# Proceedings of The 24th Seminar on Mathematical Analysis and its Applications

26-27 May, 2021

Imam Khomeini International University

Department of Pure Mathematics

In The Name of God

Proceedings of the 24th Seminar on Mathematical Analysis

and its Applications

26-27 May 2021, Qazvin, Iran

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The organizers wish to thank all PhD students of Pure Mathematics Department for their help; in particular, we thank **Vahid Keshavarz** for his technical assistance in preparing this **Book of Extended Abstracts**.

# The 24th Seminar on Mathematical Analysis and its Applications

## Day 1: Wednesday 26 May 2021

| Iran Local Time | Title | | | |
|---|---|---|---|---|
| 8:00- 8:50 | Opening Ceremony<br><br>1. Moments with the Holy Quran<br>2. National Anthem of the Islamic Republic of Iran<br>3. A short clip about Imam Khomeini International University<br>4. Speech by the Vice President for Research and Technology<br>5. Speech by the President of the Iranian Mathematical Society (IMS)<br>7. Conference Chair Report | | | |
| 8:50-9:00 | Break | Coffee at Home | | |
| 9:00-9:50 | B Djafari Rouhani | Asymptotic behavior and periodic solutions to some expansive type evolution and difference equations | | |
| 9:50-10:00 | Break | Coffee at Home | | |
| 10:00-10:50 | M.A. Ragusa | Perspectives in regularity theory of minimizers of variational functionals | | |
| 10:50-11:00 | Break | Coffee at Home | | |
| 11:00-11:50 | A. Perala | Integration operators from Bergman spaces to Hardy spaces of the unit ball | | |
| 11:50-12:00 | Break | Coffee at Home | | |
| Contributed Talks | | | | |
| | **Session A**<br>Banach Algebras, Harmonic Analysis | **Session B**<br>Differential Equations, Functional Analysis | **Session C**<br>Complex Analysis & Operator Theory | **Session D**<br>Convex Analysis, Real Analysis |
| 12:00-12:20 | M.B.Asadi | M.Fotouhi | A.Motamednejad | Z.Kefayati |
| 12:20-12:40 | M.Niazi | S. Heidary | M.Fatehi | M.Darabi |
| 12:40-13:00 | M.Alikahi | S.Haji Aghasi | M.Essmaili | P.Cheraghi |
| 13:00-14:00 | Launch at Home | | | |
| 14:00-14:20 | M.R.Abdollahpour | F.Abdollahi | F.Ayatollahzadeh | H.Pouladi |
| 14:20-14:40 | G.R.Rahimlou | D.Afraz | K.Esmaeili | M.Delfani |
| 14:40-15:00 | O.Zabeti | A.Armandnejad | M.Haji shaabani | H.Faraji |
| 15:00-15:20 | Coffee at home | | | |
| 15:20-15:40 | A.H.Sanatpour | F.Abdolrazaghi | H.Rahmatan | M.R.Ansari |
| 15:40-16:00 | M.R.Farmani | F.Safari | M.Hassanlou | M.Paknazar |
| 16:00-16:20 | E.Osgooei | N.Karamikabir | S.Rahrovi | |
| 16:20-17:00 | Break | | | |
| Iran Local Time | | Title | | |
| 17:00-17:50 | Xiaoqi Yang | Extended Newton methods for multiobjectives optimization: majorizing function technique and difference analysis | | |
| 17:50-18:00 | Break | Coffee at Home | | |
| 18:00-18:50 | H. Ammari | Wave interaction with subwavelength resonators | | |
| 18:50-19:00 | Break | Coffee at Home | | |
| 19:00-19:50 | D. Ehsani | Boundary value problems on intersection domains | | |
| End of Day 1 | | | | |

**Day 2: Thursday 27 May 2021**

| Iran Local Time | Title | | Title | |
|---|---|---|---|---|
| 8:00- 8:50 | A. Moradifam | | The sphere covering inequality and its applications | |
| 8:50-9:00 | Break | | Coffee at Home | |
| 9:00-9:50 | M.A. Khamsi | | Hahn-Banach theorem in metric spaces: An introduction to hyperconvexity | |
| 9:50-10:00 | Break | | Coffee at Home | |
| 10:00-10:50 | A.P.Schuster | | Bergman spaces with exponential weights | |
| 10:50-11:00 | Break | | Coffee at Home | |
| 11:00-11:50 | R. Espinola Garcia | | Density and smooth extension of Lipschitz functions on metric spaces | |
| 11:50-12:00 | Break | | Coffee at Home | |
| ***Contributed Talks*** | | | | |
| | **Session A** Banach Algebras, Harmonic Analysis | **Session B** Differential Equations, Functional Analysis | **Session C** Complex Analysis, Operator Theory | **Session D** Convex Analysis, Real Analysis |
| 12:00-12:20 | R.Rezavand | M T Heydari | S.Hasannejad | E.Amini |
| 12:20-12:40 | A.Sahami | V. Keshavarz | S.Eskandari | D.Haghighi |
| 12:40-13:00 | M.Asadipour | Y. Khalili | E.Najafi | H.Mehravaran |
| 13:00-14:00 | Launch at Home | | | |
| 14:00-14:20 | A.Shirinkalam | S.Shakeri | A.Zivari | A.Morassei |
| 14:20-14:40 | E.Tamimi | M.Mirzapour | Z.Kamali | M.Namjoo |
| 14:40-15:00 | A.Shekari | M.Latifi | A.Niazi Motlagh | |
| 15:00-15:20 | Coffee at home | | | |
| 15:20-15:40 | K.Sharifi | A.Pishkoo | M.Mosadeq | Ghasemzadehdibag |
| 15:40-16:00 | P.Haghmaram | M.J.Mehdipour | H.Ardakani | M.Parsamanesh |
| 16:00-16:20 | H.Ghasemi | R.Parvinianzadeh | Y.Khatib | |
| 16:20-17:00 | Break | | | |
| Iran Local Time | | | Title | |
| 17:00-17:50 | H.Hedenmalm | | Gaussian analytic functions and operator symbols of Dirichlet type | |
| 17:50-18:00 | Break | | Coffee at Home | |
| 18:00-18:50 | Q. T. Bao | | Recent developments of Ekeland's variational theorem | |
| 18:50-19:00 | Break | | Coffee at Home | |
| 19:00-19:40 | V.Colao | | Two problems related to iterative sequences and fixed points | |
| 19:40-20:10 | Closing Ceremony | | | |

