n$ or $y = x^n$ for some positive integer n. For a nonempety set X of G, The *commuting graph* $C(G, X)$ is the graph with X as the vertex set and two distinct elements of X being joined by an edge if they are commuting elements of G. The purpose of this paper is study of groups with property $\mathcal{P}(G) = C(G, G)$.

Keywords and phrases: Finite group, Power graph, Commutative elements.

2010 *Mathematics subject classification:* Primary: 05C25; Secondary: 13C20.

1. Introduction

The investigation of algebraic structures using the properties of graphs is an important topic for some researchers. The different types of graphs with respect to group are defined as: Cayley graphs, Commuting graphs and Power graphs. The power graph $\mathcal{P}(G)$ of a group G, is the graph whose vertex set is the group G so that two distinct elements are adjacent if one is a power of the other. For a nonempety set X of G, The commuting graph $C(G, X)$ is the graph with X as the vertex set and two distinct ele- ments of X being joined by an edge if they are commuting elements of G. For the first time, Kelarev and Quinn [3] have studied the directed power graph of semigroups, in which there is an arc from a vertex x to a vertex y if y is positive power of x. Some numerical properties of commuting graphs been discussed by Mahmoudifar et al. [4]. Suppose that G is a finite group with power graph $\mathcal{P}(G)$. We know that if the elements $x, y \in G$ are adjacent in the $\mathcal{P}(G)$, then $xy = yx$. Thus commutativity of the elements x, y is necessary condition for x is adjacent to y in the graph $\mathcal{P}(G)$. In this paper, we study groups that necessary and sufficient condition for adjacency elements in the $\mathcal{P}(G)$ is commutative. In other words, $\mathcal{P}(G) = C(G, G)$. We use of notation $\mathcal{P}_c(G)$ for the power graphs with this property.

* speaker

2. Main Results

Theorem 2.1. *Let G be a finite p-group with graph $\mathcal{P}_c(G)$ where p is prime. Then the group G is cyclic or generalized quaternion.*

PROOF. Suppose that G is a finite p-group with the graph $\mathcal{P}_c(G)$. Let $z \in Z(G)$ of prime order p. Since $[x,z] = 1$ for every element x of order p, then $\langle z \rangle = \langle x \rangle$. Hence the finite p-group G has a unique cyclic subgroup of order p. By ([5],5.3.6), G is cyclic or generalized quaternion. □

Lemma 2.2. *If G is a finite group with the graph $\mathcal{P}_c(G)$. Then, the elements of G are p-element for some prime number p.*

PROOF. By contradiction, assume that there exists $x \in G$ such that pq divides order x where p,q are distinct primes. Let $y,z \in \langle x \rangle$ of orders p and q. It is clearly, $[y,z] = 1$ but y is not adjacent z in the graph $\mathcal{P}_c(G)$ which is a contradiction. □

Corollary 2.3. *Suppose that G is a finite group with the graph $\mathcal{P}_c(G)$. Then, centralizer nontrivial elements is p-group for some prime number p, particularly if G is not p-group, then $Z(G)$ is trivial.*

We define class CP of finite groups in which the centralizers of all nontrivial elements contain only elements of prime power order. By the previous lemma, The finite groups with the graph $\mathcal{P}_c(G)$ are in the class CP-groups. In the next theorem Deaconescu characterized ([1]) CP-groups.

Theorem 2.4. *A group G is a CP-group if and only if one of the following holds:*

(1) G is isomorphic with $PSL(2,q)$ with $q = 4,7,8,9,17$; $PSL(3,4)$, $SZ(8)$, $SZ(32)$ or M_{10}.

(2) G has a nontrivial normal 2-subgroup P and $\frac{G}{P}$ is isomorphic with $PSL(2,4)$, $PSL(2,8)$, $SZ(8)$ or $SZ(32)$.Moreover P is elementary abelian and isomorphic with a direct sum of natural modules.

(3) G is a p-group.

(4) G is a frobenius group whose kernel is a p-group and the complement is either a cyclic q-group ($q \neq p$) or a generalized quaternion group.

(5) G is a 3-step group of order $p^a q^b$ (p,q primes, $q > 2$) i.e. $G = O_{pp'p}(G)$ and $G \supset O_{p'p}(G)$ with

(i) $O_{p'p}(G)$ is a Frobenius group with kernel $O_p(G)$ and cyclic complement.

(ii) $\frac{G}{O_p(G)}$ is a Frobenius group with kernel $\frac{O_{pp'}(G)}{O_p(G)}$.

Theorem 2.5. *Let G be a finite group with the graph $\mathcal{P}_c(G)$. Then G is isomorphism one of the following groups:*

(1) The group H_p which is a cyclic p-group or generalized quaternion group for some prime number p,

(2) $H_p \rtimes H_q$
(3) $H_p \rtimes (H_q \rtimes H_p)$.
where p and q are distinct primes.

PROOF. If G is a finite p-group, then by Theorem 2.1 G is given in the part (1). Suppose that G is not p-group. Whether G is group with mentioned property by Theorem 2.1, all Sylow p-subgroups of G are cyclic or generalized quaternion. If all Sylow p-subgroups of G are cyclic, then groups G' and $\frac{G}{G'}$ are cyclic and they have coprime orders, by Theorem ([2], 5.16) and the other hand theses groups are p-group by Lemma 2.2. Hence G is Frobenius group and $G \cong H_p \rtimes H_q$ where H_p is cyclic. Suppose that all Sylow subgroups of odd order are cyclic and Sylow 2-subgroup is generalized quaternion group. If G is not solvable, then G contains a normal subgroup G_1 with the properties that $[G : G_1] \leq 2$ and G_1 is a direct product of a Z-group (A group that is all Sylow subgroups are cyclic) and a subgroup isomorphic with $SL(2, p)$ for some odd prime number p. But $SL(2, p)$ has some element of order $2p$ which is a contradiction. Hence assume that G is solvable. Since G is CP-group, then G is not isomorphic to groups of (1) and (2) in the Theorem 2.4. Thus G is isomorphic to one of the groups 4 and 5 and the proof is complete. □

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On the annihilators of Ext modules

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Abstract
Let R be a commutative Noetherian ring, and let M and N be two finitely generated R-modules such that N is Gorenstein. For each integer t we give a bound under inclusion for the annihilator of $\mathrm{Ext}^t_R(M,N)$ in terms of minimal primary decomposition of the zero submodule of M, which is independent of the choice of minimal primary decomposition. Then, by using that bound, we compute the annihilator of $\mathrm{Ext}^t_R(M,N)$ for $t = \dim_R(N) - \dim_R(M \otimes_R N)$.

Keywords and phrases: Ext-module, annihilator, primary decomposition.
2010 *Mathematics subject classification:* 13D07, 13E05.

1. Introduction

Throughout the paper R is a commutative Noetherian ring. If S is a Gorenstein local ring and J is an ideal of S with $\mathrm{ht}_S(J) = 0$, then Lynch proved in [3, Lemma 2.1]

$$\mathrm{Ann}_S(\mathrm{Hom}_S(S/J,S)) = \bigcap_{\dim_S(S/\mathfrak{q}_i)=\dim_S(S)} \mathfrak{q}_i,$$

where $J = \bigcap_{i=1}^n \mathfrak{q}_i$ is a minimal primary decomposition of J in S. In this paper we generalize her result. More precisely, let M, N be non-zero finitely generated R-modules such that N is Gorenstein. Let $0 = \bigcap_{i=1}^n M_i$ with $\mathrm{Ass}_R(M/M_i) = \{\mathfrak{p}_i\}$ for all $1 \le i \le n$ be a minimal primary decomposition of the zero submodule of M . For each integer t we obtain the following bound for the annihilator of $\mathrm{Ext}^t_R(M,N)$

$$\mathrm{Ann}_R\Big(M/\bigcap_{\mathfrak{p}_i\in\Delta(t)} M_i\Big) \subseteq \mathrm{Ann}_R\left(\mathrm{Ext}^t_R(M,N)\right) \subseteq \mathrm{Ann}_R\Big(M/\bigcap_{\mathfrak{p}_i\in\Sigma(t)} M_i\Big)$$

for some suitable subsets $\Delta(t)$ and $\Sigma(t)$ of $\mathrm{Ass}_R(M)$. This bound is independent of the choice of minimal primary decomposition. Then, by using this bound, we compute the annihilator of $\mathrm{Ext}^t_R(M,N)$ for $t = \dim_R(N) - \dim_R(M \otimes_R N)$. We refer the reader to [4] for basic properties of primary decomposition of modules, to [5, 6] for more details about the Gorenstein modules and to [1] for the theory of local cohomology.

* speaker

2. Main Results

Assume M is an R-module. We denote the set of all associated prime ideals of M by $\mathrm{Ass}_R(M)$ and the set of its minimal elements is denoted by $\mathrm{MinAss}_R(M)$. For each $t \in \mathbb{N}_0$, we denote, respectively, the sets $\{\mathfrak{p} \in \mathrm{Ass}_R(M) : \dim_R(R/\mathfrak{p}) \geq t\}$ and $\{\mathfrak{p} \in \mathrm{Ass}_R(M) : \dim_R(R/\mathfrak{p}) = t\}$ by $\mathrm{Ass}_R^{\geqslant t}(M)$ and $\mathrm{Ass}_R^t(M)$. Similarly, the sets $\mathrm{MinAss}_R^{\geqslant t}(M)$ and $\mathrm{MinAss}_R^t(M)$ are defined as above by replacing $\mathrm{Ass}_R(M)$ by $\mathrm{MinAss}_R(M)$. Also, when $\dim_R(M) < \infty$, we denote the set $\{\mathfrak{p} \in \mathrm{Ass}_R(M) : \dim_R(R/\mathfrak{p}) = \dim_R(M)\}$ by $\mathrm{Assh}_R(M)$. We say that a subset Σ of $\mathrm{Ass}_R(M)$ is isolated if it satisfies the following condition: if $\mathfrak{q} \in \mathrm{Ass}_R(M)$ and $\mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \Sigma$, then $\mathfrak{q} \in \Sigma$. If N is a submodule of M and S is a multiplicatively closed subset of R, then we denote the contraction of $S^{-1}N$ under the canonical map $M \to S^{-1}M$ by $S_M(N)$.

Definition 2.1. *Let $(R, \mathfrak{m})$ be a local ring. A non-zero finitely generated R-module G is said to be Gorenstein if $\mathrm{depth}_R(G) = \dim_R(G) = \mathrm{id}_R(G) = \mathrm{depth}_R(R) = \dim_R(R)$. When R is not necessarily local, a non-zero finitely generated R-module G is said to be Gorenstein if $G_\mathfrak{p}$ is a Gorenstein $R_\mathfrak{p}$-module for all prime (or maximal) ideals $\mathfrak{p}$ in $\mathrm{Supp}_R(G)$; see [5,* Theorem 3.11 and Corollary 3.7*].*

When $(R, \mathfrak{m})$ is a complete Cohen-Macaulay local ring, then Gorenstein modules are the non-empty finite direct sums of the canonical module [6, Corollary 2.7].

Theorem 2.2 ([2, Theorem 2.5 and Remark 2.6]). *Let M, N be non-zero finitely generated R-modules such that N is Gorenstein. Let $0 = \bigcap_{i=1}^n M_i$ be a minimal primary decomposition of the zero submodule of M with $\mathrm{Ass}_R(M/M_i) = \{\mathfrak{p}_i\}$ for all $1 \leq i \leq n$. Let $t \in \mathbb{N}_0$ and set*

$$\Delta(t) = \{\mathfrak{p} \in \mathrm{Ass}_R(M) \cap \mathrm{Supp}_R(N) : \mathrm{ht}_R(\mathfrak{p}) \leq t\},$$

$$\Sigma(t) = \{\mathfrak{p} \in \mathrm{MinAss}_R(M) \cap \mathrm{Supp}_R(N) : \mathrm{ht}_R(\mathfrak{p}) = t\},$$

$$S^t = R \setminus \bigcup_{\mathfrak{p} \in \Delta(t)} \mathfrak{p}$$

and $T^t = R \setminus \bigcup_{\mathfrak{p} \in \Sigma(t)} \mathfrak{p}$. Then

1. *The sets $\Delta(t), \Sigma(t)$ are isolated subsets of $\mathrm{Ass}_R(M)$ and $\bigcap_{\mathfrak{p}_i \in \Delta(t)} M_i = S^t_M(0)$, $\bigcap_{\mathfrak{p}_i \in \Sigma(t)} M_i = T^t_M(0)$. In particular, $\bigcap_{\mathfrak{p}_i \in \Delta(t)} M_i$ and $\bigcap_{\mathfrak{p}_i \in \Sigma(t)} M_i$ are independent of the choice of minimal primary decomposition of the zero submodule of M.*
2. *$S^t_M(0)$ is the largest submodule L of M such that $\mathrm{Ext}^i_R(L, N) = 0$ for all $i \leq t$.*
3. $$\mathrm{Ann}_R(M/S^t_M(0)) \subseteq \mathrm{Ann}_R(\mathrm{Ext}^t_R(M, N)) \subseteq \mathrm{Ann}_R(M/T^t_M(0)).$$

4. *If* $\mathrm{Supp}_R(M) \cap \mathrm{Supp}_R(N) \neq \emptyset$ *and* $t = \dim_R(N) - \dim_R(M \otimes_R N)$, *then* $\Delta(t) = \Sigma(t)$ *and*

$$\mathrm{Ann}_R\left(\mathrm{Ext}^t_R(M,N)\right) = \mathrm{Ann}_R\left(M/T^t_M(0)\right).$$

Corollary 2.3 ([2, Remark 2.6]). *Let* $(R,\mathfrak{m})$ *be a local ring of dimension* d, *and let* M,N *be non-zero finitely generated* R*-modules such that* N *is Gorenstein. Let* $0 = \bigcap_{i=1}^n M_i$ *be a minimal primary decomposition of the zero submodule of* M *with* $\mathrm{Ass}_R(M/M_i) = \{\mathfrak{p}_i\}$ *for all* $1 \le i \le n$. *Then for each* $t \in \mathbb{N}_0$

$$\mathrm{Ann}\left(\frac{M}{\bigcap_{\mathfrak{p}_i \in \mathrm{Ass}_R^{\geqslant d-t}(M)} M_i}\right) \subseteq \mathrm{Ann}\left(\mathrm{Ext}^t_R(M,N)\right) \subseteq \mathrm{Ann}\left(\frac{M}{\bigcap_{\mathfrak{p}_i \in \mathrm{MinAss}_R^{d-t}(M)} M_i}\right).$$

In particular,

$$\mathrm{Ann}_R(\mathrm{Ext}^{d-\dim_R(M)}_R(M,N)) = \mathrm{Ann}_R(M/\textstyle\bigcap_{\mathfrak{p}_i \in \mathrm{Assh}_R(M)} M_i).$$

We end the paper by two examples showing how we can compute the above bounds for the annihilators of Ext modules. Moreover, these examples show that to improve the upper bound in Corollary 2.3 we can not replace the index set $\mathrm{MinAss}_R^{d-t}(M)$ by the larger sets $\mathrm{MinAss}_R^{\geqslant d-t}(M)$, $\mathrm{Ass}_R^{d-t}(M)$ or $\mathrm{Ass}_R^{\geqslant d-t}(M)$ and also to improve the lower bound in Corollary 2.3 we can not replace the index set $\mathrm{Ass}_R^{\geqslant d-t}(M)$ by the smaller set $\mathrm{Ass}_R^{d-t}(M)$. Also, in general, for an arbitrary integer t there is not a subset Σ of $\mathrm{Ass}_R(M)$ such that $\mathrm{Ann}_R\left(\mathrm{Ext}^t_R(M,N)\right) = \mathrm{Ann}_R\left(M/\bigcap_{\mathfrak{p}_i \in \Sigma} M_i\right)$.

Example 2.4 ([2, Example 2.7]). *Let* $R = K[[x,y]]$ *be the ring of formal power series over a field* K *in indeterminates* x,y. *Set* $M = R/\langle x^2,xy\rangle$, $M_1 = \langle x\rangle/\langle x^2,xy\rangle$ *and* $M_2 = \langle x^2,y\rangle/\langle x^2,xy\rangle$. *Then* $0 = M_1 \cap M_2$ *is a minimal primary decomposition of the zero submodule of* M *with* $\mathrm{Ass}_R(M/M_1) = \{\mathfrak{p}_1 = \langle x\rangle\}$ *and* $\mathrm{Ass}_R(M/M_2) = \{\mathfrak{p}_2 = \langle x,y\rangle\}$. *So* $\mathrm{Ass}_R(M) = \{\mathfrak{p}_1,\mathfrak{p}_2\}$ *and* $\mathrm{MinAss}_R(M) = \{\mathfrak{p}_1\}$. *Hence, we have*

$$\mathrm{Ass}_R^{\geqslant 2-t}(M) = \begin{cases} \emptyset & \text{if } t=0 \\ \{\mathfrak{p}_1\} & \text{if } t=1 \\ \{\mathfrak{p}_1,\mathfrak{p}_2\} & \text{if } t=2 \end{cases},$$

and

$$\mathrm{MinAss}_R^{2-t}(M) = \begin{cases} \emptyset & \text{if } t=0,2 \\ \{\mathfrak{p}_1\} & \text{if } t=1 \end{cases}.$$

Thus

$$\mathrm{Ann}_R\Big(M/\bigcap_{\mathfrak{p}_i \in \mathrm{Ass}_R^{\geqslant 2-t}(M)} M_i\Big) = \begin{cases} R & \text{if } t=0 \\ \langle x\rangle & \text{if } t=1 \\ \langle x^2,xy\rangle & \text{if } t=2 \end{cases},$$

$$\mathrm{Ann}_R\Big(M/\bigcap_{\mathfrak{p}_i \in \mathrm{MinAss}_R^{2-t}(M)} M_i\Big) = \begin{cases} R & \text{if } t=0,2 \\ \langle x\rangle & \text{if } t=1 \end{cases}$$

Therefore, Corollary 2.3 implies that

$$\operatorname{Hom}_R(M,R)=0, \operatorname{Ann}_R\left(\operatorname{Ext}^1_R(M,R)\right)=\langle x\rangle,$$

and

$$\langle x^2,xy\rangle \subseteq \operatorname{Ann}_R\left(\operatorname{Ext}^2_R(M,R)\right)\subseteq R.$$

Also, since $\operatorname{id}_R(R)=2$ *we deduce that* $\operatorname{Ext}^t_R(M,R)=0$ *for all* $t>2$.

Now, we directly compute $\operatorname{Ann}_R\left(\operatorname{Ext}^t_R(M,R)\right)$ *for all* t *(especially for* $t=2$*). It is straightforward to see that* $\boldsymbol{P}: 0\to R\xrightarrow{d_2} R^2\xrightarrow{d_1} R\xrightarrow{\epsilon} M\to 0$ *with* $\epsilon(f)=f+\langle x^2,xy\rangle$, $d_1(f,g)=x^2f+xyg$, $d_2(f)=(yf,-xf)$ *for all* $f,g\in R$ *being a projective resolution of* M*. Applying the functor* $\operatorname{Hom}_R(\cdot,R)$ *to the delated projective resolution* $\boldsymbol{P}_M$*, we obtain*

$$\operatorname{Ext}^1_R(M,R)\cong R/\langle x\rangle, \operatorname{Ext}^2_R(M,R)\cong R/\langle x,y\rangle,\ \operatorname{Ext}^t_R(M,R)=0 \text{ for all } t\neq 1,2.$$

It follows that $\operatorname{Ann}_R\left(\operatorname{Ext}^1_R(M,R)\right)=\langle x\rangle$ *and* $\operatorname{Ann}_R\left(\operatorname{Ext}^2_R(M,R)\right)=\langle x,y\rangle$*. Thus, there is not a subset* Σ *of* $\operatorname{Ass}_R(M)$ *such that*

$$\operatorname{Ann}_R\left(\operatorname{Ext}^2_R(M,R)\right)=\operatorname{Ann}_R\left(M/\bigcap_{\mathfrak{p}_i\in\Sigma} M_i\right).$$

Moreover, for $t=2$*, this example shows that in the second inclusion of Corollary 2.3, to obtain a better upper bound of* $\operatorname{Ann}_R\left(\operatorname{Ext}^t_R(M,R)\right)$*, we cannot replace the index set* $\operatorname{MinAss}^{d-t}_R(M)$ *by the larger sets* $\operatorname{MinAss}^{\geqslant d-t}_R(M)$*,* $\operatorname{Ass}^{d-t}_R(M)$ *or* $\operatorname{Ass}^{\geqslant d-t}_R(M)$*.*

Example 2.5 ([2, Example 2.8]). *Let* $R=K[[x,y,z,w]]$ *be the ring of formal power series over a field* K *in indeterminates* x,y,z,w*. Then* R *is a local ring with maximal ideal* $\mathfrak{n}=\langle x,y,z,w\rangle$*. Set* $\mathfrak{p}_1=\langle x,y\rangle$*,* $\mathfrak{p}_2=\langle z,w\rangle$ *and* $M=R/(\mathfrak{p}_1\cap\mathfrak{p}_2)$*. Then* $\operatorname{depth}_R(R/\mathfrak{p}_1)=\operatorname{depth}_R(R/\mathfrak{p}_2)=2$ *and hence,* $\operatorname{H}^i_\mathfrak{n}(R/\mathfrak{p}_1)=\operatorname{H}^i_\mathfrak{n}(R/\mathfrak{p}_2)=0$ *for* $i=0,1$*. Now, the exact sequence*

$$0\to M\to R/\mathfrak{p}_1\oplus R/\mathfrak{p}_2\to R/\mathfrak{n}\to 0,$$

induces the exact sequence

$$0\to\Gamma_\mathfrak{n}(M)\to\Gamma_\mathfrak{n}(R/\mathfrak{p}_1\oplus R/\mathfrak{p}_2)\to\Gamma_\mathfrak{n}(R/\mathfrak{n})\to\operatorname{H}^1_\mathfrak{n}(M)\to\operatorname{H}^1_\mathfrak{n}(R/\mathfrak{p}_1\oplus R/\mathfrak{p}_2)$$

of local cohomology modules. It follows that $\Gamma_\mathfrak{n}(M)=0$ *and* $\operatorname{H}^1_\mathfrak{n}(M)\cong R/\mathfrak{n}$*. Hence, by the Grothendieck duality [1,* Theorem 11.2.8*],*

$$\operatorname{Hom}_R(\operatorname{Ext}^3_R(M,R),E(R/\mathfrak{n}))\cong\operatorname{H}^1_\mathfrak{n}(M).$$

Thus, $\operatorname{Ann}_R\left(\operatorname{Ext}^3_R(M,R)\right)=\mathfrak{n}$*. On the other hand, if* $M_1=\mathfrak{p}_1/(\mathfrak{p}_1\cap\mathfrak{p}_2)$ *and* $M_2=\mathfrak{p}_2/(\mathfrak{p}_1\cap\mathfrak{p}_2)$*, then* $0=M_1\cap M_2$ *is a minimal primary decomposition of the zero submodule of* M*. Since* $\operatorname{Ass}^1_R(M)=\emptyset$*, we have*

$$R=\operatorname{Ann}_R\left(M/\bigcap_{\mathfrak{p}_i\in\operatorname{Ass}^1_R(M)} M_i\right)\nsubseteq\operatorname{Ann}_R\left(\operatorname{Ext}^3_R(M,R)\right).$$

Therefore in the first inclusion of Corollary 2.3, to obtain a better lower bound of $\mathrm{Ann}_R\left(\mathrm{Ext}^t_R(M,R)\right)$*, we cannot replace the index set* $\mathrm{Ass}^{\geqslant d-t}_R(M)$ *by the smaller set* $\mathrm{Ass}^{d-t}_R(M)$.

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On the annihilator of local cohomology

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Abstract

Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R, M a finitely generated R-module and t a nonnegative integer. In certain cases, we give some bounds under inclusion for the annihilator of $\mathrm{H}^t_{\mathfrak{a}}(M)$ in terms of minimal primary decomposition of the zero submodule of M, which are independent of the choice of minimal primary decomposition. Then, by using those bounds, we compute the annihilators of local cohomology modules in certain cases.

Keywords and phrases: Local cohomology, annihilator, primary decomposition.
2010 *Mathematics subject classification:* 13D45, 13D07, 13E05.

1. Introduction

Throughout this note R is a commutative Noetherian ring. The ith local cohomology of an R-module M with respect to an ideal $\mathfrak{a}$ was introduced by Grothendieck as follows:

$$\mathrm{H}^i_{\mathfrak{a}}(M) := \varinjlim_n \mathrm{Ext}^i_R(R/\mathfrak{a}^n, M).$$

We we refer the reader to [5] for more details about the local cohomology.

In this paper, we investigate the annihilator of local cohomology. Let $\mathfrak{a}$ be a proper ideal of R, M a nonzero finitely generated R-module of dimension d, and $0 = \bigcap_{i=1}^n M_i$ a minimal primary decomposition of the zero submodule of M with $\mathrm{Ass}_R(M/M_i) = \{\mathfrak{p}_i\}$ for all $1 \leq i \leq n$. We denote $\sup\{i \in \mathbb{N}_0 : \mathrm{H}^i{}_{\mathfrak{a}}(M) \neq 0\}$ by $\mathrm{cd}_R(\mathfrak{a}, M)$. If $\mathrm{cd}_R(\mathfrak{a}, M) = d < \infty$, then

$$\mathrm{Ann}_R\left(\mathrm{H}^d_{\mathfrak{a}}(M)\right) = \mathrm{Ann}_R\left(M/\bigcap_{\mathrm{cd}_R(\mathfrak{a},R/\mathfrak{p}_i)=d} M_i\right).$$

This equality is proved by Lynch whenever R is a complete local ring and $M = R$, see [7, Theorem 2.4]. In [4, Theorem 1.1], Bahmanpour et al. proved that $\mathrm{Ann}_R\left(\mathrm{H}^d_{\mathfrak{a}}(M)\right) = \mathrm{Ann}_R(M/T_R(\mathfrak{a}, M))$ whenever $\mathfrak{a} = \mathfrak{m}$ and R is a complete local ring, where $T_R(\mathfrak{a}, M)$ denotes the largest submodule N of M such

* speaker

that $\mathrm{cd}_R(\mathfrak{a},N) < \mathrm{cd}_R(\mathfrak{a},M)$. Then Bahmanpour in [3, Theorem 3.2] extended the result of Lynch for the R-module M. Next, Atazadeh et al. in [2, Proposition 3.8] proved this equality whenever R is a local ring (not necessarily complete) and finally in [1, Corollary 1.2] they extended it to the nonlocal case. We note that $T_R(\mathfrak{a},M) = \bigcap_{\mathrm{cd}_R(\mathfrak{a},R/\mathfrak{p}_i)=\mathrm{cd}_R(\mathfrak{a},M)} M_i$ [2, Remark 2.5] also, if $(R,\mathfrak{m})$ is a complete local ring and $\mathfrak{p} \in \mathrm{Ass}_R(M)$, then by the Lichtenbaum-Hartshorne Vanishing Theorem, $\mathrm{cd}_R(\mathfrak{a},R/\mathfrak{p}) = d$ if and only if $\dim_R(R/\mathfrak{p}) = d$ and $\sqrt{\mathfrak{a}+\mathfrak{p}} = \mathfrak{m}$.

When $(R,\mathfrak{m})$ is a local ring for an arbitrary integer t we give a bound for the annihilator of $\mathrm{H}^t_{\mathfrak{m}}(M)$ in Theorem 2.3. More precisely, we show that

$$\mathrm{Ann}_R\Big(M/\bigcap_{\mathfrak{p}_i\in \mathrm{Ass}_R^{\geqslant t}(M)} M_i\Big) \subseteq \mathrm{Ann}_R\left(\mathrm{H}^t_{\mathfrak{m}}(M)\right) \subseteq \mathrm{Ann}_R\Big(M/\bigcap_{\mathfrak{p}_i\in \mathrm{MinAss}_R^{t}(M)} M_i\Big),$$

where $\mathrm{Ass}_R^{\geqslant t}(M) = \{\mathfrak{p} \in \mathrm{Ass}_R(M) : \dim_R(R/\mathfrak{p}) \geq t\}$ and $\mathrm{MinAss}_R^t(M) = \{\mathfrak{p} \in \mathrm{MinAss}_R(M) : \dim_R(R/\mathfrak{p}) = t\}$. Also, whenever R is not necessarily local, in Theorem 2.4, we provide a bound for $\mathrm{Ann}_R\left(\mathrm{H}^{\mathrm{cd}_R(\mathfrak{a},M)}_{\mathfrak{a}}(M)\right)$ which implies the above equality when $\mathrm{cd}_R(\mathfrak{a},M) = d$. Finally, when M is Cohen-Macaulay, a bound for $\mathrm{Ann}_R\left(\mathrm{H}^t_{\mathfrak{a}}(M)\right)$ is given and at $t = \mathrm{grade}(\mathfrak{a},M)$, this annihilator is computed in Theorem 2.6.

We adopt the convention that the intersection of empty family of subsets of a set M is M.

2. Main Results

Let M be an R-module. The set of all associated prime ideals of R is denoted by $\mathrm{Ass}_R(M)$ and the set of all minimal elements of $\mathrm{Ass}_R(M)$ under inclusion is denoted by $\mathrm{MinAss}_R(M)$. Also, we use $\mathrm{Assh}_R(M)$ to denote the set $\{\mathfrak{p} \in \mathrm{Ass}_R(M) : \dim_R(R/\mathfrak{p}) = \dim_R(M)\}$. For each $t \in \mathbb{N}_0$, we set $\mathrm{Ass}_R^{\geqslant t}(M) = \{\mathfrak{p} \in \mathrm{Ass}_R(M) : \dim_R(R/\mathfrak{p}) \geq t\}$ and $\mathrm{MinAss}_R^t(M) = \{\mathfrak{p} \in \mathrm{MinAss}_R(M) : \dim_R(R/\mathfrak{p}) = t\}$.

Definition 2.1. *A proper submodule N of an R-module M is called a primary submodule of M if $m \in M, r \in R$ and $rm \in N$ imply that either $m \in N$ or $r^t M \subseteq N$ for some $t \in \mathbb{N}$.*

If N is a primary submodule of M, then $\mathfrak{p} = \sqrt{\mathrm{Ann}_R(M/N)}$ is a prime ideal of R and N is called a $\mathfrak{p}$-primary submodule of M. We say that a proper submodule L of M has a primary decomposition in M when $L = \bigcap_{i=1}^n M_i$ for some primary submodules $M_1, \dots, M_n$ of M. If, in addition, $\mathfrak{p}_i$'s are distinct and $\bigcap_{j\neq i} M_j \nsubseteq M_i$ for all i, then the primary decomposition is called minimal. If $L = \bigcap_{i=1}^n M_i$ is a minimal primary decomposition of L in M with M_i is $\mathfrak{p}_i$-primary, then we have $\mathrm{Ass}_R(M/L) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Over a commutative

Noetherian rings every proper submodule of a finitely generated module M has a (minimal) primary decomposition in M.

Assume N is a submodule of an R-module M. For any multiplicatively closed subset S of R, we denote the contraction of $S^{-1}N$ under the canonical map $M \to S^{-1}M$ by $S_M(N)$. Assume $\Sigma \subseteq \mathrm{Ass}_R(M)$. We say that Σ is an isolated subset of $\mathrm{Ass}_R(M)$ if it satisfies the following condition: if $\mathfrak{q} \in \mathrm{Ass}_R(M)$ and $\mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \Sigma$, then $\mathfrak{q} \in \Sigma$.

Lemma 2.2 ([6, Lemma 2.2]). *Let M be a finitely generated R-module, and N a proper submodule of M. Let $N = \bigcap_{i=1}^{n} N_i$ be a minimal primary decomposition of N in M with $\mathrm{Ass}_R(M/N_i) = \mathfrak{p}_i$ for all $1 \le i \le n$. Assume Σ is an isolated subset of $\mathrm{Ass}_R(M/N)$. Then $\bigcap_{\mathfrak{p}_i \in \Sigma} N_i = S_M(N)$, where $S = R \setminus \bigcup_{\mathfrak{p} \in \Sigma} \mathfrak{p}$. In particular, $\bigcap_{\mathfrak{p}_i \in \Sigma} N_i$ is independent of the choice of minimal primary decomposition of N in M.*

Theorem 2.3 ([6, Theorem 3.2]). *Let $(R, \mathfrak{m})$ be a local ring and $t \in \mathbb{N}_0$. Let M be a nonzero finitely generated R-module and $0 = \bigcap_{i=1}^{n} M_i$ a minimal primary decomposition of the zero submodule of M with $\mathrm{Ass}_R(M/M_i) = \{\mathfrak{p}_i\}$ for all $1 \le i \le n$. Then*

1. *$\bigcap_{\mathfrak{p}_i \in \mathrm{Ass}_R^{\geq t}(M)} M_i = S_M^t(0)$ and $\bigcap_{\mathfrak{p}_i \in \mathrm{MinAss}_R^t(M)} M_i = T_M^t(0)$, where $S^t = R \setminus \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R^{\geq t}(M)} \mathfrak{p}$ and $T^t = R \setminus \bigcup_{\mathfrak{p} \in \mathrm{MinAss}_R^t(M)} \mathfrak{p}$. In particular, $\bigcap_{\mathfrak{p}_i \in \mathrm{Ass}_R^{\geq t}(M)} M_i$ and $\bigcap_{\mathfrak{p}_i \in \mathrm{MinAss}_R^t(M)} M_i$ are independent of the choice of minimal primary decomposition of the zero submodule of M.*
2. *$S_M^t(0)$ is the largest submodule N of M such that $\dim_R(N) < t$.*
3. $$\mathrm{Ann}_R\left(M/S_M^t(0)\right) \subseteq \mathrm{Ann}_R\left(\mathrm{H}_{\mathfrak{m}}^t(M)\right) \subseteq \mathrm{Ann}_R\left(M/T_M^t(0)\right).$$

   In particular, for $t = \dim_R(M)$ there are the equalities $S_M^t(0) = T_M^t(0) = \bigcap_{\mathfrak{p}_i \in \mathrm{Assh}_R(M)} M_i$, and

   $$\mathrm{Ann}_R\left(\mathrm{H}_{\mathfrak{m}}^{\dim_R(M)}(M)\right) = \mathrm{Ann}_R\left(M/\bigcap_{\mathfrak{p}_i \in \mathrm{Assh}_R(M)} M_i\right).$$

Now, in the following theorem, we give a bound for the annihilator of top local cohomology module without the local assumption on R.

Theorem 2.4 ([6, Theorem 3.4]). *Let M be a nonzero finitely generated R-module and $\mathfrak{a}$ an ideal of R such that $\mathfrak{a}M \neq M$. Let $c = \mathrm{cd}_R(\mathfrak{a}, M)$ and $0 = \bigcap_{i=1}^{n} M_i$ be a minimal primary decomposition of the zero submodule of M with $\mathrm{Ass}_R(M/M_i) = \{\mathfrak{p}_i\}$ for all $1 \le i \le n$. Set $\Delta = \{\mathfrak{p} \in \mathrm{Ass}_R(M) : \mathrm{cd}_R(\mathfrak{a}, R/\mathfrak{p}) = c\}$ and $\Sigma = \{\mathfrak{p} \in \mathrm{Ass}_R(M) : \mathrm{cd}_R(\mathfrak{a}, R/\mathfrak{p}) = \dim_R(R/\mathfrak{p}) = c\}$. Then*

1. *$\bigcap_{\mathfrak{p}_i \in \Delta} M_i = S_M(0)$, where $S = R \setminus \bigcup_{\mathfrak{p} \in \Delta} \mathfrak{p}$. In particular, $\bigcap_{\mathfrak{p}_i \in \Delta} M_i$ is independent of the choice of minimal primary decomposition of the zero submodule of M.*
2. *$S_M(0)$ is the largest submodule N of M such that $\mathrm{cd}_R(\mathfrak{a}, N) < c$.*

3.
$$\mathrm{Ann}_R(M/\bigcap_{\mathfrak{p}_i\in\Delta} M_i) \subseteq \mathrm{Ann}_R(\mathrm{H}^c_{\mathfrak{a}}(M)) \subseteq \mathrm{Ann}_R(M/\bigcap_{\mathfrak{p}_i\in\Sigma} M_i).$$
In particular, when $c = \dim_R(M)$, there are the equalities $\Sigma = \Delta$ and
$$\mathrm{Ann}_R\left(\mathrm{H}^c_{\mathfrak{a}}(M)\right) = \mathrm{Ann}_R\left(M/S_M(0)\right).$$

When $(R,\mathfrak{m})$ is a Cohen-Macaulay local ring and $\mathfrak{a}$ is a nonzero proper ideal of R, then for $t = \mathrm{grade}(\mathfrak{a}, R)$ Bahmanpour calculated the annihilator of $\mathrm{H}^t_{\mathfrak{a}}(R)$ in [3, Theorem 2.2]. In the following theorem, we generalize his result for Cohen-Macaulay modules whenever R is not necessarily local.

Definition 2.5. *Let M be an R-module. For $\mathfrak{p} \in \mathrm{Supp}_R(M)$, the M-height of $\mathfrak{p}$, denoted $\mathrm{ht}_M(\mathfrak{p})$, is the supremum of the lengths t of strictly descending chains $\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \ldots \supset \mathfrak{p}_t$ of prime ideals in $\mathrm{Supp}_R(M)$. For an arbitrary ideal $\mathfrak{a}$ we define the M-height of $\mathfrak{a}$, denoted $\mathrm{ht}_M(\mathfrak{a})$, by $\mathrm{ht}_M(\mathfrak{a}) = \inf\{\mathrm{ht}_M(\mathfrak{p}) : \mathfrak{p} \in \mathrm{Supp}_R(M) \cap \mathrm{V}(\mathfrak{a})\}$.*

Theorem 2.6 ([6, Theorem 3.6]). *Let $\mathfrak{a}$ be an ideal of R, M a nonzero finitely generated Cohen-Macaulay R-module, and $0 = \bigcap_{i=1}^n M_i$ with $\mathrm{Ass}_R(M/M_i) = \mathfrak{p}_i$ for all $1 \le i \le n$ a minimal primary decomposition of the zero submodule of M. Then for each $t \in \mathbb{N}_0$,*
$$\mathrm{Ann}_R(\mathrm{H}^t_{\mathfrak{a}}(M)) \subseteq \mathrm{Ann}_R(M/\bigcap_{\mathrm{ht}_M(\mathfrak{a}+\mathfrak{p}_i)=t} M_i).$$
Moreover, if $M \neq \mathfrak{a}M$ and $t = \mathrm{grade}(\mathfrak{a}, M)$, then the equality holds.

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When a quotient of a distributive lattice is a Boolean algebra

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Abstract

In this article, we introduce a lattice congruence with respect to a nonempty ideal I of a distributive lattice L and a derivation d on L denoted by θ_I^d. We investigate some necessary and sufficient conditions for the quotient algebra L/θ_I^d to become a Boolean algebra.

Keywords and phrases: Distributive lattice, Boolean algebra, Congruence, Ideal, Filter .

1. Introduction

The main result of this manuscript is to obtain a necessary and sufficient condition in which the quotient lattice L/θ is a Boolean algebra.

Throughout the paper L stands for a distributive lattice. A least element, if exists, is denoted by $\bot_L(\text{or}\bot)$ and a greatest one is denoted by $\top_L(\text{or}\top)$. By a lattice map (or homomorphism), we mean a map $f : A \to B$ between two lattices which preserves binary operations $\vee$ and $\wedge$. Recall that a non-empty subset I of L is called an *ideal (filter)* of L if $a \vee b \in A$ $(a \wedge b \in A)$ and $a \wedge x \in A$ $(a \vee x \in A)$ whenever $a, b \in A$ and $x \in L$. An equivalence relation θ defined on L is said to be a lattice congruence on L if it satisfies the following conditions, $a\theta b$ implies $(a \vee c)\theta(b \vee c)$ and $(a \wedge c)\theta(b \wedge c)$, for all $a, b, c \in L$.

Definition 1.1. [1] *For a distributive lattice L, a function $d : L \to L$ is called a derivation on L, if for all $x, y \in L$:*
(i) $d(x \wedge y) = d(x) \wedge y = x \wedge d(y)$.
(ii) $d(x \vee y) = d(x) \vee d(y)$.

2. Congruences and ideals in a distributive lattice with respect to a derivation

By definition, we consider $ker_I d = d^{-1}(I) = \{x \in L \mid d(x) \in I\}$ and $(a)_I^d = \{x \in L \mid a \wedge x \in ker_I d\} = \{x \in L \mid d(a \wedge x) \in I\}$. Both of them are ideals of the lattice L.

* speaker

Now we introduce a binary relation on a distributive lattice with respect to an ideal and a derivation.

Proposition 2.1. *For an ideal I of L, a binary relation θ_I^d defined as follow is a lattice congruence.*

$$x\theta_I^d y \text{ iff } (x)_I^d = (y)_I^d$$

An element $a \in L$ is called a *kernel element* with respect to an ideal I, if $(a)_I^d = ker_I d$. Let us denote the set of all kernel elements with respect to the ideal I of L by $\mathcal{K}_I^d$.

Proposition 2.2. *For a nontrivial ideal I of L, the distributive lattice L/θ_I^d is a bounded lattice with*

(i) $\bot_{L/\theta_I^d} = ker_I d$,

(ii) $\top_{L/\theta_I^d} = \mathcal{K}_I^d$ *whenever* $\mathcal{K}_I^d \neq \emptyset$.

Proposition 2.3. *For a nontrivial ideal I of L, the congruence θ_I^d is the greatest congruence relation having $ker_I d$ as a whole class.*

Now we investigate some conditions over ideals and derivations to get a smallest congruence θ_I^d. The smallest one infer that the quotient lattice L/θ_I^d has the maximal cardinality.

Proposition 2.4. *For an ideal I and a derivation d on L, $\theta_I^{id} \subseteq \theta_I^d$.*

Lemma 2.5. *For ideals $I \subseteq J$ and a derivation d on L, if there exists a derivation d_1 on L such that $ker_I d_1 = J$, then $\theta_I^d \subseteq \theta_J^d$ and the equality holds if $d_1 = d$.*

In the rest of this section we investigate some relationships between prime ideals and ideals of the form $(x)_I^d$. First note that, if I is a prime ideal, then so is $ker_I d$.

Lemma 2.6. (i) *If I is a prime ideal of L, then $ker_I d = L$ or for each $x \notin ker_I d$, $I = ker_I d = (x)_I^d$.*

(ii) *If $(x)_I^d$ is not a subset of prime ideal $(y)_I^d$, then $x \wedge y \in ker_I d$.*

(iii) *If $(x)_I^d \neq (y)_I^d$ are prime ideals, then $x \wedge y \in ker_I d$.*

Proposition 2.7. *The quotient lattice $L/\theta_I^d = \{ker_I d, [a]_{\theta_I^d}, [b]_{\theta_I^d}\}$ such that for each $x \in [a]_{\theta_I^d}$ and $y \in [b]_{\theta_I^d}$, $x \wedge y \in ker_I d$ if and only if there exist prime ideals P_1, P_2 in L in which $P_1 \cup P_2 = L$ and $P_1 \cap P_2 = ker_I d$.*

Theorem 2.8. *Let I be an ideal of L and $a \in I$. The following assertions are equivalent:*

(i) $(a)_I^d$ *is a maximal element in the* Σ.

(ii) $(a)_I^d$ *is a prime ideal.*

(iii) $(a)_I^d$ *is a $ker_I d$-minimal prime ideal.*

For a nontrivial ideal I of L, an ideal P is called I-minimal, if it is minimal in the set of ideals containing I. From now on, we consider the set $\Sigma_1 = \{(x)_I^d \mid x \in L \setminus ker_I d\}$ which is a poset under the inclusion relations.

Lemma 2.9. *In the following assertions we have,* (i)⇒ (ii)⇒ (iii).

(i) *The set Σ_1 satisfies the descending chain condition with respect to inclusion.*

(ii) *L does not have an infinite $M \subseteq L \setminus ker_I d$ such that for each $x, y \in M$, $x \wedge y \in ker_I d$. $(*)$*

(iii) *The set Σ_1 satisfies the ascending chain condition with respect to inclusion.*

Lemma 2.10. *Let L satisfies the condition $(*)$, then L has only a finite number of distinct $ker_I d$-minimal prime ideals of the form $(a_i)_I^d (1 \le i \le n)$. Also $\bigcap_{i=1}^n (a_i)_I^d = ker_I^d$.*

The following result is an immediate consequence of Lemma 2.10.

Theorem 2.11. *If L is a distributive lattice with a bottom element $\perp$ and satisfies the condition $(*)$ for $ker_\perp(id)$, then every minimal prime ideal of L is of the form $(a)_\perp^{id}$, for some $a \in L$.*

A special case of the previous theorem is the case where L is atomic with a finite number of atoms(Inparticular L is a finite lattice).

3. when a quotient lattice is a Boolean algebra

In this section some necessary and sufficient conditions are derived for the quotient algebra L/θ to become a Boolean algebra.

Theorem 3.1. *Let L be a distributive lattice and θ a lattice congruence on L. The distributive lattice L/θ is a Boolean algebra if and only if the following conditions hold:*

(i) *There exists $a_0, b_0 \in L$ such that for each $x \in L$, $[a_0]_\theta \le [x]_\theta \le [b_0]_\theta$, which means that $\perp_{L/\theta} = [a_0]_\theta$ and $\top_{L/\theta} = [b_0]_\theta$.*

(ii) *For each $x \in L$ there exists $y \in L$ such that $(x \wedge y)\theta a_0$ and $(x \vee y)\theta b_0$.*

Theorem 3.2. *Let I be a nontrivial ideal of L. Then L/θ_I^d is a Boolean algebra if and only if for each $x \in L$, there exists $y \in (x)_I^d$ such that $x \vee y \in \mathcal{K}_I^d$.*

Corollary 3.3. *Let L/θ_I^d be a Boolean algebra. Then $[x]_{\theta_I^d}^{-1} = [y]_{\theta_I^d}$ if and only if $x \wedge y \in ker_I d$ and $x \vee y \in \mathcal{K}_I^d$.*

Proposition 3.4. (i) *If I or $ker_I d$ is a prime ideal of L, then L/θ_I^d is a Boolean algebra.*

(ii) *If each $(x)_I^d$ has a maximum element, then L/θ_I^d is a Boolean algebra.*

Lemma 3.5. *Let L be a Boolean algebra with a bottom element $\perp$ and d a derivation on L. Then $ker(d) = \theta_\perp^d$.*

Theorem 3.6. *Let I be an ideal of L and d a derivation on L. Then the following are equivalent:*

(i) $\theta_I^d = \nabla$.

(ii) $ker_I d = L$

(iii) *For each $x \in L$, $I \cap [x]_{ker(d)}$ is a singleton set.*

Proposition 3.7. *The Boolean algebra $L/\theta_I^d = \mathbf{2}$ if and only if $ker_I d$ is a prime ideal of L.*

By the following operations, the set $\Sigma = \{(x)_I^d \mid x \in L\}$ is a bounded distributive lattice. For each $x, y \in L$, $(x)_I^d \vee (y)_I^d = (x \vee y)_I^d$ and $(x)_I^d \wedge (y)_I^d = (x \wedge y)_I^d$. The bottom and the top elements in the lattice Σ are of the form, $\perp_\Sigma = (x)_I^d = L$ for each $x \in ker_I d$ and $\top_\Sigma = (x)_I^d = ker_I d$ for each $x \in \mathcal{K}_I^d$. The map $f : L \to \Sigma$ defined by $f(x) = (x)_I^d$ is a lattice epimorphism, in which $ker f = \theta_I^d$. Thus, by the Isomorphism Theorem, $L/\theta_I^d \cong \Sigma$.

Lemma 3.8. *If the quotient lattice L/θ_I^d is a Boolean algebra then for each $x \in L$, the set $\{(z)_I^d \mid z \in (x)_I^d\}$ has a maximum element.*

Consider $A_I^d(L)$, the set of all $ker_I d$-atoms of L and $A_I^d(a) = A_I^d(L) \cap \downarrow a$.

Theorem 3.9. *Let L be a $ker_I d$-atomic distributive lattice. The lattice L/θ_I^d is a Boolean algebra if and only if for each $x \in L$, there exists $y \in L$ such that $A_I^d(x)$ and $A_I^d(y)$ are a partition of $A_I^d(L)$ and $[y]_{\theta_I^d}$ is a complement of $[x]_{\theta_I^d}$ in L/θ_I^d.*

Theorem 3.10. *If L/θ_I^d is a Boolean algebra, then the congruence θ_I^d is the only congruence relation having $ker_I d$ as a whole class.*

There is still an open question concerning θ_I^d:

Is there a necessary and sufficient condition on an ideal I such that θ_I^d is the smallest congruence in which L/θ_I^d is a Boolean algebra?

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Stanley's Conjecture on the k-Cohen-Macaulay simplicial complexes of codimension 3

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Abstract

Let Δ be a simplicial complex on vertex set $[n]$. It is shown that if Δ is k-Cohen–Macaulay of codimension 3, then Δ is vertex decomposable. As a consequence we show that Δ is partitionable and Stanley's conjecture holds for $K[\Delta]$.

Keywords and phrases: Vertex decomposable, simplicial complex, Shellable.
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1. Introduction

Let Δ be a simplicial complex on vertex set $[n] = \{1, \cdots, n\}$, i.e. Δ is a collection of subsets of $[n]$ with the the property that if $F \in \Delta$, then all subsets of F are also in Δ. An element of Δ is called a *face* of Δ, and the maximal faces of Δ under inclusion are called *facets*. We denote by $\mathcal{F}(\Delta)$ the set of facets of Δ. The *dimension* of a face F is defined as $\dim F = |F| - 1$, where $|F|$ is the number of vertices of F. The dimension of the simplicial complex Δ is the maximum dimension of its facets. A simplicial complex Δ is called *pure* if all facets of Δ have the same dimension. Otherwise it is called non-pure. We denote the simplicial complex Δ with facets $F_1, \ldots, F_t$ by $\Delta = \langle F_1, \ldots, F_t \rangle$. A simplex is a simplicial complex with only one facet.

For the simplicial complexes Δ_1 and Δ_2 defined on disjoint vertex sets, the join of Δ_1 and Δ_2 is $\Delta_1 * \Delta_2 = \{F \cup G \,:\, F \in \Delta_1, G \in \Delta_2\}$.

For the face F in Δ, the link, deletion and star of F in Δ are respectively, denoted by $\text{link}_\Delta F$, $\Delta \setminus F$ and $\text{star}_\Delta F$ and are defined by $\text{link}_\Delta F = \{G \in \Delta \,:\, F \cap G = \emptyset, F \cup G \in \Delta\}$ and $\Delta \setminus F = \{G \in \Delta \,:\, F \nsubseteq G\}$ and $\text{star}_\Delta F = \langle F \rangle * \text{link}_\Delta F$.

Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates over a field K. To a given simplicial complex Δ on the vertex set $[n]$, the Stanley–Reisner ideal is the squarefree monomial ideal whose generators correspond to the non-faces of Δ. we set:

$$\mathbf{x}_F = \prod_{x_i \in F} x_i.$$

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We define the *facet ideal* of Δ, denoted by $I(\Delta)$, to be the ideal of S generated by $\{\mathbf{x}_F : \quad F \in \mathcal{F}(\Delta)\}$. The *non-face ideal* or the *Stanley-Reisner ideal* of Δ, denoted by I_Δ, is the ideal of S generated by square-free monomials $\{\mathbf{x}_F : \quad F \in \mathcal{N}(\Delta)\}$. Also we call $K[\Delta] := S/I_\Delta$ the *Stanley-Reisner ring* of Δ. We say the simplicial complex Δ is Cohen–Macaulay if $K[x_1, \ldots, x_n]/I_\Delta$ is Cohen–Macaulay. One of interesting problems in combinatorial commutative algebra is the Stanley's conjectures. The Stanley's conjectures are studied by many researchers. Let R be a $\mathbb{N}^n$- graded ring and M a $\mathbb{Z}^n$- graded R- module. Then Stanley [5] conjectured that

$$\operatorname{depth}(M) \leq \operatorname{sdepth}(M)$$

He also conjectured in [6] that each Cohen-Macaulay simplicial complex is partitionable. Herzog, Soleyman Jahan and Yassemi in [4] showed that the conjecture about partitionability is a special case of the Stanley's first conjecture. Duval, Goeckner, Klivans and Martin in [3] construct a Cohen-Macaulay complex that is not partitionable, thus disproving the partitionability conjecture. Hachimori gived an open problem as following: Whether every two dimensional Cohen-Macaulay simplicial complex is partitionable; see [8]
Ajdani and Soleyman Jahan in [1] proved the following result :

Theorem 1.1 ([1, Theorem 2.3]). *If Δ is a Cohen-Macaulay simplicial complex of codimension 2, then Δ is vertex decomposable.*

In this paper we show that any k-Cohen-Macaulay simplicial complex of codimension 3 is vertex decomposable. As a consequence we show that Δ is partitionable and Stanley's conjecture holds for $K[\Delta]$.

2. Main Results

As the main result of this section, it is shown that every k-Cohen-Macaulay simplicial complexes of codimension 3 is vertex decomposable. For the proof we need the following lemmas:

Lemma 2.1 ([7, Lemma 2.3]). *Let Δ be a simplicial complex with vertex set V. Let $W \subseteq V$ and let σ be a face in Δ. If $W \cap \sigma = \varnothing$, then $\operatorname{link}_{\Delta \setminus W}\{\sigma\} = \operatorname{link}_\Delta\{\sigma\} \setminus W$.*

Definition 2.2. *Let K be a field. A simplicial complex Δ with vertex set V is called k-Cohen-Macaulay of dimension r over K if for any subset W of V (including $\varnothing$), $\Delta \setminus W$ is Cohen-Macaulay of dimension r over K.*

Lemma 2.3. *Let Δ be a simplicial complex with vertex set V. Then the following conditions are equivalent :*
(i) *Δ is k-Cohen-Macaulay;*
(ii) *for all $\sigma \in \Delta$, $\operatorname{link}_\Delta\{\sigma\}$ is k-Cohen-Macaulay ;*

PROOF. By lemma 2.1, for any subset W of V, we have $\operatorname{link}_{\Delta \setminus W}\{\sigma\} = \operatorname{link}_\Delta\{\sigma\} \setminus W$. Since $\Delta \setminus W$ is Cohen-Macaulay so $\operatorname{link}_\Delta\{\sigma\} \setminus W$ is Cohen-Macaulay. Therefore $\operatorname{link}_\Delta\{\sigma\}$ is k-Cohen-Macaulay. □

Now, we are ready that prove the main result of this section.

Theorem 2.4 ([2, Theorem 2.4]). *Let Δ be a k-Cohen-Macaulay simplicial complex of codimension 3 on vertex set $[n]$. Then Δ is vertex decomposable.*

PROOF. We prove the theorem by induction on $|[n]|$ the number of vertices of Δ. If $|[n]| = 0$, then $\Delta = \{\}$ and it is vertex decomposable. Now Let $|[n]| > 0$ and $d \in [n]$ be a vertex of Δ. Then the simplicial complex $\text{link}_\Delta\{d\}$ is a complex on $|[n]| - 1$ vertex and its dimension is $\dim \Delta - 1$. By Lemma 2.3, $\text{link}_\Delta\{d\}$ is again k-Cohen-Macaulay of codimension 3. Therefore by induction hypothesis $\text{link}_\Delta\{d\}$ is vertex decomposable.

On the other hand since Δ is a k-Cohen-Macaulay, for each existing vertex $d \in \Delta$, $\Delta \setminus \{d\}$ is Cohen-Macaulay of codimension 2 and by Theorem 1.1, $\Delta \setminus \{d\}$ is vertex decomposable. It is easy to see that no face of $\text{link}_\Delta\{d\}$ is a facet of $\Delta \setminus \{d\}$. Therefore any vertex d is a shedding vertex and Δ is vertex decomposable. □

Stanley conjectured in [5] the upper bound for the depth of $K[\Delta]$ as the following:

$$\text{depth}\,(K[\Delta]) \leq \text{sdepth}\,(K[\Delta])$$

. Also we recall another conjecture of Stanley. Let Δ be again a simplicial complex on $\{x_1, \ldots, x_n\}$ with facets $G_1, \ldots, G_t$. The complex Δ is called partitionable if there exists a partition $\Delta = \bigcup_{i=1}^{t} [F_i, G_i]$ where $F_i \subseteq G_i$ are suitable faces of Δ. Here the interval $[F_i, G_i]$ is the set of faces $\{H \in \Delta : F_i \subseteq H \subseteq G_i\}$. In [6] and [9] respectively Stanley conjectured each Cohen-Macaulay simplicial complex is partitionable. This conjecture is a special case of the previous conjecture. Indeed, Herzog, Soleyman Jahan and Yassemi [4] proved that for Cohen-Macaulay simplicial complex Δ on $\{x_1, \ldots, x_n\}$ we have that $\text{depth}\,(K[\Delta]) \leq \text{sdepth}\,(K[\Delta])$ if and only if Δ is partitionable. Since each vertex decomposable simplicial complex is shellable and each shellable complex is partitionable. Then as a consequence of our results we obtain :

Corollary 2.5. *If Δ is a k-Cohen-Macaulay simplicial complex of codimension 3, then Δ is partitionable and Stanley's conjecture holds for $K[\Delta]$.*

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Some inequalities for the dimension of the second homology of nilpotent Leibniz algebras

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Abstract

Let L be a finite dimensional nilpotent Leibniz algebra. In this paper, we present some upper bounds for the dimension of the second homology of L, in terms of the dimension of derived subalgebra, center and some special quotients of L.

Keywords and phrases: Leibniz algebra, second homology.
2010 *Mathematics subject classification:* 17A32.

1. Introduction

The notion of a Leibniz algebra first appeared under the name of a D-algebra, introduced by A. Bloh as one of the generalizations of Lie algebras, in which multiplication by an element is a derivation. Later, they were re-considered by J.-L. Loday [6] and gain popularity under the name of Leibniz algebras. A (right) Leibniz algebra is an $\mathbb{F}$-vector space equipped with a bilinear map $[-,-] : L \times L \to L$, called the Leibniz multiplication, such that the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y],$$

holds for all $x,y,z \in L$. Note that if the bilinear mapping $[-,-]$ is also skew-symmetric, then L is a Lie algebra. The Leibniz homology (with trivial coefficients) of a Leibniz algebra L, denoted by $HL_*(L)$, is the homology of the complex $(CL_n(L) = L^{\otimes n}, \partial_n, n \geq 0)$ such that the boundary map $\partial_n : CL_n(L) \to CL_{n-1}(L)$ is defined as

$$\partial_n(x_1 \otimes \cdots \otimes x_n) = \sum_{1 \leq i < j \leq n} (-1)^j (x_1 \otimes \cdots \otimes [x_i, x_j] \otimes \cdots \otimes \widehat{x_j} \otimes \cdots \otimes x_n).$$

It can be readily checked that, similar to Chevalley-Eilenberg homology of a Lie algebra, $HL_0(L) = \mathbb{F}, HL_1(L) = L/L^2$. The first attempts for computing and developing of the homology theory of Leibniz algebras was formulated

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by Loady in calculating Leibniz homologies of the Lie algebra $gl(A)$, where A is an associative algebra over a characteristic zero field.

We remark that if L is a Leibniz algebra of dimension, then the maximal possible dimension for $HL_i(L)$ is $(\dim(L))^i$ which is met if and only if L is abelian. As an immediate consequence, one can verify that in the second step we have $\dim(L^2) + \dim(HL_2(L)) \leq (\dim(L))^2$.

Let $0 \to A \to K \to L \to 0$ be the maximal stem extension of the finite dimensional Leibniz algebra L, that is, an exact sequence such that A is a central ideal of K that contained in K^2 and $\dim(K)$ is maximal amongst all such extensions. Then Casas and Ladra in [2] showed that $A \cong HL_2(L)$.

Originally, the notion of non-abelian tensor product was introduced for groups by Brown and Loday in 1984. In 1991, Ellis extended this concept to Lie algebras. The non-abelian tensor and exterior product of Leibniz algebras was found by Gnedbaye [5] and has been used by Donadze et al. [4] to define the non-abelian exterior product. They proved that for a free Leibniz algebra F, we have $F \curlywedge F \cong [F,F]$. This implies that if $0 \to R \to F \to L \to 0$ is an arbitrary free presentation of L then $HL_2(L) \cong ([F,F] \cap R)/[R,F]$. The last quotient algebra is known as the Hopf-Schur multiplier of L.

The main goal of this paper is to present some inequalities for the dimension of the second homology of a finite dimensional nilpotent Leibniz algebra.

2. Main Results

We begin in this section by reminding the Ganea sequence for Leibniz algebras. This sequence initially obtained in [3, Proposition 4] as apart of a ten term exact sequence. Later in [1, Corollary 4.4] this sequence was described in terms of the non-abelian tensor product of Leibniz algebras. This sequence plays a key role in obtaining our next results.

Proposition 2.1. *Let L be a Leibniz algebra and N be a central ideal of L. Then the sequence*

$$\begin{aligned} HL_3(L) \to HL_3(L/N) \to \operatorname{Coker}(\tau) \to HL_2(L) \to HL_2(L/N) \\ \to N \to L/L^2 \to L/(L^2+N) \to 0, \end{aligned}$$

is exact where the map $\tau : N \otimes N \to \left(L/L^2 \otimes N\right) \oplus \left(N \otimes L/L^2\right)$ *is given by* $\tau(a \otimes b) = (\bar{a} \otimes b, -a \otimes \bar{b})$ *where* $\bar{a}$ *denotes the image of* $a \in N$ *on* L/L^2.

Corollary 2.2. *Let L be a finite dimensional Leibniz algebra and N be a central ideal of L. Then*
(i) $\dim(HL_2(L)) + \dim(L^2 \cap N) \leq 2\dim(N)\dim(L/L^2) + \dim(HL_2(L/N))$.
(ii) $\dim(HL_2(L)) + \dim(L^2 \cap N) \leq 2\dim(N)\dim(L/(N+L^2)) + 2\dim(HL_2(N)) + \dim(HL_2(L/N))$.

Loday in [7] established a Küneth-type formula for homology of Leibniz algebras. He proved that for Leibniz algebras L_1, L_2 there is a canonical isomorphism of graded vector spaces

$$HL_*(L_1 \oplus L_2) \cong HL_*(L_1) * HL_*(L_2),$$

where $*$ in this formula is a sort of non-commutative tensor product for graded modules. As a special case

$$\begin{aligned} HL_2(L_1 \oplus L_2) \cong HL_2(L_1) \oplus HL_2(L_2) \oplus (HL_1(L_1) \otimes HL_1(L_2)) \\ \oplus (HL_1(L_2) \otimes HL_1(L_1)). \end{aligned}$$

Now we are ready to state the main theorem of this paper. The proof of all inequalities presented in the following theorem are based on induction on the dimension of L or L^2. Here, as an instance, we give the proof of the first part.

Theorem 2.3. *Let L be an n-dimensional non-abelian nilpotent Leibniz algebra with* $\dim(L^2) = m$ *and* $\dim(Z(L)/(Z(L) \cap L^2)) = t$. *Then*
(i) $\dim(HL_2(L)) \leq (n-m) + m(2(n-m)-1)$.
(ii) $\dim(HL_2(L)) \leq (n+m-2)(n-m) + 1$.
(iii) $\dim(HL_2(L)) \leq (n-m)^2 + 2(m-1)(\dim(L/Z(L)) - 1) + m$.
(iv) $\dim(HL_2(L)) \leq (n-m-t)(n+m-2) + t(n-m) - m + 2$.

PROOF. (i) We proceed by induction on the dimension of L. We don't have any non-abelian Leibniz algebra of dimension one and all non-abelian nilpotent Leibniz algebra of dimension two are isomorphic to $J_1 = \langle x, y : [x, x] = y \rangle$. The statement is trivial in this case, because $\dim(HL_2(J_1)) = 1$. So suppose that $\dim(L) > 2$ and we may suppose that the statement holds for all Leibniz algebras of dimension less than $\dim(L)$. Choose $N \subseteq Z(L) \cap L^2$ to be a one dimensional ideal of L. Employing the induction hypothesis, we have

$$\begin{aligned} \dim(HL_2(L)) &\leq 2\dim(L/L^2) + \dim(HL_2(L/N)) - 1 \\ &\leq 2\dim(L/L^2) + \dim(HL_2(L/L^2)) \\ &\quad + (\dim(L^2) - 1)(2\dim(L/L^2) - 1) - 1 \\ &= \dim(HL_2(L/L^2)) + \dim(L^2)(2\dim(L/L^2) - 1). \end{aligned}$$

which completes the proof. □

Corollary 2.4. *Let L be a finite dimensional nilpotent Leibniz algebra. If L is not abelian then*

$$\dim(HL_2(L)) \leq (\dim(L) - 1)^2 + 1,$$

and equality holds if and only if $L = \langle x, y, z : [x, y] = -[y, x] = z \rangle \oplus A$, for some abelian Lie algebra A.

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Tensor product of crossed modules in Lie algebras

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Abstract

The notions of non-abelian tensor and exterior products in the category of Lie crossed modules are introduced and investigated. Also, their relationships with the homology of Lie crossed modules are established.

Keywords and phrases: Tensor product, exterior product, crossed module, lie algebra, crossed square. .

2010 *Mathematics subject classification:* Primary: 17B05, 17B99; Secondary: 18G50, 18G99.

1. Introduction

All Lie algebras are considered over a fixed field $\mathbb{F}$ and $[-,-]$ denotes the Lie bracket. In this article, using crossed squares in Lie algebras, we generalize the definitions of non-abelian tensor and exterior products for two arbitrary Lie crossed modules, similar to the works of Ellis [1] in the Lie algebra case.

Definition 1.1. *Let M and P be two Lie algebras. By an action of P on M we mean a $\mathbb{F}$-bilinear map $P \times M \longrightarrow M$, $(p,m) \longmapsto^{p} m$, satisfying (i) ${}^{[p,p']}m =^{p} ({}^{p'}m) -^{p'} ({}^{p}m)$, and (ii) ${}^{p}[m,m'] = [{}^{p}m,m'] + [m,^{p} m']$ for all $m,m' \in M$, $p,p' \in P$. For example, if P is a subalgebra of some Lie algebra L and M is an ideal in L, then Lie multiplication in L induces an action of P on M given by ${}^{p}m = [p,m]$. A **Lie Crossed module** $\mathbf{M} = (M,P,\partial)$ is a Lie homomorphism $\partial : M \longrightarrow P$ together with an action of P on M such that (i) $\partial({}^{p}m) = [p,\partial(m)]$, (ii) ${}^{\partial(m)}m' = [m,m']$ for all $m,m' \in M$, $p \in P$. If M is an ideal of P, then (M,P,i) is a Lie crossed module, in which i is the inclusion map. In This way, every Lie algebra P can be regard as Lie crossed module in the two obvious ways: $(0,P,i)$ or (P,P,id).*
*A **morphism** of Lie crossed modules $(\alpha_1,\alpha_2) : (M,P,\partial) \longrightarrow (N,Q,\sigma)$ is a pair of Lie homomorphisms $\alpha_1 : M \longrightarrow N$ and $\alpha_2 : P \longrightarrow Q$ such that $\sigma \circ \alpha_1 = \alpha_2 \circ \partial$ and for all $p \in P$, $m \in M$, $\alpha_1({}^{p}m) =^{\alpha_2(p)} \alpha_1(m)$.*

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Definition 1.2. *Let* (M,P,∂_1) *and* (N,P,∂_2) *be two Lie crossed modules. There are actions of M on N and of N on M given by* ${}^m n = {}^{\partial_1(m)} n$ *and* ${}^n m = {}^{\partial_2(n)} m$*. We take M (and N) to act on itself by Lie multiplication. The non-abelian tensor product* $M \otimes N$ *is defined in [1] as the Lie algebra generated by the symbols* $m \otimes n$ *for* $m \in M$, $n \in N$*. Let* $M \square N$ *be the submodule of* $M \otimes N$ *generated by the elements* $m \otimes n$ *with* $\partial(m) = \sigma(n)$*. One easily gets that* $M \square N$ *lies in the centre of* $M \otimes N$ *. Following G. Ellis in [1], the non-abelian exterior product* $M \wedge N$ *is defined to be* $\dfrac{(M \otimes N)}{(M \square N)}$.

Proposition 1.3. *With the above assumptions and notations, we have*

i) *The maps* $\lambda : M \otimes N \longrightarrow P$, $m \otimes n \longmapsto [\partial_1(m), \partial_2(n)]$ *are Lie crossed modules, in which the action of P on* $M \otimes N$ *is given by the equation* ${}^p(m \otimes n) = {}^p m \otimes n + m \otimes {}^p n$*, and M and N act on* $M \otimes N$ *via* ∂_1 *and* ∂_2*. Furthermore, the results holds with* $\otimes$ *replaced by* $\wedge$.

ii) *The functional homomorphism* $\partial_1 \otimes id_N : M \otimes N \longrightarrow P \otimes N$*, together with the action of* $P \otimes N$ *on* $M \otimes N$ *induced by the map* $\lambda'_P : P \otimes N \longrightarrow P$*, is a Lie crossed module.*

iii) *There is an action of P on the semidirect sum* $M \rtimes N$ *defined by the formula* ${}^p(m,n) = ({}^p m, {}^p n)$.

iv) *The map* $\beta : M \rtimes N \longrightarrow P$ *defined by* $\beta(m,n) = \partial_1(m) + \partial_2(n)$ *is a Lie homomorphism.*

v) *For any* $m \in M$, $n \in N$ *and* $x \in M \otimes N$, ${}^m x = m \otimes \lambda_N(x)$ *and* ${}^n x = -\lambda_M(x) \otimes n$.

vi) *For any* $x, y \in M \otimes N$, $\partial_1 \lambda_M(x) \otimes \lambda_N(y) = -\partial_1 \lambda_M(y) \otimes \lambda_N(x)$.

Definition 1.4. *A crossed square in Lie algebras is a triple* $\big((M,P,\partial_1),(T,L,\partial),(\lambda_1,\lambda_2)\big)$ *with properties* $\partial \circ \lambda_1 = \lambda_2 \circ \partial_1$ *of Lie endowed with actions of L on* M, T, P *(and hence actions of* M, P *via* ∂ *and of P on* M, T *via* λ_2*) and a bilinear map* $h_1 : t \times P \longrightarrow M$ *such that the following axioms hold:*

1) λ_1, ∂_1 *preserve the actions of L.*

2) $\partial, \lambda_2, \lambda_2 \circ \partial_1 = \partial \circ \lambda_1$ *are Lie crossed modules,*

3) $\lambda_1 h_1(t,p) = -{}^p t, \partial_1 h_1(t,p) = {}^t p, h_1(\lambda_1(m),p) = -{}^p m, h_1(t,\partial_1(m)) = {}^t m$

4) $h_1([t,t'],p) = h_1(t, {}^{t'} p) - h_1(t', {}^t p), \quad h_1(t,[p,p']) = h_1({}^{p'} t, p) - h_1({}^p t, p'),$

5) ${}^l h_1(t,p) = h_1({}^l t, p) + h_1(t, {}^l p),$

for all $t, t' \in T$, $p, p' \in P$, $m \in M$, $l \in L$*. It is obvious that for any ideal crossed submodule* (M,P,∂_1) *of a Lie crossed module* (T,L,∂)*, the square* $\big((M,P,\partial_1),(T,L,\partial),(i_1,i_2)\big)$ *with* $h_1(t,p) = -{}^p t$ *is a crossed square, where* i_1, i_2 *are the inclusion maps.*

Lemma 1.5. *Consider the crossed square* $\big((M,P,\partial_1),(T,L,\partial),(\lambda_1,\lambda_2)\big)$ *of Lie algebras. Then*

i) *The maps* λ_1 , ∂_1 *are Lie crossed modules.*

ii) $(ker\lambda_1, ker\lambda_2, \partial_1)$*is a central crossed submodule of* (M,P,∂_1).

iii) *For any* $t \in T$ *and* $p, p' \in P$, ${}^{p'} h_1(t,p) = h_1({}^p t, p')$.

Definition 1.6. *Consider the crossed squares* $((M,P,\partial_1),(T,L,\partial),(\lambda_1,\lambda_2))$ *and* $((N,Q,\partial_2),(T,L,\partial),(\mu_1,\mu_2))$ *together with bilinear functions* $h_1 : T\times P \longrightarrow M$ *and* $h_2 : T\times Q \longrightarrow N$*, respectively. Plainly, both Lie algebras appeared in the above crossed squares act on each other. Then we can form the non-abelian tensor products* $M\otimes N$, $M\otimes Q$, $P\otimes N$ *and* $P\otimes Q$*. Using the Lie crossed modules*

$$\lambda_M : M\otimes N \longrightarrow M\ ,\ \lambda'_M : M\otimes Q \longrightarrow M\ ,\ \lambda_P : P\otimes Q \longrightarrow P$$
$$\lambda_N : M\otimes N \longrightarrow N\ ,\ \lambda'_N : P\otimes N \longrightarrow N\ ,\ \lambda_Q : P\otimes Q \longrightarrow Q$$

We now construct the semidirect sum $(M\otimes Q)\rtimes(P\otimes N)$ *and define the maps*

$$\alpha : M\otimes N \longrightarrow (M\otimes Q)\rtimes(P\otimes N),\quad (m\otimes n)\longmapsto (m\otimes\partial_2(n),-\partial_1(m)\otimes n)$$
$$\beta : (M\otimes Q)\rtimes(P\otimes N) \longrightarrow P\otimes Q,\ (m\otimes q,p\otimes n)\longmapsto \partial_1(m)\otimes q + p\otimes\partial_2(n).$$

In the remainder of this paper, we will always assume that $\mathbf{M}=(M,P,\partial_1)$, $\mathbf{N}=(N,Q,\partial_2)$ and $\mathbf{T}=(T,L,\partial)$ are Lie crossed modules which get from crossed squares in definition 1.6.

2. Main Results

Using these assumptions, we have the following consequences.

Lemma 2.1. *i) The Lie crossed modules* $\partial\otimes id_N$, $\partial_1\otimes id_Q$ *and* $id_M\otimes\partial_2$ *preserve the actions of* P *and* Q.

ii) The maps $\mu'_M : P\otimes N \longrightarrow M\ ,\ p\otimes n\longmapsto -h_1(\mu_1(n),p)$
$\mu'_N : M\otimes Q \longrightarrow N\ ,\ m\otimes q\longmapsto h_2(\lambda_1(m),q)$

iii) There is an action of $P\otimes Q$ *on* $(M\otimes Q)\rtimes(P\otimes N)$ *by* ${}^{y}(x_1,x_2)=({}^{y}x_1,{}^{y}x_2)$*, for all* $y\in P\otimes Q$ *and* $(x_1,x_2)\in(M\otimes Q)\rtimes(P\otimes N)$.

Lemma 2.2. *If* $h_1(\mu_1(n),p)=0_M$ *and* $h_2(\lambda_1(m),q)=0_N$ *for all* $m\in M$, $n\in N$, $p\in P$, $q\in q$*, then the square* $((M\oplus n,P\oplus Q,\partial_1\oplus\partial_2),(T,L,\partial),(\rho_1,0_2))$ *with* $h(t,(p,q))=(h_1(t,p),h_2(t,q))$ *is a crossed square, where* ρ_1 *and* ρ_2 *are Lie homomorphisms defined by* $\rho_1(m,n)=\lambda_1(m)+\mu_1(n)$ *and* $\rho_2(p,q)=\lambda_1(p)+\mu_1(q)$*, and L acts on* $M\oplus N$ *and* $P\oplus Q$ *with componentwise action.*

Proposition 2.3. *i) The map* β *is a Lie homomorphism such that* $\beta(Im\alpha)=0$.

ii) The image of the map α *is an ideal of* $(M\otimes Q)\rtimes(P\otimes N)$*. put* $coker\alpha$ *to be the quotient Lie algebra of* $(M\otimes Q)\rtimes(P\otimes N)$ *by* $Im\alpha$.

iii) The maps $\mu_M : coker\alpha \longrightarrow M,\ (x_1,x_2)+Im\alpha\longmapsto \lambda'_M(x_1)+\mu'_M(x_2)$
$\mu_N : coker\alpha \longrightarrow N,\ (x_1,x_2)+Im\alpha\longmapsto \mu'_N(x_1)+\lambda'_N(x_2)$ *are Lie crossed modules, where* μ'_M *and* μ'_N *are defined in Lemma 2.1 (ii).*

iv) The homomorphism $\tilde{\beta} : coker\alpha \longrightarrow P\otimes Q$ *induced by* β*, together with the action induced by Lemma 2.1 (iii), is a Lie crossed module.*

v) If I is a subalgebra of $coker\alpha$ *generated by the elements* $(m\otimes q,p\otimes n+\partial_1(m')\otimes n')+Im\alpha$*, where* $\lambda_1(m)=\mu_1(n)$, $\lambda_1(m')=\mu_1(n')$ *and* $\lambda_2(p)=\mu_2(q)$*, then* $(I,P\square Q,\tilde{\beta})$ *is an ideal crossed submodule of* $(coker\alpha,P\otimes Q,\tilde{\beta})$.

In this part, using Proposition 2.3, we define the tensor and exterior products of Lie crossed modules and give some fundamental properties of them.

Definition 2.4. *The non-abelian tensor and exterior products of Lie crossed modules* $\mathbf{M}$ *and* $\mathbf{N}$ *are defined, respectively, as*

$$\mathbf{M} \otimes \mathbf{N} = (coker\alpha, P \otimes Q, \tilde{\beta}),$$

$$\mathbf{M} \wedge \mathbf{N} = \frac{\mathbf{M} \otimes \mathbf{N}}{\mathbf{M} \square \mathbf{N}} = \frac{(coker\alpha, P \otimes Q, \tilde{\beta})}{(I, P \square Q, \tilde{\beta})} = (\frac{coker\alpha}{I}, P \wedge Q, \bar{\beta})$$

where $\mathbf{M} \square \mathbf{N}$ *is the Lie crossed module* $(I, P \square Q, \delta)$ *introduced in Proposition* 2.3 *.*

Proposition 2.5. *i) There are two Lie crossed module morphisms* $(\mu_M, \lambda_P) : \mathbf{M} \otimes \mathbf{N} \longrightarrow \mathbf{M}$ *and* $(\mu_N, \lambda_Q) : \mathbf{M} \otimes \mathbf{N} \longrightarrow \mathbf{N}$*, where* μ_M*,* μ_N *are Lie crossed modules defined in Proposition* 2.3 *(iii).*

ii) The Lie crossed modules $ker(\mu_M, \lambda_P)$ *and* $ker(\mu_n, \lambda_Q)$ *are abelian.*

iii) If ∂_1 *and* ∂_2 *are onto, then the squares* $\big((coker\alpha, P \otimes Q, \tilde{\beta}), (M, P, \partial_1), (\mu_M, \lambda_P)\big)$ *and* $\big((coker\alpha, P \otimes Q, \tilde{\beta}), (N, Q, \partial_2), (\mu_N, \lambda_Q)\big)$ *together with bilinear functions* $h_1{}'(m, p \otimes q) = (-{}^p m \otimes q, p \otimes h_2(\lambda_1(m), q)) + Im\alpha$ *and* $h_2{}'(n, p \otimes q) = (h_1(\mu_1(n), p) \otimes q, -p \otimes^q n) + Im\alpha$*, respectively, are crossed squares.*

Lemma 2.6. *Suppose that* $\big((M, P, \partial_1), (T, L, \partial), (\lambda_1, \lambda_2)\big)$*,* $\big((N, Q, \partial_2), (T, L, \partial), (\mu_1, \mu_2)\big)$ *and* $\big((K, R, \partial_3), (T, L, \partial), (\nu_1, \nu_2)\big)$ *are crossed squares. Also, suppose that*

a) $h_1(\mu_1(n, p)) = 0$ *and* $h_2(\lambda_1(m), q) = 0$*,*

b) $m \otimes^n r = m \otimes^q r = 0_{M \otimes R}$ *and* $n \otimes^m r = n \otimes^p r = 0_{N \otimes R}$*,*

c) $p \otimes^q k = 0_{P \otimes K}$ *and* $q \otimes^p k = 0_{Q \otimes K}$*,*

for all $m \in M$*,* $n \in N$*,* $k \in K$*,* $p \in P$*,* $q \in Q$ *and* $r \in R$*.*
Then assuming $\mathbf{M} = (M, P, \partial_1)$*,* $\mathbf{N} = (N, Q, \partial_2)$ *and* $\mathbf{K} = (K, R, \partial_3)$*, there is an isomorphism* $(\mathbf{M} \oplus \mathbf{N}) \otimes \mathbf{K} \longrightarrow (\mathbf{M} \otimes \mathbf{K}) \oplus (\mathbf{N} \otimes \mathbf{K})$*.*

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Precrossed modules in Lie algebras

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Abstract

We introduce the notion of non-abelian tensor product of a given precrossed module by one of its ideals. We use this concept to describe the classical Stallings-Stammbach sequence for the schur multiplier of precrossed modules in term of the non-abelian tensor product.

Keywords and phrases: precrossed module, non-abelian tensor product, exterior product.
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1. Introduction

Throughout this paper, and are the categories of Lie algebras and vector spaces over a fixed field $\mathbb{F}$, respectively and as usual $[-,-]$ denotes the Lie algebra brackets. Non-abelin tensor product is a powerful tool in the study of extension theory of groups and Lie algebras. Precrossed modules as a natural generalization of crossed modules in their own right are subject of interest. Recently, Casas et. al in [2] introduced the actor of precrossed module and used it to derive the basic notions of action, center, semi-direct product, derivation, commutator and abelian precrossed module in Lie algebras. The main goal of the present paper is to introduce the non-abelian tensor product of a precrossed module and a precrossed ideal.

Definition 1.1. *Let M, L be two Lie algebras. A triple (M,L,∂) is a precrossed module of Lie algebras whenever $\partial\colon M \to P$ is a Lie homomorphism and there is a (left) Lie action of L on M denoted by ${}^{l}m$ for all $l \in L$ and $m \in M$, such that $\partial({}^{l}m) = [l,\partial(m)]$. In addition, ∂ is called a crossed module if ∂ satisfies the Peiffer's identity, ${}^{\partial(m)}m' = [m,m']$ for all $m,m' \in M$.*
A morphism $(f_1,f_2)\colon (M_1,L_1,\partial_1) \to (M_2,L_2,\partial_2)$ of precrossed modules is a pair of Lie algebra homomorphisms $f_1\colon M_1 \to M_2$ and $f_2\colon L_1 \to L_2$ such that $f_2\partial_1 = \partial_2 f_1$ and the morphisms preserve the actions, that is, $f_1({}^{l}m) = {}^{f_2(l)}f_1(m)$ for all $l \in L_1, m \in M_1$. Taking objects and morphisms as defined above, we obtain the category of precrossed modules in Lie algebras. One also directly check that **XLie**, *the category*

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of crossed modules of Lie algebras, is a full Birkhoff subcategory of . The category of Lie algebras can be regarded as a subcategory of the category of precrossed modules, by the inclusion functor i : *Lie* $\to$ *PXLie with* $i(L) = (0, L, 0)$*. The functor i has a right adjoint* κ : *PXLie* $\to$ *Lie,* $\kappa(M, L, \partial) = L$*.*

Definition 1.2. *Let* (N, K, ∂) *be an ideal precrossed module of a precrossed module* (M, L, ∂)*. We define the commutator precrossed module* $\gamma_2((N, K, \partial),(M, L, \partial)) = [(N, K, \partial),(M, L, \partial)]$ *to be the ideal* $([N, M] + [L, N] + [K, M], [K, L], \partial)$ *where*

$$[L, N] = \langle {}^{l}n : l \in L, n \in N\rangle \quad , \quad [K, M] = \langle {}^{k}m : k \in K, m \in M\rangle$$

In particular, the commutator precrossed submodule $\gamma_2(M, L, \partial) = [(M, L, \partial),(M, L, \partial)]$ *of* (M, L, ∂) *is the ideal* $([M, M] + [L, M], [L, L], \partial)$*. We say that* (N, K, ∂) *is central if* $\gamma_2((N, K, \partial),(M, L, \partial)) = 0$*. A precrossed module* (M, L, ∂) *is called abelian if* $\gamma_2(M, L, \partial) = 0$*. In fact,* (M, L, ∂) *is abelian if and only if M and L are abelian Lie algebras and L acts trivially on M. We will denote by* **XVect***, the full-Birkhoff subcategory of abelian percrossed modules. The abelianization functor* **Ab** *which assigns to* (M, L, ∂) *the factor precrossed module* $\mathbf{Ab}(M, L, \partial) = (M, L, \partial)/\gamma_2(M, L, \partial)$*, is left adjoint to the inclusion functor from* **XVect** *to .*

2. Main Results

Let L be a Lie algebra and N an ideal of L. The non-abelian tensor product $L \otimes N$ is the Lie algebra generated by symbols $l \otimes n$ $(l \in L, n \in N)$ with the following defining relations

(i) $\lambda(l \otimes n) = \lambda l \otimes n = l \otimes \lambda n$, $\quad (iii)$ $[l, l'] \otimes n = l \otimes ([l', n]) - l' \otimes ([l, n])$,

(ii) $(l + l') \otimes n = l \otimes n + l' \otimes n$, $\quad l \otimes [n, n'] = ([n', l]) \otimes n - ([n, l]) \otimes n'$,

$l \otimes (n + n') = l \otimes n + l \otimes n'$, $\quad (iv)$ $[(l \otimes n),(l' \otimes n')] = -([n, l]) \otimes ([l', n'])$

for all $\lambda \in \Lambda$, $l, l' \in L$ and $n, n' \in N$. Furthermore, the non-abelian exterior product $L \wedge N$ is the Lie algebra generated by the elements $l \wedge n$ subject to the relations (i)-(iv) besides the relation $n \wedge n = 0$, for all $n \in N$. There is an action of L on $L \otimes N$ defined by ${}^{l}(l_1 \wedge n) = [l, l_1] \wedge n + l_1 \wedge [l, n] \quad , \quad l, l_1 \in L, n \in N$. There is a commutator homomorphism $[-,-] : L \wedge N \to L$ which assigns any generator $l \wedge n$ to $[l, n]$ and also $(L \wedge N, L, [-,-])$ is a crossed module. It is an important note that there is an isomorphism

$$H_2(L) \cong \ker(L \wedge L \overset{[-,-]}{\to} L) \quad , \quad H_2(L, N) \cong \ker(L \wedge N \overset{[-,-]}{\to} L), \qquad (1)$$

where $H_2(L), H_2(L, N)$ are the second Cartan-Eilenberg homology of L and the second relative homology of (L, N), respectively. It is useful note that if we define the action of L on $L \wedge N$ by

$${}^{l}(l_1 \wedge n) = ([l, l_1] \wedge n + l_1 \wedge [l, n]) \quad , \quad l, l_1 \in L, n \in N$$

then $(L \wedge N, L, [-,-])$ is a crossed module.

In this part, we were greatly inspired by the works of D. Arias and M. Ladra [1], to define the non-abelian tensor product for precrossed modules and investigate the parallel applications to homology theory of precrossed modules. Recall that for a given precrossed module of Lie algebras (M, L, ∂), we can form the semidirect sum $M \rtimes L$. Moreover, there are Lie homomorphisms $\tau, \sigma \colon M \rtimes L \to L$ defined by $\tau(m,l) = l \quad , \quad \sigma(m,l) = \partial(m) + l \qquad m \in M, l \in L$. Suppose (N, K, ∂) is a precrossed ideal of (M, L, ∂). The homomorphism τ induces the Lie homomrphism $\alpha = \tau \otimes \tau$

$$\alpha \colon (N \rtimes K) \otimes (M \rtimes L) \to K \otimes L$$

which is defined on generators as $\alpha((n,k) \otimes (m,l)) = k \otimes l$. The restriction of $\sigma \otimes \sigma$ to $\ker \alpha$ defines the homomorphism $\beta : \ker \alpha \to K \otimes L$ with $\beta((n,k) \otimes (m,l)) = (\partial(n) + k) \otimes (\partial(m) + l)$.

Proposition 2.1. *With the above assumptions and notations, we have* $(\ker \alpha, K \otimes L, \beta)$ *is a precrossed module.*

Definition 2.2. *Let* (M, L, ∂) *be a precrossed module with a precrossed ideal* (N, K, ∂)*. The precrossed module* $(\ker \alpha, K \otimes L, \beta)$ *will be called the non-abelian tensor product of* (M, L, ∂) *and* (N, K, ∂) *and denoted by* $(M, L, \partial) \otimes (N, K, \partial)$*.*

In a similar way, $\alpha' \colon (N \rtimes K) \wedge (M \rtimes L) \to K \wedge L$ defined by $\alpha'((n,k) \wedge (m,l)) = k \wedge l$ is a Lie homomorphism and $\beta' \colon \ker \alpha' \to K \otimes L$ with $\beta'((n,k) \wedge (m,l)) = (\partial(n) + k) \wedge (\partial(m) + l)$ defines a precrossed module. We denote by $(M, L, \partial) \wedge (N, K, \partial)$ the precrossed module $(\ker \alpha', K \wedge L, \beta')$ and call it the non-abelian exterior product. It is obvious that the mappings $\mu_N : \ker \alpha' \to N$ and $\mu_K : K \wedge L \to K$

$$\mu_N((n,k) \wedge (m,l)) = [n,m] - {}^{l}n + {}^{k}m \quad , \quad \mu_K(k \wedge l) = [k,l]$$

which are the restrictions of the commutator maps will being Lie homomorphisms. It can be easily observed that (μ_N, μ_K) is a morphism of precrossed modules. We should note that $Im(\mu_N, \mu_K) = \gamma_2((N, K, \partial), (M, L, \partial))$.

Proposition 2.3. *Suppose* $(R_1, R_2, \delta) \rightarrowtail (F_1, F_2, \delta) \twoheadrightarrow (M, L, \partial)$ *is a projective presentation of the precrossed module* (M, L, ∂)*. Then there is an isomorphism of precrossed modules* $(M, L, \partial) \wedge (M, L, \partial) \cong \frac{\gamma_2(F_1, F_2, \delta)}{\gamma_2((R_1, R_2, \delta), (F_1, F_2, \delta))}$*. In particular* $H_2(M, L, \partial) \cong \ker((M, L, \partial) \wedge (M, L, \partial) \overset{(\mu_N, \mu_K)}{\to} (M, L, \partial))$.

Using the equations (1) and $(M, L, \partial) \wedge (M, L, \partial) \cong \big(M \wedge (M \rtimes L), L \wedge L, \partial \wedge \sigma\big)$ we can infer the following corollary.

Corollary 2.4. *Let* (M, L, ∂) *be a precrossed module. Then*

$$\kappa(H_2(M, L, \partial)) \cong \ker(H_2(M \rtimes L) \overset{H_2(\sigma)}{\to} H_2(L)).$$

Example 2.5. *(i) Let L be a Lie algebra and regard L as the precrossed module (L,L,id). By $(M,L,\partial) \wedge (M,L,\partial) \cong \big(M \wedge (M \rtimes L), L \wedge L, \partial \wedge \sigma\big)$, $(L,L,id) \wedge (L,L,id) \cong (L \wedge (L \rtimes L), L \wedge L, id \wedge \sigma)$. There is a natural isomorphism $L \oplus L \to L \rtimes L$ which sends (x,y) to $(x-y,y)$. Hence*

$$L \wedge (L \rtimes L) \cong L \wedge (L \oplus L) \cong (L \wedge L) \oplus (L/L^2 \otimes L/L^2).$$

Applying Proposition 2.3, and using an easy diagram chasing we conclude that there exist an isomorphism of abelian precrossed modules $H_2(L,L,id) \cong (H_2(L) \oplus (L/L^2 \otimes L/L^2), H_2(L), < id, 0 >)$.
(ii) For arbitrary Lie algebra L, consider the precrossed modules $(L,0,0)$ and $(0,L,0)$. Then $(L,0,0) \wedge (L,0,0) \cong (L \wedge L, 0, 0)$, $(0,L,0) \wedge (0,L,0) \cong (0, L \wedge L, 0)$. Consequently, $H_2(L,0,0) \cong (H_2(L),0,0), H_2(0,L,0) \cong (0,H_2(L),0)$.

we give a version of the classical Stallings-Stammbach sequence for homology of groups and Lie algebras.

Proposition 2.6 (Five term exact sequence). *Let $(N,K,\partial) \rightarrowtail (M,L,\partial) \twoheadrightarrow (Q,S,\mu)$ be an exact sequence of precrossed modules. There exists a natural exact sequence of abelian precrossed modules*

$$\begin{aligned}\mathcal{M}(M,L,\partial) \to \mathcal{M}(Q,S,\mu) &\to (N,K,\partial)/\gamma_2((N,K,\partial),(M,L,\partial)) \\ &\to (M,L,\partial)/\gamma_2(M,L,\partial) \twoheadrightarrow (Q,S,\mu)/\gamma_2(Q,S,\mu).\end{aligned}$$

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Polymatroidal ideals and linear resolution

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Abstract

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and $\mathfrak{m} = (x_1, \ldots, x_n)$ be the unique homogeneous maximal ideal. Let $I \subset S$ be a monomial ideal with a linear resolution and $I\mathfrak{m}$ be a polymatroidal ideal. We prove that if either $I\mathfrak{m}$ is polymatroidal with strong exchange property, or I is a monomial ideal in at most 4 variables, then I is polymatroidal.

Keywords and phrases: polymatroidal ideals, monomial localization, linear quotients, linear resolution.

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1. Introduction

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over a field K and $\mathfrak{m} = (x_1, \ldots, x_n)$ denotes the unique homogeneous maximal ideal. Let $I \subset S$ be a monomial ideal and $G(I)$ be the unique minimal set of monomial generators of I.

The monomial localization of a monomial ideal $I \subset S$ with respect to a monomial prime ideal P is the monomial ideal $I(P)$ which is obtained from I by substituting the variables $x_i \notin P$ by 1. Observe that $I(P)$ is the unique monomial ideal with the property that $I(P)S_P = IS_P$. The monomial localization $I(P)$ can also be described as the saturation $I : (\prod_{x_i \notin P} x_i)^{\infty}$. When I is a squarefree monomial ideal, we see that $I(P) = I : u$ where $u = \prod_{x_i \notin P} x_i$. Note that $I(P)$ is a monomial ideal in $S(P)$, where $S(P)$ is the polynomial ring in the variables which generate P.

It has been observed that a monomial localization of a polymatroidal is again polymatroidal ([4, Corollary 3.2]).

The author and Herzog conjectured that a monomial ideal I is polymatroidal if and only if $I(P)$ has a linear resolution for all monomial prime ideals P ([1, Conjecture 2.9]). They gave an affirmative answer to the conjecture in the following cases: 1) I is generated in degree 2; 2) I contains at least $n-1$ pure powers; 3) I is monomial ideal in at most three variables; 4) I has no embedded prime ideal and either $|\mathrm{Ass}(S/I)| \leq 3$ or $\mathrm{height}(I) = n-1$.

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Now, we consider the following statement: $(*)$ Let I be a monomial ideal with linear resolution such that $I\mathfrak{m}$ is polymatroidal. Then I is polymatroidal.

Observe that $(*)$ holds if Bandari-Herzog's conjecture is satisfied, because $I(P) = (I\mathfrak{m})(P)$ for all $P \neq \mathfrak{m}$.

In this paper, we give a positive answer to the statement $(*)$ in the following cases: 1) $I\mathfrak{m}$ is polymatroidal with strong exchange property; 2) I is a monomial ideal in at most 4 variables.

2. Main Results

Definition 2.1. *Let $I \subset S$ be a monomial ideal. We say that I has a d-linear resolution, if I has the following minimal graded free resolution:*

$$0 \to S^{m_t}(-(d+t)) \to \cdots \to S^{m_i}(-(d+i)) \to$$

$$S^{m_{i-1}}(-(d+(i-1)) \to \cdots \to S^{m_1}(-(d+1)) \to S^{m_0}(-d) \to I \to 0$$

Lemma 2.2. *([1, Page 760]) Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal and $\mathfrak{m} = (x_1, \ldots, x_n)$. If I has a linear resolution, then $I = I\mathfrak{m} : \mathfrak{m}$.*

Definition 2.3. *Let $I \subset S$ be a monomial ideal. We say that I has linear quotients, if there exists an order $u_1, \ldots, u_r$ of $G(I)$ such that for $j = 2, \ldots, r$, the minimal generators of the colon ideal $(u_1, \ldots, u_{j-1}) : u_j$ are variables.*

Definition 2.4. *Let $I \subset S$ be a monomial ideal generated in a single degree. The ideal I is polymatroidal if for any two elements $u, v \in G(I)$ such that $\deg_{x_i}(u) > \deg_{x_i}(v)$ there exists an index j with $\deg_{x_j}(u) < \deg_{x_j}(v)$ such that $x_j(u/x_i) \in I$.*

In the case that the polymatroidal ideal I is squarefree, it is called matroidal.

Any polymatroidal ideal I has linear quotients ([5, Lemma 1.3]). Then since I is generated in a single degree, it follows that I has a linear resolution ([2, Lemma 4.1]).

The author and Herzog conjectured that a monomial ideal I is polymatroidal if and only if all monomial localizations of I have a linear resolution. If the conjecture is satisfied, then the following statement holds:

$(*)$ Let I be a monomial ideal with linear resolution such that $I\mathfrak{m}$ is polymatroidal. Then I is polymatroidal.

The following example shows that the linear resolution condition of the statement $(*)$ cannot be weakened.

Example 2.5. *The ideal $I = (x_1^2, x_1x_2, x_3^2, x_2x_3) \subset S = K[x_1, x_2, x_3]$ is generated in a single degree, but it does not have a linear resolution. On the other hand $I\mathfrak{m}$ is polymatroidal, but I is not.*

Definition 2.6. *Let $I \subset S$ be a monomial ideal. We say that I satisfies the strong exchange property if I is generated in a single degree, and for all $u, v \in G(I)$ and for all i, j with $\deg_{x_i}(u) > \deg_{x_i}(v)$ and $\deg_{x_j}(u) < \deg_{x_j}(v)$, one has $x_j(u/x_i) \in I$.*

Now, we show that $(*)$ holds if $I\mathfrak{m}$ is a polymatroidal with strong exchange property.

Proposition 2.7. *Let $I \subset S$ be a monomial ideal with a linear resolution and $I\mathfrak{m}$ be polymatroidal with strong exchange property. Then I is polymatroidal with strong exchange property.*

PROOF. Let $u, v \in G(I)$ with $\deg_{x_i}(u) > \deg_{x_i}(v)$ and $\deg_{x_j}(u) < \deg_{x_j}(v)$. So $ux_k, vx_k \in I\mathfrak{m}$ for each $k = 1, \ldots, n$. Now, since $\deg_{x_i}(ux_k) > \deg_{x_i}(vx_k)$ and $\deg_{x_j}(ux_k) < \deg_{x_j}(vx_k)$, it follows that $x_j(ux_k/x_i) \in I\mathfrak{m}$ for each $k = 1, \ldots, n$. Hence $x_j(u/x_i)\mathfrak{m} \subseteq I\mathfrak{m}$. Since I has a linear resolution, it follows by Lemma 2.2, $x_j(u/x_i) \in I$. □

Lemma 2.8. *([3, Lemma 3.1]) Let $I \subset S$ be a polymatroidal ideal. Then for any monomials $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x^{b_1} \cdots x_n^{b_n}$ in $G(I)$ and for each i with $a_i < b_i$, one has j with $a_j > b_j$ such that $x_i(u/x_j) \in G(I)$.*

Lemma 2.9. *Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal with assumption $I = I\mathfrak{m} : \mathfrak{m}$. Let $u \in G(I)$ and $I\mathfrak{m}$ be a polymatroidal ideal. If for $1 \leq i \neq j \leq n$, $(u/x_j)x_i^2 \in I\mathfrak{m}$, then $(u/x_j)x_i \in I$.*

PROOF. Since $I = I\mathfrak{m} : \mathfrak{m}$, it is enough to show that $(ux_i/x_j)\mathfrak{m} \subseteq I\mathfrak{m}$. We have $(ux_i/x_j)x_j = ux_i \in I\mathfrak{m}$ and $(u/x_j)x_i^2 \in I\mathfrak{m}$. Now, let $k \neq i, j$. Then with considering Lemma 2.8 for monomials $(u/x_j)x_i^2 \in I\mathfrak{m}$ and $ux_k \in I\mathfrak{m}$, we have $(ux_i/x_j)x_k \in I\mathfrak{m}$. □

Finally, we are ready to prove that $(*)$ holds for monomial ideals in at most 4 variables.

Proposition 2.10. *Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal with $n \leq 4$. Let I has a linear resolution and $I\mathfrak{m}$ be polymatroidal. Then I is polymatroidal.*

PROOF. We have already noted that the claim is true for $n \leq 3$. Now, let $n = 4$. Since I has a linear resolution, it follows by Lemma 2.2 that $I = I\mathfrak{m} : \mathfrak{m}$. Let $\deg_{x_1}(u) > \deg_{x_1}(v)$, so there exists an index j with $\deg_{x_j}(u) < \deg_{x_j}(v)$. For convenience, we assume that $j = 2$. So $\deg_{x_2}(u) < \deg_{x_2}(v)$. Now, we consider the following cases:

Case 1: $\deg_{x_3}(u) < \deg_{x_3}(v)$ and $\deg_{x_4}(u) < \deg_{x_4}(v)$. With considering Lemma 2.8 for ux_2 and vx_2, we have $(ux_2/x_1)x_2 \in I\mathfrak{m}$. So by Lemma 2.9, it follows that $(u/x_1)x_2 \in I$.

Case 2: $\deg_{x_3}(u) > \deg_{x_3}(v)$ and $\deg_{x_4}(u) > \deg_{x_4}(v)$. With considering exchange property between ux_2 and vx_2, we have $(ux_2/x_1)x_2 \in I\mathfrak{m}$. So Lemma

2.9, implies that $(u/x_1)x_2 \in I$.

Case 3: $\deg_{x_3}(u) < \deg_{x_3}(v)$ and $\deg_{x_4}(u) > \deg_{x_4}(v)$. With considering exchange property between ux_4 and vx_4, it follows that either $(ux_4/x_1)x_2 \in I\mathfrak{m}$ or $(ux_4/x_1)x_3 \in I\mathfrak{m}$.

- Assume $(ux_4/x_1)x_2 \in I\mathfrak{m}$. With considering Lemma 2.8 for ux_2 and vx_2, we have either $ux_2^2/x_1 \in I\mathfrak{m}$, so there is nothing to prove, or $ux_2^2/x_4 \in I\mathfrak{m}$. Now with comparing $(ux_4/x_1)x_2$ and ux_2^2/x_4, we have $ux_2^2/x_1 \in I\mathfrak{m}$, which implies that $(u/x_1)x_2 \in I$.

- Assume $(ux_4/x_1)x_3 \in I\mathfrak{m}$. With considering Lemma 2.8 for ux_3 and vx_3, we have either $ux_3^2/x_1 \in I\mathfrak{m}$, so there is nothing to prove, or $ux_3^2/x_4 \in I\mathfrak{m}$. Now with comparing $(ux_4/x_1)x_3$ and ux_3^2/x_4, we have $ux_3^2/x_1 \in I\mathfrak{m}$, which implies that $(u/x_1)x_3 \in I$.

Case 4: $\deg_{x_3}(u) > \deg_{x_3}(v)$ and $\deg_{x_4}(u) < \deg_{x_4}(v)$. This follows by a similar argument of case (3). □

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Cofiniteness and Artinianness of generalized local cohomology modules

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Abstract

Let R be a commutative Noetherian ring and $I \subseteq J$ be ideals of R. Let M and N be finitely generated R-modules such that $\mathrm{pd}_R(M) < \infty$. The notion $\tilde{\mathrm{q}}_J(M,N)$ is the greatest integer i such that $H^i_J(M,N)$ is not Artinian and J-cofinite. In this paper, we give a bound for $\tilde{\mathrm{q}}_J(M,N)$ by using $\tilde{\mathrm{q}}_I(M,N)$. We show that $\tilde{\mathrm{q}}_J(M,N) \leq \tilde{\mathrm{q}}_I(M,N) + \mathrm{cd}_J(M,N/IN)$.

Keywords and phrases: cofinite module, cohomological dimension, generalized local cohomology module, Noetherian ring..

2010 *Mathematics subject classification:* Primary: 13D45; Secondary: 14B15, 13E05.