**Dear Colleagues and Friends**,

It is our pleasure to welcome you to the $24^{th}$ **Seminar on Mathematical Analysis and its Applications**, held during May $26-27$, 2021 at Department of Pure Mathematics, Imam Khomeini International University, Qazvin, Iran. The Seminar on Mathematical Analysis and its Applications is one the oldest seminars initiated by the Iranian Mathematical Society in 1979. The first seminar was hosted by Ferdowsi University of Mashhad, and the latest one by Arak University in 2019. We would like the seminar to become the platform to raise discussions and creating joint research projects. You are possibly aware that the seminar was initially planned for February 2021, but due to the outbreak of COVID-19 pandemic we have decided to postpone it to May 2021, as well as to reschedule it as a virtual one. Planning a virtual seminar has the advantage that more distinguished mathematicians from around the globe may participate in the event. Fortunately, this goal was achieved, and 13 eminent figures form Germany, Hong Kong, Italy, Spain, Sweden, Switzerland, and the United States have accepted our invitation to deliver invited talks in a variety of subjects including ODE, PDE, Linear and Nonlinear Analysis, Complex Analysis, Operator Theory in Function Spaces, Variational Analysis, and so on.
We are happy to announce that the seminar was well received by mathematicians of our country. We have got more than 150 quality papers; but the time slot was limited, so that by the recommendation of our prominent scientific committee members we were obliged to be more selective to accept just 69 papers for contributed talks. Needless to say that some of the accepted papers were transferred to poster presentation section; this does not necessarily mean that the papers lack the quality requirements of the scientific committee.
The seminar's program includes 13 invited talks, 69 contributed talks, and 38 poster presentations. We have assigned a 6 character code to every contributed talk; the first character is A, B, C, or D, followed by a five- digit number. These letters stand for Sections A, B, C, or D. The subsequent number following the letter is either 1 or 2, which indicates that the talk will be presented on Day 1 or Day 2 of the seminar. The last four-digit number represents the time of presentation.
For example, the code
**A11430** means that the talk will be presented in **Section A, on Day 1, at** $14:30$**.**
A lecture whose code begins with "$A$" will takes place at room A of

$$https://webinar.ikiu.ac.ir/ch/rooma/l/en$$

The same applies to the letters B, C, and D.
We wish to express our sincere thanks to our invited speakers and all participants for sharing their latest findings in this seminar. Last, but not least, we record our gratitude to our scientific committee members, twenty reputed analysts from across the country, for their invaluable efforts in evaluating the received papers in a reasonable amount of time. Their prompt action and accuracy of decision is greatly appreciated. We also record our thanks to our executive committee at IKIU for their cooperation and help; they have done everything to make this event pleasant.
Finally, we wish you every success in the future, and hope that you all will enjoy this event.

**Abdolrahman Razani (Chair),**
**Ali Abkar (Chair of Scientific Committee),**
**Morteza Oveisiha (Chair of Executive Committee)**

# CONTENTS

# Invited Talks

# Wave interaction with subwavelength resonators

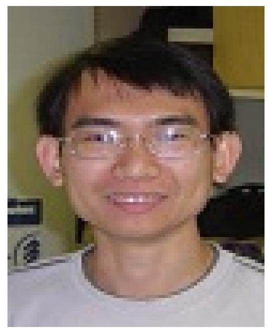
**Habib Ammari**

**Department of Mathematics, Rämistrasse 101, 8092 Zürich, Switzerland.**
**Email: habib.ammari@math.ethz.ch**

ABSTRACT

In this lecture, the speaker reviews recent results on subwavelength resonances. His main focus is on developing a mathematical and computational framework for their analysis. By characterizing and exploiting subwavelength resonances in a variety of situations, he proposes a mathematical explanation for super-focusing of waves, double-negative metamaterials, Dirac singularities in honeycomb subwavelength structures, and topologically protected defect modes at the subwavelength scale.

# Recent developments of Ekeland's variational theorem



**Truong Q. Bao**

**Department of Mathematics and Computer Science,
Northern Michigan University, Marquette, Michigan, USA.
Email: btruong@nmu.edu**

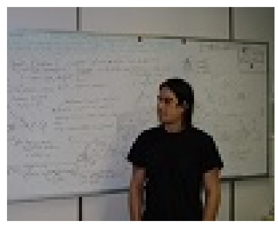
ABSTRACT

This talk discusses recent developments of Ekeland's variational theorem in many directions:

- from scalar functions to vector-valued functions and to set-valued mappings;
- from metrics to generalized distances and to pseudo-quasi-metrics;
- from vector spaces with fixed ordering cone to vector spaces with variable domination structures;
- from Pareto-type binary orders to Kuroiwa-type binary orders;
- from the lower semi-continuity to the strictly decreasing lower semi-continuity;
- from the boundedness from below to quasi-boundedness from below and to scalarized boundedness;
- from bifunctions with triangle inequality to those with cyclical antimonotonicity; etc.

We compare and contrast two known approaches known as the scalarization approach based on the so-called Tammer nonlinear scalarization function and the variational approach that does not use any scalarization technique. It is certain that a combination yields better results.

# Two problems related to iterative sequences and fixed points



**Vittorio Colao**

**Dipartimento di Matematica, Università della Calabria,
87036 Arcavacata di Rende (CS), Italy.
Email: colao@mat.unical.it**

ABSTRACT

Iterative methods for approximating fixed points had been one of the most flourishing topic in recent years. Nevertheless, there is still room for new problems.
In this talk, we will focus on two questions regarding fixed point iterations: the first is related to the approximations of fixed point for nonself mappings without using auxiliary mappings, while the other regards the convergence rate of certain strongly converging sequences. We give partial solutions to both problems and analyse further extensions.

## Boundary value problems on intersection domains

**Dariush Ehsani**

**FIZ Karlsruhe-Leibniz Institute, Department of Mathematics, Franklinstr. 1110587, Berlin**
**Email: dehsani.math@gmail.com**

ABSTRACT

A pseudodifferential calculus for studying Boundary Value Problems on smooth domains has been extensively worked out from the foundational work of Boutet de Monvel and Hörmander. We review some of these methods and discuss new methods which can be applied to domains which are intersections of smooth domains.

# Density and smooth extesion of Lipschitz functions on metric spaces

**Rafael Espinola Garcia**

**Department of Mathematical Analysis, University of Seville**
**Apdo. 1160, 41080 Sevilla, Spain.**
**Email: espinola@us.es**

ABSTRACT

The main topic of this talk is to study the density and smooth extension of Lipschitz functions defined on metric spaces. This topic has been deelply developed firstly on finite dimensional Banach spaces and, more recently, on infinite dimensional ones as well as some kinds of differential manifolds. In this talk we will walk through the history of the problem from the early 20th century to end with some new results for Lipschitz mappings defined on metric spaces with a Measurable Differentiable Structure (MDS). Measurable differentiable structures on metric spaces were first introduced by J. Cheeger in 99 as an attempt to develop a robust theory of first order differentiability on metric spaces. This work had a big impact in the theory of analysis on metric spaces as many authors worked on these structures since then. New results here presented are part of a joint work with Luis Sánchez-González.

# Gaussian analytic functions and operator symbols of Dirichlet type

**Haakan Hedenmalm**

**Department of Mathematics, KTH Royal Institute of Technology, S – 100 44 Stockholm, Sweden.**
**Email: haakanh@kth.se**

ABSTRACT

We consider two copies of the Gaussian analytic function associated with the Dirichlet space. We then consider the correlation of the two copies, and study how big the analytic correlation may be, in an average sense.

# Hahn-Banach theorem in metric spaces: An introduction to hyperconvexity

**Mohamed Amine Khamsi**

**Department of Mathematical Science, The University of Texas at El Paso, El Paso, TX 79968, USA.**
**Email: mohamed@utep.edu**

## ABSTRACT

While investigating an extension of the Hahn-Banach theorem to metric spaces, Aronszajn and Panitchpakdi (1956) discovered the concept of hyperconvexity or injectivity. The main result of their investigation is the fact that hyperconvex metric spaces are absolute nonexpansive retracts ($ANR$). In this talk, we give an elementary introduction to hyperconvexity in metric spaces. If time permits, we will mention the relationship between the fixed point problem and hyperconvexity.

# The sphere covering inequality and its applications

**Amir Moradifam**

**Department of Mathematics, University of California, Riverside, California, USA.**
**Email: moradifam@math.ucr.edu**

ABSTRACT

We show that the total area of two distinct Gaussian curvature 1 surfaces with the same conformal factor on the boundary, which are also conformal to the Euclidean unit disk, must be at least $4\pi$. In other words, the areas of these surfaces must cover the whole unit sphere after a proper rearrangement. We refer to this lower bound of total areas as the Sphere Covering Inequality. This inequality and its generalizations are applied to a number of open problems related to Moser-Trudinger type inequalities, mean field equations and Onsager vortices, etc, and yield optimal results. In particular we confirm the best constant of a Moser-Truidinger type inequality conjectured by A. Chang and P. Yang in 1987. This is a joint work with Changfeng Gui.

# Integration operators from Bergman spaces to Hardy spaces of the unit ball

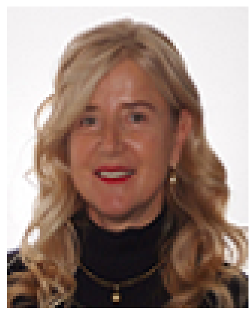
**Antti Perälä**

**Departamento de Análisis Matemático, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain.**
**Email: antti.perala@gu.se**

## ABSTRACT

We completely characterize the boundedness of the Volterra type integration operators acting from the weighted Bergman spaces to the Hardy spaces of the unit ball. A partial solution to this problem in the one-dimensional setting was previously obtained by Zhijian Wu. We solve the missing cases and extend the results to all dimensions. Our tools involve area methods from harmonic analysis, Carleson measures and Kahane-Khinchine type inequalities, as well as techniques and integral estimates related to Hardy and Bergman spaces.
This talk is based on a joint paper with Santeri Miihkinen, Jordi Pau and Maofa Wang.

# Perspectives in regularity theory of minimizers of variational functionals



**Maria Alessandra Ragusa**

**Dipartimento di Matematica e Informatica, Universitá di Catania, Italy.**
**Email: maragusa@dmi.unict.it**

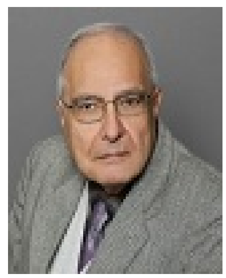
ABSTRACT

Let us consider $\Omega \subset \mathbb{R}^m$ $(m \geq 2)$ be a bounded open set. For maps $u : \Omega \to \mathbb{R}^n$ we consider the $p(x)$-energy functional defined as

$$\mathcal{E}(u,\Omega) := \int_\Omega \Big( g^{\alpha\beta}(x) G_{ij}(u) D_\alpha u^i(x) D_\beta u^j(x) \Big)^{\frac{p(x)}{2}} dx,$$

where $(g^{\alpha\beta}(x))$ and $(G_{ij}(u))$ are symmetric positive definite matrices whose entries are continuous functions defined on $\Omega$ and $\mathbb{R}^n$ respectively, and $p(x)$ a continuous function on $\Omega$ with $p(x) \geq 2$.
Main goal is the study of regularity properties, interior and up to the boundary, of the minimizers $u$ of $\mathcal{E}$ and recent developments in this direction.

# Asymptotic behavior and periodic solutions to some expansive type evolution and difference equations



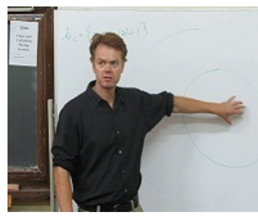
**Behzad Djafari Rouhani**

**Department of Mathematical Sciences, University of Texas at El Paso, 500 W. University Avenue, El Paso, Texas, 79968, USA.**
**Email: behzad@utep.edu**

ABSTRACT

In this talk, after reviewing some old results in nonlinear ergodic theory and their applications to the study of the asymptotic behavior of quasi-autonomous dissipative systems, we concentrate on first order expansive type evolution and difference equations and present some old and new results on the asymptotic behavior of the solutions, as well as periodic solutions to such systems.
This is joint work with Mohsen Rahimi Piranfar.

# Bergman spaces with exponential weights



**Alexander P. Schuster**

**San Francisco State University, San Francisco, CA 94132, United States.**
**Email: schuster@sfsu.edu**

## ABSTRACT

In this talk we introduce a kind of Bergman space $A^p_\varphi$ on the unit disk $\mathbf{D}$ with exponential weights, which cover those defined by Borichev, Dhuez and Kellay (in J. Funct. Anal. 242 (2007), 563-606). We obtain upper and lower bound estimates on the Bergman kernel. As an application, we discuss the Bergman projection and duality. This work is joint with Zhangjian Hu and Xiaofen Lv.

# Extended Newton methods for multiobjective optimization: Majorizing function technique and convergence analysis

**Xiaoqi Yang**

**Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong.**
**Email: xiao.qi.yang@polyu.edu.hk**

ABSTRACT

In this we will consider the extended Newton method for approaching a Pareto optimum of multiobjective optimization problems and establish its quadratic convergence criteria and estimation of radius of convergence ball under the assumption that the Hessians of objective functions satisfy an L-average condition. As applications of the obtained results, convergence theorems under the classical Lipschitz condition or the $\gamma$-condition are presented for multiobjective optimization, and the global quadratic convergence results of the extended Newton method with Armijo/Goldstein/Wolf line-search schemes are also provided.

# Contributed Talks and Posters

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**Oral Presentation** $*$ : **Speaker**

# THE NUMERICAL RANGE OF INTERVAL MATRICES

F. ABDOLLAHI* AND H. TAVAKOLIPOUR

*Department of Mathematics, College of Sciences, Shiraz University, Shiraz, Iran*
*abdollahi@shirazu.ac.ir*
*Inria and CMAP, Ecole polytechnique, Palaiseau, France*
*hanieh.tavakolipour@inria.fr*

Abstract. In this paper, we introduce the concept of the numerical range of a square interval matrix. We prove that this numerical range is always compact, but, unlike classical numerical range, the numerical range of an interval matrix is not always convex.