1. Introduction

Throughout this paper, Let R denote a commutative Noetherian ring and I be an ideal of R. Let M and N be two finitely generated R-modules. The notion of generalized local cohomology was introduced by Herzog in [4]. The ith generalized local cohomology modules of M and N with respect to I is defined as

$$H^i_I(M,N) \cong \varinjlim_{n \geq 1} \mathrm{Ext}^i_R(M/I^nM, N).$$

It is clear that $H^i_I(R,N)$ is just the ordinary local cohomology module $H^i_I(N)$. Generalized local cohomology modules have been studied by several authors (see for example [5], [7]).

Hartshorn in [3] defined an R-module M to be I-cofinite, if $\mathrm{Supp}(M) \subseteq \mathrm{V}(I)$ and $\mathrm{Ext}^i_R(R/I,M)$ is finitely generated module for all $i \geq 0$.

Recall that for an R-module M, the notion $\mathrm{cd}(I,M)$, *the cohomological dimension of M with respect to I*, is defined as:

$$\mathrm{cd}(I,M) = \sup\{i \in \mathbb{N}_0 \ : \ H^i_I(M) \neq 0\}$$

* speaker

and the notion $\mathrm{q}(I, M)$, which for first time was introduced by Hartshorne, is defined as:

$$\mathrm{q}(I, M) = \sup\{i \in \mathbb{N}_0 \ : \ H_I^i(M) \text{ is not Artinian}\},$$

with the usual convention that the supremum of the empty set of integers is interpreted as $-\infty$.

Amjadi and Naghipour in [1] defined for R-modules M and N, the notion $\mathrm{cd}_I(M, N)$, *the cohomological dimension of M and N with respect to I*, as:

$$\mathrm{cd}_I(M, N) = \sup\{i \in \mathbb{N}_0 \ : \ H_I^i(M, N) \neq 0\}.$$

Vahdanipour et al. in [6], introduced the notion $\tilde{\mathrm{q}}_I(M, N)$ as:

$$\tilde{\mathrm{q}}_J(M, N) = \sup\{i \in \mathbb{N}_0 \ : \ H_J^i(M, N) \text{ is not Artinian } J\text{-cofinite}\},$$

if there exist such i's and $-\infty$ otherwise.
In this paper, we give a bound for $\tilde{\mathrm{q}}_J(M, N)$.

The main aim of this paper is to prove the following result:

Theorem 1.1. *Let R be a Noetherian ring, $I \subseteq J$ be ideals of R and M, N be finitely generated R-modules such that $\mathrm{pd}_R(M) < \infty$. Then*

$$\tilde{\mathrm{q}}_J(M, N) \leq \tilde{\mathrm{q}}_I(M, N) + \mathrm{cd}_J(M, N/IN).$$

For any ideal I of R, we denote $\{\mathfrak{p} \in \mathrm{Spec}R : \mathfrak{p} \supseteq I\}$ by $\mathrm{V}(I)$. We refer the reader to [2] for any unexplained notion and terminology.

2. Main Results

The main purpose of this section is to prove Theorem 1.1. But first of all we need the following auxiliary lemmas.

Lemma 2.1. *Let R be a Noetherian ring, I and J be ideals of R such that $I \subseteq J$. Let M be finitely generated and N be an arbitrary module. Then*
(a) $\Gamma_I(M, N) \cong \mathrm{Hom}_R(M, \Gamma_I(N))$,
(b) $\Gamma_J(M, N) \cong \Gamma_J(M, \Gamma_I(N))$.

□

Lemma 2.2. *Let R be a Noetherian ring, I and J be ideals of R and M, N be finitely generated R-modules such that $H_J^j(M, H_I^i(N))$ is Artinian and J-cofinite, for each $i \geq 0$ and each $j \geq 0$. Then $H_J^i(M, N)$ is also Artinian and J-cofinite, for each $i \geq 0$.*

□

Lemma 2.3. *Let R be a Noetherian ring, $I \subseteq J$ be ideals of a Noetherian ring R and M, N be finitely generated R-modules. If $\tilde{q}_J(M,N) \geq 0$, then $\tilde{q}_I(M,N) \geq 0$.*

□

Lemma 2.4. *Let R be a Noetherian ring and $I \subseteq J$ be ideals of R. Let M and N be finitely generated R-modules. If $\tilde{q}_J(M,N) \geq 0$, then $\tilde{q}_J(M, \oplus_{i=0}^{\mathrm{cd}(I,N)} H_I^i(N)) \geq 0$ such that*

$$\tilde{q}_J(M, \oplus_{i=0}^{\mathrm{cd}(I,N)} H_I^i(N)) = \mathrm{Sup}\{\tilde{q}_J(M, H_I^i(N)) : i \in \mathbb{N}_0\}.$$

□

Lemma 2.5. *Let R be a Notherian ring, I be ideal of R and M, N be finitely generated R-modules such that $\mathrm{pd}_R(M) < \infty$ and $\mathrm{Supp}\, L \subseteq \mathrm{Supp}\, N$. Then $\tilde{q}_I(M,L) \leq \tilde{q}_I(M,N)$.*

□

The following proposition plays an important role in the proof of Theorem 2.7.

Proposition 2.6. *Let R be a Notherian ring, $I \subseteq J$ be ideals of R and M, N be finitely generated R-modules such that $\mathrm{pd}_R(M) < \infty$. Then*

$$\tilde{q}_J(M,N) \leq \tilde{q}_I(M,N) + \tilde{q}_J(M, \oplus_{i=0}^{\mathrm{cd}(I,N)} H_I^i(N)).$$

Now we are ready to state and prove the main result.

Theorem 2.7. *Let R be a Noetherian ring, $I \subseteq J$ be ideals of R and M, N be finitely generated R-modules such that $\mathrm{pd}_R(M) < \infty$. Then*

$$\tilde{q}_J(M,N) \leq \tilde{q}_I(M,N) + \mathrm{cd}_J(M, N/IN).$$

PROOF. Assume that $\tilde{q}_J(M,N) \geq 0$. Then by using Proposition 2.6, it follows that

$$\tilde{q}_J(M,N) \leq \tilde{q}_I(M,N) + \tilde{q}_J(M, \oplus_{i\geq 0}^{\mathrm{cd}(I,N)} H_I^i(N)).$$

Set $k := \tilde{q}_J(M, \oplus_{i\geq 0}^{\mathrm{cd}(I,N)} H_I^i(N))$. Since

$$\tilde{q}_J(M, \oplus_{i\geq 0} H_I^i(N)) \leq \mathrm{cd}_J(M, \oplus_{i\geq 0} H_I^i(N)),$$

it follows that $\mathrm{cd}_J(M, \oplus_{i\geq 0} H_I^i(N)) \geq k$ which implies that

$$H_J^k(M, \oplus_{i\geq 0}^{\mathrm{cd}(I,N)} H_I^i(N)) \neq 0.$$

Therefore, there exists a finitely generated submodule L of the R-module $\oplus_{i\geq 0}^{\mathrm{cd}(I,N)} H_I^i(N)$, such that $H_J^k(M,L) \neq 0$ concequently

$$k \leq \mathrm{cd}_J(M,L). \quad (1)$$

Since

$$\begin{aligned}\operatorname{Supp} L &\subseteq \operatorname{Supp}(\oplus_{i\geq 0}^{\operatorname{cd}(I,N)} H_I^i(N))\\ &\subseteq \operatorname{Supp}(N/IN),\end{aligned}$$

it follows from [1, Theorem B] that

$$\operatorname{cd}_J(M,L) \leq \operatorname{cd}_J(M,N/IN). \tag{2}$$

Then by relations (1) and (2) we have

$$\begin{aligned}\tilde{q}_J(M,N) &\leq \tilde{q}_I(M,N) + k\\ &\leq \tilde{q}_I(M,N) + \operatorname{cd}_J(M,L)\\ &\leq \tilde{q}_I(M,N) + \operatorname{cd}_J(M,N/IN).\end{aligned}$$

So, the assertion holds. □

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Groups which do not have four irreducible characters of degrees divisible by a prime p

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Abstract

Given a finite group G, we say that G has property $\mathcal{P}_n$ if for every prime integer p, G has at most $n-1$ irreducible characters whose degrees are multiples of p. In this paper, we classify all finite groups that have property $\mathcal{P}_4$. We show that the groups satisfying property $\mathcal{P}_4$ are exactly the finite groups with at most three nonlinear irreducible characters, one solvable group of order 168, $\mathrm{SL}_2(3)$, A_5, S_5, $\mathrm{PSL}_2(7)$ and A_6.

Keywords and phrases: Finite group; Prime divisors; Character graph. .

2010 *Mathematics subject classification:* Primary: 43A20; Secondary: 46H25.

1. Introduction

Throughout this paper, G will be a finite group, $\mathrm{Irr}(G)$ will be the set of irreducible complex characters of G and $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. We denote by $\mathrm{Irr}_{nl}(G)$ the set of nonlinear irreducible characters of G. We say that a group G satisfies property $\mathcal{P}_n$ if for every prime integer p, G has at most $n-1$ irreducible characters whose degrees are multiples of p.

A useful way to study the character set $\mathrm{Irr}(G)$ of a finite group G is to attach a graph structure on $\mathrm{Irr}_{nl}(G)$. Wr propose the following question:

Question 1. *What can be said about the structure of finite groups that have property $\mathcal{P}_n$?*

Clearly, if a finite group G has property $\mathcal{P}_n$, then it will also satisfy property $\mathcal{P}_{n+1}$. Our main goal in this paper is to classify the finite groups that have property $\mathcal{P}_4$.

Theorem 1.1. *Let G be a finite group. Then G has property $\mathcal{P}_4$ if and only if one of the followings hold:*

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* speaker

(i) $|\mathrm{Irr}_{nl}(G)| \leqslant 3$.

(ii) *G is the semidirect product of an elementary abelian group E_{2^3} by a Frobenius group of order* 21.

(iii) *G is isomorphic to one of the groups* $\mathrm{SL}_2(3)$, A_5, S_5, $\mathrm{PSL}_2(7)$ *or* A_6.

In addition, another related question has been studied by Benjamin in [1] and Ghaffarzadeh et al. in [3], respectively, for solvable and nonsolvable groups, in which instead of irreducible characters, the degrees of irreducible characters are considered. In fact, in the papers above, it is said that a group G has property $\mathcal{P}_n$ if every set of n distinct elements of $\mathrm{cd}(G)$ is setwise relatively prime. In [1], an upper bound is obtained for $|\mathrm{cd}(G)|$ when G is a nonabelian solvable group that satisfies property $\mathcal{P}_n$, and in [3], it is shown that if G is a nonsolvable group satisfying property $\mathcal{P}_4$, then $|\mathrm{cd}(G)| \leqslant 8$.

Here, we introduce some more notation. A Frobenius group with a complement H and the kernel N is denoted by (H, N). If G is a group, $N \trianglelefteq G$ and $\tau \in \mathrm{Irr}(N)$, the inertia group of τ in G is denoted by $I_G(\tau)$. We write $\mathrm{Irr}(G \mid \tau)$ for the set of irreducible constituents of τ^G and $\mathrm{cd}(G \mid \tau) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G \mid \tau)\}$. We shall write $d(G \mid \tau) = (a_0 \cdot d_0, a_1 \cdot d_1, \cdots, a_t \cdot d_t)$ to denote that $\mathrm{cd}(G \mid \tau) = \{d_0, d_1, \cdots, d_t\}$, where $d_0, d_1, \cdots, d_t$ are distinct and $\mathrm{Irr}(G \mid \tau)$ contains exactly a_i characters of degree d_i, $i \in \{0, 1, \cdots, t\}$. We also define $d(G)$ as above, in which we place $\mathrm{Irr}(G)$ and $\mathrm{cd}(G)$ instead of $\mathrm{Irr}(G \mid \tau)$ and $\mathrm{cd}(G \mid \tau)$, respectively, then $|G| = a_0 d_0^2 + a_1 d_1^2 + \cdots + a_t d_t^2$.

Finally, we will frequently make use of the following results in this paper. Let $N \trianglelefteq G$ and fix $\tau \in \mathrm{Irr}(N)$. If $\tau \in \mathrm{Irr}(N)$ is G-invariant, we have two cases to consider. First that τ extends to G, then Gallagher's theorem [4, Corollary 6.17] gives a description of $\mathrm{Irr}(G \mid \tau)$. In particular, the characters in $\mathrm{Irr}(G \mid \tau)$ are in bijection with the characters in $\mathrm{Irr}(G/N)$. Next we consider the case that τ does not extend to G. In this case, to determine the set $\mathrm{Irr}(G \mid \tau)$, one needs to use projective representations (see [4, Chapter 11]). In particular, we can find Schur representation group Γ for G/N. This implies that Γ has a central subgroup A so that $\Gamma/A \cong G/N$, A is isomorphic to the Schur multiplier for G/N, and $A \subseteq \Gamma'$. By [4, Theorem 11.28]), there exists a character $\alpha \in \mathrm{Irr}(A)$ so that the characters in $\mathrm{Irr}(G \mid \tau)$ are in bijection with the characters in $\mathrm{Irr}(\Gamma \mid \alpha)$. In particular, $\mathrm{cd}(G \mid \tau) = \{\tau(1)a \mid a \in \mathrm{cd}(\Gamma \mid \alpha)\}$. The Atlas [2], provides the character tables for the Schur representation groups of the simple groups that it includes. When τ is not G-invariant, we apply Clifford's correspondence [4, Theorem 6.11] for a description of the elements of $\mathrm{Irr}(G \mid \tau)$.

2. Solvable groups with property $\mathcal{P}_4$

In this section, we study solvable groups satisfying property $\mathcal{P}_4$. We first consider a situation in which the multiplicity of each nonlinear character degree in the group is at most two. Such groups, called DD-groups, are classified in [5].

Lemma 2.1. *Let G be a solvable group satisfying property $\mathcal{P}_4$. If G is a DD-group, then* $|\mathrm{Irr}_{nl}(G)| \leqslant 3$.

Proposition 2.2. *Let G be a solvable group satisfying property $\mathcal{P}_4$. Then one of the following holds:*

(i) $|\mathrm{Irr}_{nl}(G)| \leqslant 3$.
(ii) *There exists $K \lhd G$ such that G/K is isomorphic to* A_4 *or* (C_3, C_7)*, and G has three distinct nonlinear irreducible characters of the same degree d coprime to* 3.

Proof of Theorem A. Suppose that G is a solvable group satisfying property $\mathcal{P}_4$ and assume that (i) is false. By Proposition 2.2, there exists $K \lhd G$ such that G/K is isomorphic to A_4 or (C_3, C_7), and G contains three distinct irreducible characters of the same degree d such that $(3, d) = 1$. Let $N/K = (G/K)'$ be the kernel of the Frobenius group G/K. We consider all possibilities for $d(G)$ in each cases:

Case 1. $G/K \cong \mathrm{A}_4$. Then $d(\mathrm{A}_4) = (3 \cdot 1, 1 \cdot 3)$. Let $1_K \neq \tau \in \mathrm{Irr}(K)$ and $T = I_G(\tau)$. We will get all possibilities for $\mathrm{Irr}(G \mid \tau)$.

Case 2. $G/K \cong (C_3, C_7)$. Then $d(G/K) = (3 \cdot 1, 2 \cdot 3)$. Fix $1_K \neq \tau \in \mathrm{Irr}(K)$ and let $T = I_G(\tau)$.

3. Nonsolvable groups with property $\mathcal{P}_4$

In this section, we prove Theorem A for nonsolvable groups. We first consider the almost simple groups satisfying property $\mathcal{P}_4$. Recall that a group G is an almost simple group with socle S if S is a nonabelian simple group and $S \trianglelefteq G \leqslant \mathrm{Aut}(S)$.

Theorem 3.1. *Let S be a nonabelian simple group and G a group such that $S \trianglelefteq G \leqslant$* $\mathrm{Aut}\, S$*. Then G has property $\mathcal{P}_4$ if and only if one of the following holds:*

(i) $G = S \cong \mathrm{A}_5, \mathrm{PSL}_2(7)$ *or* A_6.
(ii) $G \cong \mathrm{S}_5$*, where* $S \cong \mathrm{A}_5$.

Theorem 3.2. *Let G be a nonsolvable group satisfying property $\mathcal{P}_4$. Let L be the solvable radical of G. Then G/L is isomorphic to one of the groups A_5,* S_5*,* $\mathrm{PSL}_2(7)$ *or* A_6.

Lemma 3.3. *Let G be a nonsolvable group satisfying property $\mathcal{P}_4$, and let L be the solvable radical of G. Let $\tau \in \mathrm{Irr}(L)$ and $T = I_G(\tau)$. If $T < G$, then* $|\mathrm{Irr}(T \mid \tau)| \leqslant 2$.

Now, we with the above results we could prove the main theorem for non solvable case.

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S-acts with finitely generated universal right congruence

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Abstract

Dandan et. all in [1], introduced universal congruences on semigroups. We generalized this concept to S-acts and consider an S-act A such that the universal right congruence ω_A, is finitely generated. Also we fined some relationships between ω_A being finitely generated and A being pseudo-finite.

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1. Introduction

A *finitary condition* for a class of algebras is a condition that is satisfied by at least all finite members of the class. Finitary conditions were very high Importance in understanding the structure and behavior of rings, groups, semigroups and many other types of algebra. The two finitary conditions we focus on them are the case where an S-act A being *pseudo-finite* and the weaker condition under which the *universal right congruence* ω_A is finitely generated. Dandan et. all in [1], introduced universal congruences on semigroups. We generalized this concept to S-acts and consider an S-act A such that the universal right congruence ω_A, is finitely generated. Also we fined some relationships between ω_A being finitely generated and A being pseudo-finite.

Throughout the paper S will denote a given monoid. A (right) S-act is a set A on which S acts unitarily from the right with the usual properties, that is, if there is an S-*action* $\mu : A \times S \to A$, denoting $\mu(a,s) := as$, such that $a(st) = (as)t$ and $a1 = a$, where 1 denotes the identity of S. In fact, an S-act is a universal algebra $(A,(\mu_s)_{s\in S})$ where each $\mu_s : A \to A$ is a unary operation on A such that $\mu_s \circ \mu_t = \mu_{st}$ for each $s,t \in S$, and $\mu_1 = id_A$.

Let A_S be a right S-act. An equivalence relation ρ on A is called a right S-act congruence or a right congruence on As, if $a\rho a'$ implies $(as)\rho(a's)$ for $a,a' \in A_S$, s ∈ S. Note that $\rho(H)$ for $H \subseteq A \times A$ denotes the congruence generated by H(i.e the smallest congruence on A containing H). Also we denote $H^{-1} = \{(a,b) \mid$

* speaker

$(b,a) \in H\}$ and it is not difficult to check that $a\rho(H)b$ if and only if either $a = b$ or there exists a sequence $a = p_1s_1, q_1s_1 = p_2s_2, q_2s_2 = p_3s_3, ..., q_ns_n = b$ where for $i = 1, ..., n, (p_i, q_i) \in H \cup H^{-1}$ and $s_1, s_2, ..., s_n \in S$. The above sequence is referred to as an H-sequence of length n. for more informations and definitions not mentioned here see [3].

2. Main Results

Definition 2.1. *For a right S-act A, the congruence $A \times A$ is said to be universal right congruence and denotes by ω_A.*

Definition 2.2. *Let A be a right S-act with ω_A being generated by a finite subset $H \subseteq A \times A$. We say that A is pseudo-finite with respect to H if there exists $n \in \mathbb{N}$ such that for any $a, b \in A$, there is an H-sequence from a to b of length at most n. We say that an S-act A is pseudo-finite with respect to $X \subseteq A$ if A is pseudo-finite with respect to $X^2 = X \times X$.*

Clearly, if an S-act A is pseudo-finite with respect to H, then ω_A is finitely generated.

Theorem 2.3. *If G is a group, then ω_G is finitely generated if and only if G is finitely generated group.*

A congruence ρ_2 is called *principal extension* of ρ_1, if there exists $(a, b) \in A \times A$ such that $\rho_2 = \rho(\rho_1 \cup \{(a, b)\})$.

Lemma 2.4. *For a right S-act A, the following are equivalent:*

(i) *ω_A is finitely generated.*

(ii) *There is a finite chain $\iota = \delta_0 \subset \delta_1 \subset \cdots \subset \delta_n = \omega_A$ of left congruences on S where each δ_i is a principal extension of δ_{i-1} for all $1 \leq i \leq n$.*

(iii) *There exists a finite subset X of A such that $\omega_A = \langle X^2 \rangle$.*

(iv) *There exists a finite subset X of A such that for any $x \in X, \omega_A = \langle \{x\} \times X \rangle$.*

(v) *For any $u \in A$ there exists a finite subset X of A such that $u \in X$ and $\omega_A = \langle \{u\} \times X \rangle$.*

Lemma 2.5. *Let A be a right S-act and H be a finite subset of A^2 which generats ω_A. Suppose $\omega_A = \langle K \rangle$ for some $K \subseteq A^2$. Then there exists a finite subset K' of K such that $\omega_A = \langle K' \rangle$.*

Further, if A is pseudo-finite with respect to H of length at most $m \in \mathbb{N}$, then it is pseudo-finite with respect to K' of length at most $m' \in \mathbb{N}$.

We now make some observations which will be very useful for later sections.

Lemma 2.6. *Let A be a non-trivial right S-act such that $\omega_A = \langle H \rangle$ for some $H \subseteq A^2$. Let $C(H) = \{x : \exists y \in A \text{ s.t } (x, y) \in H \cup H^{-1}\}$. Then,*

(i) *there exists $X \subseteq A$ such that $\omega_A = \langle X^2 \rangle$.*

(ii) *$C(H)$ is a generating subset of an S-act A.*

Proposition 2.7. *Let A be a right S-act and A' be a subact of A. Then ω_A is finitely generated if and only if there exists a finite subset X of A such that $A = XS$ and $\omega_{A'} = \rho(X^2)|_{A' \times A'}$. In addition, A is pseudo-finite if and only if there exists $n \in \mathbb{N}$ such that for any $a, b \in A'$ there is an X^2-sequence from a to b of length at most n.*

As corollary of Proposition 2.7, we have the following result.

Theorem 2.8. *The following are equivalent for an S-act A with zero.*
(i) *A is finitely generated.*
(ii) *ω_A is finitely generated.*
(iii) *A is pseudo-finite.*

Now we give a variety of alternative conditions for $S-$acts such that ω_A is finitely generated.

Proposition 2.9. *Let A be an $S-$act and B a homomorphic image of A. If ω_A is finitely generated (A is pseudo-finite), then so is ω_B(B is so).*

Corollary 2.10. *Let A and B be S-acts. If $\omega_{A \times B}$ is finitely generated ($A \times B$ is pesudo-finite), then both ω_A and ω_B are finitely generated(pseudo-finite).*

Now let A be an $S-$act and B be a $T-$act. Then $A \times B$ is a right $S \times T-$act by the action given by,

$$\mu : (A \times B) \times (S \times T) \longrightarrow A \times B$$
$$\mu((a,b),(s,t)) = (as, bt)$$

Proposition 2.11. *Let A be an $S-$act and B be a $T-$act. If ω_A and ω_B are finitely generated(pseudo-finite), then $\omega_{A \times B}$ is finitely generated ($A \times B$ is pseudo-finite $S \times T$-act).*

Definition 2.12. *Let S be a semigroup, I and J non-empty sets and P a matrix indexed by I and J with entries p_{ij} taken from S. Then the Rees matrix semigroup $T = \mathcal{M}[S; I, J; P]$ is the set $(I \times S \times J)$ together with the multiplication $(i,s,j)(k,t,l) = (i, sp_{jk}t, l)$.*
Now let A be a right S-act, then the set $\mathcal{A} = (I \times A \times J)$ is a right T-act by the action $(i,a,j)(k,s,l) = (i, ap_{jk}s, l)$ and we call it Rees matrix induced action.

Theorem 2.13. *Let $T = \mathcal{M}[S; I, J; P]$ be a Rees matrix semigroup over a semigroup S and $\mathcal{A}_T$ be the Rees matrix induced action. Then $\omega_{\mathcal{A}}$ is finitely generated if and only if the following conditions hold:*
(i) *I and J are finite;*
(ii) *there is a finite set $V \subseteq A$ such that with*

$$H = \{(ap_{j\mu}, bp_{ji}) : j \in J, i, \mu \in I, a, b \in V\}$$

every element of A is $\rho(H)$-related to an element of V.

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IA-central series

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Abstract

In this paper, we first define a new series on the IA-central subgroup and two automorphisms on this series. Then we identify the relationships of the members of these series. Finally, we study the relationships of these two new automorphisms with IA(G), $Aut_l(G)$, Ivar(G), Inn(G), and each other.

Keywords and phrases: IA-group, IA-central subgroup, autocentral automorphism, Ivar(G), inner automorphism.

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1. Introduction

The various series have many applications in algebra. In particular, they are necessary for important definitions such as nilpotency and solubility of groups. On the other hand, All kinds of automorphisms also have interesting properties. Hence, automorphisms have been the idea of many researchers articles. Let G be a group and j be any positive integer. Let us denote by G', Z(G), Aut(G) and Inn(G), respectively the commutator subgroup, the centre, the full automorphism group and the inner automorphisms. Bachmuth [1] in 1965 defined an IA-automorphism of a group G as

$$IA(G) = \left\{ \alpha \in Aut(G) \mid g^{-1}\alpha(g) = [g,\alpha] \in G', \forall\, g \in G \right\}.$$

For any group G, $Inn(G) \leqslant IA(G)$.

Hegarty [4] in 1994 introduced the absolute center

$$L(G) = \left\{ g \in G \mid g^{-1}\alpha(g) = 1, \forall\, \alpha \in Aut(G) \right\}$$

and absolute central automorphisms

$$Aut_l(G) = \left\{ \alpha \in Aut(G) \mid g^{-1}\alpha(g) \in L(G), \forall\, g \in G \right\}.$$

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On the similar lines, Ghumde and Ghate [3] in 2015 introduced the IA-central subgroup

$$S(G)=\left\{g\in G\mid g^{-1}\alpha(g)=1,\,\alpha\in IA(G)\right\}$$

and Ivar(G) group as follows:

$$Ivar(G)=\left\{\alpha\in IA(G)\mid g^{-1}\alpha(g)\in S(G),\,\forall g\in G\right\}.$$

For any group G, $L(G)\trianglelefteq S(G)\trianglelefteq Z(G)$.

2. Main results

In this section, after some new definitions, we give our main results about the automorphisms on the IA-central series.

2.1. IA-central series

Definition 2.1. *We define the IA-central series of G in the following way:*

$$\langle 1\rangle=S_0(G)\subseteq S_1(G)=S(G)\subseteq S_2(G)\subseteq\cdots\subseteq S_n(G)\subseteq\cdots$$

where

$$S_n(G)=\{g\in G\mid [g,\alpha_1,\ldots,\alpha_n]=1,\,\forall\alpha_1,\ldots,\alpha_n\in IA(G)\},\quad n\geq 1.$$

Definition 2.2. *A group G is called an S_j-group if the IA-central series stalls at some point. This means that there exists a least positive integer j for which $S_j(G)=S_{j+1}(G)=\cdots$.*

Definition 2.3. *A group G is said to be S(G)-autonilpotent(or IA-nilpotent) group of class at most n if $S_n(G)=G$, for some natural number n.*

Example 2.4. *For abelian groups, S(G)=G, so $S_n(G)=G$, for every natural number n. Therefore, for every $n\in\mathbb{N}$, the abelian groups are S(G)-autonilpotent.*

Remark 2.5. *Every S(G)-autonilpotent group of class j is trivially a S_j-group, since $G=S_j(G)=S_{j+1}(G)=\cdots$.*

2.2. The automorphisms of IA-central series

Definition 2.6. *The kernel of the natural homomorphism from Aut(G) to $Aut(G/S_j(G))$ is called the group of S_j-automorphism and denoted by $Aut_{S_j}(G)$.*

According to the above definition, A S_j-automorphism group acts as the identity on G modulo $S_j(G)$, Thus:

$$Aut_{S_j}(G)=\{\alpha\in Aut(G)\mid g^{-1}\alpha(g)\in S_j(G),\,\forall\,g\in G\}\trianglelefteq Aut(G).$$

Also, we have $Aut_l(G)\leqslant Aut_{S_j}(G)$ for every j.

Notation 2.7. *We use the notation* $IA_{S_j}(G) = IA(G) \cap Aut_{S_j}(G)$. *Another definition of* $IA_{S_j}(G)$ *is given by*

$$IA_{S_j}(G) = \{\alpha \in Aut(G) \mid g^{-1}\alpha(g) \in S_j \cap G', \forall\, g \in G\} \trianglelefteq Aut(G).$$

According to this notation, we have $IA_{S_1}(G) = Ivar(G)$.

Proposition 2.8. *For any group* G,

a) $\varphi \in Aut_{S_j}(G)$ *if and only if* $[\alpha, \varphi] \in Aut_{S_j}(G)$, *for every* $\alpha \in Aut(G)$.

b) $\dfrac{IA(G)}{IA_{S_j}(G)} \cong \dfrac{IA(G)Aut_{S_j}(G)}{Aut_{S_j}(G)}$.

PROOF. a) It is obvious by the normality of $Aut_{S_j}(G)$.

b) The result follow from the definition of $IA_{S_j}(G)$ and the third isomorphism theorem. □

Corollary 2.9. *For any group* G, $[Aut(G), Aut_{S_j}(G)] \leqslant Aut_{S_j}(G)$.

Theorem 2.10. *Let* G *be a group. If* $IA(G/S_j(G)) = Inn(G/S_j(G))$, *then*

$$IA(G) \leqslant Inn(G)Aut_{S_j}(G).$$

PROOF. Let $\alpha \in IA(G)$. By hypothesis, $IA(G/S_j(G)) = Inn(G/S_j(G))$, so there exists $g \in G$ such that for all $x \in G$,

$$\alpha(x)S_j(G) = x^g S_j(G).$$

Hence,

$$\begin{aligned} & x^{-g}\alpha(x) = \left(x^{-1}(\alpha(x))^{g^{-1}}\right)^g \in S_j(G) \\ \Longrightarrow\ & x^{-1}(\alpha(x))^{g^{-1}} \in S_j(G) \\ \Longrightarrow\ & x^{-1}g(\alpha(x))g^{-1} \in S_j(G) \\ \Longrightarrow\ & x^{-1}\varphi_g^{-1}\alpha(x) \in S_j(G) \end{aligned}$$

where $\varphi_g \in Inn(G)$.

Consequently, $\varphi_g^{-1}\alpha \in Aut_{S_j}(G)$, i.e., $\alpha = \varphi_g\varphi_g^{-1}\alpha \in Inn(G)Aut_{S_j}(G)$. □

In the special case j=1, we have the following result

Corollary 2.11. *Let* G *be a group. If* $IA(G/S(G)) = Inn(G/S(G))$, *then*

$$IA(G) \leqslant Inn(G)Ivar(G).$$

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The Torsion Theory of A Completely Prime Radical of A Module

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Abstract

This talk is about torsion theories induced by prime and completely prime radical of a module M over an arbitrary ring R. In fact, we review some basic facts and new results which have been achieved over the past years. In particular, it is shown that the class of all completely prime modules, ${}_RM$ for which ${}_RM \neq 0$ is special. Finally, some outlines about new researches of the subject under discussion are given.

Keywords and phrases: prime radical, completely prime radical, torsion theory.
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1. Introduction

All rings in this talk are associative (not necessarily with identity) and all modules are left R-modules. In [4], the authors call a proper submodule P of an R-module M to be completely prime whenever for all $r \in R$ and $m \in M$, if $rm \in P$ then $m \in P$ or $rM \subseteq P$. The terminology of radical in this talk is that of [6]. A functor γ from the category of R-mod to R-mod is called a preradical if $\gamma(M)$ is a submodule of M and $f(\gamma(M)) \subseteqq \gamma(N)$ for each homomorphism $f: M \to N$ in R-mod. A radical γ is a preradical for which $\gamma(M/\gamma(M)) = 0$ for all M in R-mod. A preradical is hereditary or left exact if $\gamma(N) = N \cap \gamma(M)$ whenever N is an arbitrary submodule of M in R-mod. We recall that for an R-module M, $\beta(M)$ is the prime radical of M which is the intersection of all prime submodules of M. Moreover, $\beta_{co}(M)$ denotes the completely prime radical of M, which is the intersection of all submodules K of M such that M/K is a completely prime module. In this talk, we discuss some basic facts and new results related to torsion theories induced by prime and completely prime radical of a module M over an arbitrary ring R, which have been achieved over the past years in [2, 4–6]. Moreover, it is shown that the class of all completely prime modules, ${}_RM$ for which ${}_RM \neq 0$ is special. Finally, some outlines about new researches of the subject under discussion are given.

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2. Main Results

We begin with a definition from [3, Page 454].

Definition 2.1. *Let γ be a functor from the category of R-mod to R-mod. Then, it is said to be Hoehnke radical if $f(\gamma(M)) \subseteqq \gamma(f(M))$ for all homomorphism $f : M \to f(M)$ and $\gamma(M/\gamma(M)) = 0$ for all $M \in$ R-mod. Moreover, γ is said to be complete if for all submodules K of M, the relation $\gamma(K) = K$ implies that $K \subseteqq \gamma(M)$. Finally, γ is said to be idempotent if $\gamma(\gamma(M)) = \gamma(M)$ for all $M \in$ R-mod.*

Definition 2.2. *A Kurosh-Amitsur radical is a complete idempotent Hoehnke radical.*

The following definition is from [6, Page 139].

Definition 2.3. *A torsion theory in the category of R-mod is a pair $(\mathfrak{T}, \mathfrak{F})$ of classes of modules in R-mod such that*

1. *$Hom(T, F) = 0$ for all $T \in \mathfrak{T}$ and $F \in \mathfrak{F}$.*
2. *If $Hom(C, F) = 0$ for all $F \in \mathfrak{F}$, then $C \in \mathfrak{T}$.*
3. *If $Hom(T, C) = 0$ for all $T \in \mathfrak{T}$, then $C \in \mathfrak{F}$.*

Definition 2.4. *We define $\mathfrak{T}_{\beta_{co}}$ to be the class of all modules M such that $\beta_{co}(M) = M$, and $\mathfrak{F}_{\beta_{co}}$ to be the class of all modules M such that $\beta_{co}(M) = 0$.*

In view of [6, Page 140], $\mathfrak{T}_{\beta_{co}}$ is a torsion class and $\mathfrak{F}_{\beta_{co}}$ is a torsion-free class and the pair $(\mathfrak{T}_{\beta_{co}}, \mathfrak{F}_{\beta_{co}})$ is a torsion theory. Moreover, $\mathfrak{T}_{\beta_{co}}$ coincides with the class of modules with no completely prime submodules. Now by Proposition 2.1 in [6], we get:

Theorem 2.5. *$\mathfrak{T}$ is a torsion class for some torsion theory exactly if it is closed under quotient objects, direct products and extensions.*

This theorem yields the following result as Corollary 4.3 in [4]:

Corollary 2.6. *$\mathfrak{T}_{\beta_{co}}$ is closed under quotients, direct products and extensions.*

We observe that in view of the following example, $\mathfrak{T}_{\beta_{co}}$ is not closed under taking submodules:

Example 2.7. *Let p be a prime number and $M = \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module. We have $\beta_{co}(M) = \mathbb{Z}_{p^\infty}$. Now, let N be a proper submodule of M. Then, N has a (maximal) completely prime submodule, say P. Thus, $\beta_{co}(N) \subset P \subset N = \beta_{co}(M) \cap N$ and $\beta_{co}(M) \cap N \nsubseteqq \beta_{co}(N)$.*

Definition 2.8. *Let M be an R-module. We define $\overline{\beta}_{co}(M)$ to be the sum of all submodules N of M such that $\beta_{co}(N) = N$. Moreover, we define $\hat{\beta}_{co}(M)$ to be the intersection of all submodules N of M such that $M/N \in \mathfrak{F}_{\beta_{co}}$.*

Now by [2, Proposition 1.1.5], we have:

Corollary 2.9. *The following statements hold.*

1. $\overline{\beta}_{co}(M)$ *is an idempotent preradical,* $\overline{\beta}_{co} \subseteqq \beta_{co}$*,* $\mathfrak{T}_{\beta_{co}} = \mathfrak{T}_{\overline{\beta}_{co}}$*.* $\overline{\beta}_{co}$ *is the largest idempotent preradical contained in* β_{co}*.*
2. $\hat{\beta}_{co}(M)$ *is radical.* $\beta_{co} \subseteqq \hat{\beta}_{co}$*,* $\mathfrak{F}_{\beta_{co}} = \mathfrak{F}_{\hat{\beta}_{co}}$*. Moreover,* $\hat{\beta}_{co}$ *is the least radical containing* β_{co}*.*

Moreover, a similar argument as [6, Proposition 2.5.], yields the following result:

Theorem 2.10. *If* $M \in \mathfrak{T}_{\beta_{co}}$ *then for each non-zero homomorphic image N of M there exists a submodule K of N such that* $0 \neq K \in \mathfrak{T}_{\beta_{co}}$*.*

In view of the fact that the results for completely prime (sub)modules are true for prime (sub)modules, we deduce that prime radical $\beta(M)$ is a complete Hoenhke radical which is neither hereditary nor idempotent (hence not Kurosh-Amistur). Furthermore, prime modules are not closed under taking essential extensions. However, if we define a faithful prime radical, as the submodules P of M such that M/P is faithful and prime, then in view of [5, Section 3] β_0 is a Kurosh-Amitsur radical. Furthermore, the class of all faithful prime modules is closed under essential extensions.

Definition 2.11. *A class* Ω *of associative rings is called a special class if it is hereditary and it consists of prime rings and it is closed under essential extensions.*

Andrunakievich and Rjabuhin [1] extended this notion to modules and showed that prime modules, irreducible modules, simple modules, modules without zero divisors, etc form special classes of modules. B. de la Rosa and S. Veldsman [3] defined a weakly special class of modules. In [4], the authors follow the definition in [3] of a weakly special class of modules to define a special class of modules.

Definition 2.12. *Let R be a ring and* $\mathfrak{K}_R$ *be a (possibly empty) class of R-modules. Let* $\mathfrak{K}$ *be the union of* $\mathfrak{K}_R$ *such that R is a ring. Then* $\mathfrak{K}$ *is called a special class of modules if it satisfies:*

1. $M \in \mathfrak{K}_R$ *and* $I \triangleleft R$ *with* $I \subseteqq (0:M)_R$ *implies* $M \in \mathfrak{K}_{R/I}$*.*
2. *If* $I \triangleleft R$ *and* $M \in \mathfrak{K}_{R/I}$*, then* $M \in \mathfrak{K}_R$*.*
3. $M \in \mathfrak{K}_R$ *and* $I \triangleleft R$ *with* $IM \neq 0$ *implies* $M \in \mathfrak{K}_I$*.*
4. $M \in \mathfrak{K}_R$ *implies* $RM \neq 0$ *and* $R/(0:M)_R$ *is a prime ring.*
5. *If* $I \triangleleft R$ *and* $M \in \mathfrak{K}_I$*, then there exists* $N \in \mathfrak{K}_R$ *such that* $(0:N)_I \subseteqq (0:M)_I$*.*

Now with some similar arguments as [7], we get:

Theorem 2.13. *Let* $\mathfrak{M} = \cup \mathfrak{M}_R$ *be a special class of modules. Then, the set* $\mathfrak{J}$ *which is the set of rings R such that there exists* $M \in \mathfrak{M}$ *with* $(0:M)_R = 0$ *with* 0 *is a special class of rings. If R is the corresponding special radical then,* $\mathfrak{R}(R)$ *is the intersection of* $(0:M)_R$ *with* $M \in \mathfrak{M}$*.*

Furthermore, by [4, Theorem 6.3] the following result is obtained:

Theorem 2.14. *Let $\mathfrak{I}$ be a special class of rings and for every ring R, let $\mathfrak{M}_R$ be the set of modules M such that M is an R-module, $RM \neq 0$ and $R/(0:M)_R \in \mathfrak{I}$. If $\mathfrak{M} = \cup \mathfrak{M}_R$, then $\mathfrak{M}$ is a special class of modules. Moreover, if r is the corresponding special radical and M is any R-module, then $r(M)$ is the intersection of all submodules P such that $M/P \in \mathfrak{M}_R$.*

For the case of completely prime modules, we have the following results as Theorem 6.4 and Corollary 6.5 in [4]:

Theorem 2.15. *Let R be any ring and let $\mathfrak{M}_R$ be the set of completely prime R-modules M such that $RM \neq 0$. If $\mathfrak{M} = \cup \mathfrak{M}_R$, then $\mathfrak{M}$ is a special class of modules.*

Corollary 2.16. *If $\mathfrak{M}_{co}$ is the special class of completely prime modules, then the special radical induced by $\mathfrak{M}_{co}$ on a ring R is given by $\beta_{co}(R)$ is the intersection of all $(0:M)_R$ such that M is a completely prime R-module. Moreover, it is the intersection of all ideals I of R such that I is a completely prime ideal.*

It is observed that some analogous investigations as this talk for the case of co-prime, completely co-prime, completely semi-prime and etc. may yield to new results.

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A Subgraph of the strongly annihilating submodule graph

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Abstract

For a module M over a commutative ring R, the strongly annihilating submodule graph of M, denoted by $\mathbb{SAG}(M)$, introduced in [4]. This graph is a generalization of a graph $\mathbb{AG}(M)$, the annihilating submodule graph of M, defined in [2]. In this note we give the more properties of $\mathbb{SAG}(M)$ and moreover we introduce and study a subgraph of the $\mathbb{SAG}(M)$.

Keywords and phrases: strongly annihilating submodule graph, coloring number, star graph.
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1. Introduction

In this presentation, all rings are commutative with nonzero identity elements and all modules are right unitary. Let M be an R-module. For any $N \leq M$, the ideal $\{r \in R \mid Mr \subseteq N\}$ is denoted by $(N : M)$. We denote $(0 : M)$ by $\operatorname{ann}_R(M)$ or simply $\operatorname{ann}(M)$. If $\operatorname{ann}(M) = 0$, then M is said to be *faithful*.
In [3] the authors introduced *The annihilating ideal graph* $\mathbb{AG}(R)$ that is a graph whose vertices are ideals of R with nonzero annihilators and in which two distinct vertices I and J are adjacent if and only if $IJ = 0$. In [2], the authors generalized the above idea to submodules of M and defined the graph $\mathbb{AG}(M)$, called *the annihilating submodule graph*, with vertices $\{0 \neq N \leq M \mid M(N : M)(K : M) = 0, \text{for some } 0 \neq K \leq M\}$, and two distinct vertices N and K are adjacent if and only if $M(N : M)(K : M) = 0$. In [4, 5], *the strongly annihilating submodule graph*, denoted by $\mathbb{SAG}(M)$, introduced and studied. In fact $\mathbb{SAG}(M)$ is an undirected (simple) graph in which a nonzero submodule N of M is a vertex if $N(K : M) = 0$ or $K(N : M) = 0$, for some $0 \neq K \leq M$ and two distinct vertices N and K are adjacent if and only if $N(K : M) = 0$ or $K(N : M) = 0$. Clearly $\mathbb{SAG}(M)$ is a subgraph of $\mathbb{AG}(M)$ and $\mathbb{SAG}(R) = \mathbb{AG}(R)$ and if M is a multiplication R-module, then $\mathbb{SAG}(M) = \mathbb{AG}(M)$. The notations of graph theory used in the sequel can be founed in [6].

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Here, we define a subgraph of $\mathbb{SAG}(M)$, denoted by $\mathbb{SAG}^*(M)$ that is a simple graph with vertices $\{0 \neq N \leq M \mid (N :_R M) \neq 0$ and there exists a nonzero submodule $K \leq M$ with $(K :_R M) \neq 0$ such that $N(K :_R M) = 0$ or $K(N :_R M) = 0\}$ and two distinct vertices N, K are adjacent if and only if $N(K :_R M) = 0$ or $K(N :_R M) = 0$. In this paper, in addition to providing the more properties of $\mathbb{SAG}(M)$, we compare the properties of $\mathbb{SAG}^*(M)$ with $\mathbb{SAG}(M)$ and $\mathbb{AG}(M)$.

2. Main Results

Example 2.1. *Let S_1 be a faithful simple R-module and S_2 be an unfaithful R-module. Setting $M = S_1 \oplus S_1 \oplus S_2$, the submodule $N = (0) \oplus (0) \oplus S_2$ is not a vertex in $\mathbb{SAG}^*(M)$, since $(N :_R M) = \mathrm{ann}_R(S_1) = 0$. But for the nonzero submodule $K = (0) \oplus S_1 \oplus (0)$ we have $N \cap K = 0$ and hence N and K are adjacent in $\mathbb{SAG}(M)$. Therefore $\mathbb{SAG}^*(M) \subsetneq \mathbb{SAG}(M)$.*

An R-module M is called prime if the annihilator of M is equal to the annihilator of any its nonzero submodule. A proper submodule N of M is called prime submodule if M/N is a prime module. One can easily check that a proper submodule N of M is prime if and only if for any $r \in R$ and any submodule K of M, the relation $Kr \subseteq N$ implies that $K \subseteq N$ or $Mr \subseteq N$. Also the set of all zero divisors of M is denoted by $Z(M) = \{r \in R \mid xr = 0$, for some $0 \neq x \in M\}$.

In the following, we show that the existence of a vertex in a graph that is connected to any other vertex is the same in both graphs $\mathbb{SAG}(M)$ and $\mathbb{AG}(M)$.

Theorem 2.2. *Let M be an R-module such that $\mathrm{ann}_R(M)$ is a nil ideal of R. Then there exists a vertex in $\mathbb{AG}(M)$ that is joined to all other vertices if and only if there exists a vertex in $\mathbb{SAG}(M)$ that is joined to all other vertices.*

Example 2.3. *Consider $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3$ as a $\mathbb{Z}_{12}$-module. Then $\mathrm{ann}_{\mathbb{Z}_{12}}(M)$ is a nilpotent ideal and $\mathbb{SAG}(M)$ is a star graph with two vertices $\mathbb{Z}_2 \oplus (0)$ and $(0) \oplus \mathbb{Z}_3$.*

Now, the existence of a vertex in $\mathbb{SAG}^*(M)$ that is connected to any other vertex is characterized

Theorem 2.4. *Let M be a faithful module. Then there exists a vertex in $\mathbb{SAG}^*(M)$ that is joined to all other vertices if and only if M can be written as $M = M_1 \oplus M_2$, where M_1 is a simple submodule and M_2 is a prime submodule of M, or $Z(R)$ is a nil ideal of R.*

Recall that a ring is called *reduced* if it has no nonzero nilpotent element.

Corollary 2.5. *Let R be a reduced ring and M be a faithful R-module. The following statements are equivalent:*

(1) *There exists a vertex in* $\mathbb{SAG}^*(M)$ *that is adjacent to every other vertex.*
(2) $\mathbb{SAG}^*(M)$ *is a star graph.*
(3) $M = M_1 \oplus M_2$*, where* M_1 *is a simple submodule and* M_2 *is a prime submodule of* M*.*

Example 2.6. $\mathbb{Q} \oplus \mathbb{Q}$ *as a* $\mathbb{Q} \oplus \mathbb{Z}$*-module is faithful and* $\mathbb{SAG}^*(\mathbb{Q} \oplus \mathbb{Q})$ *is a star graph with two adjacent vertices* $\mathbb{Q} \oplus (0)$ *and* $(0) \oplus \mathbb{Q}$*.*

Proposition 2.7. *Let* $M = M_1 \oplus M_2$*, where* $\mathrm{ann}_R(M)$ *is a nil ideal of* R*,* M_1 *is a simple submodule of* M *and* M_2 *is a prime submodule of* M*. Then there exists a vertex in* $\mathbb{AG}(M)$ *that is joined to every other vertex.*

Proposition 2.8. (a) *Let* M *be a faithful* R*-module such that it has only one nonzero proper submodule. Then* $M \cong R$ *as an* R*-modules.*
(b) *Let* R *be an Artinian ring and* M *be a finitely generated faithful* R*-module. Then any nonzero proper submodule of* M *is a vertex in* $\mathbb{SAG}^*(M)$*.*

In a graph G, a *clique* of G is a complete subgraph and the supremum of the sizes of cliques in G, denoted by $cl(G)$, is called the clique number of G. Let $\chi(G)$ denote the *chromatic number* of the graph G, that is, the minimal number of colors needed to color the vertices of G so that no two adjacent vertices have the same color.

Proposition 2.9. *Let* M *be a faithful* R*-module. Then* $\chi(\mathbb{SAG}(M)) = 1$ *if and only if* M *has only one nonzero proper submodule.*

Theorem 2.10. *For any faithful* R*-module* M*, the following are equivalent:*
(a) $\chi(\mathbb{SAG}^*(M)) = 2$.
(b) $\mathbb{SAG}^*(M)$ *is a bipartite graph with two nonempty parts.*
(c) R *is a reduced ring with exactly two minimal prime ideals or* $\mathbb{SAG}^*(M)$ *is a star graph with more than one vertex.*

Corollary 2.11. *Let* R *be an Artinian ring and* M *be a faithful* R*-module. Then the following are equivalent:*
(a) $\chi(\mathbb{SAG}^*(M)) = 2$.
(b) $\mathbb{SAG}^*(M)$ *is a bipartite graph with two nonempty parts.*
(c) $M = M_1 \oplus M_2$ *where* M_1 *and* M_2 *are homogeneous semisimple modules or* $\mathbb{SAG}^*(M)$ *is a star graph with more than one vertex.*

Corollary 2.12. *Let* R *be a reduced ring and* M *be a faithful* R*-module. The following statements are equivqlent:*
(a) $\chi(\mathbb{SAG}^*(M)) = 2$.
(b) $\mathbb{SAG}^*(M)$ *is a bipartite graph with two nonempty parts.*
(c) R *has only two minimal ideals.*

Lemma 2.13. *Let* M *be a semiprime* R*-module such that the clique number of* $\mathbb{SAG}^*(M)$ *is not infinite. Then the set of all submodules of the form* $\mathrm{ann}_M(I)$*, where* I *is an ideal of* R*, satisfies the ACC condition.*

PROOF. Assuming the contrary, there is a strictly ascending chain

$$\text{ann}_M(I_1) \subsetneq \text{ann}_M(I_2) \subsetneq ...$$

in M. Since for any $i \geq 1$, $\text{ann}_M(I_{i+1})I_i \neq 0$, there exists $r_i \in I_i$ such that $\text{ann}_M(I_{i+1})r_i \neq 0$. We set $J_i = \text{ann}_M(I_{i+1})r_i$ for $i = 1, 2, 3, ...$, and we show that for any $i < j$, $J_i \neq J_j$. Otherwise $\text{ann}_M(I_{i+1})r_i = \text{ann}_M(I_{j+1})r_j$, where $i < j$. Then

$$0 = \text{ann}_M(I_{i+1})r_i r_j = \text{ann}_M(I_{j+1})r_j^2.$$

Since M is semiprime, $\text{ann}_M(I_{j+1})r_j = 0$, a contradiction. Now for any $i < j$;

$$J_j(J_i :_R M) = \text{ann}_M(I_{j+1})r_j(\text{ann}_M(I_{i+1})r_i :_R M) \subseteq \text{ann}_M(I_{i+1})r_i r_j = 0.$$

Therefore for any $i < j$, J_i and J_j are joined in $\mathbb{SAG}^*(M)$ and hence $\mathbb{SAG}^*(M)$ has an infinite clique number which contradicts the hypothesis. □

Theorem 2.14. *For a semiprime module M, the following statements are equivalent:*
(a) $\chi(\mathbb{SAG}^*(M))$ *is finite.*
(b) $cl(\mathbb{SAG}^*(M))$ *is finite.*
(c) $\mathbb{SAG}^*(M)$ *dose not have an infinite clique number.*
(d) *There are prime submodules $P_1, P_2, \ldots, P_k$ in M such that $\bigcap_{i=1}^k (P_i :_R M) = (0)$.*

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RING MORPHISMS AND THEIR ORDERINGS

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Abstract

We associate to any ring R with identity a partially ordered set $\mathrm{Hom}(R)$, whose elements are all pairs $(\mathfrak{a}, M)$, where $\mathfrak{a} = \ker\varphi$ and $M = \varphi^{-1}(U(S))$ for some ring morphism φ of R into an arbitrary ring S. Here $U(S)$ denotes the group of units of S. The maximal elements of $\mathrm{Hom}(R)$ constitute a subset $\mathrm{Max}(R)$ which, for commutative rings R, can be identified with the Zariski spectrum $\mathrm{Spec}(R)$ of R. Every pair $(\mathfrak{a}, M)$ in $\mathrm{Hom}(R)$ has a canonical representative, that is, there is a universal ring morphism $\psi\colon R \to S_{(R/\mathfrak{a},M/\mathfrak{a})}$ corresponding to the pair $(\mathfrak{a}, M)$, where the ring $S_{(R/\mathfrak{a},M/\mathfrak{a})}$ is constructed as a universal inverting $R/\mathfrak{a}$-ring in the sense of Cohn. Several properties of the sets $\mathrm{Hom}(R)$ and $\mathrm{Max}(R)$ are studied.

Keywords and phrases: Ring morphism, Partially ordered set, Universal inverting mapping of rings. .

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1. Introduction

In this paper, the partially ordered set $\mathrm{Hom}(R)$ is considered. The elements of $\mathrm{Hom}(R)$ are ordered pairs $(\mathfrak{a}, M)$, where $\mathfrak{a} = \ker\varphi$ and $M = \varphi^{-1}(U(S))$ for some ring morphism φ of R into an arbitrary ring S. Here $U(S)$ denotes the group of units of S. It turns out that it is possible to canonically associate to any such pair $(\mathfrak{a}, M)$ a morphism of rings $\psi\colon R \to S_{(R/\mathfrak{a},M/\mathfrak{a})}$ that realizes the pair $(\mathfrak{a}, M)$, meaning that $\ker(\psi) = \mathfrak{a}$ and $\psi^{-1}(U(S_{(R/\mathfrak{a},M/\mathfrak{a})})) = M$. The ring $S_{(R/\mathfrak{a},M/\mathfrak{a})}$ is constructed as a universal inverting $R/\mathfrak{a}$-ring in the sense of Cohn [3]. With respect to a suitable partial order, the set $\mathrm{Hom}(R)$ turns out to be a meet-semilattice (Lemma 2.2). The idea is to measure and classify, via the study of the partially ordered set $\mathrm{Hom}(R)$, all ring morphisms from the fixed ring R to any other ring S.

We want to generalize the theory developed by Bavula for left Ore localizations [1, 2] to arbitrary ring morphisms. Therefore here we want to extend his idea from ring morphisms $R \to [T^{-1}]R$ that arise as left Ore localizations to arbitrary ring morphisms $\varphi\colon R \to S$.

* speaker

For a commutative ring R, the set $\mathrm{Max}(R)$ is in one-to-one correspondence with the Zarisky spectrum $\mathrm{Spec}(R)$ of R (Proposition 3.2). Thus $\mathrm{Max}(R)$ could be used as a good substitute for the spectrum of a possibly non-commutative ring R. Finally, the partially ordered set $\mathrm{Hom}(R)$ always has a least element, the pair $(0, U(R))$, which corresponds to the identity morphism $R \to R$. More generally, like in Bavula's case [2, p. 3224], the set $\mathrm{Hom}(R)$ has a natural partition into subsets $\mathrm{Hom}(R, \mathfrak{a})$ (Section 2).

Throughout, all rings are associative, with identity $1 \neq 0$, and all ring morphisms send 1 to 1. The group of (right and left) invertible elements of R will be denoted by $U(R)$.

2. The partially ordered set Hom(R)

Let R be a ring. We associate to each ring morphism $\varphi\colon R \to S$ into any other ring S the pair $(\mathfrak{a}, M)$, where $\mathfrak{a} := \ker(\varphi)$ is the kernel of φ and $M := \varphi^{-1}(U(S))$ is the inverse image of the group of units $U(S)$ of S. Recall that, if X is a set, or more generally a class, and ρ is a preorder on X, then it is possible to associate to ρ an equivalence relation $\sim_\rho$ on X and a partial order $\leq_\rho$ on the quotient set $X/{\sim_\rho}$. The equivalence relation $\sim_\rho$ on X is defined, for every $x, y \in X$, by $x \sim_\rho y$ if $x\rho y$ and $y \rho x$. The partial order $\leq_\rho$ on the quotient set $X/{\sim_\rho} := \{\, [x]_{\sim_\rho} \mid x \in X \,\}$ is defined by $[x]_{\sim_\rho} \leq_\rho [y]_{\sim_\rho}$ if $x\rho y$.

On the class $\mathcal{H}(R)$ of all morphisms $\varphi\colon R \to S$ of R into arbitrary rings S, If $\varphi\colon R \to S$, $\varphi'\colon R \to S'$ are two ring morphisms, we have a preorder ρ on $\mathcal{H}(R)$, defined setting $\varphi\rho\varphi'$ if $\ker(\varphi) \subseteq \ker(\varphi')$ and $\varphi^{-1}(U(S)) \subseteq \varphi'^{-1}(U(S))$.

Correspondingly, there is a equivalence relation $\sim$ on the class $\mathcal{H}(R)$, defined, for all ring morphisms $\varphi\colon R \to S$, $\varphi'\colon R \to S'$ with associated pairs $(\mathfrak{a}, M)$, $(\mathfrak{a}', M')$ respectively, by $\varphi \sim \varphi'$ if $(\mathfrak{a}, M) = (\mathfrak{a}', M')$. That is, $\varphi \sim \varphi'$ if and only if $\ker(\varphi) = \ker(\varphi')$ and $\varphi^{-1}(U(S)) = \varphi'^{-1}(U(S'))$. Let $\mathrm{Hom}(R) := \mathcal{H}(R)/{\sim}$ denote the set (class) of all equivalence classes $[\varphi]_\sim$ modulo $\sim$, that is, equivalently, the set of all pairs $(\ker(\varphi), \varphi^{-1}(U(S)))$. The partial order $\leq$ on $\mathrm{Hom}(R) = \mathcal{H}(R)/{\sim}$ associated to the preorder ρ on $\mathcal{H}(R)$ is defined by setting $(\mathfrak{a}, M) \leq (\mathfrak{a}', M')$ if $\mathfrak{a} \subseteq \mathfrak{a}'$ and $M \subseteq M'$.

Proposition 2.1. *Let* **Ring** *be the category of rings with identity and* **ParOrd** *the category of partially ordered sets. Then* $\mathrm{Hom}(-)\colon$ **Ring** $\to$ **ParOrd** *is a contravariant functor.*

For any fixed proper ideal $\mathfrak{a}$ of R, set

$$\mathrm{Hom}(R, \mathfrak{a}) := \{\, (\ker(\varphi), \varphi^{-1}(U(S))) \mid \varphi\colon R \to S,\ \ker(\varphi) = \mathfrak{a} \,\}.$$

Clearly, $\mathrm{Hom}(R)$ is the disjoint union of the sets $\mathrm{Hom}(R, \mathfrak{a})$:

$$\mathrm{Hom}(R) = \dot{\bigcup}_{\mathfrak{a} \lhd R} \mathrm{Hom}(R, \mathfrak{a}).$$

In particular, the partial order $\leq$ on $\mathrm{Hom}(R)$ induces a partial order on each subset $\mathrm{Hom}(R, \mathfrak{a})$.

The following lemma has an easy proof.

Lemma 2.2. *Let $(\mathfrak{a}, M), (\mathfrak{a}', M')$ be the elements of $\mathrm{Hom}(R)$ corresponding to two morphisms $\varphi\colon R \to S$ and $\varphi'\colon R \to S'$. Then the element of $\mathrm{Hom}(R)$ corresponding to the product morphism $\varphi \times \varphi'\colon R \to S \times S'$ is $(\mathfrak{a} \cap \mathfrak{a}', M \cap M')$.*

As a consequence, the partially ordered set $\mathrm{Hom}(R)$ turns out to be a meet-semilattice. In particular, with respect to the operation $\wedge$, $\mathrm{Hom}(R)$ is a commutative semigroup in which every element is idempotent and which has a zero element (= the least element $(0, U(R))$ of $\mathrm{Hom}(R)$, which corresponds to the identity morphism $R \to R$).

3. A universal construction and maximal elements in $\mathrm{Hom}(R)$

Theorem 3.1. *Let R be a ring and $(\mathfrak{a}, M)$ be an element of $\mathrm{Hom}(R)$. Then $S_{(R/\mathfrak{a}, M/\mathfrak{a})}$ is a non-zero ring, and if $\psi\colon R \to S_{(R/\mathfrak{a}, M/\mathfrak{a})}$ denotes the composite mapping of the canonical projection $\pi\colon R \to R/\mathfrak{a}$ and $\chi_{(R/\mathfrak{a}, M/\mathfrak{a})}\colon R/\mathfrak{a} \to S_{(R/\mathfrak{a}, M/\mathfrak{a})}$, then $\ker(\psi) = \mathfrak{a}$ and $\psi^{-1}(U(S_{(R/\mathfrak{a}, M/\mathfrak{a})})) = M$. Moreover, for any ring morphism $f\colon R \to S$ such that $\ker(f) \supseteq \mathfrak{a}$ and $f^{-1}(U(S)) \supseteq M$, there is a unique ring morphism $g\colon S_{(R/\mathfrak{a}, M/\mathfrak{a})} \to S$ such that $g\psi = f$.*

Proposition 3.2. *For any commutative ring R, the maximal elements of $\mathrm{Hom}(R)$ are the pairs $(P, R \setminus P)$, where P is a prime ideal.*

Proposition 3.3. *Let $\mathfrak{a}$ be an ideal of a ring R such that $(\mathfrak{a}, R \setminus \mathfrak{a}) \in \mathrm{Hom}(R)$. Then $\mathfrak{a}$ is a completely prime ideal of R, the ring $R/\mathfrak{a}$ is invertible, and $(\mathfrak{a}, R \setminus \mathfrak{a}) \in \mathrm{Hom}(R)$ is a maximal element of $\mathrm{Hom}(R)$.*

In the following example, we show that not all maximal elements of $\mathrm{Hom}(R)$ are of the form $(\mathfrak{a}, R \setminus \mathfrak{a})$ for some completely prime ideal $\mathfrak{a}$.

Example 3.4. Let R be the ring of $n \times n$ matrices with entries in a division ring D, $n > 1$. For instance, R can be the ring $\mathbb{M}_2(k)$ where k is a finite field. Then any homomorphism $\varphi\colon R \to S$, S any ring, is injective because R is simple. Every element of $M := \varphi^{-1}(U(S))$ is regular. But regular elements in R are invertible. This proves that $\mathrm{Hom}(R)$ has exactly one element, the pair $(0, U(R))$. Thus, clearly, $\mathrm{Hom}(R)$ has a greatest element, which is not of the form $(\mathfrak{a}, R \setminus \mathfrak{a})$ because R is simple, but not a domain, and R has no completely prime ideals.

Proposition 3.5. *Let R be a commutative ring. Then $\mathrm{Hom}(R)$ has a greatest element if and only if R has a unique prime ideal.*

Hence the set $\mathrm{Max}(R)$ of all maximal elements of $\mathrm{Hom}(R)$ could be used as a good substitute for the spectrum of a non-commutative ring R.

Theorem 3.6. *For every ring R, the partially ordered set $\mathrm{Hom}(R)$ has maximal elements.*

Example 3.7. As an example, we now describe the structure of the partially ordered set $\mathrm{Hom}(\mathbb{Z})$, where $\mathbb{Z}$ is the ring of integers. Assume that $\mathfrak{a} = n\mathbb{Z}$ for some $n \geq 2$ and that $(\mathfrak{a}, M)$ corresponds to some ring morphism $\varphi\colon \mathbb{Z} \to S$. Then φ induces an injective ring morphism $\overline{\varphi}\colon \mathbb{Z}/n\mathbb{Z} \to S$, and $M/n\mathbb{Z}$ is a multiplicatively closed subset of $\mathbb{Z}/n\mathbb{Z}$ that consists of regular elements and contains $U(\mathbb{Z}/n\mathbb{Z})$. Since in a finite ring all regular elements are invertible, it follows that $M/n\mathbb{Z} = U(\mathbb{Z}/n\mathbb{Z})$, so that $M = M_{\mathrm{div}(n)}$, where $\mathbb{P} := \{\, p \mid p \text{ is prime number} \,\}$ and $\mathrm{div}(n) := \{\, p \in \mathbb{P} \mid p|n \,\}$. Thus

$$\mathrm{Hom}(\mathbb{Z}) = \{\, (0, M_P) \mid P \text{ is a subset of } \mathbb{P} \,\} \,\dot\cup\, \Big\{\, (n\mathbb{Z}, M_{\mathrm{div}(n)}) \mid n \in \mathbb{Z},\ n \geqslant 2 \,\Big\}.$$

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On the triple tensor product of some class of nilpotent Lie algebras

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Abstract

In this paper, we give the explicit structure of $\otimes^3 L$ where L is a finite dimensional Lie algebra of class two such that $\dim(L/Z(L)) = d$ and $\frac{1}{2}d(d-1) - 3 \leq \dim L^2 \leq \frac{1}{2}d(d-1)$.

Keywords and phrases: nilpotent Lie algebra, tensor product, triple tensor product.
2010 *Mathematics subject classification:* Primary: 17B30, 17B05; Secondary: 17B99.

1. Introduction

Let $\mathbb{F}$ be a field. Throughout the paper, all Lie algebras are considered over a fixed field. Recall that a Lie algebra H is called a generalized Heisenberg Lie algebra of rank n if $H^2 = Z(H)$ and $\dim H^2 = n$. It is known [7] the structure of the tensor product and the triple tensor product of generalized Heisenberg Lie algebras of rank at most 2. In this note, we give the triple tensor product finite dimensional Lie algebra of class two such that $\dim(L/Z(L)) = d$ and $\frac{1}{2}d(d-1) - 3 \leq \dim L^2 \leq \frac{1}{2}d(d-1)$.
The following lemma and propositions are useful instruments in the rest.

Proposition 1.1. *[5, Proposition 2.4] Let L be a finite dimensional nilpotent Lie algebra of nilpotency class 2. Then $L = H \oplus A$ where A is abelian and H is a generalized Heisenberg Lie algebra.*

We assume that the reader is familiar with the basic definitions and properties of the tensor square $L \otimes L$ in [1]. Then

Proposition 1.2. *[1, Proposition 3] There are actions of both L and K on $L \otimes K$ given by*

$$^{l'}(l \otimes k) = [l,l'] \otimes k + l \otimes (^{l'}k),$$
$$^{k'}(l \otimes k) = (^{k'}l) \otimes k + l \otimes [k',k]$$

for all $l,l' \in L$ and $k,k' \in K$.

* speaker

We know from Proposition 1.2, L acts on $L \otimes L$. On the other hand, the tensor product $L \otimes L$ acts on L by ${}^{t}l = {}^{\lambda(t)}l$ for all $t \in L \otimes L$ and $l \in L$ such that $\lambda : L \otimes L \to L$ is a homomorphism given by $a \otimes b \mapsto [a,b]$. Now, we can construct the triple tensor product $\otimes^3 L = (L \otimes L) \otimes L$.

Lemma 1.3. *[7, Lemma 3.1] Let L be a Lie algebra of nilpotency class two. Then*
(i). $L \otimes L$ acts trivially on L.
(ii). $(L \otimes L) \otimes L$ is an abelian Lie algebra.

Let L be a nilpotent Lie algebra of class k and $\gamma_k(L)$ be the k-th term of the lower central series of L and $\varphi : \gamma_k(L) \to L$ a natural homomorphism. Let $\overline{\varphi} = (\varphi \otimes i_L) \otimes i_L : (\gamma_k(L) \otimes L) \otimes L \to \otimes^3 L$ and $\gamma : (L \otimes L) \otimes \gamma_k(L) \to \otimes^3 L$ by sending $(a \otimes b) \otimes c \mapsto (a \otimes b) \otimes c$ be homomorphisms. Then

Proposition 1.4. *[7, Proposition 3.3] If L is a nilpotent Lie algebra of class k, then*

$$(\gamma_k(L) \otimes L) \otimes L \stackrel{(\varphi\otimes i_L)\otimes i_L}{\longrightarrow} \otimes^3 L \longrightarrow \otimes^3 L/\gamma_k(L) \to 0,$$

is exact.

2. Main Results

Here, for a finite Lie algebra L of class two such that $\dim(L/Z(L)) = d$ and $\frac{1}{2}d(d-1) - 3 \leq \dim L^2 \leq \frac{1}{2}d(d-1)$, we determine the structure of the Lie algebras $\otimes^3 L$.
The following lemma is an useful instrument in the next.

Theorem 2.1. *Let L be a Lie algebra of nilpotency class two. Then*

$$(L \otimes L) \otimes L \cong (L \otimes L) \otimes L^{ab}.$$

PROOF. Let $\varphi_0 : L \otimes L \to L \otimes L$ and $\varphi_1 : L \to L^{ab}$ be homomorphisms. Then $\overline{\varphi_0} = \varphi_0 \otimes \varphi_1 : \otimes^3 L \to (L \otimes L) \otimes L^{ab}$ such that $\varphi_0 \otimes \varphi_1((x \otimes y) \otimes z) = \varphi_0(x \otimes y) \otimes \varphi_1(z)$ is an epimorphism by using [6, Proposition 1.2 (ii)]. Since L is of nilpotency class two, $L \otimes L$ is abelian by using [2, Lemma 2.8] and so $L \otimes L$ and L act trivially on each other. Thus

$$\dim((L \otimes L) \otimes L^{ab}) = \dim(L \otimes L) \dim L^{ab}. \tag{1}$$

Now, we claim that $\dim \otimes^3 L = \dim(L \otimes L) \dim L^{ab}$. First, we show that $(\varphi(L^2) \otimes L) \otimes L \cong (\varphi(L^2) \otimes L^{ab}) \otimes L^{ab}$. Since L is of nilpotency class two, it is clear that $\varphi(L^2)$ and L act trivially on each other. Hence $(\varphi(L^2) \otimes L) \otimes L \cong (\varphi(L^2) \otimes L^{ab}) \otimes L$. By a same reason, we have $\varphi(L^2) \otimes L^{ab}$ and L act trivially on each other. Thus $(\varphi(L^2) \otimes L) \otimes L \cong (\varphi(L^2) \otimes L^{ab}) \otimes L^{ab}$. By using the following exact sequence

$$L^2 \otimes L \stackrel{\varphi\otimes i_L}{\longrightarrow} L \otimes L \longrightarrow L/L^2 \otimes L/L^2 \longrightarrow 0, \tag{2}$$

we have

$$\dim L \otimes L = \dim L^{ab} \otimes L^{ab} + \dim \mathrm{Im}(\varphi \otimes i_L) = \dim L^{ab} \otimes L^{ab} + \dim \varphi(L^2) \otimes L. \quad (3)$$

Since $(\varphi(L^2) \otimes L) \otimes L \cong (\varphi(L^2) \otimes L^{ab}) \otimes L^{ab}$, we have

$$\dim(\varphi(L^2) \otimes L) \otimes L = (\dim \varphi(L^2) \otimes L^{ab}) \dim L^{ab}. \quad (4)$$

Now, by using (3), (4) we have

$$\dim(\varphi(L^2) \otimes L^{ab}) \dim L^{ab} = (\dim L \otimes L - \dim L^{ab} \otimes L^{ab}) \dim L^{ab}. \quad (5)$$

On the other hand, we have

$$\begin{aligned} \dim \otimes^3 L &= \dim \otimes^3 L^{ab} + \dim \mathrm{Im}((\varphi \otimes i_L) \otimes i_L) \\ &= \dim \otimes^3 L^{ab} + \dim(\varphi(L^2) \otimes L^{ab}) \dim L^{ab} \end{aligned} \quad (6)$$

by using Proposition 1.4. Hence $\dim \otimes^3 L = \dim(L \otimes L) \otimes L^{ab}$ by using (2) and (6) and so

$$(L \otimes L) \otimes L \cong (L \otimes L) \otimes L^{ab}.$$

□

Theorem 2.2. *Let L be an n-dimensional Lie algebra of class two such that $\dim(L/Z(L)) = d$ and $\dim L^2 = \frac{1}{2}d(d-1)$. Then $n = \frac{1}{2}d(d+1) + t$ for $t \geq 0$ and*

$$\otimes^3 L \cong A((\frac{1}{6}d(2d^2 + 3d - 5) + \frac{1}{2}(t^2 - t) + dt + \frac{1}{2}(d+t)(d+t+1))(d+t)).$$

PROOF. Proposition 1.1 implies $L = H \oplus A(t)$ for $t \geq 0$ such that the set

$$\{x_1, \ldots, x_d, y_1, \ldots, y_{\frac{1}{2}d(d-1)}\}$$

is a basis for H, hence $\dim L = \frac{1}{2}d(d+1) + t$ for $t \geq 0$. By using Lemma 1.3(i), [1, Proposition 5] and Theorem 2.1 we have $\dim \otimes^3 L = \dim(L \otimes L) \dim L^{ab}$. Also, $L \otimes L \cong A(\frac{1}{6}d(2d^2 + 3d - 5) + \frac{1}{2}(t^2 - t) + dt + \frac{1}{2}(d+t)(d+t+1))$ by using [4, Corollary 2.11] and $L^{ab} \cong A(d+t)$. Hence the result follows. □

Theorem 2.3. *Let L be an n-dimensional Lie algebra of class two such that $\dim(L/Z(L)) = d$. Then*

(i) *if $\dim L^2 = \frac{1}{2}d(d-1) - 1$, then $n = \frac{1}{2}d(d-1) - 1 + t$ for $t \geq 0$ and*

$$\otimes^3 L \cong A((\frac{1}{3}d(d^2 + 3d - 4) + \frac{1}{2}(2t^2 + 4dt + d^2 + d))(d+t)).$$

(ii) *If $\dim L^2 = \frac{1}{2}d(d-1) - 2$, then $n = \frac{1}{2}d(d-1) - 2 + t$ for $t \geq 0$ and*

$$\otimes^3 L \cong A((\frac{1}{3}d(d^2 + 3d - 7) + \frac{1}{2}(2t^2 + 4dt + d^2 + d))(d+t)).$$

(*iii*) *If* $\dim L^2 = \frac{1}{2}d(d-1)-3$, *then* $n = \frac{1}{2}d(d-1)-3+t$ *for* $t \geq 0$ *and*

$$\otimes^3 L \cong A((\frac{1}{3}d(d^2+3d-10)+\frac{1}{2}(2t^2+4dt+d^2+d))(d+t)),$$

or

$$\otimes^3 L \cong A((\frac{1}{3}d(d^2+3d-10)+\frac{1}{2}(2t^2+4dt+d^2+d)-1))(d+t)).$$

PROOF. (i). By a similar method in the proof of Theorem 2.2 we can see $n = \frac{1}{2}d(d-1)-1+t$ for $t \geq 0$. By using Lemma 1.3(i), [1, Proposition 5] and Theorem 2.1 we have $\dim \otimes^3 L = \dim(L\otimes L)\dim L^{ab}$. Let $\dim L^2 = \frac{1}{2}d(d-1)-1$. Since $L^{ab} \cong A(d+t)$ and $L\otimes L \cong A(\frac{1}{3}d(d^2+3d-4)+\frac{1}{2}(2t^2+4dt+d^2+d))$ by using [3, Theorem 2.9 (i)], we have

$$\otimes^3 L \cong A((\frac{1}{3}d(d^2+3d-4)+\frac{1}{2}(2t^2+4dt+d^2+d))(d+t)).$$

Parts of (ii) and (iii) are obtained by a similar way and by using [3, Theorem 2.9 (ii) and (iii)].

□

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Bogomolov multiplier and the Lazard correspondence

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Abstract

In this paper we extend the concept of CP covers for groups to the class of Lie algebras, and show that despite the case of groups, all CP covers of a Lie algebra are isomorphic. In addition we prove that CP covers of groups and Lie rings which are in Lazard correspondence, are in Lazard correspondence too, and the Bogomolov multipliers of the group and the Lie ring are isomorphic.

Keywords and phrases: Bogomolov multiplier, Commutativity-preserving defining pair, CP cover, Baker-Campbell-Hausdorff formula, Lazard correspondence..

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1. Introduction

The Bogomolov multiplier and the CP cover were first studied by Moravec for the class of finite groups. In the class of groups, the Bogomolov multiplier of a group is unique up to isomorphism but the corresponding CP cover is not necessarily unique. In our recent work [1], we defined the Bogomolov multiplier for Lie algebras. Here, we will introduce CP covers of Lie algebras, then we will show that all CP covers of a Lie algebra are isomorphic. Also, the Lazard correspondence that was introduced by Lazard in [2], builds an equivalence of categories between finite p-groups of nilpotency class at most $p-1$ and the finite p-Lie rings of the same order and nilpotency class. There is a close connection between many invariants of an arbitrary group and a Lie ring that is its Lazard correspondent.

2. Bogomolov multiplier and CP cover of Lie algebras

The section is devoted to introduce CP covers of Lie algebras and then we will show (unlike the situation in finite groups), all CP covers for a Lie algebra are isomorphic. Throughout this section, L will represent a Lie algebra over a field.

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Bogomolov multiplier. The Bogomolov multiplier is a group-theoretical invariant that introduced as an obstruction to the rationality problem in algebraic geometry. Let K be a field, G be a finite group and V be a faithful representation of G over K. Then there is natural action of G upon the field of rational functions $K(V)$. The Noether's problem asks whether the field of G-invariant functions $K(V)^G$ is rational over K? Saltman found some examples of groups of order p^9 with a negative answer to the Noether's problem, even when taking $K = \mathbb{C}$. His main method was the application of the unramified cohomology group $H^2_{nr}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ as an obstruction. Bogomolov proved that it is canonically isomorphic to

$$B_0(G) = \bigcap \ker\{res^A_G : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})\},$$

where A is an abelian subgroup of G. The group $B_0(G)$ is a subgroup of the Schur multiplier and Kunyavskii named it the *Bogomolov multiplier* of G. Thus vanishing the Bogomolov multiplier leads to positive answer to Noether's problem. Moravec in [3] introduced an equivalent definition of the Bogomolov multiplier. In this sense, he used a notion of the nonabelian exterior square $G \wedge G$ of a group G to obtain a new description of the Bogomolov multiplier. He showed that if G is a finite group, then $B_0(G)$ is non-canonically isomorphic to $\mathrm{Hom}(\tilde{B}_0(G), \mathbb{Q}/\mathbb{Z})$, where the group $\tilde{B}_0(G)$ can be described as a section of the nonabelian exterior square of a group G. Also, he proved that $\tilde{B}_0(G) = \mathcal{M}(G)/\mathcal{M}_0(G)$, such that the Schur multiplier $\mathcal{M}(G)$ interpreted as the kernel of the commutator homomorphism $G \wedge G \to [G, G]$ given by $x \wedge y \to [x, y]$, and $\mathcal{M}_0(G)$ is the subgroup of $\mathcal{M}(G)$ defined as $\mathcal{M}_0(G) = \langle x \wedge y \mid [x, y] = 0,\ x, y \in G \rangle$. Thus in the class of finite groups, $\tilde{B}_0(G)$ is non-canonically isomorphic to $B_0(G)$. With this definition all truly nontrivial nonuniversal commutator relations is collected into an abelian group that is called Bogomolov multiplier. Furthermore, Moravec's method relates Bogomolov multiplier to the concept of commuting probability of a group and shows that the Bogomolov multiplier plays an important role in commutativity preserving central extensions of groups, that are famous cases in K-theory.

Hopf-type formula for Bogomolov multiplier: We recall Hopf-type formula for groups and Lie algebras as follows. Let $K(F)$ denotes $\{[x, y] | x, y \in F\}$.

Theorem 2.1. *Let G be a group and L be a Lie algebra. Then*

(i) *If* $G \cong \frac{F_1}{R_1}$ *be a presentation for G, then* $\tilde{B}_0(G) \cong \frac{R_1 \cap \gamma_2(F_1)}{\langle K(F_1) \cap R_1 \rangle}$,

(ii) *If* $L \cong \frac{F_2}{R_2}$ *be a presentation for L, then* $\tilde{B}_0(L) \cong \frac{R_2 \cap F_2^2}{\langle K(F_2) \cap R_2 \rangle}$.

Definition 2.2. *Let C and* $\tilde{B}_0$ *be Lie algebras. We call a pair of Lie algebras* $(C, \tilde{B}_0)$, *a commutativity preserving defining pair (CP defining pair) for L, if*

$$L \cong C/\tilde{B}_0 \ , \ \tilde{B}_0 \subseteq Z(C) \cap C^2 \ , \ \tilde{B}_0 \cap K(C) = 0.$$

A pair $(C, \tilde{B_0})$ is called a maximal CP defining pair if the dimension of C is maximal.

Definition 2.3. *For a maximal CP defining pair $(C, \tilde{B_0})$, C is called a commutativity preserving cover or (CP cover) for L.*

Definition 2.4. *Let $c(L) = \{(C, \lambda) \mid \lambda \in Hom(C, L)\ ,\ \lambda$ surjective and $\ker\lambda \subseteq C^2 \cap Z(C)\ ,\ \ker\lambda \cap K(C) = 0\}$. (T, τ) is called a universal member in $c(L)$ if for each $(C, \lambda) \in c(L)$, there exists $h' \in Hom(T, C)$ such that $\lambda o h' = \tau$.*

Proposition 2.5. *Let L be a finite dimensional Lie algebra. Then (T, τ) is a universal element of $c(L)$ if and only if T is a CP cover.*

Proposition 2.6. *Let L be a finite dimensional Lie algebra, then all CP covers of L are isomorphic.*

3. Bogomolov multiplier and the Lazard correspondence

The section is devoted to show that the Bogomolov multiplier of a Lie ring L and a group G is isomorphic, when L is Lazard correspondent of G. Note that a Lie ring is termed a Lie algebra over that field. Also a Lie ring can be defined as a $\mathbb{Z}$-Lie algebra, and p-Lie ring is a Lie algebra over $\mathbb{Z}/p^k\mathbb{Z}$ for some positive integer k. Therefore more definitions and proofs of Lie rings can be obtained as generalizations from the Lie algebras, and there are similar results between finite Lie rings and finite dimensional Lie algebras over a field. **The Baker-Campbell-Hausdorff formula (B-C-H) and its inverse.** Let L be a p-Lie ring of order p^n and nilpotency class c with $p - 1 \geq c$ and G be a finite p-group with order p^n and the same nilpotency class c. For every $x, y \in L$, the B-C-H formula is a group multiplication in terms of Lie ring operations

$$xy := x + y + \frac{1}{2}[x, y]_L + \frac{1}{12}[x, x, y]_L + ...$$

The inverse g^{-1} of the group element g corresponds to $-g$. and the identity 1 in the group corresponds to 0 in the Lie ring. So, the B-C-H formula is used to turn Lie ring presentations into group presentations. Conversely the inverse B-C-H formula is a Lie ring addition and Lie bracket in terms of group multiplication that it is used to turn group presentations into Lie ring presentations. When $c \leq 14$, we have the general form

$$x + y := xy[x, y]_G^{\frac{-1}{2}} .... \quad , \quad [x, y]_L := [x, y]_G [x, x, y]_G^{\frac{1}{2}}.....$$

The Lazard correspondence. The B-C-H formula and it's inverse give an isomorphism between the category of nilpotent p-Lie rings of order p^n and the nilpotency class c, provided $p - 1 \geq c$ and the category of finite p-groups of the same order and nilpotency class which is known as the Lazard correspondence. By using this correspondence, in the same line of investigation, the same results on p-groups can be checked on p-Lie rings.

Proposition 3.1. *Let G be a finite p-group of class at most $p-1$, and L be its Lazard correspondent. Then every CP defining pair of G is in the Lazard correspondence with a CP defining pair of L and vice versa.*

Theorem 3.2. *Let G be a finite p-group of class at most $p-1$, and L be its Lazard correspondent. Then*

(i) *The isomorphism types of CP covers of G are in the Lazard correspondence with the isomorphism types of CP covers of L and vice versa.*

(ii) *The Bogomolov multipliers of G and L are isomorphic as abelian groups.*

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The Annihilator Graphs of Modules

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Abstract

In this paper, we introduce the annihilator graph of a module over a commutative ring with identity. We study the relations between algebraic properties of modules and graph properties of the annihilator graph. In particular, we study connectivity, girth and relations between the annihilator graph and the zero-divisor graph of a module.

Keywords and phrases: Zero divisor graph, Annihilator graph. .

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1. Introduction

Throughout, R is a commutative ring with nonzereo identity and M is an unital R-module. An element $x \in R$ is a zero-divisor if there exists a nonzero $y \in R$ such that $xy = 0$. We denote the set of zero-divisors of R as $Z(R)$, and the set of nonzero zero-divisors denoted by $Z(R)^*$. The zero-divisor graph of R, denoted by $\Gamma(R)$, is the graph with vertex set $Z(R)^*$, and for distinct elements $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. The study of the zero-divisor graph goes back to Beck, [5]. Recently, many different graphs on commutative rings have been studied by some authors, see [1, 8]. Badawi, in [2], introduced the annihilator graph of a commutative ring R, denoted AG(R). For $x \in R$, $\mathrm{ann_R(x)} = \{\mathrm{r} \in \mathrm{R} : \mathrm{rx} = 0\}$. AG(R) is a simple graph with the vertex set $Z(R)^*$ and for any two distinct elements $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $\mathrm{ann_R(x)} \cup \mathrm{ann_R(y)} \subset \mathrm{ann_R(xy)}$. Note that zero divisor graph $\Gamma(R)$ is a subgraph of annihilator graph AG(R).

As a generalization of the zero divisor elements of commutative rings to modules, Behboodi in [4], defined the set of zero divisor elements of modules. He defined three types of zero divisor elements, weak zero divisor, zero divisor and strong zero divisor. In the following, we focus on the set of weak zero divisor elements of M and as a generalization of the annihilator graph of a commutative rings [2], we define the annihilator graph of a module. Let M be an R-module and $m \in M$, the set of $\{r \in R : rM \subseteq Rm\}$ is denoted by I_m.

* speaker

Definition 1.1. *Let M be an R-module and $m \in M$. Then m is called an strong zero divisor of M, if $m = 0$ or $\operatorname{ann}(M) \subset I_m$ and there exists $0 \neq m' \in M$ that $\operatorname{ann}(M) \subset I_{m'} \subset R$ and $I_m I_{m'} M = 0$.*

For any R-module M, $Z^*(M)$ denote the set of all strong zero-divisors of M and $\tilde{Z}^*(M)$ denote the set of all non zero zero-divisors of M. By the above observation, Behboodi [4] associated a zero-divisor graph to a module that the vertices are the elements of $\tilde{Z}^*(M)$:

Definition 1.2. *Let M be an R-module. The zero-divisor graph of M, denoted $\Gamma^*(M)$, is the graph associated to M whose vertices are the elements of $\tilde{Z}^*(M)$, and two distinct vertices m, m' are adjacent if and only if $I_m I_{m'} M = 0$.*

Therefore, we can define the annihilator graph of modules as following:

Definition 1.3. *Let M be an R-module. The annihilator graph of M, denoted $\mathrm{AG(M)}$, is the graph associated to $Z^*(M)$ whose vertices are the elements of $\tilde{Z}^*(M)$, and distinct vertices m, m' are adjacent if and only if $\mathrm{ann(I_m I_{m'} M)} \neq \mathrm{ann(I_m M)} \cup \mathrm{ann(I_{m'} M)}$.*

In the following, we set up some definitions and notations of the modules and the simple graphs.

Throughout M is an R-module, $Z^*(M)$ is the set of the weak zero divisors elements of M and $\tilde{Z}^*(M) = Z^*(M) \setminus \{0\}$, $T(M) = \{m \in M : am = 0, \text{for some } 0 \neq a \in R\}$, $\operatorname{ann}(M)$ is the annihilators of M and $\sqrt{\operatorname{ann}(M)}$ is its radical ideal. For a submodule N of M $\mathrm{ann(N)} = \{\mathrm{r \in R : rN = 0}\}$. An R-module M is called a multiplication module if for any $m \in M$, $Rm = I_m M$, where $I_m = \operatorname{ann}(M/Rm)$, see [7]. Let $m \in M$, m is called a torsion element if $\mathrm{ann(m)} = \{\mathrm{r \in R : rm = 0}\}$ is not zero ideal and the set of all torsion elements of M is denoted by $T(M)$.

Let m and m' be two distinct vertices of a simple graph G. If m and m' are adjacent, then it is denoted by $m - m'$ and it is called an edge of G. A graph is called connected if there is a path between any two distinct vertices. For a vertex m of G, the set of all vertices that are adjacent with m is denoted by $N_G(m)$. The diameter and girth of a connected graph are denoted by $diam(G)$ and $gr(G)$, respectively. A complete bipartite graph is a graph G which its vertex set may be partitioned into two disjoint nonempty vertex sets V_1 and V_2 such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. If $\mid V_1 \mid = n$ and $\mid V_2 \mid = n$, the complete bipartite graph is denoted by $K^{m,n}$. If $\mid V_1 \mid = 1$ or $\mid V_2 \mid = 1$, then we call G a star graph.

Our main purpose is to compare the theoretical properties of zero-divisor graph of modules and annihilator graph of modules and to establish the some important graph theory properties of the annihilator graphs of a module. In second section, we show that AG(M) is a connected graph with $diam(\mathrm{AG(M)}) \leq 2$. we show that $gr(\mathrm{AG(M)} \leq 4$, whenever AG(M) contains a cycle. In the third section, we determine when AG(M) is identical to $\Gamma(M)$.

For notations and terminologies not given in this paper, the reader is referred to [6, 9].

2. Main Results

In this section, the properties of adjacent vertices in the annihilator graph will be studied. We determine the diameter and the girth of these graph. Also we specify the annihilator graph of a module, when its girth is not 3.

Proposition 2.1. *Let M be an R-module and m and m' be distinct elements of $\tilde{Z}^*(M)$. Then*

(i) *m, m' are not adjacent in* $\mathrm{AG(M)}$ *if and only if either* $\mathrm{ann(I_mI_{m'}M)} = \mathrm{ann(I_mM)}$ *or* $\mathrm{ann(I_mI_{m'}M)} = \mathrm{ann(I_{m'}M)}$.

(ii) *If m, m' are not adjacent in* $\mathrm{AG(M)}$, *then* $\mathrm{ann(I_mM)} \subseteq \mathrm{ann(I_{m'}M)}$ *or* $\mathrm{ann(I_{m'}M)} \subseteq \mathrm{ann(I_mM)}$.

(iii) *If* $\mathrm{ann(I_mM)} \nsubseteq \mathrm{ann(I_{m'}M)}$ *and* $\mathrm{ann(I_{m'}M)} \nsubseteq \mathrm{ann(I_mM)}$, *then $m - m'$ is an edge of* $\mathrm{AG(M)}$.

(iv) *If $d_{\Gamma^*(M)}(m, m') = 3$, then $m - m'$ is an edge of* $\mathrm{AG(M)}$.

Proposition 2.2. *Let M be an R-module. Assume that m and m' are distinct elements of $\tilde{Z}^*(M)$. Then*

(i) *If $m - m'$ is an edge of $\Gamma^*(M)$, then $m - m'$ is an edge of* $\mathrm{AG(M)}$. *Thus $\Gamma^*(M)$ is a subgraph of* $\mathrm{AG(M)}$.

(ii) *If m, m' are not adjacent vertices in* $\mathrm{AG(M)}$, *then there exists $m'' \in \tilde{Z}^*(M)$ such that $m'' \notin \{m, m'\}$ and $m - m'' - m'$ is a path in $\Gamma^*(M)$ and hence in* $\mathrm{AG(M)}$.

(iii) *If I_m and $I_{m'}$ are nilpotent ideals of R. Then $m - m'$ is an edge of* $\mathrm{AG(M)}$.

Theorem 2.3. *Let M be an R-module. Then* $\mathrm{AG(M)}$ *is a connected graph with diam*$(\mathrm{AG(M)}) \leq 2$.

PROOF. It follows from Proposition 2.2 (ii). □

Theorem 2.4. *Let M be an R-module. Then $gr(\mathrm{AG(M)}) \in \{3, 4, \infty\}$.*

Theorem 2.5. *Let M be an R-module such that* $\sqrt{\mathrm{ann(M)}} = \mathrm{ann(M)}$. *If $gr(\mathrm{AG(M)}) = 4$, then $\mathrm{AG(M)} = \mathrm{K^{n,m}}$, where $n, m \geq 2$.*

Lemma 2.6. *Let M be an R-module. Assume that $gr(\mathrm{AG(M)}) = \infty$. Then*

(i) $\mathrm{AG(M)}$ *is an star graph.*

(ii) $\Gamma^*(M) = \mathrm{AG(M)}$.

3. When the Annihilator Graph and the Zero-divisor Graph of a Module Are the Same

A proper ideal I of R is called 2-absorbing if whenever $abc \in I$ for $a, b, c \in R$, then $ab \in I$ or $bc \in I$ or $ac \in I$, see [3]. In this section, we determine some module that the annihilator graph and the zero divisor graph of these modules are the same.

Proposition 3.1. *Let M be an R-module such that* $\mathrm{ann(M)}$ *is a 2-absorbing ideal of R. Then* $\Gamma^*(M) = \mathrm{AG(M)}$.

Proposition 3.2. *Let $m \in \tilde{Z}^*(M)$ such that* $\mathrm{ann(I_m M)}$ *be a prime ideal of R. Then* $N_{\Gamma^*(M)}(m) = N_{\mathrm{AG(M)}}(m)$.

Theorem 3.3. *Let M be an non cyclic multiplication module that $M \neq T(M)$. If* $N_{\Gamma^*(M)}(m) = N_{\mathrm{AG(M)}}(m)$, *then either* $\sqrt{\mathrm{ann(I_m)}} = \mathrm{ann(I_m)}$ *or* $\sqrt{\mathrm{ann(I_m)}} = \mathrm{Nil(R)}$.

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Sums of Sylow numbers of finite groups

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Abstract

Let G be a finite group, $n_p(G)$ be the number of Sylow p-subgroups of G, and $\pi(G)$ be the set of prime divisors of $|G|$. We set $S(G) = \{p \in \pi(G) | n_p(G) > 1\}$ and define $\delta_0(G) = \sum_{p \in S(G)} n_p(G)$. In [1], the authors worked on $\delta_0(G)$, with small $\delta_0(G)$. Continuing [1], our further investigation show that if $\delta_0(G) < 57$, then G is solvable or $G/N \cong A_5$ or $G/N \cong S_5$, where N is the largest normal solvable subgroup of G.

Keywords and phrases: Sylow number, nonsolvable group. .
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1. Introduction

In this paper, all groups under consideration are finite. Denoted by $\pi(G)$, the set of prime divisors of the order of group G, and $n_p(G)$, the number of Sylow $p-$subgroups of G. All further unexplained notations are standard, and can be found in [1] (henceforth to be referred to as I).

In I, we defined the sum of the Sylow numbers and were able to use it to obtain information about the structure of finite groups. In this paper, we will continue to work and get new results.

We first need to recall some of the concepts from I. Let $S(G) = \{p \in \pi(G) | n_p(G) > 1\}$. We defined the sum of Sylow number of G as $\delta_0(G) = \sum_{p \in S(G)} n_p(G)$. By the third Sylow's theorem, we see if $p \in S(G)$, then $n_p(G) \geq 1 + p$. If G is a nonabelian simple group, then for every $p \in \pi(G)$, $n_p(G) \geq 1 + p$. Our main result is the following.

Theorem 1.1. *If G is a finite nonsolvable group with $\delta_0(G) < 29$, then $G/N \cong A_5$, where N is the largest normal solvable subgroup of G. Furthermore, if $Z(G) = 1$ then $G \cong A_5$.*

* speaker

There is probably a number greater than 29 such as m such that $\delta_0(G) < m$ and we still get $G/N \cong A_5$. Although this is still open, but we pose the following theorem. We will prove it in a slightly different way.

Theorem 1.2. *If G is a finite nonsolvable group with $\delta_0(G) < 57$, then $A_5 \leq G/N \leq S_5$, where N is the largest normal solvable subgroup of G.*

We need the following lemma to prove the theorems.

Lemma 1.3. *[3, Lemma 1] Let G be a group and N be a normal subgroup of G. Then $n_p(N)n_p(G/N) \mid n_p(G)$ for every prime p.*

2. Proof of Theorems

Proof of Theorem 1.1. Let T be a nonabelian composition factor of G. First, let $n_p(T) \geq 8$ for every prime divisor p of $|T|$. Since T is a nonsolvable group, by Feit-Thompson's theorem $2 \in \pi(T)$. Thus, by Sylow's Theorem $n_2(T) \geq 9$. Let $p, q \in \pi(T)\backslash\{2\}$. Then $n_p(T) \geq 10$ and $n_q(T) \geq 10$ by Sylow's Theorem. Hence, $\delta_0(T) \geq 29$, which is a contradiction. Therefore, $n_p(T) < 8$ for some $p \in \pi(T)$.

Assume that P is a Sylow p-subgroup of T. Then $|T : N_T(P)| = n_p(T) \leq 7$. So, $N_T(P) = H$ is a proper subgroup with $|T : H| \leq 7$. Now, T acts on $\Omega = \{Hx | x \in T\}$. For all $g \in T$ the map $\varphi_g : Hx \to Hxg$ is a permutation of Ω. Moreover, the map $\varphi_g : Hx \to Hxg$ is a homomorphism $T \to Sym(\Omega)$. Since T is a simple group, the kernel of this homomorphism is trivial. Thus, T is isomorphic to a subgroup of S_7, and so T is isomorphic to a subgroup of A_7. Hence, $T \cong A_5$, A_6, A_7, or PSL(2,7). If $T \cong A_6$, A_7, or PSL(2,7), then $\delta_0(T) \geq 29$, which is a contradiction. Therefore, $T \cong A_5$.

Suppose there exists a prime $r \geq 7$ such that $r \mid |G|$. If $n_r(G) > 1$, then by Sylow's theorem $n_r(G) \geq 1 + r \geq 8$. Since $n_2(G) \geq 5$, $n_3(G) \geq 10$, $n_5(G) \geq 6$, we have $\delta_0(G) \geq 21 + 8 = 29$, which is a contradiction. It follows that $n_r(G) = 1$ for every prime $r \geq 7$.

Since G is a finite group, it has a chief series. Suppose that

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \triangleleft N_{r-1} \trianglelefteq N_r = G$$

is a chief series of G. Since G is a nonsolvable group, there exists a maximal non-negative integer i such that N_i/N_{i-1} is a simple group or a direct product of isomorphic simple groups and N_{i-1} is a maximal solvable normal subgroup of G. Now, set $N_i := H$ and $N_{i-1} := N$. Hence, G has the normal series $1 \trianglelefteq N \triangleleft H \trianglelefteq G$ such that H/N is a direct product of isomorphic simple groups. By the above discussion H/N is a direct product of copies of A_5. By Lemma 1.3 and $\delta_0(G) < 29$, we have $H/N \cong A_5$. Set $\overline{H} := H/N \cong A_5$ and $\overline{G} := G/N$. Hence,

$$A_5 \cong \overline{H} \cong \overline{H}C_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \mathrm{Aut}(\overline{H}).$$

If $K = \{x \in G | \; xN \in C_{\overline{G}}(\overline{H})\}$, then $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$. So

$$A_5 \leq G/K \leq \mathrm{Aut}(A_5) \cong S_5.$$

Therefore, $G/K \cong A_5$ or $G/K \cong S_5$.

Suppose that $G/K \cong S_5$. We know that $n_2(S_5) = 15$, $n_3(S_5) = 10$ and $n_5(S_5) = 6$. By Lemma 1.3, $n_2(S_5) = 15 \mid n_2(G)$, $n_3(S_5) = 10 \mid n_3(G)$ and $n_5(S_5) = 6 \mid n_5(G)$, a contradiction. Therefore, $G/K \cong A_5$.

We show that $K = N$. Suppose that $K \neq N$. By Lemma 1.3, and the assumption $\delta_0(G) < 29$, $n_p(K) = 1$ for every prime $p \in \pi(G)$, so K is a nilpotent subgroup of G. Since $C_{\overline{G}}(\overline{H}) \cong K/N$ and N is a maximal solvable normal subgroup of G, K is a nonsolvable normal subgroup of G, a contradiction. Thus, $K = N$, and so $G/N \cong A_5$.

Now, let $Z(G) = 1$. We show that $N = 1$. Assume that $N \neq 1$. Let $R \in \mathrm{Syl}_r(N)$ for some $r \in \pi(N)$ and let $P \in \mathrm{Syl}_p(G)$ for some $p \in \{2,3,5\} \setminus \{r\}$. Since $R \trianglelefteq G$, we have $P \leq N_G(R) = G$. So, P normalizes R. If $N \nleq N_G(P)$, then

$$\begin{aligned} n_p(G) &= |G : N_G(P)| > |G/N : N_G(P)N/N| \\ &= |G/N : N_{G/N}(PN/N)| = n_p(G/N). \end{aligned}$$

On the other hand, $n_p(G/N) \mid n_p(G)$. So, if $p = 2$, then $n_p(G) \geq 15$, if $p = 3$, then $n_p(G) \geq 40$, and if $p = 5$, then $n_p(G) \geq 36$. Since $\delta_0(G) < 29$, we get a contradiction. Hence, $R \leq N \leq N_G(P)$. Now, we have $P \leq N_G(R)$ and $R \leq N_G(P)$, it follows that $[P,R] = 1$. Thus, $R \leq C_G(P)$. Since G/N is generated by its Sylow p-subgroups, it follows that $Z(R) \leq Z(G) = 1$, a contradiction. Therefore, $G = A_5$. □

Proof of Theorem 1.2. Since G is a finite nonsolvable group, it has the normal series $1 \trianglelefteq N \triangleleft H \trianglelefteq G$ such that H/N is a direct product of isomorphic simple groups, and N is a maximal solvable normal subgroup of G. We show that $H/N \cong A_5$. By [1, Corollary 1.8.], there exists a prime $p \in \pi(G)$ such that $n_p(H/N)^2 > |H/N|$. Since $\delta_0(G) < 57$ and $n_p(H/N) \leq n_p(G)$, we have $n_p(H/N) < 57$. Hence, $|H/N| < 57^2 = 3249$. But by [2], the simple non-abelian groups of order less than 3249 are: A_5, A_6, A_7, PSL(2,7), PSL(2,8), PSL(2,11), PSL(2,13), and PSL(2,17). It is easy to check that $\delta_0(T) \geq 57$, when $T = A_6$, A_7, PSL(2,7), PSL(2,8), PSL(2,11), PSL(2,13), and PSL(2,17). Therefore, $H/N \cong A_5$. Now, if we set $\overline{H} := H/N \cong A_5$ and $\overline{G} := G/N$, then

$$A_5 \cong \overline{H} \cong \overline{H}C_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \mathrm{Aut}(\overline{H}).$$

Put $K = \{x \in G | \; xN \in C_{\overline{G}}(\overline{H})\}$, so $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$. Hence,

$$A_5 \leq G/K \leq \mathrm{Aut}(A_5) \cong S_5.$$

Let $G/K \cong S_5$. We show that $K = N$. Suppose that $K \neq N$. Since $C_{\overline{G}}(\overline{H}) \cong K/N$ and N is a maximal solvable normal subgroup G, K is a nonsolvable normal subgroup of G. By Lemma 1.3, $n_2(S_5) = 15 \mid n_2(G)$, $n_3(S_5) = 10 \mid n_3(G)$ and $n_5(S_5) = 6 \mid n_5(G)$. Also, by Lemma 1.3, $n_p(K)n_p(S_5) \mid n_p(G)$ for every prime $p \in \pi(G)$. On the other hand, $\delta_0(G) < 57$, so $n_2(K) = 1$. Since K is a finite nonsolvable group, it has the normal series $1 \trianglelefteq N_1 \triangleleft H_1 \trianglelefteq K$ such that H_1/N_1 is a direct product of isomorphic simple groups. It follows that $n_2(K) > 1$, a contradiction. Therefore, $K = N$.

Let $G/K \cong A_5$. We also in this case show that $K = N$. Suppose that $K \neq N$. It follows that K is a nonsolvable normal subgroup of G. By Lemma 1.3, $n_2(A_5) = 5 \mid n_2(G)$, $n_3(A_5) = 10 \mid n_3(G)$ and $n_5(A_5) = 6 \mid n_5(G)$. On the other hand, by Lemma 1.3, $n_p(K)n_p(A_5) \mid n_p(G)$ for every prime $p \in \pi(G)$. Since $\delta_0(G) < 57$, we have $n_2(K) \in \{1, 3, 5\}$, $n_3(K) = 1$, and $n_5(K) = 1$.