## 1. Introduction

Let $A$ be a (bounded linear) operator on a complex Hilbert space $\mathcal{H}$. The *numerical range* of $A$ is the set

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$$

in the complex plane, where $\langle ., .\rangle$ denotes the inner product in $\mathcal{H}$. In other words, $W(A)$ is the image of the unit sphere $\{x \in \mathcal{H} : \|x\| = 1\}$ of $\mathcal{H}$ under the (bounded) quadratic form $x \mapsto \langle Ax, x\rangle$. One of the basic properties of the numerical range is that it contains all the eigenvalues of a matrix. Two other important properties of the numerical range which we discuss them in this article are convexity (Toeplitz-Hausdorff theorem) and compactness [2]. In this paper, we extend the concept of the numerical range to interval matrices.
Interval arithmetic is an approach that can be used to bound measurement errors, rounding and truncation errors in mathematical computations. As an area of numerical analysis, it has several applications in

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*Key words and phrases.* numerical range; interval matrix; interval norm; convexity; compactness.

different branches of science and engineering. One of the advantages of interval analysis is its ability to compute bounds on the range of functions. By extending interval arithmetic to vectors and matrices, it became a useful tool to find reliable and guaranteed solutions to matrix equations and optimization problems. For more details see [6, 8, 7].

The interval eigenvalue problem, which is the task of finding intervals that contain all the possible eigenvalues of the matrices in an interval matrix, has an important role in many engineering problems. See for example [4, 3] for further information about the efforts for solving interval eigenvalue problem. Due to the spectral inclusion property of the numerical range, the numerical range of interval matrices, contains all the eigenvalues of the matrices in the interval. Therefore, finding the numerical range of interval matrices can help us to solve the interval eigenvalue problems.

The paper is organized as follows. In Section 2, we consider some basic definitions, notations and properties of interval arithmetic. In Section 3, we define the concept of the numerical range for real interval matrices, show its relation with interval norm, prove its compactness and give a counterexample that demonstrates the numerical range of interval matrices is not always convex.

## 2. Definitions, notations and basic facts

In this section we give some basic definitions, notations and properties of interval arithmetic.

**Definition 2.1.** [5] An interval matrix is defined as

$$\mathbf{A} := \left[\underline{A}, \overline{A}\right] = \{A \in \mathbb{R}^{m\times n} : \underline{A} \leq A \leq \overline{A}\},$$

where $\underline{A} = (\underline{a}_{ij})$, $\overline{A} = (\overline{a}_{ij}) \in \mathbb{R}^{m\times n}$, $\underline{A} \leq \overline{A}$ , are given *point matrices*, where the concept of point matrix is used to mention any conventional matrix.

Throughout this paper, the notation $Conv(E)$ is used for the convex hull of the set $E$, the set of $m \times n$ real interval matrices is denoted by $\mathbb{IR}^{m\times n}$ and the comparison relations $\leq$ and $\geq$ are interpreted component-wise.

**Definition 2.2.** (Interval Matrix Norm) [1] A function $|||\cdot||| : \mathbb{IR}^{m\times n} \to \mathbb{R}$ is called an interval matrix norm in $\mathbb{IR}^{m\times n}$ if for each $\mathbf{A}, \mathbf{B} \in \mathbb{IR}^{m\times n}$ and $\alpha \in \mathbb{R}$ it satisfies the following properties:

(1) $|||\mathbf{A}||| \geq 0$, and $|||\mathbf{A}||| = 0$ if and only if $\mathbf{A} = [0, 0]$;

(2) $|||\mathbf{A}+\mathbf{B}||| \le |||\mathbf{A}||| + |||\mathbf{B}|||$;
(3) $|||\alpha\mathbf{A}||| = |\alpha| |||\mathbf{A}|||$,

where $|\cdot|$ is the conventional absolute value.

The following theorem shows how to construct interval matrix norms from point matrix norms.

**Theorem 2.3.** [1] *For any point matrix norm $\|\cdot\|$ in $\mathbb{R}^{m\times n}$, the function $|||\cdot||| : \mathbb{IR}^{m\times n} \to \mathbb{R}$ defined by*

$$|||\boldsymbol{A}||| = \sup\{\|\beta\| : \beta \in \boldsymbol{A}\},$$

*is an interval matrix norm in $\mathbb{IR}^{m\times n}$.*

## 3. The numerical range of $n \times n$ interval matrices

In this section after defining the concept of the numerical range and the numerical radius of interval matrices, we show that like the conventional case it is compact. Also, by an example we will show that in general we do not have the property of convexity.

**Definition 3.1.** Let $\mathbf{A} \in \mathbb{IR}^{n\times n}$. We define the numerical range of $\mathbf{A}$ by

$$W(\mathbf{A}) := \bigcup_{A\in\mathbf{A}} W(A),$$

where $W(A)$ is the conventional numerical range of the point matrix $A \in \mathbb{R}^{n\times n}$. Also, the numerical radius of $\mathbf{A}$ is defined by

$$r(\mathbf{A}) := \sup\{r(A) : A \in \mathbf{A}\},$$

where $r(A)$ is the conventional numerical radius of $A$.

**Lemma 3.2.** *Suppose that $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. Then*

$$W(\boldsymbol{A}) \subseteq \{\mu \in \mathbb{C} : |\mu| \le |||\boldsymbol{A}|||\}.$$

**Lemma 3.3.** *Let $\boldsymbol{A} = \left[\underline{A}, \overline{A}\right] \in \mathbb{IR}^{n\times n}$ and $\underline{A} \ge 0$. Then*

$$W(\boldsymbol{A}) \subseteq \{\mu \in \mathbb{C} : |\mu| \le \|\overline{A}\|_\infty\}.$$

**Theorem 3.4.** *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. Then $W(\boldsymbol{A})$ is a compact set.*

In the following example we will show that, unlike the conventional numerical range, the numerical range of interval matrices is not necessarily convex.

**Example 3.5.** Assume that $\mathbf{A} = \left\{ A_\alpha = \begin{bmatrix} 11 & 5 \\ \alpha & 0 \end{bmatrix} : \alpha \in [1,6] \right\}$. It is clear that the eigenvalues of $A_\alpha$ are

$$\lambda_{1,2} = \frac{11 \pm \sqrt{121 + 20\alpha}}{2}.$$

The numerical range of $A_\alpha, W(A_\alpha)$, is an ellipse with minor and major axis of $|\alpha - 5|$ and $\sqrt{\alpha^2 + 10\alpha + 146}$, respectively (see [2]). It is easy to check that the points $(x_1, y_1) = (\frac{11}{2}(1-\sqrt{2}), 0)$ and $(x_2, y_2) = (\frac{11}{2}, 2)$ are in $W(A_\alpha)$, for $\alpha = 6$ and $\alpha = 1$, respectively. We will show that $(x^*, y^*) = (-1, \frac{22-13\sqrt{2}}{11})$, a point on the straight line segment joining $(x_1, y_1)$ and $(x_2, y_2)$, is not in $W(\mathbf{A})$.