First, assume that $n_2(K) = 1$. Arguing as S_5, $K = N$.

Next, let $n_2(K) = 3$, or 5. By Lemma 1.3, $\delta_0(K) < 57$. Since K is a finite nonsolvable group, it has the normal series $1 \trianglelefteq N_1 \triangleleft H_1 \trianglelefteq K$ such that H_1/N_1 is a direct product of isomorphic simple groups. Similar to the above discussion, $H_1/N_1 \cong A_5$. It follows that $n_3(K) > 1$, a contradiction. Therefore, $K = N$.

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The Schröder-Bernstein Theorem for the class of Baer modules

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Abstract

The main objective of this work is to study the Schröder-Bernstein property (shortly SB property) for the class of Baer modules. Our motivation comes from Kaplansky's Theorem showing that any Baer ⋆-ring satisfies the SB property. Examples which illustrate our results are provided.

Keywords and phrases: Baer, direct summand, extending, Rickart, subisomorphism..
2010 *Mathematics subject classification:* Primary: 16D80, 16E50; Secondary: 16D99.

1. Introduction

Throughout this paper, we assume that R is an associative ring (not necessarily commutative) with unity. All modules are right and unital. Let M be an R-module. The notations $N \subseteq M$, $N \leq M$, or $N \leq_{\oplus} M$ mean that N is a subset, a submodule, or a direct summand of M, respectively. $\mathrm{End}_R(M)$ is the ring of R-endomorphisms of M. The notations $M^{(A)}$ and M^A mean $\oplus_{i\in A} M_i$ and $\prod_{i\in A} M_i$, respectively, where A is an index set and each $M_i \simeq M$. The annihilator of an element $m \in M$ will be denoted by $\mathrm{ann}_R(m)$. For other terminology and results, we refer the reader to [1] and [5].
The famous Schröder-Bernstein Theorem states that any two sets with one to one maps into each other are isomorphic. The question of whether two subisomorphic algebraic structures are always isomorphic to each other has been of interest to a number of researchers. Bumby in 1965 [2], showed that any two injective modules which are subisomorphic to each other are isomorphic. For abelian groups, Kaplansky in 1954 [4, p.12], posed the following question, also known as Kaplansky's First Test Problem: "If G and H are abelian groups such that each one is isomorphic to a direct summand of the other, are G and H necessarily isomorphic?" Negative answers have been given to this question by several authors. Besides, Kaplansky in 1968 [5, Theorem 41], showed that every Baer ⋆-ring satisfies this analogue of the Schröder-Bernstein Theorem. Recall that a ring R with an involution ⋆ is called a *Baer ⋆-ring* if the right annihilator of every nonempty subset of R is generated by a projection e (the idempotent e of the ⋆-ring R is called a *projection* if $e^\star = e$). In particular he proved the following result:

* speaker

Theorem 1.1. *[5,* Theorem 41*] Let R be a Baer $\star$-ring and e, f be projections in R. If eR is isomorphic to a direct summand of fR and fR is isomorphic to a direct summand of eR then eR is isomorphic to fR.*

Following [3], an R-module M is called to *satisfy the Schröder-Bernstein property(or SB property)* if any two d-subisomorphic direct summands of M are isomorphic (the R-modules N and K are called *d-subisomorphic to each other* whenever N is isomorphic to a direct summand of K and K is isomorphic to a direct summand of N). Moreover, a subclass C of R-modules is called to *satisfy the SB property* provided that any pair of members are isomorphic whenever they are d-subisomorphic to each other. By Kaplansky's Theorem, every Baer $\star$-ring satisfies the SB property.
Kaplansky in 1968 [5], introduced the notion of Baer rings. Recall that a ring R is called *Baer* if the right annihilator of any nonempty subset of R is generated by an idempotent. It is easy to observe that the Baer property is left and right symmetric for any ring. The notion of a Baer ring was extended to modules. An R-module M is called *Baer* if for all $N \leq M$, $\text{ann}_S(N)$ is a direct summand of S where $S = \text{End}_R(M)$ [1, Chapter 4]. Clearly R is a Baer ring if R_R is Baer. An R-module M is called *Rickart* if $\text{Ker} f = \text{ann}_M(f) \leq_{\oplus} M$ where $S = \text{End}_R(M)$ and $f \in S$. A ring R is called *right (left) Rickart* if R_R ($_RR$) is Rickart. Clearly every Baer module is Rickart. For more details see [1, Chapter 3].
Now what Kaplansky proved for Baer $\star$-rings (Theorem 1.1), motivated us to ask "*when any pair of subisomorphic or d-subisomorphic Baer modules are isomorphic to each other*".
In this paper, first we give several examples to show that subisomorphic Baer modules are not necessarily isomorphic to each other (Examples 2.2 and 2.3). We also show that two Rickart modules which are d-subisomorphic to each other are not isomorphic in general. In the main theorem, we prove that if M_R is Baer and the set of all idempotents in $\text{End}_R(M)$ forms a complete lattice then M_R satisfies the SB property.

2. Main Results

We recall that a $\star$-ring (or ring with involution) is a ring with an involution $x \to x^\star$ such that $(x^\star)^\star = x$, $(x+y)^\star = x^\star + y^\star$ and $(xy)^\star = y^\star x^\star$. A ring R with an involution $\star$ is called *a Baer $\star$-ring* if the right annihilator of every nonempty subset of R is generated by a projection. It is clear that Baer $\star$-rings are Baer rings. By Theorem 1.1, it is known that every Baer $\star$-ring satisfies the SB property. Therefore it is natural to ask ourselves whether Baer rings do satisfy the SB property. So we will be concerned with the question of when any two Baer modules which are subisomorphic or direct summand subisomorphic to each other are necessarily isomorphic. We begin with the following basic definitions in this study.
An R-module M is called *extending* if every submodule of M is essential in a direct summand of M_R. An R-module M is called *nonsingular* if $mI = 0$ implies that $m = 0$ where $m \in M$ and I is an essentail right ideal of R. A ring R is called *right nonsingular* provided that R_R is nonsingular. Next result from [1] is needed for latter uses.

Theorem 2.1. *[1,* Theorem 3.3.1*] every nonsingular extending module is Baer.*

In the following, we give some examples to show that any two subisomorphic Baer modules are not necessarily isomorphic.

Example 2.2. *Let R be a commutative domain which is not PID and I be any non principal ideal of R. By Theorem 2.1, R, I are Baer R-modules. Clearly they are subisomorphic to each other while $R \ncong I$.*

In the following, we show that even if N and K are Baer R-modules with the stronger condition: "N is isomorphic to a submodule of K and K is isomorphic to a direct summand of N", N is not isomorphic to K in general.

Example 2.3. *Let $N = \mathbb{Q}^{(\mathbb{N})} \oplus \mathbb{Z}$ and $K = \mathbb{Q}^{(\mathbb{N})}$. By Theorem 2.1, $K_{\mathbb{Z}}$ is Baer. Besides, $N_{\mathbb{Z}}$ is also Baer [1,* Theorem 4.2.18*]. Moreover, it is clear that $K \leq_{\oplus} N$ and N is isomorphic to a submodule of K, however, N is not isomorphic to K.*

We recall that a ring R is called *(von-Neumann) regular* provided that for each $r \in R$, $r \in rRr$. It is well known that regular rings R are precisely the ones whose every principal (finitely generated) right ideals are direct summands. The following result was shown in [6]:

Theorem 2.4. *[6,* Theorem 4*] Let M be an R-module and $S = \mathrm{End}_R(M)$. Then S is a regular ring if and only if for each $\varphi \in S$, $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$ are direct summands of M.*

By the above theorem, every R-module M, with the regular endomorphism ring is a Rickart module. In the following proposition, we show that Rickart modules N and K are d-subisomorphic to each other if and only if they are epimorphic images of each other.

Proposition 2.5. *Let N and K be Rickart modules. Then N and K are d-subisomorphic to each other if and only if there are R-epimorphisms $N \to K$ and $K \to N$.*

Corollary 2.6. *If N and K are Baer R-modules then N and K are d-subisomorphic to each other if and only if they are epimorphic images of each other.*

In the next example, we show that two Rickart modules which are d-subisomorphic to each other are not necessarily isomorphic.

Example 2.7. *Suppose that V is an infinite dimensional vector space over a field F with $S = \mathrm{End}_F(V)$. Let $\beta = \{v_i\}_{i \in I}$ be a basis for V_F and $R := \{(f, g) \in S \times S \mid rank(f - g) < \infty\}$. Clearly R is a subring of $S \times S$. We note that R is a regular ring and so R_R is Rickart. There exist idempotents e and g in R such that eR and gR are d-subisomorphic to each other however eR is not isomorphic to gR (see [3,* Example 2.2*] for more details). Since every direct summand of a Rickart module has the property [1,* Proposition 4.5.4*], eR and gR are Rickart R-module. Therefore the Rickart module R_R does not satisfy the SB property.*

Regarding examples 2.2, 2.3, 2.7, and Theorem 1.1 about Baer ⋆-rings, it is natural to ask the question: *"does any Baer module satisfy the SB property?"*
It is clear that any Baer module satisfies the SB property if and only if any pair of Baer modules which are d-subisomorphic to each other are isomorphic. In order to answer this question, we note that the main point in the proof of Theorem 1.1, is that the set of all projections in a Baer ⋆-ring forms a complete lattice under " $\leq$ " (if e, f are idempotents in a ring R, we write $e \leq f$ in case $ef = fe = e$, i.e., $e \in fRf$). While in a Baer ring, the set of all right ideals generated by idempotents forms a complete lattice [1, Theorem 3.1.23].
In the following Theorem, we prove that any Baer module with idempotents in its endomorphism ring forming a complete lattice has the SB property.

Theorem 2.8. *Let M be an R-module and $S = \mathrm{End}_R(M)$. If the set of all idempotents in S is a complete lattice with respect to the ordering $e \leq f$ then M satisfies the SB property.*

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Some results on 15-valent 2-arc-transitive graphs

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Abstract

Let X be a connected (G,s)-transitive graph of valency 15 for some $s \geq 2$ and $G \leq Aut(X)$. In this paper, we give a characterization of the vertex-stabilizer G_v when $G^*_{uv} = 1$.

Keywords and phrases: Arc-transitive graph, 2-arc-transitive graph, (G,s)-transitive graph, vertex-stabilizer.

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1. Introduction

In this paper, all graphs are finite, undirected and simple, i.e without loops or multiple edges. For a graph X, we use $V(X)$, $E(X)$ and $Aut(X)$ to denote its vertex set, edge set and full automorphism group, respectively. For $u,v \in V(X)$, $\{u,v\}$ is the edge incident to u and v in X. The set of all vertices adjacent to v is denoted by $X_1(v)$. Let $G \leq Aut(X)$. We denote the vertex-stabilizer of $v \in V(X)$ in G by G_v. Denote by $G_v^{X_1(v)}$ the constituent of G_v acting on $X_1(v)$ and by G^*_v the kernel of G_v acting on $X_1(v)$. Then $G_v^{X_1(v)} \cong G_v/G^*_v$. For an edge $\{u,v\} \in E(X)$, we write $G_{uv} = G_u \cap G_v$ and $G^*_{uv} = G^*_u \cap G^*_v$.

For each integer $s \geq 0$, an *s-arc* of X is an $(s+1)$-tuples $(v_0,v_1,...,v_{s-1},v_s)$ of vertices such that $\{v_{i-1},v_i\} \in E(X)$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. If $G \leq Aut(X)$ is transitive on the set of s-arcs, then X is called (G,s)*-arc-transitive*; while if in addition G is not $(G,s+1)$-arc-transitive, then X is called (G,s)*-transitive*. A graph X is called *s-arc-transitive* or *s-transitive* if it is $(Aut(X),s)$-arc-transitive or $(Aut(X),s)$-transitive, respectively. In particular, X is called *vertex-transitive* or *symmetric* if it is $(Aut(X),0)$-arc-transitive or $(Aut(X),1)$-arc-transitive, respectively.

As we all know a graph X is (G,s)-arc-transitive if and only if G is transitive on $V(X)$ and G_v is transitive on the set of s-arcs with initial vertex v. So the structure of G_v plays an important role in the study of such graphs. Interest in s-transitive graphs stems from a beautiful result of Tutte [5] in 1947 who

* speaker

proved that for any s-transitive cubic graph, $s \leq 5$. Tutte's Theorem was generalized in 1981 by Weiss [7] who proved that there exist no finite s-transitive graph for $s = 6$ and $s \geq 8$. Note that the only connected graphs of valency two are cycles which are s-arc-transitive for any positive integer s. So the valency of a s-transitive graph is greater than 2. Let X be a connected (G,s)-transitive graph. Up to now, we know the structure of G_v when X has prime or twice a prime valency [3, 4]. Furthermore, It is a well-known result that when the valency of X is prime or $s \geq 2$, the order of G_v is bounded above.

Let p a prime and n a positive integer. We denote by n the cyclic group of order n, by p^n the elementary abelian group of order p^n, by A_n and S_n the alternating group and the symmetric group of degree n. For two groups M and N, we denote by $N.M$ an extension of N by M and $N:M$ stands for a semidirect product of N by M.

All the notation and terminology used throughout this paper are standard. For group and graph theoretic concepts not defined here, we refer the reader to [1, 2].

The following proposition is about sufficient and necessary conditions for symmetric graphs.

Proposition 1.1. *Let X be a graph and $G \leq Aut(X)$. Then we have;*

(i) *X is G-arc-transitive if and only if X is G-vertex-transitive and the vertex-stabilizer G_v is transitive on $X_1(v)$ for each $v \in V(X)$.*

(ii) *X is $(G,2)$-arc-transitive if and only if X is G-vertex-transitive and G_v is 2-transitive on $X_1(v)$ for each $v \in V(X)$.*

The proof of the next lemma is straightforward

Lemma 1.2. *Let X be a (G,s)-arc-transitive graph for some $G \leq Aut(X)$ and $s \geq 1$. Let $\{u,v\} \in E(X)$. Then we have;*

(i) $G_v \cong G_v^*.G_v^{X_1(v)} \cong (G_{uv}^*.G_v^{*X_1(u)}).G_v^{X_1(v)}$.

(ii) $G_v^{*X_1(u)} \trianglelefteq G_{uv}^{X_1(u)} \cong G_{uv}^{X_1(v)}$.

We formulate the following lemma from [6–8].

Lemma 1.3. *Let X be a connected (G,s)-transitive graph with $s \geq 2$ and let $\{u,v\} \in E(X)$. Then one of the following holds:*

(i) *$s \leq 3$, $G_{uv}^* = 1$ and $G_v^* \cong G_v^{*X_1(u)} \trianglelefteq G_{uv}^{X_1(u)} \cong G_{uv}^{X_1(v)}$.*

(ii) *G_{uv}^* is a nontrivial p-group, $PSL_d(q) \trianglelefteq G_v^{X_1(v)}$, $q = p^r$ and $|X_1(v)| = \frac{q^d-1}{q-1}$.*

In view of ([1], Appendix B), we have the following observation.

Proposition 1.4. *Let H be a 2-transitive group of degree 15. Then $H \cong A_7, PSL_4(2), A_{15}$ or S_{15}.*

2. Main Results

In this section, we give our main result as follows.

Theorem 2.1. *Let X be a finite connected (G,s)-transitive graph of valency 15 for some $G \leq Aut(X)$ and $s \geq 2$. Let $\{u,v\} \in E(X)$ and $G^*_{uv} = 1$. Then $s \leq 3$ and one of the following holds:*

(i) $s = 2$, $G_v \cong A_7$, $PSL_4(2)$, A_{15} *or* S_{15}.

(ii) $s = 3$, $G_v \cong A_7 \times PSL_2(7)$, $PSL_4(2) \times (2^3 : PSL_3(2))$, $A_{15} \times A_{14}$, $S_{15} \times S_{14}$ *or* $(A_{15} \times A_{14}) : 2$ *with* $A_{15} : 2 = S_{15}$ *and* $A_{14} : 2 = S_{14}$.

PROOF. Let X be a connected (G,s)-transitive graph of valency 15 for some $G \leq Aut(X)$ and $s \geq 2$. Let $v \in V(X)$. By Proposition 1.1, we get that $G_v^{X_1(v)}$ is a 2-transitive permutation group of degree 15. So by Proposition 1.4, $G_v^{X_1(v)} \cong A_7, PSL_4(2), A_{15}$ or S_{15}. Suppose that $G^*_{uv} = 1$. Then by Lemma 1.3, $s \in \{2,3\}$ and G^*_v is isomorphic to a normal subgroup of a vertex-stabilizer of a permutation group $G_v^{X_1(v)}$. Assume that $G^*_v = 1$. Then $G_v \cong A_7$, $PSL_4(2)$, A_{15} or S_{15}. Thus, in what follows we may assume that $G^*_v \neq 1$.

Suppose that $G_v^{X_1(v)} \cong A_7$. Then $G^*_v \cong PSL_2(7)$ and $G_v \cong A_7 \times PSL_2(7)$.

Suppose that $G_v^{X_1(v)} \cong PSL_4(2)$. Then we get that $G^*_v \cong 2^3 : PSL_3(2)$ and $G_v \cong PSL_4(2) \times (2^3 : PSL_3(2))$.

Suppose that $G_v^{X_1(v)} \cong A_{15}$. Then $G^*_v \cong A_{14}$ and $G_v \cong A_{15} \times A_{14}$.

Suppose that $G_v^{X_1(v)} \cong S_{15}$. Then $G^*_v \cong A_{14}$ or S_{14}. For the former, $G_v \cong (A_{15} \times A_{14}) : 2$ with $A_{14} : 2 = S_{14}$ and $A_{15} : 2 = S_{15}$. For the latter, $G_v \cong S_{15} \times S_{14}$.

Finally, it is easy to see that $s = 2$ for $G^*_v = 1$, otherwise $s = 3$. □

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GCP-graphs

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Abstract

A GCP-graph is a suitable generalization of the Cayley graph where the vertices are elements of a polygroup. We survey some important properties on GCP-graphs in order to answer this question: which simple graph is a GCP-graph?

Keywords and phrases: Simple graph; Caylay graph; graph product; polygroup; GCP-graph .
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1. Introduction

The theory of algebraic hyperstructures which is a generalization of the concept of ordinary algebraic structures first was introduced by Marty. Since then many researchers have worked on algebraic hyperstructures and developed it. A short review of this theory appears in [3]. Application of hypergroups have mainly appeared in special subclasses. For example, polygroups which are certain subclasses of hypergroups are studied by Ioulidis and are used to study color algebra [1, 2]. Quasi-canonical hypergroups (called "polygroups" by Comer) were introduced as a generalization of canonical hypergroups.There exists a rich bibliography on polygroups [4]. This book contains the principal definitions endowed with examples and the basic results of the theory.

Cayley graphs were first introduced by Cayley as diagrams representing a group in terms of its generators. Cayley graphs, both in their directed and undirected form have been widely studied.

A connection between hyperstructure theory and graphs was found in 2019 when Heidari et al. [5, 6] studied the concept of generalized Cayley graphs over polygroups.

2. Main Results

In this section, we mention to the suitable generalization of the Cayley graph where the vertices are elements of a polygroup and introduce to some properties of them.

* speaker

Definition 2.1. [5] *Let* $\mathbf{P} = \langle P, \circ, 1, ^{-1} \rangle$ *be a polygroup and* S*, say the connection set, be a non-empty inverse closed subset (i.e.* $S^{-1} = S$*) of* P*. Then we define the generalized Cayley graph* $\mathrm{GCP}(\mathbf{P}; S)$ *as a simple graph with vertex set* P *and edge set*

$$E = \{\{x, y\} \mid x \neq y \text{ and } x \circ y^{-1} \cap S \neq \emptyset\}.$$

A graph Λ *is called a* GCP-graph if there exists a polygroup $\mathbf{P}$ and a connection set S such that $\Lambda \cong \mathrm{GCP}(\mathbf{P}; S)$.

Example 2.2. *The generalized Cayley graph of the polygroup* $\mathbf{P} = \langle \{1, 2, 3, 4\}, \circ, 1, ^{-1} \rangle$ *and connection set* $\{3, 4\}$ *is shown in Figure 1.*

$\circ$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	$\{1,3\}$	$\{2,4\}$
4	4	3	$\{2,4\}$	$\{1,3\}$

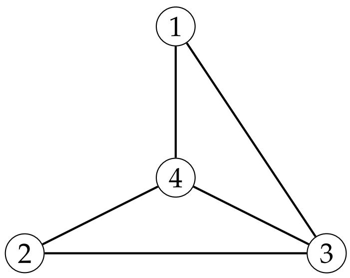

FIGURE 1. $\mathrm{GCP}(\mathbf{P_2}; \{3, 4\})$

The necessary and sufficient condition that a GCP-graph over a polygroup be connected is same as in Cayley graphs. In other words:

Theorem 2.3. [5] *Let* $\mathbf{P} = \langle P, \circ, 1, ^{-1} \rangle$ *be a polygroup and* S *be a connection set. Then, the generalized Cayley graph* $\mathrm{GCP}(\mathbf{P}; S)$ *is connected if and only if* S *generates* P*.*

In what follows, some properties of the generalized Cayley graphs over a polygroup are given.

Theorem 2.4. [6] *Let* $G_1, G_2, \ldots, G_n$ *be* GCP-graphs and $B \subseteq \{-1, 0, 1\}^n$. Then $\mathbf{Pr}(\mathrm{G_1}, \mathrm{G_2}, \ldots, \mathrm{G_n}; \mathrm{B})$ is a GCP-graph.

Corollary 2.5. [6] *The Cartesian, tensor, strong and lexicographic product of* GCP-*graphs are* GCP-*graphs.*

Lemma 2.6. *[6] Let* $\mathbf{P} = \langle P, \circ, 1, ^{-1} \rangle$ *be a polygroup and* S *be a connection set. Put* $\mathbf{Q} = \mathbf{P}\{\{v\}\}$, *where* $v \notin P$. *Then*

(I) $\mathrm{GCP}(\mathbf{Q}; \mathrm{S})$ *is connected.*

(II) $\mathrm{GCP}(\mathbf{P}; \mathrm{S})$ *is an induced subgraph of* $\mathrm{GCP}(\mathbf{Q}; \mathrm{S})$.

(III) Every Cayley graph is a GCP*-graph;*

(IV) All complete graphs and cycles are GCP*-graphs.*

(V) Every star graph S_n *is a* GCP-graph, where $n \in \mathbb{N}$.

In above lemmas we find some classes of GCP-graphs. In the next theorem, we restrict ourselves to the graphs of order at most five and prove that all simple graphs on at at most five vertices are GCP-graphs.

Theorem 2.7. *[5] All simple graphs on at most five vertices are* GCP-graphs.

Qustion. Are Theorem 2.7 hold for all simple graphs with $n \geq 6$ vertices?

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The rate of graded modules over some graded algebras

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Abstract

Let $\boldsymbol{k}$ be a field, R a standard graded $\boldsymbol{k}$-algebra and M be a finitely generated graded R-module. The rate of M, $\text{rate}_R(M)$, is a measure of the growth of the shifts in the minimal graded free resolution of M. In this paper, we find upper bounds for this invariant. More precisely, let $(A,\mathfrak{n})$ be a regular local ring and $I \subseteq \mathfrak{n}^t$ be an ideal of A, where $t \geq 2$. We prove that if $(B = A/I, \mathfrak{m} = \mathfrak{n}/I)$ is a Cohen-Macaulay local ring with multiplicity $e(B) = \binom{h+t-1}{h}$, where $h = \text{embdim}(B) - \dim B$, then $\text{rate}_{\text{gr}_\mathfrak{m}(B)}(\text{gr}_\mathfrak{m}(N)) \leq t-1$ for every B-module N which is annihilated by a minimal reduction of $\mathfrak{m}$.

Keywords and phrases: Rate, Associated graded module, Koszul algebras.
2010 *Mathematics subject classification:* Primary: 13D02 ; Secondary: 13D07, 16W50.

1. Introduction

Throughout $\boldsymbol{k}$ denotes a field and $R = \oplus_{i\geq 0} R_i$ is a commutative standard graded algebra over $\boldsymbol{k}$. We denote by $\mathfrak{m}$ the graded maximal ideal of R. Let $M = \oplus_{i\in\mathbb{Z}} M_i$ be a finitely generated graded R-module.

There are several invariants that we can associate to M. For each $i \geq 0$, we set

$$t_i^R(M) = \max\{j : \text{Tor}_i^R(M,\boldsymbol{k})_j \neq 0\}$$

provided that $\text{Tor}_i^R(M,\boldsymbol{k}) \neq 0$, otherwise we set $t_i^R(M) = -\infty$. Indeed $t_i^R(M)$ is the maximum degree of minimal generators of the i-th syzygy of M.

The regularity of M as an R-module is defined by

$$\text{reg}_R(M) = \sup\{t_i^R(M) - i : i \geq 0\}.$$

The regularity can be infinite. For example if $R = \boldsymbol{k}[x]/(x^3)$, then $\text{reg}_R(\boldsymbol{k}) = +\infty$.

Assume that M is generated by homogeneous elements of the same degree d. Then we say that M has a linear resolution if $\text{reg}_R(M) = d$. We also say that R is Koszul if $\boldsymbol{k}$ has a linear resolution that is $\text{reg}_R(\boldsymbol{k}) = 0$.

* speaker

Another important invariant is the rate of graded modules. The Backelin rate of the $\boldsymbol{k}$-algebra R is defined as

$$\mathrm{Rate}(R) = \sup\{(t_i^R(\boldsymbol{k}) - 1)/i - 1 : i \geq 2\}.$$

The notion of rate for an algebra R, introduced by Backelin ([4]) to study the Koszul property of R. He showed that $R^{(c)} = \oplus_{i \geq 0} R_{ic}$, the c-th Veronese subalgebra of R, is Koszul for all c sufficiently large.

Aramova, Bărcănescu and Herzog [3], extended the result of Backelin for modules. They defined the notion of rate for a finitely generated graded R-module M:

$$\mathrm{rate}_R(M) = \sup\{t_i^R(M)/i : i \geq 1\}.$$

For an integer d the notation $M(d)$ stands for the graded module with $M(d)_i = M_{d+i}$ for all i. A comparison with Backelin's rate shows that

$$\mathrm{Rate}(R) = \mathrm{rate}_R(\mathfrak{m}(1)).$$

Note that with the above notations $\mathrm{rate}_R(R) = -\infty$. Also, it turns out that the rate of M is finite (see [3, 1.3]).

Let $\dim_{\boldsymbol{k}} R_1 = n$, then $R \cong S/I$, where $S = \boldsymbol{k}[x_1, \ldots, x_n]$ is a polynomial ring over $\boldsymbol{k}$ and I is a graded ideal of S generated by elements of degree ≥ 2. There are lower and upper bounds for the rate of R:

$$m(I) - 1 \leq \mathrm{Rate}(R) \leq m(\mathrm{in}(I)) - 1,$$

here for a graded ideal J the notation $m(J)$ stands for the maximum degree of minimal generators of J and $\mathrm{in}(I)$ is the initial of I with respect to some term order. We refer the reader to [5] for more details and discussions on this result.

Since $m(I) \geq 2$, we have $\mathrm{Rate}(R) \geq 1$ and equality holds if and only if R is Koszul. Also, if M is generated in degree zero, one has $t_1^R(M) \geq 1$ and so by definition we have $\mathrm{rate}_R(M) \geq 1$. The equality holds if and only if M has a linear resolution.

Hence $\mathrm{rate}(M)$ can be considered as a measure of how much M deviates from having linear resolution.

Much less known about the upper bounds for the rate of graded modules. In this paper we study the rate of graded modules over some special graded algebras and find some upper bounds for the rate.

2. Main Results

2.1. Change of ring. We study the behavior of the rate of a graded module via a change of ring.

Proposition 2.1. *Let $\varphi : R \to S$ be a surjective homomorphism of standard graded $\boldsymbol{k}$-algebras. Assume that M is a finitely generated graded S-module generated by homogeneous elements of degree zero. If $\mathrm{rate}_R(S) = 1$, we have then*

$$\mathrm{rate}_R(M) \leq \mathrm{rate}_S(M)$$

Remark 2.2. *Let the situation be as in Proposition 2.1. Aramova et al. in [3,* Proposition 1.2*] showed that*

$$\text{rate}_S(M) \leq \max\{\text{rate}_R(M), \text{rate}_R(S)\}.$$

By combination of this result with Proposition 2.1 we get

$$\text{rate}_R(M) = \text{rate}_S(M).$$

2.2. Rate of modules over some special rings. The following lemma which gives an upper bound for the rate of modules over artinian algebras, will be used in the proof of the main theorem.

Lemma 2.3. *Let R be an Artinian standard graded K-algebra such that $R_i = 0$ for all $i \geq t$. Then for a finitely generated non-negatively graded R-module M,*

$$\text{rate}_R(M) \leq t_0^R(M) + t - 1.$$

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay complete local ring and $R \simeq A/I$ be a minimal Cohen presentation of R where $(A, \mathfrak{n})$ is a regular local ring and $I \subseteq \mathfrak{n}^t$ a perfect ideal of A with $t \geq 2$. It is well-known that $e(R) \geq h + 1$ where $e(R)$ is the multiplicity of R and $h = \text{embdim}(R) - \dim R$ (see for example[1]). The ring R is called of minimal multiplicity if the equality holds.

As remarked in [6], if $I \subseteq \mathfrak{n}^t$ with $t \geq 3$ the inequality $e(R) \geq h + 1$ is not sharp. Then it is shown that $e(R) \geq \binom{h+t-1}{h}$. The ideal I is called t-extremal if the equality holds.

The second author in [2, Proposition 2.14] showed that if R is of minimal multiplicity, that is 2-extremal, then $R^g := \text{gr}_{\mathfrak{m}}(R)$ is Koszul and $N^g := \text{gr}_{\mathfrak{m}}(N)$ has a linear resolution as a graded R^g-module, for every R-module N annihilated by a minimal reduction of $\mathfrak{m}$. The following theorem can be considered as a generalization of the result of [2] mentioned above.

Theorem 2.4. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a Cohen-Macaulay local ring and $e(R) = \binom{h+t-1}{h}$ where t is the initial degree of a defining ideal of R^g and $h = \text{embdim}(R) - \dim R$. Let N be an R-module and J be a minimal reduction of $\mathfrak{m}$. Then*

1. $\text{Rate}(R^g) = t - 1$,
2. *if $JN = 0$, then $\text{rate}_{R^g}(N^g) \leq t - 1$.*

As a corollary, when $t = 2$, we recover Proposition 2.14 of [2].

Corollary 2.5. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with minimal multiplicity. Then R^g is a Koszul algebra. If N is a finitely generated R-module annihilated by a minimal reduction of $\mathfrak{m}$, then N^g has a linear resolution.*

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Dominating set for bipartite graph $\Gamma(v,k,3,2)$

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Abstract

A bipartite graph (X,Y) in which X and Y are, respectively, the set of all l-subsets and all k-subsets of a v-set V and two vertices being adjacent if they have i elements in common, is denoted by $\Gamma(v,k,l,i)$. In this paper we study dominating set for $\Gamma(v,k,3,2)$, $4 \leq k \leq 6$.

Keywords and phrases: Dominating set, Bipartite graph, Steiner triple system.
2010 *Mathematics subject classification:* 05C30; 05C35; 05C69.

1. Introduction

Let t,k,v and λ be positive integers such that $0 \leq t \leq k \leq v$. Moreover, let V be a v-set. All of the i-subsets of V are denoted by $P_i(V)$. The pair $D = (V,\beta)$, where β is a subset of $P_k(V)$ (called blocks), is called a $t-(v,k,\lambda)$ design such that every t-subset of V appears in exactly λ blocks [1]. The number of blocks in D is shown by b. Moreover, a $2-(v,3,1)$ design is called a Steiner triple system and is denoted by $STS(v)$ [1].

Theorem 1.1. *[1] An $STS(v)$ exists if and only if $v \overset{6}{\equiv} 1$ or 3.*

A modified Steiner triple system on V denoted by $MSTS(v)$ is a proper subset of $P_3(V)$ such that every pairs of V occurs exactly once except for pairs $(1,2),(2,3),\cdots,(v-2,v-1),(v-1,1)$, which do not occurred at all and we have $|MSTS(v)| = \frac{(v-1)(v-2)}{6}$. A graph is a pair $G = (V,E)$, where $E \subseteq P_2(V)$ in which V is the vertex set and E is the edge set of G. Two vertices u and v are adjacent or neighbors if $\{u,v\} \in E$. A dominating set for a graph G is a subset $S \subseteq V(G)$ such that every vertex of G either is in S or is adjacent to at least one element of S. The domination number of G, is the minimum size of a dominating set in G and is denoted by $\gamma(G)$ [3]. The following theorem gives a classic bound for $\gamma(G)$:

* speaker

Theorem 1.2. *[3] Let G be an n-vertex graph with minimum degree δ, then*

$$\gamma(G) \leq \frac{n(1+ln(\delta+1))}{\delta+1}.$$

Let v,k,l be positive integers, i be a non-negative integer and $v \geq k > l \geq i$. Define a bipartite graph $\Gamma(v,k,l,i)$[2] by $V(\Gamma(v,k,l,i)) = P_k(V) \cup P_l(V)$ such that

$$\{u,w\} \in E(\Gamma(v,k,l,i)) \Leftrightarrow |u \cap w| = i, u \in P_k(V), w \in P_l(V).$$

In this paper, we study dominating sets for $\Gamma(v,k,3,2)$, where $4 \leq k \leq 6$, using design theory.

2. Main result

We begin our results with the following

Theorem 2.1. *Let $G = \Gamma(v,4,3,2)$ and $v \geq 12$, then*

$$\gamma(G) \leq \frac{7v^2 - 16v + 9}{24}.$$

PROOF. We consider two cases:
Case i. Suppose v is an odd integer and $v \geq 7$. Let

$$X = P_3(V), Y = P_4(V), V = \{a_1, a_2, \cdots, a_v\}.$$

We give a subset of X as a dominating set for Y and give a subset of Y as dominating set for X.

a) If $v \overset{6}{\equiv} 1$ or 5, then any $MSTS(v) \subset X$ is a dominating set for Y since any vertex in Y such as $B = \{a_1, a_2, a_3, a_4\}$ contains at least one non-consecutive pair, therefore B is dominated by a block of $MSTS(v)$.

b) If $v \overset{6}{\equiv} 3$, then any $STS(v)$ is a dominating set for Y, since any vertex in Y as $B = \{a_1, a_2, a_3, a_4\}$ is dominated by a block of $STS(v)$ having exactly two points in common with B.

In next step, we give a subset of Y as a dominating set of X. Let

$$C_1 = \{a_1, a_2\}, C_2 = \{a_3, a_4\}, \cdots, C_{\frac{v-1}{2}} = \{a_{v-2}, a_{v-1}\}$$

and $C = \{C_1, C_2, \cdots, C_{\frac{v-1}{2}}\}$. The set $P_2(C)$ is a dominating set for X.
Case ii. Suppose v is an even integer and $v \geq 12$. Let

$$X = P_3(V), Y = P_4(V), V = \{a_1, a_2, \cdots, a_v\}.$$

Let $V' = V \cup \{x\}$, where $x \notin V$. Similar to Case i on V' we may consider either $STS(v+1)$ or $MSTS(v+1)$ and then delete the blocks containing x.

The remaining blocks dominate Y. In next step, we give a subset of Y as the dominating set for X. Let

$$C_1 = \{a_1,a_2\}, C_2 = \{a_3,a_4\}, \cdots, C_{v/2} = \{a_{v-1},a_v\}$$

$C = \{C_1, C_2, \cdots, C_{\frac{v}{4}}\}$ and $C' = \{C_{\frac{v}{4}+1}, \cdots, C_{\frac{v}{2}}\}$. In this case $P_2(C') \cup P_2(C')$ is a dominating set for X.

Theorem 2.2. *Let $G = \Gamma(v,5,3,2)$ and $v \geq 9$, then*

$$\gamma(G) \leq \frac{4v^2 - 18v + 59}{12}.$$

PROOF. Suppose

$$X = P_3(V), Y = P_5(V), V = \{a_1, a_2, \cdots, a_v\}.$$

Let $V = A \cup B$ such that $|A \cap B| = 0$ or 1 and also $|A| \overset{6}{\equiv} 1$ or 3 and $|B| \overset{6}{\equiv} 1$ or 3. Hence $STS(|A|)$ and $STS(|B|)$ exist. The set of all blocks of these two designs is a dominating set for Y. Note that the maximum number of this dominating set occurs when $v = 12m + 11 = (6m + 3) + (6m + 9) + (-1)$. On the other side for a dominating set for X, we may consider two cases
Case i. Suppose that v be an odd integer. Let

$$C_1 = \{a_1,a_2\}, C_2 = \{a_3,a_4\}, \cdots, C_{\frac{v-3}{2}} = \{a_{v-4},a_{v-3}\}$$

and $C = \{C_1, C_2, \cdots, C_{\frac{v-3}{2}}\}$. We add a_{v-1} to all members of $P_2(C)$ to get a sets of five tuples over V. We do the same with a_{v-2} to get a similar set of five tuples. Now by adding the block $\{a_1, a_2, a_3, a_{v-2}, a_{v-1}\}$ to these later two set of five tuples we have a dominating set for X.
Case ii. Suppose that v be an even integer. Let

$$C_1 = \{a_1,a_2\}, C_2 = \{a_3,a_4\}, \cdots, C_{\frac{v-2}{2}} = \{a_{v-3},a_{v-2}\}$$

and $C = \{C_1, C_2, \cdots, C_{\frac{v-2}{2}}\}$. We add a_v to all members of $P_2(C)$ to get a set of five tuples over V. We do the same with a_{v-1} to get a similar set of five tuples. The set of these five tuples is a dominating set for X.

Theorem 2.3. *Let $G = \Gamma(v,6,3,2)$ and $v \geq 9$, then*

$$\gamma(G) \leq \frac{7v^2 - 18v + 40}{24}.$$

PROOF. Suppose

$$X = P_3(V), Y = P_6(V), V = \{a_1, a_2, \cdots, a_v\}.$$

If v is odd and $v \overset{6}{\equiv} 1$ or 5, then $MSTS(v)$ is a dominating set for Y and if $v \overset{6}{\equiv} 3$ then $STS(v)$ is a dominating set for Y. If v is even, let $V' = V - \{a_v\}$, then $|V'|$

is odd and as above we have a dominating set for Y. In next step, we give a subset of Y as a dominating set of X. If v is even, let

$$C_1 = \{a_1, a_2\}, C_2 = \{a_3, a_4\}, \cdots, C_{v/2} = \{a_{v-1}, a_v\}$$

and $C = \{C_2, C_3, \cdots, C_{v/2}\}$. We add C_1 to all members of $P_2(C)$ to get a set of six tuples over V. This set along with the set $A = \{C_2C_3C_4, C_5C_6C_7, \cdots\}$ is a dominating set for X. Note that if $|C|$ is not a multiple of 3, the last triple of A may build with the last one or two elements of C and any other member of C. If v is odd, let $V' = V - \{a_v\}$ then $|V'|$ is even and as above we have a dominating set for X.

One should note that the bounds given in this paper for $\gamma(G)$ is sharper than the bound given in Theorem 1.2.

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n-Jordan $*$-Derivations in Fréchet locally C^*-algebras

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Abstract
By using the fixed point method, we prove the Hyers-Ulam stability and the superstability of n-Jordan $*$-derivations in Fréchet locally C^*-algebras for the following generalized Jensen-type functional equation

$$f\left(\frac{a+b}{2}\right)+f\left(\frac{a-b}{2}\right)=f(a).$$

Keywords and phrases: n-Jordan $*$-derivation; Fréchet locally C^*-algebra; Fréchet algebra; fixed point method; Hyers-Ulam stability. .

2010 *Mathematics subject classification:* Primary: 17C65, 47H10; Secondary: 39B52, 39B72, 46L05..

1. Introduction

In this paper, assume that n is an integer greater than 1.

Definition 1.1. *Let $n \in \mathbb{N} - \{1\}$ and let A be a ring and B be an A-module. An additive map $D : A \to B$ is called n-Jordan derivation (n-ring derivation) if*

$$D(a^n) = D(a)a^{n-1} + aD(a)a^{n-2} + \ldots + a^{n-2}D(a)a + a^{n-1}D(a),$$

for all $a \in A$.

$$(D(\prod_{i=1}^{n} a_i) = D(a_1)a_2 \ldots a_n + a_1D(a_2)a_3 \ldots a_n + a_1a_2 \ldots a_{n-1}D(a_n)$$

for all $a_1, a_2, \ldots, a_n \in A$).

The concept of n-jordan derivations was studied by Eshaghi Ghordji.([2]).

Definition 1.2. *Let A , B be C^*-algebras. A $\mathbb{C}$-linear mapping $D : A \to B$ is called n-Jordan $*$-derivation if*

$$D(a^n) = D(a)a^{n-1} + aD(a)a^{n-2} + \ldots + a^{n-2}D(a)a + a^{n-1}D(a),$$
$$D(a^*) = D(a)^*$$

for all $a \in A$.

* speaker

Definition 1.3. *A topological vector space X is a Fréchet space if it satisfies the following three properties:*

(1) *it is complete as a uniform space,*

(2) *it is locally convex,*

(3) *its topology can be induced by a translation invariant metric, i.e., a metric $d : X \times X \to \mathbb{R}$ such that $d(x, y) = d(x + a, y + a)$ for all $a, x, y \in X$.*

For more detailed definitions of such terminologies, we can refer to [1]. Note that a ternary algebra is called a ternary Fréchet algebra if it is a Fréchet space with a metric d.

Fréchet algebras, named after Maurice Fréchet, are special topological algebras as follows.

Note that the topology on A can be induced by a translation invariant metric, i.e. a metric $d : X \times X \to \mathbb{R}$ such that $d(x, y) = d(x + a, y + a)$ for all $a, x, y \in X$.

Trivially, every Banach algebra is a Fréchet algebra as the norm induces a translation invariant metric and the space is complete with respect to this metric.

A locally C^*-algebra is a complete Hausdorff complex $*$-algebra A whose topology is determined by its continuous C^*-seminorms in the sense that a net $\{a_i\}_{i \in I}$ converges to 0 if and only if the net $\{p(a_i)\}_{i \in I}$ converges to 0 for each continuous C^*-seminorm p on A (see [4, 6]). The set of all continuous C^*-seminorms on A is denoted by $S(A)$. A Fréchet locally C^*-algebra is a locally C^*-algebra whose topology is determined by a countable family of C^*-seminorms. Clearly, any C^*-algebra is a Fréchet locally C^*-algebra.

For given two locally C^*-algebras A and B, a morphism of locally C^*-algebras from A to B is a continuous $*$-morphism φ from A to B. An isomorphism of locally C^*-algebras from A to B is a bijective mapping $\varphi : A \to B$ such that φ and φ^{-1} are morphisms of locally C^*-algebras.

Hilbert modules over locally C^*-algebras are generalization of Hilbert C^*-modules by allowing the inner product to take values in a locally C^*-algebra rather than in a C^*-algebra.

In this paper, using the fixed point method, we prove the Hyers-Ulam stability and the superstability of n-Jordan $*$-derivations in Fréchet locally C^*-algebras for the the following generalized Jensen-type functional equation

$$f\left(\frac{a+b}{2}\right) + f\left(\frac{a-b}{2}\right) = f(a).$$

2. Main Results

Lemma 2.1. ([5]) *Let A, B be C^*-algebras, and let $D : A \to B$ be a mapping such that*

$$\|D\left(\frac{a+b}{2}\right) + D\left(\frac{a-b}{2}\right)\|_B \leq \|D(a)\|_B, \tag{1}$$

for all $a, b \in A$. Then D is Cauchy additive.

Now, we prove the Hyers-Ulam stability problem for n-Jordan $*$-derivations in Fréchet locally C^*- algebras.

Theorem 2.2. *Let A, B be Fréchet locally C^*-algebras, and θ be nonegative real numbers. let $f : A \to B$ be a mapping such that*

$$\begin{aligned} \|\mu f(\frac{a+b}{2}) \quad &+ \quad \mu f(\frac{a-b}{2}) - f(\mu a) + f(c^n) - f(c)c^{n-1} + cf(c)c^{n-2} + \ldots \\ &+ \quad c^{n-2}f(c)c + c^{n-1}f(c) + f(d^*) - f(d)^*\|_B \leq \theta \end{aligned} \tag{2}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $a, b, c, d \in A$. Then the mapping $f : A \to B$ is an n-Jordan $*$-derivation.

Theorem 2.3. *Let A, B be Frechet locally C^*-algebras , and let θ be nonegative real numbers. Let $f : A \to B$ be a mapping satisfying then the mapping $f : A \to B$ is a n-Jordan $*$-derivation*

Now we prove the Hyers-Ulam stability of n-Jordan derivations in C^*-algebras.

Theorem 2.4. *Let A, B be Fréchet locally c^*-algebras. Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^4 \to \mathbb{R}^+$ such that*

$$\psi(a,b,c,d) = \sum_{i=0}^{\infty} 2^{-i}\varphi(2^i a, 2^i b, 2^i c, 2^i d) < \infty, \tag{3}$$

$$\begin{aligned} \|\mu f(\frac{a+b}{2}) \quad &+ \quad \mu f(\frac{a-b}{2}) - f(\mu a) + f(c^n) - f(c)c^{n-1} + cf(c)c^{n-2} + \ldots \\ &+ \quad c^{n-2}f(c)c + c^{n-1}f(c) + f(d^*) - f(d)^*\|_B \leq \varphi(a,b,c,d) \end{aligned} \tag{4}$$

for all $a, b, c, d \in A$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique n-Jordan $*$-derivation $D : A \to B$ such that

$$\|f(a) - D(a)\|_B \leq \psi(a, a, 0, 0) \tag{5}$$

for all $a \in A$.

Corollary 2.5. *Let A,B be Fréchet locally C^*-algebras, and let $f : A \to B$ be a mapping with $f(0) = 0$ for which there exist constants $\theta \geq 0$ and $p_1, p_2, p_3, p_4 \in (-\infty, 1)$ such that*

$$\begin{aligned} \|\mu f(\frac{a+b}{2}) &+ \mu f(\frac{a-b}{2}) - f(\mu a) + f(c^n) - f(c)c^{n-1} + cf(c)c^{n-2} + \dots \\ &+ c^{n-2}f(c)c + c^{n-1}f(c) + f(d^*) - f(d)^*\|_B \\ &\leq \theta(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3} + \|d\|^{p_4}) \end{aligned} \tag{6}$$

for all $a,b,c,d \in A$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique n-Jordan $*$-derivation $D : A \to B$ such that

$$\|f(a) - D(a)\|_B \leq \frac{2\theta\|a\|_A^{p_1}}{2 - 2^{p_1}} \tag{7}$$

for all $a \in A$.

Theorem 2.6. *Let A, B be Fréchet locally C^*-algebras. Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^4 \to \mathbb{R}^+$ such that*

$$\psi(a,b,c,d) = \sum_{i=0}^{\infty} 2^i \varphi(2^{-i}a, 2^{-i}b, 2^{-i}c, 2^{-i}d) < \infty, \tag{8}$$

$$\begin{aligned} \|\mu f(\frac{a+b}{2}) &+ \mu f(\frac{a-b}{2}) - f(\mu a) + f(c^n) - f(c)c^{n-1} + cf(c)c^{n-2} + \dots \\ &+ c^{n-2}f(c)c + c^{n-1}f(c) + f(d^*) - f(d)^*\|_B \leq \varphi(a,b,c,d) \end{aligned} \tag{9}$$

for all $a,b,c,d \in A$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique n-Jordan $*$-derivation $D : A \to B$ such that

$$\|f(a) - D(a)\|_B \leq \psi(a,a,0,0) \tag{10}$$

for all $a \in A$.

Corollary 2.7. *Let A,B be Fréchet locally C^*-algebras, and let $f : A \to B$ be a mapping with $f(0) = 0$ for which there exist constants $\theta \geq 0$ and $p_1, p_2, p_3, p_4 \in (-\infty, 1)$ such that*

$$\begin{aligned} \|\mu f(\frac{a+b}{2}) &+ \mu f(\frac{a-b}{2}) - f(\mu a) + f(c^n) - f(c)c^{n-1} + cf(c)c^{n-2} + \dots \\ &+ c^{n-2}f(c)c + c^{n-1}f(c) + f(d^*) - f(d)^*\|_B \\ &\leq \theta(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3} + \|d\|^{p_4}) \end{aligned} \tag{11}$$

for all $a,b,c,d \in A$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique n-Jordan $*$-derivation $D : A \to B$ such that

$$\|f(a) - D(a)\|_B \leq \frac{r\theta\|a\|_A^{p_1}}{2 - 2^{p_1}} \tag{12}$$

for all $a \in A$.

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On Complement to a Submodule of Multiplication Modules

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Abstract

In this paper, after recalling the definitions of multiplication modules and complement to a submodule in a module, we find some properties of associated and supported prime submodules of a multiplication module in connection with complement.

Keywords and phrases: multiplication modules, associated prime submodules, supported prime submodules, complement to a submodule .

2010 *Mathematics subject classification:* Primary: 13E05, 13E10; Secondary:13C99.

1. Introduction

In 1979, Singh and Mehdi defined the multiplication modules for the first time in [5]. Then in 1981, Barnard in [1] defined the multiplication modules in a different way. After Barnard, El-Bast and Smith in [2], study multiplication modules in more details. In this paper, the definition of multiplication modules is coincided to Barnard's definition.
In this paper all rings commutative with identity and all modules are unitary. Let S be a non-empty subset of an R-module M then the *annihilator* of S is defined as $Ann_R(S) = Ann(S) = \{r \in R | \; rS = 0\}$. By [3] and [4], A proper submodule N of an R-module M is said to be *prime* if $rx \in N$, where $r \in R$ and $x \in M$, implies that $x \in N$ or $r \in (N : M)$. If $p = (N : M)$ then p is a prime ideal of R and N is called *p-prime*. The sets of all prime ideals of R and all prime submodules of an R-module M are denoted by $Spec(R)$ and $Spec(M)$, respectively. Also the set of all maximal submodules of an R-module M is denoted by $Max(M)$.
For an R-module M, $S^{-1}R$-module $S^{-1}M$ is the module of fractions with respect to S. (Notation: $M_p = S^{-1}M$ if $S = R\backslash p$ where p is a prime ideal of R).

2. Definitions and Results

Definition 2.1. *By [1], An R-module M is called a* multiplication module *if for every submodule N of M there exists an ideal I of R such that $N = IM$. It can be shown that $N = (N : M)M$.*

* speaker

Definition 2.2. *Let M be an R-module.*
(i). The prime ideal p of R is called an associated prime ideal *of M if for some non-zero $x \in M$, $p = (0 : x) = Ann_R(x)$. The set of all associated prime ideals of M is denoted by $Ass_R(M)$.*
(ii). The prime ideal p of R is called a supported prime ideal *of M if $M_p \neq 0$. The set of all such prime ideals is denoted by $Supp_R(M)$, that is, $Supp_R(M) = \{p \in Spec(R) |\ M_p \neq 0\}$.*
It can be proved that

$$Supp_R(M) = \{p \in Spec(R) |\ p \supseteq (0 : x)\ for\ some\ x \in M,\ x \neq 0\}.$$

Also if M is finitely generated then

$$Supp_R(M) = \{p \in spec(R) |\ p \supseteq Ann_R(M)\}.$$

The following proposition is useful in the sequel.

Proposition 2.3. *Let M be an R-module and $p \in Spec(R)$, where R is a Noetherian ring. Then $p \in Supp_R(M)$ if and only if $p \supseteq q$ for some $q \in Ass_R(M)$.*

Definition 2.4. *Let M be an R-module and p a prime ideal of R. We define $M(p) = \{x \in M |\ sx \in pM\ for\ some\ s \in R \backslash p\}$. Clearly $M(p)$ is a submodule of M.*

Definition 2.5. *Let M be a weakly finitely generated R-module. The sets of* associated prime submodules *and* supported prime submodules *of M are defined, respectively, as follows:*

$$Ass_P(M) = \{M(p) |\ p \in Ass_R(M)\}\ and\ Supp_P(M) = \{M(p) |\ p \in Supp_R(M)\}.$$

Definition 2.6. *Let K be a submodule of an R-module M. A submodule $N \leq M$ is called a* complement *to K in M if N is maximal with respect to the property $L \cap K = 0$, where L is a submodule of M.*

By the Zorn's Lemma, any submodule S of M has a complement. Because if $\Omega = \{N | N\ is\ a\ submodule\ of\ M\ and\ N \cap S = 0\}$, partially ordered by inclusion, then $\Omega \neq \emptyset$ since $\langle 0 \rangle \in \Omega$. It can be shown that the Zorn's Lemma applies for Ω and therefore Ω has a maximal element. In fact any submodule C_0 of M with the property $C_0 \cap S = 0$ can be enlarged to a complement of S in M.

Definition 2.7. *A submodule N of an R-module M is said to be* strongly irreducible *if for all submodules K and L of M, the inclusion $L \cap K \subseteq N$ implies that $L \subseteq N$ or $K \subseteq N$.*

Proposition 2.8. *Let M be a multiplication R-module and let N be a prime submodule of M. Then N is strongly irreducible.*

Proposition 2.9. *Let M be a multiplication R-module and $0 \neq pM \in Supp_P(M)$ be such that it has a non-zero complement. If $pM \cap qM = 0$ for any $qM \in Supp_P(M)$ with $q \neq p$ then pM is a complement to qM in M and also pM is a maximal submodule of M.*

Theorem 2.10. *Let M be a multiplication R-module. Then either $Supp_P(M) = Max(M)$ or there exists $p_1M \in Supp_P(M)$ such that its complement in M is zero.*

Corollary 2.11. *Let M be a multiplication R-module and complement to each element of $Supp_P(M)$ in M is non-zero. Then $Supp_P(M) = Max(M) = Spec(M)$.*

Theorem 2.12. *Let M be a multiplication R-module and let $pM \in Supp_P(M)$ be such that it has a non-zero complement C in M. If $pM \cap qM = 0$ for each $qM \in Supp_P(M)$ with $q \neq p$, then*

$$Supp_P(M) = Max(M) = \{pM, C\}.$$

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On the Center and Automorphisms of Crossed Modules

M. A. DEHGHANIZADEH*

Abstract

The term crossed module was introduced by J. H. C. Whitehead in his work on combinatorial homotopy theory. Crossed modules and its applications play very important roles in category theory, homotopy theory, homology and cohomology of groups, algebra, ktheory etc. Actor crossed module of algebroid was defined by M. Alp. Nilpotent, Solvable, n-Complete and Representations of crossed modules was studied by M. A. Dehghanizadeh and B. Davvaz. In this paper we examine the center, n-center, central automorphisms and n-central automorphisms groups to crossed modules and obtain some results and theorem.

Keywords and phrases: center, n-center, automorphism, crossed module. .
2010 *Mathematics subject classification:* Primary: 18D35, 20L05; Secondary: 55U35 .

1. Introduction

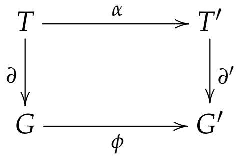
We recall some definitions and properties of the crossed module category. A crossed module (T, G, ∂) consist of a group homomorphism $\partial : T \longrightarrow G$ together with an action $(g, t) \longrightarrow {}^{g}t$ of G on T satisfying $\partial({}^{g}t) = g\partial(t)g^{-1}$ and ${}^{\partial(s)}t = sts^{-1}$, for all $g \in G$ and $s, t \in T$ [1–6]. In addition to the inner automorphism map $\tau : N \to Aut(N)$ already mentioned; other standard examples of crossed modules are:

- The inclusion of a normal subgroup $N \to G$;
- A G-module M with the zero homomorphism $M \to G$
- And any epimorphism $E \to G$ with central kernel.

2. Main Results

Definition 2.1. *A crossed module morphism* $< \alpha, \phi >: (T, G, \partial) \to (T', G', \partial')$ *is a commutative diagram of homomorphisms of groups*

* speaker



such that for all $x \in G$ and $t \in T$; we have $\alpha({}^{x}t) =^{\phi(x)} \alpha(t)$.

Definition 2.2. *Suppose that (T, G, ∂) be a crossed module. Center of (T, G, ∂) is the crossed module kernel $Z(T, G, \partial)$ of $< \eta, \gamma >$. Thus $Z(T, G, \partial)$ is the crossed module $(T^G, St_G(T) \cap Z(G), \partial)$ where T^G denotes the fixed point subgroup of T; that is,*

$$T^G = \{ t \in T \mid \ {}^{x}t = t \text{ for all } x \in G \}.$$

$St_G(T)$ is the stabilizer in G of T, that is:

$$St_G(T) = \{ x \in G \mid \ {}^{x}t = t \text{ for all } t \in T \}$$

and $Z(T)$ is the center of G. Note that T^G is central in T.

Definition 2.3. *Suppose that (T, G, ∂) be a crossed module. n-center of (T, G, ∂), $Z^n(T, G, \partial)$, for n a nonnegative integer g is the crossed module $\big((T^G)^n, Z^n(G) \cap St_G(T), \partial\big)$ where*

$$(T^G)^n = \{ t \in T \mid \ t^n = 1 \quad \text{and} \quad {}^{g}t = t; \ \forall g \in G \}$$

$$Z^n(G) = \{ g \in Z(G) \mid \quad g^n = 1 \}$$

$$St_G(T) = \{ g \in G \mid \ {}^{g}t = t, \ \forall t \in T \}$$

The n-central of (T, G, ∂) is a normal crossed submodule called n-central crossed submodule of (T, G, ∂).

Definition 2.4. *Suppose that (T, G, ∂) be a crossed module and $Z(T, G, \partial)$; center of it and $< \alpha, \phi > \in Aut(T, G, \partial)$. If $< \bar{\alpha}, \bar{\phi} >$ induced of $< \alpha, \phi >$ in $Aut\left(\frac{T}{T^G}, \frac{G}{St_G(T) \cap Z(G)}, \bar{\partial}\right)$; is identity, then $< \alpha, \phi >$ is called central automorphism of crossed module (T, G, ∂).*

Definition 2.5. *Suppose that (T, G, ∂) be a crossed module and $Z^n(T, G, \partial)$; n-central of it; $Z^n(T, G, \partial) = \big((T^G)^n, Z^n(G) \cap St_G(T), \partial\big)$; and $< \alpha, \phi > \in$ $\mathrm{Aut}(T, G, \partial)$. If $< \alpha, \phi >$ induces $< \bar{\alpha}, \bar{\phi} >$ in $Aut\left(\frac{T}{(T^G)^n}, \frac{G}{St_G(T) \cap Z^n(G)}, \bar{\partial}\right)$; is identity; then $< \alpha, \phi >$ is called n-central automorphism of crossed module (T, G, ∂).*

Definition 2.6. *A Adeny-Yen crossed module map is a of into the such that and is $< \phi_1, \phi_2 >$ of $Aut_C(T, G, \partial)$ into the $Hom((T, G, \partial), Z(T, G, \partial))$ such that*

$$< \phi_1, \phi_2 >< \alpha, \theta >=< \phi_1, \phi_2 >_{< \alpha, \theta >}$$

and $<\phi_1,\phi_2>_{<\alpha,\theta>}$ *is crossed module homomorphism of* (T,G,∂) *into* $Z(T,G,\partial) = (T^G, St_G(T)\cap Z(G),\partial)$ *such that* $<\phi_1,\phi_2>_{<\alpha,\theta>} = <\phi_{1<\alpha,\theta>},\phi_{2<\alpha,\theta>}>$*;*

$$\phi_{1<\alpha,\theta>} \; : \; T \longrightarrow T^G$$
$$\phi_{1<\alpha,\theta>}(t) = t^{-1}\alpha(t)$$

and

$$\phi_{2<\alpha,\theta>} \; : \; G \longrightarrow St_G(T)\cap Z(G)$$
$$\phi_{2<\alpha,\theta>}(g) = g^{-1}\theta(g)$$

Let C^* be the set of all central automorphisms of (T,G,∂) fixing $Z(T,G,\partial)$ element wise.

Theorem 2.7. *For purely non-abelian groups T and G , Adeny-Yen crossed module map is one-to-one correspondence of* $Aut_C(T,G,\partial)$ *onto* $Hom((T,G,\partial), Z(T,G,\partial))$.

Theorem 2.8. *For any non-abelian groups T and G the restriction of the Adeny-Yen crossed module map* $<\phi_1,\phi_2>: C^* \longrightarrow Hom((T,G,\partial),(Z(T,G,\partial))$ *is a homomorphism crossed module.*

Definition 2.9. *Given a crossed module* $\mathcal{X} = (\partial : T \to G)$. *We denote by* $Der(\mathcal{X})$ *the set of all derivations from G to T, i.e. all maps* $\chi : G \to T$ *such that for all* $q,r \in G$ *,*

$$\chi(qr) = (\chi q)^r \chi(r).$$

Definition 2.10. *The Whitehead group* $\mathcal{W}(\mathcal{X})$ *is defined to be group of units of* $Der(\mathcal{X})$. *The elements of* $\mathcal{W}(\mathcal{X})$ *will be called regular derivations.*

Example 2.11. *If T is a G-module, then the trivial homomorphism* $T \to G$ *is a crossed module and* $Der(\mathcal{X})$ *is the usual abelian group of derivations.*

Example 2.12. *Together with the conjugation action of a group G of itself, the identity map* $\mathcal{X} = (id : G \to G)$ *is a crossed mudule. An automorphism* α *of G determines its displacement derivation* $\delta_\alpha \in \mathcal{W}(\mathcal{X})$ *given by* $\delta_\alpha(r) = \alpha(r)r^{-1}$ *and the correspodence* $\alpha \to \delta_\alpha$ *is an isomorphism* $\delta : Aut(G) \to \mathcal{W}(\mathcal{X})$.

Definition 2.13. *The actor crossed module* $\mathcal{A}(\mathcal{X})$ *is defined to be the crossed module*

$$\mathcal{A}(\mathcal{X}) = (\Delta : \mathcal{W}(\mathcal{X}) \to Aut(\mathcal{X})).$$

Theorem 2.14. *Let* (T,G,∂) *has trivial n-central. then its actor* $\mathcal{A}(T,G,\partial)$ *also has trivial n-central.*

Theorem 2.15. *There is a homomorphism of groups*

$$\Delta : \mathcal{W}(\mathcal{X}) \to Aut(\mathcal{X})$$
$$\chi \mapsto <\sigma,\rho>$$

and with the action $\chi^{<\alpha,\phi>} = \alpha^{-1}\chi\phi$, $\mathcal{A} <\mathcal{X}> = (\Delta : \mathcal{W}(\mathcal{X}) \to Aut(\mathcal{X}))$, *is a crossed module.*

Theorem 2.16. *Let χ be a crossed module and $\mathcal{W}(\chi)$, whitehead group of χ. Then* $Aut_{C_n}(\mathcal{W}) = Aut(\mathcal{W})$.

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On skew Armendariz ideals of rings

F. Fatahi*, A. Moussavi and R. Safakish

Abstract

Let R be a ring with an endomorphism α and $R[x;\alpha]$ be the ring of skew polynomials. In this paper we study the skew Armendariz property on ideals of rings, introducing a new concept which unifies the various Armendariz properties for rings. A ring R is weak skew Armendariz if and only if every left ideal of R is weak skew Armendariz. We determine weak skew Armendariz ideals of some ring extensions and study related properties.

Keywords and phrases: Armendariz ideal, weak skew Armendariz ideal, weak annihilator..
2010 *Mathematics subject classification:* Primary 16S36; Secondary 16D25.

1. Introduction

Throught this paper, all rings are associative with an identity. Given a ring R with an endomorphism α, the skew polynomial ring over R is denoted by $R[x;\alpha]$ whose elements are the polynomials over R, the addition is defined as usual and the multiplication subject to the relation $xr = \alpha(r)x$ for any $r \in R$.

In [7], a ring R is called *Armendariz* if whenever the product of any two polynomials in $R[x]$ over R is zero then so the product of any pair of coefficients from the two polynomials. This definition was given by Rege and Chhawchharia in [7].

Several types of generalizations of Armendariz rings have been introduced for some of which variations of the previous results are also valid. The Armendariz property of rings was extended to skew polynomial ring in [2] for an endomorphism α of a ring R. A ring R is called *skew α-Armendariz* if for $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha]$, $f(x)g(x) = 0$ implies that $a_i\alpha^i(b_j) = 0$ for all $0 \leq i \leq n$ and $0 \leq j \leq m$. In 2006, Liu and Zhao [3] introduced the notion of a weak Armendariz ring and following that, C. Zhang and J. Chen [8] say a ring R with an endomorphism α is *weak α-skew Armendariz* if two polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha]$ satisfy $f(x)g(x) = 0$ then

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$a_i\alpha^i(b_j) \in nil(R)$ for each i and j. A ring R is said to be *nil α-skew Armendariz* if whenever polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha]$ satisfy $f(x)g(x) \in nil(R)[x;\alpha]$ then $a_i\alpha^i(b_j) \in nil(R)$ for each i, j.

For a nonempty subset X of a ring R, the left and right annihilator of R which is denoted by $r_R(X) = \{r \in R \mid Xr = 0\}$ and $l_R(X) = \{r \in R \mid rX = 0\}$.

The concept of *Armendariz ideal* is introduced and studied by Ghalandarzadeh et al., in [1]. A one-sided ideal I of a ring R is said to be Armendariz if whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) \in r_{R[x]}(I[x])$, then, $a_i b_j \in r_R(I)$ for each i, j. According to Nikmehr [4], a one-sided ideal I of a ring R is said to be *α-skew Armendariz* if for $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha]$, $f(x)g(x) \in r_{R[x;\alpha]}(I[x])$ implies $a_i\alpha^i(b_j) \in r_R(I)$, and that, for each $a, b \in R$, $ab \in r_R(I)$ if and only if $a\alpha(b) \in r_R(I)$.

For a subset X of a ring R, Ouyang and Birkenmeier [5] define the notion of *weak annihilator* of X in R, $N_R(X) = \{a \in R \mid xa \in nil(R), \text{ for all} \, x \in X\}$, and investigate the properties of the weak annihilator over ring extensions. In the present paper we study the Armendariz property on ideals of rings, introducing a new concept which unifies the various Armendariz properties for rings.

2. Main Results

If X is a singleton, say $X = \{r\}$, $N_R(r)$ is used in place of $N_R(\{r\})$. Obviously, for any nonempty subset X of R, we have $N_R(X) = \{a \in R \mid xa \in nil(R), \text{for all } x \in X\} = \{\text{b} \in \text{R} \mid \text{bx} \in \text{nil(R)}, \text{for all x} \in \text{X}\}$, $r_R(X) \subseteq N_R(X)$ and $l_R(X) \subseteq N_R(X)$.

Let $T_2(\mathbb{Z})$ be the triangular matrix ring over the ring of integers $\mathbb{Z}$ and let $X = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\}$. Then $r_{T_2(\mathbb{Z})}(X) \subsetneq N_{T_2(\mathbb{Z})}(X)$ and $l_{T_2(\mathbb{Z})}(X) \subsetneq N_{T_2(\mathbb{Z})}(X)$.

If R is a reduced ring, then $r_R(X) = N_R(X) = l_R(X)$ for any subset X of R. It is easy to see that for any subset $X \subseteq R$, $N_R(X)$ is an ideal of R in case $nil(R)$ is an ideal. For more details and results of weak annihilators, see [6].

The next lemmas appear in [6] and will be helpful in the sequel.

Lemma 2.1. *Let X, Y be subsets of R. Then, we have the followings:*
(i) $X \subseteq Y$ implies $N_R(X) \supseteq N_R(Y)$.
(ii) $X \subseteq N_R(N_R(X))$.
(iii) $N_R(X) = N_R(N_R(N_R(X)))$.

Lemma 2.2. *Let R be a subring of S. Then, for any subset X of R, we have $N_R(X) = N_S(X) \cap R$.*

Definition 2.3. *We say a left ideal I of R is weak α-skew Armendariz if whenever polynomials $f(x), g(x) \in R[x;\alpha]$ satisfy $f(x)g(x) \in r_R(I)[x;\alpha]$ we have $a_i\alpha^i(b_j) \in N_R(I)$ for all $a_i \in C_{f(x)}$ and $b_j \in C_{g(x)}$.*

The following result shows that our definition of a weak α-skew Armendariz left ideal is a generalization of Zhang and J. Chen [8], to the more general setting.

Theorem 2.4. *A ring R is weak α-skew Armendariz if and only if every left ideal of R is weak α-skew Armendariz.*

Proof. Let R be a weak α-skew Armendariz ring and I be a left ideal of R. If $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j$ are element of $R[x;\alpha]$ such that $f(x)g(x) \in r_R(I)[x;\alpha]$. Thus we have $d(x)f(x)g(x) = 0$ for any $d \in I$. Since R is weak α-skew Armendariz, $da_i\alpha^i(b_j) \in nil(R)$ for all i, j. Thus $a_i\alpha^i(b_j) \in N_R(I)$. This show that I is a weak α-skew Armendariz ideal.

Clearly, the other condition is hold, because R is an ideal of R and $N_R(R) \subseteq nil(R)$. □

Theorem 2.5. *Let R be a reversible ring with an endomorphism α. If $r_R(I)$ is an α-compatible ideal of R, then I is a weak α-skew Armendariz ideal.*

Corollary 2.6. *Each left ideal of an α-compatible semicommutative ring is weak α-skew Armendariz.*

Theorem 2.7. *Let I be a left ideal of R and Δ be a multiplicative closed subset in R containing of central regular elements. Then I is a weak α-skew Armendariz ideal of R if and only if $\Delta^{-1}I$ is an $\overline{\alpha}$-skew Armendariz ideal of $\Delta^{-1}R$.*

By Theorem 2.7 we have the following results.

Corollary 2.8. *[8, Proposition 2.11] A ring R is weak α-skew Armendariz if and only if $\Delta^{-1}R$ is weak $\overline{\alpha}$-skew Armendariz.*

Corollary 2.9. *Let I be an ideal of R. Then $I[x]$ is a weak α-skew Armendariz ideal of $R[x]$ if and only if $I[x;x^{-1}]$ is a weak α-skew Armendariz ideal of $R[x;x^{-1}]$.*

By Corollary 2.9 we have the following.

Corollary 2.10. *[8, Corollary 2.12] For a ring R and an automorphism α of $R, R[x]$ is weak $\overline{\alpha}$-skew Armendariz if and only if $R[x;x^{-1}]$ is weak $\overline{\alpha}$-skew Armendariz.*

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ϕ-primary subsemimodules

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Abstract

Let R be a commutative semiring with identity and M be a unitary R-semimodule. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function, where $\mathcal{S}(M)$ is the set of all subsemimodules of M. A proper subsemimodule N of M is called ϕ-primary subsemimodule, if whenever $r \in R$ and $x \in M$ with $rx \in N - \phi(N)$, implies that $r \in \sqrt{(N :_R M)}$ or $x \in N$. So if we take $\phi(N) = \emptyset$ (resp., $\phi(N) = \{0\}$), a ϕ-primary subsemimodule is primary (resp., weakly primary). In this paper, we study the concept of ϕ-primary subsemimodule which is a generalization of ϕ-prime subsemimodule in a commutative semiring.

Keywords and phrases: Semiring, Semimodule, ϕ-primary subsemimodule, M-subtractive subsemimodule..

2010 *Mathematics subject classification:* Primary:16Y60 .

1. Introduction

Anderson and Bataineh [2] have introduced the concept of ϕ-prime ideals in a commutative ring as a generalization of weakly prime ideals in a commutative ring introduced by Anderson and Smith [1]. After that several authors [6, 11], etc. explored this concept in different ways either in commutative ring or semiring. Recently, we generalized the above mentioned concepts to semiring theory; for example see [7], [9]. In this paper, we introduce the notion of ϕ-primary subsemimodules of a commutative semiring as a generalization of all the above mentioned definitions and prove several results connected with ring theory.

For the definition of monoid, semirings, semimodules and subsemimodules of a semimodule we refer [6, 8]. All semirings in this paper are commutative with non-zero identity. The semiring R is to be also a semimodule over itself. In this case, the subsemimodules of R are called ideals of R. Let M be a semimodule over a semiring R. A subtractive subsemimodule ($= k$-subsemimodule) N is a k-subsemimodule of M such that if x, $x + y \in N$, then $y \in N$. If N is a proper subsemimodule of an R-semimodule M, then we denote

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$(N :_R M) = \{r \in R : rM \subseteq N\}$ and $\sqrt{(N :_R M)} = \{r \in R : r^n M \subseteq N$ for some n $\in \mathbb{N}\}$. Clearly, $(N :_R M)$ and $\sqrt{(N :_R M)}$ are ideals of R.

2. Main Results

Definition 2.1. *Let $\mathcal{S}(M)$ be the set of subsemimodule of M and $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\varnothing\}$ be a function. The proper subsemimodule N of M is called a ϕ-primary semimodule if $r \in R$, $x \in M$ and $rx \in N - \phi(N)$, then $r \in \sqrt{(N :_R M)}$ or $x \in N$.*

Since $N - \phi(N) = N - (N \cap \phi(N))$, so without loss of generality, throughout this article we will consider $\phi(N) \subseteq N$. In the rest of the article we use the following functions $\phi_\alpha : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\varnothing\}$.

$$\begin{aligned}
\phi_\varnothing(N) &= \varnothing, \quad N \in \mathcal{S}(M),\\
\phi_0(N) &= \{0\}, \quad N \in \mathcal{S}(M),\\
\phi_1(N) &= (N :_R M)N, \quad N \in \mathcal{S}(M),\\
\phi_2(N) &= (N :_R M)^2 N, \quad N \in \mathcal{S}(M),\\
\phi_w(N) &= \cap_{i=1}^{\infty} (N :_R M)^i N, \quad N \in \mathcal{S}(M).
\end{aligned}$$

Then it is clear that $\phi_\varnothing$, ϕ_0-primary subsemimodules are primary, weakly primary subsemimodules respectively. Evidently for any subsemimodule and every positive integer n, we have the following implications:

$$primary \Rightarrow \phi_w - primary \Rightarrow \phi_n - primary \Rightarrow \phi_{n-1} - primary.$$

For functions $\phi, \psi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\varnothing\}$, we write $\phi \leqslant \psi$ if $\phi(N) \subseteq \psi(N)$ for each $N \in \mathcal{S}(M)$. So whenever $\phi \leqslant \psi$, any ϕ-primary subsemimodule is ψ-primary.

Definition 2.2. *[7, Definition 2.2] A proper subsemimodule N of M is called M-subtractive, if N and $\phi(N)$ are subtractive subsemimodules of M.*

The following theorems asserts that under some conditions ϕ-primary subsemimodules are primary.

Theorem 2.3. *If N is a ϕ-primary subsemimodule of M and $\phi(N)$ is a primary subsemimodule, then N is a primary subsemimodule of M.*

Theorem 2.4. *Let N be a ϕ-primary M-subtractive subsemimodule of M such that $(N :_R M)N \nsubseteq \phi(N)$. Then N is a primary subsemimodule of M.*

Proof. Let $ax \in N$ for some $a \in R$ and $x \in M$. If $ax \notin \phi(N)$, then $ax \in N - \phi(N)$, which implies that $a \in \sqrt{(N :_R M)}$ or $x \in N$, as N is a ϕ-primary subsemimodule of M. Therefore, N is primary. So, let $ax \in \phi(N)$.

Let $aN \nsubseteq \phi(N)$. Then there is $n \in N$ such that $an \notin \phi(N)$ and $an \in N$. Therefore, $a(x + n) \in N - \phi(N)$. Thus we have either $a \in \sqrt{(N :_R M)}$ or $x +$

$n \in N$, that is, $a \in \sqrt{(N :_R M)}$ or $x \in N$, as N is M-subtractive subsemimodule of M. So N is a primary subsemimodule of M. Now we can assume that $aN \subseteq \phi(N)$.

Suppose that $(N :_R M)x \nsubseteq \phi(N)$. Then there exists $u \in (N :_R M)$ such that $ux \in (N :_R M)x$ but $ux \notin \phi(N)$. Therefore $(a+u)x \in N - \phi(N)$. Since N is ϕ-primary, we have either $a + u \in \sqrt{(N :_R M)}$ or $x \in N$. Now it follows by [6, Lemma 2.10] that $a \in \sqrt{(N :_R M)}$ or $x \in N$, and therefore, N is primary. So we may assume that $aN \subseteq \phi(N)$ and $(N :_R M)x \subseteq \phi(N)$.

Since $(N :_R M)N \nsubseteq \phi(N)$, then there exist some $r \in (N :_R M)$ and $n_1 \in N$ such that $rn_1 \notin \phi(N)$. So $(a+r)(x+n_1) \in N - \phi(N)$ and hence $(a+r) \in \sqrt{(N :_R M)}$ or $x + n_1 \in N$, that is, $a \in \sqrt{(N :_R M)}$ or $x \in N$. Therefore, in any case, we have N is a primary subsemimodule of M. □

Corollary 2.5. [6, Theorem 2.11] *Let N be a weakly primary subtractive subsemimodule of M such that $(N :_R M)N \neq \{0\}$. Then N is primary.*

Theorem 2.6. *Let N be a proper M-subtractive subsemimodule of M. Then the following statements are equivalent:*

(i) N is a ϕ-primary subsemimodule of M.

(ii) For any $m \in M - N$, $\sqrt{(N :_R m)} = \sqrt{(N :_R M)} \cup (\phi(N) :_R m)$.

(iii) For any $r \in R - \sqrt{(N :_R M)}$, $(N :_M r) = N \cup (\phi(N) :_M r)$.

(iv) For any $r \in R - \sqrt{(N :_R M)}$, $(N :_M r) = N$ or $(N :_M r) = (\phi(N) :_M r)$.

(v) If $IP \subseteq N - \phi(N)$ for some ideal I of R and a subsemimodule P of M, then either $I \subseteq \sqrt{(N :_R M)}$ or $P \subseteq N$.

Corollary 2.7. [6, Theorem 2.9] *Let N be a proper subtractive subsemimodule of M. Then the following statements are equivalent:*

(i) N is weakly primary.

(ii) For $m \in M - N$, $\sqrt{(N :_R m)} = \sqrt{(N :_R M)} \cup (0 :_R m)$.

Proposition 2.8. *Let N be a proper subsemimodule of M. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ and $\psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be two functions such that $(\phi(N) :_R m) \subseteq \psi((N :_R M))$ for every $m \in M - N$. If N is a ϕ-primary subsemimodule of M, then $(N :_R M)$ is a ψ-primary ideal of R.*

Proof. Let $ab \in (N :_R M) - \psi((N :_R M))$ for some $a, b \in R$ and $a \notin (N :_R M)$. Then there is $0 \neq m \in M$ such that $am \notin N$. We have $abm \in N - \phi(N)$, and so $b \in \sqrt{(N :_R M)}$ since N is a ϕ-primary subsemimodule. Consequently $(N :_R M)$ is a ψ-primary ideal. □

Corollary 2.9. [4, Lemma 2] *Let R be a semiring. If N is a primary subsemimodule of an R-semimodule M, then $(N :_R M)$ is a primary ideal.*

Corollary 2.10. [6, Proposition 2.4] *Let M be an entire R-semimodule and N a weakly primary subsemimodule of M. Then $(N :_R M)$ is a weakly primary ideal of R.*

Recall from [3, Definition 2] that, an R-subsemimodule N of M is said to be a strong subsemimodule if for each $x \in N$, there exists $y \in N$ such that $x + y = 0$.

Theorem 2.11. *Let $f : M \to M'$ be an epimorphism of R-semimodules with $f(0) = 0$ and let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ and $\phi' : \mathcal{S}(M') \to \mathcal{S}(M') \cup \{\emptyset\}$ be two functions. Then the following statements hold:*

(i) If N' is a ϕ'-primary subsemimodule of M' and $\phi(f^{-1}(N')) = f^{-1}(\phi'(N'))$, then $f^{-1}(N')$ is a ϕ-primary subsemimodule of M.

(ii) If N is a subtractive strong ϕ-primary subsemimodule of M containing $Ker(f)$ and $\phi'(f(N)) = f(\phi(N))$, then $f(N)$ is a ϕ'-primary subsemimodule of M'.

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One-Sided Repeated-Root Two-Dimensional Constacyclic Codes

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Abstract

In this paper, we study some repeated-root two-dimensional constacyclic codes over a finite field $\mathbb{F} = \mathbb{F}_q$. We obtain the generator matrices and generator polynomials of these codes and their duals. We also investigate when such codes are self-dual.

Keywords and phrases: Two-dimensional constacyclic codes, Self-dual codes. .
2010 *Mathematics subject classification:* Primary: 94B05, 11T71, 94B15.

1. Introduction

Two-dimensional (2D, for short) cyclic codes which have a long history, see for example [2, 3], still gain attention, see [6] and the references there in. Constacyclic codes which are a generalization of cyclic codes are investigated over finite fields and some other types of rings, see [1] and its references. In [5], 2D constacyclic codes were introduced and studied as a generalization of 2D cyclic codes.

We recall the definition of 2D constacyclic codes. We always assume that p is a prime number, $\mathbb{F} = \mathbb{F}_q$ is a finite field with $q = p^r$ elements and λ and δ are units in $\mathbb{F}$. Consider

$$\begin{array}{rcl} \tau_\lambda : \mathbb{F}^n & \longrightarrow & \mathbb{F}^n \\ (d_0, d_1, \ldots, d_{n-1}) & \longmapsto & (\lambda d_{n-1}, d_0, \ldots, d_{n-2}), \quad \text{where } d_j \in \mathbb{F} \end{array}$$

and

$$\begin{array}{rcl} \Upsilon_\delta : (\mathbb{F}^n)^m & \longrightarrow & (\mathbb{F}^n)^m \\ (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{m-1}) & \longmapsto & (\delta \mathbf{a}_{m-1}, \mathbf{a}_0, \ldots, \mathbf{a}_{m-2}), \quad \text{where } \mathbf{a}_j \in \mathbb{F}^n. \end{array}$$

Assume that $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{m-1})$ is an element of $\mathbb{F}^{nm}$, where $\mathbf{a}_j = (a_{j0}, a_{j1}, \ldots, a_{j\,n-1}) \in \mathbb{F}^n$. For any i, j, $0 \le j \le m-1$ and $0 \le i \le n-1$, define

$$\Theta^{j,i}_{\delta,\lambda}(\mathbf{a}) = \Upsilon^j_\delta(\tau^i_\lambda(\mathbf{a}_0), \tau^i_\lambda(\mathbf{a}_1), \ldots, \tau^i_\lambda(\mathbf{a}_{m-1})).$$

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A 2D linear code D of length nm is called (λ,δ)-*constacyclic code* over $\mathbb{F}$, if $\Theta^{j,i}_{\delta,\lambda}(D) = D$ for any $0 \le j \le m-1$ and $0 \le i \le n-1$. In $\mathbb{F}^{nm} \simeq M_{m\times n}(\mathbb{F})$, any nm-array $(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{m-1})$ corresponds to a polynomial in $\mathbb{F}[x,y]$ with x-degree less than n and y-degree less than m, say $a(x,y) = \sum_{j=0}^{m-1}\sum_{i=0}^{n-1} a_{ji}x^i y^j$. With this correspondence, any (λ,δ)-constacyclic code of length nm over $\mathbb{F}$ is identified with an ideal of the quotient ring $\mathcal{S} = \frac{\mathbb{F}[x,y]}{\langle x^n-\lambda, y^m-\delta\rangle}$.

2. One-sided repeated-root 2D constacyclic codes and their duals

In this paper, we deal with 2D constacyclic codes which are either simple root or have repeated roots in at most one direction. We call such codes one-sided repeated-root codes, as defined below.

Definition 2.1. We call a two-dimensional (λ,δ)-constacyclic code D of length nm over $\mathbb{F}_{p^r}$, one-sided repeated root, if either $\gcd(n,p) = 1$ or $\gcd(m,p) = 1$.

From now on, we assume that n, m are two integers, such that $\gcd(n,p) = 1$, $m = m'p^s$ and $\gcd(m',p) = 1$. Also we assume that λ, δ are non-zero elements of $\mathbb{F}$. We let $\mathcal{S} = \frac{\mathbb{F}[x,y]}{\langle x^n-\lambda, y^m-\delta\rangle}$. Moreover, we assume that $x^n - \lambda = \prod_{j=1}^{\eta} f_j(x)$, where $f_j(x)$, $1 \le j \le \eta$, are monic irreducible coprime polynomials in $\mathbb{F}[x]$. Also we set $d_j = \deg f_j$, $K_j = \frac{\mathbb{F}[x]}{\langle f_j(x)\rangle} \cong \mathbb{F}_{q^{d_j}}$ and $\mathcal{S}_j = \frac{K_j[y]}{\langle y^m-\delta\rangle}$. We consider elements of $\mathcal{S}$ as those elements of $\mathbb{F}[x,y]$ whose x-degree and y-degree is less than n and m, respectively.

Now, we can determine the general form of ideals of $\mathcal{S}$.

Theorem 2.2. *Let C be a (λ,δ)-constacyclic code over $\mathbb{F}$. Then there exist unique polynomials $g_j(x,y)$ such that $g_j(x,y) \mid y^m - \delta$ in $K_j[y]$, $g_j(x,y)$ is monic when considered as a polynomial in y and as an ideal of $\mathcal{S}$,*

$$C = \langle g_1(x,y)\prod_{i\neq 1} f_i(x), g_2(x,y)\prod_{i\neq 2} f_i(x), \ldots, g_\eta(x,y)\prod_{i\neq \eta} f_i(x)\rangle.$$

Moreover, $\dim(C) = mn - \sum_{j=1}^{\eta} d_j t_j$, *where* $t_j = \deg_y g_j$.

In what follows we assume that C and g_j, $0 \le j \le \eta$, are as in Theorem 2.2.

Corollary 2.3. *The following set is a basis for C over $\mathbb{F}$.*

$$\Delta = \bigcup_{j=1}^{\eta} \{x^r y^l g_j(x,y)\prod_{i\neq j} f_i(x) \mid 0 \le r < d_j, 0 \le l < m - t_j\},$$

where $t_j = \deg_y g_j$.

For any (λ,δ)-constacyclic code $C \subseteq \mathbb{F}^{nm}$, let

$$C^{\perp} = \{\mathbf{u} \in \mathbb{F}^{nm} \mid \mathbf{u}\cdot\mathbf{w} = 0 \text{ for any } \mathbf{w} \in C\}$$

be the dual of the code C. By [5, Propositin 2.2], $C^{\perp}$ is a $(\lambda^{-1},\delta^{-1})$-constacyclic code over $\mathbb{F}$. We shall determine the unique generating set of the dual of C as an ideal of $\mathcal{T}=\frac{\mathbb{F}[x,y]}{<x^n-\lambda^{-1},y^m-\delta^{-1}>}$.

If $f(x)$ is a non-zero polynomial of degree d in $\mathbb{F}[x]$, we define the reciprocal of $f(x)$ by $f^*(x)=x^d f(x^{-1})$. Since $x^n-\lambda=\prod_{i=1}^{\eta} f_i(x)$, we have $x^n-\lambda^{-1}=u\prod_{i=1}^{\eta} f_i^*(x)$ for some $u\in\mathbb{F}$. Suppose that $f(x,y)\in\mathbb{F}[x,y]$ has x- and y-degree less than n and m, respectively. Now for any polynomial $f(x,y)\in\mathbb{F}[x,y]$ define $f^*(x,y)=x^{\deg_x f}y^{\deg_y f}f(\frac{1}{x},\frac{1}{y})$. If $f\in\mathcal{S}$ (resp. in $\mathcal{T}$), we consider f^* as an element of $\mathcal{T}$ (resp. $\mathcal{S}$).

Assume that $h_i(x,y)=\frac{y^m-\delta}{g_i(x,y)}$ in $K_i[y]$. If $g_i(x,y)=0$, we assume that $h_i(x,y)=1$. Suppose that $h_i^{\sharp}(x,y)$ is the monic polynomial in $\mathbb{F}[x,y]$ such that $h_i^{\sharp}(x,y)=\frac{h_i^*(x,y)}{h_i(x,0)}$ in $\frac{\mathbb{F}[x]}{\langle f_i^{\sharp}(x)\rangle}[y]$, where $f_i^{\sharp}(x)=\frac{f_i^*(x)}{f_i(0)}$. With this notations, we have the following theorem that gives the generating set of the dual of the code C.

Theorem 2.4. *Let C be a (λ,δ)-constacyclic code over $\mathbb{F}$. Then*

$$C^{\perp}=\langle h_1^{\sharp}(x,y)\prod_{i\neq 1} f_i^{\sharp}(x), h_2^{\sharp}(x,y)\prod_{i\neq 2} f_i^{\sharp}(x),\ldots,h_{\eta}^{\sharp}(x,y)\prod_{i\neq \eta} f_i^{\sharp}(x)\rangle.$$

Next we study when C is self-dual, that is, $C=C^{\perp}$. To see why it is important to study and find self-dual codes see for example [4, Section 3]. Note that if C is self-dual, then it is both (λ,δ)-constacyclic and $(\lambda^{-1},\delta^{-1})$-constacyclic. Here, we just consider the cases that $\lambda=\lambda^{-1}$, $\delta=\delta^{-1}$ and $f_i(x)=f_i^{\sharp}(x)$. Let in $K_j[y]$, $y^{m'}-\delta=\prod_{l=1}^{t_j} h_{jl}(x,y)$, where $h_{jl}(x,y), 1\leq l\leq t_j$, are monic irreducible coprime polynomials in $K_j[y]$. Assume that $h_{jl}(x,y)=h_{jl}^{\sharp}(x,y)$ for $1\leq l\leq a_j$ and $h_{jl}^{\sharp}\neq h_{jl}$ for $a_j<l$. Since $(y^{m'}-\delta)^{\sharp}=y^{m'}-\delta^{-1}=y^{m'}-\delta$, so for each $1\leq l\leq t_j$, we have $h_{jl}^{\sharp}=h_{jl'}$ for some $1\leq l'\leq t_j$. Thus we can suppose that

$$y^{m'}-\delta=\prod_{l=1}^{a_j} h_{jl}(x,y)\prod_{l=a_j+1}^{b_j} h_{jl}(x,y)\prod_{l=a_j+1}^{b_j} h_{jl}^{\sharp}(x,y). \tag{2.1}$$

Theorem 2.5. *Let $p=2$, $s>0$, $f_i^{\sharp}(x)=f_i(x)$ for all i, and C be a (λ,δ)-constacyclic code of length $n(2^s m')$ over $\mathbb{F}$, where $\lambda^2=\delta^2=1$. The code C is self-dual if and only if for every j,*

$$g_j(x,y)=\prod_{l=1}^{a_j} h_{jl}^{2^{s-1}}(x,y)\prod_{l=a_j+1}^{b_j} h_{jl}^{\alpha_{jl}}(x,y)\prod_{l=a_j+1}^{b_j}(h_{jl}^{\sharp})^{2^s-\alpha_{jl}}(x,y), \tag{2.2}$$

for some α_{jl}, $0\leq\alpha_{jl}\leq 2^s$.

Theorem 2.6. *Assume that* $f_i^\sharp(x) = f_i(x)$, *for all* i *and* $\lambda^2 = \delta^2 = 1$. *Let* p *be an odd prime number or* $s = 0$. *There exists a self-dual 2D* (λ, δ)*-constacyclic code of length* $nm = n(p^s m')$ *over* $\mathbb{F}$ *if and only if in* (2.1), $a_j = 0$ *for all* j. *In this case, a code* C *is self-dual if and only if*

$$g_j(x,y) = \prod_{l=1}^{b_j} h_{jl}^{\alpha_{jl}}(x,y) \prod_{l=1}^{b_j} (h_{jl}^\sharp)^{p^s - \alpha_{jl}}(x,y),$$

for some α_{jl}, $0 \leq \alpha_{jl} \leq p^s$.

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On Differential Semigroups and Radical Differential Ideals

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Abstract

Our objective in this paper is to define a notion of derivation in a semigroup by using its ideals. We call a semigroup with such a derivation, a differential semigroup. We study some properties of the derivations in (commutative) semigroups. Also we determine the radical differential ideals of any differential monoid by using prime differential ideals.

Keywords and phrases: prime radical, differential semigroup, differential ideal.

2010 *Mathematics subject classification:* Primary: 20M11, 20M12; Secondary: 20M14.

1. Introduction

The notion of differential algebra was introduced in the work [2] of Kolchin and [5] of Ritt. This notion has a huge applications in algebraic geometry and topology. Specially Kolchin in [2] and [3] used this notion for rings to give a generalization of the notion of derivation in the field of rational functions and ordinary polynomial rings to any arbitrary ring. A derivation on a ring R was defined as a mapping on elements of R with $r \longrightarrow r'$ for which $(xy)' = x'y + xy'$, for any $x, y \in R$. In this paper we use ideals of S (instead of elements) to define a derivation in S. We introduce the notion of differential semigroups and study prime and radical ideals of them in Section 2. Also we define differential ideals as ideals each of which contains its derivative. We will show that any radical differential ideal in a differential semigroup is an intersection of prime differential ideals. This is a reconstruction of the well-known result in classical ring theory (Krull's theorem for prime ideals) that any radical ideal in a ring is an intersection of prime ideals. Indeed we prove a similar result for radical differential ideals and prime differential ideals in a commutative monoid.

For basic results and definition relating to semigroups and radical of ideals in this paper, we refer the reader to [1] and [4], respectively.

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2. Main Results

Suppose that S is a semigroup and $(\mathcal{I}(S),\cup,\cap)$ is the lattice of all ideals of S and $IJ = \{ij|i \in I, j \in J\}$ for every $I,J \in \mathcal{I}(S)$. Clearly $\mathcal{I}(S)$ with this multiplication of ideals forms a semigroup.

Definition 2.1. *By a derivation on ideals of S we mean a map*

$$d : \mathcal{I}(S) \longrightarrow \mathcal{I}(S)$$

preserving unions and intersections of ideals and satisfying $d(IJ) = d(I)J \cup Id(J)$ for any $I,J \in \mathcal{I}(S)$.

We denote the image of any $I \in \mathcal{I}(S)$, under a distinguished derivation d with $d(I) = I'$. In this notation I' is called the derivative of I. Also we use the notations $I'', I''', \dots I^{(n)}$ (for any $4 \leq n \in \mathbb{N}$) to show higher order derivatives of I. Indeed $I^{(n)} = d^n(I)$, is the composition of d by itself n times.

Always there exist trivial derivations in any semigroup S. For example the identity map on $\mathcal{I}(S)$ is a derivation. Also if S contains a zero the constant zero map is a derivation on $\mathcal{I}(S)$.

A homomorphisms $f : S \longrightarrow T$ between differential semigroups S and T is defined as a homomorphism of semigroups for which $f(I') = (f(I)')$, for any $I \in \mathcal{I}(S)$.

In the continuation of this article we consider commutative semigroups, however many of the results may be hold in arbitrary semigroups similarly. Also by a *differential semigroup* we mean a commutative semigroup with a derivation on the semigroup of its ideals. Moreover, an ideal I of a semigroup S is called a *differential ideal* if $I' \subseteq I$. The followings are some of the preliminary rules for computing derivations that are routinely checked. For every $I,J \in \mathcal{I}(S)$ and $n \in \mathbb{N}$ we have:

$$(I \cup J)' = I' \cup J',$$

$$(I \cap J)' = I' \cap J',$$

$$(IJ)' = I'J \cup IJ',$$

$$(IJ)'' = I''J \cup I'J' \cup IJ'',$$

$$(IJ)^{(n)} = I^{(n)}J \cup I^{(n-1)}J' \cup I^{(n-2)}J'' \cup \ldots \cup I'J^{(n-1)} \cup IJ^{(n)},$$

$$(I^n)' = I^{n-1}I'.$$

The following two lemmas can be easily proved.

Lemma 2.2. *If S is a differential monoid with identity 1 then for any $I \in \mathcal{I}(S)$, $S'I \subseteq I'$.*

Lemma 2.3. *If I is a differential ideal of a commutative semigroup S and $JK \subseteq I$, for some $J,K \in \mathcal{I}(S)$, then $JK' \subseteq I$ and $J'K \subseteq I$.*

An ideal P of a semigroup S is called prime if $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$, for any ideals I and J of S. It is easy to see that in any commutative semigroup this is equivalent to $xy \in P$ implies $x \in P$ or $y \in P$, for any $x, y \in S$.

Proposition 2.4. *Suppose that I is a differential ideal of a differential semigroup S and $\mathcal{I} \subseteq \mathcal{I}(S)$. Then the set*

$$\mathcal{J} = \{J \in \mathcal{I}(S) | JK \subseteq I, \textit{for any } K \in \mathcal{I}\},$$

is an ideal of $\mathcal{I}(S)$ containing some ideals of S and their derivatives. Also if I is a prime ideal of S then $\mathcal{J}$ is a prime ideal of $\mathcal{I}(S)$. Moreover,

$$A = \bigcup_{J \in \mathcal{J}} J = \{x \in S | xk \in I \textit{ for any } k \in K, \textit{ for any } K \in \mathcal{I}\},$$

is a differential ideal of S, and if S contains an identity 1, $S \in \mathcal{I}$ and I is a prime ideal of S then A is also prime.

Proof. Clearly $\mathcal{J}$ is an ideal of $\mathcal{I}(S)$. If $J \in \mathcal{J}$ then for any $K \in \mathcal{I}$, $JK \subseteq I$. So by Lemma 2.3, $J'K \subseteq I$, which implies $J' \in \mathcal{J}$. Thus derivative of any ideal in $\mathcal{J}$ is also in $\mathcal{J}$.

To prove $\mathcal{J}$ is prime, suppose that $MN \in \mathcal{J}$ for some ideals $M, N \in \mathcal{I}(S)$. Then $MNK \subseteq I$, for any $K \in \mathcal{I}$. Since I is a prime ideal of S, $M \subseteq I$ or $NK \subseteq I$, for any $K \in \mathcal{I}$. So $M \in \mathcal{J}$ or $N \in \mathcal{J}$, that is, $\mathcal{J}$ is a prime ideal in $\mathcal{I}(S)$.

To prove the last part, first note that A is a union of ideals and so it is an ideal of S. Also $A' = (\bigcup_{J \in \mathcal{J}} J)' = \bigcup_{J \in \mathcal{J}} J'$. But by the first part $J' \in \mathcal{J}$, for any $J \in \mathcal{J}$ which implies $A' \subseteq A$, that is, A is a differential ideal of S. If $st \in A$ for some $s, t \in S$ then $st \in J$ for some $J \in \mathcal{J}$. Since $S \in \mathcal{I}$, $st \in I$. Thus $s \in I$ or $t \in I$, for I is a prime ideal of S. Also clearly $I \in \mathcal{J}$ that implies $I \subseteq A$. So $s \in A$ or $t \in A$. Therefore A is a prime ideal of S. □

For any ideal I of a semigroup S the radical of I is denoted by $rad(I)$ is the set of all $x \in S$ for which there exists $n \in \mathbb{N}$ such that $x^n \in I$. Also I is called a radical ideal if $I = rad(I)$, that is, for any $x \in S$ and $n \in \mathbb{N}$, $x^n \in I$ implies $x \in I$. Clearly every prime ideal is radical and the intersection of radical ideals is also radical. Also we can see that in a commutative monoid any radical ideal is an intersection of prime ideals. We will find a similar result for differential ideals as a substitute for any ideals.

The intersection of all radical differential ideals of a differential semigroup S which contain a subset T of S, is denoted by $drad(T)$.

Remark 2.5. *It is easily checked that Proposition 2.4 can be rewritten for radical ideals as an alternative to prime ideals. Also for the last part of the new revision we do not need the condition $S \in \mathcal{I}$.*

Lemma 2.6. *Let I be an ideal of a differential monoid S and $T \subseteq S$. Then $I\,drad(T) \subseteq drad(IT)$.*

Proposition 2.7. *Suppose that S is a differential monoid and $T \subset S$. Let*

$$\Sigma = \{I \in \mathcal{I}(S) | I \cap T = \emptyset \text{ and } I \text{ is a radical differential ideal}\}.$$

If $\Sigma \neq \emptyset$ then it has a maximal element with respect to the inclusion which is also a prime ideal of S.

Using the Axiom of Choice we have the following revision of Krull's theorem for differential ideals as we have promised before.

Theorem 2.8. *Suppose that I is a radical differential ideal of a differential monoid S. Then I is an intersection of prime differential ideals of S.*

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On EL^2-semihypergroups of order 2

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Abstract

EL^2-semihypergroups obtained from quasi ordered semihypergroups using "Ends lemma". In this paper we classify all EL^2-semihypergroups over sets with two elements obtained from quasi ordered semihypergroups.

Keywords and phrases: Semihypergroup, EL^2-semihypergroup, Ends lemma. .
2010 *Mathematics subject classification:* Primary: 20N20, 16Y99.

1. Introduction

The concept of EL-hyperstructures first described by Chvalina [1] when he was investigated quasi ordered sets and hypergroups. Then Novak in [6, 7] studied some properties of EL-hyperstructures. Ghazavi et al. introduced a new class of EL-hyperstructures called EL^2-hyperstructures in [4]. EL^2-hyperstructures are hyperstructures based on (partially) quasi ordered (semi)hypergroups instead of a (partially) quasi ordered (semi)groups. Moreover, Ghazavi and Mirvakili computed EL-hypergroups of order 2 [5].

In this paper, first we characterize all quasi ordered semihypergroups of order 2. Then, we concentrate on quasi ordered semigroups and in order to find and classify all EL^2-semihypergroups and EL^2-hypergroups of order 2.

A hypergroupoid is a pair $(H, \circ)$ where H is a nonempty set and $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ is a hyperoperation when $\mathcal{P}^*(H)$ is the family of non-empty subsets of H. A semihypergroup is an associative hypergroupoid, i.e. hypergroupoid satisfying the equality $a \circ (b \circ c) = (a \circ b) \circ c$ for every $a, b, c \in H$.

If A and B are two non-empty subsets of H and $x \in H$, then $x \circ A = \{x\} \circ A$, $A \circ x = A \circ \{x\}$ and $A \circ B = \bigcup \{a \circ b | a \in A, b \in B\}$.

If the semihypergroup $(H, \circ)$ satisfies $a \circ H = H = H \circ a$, for all $a \in H$, it is called a *hypergroup*.

A semihypergroup $(S, \circ, R)$ is called a (partially) quasi ordered semihypergroup if $(S, \circ)$ is a semihypergroup and "R" is a (partially) quasi order relation

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on S such that for all $a, b, c \in S$ with the property aRb there holds $a \circ c\overline{R}b \circ c$ and $c \circ a\overline{R}c \circ b$ (monotone condition), where if A and B are non-empty subsets of S, then we say $A\overline{R}B$ whenever for all $a \in A$, there exists $b \in B$ and for all $b \in B$ there exists $a \in A$ such that aRb [4].

Moreover, the notation $[x)_R$ used below stands for the set $\{s \in S; xRs\}$ and also $[A)_R = \bigcup\limits_{x \in A} [x)_R$. Similarly, $(x]_R = \{s \in S; sRx\}$ and $(A]_R = \bigcup\limits_{x \in A} (x]_R$. The EL-hyperstructures or *Ends lemma* based hyperstructures are hyperstructures constructed from a (partially) quasi ordered (semi)groups using "Ends lemma".

This concept was first introduced by Chvalina in 1995 [1]. In particular, Chvalina proved that:

Lemma 1.1. *([1], Theorem 1.3) Let $(S, \cdot, R)$ be a partially ordered semigroup. Binary hyperoperation $\circ : S \times S \longrightarrow \mathcal{P}^*(S)$ defined by $a \circ b = [a \cdot b)_R = \{x \in S, a \cdot bRx\}$ is associative. The semihypergroup $(S, \circ)$ is commutative if and only if the semigroup $(S, \cdot)$ is commutative.*

Theorem 1.2. *([1], Theorem 1.4) Let $(S, \cdot, R)$ be a partially ordered semigroup. The following conditions are equivalent:*

(1) *For any pair $(a, b) \in S^2$ there exists a pair $(c, c_1) \in S^2$ such that $(b \cdot c)Ra$ and $(c_1 \cdot b)Ra$.*
(1) *The associated semihypergroup $(S, \circ)$ is a hypergroup.*

We need the following theorem.

Theorem 1.3. *Let $S = \{a, b\}$. Then, there are 4 quasi order relations on S as follows:.*
$R_1 = \{(a, a), (b, b)\}$,
$R_2 = \{(a, a), (b, b), (a, b), (b, a))\} = S \times S$,
$R_3 = \{(a, a), (b, b), (b, a), \}$,
$R_4 = \{(a, a), (b, b), (a, b)\}$.

Definition 1.4. *[1] Suppose $(S, \circ, R)$ is a (partially) quasi ordered hypergroupoid. For $a, b \in S$, we define the new hyperoperation $* : S \times S \longrightarrow \mathcal{P}^*(S)$ as follows:*

$$a * b = [a \circ b)_R = \bigcup_{m \in a \circ b} [m)_R.$$

Remark 1.5. *From now on, we name $(S, *)$ as the EL^2-hypergroupoid associated to (partially) quasi ordered hypergroupoid $(S, \circ, R)$.*

Theorem 1.6. *[1] Let $(S, \circ, R)$ be a (partially) quasi ordered semihypergroup i.e. the hyperoperation $\circ$ is associative. Then, the hyperoperation $*$ on S, defined in Definition 1.4, is associative and therefore $(S, *)$ is a semihypergroup.*

2. EL^2-semihypergroups of order 2

Now, in order to find and study EL^2-semihypergroups of order 2, we need all semihypergroups with two elements. Then, we obtain the next theorem:

Theorem 2.1. *There are, up to isomorphism, 17 semihypergroups of order 2 are give in Table 1. In Table 8 the Cayley table $(abcd)$ of semihypergroup $(S = \{a,b\},\circ)$ means that $a = 1\circ 1$, $b = 1\circ 2$, $c = 2\circ 1$ and $d = 2\circ 2$. Also, $S_i = (abcd)$ means that the semihypergroup $(S = \{a,b\},\circ_i)$*

TABLE 1. Semihypergroup of order 2

$S_1 = (a,a,a,a)$	$S_7 = (a,S,a,b)$	$S^*_{13} = (a,S,S,b)$
$S_2 = (a,a,a,b)$	$S_8 = (a,S,b,b)$	$S_{14} = (S,b,S,b)$
$S_3 = (a,a,b,b)$	$S_9 = (S,b,b,b)$	$S^*_{15} = (S,S,S,b)$
$S^*_4 = (a,b,b,a)$	$S^*_{10} = (S,S,a,b)$	$S^*_{16} = (a,S,S,S)$
$S_5 = (a,b,a,b)$	$S^*_{11} = (S,a,S,b)$	$S^*_{17} = (S,S,S,S)$
$S^*_6 = (S,a,a,b)$	$S_{12} = (S,S,b,b)$	

Among these 17 semihypergroups there are 8 ones which are hypergroups. We mention them by a “ $*$ ” sign in the related Cayley tables of Table 1.

By Theorem 1.3 there are 4 quasi ordered relations on a set with two elements. Hence there are 4*17=68 triple $(S,\circ_i,R_j)$ for $1\le i\le 17$ and $1\le j\le 4$. Now, we look after the ones which are quasi ordered semihypergroups. (i.e. those which has the monotone condition).

Theorem 2.2. *For all $i\in\{1,2,\dots,17\}$ and $j\in\{1,2\}$, the triple $(S,\circ_i,R_j)$ is a quasi ordered semihypergroup.*

Theorem 2.3. *For all $i\in\{1,2,3,5,7,8,9,12,13,14,15,16,17\}$ and $j\in\{3,4\}$, triples $(S,\circ_i,R_j)$ are quasi ordered semihypergroups.*

Proposition 2.4. *For all $i\in\{4,6,10,11\}$ and $j\in\{3,4\}$, the triple $(S,\circ_i,R_j)$ is not a quasi ordered semihypergroup.*

Now, by Theorems 2.2 and 2.3, we have:

Corollary 2.5. *There are 56 quasi ordered semihypergroups of order 2.*

Definition 2.6. *Suppose $(S,\circ)$ is an semihypergroup. Then, $(S,*)$ is said to be a nontrivial semihypergroup if it is not total semihypergroup (i.e. $a*b = S$ for all $(a,b)\in S$) nor it is not associated to $(S,\circ_i,R_1)$, $i\in\{1,2,\cdots,17\}$ in EL^2-construction.*

Theorem 2.7. *There are 7 non-trivial EL^2-semihypergroups of order 2. (S_i has the EL^2-construction for $i\in\{1,6,9,12,14,15,16\}$.)*

Definition 2.8. *The semihypergroup* $(S,*)$ *is said to be a proper semihypergroup if it is not a hypergroup. (i.e. the hyperoperation* $*$ *is not reproductive.)*

Theorem 1. *There are 3 proper* EL^2*-semihypergroups created by semihypergroups.(* S_9, S_{12}, S_{14} *are proper* EL^2*-semihypergroups).*

Corollary 2.9. *There are 4 non-trivial hypergroup with* EL^2*-construction.*

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Some relations between the distinguishing and some graph parameters

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Abstract

The distinguishing number of a simple graph G is the least number $D(G)$ of colors needed for a coloring of G which is preserved only by the identity automorphism. Similar parameters have been defined whose concern is breaking the symmetries of a graph. In this paper, we present interesting connections between these parameters and some other graph parameters such as the independence number. In particular, we study conditions under which a given graph G is (D,α)-ordinary, that is, for which $D(G) \leq \alpha(G)$.

Keywords and phrases: graph, distinguishing, independence number, fixing number.
2010 *Mathematics subject classification:* Primary: 05C09, 05C15; Secondary: 05C25, 05C30.

1. Introduction

Throughout the paper we assume $n = |V(G)|$ is the number of vertices of G unless otherwise stated. All graphs are assumed to be undirected, simple and finite, and by "coloring" we mean vertex coloring. Let G be a graph and c be a coloring of G. We say that an $\alpha \in Aut(G)$ preserves the coloring c if for any pair of vertices u and v in $V(G)$, for which $\alpha(u) = v$, the vertices u and v have the same color. For a graph G, a *distinguishing coloring* is a coloring of G such that the only automorphism which preserves the coloring, is the identity automorphism. Then the *distinguishing number* of G, denoted by $D(G)$, is defined to be the smallest number of colors for a distinguishing coloring of G. As stated in [2], it is easy to see that $D(K_n) = n$, $D(K_{n,n}) = n+1$, $D(P_n) = 2$ for $n \geq 2$, $D(C_3) = D(C_4) = D(C_5) = 3$, while $D(C_n) = 2$, for $n \geq 6$.

It has been showed in [2] that if the automorphism group of a graph G is abelian, then $D(G) \leq 2$. Indeed, it is stated in [4] that the distinguishing number of "almost" all connected graphs is at most 2. On the other hand, the independence number of a graph G is at least 2 unless G is a complete graph on more than 1 vertices. It follows that for almost all connected graphs G, $D(G) \leq \alpha(G)$. We call the graphs for which this inequality holds (D,α)-*ordinary* graphs. It is, therefore, an interesting problem to see under which

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conditions a graph is (D,α)-ordinary. In this paper, along with studying some relationships between the distinguishing number and some other graph parameters such as the independence number and the *fixing* number of graphs, we consider the problem of identifying conditions under which a graph is (D,α)-ordinary/nonordinary. In [1], along with some new parameters related to distinguishing colorings, the authors introduced the *distinguishing threshold* $\theta(G)$ as the minimum number k of colors such that any coloring of the graph G with k colors is distinguishing. Obviously $D(G) \leq \theta(G) \leq n$. They also showed that $\theta(K_n) = \theta(\overline{K_n}) = n$, $\theta(K_{m,n}) = m+n$, $\theta(P_n) = \lceil \frac{n}{2} \rceil + 1$, for $n \geq 2$, and $\theta(C_n) = \lfloor \frac{n}{2} \rfloor + 2$, for $n \geq 3$. In addition, it has been proved [5] that

$$\theta(G) = \max\{|\alpha| \mid \alpha \in \mathrm{Aut}(G) \setminus \{\mathrm{id}\}\} + 1, \tag{1}$$

where $|\alpha|$ stands for the number of cycles in the cycle decomposition of α acting on the set of vertices of G. In [6] authors have defined the concept of a *steady* vertex which plays an important role in evaluating the distinguishing number of some graphs A vertex v in a graph G is said to be steady if $\mathrm{Stab}_{\mathrm{Aut}(G)}(u) \cong \mathrm{Aut}(G-u)$. The authors of [6], also prove the following.

Theorem 1.1. *A vertex v of a graph G is steady if and only if every distinguishing coloring of G induces a distinguishing coloring on $G - v$.* □

2. Main Results

2.1. Distinguishing and fixing number As stated in [6], the concept of a steady vertex, can be generalized to a *steady set*. To define it, we make use of the following notation. For any non-empty subset $A \subseteq V(G)$, in this paper, we denote by $\mathrm{Stab}_{\mathrm{Aut}(G)}(A)$ the set of all automorphism of G which point-wise fix A. Then, we say that a non-empty set $A \subseteq V(G)$ is a steady set in G if $\mathrm{Stab}_{\mathrm{Aut}(G)}(A) \cong \mathrm{Aut}(G-A)$. Using a similar approach as in [6] we can generalize Theorem 1.1 to steady sets as follows.

Proposition 2.1. *A subset A of the vertices of a graph G is steady if and only if every distinguishing coloring of G induces a distinguishing coloring on $G - A$.* □

The concept of a distinguishing coloring is closely related to the concept of a *fixing set* in a graph G which was introduced in [3] as they are both "symmetry breaking" tools. A non-empty set $A \subseteq V(G)$ is a called a fixing set of G if $\mathrm{Stab}_{\mathrm{Aut}(G)}(A) = \{\mathrm{id}\}$. If G is asymmetric, i.e. if $\mathrm{Aut}(G) = \{\mathrm{id}\}$, then as a convention, we assume that the empty set is a fixing set for G. Note that $V(G)$ is both a fixing set and a steady set for G. Furthermore, the fixing number of G, denoted by $\mathrm{Fix}(G)$, is the minimum size of a fixing set of G. Thus, for an asymmetric graph G, we have $\mathrm{Fix}(G) = 0$. It is pointed out in [3] that $\mathrm{Fix}(K_n) = n-1$, $\mathrm{Fix}(P_n) = 1$, for $n \geq 2$, and $\mathrm{Fix}(C_2) = 2$, for $n \geq 3$.

Proposition 2.2. *For any graph G, we have $D(G) \leq \mathrm{Fix}(G) + 1 \leq \theta(G)$.* □

For the next result, we recall that a set $A \subseteq V(G)$ is a *vertex cover* of G if every edge of G has one of its vertices in A. The *vertex covering number* $\beta(G)$ is the minimum cardinality of a vertex cover of G. Note that $\alpha(G) + \beta(G) = n$.

Theorem 2.3. *For any graph G if $\theta(G) \neq n$, then $\mathrm{Fix}(G) < \beta(G)$.*

In the rest of this subsection, we study the connections between steady vertices (sets) and fixing sets.

Proposition 2.4. *Let $v \in V(G)$ be an steady vertex. If A is a fixing set of G, then $A - \{v\}$ is also a fixing set of $G - \{v\}$.*

Now, we conclude the subsection by generalizing Proposition 2.4 to steady sets which, in turn, is the fixing set variant of Proposition 2.1.

Theorem 2.5. *Let G be a graph and $A, B \subseteq V(G)$ be a steady and a fixing set of G, respectively. Then $B - A$ is a fixing set of $G - A$.*

2.2. Distinguishing and independence number In this subsection we study situations in which the distinguishing number of a graph is bounded above by its independence number. We say that a graph G is (D,α)-ordinaryif $D(G) \leq \alpha(G)$ and a graph is (D,α)-nonordinary if it is not (D,α)-ordinary. It is easy to see that the graphs $\overline{K_n}$, C_5, P_n, for $n \geq 5$, and all asymmetric graphs are (D,α)-ordinary while the graphs K_n, $K_{n_1,\ldots,n_t}$ and P_4 are (D,α)-nonordinary, and that the set of all (D,α)-nonordinary graphs is closed under the join operation which provides an infinite family of such graphs. This shows that it is an interesting problem to determine which graphs are (D,α)-ordinary/nonordinary. Note that according to Proposition 2.3, for the graphs G with $\theta(G) \neq n$, if $\alpha(G)$ is large, then $\mathrm{Fix}(G)$ must be small. Another consequence of Proposition 2.3 is the following result which provides some necessary condition for a graph to be (D,α)-nonordinary.

Theorem 2.6. *If G is (D,α)-nonordinary then either $\theta(G) = n$ or $\alpha(G) \leq \frac{n-1}{2}$.*

Corollary 2.7. *If a graph G satisfies $\theta(G) \neq n$ and $\alpha(G) > \frac{n-1}{2}$, then $D(G) < \alpha(G)$.*

We now investigate some families of graphs to see whether they are (D,α)-ordinaryẆe first consider forests. The following result has been proved in [7].

Theorem 2.8. *For any tree T, we have $D(T) \leq \Delta(T)$, except for K_2.* □

In order to prove the next result, we will make use of the following fact.

Lemma 2.9. *Let F be a forest consisting of a_i copies of the tree T_i, for $i = 1, \ldots, m$, where a_i are positive integers. Then $D(F) \leq \max_i \{D(T_i) + a_i - 1\}$.* □

Proposition 2.10. *All forests with more than 2 vertices are (D,α)-ordinary.*

In addition, for the case of cycles, we observe the following.

Proposition 2.11. *A cycle C_n is (D,α)-ordinary if and only if $n \geq 6$.* □

As another family of graphs whose distinguishing numbers have been studied thoroughly, we consider the so-called *generalized Johnson graphs*. The following result is due to Kim et al. and we state the rephrased version in [5].

Theorem 2.12. *Assume that $2 \leq k \leq n/2$ and set $e = \frac{1}{2}\binom{n}{k}$.*

(a) *If $n = 5$ and $k = 2$, then $D(J(n,k,1)) = D(J(n,k,2)) = 3$.*

(b) *If $n \neq 5$ and $2 \leq k < \frac{n}{2}$, then $D(J(n,k,i)) = 2$, for each $i = 1,\dots,k$.*

(c) *If $k = \frac{n}{2}$ and $i \notin \{\frac{k}{2}, k\}$, then $D(J(n,k,i)) = 2$.*

(d) *If $k = \frac{n}{2}$ and $i = \frac{k}{2}$, then $D(J(n,k,i)) = 3$.*

(e) *If $k = \frac{n}{2}$ and $i = k$, then $D(J(n,k,i)) = \lceil \frac{1+\sqrt{1+8e}}{2} \rceil$.* □

Using Theorem 2.12, we can see which generalized Johnson graphs are (D,α)-ordinary. The result appears in the following proposition.

Theorem 2.13. *Assume that $2 \leq k \leq n/2$. The graph $J(n,k,i)$ is (D,α)-ordinary if and only if $(n,k,i) \neq (4,2,1),(5,2,1)$.*

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CFGH: A hypergroup for the control flow graph

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Abstract

A Control Flow Graph (CFG) is a directed graph that represents all paths that might be traversed through a program during its execution. This graph is used to generate test cases for a program. In this paper, we define a hyper-operation on the vertex set of a CFG. Consequently, it is proved that (1) the generated hyperstructure is a quasi-ordering hypergroup, (2) the connectivity in a CFG is equivalent to the inner irreducibility in the hypergroup, and (3) each sub-graph in a CFG is a sub-hypergroup.

Keywords and phrases: Hyper-operation; Quasi-ordering Hypergroup; Control Flow Graphs.
2010 *Mathematics subject classification:* 20N20, 68Q45, 68Q70.

1. Introduction

Software testing is an important task in the life cycle of software development process. A Control Flow Graph (CFG) is utilized to generate the test cases of a program [4, 6]. In a CFG, each node represents a basic program block and edges are the control dependencies between these blocks. A basic block is a straight-line code sequence with no branches in except to the entry and no branches out except at the exit [1]. In fact, CFG shows all paths that might be traversed through a program during its execution, and each test case is generated by the software testing task to trace one of these paths [7]. In this paper, we construct a quasi-ordering hypergroup on the set of CFG vertexes and state the relationship between a CFG and its corresponding hypergroup (called CFGH). Consequently, we prove that the connectivity in a CFG is equivalent to the inner irreducibility in CFGH, and each sub-graph of a CFG is a sub-hypergroup of CFGH.

The theory of algebraic hyperstructures which is a generalization of the concept of ordinary algebraic structures first was introduced by Marty [5]. Since then many researchers have worked on algebraic hyperstructures and developed it [2].

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2. Main Results

In this section, the definition of a CFG is presented and then a hypergroup associated with an CFG is constructed and some properties are proved.

Definition 2.1. *A CFG G is a quintuple* $(Q, \Sigma, \delta, q_0, F)$ *where Q is a non-empty finite set of basic program blocks and* $\Sigma = \{t, f\}$ *is the set of jump symbols when t and f are "true" and "false" conditions. The transition function* $\delta : Q \times \Sigma \to Q$ *shows the control dependencies between blocks based on the conditions. Vertex* $q_0 \in Q$ *is the initial block, and* $F \subseteq Q$ *is a set of final blocks.*

Note that for two sequential blocks q_i and q_j that have a non-conditional dependency, we can consider t, f as its condition which states that the control flow can move from q_i to q_j without satisfying a true or false condition. Since the transition function δ is defined as a total function, we consider a non-conditional edge from each final statement to itself. However, we do not label a non-conditional edge to increase the readability of a CFG. For more clarity, we use a simple example to illustrate the CFG of a program. Fig. 1 shows the CFG of the program of Listing 1 which calculates the sum of 1 to n. In this example, program block 1 contains statements 1, 2, 3, 4, 5, block 2 contains 6,7,8, and block 3 contains statements 9 and 10.

```
1   void main () {
2   int n,s=1,counter=1;
3   cout<<"enter an integer number:";
4   cin>>n;
5   while(counter<=n) {
6   sum=sum+counter;
7   counter=counter+1;
8   }
9   cout<<sum;
10  }
```

Listing 1. An example program to creating CFG

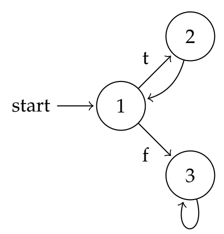

FIGURE 1. The CFG of Listing 1

In the following, we correspond a hypergroup to a CFG independent of initial and final blocks, so for more simplicity the notion (Q,Σ,δ) is considered for a CFG. Moreover, we define δ^* as an extension of δ to the domain $Q \times \Sigma^*$ by $\delta^*(q,\lambda) = q$ for each $q \in Q$; $\delta^*(q,xa) = \delta(\delta^*(q,x),a)$ for each $x \in \Sigma^*, a \in \Sigma$.

For two CFGs $\mathcal{G}_1 = (Q_1,\Sigma,\delta_1)$ and $\mathcal{G}_2 = (Q_2,\Sigma,\delta_2)$, $\mathcal{G}_2$ is a sub-CFG of $\mathcal{G}_1$ if $Q_2 \subseteq Q_1$ and $\delta_2 = \delta_1 |_{Q_2\times\Sigma}$, i.e., δ_2 is the restriction of δ_1 to $Q_2 \times \Sigma$. Moreover, the nonempty sub-CFG $\mathcal{G}_2$ is called separated if the subscriptions of $\delta_1(Q_1 \setminus Q_2, \Sigma^*)$ and Q_2 be empty. Based on the definitions, a CFG is connected if it does not possess any separated proper sub-CFG.

Suppose $(H,\circ)$ be a hypergroup, then a^n is a non-empty subset of H where a^n is $a \circ a \circ \cdots \circ a$ in which a belongs to H and n is an integer number. If for every $a,b \in H$, we have $a \in a^2 = a^3$ and $a \circ b = a^2 \cup b^2$, then $(H,\circ)$ is a quasi-ordering hypergroup. Note that $(H,\circ)$ is an ordering hypergroup when for every $a,b \in H$, if $a^2 = b^2$ implies $a = b$.

Definition 2.2. *Suppose $\mathcal{G} = (Q,\Sigma,\delta)$ be a CFG. Then we define a hyper-operation on Q by $q_1 \circ_{\mathcal{G}} q_2 = \sigma(\{q_1,q_2\})$ for any $q_1,q_2 \in Q$, where $\sigma(P) = \{q \in Q : \delta^*(q,x) \in P,\ \text{for some } x \in \Sigma^*\}$ for every subset P of Q. Moreover, the hypergroupoid $(Q,\circ_{\mathcal{G}})$ is denoted by $\mathcal{H}(\mathcal{G})$.*

Theorem 2.3. *[3] If $\mathcal{G} = (Q,\Sigma,\delta)$ be a CFG, then $\mathcal{H}(\mathcal{G})$ is a quasi-ordering hypergroup.*

Consider $\mathcal{G}$ as the CFG of Listing 1. For $x = 1$, $x = 2$, and $x = 3$, $x \circ_{\mathcal{G}} x$ is equal to $\{1\}$, $\{1,2\}$, and $\{1,2,3\}$ respectively. Consequently, $\mathcal{H}(\mathcal{G})$ is a quasi-ordering hypergroup.

Proposition 2.4. *[3] Let $\mathcal{G} = (Q,\Sigma,\delta)$ be a CFG, T be a non-empty subset of Q, and T^c be the complement of T in Q. Then the following assertions hold:*

(I) If $(T,\Sigma,\delta |_{T\times\Sigma})$ is a sub-CFG of $\mathcal{G}$, then T^c is a sub-hypergroup of $\mathcal{H}(\mathcal{G})$.

(II) If $(T,\Sigma,\delta |_{T\times\Sigma})$ is a separated sub-CFG of $\mathcal{G}$, then T^c is a sub-CFG of $\mathcal{G}$, and T is a sub-hypergroup of $\mathcal{H}(\mathcal{G})$.

(III) If T is a sub-hypergroup of $\mathcal{H}(\mathcal{G})$, then $(T^c,\Sigma,\delta |_{T\times\Sigma})$ is a sub-CFG of $\mathcal{G}$.

For sub-hypergroups $(H_1,\circ)$ and $(H_2,\circ)$ of the commutative hypergroup $(H,\circ)$, if $H = H_1 \circ H_2$ and the subscriptions of H_1 and H_2 be empty, then the hypergroup is an inner disjoint product of its sub-hypergroups. Moreover, the commutative hypergroup $(H,\circ)$ is inner irreducible if for any pair H_1 and H_2 of its sub-hypergroups such that $H_1 \circ H_2 = H$, we have $H_1 \cap H_2 \neq \emptyset$.

Theorem 2.5. *[3] Let $\mathcal{G} = (Q,\Sigma,\delta)$ ba a CFG, then $\mathcal{G}$ is connected if and only of $\mathcal{H}(\mathcal{G})$ is inner irreducible.*

For example, consider $\mathcal{G}$ as the CFG of Listing 1. Based on the Theorem 2.5 the hypergroup of this CFG (called CFGH) is inner irreducible since there are no inner disjoint product of its sub-hypergroups, thus this CFGH is connected.

Consider the CFG $\mathcal{G} = (Q, \Sigma, \delta)$. It is strongly connected if for any $q_1$and q_2 belong to Q, there exist $u, v \in \Sigma^*$ such that $\delta^*(q_1, u) = q_2$ and $\delta^*(q_2, v) = q_1$. In other words, $\mathcal{G}$ is strongly connected if it consists of a unique layer. Moreover, if hypergroup H is equal to $\{h\} \cup h^2$ for some $h \in H$, then it is called 2-single-power cyclic [2].

Theorem 2.6. *[3] The following assertions are equivalent for the CFG $\mathcal{G} = (Q, \Sigma, \delta)$:*
(I) $\mathcal{G}$ is strongly connected.
(II) The hypergroup $\mathcal{H}(\mathcal{G})$ is 2-single-power cyclic.
(III) The hypergroup $\mathcal{H}(\mathcal{G})$ is the total hypergroup on Q.

Consider the CFG $\mathcal{G} = (Q, \Sigma, \delta)$. If for any state $q \in Q$ and any word $x \in \Sigma^*$ there exists a word $y \in \Sigma^*$ such that $\delta^*(q, xy) = q$, then $\mathcal{G}$ is called retrievable.

Theorem 2.7. *[3] A CFG $\mathcal{G}$ is retrievable if and only if the state set of each sub-CFG is a sub-hypergroup.*

Corollary 2.8. *[3] For the retrievable CFG $\mathcal{G}$, the following assertions are equivalent: (1) $\mathcal{G}$ is connected, (2) $\mathcal{G}$ is simple, and (3) $\mathcal{H}(\mathcal{G})$ is simple.*

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Distance spectral of the unitary Cayley graphs of commutative rings

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Abstract

Let R be a commutative ring with unity $1 \neq 0$ and let $R^{\times}$ be the set of all unit elements of R. The unitary Cayley graph of R, denoted by $G_R = \mathrm{Cay}(R, R^{\times})$, is a simple graph whose vertex set is R and there is an edge between two distinct vertices x and y if and only if $x - y \in R^{\times}$. This paper involves determining the distance, distance Laplacian and distance signless Laplacian spectrum of the unitary Cayley graphs with diameter at most 2.

Keywords and phrases: Unitary Cayley graph, Distance spectrum, Distance Laplacian spectrum, Distance signless Laplacian spectrum..

2010 *Mathematics subject classification:* Primary: 05C50, 13M05.

1. Introduction

Let R be a commutative ring with unity $1 \neq 0$ and let $R^{\times}$ be the set of all unit elements of R. In this paper, we consider the unitary Cayley graph of R, denoted by $G_R = \mathrm{Cay}(R, R^{\times})$, which is a simple graph whose vertex set is R and there is an edge between two distinct vertices x and y if and only if $x - y \in R^{\times}$. The following proposition is a basic consequence of the definition and it was illustrated in [1, Proposition 2.2].

Proposition 1.1. *Let R be a commutative ring.*

(a) *Then G_R is a $|R^{\times}|$-regular graph.*

(b) *If R is a local ring with maximal ideal $\mathfrak{M}$, then G_R is a complete multipartite graph whose partite sets are the cosets of $\mathfrak{M}$ in R. In particular, G_R is a complete graph if and only if R is a field.*

(c) *If R is an Artinian ring and $R \cong R_1 \times \ldots \times R_t$ as a product of local rings, then $G_R \cong \otimes_{i=1}^{t} G_{R_i}$. Hence, G_R is a direct product of complete multi-partite graphs.*

For two distinct vertices u and v in a connected graph G, $d(u, v)$ denotes the distance between u and v, i.e., the length of a shortest path between u and v. The maximum distance between two vertices is called the diameter of G and

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denoted by $\operatorname{diam}(G)$, i.e., $\operatorname{diam}(G) = \max\{d(u,v) : u,v \in G\}$. In [1], Akhtar et al. calculated the diameter G_R when R is a finite ring.

Theorem 1.2. *[1, Theorem 3.1] Let $R = R_1 \times \ldots \times R_t$ be an Artinian ring. Then*

$$diam(G_R) = \begin{cases} 1 & t = 1 \text{ and } R \text{ is a field} \\ 2 & t = 1 \text{ and } R \text{ is not a field} \\ 2 & t > 1 \text{ and } f_1 > 3 \\ 3 & t > 2 \text{ and } f_1 = 2, f_2 > 3 \\ \infty & t > 2 \text{ and } f_1 = f_2 = 2. \end{cases}$$

The transmission of a vertex v, denoted by $Tr(v)$, is defined to be the sum of the distances from v to all other vertices in G, i.e., $Tr(v) = \sum_{u \in V(G)} d(u,v)$. The distance matrix D of a graph G is the matrix indexed by the vertices of G where $D = [d_{ij}]_{n\times n}$ with $d_{ij} = d(v_i, v_j)$ denotes the distance between the vertices v_i and v_j. The spectrum of D, denoted by $\operatorname{Spec}_D(G)$, is called the distance spectrum of G. Let $\operatorname{Diag}(Tr)$ denote the diagonal matrix of the vertex transmissions in G. Similarly to the Laplacian matrix of a connected graph G, we define the distance Laplacian as the matrix $D^L = \operatorname{Diag}(Tr) - D$. The spectrum of D^L, denoted by $\operatorname{Spec}_{D^L}(G)$, is called the distance Laplacian spectrum of the graph G. Also, the distance signless Laplacian matrix of a connected graph G is defined to be $D^{|L|} = \operatorname{Diag}(Tr) + D$. The spectrum of $D^{|L|}$, denoted by $\operatorname{Spec}_{D^{|L|}}(G)$, is called the distance signless Laplacian spectrum of the graph G. In this paper, we study the distance, distance Laplacian and distance signless Laplacian spectral of G_R when $\operatorname{diam}(G_R)$ is at most 2.

2. Main Results

In this section, we present our results about the distance, distance Laplacian and distance signless Laplacian spectral of G_R when $\operatorname{diam}(G_R)$ is at most 2. We start our work with investigating the distance spectral of G_R.

It is well known that the distance matrix of a graph G with diameter 2 can be written in terms of the adjacency matrices of G and its complement $\overline{G}$. So, if G is a regular graph with diameter 2, then the distance spectrum of G can be obtained from its adjacency spectrum as stated in the next theorem.

Theorem 2.1. *[4] Let G be a k-regular graph on n vertices with diameter at most 2 and adjacency spectrum $k = \lambda_1, \lambda_2, \ldots, \lambda_n$. Then the distance spectrum of G is $2n - 2 - k, -(2 + \lambda_2), -(2 + \lambda_3), \ldots, -(2 + \lambda_n)$.*

The adjacency spectrum of the graph G_R has been studied in [1, 5]. With using their results about the adjacency spectrum and this fact that the graph G_R is a regular graph, the following result is an immediate consequences from the above.

Theorem 2.2. *Let R be a finite commutative ring.*

(a) If $(R,\mathfrak{M})$ is a local ring with $|\mathfrak{M}| = m$ and $|\frac{R}{\mathfrak{M}}| = f$, then

$$Spec_D(G_R) = \begin{pmatrix} 0 & m-2 & |R|+m-2 \\ |R|-f & f-1 & 1 \end{pmatrix}.$$

(b) If $R \cong R_1 \times R_2 \times \ldots \times R_t$ where $t \geqslant 2$, R_i is a local ring with maximal ideal $\mathfrak{M}_i$ such that $|\mathfrak{M}_i| = m_i$, $|\frac{R}{\mathfrak{M}_i}| = f_i$ for all $1 \leqslant i \leqslant t$ and $3 < f_1 \leqslant f_2 \leqslant \cdots \leqslant f_t$, then the distance spectrum of G_R consists of:

(i) $2|R| - |R^\times| - 2$ with multiplicity 1.

(ii) $-\left(2 + (-1)^{|C|}\dfrac{|R^\times|}{\prod_{j\in C}|R_j^\times|/m_j}\right)$ with multiplicity $\prod_{j\in C}|R_j^\times|/m_j$ for all non-empty subsets C of the set $\{1,2,\ldots,t\}$.

(iii) -2 with multiplicity $|R| - \prod_{i=1}^{t}(1+|R_i^\times|/m_i)$

The distance Laplacian spectral of the graphs with diameter at most 2 has been investigated in [3].

Theorem 2.3. *[3, Theorem 3.1.] Let G be a connected graph on n vertices with diameter at most 2. Let $\lambda_1^L \geqslant \lambda_2^L \geqslant \cdots \geqslant \lambda_n^L = 0$ be the Laplacian spectrum of G. Then the distance Laplacian spectrum of G is $\mu_1 = 2n - \lambda_{n-1}^L \geqslant \mu_2 = 2n - \lambda_{n-2}^L \geqslant \cdots \geqslant \mu_{n-1} = 2n - \lambda_1^L \geqslant \mu_n = 0$.*

The Laplacian spectral of the graph G_R has been calculated in [6]. So, we can deal with the distance Laplacian matrix of G_R and calculate its spectral.

Theorem 2.4. *Let R be a finite commutative ring.*

(a) Let $(R,\mathfrak{M})$ be a local ring with $|\mathfrak{M}| = m$ and $|\frac{R}{\mathfrak{M}}| = f$. Then

$$Spec_{D^L}(G_R) = \begin{pmatrix} 0 & |R| & |R|+m \\ 1 & f-1 & |R|-f \end{pmatrix}.$$

(b) Let $R \cong R_1 \times R_2 \times \ldots \times R_t$ where $t \geqslant 2$, R_i is a local ring with maximal ideal $\mathfrak{M}_i$ such that $|\mathfrak{M}_i| = m_i$, $|\frac{R}{\mathfrak{M}_i}| = f_i$ for all $1 \leqslant i \leqslant t$ and $3 < f_1 \leqslant f_2 \leqslant \cdots \leqslant f_t$. Then the distance spectrum of G_R consists of:

(i) $2|R| - |R^\times| - 2$ with multiplicity 1.

(ii) $-\left(2 + (-1)^{|C|}\dfrac{|R^\times|}{\prod_{j\in C}|R_j^\times|/m_j}\right)$ with multiplicity $\prod_{j\in C}|R_j^\times|/m_j$ for all non-empty subsets C of the set $\{1,2,\ldots,t\}$.

(iii) -2 with multiplicity $|R| - \prod_{i=1}^{t}(1+|R_i^\times|/m_i)$

The following result concerns with the distance signless Laplacian spectrum of a graph with diameter at most 2.

Theorem 2.5. *[2, Theorem 6] Let G be a connected k-regular graph on n vertices with diameter at most 2. If $\{2k,\lambda_2^+,\ldots,\lambda_n^+\}$ are the eigenvalues of the signless*

Laplacian matrix $|L|(G)$ of G, then the distance signless Laplacian eigenvalues of G are $4n-2k-4$ and $2n-4-\lambda_i^+$ for all $2 \leqslant i \leqslant n$.

In last result, we derive the distance signless Laplacian spectrum of the graph G_R with diameter at most 2. By Theorem 2.5, we only need to know about the signless Laplacian spectral of G_R, which has been studied in [6].

Theorem 2.6. *Let R be a finite ring.*

(a) If $(R,\mathfrak{M})$ is a local ring with $|\mathfrak{M}| = m$ and $|\frac{R}{\mathfrak{M}}| = f$, then

$$Spec_{D^{|L|}}(G_R) = \begin{pmatrix} |R|+m-4 & |R|+2m-4 & 2(|R|+m)-4 \\ |R|-f & f-1 & 1 \end{pmatrix}.$$

(b) Let $t \geqslant 2$ and $f_1 > 3$. If $R \cong R_1 \times R_2 \times \ldots \times R_t$, then the distance signless Laplacian spectrum of G_R consists of:

(i) $4|R| - 2|R^\times| - 4$ with multiplicity 1,

(ii) $2|R| - |R^\times| - 4$ with multiplicity $|R| - \prod_{i=1}^t f_i$,

(iii) $2|R| - 4 - \lambda_A$ with multiplicity $\prod_{i\in A'}(f_i - 1)$ for all $A \subsetneq \{1,2,\ldots,t\}$, where

$$\lambda_A = |R^\times| + (-1)^{|A'|} \prod_{i\in A} |R_i^\times| \prod_{j\in A'} |m_j|.$$

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Supplemented acts over monoids and their properties

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Abstract

In this talk, we generalize the notion of supplement in modules to S-acts for a monois S. In contrast to the case of modules, here we show that supplements of S-acts always exist and uniquely characterize the supplement of a proper subact of an S-act. We introduce supplemented acts as acts whose proper subacts all have proper supplements and study some connections between the property of being supplemented and some other properties of acts. Among other results, it is proved that supplemented acts are exactly completely reducible ones.

Keywords and phrases: S-act, supplement, supplemented, hollow.
2010 *Mathematics subject classification:* 20M30, 20M50.

1. Introduction

Miyashita [6] initiated the study of supplemented modules. This notion was also studied by Kasch and Mares [3] and continued in many papers (see, for example, [1, 2, 5]). A module M is called supplemented if every submodule N of M has a supplement in M, that is, a submodule K of M which is minimal with respect to $M = N + K$. Supplements are applied to get projective covers of modules. Here we generalize the concept of supplement in modules to acts over monoids. First we explicitly characterize supplements of proper subacts of an S-act so that, in contrast to the case of modules, they always uniquely exist. We show that S-acts whose proper subacts have improper supplements are exactly the hollow acts, the acts whose proper subacts are superfluous. Thereafter, we consider those S-acts for which the supplement of any proper subact is proper, namely supplemented acts, and study some relationship between such acts and some other classes of acts. In particular, it is proved that supplemented acts coincide with completely reducible acts, the acts which are disjoint unions of simple subacts.

Let S be a monoid. A non-empty set A is called a *(right) S-act* if there is a mapping $\lambda : A \times S \to A$, denoting $\lambda(a, s)$ by as, satisfying $a(st) = (as)t$ and $a1 = a$ for all $a \in A$ and $s, t \in S$. A non-empty subset B of A is called a *subact*

* speaker

of A if $bs \in B$ for every $s \in S$ and $b \in B$. By a *simple* S-act we mean an S-act with no proper subact. An S-act A is called *completely reducible* if it is a disjoint union of simple subacts. Also, A is said to be *decomposable* if it is a disjoint union of two (proper) subacts; otherwise, it is called *indecomposable*.

For undefined terms and notations concerning S-acts, one may consult [4].

2. Main Results

We first extend the notion of supplement in modules to acts.

Definition 2.1. *Let B be a subact of an S-act A. A subact C of A is said to be a supplement of B in A if C is minimal with respect to $A = B \cup C$, that is, $A = B \cup C$ and if $A = B \cup D$ for some subact D of C, then $D = C$.*

It is clear from the definition of supplement that a subact B of an S-act A is a supplement of A in A if and only if B is simple. So the supplement of an S-act A in itself is not unique and does not exist in general. However, we show that supplements of proper subacts of an S-act always uniquely exist and characterize them in the following:

Theorem 2.2. *Let B be a proper subact of an S-act A. Then a subact C of A is a supplement of B in A if and only if $C = (A \setminus B)S$, i.e. the subact generated by $A \setminus B$.*

From now on, the word "supplement" stands only for supplements of *proper* subacts and the (unique) supplement $(A \setminus B)S$ of a proper subact B of an S-act A in A is denoted by B^s_A.

Proposition 2.3. *The supplement of any maximal subact of an S-act is cyclic.*

Let B be a subact of an S-act A. Then B is called *superfluous* if $B \cup C \neq A$ for each proper subact C of A. An S-act A is said to be *hollow* if any proper subact of A is superfluous.

Here we present an equivalent condition for an act to be hollow in terms of supplements.

Theorem 2.4. *An S-act A is hollow if and only if $B^s_A = A$ (equivalently, $B \subseteq B^s_A$) for any proper subact B of A.*

Corollary 2.5. *An S-act A is hollow if and only if for any proper subact B of A and any $b \in B$, we have $b = as$ for some $a \in A \setminus B$ and $s \in S$.*

In what follows, the notion of supplemented acts is introduced and studied. Moreover, some connections between the property of being supplemented and other properties of acts are investigated.

Recall that a module M is supplemented if each submodule of M has a supplement in M. As for acts, we make the following definition.

Definition 2.6. *An S-act A is called supplemented if the supplement of any proper subact B of A is proper in A, that is, $B^s_A < A$.*

Lemma 2.7. *Let A be a supplemented S-act. Then each subact of A is supplemented.*

Corollary 2.8. *Every cyclic subact of a supplemented act is simple.*

In the following, we present some equivalent conditions for an act to be supplemented.

Theorem 2.9. *Let A be an S-act. Then the following are equivalent:*

(i) *A is supplemented.*

(ii) *A is completely reducible.*

(iii) *Every cyclic subact of A is simple.*

(iv) *For any proper subact B of A, there is a proper subact C of A with $A = B \cup C$.*

(v) *For any proper subact B of A, the intersection $B \cap B^s_A$ is empty, i.e. $B^s_A = A \setminus B$.*

Proposition 2.10. *Let B be a proper subact of a supplemented S-act A. Then $(B^s_A)^s_A = B$.*

Corollary 2.11. *An S-act A is supplemented if and only if any proper subact of A is a supplement of a proper subact of A.*

Proposition 2.12. *Let $A = \bigcup_{i \in I} A_i$ be an S-act where each $A_i, i \in I$, is a subact of A. Then A is supplemented if and only if each $A_i, i \in I$, is supplemented.*

The unique decomposition theorem for acts states that every S-act can be uniquely decomposed into a disjoint union of indecomposable S-acts. Thus, from Proposition 2.12 we obtain:

Corollary 2.13. *An S-act A is supplemented if and only if every indecomposable subact of A is supplemented.*

We say that an S-act A is *Artinian* (*Noetherian*) if every descending (ascending) chain of subacts of A terminates.

Theorem 2.14. *Let A be a supplemented S-act. Then A is Noetherian if and only if it is Artinian.*

Proposition 2.15. *Let A be a finitely generated S-act and B be a proper subact of A with proper supplement and $(B^s_A)^s_A = B$. Then B is finitely generated.*

Corollary 2.16. *The following assertions hold:*

(i) *A supplemented S-act is Noetherian if and only if it is finitely generated.*

(ii) *Let A be a finitely generated S-act and $B^s_A < A$ for some proper subact B of A. Then B^s_A is finitely generated.*

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A cotorsion theory in the homotopy category of complexes of flat R-modules

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Abstract

This note is devoted to the study of cotorsion theory in the homotopy category of flat R-modules, $\mathbb{K}(\text{Flat-}R)$. Let R be an arbitrary ring and $\mathbb{K}(\text{dg-CotF-R})$ be the homotopy category of all dg-cotorsion complexes of flat R-modules. It is proved that $\left(\mathbb{K}_{\text{p}}(\text{Flat-}R), \mathbb{K}(\text{dg-CotF-R})\right)$ forms a complete cotorsion pair in $\mathbb{K}(\text{Flat-}R)$, where $\mathbb{K}_{\text{p}}(\text{Flat-}R)$ is the subcategory of all flat complexes.

Keywords and phrases: Homotopy category; flat complex; dg-cotorsion complex.
2010 *Mathematics subject classification:* Primary: 18E30, 55U35; Secondary: 18G35, 18G20, 16E05.

1. Introduction

Cotorsion pairs (or cotorsion theories), originally defined by Salce in [7], have now appeared in various contexts and play significant role in various fields of mathematics. Their capability in proving the Flat Cover Conjecture in Mod-R and in $\mathcal{C}(R)$ [1, 3] is worthwhile.

A couple of works concerning cotorsion theories in the category of complexes are worth recalling: the first one is a paper by Gillespie [4] where certain cotorsion theories in the category of unbounded complexes over an abelian category $\mathcal{C}$ arise from cotorsion theories of $\mathcal{C}$ and this applies to get flat cover conjecture over $\text{Ch}(\mathcal{C})$. The second is [2] in which Enochs and his colleagues showed that $\mathcal{C}_{\text{p}}(\text{Flat-}R)$ is a covering class and $\mathcal{C}_{\text{p}}(\text{Flat-}R)^{\perp}$ is an enveloping class in the category $\mathcal{C}(R)$, where $\mathcal{C}_{\text{p}}(\text{Flat-}R)$ is the full subcategory of $\mathcal{C}(R)$ consisting of all flat complexes. The aim of the present paper is to prove the existence of a complete cotorsion pair $(\mathbb{K}_{\text{p}}(\text{Flat-}R), \mathbb{K}(\text{dg-CotF-R}))$ in the homotopy category $\mathbb{K}(\text{Flat-}R)$ of complexes of flat R-modules, for arbitrary R, where $\mathbb{K}_{\text{p}}(\text{Flat-}R)$ is the subcategory of all flat complexes and $\mathbb{K}(\text{dg-CotF-R})$ is the homotopy category of dg-cotorsion complexes of flat R-modules. In the setting of quasi coherent sheaves over a Noetherian scheme, this cotorsion pair was discovered in [5]. The approach taken there is based on set-theoretic

* speaker

arguments that are typically applied by many authors, particularly in the stage of proof of the existence of various covers and envelopes. We prove a version of this result in the setting of modules over an arbitrary ring. Our approach is a simpler one, based on the theory of homotopic chain maps. The other privilege is that one does not need to restrict to commutative Noetherian rings, as a prerequisite of passing from the context of Noetherian schemes to that of rings.

2. Main Results

In this paper R denotes an associative ring with identity and by default all modules are left R-modules. If we say that $\mathbf{X}$ is a complex, we mean that it is a complex of R-modules, that is, a sequence of (left) R-modules X^i and R-linear maps $\partial^i : X^i \to X^{i+1}$, $i \in \mathbb{Z}$, such that $\partial^{i+1}\partial^i = 0$.

We denote by $\mathcal{C}(R)$ the category of complexes over R whose morphisms are the usual chain maps between complexes.

The homotopy category $\mathbb{K}(R)$ has as objects the complexes in R and the morphisms are the homotopy equivalences of morphisms in $\mathcal{C}(R)$. Let $\mathcal{X}$ be a class of R-modules. We denote by $\mathbb{K}(\mathcal{X})$ the homotopy category of complexes over $\mathcal{X}$, which is a triangulated subcategory of $\mathbb{K}(R)$ see, e.g., [8].

Definition 2.1. *A pair $(\mathcal{S},\mathcal{C})$ of full subcategories of $\mathcal{T}$ is called a cotorsion pair in $\mathcal{T}$ if ${}^{\perp}\mathcal{C} = \mathcal{S}$ and $\mathcal{S}^{\perp} = \mathcal{C}$, where the left orthogonal of $\mathcal{C}$ in $\mathcal{T}$ is defined by ${}^{\perp}\mathcal{C} = \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(X,C) = 0 \text{ for all } C \in \mathcal{C}\}$. The right orthogonal of $\mathcal{S}$ in $\mathcal{T}$ is defined similarly. A cotorsion pair $(\mathcal{S},\mathcal{C})$ is called complete if any object X of $\mathcal{T}$ fits into a triangle $S \longrightarrow X \longrightarrow C \longrightarrow T(S)$ where $S \in \mathcal{S}$ and $C \in \mathcal{C}$.*

Definitions 2.2.

- *(i) An acyclic complex $\mathbf{F}$ of flat R-modules is called a flat complex if all its syzygies are also flat R-modules. We denote by $\mathbb{K}_{\mathrm{p}}(\text{Flat-}R)$ the full subcategory of $\mathbb{K}(\text{Flat-}R)$ consisting of flat complexes.*
- *(ii) A complex $\mathbf{C}$ of cotorsion R-modules is called dg-cotorsion if $\mathrm{Hom}_R(\mathbf{F},\mathbf{C})$ is exact, whenever $\mathbf{F}$ is a flat complex.*
- *(iii) A complex $\mathbf{C}$ of cotorsion flat modules is said to be dg-cotorsion flat if it is dg-cotorsion. We denote by $\mathbb{K}(\text{dg-CotF-R})$ the corresponding homotopy category.*
- *(iv) A complex $\mathbf{C}$ is cotorsion if it is exact and $\ker(C^i \to C^{i+1})$ is cotorsion R-module for all $i \in \mathbb{Z}$.*

Remark 2.3. *By [6, Theorem 8.6], $\mathbb{K}_{\mathrm{p}}(\text{Flat-}R)$ is the right orthogonal of $\mathbb{K}(\text{Proj-}R)$ in $\mathbb{K}(\text{Flat-}R)$, that is, $\mathbb{K}_{\mathrm{p}}(\text{Flat-}R) = \mathbb{K}(\text{Proj-}R)^{\perp}$.*

Proposition 2.4. *Let $\mathbf{F} \in \mathbb{K}(\text{Flat-}R)$ satisfy $\mathrm{Hom}_{\mathbb{K}(R)}(\mathbf{F},\mathbf{C}) = 0$ for any $\mathbf{C} \in \mathbb{K}(\text{dg-CotF-R})$. Then $\mathbf{F}$ is exact.*

Lemma 2.5. *The inclusion* $\mathbb{K}(\text{Proj-}R) \longrightarrow \mathbb{K}(\text{Flat-}R)$ *has a right adjoint; that is, for any complex* $\mathbf{F}$ *of flat* R*-modules, there exists a triangle* $\mathbf{P} \longrightarrow \mathbf{F} \longrightarrow \mathbf{L} \longrightarrow \Sigma\mathbf{P}$ *with* $\mathbf{P} \in \mathbb{K}(\text{Proj-}R)$ *and* $\mathbf{L} \in \mathbb{K}_{\text{p}}(\text{Flat-}R)$.

PROOF. See [6, Proposition 8.1]. □

Proposition 2.6. *For any ring* R*,* $(\mathbb{K}_{\text{p}}(\text{Flat-}R), \mathbb{K}(\text{dg-CotF-R}))$ *is a cotorsion pair in the homotopy category of complexes of flat* R*-modules.*

PROOF. Here we give a sketch of the proof .

Step 1. According to the definitions, in order to show $\mathbb{K}_{\text{p}}(\text{Flat-}R)^{\perp} = \mathbb{K}(\text{dg-CotF-R})$, one only needs to verify $\mathbb{K}_{\text{p}}(\text{Flat-}R)^{\perp} \subseteq \mathbb{K}(\text{dg-CotF-R})$. Choose

$$\mathbf{X} : \cdots \longrightarrow X^i \xrightarrow{\partial^i} X^{i+1} \xrightarrow{\partial^{i+1}} X^{i+2} \longrightarrow \cdots$$

in $\mathbb{K}(\text{Flat-}R)$ such that it lies in $\mathbb{K}_{\text{p}}(\text{Flat-}R)^{\perp}$. We must check that for any i, X^i is a cotorsion R-module and, with no lose in generality, we may set $i = 0$.

Step 2. The inclusion $\mathbb{K}_{\text{p}}(\text{Flat-}R) \subseteq^{\perp} \mathbb{K}(\text{dg-CotF-R})$ is just the definition. To settle the reverse inclusion, pick an object $\mathbf{F} \in \mathbb{K}(\text{Flat-}R)$ that lies inside ${}^{\perp}\mathbb{K}(\text{dg-CotF-R})$. By virtue of Proposition 2.4, we deduce at the first pace that it is exact. Take the triangle $\mathbf{P} \longrightarrow \mathbf{F} \longrightarrow \mathbf{L} \longrightarrow \Sigma^{-1}\mathbf{P}$ from Lemma 2.5 where $\mathbf{P} \in \mathbb{K}(\text{Proj-}R)$ and $\mathbf{L} \in \mathbb{K}_{\text{p}}(\text{Flat-}R)$. The complex $\mathbf{P}$ is then exact, because $\mathbf{L}$ is so, and lies in ${}^{\perp}\mathbb{K}(\text{dg-CotF-R})$, as $\mathbf{L}$ and $\mathbf{F}$ do. In view of the fact that $\mathbb{K}(\text{Proj-}R)^{\perp} = \mathbb{K}_{\text{p}}(\text{Flat-}R)$, [6, Theorem 8.6], it suffices to show that $\mathbf{P} \in \mathbb{K}_{\text{p}}(\text{Flat-}R)$. □

Theorem 2.7. *For any ring* R*, the pair* $\Big(\mathbb{K}_{\text{p}}(\text{Flat-}R), \mathbb{K}(\text{dg-CotF-R})\Big)$ *is a complete cotorsion pair in* $\mathbb{K}(\text{Flat-}R)$.

PROOF. A well-known result says that every $\mathbf{X} \in \mathcal{C}(R)$ admits a flat cover with a dg-cotorsion kernel. So if $\mathbf{X} \in \mathbb{K}(\text{Flat-}R)$, then there exists a short exact sequence $0 \longrightarrow \mathbf{C} \longrightarrow \mathbf{F} \longrightarrow \mathbf{X} \longrightarrow 0$ in $\mathcal{C}(R)$ with $\mathbf{F} \in \mathbb{K}_{\text{p}}(\text{Flat-}R)$ and $\mathbf{C} \in \mathbb{K}(\text{dg-CotF-R})$. But this sequence will then split at the module level, and consequently transforms into a triangle $\mathbf{C} \longrightarrow \mathbf{F} \longrightarrow \mathbf{X} \longrightarrow \Sigma^{-1}\mathbf{C}$ in $\mathbb{K}(\text{Flat-}R)$. Therefore, by definition, the pair $\Big(\mathbb{K}_{\text{p}}(\text{Flat-}R), \mathbb{K}(\text{dg-CotF-R})\Big)$ is a complete cotorsion pair in the homotopy category of complexes of flat R-modules. □

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A Note on Rees Large Subacts

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Abstract

In this paper, Rees large subacts and Rees Socle of an S-act based on Rees congruences are studied. We also investigate when S-acts satisfy the descending or ascending chain condition on non-Ress large subacts.

Keywords and phrases: Rees Large, Essential, S-acts, Rees Artinian, Cocyclic .
2010 *Mathematics subject classification:* 20M30.

1. Introduction

Throughout this paper, S will denote a monoid. In this section, we recall some notions which will be needed in the sequel. Recall that an equivalence relation ρ on an S-act A_S is said to be a *congruence* on A_S if $a\rho a'$ implies $as\ \rho\ a's$ for any $a, a' \in A_S$ and $s \in S$. The set of all congruences on A_S is denoted by $Con(A)$. We recall that for a subact B of an S-act A, the *Rees congruence* ρ_B is defined by $(a, b) \in \rho_B$ if $a, b \in B$ or $a = b$. The set of all Rees congruences on A_S is denoted by $RCon(A)$. We recall from [5] that an S-act A_S is *finitely cogenerated* if for any family of congruences $\{\rho_i |\ i \in I\}$ on A_S, if $\cap_{i\in I}\rho_i = \Delta_A$, then $\cap_{j\in J}\rho_j = \Delta_A$ for some finite subset J of I. We also call A_S *finitely Rees cogenerated* whenever for any family of Rees congruences $\{\rho_{B_i} |\ i \in I\}$ on A_S, if $\cap_{i\in I}\rho_{B_i} = \Delta_A$, then $\cap_{j\in J}\rho_{B_j} = \Delta_A$ for some finite subset J of I.

We recall from [6] that an S-act A_S is called *Rees artinian* (*Rees noetherian*) if it satisfies the descending (ascending) chain condition on its Rees congruences, equivalently, on its subacts (or, equivalently, the minimal (maximal) condition on its subacts). By [6, Proposition 7], Rees artinian (Rees noetherian) S-acts are those which all their factor acts (subacts) are finitely Rees cogenerated (generated). Moreover, an S-act A_S is called *artinian* (*noetherian*) in case $Con(A)$ satisfies the descending (ascending) chain condition, equivalently, the minimal (maximal) condition. By [6, Theorems 5 and 6], artinian S-acts are those which all their factor acts are finitely cogenerated, also noetherian S-acts are those which all their congruences are finitely generated.

* speaker

Recall from [1] that a monomorphism $f : A \to B$ of S-acts is said to be *essential* if for each homomorphism $g : B \to C$, g is a monomorphism whenever gf is. If f is an inclusion map, then B is said to be an *essential extension* of A, or A is called *large* in B. In this situation, we write $A \subseteq' B$. It follows from [3, Lemma 3.1.15] that $A \subseteq' B$ if and only if for every non-trivial $\theta \in Con(B)$, $\theta \cap \rho_A \neq \Delta_B$. Moreover, recall from [4] that if S contains a zero, a non-zero subact B of A_S is called *intersection large* if for all non-zero subact C of A_S, $B \cap C \neq \Theta$, and will denoted by B is $\cap$-large in A_S. In [2], the authors proved that every large subact of A_S is $\cap$-large, but the converse is not true.

In the category Act — S, we get $\coprod_{i \in I} A_i = \dot{\cup}_{i \in I} A_i$. If S contain a zero, In fact in the category Act_0— S, $\coprod_{i \in I} A_i = \cup_{i \in I} A_i$ where $A_i \cap A_j = \Theta$. Now, We merge both cases and express them as $\coprod_{i \in I} A_i = \cup_{i \in I} A_i$ where $|A_i \cap A_j| \leq 1$. We refer the reader to [3] for preliminaries and basic results related to S-acts.

2. Main Results

In this section, We begin with the following definition that generalizes the notion of $\cap$-large for an arbitrary S (can be without zero).

Definition 2.1. *Let A_S be an S-act. A subact B is called Rees large (Rees essential) in A if for every non-trivial Rees congruence ρ_C, $\rho_C \cap \rho_B \neq \Delta_A$, which is denoted by $B \subseteq_{RL} A$.*

It is easily checked that B is Rees large in A if for every proper subact C, $|C \cap B| > 1$. In module theory, the socle of a module is defined to be the sum of the minimal nonzero its submodules. Equivalently, the intersection of essential submodules. For S-acts, socle and Rees socle defined as follows.

Definition 2.2. *Let A_S be an S-act. Socle of A is defined by*

$$\mathrm{Soc(A)} = \cap\{\mathrm{L} \subseteq \mathrm{A} |\ \mathrm{L} \subseteq' \mathrm{A}\},$$

and Rees socle is defined by

$$\mathrm{RSoc(A)} = \cap\{\mathrm{L} \subseteq \mathrm{A} |\ \mathrm{L} \subseteq_{\mathrm{RL}} \mathrm{A}\}.$$

If $\mathrm{Soc(A)}, \mathrm{RSoc(A)} \neq \emptyset$, then $\mathrm{Soc(A)}$ and $\mathrm{RSoc(A)}$ are subacts of A. By an argument closely resembles the proof in module theory, one can show the following proposition.

Proposition 2.3. *Let A_S be an S-act. Then* $\mathrm{RSoc(A)}$ *is the union of simple or θ-simple subacts of A.*

Obviously, $\mathrm{RSoc(A)} \subseteq \mathrm{Soc(A)}$. But, unlike the case for module theory, the converse can not be valid in general. For instance, if $S = (\mathbb{N}, max) \cup \{\infty\}$, it is not difficult to see that $\mathrm{RSoc(S_S)} = \{\infty\} \subsetneq \mathrm{Soc(S_S)} = \mathrm{S}$.

The next result presents some general properties of the essentiality and socle.

Proposition 2.4. *For a monoid S, the following statements are true.*
(i) *If $B_1 \subseteq' A_1$ and $B_2 \subseteq' A_2$, then $B_1 \cap B_2 \subseteq' A_1 \cap A_2$.*
(ii) *If $f : A_S \to B_S$ is an S-morphism and $B' \subseteq' B$, then $f^{-1}(B') \subseteq' A_S$.*
(iii) *If $B \subseteq' A$ and B is indecomposable, then A is indecomposable or $A = A' \cup \Theta$ in which A' is indecomposable.*
(iv) *If $A = A_1 \coprod A_2$, then* $\mathrm{Soc}(A_1) \cup \mathrm{Soc}(A_2) \subseteq \mathrm{Soc}(A)$.

The previous proposition is also valid where "$\subseteq'$" is replaced by "$\subseteq_{RL}$" and "socle" by "Rees socle".

Proposition 2.5. *Every Rees essential extension of a finitely Rees cogenerated S-act is again finitely Rees cogenerated.*

Now, we use the concepts of Rees large and Rees socle to characterize finitely Rees cogenerated S-acts.

Theorem 2.6. *An S-act A_S is finitely Rees cogenerated if and only if $RSoc(A)$ is a finitely generated Rees large subact of A_S.*

A non-Rees large subact means a subact that can not be a Rees large subact, similarly non-finitely generated or non-large can be defined.

Lemma 2.7. *Let A_S be an S-act. Then each non-finitely generated subact of A_S is Rees large in A_S if and only if each non-Rees large subact of A_S is Rees noetherian.*

Corollary 2.8. *If each non-finitely generated subact of A_S is Rees large in A_S, then A_S is a finite coproduct of indecomposable S-acts.*

Recall from [7] that an S-act A is called *cocyclic* if the intersection of its non-zero subacts is non-zero. So A is cocyclic if and only if every its non-zero subact is Rees large. Now we characterize S-acts which satisfy DCC (descending chain condition) on non-Rees large subacts . Clearly cocyclic S-acts and Rees artinian S-acts satisfy DCC on non-Rees large subacts.

Theorem 2.9. *The following statements are equivalent for an S-act A_S.*
(i) *A_S satisfies DCC on non-Rees large subacts.*
(ii) *Every non-Rees large subact of A_S is Rees artinian.*
(iii) *Every decomposable subact of A_S is Rees artinian.*
In this case, A_S is either cocyclic or finitely Rees cogenerated.

Theorem 2.10. *The following statements are equivalent for an S-act A_S.*
(i) *A_S satisfies ACC on non-Rees large subacts.*
(ii) *Every non-Rees large subact of A_S is Rees noetherian.*
(iii) *Every decomposable subact of A_S is Rees noetherian.*
In this case A_S is a finite coproduct of indecomposable subacts.

Proposition 2.11. *If an S-act A_S satisfies the ascending chain condition on non-Rees large subacts, then A_S is cocyclic or it has a Rees noetherian subact B_S such that B is a Rees large subact of A.*

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Pseudo o-minimaliy for double stone algebras

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Abstract

Lei Chen, Niandong Shi and Guohua Wu introduced the notion of pseudo o-minimality for stone algebras. They then described definable sets in stone algebras using P.H Schmitt's result for model completion and quantifier elimination, and proved that an extension of the theory of stone algebras is *Pseudo* $o-minimal$. In this paper we investigate pseudo o-minimality in double stone algebras using David M. Clark results and prove that the theory of double stone algebras **DBS** is pseudo o-minimal.

Keywords and phrases: Double stone algebra, Definable set, Quantifier elimination, Pseudo o-minimality. .

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1. Introduction

The o-minimal linear ordered structures introduced by Van Den Dries in [7] have been extensively studied in the last four decades. In [6], Toffalory generalized the concept of o-minimality to partially ordered structures. Then Lei Chen, Niandong Shi and Guohua Wu, using this generalization, introduced the notion of pseudo o-minimality in stone algebras[1]. They investigated definable sets in stone algebras by using schmitt's results in [5]. In this paper, we investigate the definable sets in double stone algebras, examine the o-minimality feature and some model theoric features for double stone algebras . Then we prove that the theory of double stone algebras is pseudo o-minimal.

Definition 1.1. *A first order structure* $\mathcal{S}_* = (S, \wedge, \vee, *, 0, 1. \leq)$ *is called a stone algebra if* $(S, \wedge, \vee, 0, 1)$ *is a bounded distributive lattice and the operation* $*$ *of pseudo complementation satisfies* $\{\forall a(a \wedge a^* = 0), \forall a \forall b(a \wedge b = 0 \to b \leq a^*), \forall a(a^* \vee a^{**} = 1)\}$. $\mathcal{S}_+ = (S, \wedge, \vee, +, 0, 1, \leq)$ *is called a dual stone algebra if* $(S, \wedge, \vee, 0, 1)$ *is a bounded distributive lattice and the operation* $+$ *of dual pseudo complementation satisfies* $\{\forall a(a \vee a^+ = 1), \forall a \forall b(a \vee b = 1 \to a^+ \leq b), \forall a(a^+ \wedge a^+ = 0)\}$. $\mathcal{DS} = (S, \wedge, \vee, *, +, 0, 1, \leq)$ *is called a double stone algebra if* $(S, \wedge, \vee, *, 0, 1, \leq)$ *is a stone algebra and* $(S, \wedge, \vee, +, 0, 1, \leq)$ *is a dual stone algebra.*

* speaker

The subalgebra $Sk(\mathcal{DS}) = \{x^*|x \in \mathcal{DS}\} = \{x \in \mathcal{DS}|x = x^{**}\}$ and it's dual $Sk(\mathcal{DS}) = \{x^+|x \in \mathcal{DS}\} = \{x \in \mathcal{DS}|x = x^{++}\}$ play an important roul in the study of double stone algebras. In fact $Sk(\mathcal{DS}) = Sk(\mathcal{DS})$, and $(Sk(\mathcal{DS}), \wedge, \vee, 0, 1)$ is a boolean algebra. The dense set of $\mathcal{DS}$, $D(\mathcal{DS}) = \{x \in DS|x^* = 0\}$ is a filter of $\mathcal{DS}$, and the set $DD(\mathcal{DS}) = \{x \in DS|x^* = 0, x^+ = 1\}$ is called the doubly dense set of $\mathcal{DS}$.

Lemma 1.2. *Every Double ston algebra $\mathcal{DS}$ has the following properties:*

i) $x \leq x^{**}, x \leq y \to y^* \leq x^*, x = y \to x^* = y^*$, $x^{++} \leq x, x \leq y \to y^+ \leq x^+, x = y \to x^+ = y^+$.

ii) $(x \vee y)^* = x^* \wedge y^*, (x \wedge y)^* = x^* \vee y^*$, $(x \vee y)^+ = x^+ \wedge y^+, (x \wedge y)^+ = x^+ \vee y^+$.

iii) $x^* = x^{***}, x^+ = x^{+++}$.

iv) $0^* = 1, 1^* = 0, 0^+ = 1, 1^+ = 0$.

v) $(x \vee y)^{**} = x^* * \vee y^{**}, (x \wedge y)^{**} = x^{**} \wedge y^{**}$, $(x \vee y)^{++} = x^{++} \vee y^{++}, (x \wedge y)^{++} = x^{++} \wedge y^{++}$.

Lemma 1.3. *For any $x \in \mathcal{DS}$:*

i) $x^{++} \leq x^{**}$.

ii) $x^{+*} = x^{++}$.

iii) $x^{*+} = x^{**}$.

iv) $x^* \leq x^+$.

Theorem 1.4. *(Clarc and Krauss [3]) $(\mathfrak{D}, \mathfrak{E})$ is a full duality between $ISP(\mathcal{DS})$ and $IS_cP(\mathfrak{D}\mathfrak{S})$.*

If $\mathcal{DS} = \mathfrak{E}(\mathfrak{X})$ is a double Stone algebra, then:

$$Sk(\mathcal{DS}) = \{\sigma \in DB|\sigma^{**} = \sigma\} = \{\sigma \in DS|\sigma^{-1}\{a, b\} = \emptyset\},$$
$$DD(\mathcal{DS}) = \{\delta \in DB|\delta^* = 0 \text{ and } \delta^+ = 0\} = \{\delta \in DS|\delta^{-1}\{0, 1\} = \emptyset\}$$

are the skeleton of $\mathcal{DS}$, and the sublattice of doubly dense elements of $\mathcal{DS}$ [4].

Theorem 1.5. *(David clark[2]) For a double stone algebra $\mathcal{DS} = E(\mathfrak{X})$, the following are equivalent:*

i) *$\mathcal{DS}$ is existentially closed.*

ii) *$\mathcal{DS}$ satisfies the following $\forall\exists$-axioms:*

(DS1) $DD(\mathcal{DS})$ is nonempty and form a relatively complemented sublattice of $\mathcal{DS}$.

(DS2) for every $\gamma, \delta \in DD(\mathcal{DS})$. there is a $\sigma \in \mathcal{DS}$ such that $(\gamma \wedge \delta) \vee (\delta \wedge \sigma^*) = \gamma \vee \delta$.

(DS3) $DD(\mathcal{DS})$ contain no covers.

(DS4) if $\delta^* = 0$ and $\delta < 1$, then there is a $\gamma > \delta$ such that $\gamma^+ = \delta^+$.

(DS5) if $\delta^+ = 1$ and $\delta > 0$, then there is a $\gamma < \delta$ such that $\gamma^* = \delta^*$.

Given $\mathbf{Th}(\mathcal{DS})$, as the complete theory of double stone algebras, we introduce the theory **DBS** , which is $\mathbf{Th}(\mathcal{DS})$ with the additional axioms $DS1, ..., DS5$ in theorem 1.5. Since every model of **DBS** is existentially closed, **DBS** is model complete. And since **DBS** has the amalgamation property , we conclude that **DBS** has the quantifier elimination property.

2. pseudo o-minimality for $\mathcal{DBS}$

Definition 2.1. *Let $\mathcal{A} = (A, \leq, ...)$ be a structure partially ordered by $\leq$. $\mathcal{A}$ is said to be pseudo o-minimal if and only if the only subsets of A definable in $\mathcal{A}$ are the finite Boolean combinations of sets defined by formulas $a \leq v$ or $v \leq b$ with a and b in A. i.e if every definable subset of A is a Boolean combination of finitely many strongly connected subsets of A. A set $S \subseteq A$ is said to be strongly connected if for any $x, y \in S$, $x \wedge y \in S$ or $x \vee y \in S$. If every model of a theory **T** is a pseudo o-minimal structure, then **T** is a pseudo o-minimal theory.*

Lemma 2.2. *Every atomic formula of **DBS** is equivalent to a conjunction of the formulas $\tau \leq \nu$, where $\tau \in \{a, x \wedge a, x^* \wedge a, x^+ \wedge a, x^{**} \wedge a, x^{++} \wedge a, x \wedge x^+ \wedge a\}$ and $\nu \in \{b, x \vee b, x^* \vee b, x^+ \vee b, x^{**} \vee b, x^{++} \vee b, x \vee x^* \vee b\}$, where a, b are terms of **DBS** and do not contain variable x.*

PROOF. By lemma 1.2 and lemma 1.3 and distributivity, every atomic formula is equivalent to a conjunction of formulas $\wedge_{j=1}^{m} \tau_j \leq \vee_{k=1}^{n} \nu_k$ such that τ_js and ν_ks are one of $x_j, x_j, x_j^{**}, a, x_k, x_k^{+}, x_k^{++}, b$ where a and b are constants. ☐

Corollary 2.3. *Each atomic formula of **DBS** is equivalent to a conjunction of some of the following formulas:*

$a \leq b$	$a \leq x \vee b$	$a \leq x^* \vee b$
$a \leq x^+ \vee b$	$a \leq x^{**} \vee b$	$a \leq x^{++} \vee b$
$a \leq x \vee x^* \vee b$	$x \wedge a \leq b$	$x \wedge a \leq x \vee b$
$x \wedge a \leq x^* \vee b$	$x \wedge a \leq x^+ \vee b$	$x \wedge a \leq x^{**} \vee b$
$x \wedge a \leq x^{++} \vee b$	$x \wedge a \leq x \vee x^* \vee b$	$x^* \wedge a \leq b$
$x^* \wedge a \leq x \vee b$	$x^* \wedge a \leq x^* \vee b$	$x^* \wedge a \leq x^+ b$
$x^* \wedge a \leq x^{**} \vee b$	$x^* \wedge a \leq x^{++} \vee b$	$x^* \wedge a \leq x \vee x^* \vee b$
$x^+ \wedge a \leq b$	$x^+ \wedge a \leq x \vee b$	$x^+ \wedge a \leq x^* \vee b$
$x^+ \wedge a \leq x^+ \vee b$	$x^+ \wedge a \leq x^{**} \vee b$	$x^+ \wedge a \leq x^{++} \vee b$
$x^+ \wedge a \leq x \vee x^* \vee b$	$x^{**} \wedge a \leq b$	$x^{**} \wedge a \leq x \vee b$
$x^{**} \wedge a \leq x^* \vee b$	$x^{**} \wedge a \leq x^+ \vee b$	$x^{**} \wedge a \leq x^{**} \vee b$
$x^{**} \wedge a \leq x^{++} \vee b$	$x^{**} \wedge a \leq x \vee x^* \vee b$	$x^{++} \wedge a \leq b$
$x^{++} \wedge a \leq x \vee b$	$x^{++} \wedge a \leq x^* \vee b$	$x^{++} \wedge a \leq x^+ \vee b$
$x^{++} \wedge a \leq x^{**} \vee b$	$x^{++} \wedge a \leq x^{++} \vee b$	$x^{++} \wedge a \leq x \vee x^* \vee b$
$x \wedge x^+ \wedge a \leq b$	$x \wedge x^+ \wedge a \leq x \vee b$	$x \wedge x^+ \wedge a \leq x^* \vee b$
$x \wedge x^+ \wedge a \leq x^+ \vee b$	$x \wedge x^+ \wedge a \leq x^{**} \vee b$	$x \wedge x^+ \wedge a \leq x^{++} \vee b$
$x \wedge x^+ \wedge a \leq x \vee x^* \vee b$		

Theorem 2.4. *Each of the sets defined by formulas in corollary 2.3, are strongly connected sets in a double Stone algebra.*

PROOF. Using lemma 1.2, one can make straightforward proofs for all of the listed formulas. □

Theorem 2.5. **DBS** *is a pseudo o-minimal theory.*

PROOF. Let $\mathcal{DS} = (DS, \wedge, \vee *, +, 0, 1)$ be a double stone algebra and a model of **DBS**. Since **DBS** has the quantifier elimination property, every definable set in $\mathcal{DS}$ is a boolean combination of strongly connected sets. On the other hand, by lemma 2.2 and theorem 2.4, strongly connected sets are defined exactly by forty nine formulas listed in corollary 2.3. Therefore, every definable set of $\mathcal{DS}$ is a finitely boolean combination of strongly connected sets. Hence $\mathcal{DS}$ is a pseudo o-minimal double stone algebra. So we have the pseudo o-minimality for the theory **DBS**. □

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Co-Intersection graph of act

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Abstract

Here, we define the co-intersection graph $Coint(A)$ of an S-act A which is a graph whose vertices are non-trivial subacts of A and two distinct vertices B_1 and B_2 are adjacent if $B_1 \cup B_2 \neq A$. We investigate the relationship between the algebraic properties of an S-act A and the properties of the graph .

Keywords and phrases: S-Act, co-intersection graph .
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1. Introduction

Let S be a semigroup. A non-empty set A is said to be a (*left*) S*-act* if there is a mapping $\lambda : S \times A \to A$, denoting $\lambda(s,a)$ by sa, satisfying $(st)a = s(ta)$ and, if S is a monoid with 1, $1a = a$, for all $a \in A$, $s,t \in S$.

Definition 1.1. Let A be an S-act. The *co-intersection graph* of A, $Coint(A)$, is a graph whose vertices are all non-trivial subacts of A such that two distinct vertices B_1 and B_2 are adjacent if and only if $B_1 \cup B_2 \neq A$.

2. Main Results

2.1. Some properties of the graph $Coint(A)$ It is clear that if A and B are isomorphic S-acts, then the graphs $Coint(A)$ and $Coint(B)$ are isomorphic. The converse is not true in general. This result is illustrated in the following example.

Example 2.1. Take the monoid $S = \{1,s\}$, where $s^2 = 1$. Consider two S-acts $A = \{a,b,c\}$ with trivial action and $B = \{a,b,c,d\}$ presented by the following action table:

	a	b	c	d
1	a	b	c	d
s	a	b	d	c

* speaker

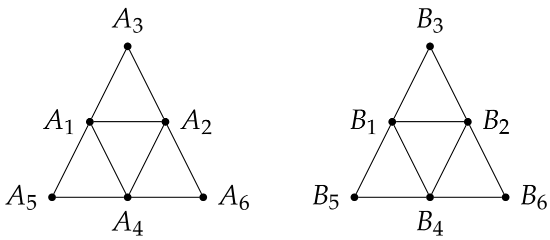
The non-trivial subacts of A and B are:

$$A_1 = \{a\}, A_2 = \{a,b\}, A_3 = \{b\}, A_4 = \{b,c\}, A_5 = \{c\}, A_6 = \{a,c\}$$

and

$$B_1 = \{a\}, B_2 = \{a,b\}, B_3 = \{b\}, B_4 = \{b,c,d\}, B_5 = \{c,d\}, B_6 = \{a,c,d\},$$

respectively. Then $Coint(A)$ and $Coint(B)$ are isomorphic which are given in the following:



$Coint(A) \cong Coint(B)$ whereas A and B are not isomorphic S-acts.

In the following, we give some conditions on two S-acts A, B under which A and B are isomorphic S-acts when $Coint(A) \cong Coint(B)$.

Lemma 2.2. *Let A be a free S-act with a basis X where S is a group. Then $Coint(A) \cong Coint(X)$ in which X is considered as an S-act with trivial action.*

Theorem 2.3. *Let A and B be two free S-acts and $Coint(A) \cong Coint(B)$. Then $A \cong B$ under each of the following conditions:*
(i) *S is a group.*
(ii) *S has only finitely many left ideals, and A and B have finite bases.*

Example 2.4. The bicyclic monoid $S = \langle u, v \mid uv = 1 \rangle = \{v^m u^n : m, n \geq 0\}$ has a complete co-intersection graph. To see this, let I and J be two non-trivial left ideals of S such that $v^m u^n \notin I$ and $v^k u^l \notin J$ for some non-negative integers m, n, k and l. First, suppose that $n \geq l$. We show that $v^m u^l \notin I \cup J$. Assume on the contrary that $v^m u^l \in I \cup J$, then either $v^m u^l \in I$ or $v^m u^l \in J$. If $v^m u^l \in I$, then $(v^m u^{m+n-l})(v^m u^l) = v^m u^n \in I$ and if $v^m u^l \in J$, then $(v^k u^m)(v^m u^l) = v^k u^l \in J$, which are contradictions. Therefore, $v^m u^l \notin I \cup J$ and $I \cup J \neq S$. Now suppose that $n < l$. We show that $v^k u^n \notin I \cup J$. Let $v^k u^n \in I \cup J$, then either $v^k u^n \in I$ or $v^k u^n \in J$. If $v^k u^n \in I$, then $(v^m u^k)(v^k u^n) = v^m u^n \in I$ and if $v^k u^n \in J$, then $(v^k u^{l+k-n})(v^k u^n) = v^k u^l \in J$, which are contradictions in both cases. Therefore, $v^k u^n \notin I \cup J$ and $I \cup J \neq S$. Hence, the graph $Coint(S)$ is complete.

Now, we give a necessary and sufficient condition for an S-act A to have a co-intersection complete graph.

Theorem 2.5. *Let A be a Noetherian S-act. Then $Coint(A)$ is complete if and only if A contains a unique maximal subact.*

Theorem 2.6. *Let G be a non-null bipartite graph. Then G is a co-intersection graph of an S-act if and only if $G = P_i$, where $i \in \{2,3\}$.*

Theorem 2.7. *The cycle graph C_n is a co-intersection graph of an S-act if and only if $n = 3$.*

2.2. Connectivity, diameter and girth Here, we characterize all S-acts A for which the associated co-intersection graphs are connected. Using these results, the diameter and the girth of co-intersection graphs of S-acts are obtained.

Theorem 2.8. *Let A be an S-act. Then the graph $Coint(A)$ is disconnected if and only if A is a coproduct of two simple subacts.*

Corollary 2.9. *Let A be an S-act and have at least one edge. Then $Coint(A)$ is connected.*

Theorem 2.10. *Let A be an S-act. Then the following assertions hold:*
(i) *If $Coint(A)$ is connected, then diam $(Coint(A)) \leq 3$.*
(ii) *If $Coint(A)$ contains a cycle, then girth $(Coint(A)) = 3$.*

2.3. Some finiteness conditions Here, we study finiteness conditions of some parameters of co-intersection graphs of S-acts such as clique number, chromatic number, independence number and domination number.

Theorem 2.11. *Let A be an S-act. Then the following are equivalent:*
(i) $\deg(B) < \infty$ *for each vertex B in $Coint(A)$.*
(ii) $\deg(B) < \infty$ *for some vertex B in $Coint(A)$.*
(iii) $|Coint(A)| < \infty$.
(iv) $\chi(Coint(A)) < \infty$.
(v) $\omega(Coint(A))) < \infty$.

Corollary 2.12. *Let A be an S-act and B be non-trivial subact of A with* $\deg(B) < \infty$. *Then A is both Artinian and Noetherian.*

Theorem 2.13. *Let A be a Noetherian S-act. Then the following assertions hold:*
(i) $\mathrm{Max}(A)$ *is both independent and dominating set in $Coint(A)$.*
(ii) $\alpha(Coint(A)) = |\mathrm{Max}(A)|$.
(iii) $\gamma(Coint(A)) \leq \alpha(Coint(A))$.

Theorem 2.14. *Let A be an Artinian S-act. Then $\gamma(Coint(A)) = 1$ or 2.*

Theorem 2.15. *Let A be an S-act and e be a cut edge with end-point B_1 and B_2. Then one end-point is a minimal subact and the other one is a maximal subact .*

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Bounds for the index of the second center subgroup of a pair of finite groups

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Abstract

By a pair of groups, we mean a group G and a normal subgroup N. In the present work, we give an upper bound for $|N/Z_2(G,N)|$ in which $Z_2(G,N)$ denote the second center subgroup of a pair (G,N) of finite groups where N is a subgroup of G. As a consequence, we obtain an upper bound for $K/Z(H,K)$ where $H \cong G/Z(G,N)$ and $K \cong N/Z(G,N)$, for a pair (G,N) of finite groups.

Keywords and phrases: Pair of groups, Upper bound, Second center..
2010 *Mathematics subject classification:* Primary: 20F14; 20E22; 20F05.

1. Introduction

A basic theorem of Schur [13] asserts that if the center of a group G has finite index, then the derived subgroup of G is finite. A question that naturally arises from Schur's theorem is whether the converse of theorem is valid. An extra special p-group of infinite order shows that the answer is negative. One of the remarkable problems is finding conditions under which the converse of Schur's theorem holds. Neumann [8] provided a partial converse of Schur's theorem as follows:
If G is finitely generated by k elements and $\gamma_2(G)$ is finite, then $G/Z(G)$ is finite and $|G/Z(G)| \leq |\gamma_2(G)|^k$.
This result was recently generalized by P .Niroomand [9]. He proved that if G' is finite and $G/Z(G)$ is finitely generated, then $G/Z(G)$ is finite and $|G/Z(G)| \leq |G'|^{d(G/Z(G)}$, in which $d(X)$ is the minimal number of generators of a group X. B. Sury [14] gave a completely elementary short proof of a further generalization of the Niroomand's result. Yadav [15] states another extension of the Neumann's result when $Z_2(G)/Z(G)$ is finitely generated. He [17] also provided other modifications of the converse of Schur's theorem as follows : For a group G the factor group $G/Z(G)$ is finite if any of the following holds true.

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(i) G' is finite and $Z_2(G)$ is abelian.
(ii) G' is finite and $Z_2(G) \leq G'$.
(iii) G' is finite and $Z_2(G)/Z(Z_2(G))$ is finitely generated.
(iv) $G/G'Z_2(G)$ is finite and $G/Z(Z_2(G))$ is finitely generated.

Another modification of the converse of the Schur's theorem may be concluded from a more general theorem of P. Hall (see Theorem 2 in [5]), as follows:

For a group G, if G' is finite then $G/Z_2(G)$ is finite.

The first explicit bound for the order of $G/Z_2(G)$ in terms of the order of G' was given by I.D. Macdonald [6], in 1961. He proved that for a group G, if G' is finite of order n, then $|G/Z_2(G)| \leq n^{\log_2 n(1+\log_2 n)}$.
Considering the modifications of the converse of Schur's theorem, finding upper bounds for the orders $|G/Z(G)|$ and $|G/Z_2(G)|$ in terms of $|G'|$, is a noticeable and interesting problem. I. M. Isaacs and K. Podoski and B. Szegedy gave different answers for this problem (see [3], [12], [10], [11]).
Ellis extended the concepts of capability, Schur multipliers and central series of groups for pairs of groups. By a pair of groups,we mean a group G and a normal subgroup N. An excellent introduction to the extended concepts capable pairs and Schur multiplier of pairs of groups appear in [2] and [1], respectively. Ellis [2] also define the concept of relatively capable groups. A group K is relatively capable if and only if there exists a pair (G, N) of groups such that $K \cong N/Z(G, N)$.
Recall that, for a pair (G, N) of groups, the center subgroup and the second center subgroup, denoted by $Z(G, N)$ and $Z_2(G, N)$ respectively, are defined as follows:

$$Z(G, N) = \{x \in N | x^g = x, \forall g \in G\},$$

$$\frac{Z_2(G, N)}{Z(G, N)} = Z(\frac{G}{Z(G, N)}, \frac{N}{Z(G, N)}).$$

The author generalized some result of [11] for pairs of finite groups in [7] and obtained an upper bound for $|N/Z_2(G, N)|$ in terms of $|[N, G]|$ and $rank(G')$,where $rank(G)$ is, the minimal number r such that every subgroup of G can be generated by r elements.

In the present research we apply the result of [7] and give a better upper bound for $|N/Z_2(G, N)|$ in terms of $|[N, G]|$ and $rank(G')$, for any pair (G, N) of finite groups. Also we use this result and obtain an upper bound for $K/Z(H, K)$ which appears as $H \cong G/Z(G, N)$ and $K \cong N/Z(G, N)$, for another pair (G, N) of finite groups.

2. Main Results

First we state the following results which are needed to prove the main theorem of the paper.

Lemma 2.1. *Let H and K be two subgroups of a group G, such that $K \lhd G$ and H can be generated by d elements. Then*

$$|K : C_K(H)| \leq |[H,K]|^d.$$

PROOF. See the proof of Lemma 10 in [11]. □

Theorem 2.2. *Let (G,N) be a pair of finite groups. Suppose that $Z = Z(G,N) \cap [N,G]$ and $rank([N,G]/Z) = r$. Then*

$$|C_N(G') : Z_2(G,N)| \leq |\frac{[N,G]}{Z}|^r.$$

PROOF. See the proof of Theorem 2.4 in [7]. □

Now we are going to obtain an upper bound for $|N : Z_2(G,N)|$. For this we need to prove some lemmas.

Lemma 2.3. *Let (G,N) be a pair of finite groups. Suppose that $Z = Z(G,N) \cap [N,G]$ and $A/Z = C_{N/Z}(G'/Z)$. Then A is a nilpotent group and $G'/C_{G'}(P)$ is a p-group, for any Sylow p-subgroup P of A.*

Lemma 2.4. *Let (G,N) be a pair of finite groups. Suppose that $Z = Z(G,N) \cap [N,G]$ and $A/Z = C_{N/Z}(G'/Z)$. Then*

$$|A : C_N(G')| \leq |[N,G]/Z|^{2r},$$

in which $r = rank(G'/Z(G) \cap G')$.

Theorem 2.5. *Let (G,N) be a pair of finite groups. Then*

$$|N : Z_2(G,N)| \leq |\frac{[N,G]}{[N,G] \cap Z(N,G)}|^{4r},$$

in which $r = rank(G'/G' \cap Z(G))$.

The next main result of the paper is an immediate consequence of the above theorem.

Corollary 2.6. *Let (H,K) be a pair of groups such that $H \cong G/Z(G,N)$ and $K \cong N/Z(G,N)$, for a pair (G,N) of finite groups. Then*

$$|K : Z(H,K)| \leq |[K,H]|^{4r},$$

in which $r = rank(H')$.

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Connected Domination Number of Central Trees

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Abstract

Let $G = (V, E)$ be a graph. A subset S of V is called a dominating set of G if every vertex not in S is adjacent to some vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets of G. A dominating set S is called a connected dominating set if the induced subgraph $< S >$ is connected. The minimum cardinality taken over all connected dominating sets of G is called the connected domination number of G and is denoted by $\gamma_c(G)$. In this paper, we study the connected domination number of central trees. Indeed, we obtain some tight bounds for the connected domination number of a central trees C(T) in terms of some invariants of the graph T. Also we characterize the connected domination number of the central of some families of trees.

Keywords and phrases: Connected domination number, Central trees .
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1. Introduction

The notion of domination and its many generalizations have been intensively studied in graph theory and the literature on this subject is vast, see for example [2], [3] and [4]. Throughout this paper, we use standard notation for graphs and we assume that each graph is non-empty, finite, undirected and simple. For the standard graph theory terminology not given here we refer to [1]. Throughout this paper, G is a non-empty, finite, undirected and simple graph with the vertex set $V(G)$ and the edge set $E(G)$.

Let G be a graph with the vertex set $V(G)$ of *order* n and the edge set $E(G)$ of *size* m. The *open neighborhood* and the *closed neighborhood* of a vertex $v \in V(G)$ are $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. The *degree* of a vertex v is defined as $deg_G(v) = |N_G(v)|$. The *minimum* and *maximum degree* of a vertex in G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We write $K_{1,n-1}$ and P_n for a *star* graph and a *path* graph of order n, respectively, while The *m-corona* $G \circ P_m$ of a graph G is the graph of order $(m+1)|V(G)|$ obtained from G by adding a path of order m

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to each vertex of G. A double star graph $S_{1,n,n}$ is obtained from the star graph $K_{1,n}$ by replacing every edge with a path of length 2.

$G[S]$ denote the subgraph of G induced on the vertex set S. The *complement* of a graph G, denoted by $\overline{G}$, is a graph with the vertex set $V(G)$ such that for every two vertices v and w, $vw \in E(\overline{G})$ if and only if $vw \notin E(G)$. A *vertex cover* of the graph G is a set $D \subseteq V(G)$ such that every edge of G is incident to at least one element of D. The *vertex cover number* of G, denoted by $\tau(G)$, is the minimum cardinality of a vertex cover of G.

For a tree graph G, any vertex of degree one is called a *leaf* and the neighbour of a leaf is called a *support vertex* of G.

Vernold et al., in [7] by doing an operation on a given graph G obtained the central graph of G as follows.

Definition 1.1. [7] *The central graph $C(G)$ of a graph G of order n and size m is a graph of order $n+m$ and size $\binom{n}{2}+m$ which is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G in $C(G)$.*

We fix a notation for the vertex set and the edge set of the central graph $C(G)$ to work with throughout the paper. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$. We set $V(C(G)) = V(G) \cup \mathcal{C}$, where $\mathcal{C} = \{c_{ij} : v_iv_j \in E(G)\}$ and $E(C(G)) = \{v_ic_{ij}, v_jc_{ij} : v_iv_j \in E(G)\} \cup \{v_iv_j : v_iv_j \notin E(G)\}$.

Definition 1.2. *A subset S of V is called a dominating set of G if every vertex not in S is adjacent to some vertex in S. A dominating set S is called connected dominating set if the induced subgraph $< S >$ is connected. The minimum cardinality taken over all connected dominating sets in G is called the connected domination number of G and is denoted by $\gamma_c(C(G))$. Moreover, a connected dominating set of G of cardinality $\gamma_c(C(G))$is called a γ_c - set of G.*

Definition 1.3. *A* total dominating set, *briefly TDS, of a graph G is a set $S \subseteq V(G)$ such that $N_G(v) \cap S \neq \emptyset$, for any vertex $v \in V(G)$. The* total domination number *of G is the minimum cardinality of a TDS of G and is denoted by $\gamma_t(G)$. Moreover, a total dominating set of G of cardinality $\gamma_t(G)$ is called a γ_t−set of G.*

The concept of connected domination in graphs was introduced by Sampathkumar and Walikar [6] in 1979. In this paper, we study the connected domination number of central trees.

The paper proceeds as follows. In Section 2, first we determine $\gamma_c(C(T))$ explicitly, when T is $K_{1,n-1}$, P_n, corona graph $G \circ P_1$, 2-corona graph $G \circ P_2$, double star graph $S_{1,n,n}$. In continue, we present some upper and lower bounds for $\gamma_c(C(T))$.

2. Main Results

In this section, we obtain the connected domination number of the central trees. The connected domination number of the central graph of star graph is given in the first Theorem.

Theorem 2.1. *For a star graph $K_{1,n-1}$ of order $n \geq 3$, $\gamma_c(C(K_{1,n-1})) = 3$.*

Theorem 2.2. *For any path P_n of order $n \geq 3$,*

$$\gamma_c(C(P_n)) = \begin{cases} 3 & \text{if } n = 3, 4, \\ \lfloor \frac{n}{2} \rfloor & \text{otherwise.} \end{cases}$$

Theorem 2.3. *For any integer $n \geq 2$, $\gamma_c(C(S_{1,n,n})) = n + 1$.*

Theorem 2.4. *For any tree T of order $n \geq 4$,*

$$\gamma_c(C(T \circ P_1)) = n.$$

Theorem 2.5. *For any tree T of order $n \geq 4$,*

$$\gamma_c(C(T \circ P_2)) = n + \tau(T).$$

In continue, we obtain a lower bound and an upper bound for the connected domination number of the central graph of a tree.

Theorem 2.6. *For any tree T of order $n \geq 3$ with $\Delta(T) \geq n - 3$, $\gamma_c(C(T) = 3$.*

Corollary 2.7. *For any tree T of order $3 \leq n \leq 6$, $\gamma_c(C(T) = 3$.*

Theorem 2.8. *For any tree T of order $n \geq 7$ with $\Delta(T) \leq n - 4$,*

$$\gamma_c(C(T) \leq \tau(T) + 2$$

.

By Theorem 2.6, Corollary 2.7 and Theorem 2.8, we have the following result.

Corollary 2.9. *For any tree T of order $n \geq 3$,*

$$3 \leq \gamma_c(C(T) \leq \tau(T) + 2$$

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Energy of Monad Graphs Generated by Cubic Function

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Abstract

There are a limit numbers of methods which associate group theory to graph theory. In 2003, V.I. Arnold introduced a very important phenomena named by monad. The monad graph is a directed graph involved to the finite group G, where every vertex of the elements g of G is adjacent to its image by a directed edge under the action of the map f In this work, we will calculate the energy of some monad graphs generated by cubic function, i.e. $f(g) = g^3$ for all $g \in G$.

Keywords and phrases: Directed graphs, graph energy, finite group.

1. Introduction

Since 1978, when the concept of graph energy based on the eigenvalues of the adjacency matrix was conceived [6], a large number of other "graph energies" has been put forward. Nowadays, their number is near to 200 [7, 8]. Almost all of these "graph energies" are based on the eigenvalues of various graph matrices, different from the adjacency matrix. In the present paper we consider one more "graph energy", which – in contrast to the earlier ones – has its roots from group theory and uses the eigenvalues of the adjacency matrix. Monad is discrete dynamical systems, for more details, we refer to [1],[2],[3],[4].

Let $\mathcal{G}$ be a digraph (directed graph) of order n. Let $V(\mathcal{G}) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set and $E(\mathcal{G})$ the edge set of $\mathcal{G}$. By e_{ij} is denoted the directed edge of $\mathcal{G}$ starting at vertex v_i and ending at vertex v_j. The adjacency matrix of $\mathcal{G}$ is the $n \times n$ matrix $A(\mathcal{G})$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } e_{ij} \in E(\mathcal{G}) \\ 0 & \text{otherwise.} \end{cases}$$

$\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A(\mathcal{G})$. In the case of digraphs, some of the eigenvalues may be complex numbers. Therefore, the *energy of digraphs* is

* speaker

defined as the sum of absolute values of the real parts of the eigenvalues, i.e.,

$$E(\mathcal{G}) = \sum_{i=1}^{n} |Re\lambda_i| \,.$$

In this paper, we are interested in the energy of the so-called *monad graphs*.

2. Main Results

In [2], a very interesting phenomena termed as *monad* was introduced by Arnold. Let G be a finite group. A monad function is a mapping of each element from G into itself, i.e. $f : G \to G$ for all $g \in G$. The monad graph $\Gamma(G)$ is a directed graph such that every vertex of G is adjacent to its image by a directed connected edge under the action of f. In fact, the monad function considered in [2] was a square function.

In Table 1, the additive notation for the group operation have been used to show the monad graphs of the first few residue classes of cyclic groups.

In order to have our results, we will consider the following lemmas: Lemma 2.1 is an immediate consequence of the Sachs coefficient theorem [5]. Recall that for digraphs, this theorem reads:

Lemma 2.1. *Let $\mathcal{G}$ be a digraph.*

(a) *If the directed edge e does not belong to any cycle of G, then e does not contribute to the spectrum of G. In other words, by deleting e from G, neither the spectrum nor the energy of $\mathcal{G}$ will change.*

(b) *If the vertex v does not belong to any cycle of G, then v contributes to the spectrum of $\mathcal{G}$ by a zero. Therefore, by deleting v from $\mathcal{G}$, the energy of G will not change.*

Lemma 2.2. *Let $\mathcal{G}$ be a digraph with characteristic polynomial*

$$\phi(\mathcal{G}) = \sum_{k=0}^{n} a_k \, x^{n-k} \,.$$

Then $a_0 = 1$ and for $k \geq 1$,

$$a_k = \sum_{S \in L_k} (-1)^{\omega(S)}$$

where L_k denotes the set of k-vertex subgraphs of G, in which every component is a directed cycle. $\omega(S)$ is the number of connected components of S.

According to Lemma 2.2, the characteristic polynomial of the directed cycle O_n is

$$\phi(O_n, \lambda) = \lambda^n - 1 \,.$$

Thus the eigenvalues of O_n are

$$\lambda_j = e^{2\pi i j/n} \,, \quad j = 0, 1, 2, \ldots, n-1$$

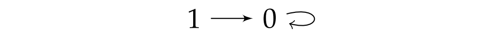
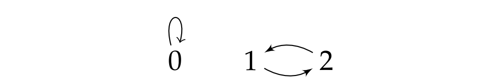
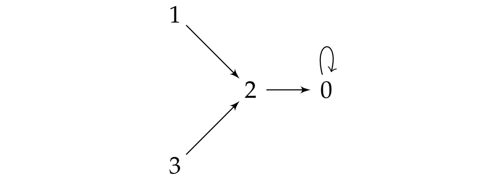
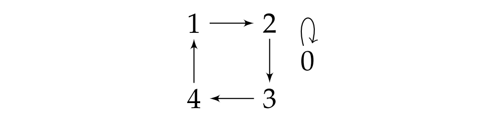
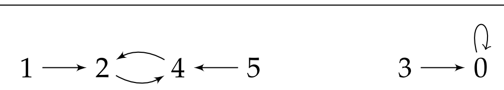
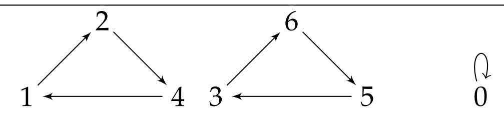
$\Gamma(\mathbb{Z}_2)$		A_1
$\Gamma(\mathbb{Z}_3)$		$O_1 + O_2$
$\Gamma(\mathbb{Z}_4)$		T_4
$\Gamma(\mathbb{Z}_5)$		$O_1 + O_4$
$\Gamma(\mathbb{Z}_6)$		$A_1 + A_2$
$\Gamma(\mathbb{Z}_7)$		$O_1 + 2O_3$

TABLE 1. Monad graphs of the residue class cyclic groups for $n \leq 8$. O_n is the directed cycle on n vertices. A_n is the connected graph on $2n$ vertices, consisting of a directed cycle of length n to which n one-edge branches are attached, each for every vertex of the cycle. D_n is the $4n$-vertex graph consisting of the cycle O_n, to each of its vertices a three-edge branch is attached; for examples see the 16-vertex digraph on Fig. 1 and the 80-vertex digraph on Fig. 2. T_{2^n} is the rooted binary tree on 2^n vertices and n leaves. For more details see [3].

implying

$$E(O_n) = \sum_{j=0}^{n-1} \left| \cos \frac{2\pi j}{n} \right|. \tag{1}$$

By direct calculation we get $E(O_1) = 1,\ E(O_2) = 2,\ E(O_3) = 2,\ E(O_4) = 2,\ E(O_5) = 1 + \sqrt{5} \approx 3.236,\ E(O_6) = 4.$

The following results were obtained in [9]

Theorem 2.3. *Let A_n, D_n, and T_{2^n} be the digraphs described in the caption of Table 1. Then*

$$E(A_n) = E(O_n);$$

$$E(D_n) = E(O_n);$$

$$E(T_{2^n}) = 1.$$

n	$\Gamma(\mathbb{Z}_n)$	$E(\Gamma(\mathbb{Z}_n))$
2	A_1	1
3	$O_1 + O_2$	3
4	T_4	1
5	$O_1 + O_4$	3
6	$A_1 + A_2$	3
7	$O_1 + 2O_3$	5
8	T_8	1
9	$O_1 + O_6$	5
10	$A_1 + A_4$	3
11	$O_1 + O_{10}$	7.472
12	$T_4 + (T_4 * O_2)$	3
13	$O_1 + O_{12}$	8.464
14	$A_1 + 2A_3$	5
15	$O_1 + O_2 + 3O_4$	9
16	T_{16}	1
17	$O_1 + 2O_8$	9.6568
18	$A_1 + A_2 + A_6$	7
19	$O_1 + O_{18}$	12.517
20	$T_4 + (T_4 * O_4)$	3
21	$O_1 + 2O_3 + 2O_6$	13
22	$A_1 + A_{10}$	7.472
23	$O_1 + 2O_{11}$	15.0536

TABLE 2. Energies of monad graphs pertaining to cyclic groups $\mathbb{Z}_n$ for the first few values of n

Theorem 2.4. *The energy of the monad graph pertaining to an additive cyclic group O_n of odd order n is given by*

$$E(\Gamma(G_n)) = \sum_m O_m = \sum_m \sum_{j=0}^{m-1} \left| \cos \frac{2\pi j}{m} \right|$$

for some (not necessarily mutually distinct) values of m , $1 \leq m \leq n-1$. For details see Table 2.

Now, let us consider the case of cubic map, i.e. monad map is $f(g) = g^3$ for all $g \in G$. In the following table, for the simplest abelian groups of residue class groups $n \leq 11$ and additive notation for the group operation, we show the monad graphs generated by map $f(g) = g^3$ as:

Theorem 2.5. *Energy of the monad graphs pertaining to a group of order 3^r for $r > o$ is given by*

$$E(\Gamma(G_{3^r})) = E(T_{3^r}) * (E(2\sum_{i=2}^{r} O_{2^i}) + E(O_2)) = 1 * (\sum_{i=2}^{r} \sum_{j=0}^{n-1} \left| \cos \frac{2\pi j}{2^{i-1}} \right| + 2)).$$

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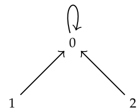
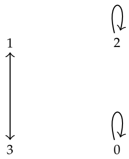
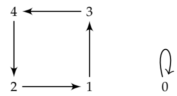
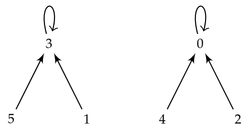
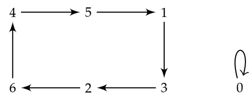
Group	*Monad Graph*	Graph's symbol	*Group*	*Monad Graph*	*Graph's symbol*
Z_2		O_1+O_1	Z_3		T_3
Z_4		O_2+2O_1	Z_5		O_4+O_1
Z_6		$2T_3$	Z_7		O_6+O_1

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Definable Monotone Functions in Type Complete Ordered Fields

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Abstract

Type complete ordered structures have been studied in [2] and [4] within many remarkable results. The main results were achieved under the additional definable completeness named *DC*. An ordered structure $\mathcal{M} = (M, <, \dots)$ satisfies *DC* if every definable subset of M has a least upper bound in $M \cup \{\pm\infty\}$. Here, we study type-complete structures in which definable bounded monotone functions converge.

Keywords and phrases: type-complete, definable monotone function.
2010 *Mathematics subject classification:* Primary: 12J15, 03C64; Secondary: 06F30.

1. Introduction

Let $\mathcal{M} = (M, <, \dots)$ be a first order expansion of a dense linear order $(M, <)$ which has no end points. For $a \in M$, let a^- and a^+ denote the partial types $\{b < x < a |\ b \in M\ ,\ b < a\}$ and $\{a < x < b |\ b \in M\ ,\ b > a\}$, respectively. The structure $\mathcal{M}$ is said to be type-complete if for any $a \in M$, a^- is a complete type, equivalently, for every definable set $X \subseteq M$ and any $a \in M_\infty (= M \cup \{\infty\})$, defining formula $\varphi(x)$ of X is in a^- or $\neg\varphi \in a^-$. Type-completeness (abbreviated by *TC*) is a first order property in the language of $\mathcal{M}$. Let *TC* denote the theory of type-complete expansions of dense linear orders in a first order language $\mathcal{L} = \{<, \dots\}$. Also the structure $\mathcal{M}$ is called definably complete if every definable subset of M has a least upper bound in $M \cup \{\pm\infty\}$. Definable completeness of $\mathcal{L}$-structures is a first order property that is denoted by *DC* in [4]. For details of *DC*-structures, see [1] and [3]. Models of *TC* + *DC* have been studied in [2] and [4]. Note that models of *TC* are called *locally o-minimal* structures in [2]. Here, we replace the assumption of being definably complete with the weaker assumption that definable bounded monotone functions converge in the ordered structure. In the following, we fix an $\mathcal{L}$-structure $\mathcal{M}$ which is a model of *TC* and assume that every definable bounden monotone function $f : M \to M$ converges in M.