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# $\alpha$-DUAL OF G-FRAMES IN FINITE DIMENSIONAL HILBERT SPACES

MOHAMMAD REZA ABDOLLAHPOUR

*Department of Mathematics, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran*
*mrabdollahpour@yahoo.com, m.abdollah@uma.ac.ir*

Abstract. In this paper we introduce $\alpha$-dual of a $g$-frame and we get some results when we use the Hilbert-Schmidt norm for the members of a $g$-frame in a finite dimensional Hilbert space.

## 1. Introduction

G-frames for Hilbert spaces defined by Sun in 2006 [3]. Let $\mathcal{H}$ be a separable Hilbert space and $\{\mathcal{H}_i\}_{i\in I}$ be a sequence of separable Hilbert spaces. We wall a sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ a $g$-Bessel sequence for $\mathcal{H}$ if there exists $B > 0$ such that

$$\sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \qquad f \in \mathcal{H}. \tag{1.1}$$

Let us define

$$\left(\sum_{i\in I} \oplus \mathcal{H}_i\right)_{l_2} = \left\{\{g_i\} : g_i \in \mathcal{H}_i, \sum_{i\in I} \|g_i\|^2 < \infty\right\}$$

with the inner product given by $\langle\{f_i\}, \{g_i\}\rangle = \sum_{i\in I}\langle f_i, g_i\rangle$. It is clear that $\left(\sum_{i\in I} \oplus\mathcal{H}_i\right)_{l_2}$ is a Hilbert space with respect to the pointwise

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operations. It is proved in [2] , if $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a $g$-Bessel sequence for $\mathcal{H}$, then the operator

$$T : \left( \sum_{i \in I} \oplus \mathcal{H}_i \right)_{l_2} \to H, \quad T(\{g_i\}) = \sum_{i \in I} \Lambda_i^*(g_i) \tag{1.2}$$

is well defined, bounded and $T^* f = \{\Lambda_i f\}_{i \in I}$.

**Definition 1.1.** We call a sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ a $g$-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$ if there exist two positive constants $A$ and $B$ such that

$$A\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B\|f\|^2 \tag{1.3}$$

for all $f \in \mathcal{H}$. We call $A$ and $B$ the lower and upper $g-$frame bounds, respectively.
We call $\{\Lambda_i\}_{i \in I}$ a tight $g$-frame if $A = B$ and Parseval $g$-frame if $A = B = 1$.

The sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a $g$-frame for $\mathcal{H}$ if and only if the operator $T$ defined by (1.2) is bounded and onto (see [2]). The operators $T$ and $T^*$ are called the synthesis and analysis operators, respectively.

**Proposition 1.2.** [3] *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a $g$-frame for $\mathcal{H}$. The operator*

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{i \in I} \Lambda_i^* \Lambda_i f$$

*is a positive, bounded and invertible operator.*

The operator $S$ is called the $g$-frame operator of $\{\Lambda_i\}_{i \in I}$.
It is easy to check that if $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a $g$-Bessel sequence, then $S$ is well defined and $S = TT^*$.

## 2. Main results

In this section $\mathcal{H}$ denotes a finite dimensional Hilbert space. We also denote the norm of a Hilbert-Schmidt operator $T$ by $\|T\|_2$.

**Definition 2.1.** Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be $g$-frames for $\mathcal{H}$. We say that $\{\Theta_i\}_{i \in I}$ is a dual $g$-frame (or simply dual) of $\{\Lambda_i\}_{i \in I}$ if

$$f = \sum_{i \in I} \Lambda_i^* \Theta_i f, \;\; f \in \mathcal{H}.$$

Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a $g$-frame for $\mathcal{H}$ with $g$-frame operator $S$. Then $\{\, \Lambda_i S^{-1}\}_{i\in I}$ is a dual $g$-frame of $\{\, \Lambda_i \,\}_{i\in I}$. $\{\, \Lambda_i S^{-1} \,\}_{i\in I}$ is called *canonical dual* $g$-frame of $\{\, \Lambda_i \,\}_{i\in I}$. We refer to [1], for the proof of the following results.

**Proposition 2.2.** *Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a dual of $g$-frame $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ for $\mathcal{H}$. Then*

$$\sum_{i\in I} \|\Lambda_i - \Theta_i\|_2^2 = \sum_{i\in I} \|\Lambda_i\|_2^2 + \sum_{i\in I} \|\Theta_i\|_2^2 - 2\dim \mathcal{H}.$$

**Corollary 2.3.** *Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be two Parseval $g$-frames for $\mathcal{H}$. If $\{\Theta_i\}_{i\in I}$ is a dual of $\{\Lambda_i\}_{i\in I}$, then $\Lambda_i = \Theta_i$ for all $i \in I$.*

**Corollary 2.4.** *Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a dual of $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ for $\mathcal{H}$, where $\{\Theta_i\}_{i\in I}$ and $\{\Lambda_i\}_{i\in I}$ are two tight $g$-frames for $\mathcal{H}$ with bounds $B_\Theta$ and $B_\Lambda$, respectively. Then $B_\Theta + B_\Lambda = 2$ if and only if $\Lambda_i = \Theta_i$ for all $i \in I$.*

*Remark* 2.5. Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be two $g$-frames for $\mathcal{H}$ with the associated synthesis operators $T_\Lambda$ and $T_\Theta$, respectively. Then

$$\sum_{i\in I} \|\Lambda_i - \Theta_i\|_2^2 = \sum_{i\in I} \|\Lambda_i\|_2^2 + \sum_{i\in I} \|\Theta_i\|_2^2 - \mathrm{tr}(\mathrm{T}_\Theta \mathrm{T}_\Lambda^*) - \mathrm{tr}(\mathrm{T}_\Lambda \mathrm{T}_\Theta^*).$$

Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a $g$-frame for $\mathcal{H}$ with the frame operator $S$. It is clear that $\{\Lambda_i S^{\alpha-1}\}_{i\in I}$ is a $g$-frame for $\mathcal{H}$ with the property $\sum_{i\in I} \Lambda_i^* \Lambda_i S^{\alpha-1} = S^\alpha f$ for all $f \in \mathcal{H}$. For $\alpha = 0$ we get the canonical dual $g$-frame of $\{\Lambda_i\}_{i\in I}$.