* speaker

2. Main Results

Lemma 2.1. *Let $Y \subseteq M$ be definable and $K \subseteq M$ compact. If K is open or $Y \subseteq K$, then $K \cap Y$ is a finite union of intervals in M.*

PROOF. Given a definable subset $Y \subseteq M$, and suppose that K is open, then for each $x \in K$, there is an open interval $I_x \subseteq K$ such that $I_x \cap Y$ is an interval. since K is compact and the I_x cover K, there exist finitely many points $x_1, \cdots, x_n \in K$ such that $K = I_{x_1} \cup \cdots \cup I_{x_n}$ and hence $K \cap Y$ is a finite union of intervals. If K is arbitrary, then we cannot arrange for all I_x to be contained in K, and so we only have $K \subseteq I_{x_1} \cup \cdots \cup I_{x_n}$. But since $Y \subseteq K$ so $Y \subseteq I_{x_1} \cup \cdots \cup I_{x_n}$ so $Y = Y \cap (I_{x_1} \cup \cdots \cup I_{x_n})$, and finally $Y \cap K = Y = \cup_{i=1}^{n} (Y \cap I_{x_i})$. □

Definition 2.2. *An ordered structure $\mathcal{M}$ is said to be ordered-minimal (abbreviated by o-minimal) if every definable set $X \subseteq M$ is a finite union of open intervals and points in M.*

It is worth mentioning that o-minimal structures have been extensively studied in the four last decades. The most important examples of o-minimal structures are dense linear orders, divisible ordered abelian groups, and real closed fields.

Theorem 2.3. *If every closed and bounded subset of M is compact (Heine-Borel property), then the structure $\mathcal{M}$ is o-minimal.*

PROOF. suppose that $Y \subseteq M$ is definable. Since $\mathcal{M}$ is type-complete, then $\infty^- \subseteq Y$ or $\infty^- \subseteq M \setminus Y$. Also, $(-\infty)^+ \subseteq Y$ or $(-\infty)^+ \subseteq M \setminus Y$. Hence, in order to prove that M is o-minimal, we may assume after removing one or two unbounded intervals that Y is bounded, whence contained in some closed bounded interval $K := [a, b]$. Hence $Y = Y \cap K$ is a finite union of intervals by lemma 2.1

□

Corollary 2.4. *If $\mathcal{M}$ is a substructure of real ordered structure $(\mathbb{R}, <, \dots)$ with the assumptions above, then $\mathcal{M}$ is o-minimal.*

For example the ordered field of real algebraic numbers is o-minimal.

Proposition 2.5. *If $X \subseteq M$ is definable, then it is discrete or otherwise has a nonempty interior.*

Lemma 2.6. *Let $f : M \to M$ be a definable increasing function. Then, we have the following.*

- *If $\inf f(M)$ exists in M, then it is a boundary point of $f(M)$.*
- *If $f(M)$ is definably connected, then it is an interval in M.*
- *If $f(M)$ is discrete, then it is closed and bounded, and so has minimum and maximum elements.*

Theorem 2.7. *Every finite union of definable discrete subsets of M is discrete.*

PROOF. It is enough to show that the union of two definable discrete subsets of M is discrete. Let $X, Y \subseteq M$ be definable discrete sets and suppose that $X \cup Y$ is not discrete, so it has nonempty interior (by proposition 2.5) and contains an open interval I. Assume that $a \in I$ then a^- and a^+ don't belong to X nor Y (since they are discrete), so both belong to $M \setminus X$ and $M \setminus Y$. hence they belong to $(M \setminus X) \cap (M \setminus Y) = M \setminus (X \cup Y)$. Therefore a would be a isolated point of $X \cup Y$ and it is a contradiction. □

Proposition 2.8. *The set of boundary points of every definable subset of M is closed, bounded, and discrete.*

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A note on mono-covered acts

MOHAMMAD ROUEENTAN* and ROGHAIEH KHOSRAVI

Abstract

The main purpose of this article is an introduction and investigation of new kinds of acts namely mono-covered acts. Some general properties of these kinds of acts are presented and their relations with some other concepts are studied.

Keywords and phrases: Trace, act, monomorphism .
2010 *Mathematics subject classification:* Primary: 20M30.

1. Introduction

Throughout this article S will denote a monoid and an S-act A_S (or A) is a right S-act. From [3], the trace of an S-act B in an S-act A is defined by $Tr(B,A) := \bigcup_{\varphi\in Hom(B,A)} \varphi(B)$. Also by modeling trace concept the notion of mono-trace is defined in [5]. For any S-acts A, B the mono-trace of B in A is defined by $MTr(B,A) := \bigcup_{\varphi\in Mon(B,A)} \varphi(B)$ where $Mon(B,A) = \{f : B \hookrightarrow A | f \textit{ is a monomorphism}\}$. A right S-act A is called *mono-covered* if for any subact B of $A, MTr(B,A) = A$. From [1] for an element a of an S-act A, the annihilator of a is defined by $ann(a) := \{(s,t) \in S\times S | \, as = at\} = ker(\lambda_a)$ where $\lambda_a : S_S \longrightarrow A$ is defined by $\lambda_a(s) = as$ for every $s \in S$. Moreover, for an S-act A, the annihilator of A is defined by $ann(A) = \bigcap_{a\in A} ann(a)$. Recall that a non-zero S-act A is called $uniform$ if every non-zero subact is large in A i.e., for any non-zero subact B of A, any S-homomorphism $g : A \longrightarrow C$ such that $g|_B$ is a monomorphism is itself a monomorphism. We denote this situation by $B \subseteq' A$. We encourage the reader to see [3] for basic results and definitions related to acts not defined here.

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* speaker

2. Main Results

Definition 2.1. *Let S be a monoid. A right S-act A is called mono-covered if for any element $a \in A$ and any subact B of A, there exists a monomorphism $f : B \hookrightarrow A$ such that $a = f(b)$ for some $b \in B$.*

It is clear that a right S-act A is mono-covered if and only if for any subact B of $A, MTr(B, A) = A$. Also it is easy to check that any retract of any mono-covered act is a mono-covered. Moreover, the right S-act S_S is mono-covered if and only if every projective (free) S-act is a mono-covered.

The following proposition contains some general properties of mono-trace. Recall that an S-act A is called *injective* if for any S-act B, any subact C of B and any homomorphism $f : C \longrightarrow A$, there exists a homomorphism $\bar{f} : B \longrightarrow A$ such that $\bar{f} |_C = f$. Also the S-act A is called *cyclic quasi-injective* if it is injective relative to all inclusions from its cyclic subacts. We denote in short "cyclic quasi-injective", by "CQ-injective".

For any S-act A, by $E(A)$ we denote the injective envelope of A.

Proposition 2.2. *Let S be a monoid and $B \subseteq C \subseteq A$ be S-acts. Then the following hold:*

(i) *If A is CQ-injective and $A = \bigcup_{b \in B} \{f(b) \mid f : bS \hookrightarrow A\}$, then $MTr(B, A) = A$.*

(ii) *If $MTr(B, A) = A$ and A is CQ-injective, then $MTr(C, A) = A$.*

(iii) *If $MTr(B, A) = A$ and C is CQ-injective, then $MTr(B, C) = C$.*

(iv) *If $MTr(I, S_S) = S_S$, then $MTr(E(I), E(S)) = E(S)$, where I is a right ideal of S.*

Corollary 2.3. *Let S be a monoid. Then the following hold:*

(i) *Any CQ-injective subact of any mono-covered act is mono-covered.*

(ii) *If A is a CQ-injective act and for any elements $a, b \in A, ann(a) = ann(b)$, then A is mono-covered.*

Proposition 2.4. *Suppose S is a monoid and A is a mono-covered act. Then for any subact B of $A, ann(A) = ann(B)$. Also if $ann(a) = ann(b)$ for any elements $a, b \in A$ and $A(S)$ satisfies the descending chain condition on cyclic subacts (principal right ideals), then A is a mono-covered act.*

From [2] an S-act A is called uniserial if the set of its subacts is linearly ordered by inclusion.

Proposition 2.5. *Let S be a monoid and A be a right S-act which satisfies the descending chain conditions on subacts. Then A is a uniserial mono-covered act if and only if A is a simple S-act.*

Proposition 2.6. *Suppose S is a commutative monoid and A is a CQ-injective S-act. If $T = End(A)$, then the following conditions are equivalent:*

(i) *For any elements* $a,b \in A, ann(a) = ann(b)$.

(ii) *A is a mono-covered act.*

(iii) *If* $a \in A$, *then* $A = Ta$, *where* $Ta = \{f(a) \mid f \in T\}$.

(iv) *A is a simple* T*-act.*

Proposition 2.7. *Suppose that* S_S *is mono-covered. Then for every right ideal* I *of* S, *there exists* $x \in E(I)$ *such that* $E(I) = Tx$ *where* $T = Hom(E(I), E(I))$.

PROOF. Suppose I is a right ideal of S. Since S_S is a cyclic right S-act, S embedded in I and so $E(S)$ and $E(I)$ are retract of each other. Thus there exists an epimorphism $h : E(I) \longrightarrow E(S)$. Now, projectivity of S implies the existence of a homomorphism $f : S \longrightarrow E(I)$ such that $hof = i$ where i is the inclusion map. Thus f is a monomorphism. If $m \in E(I)$, then for the homomorphism $\lambda_m : S \longrightarrow E(I), \lambda_m(1) = m$. Again, injectivity of $E(I)$ implies the existence of a homomorphism $g : E(I) \longrightarrow E(I)$ such that $gof = \lambda_m$. If $f(1) = x$, then $g(f(1)) = m$ and hence $g(x) = m$. Consequently, $E(I) = Tx$. □

Proposition 2.8. *Over a commutative monoid* S, *any cyclic mono-covered act is uniform. In particular if* S_S *is a mono-covered act, then* S *is a uniform monoid.*

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The autocentralizer automorphism of groups

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Abstract

Let G be a finite group and let $Aut^{H_a(G)}(G)$ be the group of autocommutator automorophisms of G where $H_a(G)$ is an autocentralizer subgroup of G.
In this paper, we find necessary and suffcient conditions on the finite group G such that this subgroup of automorphisms be equal of Inn(G) or C^*. We give some properties of these automorphisms.

Keywords and phrases: Centralizer subgroup, autocentralizer subgroup and autocentralizer automorphism. .

2010 *Mathematics subject classification:* Primary: 22D15, 43A10; Secondary: 43A20, 46H25.

1. Introduction

In this paper our notations are standard. Let G be a finite group, by $Z(G)$, $G^{'}$, $C_G(a)$, $Hom(G,H)$, $Aut(G)$, $Inn(G)$ respectively the center, the commutator subgroup, the centralizer subgroup, the homomorphism group of G into an abelian group H, the full automorphism group and the inner automorphisms of G.

Arora and Karan [1] defined the autonormalizer subgroup of H in G, $\overline{N_G(H)} = \{x \in G \mid [x,\alpha] \in H, \ \ for\ all\ \alpha \in Aut(G)\}$. In the paper we denote the autocentralizer subset $H_a(G)$ of G for a some $a \in G$ as:

$$H_a(G) = \{x \in G \mid [x,\alpha] \in C_G(a),\ \ for\ all\ \alpha \in Aut(G)\}$$

and

$$H_Z(G) = \{x \in G \mid [x,\alpha] \in Z(G),\ \ for\ all\ \alpha \in Aut(G)\}.$$

An automorphism α of G is called central if $x^{-1}\alpha(x) \in Z(G)$ for each $x \in G$. The set of all central automorphisms of G, denoted by $Aut_c(G)$, fix G' elementwise and form a normal subgroup of the full automorphism group of G.(see [3]). The group of all central automorphism of G is defined as follows:

$$C^* = \{\alpha \in Aut(G) \mid [x,\alpha] \in Z(G), \alpha(z) = z\ \ for\ all\ z \in Z(G)\ and\ x \in G\}.$$

* speaker

Similarly in this paper we introduce autocentralizer automorphism. An automorphism α of G denote autocentralizer if $x^{-1}\alpha(x) \in H_a(G)$ for eachl $x \in G$ for some $a \in G$. The set of all autocentralizer automorphisms of G, denote by $Aut^{H_a(G)}(G)$. There are some well-known results about autocentralizer automorphism of finite groups. We prove there exists a bijection between $Aut^{H_a(G)}_{C_G(a)}(G)$ and $Hom(\frac{G}{C_G(a)}, H_a(G))$. Also we prove if G be a finite group, $C_G(a) \leq Z(G)$, $G' \leq H_a(G)$ and $Hom(\frac{G}{C_G(a)}, H_a(G)) \simeq \frac{G}{Z(G)}$, then $Aut^{H_a(G)}_{C_G(a)}(G) = Inn(G)$.

2. Main Results

Definition 2.1. *Let G be a group.The autocentralizer $H_a(G)$ of G for some $a \in G$ define by*

$$H_a(G) = \{x \in G \mid [x, \alpha] \in C_G(a),\ for\ all\ \alpha \in Aut(G)\}$$

Note that by the outonormalizer subgroup definition, $H_a(G)$ is a subgroup of G. It is easy to see that if $C_G(a)$ be a characteristic subgroup of G then $H_a(G)$ is a characteristic subgroup of G. Also it is clear $\bigcap H_a(G) = H_Z(G)$ for any $a \in G$.

Definition 2.2. *An automorphism α of G define autocentralizr, if $x^{-1}\alpha(x) \in H_a(G)$ for some $a \in G$ and for each $x \in G$. We denote the set of all autocentralizer automorphisms of G by $Aut^{H_a(G)}(G)$, ie*

$$Aut^{H_a(G)}(G) = \{\alpha \in Aut(G) \mid [x, \alpha] \in H_a(G),\ for\ all\ x \in G\}.$$

Notice that if $H_a(G)$ be a normal subgroup of G then $Aut^{H_a(G)}(G)$ is the subgroup of $Aut(G)$. Also if $C_G(a)$ be a characteristic subgroup of G then $Aut^{H_a(G)}(G)$ is the subgroup of $Aut(G)$. It is very interesting to characterize if $C_G(a) = Z(G)$ then in which $Aut^{H_a(G)}(G)$ be equal to $Aut_c(G)$, the group of all central automorphisms of G. We mean $Aut_{C_G(a)}(G)$ the subgroup of $Aut(G)$ consisting of all automorphisms which fix $C_G(a)$ pointwise. We denote $Aut^{H_a(G)}(G) \cap Aut_{C_G(a)}(G)$ by $Aut^{H_a(G)}_{C_G(a)}(G)$.

Azhdari and Akhavan-Malayeri [2] showed, if M, N be two normal subgroups of G and M be a central subgroup of G then $Aut^M_N(G) \leq Aut_c(G)$. they proved if $M \leq Z(G) \bigcap N$, then $Aut^M_N(G) \simeq Hom(\frac{G}{N}, M)$. By substituting M by $H_a(G)$ and N by $C_G(a)$, we have a similar result for $Aut^{H_a(G)}_{C_G(a)}(G)$. We prove $Aut^{H_a(G)}_{C_G(a)}(G) \simeq Hom(\frac{G}{C_G(a)}, H_a(G))$.

Theorem 2.3. *Let G be a finite group.*

(i) If $C_G(a)$ be a normal subgroup of G and $H_a(G)$ be a central subgroup of G, then

$$Aut^{H_a(G)}_{C_G(a)}(G) \simeq Hom(\frac{G}{C_G(a)}, H_a(G)).$$

(ii) If $H_a(G)$ be a central subgroup and $C_G(a)$ be abelian subgroup of G, then $Aut^{H_a(G)}_{C_G(a)}(G) \simeq \frac{K \cap C_G(a)}{Z(G)}$ where $Z(\frac{G}{H_a(G)}) = \frac{K}{H_a(G)}$.

PROOF.

(i) Let $\theta : Aut^{H_a(G)}_{C_G(a)}(G) \longrightarrow Hom(\frac{G}{C_G(a)}, H_a(G))$, defined by $\theta(\alpha) = \alpha^*$ where $\alpha^*(xC_G(a)) = x^{-1}\alpha(x)$ for each $\alpha \in Aut^{H_a(G)}_{C_G(a)}(G)$. Since α is an automorphism fixing $C_G(a)$ elementwise α^* is a well-defined homomorphism of $\frac{G}{C_G(a)}$ to $H_a(G)$. Therefore θ is a well-defined map. Clearly, θ is one-to-one.
In the first place, θ is a homomorphism: for if $\alpha_1, \alpha_2 \in Aut^{H_a(G)}_{C_G(a)}(G)$ and $x \in G$, then

$$\begin{aligned}
(\alpha_1\alpha_2)^*(xC_G(a)) &= x^{-1}\alpha_1\alpha_2(x) = x^{-1}\alpha_1(\alpha_2(x)) \\
&= x^{-1}\alpha_1(xx^{-1}\alpha_2(x)) = x^{-1}\alpha_1(x).\alpha_1(x^{-1}\alpha_2(x)) \\
&= x^{-1}\alpha_1(x).x^{-1}\alpha_2(x) = \alpha_1^*(xC_G(a))\alpha_2^*(xC_G(a)).
\end{aligned}$$

Our homomorphism is also surjective, for this let $\beta \in Hom(\frac{G}{C_G(a)}, H_a(G))$, we define the map

$$\begin{aligned}
&\alpha : G \longrightarrow G \\
&x \longmapsto x\beta(xC_G(a))
\end{aligned}$$

evidently α is a well-defined homomorphism. By Lemma 1.1 in [2], α is an isomorphism. Furthermore α centralizes $\frac{G}{C_G(a)}$ and $H_a(G)$ and consequently $\alpha \in Aut^{H_a(G)}_{C_G(a)}(G)$. Also by the definition of θ, we have $\alpha^* = \beta$ and it follows that θ is an isomorphism of $Aut^{H_a(G)}_{C_G(a)}(G)$ to $Hom(\frac{G}{C_G(a)}, H_a(G))$, as required.

(ii) It is straightforward to see that $g \in K \cap C_G(a)$ if and only if $I_g \in Aut^{H_a(G)}_{C_G(a)}(G)$. And a quick calculation shows that the map $\phi : K \cap C_G(a) \longrightarrow Aut^{H_a(G)}_{C_G(a)}$ defined by $\phi(x) = I_{x^{-1}}$, for all $x \in K \cap C_G(a))$ is an epimorphism with the kernel equal to $Z(G)$, as required. □

As an immediate consequence of this result, one readily gets the following corollaries.

Corollary 2.4. *$Inn(G) = Aut^{H_a(G)}_{C_G(a)}(G)$ if and only if $C_G(a) \leq Z(G)$, $G' \leq H_a(G)$ and $Hom(\frac{G}{C_G(a)}, H_a(G)) \simeq \frac{G}{Z(G)}$.*

PROOF. If $Inn(G) = Aut^{H_a(G)}_{C_G(a)}(G)$, then $Inn(G) \leq Aut^{H_a(G)}_{C_G(a)}(G)$ and so $G' \leq H_a(G)$, $C_G(a) \leq Z(G)$ and $Hom(\frac{G}{N_G(C_G(a))}, H_a(G)) \simeq \frac{G}{Z(G)}$. Conversely, if $G' \leq H_a(G), C_G(a) \leq Z(G)$ therefore $Inn(G) \leq Aut^{H_a(G)}_{N_G(C_G(a))}(G)$ and $Hom(\frac{G}{C_G(a)}, H_a(G)) \simeq \frac{G}{Z(G)}$. So the equality holds. □

Corollary 2.5. *Let G is a finite group. If $H_a(G) = H_Z(G)$ and $C_G(a)$ be a characteristic subgroup of G, then*

$$Aut^{H_a(G)}_{C_G(a)}(G) = C^*.$$

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On characters of polygroups

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Abstract

In this paper we introduce characters of polygroups over hyperrings and show such characters induce characters of the fundamental group over corresponding fundamental ring.

Keywords and phrases: character, polygroup, hyperring.
2010 *Mathematics subject classification:* Primary: 22D15, 43A10; Secondary: 43A20, 46H25.

1. Introduction

The concept of hypergroup, which is a generalization of the concept of ordinary group, we first introduced by Marty [6]. A hypergroup is a set H equipped with an associative hyperoperation $\cdot : H \times H \longrightarrow \mathcal{P}^*(H)$ which satisfies the property $x \cdot H = H \cdot x = H$, for all $x \in H$. If the hyperoperation $\cdot$ is associative then H is called a semihypergroup.

In the above definition if $A, B \subseteq H$ and $x \in H$ then we define

$$A \cdot B = \bigcup_{a \in A, b \in B} a \cdot b, \;\; x \cdot B = \{x\} \cdot B \;\; and \;\; A \cdot x = A \cdot \{x\}.$$

A polygroup is a special case of hypergroup. According to [2] and [3] a polygroup is a system $\mathcal{P} = \langle P, \cdot, e, ^{-1} \rangle$, where $e \in P$, $^{-1}$ is a unary operation on P, $\cdot$ maps $P \times P$ into non-empty subsets of P, and the following axioms hold for all $x, y, z \in P$:

(P_1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;
(P_2) $x \cdot e = e \cdot x = x$;
(P_3) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.

A hyperring is a hyperstructure with two hyperoperation $+$ and $\cdot$ that satisfies the ring-like axiomes: $(R, +, \cdot)$ is a hyperring if $(R, +)$ is a commutative polygroup, $\cdot$ is an associative hyperoperation and the distributive laws $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$, $(x + y) \cdot z \subseteq x \cdot z + y \cdot z$ are satisfied for every $x, y, z \in R$. The element 0 is called zero element of R if $0 \cdot x = x \cdot 0 = 0$ for all $x \in R$.

* speaker

$(R,+,\cdot)$ is called a semihyperring if $+$, $\cdot$ are associative hyperoperations where $\cdot$ is distributive whith respect to $+$.

The character is a very important function in the theory of representations because it characterizes the representation. Thus it is natural to define this for hypermatrix representations as well. But first we need some definitions. In the following M_n denotes the set of all $n \times n$ hypermatrices over a given hyperring.

Definition 1.1. *Let $(R,+,\cdot)$ be hyperring endowed with a zero element 0, and the set of unite elements $U_R = \{u \in R \,|\, r \in (ur) \cap (ru) \text{ for all } r \in R\}$ is non-empty. A hypermatrix $I_n \in M_n$ is called unit hypermatrix if it is of the form $I_n = diag(u_1,...,u_n)$ where $u_i \in U_R$ for all $i \in 1,...,n$. So one has $A \in (AI_n \cap I_nA)$ for all $A \in M_n$. Remark that the above relation can be also valid for non-diagonal hypermatrices, but the set of identities becomes greater. An $A \in M_n$ will be called invertible if there exists a hypermatrix $A^{-1} \in M_n$, called inverse of A, such that $I_n \in (A^{-1}A \cap AA^{-1})$ where I_n is a unit matrix.*

These definitions may give enormous number of identities and inverses, however, more we are interested in hyperrings endowed unique 0 and 1. A hypermatrix $A \in M_n$ is said to quasi-diagonal if it is of the block form:

$$A = diag(A_1, A_2, ..., A_k)$$

where $A_i \in M_{n_i}$, $i = 1,...,k$, $n_i \in N^*$ and $n_1 + ... + n_k = n$. So A is the direct sum of the hypermatrices A_i and we write

$$A = A_1 \oplus ... \oplus A_k$$

$A \in M_n$ will be called reducible if there is an invertible hypermatrix B such that $(B^{-1}A)B$ or $B^{-1}(AB)$ contains a proper quasi-diagonal $A' \in M_n$, i.e. A is similar to a quasi-diagonal hypermatrix. If A is not reducible, then it is called irreducible.

In this paper we introduce characters of polygroups over hyperrings and show such characters induce characters of the fundamental group over corresponding fundamental ring.

2. Main Results

Definition 2.1. *[7] Let $A = (a_{ij})$ be hypermatrix over the commutative hyperring $(R,+,\cdot)$, then we can define the following traces:*

1. $Tr : M_n \longrightarrow \mathcal{P}(R) \;:\; TrA = \sum_{i=1}^{n} a_{ii}$.
2. $tr_x : M_n \longrightarrow R \;:\; tr_x \in TrA,\ (x \in R)$.
3. $tr_\varphi : M_n \longrightarrow R/\gamma^* \;:\; tr_\varphi A = \gamma^*(\sum a_{ii}) = tr\varphi(A) \in R/\gamma^*$, *where φ is the fundamental map (fundamental trace).*

Theorem 2.2.

1. $tr_\varphi(I_n) = n$ *for all* $n \times n$ *unit hypermatrices on* R.
2. $tr_\varphi(AB) = tr_\varphi(BA)$ *for all* $A, B \in M_n$.
3. $tr_\varphi(B^{-1}(AB)) = tr_\varphi((B^{-1}A)B) = tr_\varphi A$ *for all* $A, B \in M_n$.

Definition 2.3. *Let* T *be a representation of a polygroup* P *by hypermatrices over* R, *we shall call fundamental character of* T *the mapping*

$$X_T : H \longrightarrow R/\gamma^*;\ x \longmapsto X_T(x) = tr_\varphi(T(x)) = tr(T^*(x)).$$

Definition 2.4. *Let* T *be a hypermatrix representation of degree* n *of* P *over* R. *Let denote by* $diag(M_{n_1}, ..., M_{n_k})$ *where* $n_i \in N^*$ *and* $n_1 + ... + n_k = n$ *the set of all quasi-diagonal hypermatrices over* R, $diag(A_1, ..., A_k)$ *where* $A_{n_i} \in M_{n_i}$, $\forall i = 1, ..., k$. *If every matrix* $T(x)$, $x \in P$ *has a similar matrix which belongs to a fixed set*

$$diag(M_{n_1}, ..., M_{n_k}),$$

then T *is called reducible. If* T *is not reducible, then it is called irreducible. If* T *is reducible then for every* $x \in P$ *we can write*

$$T(x) = diag(T_1(x), ..., T_k(x)),$$

where $T_i(x) \in M_{n_i}$, $i = 1, ..., k$. *The components* T_i, $i = 1, ..., k$ *are also representations of* P *over* R. *Indeed, it is clear that for all* $x, y \in P$ *we have*

$$\begin{aligned} T(xy) &= diag(T_1(xy), ..., T_k(xy)) \\ &= diag(T_1(x), ..., T_k(x)) \cdot diag(T_1(y), ..., T_k(y)) \\ &= T(x)T(y) \end{aligned}$$

We write $T = T_1 \oplus ... \oplus T_k$ *and* T *is called direct sum of the representations* $T_1, ..., T_k$. *In a direct sum it is immediate that*

$$tr_\varphi T(x) = tr_\varphi T_1(x) + ... + tr_\varphi T_k(x).$$

Therefore, the character X_T *can be written as*

$$X_T = X_{T_1} + ... + X_{T_k}.$$

A character is called reducible (resp. irreducible) if it corresponds to reducible (resp. irreducible) representation.

Theorem 2.5. *Let* T *be any inclusion hypermatrix representation of* P *over* R, *of degree* n. *Then there exists a unique group character* X_T *such that* $X_T = X_{T_\varphi} \circ \varphi$, *of the fundamental group over the fundamental ring.*

Example 2.6. *Suppose that the multiplication table for polygroup* $\mathcal{P} = \langle P, \cdot, e, ^{-1} \rangle$ *where* $P = \{e, a, b\}$ *is*

$\cdot$	e	a	b
e	e	a	b
a	a	$\{e, b\}$	$\{a, b\}$
b	b	$\{a, b\}$	$\{e, a\}$

In $\mathbb{Z}_3$, we define a hyperoperation $\oplus$ as follows:
$1 \oplus 1 = \{0,2\}, 2 \oplus 2 = \{0,1\}, 1 \oplus 2 = \{1,2\}$ *and $\oplus$ be the usual sum for the other cases, and let $\odot$ be the usual product in $\mathbb{Z}_3$. One can see that $(\mathbb{Z}_3, \oplus, \odot)$ is a semihyperring.*

If we choose i_0, j_0, $i_0 \neq j_0$, $0 \leq i_0$, $j_0 \leq n$ and then put $T(e) = I_n$, $T(a) = A_n$ and $T(b) = B_n$ where

$$A_n = (a_{ij}) \ with \ \begin{cases} a_{ii} = 1 \ \ i = 1, ..., n \\ a_{i_0 j_0} = 1 \\ a_{ij} = 0 \ \ otherwise. \end{cases}$$

$$B_n = (b_{ij}) \ with \ \begin{cases} b_{ij} = a_{ij} \ \ if \ \ i \neq i_0, \ j \neq j_0 \\ b_{i_0 j_0} = 2, \end{cases}$$

then T is a representation of P. Characters of P over R are:
$TrA_n = 1 \oplus 1 \oplus 1 = (1 \oplus 1) \oplus 1 = \{0,2\} \oplus 1 = \{0 \oplus 1, 2 \oplus 1\} = \{1, \{1,2\}\}$ *and* $TrB_n = TrA_n = \{1, \{1,2\}\}$.

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Commutativity degree of crossed modules

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Abstract

In this article, we extend the notion of commutativity degree to the class of all finite crossed modules. We shall state some results concerning commutativity degree of crossed modules and obtain some upper and lower bounds for commutativity degree of finite crossed modules. Finally we show that, if two crossed modules are isoclinic, then they have the same commutativity degree.

Keywords and phrases: Crossed Module; Commutativity Degree; Isoclinism. .
2020 *Mathematics subject classification:* Primary: 20E34, Secondary: 20N99, 18B99.

1. Introduction

In 1968, Erdös and Turán, [2] introduced the concept of commutativity degree of groups, when they worked on symmetric groups. Let G be a finite group, the commutativity degree of G, denoted by $d(G)$ is defined as

$$d(G) = \frac{|\{(x,y) \in G \times G : xy = yx\}|}{|G|^2}.$$

Note that $d(G) > 0$ and $d(G) = 1$ if and only if G is abelian. In 1973, Gustafson [3] obtained an upper bound for $d(G)$, when G is a non-abelian finite group. Few years later, Rusin [6] computed the value of $d(G)$, when $G' \subseteq Z(G)$ and $G' \cap Z(G)$ is trivial and classified all finite groups G for which $d(G)$ is greater than $\frac{11}{32}$. A crossed module (T,G,δ) is a group homomorphism $\delta : T \longrightarrow G$ together with an action of G on T satisfying certain conditions. In [5] and [7] the concept of isoclinism has been generalized for crossed modules. In this article, we generalize the concept of commutativity degree for the finite crossed modules and show that two isoclinic crossed modules have the same commutativity degree.

* speaker

2. Main Results

A crossed module (T, G, δ) is a pair of groups T and G together with an action of G on T and a homomorphism $\delta : T \longrightarrow G$ called the boundary map, satisfying the following axioms:

$i)$ $\delta({}^{g}t) = g\delta(t)g^{-1}$ for all $g \in G$, $t \in T$,

$ii)$ ${}^{\delta(t)}s = tst^{-1}$ for all $t, s \in T$.

We will denote such a crossed module by $T \xrightarrow{\delta} G$. A crossed module (T, G, δ) is said to be finite, if the groups T and G are both finite. A crossed module (S, H, δ') is a subcrossed module of (T, G, δ), when

$i)$ S is a subgroup of T and H is a subgroup of G,

$ii)$ $\delta' = \delta|_S$, the restriction of δ to S,

$iii)$ the action of H on S is induced by the action of G on T.

In this case, we write $(S, H, \delta') \leqslant (T, G, \delta)$. A subcrossed module (S, H, δ) of (T, G, δ) is a normal subcrossed module, if

$i)$ H is a normal subgroup of G,

$ii)$ ${}^{g}s \in S$ for all $g \in G$, $s \in S$,

$iii)$ ${}^{h}tt^{-1} \in S$ for all $h \in H$, $t \in T$.

This is denote by $(S, H, \delta) \trianglelefteq (T, G, \delta)$. Let (S, H, δ) be a normal subcrossed module of (T, G, δ). Consider the triple $(\frac{T}{S}, \frac{G}{H}, \bar{\delta})$, where $\bar{\delta} : \frac{T}{S} \longrightarrow \frac{G}{H}$ is induced by δ. There is the action of $\frac{G}{H}$ on $\frac{T}{S}$ given by ${}^{gH}(tS) = ({}^{g}t)S$. It is called the quotient crossed module of (T, G, δ) by (S, H, δ) and denoted by $\frac{(T,G,\delta)}{(S,H,\delta)}$. Let (T, G, δ) be a crossed module. The center of (T, G, δ) is the crossed module $Z(T, G, \delta) : T^G \longrightarrow St_G(T) \cap Z(G)$, where $T^G = \{t \in T : {}^{g}t = t \quad for\ all\ g \in G\}$ and $St_G(T) = \{g \in G : {}^{g}t = t \quad for\ all\ t \in T\}$. A crossed module (T, G, δ) is abelian, if $(T, G, \delta) = Z(T, G, \delta)$. In addition, the commutator subcrossed module $[(T, G, \delta), (T, G, \delta)]$ of (T, G, δ) is $[(T, G, \delta), (T, G, \delta)] : D_G(T) \longrightarrow [G, G]$, where $D_G(T)$ is the subgroup generated by $\{{}^{g}tt^{-1} : t \in T,\ g \in G\}$ and $[G, G]$ is the commutator subgroup of G. The (T, G, δ) is abelian if and only if G is abelian and the action of the crossed module is trivial [4].

Remark 2.1. *let (T, G, δ) be a crossed module. We denote $\frac{(T,G,\delta)}{Z(T,G,\delta)}$ by $\bar{T} \xrightarrow{\bar{\delta}} \bar{G}$, where $\bar{T} = \frac{T}{T^G}$ and $\bar{G} = \frac{G}{St_G(T) \cap Z(G)}$, for shortness.*

Lemma 2.2. ([5]). *Let (T, G, δ) be a crossed module. Define the maps $c_1 : \bar{T} \times \bar{G} \longrightarrow D_G(T)$, where $(tT^G, g(St_G(T) \cap Z(G))) \mapsto {}^{g}tt^{-1}$ and $c_0 : \bar{G} \times \bar{G} \longrightarrow [G, G]$, where $(g(St_G(T) \cap Z(G)), g'(St_G(T) \cap Z(G))) \mapsto [g, g']$, for all $t \in T$, $g, g' \in G$. Then the maps c_1 and c_0 are well-defined.*

Definition 2.3. ([7]). *The crossed modules (T_1, G_1, δ_1) and (T_2, G_2, δ_2) are isoclinic, if there exist isomorphisms*

$$(\eta_1, \eta_0) : (\bar{T_1}, \bar{G_1}, \bar{\delta_1}) \longrightarrow (\bar{T_2}, \bar{G_2}, \bar{\delta_2})$$

and

$$(\epsilon_1,\epsilon_0) : (D_{G_1}(T_1) \to [G_1,G_1]) \longrightarrow (D_{G_2}(T_2) \to [G_2,G_2])$$

such that the diagrams

$$\begin{array}{ccc} \bar{T}_1 \times \bar{G}_1 & \xrightarrow{c_1} & D_{G_1}(T_1) \\ \Big\downarrow \eta_1\times\eta_0 & & \Big\downarrow \epsilon_1 \\ \bar{T}_2 \times \bar{G}_2 & \xrightarrow{c_1'} & D_{G_1}(T_1) \end{array}$$

and

$$\begin{array}{ccc} \bar{G}_1 \times \bar{G}_1 & \xrightarrow{c_0} & [G_1,G_1] \\ \Big\downarrow \eta_0\times\eta_0 & & \Big\downarrow \epsilon_0 \\ \bar{G}_2 \times \bar{G}_2 & \xrightarrow{c_0'} & [G_2,G_2] \end{array}$$

are commutative, where (c_1,c_0) and (c_1',c_0') are commutator maps of crossed modules (T_1,G_1,δ_1) and (T_2,G_2,δ_2), that introduced in Lemma 2.3. The pair $((\eta_1,\eta_0),(\epsilon_1,\epsilon_0))$ will be called an isoclinism from (T_1,G_1,δ_1) to (T_2,G_2,δ_2) and this situation will be denoted by $((\eta_1,\eta_0),(\epsilon_1,\epsilon_0)) : (T_1,G_1,\delta_1) \sim (T_2,G_2,\delta_2)$.

Definition 2.4. ([1]) *Let (T,G,δ) be a finite crossed module. The commutativity degree $d(T,G,\delta)$ of (T,G,δ) is defined by*

$$d(T,G,\delta) = \frac{|\{(x,y) \in G \times G : xy = yx, \quad x,y \in St_G(T)\}|}{|G|^2}.$$

It is clear that, (T,G,δ) is abelian if and only if $d(T,G,\delta) = 1$.

Theorem 2.5. *Let (T,G,δ) be a crossed module. Then $d(T,G,\delta) \leq \frac{K(G)}{|G|}$, where $K(G)$ is the number of conjugacy classes of G.*

Corollary 2.6. *If (T,G,δ) is a crossed module and the action of G on T is trivial, then $d(T,G,\delta) = \frac{K(G)}{|G|}$ and $\frac{1}{|G'|} \leq d(T,G,\delta)$.*

Theorem 2.7. *Let (T,G,δ) be a crossed module. Then $d(T,G,\delta) \leq \frac{1}{4}(1 + \frac{3}{|G'|})$.*

Theorem 2.8. *Let (T,G,δ) be a crossed module. If G is a non-abelian finite group, then $d(T,G,\delta) \leq \frac{5}{8}$.*

Example 2.9. *Let $D_{pq} =< a,b : a^p = b^q = e, bab^{-1} = a^r >$ such that p is prime, $q|p-1$ and r has order q mod p. This type of group is called a generalized dihedral group. Conjugacy classes type are $[e]$, $[a^u]$ and $[b^w]$ so that no classes are 1, $\frac{p-1}{q}$ and $q-1$, respectively and $Z(D_{pq}) = \{e\}$. Consider the map $i : D_{pq} \longrightarrow D_{pq}$. If the action of D_{pq} on D_{pq} is conjugacy, then $St_{D_{pq}}(D_{pq}) = Z(D_{pq})$ and*

$d(D_{pq},D_{pq},i) = \frac{|Z(D_{pq})|^2}{|D_{pq}|^2} = \frac{1}{(pq)^2}$. *If the action of* D_{pq} *on* D_{pq} *is trivial, then* $d(D_{pq},D_{pq},i) = \frac{K(D_{pq})}{|D_{pq}|} = \frac{1+\frac{p-1}{q}+q-1}{pq} = \frac{q^2+p-1}{pq^2}$. *If the action of* D_{pq} *on* D_{pq} *is faithful, then* $d(D_{pq},D_{pq},i) = \frac{1}{|D_{pq}|^2} = \frac{1}{(pq)^2}$.

Theorem 2.10. *Let* $(T_1,G_1,\delta_1),(T_2,G_2,\delta_2)$ *be two isoclinic finite crossed modules. Then* $d(T_1,G_1,\delta_1) = d(T_2,G_2,\delta_2)$.

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On hyper CI-algebras: as a generalization of hyper BE-algebras

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Abstract

In this paper, we define the notion of hyper CI-algebras as a generalization of hyper BE-algebras and present some properties. Also, we define the commutative hyper CI-algebra and find the number of commutative hyper CI-algebra of order less than 3.

Keywords and phrases: CI-algebra, hyper BE-algebra, hyper (commutative) CI-algebra.
2010 *Mathematics subject classification:* Primary: 06F35, 03G25; Secondary: 20N20.

1. Introduction and Preliminaries

The hyper algebraic structure theory was introduced in 1934, by F. Marty at the 8^{th} congress of Scandinavian Mathematiciens [3]. H.S. Kim et al. defined the notion of a BE-algebra as a generalization of a dual BCK-algebra [2]. B.L. Meng introduced the notion of CI-algebras, and studied some relations with BE-algebras [4]. A. Radfar et al. introduced the notion of hyper BE-algebra and defined some types of hyper filters in hyper BE-algebras. They showed that under special condition hyper BE-algebras are equivalent to dual hyper K-algebras [6]. A. Rezaei et al. characterized the relation between dual hyper K-algebras and commutative hyper BE-algebras and some types of commutative hyper BE-algebras [7]. F. Iranmanesh et al. studied some types of Hv-BE-algebras and investigate the relationship between them [1]. Recently, R. Naghibi et al. introduced the new class of H_v-BE-algebra as a generalization of a (hyper) BE-algebra and they construct the H_v-BE-algebra associated to a BE-algebra [5]. In this paper, we introduce notions of (commutative) hyper CI-algebra and study its properties.

An algebra $\mathbb{A} = (A; *, 1)$ of type (2,0) is called a CI-algebra if following axioms hold ([4]):
(CI1) $x * x = 1$,
(CI2) $1 * x = x$,
(CI3) $x * (y * z) = y * (x * z)$, for all $x, y, z \in A$.

* speaker

CI-algebra $\mathbb{A} = (A; *, 1)$ is said to be BE-algebra if satisfies (BE) $x * 1 = 1$, for all $x \in A$ ([2]).

Let H be a nonempty set and $\circ : H \times H \to P^*(H)$ be a hyperoperation, where $P^*(H) = P(H) \setminus \emptyset$. Then $\mathbb{H} = (H; \circ, 1)$ is called a hyper BE-algebra, if it satisfies the following axioms ([6]):
(HBE$_1$) $x \prec 1$ and $x \prec x$,
(HBE$_2$) $x \circ (y \circ z) = y \circ (x \circ z)$,
(HBE$_3$) $x \in 1 \circ x$,
(HBE$_4$) $1 \prec x$ implies $x = 1$, for all $x, y, z \in H$.

A hyper BE-algebra $\mathbb{H} = (H; \circ, 1)$ is said to be commutative if

$$(x \circ y) \circ y = (y \circ x) \circ x, \text{ for all } x, y \in H \text{ ([7])}.$$

2. On hyper CI-algebras

In this section, as a generalization of hyper BE-algebra, we define the notion of hyper CI-algebra and investigate some results.

Definition 2.1. *Let H be a nonempty set and $\circ : H \times H \to P^*(H)$ be a hyperoperation. Then $\mathbb{H} = (H; \circ, 1)$ is called a hyper CI-algebra, if it satisfies the following axioms:*
(HCI$_1$) $x \prec x$,
(HCI$_2$) $x \circ (y \circ z) = y \circ (x \circ z)$,
(HCI$_3$) $x \in 1 \circ x$, for all $x, y, z \in H$.

Where the relation $'' \prec ''$ is defined by $x \prec y \Leftrightarrow 1 \in x \circ y$. For any two nonempty subsets A and B of H, we define $A \prec B$ if and only if there exist $a \in A$ and $b \in B$ such that $a \prec b$ and $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b$.

We will also refer to the hyper CI-algebra $\mathbb{H} = (H; \circ, 1)$ by $\mathbb{H}$.

Example 2.2. *Consider $\mathbb{R}$ as the set of real numbers. Define the hyper operation $'' \circ ''$ on $\mathbb{R}$ as follows:*

$$x \circ y = \begin{cases} \{1, y\} & \text{if } x = 1; \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

Then $(\mathbb{R}; \circ, 1)$ is a hyper CI-algebra.

Example 2.3. *Let $H = \{1, a, b, c\}$. Define the hyperoperation $'' \circ ''$ on H as follows:*

$\circ$	1	a	b	c
1	$\{1\}$	$\{a\}$	$\{1, b\}$	$\{a, c\}$
a	$\{a\}$	$\{1, a\}$	$\{a, c\}$	$\{1, a, b, c\}$
b	$\{1\}$	$\{a\}$	$\{1\}$	$\{1, a\}$
c	$\{1\}$	$\{1\}$	$\{a\}$	$\{1, a\}$

Then $(H; \circ, 1)$ is a hyper CI-algebra.

Proposition 2.4. *Any hyper BE-algebra is a hyper CI-algebra.*

The following example shows that every hyper CI-algebra is not a hyper BE-algebra, in general.

Example 2.5. *Let $H = \{1, a, b, c\}$. Define the hyperoperation " $\circ$ " on H as follows:*

$\circ$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{b\}$	$\{1, a, b\}$	$\{1, a, b\}$
b	$\{b\}$	$\{a, b\}$	$\{1, a, b\}$

Then $(H; \circ, 1)$ is a hyper CI-algebra. Since $a \nless 1$, we get (HBE_1) does not hold. Thus it is not a hyper BE-algebra.

Proposition 2.6. *Let $(A; *, 1)$ be a CI-algebra. If define $x \circ y = \{x * y\}$, for all $x, y \in H$, then $(A; \circ, 1)$ is a hyper CI-algebra.*

Proposition 2.7. *Let $\mathbb{H}$ be a hyper CI-algebra and $x, y \in H$. If $x \prec y$, then $x \prec (x \circ y) \circ x$ and $y \prec (x \circ y) \circ y$.*

Proposition 2.8. *Let $\mathbb{H}$ be a hyper CI-algebra. Then*
(i) $1 \circ x \prec x$,
(ii) $x \prec 1 \circ x$,
(iii) $x \prec 1 \circ (1 \circ (\cdots (1 \circ x) \cdots))$,
(iv) $y \prec ((y \circ x) \circ x)$, *for all* $x, y \in H$.

Theorem 2.9. *Let $\mathbb{H}$ be a hyper CI-algebra. Then*
(i) $A \circ (B \circ C) = B \circ (A \circ C)$,
(ii) $A \prec A$,
(iii) $x \prec y \circ z$ *implies* $y \prec x \circ z$,
(vi) $z \in x \circ y$ *implies* $x \prec z \circ y$, *for all* $x, y, z \in H$ *and* $A, B, C \subseteq H$.

Theorem 2.10. *There exist 16 hyper CI-algebras of order less than 3 up to isomorphism.*

3. On commutative hyper CI-algebras

In this section, we discuss on commutative hyper CI-algebras, and enumerate them of order less than 3.

Definition 3.1. *A hyper CI-algebra $\mathbb{H}$ is said to be commutative if*

$$(y \circ x) \circ x = (x \circ y) \circ y, \text{ for all } x, y \in H.$$

Example 3.2. *(i) In Example 2.2, $(\mathbb{R}; \circ, 1)$ is a commutative hyper CI-algebra.*
(ii) Let $H = \{1, a, b\}$. Define the hyper operation " $\circ$ " as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{1, a\}$	$\{b\}$
a	$\{1, a, b\}$	$\{1, a, b\}$	$\{b\}$
b	$\{1, a\}$	$\{1, a, b\}$	$\{1, a, b\}$

Then $(H;\circ,1)$ *is a commutative hyper CI-algebra.*

Proposition 3.3. *Let* $\mathbb{H}$ *be a commutative hyper CI-algebra such that* $1\circ x=\{x\}$. *Then* $x\circ y=y\circ x=\{1\}$ *implies* $x=y$.

Proposition 3.4. *Let* $\mathbb{H}$ *be a commutative hyper CI-algebra. Then*
(i) $1\in(x\circ 1)\circ 1$,
(ii) $(x\circ 1)\prec 1$, *for all* $x\in H$.

The following example shows that, in the Proposition 3.4, condition commutativity is necessary.

Example 3.5. *Consider the Example 2.5, hyper CI-algebra* $(H;\circ,1)$ *is not commutative, since* $H=(1\circ a)\circ a\neq(a\circ 1)\circ 1=\{b\}$. *Also, we have* $1\notin(b\circ 1)\circ 1=\{b\}$, *and* $b=(b\circ 1)\not\prec 1$.

Theorem 3.6. *There exist 7 commutative hyper CI-algebras of order less than 3 up to isomorphism.*

4. Conclusions and future works

In the present paper, we have introduced the concept of hyper CI-algebras, and presented some of their useful properties. It is shown that there exist 16 hyper CI-algebra, and 7 commutative hyper CI-algebra of order less than 3 up to isomorphism. In our future work, we will investigate among filters in hyper CI-algebras and characterization of hyper CI-algebras in cases $|H|=3$ and 4.

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Reconstructing normal edge-transitive Cayley graphs of abelian groups

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Abstract

Cayley graphs of groups have been used extensively for designing interconnection networks with optimal fault tolerance. On the other hand, normal edge-transitive Cayley graphs have been extensively studied by many authors and they are characterized in some classes of groups. In this paper, first we focus on reconstruction problem posed by Praeger and give a necessary and sufficient condition for a Cayley graph of an abelian group to be normal edge-transitive. Then we investigate the main properties of these graphs as interconnection networks and we show that they have several supremacies comparing with many other known networks.

Keywords and phrases: Normal edge-transitive Cayley graphs, Factorization of groups, Optimal fault-tolerance..

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1. Introduction

To design very fast computers, a lot of processors need to work together and communicate with each other. These processors must be interconnected with each other such that the time of communication between them would be as short as possible and compatible with hardware technology restrictions. To solve these issues, algebraic and geometric tools and graph theory methods have been applied and several studies have been done in Mathematics, Computer Science and Hardware Engineering.

Recall that an interconnection network is a network of interconnected devices and refers to the network used to route data between the processors in a multiprocessor computing system. The interconnection network is often modeled as a graph. The vertices of the graph correspond to processors, and two vertices are adjacent in the graph whenever there is a direct communication link between the two corresponding processors (see [1] and [6]). In the rest of the paper, similarly to [6], we use the terms interconnection networks and graphs interchangeably. This easy model enables us to use graph theory in

* speaker

designing interconnection networks, and also using graph theoretical parameters for comparing the performances of different networks with each other.

Note that the main theoretical properties of a good graph with high performance as an interconnection network (and definitely not all in reality) are the following: low degree, small diameter, small mean distance, high vertex-connectivity, high edge-connectivity, algebraic and easy construction, and having easy routing and alternate path algorithms (see [1]). Clearly, designing of a network with all the above properties is not an easy task, because some of these properties are in conflict with each other. So we have to use very advanced tools in Mathematics (specially, in graph theory and algebraic graph theory).

Cayley graphs of groups are very algebraic structures which have many applications in Mathematics and other research areas. Akers and Krishnamurthy in [1] were first who suggested Cayley graphs in designing interconnection networks. Later, in [18] this study was continued and specially, Cayley graphs of simple groups are suggested for presenting networks with better performance rather than several other known networks. In fact, the algebraic structure of a Cayley graph makes working with them easy, specially in presenting very effective routing algorithms. Furthermore, edge-transitive Cayley graphs are more interesting because they are symmetric (vertex- and edge-transitive) and so they have optimal fault-tolerance. Therefore normal edge-transitive Cayley graphs of groups which have been extensively studied by many authors (for example see [2],[3], [4], [5], [8], [12] and [19]) are very good choices for designing interconnection networks.

To state our main results in this paper, first we recall some facts and notions. For every set S, by $|S|$ we mean the cardinality of S. For every non-empty set I and every set $A = \prod_{i\in I} A_i$, by $\pi_j : A \to A_j$ we mean the natural projection onto the j-th component for $j \in I$. The normalizer of a subgroup K in a group G is denoted by $N_G(K)$. For every group G, let $\mathrm{Aut}(G)$ denote the automorphism group of G. Also note that by id_G, we mean the identity map on G.

Let G be a group and $C \subseteq G \setminus \{1_G\}$, where 1_G denotes the identity element of G. The *Cayley graph* $\mathrm{Cay}(G,C)$ *of* G *with respect to* C is defined as the graph with vertex set G and arc set $E(\mathrm{Cay}(G,C))$ consisting of those ordered pairs (g,g') such that $cg = g'$, for some $c \in C$ (equivalently, $g'g^{-1} \in C$). The set C is called *the connection set of* $\mathrm{Cay}(G,C)$. Note that Cayley graphs of groups are always vertex-transitive but they are not necessarily edge-transitive. The graph $\mathrm{Cay}(G,C)$ is undirected if and only if C is an inverse-closed subset of G (i.e. $C = C^{-1}$). Also note that $\mathrm{Cay}(G,C)$ is connected if and only if C is a generating set of G. From now on, we denote $\mathrm{Aut}(\mathrm{Cay}(G,C))$ by $A_C(G)$. For every $g \in G$, the function $\rho_g : G \to G$ defined by $h \mapsto hg$, for every $h \in G$, belongs to $A_C(G)$. The group $\{\rho_g \mid g \in G\}$ is called *the right regular representation of* G and it is denoted by G_R.

Following the terminology in [17] we denote the group $\{\sigma \in \mathrm{Aut}(G) \mid \sigma(C) = C\}$, by $\mathrm{Aut}(G;C)$. Recall that the Cayley graph $\mathrm{Cay}(G,C)$ is said to be *normal edge-transitive*, if $N_{A_C(G)}(G_R)$ is transitive on edges (see [17]). Equivalently, by [17, Proposition 1], for an inverse-closed generating set C of a group G, the Cayley graph $\mathrm{Cay}(G,C)$ is normal edge-transitive if and only if $\mathrm{Aut}(G;C)$ acts transitively on $\widehat{C}$, where $\widehat{C} = \{\{c,c^{-1}\} \mid c \in C\}$.

Let $\mathcal{P}$ be the collection of pairs (G,C), where C is a generating set of G and $\mathrm{Cay}(G,C)$ is a normal edge-transitive Cayley graph such that G has no pair of non-trivial subgroups H_1, H_2 which are $\mathrm{Aut}(G;C)$-invariant and $G = H_1 \times H_2$. Note that for every normal edge-transitive Cayley graph $\mathrm{Cay}(G,C)$, where G is a simple group and C is a generating set of G, the pair (G,C) belongs to $\mathcal{P}$.

Let G be a group and $\mathbf{P}$ be a vertex partition of $\Gamma = \mathrm{Cay}(G,C)$, where C is a subset of G. Recall that for a finite group G and a generating set C of G, $N_{A_C(G)}(G_R)$ is equal to $G_R \rtimes \mathrm{Aut}(G;C)$, the semidirect product of G_R and $\mathrm{Aut}(G;C)$ (see [7]). We denote the quotient of Γ relative to $\mathbf{P}$ by $\Gamma_{\mathbf{P}}$. Recall that the group G induces a group of automorphisms of $\Gamma_{\mathbf{P}}$ if and only if $\mathbf{P}$ is the set of cosets of a subgroup H of G. Furthermore the quotient is a Cayley graph for a quotient group of G if and only if H is a normal subgroup of G (see [17, Theorem 3]). On the other hand, note that if G is finite and $N_{A_C(G)}(G_R)$ is transitive on the edges of $\mathrm{Cay}(G,C)$ (or on unordered edges of $\mathrm{Cay}(G,C)$), then $N_{A_C(G)}(G_R)$ acts transitively on the edges (or unordered edges) of the quotient of $\mathrm{Cay}(G,C)$ if and only if H is $\mathrm{Aut}(G;C)$-invariant (see [17, Theorem 3]). If $\mathrm{Cay}(G,C)$ is a finite normal edge-transitive Cayley graph, then the set

$$\mathcal{C}(G,C) = \{\mathrm{Cay}(\frac{G}{H}, \frac{CH}{H}) \mid H \text{ is an } \mathrm{Aut}(G;C)\text{-invariant normal subgroup of } G\}$$

is a non-empty set. Having these notions and facts, we are able to state our main questions in this paper.

Question 1. *([17, Question 2. Reconstruction]) Given a normal edge-transitive Cayley graph $\mathrm{Cay}(G,C)$, under what conditions is it determined by its quotient graphs in $\mathcal{C}(G,C)$?*

2. Main Results

The main idea of this paper is to use the notion of normal edge-transitive Cayley graphs to provide an algebraic algorithm for constructing symmetric graphs and using it to continue the study of these graphs. For this purpose, using the idea of reconstruction question posed in [17] about normal edge-transitive Cayley graphs (see Question 1), we present a special factorization of groups which is well-behavior with respect to normal edge-transitivity (see Theorem 2.1). Note that edge-transitive graphs are not very well-behaviour with respect to different products of graphs (see [9]). Then we show that using our factorization, every normal edge-transitive Cayley graph of an abelian group can be decomposed graph theoretically to normal edge-transitive Cayley graphs of its Sylow subgroups (see Theorem 2.2 and Proposition 2.3).

Theorem 2.1. *[11, Theorem 1.3] Let G be a finite group and C be an inverse-closed generating set of G. Then $Cay(G,C)$ is normal edge-transitive if and only if there exists a family $\{(G_i,C_i)\}_{i=1}^{n} \subseteq \mathcal{P}$ such that the following conditions hold.*

(i) $G = G_1 \times \cdots \times G_n$;

(ii) *for every $1 \leq i \leq n$, $C_i = \pi_i(C) \setminus \{1_{G_i}\}$ and $Cay(G_i,C_i)$ is normal edge-transitive;*

(iii) *$Aut(G_1;C_1) \times \cdots \times Aut(G_n;C_n)$ has a subgroup which has an orbit O on G such that $G = O \cup O^{-1}$.*

Then we continue the study of normal edge-transitive Cayley graphs of abelian groups. Recall that in [8], normal edge-transitivity of Cayley graphs of $\mathbb{Z}_{p^2}$, $\mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_p \times \mathbb{Z}_q$ were determined, where p and q are prime numbers. In the following theorem, we continue this study with similar strategy and we show that by determining normal edge-transitive Cayley graphs of abelian p-groups, we can determine the other normal edge-transitive Cayley graphs of abelian groups.

Theorem 2.2. *[11, Theorem 1.4] Let G be a finite abelian group and C be an inverse-closed generating set of G. Then $Cay(G,C)$ is normal edge-transitive if and only if there exists a family $\{G_i\}_{i=1}^{n}$ of groups such that the following conditions hold.*

(i) *$G_1, \cdots, G_n$ are all Sylow subgroups of G;*

(ii) *for every $1 \leq i \leq n$, $C_i = \pi_i(C) \setminus \{1_{G_i}\}$ and the Cayley graph $Cay(G_i,C_i)$ is normal edge-transitive;*

(iii) *$Aut(G_1;C_1) \times \cdots \times Aut(G_n;C_n)$ has a subgroup which has an orbit O on G such that $C = O \cup O^{-1}$.*

Proposition 2.3. *[11, Proposition 3.6] For a group G and an inverse-closed generating set C of G, suppose that $Cay(G,C)$ is normal edge-transitive and $G = G_1 \times \cdots \times G_n$, where G_i is $Aut(G;C)$-invariant for every $1 \leq i \leq n$. Let $C_i = \pi_i(C) \setminus \{1_G\}$ for every $1 \leq i \leq n$. Then the following results hold.*

(i) $(G_i,C_i) \in \mathcal{P}$;

(ii) *$Cay(G,C)$ is a spanning subgraph of the strong product of $\{Cay(G_i,C_i)\}_{i=1}^{n}$;*

(iii) *For every $1 \leq i \leq n$, the natural projection $\pi_j : Cay(G,C) \to Cay(G_j,C_j)$ is a full epimorphism.*

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Deficient square graph of finite group

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Abstract

In this paper, we define thedeficient square graph $\Gamma_{ds}(G)$ which is a graph associated to a non-abelian finite group with the vertex set $G \setminus Z(G)$, where $Z(G)$ denotes the center of G, and two vertices x and y are joined whenever $|\{x,y\}^2| < 4$. We investigate how how the graph theoretical properties of $\Gamma_{ds}(G)$ can be effected on the group theoretical properties of G. We claim that if G and H are two non-abelian finite groups such that $\Gamma_{ds}(G) \cong \Gamma_{ds}(H)$, then $|G| = |H|$.

Keywords and phrases: Deficient square graph, Planar and Capable group. .
2010 *Mathematics subject classification:* Primary: 05C25; Secondary: 20P05.

1. Introduction

The study of algebraic structure, using the properties of graphs, becomes an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring or a group and investigation of algebraic properties of ring or group and investigation of algebraic properties of ring or a group using the associated graph, for instance, see [3]. In the present article, to any non-abelian group G we assign a graph and investigate algebraic properties of the group using the graph theoretical concepts. Before starting, let us introduce some necessary notation and definitions.

We consider the following way: let $Z(G)$ be the center of G, associate a graph $\Gamma_{ds}(G)$ with G as follows: Take $G \setminus Z(G)$ as the vertices of $\Gamma_{ds}(G)$ and join two distinct vertices x and y whenever $|\{x,y\}^2| < 4 := \{xy = yx \ or \ x^2 = y^2\}$. Two elements x and y of group G satisfy the deficient square property on $2-$subsets if $|\{x,y\}^2| < 4$, see[3] and [3]. Let $ds(G)$ be the probability that two randomly chosen elements x and y of G satisfy the deficient square property, that is, $xy = yx$ or $x^2 = y^2$.

Let $F_G(x)$ be the Freiman centralizer of an element x of a group G, that is, $F_G(x) = \{y \in G : |\{x,y\}^2| < 4\} = \{y \in G : xy = yx \ or \ x^2 = y^2\}$. We denote $F_G(x)$ simply by $F(x)$. It is clear that $C(x) \subseteq F(x)$, in which $C(x)$ is the centralizer of x.

* speaker

2. Deficient square graph

Throughout this section, G is a finite group and C_n is a cyclic group of order n. Here, we define the deficient square graph $\Gamma_{ds}(G)$ and then we state some basic graph theoretical properties of $\Gamma_{ds}(G)$, such as domination number. Moreover, we give its effect on the group theoretical properties of G.

Definition 2.1. *A graph $\Gamma_{ds}(G)$ associate to G may be defined as follows: Take $G \setminus Z(G)$ as vertices of $\Gamma_{ds}(G)$ and join two distinct vertices x and y, whenever $|\{x,y\}|^2 < 4$. The graph $\Gamma_{ds}(G)$ is called the deficient square graph of G.*

According to the definition, $d(x) = |F(x)| - |Z(G)| - 1$ for every vertex x. Clearly $\Gamma_{ds}(G)$ is precisely the null graph if and only if G is abelian. There is no group with deficient square empty graph, the otherwise it implies that $Z(G) = 1$ and $|C(x)| = |F(x)| = 2$ for every $x \in G \setminus Z(G)$ which is impossible.

Lemma 2.2. *For each group with odd order, $F(x) = C(x)$ for every $x \in G \setminus Z(G)$.*

Corollary 2.3. *If the order of G is odd, then $\Gamma_{ds}(G) = \Gamma_c(G)$.*

We want to express what the graph properties $\Gamma_{ds}(G \times A)$ can inherit from $\Gamma_{ds}(G)$, where A and G are finite abelian group and order of odd, respectively.

Theorem 2.4. *Let the order of G be odd.*
(i) If $\Gamma_{ds}(G)$ is complete, then $\Gamma_{ds}(G \times A)$ is also.
(ii) If $\Gamma_{ds}(G)$ is k-regular, then $\Gamma_{ds}(G \times A)$ is $|A|(k+1) - 1$-regular.
(iii) If $\Gamma_{ds}(G)$ is connected, then $\Gamma_{ds}(G \times A)$ is also.

Theorem 2.5. *With the above notations and assumptions,*
(i) if order of G is odd, then $\gamma(\Gamma_{ds}(G)) > 1$.
(ii) if order of G is even, then $\{x\}$ is dominating set for $\Gamma_{ds}(G)$ if and only if $Z(G)$ is elementary $2-$group and $o(x) = o(y) = 4$ for every $y \in G \setminus C(x)$.

Corollary 2.6. *There is no group with deficient square star graph.*

Proposition 2.7. *If $Z(G)$ is elementary abelian $2-$group, then $|Z(G)|$ divides $|F(x)|$, for every $x \in G \setminus Z(G)$.*

Theorem 2.8. *Let G be non-abelian group and $Z(G)$ be elementary abelian $2-$group such that $\Gamma_{ds}(G) \cong \Gamma_{ds}(H)$, for some group H.*
(i) If $\gamma(\Gamma_{ds}(G)) > 1$, then $|Z(G)|$ divides

$$(|G| - |Z(G)|, |G| - |F(x)|, |F(x)| - |Z(G)|),$$

for very $x \in G \setminus Z(G)$.
(ii) If $\gamma(\Gamma_{ds}(G)) = 1$, then $|Z(G)|$ divides

$$(|G| - |Z(G)|, |G| - |F(x)|, |F(x)| - |Z(G)|),$$

for very $x \in G \setminus Z(G)$ such that $o(x) \neq 4$.

We define the deficient square probability $ds(G)$ of a finite group G to be the probability that a randomly chosen ordered pair of elements of G has the $DS-property$, that is,

$$ds(G) = \frac{|\{(x,y) \in G \times G \colon xy=yx \text{ or } x^2=y^2\}|}{|G|^2}.$$

It is clear that if G is abelian, then $ds(G) = 1$.
We recall [**??**, *Theorem* 2] as below, which is on essential tool in the next lemma.

Theorem 2.9. *If $P_4 = 0$, then $G = Q_8 \times E$, where Q_8 is quaternion group of order 8 and E an elementary abelian 2-group, (G is assumed to be non-abelian).*

Freiman in the above theorem showed that $ds(G) = 1$, for a finite non-abelian group G if and only if G is a direct product of the quaternion group of order 8 with an elementary abelian 2-group.

Lemma 2.10. *$\Gamma_{ds}(G)$ is complete graph if and only if G is a direct product of the quaternion group of order 8 with an elementary abelian 2-group.*

We want to express what the graph properties $\Gamma_{ds}(G \times E)$ can inherit from $\Gamma_{ds}(G)$, where E is elementary abelian 2-group of rank of n.

Theorem 2.11. *Let G be a finite non-abelian group.*
(i) If $\Gamma_{ds}(G)$ is complete, then $\Gamma_{ds}(G \times E)$ is complete.
(ii) If $\Gamma_{ds}(G)$ is k-regular, then $\Gamma_{ds}(G \times E)$ is $2^n(k+1) - 1$-regular.
(iii) If $\Gamma_{ds}(G)$ is connected, then $\Gamma_{ds}(G \times E)$ is connected.

We give some groups with unique the deficient square graph of G, i.e. groups G with the property that if $\Gamma_{ds}(G) \cong \Gamma_{ds}(H)$ for some H, then $G \cong H$. As expected, and as we shall show, the deficient square graph, in general, is not unique and there are non-isomorphic groups with the same deficient square graph. We concentrate on the following question.

Question 1. *Let G and H be two groups such that $\Gamma_{ds}(G) \cong \Gamma_{ds}(H)$, can we prove $|G| = |H|$?*

Question 1 has affirmative answer when one of groups is S_n and A_n.

Lemma 2.12. *Let $\Gamma_{ds}(G) \cong \Gamma_{ds}(S_3)$. Then $G \cong S_3$.*

Lemma 2.13. *Let G and H be two non-cyclic groups with $\Gamma_{ds}(G) \cong \Gamma_{ds}(H)$ and $|V(\Gamma_{ds}(G)|$ be prime. Then $|G| = |H|$.*

Theorem 2.14. *Let G be a non-cyclic groups with $\Gamma_{ds}(G) \cong \Gamma_{ds}(S_n)$. Then $|G| = |S_n|$.*

Theorem 2.15. *Let G be a non-cyclic groups with $\Gamma_{ds}(G) \cong \Gamma_{ds}(A_n)$. Then $|G| = |A_n|$.*

3. K_n-free

In this section, we define $K_n - free$ and then we examine K_n-freeness, planarity and regularity of $\Gamma_{ds}(G)$.

Definition 3.1. *A graph that does not contain K_n is called a K_n-free graph.*

Theorem 3.2. *Let G be a non-abelian group. Then $\Gamma_{ds}(G)$ is*
(i) $K_5 - free$ if and only if G is isomorphic to one of the groups D_8, S_3, A_4, $C_2^2 \rtimes C_4$ or $M_4(2) = \langle a,b : a^2 = 1, aba = b^{-3}\rangle$.
(ii) $K_6 - free$ if and only if G is isomorphic to one of the groups D_8, S_3, A_4, $C_2^2 \rtimes C_4$, F_5 or $M_4(2) = \langle a,b : a^2 = 1, aba = b^{-3}\rangle$.
(iii) $K_7 - free$ if and only if G is isomorphic to one of the groups D_8, S_3, A_4, $C_2^2 \rtimes C_4$, F_5, Q_8, Q_{12}, D_6, $Sl(2,3)$, $A_4 \times C_2$, $C_4 \circ D_4$, SD_{16}, He_3, 3_-^{1+2}, $\langle a,b : a^7 = b^3 = 1, bab^{-1} = a^4\rangle$ or $M_4(2) = \langle a,b : a^2 = 1, aba = b^{-3}\rangle$.
(iv) $K_8 - free$ if and only if G is isomorphic to one of the groups D_8, S_3, A_4, $C_2^2 \rtimes C_4$, F_5, Q_8, Q_{12}, D_6, $Sl(2,3)$, $A_4 \times C_2$, $C_4 \circ D_4$, SD_{16}, He_3, 3_-^{1+2}, $\langle a,b : a^7 = b^3 = 1, bab^{-1} = a^4\rangle$ or $M_4(2) = \langle a,b : a^2 = 1, aba = b^{-3}\rangle$.
(v) $K_9 - free$ if and only if G is isomorphic to one of the groups D_8, S_3, A_4, $C_2^2 \rtimes C_4$, F_5, Q_8, Q_{12}, D_6, $Sl(2,3)$, $A_4 \times C_2$, $C_4 \circ D_4$, SD_{16}, He_3, 3_-^{1+2}, $\langle a,b : a^4 = b^4 = 1, a^{-1}ba = b^{-1}\rangle$, $\langle a,b : b^2 = 1, bab = a^3\rangle$, $C_7 \rtimes C_3 = \langle a,b : a^7 = b^3 = 1, bab^{-1} = a^4\rangle$, $C_3 \times S_3$, Dic_5, $C_4 \rtimes C_8$, $M_5(2)$, $C_8 \circ D_4$, $C_2^2 \rtimes C_8$, $C_2 \times (C_2^2 \rtimes C_4)$, $C_2.C_4^2$, $C_4^2 \rtimes C_2$, $C_8 \rtimes C_4$, $C_3 \rtimes C_8$, $C_4 \times S_3$, $C_4 \times A_4$, $C_4.A_4$ or $M_4(2) = \langle a,b : a^2 = 1, aba = b^{-3}\rangle$.

Lemma 3.3. *(i) If $\Gamma_{ds}(G)$ contains $K_{3,3}$, then it contains K_5.*
(ii) Let G be a non-abelian group. Then $\Gamma_{ds}(G)$ is planer if and only if G is isomorphic to one of the groups D_8, S_3, A_4, $C_2^2 \rtimes C_4$ or $M_4(2) = \langle a,b : a^2 = 1, aba = b^{-3}\rangle$.

Corollary 3.4. *Let G be a non-abelian group. Then $girth(\Gamma_{ds}(G)) = 3$.*

In the following, we consider K_n-free graph that can be at most $n-2$-regular. On the other hand, if $\Gamma_{ds}(G)$ contins K_n, then it is at least $r-$regular when $r \geq n-1$. Thus we obtained the following results.

rank of regularity height3-regular	$\Gamma_{ds}(C_2^2 \rtimes C_4)$
5-regular	$\Gamma_{ds}(He_3)$ $\Gamma_{ds}(3_-^{1+2})$ $\Gamma_{ds}(Q_8)$
7-regular	$\Gamma_{ds}(C_4 \circ D_4)$ $\Gamma_{ds}(M_5(2))$ $\Gamma_{ds}(C_8 \circ D_4)$ $\Gamma_{ds}(C_2^2 \rtimes C_8)$ $\Gamma_{ds}(C_2 \times (C_2^2 \rtimes C_4))$ $\Gamma_{ds}(C_2.(C_4^2)$ $\Gamma_{ds}(C_8 \rtimes C_4))$

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Z-Scott topology and Z-refinement property

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Abstract

In this paper, we first recall the concept of a Z-poset and then introduce a topology on it called Z-Scott topology. Finally, we investigate the properties of this topology and also define especial kind of maps between Z-posets and give some sufficient conditions under which an arbitrary map is of the form of such maps.

Keywords and phrases: Subset systems; Z-Scott topology; Z-continuous posets; Z-refinement property.

2010 *Mathematics subject classification:* 06A15, 06B35.

1. Introduction

The concept of a subset system Z on the category **Pos** of posets with order-preserving maps as morphisms, is defined by Wright et al. in [14]. There the authors suggested a way to generalize Dana Scott's continuous lattices [13]. Markowsky [5] had already generalized Scott's continuous lattices to continuous posets. In both Scott's and Markowsky's defintions directed sets played a fundamental role. In [14], instead of confinding themselves to directed sets, the authors introduced a more general concept, that of a subset system. The results in this paper are presented as pure mathematics, that is without applications. However the posets with Z-set structures have the applications to problems in computer science and, in particular, to fixed point semantics for programming languages, see for example [2].

2. Main Results

First we recall from [1], some concepts that will be needed in the sequel.

Definition 2.1. Let P be an ordered set and $Q \subseteq P$.

(i) Q is a *down-set* if, $x \in Q$, $y \in P$ and $y \leq x$ then we have $y \in Q$.
(ii) Dually, Q is an *up-set* if, $x \in Q$, $y \in P$ and $x \leq y$ then we have $y \in Q$.

* speaker

For an arbitrary subset Q of P and $x \in P$, we define

$$\downarrow Q := \{y \in P \mid (\exists x \in Q)\ y \leq x\} \text{ and } \uparrow Q := \{y \in P \mid (\exists x \in Q)\ x \leq y\},$$

$$\downarrow x := \{y \in P \mid y \leq x\} \text{ and } \uparrow x := \{y \in P \mid x \leq y\}.$$

Definition 2.2. A *Galois connection* between two posets P and Q is a pair $(\alpha;\beta)$ of order-preserving maps $\alpha\colon P \to Q$ and $\beta\colon Q \to P$ such that $\alpha(x) \leq y$ if and only if $x \leq g(y)$ for all $x \in P$, $y \in Q$. The map α (resp. β) is the *left map* (resp. *right map*) of the Galois connection. We refer the reader to Ern´e et al. [10].

We recall form [14] the definition of a subset system.

Definition 2.3. A *subset system* is a function Z hat assigns to each poset P a set $\mathsf{Z}(P)$ of subsets of P called Z-*sets* such that

(i) for all $x \in P$, $\{x\} \in \mathsf{Z}(P)$;

(ii) if $\varphi\colon P \to Q$ be an order-preserving map between posets, then $\varphi(Y) \in \mathsf{Z}(Q)$ for all $Y \in \mathsf{Z}(P)$. In other words, each order-preserving map between posets preserves Z-sets.

Remark 2.0.1. Each subset system Z defines a functor on the category **Pos**.

Here are some examples of subset systems:

(1) a (resp. a^*) selects all (resp. nonempty) subsets. It works well for investigating completely distributive lattices, see Raney [11, 12], Ern´e et al. [10].

(2) b selects upper-bounded subsets.

(3) c selects chains (i.e. subsets C such that $x \leq y$ or $y \leq x$ whenever $x, y \in C$). See Markowsky and Rosen [8], and Markowsky [3, 4, 6, 7]. See also Erne [[9], p. 54].

(4) e^* selects singletons.

Definition 2.4. Let Z be a subset system. A poset P is called Z-*complete* if, every Z-set of P has a least upper bound. A morphism $\alpha\colon P \to Q$ is Z-continuous if for every Z-set S in P such that $\bigvee S$ exists, we have $\bigvee \alpha(S)$ exists in Q and $\alpha(\bigvee S) = \bigvee \alpha(S)$.

Definition 2.5. Let Z be a subset system and P be a poset. We say that $x \in P$ is Z-way-below $y \in P$, written $x \ll^{\mathsf{Z}} y$, if, for every Z-subset S with sup, $y \leq \bigvee S$ implies $x \in \downarrow S$. We write $\Downarrow^{\mathsf{Z}}_x$ and $\Uparrow^{\mathsf{Z}}_x$ for the subsets $\{y \in P \mid y \ll^{\mathsf{Z}} x\}$ and $\{y \in P \mid x \ll^{\mathsf{Z}} y\}$, respectively. A element $x \in P$ is called Z-*compact* if $x \ll^{\mathsf{Z}} x$. The poset P is called Z-*continuous* if $\Downarrow^{\mathsf{Z}}_x$ contains a Z-subset whose sup is x, for all $x \in P$.

Definition 2.6. Let Z be a subset system and P be a poset. A subset U is Z-*Scott open* if it is up-closed $Z \cap U$ is non-empty whenever Z is a Z-subset of P with sup such that $\bigvee Z \in U$. We denote the set of all Z-Scott open subsets with σ^{Z}_P.

Theorem 2.7. *Let* Z *be a subset system and* P *be a poset. Then* σ_P^{Z} *is a topology on* P.

The topology σ_P^{Z} is called Z-Scott topology.

Theorem 2.8. *Let* Z *be a subset system and* P *be a poset. A subset* F *of* P *is* Z*-Scott closed if and only if* F *is down closed and closed under all existing suprema of all* Z*-subsets.*

Theorem 2.9. *Let* Z *be a subset system and* P *be a poset. Then* σ_P^{Z} *is* T_0.

Theorem 2.10. *Let* Z *be a subset system and* P, Q *be two posets. A map* $\alpha\colon P \to Q$ *is* Z*-continuous if and only if it is continuous with respect to* Z*-Scott topologies on* P *and* Q.

Theorem 2.11. *Let* P *be* Z*-continuous. Then* $\{\Uparrow_x^{\mathsf{Z}} | \, x \in P\}$ *forms a basis for the* Z*-Scott topology* σ_P^{Z}.

Definition 2.12. A map $\alpha\colon P \to Q$ has the Z-refinement property if, whenever $x \in P$ and $\alpha(x) \leq \bigvee Z'$ for some Z-subset Z' of Q with sup, there exists a Z-subset Z of P with sup such that $\alpha(Z) \subseteq \downarrow Z'$ and $x \leq \bigvee Z$.

Definition 2.13. A subset A of a poset P has the Z-*refinement property* if the inclusion map $A \to P$ has Z-refinement property; equivalently, whenever $a \in A$ and $a \leq \bigvee Z$ for some Z-subset Z of P with sup, there exists a Z-subset Z' of Aincluded in $\downarrow Z$, with sup in A such that $a \leq \bigvee_A Z'$.

Theorem 2.14. *Let* $\alpha\colon P \to Q$ *be an order-preserving map. Then* $\alpha(P)$ *satisfies the* Z*-refinement property in* Q *in any of the following cases:*

(1) α *is surjective;*
(2) α *is a* Z*-continuous map with the* Z*-refinement property;*
(3) α *is* Z*-continuous and there exists a* Z*-continuous map* $\beta\colon Q \to P$ *such that* $\alpha \circ \beta circ\alpha = \alpha$ *and* $\alpha(\beta(y)) \leq y$ *for all* $y \in Q$;
(4) α *is a left map whose right map* β *is* Z*-continuous.*

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Some Results on Internal state Residuated Lattices

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Abstract

In this paper, we investigate the notion of state operators on residuated lattices and some of the features associated with these operators. Also, we characterize the filters generated by a subset in state residuated lattices by studying state operators on divisible residuated lattices and Heyting algebras.

Keywords and phrases: residuated lattice; state residuated lattice; internal state filter; state congruence.

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1. Introduction

The notion of a state is an analogue of probability measure. Such a notion plays a crucial role in the theory of quantum structures which generalizes the Kolmogorov probabilistic space [1]. Residuated lattices are the algebraic counterpart of logics without contraction rule. The concept of residuated lattices introduced by Krull on1924 who discussed decomposition into isolated component ideals. After him, they were investigated by Ward 1938, as the main tool in the abstract study of ideal lattices in ring theory. For a survey of residuated lattices we refer to [4, 6]. In this work, the notion of state residuated lattices are investigated and some results of [2, 5] are generalized in this class of algebras.

2. Main Results

2.1. Residuated Lattices An algebra $\mathfrak{A} = (A;\vee,\wedge,\odot,\rightarrow,0,1)$ is called a *residuated lattice* if $\ell(\mathfrak{A}) = (A;\vee,\wedge,0,1)$ is a bounded lattice, $(A;\odot,1)$ is a commutative monoid and $(\odot,\rightarrow)$ is an adjoint pair, i.e. $a \odot b \leq c$ iff $a \leq b \rightarrow c$, for all $a,b,c \in A$. In a residuated lattice $\mathfrak{A}$, for any $a \in A$, we put $\neg a := a \rightarrow 0$. We denote by $\mathcal{RL}$ the class of residuated lattices. Following the results of [1], we deduce that the class $\mathcal{RL}$ is equational, hence it forms a variety. A residuated lattice $\mathfrak{A}$ is called a *divisible residuated lattice* if it satisfies the divisibility condition (denoted by (div)):

* speaker

$(div)\, x \odot (x \to y) = x \wedge y.$

The following remark provides some rules of calculus in a residuated lattice which will be used in this paper (see [3]).

Remark 2.1. *Let $\mathfrak{A}$ be a residuated lattice. Then the following assertions are satisfied for any $x, y, z \in A$:*

- r_1 $x \leq y \Leftrightarrow x \to y = 1$;
- r_2 $x \to x = 0 \to x = x \to 1 = 1$ *and* $1 \to x = x$;
- r_3 $x \to (y \to z) = (x \odot y) \to z$;
- r_4 $x \odot y \leq x \odot (x \to y) \leq x \wedge y$;
- r_5 $x \leq y \to (x \odot y)$;
- r_6 $x \leq y \Rightarrow x \odot z \leq y \odot z$;
- r_7 $x \leq y \Rightarrow z \to x \leq z \to y$ *and* $y \to z \leq x \to z$;
- r_8 $x \to y \leq (y \to z) \to (x \to z)$;
- r_9 $x \to y \leq (z \to x) \to (z \to y)$;
- r_{10} $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$;
- r_{11} $x \to (y \to z) = y \to (x \to z)$;
- r_{12} $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$. *In particular,* $x^n \vee y^m \geq (x \vee y)^{nm}$, *for any integers* n, m;
- r_{13} $(x \vee y) \to z = (x \to z) \wedge (y \to z)$;
- r_{14} $\neg x \odot x = 0$. *In particular,* $x \leq y$ *implies* $x \odot \neg y = 0$.

2.2. State residuated lattice In this section, the required definitions and basic concepts are given from [5]. Let $\mathfrak{A}$ be a residuated lattice and $\nu : A \to A$ is a function. For convenience, we enumerate some conditions which will be used in this paper:

- $\mathfrak{s}_1$ $\nu(0) = 0$;
- $\mathfrak{s}_2$ ν is monotone;
- $\mathfrak{s}_3$ $\nu(x \to y) \leq \nu(x) \to \nu(y)$;
- $\mathfrak{s}_4$ $\nu(x \to y) = \nu(x) \to \nu(x \wedge y)$;
- $\mathfrak{s}_5$ $\nu(\nu(x) \odot \nu(y)) = \nu(x) \odot \nu(y)$;
- $\mathfrak{s}_6$ $\nu(\nu(x) \vee \nu(y)) = \nu(x) \vee \nu(y)$;
- $\mathfrak{s}_7$ $\nu(\nu(x) \wedge \nu(y)) = \nu(x) \wedge \nu(y)$.

Lemma 2.2. *Let $\mathfrak{A}$ be a residuated lattice and $\nu : A \to A$ be a function. The following assertions hold:*

(1) *if ν satisfies $\mathfrak{s}_1$ and $\mathfrak{s}_4$, then ν satisfies the following assertion:*

$\mathfrak{s}_8$ $\nu(1) = 1$;

(2) *if ν satisfies $\mathfrak{s}_2$ and $\mathfrak{s}_4$, then ν satisfies $\mathfrak{s}_3$;*

(3) *if ν satisfies $\mathfrak{s}_3$ and $\mathfrak{s}_8$, then ν satisfies $\mathfrak{s}_2$;*

(4) *if ν satisfies $\mathfrak{s}_1$ and $\mathfrak{s}_4$, then ν satisfies $\mathfrak{s}_2$ if and only if ν satisfies $\mathfrak{s}_3$.*

Definition 2.3. *Let $\mathfrak{A}$ be a residuated lattice. A mapping $\nu : A \to A$ is called a state operator on $\mathfrak{A}$ if it satisfies $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{s}_4$, $\mathfrak{s}_5$, $\mathfrak{s}_6$ and $\mathfrak{s}_7$.*

Example 2.4. *Let $\mathfrak{A}$ be a residuated lattice. Clearly Id_A is a state operator. So $\mathfrak{A}_{Id_A}$ is a state residuated lattice.*

Definition 2.5. *Let $\mathfrak{A}$ be a residuated lattice. A mapping $\nu : A \to A$ is called a good state operator on $\mathfrak{A}$ if it satisfies $\mathfrak{s}_1$, $\mathfrak{s}_5$, $\mathfrak{s}_6$, $\mathfrak{s}_7$ and the following assertion:*

$\mathfrak{gs}$ $\quad \nu(x \to y) = \nu(x) \to \nu(y)$.

A good state residuated lattice is a state residuated lattice $\mathfrak{A}_\nu$, where ν is a good state operator on $\mathfrak{A}$. By [7] we can obtain that a state operator on a linear residuated lattice is a good state operator.

Definition 2.6. *Let $\mathfrak{A}$ be a divisible residuated lattice. A mapping $\nu : A \to A$ is called a state operator on $\mathfrak{A}$ if it satisfies $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{s}_4$, $\mathfrak{s}_5$ and $\mathfrak{s}_6$.*

The following proposition characterizes divisible residuated lattices in terms of state operators.

Proposition 2.7. *[5] Let $\mathfrak{A}$ be a residuated lattice. Then the following assertions are equivalent:*

(1) *$\mathfrak{A}$ is a divisible residuated lattice;*
(2) *every state operator on $\mathfrak{A}$ satisfies the following condition:*

$$\nu(x \wedge y) = \nu(x) \odot \nu(x \to y).$$

In the following proposition we characterize Heyting algebras in terms of state operators.

Proposition 2.8. *Let $\mathfrak{A}$ be a residuated lattice. Then $\mathfrak{A}$ is a Heyting algebra if and only if every state operator on $\mathfrak{A}$ satisfies the condition $\nu(x^2) = \nu(x)$.*

The following theorems give relations between state operators and states on residuated lattices.

Theorem 2.9. *Let $\mathfrak{A}_\nu$ be a good state residuated lattice. If s is a Bosbach state on $\nu(A)$, then the mapping $s_\nu : A \longrightarrow [0,1]$, defined by $s_\nu(a) = s(\nu(a))$, is a Bosbach state on $\mathfrak{A}$.*

2.3. State maximal and state prime filters

Definition 2.10. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. A filter F of $\mathfrak{A}$ is called a ν-filter of $\mathfrak{A}_\nu$ if $\nu(F) \subseteq F$. The set of all ν-filters will be denoted by $\mathcal{F}(\mathfrak{A}_\nu)$.*

Corollary 2.11. *Let $\mathfrak{A}_\nu$ be a state residuated lattice, F be a filter of $\mathfrak{A}_\nu$ and $x, y \in A$. The following assertions hold:*

(1) $\mathcal{F}^\nu(x) = \mathcal{F}(x \odot \nu(x)) = \mathcal{F}(x \wedge \nu(x)) = \mathcal{F}(x, \nu(x)) = \mathcal{F}(x) \veebar \mathcal{F}(\nu(x))$;
(2) $\mathcal{F}^\nu(F, x) = \mathcal{F}(F \cup (x \odot \nu(x))) = \{a \in A | f \odot (x \odot \nu(x))^n \leq a,\ f \in F,\ n \geq 1\}$;
(3) *if $x \leq y$ then $\mathcal{F}^\nu(y) \subseteq \mathcal{F}^\nu(x)$;*
(4) $\mathcal{F}^\nu(\nu(x)) \subseteq \mathcal{F}^\nu(x)$.

In the following corollary, we give another proof for characterizing of simple state residuated lattices.

Corollary 2.12. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. Then $\mathfrak{A}_\nu$ is simple if and only if $\nu(A)$ is a simple residuated lattice and ν is faithful.*

Corollary 2.13. *Let $\mathfrak{A}_\nu$ be a state residuated lattice and $x,y \in A$. The following assertions hold:*

(1) $\mathcal{F}^\nu(x) \cap \mathcal{F}^\nu(y) = \mathcal{F}^\nu((x \odot \nu(x)) \vee (y \odot \nu(y)))$;
(2) $\mathcal{F}^\nu(x) \veebar \mathcal{F}^\nu(y) = \mathcal{F}^\nu(x \odot y)$.

Corollary 2.14. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. Then $\mathcal{PF}(\mathfrak{A}_\nu)$ is a sublattice of $\mathcal{F}(\mathfrak{A}_\nu)$.*

Definition 2.15. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. A proper ν-filter M is called maximal, if it is not strictly contained in any ν-filter. We use $Max(\mathfrak{A}_\nu)$ to denote the set of all maximal ν-filters.*

Proposition 2.16. *Any proper ν-filter of a state residuated lattice $\mathfrak{A}_\nu$ can be extended to a maximal ν-filter.*

Corollary 2.17. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. Then $\mathfrak{A}_\nu$ is local if and only if $\nu(A)$ is local.*

Definition 2.18. *Let $\mathfrak{A}_\nu$ be a state residuated lattice and α be a cardinal. A proper ν-filter G of $\mathfrak{A}_\nu$ is called α-irreducible if for any family of ν-filters $\mathcal{F}$ of cardinal α, $G = \bigcap \mathcal{F}$ implies $G = F$ for some $F \in \mathcal{F}$. A ν-filter G is called (finite) irreducible if it is α-irreducible for any (finite) cardinal α. A ν-filter P is called prime if it is finite irreducible. It is obvious that a ν-filter P is prime if and only if $F_1 \cap F_2 = P$ implies $F_1 = P$ or $F_2 = P$ for any ν-filters F_1, F_2. The set of prime ν-filters of $\mathfrak{A}_\nu$ is called the prime spectrum of $\mathfrak{A}_\nu$ and denoted by $Spec(\mathfrak{A}_\nu)$. It is obvious that any maximal ν-filter of a residuated lattice $\mathfrak{A}_\nu$ is irreducible and so is a prime ν-filter.*

Now, we characterize state prime filters in residuated lattices.

Proposition 2.19. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. For any ν-filter P, the following assertions are equivalent:*

(1) *P is a prime ν-filter.*
(2) *If F_1 and F_2 are ν-filters and $F_1 \cap F_2 \subseteq P$, then $F_1 \subseteq P$ or $F_2 \subseteq P$.*
(3) *If $x, y \in A$ such that $(x \odot \nu(x)) \vee (y \odot \nu(y)) \in P$, then $x \in P$ or $y \in P$.*

Proposition 2.20. *Let $\mathfrak{A}_\nu$ be a state residuated lattice and P be a proper ν-filter of $\mathfrak{A}_\nu$. If $\{F \in \mathcal{F}(\mathfrak{A}_\nu) | P \subseteq F\}$ is a chain, then P is ν-prime.*

Theorem 2.21. *Let $\mathfrak{A}_\nu$ be a state residuated lattice, F be a ν-filter and I be a $\vee$-closed subset of $\mathfrak{A}_\nu$ such that $F \cap I = \emptyset$. There is a prime ν-filter P containing F such that $P \cap I = \emptyset$.*

Corollary 2.22. *Let $\mathfrak{A}_\nu$ be a state residuated lattice and F be a ν-filter. The following assertions hold:*

(1) *If $a \notin F$, there exists $P \in Spec(\mathfrak{A}_\nu)$ such that $F \subseteq P$ and $a \notin P$;*

(2) *if* $a \neq 1$, *there exists* $P \in Spec(\mathfrak{A})$ *such that* $a \notin P$;
(3) $F = \cap\{P \in Spec(\mathfrak{A}_\nu) | F \subseteq P\}$;
(4) $\cap Spec(\mathfrak{A}_\nu) = \mathbf{1}$.

Theorem 2.23. *Any state residuated lattice* $\mathfrak{A}_\nu$ *is isomorphic to a subdirect product of state residuated lattices* $\{(\mathfrak{A}/P)_{\nu/P} | P \in Spec(\mathfrak{A}_\nu)\}$.

Theorem 2.24. *Let* $\mathfrak{A}_\nu$ *be a faithful state residuated lattice. Then* $\mathfrak{A}_\nu$ *is subdirectly irreducible if and only if* $\nu(A)$ *is subdirectly irreducible.*

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Some remarks on regular association schemes of order pqr

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Abstract

Let $\mathcal{C}$ be a non-thin regular association scheme of order pqr, where p, q and r are any prime numbers. Using the thin radical and thin residue, we give sufficient conditions for such association scheme to be schurian. Also, we show that $\mathcal{C}$ is isomorphic to the wreath product of two thin regular association schemes of order r and pq, if thin radical and thin residue are equal of order pq.

Keywords and phrases: Association scheme, Regular scheme, Wreath product .
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1. Introduction

Suppose that $\mathcal{C}$ is a regular association scheme; this implies that $\mathcal{C}$ has a nontrivial thin radical and so $\mathcal{C}$ has a normal thin closed subset of prime valency [2, Theorem 29]. All regular association schemes of degree p are thin. In [3, Theorem 15], it was shown that each regular association scheme of order pq is thin or the wreath product of two thin cyclic schemes of order p and q. Our main results show that any regular association scheme of order pqr whose thin radical and thin residue are equal is isomorphic to the wreath product of two thin regular association schemes of order r and pq.

2. Preliminaries

In this section, we prepare some notations and results for association schemes. For general introduction to association schemes and regular schemes, we refer the reader to [2, 6].
An association scheme $\mathcal{C} = (V, \mathcal{R} = \{R_i\}_{i\in I})$ on a finite set V is a pair consisting of V and a partition $\mathcal{R}$ of $V \times V$ into $|I|$ binary relations R_i satisfying the following conditions:
1- $1_x = \{(x,x) : x \in V\} \in \mathcal{R}$;
2- For each $R_i \in \mathcal{R}$, $R_{i^*} = \{(y,x) : (x,y) \in R_i\} \in \mathcal{R}$;

* speaker

3- $p^k_{i,j} = |xR_i \cap yR_{j'}|$ which is independent of the choice of $(x,y) \in R_k$ for all $i,j,k \in I$, where
$xR_i = \{y \in V : (x,y) \in R_i\}$.

The numbers $p^k_{i,j}$ are called *intersection numbers* of the association scheme $\mathcal{C}$. For each relation $R_i \in \mathcal{R}$, the integer $n_{R_i} = p^0_{i,i'}$ is called the *valency* of $\mathcal{R}$. The numbers $|V|$ and $|I|$ are called the *order* and the *rank* of $\mathcal{C}$, respectively. Let P and Q be nonempty subsets of $\mathcal{R}$. We define PQ to be the set of all elements R_k in $\mathcal{R}$ for which there exist elements $R_i \in P$ and $R_j \in Q$ satisfying $p^k_{i,j} \geq 1$. A subset P of $\mathcal{R}$ is called *closed* if $RS^* \subseteq P$ holds for all $R,S \in P$. The relation R of a scheme $\mathcal{C}$ is called *thin* if its valency n_R is 1. The set $O_\vartheta(\mathcal{C}) = \{R \in \mathcal{R} : n_R = 1\}$ is called the *thin radical* of $\mathcal{C}$ and $\mathcal{C}$ is called thin if $O_\vartheta(\mathcal{C}) = \mathcal{C}$. Let $O^\vartheta(\mathcal{C})$ be the smallest closed subset of $\mathcal{R}$ that contains RR^* for any $R \in \mathcal{R}$. $O^\vartheta(\mathcal{C})$ is called the *thin residue* of $\mathcal{C}$.

A relation R of $\mathcal{R}$ is called *regular* if $R^*RR = \{R\}$, and an association scheme is called regular if each of its relations is regular.

Let T be a closed subset of $\mathcal{C}$. For each relation $R \in \mathcal{R}$, we define $R^T = \{(xT, yT) : y \in xR\}$. Setting

$$V/T = \{xT | x \in V\} \text{ and } \mathcal{R}//T = \{R^T | R \in \mathcal{R}\}$$

one obtains that $\mathcal{C}//T = (V/T, \mathcal{R}//T)$ is a scheme. The scheme $\mathcal{C}//T$ is called the *quotient scheme* of $\mathcal{C}$ over T. Let $\mathcal{C}_1 = (V_1, \mathcal{R}_1)$ and $\mathcal{C}_2 = (V_2, \mathcal{R}_2)$ be two association schemes. The wreath product of $\mathcal{C}_1 = (V_1, \mathcal{R}_1)$ with $\mathcal{C}_2 = (V_2, \mathcal{R}_2)$ is defined as follows:

$$\mathcal{C}_1 \wr \mathcal{C}_2 = (V_1 \times V_1, \mathcal{R}_1 \wr \mathcal{R}_2),$$

where $\mathcal{R}_1 \wr \mathcal{R}_2 = \{R_0 \otimes S : R_0 \in \mathcal{R}_1, S \in \mathcal{R}_2\} \cup \{S \otimes V_2 \times V_2 : S \in \mathcal{R}_1 \backslash \{R_0\}\}$.

Theorem 2.1 ([1]). *Assume that $O^\vartheta(\mathcal{C}) \subseteq O_\vartheta(\mathcal{C})$ and that $\{RR^* | R \in \mathcal{C}\}$ is linearly ordered with respect to set-theoretic inclusion. Then $\mathcal{C}$ is schurian.*

Theorem 2.2 ([5]). *Let $\mathcal{C}$ be a scheme whose thin radical and thin residue are equal. $\mathcal{C}$ is then isomorphic to a fission of the wreath product of 2 thin schemes.*

Theorem 2.3 ([5]). *Let $\mathcal{C}$ be a p-scheme of degree p^n . The degrees of the thin radical and the thin residue of $\mathcal{C}$ are then equal to p if and only if $\mathcal{C}$ is isomorphic to the wreath product of a thin scheme of degree p and a thin scheme of degree p^{n-1}.*

Theorem 2.4 ([2]). *Let $\mathcal{C}$ be a regular association scheme. Then, $\mathcal{C}$ has non-trivial thin radical.*

Theorem 2.5 ([2]). *Let $\mathcal{C}_1 = (V_1, \mathcal{R}_1)$ and $\mathcal{C}_2 = (V_2, \mathcal{R}_2)$ be two regular association schemes. Then the wreath product $\mathcal{C}_1 \wr \mathcal{C}_2$ is also regular.*

Theorem 2.6 ([3]). *Let p and q be primes. A non-thin regular association scheme of order pq is the wreath product of two thin cyclic schemes of order p and q.*

3. Main results

In this section, assume that p, q and r are any primes. To prove the main theorems, we need to the following lemmas:

Lemma 3.1. *Assume that $\mathcal{C} = (V, \mathcal{R})$ is a non-thin regular association scheme of order pqr. Then*
$|O^{\vartheta}(\mathcal{C}) \cap O_{\vartheta}(\mathcal{C})| > 1$.

PROOF. First, suppose that for all $R \in \mathcal{R}$, $|\pi(n_R)| = 2$, where $\pi(n_R)$ denote the set of prime divisors of n_R. By Theorem 2.4, $|O_{\vartheta}(\mathcal{C})| > 1$. It is easy to check that in this case $|O_{\vartheta}(\mathcal{C})| \in \{p, pq, pr\}$. Since $\mathcal{C}$ is regular, $n_R = n_{RR^*}$ and $RR^* \subseteq O^{\vartheta}(\mathcal{C})$ is a thin closed subset of $\mathcal{C}$ and the proof in this case is complete. Thus, we assume that there exist $R \in \mathcal{R}$ such that $|\pi(n_R)| = 1$. With loss of generality, Assume that $n_R = r$. Similarly, since $\mathcal{C}$ is regular and RR^* is a closed subset. Then, $R_0 \in RR^*$ and $n_{RR^*} = r$. Thus, $O^{\vartheta}(\mathcal{C})$ contains thin closed subset RR^*. □

Lemma 3.2. *Assume that $\mathcal{C}$ is a regular association scheme of order pqr. Then, the degrees of the thin radical and the thin residue of $\mathcal{C}$ is equal to r if and only if $\mathcal{C}$ is isomorphic to the wreath product of two regular association schemes of order r and pq.*

PROOF. By [5], for each relation $R \in \mathcal{R}$ we have $n_R \leq n_{O^{\vartheta}(\mathcal{C})} = r$. Moreover, since $\mathcal{C}$ is a regular association scheme, RR^* is a closed subset and $n_R = n_{RR^*}$. It follows that, for each relation $R \in \mathcal{R}$, we have $n_R = \{1, r\}$. Then, the proof is similar to the proof of Theorem 2.3. □

Theorem 3.3. *Let $\mathcal{C} = (V, \mathcal{R})$ be a non-thin regular commutative association scheme of order pqr. Assume that one of the following conditions holds.*
1. $O^{\vartheta}(\mathcal{C}) = O_{\vartheta}(\mathcal{C})$, $n(O_{\vartheta}(\mathcal{C})) = r$ *and* $\mathcal{C}//O^{\vartheta}(\mathcal{C}) \cong C_{pq}$.
2. $n(O_{\vartheta}(\mathcal{C})) = pq$.

Then $\mathcal{C}$ is schurian.

Corollary 3.4. *A non-thin regular association scheme of order pqr whose thin radical and thin residue are equal, of order pq, is isomorphic to the wreath product of a thin cyclic association scheme of prime order r and the regular association scheme of order pq.*

PROOF. Let $\mathcal{C}$ be a non-thin regular association scheme of order pqr such that $|O^{\vartheta}(\mathcal{C})| = |O_{\vartheta}(\mathcal{C})| = pq$. In this case, $\mathcal{C}$ is isomorphic to a fission of the wreath product of two thin schemes. We may assume that $\mathcal{C}$ is isomorphic to the wreath product of $\mathcal{C}_{O^{\vartheta}(\mathcal{C})}$ and $\mathcal{C}//O^{\vartheta}(\mathcal{C})$. By the definition of $O^{\vartheta}(\mathcal{C})$ and $O_{\vartheta}(\mathcal{C})$, clearly two association schemes $\mathcal{C}_{O^{\vartheta}(\mathcal{C})}$ and $\mathcal{C}//O^{\vartheta}(\mathcal{C})$ are thin and so regular. Finally, by Theorem 2.5, $\mathcal{C}$ is isomorphic to the wreath product of two regular association schemes. □

Example 3.5. *Let $\mathcal{C}$ be the adjacency algebra of association scheme with order* 30 *([4], No.223). Then*

$$O^{\vartheta}(\mathcal{C}) = O_{\vartheta}(\mathcal{C}) \cong C_{15}$$

and $\mathcal{C}//O^{\vartheta}(\mathcal{C}) \cong C_2$.

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A survey on some subclasses of residuated lattices

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Abstract

Notions of quasicomplemented, mp, and Rickart residuated lattices is investigated as some important subclasses of the variety of residuated lattices. A combination of algebraic and topological methods is applied to obtain new and structural results on these subclasses.

Keywords and phrases: Rickart residuated lattice; mp-residuated lattice; quasicomplemented residuated lattice; generalized Stone residuated lattice. .

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1. Introduction

Let $\mathfrak{A}$ be a residuated lattice, $\mathscr{F}(\mathfrak{A})$ the lattice of filters, and $\mathscr{PF}(\mathfrak{A})$ the lattice of principal filters of $\mathfrak{A}$. Then $\Gamma(\mathfrak{A})$, the lattice of coannihilators of $\mathfrak{A}$, is the skeleton of $\mathscr{F}(\mathfrak{A})$, and $\gamma(\mathfrak{A})$, the lattice of coannulets of $\mathfrak{A}$, is the skeleton of $\mathscr{PF}(\mathfrak{A})$. So $(\Gamma(\mathfrak{A}); \vee^{\Gamma}, \cap, \{1\}, A)$ is a complete Boolean lattice, in which $\vee^{\Gamma}$ is the join in the skeleton, and $\gamma(\mathfrak{A})$ is a sublattice of $\Gamma(\mathfrak{A})$. $\mathfrak{A}$ is said to be *Baer* provided that $\Gamma(\mathfrak{A})$ is a sublattice of $\mathscr{F}(\mathfrak{A})$, and *Rickart* provided that $\gamma(\mathfrak{A})$ is a Boolean sublattice of $\mathscr{F}(\mathfrak{A})$. Obviously, $\mathfrak{A}$ is Rickart if and only if $\gamma(\mathfrak{A})$ is both Boolean and a sublattice of $\mathscr{F}(\mathfrak{A})$. The class of residuated lattices which satisfies the former condition is called *quasicomplemented* and the class of residuated lattices which satisfies the latter condition, and can be characterized by a property that can be formulated in terms of universal algebra; namely that each prime filter contains a unique minimal prime filter, is called *mp*. I assume the reader is familiar with the rudimentary properties of residuated lattices. For the basic facts concerning mp-, quasicomplemented, and Rickart residuated lattices I refer to [2], [3], and [5], respectively. Proposition 2.2, which characterizes the direct summands of a residuated lattice, is the heart, and Theorem 2.10, which characterizes Rickart residuated lattice, is the main theorem of this section.

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2. Main Results

Definition 2.1. *Let $\mathfrak{A}$ be a residuated lattice. The set of complemented elements of $\mathscr{F}(\mathfrak{A})$ shall be denoted by $\beta(\mathscr{F}(\mathfrak{A}))$, and its elements called the direct summands of $\mathfrak{A}$.*

For a residuated lattice $\mathfrak{A}$, the set of complemented elements of $\ell(\mathfrak{A})$ is denoted by $\beta(\mathfrak{A})$ and called *the Boolean center* of $\mathfrak{A}$. In residuated lattices, however, although the underlying lattices need not be distributive, the complements are unique. In the following, we set $\mathscr{F}(\beta(\mathfrak{A})) = \{\mathscr{F}(e) \mid e \in \beta(\mathfrak{A})\}$.

Proposition 2.2. *Let $\mathfrak{A}$ be a residuated lattice and F a filter of $\mathfrak{A}$. The following assertions are equivalent:*

(1) $F \in \beta(\mathscr{F}(\mathfrak{A}))$;
(2) $F \veebar F^{\perp} = A$;
(3) $F \in \mathscr{F}(\beta(\mathfrak{A}))$.

Let $\mathfrak{A}$ be a residuated lattice. Recall [1] that for any subset X of A, we set $X^{\perp} = kd(X)$, $\Gamma(\mathfrak{A}) = \{X^{\perp} \mid X \subseteq A\}$, $\gamma(\mathfrak{A}) = \{x^{\perp} \mid x \in A\}$, and $\lambda(\mathfrak{A}) = \{x^{\perp\perp} \mid x \in A\}$. Elements of $\Gamma(\mathfrak{A})$, $\gamma(\mathfrak{A})$, and $\lambda(\mathfrak{A})$ are called *coannihilators*, *coannulets*, and *dual coannulets* of $\mathfrak{A}$, respectively.

Definition 2.3. *[3] a residuated lattice $\mathfrak{A}$ is called quasicomplemented provided that $\lambda(\mathfrak{A}) \subseteq \gamma(\mathfrak{A})$.*

Proposition 2.4. *[3, Proposition 3.3,Corollary 3.4] Let $\mathfrak{A}$ be a residuated lattice. $\mathfrak{A}$ is quasicomplemented provided that $\gamma(\mathfrak{A}) \subseteq \mathscr{P}\mathscr{F}(\mathfrak{A})$. In particular, any finite residuated lattice is quasicomplemented.*

A filter F of a residuated lattice $\mathfrak{A}$ is called an *α-filter* provided that for any $x \in F$ we have $x^{\perp\perp} \subseteq F$. The set of α-filters of $\mathfrak{A}$ is denoted by $\alpha(\mathfrak{A})$. For any subset X of A, the α-filter generated by X is denoted by $\alpha(X)$. By [3, Proposition 5.3] follows that $(\alpha(\mathfrak{A}); \cap, \vee^{\alpha}, \{1\}, A)$ is a frame, in which $\vee^{\alpha}\mathcal{F} = \alpha(\veebar\mathcal{F})$, for any $\mathcal{F} \subseteq \alpha(\mathfrak{A})$. For the basic facts concerning α-filters and quasicomplemented residuated lattices we refer to [3].

Proposition 2.5. *Let $\mathfrak{A}$ be residuated lattice. The following assertions hold:*

(1) $Min(\mathfrak{A}) \subseteq \alpha(\mathfrak{A})$;
(2) *any prime filter contains a prime α-filter.*

Let $\mathfrak{A}$ be a residuated lattice and Π a collection of prime filters of $\mathfrak{A}$. For a subset π of Π we set $k(\pi) = \bigcap \pi$, and for a subset X of A we set $h_{\Pi}(X) = \{P \in \Pi \mid X \subseteq P\}$ and $d_{\Pi}(X) = \Pi \setminus h_{\Pi}(X)$. The collection Π can be topologized by taking the collection $\{h_{\Pi}(x) \mid x \in A\}$ as a closed (an open) basis, which is called *the (dual) hull-kernel topology* on Π and denoted by $\Pi_{h(d)}$. Also, the generated topology by $\tau_h \cup \tau_d$ is called *the patch topology* and denoted by τ_p. For a subset X of A, we set $H_{\Pi}(X) = \{h_{\Pi}(x) \mid x \in X\}$ and $D_{\Pi}(X) = \{d_{\Pi}(x) \mid x \in X\}$. As usual, the Boolean lattice of all clopen subsets of a topological space A_τ shall be denoted by $Clop(A_\tau)$. For a detailed discussion on the (dual) hull-kernel and patch topologies on a residuated lattice, we refer to [4].

The following proposition gives a topological characterization for quasicomplemented residuated lattices.

Theorem 2.6. *[4, Theorem 5.9] Let $\mathfrak{A}$ be a residuated lattice. The following assertions are equivalent:*

(1) *$\mathfrak{A}$ is quasicomplemented;*
(2) *$Min_h(\mathfrak{A})$ is compact;*
(3) *$D_m(\mathfrak{A}) = Clop(Min_h(\mathfrak{A}))$;*
(4) *$D_m(\mathfrak{A})$ is a Boolean lattice with the set theoretic operations.*

Let $\mathfrak{A}$ be a residuated lattice. For an ideal I of $\ell(\mathfrak{A})$, set $\omega(I) = \{a \in A | a \vee x = 1$, for some $x \in I\}$, and $\Omega(\mathfrak{A}) = \{\omega(I) | I \in \ell(\mathfrak{A})\}$. Using Proposition 3.4 of [2], it follows that $\Omega(\mathfrak{A}) \subseteq \mathscr{F}(\mathfrak{A})$, and so elements of $\Omega(\mathfrak{A})$ are called *ω-filters* of $\mathfrak{A}$. For an ω-filter F of $\mathfrak{A}$, I_F denoted an ideal of $\ell(\mathfrak{A})$, which satisfies $F = \omega(I_F)$. It is shown that $(\Omega(\mathfrak{A}); \cap, \vee^\omega, \{1\}, A)$ is a bounded distributive lattice, in which $F \vee^\omega G = \omega(I_F \curlyvee I_G)$, for any $F, G \in \Omega(\mathfrak{A})$ (by $\curlyvee$, we mean the join operation in the lattice of ideals of $\ell(\mathfrak{A})$). For a prime filter $\mathfrak{p}$ of $\mathfrak{A}$, set $D(\mathfrak{p}) = \omega(A \setminus \mathfrak{p})$, and $D(\mathfrak{A}) = \{D(\mathfrak{p}) \mid \mathfrak{p} \in Spec(\mathfrak{A})\}$. For the basic facts concerning ω-filters of a residuated lattice we refer to [2].

Definition 2.7. *[2] A residuated lattice $\mathfrak{A}$ is called mp provided that any prime filter of $\mathfrak{A}$ contains a unique minimal prime filter of $\mathfrak{A}$.*

The following theorem gives some algebraic criteria for mp-residuated lattices.

Theorem 2.8. *(Cornish's characterization) Let $\mathfrak{A}$ be a residuated lattice. The following assertions are equivalent:*

(1) *Any two distinct minimal prime filters are comaximal;*
(2) *$\mathfrak{A}$ is mp;*
(3) *for any prime filter $\mathfrak{p}$ of $\mathfrak{A}$, $D(\mathfrak{p})$ is a minimal prime filter of $\mathfrak{A}$;*
(4) *$(\Omega(\mathfrak{A}); \cap, \veebar)$ is a frame;*
(5) *$(\gamma(\mathfrak{A}); \cap, \veebar)$ is a lattice;*
(6) *$Min_d(\mathfrak{A})$ is Hausdorff;*
(7) *$Min_d(\mathfrak{A})$ is a retraction of $Spec_d(\mathfrak{A})$;*
(8) *$Spec_d(\mathfrak{A})$ is a normal space.*

Let $\mathfrak{A}$ be a $\wedge$-semilattice with zero. Recall that an element $a^* \in A$ is a *pseudocomplement* of $a \in A$ if $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \leq a^*$. An element can have at most one pseudocomplement. $\mathfrak{A}$ is called *pseudocomplemented* if every element of A has a pseudocomplement. The set $S(\mathfrak{A}) = \{a^* \mid a \in A\}$ is called *the skeleton of* $\mathfrak{A}$ and we have $S(\mathfrak{A}) = \{a \in A \mid a = a^{**}\}$. It is well-known that if $\mathfrak{A}$ is a pseudocomplemented complete $\wedge$-semilattice, then $S(\mathfrak{A})$

is a complete Boolean lattice, where the meet in $S(\mathfrak{A})$ is calculated in $\mathfrak{A}$, the join in $S(\mathfrak{A})$ is given by $\vee X = (\wedge\{x^* \mid x \in X\})^*$, for any $X \subseteq S(\mathfrak{A})$, and $1 \overset{def.}{=} 0^*$.

Applying Proposition 2.11 of [3], it follows that $\Gamma(\mathfrak{A})$ is the skeleton of $\mathscr{F}(\mathfrak{A})$, and $\gamma(\mathfrak{A})$ is the skeleton of $\mathscr{PF}(\mathfrak{A})$. So $(\Gamma(\mathfrak{A}); \vee^\Gamma, \cap, \{1\}, A)$ is a complete Boolean lattice, in which $\vee^\Gamma$ is the join in the skeleton, and $\gamma(\mathfrak{A})$ is a sublattice of $\Gamma(\mathfrak{A})$. $\mathfrak{A}$ is said to be *Baer* provided that $\Gamma(\mathfrak{A})$ is a sublattice of $\mathscr{F}(\mathfrak{A})$, and *Rickart* provided that $\gamma(\mathfrak{A})$ is a Boolean sublattice of $\mathscr{F}(\mathfrak{A})$.

Definition 2.9. *[5] A residuated lattice $\mathfrak{A}$ is called Rickart provided that $\gamma(\mathfrak{A}) = \mathscr{F}(\beta(\mathfrak{A}))$.*

The following theorem provides some criteria for a residuated lattice to be Rickart.

Theorem 2.10. *Let $\mathfrak{A}$ be a residuated lattice. The following assertions are equivalent:*

(1) *$\mathfrak{A}$ is Rickart;*
(2) *$\mathfrak{A}$ is quasicomplemented and normal;*
(3) *$\mathfrak{A}$ is generalized Stone, i.e. $x^\perp \veebar x^{\perp\perp} = A$, for any $x \in A$.;*
(4) *$\gamma(\mathfrak{A})$ is a Boolean sublattice of $\mathscr{PF}(\mathfrak{A})$;*
(5) *any prime filter of $\mathfrak{A}$ contains a unique prime α-filter;*
(6) *for any $x \in A$ there exists $e \in \beta(\mathfrak{A})$ such that $d_m(x) = d_m(e)$.*

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On the isoclinism of a pair of Hom-Lie algebras

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Abstract

In 1940, P. Hall introduced the notion of isoclinism on the class of all groups. In this article, we first, introduce pairs of Hom-Lie algebras and then define the concept of isoclinism for them. As the main result, we state some conditions under them, two pairs of Hom-Lie algebras are isoclinic.

Keywords and phrases: Hom-Lie algebras, Isoclinism, Pairs of Hom-Lie algebras.
2010 *Mathematics subject classification:* Primary: 22D15, 43A10; Secondary: 43A20, 46H25.

1. Introduction

The concept of group isoclinism was introduced by P. Hall in 1940 [1]. In 1993, Kay Moneyhun used this notion on Lie algebras [4] and in 2010 Salemkar and Mirzaei generalized it to n-isoclinism [5]. The notion of isoclinism for pairs of Lie algebras was studied by Moghaddam and Parvaneh in 2009 [3]. Also, in [2], Hartwig, Larsson, and Silvestrov introduced the notion of Hom-Lie algebras.

In this paper, we introduce the concept of pairs of Hom-Lie algebras and investigate some properties of isoclinism for these algebraic structures.

Throughout this paper, we fix F as a ground field and all the vector spaces are considered over F and linear maps are F-linear maps. We begin by reviewing some basic concepts and recalling known facts which will be used in the article.

Definition 1.1. *A Hom-Lie algebra is a triple* $(V, [-,-], \varphi)$ *consisting of a vector space* V*, a bilinear map* $[-,-] : V \times V \longrightarrow V$ *and linear map* $\varphi : V \longrightarrow V$ *provided*

(i) $[x,y] = -[y,x]$, $(skew - symmetry)$

(ii) $[\varphi(x),[y,z]] + [\varphi(y),[z,x]] + [\varphi(z),[x,y]] = 0$, $(Hom - Jacobi\ identity)$

for all $x, y, z \in V$.

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A *Hom-Lie subalgebra* of (V, φ) is a vector subspace W of V, which is closed by bracket and φ, i.e. $[w, w'], \varphi(w) \in W$ for all $w, w' \in W$. A Hom-Lie subalgebra $(W, \varphi_|)$ is said to be *ideal* if $[w, v] \in W$ for all $w \in W$, $v \in V$ in which $\varphi_|$ is the restriction of φ to W. For any ideal W of (V, φ), we can naturally, define the quotient Hom-Lie algebra on the quotient vector space V/W with $\widetilde{\varphi}$: $V/W \longrightarrow V/W$ which induced naturally by φ.

In the whole paper, we assume that φ preserves the product which is called *multiplicative*, i.e. $\varphi([v_1, v_2]) = [\varphi(v_1), \varphi(v_2)]$ for all $v_1, v_2 \in V$. Taking $\varphi = id_V$, we exactly recover the Lie algebras. A vector space endowed with a trivial bracket and any linear map are called *abelian* Hom-Lie algebra. Let (V, φ_1) and (W, φ_2) be two Hom-Lie algebras. A linear map $f : V \longrightarrow W$ is a *Hom-Lie algebra morphism*, if for all $v_1, v_2 \in V$, $f([v_1, v_2]) = [f(v_1), f(v_2)]$ and $f \circ \varphi_1 = \varphi_2 \circ f$. In other words, the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \varphi_1 \downarrow & & \downarrow \varphi_2 \\ V & \xrightarrow[f]{} & W \end{array}$$

Definition 1.2. *Let $(I, \varphi_|)$ be an ideal of Hom-Lie algebra (V, φ), then $(I, (V, \varphi))$ is considered to be a pair of Hom-Lie algebras and the commutator and the φ-center of the pair $(I, (V, \varphi))$ is defined respectively, as follows*

$$[I, V] = \langle [i, v] \mid i \in I, v \in V \rangle,$$
$$Z_\varphi(I, V) = \{i \in I \mid [\varphi^k(i), v] = 0, \ \ \forall v \in V, k \geq 0\}.$$

Clearly, $[I, V]$ and $Z_\varphi(I, V)$ are both ideals of (V, φ) contained in I. If $I = V$, then we get the derived Hom-Lie subalgebra V^2 and the φ-center of (V, φ), respectively.

Now, we introduce the notion of isoclinism for the pairs of Hom-Lie algebras $(I_1, (V_1, \varphi_1))$ and $(I_2, (V_2, \varphi_2))$ in the following way.

Definition 1.3. *Let $(I_1, (V_1, \varphi_1))$ and $(I_2, (V_2, \varphi_2))$ be two pairs of Hom-Lie algebras. Then (α, β) is called a pair of isoclinisms between $(I_1, (V_1, \varphi_1))$ and $(I_2, (V_2, \varphi_2))$, in which $\alpha : \overline{V}_1 \longrightarrow \overline{V}_2$, with $\alpha(I_1) = I_2$ and $\beta : [I_1, V_1] \longrightarrow [I_2, V_2]$ are both isomorphisms such that the following diagram commutes*

$$\begin{array}{ccc} \overline{I}_1 \times \overline{V}_1 & \longrightarrow & [I_1, V_1] \\ \alpha_| \times \alpha \downarrow & & \downarrow \beta \\ \overline{I}_2 \times \overline{V}_2 & \longrightarrow & [I_2, V_2] \end{array}$$

given by

$$\begin{array}{ccc} (\bar{i}_1, \overline{v}_1) & \longmapsto & [i_1, v_1] \\ \alpha_| \times \alpha \downarrow & & \downarrow \beta \\ (\bar{i}_2, \overline{v}_2) & \longmapsto & [i_2, v_2] \end{array}$$

where $^-$ *is the congruence modulo* $Z_{\varphi_i}(I_i, V_i)$, *for* $i = 1, 2$. *In fact, for all* $\bar{i}_1 \in \bar{I}_1, \bar{v}_1 \in \bar{V}_1$, *we have* $\beta([i_1, v_1]) = [i_2, v_2]$, *where* $i_2 \in \alpha(\bar{i}_1)$ *and* $v_2 \in \alpha(\bar{v}_1)$.

In this case, we say that $(I_1, (V_1, \varphi_1))$ *and* $(I_2, (V_2, \varphi_2))$ *are isoclinic and it is denoted by* $(I_1, (V_1, \varphi_1)) \sim (I_2, (V_2, \varphi_2))$.

2. Main Results

The following lemmas are devoted to show some properties of pairs of Hom-Lie algebras are used to prove our main result.

Lemma 2.1. *Let* (α, β) *be a pair of isoclinism between the pairs of Hom-Lie algebras* $(I_1, (V_1, \varphi_1))$ *and* $(I_2, (V_2, \varphi_2))$. *For all* $x \in [I_1, V_1]$ *and* $v \in V_1$
(i) $\alpha(x + Z_{\varphi_1}(I_1, V_1)) = \beta(x) + Z_{\varphi_2}(I_2, V_2)$;
(ii) $\beta([x, v]) = [\beta(x), v']$, *where* $v' \in \alpha(v + Z_{\varphi_1}(I_1, V_1))$.

Lemma 2.2. *Two pairs of Hom-Lie algebras* $(I_1, (V_1, \varphi_1))$ *and* $(I_2, (V_2, \varphi_2))$ *are isoclinic if and only if there exist Hom-ideals* J_1 *and* J_2 *of* V_1 *and* V_2 *contained in* $Z_{\varphi_1}(I_1, V_1)$ *and* $Z_{\varphi_2}(I_2, V_2)$ *respectively, and isomorphisms*

$$\alpha : \frac{V_1}{J_1} \longrightarrow \frac{V_2}{J_2}$$

with $\alpha(I_1/J_1) = I_2/J_2$ *and*

$$\beta : [I_1, V_1] \longrightarrow [I_2, V_2]$$

such that for all $i_1 \in I_1$, $i_2 \in \alpha(i_1 + Z_{\varphi_1}(I_1, V_1))$ *and* $v_2 \in \alpha(v_1 + Z_{\varphi_1}(I_1, V_1))$,

$$\beta([i_1, v_1]) = [i_2, v_2].$$

The following theorems state some useful properties of pairs of Hom-Lie algebras.

Theorem 2.3. *Let* (α, β) *be a pair of isoclinisms between two pairs of Hom-Lie algebras* $(I, (V, \varphi))$ *and* $(J, (W, \psi))$
(i) If V_1 *is a Hom-Lie subalgebra of* (V, φ) *containing* $Z_\varphi(I, (V, \varphi))$ *and* $\alpha(V_1/Z_\varphi(I, (V, \varphi))) = W_1/Z_\psi(J, (W, \psi))$, *for some Hom-Lie subalgebra* W_1 *of* W, *then*

$$(V_1 \cap I, (V_1, \varphi)) \sim (J \cap W_1, (W_1, \psi)).$$

(ii) If I_1 *is a Hom-Lie subalgebra of* I *containing in* $[I, V]$, *then*

$$(\frac{I}{I_1}, (\frac{V}{I_1}, \varphi)) \sim (\frac{J}{\beta(I_1)}, (\frac{W}{\beta(I_1)}, \psi)).$$

PROOF. (i) Assume that $I_1 = Z_\varphi(V_1 \cap I, (V, \varphi))$ and $J_1 = Z_\psi(W_1 \cap J, (W, \psi))$ which are contained in $Z_\varphi(V_1 \cap I, (V_1, \varphi))$ and $Z_\psi(W_1 \cap J, (W_1, \psi))$, respectively. Consider the following natural maps

$$\bar{\alpha} : \frac{V_1}{I_1} \longrightarrow \frac{W_1}{J_1},$$
$$\bar{\beta} : [V_1 \cap I, (V_1, \varphi)] \longrightarrow [W_1 \cap J, (W_1, \psi)].$$

Now, by using Lemma 2.2, the result will be obtained.
(ii) Since β is an isomorphism, $\beta(I_1)$ is an ideal of W. Put $\overline{V} = V/I_1$, $\overline{I} = I/I_1$, $\widetilde{W} = W/\beta(I_1)$ and $\widetilde{J} = J/\beta(I_1)$. Now, define $\overline{\alpha} : \overline{V}/\overline{I}_1 \longrightarrow \widetilde{W} / \widetilde{J}_1$ and $\overline{\beta} : [\overline{I}, \overline{V}] \longrightarrow [\widetilde{J}, \widetilde{W}]$. Now, by using Lemma 2.2 the assertion holds. □

Theorem 2.4. *Let $(I, (V, \varphi))$ be a pair of Hom-Lie algebras, J a Hom-Lie subalgebra and W an ideal of V contained in I, then*
(i) $(J \cap I, (J, \varphi)) \sim (J \cap I + Z_\varphi(I, V), (J + Z_\varphi(I, V), \varphi))$. In particular, if $V = J + Z_\varphi(I, V)$, then $(J \cap I, (J, \varphi)) \sim (I, (V, \varphi))$.
(ii) $(I/W, (V/W, \varphi)) \sim (I/W \cap [I, V], (V/W \cap [I, V], \varphi))$. In particular, if $W \cap [I, V] = 0$, then $(I/W, (V/W, \varphi)) \sim (I, (V, \varphi))$.

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Cubic edge-transitive graphs of order $40p$

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Abstract

A simple graph is called semisymmetric if it is regular and edge-transitive but not vertex-transitive. Let p be a prime. Folkman in [1] proved that a regular edge- transitive graph of order $2p$ or $2p^2$, necessarily vertex-transitive. We prove that if Γ is a connected cubic edge-transitive graph of order $40p$, p a prime, then either is semisymetric for, $p = 3$ and Γ is isomorphic to the cubic semisymmetric graph of order 120 in [2] or $p = 31$ and $\Gamma \cong C(L_2(31); S_4, S_4)$. and for $p \neq 3,31$ Γ is vertex-transitive.

Keywords and phrases: semisymmetric graph, edge-transitive graph, vertex-transitive graph..
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1. Introduction

In this paper all graphs are finite, undirected and simple, i.e. without loops or multiple edges. A graph is said semisymmetric if it is regular and edge-transitive, but not vertex-transitive. An interesting research question is to classify connected cubic edge-transitive graphs of various types of orders. Folkman proved in [1] that a cubic semisymmetric graph of order $2p$ or $2p^2$ is vetex-transitive. Connected cubic edge-transitive graphs of orders $6p^2$, $2p^3$, $6p^3$, $8p^3$, $20p^2$ have been classified in diferent articles.In this paper we will characterize connected cubic edge-transitve graphs of order $40p$. In fact we prove that if Γ is a connected cubic semisymmetric graph of order $40p$, p prime, then either $p = 3$ and Γ is isomorphic to the cubic semisymmetric graph of order 120 in [2] or $p = 31$ and $\Gamma \cong C(L_2(31); S_4, S_4)$. So precisely, we shall prove the following theorem.

Theorem 1.1. *Let p be a prime and $p \neq 3,31$. Then a connected cubic edge-transitive graph of order $40p$, is vertex-transitive.*

2. preliminaries

Let G be a subgroup of $Aut(\Gamma)$, if the action of G on $V(\Gamma), E(\Gamma)$ and $arc(\Gamma)$ be transitive, Γ is called respectively G-vertex transitive, G-edge transitive and

* speaker

G-arc transitive. Γ is called *G-semisymmetic* if it is regular and G- edge transitive but not G-vertex transitive. Furthermore, Γ is called *symmetric* if it is both G-vertex transitive and G-arc transitive. When $G = Aut(\Gamma)$, we usually remove G and say Γ is vertex-transitive, edge transitive, arc transitive, semisemmetric or symmetric. A G-semisymmetric graph is bipartite, let U_G and W_G be its bipartitions, then $|U_G| = |W_G|$. There is only one cubic symmetric graph of order 40 which is denoted by $F40$ (see [2]) and it is bipartite. According to [2], for $p = 3$ there is only one cubic semisymmetric graph of order 120 and for $p = 5, 7, 11, 13$ and 17 there is no cubic semisymmetric graph of order $40p$. Thus we can assume that $p > 19$. Here are some important results that we will use in this article. A finite simple group is called *K_n-group*, when its order is divisible by exactly n distinct primes. In the following theorem, we determine all K_3 and K_4-groups.

Theorem 2.1. *We have*
i) A K_3-group is isomorphic to one of the following groups:
$A_5, A_6, L_2(7), L_2(2^3), L_2(17), L_3(3), U_3(3), U_4(2)$
ii) A K_4-group is isomorphic to one of the following groups:
(1) $A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, L_2(2^4), L_2(5^2), L_2(7^2)$,
$L_2(3^4), L_2(97), L_2(3^5), L_2(577), L_3(2^2), L_3(5), L_3(7)$,
$L_3(2^3), L_3(17), L_4(3), U_3(2^2), U_3(5), U_3(7), U_3(2^3)$,
$U_3(3^2), U_4(3), U_5(2), S_4(2^2), S_4(5), S_4(7), S_4(3^2)$,
$S_6(2), O_8^+(2), G_2(3), S_Z(2^3), S_Z(2^5), {}^3D_4(2), {}^2F_4(2)$;
(2) $L_2(r)$ *where r is a prime and* $r^2 - 1 = 2^a.3^b.s, s > 3$ *is a prime,* $a, b \in \mathbb{N}$;
(3) $L_2(2^m)$, *where* $m, 2^m, \frac{2^m+1}{3}$ *are primes grater* 3;
(4) $L_2(3^m)$ *where* $m, \frac{3^m+1}{4}, \frac{3^m-1}{2}$ *are odd primes.*

Corollary 2.2. *There are only three simple K_4-groups of order $2^i.3.5.p$, for some prime p, $p > 5$ and $i \in \mathbb{N}$, $2 \leq i \leq 9$ and they are $L_2(2^4), L_2(11)$ and $L_2(31)$.*

Proposition 2.3. *Let G be a finite group and $N \trianglelefteq G$. If $|N|$ and $|\frac{G}{N}|$ are relatively prime, then G has a subgroup H such that $G = NH, N \cap H = 1$ i.e. $G \cong H \ltimes_\phi N$.*

Theorem 2.4. *Every group of order $p^a q^b$ is solvable, where p, q are distinct primes and $a, b \in \mathbb{N}$*

Theorem 2.5. *[3] Let Γ be a connected cubic G-semisymmetric graph. Then order of the stabilizer of each vertex v is $2^r.3$, where $0 \leq r \leq 7$. Furthermore, for each edge u, v, (G_u, G_v) is one of the following pairs or their inverses and $G_u \cap G_v$ is of index 3 in G_u and G_v:*
$(Z_3, Z_3), (S_3, S_3), (S_3, Z_6), (D_{12}, D_{12}), (D_{12}, A_4), (S_4, D_{24}), (S_4, Z_3 \ltimes D_8)$,
$(A_4 \times Z_2, D_{12} \times Z_2), (S_4, S_4), (S_4 \times Z_2, D_8 \times S_3), (S_4 \times Z_2, S_4 \times Z_2)$,
$(A_{96}, B_{96}), (A_{192}, B_{192}), (C_{192}, D_{192}), (A_{384}, B_{384})$.

Theorem 2.6. *[4] Let* Γ *be a connected cubic* G*-semisymmetric graph for some* $G \leq Aut(\Gamma)$ *and* $N \trianglelefteq G$*. If* $\frac{|G|}{|N|}$ *is not divisible by* 3*, then* Γ *is* N*-semisymmetric graph.*

Theorem 2.7. *[5] Let* Γ *be a connected cubic* G*-semisymmetric graph for some* $G \leq Aut(\Gamma)$*. Then* $\Gamma \cong K_{3,3}$ *or* G *acts faithfully on each of the bipartition sets of* Γ*.*

Theorem 2.8. *[6] Let* Γ *be a connected cubic* G*-semisymmetric graph,* $\{U, W\}$ *be a bipartition of* Γ*, and* $N \trianglelefteq G$*. If The actions of* N *on both* U *and* W *are intransitive, then* N *acts semiregularly on both* U *and* W *and* Γ_N *is* $\frac{G}{N}$*-semisymmetric.*

The next corollary drives directly from theorem 2.8.

Corollary 2.9. *Let* Γ *be a connected cubic* G*-semisymmetric graph with* $\{U, W\}$ *as a bipartition and* $N \trianglelefteq G$*. Then either* $|N|$ *divides* $|U|$ *or* $|U|$ *divides* $|N|$*.*

In the following, we will introduce *the coset graphs* and mention some important properties about them. Let G be a group and H, K be two finite subgroups of G. The coset graph $C(G; H, K)$ of G is a bipartite graph with sets of vertices $\{H_g, g \in G\} \cup \{K_g, g \in\}$ and two vertices, H_g and $K_{g'}$ are adjacent if and only if $H_g \cap K_{g'} \neq \emptyset$. The following theorems can be extracted from [7] and [8].

Proposition 2.10. *Let* G *be a finite group and* H, K *be two subgroups of* G*. The coset graph* $C(G; H, K)$ *has the following propeties:*
(i) $C(G; H, K)$ *is regular if and only if* $\frac{|H|}{|H \cap K} = \frac{|K|}{|H \cap K|} = d$*;*
(ii) $C(G; H, K)$ *is connected if and only if* $G =< H, K >$*;*
(iii) G *acts on* $C(G; H, K)$ *with multiple of right and this action faithfull if and only if*
$Core_G(H \cap K) = 1$*, in this case* $C(G; H, K)$ *is* G*-semisymmetric.*

Theorem 2.11. *Let* Γ *be a regular graph and* $G \leq Aut(\Gamma)$*. If* Γ *is* G*-semisymmetric, then* $\Gamma \cong C(G; G_u, G_v)$ *where* u, v *are adjacent vertices.*

3. main result

In this section, we prove theorem 1.1. First we state and prove some lemmas.

Notation and Assumptions: In the remaining of this paper Γ is a cubic connected semisymmetric graph of order $40p$, where $p > 19$ is a prime. Set $A = Aut(\Gamma)$.

Lemma 3.1. *If* $O_p(A) = 1$*, then* A *does not have normal subgroup of orders* 10 *and* 20*.*

Lemma 3.2. *We have either $|O_p(A)| = p$ or $p = 31$ and $\Gamma \cong C(L_2(31); S_4, S_4)$.*

By lemma above, in the remaining of this paper we assume that $p \neq 31$. Then a Sylow p-subgroup of A is normal in A.

Lemma 3.3. *Let M be the Sylow p-subgroup of A and $\frac{A}{M} \cong G$. Then we have*
(i) For each vertex u, A_u is isomorphic to a subgroup of G.
(ii) $A \cong M \rtimes_\phi G$ for some homomorphism $\phi : G \to Aut(M)$.

Lemma 3.4. *Let M be the Sylow p-subgroup of A. Then $\frac{A}{M}$ is not isomorphic to A_5.*

Lemma 3.5. *Let M be the Sylow p-subgroup of A. Then $\frac{A}{M}$ is not isomorphic to S_5.*

Now we can prove our main theorem. We note that by F40 we mean the Foster graph of order 40 which is the unique cubic symmetric graph of this order.

***The proof of theorem 1.1*:** Let Γ be a connected cubic semisymmetric graph of order $40p$. By [2] we assume that $p > 19$. Let $A = Aut(\Gamma)$ and M be a Sylow p-subgroup of A. We have $|A| = 2^{r+2}.3.5.p$ and by lemma 3.2, either $M \trianglelefteq A$ or $\Gamma \cong C(L_2(31); S_4, S_4)$. So we assume that M is normal in A. Let U, W be a bipartition of Γ. Then we have $|U| = |W| = 20p$ and M is on both U and W intransitive. Now by theorem 2.8, Γ_M is a connected cubic $\frac{A}{M}$-semisymmetric graph of order 40. Set $G = \frac{A}{M}$. Since Γ_M is G-semisymmetric, we get by [2] that it is G-edge transitive and hence it is symmetric. Thus $\Gamma_M \cong F40$ and G is isomorphic to a subgroup of $Aut(F40)$. By [2] we get that $|Aut(F40)| = 480$, so we have $|G| = 2^{r+2}.3.5 \leq 480$. This implies that $2^{r+2} \leq 32$ and G is intransitive on U. Therefore $|G| < 480$ and $2 \leq r \leq 4$. Hence $|G| = 60, 120$ or 240 and G is transitive on both U_M and W_M. This gives us that G is a transitive permutation group of degree 20 and of order 60, 120 or 240. We note that $Aut(F40)$ has a subgroup $H \cong A_5 \times Z_2 \times Z_2$ of index 2 and G is a subgroup of $Aut(F40)$. So $G \cap H$ is a subgroup of index at most 2 in G. Assume that $|G| = 60$. Then using Gap we get that $G \cong A_5$ or $Z_5 \times A_4$. The first case due to lemma 3.4 is not true and by the structure of Aut(F40) we get that G is not isomorphic to $Z_5 \times A_4$. Now assume that $|G| = 120$. Again using Gap we obtain that $G \cong S_5, Z_2 \times A_5, Z_5 \rtimes S_4, Z_5 \times S_4$ or $D_{10} \times A_4$. By lemma 3.5, G is not isomorphic to S_5. By the structure of $Aut(F40)$, we get that the only possibility for G is to be isomorphic to $Z_2 \times A_5$. If $G \cong Z_2 \times A_5$, then G has a normal subgroup $K \cong A_5$ and 3 does not divide the order of $\frac{G}{K}$. This implies that Γ_M is K-semisymmetric which is impossible. Now assume that $|G| = 240$. Then G is a subgroup of index 2 in $Aut(F40)$. By this and Gap we get that $G \cong Z_2 \times S_5, Z_4 \times A_5, Z_2 \times Z_2 \times A_5$ or $Z_4 \rtimes A_5$. Assume that $G \cong Z_2 \times S_5$ and set $T = Z(G)$. Then $T \cong Z_2$ and $(\Gamma_M)_T$ is $\frac{G}{T}$-semisymmetric. Now let $B = \frac{G}{T}$. Then $B \cong S_5$ and for each edge $\{u, v\}$ in Γ we have $|B_u| = |B_v| = 12$, also each subgroup of B of order 12 is isomorphic to A_4. This gives us that $(B_u, B_v) = (A_4, A_4)$ that by theorem 2.5 is impossible. A similar argument shows that G is not isomorphic to $Z_4 \rtimes A_5, Z_4 \times A_5$ or $Z_2 \times Z_2 \times A_5$. This completes the proof of theorem 1.1. □

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W-neat rings

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Abstract

In this paper, we offer a new generalization of the neat ring that is called a w-neat ring. A ring R is said to be weakly clean if every $r \in R$ can be written as $r = u + e$ or $r = u - e$ where $u \in U(R)$ and $e \in Id(R)$. We define a w-neat ring to be one for which every proper homomorphic image is weakly clean. We obtain some properties of w-neat rings.

Keywords and phrases: Weakly clean ring, W-neat ring.
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1. Introduction

Let R be a commutative ring with identity. The ring R is said to be clean if for each $r \in R$ there exist $u \in U(R)$ and $e \in Id(R)$ such that $r = u + e$ [6]. Clean rings were introduced as a class of exchange rings [6]. A ring R is said to be neat if every proper homomorphic image is clean [7]. A ring R is said to be a weakly clean if for each $r \in R$ there exist $u \in U(R)$ and $e \in Id(R)$ such that $r = u + e$ or $r = u - e$ [1, 3–5]. In [1] it is shown that every homomorphic image of a weakly clean ring is again weakly clean. This leads to our definition of a w-neat ring. We define a w-neat ring to be one for which every proper homomorphic image is weakly clean. We obtain some properties of w-neat rings.

2. Main Results

In [1] it is shown that every homomorphic image of a weakly clean ring is again weakly clean. This leads to our definition of a w-neat rings.

Definition 2.1. *Let R be a ring. Then R is said to be w-neat if every proper homomorphic image is a weakly clean ring.*

Example 2.2. *Let $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)} = \{r/s \mid r, s \in \mathbb{Z},\ s \neq 0, 3 \nmid s,\ 5 \nmid s\}$. Thus R is a weakly clean ring, by [1]. Since every homomorphic image of a weakly clean ring is again weakly clean, R is a w-neat ring.*

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* speaker

It is clear that every neat ring is a w-neat ring but every w-neat ring is not a neat ring. The following exapmle shows that every w-neat ring need not to be a neat ring.

Example 2.3. *Let* $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$. *Thus by Example 2.2, R is a w-neat ring but R is not clean since an indecomposable clean ring is quasilocal, by [2,* Theorem 3*]. Therefore R is not a neat ring.*

Lemma 2.4. *Every homomorphic image of a w-neat ring is a w-neat ring.*

PROOF. It is straightforward. □

Lemma 2.5. *Let R be a domain with* $dim(R) = 1$. *Then R is w-neat.*

PROOF. Since R is a domain with $dim(R) = 1$. Thus every homomorphic image of R is a zero-dimensional ring. Then every homomorphic image of R is weakly clean. Thus R is w-neat. □

Corollary 2.6. *Every PID is a w-neat ring.*

PROOF. Follows from Lemma 2.5. □

The following exapmle shows that every w-neat ring need not to be a weakly clean ring.

Example 2.7. *Let F be a field and* $R = F[x,y]$. *Since by [1,* Theorem 1.9*],* $R/Ry \cong F[x]$ *is not a weakly clean ring, R is not w-neat.* $F[x]$ *is w-neat by the Lemma 2.5 which is not a weakly clean ring.*

Lemma 2.8. *Let R be a w-neat ring which is not weakly clean. Then R is reduced.*

PROOF. Assume that R is a w-neat ring which is not weakly clean and $Nil(R) \neq 0$. Since R is a w-neat ring, $R/Nil(R)$ is weakly clean. Thus R is weakly clean by [1, Theorem 1.9], a contradiction. Then $Nil(R) = 0$. □

Theorem 2.9. *Let R be a ring. Then the following statements are equivalent.*
(1) R is a w-neat ring.
(2) The ring R/rR *is weakly clean for every* $0 \neq r \in R$.
(3) Let $\{P_\lambda\}_{\lambda\in\Lambda}$ *be a collection of nonzero prime ideals of R and* $I = \bigcap_{\lambda\in\Lambda} P_\lambda \neq 0$. *Then the ring* R/I *is weakly clean.*
(4) The ring R/rR *is w-neat for every* $r \in R$.
(5) The ring R/I *is weakly clean for every nonzero semiprime ideal I of R.*

PROOF. The proof is similar to [7, Proposition 2.1]. □

Corollary 2.10. *Let R be a w-neat ring which is not weakly clean. Then R is semiprime.*

PROOF. Follows from Theorem 2.9. □

Proposition 2.11. *Let $R = I \oplus J$ for some ideals I and J of R such that at most one I and J is not clean. Then R is w-neat if and only if R is weakly clean.*

PROOF. Suppose that there is are ideals I and J of R such that $R = I \oplus J$. Assume that R is a w-neat ring. Thus by Theorem 2.9 $J \cong R/I$ and $I \cong R/J$ are weakly clean. Thus R is a direct product of weakly clean rings. Therefore R is weakly clean, by [1, Theorem 1.7]. Conversely, is clear. □

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The conductor ideal of simplicial affine semigroups

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Abstract

In this paper we investigate the normality of R and, to detect a generating set for the conductor of R, $C_R = (R :_T \bar{R})$, where T denotes the total ring of fractions of R.

Keywords and phrases: simplicial affine semigroup, conductor, normality, Apéry set .
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1. Introduction

Thorough this section, $S \subseteq \mathbb{N}^d$ is a simplicial affine semigroup with $\mathrm{mgs}(S) = \{\mathbf{a}_1, \dots, \mathbf{a}_{d+r}\}$, where $\mathbf{a}_1, \dots, \mathbf{a}_d$ are the extremal rays of S. Let $R = \mathbb{K}[S]$ be the affine semigroup ring. Recall that the normalization of an integral domain R is the set of elements in its field of fractions satisfying a monic polynomial in $R[y]$. Then $R = \mathbb{K}[S]$ is an integral domain with normalization $\bar{R} = \mathbb{K}[\mathrm{group}(S) \cap \mathrm{cone}(S)]$ [4, Proposition 7.25]. Recall that the conductor of R, $C_R = (R :_T \bar{R})$, where T denotes the total ring of fractions of R, is the largest common ideal of R and $\bar{R}$, [3, Exercise 2.11].

The integral closure of S in $\mathrm{group}(S)$, $\bar{S} = \{\mathbf{a} \in \mathrm{group}(S) \,;\, n\mathbf{a} \in S$ for some $n \in \mathbb{N}\}$, is called the *normalization* of S. As a geometrical interpretation, one can see that $\bar{S} = \mathrm{cone}(S) \cap \mathrm{group}(S)$. The semigroup S is *normal* when $S = \bar{S}$, equivalently $\mathbb{K}[S]$ is a normal ring, [1, 2]. Since S is finitely generated, $\mathrm{cone}(S)$ is generated by finitely many rational vectors, i.e. it is the intersection of finitely many rational vector halfspaces, [5, Corollary 7.1(a)]. By Gordan's lemma, $\bar{S}$ is also finitely generated.

The *conductor* of S is defined as $\mathfrak{c}(S) = \{\mathbf{b} \in S \,;\, \mathbf{b} + \bar{S} \subseteq S\}$. The conductor, $\mathfrak{c}(S)$, is the largest ideal of S that is also an ideal of $\bar{S}$, [1, Exercise 2.9]. Recall that The *Apéry set* of an element $\mathbf{b} \in S$ is defined as $\mathrm{Ap}(S, \mathbf{b}) = \{\mathbf{a} \in S \,;\, \mathbf{a} - \mathbf{b} \notin S\}$. Throughout the paper, $E = \{\mathbf{a}_1, \dots, \mathbf{a}_d\}$ will denote the set of extremal rays of S. Then $\mathrm{Ap}(S, E) = \cap_{i=1}^{d} \mathrm{Ap}(S, \mathbf{a}_i)$.

* speaker

The *fundamental (semi-open) parallelotope* of S is the set

$$P_S = \{\sum_{i=1}^{d} \lambda_i \mathbf{a}_i \ ; \ \lambda_i \in \mathbb{Q}\ ,\ 0 \le \lambda_i < 1 \text{ for } i = 1,\ldots,d\}.$$

2. Main Results

Remark 2.1. *As $\mathfrak{c}(S)$ is an ideal of S, we have $S = \bar{S}$ precisely when $0 \in \mathfrak{c}(S)$. In other words, S is normal if and only if $\mathfrak{c}(S) = S$.*

Lemma 2.2. *As an affine semigroup, $\bar{S}$ is generated by $(P_S \cap \operatorname{group}(S)) \cup \{\mathbf{a}_1,\ldots,\mathbf{a}_d\}$, and $P_S \cap \operatorname{group}(S) = \{r(\mathbf{w})\ ;\ \mathbf{w} \in \operatorname{Ap}(S,E)\}$.*

As an immediate consequence of Lemma 2.2,

$$\mathfrak{c}(S) = \{\mathbf{a} \in S\ ;\ \mathbf{a} + r(\mathbf{w}) \in S \text{ for all } \mathbf{w} \in \operatorname{Ap}(S,E)\}.$$

Definition 2.3. *The element $\mathbf{b} - \sum_{i=1}^{d} \mathbf{a}_i$, where $\mathbf{b} \in \operatorname{Max}_{\preceq_S} \operatorname{Ap}(S,E)$, is called a* Quasi-Frobenius *element. The set of Quasi-Frobenius elements of S is denoted by* $\operatorname{QF}(S)$.

Let $\operatorname{relint}(S)$ denote the elements of $\mathbb{R}^d$ that belong to the relative interior of $\operatorname{cone}(S)$,

$$\operatorname{relint}(S) = \left\{ \mathbf{b} \in \operatorname{cone}(S)\ ;\ \mathbf{b} = \sum_{i=1}^{d} \lambda_i \mathbf{a}_i \text{ with } \lambda_i \in \mathbb{R}_{>0} \text{ for all } i = 1,\ldots,d \right\}.$$

Theorem 2.4. *The following statements are equivalent.*

1. *S is normal;*
2. *$-\operatorname{QF}(S) \subseteq S \cap \operatorname{relint}(S)$;*
3. *$-\operatorname{QF}(S) \subseteq \operatorname{relint}(S)$;*
4. *$\operatorname{Ap}(S,E) \subseteq P_S$.*

Our next aim is to find a generating set for $\mathfrak{c}(S)$ as an ideal of $\bar{S}$. Suppose that $C_j = \{\mathbf{w} \in \operatorname{Ap}(S,E)\ ;\ r(\mathbf{w}) = \mathbf{b}_j\}$, for $j = 0,\ldots,k$, where $r(\operatorname{Ap}(S,E)) = \{r(\mathbf{w})\ ;\ \mathbf{w} \in \operatorname{Ap}(S,E)\} = \{0 = \mathbf{b}_0, \mathbf{b}_1,\ldots,\mathbf{b}_k\}$. For any $(\mathbf{w}_1,\ldots,\mathbf{w}_k) \in C_1 \times \cdots \times C_k$, we consider the vector

$$\mathbf{f}_{(\mathbf{w}_1,\ldots,\mathbf{w}_k)} = \sum_{i=1}^{d} f_i \mathbf{a}_i,$$

where $f_i = \max\{[\mathbf{w}_j - r(\mathbf{w}_j)]_i\ ;\ j = 1,\ldots,k\}$, for $i = 1,\ldots,d$. Note that

$$f_i = \max\{\lfloor [\mathbf{w}_j]_i \rfloor\ ;\ j = 1,\ldots,k\},$$

for $i = 1,\ldots,d$, where $\lfloor [\mathbf{w}_j]_i \rfloor$ denotes the greatest integer less than or equal to $[\mathbf{w}_j]_i$.

Theorem 2.5. *Let $\mathbf{c}$ be a minimal generator of $\mathfrak{c}(S)$. Then there exist $(\mathbf{w}_1,\dots,\mathbf{w}_k) \in C_1 \times \cdots \times C_k$ such that $\mathbf{c} = \mathbf{f}_{(\mathbf{w}_1,\dots,\mathbf{w}_k)} - \mathbf{b}_j + \sum_{i=1}^{d} l_i\mathbf{a}_i$ for some $l_i \in \{0,1\}$ and $j \in \{0,\dots,k\}$. Moreover, at least for one i, we have $l_i = 0$.*

Example 2.6. *Let $\mathbf{a}_1 = (3,0), \mathbf{a}_2 = (0,3), \mathbf{a}_3 = (5,2), \mathbf{a}_4 = (2,5)$. we have*

$$\begin{aligned}\mathrm{Ap}(S,E) &= \{0,\mathbf{a}_3,\mathbf{a}_4,\mathbf{a}_3+\mathbf{a}_4,2\mathbf{a}_3,2\mathbf{a}_4\}\\ &= \{0,\mathbf{w}_1=(5,2),\mathbf{w}_2=(2,5),\mathbf{w}_3=(7,7),\mathbf{w}_4=(10,4),\mathbf{w}_5=(4,10)\}\end{aligned}$$

and $r(\mathrm{Ap}(S,E)) = \{0,\mathbf{b}_1 = (1,1),\mathbf{b}_2 = (2,2)\}$. Note that $C_1 = \{\mathbf{w}_3,\mathbf{w}_4,\mathbf{w}_5\}$, $C_2 = \{\mathbf{w}_1,\mathbf{w}_2\}$ and $\mathbf{f}_{(\mathbf{w}_3,\mathbf{w}_i)} = 2\mathbf{a}_1 + 2\mathbf{a}_2 = (6,6)$, $\mathbf{f}_{(\mathbf{w}_4,\mathbf{w}_i)} = 3\mathbf{a}_1 + \mathbf{a}_2 = (9,3)$, $\mathbf{f}_{(\mathbf{w}_5,\mathbf{w}_i)} = \mathbf{a}_1 + 3\mathbf{a}_2 = (3,9)$, for $i = 1,2$.

As $\{(6,6)-(1,1),(9,3)-(1,1),(3,9)-(1,1)\} + r(\mathrm{Ap}(S,E)) \subset S$, we have

$$\{(5,5),(8,2),(2,8)\} \subset \mathfrak{c}(S).$$

If $\mathfrak{c}(S) \neq \{(5,5),(8,2),(2,8)\} + \bar{S}$, the other generators of $\mathfrak{c}(S)$ are among

$$\{(9,3),(3,9),(6,6)\} + \{l_i\mathbf{a}_i - (2,2)\ ;\ l_i \in \{0,1\}, i = 1,2\},$$

by Theorem 2.5. Since the above set which equals

$$\{(7,1),(1,7),(4,4),(10,1),(1,10),(4,7),(7,4)\},$$

has no element in S, $\mathfrak{c}(S)$ is generated by $\{(5,5),(8,2),(2,8)\} = \{\mathbf{f}_{(\mathbf{w}_3,\mathbf{w}_1)} - \mathbf{b}_1, \mathbf{f}_{(\mathbf{w}_4,\mathbf{w}_1)} - \mathbf{b}_1, \mathbf{f}_{(\mathbf{w}_5,\mathbf{w}_1)} - \mathbf{b}_1\}$, as an ideal of $\bar{S} = \langle (3,0),(0,3),(1,1)\rangle$.

The following example shows that the summand $\sum_{i=1}^{d} l_i\mathbf{a}_i$ in the statement of Theorem 2.5 can not be removed.

Example 2.7. *Let $\mathbf{a}_1 = (5,2), \mathbf{a}_2 = (2,2), \mathbf{a}_3 = (2,1), \mathbf{a}_4 = (5,3)$. Then $\mathrm{Ap}(S,E) = \{0,\mathbf{w}_1 = (2,1),\mathbf{w}_2 = (4,2),\mathbf{w}_3 = (6,3),\mathbf{w}_4 = (8,4),\mathbf{w}_5 = (5,3)\}$ and $r(\mathrm{Ap}(S,E)) = \{0,\mathbf{b}_1 = (2,1),\mathbf{b}_2 = (4,2),\mathbf{b}_3 = (1,1),\mathbf{b}_4 = (3,2),\mathbf{b}_5 = (5,3)\}$. Note that $C_i = \{\mathbf{w}_i\}$ for $i = 1,\dots,5$ and $\mathbf{f}_{(\mathbf{w}_1,\dots,\mathbf{w}_5)} = \mathbf{a}_1$. By Theorem 2.5, the generators of $\mathfrak{c}(S)$ are among*

$$\{\mathbf{a}_1 - \mathbf{b}_i, 2\mathbf{a}_1 - \mathbf{b}_i, \mathbf{a}_1 + \mathbf{a}_2 - \mathbf{b}_i\ ;\ i = 0,\dots,5\}.$$

The only elements of the above set, that belong also to S are

$$\{(5,2),(10,4),(5,3),(2,1),(7,4),(4,2),(6,3)\}.$$

Note that $(2,1)+(1,1) \notin S$, $\{(5,2),(4,2)\} + r(\mathrm{Ap}(S,E)) \subseteq S$, $\{(10,4),(7,4),(6,3)\} \subset (5,2) + \bar{S}$ and $(5,3) = (4,2) + \mathbf{b}_3$. Therefore, $\mathfrak{c}(S)$ is generated by $\{(5,2),(4,2)\} = \{\mathbf{f}_{(\mathbf{w}_1,\dots,\mathbf{w}_5)}, \mathbf{f}_{(\mathbf{w}_1,\dots,\mathbf{w}_5)} + \mathbf{a}_2 - \mathbf{b}_4\}$, as an ideal of $\bar{S} = \langle (1,1),(2,1),(5,2)\rangle$.

Proposition 2.8. *Assume that there is a fixed class C_j such that for any $\mathbf{w} \in C_j$ and $\mathbf{w}' \in \mathrm{Ap}(S,E) \setminus C_j$, one has $\max_{\preceq_c}(\mathbf{w},\mathbf{w}') = \mathbf{w}$. If either C_j is a singleton or $\mathbf{b}_j = \min_{\preceq_c}(r(\mathrm{Ap}(S,E)) \setminus \{0\})$, then $\mathfrak{c}(S)$ is generated by*

$$\{\mathbf{w} - \mathbf{b}\ ;\ \mathbf{w} \in C_j\ ,\ \mathbf{b} \in \max_{\preceq_c}\{\mathbf{b}_1,\dots,\mathbf{b}_k\}\},$$

as an ideal of $\bar{S}$.

Applying the above proposition to the semigroup in Example 2.6, provides an easier argument to find the minimal generating set of $\mathfrak{c}(S)$.

Example 2.9. *Let* $\mathbf{a}_1 = (3,0), \mathbf{a}_2 = (0,3), \mathbf{a}_3 = (5,2), \mathbf{a}_4 = (2,5)$. *As we have seen in Example 2.6,* $\mathrm{Ap}(S,E) = \{0, \mathbf{w}_1 = (5,2), \mathbf{w}_2 = (2,5), \mathbf{w}_3 = (7,7), \mathbf{w}_4 = (10,4), \mathbf{w}_5 = (4,10)\}$, $r(\mathrm{Ap}(S,E)) = \{0, \mathbf{b}_1 = (1,1), \mathbf{b}_2 = (2,2)\}$, $C_1 = \{\mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$ *and* $C_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$. *Note that* $\max_{\preceq_c}\{\mathbf{w}_i, \mathbf{w}_j\} = \mathbf{w}_j$ *for* $i = 1,2$ *and* $j = 3,4,5$. *Therefore,* $\mathfrak{c}(S)$ *is generated by* $\{\mathbf{w}_3 - (2,2), \mathbf{w}_4 - (2,2), \mathbf{w}_5 - (2,2)\} = \{(5,5), (8,2), (2,8)\}$, *as an ideal of* $\bar{S} = \langle (3,0), (0,3), (1,1) \rangle$.

Corollary 2.10. *If* $\mathbb{K}[S]$ *is a Gorenstein ring and* $\max_{\preceq_c}(r(\mathrm{Ap}(S,E)))$ *has a single element, then* $\mathfrak{c}(S)$ *is a principal ideal of* $\bar{S}$ *.*

The following is an example of a Cohen-Macaulay simplicial affine semigroup, for which $\max_{\preceq_c}\mathrm{Ap}(S,E)$ is a singleton but $\mathfrak{c}(S)$ is not principal.

Example 2.11. *Let* $\mathbf{a}_1 = (3,0), \mathbf{a}_2 = (0,3), \mathbf{a}_3 = (2,1)$. *Then* $\mathrm{Ap}(S,E) = \{0, \mathbf{w}_1 = (2,1), \mathbf{w}_2 = (4,2)\}$ *and* $r(\mathrm{Ap}(S,E)) = \{0, \mathbf{b}_1 = (2,1), \mathbf{b}_2 = (1,2)\}$. *Since* $C_i = \{\mathbf{w}_i\}$ *for* $i = 1,2$, $\mathbb{K}[S]$ *is Cohen-Macaulay. Moreover,* $\max_{\preceq_c}\{\mathbf{w}_1, \mathbf{w}_2\} = \{\mathbf{w}_2\}$ *and* $\max_{\preceq_c} r(\mathrm{Ap}(S,E)) = \{\mathbf{b}_1, \mathbf{b}_2\}$. *By Proposition 2.8,* $\mathfrak{c}(S)$ *is generated by* $\{\mathbf{w}_2 - \mathbf{b}_1, \mathbf{w}_2 - \mathbf{b}_2\} = \{(2,1), (3,0)\}$ *as an ideal of* $\bar{S} = \langle (3,0), (0,3), (1,2), (2,1) \rangle$.

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New results on Condition (P') and (PF'')-cover

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Abstract

In this paper, we give a necessary and sufficient condition for cyclic act to have a (PF'')-cover and give some classes of monoids that all cyclic right S-acts have a Condition (PF'')-cover. We show that every weakly pullback flat cover is also (PF'')-cover and every (PF'')-cover is (P')-cover, but the converses are not true.

Keywords and phrases: Acts, Condition (PF''), Covers.
2010 *Mathematics subject classification:* Primary: 20M30; Secondary: 20M50.

1. Introduction

For a monoid S, with 1 as its identity, a set A (we consider nonempty) is called a right S-act, usually denoted by A_S (or simply A), if S acts on A unitarian from the right, that is, there exists a mapping $A \times S \to A$, $(a,s) \mapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A$ and $s, t \in S$. Left acts are defined dually. The study of flatness properties of S-acts in general began in the early 1970s.

In [4] the authors defined Condition (PF''), which lies strictly between weak pullback flatness and Condition (P'), and proved that Condition (PF'') coincide with the conjunction of Condition (P') and Condition (E').

In [5] Qiao and Wang investigated the weak pullback flatness cover of cyclic acts over monoids, and in [2] Irannezhad and Madanshekaf considered Condition (P')-cover of cyclic acts over monoids. Naturally, we restrict our attention to (PF'')-covers.

Let $\mathcal{X}$ be a class of right S-acts. We assume that $\mathcal{X}$ is closed under isomorphisms, i.e., if $A \in \mathcal{X}$ and $B \cong A$, then $B \in \mathcal{X}$. For a right S-act A, an S-act $X \in \mathcal{X}$ is called an $\mathcal{X}$-cover of A if there is a homomorphism $\varphi : X \to A$ such that the following hold:
(1) for any homomorphism $\psi : X' \to A$ with $X' \in \mathcal{X}$, there exists a homomorphism $f : X' \to X$ with $\psi = \varphi f$. In other words, the following diagram

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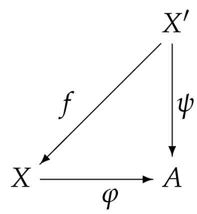
commutes:



(2) If an endomorphism $f : X \to X$ is such that $\varphi = \varphi f$, then f must be an automorphism.

If (1) holds, we call $\varphi : X \to A$ an $\mathcal{X}$-precover.

2. Main Results

In this section, at first, we give a necessary and sufficient condition for a cyclic act to have a (PF'')-cover.

Definition 2.1. *We say a right S-act A_S satisfies Condition (PF'') if for any $a, a' \in A$ and $s, s', t, t', z, w \in S$, $as = a's'$, $at = a't'$, and $sz = tw = t'w = s'z$ imply $a = a''u, a' = a''v$ for some $a'' \in A$, $u, v \in S$ with $us = vs'$ and $ut = vt'$.*

Lemma 2.2. *Let ρ be a right congruence on a monoid S. Then the cyclic right S-act S/ρ satisfies Condition (PF'') if and only if it satisfies Condition (P') and*

$$(\forall s, s', z \in S)(s\rho s' \wedge sz = s'z \Longrightarrow (\exists u \in S)(u\rho 1 \wedge us = us')).$$

Lemma 2.3. *Let ρ be a right congruence on a monoid S such that the right S-act S/ρ satisfies Condition (PF'') and $R = [1]_\rho$. Then R is a weakly left collapsible.*

Theorem 2.4. *Let R be a weakly left collapsible submonoid of S. Set*

$$H = \{(p, q) \in R \times R | \exists z \in S; pz = qz\} \cup \{(p, 1) | p \in R\}$$

and let $\sigma = \sigma(H)$ be the right congruence on S generated by H. Then S/σ satisfies Condition (PF'').

Theorem 2.5. *Let S be a monoid. Then the cyclic S-act S/σ has a (PF'')-cover if and only if $[1]_\rho$ contains a weakly left collapsible submonoid R such that for all $u \in [1]_\rho, uS \cap R \neq \emptyset$.*

Proposition 2.6. *Let S be a monoid. Then the every cyclic S-act has a (PF'')-cover if and only if every left unitary submonoid T of S contains a weakly left collapsible submonoid R such that for all $u \in T, uS \cap R \neq \emptyset$.*

Since commutative monoids are necessarily weakly left collapsible, thus every cyclic S-act, for commutative monoid S, has a (PF'')-cover.

Definition 2.7. *We say a right S-act A_S satisfies Condition (P') if for any $a,a' \in A$ and $s,t,z \in S$, $as = a't$ and $sz = tz$ imply $a = a''u, a' = a''v$ for some $a'' \in A$, $u,v \in S$ with $us = vt$.*

Remark 2.8. *Let S be a monoid.*
(1) *If S is idempotent, then every cyclic right S-act satisfying Condition (P') satisfies Condition (PF'').*
(2) *If S is right collapsible, then every cyclic right S-act satisfying Condition (PF'') is weakly pullback flat.*
(3) *The one-element S-act Θ_S satisfies Condition (PF'') if and only if S is a weakly left collapsible monoid.*

It is clear that every weakly pullback flat cover is (PF'')-cover and every (PF'')-cover is (P')-cover. Now we give examples to show that the converses are not true.

Example 2.9. *Let $X = \{x,y\}$ and $S = X^*$. Then S is a cancellative monoid and so it is weakly left collapsible. By Remark 2.8 the one-element S-act Θ_S has (PF'')-cover, but Θ_S has no weakly pullback flat cover, since if the one-element S-act Θ_S has weakly pullback flat cover, then the one-element S-act Θ_S has (P)-cover, by [2,* Lemma 2.9*], which is a contradiction.*

Example 2.10. *From [4,* Example 12*], Condition (P') does not imply Condition (PF''). Therefore, not every (P')-cover is a (PF'')-cover.*

Theorem 2.11. *Let S be an idempotent monoid. Then every cyclic S-act S/ρ has a (P')-cover if and only if S/ρ has a (PF'')-cover.*

Theorem 2.12. *Let S be a right collapsible monoid. Then every cyclic S-act S/ρ has a (PF'')-cover if and only if S/ρ has a weakly pullback flat cover.*

Theorem 2.13. *Let S be a monoid. Then every indecomposable S-act satisfying Condition (PF'') is locally cyclic if S satisfies any of the following condition:*
(1) S is a right collapsible monoid.
(2) S is a right regular band, that is, $a^2 = a$ and $aba = ba$, for $a,b \in S$.

PROOF. (1) By Remark 2.8, we see that Condition (PF'') implies weakly pullback flat, so the result is obvious.
(2) Let A be an indecomposable S-act satisfying Condition (PF''), and S be a right regular band. Let $a,a' \in A$. Since A is indecomposable, there exists a set of equations

$$\begin{aligned} a &= a_1u_1 \\ a_1v_1 &= a_2u_2 \\ &\cdots \\ a_nv_n &= a'. \end{aligned}$$

Since A satisfies Condition (PF''), from $a = a_1u_1$ and $1u_1 = u_1 = u_1u_1$, we conclude that there exist $b_1 \in A_S, s_1, t_1 \in S$ with $a = b_1s_1, a_1 = b_1t_1$ and $s_1 = t_1u_1$. Hence $(b_1t_1)v_1 = a_2u_2$, and since S is a right regular band, we have $u_2(t_1v_1u_2) = u_2(t_1v_1)u_2 = (t_1v_1)u_2 = (t_1v_1t_1v_1)u_2 = t_1v_1(t_1v_1u_2)$. Hence there exist $b_2 \in A_S, s_2, t_2 \in S$ with $b_1 = b_2s_1, a_2 = b_2t_2$ and $s_2t_1v_1 = t_2u_2$. Continuing in this way, we deduce that $u_{n+1}(t_nv_nu_{n+1}) = (t_nv_n)u_{n+1} = (t_nv_nt_nv_n)u_{n+1} = t_nv_n(t_nv_nu_{n+1})$. Therefore there exist $b_{n+1} \in A_S$, $s_{n+1}, t_{n+1} \in S$ with $b_n = b_{n+1}s_{n+1}, a = b_{n+1}t_{n+1}$ and $s_{n+1}(t_nv_n) = t_{n+1}$. Consequently, $a = b_1s_1 = b_2s_2s_1 = ... = b_ns_n...s_2s_1 = b_{n+1}s_{n+1}s_n...s_2s_1$ and $a' = b_{n+1}t_{n+1}$, as required. Therefore A is locally cyclic. □

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A note on automorphism groups of cubic semisymmetric graphs of special order

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Abstract

A simple graph is called semisymmetric if it is regular and edge-transitive but not vertex-transitive. The class of semisymmetric graphs was first introduced by Folkman [2]. By using group theoretic methods, Iofinova and Ivanov [4] in 1985 classified cubic semisymmetric graphs whose automorphism group acts primitively on both biparts. This was the first classification theorem for such graphs. In this paper we examine the results for automorphism groups of semisymmetric connected cubic graph of order $44p$, p prime.

Keywords and phrases: edge-transitive, vertex-transitive, Semisymmetric, Automorphism groups, Cubic graphs .

2010 *Mathematics subject classification:* Primary: 22D15, 43A10; Secondary: 43A20, 46H25.

1. Introduction

Group and graph theory both provide interesting and meaninful ways of examining relationships between elements of a given set. This investigation begins with automorphism groups of common graphs and an introduction of Frucht's Theorem, followed by an in-depth examination of the automorphism groups of generalized Petersen graphs and cubic Hamiltonian graphs in LCF notation. In the present study, S_n, A_n, Z_n and D_{2n} represent the symmetric and the alternating groups of degree n, the cyclic groups of order n and the dihedral groups of order 2n respectively. In addition, we denote a projective special linear group by $L_n(q)$ and $U_n(q)$ refers to aprojective special unitary group. Let G be a subgroup of $Aut(X)$. If action G on $V(X), E(X)$ and $Arc(X)$ be transitive, X is called respectively G-vetex transitive, G-edge transitive and G-Arc transitive. X is called G-semisymmetic if it is regular and G-edge transitive but not G-vertex transitive. Furtheremore X is called symmetric if both G-vertex transitive and G-arc transitive. For $G = Aut(X)$, we usually remove G and say X is vertex-trasitive, edge transitibe, arc transitive, semisemmetric or symmetric.

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In the following we discussed about some important findings that are used in the present study.

Theorem 1.1. *[1, 3]*
i) A K_3-group is isomorphic to one of the following groups:

$$A_5, A_6, L_2(7), L_2(2^3), L_2(17), L_3(3), U_3(3), U_4(2)$$

ii) A K_4-group is isomorphic to one of the following groups:

$$(1) A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, L_2(2^4), L_2(5^2), L_2(7^2), L_2(3^4), L_2(97), L_2(3^5), L_2(577), L_3(2^2), L_3(5), L_3(7), L_3(2^3), L_3(17), L_4(3), U_3(2^2), U_3(5), U_3(7), U_3(2^3), U_3(3^2), U_4(3), U_5(2), S_4(2^2), S_4(5), S_4(7), S_4(3^2), S_6(2), O_8^+(2), G_2(3), S_Z(2^3), S_Z(2^5), ^3D_4(2), {}^2F_4(2);$$

(2) $L_2(r)$*where r is a prime and* $r^2 - 1 = 2^a.3^b.s, s > 3$ *is a prime,* $a, b \in \mathbb{N}$;
(3) $L_2(2^m)$, *where* $m, 2^m, \frac{2^m+1}{3}$ *are primes grater* 3;
(4)$L_2(3^m)$ *where* $m, \frac{3^m+1}{4}, \frac{3^m-1}{2}$ *are odd primes.*

It is important to note that only nonabelian simple groups of order less than 300 are A_5 and $L_2(7)$.

The theorem below is also well-known see [6] Let G a finite group and $N \trianglelefteq G$. If $|N|$ and $|\frac{G}{N}|$ are relatively prime, then G has a subgroup H such that $G = NH, N \cap H = 1$ i.e. $G \cong H \ltimes_\phi N$.

Theorem 1.2. *[5] Let X be a connected cubic G-semisymmetric graph and* $\{U, W\}$ *be a bipartition of X furthermore,* $N \trianglelefteq G$. *If The actions of N on both U and W are intransitive, then N acts semiregularly on both U and W ,* X_N *is* $\frac{G}{N}$*-semisymmetric and X is a regular N covering of* X_N.

2. Main Results

We first provide a general discussion on G-semisymmetric graphs. Let X be a connected cubic G-semisymmetric graph of order n. It is very clear that X is regular and bipartite. Moreover, it is G-edge transitive and hence edge-transitive. If X is vertex transitive, then it is symmetric cubic of order n, since according to [7] a cubic vertex and edge-transitive graph is necessarily symmetric. Therefore, X is either a bipartite cubic symmetric graph of order n or it is a cubic semisymmetric graph of order n.
In the following, we will examine the largest normal p-subgroup of automorphism X.

Theorem 2.1. *There are only two groups, simple K_4-group whose orders of the form* $2^i.3.11.p$ *for some prime p,* $p > 5$ *and* $i \in \mathbb{N}, 1 \leq i \leq 8 : L_2(11), L_2(23)$.

Theorem 2.2. *If X be a connected cubic semisymmetric graph of order* $44p$ *and* $Aut(X) = A$*, then We have* $|O_p(A)| = p$ *or* $p = 23$.

Let $\{U, W\}$ *be a bipartition for X. Then* $|U| = |W| = 20p$ *and* $|A| = 2^{r+2}.3.11.p$ *some r,* $0 \leq r \leq 7$. *Let N be a minimal normal subgroup of A then* $N \cong T^k$*, where T be a simple group. If T is nonabelian, and since the powers* $3, 5$ *in* $|A|$ *equal* 1*, then* $k = 1$ *and* $N \cong T$. *Eithere* $|N|$ *divides* $|U| = 20p$ *or* $|U|$ *divides* $|N|$. *In first case, since* $|N|$ *is divisible by at least three distinct primes so* $|N| = 2^i.11.p$ *and hence N is a simple* K_3*-group but the order of such groups, listed in 1.1 is divisible by* 3. *Therfore* $|N|$ *is divisible by* $22p$. *According to 1.1, N must be a simple* K_4*-group of order* $2^i.3.11.p$. $N \cong L_2(11), L_2(23)$*, these groups correspond to* $p = 5, p = 23$. *Since* $p > 19, N \cong L_2(23)$ *and the order of* $\frac{A}{N}$ *does not divisible by* 3. *X is N-semisymmetric graph for each* $u \in U$ *and* $v \in W$ *and we have* $N_u = N_v \cong D_{12}$. *We conclude that* $N =< D_{12}, D_{12} >$ *which is imposible. So N is solvable and we result it is elementary abelian and hence it follows* X_N *is a connected cubic* $\frac{A}{N}$*-semisymmetric graph of order* $\frac{44p}{|N|}$. *We claim* $|O_p(A)| = p$. *Suppose that X is a semisymmetric cubic graph of the order* $44p$*, p prime, which* $p > 7$ *is an odd number. Consider* $A = Aut(x)$. *Also, suppose that* $Q = Op(A)$*, then the order of Q is equal to p. Take* $\{U, W\}$ *to be a bipartition for X. Then* $|U| = |W| = 22p$. *Since A acts transitionally on X- bipartition using Orbit-Stabilizer theorem and according to theorem 1, it can be concluded that* $|A| = 2^{r+1}.3.11.p$. *Now suppose that* $Q = Op(A)$ *is a normal maximal subgroup of A, we are intended to solve* $|Q| = p$. *First, we assume that* $|Q| = 1$ *and N is a normal minimal subgroup of A. We claim that A is solvable. As if not, the factor of composition series A must be a simple non-abelian group with the first* 4 *factors of* $p, 2, 3, 11$. *Therefore, according to the classification of simple finite groups, these factors should be isomorphism with one of the simple groups of* $L_2(11)$ *,*$L_2(23)$. *But this is not possible, as* $p > 7$. *Therefore, A is soluble and N is soluble. So N is elementary abelian group. Clearly,* $22p \nmid N$*, hence N operates non-transitionally on both bipartition X. As a result on both of them, N should be semi-regular. By counting the number of N, it can be concluded that the order N is equal to* 2 *or p. But as* $|Q| = 1$*, N must be equal to* 2. *Now we consider* $|X_N| = 22p$. X_N *is a semi-symmetric cubic graph and* $\{U(X_N), W(X_N)\}$ *are collections of* X_N*, each with two bipartitions and* $|U(X_N)| = |W(X_N)|$. *Suppose* $\dfrac{M}{N}$ *is a* $\dfrac{A}{N}$ *minimal subgroup. As* $\dfrac{A}{N}$ *is solvable,* $\dfrac{M}{N}$ *is also solvable and is elementary abelian group. Hence, we should have* $|\dfrac{M}{N}| = p$. *Hence, M is a normal sub-collection of A, ranked* $22p$. *Suppose P is a* $p -$ *Sylow subgroup of M, then we can simply prove that P is normal in M, and is an characteristic subgroup of M. As* $M \trianglerighteq A$*, P is normal in A. Then, A has a normal sub-group, ranked as p, which is in contrast to the* $Q = 1$. *Therefore,* $|Q| = p$. □

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When is a local homeomorphism a full subsemicovering?

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Abstract

In this paper, by reviewing the concept of subsemicovering maps and full subsemicovering maps, we present some conditions under which a local homeomorphism becomes a full subsemicovering map.

Keywords and phrases: fundamental group, semicovering map, subsemicovering map.
2010 *Mathematics subject classification:* Primary: 57M10; Secondary: 57M12, 57M05.

1. Introduction

Let $p : \tilde{X} \to X$ be a local homeomorphism. We are interested in finding some conditions on p or $\tilde{X}$ under which the map p can be extended to a semicovering map $q : \tilde{Y} \to X$. We recall that Steinberg [3, Section 4.2] defined a map $p : \tilde{X} \to X$ of locally path connected and semilocally simply connected spaces as a *subcovering map (and $\tilde{X}$ a subcover)* if there exist a covering map $p' : \tilde{Y} \to X$ and a topological embedding $i : \tilde{X} \to \tilde{Y}$ such that $p' \circ i = p$.

The following definition is stated in [2, Definition 3.1].

Definition 1.1. *Let $p : \tilde{X} \to X$ be a local homeomorphism. We say that p can be extended to a local homeomorphism $q : \tilde{Y} \to X$, if there exists an embedding map $\varphi : \tilde{X} \hookrightarrow \tilde{Y}$ such that $q \circ \varphi = p$. In particular, if q is a covering map, then p is called a subcovering map (see [3, Section 4.2]) and if q is a semicovering map, then we call the map p a subsemicovering map. Moreover, if $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = q_*(\pi_1(\tilde{Y}, \tilde{y}_0))$, then we call the map p full subcovering and full subsemicovering, respectively.*

Note that since every covering map is a semicovering map, every subcovering map is a subsemicovering map. Also, if $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ can be extended to $q : (\tilde{Y}, \tilde{y}_0) \to (X, x_0)$ via $\varphi : (\tilde{X}, \tilde{x}_0) \to (\tilde{Y}, \tilde{y}_0)$, then $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a subgroup of $q_*(\pi_1(\tilde{Y}, \tilde{y}_0))$.

The following theorem can be found in [2, Theorem 3.8].

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Theorem 1.2. *Let* $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ *be a map such that* $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ *is an open subgroup of* $\pi_1^{qtop}(X, x_0)$*. Then* p *is a subsemicovering map if and only if*

1. $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ *is a local homeomorphism;*
2. *if* f *is a path in* $\tilde{X}$ *with* $p \circ f$ *null homotopic (in* X*), then* $f(0) = f(1)$*.*

The following theorem is stated in [4, Theorem 3.7].

Theorem 1.3. *For a connected, locally path connected space* X*, there is a one-to-one correspondence between its equivalent classes of connected covering spaces and the conjugacy classes of subgroups of its fundamental group* $\pi_1(X, x_0)$ *with open core in* $\pi_1^{qtop}(X, x_0)$*.*

The following theorem can be found in [1, Theorem 2.21].

Theorem 1.4. *Suppose that* X *is locally path connected and* $x_0 \in X$*. A subgroup* $H \subseteq \pi_1(X, x_0)$ *is open in* $\pi_1^{qtop}(X, x_0)$ *if and only if* H *is a semicovering subgroup of* $\pi_1(X, x_0)$*.*

The following corollary is a consequence of the above theorem (see [1, Corollary 3.4]).

Corollary 1.5. *Every semicovering subgroup of* $\pi_1(X, x_0)$ *is open in* $\pi_1^{qtop}(X, x_0)$*.*

Theorem 1.6. *A map* $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ *is a full subsemicovering map if and only if*

1. $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ *is a local homeomorphism;*
2. *if* f *is a path in* $\tilde{X}$ *with* $p \circ f$ *null homotopic (in* X*), then* $f(0) = f(1)$*;*
3. $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ *is an open subgroup of* $\pi_1^{qtop}(X, x_0)$*.*

PROOF. Since every full subsemicovering map is a subsemicovering map, the necessity of conditions (1) and (2) are obtained by Theorem 1.2. To prove condition (3), let p can be extended to a semicovering map $q : (\tilde{Y}, \tilde{y}_0) \to (X, x_0)$ such that $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = q_*(\pi_1(\tilde{Y}, \tilde{y}_0))$. Hence $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is open in $\pi_1^{qtop}(X, x_0)$ since $q_*(\pi_1(\tilde{Y}, \tilde{y}_0))$ is open in $\pi_1^{qtop}(X, x_0)$ by Corollary 1.5. Sufficiency is obtained similar to the proof of Theorem 1.2. □

The following theorem can be concluded by the classification of connected covering spaces of X, Theorem 1.3, and Theorem 1.2.

Theorem 1.7. *A map* $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ *is a full subcovering map if and only if*

1. $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ *is a local homeomorphism;*
2. *if* f *is a path in* $\tilde{X}$ *with* $p \circ f$ *null homotopic (in* X*), then* $f(0) = f(1)$*;*
3. $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ *contains an open normal subgroup of* $\pi_1^{qtop}(X, x_0)$*.*

2. Main Results

In the following, we are going to find a sufficient condition for extending a local homeomorphism to a semicovering map. For this purpose first, note that Steinberg in [3, Theorem 4.6] presented a necessary and sufficient condition for a local homeomorphism $p : \tilde{X} \to X$ to be subcovering. More precisely, he proved that a continuous map $p : \tilde{X} \to X$ of locally path connected and semilocally simply connected spaces is subcovering if and only if $p : \tilde{X} \to X$ is a local homeomorphism and any path f in $\tilde{X}$ with $p \circ f$ null homotopic (in X) is closed, that is, $f(0) = f(1)$. We show that the latter condition on a local homeomorphism $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a sufficient condition for p to be full subsemicovering provided that $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is an open subgroup of the quasitopological fundamental group $\pi_1^{qtop}(X, x_0)$.

The following theorem can be concluded by Theorem 1.4 and Theorem 1.6.

Theorem 2.1. *Let $\tilde{X}$ be simply connected, locally path connected and connected, then an onto map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a full subsemicovering map if and only if*

1. *$p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is an local homeomorphism;*
2. *if f is a path in $\tilde{X}$ with $p \circ f$ null homotopic (in X), then $f(0) = f(1)$.*

PROOF. We must show that $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ contains an open subgroup of $\pi_1^{qtop}(X, x_0)$. It is enogh to show that $\pi_1^{qtop}(X, x_0)$ is semilocally simply connected space. Let x be a point of X. Since $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is an onto local homeomorphism, there exist an open neighborhood U of $\tilde{x} \in p^{-1}(x)$ such that $p(U)$ is an open neighborhood of x and $p|_U : U \to p(u)$ is a homeomorphism. If α is an arbitrary loop in $p(U)$, then $[p^{-1}(\alpha)]$ is an loop in U. Since $\tilde{X}$ is simply connected, $[p^{-1}(\alpha)] = 1$ and so $1 = p_*[p^{-1}(\alpha)] = [p \circ p^{-1}(\alpha)] = [\alpha]$. Therefore $\pi_1^{qtop}(X, x_0)$ is semilocally simply connected space. □

We need the following proposition for the next example.

Proposition 2.2. *Let $p : \tilde{X} \to X$ be a local homeomorphism. Suppose that $\tilde{X}$ is Hausdorff and that every null homotopic loop α in X is of the form $\Pi_{i=1}^{n} \alpha_i$, where*

$$\alpha_i(t) = \begin{cases} (f_i \circ \lambda_i)(t), & t \in [0, a_i], \\ (f_i^{-1} \circ \gamma_i)(t), & t \in [a_i, 1], \end{cases}$$

in which $0 \leq a_i \leq 1$, f_i is a path in X, $\lambda_i : [0, a_i] \to [0, 1]$ is defined by $\lambda_i(t) = \frac{t}{a_i}$, and $\gamma_i : [a_i, 1] \to [0, 1]$ is defined by $\gamma_i(t) = \frac{t - a_i}{1 - a_i}$, for every $i \in \mathbb{N}$. Then p has the condition ($\bigstar$) in Theorem 1.2.

The following example shows that the condition ($\bigstar$) is not a sufficient condition for p to be subsemicovering. Hence we cannot omit openness of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ from the hypotheses of Theorem 1.6.

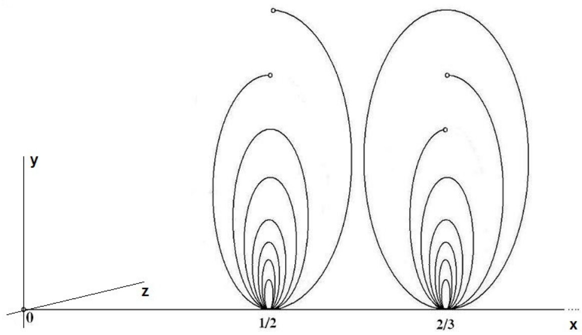

FIGURE 1. $\tilde{X}$

Example 2.3. *Let* $X = \mathbb{HE} = \bigcup_{n\in\mathbb{N}}\{(x,y) \in \mathbb{R}^2 | (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$ *be the Hawaiian Earring space. Put* $W_i = \bigcup_{n\in\{\mathbb{N}\setminus\{i,i+1\}\}}\{(y,z) \in \mathbb{R}^2 | (y - \frac{1}{n})^2 + z^2 = \frac{1}{n^2}\}$ *and*

$$S_i = \{(y,z) | (y - (1 - \frac{1}{i}))^2 + z^2 = (\frac{1}{i})^2, z > 0\} \bigcup \{(y,z) | (y - (1 - \frac{1}{i+1}))^2 + z^2 = (\frac{1}{(i+1)})^2, z < 0\}$$

for every $i \in \mathbb{N}$. *Let* $\tilde{X} = ((0,1) \times \{0\} \times \{0\}) \bigcup_{i=1}^{\infty}(\{1 - \frac{1}{i+1}\} \times (W_i \cup S_i))$ *be a subset of* $\mathbb{R}^3$ *(see Figure 1). We define* $p : \tilde{X} \to X$ *by*

$$p(x,y,z) = \begin{cases} (y,z), & x = 1 - \frac{1}{i+1},\ i \in \mathbb{N}, \\ \\ \frac{1}{i}(1 + \cos(\frac{2\pi}{1-x}), \sin(\frac{2\pi}{1-x})), & 1 - \frac{1}{i} < x < 1 - \frac{1}{i+1},\ i \in \mathbb{N}. \end{cases}$$

It is routine to check that p *is a local homeomorphism that has UPLP. Let* $\alpha : I \to X$ *be a loop defined by*

$$\alpha(t) = \begin{cases} (0,0), & t \in [0, \frac{1}{2}] \cup \{1\}, \\ \frac{1}{i}(1 + \cos(\frac{2\pi}{1-t}), \sin(\frac{2\pi}{1-t})), & 1 - \frac{1}{i} \le t \le 1 - \frac{1}{i+1},\ i \in \mathbb{N} \setminus \{1\}. \end{cases}$$

The loop α *has no lifting with starting point* $(\frac{1}{2}, 0, 0)$ *and the incomplete lifting of* α *with starting point* $(\frac{1}{2}, 0, 0)$ *is* $\tilde{\alpha} : [0,1) \to \tilde{X}$ *defined by*

$$\tilde{\alpha}(t) = \begin{cases} (\frac{1}{2}, 0, 0), & t \in [0, \frac{1}{2}], \\ (t, 0, 0), & t \in [\frac{1}{2}, 1). \end{cases}$$

By using Proposition 2.2, p has the condition $(\bigstar)$.Also, the path α has no lifting with starting point $(\frac{1}{2},0,0)$ and the incomplete lifting of α with starting point $(\frac{1}{2},0,0)$ is $\tilde{\alpha} : [0,1) \to \tilde{X}$. Hence $\tilde{\alpha}$ does not have any strong neighborhood. Therefore p is not subsemicovering.

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Principal right congruences over completely 0-simple semigroups

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Abstract

Regarding that completely simple and completely 0-simple semigroups involve classes namely right groups, left groups, right zero semigroups, left zero semigroups and rectangular bands, in this paper we identify principal right congruences on such semigroups.

Keywords and phrases: completely 0-simple semigroup, principal right congruence.
2010 *Mathematics subject classification:* 20M20, 18A32.

1. Introduction

Considering Birkhoff's theorem, stating that any algebra A is isomorphic to a subdirect product of subdirectly irreducible algebras (which are homomorphic images of A), structure of subdirectly irreducible semigroups (semigroups with least nondiagonal congruences) was a matter of interest in semigroup theory. Accordingly, investigating semigroups possessing least nondiagonal right congruences, termed right subdirectly irreducible semigroups, was initiated by Rankin et al. [3], who presented a general account on such semigroups. This class of semigroups is indeed a subclass of subdirectly irreducible semigroups on which the first investigations were pioneered by the efforts of Thierrin [7] and Schein [6]. This work is a basic part of the project characterizing right subdirectly irreducible completely (0-)simple semigroups in terms of identifying principal right congruences on such semigroups. We recall from [1] that a completely (0-)simple semigroup is indeed a (0-)simple semigroup containing a (0-)minimal left ideal and a (0-)minimal right ideal. In what follows we present preliminary notions and terminologies needed in the sequel.

Throughout this paper, S will denote a semigroup. To every semigroup S we can associate the monoid S^1 with the identity element 1 adjoined if necessary. Indeed, $S^1 = \begin{cases} S & \text{if } S \text{ has an identity element,} \\ S \cup \{1\} & \text{otherwise,} \end{cases}$
where $11 = 1$, $1s = s = s1$ for all $s \in S$.

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Recall that a semigroup S is called *simple (θ-simple)* if S contains no (nonzero) ideal other than itself. An equivalence relation ρ on a semigroup S is called a *right congruence* if $a\ \rho\ a'$ implies $(as)\ \rho\ (a's)$ for every $a, a', s \in S$ and the class of a under ρ is denoted by a_ρ. For a semigroup S the diagonal relation $\{(s,s) \mid s \in S\}$ on S is a right congruence on S which is denoted by Δ_S. Also if I is a right ideal of S, then the right congruence $(I \times I) \cup \Delta_S$ on S is denoted by ρ_I and is called the *Rees congruence* by the right ideal I. For $a, b \in S$, the principal right congruence on S generated by the pair (a,b) is denoted by $\rho(a,b)$. The following known result is frequently applied in the next arguments.

Lemma 1.1. *Let S be a semigroup and $a, b \in S$. Then for $x, y \in S, x\rho(a,b)y$ if and only if $x = y$ or there exist $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n, w_1, w_2, \ldots, w_n \in S^1$ where for every $i = 1, 2, \ldots, n$, $(p_i, q_i) \in \{(a,b), (b,a)\}$, with the following sequence of equalities:*

$$\begin{array}{llll} x = p_1 w_1 & q_2 w_2 = p_3 w_3 & \cdots & q_n w_n = y, \\ \quad q_1 w_1 = p_2 w_2 & & \cdots & \end{array} \tag{1}$$

between x and y, which shall be called of length n.

For a thorough account on the preliminaries, the reader is referred to [1, 2, 5].

2. Main Results

This section is devoted to characterize principal right congruences over completely 0-simple semigroups. First we recall Rees matrix semigroups briefly to present completely 0-simple semigroups.

Let G be a group with the identity element e, and let I, Λ be nonempty sets. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in the 0-group $G^0 (= G \cup \{0\})$. Let $S = (I \times G \times \Lambda) \cup \{0\}$ and define a composition on S by

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, a p_{\lambda j} b, \mu) & \text{if } p_{\lambda j} \neq 0, \\ 0 & \text{if } p_{\lambda j} = 0, \end{cases}$$
$$(i, a, \lambda)0 = 0(i, a, \lambda) = 00 = 0.$$

The semigroup S constructed in this fashion is called an $I \times \Lambda$ Ress matrix semigroup over the 0-group G^0 and is denoted by $\mathcal{M}^0[G; I, \Lambda; P]$. The matrix P is called *regular* if no row or column of P consists entirely zeros. It can be routinely checked that $\mathcal{M}^0[G; I, \Lambda; P]$ is regular if and only if P is regular. Moreover, a Rees matrix semigroup without zero element over a group is constructed in the same fashion and is denoted by $\mathcal{M}[G; I, \Lambda; P]$. Recalling Rees Theorem [4], any completely 0-simple semigroups is isomorphic to a regular Rees matrix semigroup over a 0-group. Therefore in the sequel, the term $\mathcal{M}^0[G; I, \Lambda; P]$ with regular matrix P, stands for a completely 0-simple semigroup. It is known that there is a one to one order preserving correspondence between subsets of I and right ideals of $\mathcal{M}^0[G; I, \Lambda; P]$, given by $\emptyset \mapsto \{0\}$ and $\emptyset \neq I' \mapsto T_{I'} = \{(i, a, \lambda) \,|\, i \in I', a \in G, \lambda \in \Lambda\} \cup \{0\}$ for any $\emptyset \neq I' \subseteq I$. The next result follows immediately from the rule of multiplication defined in Rees matrix semigroups over 0-groups.

Lemma 2.1. *Let* $S = \mathcal{M}^0[G; I, \Lambda; P]$ *and* $\mathfrak{m} = (i, a, \lambda), \mathfrak{n} = (i, b, \eta), \mathfrak{p} = (i, c, \theta) \in S$. *Then* $\mathfrak{m}\mathfrak{n} = \mathfrak{m}\mathfrak{p} \neq 0$ *implies that* $\mathfrak{n} = \mathfrak{p}$.

Take the binary relation ε_Λ on Λ given by $(\lambda, \mu) \in \varepsilon_\Lambda$ if $\{i \in I \,|\, p_{\lambda i} = 0\} = \{i \in I \,|\, p_{\mu i} = 0\}$. It is known that ε_Λ is an equivalence relation on Λ ([1, Section 3.5]).

Lemma 2.2. *Let* ρ *be the principal right congruence on* $S = \mathcal{M}^0[G; I, \Lambda; P]$ *generated by the pair* $(\mathfrak{m}, \mathfrak{n})$ *where* $\mathfrak{m} = (i, a, \lambda)$ *and* $\mathfrak{n} = (j, b, \mu)$. $0_\rho = 0$ *if and only if* $(\lambda, \mu) \in \varepsilon_\Lambda$. *In the case that* $(\lambda, \mu) \notin \varepsilon_\Lambda$, ρ *is the Rees congruence* $\rho_{T_{\{i,j\}}}$.

So in what follows, we will identify principal right congruences generated by the pairs $(\mathfrak{m}, \mathfrak{n})$ where $\mathfrak{m} = (i, a, \lambda)$, $\mathfrak{n} = (j, b, \mu)$, $\mathfrak{m} \neq \mathfrak{n}$ and $(\lambda, \mu) \in \varepsilon_\Lambda$. In such situations all terms $p_i w_i$ and $q_i w_i$ in (1) are nonzero. Thus for an arbitrary element $t \in I$, if $p_{\lambda t} \neq 0$ then $p_{\mu t} \neq 0$ and in this case the elements $a p_{\lambda t} (b p_{\mu t})^{-1}$ and $p_{\lambda t} p_{\mu t}^{-1}$ in G are denoted by X_t and Y_t respectively.

To reach our target we will identify class of an arbitrary element $s \in S$ for the principal right congruence $\rho(\mathfrak{m}, \mathfrak{n})$. Note that regarding the argument after Lemma 1.1, for any element $\mathfrak{z} = (k, z, \theta)$ where $z \in G$ and $\theta \in \Lambda$ and $k \in I \backslash \{i, j\}$, $\rho_\mathfrak{z} = \{\mathfrak{z}\}$. So in the following arguments we just identify classes of elements of the form $\mathfrak{z} = (k, z, \theta)$ where $k = i$ or j.

We proceed in two cases.

Case 1: $i \neq j$. First we identify the class $\mathfrak{m}$ (identically $\mathfrak{n}$). Let $\mathfrak{p} \in \mathfrak{m}_\rho$.

Set $\mathfrak{Y} = \{Y_t \,|\, t \in I\}$, $\mathfrak{Y}^{-1} = \{Y_t^{-1} \,|\, t \in I\}$ and $\mathfrak{N} = \langle \mathfrak{Y}\mathfrak{Y}^{-1} \rangle$, the subsemigroup of G generated by $\mathfrak{Y}\mathfrak{Y}^{-1}$ which is indeed a subgroup of G. We have $\mathfrak{p} \in \mathfrak{m}_\rho$ if and only if

$$\mathfrak{p} = \begin{cases} (i, an, \lambda) & \text{or} \\ (i, anx, \mu) & \text{or} \\ (j, bx^{-1}n, \lambda) & \text{or} \\ (j, by^{-1}nx, \mu) \end{cases} \tag{2}$$

for some $x, y \in \mathfrak{Y}, n \in \mathfrak{N}$.

Now we identify the class of an arbitrary element $\mathfrak{z} = (i, z, \theta)$ not involved in $\mathfrak{m}_\rho$ where $z \in G$ and $\theta \in \Lambda$. We have $\mathfrak{p} \in \mathfrak{z}_\rho$ if and only if

$$\mathfrak{p} = \begin{cases} (i, ana^{-1}z, \theta) & \text{or} \\ (j, bx^{-1}na^{-1}z, \theta) \end{cases} \tag{3}$$

for some $x, y \in \mathfrak{Y}, n \in \mathfrak{N}$. Similarly we can prove that for $\mathfrak{z} = (j, z, \theta)$ not involved in $\mathfrak{m}_\rho$ where $z \in G$ and $\theta \in \Lambda$, $\mathfrak{p} \in \mathfrak{z}_\rho$ if and only if

$$\mathfrak{p} = \begin{cases} (i, anxb^{-1}z, \theta) & \text{or} \\ (j, bx^{-1}nyb^{-1}z, \theta) \end{cases} \tag{4}$$

for some $x, y \in \mathfrak{Y}, n \in \mathfrak{N}$.

Case 2: $i = j$.

Setting $\mathfrak{X} = \{X_t \,|\, t \in I\}$ and $\mathfrak{M} = \langle \mathfrak{X} \rangle$, the subgroup of G generated by $\mathfrak{X}$, $\mathfrak{p} \in \mathfrak{m}_\rho$ if and only if

$$\mathfrak{p} = \begin{cases} (i, ma, \lambda) & \text{or} \\ (i, mb, \mu) \end{cases} \tag{5}$$

for some $m \in \mathfrak{M}$.

Now we identify the class of an arbitrary element $\mathfrak{z} = (i, z, \theta)$ not involved in $\mathfrak{m}_\rho$ where $z \in G$ and $\theta \in \Lambda$. We get $\mathfrak{p} \in \mathfrak{z}_\rho$ if and only if $\mathfrak{p} = (i, mz, \theta)$ for some $m \in \mathfrak{M}$. Now we present the main result of the paper.

Theorem 2.3. *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a completely 0-simple semigroup and $\mathfrak{a}, \mathfrak{b} \in S$. Let ρ be the principal right congruence on S generated by the pair $(\mathfrak{m}, \mathfrak{n})$, where $\mathfrak{m} = (i, a, \lambda) \neq \mathfrak{n} = (j, b, \mu)$. If $(\lambda, \mu) \notin \varepsilon_\Lambda$ then ρ is the Rees congruence on S by the right ideal $\mathfrak{m}S \cup \mathfrak{n}S$. If $(\lambda, \mu) \in \varepsilon_\Lambda$, ρ is identified as follows:*

i) *If $i \neq j$, $(\mathfrak{a}, \mathfrak{b}) \in \rho$ if and only if $\mathfrak{a} = \mathfrak{b}$ or both $\mathfrak{a}$ and $\mathfrak{b}$ are elements of the form stated in 2 or 3 or 4.*

ii) *If $i = j$, $(\mathfrak{a}, \mathfrak{b}) \in \rho$ if and only if $\mathfrak{a} = \mathfrak{b}$ or both $\mathfrak{a}$ and $\mathfrak{b}$ are elements of the form stated in 5 or $\mathfrak{a} = (i, m_1 z, \theta), \mathfrak{b} = (i, m_2 z, \theta)$, for some $m_1, m_2 \in \mathfrak{M}, z \in G, \theta \in \Lambda$.*

The next theorem is an straightforward result of the above theorem.

Theorem 2.4. *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a completely simple semigroup and $\mathfrak{a}, \mathfrak{b} \in S$. Then the principal right congruence ρ on S generated by the pair $(\mathfrak{m}, \mathfrak{n})$, where $\mathfrak{m} = (i, a, \lambda) \neq \mathfrak{n} = (j, b, \mu)$ is identified as follows:*

i) *If $i \neq j$, $(\mathfrak{a}, \mathfrak{b}) \in \rho$ if and only if $\mathfrak{a} = \mathfrak{b}$ or both $\mathfrak{a}$ and $\mathfrak{b}$ are elements of the form stated in 2 or 3 or 4.*

ii) *If $i = j$, $(\mathfrak{a}, \mathfrak{b}) \in \rho$ if and only if $\mathfrak{a} = \mathfrak{b}$ or both $\mathfrak{a}$ and $\mathfrak{b}$ are elements of the form stated in 5 or $\mathfrak{a} = (i, m_1 z, \theta), \mathfrak{b} = (i, m_2 z, \theta)$, for some $m_1, m_2 \in \mathfrak{M}, z \in G, \theta \in \Lambda$.*

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On submodules of the set of rational numbers

BABAK AMINI* and AFSHIN AMINI

Abstract

In this note, we completely determine all submodules of the set of rational numbers.

Keywords and phrases: Rational numbers, Submodules.
2010 *Mathematics subject classification:* Primary: 16D70, 16D10.

1. Introduction

The study of the set of rational numbers and its submodules is an interesting subject for mathematicians. In this work, we investigate submodules of the set of rational numbers in some new aspect. First, we recall some basic terminologies and results. We denote the set of rational numbers and integers respectively by $\mathbb{Q}$ and $\mathbb{Z}$. Note that every abelian group can be viewed as a $\mathbb{Z}$-module and so its subgroups are exactly its $\mathbb{Z}$-submodules. We refer the reader to [1] and [2] for undefiend terms and notions.

Definition 1.1. *A submodule K of a nonzero $\mathbb{Z}$-module M is said to be essential in M, in case for any nonzero submodule L of M one has $K \cap L \neq (0)$.*

Definition 1.2. *A $\mathbb{Z}$-module M is called torsion, if for every $x \in M$, there exist a positive integer n such that $nx = 0$.*

It is well known that any torsion $\mathbb{Z}$-module can be decomposed into a direct sum of its p-primary components.

Theorem 1.3. *Let M be a nonzero torsion $\mathbb{Z}$-module and P be the set of prime numbers, then $M = \bigoplus_{p \in P} M(p)$, where*

$$M(p) = \{x \in M \mid p^n x = 0 \text{ for some positive integer } n\}.$$

In the theorem above, $M(p)$ is calle the p-primary component of M. For an explicite example, consider the torsion $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. Then we have $\mathbb{Q}/\mathbb{Z} = \bigoplus_{p \in P} \mathbb{Z}_{p^\infty}$, where

* speaker

$$\mathbb{Z}_{p^\infty} = \frac{\mathbb{Q}}{\mathbb{Z}}(p) = \{\frac{m}{p^n} + \mathbb{Z} \mid m, n \in \mathbb{Z} \text{ and } n \geq 0\}.$$

We know that for any number p, the proper submodules of $\mathbb{Z}_{p^\infty}$ are cyclic and form a chain

$$H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots,$$

where $H_0 = (0)$ and $H_n = (\frac{1}{p^n} + \mathbb{Z})$, for each $n \geq 1$. Note that for each positive inreger n, we have $H_n \cong \mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}_{p^n}$.

Let $f \colon \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ be the natural epimorphism and p be a prime number, then

$$f^{-1}(\mathbb{Z}_{p^\infty}) = \{\frac{m}{p^n} \mid m, n \in \mathbb{Z} \text{ and } n \geq 0\}$$

and for each positive integer k,

$$f^{-1}(H_k) = \{\frac{m}{p^n} \mid m, n \in \mathbb{Z} \text{ and } 0 \leq n \leq k\}.$$

2. Main Results

Now we are ready to characterize the submodules of $\mathbb{Q}$ elementwise. Let K be a proper submodule of $\mathbb{Q}$ containing $\mathbb{Z}$. Then $K/\mathbb{Z}$ is a proper submodule of $\mathbb{Q}/\mathbb{Z}$ and so $K/\mathbb{Z}$ is a torsion $\mathbb{Z}$-module and we have

$$\frac{K}{\mathbb{Z}} = \bigoplus_{p \in P} \frac{K}{\mathbb{Z}}(p).$$

It is easily seen that $\frac{K}{\mathbb{Z}}(p) \subseteq \frac{\mathbb{Q}}{\mathbb{Z}}(p) = \mathbb{Z}_{p\infty}$. Thus $\frac{K}{\mathbb{Z}}(p) = \mathbb{Z}_{p\infty}$ or for some integer $n \geq 0$, we have $\frac{K}{\mathbb{Z}}(p) = (\frac{1}{p^n} + \mathbb{Z})$. For each prime number p let $g(K, p) = \infty$ if $\frac{K}{\mathbb{Z}}(p) = \mathbb{Z}_{p\infty}$ and $g(K, p) = n$ if $\frac{K}{\mathbb{Z}}(p) = (\frac{1}{p^n} + \mathbb{Z})$ for some integer $n \geq 0$. So,

$$\frac{K}{\mathbb{Z}}(p) = \{\frac{m}{p^n} + \mathbb{Z} \mid m, n \in \mathbb{Z}, 0 \leq n \leq g(K, p)\}.$$

Thus by considering the natural map $f : \mathbb{Q} \to \frac{\mathbb{Q}}{\mathbb{Z}}$ we have

$$K = f^{-1}(\frac{K}{\mathbb{Z}}) = f^{-1}(\bigoplus_{p \in P} \frac{K}{\mathbb{Z}}(p)) = \sum_{p \in P} f^{-1}(\frac{K}{\mathbb{Z}}(p))$$

and so

$$K = \sum_{p \in P} \{\frac{m}{p^n} \mid m, n \in \mathbb{Z}, 0 \leq n \leq g(K, p)\}.$$

Hence

$$K = \{\frac{m}{{p_1}^{n_1}{p_2}^{n_2}\dots{p_r}^{n_r}} \mid m, n_1, n_2, \dots, n_r \in \mathbb{Z}, 0 \le n_i \le g(K, p_i), r \ge 1\},$$

where $P = \{p_1, p_2, p_3, \dots\}$ is the set of prime numbers. Therefore, we have poved the following theorem.

Theorem 2.1. *Let K be a submodule of $\mathbb{Q}$ containing $\mathbb{Z}$. Then*

$$K = \{\frac{m}{{p_1}^{n_1}{p_2}^{n_2}\dots{p_r}^{n_r}} \mid m, n_1, n_2, \dots, n_r \in \mathbb{Z}, 0 \le n_i \le g(K, p_i), r \ge 1\},$$

where $P = \{p_1, p_2, p_3, \dots\}$ is the set of prime numbers and for any $p \in P$, we have $\frac{K}{\mathbb{Z}}(p) = \{\frac{m}{p^n} + \mathbb{Z} \mid m, n \in \mathbb{Z}, 0 \le n \le g(K, p)\}$.

Corollary 2.2. *Let K be a submodule of $\mathbb{Q}$ containing $\mathbb{Z}$. Then K is cyclic if and only if for each prime number p, one has $g(K, p) \neq \infty$ and $\{p \in P \mid g(K, p) \neq 0\}$ is a finite set.*

Note that in the corollary above, if $\{p \in P \mid g(K, p) \neq 0\} = \{q_1, q_2, \dots, q_s\}$, then

$$K = (\frac{1}{{q_1}^{g(K,q_1)}{q_2}^{g(K,q_2)}\dots{q_s}^{g(K,q_s)}}).$$

Now we consider the general case for submodules of $\mathbb{Q}$. Let K be a nonzero submodule of $\mathbb{Q}$. Since $\mathbb{Z}$ is an essential submodule of $\mathbb{Q}$, there exist a positive integer t such that $K \cap \mathbb{Z} = t\mathbb{Z}$. Thus $t\mathbb{Z} \le K$ and so $\mathbb{Z} \le t^{-1}K$ and $t^{-1}K$ is a submodule of $\mathbb{Q}$ containing $\mathbb{Z}$, then by Theorem 2.1, we have

$$t^{-1}K = \{\frac{m}{{p_1}^{n_1}{p_2}^{n_2}\dots{p_r}^{n_r}} \mid m, n_1, n_2, \dots, n_r \in \mathbb{Z}, 0 \le n_i \le g(t^{-1}K, p_i), r \ge 1\}.$$

So we have the following theorem.