**Definition 2.6.** Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a $g$-frame for $\mathcal{H}$ with $g$-frame operator $S_\Lambda$. A $g$-frame $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a $\alpha$-dual of $\{\Lambda_i\}_{i\in I}$ if $\sum_{i\in I} \Lambda_i^* \Theta_i = S_\Lambda^\alpha f$ for all $f \in \mathcal{H}$.

It is clear that $\{\Lambda_i S_\Lambda^{\alpha-1}\}_{i\in I}$ is a $\alpha$-dual of $\{\Lambda_i\}_{i\in I}$. The canonical dual $g$-frame of $\{\Lambda_i\}_{i\in I}$ has some interesting properties between other dual $g$-frames of $\{\Lambda_i\}_{i\in I}$ (see [3]). We will show that the $\alpha$-dual frame $\{\Lambda_i S_\Lambda^{\alpha-1}\}_{i\in I}$ has some minimal properties between other $\alpha$-dual frames of $\{\Lambda_i\}_{i\in I}$.

**Proposition 2.7.** *Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a $\alpha$-dual of $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ with $g$-frame operator $S_\Theta$. Then*

$$\sum_{i\in I} \|\Lambda_i S_\Lambda^{\alpha-1}\|_2^2 = \left\|S_\Lambda^{\frac{2\alpha-1}{2}}\right\|_2^2 \leq \|S_\Theta^{\frac{1}{2}}\|_2^2 = \sum_{i\in I} \|\Theta_i\|_2^2. \tag{2.1}$$

*The equality in (2.1) holds if and only if $\Theta_i = \Lambda_i S_\Lambda^{\alpha-1}$ for all $i \in I$.*

**Corollary 2.8.** *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for $\mathcal{H}$ with g-frame operator $S_\Lambda$. Then*

$$\sum_{i\in I} \|\Lambda_i - \Lambda_i S_\Lambda^{\alpha-1}\|_2^2$$
$$= \min\Big\{ \sum_{i\in I} \|\Lambda_i - \Theta_i\|_2^2 : \{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\} \text{ is a } \alpha\text{-dual of } \{\Lambda_i\}_{i\in I} \Big\}.$$

*Moreover, if $\{\Theta_i\}_{i\in I}$ is a $\alpha$-dual of $\{\Lambda_i\}_{i\in I}$, then $\sum_{i\in I} \|\Lambda_i - \Theta_i\|_2^2 = \sum_{i\in I} \|\Lambda_i - \Lambda_i S_\Lambda^{\alpha-1}\|_2^2$ if and only if $\Theta_i = \Lambda_i S_\Lambda^{\alpha-1}$ for all $i \in I$.*

**Corollary 2.9.** *Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a dual of g-frame $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ for $\mathcal{H}$. Then*

$$\sum_{i\in I} \|\Theta_i\|_2^2 = \|S_\Theta^{\frac{1}{2}}\|_2^2 \geq \|S_\Lambda^{-\frac{1}{2}}\|_2^2 = \sum_{i\in I} \|\Lambda_i S_\Lambda^{-1}\|_2^2 \tag{2.2}$$

*where $S_\Lambda$ and $S_\Theta$ are the g-frame operators of $\{\Lambda_i\}_{i\in I}$ and $\{\Theta_i\}_{i\in I}$, respectively. Moreover, the following are equivalent*

(i) $\sum_{i\in I} \|\Theta_i\|_2^2 = \sum_{i\in I} \|\Lambda_i S_\Lambda^{-1}\|_2^2$;
(ii) $\Theta_i = \Lambda_i S_\Lambda^{-1}$ *for all* $i \in I$;
(iii) $S_\Theta = S_\Lambda^{-1}$.

**Proposition 2.10.** *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame and $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a Parseval g-frame for $\mathcal{H}$. Then*

(i) $\mathrm{tr}(\mathrm{T}_\Theta \mathrm{T}_\Lambda^*) + \mathrm{tr}(\mathrm{T}_\Lambda \mathrm{T}_\Theta^*) \leq 2\|\mathrm{S}_\Lambda^{\frac{1}{4}}\|_2^2$;
(ii) $\mathrm{tr}(\mathrm{T}_\Theta \mathrm{T}_\Lambda^*) + \mathrm{tr}(\mathrm{T}_\Lambda \mathrm{T}_\Theta^*) = 2\|\mathrm{S}_\Lambda^{\frac{1}{4}}\|_2^2$ *if and only if* $\Theta_i = \Lambda_i S_\Lambda^{-\frac{1}{2}}$ *for all* $i \in I$.

**Corollary 2.11.** *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for $\mathcal{H}$ with g-frame operator $S_\Lambda$. Then*

$$\max\{\Re\mathrm{tr}(\mathrm{T}_\Theta \mathrm{T}_\Lambda^*) : \{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\} \text{ is a Parseval g-frame for } \mathcal{H}\} = \|S_\Lambda^{\frac{1}{4}}\|_2^2.$$

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# UNIQUE WEAK SOLUTION TO THE SYSTEM OF FRACTIONAL SCHRÖDINGER EQUATIONS

F. ABDOLRAZAGHI* AND A. RAZANI

*Department of Mathematics, Imam Khomeini International University, Qazvin, Iran*
*fatemeh_abdolrazaghi@yahoo.com*

Abstract. In this paper, using Minty-Browder theorem, we prove the existence of unique weak solution for a class of fractional system of Schrödinger equations.

## 1. Introduction and the problem formulation

Consider the following fractional Schrödinger system

$$(-\Delta)^s u_i + V_i(x)u_i + f_i(x, u_1, \cdots, u_n) = \lambda_i u_i, \tag{1.1}$$

for all $x \in \mathbb{R}^N$, where $N \geq 2$, $i = 1, \cdots, n$, $s \in (0,1)$, $(-\Delta)^s$ denote the fractional Laplacian and $f_i \in C(\mathbb{R}^N \times \mathbb{R}^n, \mathbb{R})$. Recall that fractional Laplacian, $(-\Delta)^s$, with $s \in (0,1)$ of a function $\phi \in \varphi$ is defined by $\Lambda((-\Delta)^s\phi)(k) = |k|^{2s}\Lambda(\phi)(k)$, for all $s \in (0,1)$, [4, 1]. Here $\varphi$ denotes the Schwartz space consist of rapidly decreasing $C^\infty$ functions in $\mathbb{R}^N$ and $\Lambda$ is the Fourier transform, i.e.

$$\Lambda(\phi)(k) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp\{-2\pi ikx\}\phi(x)dx.$$

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## 2. Weak solution

First, we present some definitions and notations that we may need them through this paper. Assume

(i) $V \in C(\mathbb{R}^N, \mathbb{R})$, $\mathcal{V} = \inf_{x\in\mathbb{R}^N} V(x) > 0$

(ii) $\exists r_0 > 0 : \forall M > 0$, $meas\{x \in B(y; r_0) : V(x) \le M\} \to 0$ $(|y| \to \infty)$.