Theorem 2.3. *Let K be a nonzero submodule of $\mathbb{Q}$ such that $K \cap \mathbb{Z} = t\mathbb{Z}$ for some positive integer t. Then*

$$K = \{\frac{tm}{{p_1}^{n_1}{p_2}^{n_2}\dots{p_r}^{n_r}} \mid m, n_1, n_2, \dots, n_r \in \mathbb{Z}, 0 \le n_i \le g(t^{-1}K, p_i), r \ge 1\},$$

where $P = \{p_1, p_2, p_3, \dots\}$ is the set of prime numbers and for any $p \in P$, we have $\frac{t^{-1}K}{\mathbb{Z}}(p) = \{\frac{m}{p^n} + \mathbb{Z} \mid m, n \in \mathbb{Z}, 0 \le n \le g(t^{-1}K, p)\}$.

Similar to Corollary 2.2, we have the following result.

Corollary 2.4. *Let K be a nonzero submodule of $\mathbb{Q}$ such that $K \cap \mathbb{Z} = t\mathbb{Z}$ for some positive integer t. Then K is cyclic if and only if for each prime number p, one has $g(t^{-1}K, p) \neq \infty$ and $\{p \in P \mid g(t^{-1}K, p) \neq 0\}$ is a finite set.*

Observing that in the above corollary, if $\{p \in P \mid g(t^{-1}K, p) \neq 0\} = \{q_1, q_2, \ldots, q_s\}$, then

$$K = \left(\frac{t}{{q_1}^{g(t^{-1}K,q_1)}{q_2}^{g(t^{-1}K,q_2)}\ldots{q_s}^{g(t^{-1}K,q_s)}}\right).$$

In this case the integer t is relatively prime to each of $q_1, q_2, \ldots, q_s$.

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The first Zagreb indices of Scalar Product Graph of some Modules

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Abstract

Let R be a commutative ring with identity and M an R-module. The Scalar Product Graph of M is the graph with M as the vertex set and every edge in this graph is xy such that $x = ry$ or $y = rx$ for some r belong to R . In this paper exact formula for first Zagreb index of Scalar-product graph of some modules will be presented.

Keywords and phrases: Scalar Product, Graph join, Zagreb Index, Module. .
2010 *Mathematics subject classification:* 05C25, 13CXX.

1. Introduction

Throughout this paper we consider connected graphs without loops and multiple edges. Let $G = (V, E)$ be a graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, $E \subseteq P_2(V)$ and $|E| = m$. For a graph G, the degree of a vertex v is the number of edges incident to v and denoted by $deg_G(v)$. The distance between the vertices u and v of G is denoted by $d_G(u, v)$. The join $G_1 + G_2$ of two graphs G_1 and G_2 is a graph obtained from G_1 and G_2 by joining every vertex of G_1 to all vertices of G_2. The automorphism group of a graph G is denoted by $Aut(G)$.
A topological index is a numeric quantity derived from the structure of a graph which is invariant under automorphisms of the considered graph. Suppose Σ denotes the class of all graphs, then a function $\Lambda : \Sigma \to \mathbb{R}^+$ is called a topological index if $G \cong H$ implies $\Lambda(G) = \Lambda(H)$. Usage of topological indices in chemistry began in 1947 [6] when chemist Harold Wiener developed the most widely known topological descriptor. The first Zagreb indices [2] of a graph G are defined as:

$$M_1(G) = \sum_{v \in V(G)} deg(v)^2 \tag{1}$$

* speaker

Let R and M be a commutative ring with identity and an R-module, Also $W(R)$ is the set of all non-unit elements of R. In [1] authors investigate cozero-divisor graphs on R-module M which vertices from $W_R(M)^* = W_R(M) \setminus \{0\}$ and two distinct vertices m and n are adjacent if and only if $m \notin Rn$ and $n \notin Rm$, and they studied girth, independent number, clique number and planarity of this graph. In [4] Nouri-Jouybari and et al. introduce a new class of graphs arising from modules, namely Scalar-product graph of R-module, denoted by $G_R(M)$. In $G_R(M)$, vertices are elements of M and two distinct vertices x and y are adjacent if and only if there exists r belong to R that $x = ry$ or $y = rx$. Properties of these graph have been expressed in [4], [5]. In next section, we present computing first Zagreb indices of scalar product graphs of some $\mathbb{Z}$-modules by join of two graphs.

2. Main Results

In this section, we compute first Zagreb index of $G_{\mathbb{Z}}(\mathbb{Z}_{2p})$ which p is prime number.

Lemma 2.1. *Let G and H be graphs. Then we have:*

1. $|E(G+H)| = |E(G)| + |E(H)| + |V(G)|\,|V(H)|$

2. $$d_{G+H}(u,v) = \begin{cases} 0 & u = v \\ 1 & uv \in E(G) \text{ or } uv \in E(H) \text{ or } (u \in V(G) \& v \in V(H)) \\ 2 & \text{otherwise} \end{cases}$$

Theorem 2.2. *[3] Let $G_1, G_2, \ldots, G_n$ be graphs with $V_i = |V(G_i)|$, $E_i = |E(G_i)|$, $1 \le i \le n$, $G = G_1 + G_2 + \ldots + G_n$ and $V = |V(G)|$.Then:*

$$M_1(G) = \sum_{i=1}^{n} (M_1(G_i) + |V_i|(|V| - |V_i|)^2 + 4|E_i|(|V| - |V_i|)) \quad (2)$$

Corollary 2.3. *Let G_1, G_2 be two graphs with $V_i = |V(G_i)|$, $E_i = |E(G_i)|$, $i = 1,2$. Then:*

$$\begin{aligned} M_1(G_1 + G_2) = & M_1(G_1) + |V_1||V_2|^2 + 4|E_1||V_2| \\ & + M_1(G_2) + |V_2||V_1|^2 + 4|E_2||V_1| \end{aligned}$$

Definition 2.4. *Let R be a commutative ring with non-zero identity and M be an unitary R module. We define Scalar-product graph of R-module M, namely $G_R(M)$, which vertices of $G_R(M)$ are elements of M and two distinct vertices x and y are adjacent if and only if there exists r belong to R that $x = ry$ or $y = rx$. For example of this type of graphs see Fig 1.*

Remark 2.5. *According to definition of cozero-divisor graph over modules in above we have the followings:*

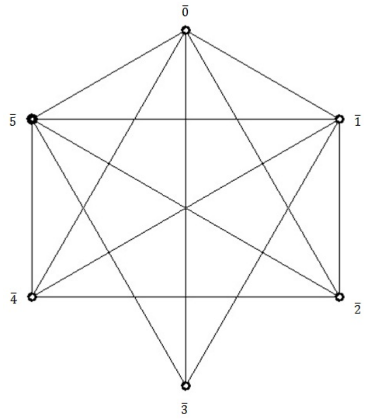

FIGURE 1. Scalar-product graph of $\mathbb{Z}$-modules $\mathbb{Z}_6$

(1) If M is an R-module, the subgraph of $G_R(M)$ which vertices are $W_R(M)^*$ is complement of cozero-divisors graph of M.
(2) We have $G_R(M) = G_1 + G_2$ where G_1 is a complete graph with $\mid W_R(M)^* \mid$ vertices and G_2 is complement of cozero-divisor graph of M.

Lemma 2.6. *Let p be a prime number. Then, $M_1(K_p) = p(p-1)^2$, $M_1(\overline{K_{1,p-1}}) = (p-1)(p-2)^2$*

PROOF. Every vertices of K_p has $p-1$ degrees. By definition, the first Zagreb index of K_p is $p(p-1)^2$. Degrees of vertices of $\overline{K_{1,p-1}}$ is 0 or $p-2$. By definition, the first Zagreb index of $\overline{K_{1,p-1}}$ is $(p-1)(p-2)^2$. □

Theorem 2.7. *Let $p \geq 3$ be a prime number and G be Scalar-product graph of $\mathbb{Z}$-modules $\mathbb{Z}_{2p}$. Then $M_1(G) = 8p^3 - 15p^2 + 13p - 4$.*

PROOF. By remark 2.5 we have $G_{\mathbb{Z}}(\mathbb{Z}_{2p}) = K_p + G_2$ that K_p is complete graph with p vertices and G_2 is complement of cozero-divisor graph of $\mathbb{Z}_{2p}$ which be $\overline{K_{1,p-1}}$. By corollary 2.3 we have:

$$\begin{aligned} M_1(G_{\mathbb{Z}}(\mathbb{Z}_{2p})) =& M_1(K_p + \overline{K_{1,p-1}}) = M_1(K_p) + \mid V(K_p) \mid\mid V(\overline{K_{1,p-1}})^2 \mid \\ &+ 4 \mid E(K_p) \mid\mid V(\overline{K_{1,p-1}}) \mid + M_1(\overline{K_{1,p-1}}) \\ &+ \mid V(\overline{K_{1,p-1}}) \mid\mid V(K_p) \mid^2 + 4 \mid E(\overline{K_{1,p-1}}) \mid\mid V(K_p) \mid \end{aligned}$$

by lemma 2.6 we have:

$$M_1(K_p) = p(p-1)^2, M_1(\overline{K_{1,p-1}}) = (p-1)(p-2)^2$$

By computing we have:
$M_1(G_{\mathbb{Z}}(\mathbb{Z}_{2p})) = 8p^3 - 15p^2 + 13p - 4$ □

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Results on signless Laplacian spectral characterization of broken graphs

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Abstract

A graph G is said to be (DLS) DQS if there is no other non-isomorphic graph with the same (Laplacian spectrum) signless Laplacian spectrum as G. A sun graph $SG(p)$ is obtained by appending a pendant vertex to any vertices of a cycle C_p and a broken sun graph $BSG(p,q)$ is a graph obtained by deleting $p-q$ pendant vertices of a sun $SG(p)$. We obtain some results related to the graphs that their signless Laplacian eigenvalues are the same as the signless Laplacian eigenvalues of a broken sun graph.

Keywords and phrases: Broken graph; *DQS* graph.
2010 *Mathematics subject classification:* Primary: 05C50.

1. Introduction

Graphs that are determined by their spectrum have received more attention, since they have been applied to several fields, such as randomized algorithms, combinatorial optimization problems and machine learning. An important part of spectral graph theory is devoted to determining whether given graphs or classes of graphs are determined by their spectra or not. So, finding and introducing any class of graphs which are determined by their spectra can be an interesting and important problem. Let $G=(V,E)$ be a simple graph with vertex set $V=V(G)=\{v_1,\cdots,v_n\}$ and edge set $E=E(G)=\{e_1,\cdots,e_m\}$, where $|V(G)|=n$ and $|E(G)|=m$.

Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of degree sequence of a graph G, respectively. The signless Laplacian matrix of G denoted by $Q(G)$ is the matrix $Q(G)=D(G)+A(G)$ (also $D(G)-A(G)$ is called the Laplacian matrix of G). The multiset of eigenvalues of $Q(G)$ is called the signless Laplacian spectrum of G. A graph G is said to be (DLS)

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DQS if there is no other non-isomorphic graph with the same (Laplacian spectrum) signless Laplacian spectrum as G.

A sun graph $SG(p)$ is obtained by appending a pendant vertex to any vertices of a cycle C_p and a broken sun graph $BSG(p,q)$ is a graph obtained by deleting $p-q$ pendant vertices of a sun $SG(p)$. A consecutive broken sun graph, $CBSG(p,q)$, is a broken sun graph such that subgraph induced by the vertices with degree 2 is a path on $p-q$ vertices. See the references for more details. Here we obtain that some kinds of broken graphs that are DQS.

2. Main Results

First we recall some results. The first result shows that two Q-spectral graphs have the same number of vertices and the same number of edges.

Lemma 2.1. *Let H and G be two Q-spectral graphs. Then the following hold:*

(i) G and H have the same number vertices.

(ii) G and H have the same number edges.

(iii) $\sum_{i=1}^{n} d_i^2(G) = \sum_{i=1}^{n} d_i^2(H)$.

(iv) $6N_G(C_3) + \sum_{i=1}^{n} d_i^3(G) = 6N_H(C_3) + \sum_{i=1}^{n} d_i^3(H)$.

We note that all signless Laplacian eigenvalues of any graph are non-negative. There is a well known result related to the multiplicity of zero as an signless Laplacian eigenvalue of graphs.

Lemma 2.2 ([3]). *For every graph the multiplicity of the eigenvalue 0 in the Q-spectrum is equal to the number of the bipartite components.*

Now let us state our main results without proof.

Lemma 2.3. *If H is Q-cospectral with $\Gamma = BSG(p,q)$, then $det(H) \in \{0,4\}$.*

Lemma 2.4. *If H is Q-cospectral with $\Gamma = BSG(p,q)$ and $p \geq 3$ is odd, then H is a connected graph.*

Corollary 2.5. *If H is Q-cospectral with $\Gamma = BSG(p,q)$ and $p \geq 3$ is odd, then H has at most one triangle.*

Theorem 2.6. *If H is Q-cospectral with $\Gamma = BSG(p,q)$ and $p \geq 5$ is odd, then they have the same degree sequence.*

Lemma 2.7. *If H is Q-cospectral with $\Gamma = BSG(p,q)$ and $p = 3$, then they have the same degree sequence.*

Corollary 2.8. *If H is Q-cospectral with $\Gamma = BSG(p,q)$ and $p \geq 3$ is odd, then H is a broken sun graph.*

We close the paper by the following result.

Theorem 2.9. *For $p > 2$ even and $0 < q < p$, the consecutive broken sun graph $\Pi = CBSG(p,q)$ is DQS.*

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Fair domination polynomial of a graph

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Abstract

A dominating set of a simple graph $G = (V, E)$ is a subset $D \subseteq V$ such that every vertex not in D is adjacent to at least one vertex in D. The cardinality of a smallest dominating set of G, denoted by $\gamma(G)$, is the domination number of G. The neighbourhood of a vertex v in G, $N(v)$ is the set of all of the vertices adjacent to v. For $k \geq 1$, a k-fair dominating set (kFD-set) in G, is a dominating set S such that $|N(v) \cap D| = k$ for every vertex $v \in V \setminus D$. A fair dominating set, in G is a kFD-set for some integer $k \geq 1$. Fair domination polynomial of G is denoted by $D_f(G, x)$ is defined as $\sum d_f(G, i)x^i$, where $d_f(G, i)$ is the number of fair dominating sets of G of size i. In this paper, after presenting preliminaries, we study this polynomial for some specific graphs.

Keywords and phrases: domination number, fair domination polynomial, cycle.
2010 *Mathematics subject classification:* Primary: 05C25 .

1. Introduction

Let $G = (V, E)$ be a simple graph with n vertices. A set $D \subseteq V$ is a dominating set, if every vertex in $V \backslash D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. The neighbourhood of a vertex v in G, $N(v)$ is the set of all of the vertices adjacent to v. For $k \geq 1$, a k-fair dominating set (kFD-set) in G, is a dominating set D such that $|N(v) \cap D| = k$ for every vertex $v \in V \setminus D$. The k-fair domination number of G, denoted by $fd_k(G)$, is the minimum cardinality of a kFD-set. A kFD-set of G of cardinality $fd_k(G)$ is called a $fd_k(G)$-set. A fair dominating set, abbreviated FD-set, in G is a kFD-set for some integer $k \geq 1$. The fair domination number, denoted by $fd(G)$, of a graph G that is not the empty graph is the minimum cardinality of an FD-set in G. An FD-set of G of cardinality $fd(G)$ is called a $fd(G)$-set. By convention, if $G = \overline{K_n}$, we define $fd(G) = n$. By the definition it is easy to see that for any graph G of order n, $\gamma(G) \leq fd(G) \leq n$ and $fd(G) = n$ if and only if $G = \overline{K_n}$. Caro, Hansberg and Henning in [5] showed that for a disconnected graph G (without isolated vertices) of order $n \geq 3$, $fd(G) \leq n - 2$, and they constructed an

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infinite family of graphs achieving equality in this bound. The corona $G \circ K_1$, is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added. Caro, Hansberg, and Henning in [5] proved that if T is a tree of order $n \geq 2$, then $fd(T) \leq \frac{n}{2}$ with equality if and only if $T = T' \circ K_1$ for some tree T'. We know that if S is a dominating set of G and $S \subseteq S'$, then S' is a dominating set, too. But this is not true for the fair dominating sets. As an example consider the cycle C_9 with $V(C_9) = \{1, 2, ..., 9\}$. Observe that there are three fair dominating sets with cardinality three for C_9, but there is no dominating set of C_9 with cardinality four. This notation shows that study the fair dominating sets and finding the number of fair dominating sets of a graph with arbitrary cardinality is not easy problem. Regarding to enumerative side of dominating sets, Alikhani and Peng have introduced the domination polynomial of a graph. The domination polynomial of graph G is the generating function for the number of dominating sets of G, i.e., $D(G, x) = \sum_{i=1}^{|V(G)|} d(G, i)x^i$ (see [1, 3]). This polynomial and its roots has been actively studied in recent years (see for example [4, 7]). It is natural to count the number of another kind of dominating sets ([2]). In this paper we consider the fair domination polynomial of a graph and count the number the fair dominating sets of certain graphs. We denote the set $\{1, 2, ..., n\}$ simply by $[n]$.

2. Main Results

In this section, similar to the domination polynomial, we state the definition of the fair domination polynomial. Then, we count the number of fair dominating sets of specific graphs such as complete bipartite graph $K_{n,n}$, and cycles.

2.1. Fair domination polynomial In this subsection, we state the definition of the fair domination polynomial.

Definition 2.1. *Let $\mathcal{D}_f(G, i)$ be the family of the fair dominating sets of a graph G with cardinality i and let $d_f(G, i) = |\mathcal{D}_f(G, i)|$. The fair domination polynomial $D_f(G, x)$ of G is defined as*

$$D_f(G, x) = \sum_{i=fd(G)}^{|V(G)|} d_f(G, i)x^i,$$

where $fd(G)$ is the fair domination number of G.

2.2. Results for $K_{n,n}$ and C_n In this subsection, we study the number of fair dominating sets of complete bipartite graph $K_{n,n}$ and the cycle graph C_n. We start with $K_{n,n}$.

Theorem 2.2. (i) *If $r > 2$ is odd, then*

$$d_f(K_{n,n}, r) = \begin{cases} 2; & \text{if } r = n, \\ 0; & \text{if } r < n, \\ 2\binom{n}{r-n}; & \text{if } r > n. \end{cases}$$

(ii) *If $r \geq 2$ is even, then*

$$d_f(K_{n,n}, r) = \begin{cases} \binom{n}{r/2}^2; & \text{if } r < n, \\ \binom{n}{r/2}^2 + 2; & \text{if } r = n, \\ \binom{n}{r/2}^2 + \binom{n}{r-n}^2; & \text{if } r > n. \end{cases}$$

A partition of a positive integer n, is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition. A summand in a partition is also called a part. The number of partitions of n is given by the partition function $p(n)$. A partition of n into exactly k parts is an unordered sum of n that uses exactly k positive integers. The number of such partitions will henceforth be denoted by $p(n,k)$. For example, $p(5,3) = 2$ as $5 = 1+2+2$ and $5 = 1+1+3$ are the only two sums of 5 that can be formed using three positive integers. Let to denote by $P(n; x_1^{t_1}, x_2^{t_2}, ..., x_k^{t_k})$, the partition of n into exactly k parts $x_1, ..., x_k$, where t_i is the number of x_i in the partition. Note that the equality $n = t_1x_1 + t_2x_2 + ... + t_kx_k$ is true. For example $P(5, 1^1, 2^2)$ is correspond to the partition of $5 = 1+2+2$. To construct the fair dominating sets of C_n with $V(C_n) = [n]$ of size k, we consider the partition of the number k to $n-k$ natural numbers, when $n-k$ is odd or $n-k = k$ and consider the partition of the number k to $\frac{n-k}{2}$ natural numbers, when $n-k$ is even. We consider $P(k; x_1^{t_1}, x_2^{t_2}, ..., x_{n-k}^{t_{n-k}})$. We define the family $\mathcal{A} \subseteq [n]$, based on $P(k; x_1^{t_1}, x_2^{t_2}, ..., x_{n-k}^{t_{n-k}})$ as follows:

$$\mathcal{A} = A_1 \cup A_2 \cup ... \cup A_{n-k},$$

where the set A_1 contains x_1 consecutive numbers from $[n]$, the set A_2 contains x_2 consecutive numbers from $[n] \setminus A_1$ and finally the set A_{n-k} contains x_{n-k} consecutive numbers from $[n] \setminus \big(A_1 \cup A_2 \cup ... \cup A_{n-k-1}\big)$ and also $d(G[A_i], G[A_{i+1}]) = 2$.

We define the family $\mathcal{B} \subseteq [n]$, based on $P(k; x_1^{t_1}, x_2^{t_2}, ..., x_{\frac{n-k}{2}}^{t_{\frac{n-k}{2}}})$, as follows:

$$\mathcal{B} = B_1 \cup B_2 \cup ... \cup B_{\frac{n-k}{2}},$$

where the set B_1 contains x_1 consecutive numbers from $[n]$, the set B_2 contains x_2 consecutive numbers from $[n] \setminus B_1$ and finally the set $B_{\frac{n-k}{2}}$ contains $x_{\frac{n-k}{2}}$ consecutive numbers from $[n] \setminus \left(B_1 \cup B_2 \cup ... \cup B_{\frac{n-k}{2}-1}\right)$ and also $d(G[B_i], G[B_{i+1}]) = 3$.

The following theorem gives the number of fair dominating sets of cycles:

Theorem 2.3. *Let $C_n, n \geq 3$, be the cycle of order n.*

(i) *If $n - k$ is even and $n \leq 2k$, then $d_f(C_n, k) = n(|\mathcal{A}| + |\mathcal{B}|)$.*
(ii) *If $n - k$ is even and $n > 2k$, then $d_f(C_n, k) = n|\mathcal{B}|$.*
(iii) *If $n - k$ is odd and $n \leq 2k$, then $d_f(C_n, k) = n|\mathcal{A}|$.*
(iv) *If $n - k$ is odd and $n > 2k$, then $d_f(C_n, k) = 0$.*

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On eccentric adjacency index of graphs and trees

VAHID HASHEMI*, FATEME PARSANEJAD and REZA SHARAFDINI

Abstract

The eccentric adjacency index (for short, EAI) of a connected graph G is defined as

$$\xi^{ad}(G) = \sum_{u \in V(G)} \mathbf{S}_G(u) \varepsilon_G(u)^{-1},$$

where $\mathbf{S}_G(u)$ denotes sum of degrees of vertices adjacent to the vertex u and $\varepsilon_G(u)$ is defined as the maximum length of any minimal path connecting u to any other vertex of G. Inspired from [Jelena Sedlar, On augmented eccentric connectivity index of graphs and trees, MATCH Commun. Math. Comput. Chem. 68 (2012) 325–342.], we establish all extremal graphs with respect to EAI among all (simple connected) graphs, among trees and among trees with perfect matching.

Keywords and phrases: eccentricity, tree, eccentric adjacency in index..

1. Introduction

Let G be any simple connected graph with the vertex set $V(G)$ and the edge set $E(G)$. For two vertices u and v in $V(G)$ their distance $d(u,v)$ is defined as the length of a shortest path connecting u and v in G.

The degree $d(u)$ of the vertex u in G is defined as the number of neighbors of u in G, i.e., $d(u) = |\{v \in V(G) | d(u,v) = 1\}|$. The eccentricity $\varepsilon(u)$ of the vertex u of G is the distance from u to any vertex farthest away from it in G, i.e., $\varepsilon(u) = \max_{v \in V(G)} d(u,v)$. The maximum eccentricity over all vertices of G is called the diameter of G and is denoted by $D(G)$; the minimum eccentricity among the vertices of G is called the radius of G and is denoted by $R(G)$. The set of all vertices of minimum eccentricity is called the center of G and such vertices are called *central*.

The eccentric adjacency index of a connected graph G is defined as [3]

$$\xi^{ad}(G) = \sum_{u \in V(G)} \mathbf{S}_G(u) \varepsilon_G(u)^{-1},$$

* speaker

where $\mathbf{S}_G(u)$ denotes sum of degrees of vertices adjacent to the vertex u and $\varepsilon_G(u)$ is defined as the maximum length of any minimal path connecting u to any other vertex of G.

Let us now define some special kinds of graphs. First, K_n will denote a *complete graph* on n vertices. Special class of graphs which will be of interest are trees. A *tree* T is a simple connected graph with no cycles. It is easily seen that a tree has only one central vertex if $D(T)$ is even, and two central vertices if $D(T)$ is odd. We say that a vertex in tree T is a leaf if its degree is 1, otherwise we say that a vertex is non-leaf. Also, we say that a vertex in a tree is *branching* if its degree is equal or greater than 3. Now, P_n will denote a *path* on n vertices and S_n will denote a *star* on n vertices.

2. Extremal trees

In this section, we want to establish trees with minimum and maximum value of EAI.

First, we will do the minimum. To do so, we recall the following transformation of trees which increases the diameter, but decreases the value of EAI. This transformation was inspired from [4] in which the augmented eccentric connectivity index of trees was considered.

Transformation A ([4]). *Let $T \neq P_n$ be a tree of order n and with diameter D, and let $P_D = v_0v_1 \cdots v_D$ be a diametric path in T chosen so that the first branching vertex is furthest possible from v_0. We consider the following cases:*

(A1) *v_1 is branching, while v_2 is not. In this case, we set $u = v_1$;*
(A2) *Both v_1 and v_2 are branching vertices. In this case, we set $u = v_2$;*
(A3) *v_1 is not branching. In this case, we set $u = v_i$, where v_i, $i \neq 1$ is the first branching vertex on P_D.*

Let $w_1, \ldots, w_k$ be the $k = d(u) - 2$ neighbors of u outside of P_D. Let T' be the tree obtained from T by deleting edges $uw_1, \ldots, uw_k$ and adding edges $v_0w_1, \ldots, v_0w_k$, See Figure 1.

Lemma 2.1. *Let T' be a tree obtained from tree $T \neq P_n$ applying Transformation A. Then*

$$\xi^{ad}(T) > \xi^{ad}(T').$$

Applying Transformation A consecutively, we arrive at a tree of order n having no branching vertex, namely P_n, which by Lemma 2.1 has the least value of ξ^{ad}.

Lemma 2.2. *Let $T \neq P_n$ be a tree with n vertices. Then*

$$\xi^{ad}(T) > \xi^{ad}(P_n).$$

Now, we can summarize our results in the following theorem.

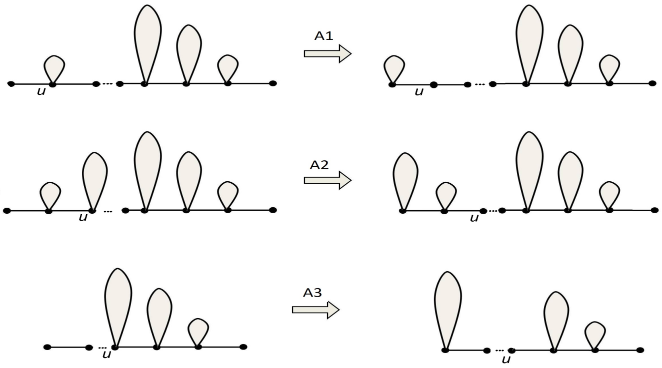

FIGURE 1. Transformations A

Theorem 2.3. *Let T be a tree on n vertices. Then*

$$\xi^{ad}(T) \geq \frac{6}{n-2} + 4(\frac{1}{n-1} + 2H_{n-3} - H_{\lfloor \frac{n}{2} \rfloor - 1} - H_{\lfloor \frac{n-1}{2} \rfloor}),$$

with equality if and only if $T = P_n$.

Let us find the tree with maximum value of EAI. To do so, we need to introduce a transformation that increases the value EAI of trees.

Transformation B. *Let T be a tree of diameter $D \geqslant 4$ with a center u. If w is a non-leaf and non-central vertex adjacent to u. Let $w_1, \ldots, w_k$ be the non-central neighbors of w, Let T' be the tree obtained from T by deleting edges $ww_1, \ldots, ww_k$ and adding edges $uw_1, \ldots, uw_k$.*

Lemma 2.4. *Suppose that T is a tree of order n with diameter $D(T) \geqslant 4$. Let T' be a tree obtained from T applying Transformation B. Then*

$$\xi^{ad}(T) < \xi^{ad}(T').$$

Applying Transformation B consecutively, we arrive at a tree of order n and diameter 3, but it is not a maximal tree with respect to EAI. Therefore, we need to introduce another transformation.

Transformation C. *Let T be a tree of diameter $D(T) = 3$, where u and v are the central vertices. Let $w_1, ..., w_k$ and $z_1, ..., z_j$ be the neighbors of u and v respectively. The star T' is obtained by deleting edges $uw_1, ..., uw_k$ and adding edges $vw_1, ..., vw_k$.*

Lemma 2.5. *Suppose that T is a tree of order n with diameter $D(T) = 3$. Let T' be a tree obtained from T applying Transformation C. Then*

$$\xi^{ad}(T) < \xi^{ad}(T').$$

Theorem 2.6. *Let T be a tree on n vertices. Then*

$$\xi^{ad}(T) \leq \frac{(n-1)^2 + 2(n-1)}{2},$$

with equality if and only if $T = S_n$.

3. Extremal graphs

Let us now establish extremal graphs among all simple connected graphs. Those results will follow easily from results for trees.

Proposition 3.1. *For a connected graph G on n vertices, we have*

$$\xi^{ad}(G) \leq n(n-1)^2,$$

with equality if and only if $G = K_n$.

In the following proposition we establish minimal graphs.

Proposition 3.2. *For a connected graph G on n vertices, we have*

$$\xi^{ad}(G) \geq \frac{6}{n-2} + 4\left(\frac{1}{n-1} + 2H_{n-3} - H_{\lfloor \frac{n}{2} \rfloor - 1} - H_{\lfloor \frac{n-1}{2} \rfloor}\right),$$

with equality if and only if $G = P_n$.

4. Extremal trees with perfect matching

A *matching* in a graph G is collection of edges S from G such that no vertex from G is incident to two edges from S. We say that a matching S is *perfect* if every vertex from G is incident to one edge from S. Obviously, only graphs with even number of vertices can have a perfect matching.

J. Sedlar in [4] introduces some interesting classes of graphs with diameter 4. We say that a tree T is *degree balanced* if its diameter is 4 and all neighbors of (the only) central vertex differ in degree by at most one. With $TB_{n,k}$ we will denote the degree balanced tree on n vertices in which the degree of its central vertex is k, see for example Figure 2.

Now let us introduce a transformation that increases the EAI of trees with a perfect matching.

Transformation D. *Let T be a tree with a perfect matching and P_D a diametric path in T. We label the vertices in P_D in a way that v_i and v_{i+1} are adjacent, v_0 is a leaf vertex, and we name the central vertex with the smallest index, u. Let $w_1, \ldots, w_k$ be all the vertices adjacent to v_2 that $d(w_i) = 2$, except for v_3. The tree T' is obtained by deleting edges $v_2 w_i$ and adding edges $u w_i$.*

Lemma 4.1. *By applying Transformation D on a tree T of diameter $D(T) \geq 5$:*

$$\xi^{ad}(T) < \xi^{ad}(T').$$

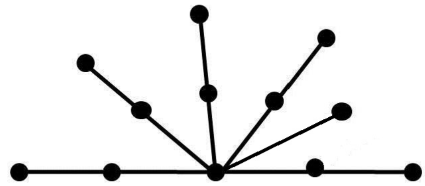

FIGURE 2. The degree balanced tree $TB_{12,6}$

Theorem 4.2. *Let T be a tree with perfect matching on $n \geq 6$ vertices. Then*

$$\xi^{ad}(T) \leq \frac{n^2 + 11n - 16}{12},$$

with equality if and only if $T = TB_{n,\frac{n}{2}}$.

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On finite p-groups whose absolute central automorphisms are all nth autoclass-preserving

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Abstract

Let G be a group and $L(G)$ denotes the absolute center of G. An automorphism α of G is called an absolute central automorphism if $x^{-1}x^{\alpha} \in L(G)$, for each $x \in G$. Let G be an autonilpotent finite p-group of class $n+1$, where $n \geq 1$. We call an automorphism α of G an nth autoclass-preserving if for all $x \in G$, there exists an element $g_x \in K_{n-1}(G)$ such that $x^{\alpha} = g_x^{-1}xg_x$, where $K_{n-1}(G)$ is the $n-1$th autocommutator subgroup of G. In this paper we obtain a necessary and sufficient condition for an autonilpotent finite p-group of class $n+1$ such that each absolute central automorphism an nth autoclass-preserving.

Keywords and phrases: Automorphism group, nth autoclass-preserving automorphism, absolute center, finite p-group.

2010 *Mathematics subject classification:* Primary: 20D45; Secondary: 20D25, 20D15.

1. Introduction

Throughout the paper all groups are assumed to be finite and p denotes a prime number. By $G', Z(G)$, $\mathrm{Inn}(G)$ and $\mathrm{Aut}(G)$, respectively we denote the commutator subgroup, the center, the group of all inner automorphisms and the group of all automorphisms of G. For each $x \in G$ and $\alpha \in \mathrm{Aut}(G)$, the element $[x, \alpha] = x^{-1}x^{\alpha}$ is called the autocommutator of x and α. Also for $n \geq 1$, the autocommutator of higher weight inductively as follows:

$$[x, \alpha_1, \alpha_2, ..., \alpha_n] = [[x, \alpha_1, \alpha_2, ..., \alpha_{n-1}], \alpha_n],$$

for all $\alpha_1, \alpha_2, ..., \alpha_n \in \mathrm{Aut}(G)$ and $x \in G$. In 1994, Hegarty [2] introduced the concepts of absolute center and autocommutator subgroups of a group G, as follows:

$$L(G) = \{x \in G \mid [x, \alpha] = 1, \forall \alpha \in \mathrm{Aut}(G)\},$$

$$K(G) = [G, \mathrm{Aut}(G)] = \langle [x, \alpha] \mid x \in G, \alpha \in \mathrm{Aut}(G) \rangle.$$

* speaker

It is easy to check that these are characteristic subgroups of G. Assume that $K_0(G) = G$ and $K_1(G) = K(G)$, then for $n \geq 1$,

$$K_n(G) = [K_{n-1}(G), \mathrm{Aut}(G)] = \langle [x, \alpha_1, \alpha_2, ..., \alpha_n] \mid x \in G, \alpha_i \in \mathrm{Aut}(G) \rangle,$$

which is called nth autocommutator subgroup of G. One can easily see that for $n \geq 0$, $\gamma_{n+1}(G) \leq K_n(G)$, where $\gamma_{n+1}(G)$ is the $(n+1)$th term of the lower central series of G, and also $K_n(G)$ is a characteristic subgroup of G. Therefore, we obtain the following descending series of G,

$$G = K_0(G) \supseteq K_1(G) \supseteq K_2(G) \supseteq ... \supseteq K_n(G) \supseteq ...,$$

which is called the lower autocentral series of G.
Next, let $L_1(G) = L(G)$ and for $n \geq 2$, the nth absolute center of G is defined inductively as

$$L_n(G) = \{x \in G \mid [x, \alpha_1, \alpha_2, ..., \alpha_n] = 1, \forall \alpha_1, \alpha_2, ..., \alpha_n \in \mathrm{Aut}(G)\}.$$

Hence, we obtain an ascending chain of characteristic subgroups of G as follows:

$$\{1\} = L_0(G) \subseteq L_1(G) \subseteq L_2(G) \subseteq ... \subseteq L_n(G) \subseteq ....$$

It is easy to see that $L_n(G) \leq Z_n(G)$, where $n \geq 0$ and $Z_n(G)$ is the nth center of G. Also $K_n(G) = \gamma_{n+1}(G)$ and $L_n(G) = Z_n(G)$, when all the automorphisms $\alpha_i, (1 \leq i \leq n)$, are taken to be the inner automorphisms of G. A group G is said to be autonilpotent of class n if n is the smallest natural number such that $L_n(G) = G$. Moreover $L_n(G) = G$ if and only if $K_n(G) = 1$. Let us denote by $M_p(n, m)$ for the minimal non-abelian p-group of order p^{n+m} defined by

$$\langle a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}} \rangle,$$

where $n \geq 2$, $m \geq 1$ and x^H the conjugacy class of all $x^h = h^{-1}xh$, where H is a subgroup of G, $x \in G$ and $h \in H$. Recall an abelian finite p-group A has invariants or is of type $(n_1, n_2, ..., n_k)$ if it is the direct product of cyclic subgroups of orders $p^{n_1}, p^{n_2}, ..., p^{n_k}$, where $n_1 \geq n_2 \geq ... \geq n_k > 0$.

An automorphism α of G is called an absolute central automorphism if $[x, \alpha] \in L(G)$, for all $x \in G$. The absolute central automorphisms of G, denoted by $\mathrm{Aut}^L(G)$, fix G' element-wise and form a normal subgroup of the automorphism group of G. An automorphism α of G is called class preserving automorphism if $x^\alpha \in x^G$, for all $x \in G$. The set of all class preserving automorphisms of G, denoted by $\mathrm{Aut}_c(G)$.

Now, let G be an autonilpotent finite p-group of class $n+1$, where $n \geq 1$. We call an automorphism α of G an nth autoclass-preserving if for each $x \in G$, $x^\alpha \in x^{K_{n-1}(G)}$, where $K_{n-1}(G)$ is the $n-1$th autocommutator subgroup of G. We note that the set $\mathrm{Aut}^n_{ac}(G)$ of all nth autoclass-preserving automorphisms of G is a normal subgroup of $\mathrm{Aut}(G)$. There are some results on the absolute center and autocommutator subgroups of a finite group G, see for example [2], [3], [4] and [5].

2. Main Results

In this section, we give a necessary and sufficient condition for an autonilpotent finite p-group G of class $n+1$ where $n \geq 1$, such that every absolute central automorphism an nth autoclass-preserving. We observe that $\mathrm{Aut}^n_{ac}(G) = \mathrm{Aut}_c(G)$ for $n = 1$.

Lemma 2.1. *Let G be an autonilpotent finite p-group of class $n+1$. Then $K_n(G) \leq L(G)$.*

Lemma 2.2. *Let G be an autonilpotent finite p-group of class $n+1$ and $\mathrm{Aut}^n_{ac}(G) = \mathrm{Aut}^L(G)$. Then*

$$\mathrm{Aut}^n_{ac}(G) \cong \mathrm{Hom}(G/G', K_n(G)) \cong \mathrm{Hom}(G/Z(G), K_n(G)) \cong \mathrm{Hom}(G/K_n(G), K_n(G)).$$

Let G be an autonilpotent finite p-group of class $n+1$. Also G/G', $L(G)$ and $K_n(G)$ are of types $(a_1, a_2, ..., a_k)$, $(b_1, b_2, ..., b_l)$ and $(c_1, c_2, ..., c_m)$.

By fixed the above notation, we have the following result.

Theorem 2.3. *Let G be an autonilpotent finite p-group of class $n+1$. Then the following statements are equivalent:*

(i) $\mathrm{Aut}^n_{ac}(G) = \mathrm{Aut}^L(G)$;

(ii) $\mathrm{Aut}^n_{ac}(G) \cong \mathrm{Hom}(G, K_n(G))$ *and one of the following conditions holds:*

- *(1)* $K_n(G) = L(G)$ *or*
- *(2)* $K_n(G) < L(G)$, $m = l$ *and* $a_1 \leq c_t$, *where t is the largest integer between 1 and m such that $b_t > c_t$.*

Corollary 2.4. *Let G be an autonilpotent finite p-group of class 2. Then $\mathrm{Aut}_c(G) = \mathrm{Aut}^L(G)$ if and only if $\mathrm{Aut}_c(G) \cong \mathrm{Hom}(G/G', K(G))$ and $G' = K(G) = L(G)$.*

In the following theorem, let $\mathrm{Aut}^K(G)$ denote the set of all automorphisms of G, which centralizes $G/K(G)$ element-wise.

Theorem 2.5. *Let G be a non-abelian autonilpotent finite p-group of class 2. Then $\mathrm{Aut}^K(G) = \mathrm{Inn}(G)$ if and only if $K(G)$ is cyclic and $Z(G) = K(G)G^{p^n}$ where $\exp(K(G)) = p^n$.*

Lemma 2.6. *Let G be an autonilpotent finite p-group of class 2. Then* $\exp(G/L(G))|\exp(K(G))$

As an application of Theorem 2.5, we have the main result of [6].

Corollary 2.7. *[6, Theorem 3.2] Let G be a non-abelian autonilpotent finite p-group of class 2. Then $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$ if and only if $L(G) = Z(G)$ and $L(G)$ is cyclic.*

We end the paper by giving an example of a group which satisfies the hypothesis of Theorem 2.3. Its GAP id is 452 ([1]).

Example 2.8. *Let* $G = \langle f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8 \rangle$ *be a* 2*-group of order* 2^8 *with the following relations:*

$f_8^2 = 1, f_1^2 = f_4, f_2^4 = f_5, f_2^6 = f_3, f_4^2 = f_6, f_5^{-2} = f_7, f_6^2 = f_8, f_7^2 = f_8, [f_2, f_6] = [f_2, f_7] = [f_3, f_4] = [f_3, f_5] = [f_3, f_7] = [f_4, f_5] = [f_4, f_7] = 1, [f_1, f_2] = f_3 f_5 f_8, [f_1, f_3] = f_5 f_7, [f_1, f_5] = f_7 f_8, [f_1, f_7] = f_8, [f_2, f_4] = f_8.$

In this group, $Z(G) = \langle f_6 \rangle \cong C_4$*. Also* $L(G) = \langle f_8 \rangle \cong C_2, L_2(G) = \langle f_6, f_7 \rangle \cong C_4 \times C_2, L_3(G) = \langle f_4, f_5 \rangle \cong C_8 \times C_4, L_4(G) = \langle f_3, f_4, f_5 \rangle \cong C_{16} \times C_4, L_5(G) = \langle f_2, f_4 \rangle \cong M_2(5, 2)$*. Finally,* $L_6(G) = G$*, which shows that* G *be an autonilpotent group of class* 6*.*

On the other hand, $K(G) = \langle f_2, f_4 \rangle \cong M_2(5, 2), K_2(G) = \langle f_3, f_5, f_6 \rangle \cong C_{16} \times C_2, K_3(G) = \langle f_5 \rangle \cong C_8, K_4(G) = \langle f_7 \rangle \cong C_4, K_5(G) = \langle f_8 \rangle \cong C_2$ *and* $K_6(G) = \langle 1 \rangle$*. Hence* $K_5(G) = L(G)$*.*

Now we observe that $\mathrm{Aut}_{ac}^5(G) = \mathrm{Aut}^L(G)$*.*

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Relation between Sylowility degree and Sylow Graph

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Abstract

Let G be a finite group. The sylowility degree is given by $P_{rSy}(G) = \frac{|\{(x,y)\in G\times G|\langle x,y\rangle \leq_{Sy} G, \forall x,y\in G\}|}{|G|^2}$ and the sylow graph $\Gamma_G(V,E) = \Gamma_{Sy}$ is defined by the set of all vertices of $E(\Gamma_S y) = \{\{x,y\} \mid \langle x,y\rangle \leq_{Sy} G\}$. In this seminar, we establish some properties of the sylow graph defined by group D_{2n} and study the relation between Γ_{Sy} and $P_{rSy}(G)$.

Keywords and phrases: Sylow Graph, Sylowility Degree, Dihedral Group.
2010 *Mathematics subject classification:* Primary: 22D15, 43A10; Secondary: 43A20, 46H25.

1. Introduction

It will-known the cyclic group is define by $C_n = \langle a^i | 1 \leq i \leq n\rangle$, Shelash and Ashrafi introduced all Sylow subgroups of some of finite groups. We will present in this paper in this seminar parameter to compute the Sylowility degree $Sy(C_n)$ of cyclic group.

2. Main Results

In this seminar we will present a new parameters about dihedral group to study the relationship between those parametrise.

Definition 2.1. $|Syl_G(g)| = |\{y \in G|\langle g,y\rangle \leq_{Sy} G\}|$ *is the number of all elements in group G such that $\langle g,y\rangle$ is a sylow subgroup on G.*

Definition 2.2. $|Syl_p(G)| = |\{y \in G|\langle g,y\rangle \leq_{Sy} G\}|$.

Definition 2.3. $|Syl(G)| = \sum_{i=1}^{s} |Syl_{p_i}(G)|$ *is the number of all elements in group G such that $\langle g,y\rangle$ is a sylow subgroup on G.*

Proposition 2.4. *If $n = p^\alpha$, then $|Syl(C_{p^\alpha})| = \frac{p^2-1}{p^2}$;*

Theorem 2.5. *Let $n = \prod_{i=1}^{s} p_i^{\alpha_i}$ be an integer number, the following are held:*

* speaker

1. For any $p_i \mid n$, the degree $|Syl_{(C_n,p_i)}(e)| = \varphi(p^{\alpha_i})$;
2. For any $p_i^{\alpha_i} \mid n$, the degree $|Syl_{(C_n,p_i)}(a^{\frac{n}{p_i^{\alpha_i}}})| = \sum_{t=1}^{\alpha} \varphi(p^t)$;
3. For any $p_i^t \mid n$ and $t < \alpha_i$, the degree $|Syl_{(C_n,p_i)}(a^{\frac{n}{p_i^t}})| = \varphi(p^\alpha)$;

Proposition 2.6. *Let Γ_{Sy} be a simple and sylow graph, The number of degree vertices $deg(v)$ for any $v \in V(\Gamma_{Sy})$ when $G \cong C_n$ is given by the following: if $v = a^i$, then $deg(v) = |N_G(v)| - 1$;*

Theorem 2.7.

$$P_{rSy}(G) = \frac{2|E(\Gamma_{Sy})| + n}{|G|^2}$$

Lemma 2.8. *Let $n = p^\alpha$ be an integer number, the following are held:*

1. If $p \mid i$, then $Sy_{C_n}(a^i) = \{a^j \mid Gcd(i,j) = 1\}$, $|Sy_{C_n}(a^i)| = \varphi(p^\alpha)$ and $\#a^i = p^{\alpha-1}$;
2. If $p \nmid i$, then $Sy_{C_n}(a^i) = \{a^j \mid 1 \le j \le n\}$, $|Sy_{C_n}(a^i)| = p^\alpha$ and $\#a^i = \varphi(p^\alpha)$.

Example 2.9. *Consider the cyclic group C_{11}:*

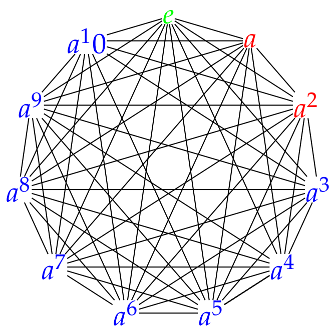
$\|g\|$	10	11	11	11	11	11	11	11	11	11	11
	e	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}
e	0	1	1	1	1	1	1	1	1	1	1
a	1	1	1	1	1	1	1	1	1	1	1
a^2	1	1	1	1	1	1	1	1	1	1	1
a^3	1	1	1	1	1	1	1	1	1	1	1
a^4	1	1	1	1	1	1	1	1	1	1	1
a^5	1	1	1	1	1	1	1	1	1	1	1
a^6	1	1	1	1	1	1	1	1	1	1	1
a^7	1	1	1	1	1	1	1	1	1	1	1
a^8	1	1	1	1	1	1	1	1	1	1	1
a^9	1	1	1	1	1	1	1	1	1	1	1
a^{10}	1	1	1	1	1	1	1	1	1	1	1
$\|Sy_G(g)\|$	10	11	11	11	11	11	11	11	11	11	11
$\|deg(g)\|$	9	10	10	10	10	10	10	10	10	10	10

$$|Sy_{C_{11}}(e)| = 10, |Sy_{C_{11}}(a^i)| = 11, \textit{for each} 1 \le i \le 11,$$

and

$$|Syl_p(C_{11})| = 10 + (10) * 11 = 120$$

then $Syl(C_{11}) = \dfrac{120}{121}$.



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The Solvablitiy Degree of the Alternation Group

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Abstract

Let G be a finite non-solvable group with solvable radical $Sol(G)$. The solvable graph $\Gamma_{sol}(G)$ of group G is a graph with vertex set $V(\Gamma_{sol}) = \{\sigma \mid \sigma \in G\}$ and two distinct vertices σ_1 and σ_2 are adjacent if and only if $\langle\sigma_1,\sigma_2\rangle$ is solvable group, so the solvability degree of G is define by the number of all elements such that $\{(\sigma_1,\sigma_2) \in G \times G \mid \langle\sigma_1,\sigma_2\rangle \leq_{Sol} G\}$ on the number $(G)^2$. We show that the relation between $\Gamma_{sol}(G)$ and the solvability degree of G.

Keywords and phrases: Solvable group, Solvable graph, Solvability degree. .

2010 *Mathematics subject classification:* Primary: 22D15, 43A10; Secondary: 43A20, 46H25.

1. Introduction

Let $\Gamma(V,E)$ be a simple graph. The set of vertices denoted by $V(\Gamma)$ and the set of edges denoted by $E(\Gamma)$.

The solvable Graph of a finite group G denoted by $\Gamma_{sol}(G)$ was introduced by Ma et. all in [?] in the year 2014. The graph $\Gamma_{sol}(G)$ has vertex set as elements of the non-solvable group G and any two vertices σ_i and σ_j are adjacent in $\Gamma_{sol}(G)$ if and only if $\langle\sigma_i,\sigma_j\rangle \leq_{Sol}$ is solvable subgroup of G. In this paper we take the generalizer of non-solvable group of type $C_p \times A_5$ it is will-known the A_5 is smallest non-solvable group, thus $C_p \times A_5$ is non-solvable group. It is clear that if group G is a solvable, then $\Gamma_{sol}(G) \cong K_{|G|}$ since for any two elements a,b of G the subgroup $\langle a,b\rangle$ is solvable in G.

In this paper, we consider a simple graph which is undirected, with no loops or multiple edges. Let Γ be a graph. We will denote by $V(\Gamma)$ and $E(\Gamma)$, the set of vertices and edges of Γ, respectively. The degree of a vertex $v \in V(\Gamma)$ is denoted by $deg(v)$, and it well-known that $deg(v) = |N(v)|$. The degree sequence of a graph with vertices $v_1,\cdots,v_n$ is $d = (deg(v_1),\cdots,deg(v_n))$. Every graph with the degree sequence d is a realization of d. A degree sequence is unigraphic if all its realizations are isomorphic. We can present it by $\Delta(\Gamma) = \begin{pmatrix} n_1 & n_2 & \cdots & n_s \\ \mu(n_1) & \mu(n_2) & \cdots & \mu(n_s) \end{pmatrix}$, where n_i are degree vertices and $\mu(n_i)$

* speaker

are multiplicities. The split graph is a graph in which the vertices can be partitioned into a clique and an independent set.

Suppose that g an element of group G, the solvabilizer of g define by $\{y \in G \mid \langle g,y\rangle\} \leq_{sol}$ in G and denoted by $Sol_G(g)$ and the centralizer of g is given by $Cent_G(g) = \{y \in G \mid gy = yg\}$ where $Cent_G(g) \subset Sol_G(g)$ and $|Sol_G(g)|$ divided $Cent_G(g)$ for each $g \in G$ for more see [1, 2]. It is clear that is not necessarily a subgroup of G. It is easy to see that $Sol(G) = \{(u,v) \in G \times G, \langle u,v\rangle \leq_{sol} G\} = \bigcup_{\forall u \in G} Sol_G(u)$. Also, $Sol(G)$ is the solvable radical of G (see [3]).

Let G be a finite non-solvable group. Then the probability that a randomly chosen pair of elements of G generates a solvable group is defined by:

$$P_{sol}(G) = \frac{|\{(g,y) \in G \times G \mid \langle g,y\rangle \leq_{sol} G\}|}{|G|^2}.$$

Note that $P_{sol}(G)$ is the probability that a randomly chosen pair of elements of G generates a solvable group (see [4, 5]).

We can present the conjugate definition using the conjugacy class $Cl_G(g)$, as follows:

$$\begin{aligned} |Sol(G)| &= |\{(u,v) \in G \times G \mid \langle u,v\rangle \leq_{sol} G\}| \\ &= \bigcup_{\forall u \in G} |\{v \in G \mid \langle u,v\rangle \leq_{sol} G\}| \\ &= \sum |cl_G(u)||sol_G(u)| \end{aligned}$$

We introduce in this paper, some important relations between the solvable graph $\Gamma_{sol}(G)$ of G and the probability that a randomly chosen pair of elements of G generates a solvable group $P_{sol}(G)$.

2. Main Results

Proposition 2.1. *Suppose that the general element of A_5 is defined by $(abcde)$. Then the solvability degree of elements are given as follows: Let $G \cong A_5$, the solvability degree is given by:*

1. $Sol_{A_5}(e) = \{g \mid \forall g \in A_5\}$
2.

$$Sol_{A_5}((ab)(cd)) = \begin{cases} g & \#(g) \\ Identity & 1 \\ (bc)(de),(bd)(ce),(be)(cd),(ab)(ce), & \\ (ab)(cd),(ab)(ce),(ac)(de),(ac)(bd), & \\ (ac)(be),(ad)(ce),(ad)(bc),(ad)(be), & \\ (ae)(cd),(ae)(bc),(ae)(bd) & 15 \\ (abc)^{\pm},(abd)^{\pm},(abe)^{\pm},(cda)^{\pm},(cdb)^{\pm},(cde)^{\pm} & 12 \\ (abced)^{\pm},(abdec)^{\pm},(acdbe)^{\pm},(aebcd)^{\pm} & 8 \end{cases}$$

3. $Sol_{A_5}((abc)) = \begin{cases} g & \#(g) \\ Identity & 1 \\ (ab)(ij)_{i<j, i\neq j\neq a,b}, (ac)(ij)_{i<j, i\neq j\neq a,b} (bc)(ij)_{i<j, i\neq j\neq a,b} & 9 \\ (abi)^{\pm}_{i=c,d,e}, (acj)^{\pm}_{j=b,e}, (bcj)^{\pm}_{j=d,e} & 14 \end{cases}$

4. $Sol_{A_5}((abcde)) = \begin{cases} g & \#(g) \\ Identity & 1 \\ (be)(cd), (ab)(ce), (ac)(de), (ad)(bc), (ae)(bd) & 5 \\ (abcde), (acebd), (adbec), (aedcb) & 4 \end{cases}$

5. $Sol_{A_5}((abced)) = \begin{cases} g & \#(g) \\ Identity & 1 \\ (bd)(ce), (ab)(cd), (ac)(de), (ad)(bc), (ae)(bc) & 5 \\ (abced), (acdbe), (adecb), (aebdc) & 4 \end{cases}$

Corollary 2.2. *1. If $g = e$, then $|Sol_{A_5}(e)| = 60$*
2. If $g = (ab)(cd)$, then $|Sol_{A_5}((ab)(cd))| = 36$
3. If $g = (abc)$, then $|Sol_{A_5}((abc))| = 24$
4. If $g = (abcde)$, then $|Sol_{A_5}((abcde))| = 10$
5. If $g = (abced)$, then $|Sol_{A_5}((abced))| = 10$

Proposition 2.3. *The following hold:*

$$Con_{A_5}(e) = \{g \mid \forall g \in A_5\}$$

$$Con_{A_5}((ab)(cd)) = \begin{cases} g & \#(g) \\ (bc)(de), (bd)(ce), (be)(cd), (ab)(ce), & \\ (ab)(cd), (ab)(ce), (ac)(de), (ac)(bd), & \\ (ac)(be), (ad)(ce), (ad)(bc), (ad)(be), & \\ (ae)(cd), (ae)(bc), (ae)(bd) & 15 \end{cases}$$

$$Con_{A_5}((abc)) = \begin{cases} g & \#(g) \\ (cde), (ced), (bcd), (bce), (bdc), (bde), (bec), (bed), & \\ (abc), (abd), (abe), (acb), (acd), (ace), (adb), (adc), & \\ (ade), (aeb), (aec), (aed) & 20 \end{cases}$$

$$Con_{A_5}((abcde)) = \begin{cases} g & \#(g) \\ (abcde), (abdec), (abecd), (acedb), (acbed), (acdbe), & \\ (adceb), (adebc), (adbce), (aedcb), (aebdc), (aecbd) & 12 \end{cases}$$

$$Con_{A_5}((abced)) = \begin{cases} g & \#(g) \\ (abced), (abdce), (abedc), (acdeb), (acbde), (acebd), & \\ (adecb), (adbec), (adcbe), (aecdb), (aedbc), (aebcd) & 12 \end{cases}$$

Corollary 2.4. *The following held:*
1. If $g = e$, then $|Con_{A_5}(e)| = 1$
2. If $g = (ab)(cd)$, then $|Con_{A_5}((ab)(cd))| = 15$
3. If $g = (abc)$, then $|Con_{A_5}((abc))| = 20$
4. If $g = (abcde)$, then $|Con_{A_5}((abcde))| = 12$

5. *If* $g = (abced)$*, then* $|Con_{A_5}((abced))| = 12$

Corollary 2.5. *The following are obtained for* A_5*:*

$typesofelement$	order	ConjugacyClass(y)	Size	Sol(y)
C_1	1	$()$	1	60
C_2	2	$(ab)(cd)$	15	36
C_3	3	(abc)	20	24
C_5	5	$(abcde)$	12	10
D_{10}	10	$(abced)$	12	10

Theorem 2.6.

$$P_{sol}(G) = \frac{2|E(\Gamma_{sol}(G))|}{|G|^2}$$

PROOF. In the first, the parameters solvablitiy degree is define by $P_{sol}(G) = \frac{\{(u,v) \in G \times G, \langle u,v \rangle \leq_{sol} G\}}{|G|^2}$, Let $|G| = n$, suppose that u_i and u_j are elements in G and $cl_G(u_i)$ where $1 \leq i \leq r$, we can used this definition by

$$\begin{aligned}
P_{sol}(G) &= \frac{|\{(u_i,u_j) \in G \times G, \langle u_i,u_j \rangle \leq_{sol} G\}|}{|G|^2} \\
&= \frac{|Sol_G(u_1) \cup Sol_G(u_2) \cup \cdots \cup Sol_G(u_n)|}{|G|^2} \\
&= \frac{|Sol_G(u_1)| + |Sol_G(u_2)| + \cdots + |Sol_G(u_n)|}{|G|^2} \\
&= \frac{|cl_G(u_1)||Sol_G(u_1)| + |cl_G(u_2)||Sol_G(u_2)| + \cdots + |cl_G(u_r)||Sol_G(u_r)|}{|G|^2} \\
&= \frac{\sum_{1 \leq i \leq r} |cl_G(u_i)||Sol_G(u_i)|}{|G|^2} = \frac{2|E(\Gamma_{sol}(G))|}{|G|^2}.
\end{aligned}$$

□

Proposition 2.7. *The matrix degree sequences of solvable graph is given by:*

$$\Delta(\Gamma_{sol}(A_5)) = \begin{pmatrix} 59 & 35 & 23 & 9 \\ 1 & 15 & 20 & 24 \end{pmatrix}$$

Proposition 2.8. *The number of edges of solvable graph is given by:*

$$E(\Gamma_{sol}(A_5)) = 630$$

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۱۸ تا ۱۹ اسفند ۱۴۰۰

نتایجی روی همریختی‌ها و مدول‌های تقریباً یکدست

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چکیده. در این مقاله مدول‌های تقریباً یکدست را مورد بررسی قرار می‌دهیم. فرض می‌کنیم $(R, \mathfrak{m})$ یک حلقه موضعی نوتری از بعد d، T یک جبر جابجایی اکید با عنصر یکه 1_T روی R باشدکه $mT \neq T$. دنباله‌های تقریباً دقیق از T-مدول‌ها را تعریف کرده و T-مدول‌های تقریباً یکدست را تعریف می‌کنیم.همچنین، همریختی‌های تقریباً یکدست صادق را بین R-جبرهای T و W تعریف می‌کنیم که در اینجا w دارای ویژگی‌های مشابه به T به عنوان R-جبر است و نتایجی در رابطه با مدول‌های تقریباً یکدست و همریختی‌های تقریباً یکدست ارائه می‌دهیم.

واژه‌های کلیدی: حلقه‌های تقریبی، مدول‌های تقریباً یکدست، همریختی‌های تقریباً یکدست .
طبقه‌بندی موضوعی [۲۰۱۰]: (۱ تا ۳ مورد) 43A60, 43A22.

۱. پیش‌گفتار

فرض کنیم $(R, \mathfrak{m})$ یک حلقه موضعی نوتري از بعد d با دستگاه پارامترهاي $\underline{x} := x_1, \ldots, x_d$ باشد. حدسیه تک‌جمله‌اي هوچستر[1] بیان می‌کند که براي هر $t \geq 0$، داریم $x_1^t \cdots x_d^t \notin (x_1^{t+1} \cdots x_d^{t+1})$. حدسیه تک‌جمله‌اي براي تمام حلقه‌هاي موضعی با مشخصه‌هاي یکسان و براي تمام حلقه‌هاي موضعی با بعد حداکثر ۳ برقرار است. مطالعات روي این حدسیه باعث پیدایش مفهوم جدیدي تحت عنوان نظریه حلقه‌هاي تقریبی شد که اطلاعات لازم را می‌توان در [۱]پیدا کرد. نتایجی مهم و قابل توجه را در رابطه با حلقه‌هاي تقریبا کوهن-مکالی را می‌توان در [۴] یافت. فرض کنیم T یک جبر روي حلقه R با ویژگی‌هاي یاد شده بالا باشد که مجهز به یک نگاشت تحت عنوان نگاشت ارزیابی باشد. تقریبا کوهن-مکالی بودن T در [۷] تعریف شد وبعد از آن نتایجی در رابطه با جبرهاي تقریبا کوهن-مکالی روي حلقه موضعی R در [۱، ۸] ارائه شده‌اند.

۲. تعاریف

قضیه ۱.۲. [۹، قضایای ۲.۳۱ و ۲.۵۶] فرض کنیم M یک R-مدول و $\{M_i\}_{i\in I}$ خانواده‌اي ناتهی از R-مدول‌ها باشد. دراین‌صورت عبارت‌هاي زیر برقرار هستند:

(۱) $\mathrm{Hom}_R\left(\bigoplus_{i\in I} M_i, M\right) \cong \prod_{i\in I} \mathrm{Hom}_R\left(M_i, M\right)$.

(۲) $M \otimes_R \left(\bigoplus_{i\in I} M_i\right) \cong \bigoplus_{i\in I}(M \otimes_R M_i)$.

تعریف ۲.۲. فرض کنیم M یک R-مدول باشد. گوییم M یک R-مدول یکدست است اگر $M \otimes_R \square$ یک تابعگون دقیق باشد، یعنی اگر

$$0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$$

یک دنباله دقیق از R-مدول‌ها باشد، دراین‌صورت

$$0 \longrightarrow M \otimes_R N' \overset{\mathrm{id}_M \otimes f}{\longrightarrow} M \otimes_R N \overset{\mathrm{id}_M \otimes g}{\longrightarrow} M \otimes_R N'' \longrightarrow 0$$

دنباله‌اي دقیق باشد.

گزاره ۳.۲. [۹، گزاره ۳.۶۴] فرض کنیم R یک حلقه جابجایی دلخواه (نه لزوما نوتري) باشد. دراین‌صورت عبارت‌هاي زیر برقرار هستند:

(۱) R یک R-مدول یکدست است.
(۲) حاصل‌جمع مستقیم $\bigoplus_{i\in I} M_i$ از R-مدول‌ها یکدست است اگروتنهااگر هر M_i یکدست باشد.
(۳) هر R-مدول تصویري P، یک R-مدول یکدست است.

گزاره ۴.۲. [۹، گزاره ۳.۸۴] اگر هر زیرمدول با تولید متناهی از R-مدول M یکدست باشد، آن‌گاه M یکدست است.

* سخنران

[1] Hochster

توجه کنید که اگر درگزاره ۳.۲، R نوتري باشد، آنگاه عکس قسمت (۳) براي هر R-مدول با توليد متناهی نيز برقرار است يعنی هر R-مدول با توليد متناهی Mیکدست است اگروتنهااگر تصويري باشد [۹، نتيجه ۳.۷۵].

تعریف ۵.۲. فرض کنیم M یک R-مدول باشد. دراین‌صورت تکیه‌گاه M را با نماد $\mathrm{Supp}_R(M)$ نمایش داده و به‌صورت زیر تعریف می‌شود:

$$\mathrm{Supp}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) : M_{\mathfrak{p}} \neq ۰\}.$$

تعریف ۶.۲. فرض کنیم M یک R-مدول باشد. دراین‌صورت پوچساز M را با نماد $\mathrm{Ann}_R(M)$ نمایش داده و به‌صورت زیر تعریف می‌شود:

$$\mathrm{Ann}_R(M) = \{r \in R : rm = ۰,\ \text{برای هر } m \in M\}.$$

تعریف ۷.۲. فرض کنیم X یک R-مدول باشد. دراین‌صورت مجموعه ایده‌آل‌های اول وابسته به X را با نماد $\mathrm{Ass}_(X)$ (با تاکید بر حلقه R با $Ass_R(X)$) نشان داده و به‌صورت زیر تعریف می‌کنیم:

$$\mathrm{Ass}_R(X) = \{p \in Spec(R) : \exists x \in X \setminus \{۰\}\ s \cdot t \cdot\ p = (۰ :_K x)\}.$$

قضیه ۸.۲. فرض کنیم M یک R-مدول باشد. دراین‌صورت
(۱) $\mathrm{Ass}_R(M) \subseteq \mathrm{Supp}_R(M)$.
(۲) اگر M با تولید متناهی باشد، آنگاه $\mathrm{Ass}_R(M)$ مجموعه‌اي متناهی است.
(۳) مجموعه عناصر مینیمال $\mathrm{Ass}_R(M)$ و $\mathrm{Supp}_R(M)$ یکسان هستند.

تعریف ۹.۲. هرگاه $f : R \to S$ یک همریختی حلقه‌های تعویض‌پذیر باشد و $s_۱, s_۲, \cdots, s_n \in S$، آنگاه $R[x_۱, x_۲, \cdots, x_n] \to S$ داده شده با نگاشت $f \longmapsto \phi f(s_۱, \cdots, s_n)$ یک همریختی حلقه‌ها می‌باشد. نگاشت $R[x_۱, \cdots, x_n] \to S$ نگاشت ارزیابی یا همریختی جانشانی نام دارد.

۳. نتایجی روي همریختی‌ها و مدول‌هاي تقریبا یکدست

در این مقاله، نتایجی را در رابطه با مدول‌هاي تقریبا یکدست و همریختی‌هاي تقریبا یکدست ارائه می‌دهیم. همچنین، همریختی‌هاي تقریبا یکدست و سپس T-همریختی‌هاي تقریبا یکدست صادق را بین T-مدول‌ها تعریف می‌کنیم و نتایجی درباره آن‌ها بیان می‌کنیم.

نکته ۱.۳. فرض کنیم M و N، T-مدول باشند. اگر $M \approx N$، آنگاه یک T-همریختی مانند $f : M \longrightarrow N$ یا یک T-همریختی مانند $g : N \longrightarrow M$ وجود دارد که $\ker f \approx ۰$ و $\mathrm{coker}\ f \approx ۰$ یا $\ker g \approx ۰$ و $\mathrm{coker}\ g \approx ۰$. در این رساله منظور از $M \approx N$، یعنی T-همریختی‌های $f : M \longrightarrow N$ و $g : N \longrightarrow M$ وجود دارند که $f \circ g \approx i_N$ و $g \circ f \approx i_M$.

تعریف ۲.۳. دنباله متناهی یا نامتناهی از T همریختی‌ها و T-مدول‌های

$$\cdots \longrightarrow M_{n-۱} \xrightarrow{\varphi_n} M_n \xrightarrow{\varphi_{n+۱}} M_{n+۱} \longrightarrow \cdots,$$

را تقریبا دقیق گوییم هرگاه $\mathrm{im}\varphi_n$ و $\ker \varphi_{n+۱}$ تقریبا یکریخت باشند.

تعریف ۳.۳. گوییم T-مدول M تقریبا یکدست است، هرگاه

$$۰ \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow ۰ \qquad (۱)$$

یک دنباله تقریبا دقیق از T-مدول‌ها باشد، آنگاه

$$۰ \longrightarrow M \otimes_T A \xrightarrow{i_M \otimes \varphi} M \otimes_T B \xrightarrow{i_M \otimes \psi} M \otimes_T C \longrightarrow ۰ \qquad (۲)$$

یک دنباله تقریبا دقیق باشد. T-مدول M را یک T-مدول تقریبا یکدست صادق گوییم، هرگاه تقریبا دقیق بودن دنباله (۱) معادل با تقریبا دقیق بودن دنباله (۲) باشد.

لم ۴.۳. فرض کنیم I، J دو ایده‌آل از T و M یک T-مدول تقریبا یکدست باشد. دراین‌صورت $IM \cap JM \approx (I \cap J)M$.

اثبات. دنباله دقیق $۰ \longrightarrow I \cap J \longrightarrow T \longrightarrow T/I \oplus T/J \longrightarrow ۰$ را در نظر می‌گیریم. چون M تقریبا یکدست است، بنابراین دنباله تقریبا دقیق از T-مدول‌های زیر را داریم:

$$۰ \longrightarrow M \otimes_T (I \cap J) \longrightarrow M \otimes_T T = M \longrightarrow M/IM \oplus M/JM \longrightarrow ۰.$$

هسته $g : M \longrightarrow M/IM \oplus M/JM$ برابر با $IM \cap JM$ و برد $f : M \otimes_T (I \cap J) \longrightarrow M$ برابر با $(I \cap J)M$ است. چون $\mathrm{im}\ f \approx \ker g$، پس داریم $IM \cap JM \approx (I \cap J)M$. □

تعریف ۵.۳. فرض کنیم T و W دو جبر روی R باشند که مجهز به نگاشت‌های ارزیابی v_T و v_W هستند. گوییم نگاشت $\varphi : T \longrightarrow W$ تقریبا یکدست است، اگر W به عنوان T-مدول یک T-مدول تقریبا یکدست باشد. همچنین، گوییم φ تقریبا یکدست صادق است، اگر W یک T-مدول تقریبا یکدست صادق باشد.

گزاره ۶.۳. فرض کنیم T و W دو جبر با یکه‌های 1_T و 1_W روی R با نگاشت‌های ارزیابی v_T و v_W باشند. همچنین $\varphi : T \longrightarrow W$ یک نگاشت تقریبا یکدست باشد و M یک T-مدول با تولید متناهی باشد. دراین‌صورت $\mathrm{Ann}_T(M)W \approx \mathrm{Ann}_W(M \otimes_T W)$.

گزاره ۷.۳. فرض کنیم T و W دو جبر با یکه‌های 1_T و 1_W روی R با نگاشت‌های ارزیابی v_T و v_W باشند. همچنین $\varphi : T \longrightarrow W$ یک نگاشت تقریبا یکدست (صادق) باشد و M یک W-مدول تقریبا یکدست (صادق) باشد. دراین‌صورت M یک T-مدول تقریبا یکدست (صادق) است.

لم ۸.۳. فرض کنیم T، W دو جبر با یکه‌های به ترتیب 1_T، 1_W روی R با نگاشت‌های ارزیابی v_T، v_W و $\varphi : T \longrightarrow W$ یک همریختی بین حلقه‌ها باشد. اگر M یک T-مدول تقریبا یکدست (صادق) باشد، آنگاه $M \otimes_T W$ یک W-مدول تقریبا یکدست (صادق) است.

اثبات. فرض کنیم $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ یک دنباله تقریبا دقیق از W-مدول‌ها باشد. دراین‌صورت با تانسور کردن $M \otimes_T W$ به دنباله دقیق بالا، با استفاده از یکریختی $(M \otimes_T W) \otimes A = M \otimes_T A$ و تقریبا یکدست بودن M به عنوان T-مدول، دنباله تقریبا دقیق زیر را داریم:

$$0 \longrightarrow M \otimes_T A \overset{i_M \otimes f}{\longrightarrow} M \otimes_T B \overset{i_M \otimes g}{\longrightarrow} M \otimes_T C \longrightarrow 0.$$

بنابراین

$$0 \longrightarrow (M \otimes_T W) \otimes_W A \overset{i_M \otimes f}{\longrightarrow} (M \otimes_T W) \otimes_W B \overset{i_M \otimes g}{\longrightarrow} (M \otimes_T W) \otimes_W C \longrightarrow 0$$

یک دنباله تقریبا دقیق است و در نتیجه $M \otimes_T W$ یک W-مدول تقریبا یکدست است. با برگشت این استدلال، می‌بینیم که اگر M یک T-مدول تقریبا یکدست صادق باشد، آنگاه $M \otimes_T W$ یک W-مدول تقریبا یکدست صادق است. □

گزاره ۹.۳. فرض کنیم T، W دو جبر با یکه‌های به ترتیب 1_T، 1_W روی R با نگاشت‌های ارزیابی v_T، v_W و $\varphi : T \longrightarrow W$ یک همریختی بین حلقه‌ها باشد. اگر φ یک نگاشت یکدست صادق باشد، آنگاه M یک T-مدول تقریبا یکدست است اگروتنهااگر $M_W = M \otimes_T W$ یک W-مدول تقریبا یکدست باشد.

اثبات. اگر M یک T-مدول تقریبا یکدست باشد، آنگاه بنابر لم ۸.۳، M_W یک W-مدول تقریبا یکدست است.
برعکس، فرض کنیم M_W یک W-مدول تقریبا یکدست باشد و $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ یک دنباله تقریبا دقیق از T-مدول‌ها باشد. از تقریبا یکدست بودن φ و تقریبا یکدست بودن W به عنوان T-مدول، نتیجه می‌گیریم که دنباله زیر تقریبا دقیق است:

$$0 \longrightarrow W \otimes_T A \longrightarrow W \otimes_T B \longrightarrow W \otimes_T C \longrightarrow 0$$

تقریبا یکدست بودن M_W به عنوان W-مدول نتیجه می‌دهد که

$$0 \longrightarrow M_W \otimes_W W \otimes_T A \longrightarrow M_W \otimes_W W \otimes_T B \longrightarrow M_W \otimes_W W \otimes_T C \longrightarrow 0$$

یک دنباله تقریبا دقیق است. بنابراین می‌توانیم دنباله بالا را به‌صورت زیر بنویسیم که یک دنباله تقریبا دقیق است:

$$0 \longrightarrow W \otimes_T M \otimes_T A \longrightarrow W \otimes_T M \otimes_T B \longrightarrow W \otimes_T M \otimes_T C \longrightarrow 0,$$

با استفاده از تقریبا یکدست صادق بودن W به عنوان T-مدول، نتیجه می‌گیریم که دنباله زیر یک دنباله تقریبا دقیق از T-مدول‌ها است:

$$0 \longrightarrow M \otimes_T A \longrightarrow M \otimes_T B \longrightarrow M \otimes_T C \longrightarrow 0.$$

این یعنی M تقریبا یکدست است. □

قضیه ۱۰.۳. هرگاه $(P, \{x_i\})$ و $(Q, \{\psi_i\})$ هر دو حاصلضرب برای خانواده $\{A_i | \ i \in I\}$ از اشیا رسته e باشند، آنگاه P و Q هم ارزند (با تقریب یکریختی معادلند).

قضیه ۱۱.۳. فرض کنیم $\{G_i | \ i \in I\}$ خانواده‌ای از گروه‌ها بوده و $\{\phi_i : H \to G_i | \ i \in I\}$ خانواده‌ای از همریختی‌های گروه‌ها باشد. در این صورت همریختی منحصر بفردی مانند $\phi : H \to \prod_{i \in I} G_i$ هست به طوری که به ازای هر $i \in I$، $\prod_i \phi = \phi_i$ و این خاصیت $\prod_{i \in I} G_i$ را با تقریب یکریختی معین می‌کند.

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بررسی کدهای MRD روی حلقه‌های ایده‌آل اصلی متناهی با استفاده از گراف‌های ماتریسی

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چکیده. در این مقاله به بررسی کدهای MRD (ماکزیمم فاصله رتبه) روی حلقه ایده‌آل اصلی متناهی R می‌پردازیم. برای مطالعه کدهای MRD روی حلقه R از گراف‌های ماتریسی که راس‌های آن ماتریس‌های $m \times n$ روی R هستند، استفاده شده است. ابتدا ثابت می‌کنیم، مجموعه‌ی مستقل ماکسیمال گراف ماتریسی یک MRD کد است و در ادامه وجود کدهای خطیMRD را روی حلقه R نشان می‌دهیم.

واژه‌های کلیدی: حلقه ایده‌آل اصلی، گراف‌های ماتریسی، کدهای MRD .
طبقه‌بندی موضوعی [۲۰۱۰]: 05C25, 05C50, 94B60 .

۱. مقدمه

درسراسر این مقاله حلقه R را یک حلقه یکدار و جابجایی در نظر می‌گیریم.

ماتریس‌ها و رتبه ماتریس‌ها نقش مهمی در تئوری کدگذاری و تئوری گراف دارند. آنها برای مطالعه انواع گراف‌ها کاربرد بسیاری دارند.

گرافی که از رتبه‌ی ماتریس‌ها بدست می‌آید، گراف دو خطی یا گراف ماتریسی می‌نامیم، که رأس‌های آن ماتریس‌های $m \times n$ روی یک میدان متناهی می‌باشند و دو رأس A, B را مجاور گوییم اگر و تنها اگر $rank(A-B) = 1$ [۲]. گراف‌های $\Gamma_2(F_q^{m\times n})$، $\Gamma_2(Z_{p^s}^{m\times n})$ و $\Gamma_d(Z_{p^s}^{m\times n})$ گراف‌های دو خطی یا گراف‌های ماتریسی هستند که در [۶] و [۵] و [۴] مورد مطالعه قرار گرفته است. هوانگ و همکارانش [۱] گراف ماتریسی را روی حلقه Z_{p^s} مورد مطالعه قرار دادند. انها با تغییر در شرط مجاور بودن دو راس، گراف مذکور را تعمیم دادند. گراف تعمیم یافته دو خطی روی حلقه Z_{p^s} گرافی است که مجموعه‌ی راس‌های آن، مجموعه‌ای از ماتریس‌های $m \times n$ روی حلقه Z_{p^s} می‌باشد و دو راس A و B مجاورند، اگر و تنها اگر $rank(A-B) < d$ که m، n و d اعداد صحیح و مثبت و $2 \leq d \leq \min\{m,n\}$. این گراف تعمیم‌یافته دو خطی برای اثبات وجود کدهای MRD کاربرد دارد. دلسرت [۳] مجموعه‌ای از ماتریس‌ها را روی یک میدان متناهی به عنوان یک کد در نظر گرفت. او نشان داد این کدها، کرانی مشابه کران سینگلتون دارند، این کد را ماکزیمم فاصله رتبه (MRD) نامید که در تصحیح خطای کدها کاربرد دارد. مطالعات گسترده‌ای در مورد کدهای MRD روی میدان‌های متناهی و حلقه‌هایZ_{p^s} که از گراف تعمیم یافته‌ی دوخطی ناشی می‌شود، در [۴]و [۳]صورت گرفته است.

در این مقاله ما کدهای MDR را روی حلقه‌های متناهی ایده‌آل اصلی (PIR) با استفاده از گراف‌های تعمیم‌یافته روی حلقه Z_{p^s} مطالعه می‌کنیم و از مفهوم رتبه ماتریسی روی حلقه‌های جابجایی متناهی برای تعریف گراف ماتریسی استفاده می‌کنیم و ثابت می‌کنیم کدهای MRD، همان مجموعه‌های مستقل ماکسیمال، گراف دو خطی یا ماتریسی هستند و در پایان وجود کدهای MRD را روی حلقه ایده‌ال اصلی متناهی بررسی می‌کنیم.

۲. رتبه ماتریس‌ها

حلقه R را یک حلقه جابجایی و متناهی و R^* را مجموعه‌ی عناصر یکه در R در نظر می‌گیریم. مجموعه‌ای از ماتریس‌های $m \times n$ با درایه‌هایی از R را با $R^{m\times n}$ نشان می‌دهیم.

مطالعات گسترده‌ای از مفهوم رتبه ماتریس روی حلقه‌های جابجایی و تعمیم آن روی میدان‌های متناهی صورت گرفته‌است [۱]. رتبه‌ی یک ماتریس ناصفر مانند A در $R^{m\times n}$ را با $rankA$ نشان می‌دهیم، که برابر است با کوچکترین عدد صحیح مثبت t به‌طوریکه $A = BC$ که $B \in R^{m\times t}$ و $C \in R^{t\times n}$می‌باشند و داریم $rankA \leq min\{m,n\}$. حال فرض می‌کنیم R یک حلقه جابجایی و متناهی باشد در این صورت داریم: $R = R_1 \times R_2 \times \cdots \times R_l$ که R_i ها حلقه‌های موضعی متناهی با نگاشت تصویری $\rho_i : (r_1, r_2, \cdots, r_l) \to r_i$، برای هر $i \in \{1, 2, \cdots, l\}$ که حلقه‌هایی جابجایی دارای ایده‌ال ماکسیمال یکتایی مانند M هستند به یاد داریدکه اگر R یک حلقه موضعی با ایده‌ال ماکسیمال M باشد آنگاه $R^* = R - M$ و میدان $\dfrac{R}{M}$ را یک میدان خارج قسمتی

* سخنران

می‌نامیم، که دارای نگاشت طبیعی $\pi: R \to \dfrac{R}{M}$ باضابطه $\pi(R) = r + M$، برای هر $r \in R$ است. و در نهایت اگر $A \in R^{m\times n}$ آنگاه:

$rankA = \max\{rank\rho_i(A)\}_{1\leq i\leq l}$

۳. کدهای MRD

در این بخش به مطالعه کدهای MRD روی حلقه‌های ایده‌آل اصلی متناهی (PIR) می‌پردازیم، که PIR ها جابجایی در نظر گرفته شده است.
ابتدا مفاهیم مقدماتی کدهای ماتریسی و فاصله رتبه کدهای ماتریسی را گفته و سپس ثابت می‌کنیم که مجموعه‌های مستقل از گراف‌های ماتریسی، کدهای MRD هستند و در ادامه وجود کدهای خطی MRD روی یک PIR را با استفاده از کدهای MRD روی حاصل‌ضرب مستقیمی از میدان‌های متناهی، نشان می‌دهیم.
فرض کنیم R یک حلقه جابجایی و متناهی و کد (ماتریس) A از مرتبه $m \times n$ زیرمجموعه‌ای از C در $R^{m\times n}$ باشد. در این صورت فاصله رتبه بین دو ماتریس $A, B \in R^{m\times n}$ را که با $d_{rk}(A, B)$ نشان می‌دهیم برابر است با $rank(A - B)$. پس فاصله رتبه کد C از مرتبه $m \times n$ روی R را به صورت $d_{rk}(C) = \min\{d_{rk}(A, B) : A, B \in C,\ A \neq B\}$ تعریف می‌کنیم [۱] و [۲].
کد C از مرتبه $m \times n$ با فاصله رتبه d را یک $(m \times n, d)$–کد می‌نامیم. اگر $C \subseteq R^{m\times n}$ یک زیرمدولی از $R^{m\times n}$ روی R باشد، آنگاه C را یک کد خطی می‌نامیم. فرض کنیم $n \leq m$ و C یک $(m \times n, d)$–کد باشد، ماتریس A را در C به صورت، $A = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \cdots, \bar{x}_m)$ در نظر می‌گیریم که $\bar{x}_i \in R^n$ سطر i–ام ماتریس A می‌باشد، این بدان معناست که ما می‌توانیم $C \subseteq (R^n)^m$ را به عنوان یک کد به طول m روی یک مجموعه‌ی دلخواهی از R^n مطالعه و فاصله همینگ این کد را پیدا کنیم، که فاصله همینگ $d_H(C)$ با کران سینگلتون مطابقت دارد [۳]. داریم:

$$d_H(C) \leq m - \log_{|R|^n} |c| + 1 \Rightarrow |C| \leq |R|^{n(m-d_H(C)+1)}$$

بنابراین یک کد ماتریسی C از مرتبه‌ی $m \times n$ با فاصله رتبه‌ی $d_{rk}(C)$ روی میدان F_q، کرانی به شکل کران سینگلتون دارد که برابر است با :

$$|C| \leq q^{n(m-d_{rk}(C)+1)}$$

حال ما با استفاده از مجموعه‌های مستقل گراف‌های ماتریسی نشان می‌دهیم که کدهای ماتریسی روی حلقه‌های ایده‌آل اصلی متناهی و جابجایی،کرانی مشابه کران سینگلتون دارند. اگر R یک حلقه ایده‌آل اصلی متناهی و $C \subseteq R^{m\times n}$ باشد آنگاه C هم به عنوان یک کد ماتریسی و هم، مجموعه‌ای از رئوس یک گراف ماتریسی $\Gamma(R^{m\times n})$ است. علاوه بر آن اگر $d \geq 2$ آنگاه برای هر $A, B \in C$ و $A \neq B$ داریم:
$d_{rk}(A, B) = rank(A - B) \geq d$ اگر و تنها اگر راس‌های A و B در $\Gamma_d(R^{m\times n})$ مجاور نباشند.گزاره زیر بیانگر این مطلب است.

گزاره ۱.۳. اگر R یک حلقه متناهی ایده‌آل اصلی *(PIR)* و $2 \leq d \leq m \leq n$ باشد، کد $C \subseteq R^{m\times n}$، $d_{rk}(C) \geq d$ است اگر و تنها اگر C یک مجموعه‌ی مستقل، گراف $\Gamma_d(R^{m\times n})$ باشد.

قضیه ۲.۳. [۴] اگر R یک PIR متناهی و جابجایی باشد. آنگاه:

$$\alpha(\Gamma_d(R^{m\times n})) = |R|^{\max\{m,n\}(\min\{m,n\}-d+1)}$$

به α عدد استقلال گراف $\Gamma_d(R^{m\times n})$ می‌گوییم که مرتبه مجموعه مستقل ماکسیمال گراف ماتریسی $\Gamma_d(R^{m\times n})$ می‌باشد.

با توجه به گزاره ۱.۳ و عدد استقلال در قضیه ۲.۳ نتیجه می‌گیریم که اگر C کدی با $d_{rk}(C) = d$ و $d \geq 2$ باشد، آنگاه داریم:

$$|C| \leq \alpha(\Gamma_d(R^{m\times n})) = |R|^{n(m-d+1)}$$

و اگر $d_{rk}(C)=1$ باشد، داریم: $|C|\leq |R|^{nm}$.
پس ما کرانی به شکل کران سینگلتون برای کدهای ماتریسی روی حلقه‌های متناهی با ایده‌آل اصلی داریم که نتیجه زیر حاصل می‌شود.

نتیجه ۳.۳. اگر R یک حلقه متناهی ایده‌آل اصلی و $m\leq n$ باشد، برای یک کد $C\subseteq R^{m\times n}$ داریم:

$$|C|\leq |R|^{n(m-d_{rk}(C)+1)}.$$

یک $(m\times n, d)$-کد C را روی یک حلقه متناهی ایده‌آل اصلی، یک کد ماکزیمم فاصله رتبه (MRD *code*) می‌نامیم، اگر داشته باشیم:

$$|C|=|R|^{n(m-d+1)}.$$

با فرض اینکه حلقه R یک حلقه متناهی اصلی و $2\leq d\leq m\leq n$ و $C\subseteq R^{m\times n}$ باشد، اگر C یک مجموعه مستقل ماکسیمال از گراف $\Gamma_d(R^{m\times n})$ و هم یک $(m\times n,d)-MRD$ کد باشد آنگاه داریم: $|C|=|R|^{n(m-d+1)}=\alpha(\Gamma_d(R^{m\times n}))$.
از $|C|=|R|^{n(m-d+1)}$ و با توجه به نتیجه ۳.۳ که داریم $|C|\leq |R|^{n(m-d_{rk}(C)+1)}$، ثابت می‌شود: $d\geq d_{rk}(C)$
حال با توجه به گزاره ۱.۳ نتایج زیر را داریم:
C یک $(m\times n,d)-MRD$ کد است $\Leftrightarrow d_{rk}(C)=d$ و $|C|=|R|^{n(m-d+1)}\Leftrightarrow C$ یک مجموعه‌ی مستقل از گراف $\Gamma_d(R^{m\times n})$ است و $C|C|=|R|^{n(m-d+1)}\Leftrightarrow$ یک مجموعه‌ی مستقل ماکسیمال گراف $\Gamma_d(R^{m\times n})$ است.
بنابراین ما نشان دادیم:

قضیه ۴.۳. اگر R یک حلقه متناهی ایده‌آل اصلی و $2\leq d\leq m\leq n$ و $C\subseteq R^{m\times n}$ باشد. آنگاه C یک $(m\times n,d)-MRD$ کد است، اگر و تنها اگر، C یک مجموعه‌ی مستقل ماکسیمال گراف $\Gamma_d(R^{m\times n})$ باشد.

حال کدهای MRD را روی حلقه‌های متناهی ایده‌آل اصلی، با استفاده از مجموعه‌های مستقل گراف‌ها بدست می‌آوریم.

گزاره ۵.۳. [۷] اگر R یک حلقه زنجیر متناهی با ایده‌آل ماکسیمال $M=R\theta$ و میدان خارج قسمتی F_q، پوچتوانی e و $V=\{v_1,v_2,\cdots,v_q\}$ یک مجموعه از نمایندهای همه همدسته‌های M در R باشد، آنگاه برای هر $r\in R$ می‌توان نوشت:

$$r=r_0+r_1\theta+r_2\theta^2+\cdots+r_{\theta-1}\theta^{\theta-1}\ \ : r_i\in V$$

گزاره ۶.۳. ۱- فرض کنید R یک حلقه زنجیر متناهی با ایده‌آل ماکسیمال $R\theta$ و پوچتوانی e باشد. اگر A یک مجموعه مستقل ماکسیمال از $\Gamma_d\left(\left(\frac{R}{R\theta}\right)^{m\times n}\right)$ باشد آنگاه:

$$I=A+A\theta+A\theta^2+\cdots+A\theta^{\theta-1}$$

یک مجموعه‌ی مستقل ماکسیمال از $\Gamma_d(R^{m\times n})$ است.
۲- فرض کنید R یک حلقه متناهی ایده‌آل اصلی باشد و $R=R_1\times R_2\times\cdots\times R_l$. اگر I_i یک مجموعه‌ی مستقل ماکسیمال از $\Gamma_d(R_i^{m\times n})$ برای هر $i\in\{1,2,\cdots,l\}$ باشد، آنگاه:

$$I=I_1\times I_2\times\cdots\times I_i=\{(A_1,A_2,\cdots,A_i):\ A_i\in I_i\}$$

یک مجموعه‌ی مستقل ماکسیمال از $\Gamma_d(R^{m\times n})$ است.

قضیه ۷.۳. اگر R یک حلقه متناهی ایده‌آل اصلی باشد و به صورت $R = R_۱ \times R_۲ \times \cdots \times R_l$ تجزیه شود، که R_i یک حلقه زنجیر متناهی با ایده‌آل ماکسیمال $R\theta_i$، پوچتوانی e_i و میدان خارج‌قسمتی F_{q_i} برای هر $i \in \{۱, ۲, \cdots, l\}$ باشند. برای هر m و n و d که $۲ \leq d \leq \min\{m, n\}$، یک $(m \times n, d) - MRD$ کد خطی روی R وجود دارد.
علاوه بر آن این $(m \times n, d) - MRD$ کد به صورت $C = C_۱ \times C_۲ \times \cdots \times C_l$ که هر C_i یک $(m \times n, d) - MRD$ کد خطی روی R_i است که به شکل $C_i = \bar{C}_i + \bar{C}_i\theta_i + \bar{C}_i\theta_i^۲ + \cdots + \bar{C}_i\theta_i^{e_i-۱}$ است که $\bar{C}_i$ یک $(m \times n, d) - MRD$ کد خطی روی F_{q_i} است.

اثبات. اگر m و n و d اعداد صحیح مثبت که $۲ \leq d \leq m \leq n$ و اگر R یک حلقه زنجیر متناهی با ایده‌آل ماکسیمال $R\theta$، پوچتوانی e_i و میدان خارج‌قسمتی $R/R\theta \cong F_q$ باشد، یک $(m \times n, d) - MRD$ کد خطی روی F_q وجود دارد [۳]. ما با استفاده از کد خطی MRD، $\bar{C}$ روی F_q یک کد خطی MRD روی R بدست می‌آوریم. که بخشی از نتایج مقاله است. با توجه به قضیه ۴.۳ $\bar{C}$ یک مجموعه‌ی مستقل گراف $\Gamma_d(R^{m\times n})$ است. از گزاره۶.۳(۱) داریم:

$$C := \bar{C} + \bar{C}\theta + \bar{C}\theta^۲ + \cdots + \bar{C}\theta^{e-۱} = \{A_۰ + A_۱\theta + A_۲\theta^۲ + \cdots + A_{e-۱}\theta^{e-۱} : A_i \in \bar{C}\}$$

یک مجموعه مستقل ماکسیمال گراف $\Gamma_d(R^{m\times n})$ است. از طرفی با توجه به قضیه ۴.۳ و گزاره ۵.۳، C یک $(m \times n, d) -$ MRD کد خطی روی R است. بنابراین $\bar{C}$ یک کد خطی روی F_d است. فرض کنید R یک PIR متناهی که به صورت $R = R_۱ \times R_۲ \times \cdots \times R_l$ است و R_i یک حلقه زنجیر متناهی باشد، آنگاه یک $(m \times n, d) - MRD$ کد خطی C_i روی R_i برای هر $i \in \{۱, ۲, \cdots, l\}$ وجود دارد. از قضیه ۴.۳ داریم، C_i یک مجموعه مستقل خطی از $\Gamma_d(R^{m\times n})$ است. با استفاده از گزاره ۶.۳(۲) داریم:

$$C = C_۱ \times C_۲ \times \cdots C_l = \{(A_۱, A_۲, \cdots, A_l) : A_i \in C_i\}$$

یک مجموعه مستقل ماکسیمال گراف $\Gamma_d(R^{m\times n})$ است و C_i یک $(m \times n, d) - MRD$ کد خطی روی R_i برای هر $i \in \{۱, ۲, \cdots, l\}$ می‌باشد. بنابراین C یک $(m \times n, d) - MRD$ کد خطی روی R است و اثبات تمام است. □

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مدول‌های کوهمولوژی موضعی غیرآرتینی از بعد صفر

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چکیده. اولین‌بار هارتشورن در پاسخ به حدس گروتندیک، مثالی یافت که نشان می‌داد مدول کوهمولوژی موضعی برای مدول متناهی‌مولد، روی حلقه موضعی و نوتری از بعد سوکل نامتناهی است. در این مقاله، شرایطی را بیان می‌کنیم که مدول‌های کوهمولوژی موضعی تحت این شرایط از بعد سوکل نامتناهی باشند.
واژه‌های کلیدی: بعد کوهمولوژیک، حلقه نوتری، کوهمولوژی موضعی، مدول آرتینی.
طبقه‌بندی موضوعی [۲۰۱۰]: 13D45, 14B15, 13E05.

۱. مقدمه

در سراسر این مقاله R یک حلقه جابجایی یکدار و نوتری است.

مدول‌های کوهمولوژی موضعی اولین بار توسط گروتندیک[1] در سال ۱۹۶۶ معرفی شد.

فرض کنیم M یک R-مدول باشد. برای هر $i \geq 0$، i-امین مدول کوهمولوژی موضعی M نسبت به ایده‌آل I به صورت زیر تعریف می‌شود:

$$H_I^i(M) \cong \varinjlim_{n\geq 1} \operatorname{Ext}_R^i(R/I^n, M).$$

یادآوری می‌کنیم، اگر $(R, \mathfrak{m}, k)$ حلقه موضعی باشد، آنگاه برای هر R-مدول L، سوکل L را با نماد $\operatorname{Soc}_R L$ نشان داده و به صورت زیر تعریف می‌کنیم:

$$\operatorname{Soc}_R L = (0 :_L \mathfrak{m}) \simeq \operatorname{Hom}_R(k, L),$$

که یک فضای k-برداری است.

همچنین یادآوری می‌کنیم، R-مدول ناصفر M، آرتینی است اگر و تنها اگر $\operatorname{Supp} M = \{\mathfrak{m}\}$ و سوکل M متناهی‌مولد باشد.

می‌دانیم که اگر $(R, \mathfrak{m}, k)$ حلقه موضعی و نوتری باشد، آنگاه برای هر R-مدول متناهی‌مولد M و هر $i \in \mathbb{N}_0$، R-مدول $H_{\mathfrak{m}}^i(M)$ آرتینی است. بنابراین R-مدول $\operatorname{Hom}_R(k, H_{\mathfrak{m}}^i(M))$ متناهی‌مولد است. با توجه به این مطلب، گروتندیک در [۲]، حدس زیر را مطرح کرد:

حدس: برای هر ایده‌آل I از حلقه نوتری R و هر R-مدول متناهی‌مولد M، R-مدول $\operatorname{Hom}_R(R/I, H_I^i(M))$ برای هر $i \in \mathbb{N}_0$، متناهی‌مولد است.

هارتشورن در [۳]، با بیان مثال نقض، حدس گروتندیک را رد کرد و نشان داد که این حدس حتی در حالت منظم موضعی بودن حلقه نیز صحیح نیست. این اولین مثالی است که نشان می‌دهد مدول کوهمولوژی موضعی برای مدول متناهی‌مولد، روی حلقه موضعی و نوتری از بعد سوکل نامتناهی است.

مثال هارتشورن: فرض کنیم k یک میدان، $R = k[[u, v]][x, y]$، $I = (x, y)R$، $P = (u, v, x, y)R$ و $f = ux + vy$.
در این صورت $\operatorname{Soc}_{R_P} H_{IR_P}^2(R_P/fR_P)$ از بعد نامتناهی است.

در سال ۲۰۰۴، مثال‌های مشابهی توسط مارلی و واسیلو در [۸]، ارائه شد. اما با این حال، نتایج کمی موجود است که نشان می‌دهد دقیقاً چه زمانی مدول کوهمولوژی موضعی برای یک مدول متناهی‌مولد روی یک حلقه موضعی و نوتری از بعد سوکل نامتناهی است.

هونیکه در [۴]، حدسی به صورت زیر مطرح کرد که برای هر ایده‌آل I در حلقه منظم موضعی $(R, \mathfrak{m})$، R-مدول $H_I^i(R)$ برای هر $i \geq 0$، از بعد سوکل متناهی است.

هونیکه و شارپ در [۵] و لیوبزنیک در [۶] و [۷]، نشان دادند این حدس برای هر حلقه منظم موضعی که شامل یک میدان است، برقرار می‌باشد. اما در حالت کلی، این حدس هنوز یک مساله باز در جبر موضعی است.

در این مقاله در ارتباط با مدول‌های کوهمولوژی موضعی از بعد سوکل نامتناهی، نتیجه زیر را نشان می‌دهیم:

قضیه ۱.۱. فرض کنیم $(R, \mathfrak{m}, k)$ حلقه موضعی و نوتری از بعد $d \geq 4$ باشد. فرض کنیم $2 \leq i \leq d-2$ عدد صحیح و $x_1, \cdots, x_i$ قسمتی از دستگاه پارامتری R باشد. همچنین فرض می‌کنیم Υ_i مجموعه همه ایده‌آل‌های اول $\mathfrak{p}$ از R باشد به‌طوریکه

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[1] Grothendieck

$\dim R/\mathfrak{p} = i+1$، $\operatorname{Supp} H^i_{(x_1,\cdots,x_i)R}(R/\mathfrak{p}) = \{\mathfrak{m}\}$ و $\dim_k \operatorname{Soc}_R H^i_{(x_1,\cdots,x_i)R}(R/\mathfrak{p}) = \infty$. در این صورت Υ_i یک مجموعه نامتناهی است.

برای هر ایده‌آل I از R، $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq I\}$ را با $V(I)$ نشان می‌دهیم. برای مفاهیم و اصطلاحاتی که در این مقاله استفاده شده است، خواننده می‌تواند به مرجع [۱] مراجعه نماید.

۲. نتایج اصلی

فرض کنیم $(R, \mathfrak{m}, k)$ حلقه موضعی باشد. در این صورت برای هر R-مدول L، سوکل L را به صورت زیر تعریف می‌کنیم:

$$\operatorname{Soc}_R L = (0 :_L \mathfrak{m}) \simeq \operatorname{Hom}_R(k, L),$$

که یک فضای k-برداری است.

هدف ما در این مقاله اثبات قضیه ۱.۱ است اما قبل از اثبات باید تعاریف و لم‌های مورد نیاز را بیان کنیم.

تعریف ۱.۲. اگر I ایده‌آلی از حلقه R و M یک R-مدول باشد، آنگاه بعد کوهمولوژیک[2] M نسبت به I به صورت زیر تعریف می‌شود:

$$\operatorname{cd}(I, M) = \sup\{i \in \mathbb{N}_0 : H^i_I(M) \neq 0\}.$$

لم ۲.۲. فرض کنیم I ایده‌آلی از R و M و N، R-مدول‌های متناهی‌مولد باشند به‌طوریکه $\operatorname{Supp} N \subseteq \operatorname{Supp} M$. دراین‌صورت $\operatorname{cd}(I, N) \leq \operatorname{cd}(I, M)$.

تعریف ۳.۲. فرض کنیم M یک R-مدول متناهی‌مولد و I ایده‌آلی از R باشد. دراین‌صورت $q(I, M)$ را به صورت زیر تعریف می‌کنیم:

$$q(I, M) = \sup\{i \in \mathbb{N}_0 : \text{آرتینی نیست } H^i_I(M)\},$$

مشروط بر اینکه این مجموعه کوچکترین کران بالا داشته باشد در غیر این صورت، $-\infty$ تعریف می‌کنیم.

لم ۴.۲. فرض کنیم I ایده‌آلی از R و M و N، R-مدول‌های متناهی‌مولد باشند به‌طوریکه $\operatorname{Supp} N \subseteq \operatorname{Supp} M$. دراین‌صورت $q(I, N) \leq q(I, M)$.

لم ۵.۲. فرض کنیم I و J ایده‌آل‌های حلقه R باشند. فرض کنیم M یک R-مدول J-تابدار باشد. دراین‌صورت برای هر $i \in \mathbb{N}_0$، $H^i_{I+J}(M) \cong H^i_I(M)$.

لم ۶.۲. فرض کنیم R حلقه نوتری و I ایده‌آلی از R باشد بطوریکه I توسط t عضو تولید می‌شود. دراین‌صورت برای هر $i > t$ و هر R-مدول M، $H^i_I(M) = 0$.

گزاره زیر نقش اساسی در اثبات نتیجه اصلی دارد.

گزاره ۷.۲. فرض کنیم $(R, \mathfrak{m}, k)$ حلقه موضعی و نوتری از بعد $d \geq 4$ و $x_1, \cdots, x_d$ یک دستگاه پارامتری از R باشد. فرض کنیم $2 \leq i \leq d-2$ یک عدد صحیح باشد و قرار می‌دهیم $a := x_{i-1}x_{i+1} + x_i x_{i+2}$. در این صورت شرایط زیر برقرارند.

(۱) $\dim R/(a, x_{i+3}, \cdots, x_d)R = i+1$.

(۲) $\operatorname{cd}((x_1, \cdots, x_i)R, R/(a, x_{i+3}, \cdots, x_d)R) = i$.

(۳) $\operatorname{Supp} H^i_{(x_1,\cdots,x_i)R}(R/(a, x_{i+3}, \cdots, x_d)R) = \{\mathfrak{m}\}$.

(۴) $\dim_k \operatorname{Soc}_R H^i_{(x_1,\cdots,x_i)R}(R/(a, x_{i+3}, \cdots, x_d)R) = \infty$.

(۵) $\operatorname{q}((x_1, \cdots, x_i)R, R/(a, x_{i+3}, \cdots, x_d)R) = i$.

(۶) عضوی مانند $\mathfrak{p} \in \operatorname{Assh}_R R/(a, x_{i+3}, ..., x_d)R$ موجود است بطوریکه

$$\operatorname{Supp} H^i_{(x_1,\ldots,x_i)R}(R/\mathfrak{p}) = \{\mathfrak{m}\}$$

و $\dim_k \operatorname{Soc}_R H^i_{(x_1,\ldots,x_i)R}(R/\mathfrak{p}) = \infty$.

حال نتیجه اصلی را بیان می‌کنیم.