**Definition 2.1.** The space $H^s(\mathbb{R}^N)$ is defined by $H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N}(|\xi|^{2s}\hat{u}^2 + \hat{u}^2)d\xi < \infty\}$, where $\hat{u} = \Lambda(u)$ with respect to the norm $\|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N}(|\xi|^{2s}\hat{u}^2 + \hat{u}^2)d\xi\right)^{\frac{1}{2}}$.

Due to the appearance of potential energies in the system of equations (1.1), we introduce the subspace $E = \{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2dx < \infty\}$, with corresponding to the norm $\|u\|_E = (\int_{\mathbb{R}^N}(|\xi|^{2s}\hat{u}^2 + \hat{u}^2)d\xi + \int_{\mathbb{R}^N} V(x)u^2dx)^{\frac{1}{2}}$. We make $E$ an inner product space by introducing the Sobolev inner product $\prec \cdot, \cdot \succ_E$ defined by

$$\prec u, v \succ_E = \int_{\mathbb{R}^N} \left((-\Delta)^{\frac{s}{2}}u(x)(-\Delta)^{\frac{s}{2}}v(x) + V(x)u(x)v(x)\right)dx,$$

for all $u, v \in E$. By Plancherel's theorem and condition (i), it is easily seen that $\|\cdot\|_E$ is equivalent to $\|u\| = (\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}u|^2 + V(x)u^2)dx)^{\frac{1}{2}}$. Let $X$ be the Cartesian product $X = E^n$ which is a Hilbert space with the corresponding product norm $\|(u_1, \cdots, u_n)\|_X = \sum_{i=1}^n(\|u_i\|^2)^{\frac{1}{2}}$. As a consequence, we can extend [3, Lemma 2.2] for general space $X$ as

$$\sum_{i=1}^n \|u_i\|_p^2 \le S\sum_{i=1}^n \| u \|^2 = S\|(u_1, \cdots, u_n)\|_X^2, \tag{2.1}$$

where $S$ is the maximum of Sobolev constants $c_p$ corresponding with $u_i$. Moreover, according to [theorem 2.2,[4]], it is obvious that $X$ is compactly embedded into $\left(L^p(\mathbb{R}^N)\right)^n$ for $p \in [2, 2_s^*)$.

**Definition 2.2.** We say that $U = (u_1, \cdots, u_n) \in X$ is a weak solution of the system of equations (1.1), if

$$\int_{\mathbb{R}^N} \left((-\triangle)^{\frac{s}{2}}u(-\triangle)^{\frac{s}{2}}\phi_i + V_i(x)u_i\phi_i\right)dx + \sum_{i=1}^n \int_{\mathbb{R}^N} f_i(x, u_1, \cdots, u_n)\phi_i dx - \sum_{i=1}^n \left(\lambda_i \int_{\mathbb{R}^N} u_i\phi_i dx\right) = 0,$$

for all $\phi = (\phi_1, \cdots, \phi_n) \in X$.

The next result, that is due to Minity and Browder, is useful for reaching to our purpose.

**Theorem 2.3.** *([2]) Let $E$ be a reflexive banach space and $A : X \to X^*$ be nonlinear, continuous, strictly monotone and coercive operator. Then $A$ is bijective mapping.*

We are now ready to prove the main theorem:

**Theorem 2.4.** *Assume that the following conditions hold:*
**($\mathbf{H_1}$)** *There exist $c_i > 0$ $(i = 1, \cdots, n)$ and $p \in [2, 2_s^*)$ such that*

$$|f(x, s_1, \cdots, s_n)| \leq \alpha_i(x) + \sum_{i=1}^{n} c_i |s_i|^{p-1},$$

*for all $(x, s_1, \cdots, s_n) \in \mathbb{R}^N \times \mathbb{R}^n$, where $\alpha_i \in L^q(\mathbb{R}^N)$, with $i = 1, \cdots, n$, and $q \in (\frac{2N}{N+2s}, 2]$. Also $f_i(x, 0, u_2, \cdots, u_n)$, $f_i(x, u_1, 0, \cdots, u_n)$, $\cdots$, $f_i(x, u_1, \cdots, u_{n-1}, 0) \in L^q(\mathbb{R}^N \times \mathbb{R}^{n-1})$.*
**($\mathbf{H_2}$)** *$\frac{f_i(x,s_1,\cdots,s_n)-f_i(x,t_1,\cdots,t_n)}{s_i-t_i} \geq \mu_i^*$ for all $x \in \mathbb{R}^N$, for all $s_i, t_i \in \mathbb{R}$, $s_i \neq t_i$, $i = 1, \cdots, n$.*
*Then problem (1.1) has a unique nontrivial weak solution $U = (u_1, \cdots, u_n) \in X$, for $\lambda_i = \mu_i^*$.*

**Proof.** Let the operator $A : X \to X^*$ be as follows:

$$\begin{aligned} &\prec A(u_1, \cdots, u_n), (\phi_1, \cdots, \phi_n) \succ \\ &= \sum_{i=1}^{n} \int_{\mathbb{R}^N} \left((-\triangle)^{\frac{s}{2}} u_i (-\triangle)^{\frac{s}{2}} \phi_i + V_i(x) u_i \phi_i\right) dx \\ &\quad + \sum_{i=1}^{n} \int_{\mathbb{R}^N} f_i(x, u_1, \cdots, u_n) \phi_i dx - \sum_{i=1}^{n} \lambda_i \left(\int_{\mathbb{R}^N} u_i \phi_i dx\right), \end{aligned}$$

for all $u_i, \phi_i \in E$, $i = 1, \cdots, n$.
It is easy to see that $\prec AU, \phi \succ \in X^*$. So $A$ is well defined. It is sufficient to investigate that $A$ satisfies in Theorem 2.3. We have

$$\begin{aligned} &\prec AU - AV, U - V \succ \\ &= \textstyle\sum_{i=1}^{n} \int_{\mathbb{R}^N} \left(|(-\triangle)^{\frac{s}{2}} (u_i - v_i)|^2 + V_i(x)|u_i - v_i|^2\right) dx \\ &+ \textstyle\sum_{i=1}^{n} \int_{\mathbb{R}^N} (f_i(x, u_1, \cdots, u_n) - f_i(x, v_1, \cdots, v_n))(u_i - v_i) dx \\ &- \textstyle\sum_{i=1}^{n} \left(\int_{\mathbb{R}^N} \mu_i^* (|u_i - v_i|^2) dx\right) \geq \sum_{i=1}^{n} \|u_i - v_i\|^2. \end{aligned}$$