[2] Cohomological dimension

قضیه ۸.۲. فرض کنیم $(R,\mathfrak{m},k)$ حلقه موضعی و نوتری از بعد $d \geq ۴$ باشد. فرض کنیم $۲ \leq i \leq d-۲$ عدد صحیح و $x_۱$، $\cdots$، x_i قسمتی از دستگاه پارامتری R باشد. همچنین فرض می‌کنیم Υ_i مجموعه همه ایده‌آل‌های اول $\mathfrak{p}$ از R باشد بطوریکه $\dim R/\mathfrak{p} = i+۱$، $\operatorname{Supp} H^i_{(x_۱,\cdots,x_i)R}(R/\mathfrak{p}) = \{\mathfrak{m}\}$ و $\dim_k \operatorname{Soc}_R H^i_{(x_۱,\cdots,x_i)R}(R/\mathfrak{p}) = \infty$. در این صورت Υ_i یک مجموعه نامتناهی است.

اثبات. چون طبق فرض، $x_۱,\cdots,x_i$ قسمتی از یک دستگاه پارامتری حلقه R است پس می‌توانیم عناصر $x_{i+۱},x_{i+۲},...,x_d \in \mathfrak{m}$ را بیابیم بطوریکه $x_۱,...,x_d$ یک دستگاه پارامتری R باشد. قرار می‌دهیم $a := x_{i-۱}x_{i+۱} + x_i x_{i+۲}$. در این صورت طبق گزاره ۷.۲،

$$\dim R/(a,x_{i+۳},...,x_d)R = i+۱$$

و عضو $\mathfrak{p} \in \operatorname{Assh}_R R/(a,x_{i+۳},...,x_d)R$ موجود است بطوریکه $\mathfrak{p} \in \Upsilon_i$ لذا $\Upsilon_i \neq \emptyset$. حال ادعا می‌کنیم Υ_i یک مجموعه نامتناهی است. فرض کنیم چنین نباشد پس Υ_i یک مجموعه متناهی است. فرض کنیم $\Upsilon_i = \{Q_۱,...,Q_n\}$. دو حالت در نظر می‌گیریم:

حالت اول: فرض کنیم $۲ \leq i \leq d-۳$. عضوهای $y_{i+۱},y_{i+۲} \in \mathfrak{m}$ را چنان انتخاب می‌کنیم که $x_۱,...,x_i,y_{i+۱},y_{i+۲}$ قسمتی از دستگاه پارامتری R باشد. در این صوت طبق قضیه اجتناب از ایده‌آل‌های اول،

$$\mathfrak{m} \nsubseteq \left(\bigcup_{P \in \operatorname{Assh}_R R/(x_۱,...,x_i,y_{i+۱},y_{i+۲})R} P\right) \bigcup \left(\bigcup_{j=۱}^{n} Q_j\right).$$

بنابراین $y_{i+۳} \in \mathfrak{m}$ موجود است بطوریکه

$$y_{i+۳} \notin \left(\bigcup_{P \in \operatorname{Assh}_R R/(x_۱,...,x_i,y_{i+۱},y_{i+۲})R} P\right) \bigcup \left(\bigcup_{j=۱}^{n} Q_j\right).$$

واضح است $x_۱,...,x_i,y_{i+۱},y_{i+۲},y_{i+۳}$ قسمتی از دستگاه پارامتری R است. حال می‌توانیم عناصر $y_{i+۴},...,y_d \in \mathfrak{m}$ را بیابیم بطوریکه $x_۱,...,x_i,y_{i+۱},...,y_d$ یک دستگاه پارامتری R باشد. قرار می‌دهیم $b := x_{i-۱}y_{i+۱} + x_i y_{i+۲}$. در این صورت طبق گزاره ۷.۲،

$$\dim R/(b,y_{i+۳},...,y_d)R = i+۱.$$

عضو $\mathfrak{p} \in \operatorname{Assh}_R R/(b,y_{i+۳},...,y_d)R$ موجود است بطوریکه $\mathfrak{p} \in \Upsilon_i$. بنابراین $۱ \leq t \leq n$ موجود است بطوریکه $\mathfrak{p} = Q_t$. در نتیجه $y_{i+۳} \in Q_t$ که یک تناقض است.

حالت دوم: فرض کنیم $i = d-۲$. برای هر $۱ \leq j \leq n$، طبق لم ۵.۲ و از رابطه

$$H^{d-۲}_{(x_۱,...,x_{d-۲})R}(R/Q_j) \neq ۰,$$

نتیجه می‌شود $x_{d-۲} \notin Q_j$. از رشته دقیق کوتاه

$$۰ \longrightarrow R/Q_j \xrightarrow{x_{d-۲}} R/Q_j \longrightarrow R/(Q_j + Rx_{d-۲}) \longrightarrow ۰,$$

رشته دقیق زیر حاصل می‌شود:

$$H^{d-۳}_{(x_۱,...,x_{d-۲})R}(R/(Q_j+Rx_{d-۲})) \longrightarrow H^{d-۲}_{(x_۱,...,x_{d-۲})R}(R/Q_j) \xrightarrow{x_{d-۲}} H^{d-۲}_{(x_۱,...,x_{d-۲})R}(R/Q_j).$$

لذا رشته دقیق زیر را داریم:

$$H^{d-۳}_{(x_۱,...,x_{d-۲})R}(R/(Q_j+Rx_{d-۲})) \longrightarrow (۰ :_{H^{d-۲}_{(x_۱,...,x_{d-۲})R}(R/Q_j)} x_{d-۲}) \longrightarrow ۰.$$

علاوه بر این از رابطه

$$\operatorname{Supp} H^{d-۲}_{(x_۱,...,x_{d-۲})R}(R/Q_j) = \{\mathfrak{m}\},$$

نتیجه می‌شود

$$(۰ :_{H^{d-۲}_{(x_۱,...,x_{d-۲})R}(R/Q_j)} x_{d-۲}) \neq ۰.$$

بنابراین طبق لم ۵.۲،

$$H^{d-۳}_{(x_۱,...,x_{d-۲})R}(R/(Q_j+Rx_{d-۲})) \simeq H^{d-۳}_{(x_۱,...,x_{d-۳})R}(R/(Q_j+Rx_{d-۲})) \neq ۰.$$

لذا $\operatorname{cd}((x_۱,...,x_{d-۳})R, R/(Q_j+Rx_{d-۲})) \geq d-۳$. چون ایده‌آل $(x_۱,...,x_{d-۳})R$ توسط $d-۳$ عضو تولید می‌شود پس طبق لم ۶.۲،

$$\operatorname{cd}((x_۱,...,x_{d-۳})R, R/(Q_j+Rx_{d-۲})) \leq d-۳.$$

لذا $\operatorname{cd}((x_۱,...,x_{d-۳})R, R/(Q_j+Rx_{d-۲})) = d-۳$.

قرار می‌دهیم:

$$T_j := \bigoplus_{P\in \operatorname{mAss}_R R/(Q_j+Rx_{d-۲})} R/P.$$

لذا T_j یک R–مدول متناهی‌مولد است بطوریکه $\operatorname{Supp} T_j = \operatorname{Supp} R/(Q_j+Rx_{d-۲})$. بنابراین طبق لم ۲.۲،

$$\begin{aligned} d-۳ &= \operatorname{cd}((x_۱,...,x_{d-۳})R, R/(Q_j+Rx_{d-۲})) \\ &= \operatorname{cd}((x_۱,...,x_{d-۳})R, T_j) \\ &= \max\{\operatorname{cd}((x_۱,...,x_{d-۳})R, R/P) \ : \ P\in \operatorname{mAss}_R R/(Q_j+Rx_{d-۲})\}. \end{aligned}$$

لذا عضو $\mathfrak{q}_j \in \operatorname{mAss}_R R/(Q_j+Rx_{d-۲})$ موجود است بطوریکه

$$\operatorname{cd}((x_۱,...,x_{d-۳})R, R/\mathfrak{q}_j) = d-۳.$$

برای هر $۱\leq j\leq n$، طبق لم ۵.۲ و از رابطه

$$H^{d-۳}_{(x_۱,...,x_{d-۳})R}(R/\mathfrak{q}_j) \neq ۰,$$

نتیجه می‌شود $x_{d-۳}\notin \mathfrak{q}_j$ و $\mathfrak{q}_j\neq \mathfrak{m}$. طبق قضیه اجتناب از ایده‌آل‌های اول، می‌توان عضو $y_۱\in\mathfrak{m}$ را انتخاب کرد بطوریکه

$$y_۱ \notin \left(\bigcup_{P\in \operatorname{Assh}_R R/(x_۱,...,x_{d-۲})R} P\right) \bigcup \left(\bigcup_{j=۱}^{n} \mathfrak{q}_j\right).$$

واضح است که $x_۱,...,x_{d-۲},y_۱$ بخشی از دستگاه پارامتری R است. عضوی مانند $y_۲\in\mathfrak{m}$ را انتخاب می‌کنیم بطوریکه $x_۱,...,x_{d-۲},y_۱,y_۲$ یک دستگاه پارامتری برای R است. قرار می‌دهیم

$$c := x_{d-۳}y_۱ + x_{d-۲}y_۲.$$

طبق گزاره ۷.۲، عضوی مانند $\mathfrak{p}\in \operatorname{Assh}_R R/cR$ موجود است بطوریکه $\mathfrak{p}\in\Upsilon_{d-۲}$. بنابراین $۱\leq t\leq n$ موجود است بطوریکه $\mathfrak{p}=Q_t$. چون $\mathfrak{q}_t \subseteq Q_t \subseteq \mathfrak{p} \in c$ و $x_{d-۲}\in\mathfrak{q}_t$ لذا $x_{d-۳}y_۱\in\mathfrak{q}_t$. چون $x_{d-۳}\notin\mathfrak{q}_t$ پس $y_۱\in\mathfrak{q}_t$ که یک تناقض است. □

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مدول کوهمولوژی موضعی تعمیم یافته آرتینی با چند جمله ای هیلبرت-کیربی

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چکیده. فرض کنید $R=\oplus_{n\in\mathbb{N}_\circ}R_n$ یک حلقه همگن نوتری با ایده آل نامرتبط $R_+=\oplus_{n\in\mathbb{N}}R_n$ و حلقه پایه موضعی $(R_\circ,\mathfrak{m}_\circ)$ باشد و M و N دو R-مدول $\mathbb{Z}$-مدرج با تولید متناهی باشند. در این مقاله نشان می دهیم که اگر u کوچکترین عدد صحیح iای باشد که $H^i_{R_+}(M,N)$ آرتینی نیست و $\mathfrak{q}_\circ$ یک ایده آل $\mathfrak{m}_\circ$-اولیه از $R_\circ$ باشد آنگاه $H^u_{R_+}(M,N)/\mathfrak{q}_\circ H^u_{R_+}(M,N)$ آرتینی با چند جمله ای هیلبرت- کیربی از درجه کمتر از u است.

واژه‌های کلیدی: کوهمولوژی موضعی تعمیم یافته، ایده آل نامرتبط، آرتینی، چندجمله ای هیلبرت - کیربی.
طبقه‌بندی موضوعی [۲۰۱۰]: 13D45, 14B15.

۱. پیش‌گفتار

در این مقاله، فرض می‌کنیم $R=\oplus_{n\in\mathbb{N}_\circ}R_n$ یک حلقه همگن نوتری با حلقه پایه موضعی $(R_\circ,\mathfrak{m}_\circ)$ باشد که $R_\circ$ یک حلقه نوتری است؛ یعنی تعداد متناهی عضو $u_1,\dots,u_r\in R_1$ وجود دارد بطوریکه $R=R_\circ[u_1,\dots,u_r]=R_\circ[R_1]$. فرض کنید N,M دو R-مدول $\mathbb{Z}$-مدرج با تولید متناهی باشند. مشخص است که برای هر $i\in\mathbb{N}_\circ$ و هر ایده‌آل مدرج $\mathfrak{a}$ از R، i-امین مدول کوهمولوژی موضعی تعمیم یافته $H^i_\mathfrak{a}(M,N)=\varinjlim_{n\in\mathbb{N}} Ext^i_R\left(\frac{M}{\mathfrak{a}^nM},N\right)$ یک ساختار مدولی $\mathbb{Z}$-مدرج به‌صورت $H^i_\mathfrak{a}(M,N)=\oplus_{n\in\mathbb{Z}}H^i_\mathfrak{a}(M,N)_n$ دارد و هرگاه $\mathfrak{a}=R_+$ باشد، $R_\circ$-مدول $H^i_{R_+}(M,N)_n$ به‌ازای هر $n\in\mathbb{Z}$، با تولید متناهی است و برای هر $n\gg 0$ صفر است.
در این مقاله بحث آرتینی بودن کوهمولوژی موضعی تعمیم یافته در سطح $u_{R_+}(M,N)$ ادامه می‌دهیم. توجه کنید که اگر R یک حلقه نوتری و N,M دو R-مدول با تولید متناهی باشند، آنگاه

$$u:=u_{R_+}(M,N)=\inf\{i\mid H^i_{R_+}(M,N)\text{ is not Artinian}\}$$

دقت کنید که u یک عدد صحیح نامنفی است. یادآوری می‌شود که با فرض $M=R$، مدول کوهمولوژی موضعی معمولی $H^i_{R_+}(N)=H^i_{R_+}(M,N)$ و همینطور $u(N):=u_{R_+}(R,N)$ بدست می‌آید. در[۴، قضیه۱.۲] ، ثابت شده است که $H^n_{R_+}(N)/\mathfrak{m}_\circ H^n_{R_+}(N)$ آرتینی است که در آن $n=\dim(N/\mathfrak{m}_\circ N)$ کمترین عدد صحیح iای است که $H^i_{R_+}(N)$ غیرصفر است.

۲. نتایج اصلی

لم ۱.۲. فرض کنید R یک حلقه نوتری، M یک R-مدول و $\mathfrak{a}$ ایده‌آلی از R باشد. در اینصورت

(۱) اگر $\mathfrak{a}$ شامل یک عنصر نامقسوم‌علیه صفر روی M باشد، آنگاه M، $\mathfrak{a}$-آزاد تاب است؛ یعنی $\Gamma_\mathfrak{a}(M)=0$.
(۲) اگر M متناهی مولد باشد، آنگاه M، $\mathfrak{a}$-آزاد تاب است اگر و فقط اگر $\mathfrak{a}$ یک عنصر نامقسوم‌علیه صفر نسبت به M داشته باشد.

اثبات. [۱، لم۱.۱.۲] □

لم ۲.۲. فرض کنید R یک حلقه نوتری، $\mathfrak{a}$ یک ایده‌آل R و M یک R-مدول $\mathfrak{a}$-تاب باشد. در اینصورت یک تحلیل اینجکتیو برای M وجود دارد که هر جمله آن یک R-مدول $\mathfrak{a}$-تاب است.

اثبات. [۱، نتیجه ۶.۱.۲] □

قضیه ۳.۲. فرض کنید $M=\sum_{-\infty}^{+\infty}M_n$ یک $R[x_1,\dots,x_s]$-مدول مدرج باشد. در اینصورت

* سخنران

(۱) اگر $M \in \mathcal{N}_s$، آنگاه نگاشت $f_M : \mathbb{Z} \to G$ با ضابطه $f_M(n) = \varphi(M_n)$ یک تابع چندجمله‌ای از درجه حداکثر $s - ۱$ است.

(۲) اگر $M \in \mathcal{N}'_s$، آنگاه نگاشت $f'_M : \mathbb{Z} \to G$ با ضابطه $f'_M(n) = \varphi(M_{-n})$ یک تابع چندجمله‌ای از درجه حداکثر $s - ۱$ است.

اثبات. [۳، قضیه۲] □

لم ۴.۲. فرض کنید R یک حلقه نوتری، $\mathfrak{a}$ یک ایده‌آل R و M یک R-مدول با تولید متناهی باشد. در اینصورت

(۱) اگر N یک R-مدول $\mathfrak{a}$-تاب باشد، آنگاه برای هر $i \geq ۰$ داریم $H^i_{\mathfrak{a}}(M,N) \cong Ext^i_R(M,N)$. به‌علاوه اگر N با تولید متناهی باشد، آنگاه برای هر $i \geq ۰$، $H^i_{\mathfrak{a}}(M,N)$ نیز با تولید متناهی است.

(۲) اگر $pd(M)$ متناهی باشد، آنگاه برای هر $i > pd(M) + ara(\mathfrak{a})$، $H^i_{\mathfrak{a}}(M,N) = ۰$.

اثبات. [۵، لم۱.۲] □

گزاره ۵.۲. فرض کنید R یک حلقه مدرج همگن، $(R_۰, \mathfrak{m}_۰)$ موضعی و M و N یک R-مدول مدرج با تولید متناهی می‌باشد. فرض کنید $pd(M)$ متناهی باشد. در اینصورت

(۱) برای هر $i \in \mathbb{N}_۰$ و هر $n \in \mathbb{Z}$، $R_۰$-مدول $H^i_{R_+}(M,N)_n$ با تولید متناهی است.

(۲) $r \in \mathbb{Z}$ وجود دارد بطوریکه برای هر $i \in \mathbb{N}_۰$ و هر $n \geq r$، $H^i_{R_+}(M,N)_n = ۰$.

اثبات. [۵، گزاره۳.۲] □

لم ۶.۲. فرض کنید R یک حلقه مدرج همگن، M و N دو R-مدول با تولید متناهی و $R_۰$ موضعی با ایده‌آل ماکسیمال $\mathfrak{m}_۰$ باشد. در این‌صورت برای هر $i \geq ۰$، R-مدول $H^i_{R_+}(M, \Gamma_{\mathfrak{m}_۰ R}(N))$ آرتینی است.

اثبات. [۶، لم۳.۵] □

نکته ۷.۲. فرض کنید $R = \oplus_{n\in\mathbb{N}_۰} R_n$ یک حلقه مدرج همگن و $A = \oplus_{n\in\mathbb{N}_۰} A_n$ یک R-مدول مدرج باشد. در اینصورت

(۱) اگر $A = \oplus_{n\in\mathbb{Z}} A_n$ یک R-مدول آرتینی باشد بطوریکه برای هر $n \ll ۰$، $R_۰$-مدول A_n با تولید متناهی باشد، آنگاه $\ell_{R_۰}(A_n) < \infty$ برای هر $n \ll ۰$. با شرایط مشابه در قضیه ۳.۲، می‌بینیم که چندجمله‌ای منحصربه‌فرد $P_A(X) \in Q[X]$ از درجه حداکثر $\dim_{K_۰}(R_۱/\mathfrak{m}_۰ R_۱) - ۱$ وجود دارد بطوریکه $\ell_{R_۰}(A_n) = P_A(n)$ برای هر $n \ll ۰$. در این حالت می‌گوییم A یک R-مدول مدرج آرتینی با چندجمله‌ای هیلبرت-کیربی $P_A(X)$ است.

(۲) به آسانی دیده می‌شود اگر A یک R-مدول مدرج آرتینی با چندجمله‌ای هیلبرت-کیربی باشد، آنگاه $Tor^R_i(A,B)$ و $Ext^i_R(B,A)$ برای هر R-مدول مدرج متناهی مولد B و هر $i \in \mathbb{N}_۰$، R-مدول‌های مدرج آرتینی با چندجمله‌ای‌های هیلبرت-کیربی خواهند بود.

تعریف ۸.۲. فرض کنید R یک حلقه نوتری و M, N دو R-مدول با تولید متناهی باشند، تعریف میکنیم

$$u_{R_+}(M,N) = \inf\{i \mid H^i_{R_+}(M,N) \text{ is not Artinian}\}$$

لم ۹.۲. فرض کنید R یک حلقه مدرج همگن و M, N، R-مدول‌های مدرج با تولید متناهی باشند. اگر x یک عنصر N-منظم همگن باشد، آنگاه $u_{R_+}(M, N/xN) = u - ۱$.

اثبات. فرض کنید $\deg(x) = j$. از رشته دقیق کوتاه

$$۰ \longrightarrow N \xrightarrow{x} N(j) \longrightarrow \frac{N}{xN}(j) \longrightarrow ۰$$

از R-مدول‌های مدرج، رشته دقیق طولانی

$$H^i_{R_+}(M,N) \xrightarrow{x} H^i_{R_+}(M,N)(j) \longrightarrow H^i_{R_+}(M,N/xN)(j) \xrightarrow{\alpha_i} H^{i+۱}_{R_+}(M,N) \xrightarrow{x} H^{i+۱}_{R_+}(M,N)(j) \quad (۱)$$

به‌دست می‌آید. که در آن $H^i_{R_+}(M,N/xN)$ برای هر i، $i < u - ۱$ یک R-مدول آرتینی است. چون $H^u_{R_+}(M,N)$ آرتینی نیست بنابراین طبق قضیه ملکرسون [۱، قضیه۲.۱.۷]، $(۰ :_{H^u_{R_+}(M,N)} x)$ آرتینی نیست و در نتیجه $Im(\alpha_{u-۱})$، یک R-مدول آرتینی نیست. بنابراین از رشته دقیق طولانی (۱) نتیجه می‌شود $H^{u-۱}_{R_+}(M,N/xN)$ آرتینی نیست. لذا $u_{R_+}(M,N/xN) = u - ۱$ و اثبات تمام است. □

قضیه ۱۰.۲. فرض کنید R یک حلقه مدرج همگن و N, M دو R-مدول مدرج با تولید متناهی باشند. در اینصورت $H^u_{R_+}(M,N)/\mathfrak{q}_\circ H^u_{R_+}(M,N)$ یک R-مدول مدرج آرتینی با چندجمله‌ای هیلبرت-کیربی از درجه کمتر از u است.

اثبات. بدون کاستن از کلیت مسئله میتوان فرض کرد $\mathfrak{m}_\circ = \mathfrak{q}_\circ$. ادعا را با استقراء بر روی $d = \dim N$ اثبات خواهیم کرد. اگر $\dim N = \circ$ باشد، آنگاه N آرتینی است و بنابراین $\mathfrak{m}$-تاب است. طبق لم ۲.۲، یک تحلیل اینجکتیو N که تمام جملاتش $\mathfrak{m}$-تاب هستند وجود دارد. بنابر لم ۴.۲ داریم $H^u_{R_+}(M,N) = Ext^u_{R_+}(M,N)$ که بوضوح یک R-مدول آرتینی است.
بنابراین فرض می‌کنیم $d = \dim N > \circ$ و حکم برای تمامی R-مدول‌های مدرج با تولید متناهی از بُعد کمتر از d برقرار باشد. رشته دقیق کوتاه

$$\circ \longrightarrow \Gamma_{\mathfrak{m}_\circ R}(N) \longrightarrow N \longrightarrow \frac{N}{\Gamma_{\mathfrak{m}_\circ R}(N)} \longrightarrow \circ$$

رشته دقیق طولانی

$$\cdots \longrightarrow H^u_{R_+}(M,\Gamma_{\mathfrak{m}_\circ R}(N)) \xrightarrow{\alpha} H^u_{R_+}(M,N) \xrightarrow{\beta} H^u_{R_+}(M,N/\Gamma_{\mathfrak{m}_\circ R}(N)) \xrightarrow{\theta} H^{u+1}_{R_+}(M,\Gamma_{\mathfrak{m}_\circ R}(N)) \longrightarrow \cdots$$

را به‌دست می‌دهد. از رشته دقیق فوق رشته‌های دقیق کوتاه زیر بدست می‌آید.

$$\circ \longrightarrow im(\alpha) \longrightarrow H^u_{R_+}(M,N) \longrightarrow im(\beta) \longrightarrow \circ$$

و

$$\circ \longrightarrow im(\beta) \longrightarrow H^u_{R_+}(M,N/\Gamma_{\mathfrak{m}_\circ R}(N)) \longrightarrow im(\theta) \longrightarrow \circ$$

رشته‌های دقیق کوتاه فوق، رشته‌های دقیق زیر را القاء میکنند.

$$im(\alpha) \otimes_R \frac{R}{\mathfrak{m}_\circ R} \longrightarrow H^u_{R_+}(M,N) \otimes_R \frac{R}{\mathfrak{m}_\circ R} \longrightarrow im(\beta) \otimes_R \frac{R}{{}_\circ R} \longrightarrow \circ \qquad (۲)$$

و

$$Tor^1_R\left(im(\theta), \frac{R}{\mathfrak{m}_\circ R}\right) \longrightarrow im(\beta) \otimes_{R_\circ} \frac{R_\circ}{\mathfrak{m}_\circ} \longrightarrow H^u_{R_+}\left(M, \frac{N}{\Gamma_{\cdot R}(N)}\right) \otimes_{R_\circ} \frac{R_\circ}{\mathfrak{m}_\circ} \longrightarrow im(\nu) \otimes_{R_\circ} \frac{R_\circ}{\mathfrak{m}_\circ} \longrightarrow \circ \qquad (۳)$$

توجه کنید که برای هر $i \in \mathbb{N}_\circ$، طبق لم ۶.۲، $H^i_{R_+}(M,\Gamma_{\cdot R}(N))$ یک R-مدول آرتینی است. بنابراین هر دو $im(\alpha)$ و $im(\beta)$ آرتینی هستند. اگر $H^u_{R_+}(M,N)/\mathfrak{m}_\circ H^u_{R_+}(M,N)$ یک R-مدول آرتینی باشد، آنگاه از رشته دقیق (۲) نتیجه می‌شود $im(\beta) \otimes_R \dfrac{R}{\mathfrak{m}_\circ R}$ یک R-مدول آرتینی است و از رشته دقیق (۳) نتیجه می‌شود

$$H^u_{R_+}(M,N/\Gamma_{\mathfrak{m}_\circ R}(N))/\mathfrak{m}_\circ H^u_{R_+}(M,N/\Gamma_{\mathfrak{m}_\circ R}(N))$$

یک R-مدول آرتینی است.
به طور مشابه $H^u_{R_+}(M,N)/\mathfrak{m}_\circ H^u_{R_+}(M,N)$ آرتینی است اگر $H^u_{R_+}(M,N/\Gamma_{\mathfrak{m}_\circ R}(N))/\mathfrak{m}_\circ H^u_{R_+}(M,N/\Gamma_{\mathfrak{m}_\circ R}(N))$ یک R-مدول آرتینی باشد. همچنین به‌آسانی دیده می‌شود که $u = u_{R_+}(M,N/\Gamma_{\mathfrak{m}_\circ R}(N))$.
در نتیجه بدون کاستن از کلیت مسئله می‌توان فرض کرد $\Gamma_{\mathfrak{m}_\circ R}(N) = \circ$. در نتیجه، طبق لم ۱.۲، یک عنصر N-منظم $x \in \mathfrak{m}_\circ$ وجود دارد. حال از رشته دقیق کوتاه

$$\circ \longrightarrow N \xrightarrow{x} N \longrightarrow \frac{N}{xN} \longrightarrow \circ$$

رشته دقیق زیر بدست می‌آید.

$$\cdots \xrightarrow{x} H^{u-1}_{R_+}(M,N) \longrightarrow H^{u-1}_{R_+}(M,N/xM) \longrightarrow H^u_{R_+}(M,N) \xrightarrow{x} H^u_{R_+}(M,N), \qquad (۴)$$

که از آن رشته دقیق

$$\frac{H^{u-1}_{R_+}(M,N/xN)}{\mathfrak{m}_\circ H^{u-1}_{R_+}(M,N/xN)} \longrightarrow \frac{H^u_{R_+}(M,N)}{\mathfrak{m}_\circ H^u_{R_+}(M,N)} \xrightarrow{x} \frac{H^u_{R_+}(M,N)}{\mathfrak{m}_\circ H^u_{R_+}(M,N)} \qquad (۵)$$

به‌دست میاید.
چون $x \in \mathfrak{m}_\circ$، این رشته دقیق، اپی‌مرفیسم

$$H^{u-1}_{R_+}(M,N/xN)/\mathfrak{m}_\circ H^{u-1}_{R_+}(M,N/xN) \longrightarrow H^{u-1}_{R_+}(M,N)/\mathfrak{m}_\circ H^u_{R_+}(M,N) \longrightarrow \circ \qquad (۶)$$

را به‌دست می‌دهد. طبق لم ۹.۲، داریم $u_{R_+}(M, N/xN) = u - 1$، چون x یک عنصر N-منظم است بنابراین طبق فرض استقراء $H^{u-1}_{R_+}(M, N/xN)/\mathfrak{m}_0 H^{u}_{R_+}(M, N/xN)$ یک R-مدول آرتینی است. لذا از رشته دقیق (۶) نتیجه می‌شود که $H^{u}_{R_+}(M, N)/\mathfrak{m}_0 H^{u}_{R_+}(M, N)$ نیز آرتینی است. حال طبق قسمت (۱) تبصره ۷.۲ نتیجه می‌گیریم $H^{u}_{R_+}(M, N)/\mathfrak{q}_0 H^{u}_{R_+}(M, N)$ یک R-مدول آرتینی با چندجمله‌ای هیلبرت-کیربی است. با استقراء بر روی $\dim N$ اثبات می‌کنیم که درجه این چندجمله‌ای کمتر از u است. در حالتی که $\dim N = 0$ همچنانکه در ابتدای اثبات نشان دادیم $H^{u}_{R_+}(M, N) = Ext^{u}_{R}(M, N)$ و بنابراین در این حالت نتیجه بوضوح بدست می‌آید. پس فرض کنید $\dim N > 0$ و حکم برای تمام R-مدول‌های با بُعد کمتر از $\dim N$ برقرار باشد. در باقی اثبات، قرار می‌دهیم $U := H^{u-1}_{R_+}(M, N)/\mathfrak{m}_0 H^{u-1}_{R_+}(M, N)$ و $V := H^{u}_{R_+}(M, N)/\mathfrak{m}_0 H^{u}_{R_+}(M, N)$.

از اپی‌مرفیسم (۵) (با تانسور کردن رشته دقیق (۴) با $\frac{R_0}{\mathfrak{q}_0}$ و جایگذاری x با توانی که $x^n \in \mathfrak{q}_0$) و قسمت (۱) گزاره ۵.۲، نتیجه می‌گیریم

$$P_V(n) = \ell_{R_0}(V_n) \leq \ell_{R_0}(u_n) = P_U(N)$$

برای هر $n \gg 0$. این نتیجه می‌دهد $\deg P_V(X) \leq \deg P_V(X) \leq u - 1$ که در آن تساوی آخری طبق فرض استقراء برقرار است و حکم تمام است. □

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مواردی از مدولهای کوهمولوژی موضعی تعمیم یافته مدرج آرتینی

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چکیده. فرض کنید $R=\oplus_{n\in\mathbb{N}_0}R_n$ یک حلقه همگن نوتری با ایده آل نامرتبط $R_+=\oplus_{n\in\mathbb{N}}R_n$ و حلقه پایه موضعی $(R_0,\mathfrak{m}_0)$ باشد و M و N دو R-مدول $\mathbb{Z}$-مدرج با تولید متناهی باشند. در این مقاله نشان می دهیم R-مدول مدرج $\Gamma_{\mathfrak{m}_0R}(H^i_{R_+}(M,N))$ برای $0\leq i\leq g(M,N)$، آرتینی با چندجملهای هیلبرت-کیربی است. که در آن

$$g(M,N):=\inf\{i\in\mathbb{N}_0 \mid \#\{n\mid \ell_{R_0}(H^i_{R_+}(M,N)_n)=\infty\}=\infty\}.$$

واژه‌های کلیدی: کوهمولوژی موضعی تعمیم یافته، ایده آل نامرتبط، آرتینی، چندجمله ای هیلبرت - کیربی.
طبقه‌بندی موضوعی [۲۰۱۰]: 13D45, 14B15.

۱. پیش‌گفتار

در این مقاله، فرض می‌کنیم $R=\oplus_{n\in\mathbb{N}_0}R_n$ یک حلقه همگن نوتری با حلقه پایه موضعی $(R_0,\mathfrak{m}_0)$ باشد که R_0 یک حلقه نوتری است؛ یعنی تعداد متناهی عضو $u_1,\dots,u_r\in R_1$ وجود دارد بطوریکه $R=R_0[u_1,\dots,u_r]=R_0[R_1]$. فرض کنید N,M دو R-مدول $\mathbb{Z}$-مدرج با تولید متناهی باشند. مشخص است که برای هر $i\in\mathbb{N}_0$ و هر ایده‌آل مدرج $\mathfrak{a}$ از R،در i-امین مدول کوهمولوژی موضعی تعمیم یافته $H^i_{\mathfrak{a}}(M,N)=\varinjlim_{n\in\mathbb{N}} Ext^i_R\left(\frac{M}{\mathfrak{a}^nM},N\right)$ یک ساختار مدولی $\mathbb{Z}$-مدرج به‌صورت $H^i_{\mathfrak{a}}(M,N)=\oplus_{n\in\mathbb{Z}}H^i_{\mathfrak{a}}(M,N)_n$ دارد و هرگاه $\mathfrak{a}=R_+$ باشد، R_0-مدول $H^i_{R_+}(M,N)_n$ به‌ازای هر $n\in\mathbb{Z}$، با تولید متناهی است و برای هر $n\gg 0$ صفر است. در [۶، قضیه۳.۲] اثبات شده است، اگر $\dim R_0\leq 1$، آنگاه $\Gamma_{\mathfrak{m}_0R}(H^i_{R_+}(M,N))$ یک R-مدول آرتینی است.

در این مقاله نشان می‌دهیم بدون محدودیت روی $\dim(R_0)$، همان نتیجه، در حالتی که $0\leq i\leq g(M,N)$، برقرار است.

۲. نتایج اصلی

لم ۱.۲. فرض کنید R یک حلقه نوتری، M یک R-مدول و $\mathfrak{a}$ ایده‌آلی از R باشد. در اینصورت

(۱) اگر $\mathfrak{a}$ شامل یک عنصر نامقسوم‌علیه صفر روی M باشد، آنگاه M، $\mathfrak{a}$-آزاد تاب است؛ یعنی $\Gamma_{\mathfrak{a}}(M)=0$.

(۲) اگر M متناهی مولد باشد، آنگاه M، $\mathfrak{a}$-آزاد تاب است اگر و فقط اگر $\mathfrak{a}$ یک عنصر نامقسوم‌علیه صفر نسبت به M داشته باشد.

اثبات. [۱، لم۲.۱.۱۰] □

قضیه ۲.۲. فرض کنید $M=\sum_{-\infty}^{+\infty}M_n$ یک $R[x_1,\dots,x_s]$-مدول مدرج باشد. در اینصورت

(۱) اگر $M\in\mathcal{N}_s$، آنگاه نگاشت $f_M:\mathbb{Z}\to G$ با ضابطه $f_M(n)=\varphi(M_n)$ یک تابع چندجمله‌ای از درجه حداکثر $s-1$ است.

(۲) اگر $M\in\mathcal{N}'_s$، آنگاه نگاشت $f'_M:\mathbb{Z}\to G$ با ضابطه $f'_M(n)=\varphi(M_{-n})$ یک تابع چندجمله‌ای از درجه حداکثر $s-1$ است.

اثبات. [۳، قضیه۲] □

لم ۳.۲. فرض کنید R یک حلقه نوتری، $\mathfrak{a}$ یک ایده‌آل R و M یک R-مدول با تولید متناهی باشد. در اینصورت

(۱) اگر N یک R-مدول $\mathfrak{a}$-تاب باشد، آنگاه برای هر $i\geq 0$ داریم $H^i_{\mathfrak{a}}(M,N)\cong Ext^i_R(M,N)$. به‌علاوه اگر N با تولید متناهی باشد، آنگاه برای هر $i\geq 0$، $H^i_{\mathfrak{a}}(M,N)$ نیز با تولید متناهی است.

(۲) اگر $pd(M)$ متناهی باشد، آنگاه برای هر $i>pd(M)+ara(\mathfrak{a})$، $H^i_{\mathfrak{a}}(M,N)=0$.

* سخنران

اثبات. [۵، لم۲.۱۰] □

گزاره ۴.۲. فرض کنید R یک حلقه مدرج همگن، $(R_\circ, \mathfrak{m}_\circ)$ موضعی و M و N یک R-مدول مدرج با تولید متناهی می‌باشد. فرض کنید $pd(M)$ متناهی باشد. در اینصورت

(۱) برای هر $i \in \mathbb{N}_\circ$ و هر $n \in \mathbb{Z}$، $R_\circ$-مدول $H^i_{R_+}(M,N)_n$ با تولید متناهی است.

(۲) $r \in \mathbb{Z}$ وجود دارد بطوریکه برای هر $i \in \mathbb{N}_\circ$ و هر $n \geq r$، $H^i_{R_+}(M,N)_n = \circ$.

اثبات. [۵، گزاره۳.۲] □

نکته ۵.۲. فرض کنید $R = \oplus_{n\in\mathbb{N}_\circ} R_n$ یک حلقه مدرج همگن و $A = \oplus_{n\in\mathbb{N}_\circ} A_n$ یک R-مدول مدرج باشد. در اینصورت

(۱) اگر $A = \oplus_{n\in\mathbb{Z}} A_n$ یک R-مدول آرتینی باشد بطوریکه برای هر $n \ll \circ$، $R_\circ$-مدول A_n با تولید متناهی باشد، آنگاه $\ell_{R_\circ}(A_n) < \infty$ برای هر $n \ll \circ$. با شرایط مشابه در قضیه ۲.۲، می‌بینیم که چندجمله‌ای منحصربه‌فرد $P_A(X) \in Q[X]$ از درجه حداکثر $\dim_{K_\circ}(R_1/\mathfrak{m}_\circ R_1) - 1$ وجود دارد بطوریکه $\ell_{R_\circ}(A_n) = P_A(n)$ برای هر $n \ll \circ$. در این حالت می‌گوییم A یک R-مدول مدرج آرتینی با چندجمله‌ای هیلبرت-کیربی $P_A(X)$ است.

(۲) به آسانی دیده می‌شود اگر A یک R-مدول مدرج آرتینی با چندجمله‌ای هیلبرت-کیربی باشد، آنگاه $Tor_i^R(A,B)$ و $Ext^i_R(B,A)$ برای هر R-مدول مدرج متناهی مولد B و هر $i \in \mathbb{N}_\circ$، R-مدول‌های مدرج آرتینی با چندجمله‌ای‌های هیلبرت-کیربی خواهند بود.

نکته ۶.۲. اگر $(R_\circ, \mathfrak{m}_\circ) \to (S_\circ, \mathfrak{n}_\circ)$ یک همریختی یکدست از حلقه‌های موضعی بطوریکه $\mathfrak{m}_\circ S_\circ$ یک ایده‌آل $\mathfrak{n}_\circ$-اولیه از $S_\circ$ و A یک R-مدول مدرج آرتینی با چندجمله‌ای هیلبرت-کیربی باشد، آنگاه، طبق [۳، قضیه۱]، براحتی مشاهده می‌شود که A یک R-مدول آرتینی است اگر و فقط اگر $A \otimes_{R_\circ} S_\circ = \oplus_{n\in\mathbb{Z}}(A_n \otimes_{R_\circ} S_\circ)$ یک مدول مدرج آرتینی روی حلقه مدرج همگن $R \otimes_{R_\circ} S_\circ = \oplus_{n\in\mathbb{N}_\circ}(R_n \otimes_{R_\circ} S_\circ)$ باشد. با توجه به [۲، لم۲۵.۲.۱] داریم

$$\ell_{S_\circ}(A_n \otimes_{R_\circ} S_\circ) = \ell_{R_\circ}(A_n)\ell_{S_\circ}(S_\circ/\mathfrak{m}_\circ S_\circ)$$

که نتیجه می‌دهد

$$P_{A\otimes_{R_\circ} S_\circ}(X) = \ell_{S_\circ}(S_\circ/\mathfrak{m}_\circ S_\circ)P_A(X).$$

بنابراین خود چندجمله‌ای هیلبرت-کیربی تحت این دسته از توسیع‌های موضعی یکدست، با احتساب ضربی ثابت، پایا است.

تعریف ۷.۲. بُعد طول متناهی کوهمولوژیکال N, M را، کوچکترین i که $R_\circ$-مدول $H^i_{R_+}(M,N)_n$ از طول نامتناهی برای تعداد نامتناهی اعداد صحیح n باشد می‌نامیم و با نماد $g(M,N)$ نشان می‌دهیم. در واقع

$$g(M,N) := \inf\{i \in \mathbb{N}_\circ \mid \#\{n \mid \ell_{R_\circ}(H^i_{R_+}(M,N)_n) = \infty\} = \infty\}.$$

نکته ۸.۲. مفروضات تعریف ۷.۲ را در نظر بگیرید. در اینصورت

(۱) فرض کنید $x \in R_+$ یک عنصر همگن باشد بطوریکه $\Gamma_{R_+}(\circ :_N x) = (\circ :_N x)$. از رشته دقیق کوتاه

$$\circ \longrightarrow (\circ :_N x) \longrightarrow N \longrightarrow \frac{N}{(\circ :_N x)} \longrightarrow \circ$$

رشته دقیق طولانی

$$\cdots \longrightarrow H^i_{R_+}(M,(\circ :_N x)) \longrightarrow H^i_{R_+}(M,N) \longrightarrow H^i_{R_+}(M,N/(\circ :_N x)) \longrightarrow H^{i+1}_{R_+}(M,(\circ :_N x)) \longrightarrow \cdots$$

از R-مدول‌ها و R-همریختی‌ها بدست می‌آید. طبق لم ۳.۲ و نکته ۵.۲ قسمت(۲)، نتیجه می‌گیریم $H^i_{R_+}(M,(\circ :_N x))$ تنها تعداد متناهی مولفه‌های غیرصفر برای هر $i \in \mathbb{N}_\circ$ دارد. بنابراین $g(M,N) = g(M,xN)$. حال از رشته دقیق کوتاه

$$\circ \longrightarrow xN \longrightarrow N \longrightarrow \frac{N}{xN} \longrightarrow \circ$$

و رشته دقیق طولانی

$$\cdots \longrightarrow H^i_{R_+}(M,N) \longrightarrow H^i_{R_+}(M,N/xN) \longrightarrow H^{i+1}_{R_+}(M,N) \longrightarrow \cdots$$

نتیجه می‌شود که $g(M,N/xN) \geq g(M,N) - 1$.

(۲) از رشته دقیق کوتاه $\circ \longrightarrow \Gamma_{R_+}(N) \longrightarrow N \longrightarrow N/\Gamma_{R_+}(N) \longrightarrow \circ$ و لم ۳.۲، داریم $g(M,N) = g(M,N/\Gamma_{R_+}(N))$. فلذا، در محاسبه $g(M,N)$، بدون کاستن از کلیت مسئله، می‌توان فرض کرد $\Gamma_{R_+}(N) = \circ$.

قضیه ۹.۲. فرض کنید R یک حلقه مدرج همگن، $(R_\circ, \mathfrak{m}_\circ)$ موضعی و M, N دو R-مدول مدرج با تولید متناهی باشند. در اینصورت برای $\circ \leq i \leq g(M,N)$، R-مدول مدرج $\Gamma_{\mathfrak{m}_\circ R}(H^i_{R_+}(M,N))$ آرتینی با چندجمله‌ای هیلبرت-کیربی است.

اثبات. فرض کنید $i \in \mathbb{N}_\circ$، بوضوح $\Gamma_{\mathfrak{m}_\circ R}(H^i_{R_+}(M,N)_n) = \Gamma_{\mathfrak{m}_\circ}(H^i_{R_+}(M,N)_n)$. بنابراین طبق گزاره ۴.۲، $\Gamma_{\mathfrak{m}_\circ R}(H^i_{R_+}(M,N)_n) = \Gamma_{\mathfrak{m}_\circ}(H^i_{R_+}(M,N)_n)$، به عنوان یک $R_\circ$-زیرمدول $H^i_{R_+}(M,N)_n$، توسط توانی از $\mathfrak{m}_\circ$ صفر می‌شود و بنابراین یک $R_\circ$-مدول آرتینی است. لذا بنابر نکته ۵.۲ قسمت(۱)، کافی است نشان دهیم برای $\circ \leq i \leq g(M,N)$، $\Gamma_{\mathfrak{m}_\circ R}(H^i_{R_+}(M,N))$ یک R-مدول آرتینی است. این کار را با استقراء روی i انجام می‌دهیم. اگر $i = \circ$ باشد، آنگاه طبق [۴، گزاره۱.۲]، داریم

$$\Gamma_{\mathfrak{m}_\circ R}(H^\circ_{R_+}(M,N)) = \Gamma_{\mathfrak{m}_\circ R}(\Gamma_{R_+}(Hom_R(M,N))) = \Gamma_{\mathfrak{m}}(Hom_R(M,N))$$

که بوضوح یک R-مدول آرتینی است که $\mathfrak{m} = \mathfrak{m}_\circ + R_+$. فرض کنید $\circ < i \leq g(M,N)$ و حکم برای مقادیر کوچکتر از i برقرار باشد. به‌علاوه، فرض کنید X یک مجهول روی حلقه $R_\circ$، $S_\circ = R_\circ[X]_{\mathfrak{m}_\circ R_\circ[X]}$، $\mathfrak{n}_\circ = \mathfrak{m}_\circ S_\circ$، $S = S_\circ \otimes_{R_\circ} R$، $M' = S \otimes_R M = S_\circ \otimes_{R_\circ} M$ و $N' = S \otimes_R N = S_\circ \otimes_{R_\circ} N$. به سادگی دیده می‌شود که $S_\circ$ یک $R_\circ$-جبر یکدست باوفای نوتری با ایده‌آل ماکسیمال $\mathfrak{n}_\circ$ با میدان مانده نامتناهی است. بنابر قضیه تغییر پایه یکدست برای کوهمولوژی موضعی تعمیم یافته داریم

$$S_\circ \otimes_{R_\circ} \Gamma_{\mathfrak{m}_\circ R}(H^i_{R_+}(M,N)) \cong \Gamma_{\mathfrak{m}_\circ S}(S_\circ \otimes_{R_\circ} H^i_{R_+}(M,N)) \cong \Gamma_{\mathfrak{n}_\circ S}(H^i_{S_+}(M',N'))$$

بنابراین طبق نکته ۶.۲ ، کافی است نشان دهیم $\Gamma_{\mathfrak{n}_\circ S}(H^i_{S_+}(M',N'))$ یک S-مدول مدرج آرتینی با چندجمله‌ای هیلبرت-کیربی است. بنابراین بدون کاستن از کلیت مسئله، می‌توان به‌جای R، N, M به‌ترتیب S، N', M' قرارداد و فرض کرد $k_\circ$ نامتناهی است. همچنین به‌ازای هر عدد صحیح i، R-مدول $\Gamma_{\mathfrak{m}_\circ R}(H^i_{R_+}(M,N))$ آرتینی است اگر و فقط اگر R-مدول $\Gamma_{\mathfrak{m}_\circ R}(H^i_{R_+}(M,N/\Gamma_{\mathfrak{m}_\circ}(N)))$ آرتینی باشد. بنابراین طبق نکته ۸.۲ قسمت(۲)، با جایگذاری $\dfrac{N}{\Gamma_{R_+}(N)}$ به‌جای N می‌توان فرض کرد $\Gamma_{R_+}(N) = \circ$. لذا طبق لم ۱.۲، عنصر N-منظم x متعلق به R_1 وجود دارد. حال از رشته دقیق کوتاه

$$\circ \longrightarrow N(-1) \xrightarrow{x} N \longrightarrow \frac{N}{xN} \longrightarrow \circ$$

رشته دقیق طولانی (از R-مدول‌ها و R-همریختی‌های مدرج)

$$\xrightarrow{x} H^{i-1}_{R_+}(M,N) \longrightarrow H^{i-1}_{R_+}(M,N/xN) \longrightarrow H^i_{R_+}(M,N)(-1) \xrightarrow{x} H^i_{R_+}(M,N)$$

بدست می‌آید که از آن رشته دقیق کوتاه

$$\circ \longrightarrow \frac{H^{i-1}_{R_+}(M,N)}{xH^{i-1}_{R_+}(M,N)} \longrightarrow H^{i-1}_{R_+}(M,N/xN) \longrightarrow (\circ :_{H^i_{R_+}(M,N)} x)(-1) \longrightarrow \circ \qquad (1)$$

حاصل می‌شود.

با اعمال فانکتور $\Gamma_{\mathfrak{m}_\circ R}(-)$ روی رشته دقیق کوتاه (۱) رشته دقیق زیر از R-مدول‌ها و R-همریختی‌ها مدرج بدست می‌آید

$$\Gamma_{\mathfrak{m}_\circ R}(H^{i-1}_{R_+}(M,N/xN)) \longrightarrow \Gamma_{\mathfrak{m}_\circ R}(\circ :_{H^i_{R_+}(M,N)} x)(-1) \longrightarrow H^1_{\mathfrak{m}_\circ R}\left(\frac{H^{i-1}_{R_+}(M,N)}{xH^{i-1}_{R_+}(M,N)}\right) \qquad (2)$$

چنانچه $i-1 < g(M,N)$، آنگاه طبق تعریف $g(M,N)$، داریم $\ell_{R_\circ}(H^i_{R_+}(M,N)_n) < \infty$ برای هر n، بجز تعداد متناهی. بنابراین $R_\circ$-مدول $H^{i-1}_{R_+}(M,N)_n/xH^{i-1}_{R_+}(M,N)_n$ برای هر $n \ll \circ$ با طول متناهی خواهد بود. بنابراین طبق [۱، نتیجه۱۳.۴.۳] داریم

$$\begin{aligned} H^1_{\mathfrak{m}_\circ R}(H^{i-1}_{R_+}(M,N)/xH^{i-1}_{R_+}(M,N)) &\cong \bigoplus_{n\in\mathbb{Z}} (H^1_{\mathfrak{m}_\circ R}(H^{i-1}_{R_+}(M,N)/xH^{i-1}_{R_+}(M,N))_n \\ &\cong \bigoplus_{n\in\mathbb{Z}} H^1_{\mathfrak{m}_\circ}(H^{i-1}_{R_+}(M,N)_n/xH^{i-1}_{R_+}(M,N)_n) \end{aligned}$$

پس آخرین مدول از تعداد متناهی درجه تشکیل شده است و مولفه‌های مدرج آن $R_\circ$-مدول آرتینی هستند.
لذا $H^1_{\mathfrak{m}_\circ R}(H^{i-1}_{R_+}(M,N)/xH^{i-1}_{R_+}(M,N))$ یک R-مدول آرتینی است. از طرف دیگر، طبق نکته ۸.۲ قسمت(۱)، می‌بینیم که

و در نتیجه، بنابر فرض استقراء $\Gamma_{\mathfrak{m}_\circ R}(H^{i-1}_{R_+}(M,N/xN))$ یک R-مدول آرتینی $g(M,N/xN) \geq g(M,N)-1 \geq i-1$ است. لذا از رشته دقیق (۲) نتیجه می‌گیریم $\Gamma_{\mathfrak{m}_\circ R}(\circ :_{H^i_{R_+}(M,N)} x)$ آرتینی است. چون

$$\Gamma_{\mathfrak{m}_\circ R}(\circ :_{H^i_{R_+}(M,N)} x) = (\circ :_{\Gamma_{\mathfrak{m}_\circ R}(H^i_{R_+}(M,N))} x)$$

پس طبق قضیه ملکرسون [۱، قضیه۲.۱.۰۷]، نتیجه می‌گیریم $\Gamma_{\mathfrak{m}_\circ R}(H^i_{R_+}(M,N))$ نیز یک R-مدول آرتینی است. □

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گراف‌های نامتباین گروه‌های دودوری

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چکیده. گراف نامتباین گروه G نسبت به زیر گروه H که آن را با $\Pi_{G,H}$ نشان می دهیم گرافی است با مجموعه رئوس $G-\{e\}$ و دو راس g و h از این گراف به هم متصل اند اگر $(|g|,|h|)\neq ۱$ و g یا h متعلق به H باشند. در این مقاله گراف‌های نامتباین گروه های دودوری را معرفی و مورد بررسی قرار داده ایم.

واژه‌های کلیدی: گراف های نامتباین، گروه های دودوری.
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۱. پیش‌گفتار

در این مقاله تمامی گراف ها ساده هستند. فرض کنیم G یک گروه متناهی و H زیرگروهی از G باشد. گراف نامتباین G نسبت به H که آن را با $\Pi_{G,H}$ نشان می دهیم گرافی است با مجموعه راسهای $G-\{e\}$ و دو راس g و h از این گراف به هم متصل اند اگر $(|g|,|h|)\neq ۱$ و g یا h متعلق به H باشند. اگر $G=H$ باشد $\Pi_{G,H}$ را با Π_G نشان می دهیم و به آن گراف نامتباین گروه G می گوییم.
در سال ۲۰۱۴ در ([۶]) گراف‌های متباین و نامتباین روی گروه‌ها تعریف و برخی از مشخصات این گراف‌ها مورد بررسی قرار داده شدند. همچنین برخی ویژگی‌های این گراف‌ها توسط منصوری، عرفانیان و طلوعی مورد بررسی قرار گرفتند.([۷])
فرض کنیم u و v دو رأس از گراف Π باشند. در این صورت فاصله بین u و v که با $d(u,v)$ نمایش داده می‌شود، طول کوتاه‌ترین مسیر بین u و v است. قطر گراف Π را با $diam(\Pi)$ نمایش داده و به صورت زیر تعریف می‌کنیم.

$$diam(\Pi)=\sup\{d(u,v)\mid u,v\in V(\Pi)\};$$

عدد رنگی گراف Π برابر مینیمم تعداد رنگ‌های مورد نیاز برای رنگ آمیزی رئوس گراف Π است به طوری که هیچ دو رأس مجاور هم رنگ نباشند. عدد رنگی گراف Π را با $\chi(\Pi)$ نشان می‌دهیم. مرتبه یک خوشه ماکسیمم از گراف Π را عدد خوشه ای گراف Π می گوییم و با $w(\Pi)$ نشان می دهیم. یک گراف را مسطح می‌نامیم هرگاه بتوان آن را طوری در صفحه رسم کرد که یال‌های گراف یکدیگر را به‌جز در رئوس مشترک قطع نکنند. یک گراف مسطح است اگر و تنها اگر دارای هیچ زیرگراف همانریخت با $K_۵$ یا $K_{۳,۳}$ نباشد. اگر Π یگ گراف ساده باشد گراف خط $L(\Pi)$ در تناظر یک به یک با مجموعه یال‌های گراف Π است و دو راس در $L(\Pi)$ را با یک یال به هم وصل می کنیم اگر و تنها اگر یال‌های متناظر در Π با یکدیگر مجاور باشند. یک گراف غیرتهی گراف خط است اگر و تنها اگر بتوان یال‌های آن را به مجموعه‌ای از خوشه‌ها افراز کرد به طوری که هر رأس آن حداکثر در دو خوشه قرار گیرد. در این مقاله گراف های نامتباین گروه های دودوری را با استفاده از نرم افزار گپ تعریف و برخی ویژگی های این گراف‌ها را ارائه می‌کنیم. همچنین ثابت می‌کنیم عدد رنگی و عدد خوشه ای گراف‌های دودوری با هم برابرند.

۲. نتایج اصلی

گروه دودوری عبارت است از

$$T_{۴n}=< a\ ,\ b|a^{۲n}=۱a^n=b^۲,b^{-۱}ab=a^{-۱}>$$

در این قسمت به بررسی خواص گراف $\Pi_{T_{۴n}}$ می‌پردازیم.

قضیه ۱.۲. گراف $\Pi_{T_{۴n}}$:
(۱) همبند است؛
(۲) اویلری نیست مگر در حالت $n=۲^k$؛
(۳) دارای زیرگراف فراگیر اویلری است؛

* سخنران

(۴) $\gamma(\Pi_{T_{4n}}) = 1$.

(۵) مسطح نیست مگر در حالت $n = 1$.

اثبات. (۱) ساختار گراف $\Pi_{T_{4n}}$ همان گراف $\Pi_{Z_{2n}}$ است که تعداد $2n$ رأس از مرتبه ۴ به آن اضافه شده است و این $2n$ رأس به یکدیگر متصل ند و همچنین به رئوس مرتبه زوج متصل هستند. حال از آنجا که $\Pi_{Z_{2n}}$ همبند است پس $\Pi_{T_{4n}}$ نیز همبند است؛

(۲) در حالتی که $n = 2^k$ با شد گراف $\Pi_{T_{4n}}$ یک گراف کامل از مرتبه $4n - 1$ می‌باشد که مرتبه هر رأس آن $4n - 2$ می‌باشد که عددی زوج است پس در این حالت گراف اویلری است. در صورتی که $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$ درجه رأس دارای مرتبه p_i برابر است با

$$\frac{2n({p_i}^{k_i} - 1)}{{p_i}^{k_i}} - 1$$

که همواره عددی فرد است بنابراین در حالتهای دیگر بجز $n = 2^k$ گراف $\Pi_{T_{4n}}$اویلری نیست؛

(۳) از آنجا که $\Pi_{Z_{2n}}$ دارای زیرگراف فراگیر اویلری است یعنی هریال درون یک مثلث قرار دارد کافی است ثابت کنیم یالی که یک سر آن یا دو سر آن رأس‌های مرتبه ۴ هستند درون مثلث قرار دارد. در حالتی که $n = 1$ باشد Π_{T_4}خود یک گراف کامل مرتبه ۳ است. در سایر حالتها حداقل ۶ رأس از مرتبه ۴ وجود دارد و اگر e یالی باشد که یک سر آن رأس v_1 رأس مرتبه ۴ باشد بنابراین سر دیگر آن که آن را v_2 نامگذار ی می‌کنیم باید رأسی از مرتبه زوج باشد پس v_2 به رأس دیگری از مرتبه ۴ مانند v_3 متصل است که v_1 و v_2و v_3 تشکیل یک مثلث می‌دهند. حال فرض کنیم یال e با دو سر v_1 و v_2 باشد که v_1 و v_2 هر دو از مرتبه ۴ هستند چون حداقل ۶ رأس از مرتبه ۴ وجود داشت پس رأسهای دیگری از مرتبه ۴ مانند v_3 وجود دارد که $v_3 \sim v_2 \sim v_1 \sim v_3$ تشکیل یک مثلث می‌دهند بنابراین داری زیرگراف فراگیر اویلری است؛

(۴) از آنجا که $\Pi_{T_{4n}}$ دارای عنصری از مرتبه $2n$ است که به تمامی رئوس متصل است پس $\gamma(\Pi_{T_{4n}}) = 1$؛

(۵) در حالت $n = 1$ داریم $\Pi_{T_{4n}}$ یک گروه کامل از مرتبه ۳ و لذا مسطح است. در سایر حالت‌ها چون حد اقل ۶ رأس از مرتبه ۴ وجود دارد پس $k_5 \leqslant \Pi_{T_{4n}}$ و لذا مسطح نیست.

□

قضیه ۲.۲. فرض کنیم $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$ که p_i ها اعداد اول متمایز باشند:

(۱) اگر n عددی فرد باشد، آنگاه :

$$\omega(\Pi_{T_{4n}}) = 3n;$$

(۲) اگر nعددی زوج باشد یعنی $p_1 = 2$ آنگاه

$$\omega(\Pi_{T_{4n}}) = 4n - p_2^{k_2} p_3^{k_3} \ldots p_r^{k_r}.$$

اثبات. در هر دو حالت با توجه به اینکه گروه T_{4n} دارای $2n$ عضو از مرتبه ۴ می‌باشد لذا خوشه‌ای که رئوس آن دارای مرتبه زوج می‌باشد بزرگ‌ترین خوشه در $\Pi_{T_{4n}}$ می‌باشد. کافی است در هر دو حالت تعداد رئوسی از T_{4n} را که مرتبه آنها را ۲ می‌شمارد را به دست آوریم. در حالت ۱ تعداد عناصری که در Z_{2n} دارای مرتبه زوج هستند، برابر است با $n = \frac{2n(2-1)}{2}$. لذا تعداد عناصر مرتبه زوج $\Pi_{T_{4n}}$ برابر است با $2n + n = 3n$. در حالت ۲ تعداد عناصری که در z_{2n} دارای مرتبه زوج هستند، برابر است با $\frac{2n(2^{k_1+1} - 1)}{2^{k_1+1}}$ در نتیجه :

$$\omega(\Pi_{T_{4n}}) = 2n + \frac{2n(2^{k_1+1} - 1)}{2^{k_1+1}} = 2n(2 - \frac{1}{2^{k_1+1}}) = 4n - p_2^{k_2} p_3^{k_3} \ldots p_r^{k_r}.$$

□

قضیه ۳.۲.

$$\chi(\Pi_{T_{4n}}) = \omega(\Pi_{T_{4n}}).$$

اثبات. اگر n عددی فرد باشد $\omega(\Pi_{T_{4n}}) = 3n$ که در این حالت $n - 1$ رأس باقیمانده از گراف $\Pi_{T_{4n}}$ را می‌توان با همین $3n$ رنگ، رنگ آمیزی نمود.
اگر n عددی زوج باشد آنگاه بنا بر قضیه قبل $\omega(\Pi_{T_{4n}}) = 4n - p_2^{k_2} p_3^{k_3} \ldots p_r^{k_r}$ که $n = 2^{k_1} p_2^{k_2} \ldots p_r^{k_r}$ لذا $p_2^{k_2} p_3^{k_3} \ldots p_r^{k_r} <$ n پس

$$۴n - p_۲^{k_۲} p_۳^{k_۳} \ldots p_r^{k_r} > ۳n$$

و با استدلال قبل حد اکثر $n - ۲$ رأس باقی می‌ماند که این رئوس را می‌توان با $\omega(\Pi_{T_{۴n}})$ رنگ، رنگ آمیزی کرد. □

قضیه ۴.۲. در گراف $\Pi_{T_{۴n}}$:

(۱) $diam(\Pi_{T_{۴n}}) \leqslant ۲$؛

(۲) گراف $\Pi_{T_{۴n}}$ یک گراف خط است اگر و تنها اگر $n = p^k$ یا $n = ۲^k$ یا $n = ۲^k p^k$ که عددی اول است؛

(۳) اگر $n = p_۱^{k_۱} p_۲^{k_۲} \ldots p_r^{k_r}$ تعداد رئوس end−vertices گراف $\Pi_{T_{۴n}}$ برابر است با

(آ) اگر n عددی فرد باشد:

$$(p_۱^{k_۱} - ۱)(p_۲^{k_۲} - ۱) \ldots (p_r^{k_r} - ۱)$$

(ب) اگر n عددی زوج باشد یعنی $p_۱ = ۲$:

$$(۲^{k_۱ + ۱} - ۱)(p_۲^{k_۲} - ۱) \ldots (p_r^{k_r} - ۱)$$

اثبات. (۱) چون $diam(\Pi_{Z_{۲n}}) \leqslant ۲$ پس بنابر تعریف گراف $\Pi_{Z_{۲n}}$ داریم $diam(\Pi_{T_{۴n}}) \leqslant ۲$؛

(۲) از آنجا که یالهای $\Pi_{T_{۴n}}$ را می‌توان به دو خوشه چنان افراز کرد که هر رأس در دو خوشه قرار گیرد اگر و تنها اگر $۴n = ۲^{k_۱} p^{k_۲}$ بنابراین حالتهایی که n می‌تواند اختیار کند عبارتند از ۲^k و p^k و $۲^{k_۱} p^{k_۲}$؛

(۳) تعداد رئوس end−vertices گراف $\Pi_{T_{۴n}}$ برابر است با تعداد رئوس end−vertices گراف $\Pi_{Z_{۲n}}$ که برابر است با تعداد رئوسی که عدد $p_۱ p_۲ \cdots p_r$ مرتبه آنها را می‌شمارند که این تعداد برابر است با $(p_۱^{k_۱} - ۱)(p_۲^{k_۲} - ۱) \ldots (p_r^{k_r} - ۱)$ در حالتی که n فرد است و $(۲^{k_۱+۱} - ۱)(p_۲^{k_۲} - ۱) \ldots (p_r^{k_r} - ۱)$ در حالتی که n زوج است.

□

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مشتق BL-جبر و شبه BL-جبرها

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چکیده. در این مقاله مشتق روی ساختارهای BL-جبرها و شبه BL-جبرها مورد مطالعه قرار گرفته است.
واژه‌های کلیدی: مشتق، BL-جبر، شبه BL-جبر.
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۱. پیش‌گفتار

مفهوم MV-جبر به وسیله چانگ سال ۱۹۵۸ در مقاله [۲] معرفی شد. بعد از MV-جبرها، جبرهای ضربی و جبرهای گودل معرفی و مورد مطالعه قرار گرفتند. این سه جبر به ترتیب مدل های جبری سه منطق فازی مهم لوکاسیویچ، ضربی و گودل هستند. هایک در سال ۱۹۹۶ منطق پایه این سه منطق را که به طور مختصر به آن منطق BL می گویند و ساختار جبری متناظر با آن به نام BL-جبر را معرفی کرد. در سال ۱۹۹۹ توسط جورج ایسکیو و یورگلسکیو شبه MV-جبرها به عنوان MV-جبر که خاصیت جابجایی ندارند در مقاله [۵] معرفی شدند. مقایسه ساختار شبه MV-جبرها و BL-جبرها، منجر به معرفی شبه BL-جبرها در سال ۲۰۰۰ توسط دی نولا ،جورج ایسکیو و یورگلسکیو در مقاله [۳، ۴] شد و توسط دانشمندان بسیاری مورد مطالعه قرار گرفت. شبه BL-جبر تعمیمی از BL-جبر است که تمام خواص BL-جبر را شامل می شود ولی خاصیت جابجایی را ندارد. در حالت کلی مفهوم مشتق را می توان برای هر جبر از نوع (۲، ۲) ، از جمله در ساختار مشبکه‌ها تعریف کرد . نخستین بار شاژ در [۸] این تعریف جبری مشتق را در مشبکه ها معرفی کرد. بعد از وی تعدادی از ریاضیدانان در زمینه مشتق روی مشبکه ها و تعمیم هایی از آن، MV-جبر و BL-جبر به پژوهش و بررسی ویژگی‌های آن پرداختند.

۲. نتایج اصلی

تعریف ۱.۲. [۳] جبر $(L,\vee,\wedge,\otimes,\to,0,1)$ با اعمال دوتایی $\vee,\wedge,\otimes,\to$ و ثابت های ۰ و ۱ را یک BL-جبر گوییم هر گاه برای هر $x,y,z\in L$ داشته باشیم:

(BL-۱) $(L,\vee,\wedge,0,1)$ یک مشبکه کراندار است؛
(BL-۲) $(L,\otimes,1)$ یک تکواره جابجایی است؛
(BL-۳) اعمال $\otimes$ و $\to$ یک زوج الحاقی باشند یعنی $x\otimes y\ \le z$ اگر و فقط اگر $x\le y\to z$ ؛
(BL-۴) $x\wedge y=(x\to y)\otimes x$؛
(BL-۵) $(x\to y)\vee(y\to x)=1$، برای هر $x,y\in L$.

تعریف ۲.۲. [۳] جبر $(L,\vee,\wedge,\otimes,\to,\rightsquigarrow,0,1)$ با اعمال دوتایی $\vee,\wedge,\otimes,\to,\rightsquigarrow$ و ثابت های ۰ و ۱ را یک شبه BL-جبر گوییم هر گاه برای هر $x,y,z\in L$ داشته باشیم:

(PBL-۱) $(L,\vee,\wedge,0,1)$ یک مشبکه کراندار است؛
(PBL-۲) $(L,\otimes,1)$ یک تکواره است؛
(PBL-۳) $x\otimes y\ \le z$ اگر و فقط اگر $x\le y\to z$ اگر و فقط اگر $y\le x\rightsquigarrow z$؛
(PBL-۴) $x\wedge y=(x\to y)\otimes x=x\otimes(x\rightsquigarrow y)$؛
(PBL-۵) $(x\to y)\vee(y\to x)=(x\rightsquigarrow y)\vee(y\rightsquigarrow x)=1$ برای هر $x,y\in L$.

تعریف ۳.۲. اگر $(L,\vee,\wedge,\otimes,\to,0,1)$ یک BL-جبر باشد مشتق های تعریف شده روی L به صورت زیر می باشند: [۹، ۱، ۶]:

(۱) $D(x\otimes y)=(D(x)\otimes y)\vee(x\otimes D(y))$ برای هر $x,y\in L$؛
(۲) $D(x\ominus y)=(D(x)\ominus y)\otimes(x\ominus D(y))$ برای هر $x,y\in L$؛
(۳) $D(x\to y)=(D(x)\to y)\vee(x\to\ D(y))$ برای هر $x,y\in L$.
(۴) $D(x\wedge y)=(D(x)\wedge y)\otimes(x\wedge\ D(y))$ برای هر $x,y\in L$.

* سخنران

تعریف ۴.۲. اگر $(L,\vee,\wedge,\otimes,\to,\rightsquigarrow,\circ,۱)$ یک شبه BL -جبر باشد. آنگاه نگاشت $D:A\to A$ را
(۱) یک مشتق نوع ۱ است، اگر $D(x\otimes y)=(D(x)\otimes y)\vee(x\otimes D(y))$ برای هر $x,y\in A$؛
(۲) یک مشتق نوع ۲ است، اگر $D(x\ominus y)=(D(x)\ominus y)\otimes(x\ominus D(y))$ برای هر $x,y\in A$؛
(۳) یک مشتق نوع ۳ است، اگر $D(x\oslash\ y)=(D(x)\oslash\ y)\otimes(x\oslash\ D(y))$ برای هر $x,y\in A$؛
می نامیم.

برای شبه BL -جبر A مشتق های نوع ۱، ۲ و ۳ را به ترتیب به اختصار $D_۱, D_۲$ و $D_۳$ می نامیم.

تعریف ۵.۲. فرض کنید $(A,\vee,\wedge,\otimes,\to,\rightsquigarrow,\circ,۱)$ شبه BL -جبر باشد. در اینصورت نگاشت $D:A\to A$ مشتق پیکانی است و
(۴) یک $(\to,\vee)$ -مشتق است اگر $D(x\to y)=(Dx\to y)\vee(x\to Dy)$ برای هر $x,y\in A$؛
(۵) یک $(\rightsquigarrow,\vee)$ -مشتق است اگر $D(x\to y)=(Dx\to y)\vee(x\to Dy)$ برای هر $x,y\in A$.

نمادهای $\overrightarrow{D}$ و $\overset{\rightsquigarrow}{D}$ برای $(\to,\vee)$ -مشتق و $(\rightsquigarrow,\vee)$ -مشتق در تعاریف بالا استفاده می گردد.

قضیه ۶.۲. D یک $(\to,\vee)$ -مشتق روی BL -جبر A است اگر و فقط اگر $D(x\longrightarrow y)=x\longrightarrow D(y)$ برای هر $x,y\in A$.

قضیه ۷.۲. اگر D یک $(\rightsquigarrow,\vee)$ -مشتق روی BL -جبر A باشد آنگاه برای هر $x\in A$ موارد زیر برقرار می باشند:
(۱) $D(۱)=۱$؛
(۲) $x\leq D(x)$؛
(۳) $(D(x))^-\leq D(x^-)$؛
(۴) $D(x)=x\vee D(x)$؛
(۵) اگر $D(x)=\circ$، آنگاه $x=\circ$؛
(۶) $D^n(x\rightsquigarrow y)=x\rightsquigarrow D^n(y)$؛
(۷) $D(x)\otimes D(y)\leq D(x)\rightsquigarrow D(y)\leq D(x\rightsquigarrow y)$؛
(۸) $D(x\rightsquigarrow y)\vee D(y\rightsquigarrow x)=۱$.

گزاره ۸.۲. فرض کنید $D_۱$ و $D_۲$ $(\rightsquigarrow,\vee)$ -مشتق روی BL -جبر A باشند. در این صورت $D_۱oD_۲$ هم یک $(\rightsquigarrow,\vee)$ -مشتق روی A است.

قضیه ۹.۲. اگر A یک BL -جبر و D یک $(\wedge,\otimes)$ -مشتق روی A باشد. آنگاه عبارت های زیر برای هر $x,y\in A$ برقرار است:
(۱) $D(\circ)=\circ$؛
(۲) $D(x)\leq x$؛
(۳) $D(x)=D(x)\otimes x$؛
(۴) اگر $x\geq D(۱)$، آنگاه $D(x)=D(x)\otimes D(۱)$؛
(۵) $D(x)=D(x)^n$؛
(۶) $(D(x)\wedge D(y))^۲\leq D(x\wedge y)\leq(x\wedge y)^۲$.

گزاره ۱۰.۲. فرض کنید D یک $(\wedge,\otimes)$- مشتق روی BL -جبر A باشد. آنگاه $Fix_{D(A)}\subseteq G(A)$.

گزاره ۱۱.۲. فرض کنید A یک BL -جبر باشد و $A=G(A)$. در این صورت D تابع همانی روی A است اگر و فقط اگر D یک $(\rightsquigarrow,\vee)$ -مشتق و $(\wedge,\otimes)$- مشتق روی A است.

قضیه ۱۲.۲. فرض کنید $(A,\vee,\wedge,\otimes,\to,\rightsquigarrow,\circ,۱)$ یک شبه BL -جبر و D مشتق نوع i روی A که $۱\leq i\leq ۳$ است. در این صورت برای هر $۱\leq i\leq ۳$، داریم:
(۱) $D_i(\circ)=\circ$؛
(۲) $D_i(x)=D_i(x)\otimes x$ آنگاه $D_i(x)\leq x$، برای هر $x\in A$؛
(۳) $D_i(x^\sim)\leq(D_i(x))^\sim$ برای $i=۲,۳$؛
(۴) $D_i(x^-)\leq(D_i(x))^-$ بعلاوه $x\in B(A)$ نتیجه می دهد $D_۱(x)\leq x$؛
(۵) $D_۱(x)=۱$ نتیجه می دهد $x^-=\circ$، و برای $i=۲,۳$ و $D_i(x)=۱$ نتیجه می دهد که $x=۱$.

قضیه ۱۳.۲. فرض کنید A یک شبه BL -جبر باشد. اگر D مشتق نوع ۱ ایزوتون باشد به طوریکه برای هر $x\in A$ $D(x)\leq x$ و $D(x)=D(x)\otimes D(x)$ در این صورت برای هر $x,y\in A$ شرایط زیر برقرار می باشند:
(۱) $D(x)=D(۱)\otimes x=x\otimes D(۱)$؛

(۲) $D(x \otimes y) = D(x) \otimes D(y)$؛
(۳) $D(x \ominus y) \leq D(x) \ominus D(y), D(x \oslash y) \leq D(x) \oslash D(y)$؛
(۴) $D(x \vee y) = D(x) \vee D(y)$؛

گزاره ۱۴.۲. فرض کنید D مشتق نوع ۳ روی شبه BL -جبر A باشد. آنگاه برای هر $x, y \in A$ داریم $D(x \oslash y) \leq D(x) \oslash D(y)$.

قضیه ۱۵.۲. فرض کنید D یکی از مشتق های پیکانی روی شبه BL -جبر A باشد. در این صورت برای هر $x, y \in A$ شرایط زیر برقرار هستند:

(۱) $\overset{\rightsquigarrow}{D}(۱) = ۱$ ، $\overrightarrow{D}(۱) = ۱$؛
(۲) اگر $x \leq y$ آنگاه $\overset{\rightsquigarrow}{D}(x \rightsquigarrow y) = ۱$ و $\overrightarrow{D}(x \rightarrow y) = ۱$؛
(۳) $\overset{\rightsquigarrow}{D}(x) = x \vee \overset{\rightsquigarrow}{D}(x)$ و سپس $\overset{\rightsquigarrow}{D}(x) \geq x$، هم چنین $\overrightarrow{D}(x) = x \vee \overrightarrow{D}(x)$ و سپس $\overrightarrow{D}(x) \geq x$؛
(۴) $(\overset{\rightsquigarrow}{D}x)^{\sim} \leq \overset{\rightsquigarrow}{D}(x^{\sim}), (\overrightarrow{D}x)^{-} \leq \overrightarrow{D}(x^{-})$؛
(۵) $y \leq \overset{\rightsquigarrow}{D}(x \rightsquigarrow y)$ ، $y \leq \overrightarrow{D}(x \rightarrow y)$؛
(۶) $\overset{\rightsquigarrow}{D}(x \rightsquigarrow y) = x \rightsquigarrow \overset{\rightsquigarrow}{D}y, \overrightarrow{D}(x \rightarrow y) = x \rightarrow \overrightarrow{D}y$.

لم ۱۶.۲. فرض کنید $\overset{\rightsquigarrow}{D}$ و $\overrightarrow{D}$ مشتق های پیکانی روی شبه BL -جبر A باشند. در این صورت

(۱) اگر $\overset{\rightsquigarrow}{D}$ ایزوتون باشد، آنگاه $\overset{\rightsquigarrow}{D}(x) \geq \overset{\rightsquigarrow}{D}(۰) \vee x$؛ و اگر $\overrightarrow{D}$ ایزوتون باشد آنگاه $\overrightarrow{D}(x) \geq \overrightarrow{D}(۰) \vee x$.
(۲) اگر $\overset{\rightsquigarrow}{D}(x) = \overset{\rightsquigarrow}{D}(۰) \vee x$ آنگاه $\overset{\rightsquigarrow}{D}$ ایزوتون است. اگر $\overset{\rightsquigarrow}{D}(x) = \overset{\rightsquigarrow}{D}(۰) \vee x$ آنگاه $\overset{\rightsquigarrow}{D}$ ایزوتون است.

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از چه مرتبه‌هایی دقیقا چهار یا پنج گروه وجود دارند؟

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چکیده. تعداد گروه‌های غیر یکریخت از مرتبه n را با $\nu(n)$ نشان می‌دهیم. حل معادله $\nu(n) = k$ برای اعداد صحیح مثبت کوچک k جالب خواهد بود. حالات $۳ \leq \nu(n)$ قبلا حل شده است. هدف از این مقاله مشخصه‌سازی تمام اعداد n با شرط $\nu(n) = ۴, ۵$ است.

واژه‌های کلیدی: عدد دوری، عدد آبلی، مرتبه خالی از مربع گروه، گروه با مرتبه خالی از مکعب.
طبقه‌بندی موضوعی [۲۰۱۰]: 20D99.

۰. پیش‌گفتار

تمامی گروه‌های این مقاله متناهی فرض می‌شوند و برای مفاهیم و نمادهای مقاله به [۸] ارجاع می‌دهیم. در سراسر مقاله تعداد گروه‌های غیر یکریخت از مرتبه n را با $\nu(n)$ نشان داده و $\phi(n)$ تابع فی اویلر را نشان می‌دهد. تمامی محاسبات ما با نرم‌افزار گپ [۹] انجام شده است.

محاسبه دقیق $\nu(n)$ مسئله‌ای بسیار مهم و مشکل در نظریه گروه‌های متناهی است. هولدر بیش از صد سال قبل و در مرجع [۴] مقدار دقیق این تابع را برای تمام nهای خالی از مربع بدست آورده است. به زبان دقیق‌تر، اگر $n = p_۱p_۲...p_r$ که $p_۱ > p_۲ > ... > p_r$ اعداد اول هستند، آنگاه

$$\nu(n) = \sum_S \left(\prod_{j=۱}^{n} \frac{p_{S(j)}^{c_{S(j)}} - ۱}{p_{S(j)} - ۱} \right) \tag{۱}$$

که در آن جمع روی تمام زیر مجموعه‌های $S = \{S(۱), S(۲), \ldots, S(r)\}$ از $\{۲, ۳, \ldots, n\}$ برداشته می‌شود و $c_S(j)$ معرف تعداد اعداد مختلف $p_i - ۱$ می‌باشد که $i \in S$ و این اعداد بر p_j بخش‌پذیر هستند.

اگر تعریف کنیم $f(n, m) = \prod_{q|m}(n, q - ۱)$ آنگاه می‌توان دید که

$$\nu(n) = \sum_{d|n} \prod_{p|d} \frac{f(p, \frac{n}{d} - ۱)}{p - ۱} \tag{۲}$$

زمانی که p و q اعداد اول متمایز می‌باشند. خواننده برای مطالعه جزئیات بیشتر به [۱] ارجاع داده می‌شود.

فرض کنید $n = \prod_{i=۱}^{r} p_i^{\alpha_i}$. تعریف می‌کنیم $\Phi(n) = \prod_{i=۱}^{r} (p_i^{\alpha_i} - ۱)$. رِدِی [۷، Satz ۱۰] اعداد صحیح n که تمام گروه‌های از این مرتبه آبلی باشند را دسته بندی نموده است. چنین عددی را آبلی می‌نامیم. رِدِی ثابت نمود:

قضیه ۱.۰. $n = \prod_{i=۱}^{r} p_i^{\alpha_i}$ عدد آبلی است اگر و تنها اگر $(n, \Phi(n)) = ۱$.

هم‌چنین دیکسون [۳] ثابت کرده است:

قضیه ۲.۰. اگر $n = p_۱^{r_۱} \ldots p_k^{r_k}$ تجزیه n به عوامل اول باشد آنگاه هر گروه از مرتبه n آبلی است اگر و تنها اگر برای هر i و j که $۱ \leq i, j \leq k$ داشته باشیم $r_i \leq ۲$ و $p_i \nmid p^{r_j} - ۱$.

نتیجه بلافصل قضیه اخیر این است که $\nu(n) = ۱$ اگر و تنها اگر $(n, \phi(n)) = ۱$. مورتی [۵] ثابت کرده است اگر n خالی از مربع باشد آنگاه $\nu(n) \leq \phi(n)$. همچنین تمام اعداد صحیح مثبت n که $\nu(n) = ۲$ را بدست آورده‌اند.

اُلسان [۶] در مقاله چاپ نشده خود معادله $\nu(n) = ۳$ را حل کرده است. در [۲] برای معادله $\nu(n) = ۲, ۳$ حل متفاوتی به صورت زیر ارائه شده است.

قضیه ۳.۰. فرض کنید $n = \prod_{i=۱}^{r} p_i^{\alpha_i}$. در این صورت

(۱) $\nu(n) = ۲$ اگر و تنها اگر یکی از حالات زیر اتفاق بیفتد:

* سخنران

(a) $r=۱$ و $\alpha_۱=۲$؛

(b) $r\geq ۲$، n خالی از مربع باشد و دوتایی منحصر بفرد (i,j) موجود باشد به طوری که $p_i|p_j-۱$؛

(c) $r\geq ۲$، $n=p_۱^۲p_۲...p_r$ و برای هر i، j و k که $۱\leq i,j\leq r$ و $۲\leq k\leq r$، داشته باشیم $p_i\nmid p_j-۱$ و $p_k\nmid p_۱^۲-۱$.

(۲) $\nu(n)=۳$ اگر و تنها اگر یکی از حالات زیر اتفاق بیفتد:

(a) $r\geq ۳$، n خالی از مربع بوده و سه تایی منحصر بفرد (i,j,k) موجود باشد که $p_i|p_j-۱$ و $p_j|p_k-۱$؛

(b) $r\geq ۲$، $n=p_۱^۲p_۲...p_r$ و برای هر i و j که $۱\leq i,j\leq r$ داشته باشیم $p_i\nmid p_j-۱$؛ و عدد منحصر بفرد k، $۲\leq k\leq r$، موجود باشد که $p_k|p_۱^۲-۱$.

این نتیجه، انگیزه ما برای حل معادله $\nu(n)=۴,۵$ می‌باشد. به عبارت دقیق‌تر:

سوال ۴.۱. برای کدام اعداد صحیح مثبت n داریم $\nu(n)=۴,۵$؟

۲. نتایج اصلی

هدف اصلی این بخش ارائه نتیجه اصلی این مقاله است. قسمت اول این قضیه یک مشخصه‌سازی از اعداد n ارائه می‌کند که برای آن‌ها دقیقا چهار گروه از مرتبه n وجود دارند. قسمت دوم قضیه یک مشخصه‌سازی از اعداد n بدست می‌دهد که برای آن‌ها دقیقا پنج گروه از مرتبه n وجود دارند.

قضیه ۲.۱. فرض کنید $n=\prod_{i=۱}^{r}p_i^{\alpha_i}$ که p_iها اعداد اول متمایز هستند، در این صورت

(۱) $\nu(n)=۴$ اگر و تنها اگر یکی از حالات زیر رخ دهد:

(الف) $n=۲p_۱p_۲$، که $p_۱$ و $p_۲$ اعداد اول فرد $p_۱<p_۲$ و $p_۱\nmid p_۲-۱$؛

(ب) $r\geq ۳$، n خالی از مربع بوده و سه‌تایی منحصر بفرد (i,j,k) موجود باشد که $p_i|p_k-۱$، $p_j|p_k-۱$ و $p_i\nmid p_j-۱$؛

(پ) $r\geq ۴$، n خالی از مربع بوده و دقیقا دو عدد دوتایی (i,j) و (k,l) موجود باشد که $p_i|p_j-۱$ و $p_k|p_l-۱$؛

(ت) $n=۴p$، $p>۳$ و $۴\nmid p-۱$؛

(ث) $r\geq ۲$، $n=p_۱^۲p_۲\ldots p_r$ و برای هر i که $۲\leq i\leq r$، داشته باشیم $p_i\nmid p_۱^۲-۱$، $p_۱^۲\nmid p_i-۱$ و دوتایی منحصر بفرد (i,j) که $j\neq ۱$ موجود باشد، به طوری که $p_i|p_j-۱$؛

(د) $r\geq ۲$، $n=p_۱^۲p_۲^۲p_۳\ldots p_r$ و برای هر i و j که $۱\leq i,j\leq r$، داشته باشیم $p_i\nmid p_j-۱$، $p_i\nmid p_j^۲-۱$، $p_۱^۲\nmid p_j-۱$ و $p_۲^۲\nmid p_j-۱$.

(۲) $\nu(n)=۵$ اگر و تنها اگر یکی از حالات زیر رخ دهد:

(الف) $r\geq ۳$، $n=۳p_۱p_۲...p_r$، n خالی از مربع بوده و دوتایی منحصر بفرد (i,j) موجود باشد که $۳|p_i-۱$، $۳|p_j-۱$ و برای هر s و t که $۱\leq s,t\leq r$، داشته باشیم $p_s\nmid p_t-۱$؛

(ب) $r\geq ۴$، $n=p_۱p_۲...p_r$، n خالی از مربع بوده و چهارتایی منحصر بفرد (i,j,k,l) موجود باشد به طوری که $p_ip_j|p_k-۱$ و $p_k|p_l-۱$ یا اینکه $p_i|p_j-۱$، $p_j|p_k-۱$ و $p_k|p_l-۱$؛

(ت) $n = ۲p^۲$؛

(ث) $n = ۴p$ که $p > ۳$ و $۴|p - ۱$؛

(د) $r \geq ۲$، $n = p_۱^۲p_۲...p_r$ و برای هر i و j که $۱ \leq i, j \leq r$، داشته باشیم $p_i \nmid p_j - ۱$، و عدد منحصر بفرد k، $۲ \leq k \leq r$، موجود باشد به طوری که $p_۱^۲|p_k - ۱$؛

(ذ) $r \geq ۲$، $n = p_۱^۲p_۲...p_r$ و برای هر i که $۲ \leq i \leq r$، داشته باشیم $p_۱^۲ \nmid p_i - ۱$ و دوتایی منحصر بفرد (k, l) چنان موجود باشد که $p_۱|p_k - ۱$ و $p_l|p_۱^۲ - ۱$؛

(ر) $n = p_۱^۳p_۲...p_r$ و برای هر i و j که $۱ \leq i, j \leq r$، داشته باشیم $p_i \nmid p_j - ۱$.

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بررسی دو تعمیم از حدس هوپرت برای برخی گروه‌های متناهی

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چکیده. پس از حدس هوپرت درباره‌ی تشخیص‌پذیری گروه‌های ساده‌ی متناهی، دو حدس جدید درباره‌ی تشخیص‌پذیری گروه‌های شبه‌ساده و تقریباً ساده‌ی متناهی مطرح شد. در این مقاله به تشخیص‌پذیری گروه‌های شبه ساده‌ی پراکنده و همچنین گروه سیمپلکتیک همدیس تصویری در بعد چهار خواهیم پرداخت.

واژه‌های کلیدی: گروه‌های متناهی-گروه‌های شبه ساده-گروه‌های تقریباً ساده-درجه سرشت‌های تحویل‌ناپذیر-حدس هوپرت.
طبقه‌بندی موضوعی [۲۰۱۰]: 20C34, 20C33, 20C15.

۱. پیش‌گفتار

مبحث درجه سرشت یکی از موضوعات مهم در نظریه‌ی سرشت‌ها است. سوال مهم در این موضوع بررسی تاثیرپذیری و تاثیرگذاری مجموعه‌ی درجه سرشت‌های تحویل‌ناپذیر گروه G روی آن گروه می‌باشد. مجموعه‌ی درجه سرشت‌های تحویل‌ناپذیر گروه G اطلاعاتی در مورد ساختار گروه به ما می‌دهد. به عنوان مثال گروه G آبلی است اگر و تنها اگر $cd(G) = \{۱\}$. اما این مجموعه ساختار کلی گروه G را لزوماً به طور دقیق مشخص نمی‌کند. به عبارت دیگر ممکن است دو گروه غیر یکریخت دارای مجموعه‌ی درجه سرشت‌های تحویل‌ناپذیر یکسان باشند.

بنابراین گروه‌های متناهی به واسطه‌ی مجموعه‌ی درجه سرشت‌های تحویل‌ناپذیرشان لزوماً قابل شناسایی نیستند. به طور مثال توجه کنید که برای هر گروه آبلی متناهی مجموعه‌ی درجه سرشت‌های تحویل‌ناپذیر تنها شامل عدد ۱ است، پس گروه‌های آبلی متناهی توسط درجه سرشت‌هایشان تشخیص‌پذیر نیستند. همچنین در مورد گروه‌های غیرساده‌ی ناآبلی، مجموعه‌ی درجه سرشت‌های تحویل‌ناپذیر یک گروه لزوماً ساختار آن را تعیین نمی‌کند. به عنوان مثال داریم $cd(Q_۸) = cd(\mathbb{S}_۳)$، درحالیکه $\mathbb{S}_۳$ پوچتوان نیست اما $Q_۸$ پوچتوان است. به علاوه دو گروه $Q_۸$ و $D_۸$ نه تنها مجموعه‌ی درجه سرشت‌های برابر، بلکه جدول سرشت‌های برابر دارند. بنابراین نه تنها توسط مجموعه‌ی درجه سرشت‌هایشان بلکه حتی به واسطه‌ی جدول سرشتشان نیز قابل شناسایی نیستند. اما مسئله تشخیص پذیری گروه‌ها توسط مجموعه‌ی درجه سرشت‌های تحویل‌ناپذیر آن‌ها، در مورد گروه‌های ساده‌ی ناآبلی متناهی متفاوت است.

در سال ۲۰۰۰ هوپرت در [۱] حدسی را مبنی بر تشخیص‌پذیری گروه‌های ساده‌ی ناآبلی مطرح کرد. حدس هوپرت بیان می‌کند که هر گروه ساده‌ی ناآبلی با تقریب ضرب مستقیم در یک گروه آبلی، به واسطه‌ی مجموعه‌ی درجه سرشت‌های تحویل‌ناپذیر خود به طور یکتا مشخص می‌شود. به هر حال حدس هوپرت هنوز یک حدس باز است و برای تمامی گروه‌های ساده‌ی ناآبلی به اثبات نرسیده است. در عین حال این حدس برای بسیاری از گروه‌های ساده‌ی ناآبلی اثبات شده و تا کنون هیچ مثال نقضی برای آن یافت نشده است.
از این پس مجموعه‌ی درجه سرشت‌های تحویل‌ناپذیر گروه G را با نماد $cd(G)$ نمایش می‌دهیم.

حدس هوپرت: فرض کنید S گروه ساده‌ی ناآبلی متناهی و G یک گروه متناهی باشد. در این صورت $cd(G) = cd(S)$ اگر و تنها اگر $G \cong S \times A$ به‌طوری‌که A یک زیرگروه آبلی از G است.

هوپرت حدس خود را برای بسیاری از گروه‌های ساده‌ی ناآبلی متناهی در [۱]، [۲] و [۳] اثبات کرد. بعد از او، ریاضی‌دانان بسیار دیگری نیز به بررسی این حدس برای گروه‌های ساده‌ی ناآبلی متناهی مختلف پرداختند. به طور مثال:

- هوپرت حدس خود را برای $PSp_۴(q)$ زمانی که q میدان ۳، ۴، ۵ یا ۷ عضوی باشد در [۳] اثبات کرد.
- ویکفیلد حدس هوپرت را برای $PSp_۴(q)$ زمانی که $q > ۷$ باشد در [۴] اثبات کرد.
- همچنین حدس هوپرت برای تمام گروه‌های ساده‌ی پراکنده در مقالات مختلف توسط ریاضی‌دانان مختلف به اثبات رسیده است.

گروه‌های متناهی تقریباً ساده و شبه ساده پس از گروه‌های ساده‌ی ناآبلی متناهی، نزدیکترین رده از گروه‌های متناهی هستند که مسئله تشخیص‌پذیری برای آن‌ها امکان‌پذیر می‌نماید.

نوان و همکارانش در سال ۲۰۱۵ در [۵] تعمیمی برای حدس هوپرت ارائه کردند که در آن به جای در نظر گرفتن گروه‌های ساده‌ی ناآبلی متناهی، گروه‌های شبه ساده را در نظر گرفتند. آن‌ها در این مقاله تعمیم خود را برای گروه $SL(۲, q)$ با $q \geqslant ۵$ و $SL(۳, q)$ با $q \geqslant ۲$ اثبات نمودند. این حدس جدید ادعا می‌کند که اگر G گروهی متناهی و H گروه متناهی شبه ساده باشد به طوری که ضربگرد

* سخنران

شور $\frac{H}{Z(H)}$، یعنی $Mult(\frac{H}{Z(H)})$ گروهی دوری باشد، در این صورت داریم $cd(G)=cd(H)$ اگر وتنها اگر $G \cong HoA$، یعنی G ضرب مرکزی H و گروه آبلی A باشد.

حدس نوان و همکاران: فرض کنید G یک گروه متناهی و H گروه شبه‌ساده‌ی متناهی با $Mult(\frac{H}{Z(H)})$ دوری باشد. در این صورت $cd(G)=cd(H)$ اگر و تنها اگر $G \cong HoA$.

همان طور که گفته شد حدس بالا را می‌توان به عنوان تعمیمی از حدس هوپرت، برای گروه‌های شبه‌ساده مختلف بررسی نمود. توجه کنید که از ضرب مرکزی $G \cong HoA$ می‌توان نتیجه گرفت که G تعمیمی از H است و این تعمیم، حالت کلی‌تر از ضرب مستقیم $G \cong H \times A$ است. همچنین می‌دانیم هر گروه ساده متناهی، شبه‌ساده نیز هست. بنابراین حدس فوق واقعاً تعمیمی از حدس هوپرت می‌باشد. نویسندگان در مقاله‌ی یاد شده تاکید کردند که شرط دوری بودن ضربگرد شور الزامی است. آن‌ها مثال $G=۲^{۲}{\cdot}\Omega_{۸}^{+}(۲)$ و $H=G/Z$ که Z زیرگروه مرکزی G از مرتبه‌ی ۲ است را قید کردند. در این مثال $cd(G)=cd(H)$، اما G ضرب مرکزی از H با هیچ زیرگروه آبلی نیست.

در سال ۲۰۱۵ علوی و همکارانش در [۶] موفق گردیدند با الهام از حدس هوپرت، شکل متفاوتی از آن را برای گروه‌های تقریباً ساده از نوع ماتیو مطرح و به این روش تشخیص‌پذیری این دسته از گروه‌ها را به واسطه‌ی مجموعه‌ی درجه سرشت‌های تحویل‌ناپذیرشان، اثبات کنند. این مقاله در سال ۲۰۱۸ به چاپ رسید. نتیجه‌ی اصلی این مقاله اثبات قضیه‌ای است که بیان می‌کند اگر G گروهی متناهی و S یک گروه ماتیو باشد به طوری که داشته باشیم $S \leqslant H \leqslant Aut(S)$، آنگاه تساوی $cd(G)=cd(H)$ نتیجه می‌دهد زیرگروه نرمال و آبلی A موجود است که $H \cong \frac{G}{A}$. علوی و همکارانش همین قضیه را در سال ۲۰۱۶ در [۷] برای گروه‌های تقریباً ساده با بنیان گروه‌های ساده پراکنده نیز ثابت کردند. همچنین آن‌ها با بیان مثالی نشان دادند که حدس هوپرت به طور مستقیم قابل تعمیم به گروه‌های تقریباً ساده نیست. به طور مثال چهار گروه $\mathbb{Z}_۴ : \mathbb{A}_۵$، $\mathbb{S}_۵ \times \mathbb{Z}_۲$، $SL_۲(۵) : \mathbb{Z}_۲$ و $SL_۲(۵){\cdot}\mathbb{Z}_۲$ همگی از مرتبه‌ی ۲۴۰ و با مجموعه درجه سرشت‌های برابر با گروه $Aut(\mathbb{S}_۵)=\mathbb{A}_۵$ هستند. اما تمامی آن‌ها را نمی‌توان به صورت ضرب مستقیم $\mathbb{S}_۵$ با یک گروه آبلی نوشت.

در سال ۲۰۲۰ در [۸] شیرژیان و ایرانمنش تعمیمی کلی از حدس هوپرت را برای گروه‌های تقریباً ساده ارائه و در همین مقاله حدس خود را برای گروه‌های تقریباً ساده با بنیان گروه‌های لی $PSL(۳,q)$ و $PSU(۳,q^{۲})$ اثبات کردند. این حدس را می‌توان در حالت کلی به عنوان تعمیمی برای حدس هوپرت در نظر گرفت و آن را برای گروه‌های تقریباً ساده مختلف بررسی کرد. توجه کنید که از $H \cong \frac{G}{A}$ می‌توان نتیجه گرفت که G تعمیمی از H است و این تعمیم، حالت کلی‌تر از ضرب مرکزی $G \cong H \times A$ است. همچنین می‌دانیم هر گروه ساده متناهی، تقریباً ساده نیز هست. بنابراین حدس فوق واقعاً تعمیمی از حدس هوپرت می‌باشد.

حدس شیرژیان-ایرانمنش: فرض کنید G یک گروه متناهی و H گروه تقریباً ساده با بنیان گروه ساده‌ی نوع لی باشد به‌طوری‌که $cd(G)=cd(H)$. در این صورت $G/A \cong H$ است جایی‌که A زیرگروه آبلی از G است.

نویسندگان با ارائه‌ی مثالی نشان دادند که بر خلاف حدس هوپرت، عکس این حدس جدید لزوماً برقرار نیست. آن‌ها مثالی ساختند که در آن، گروه خارج قسمتی G/A که A گروهی آبلی است، یکریخت با گروه تقریباً ساده‌ی H است در حالی که $cd(G) \neq cd(H)$.

در این مقاله ما درباره‌ی دو تعمیم مطرح شده از حدس هوپرت برای گروه‌های شبه ساده‌ی پراکنده و همچنین گروه سیمپلکتیک همدیس تصویری چهاربعدی $PCSp_۴(q)$ صحبت خواهیم کرد.

لازم به ذکر است که گروه $PCSp_۴(q)$، در واقع یک گروه تقریباً ساده با بنیان $PSp_۴(q)$ است که توسط خودریختی قطری این گروه گسترش داده شده است. توجه کنیم که روی میدان‌های زوج عضوی $PCSp_۴(q) \cong PSp_۴(q)$، از طرفی حدس هوپرت برای گروه ساده‌ی $PSp_۴(q)$ قبلا در مقالات [۳] و [۴] اثبات شده است، در نتیجه ما به اثبات تعمیم حدس هوپرت برای $PCSp_۴(q)$ تنها روی میدان‌های فرد عضوی پرداخته‌ایم.

۲. نتایج اصلی

در اولین قضیه از این مقاله، ما به بررسی حدس نوان و همکارانش برای گروه‌های شبه‌ساده‌ی پراکنده به جز $۲{\cdot}M_{۱۲}$ می‌پردازیم. نتایج مورد بحث در این بخش، در مقاله‌ی چاپ شده‌ی [۹] ارائه گردیده اند.

قضیه ۱.۲. فرض کنید G یک گروه متناهی و H یک گروه شبه ساده‌ی پراکنده به جز $۲{\cdot}M_{۱۲}$ باشد، در این صورت $cd(G)=cd(H)$ اگر و تنها اگر $G \cong HoA$، ضرب مرکزی از H با یک گروه آبلی A باشد.

طرح اثبات. برای اثبات قضیه‌ی فوق مراحل زیر را به ترتیب طی کرده‌ایم:

مرحله‌ی اول: نشان داده‌ایم $G'=G''$.

مرحله‌ی دوم: فرض کنیم G'/M یک عامل اصلی از G است. با توجه به تام بودن گروه G'، می‌توان گفت $G'/M \cong S^k$ به‌طوری‌که S گروه ساده‌ی ناآبلی و k یک عدد صحیح بزرگتر یا مساوی یک است. نشان داده‌ایم $k = 1$ و گروه ساده‌ی S نیز کاملا قابل تشخیص است و داریم $G'/M \cong H/Z(H)$.

نکته ۲.۲. علت حذف شدن گروه شبه‌ساده‌ی $2{\cdot}M_{12}$ دقیقاً به همین مرحله برمی‌گردد. در این مرحله باید نشان دهیم $G'/M \cong M_{12}$. به عبارت دیگر باید تمامی گروه‌های ساده‌ی ناآبلی دیگر را به کمک درجه سرشت مناسبی از آن‌ها، از کاندید بودن برای S حذف کنیم. روش کار به این صورت است که درجه سرشت مناسبی از گروه ساده‌ی ناآبلی $S \neq M_{12}$ را به گونه‌ای انتخاب می‌کنیم که گسترش‌پذیر به $Aut(S)$ باشد. سپس نتیجه می‌گیریم که درجه سرشت انتخابی، در واقع باید درجه سرشتی از گروه G نیز باشد، اما چون درجه سرشت‌های G به طور کامل در دسترس هستند و چنین درجه‌ای در بین آن‌ها یافت نمی‌شود به تناقض می‌رسیم. این در حالی است که $cd(M_{11}) \subset cd(G)$. همچنین $cd(PSL_2(11)) - \{5\} \subset cd(G)$ که البته این درجه سرشت ۵ نیز کمکی به حذف گروه ساده‌ی $PSL_2(11)$ نخواهد کرد چرا که گسترش پذیر به $Aut(PSL_2(11))$ و در نتیجه گسترش‌پذیر به G نیست که منجر به ایجاد تناقض شود. بنابراین در این مرحله هر سه گروه $PSL_2(11)$، M_{11} و M_{12} نامزدهای مناسبی برای G'/M هستند و در نتیجه نمی‌توان نشان داد که $G'/M \cong M_{12}$. ما به اثبات حدس نوان و همکارانش برای گروه $2{\cdot}M_{12}$ در مقاله‌ی دیگر و از طریق مسیر دیگر خواهیم پرداخت.

مرحله‌ی سوم: نشان داده‌ایم که G' یکریخت با یک پوشش مرکزی تام برای $H/Z(H)$ است.

مرحله‌ی چهارم: نشان داده‌ایم که $G = G'oC_G(G')$ به طوری که $C_G(G')$ آبلی است. بنابراین $cd(G) = cd(G')$ و در نتیجه $cd(G') = cd(H)$.

مرحله‌ی پنجم: نشان داده‌ایم پوشش‌های مرکزی تام $H/Z(H)$ دارای مجموعه درجه‌سرشت‌های منحصر به فرد هستند. بنابراین $H \cong G'$ و $G = HoC_G(G')$.

در دومین قضیه از این مقاله، ما به بررسی حدس شیرژیان–ایرانمنش برای گروه تقریباً ساده‌ی $PCSp_4(q)$ می‌پردازیم. نتایج مورد بحث در این بخش، در مقاله‌ی در دست چاپ [۱۰] ارائه گردیده اند.

قضیه ۳.۲. فرض کنید G یک گروه متناهی و H گروه سیمپلکتیک همدیس تصویری $PCSp_4(q)$، توسعه یافته از گروه ساده‌ی سیمپلکتیک تصویری $PSp_4(q)$ توسط خودریختی قطری d از آن باشد. در این صورت اگر $cd(G) = cd(H)$، آنگاه $G/Z(G)$ یکریخت با H است.

۳. دست‌آوردهای پژوهش

قضایای اثبات شده در این پژوهش در واقع گامی در زمینه‌ی تشخیص‌پذیری گروه‌های متناهی توسط درجه سرشت‌های آن‌هاست که در راستای سه حدس مهم در این زمینه انجام شده‌است.

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Index