Since $u_i \neq v_i$ this implies $\prec AU - AV, U - V \succ > 0$. It is sufficient to show that $A$ is coercive. By definition

$$\begin{aligned} < AU, U > &= \sum_{i=1}^{n} \int_{\mathbb{R}^N} \left(|(-\triangle)^{\frac{s}{2}} u_i|^2 + V_i(x) u_i^2\right) dx \\ &\quad + \sum_{i=1}^{n} \int_{\mathbb{R}^N} f_i(x, u_1, \cdots, u_n) u_i dx - \left(\int_{\mathbb{R}^N} (\sum_{i=1}^{n} \mu_i^* |u_i|^2) dx\right). \end{aligned}$$

Setting $s_i = 0$ $(i = 1, \cdots, n)$ in condition $(\mathbf{H_2})$, we have

$$s_i f_i(x, s_1, \cdots, s_n) \geq s_i f_i(x, s_1, \cdots, s_{i-1}, 0, s_{i+1}, \cdots, s_n) + \mu_i^* |s_i|^2. \quad (2.2)$$

Hence we get

$$\begin{aligned}
&\prec AU,U \succ \\
&\quad \geq \sum_{i=1}^{n} \|u_i\|^2 - \int_{\mathbb{R}^N} \big(\sum_{i=1}^{n} \mu_i^* |u|^2\big) dx \\
&\qquad + \sum_{i=1}^{n} \big( \int_{\mathbb{R}^N} (u_i f_i(x, u_1, \cdots, u_{i-1}, 0, u_{i+1}, \cdots, u_n) + \mu_i^* |u_i|^2) dx \big) \\
&\quad \geq \|U\|_X^2 - \sum_{i=1}^{n} \|u_i\|_p \big( \int_{\mathbb{R}^N} |f_i(x, u_1, \cdots, u_{i-1}, 0, u_{i+1}, \cdots, u_n)|^q dx \big)^{\frac{1}{q}} \\
&\qquad - \sum_{i=1}^{n} k_i \|u_i\| \big( \int_{\mathbb{R}^N} |f_i(x, u_1, \cdots, u_{i-1}, 0, u_{i+1}, \cdots, u_n)|^q dx \big)^{\frac{1}{q}},
\end{aligned}$$

where $k_i$ is Sobolev constant corresponding to $u_i$. But we know that one other norm equivalent to the product norm is $\|U\|_X = \max_{1 \leq i \leq n}\{\|u_i\|\}$. So inequality (2.1) implies

$$\begin{aligned}
&\lim_{\|U\|_X \to \infty} \frac{\prec AU, U \succ}{\|U\|_X} \\
&\quad \geq \|U\|_X - S \sum_{i=1}^{n} \left( \int_{\mathbb{R}^N} |f_i(x, u_1, \cdots, u_{i-1}, 0, u_{i+1}, \cdots, u_n)|^q dx \right)^{\frac{1}{q}} \right).
\end{aligned}$$

where $S$ is maximum of Sobolev constants $k_i$ for $i = 1, \cdots, n$. It is clearly observed that the right hand side of the preceding inequality tends to infinity so does the left hand side. Then, we get desired result and the proof is completed. □

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# BEST APPROXIMATION IN QUOTIENT BANACH LATTICE SPACES

MAJID ABRISHAMI-MOGHADDAM

*Department of Mathematics ,Birjand Branch, Islamic Azad University, Birjand, Iran.*
*m.abrishami.m@gmail.com*

Abstract. In this research we develop the theory of best approximation in quotient of Banach lattice spaces and discuss about the relationship between the proximinal elements of a given space and its quotient space.

## 1. Introduction and preliminaries

The problem of best approximation in Banach lattices has been studied with many authors (e.g.[1],[3],[4],[5]). In this paper we obtain some results on quotient of Banach lattice spaces about proximinality and approximate sets.

A real vector space $X$ is said to be an ordered vector space whenever it is equipped with an order relation $\leq$ (i.e., $\leq$ is a reflexive, antisymmetric, and transitive binary relation on $X$). A vector lattice space (or a Riesz space) is an ordered vector space $X$ with the additional property that for each pair of vectors $x, y \in X$, the $\sup\{x, y\}$ and the $\inf\{x, y\}$ both exist in $X$. As usual, $\sup\{x, y\}$ is denoted by $x \vee y$ and $\inf\{x, y\}$ by $x \wedge y$. Recall that a vector subspace $S$ of a vector lattice space $X$ is said to be a vector sublattice, whenever $S$ is closed under the

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lattice operations of $X$, i.e., whenever for each pair $x, y \in S$ the vector $x \vee y$ and $x \wedge y$ (taken in $X$) belongs to $S$. A subset $A$ of a vector lattice space is called solid whenever $|x| \leq |y|$ and $y \in A$ imply $x \in A$. A solid vector subspace of a vector lattice space is referred to as an ideal. For any vector $x$ in a vector lattice space define $x^+ := x \vee 0,\ x^- := x \wedge 0$ and $|x| := x \vee (-x)$. The element $x^+$ is called the positive part, $x^-$ is called the negative part, and $|x|$ is called the absolute value of $x$. If $X$ is an ordered vector space, then the set $X^+ = \{x \in X\ :\ x \geq 0\}$ is called a positive cone of $X$, and its members are called the positive elements of $X$. An element $\mathbf{1} \in X$ is called a strong unit if for each $x \in X$ there exists $0 < \lambda \in \mathbb{R}$ such that $x \leq \lambda \mathbf{1}$. Then for each $x \in X$ there exists $0 < \lambda \in \mathbb{R}$ such that $|x| \leq \lambda \mathbf{1}$. Using $\mathbf{1}$ we can define a norm on $X$ by

$$\|x\| = \inf\{\lambda > 0\ :\ |x| \leq \lambda \mathbf{1}\}. \tag{1.1}$$

Recall that a norm $\|.\|$ on a vector lattice space is said to be a lattice norm whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. A vector lattice space equipped with a lattice norm is known as a normed vector lattice space. If a normed vector lattice space is also norm complete, then it is referred to as a Banach lattice. It is well known that $X$ equipped with the norm (1.1) is a Banach lattice which is called a Banach lattice with strong unit $\mathbf{1}$.

The closed ball with center at $x$ and radius $r$ defined on Banach lattice $X$ as follows:

$$B(x, r) = \{y \in X\ :\ \|y - x\| \leq r\} = \{y \in X\ :\ x - r\mathbf{1} \leq y \leq x + r\}.$$

Let $A$ be an ideal in Banach lattice space $X$. We recall that the equivalence class determined by $x$ in $\frac{X}{A}$ will be denoted by $\dot{x} = x + A$. In $\frac{X}{A}$ we introduce a relation $\dot{\leq}$ by letting $\dot{x} \dot{\leq} \dot{y}$ whenever there exist $x_1 \in \dot{x}$ (i.e., $x_1 - x \in A$) and $y_1 \in \dot{y}$ with $x_1 \leq y_1$. Clearly, $\frac{X}{A}$ under the relation $\dot{\leq}$ is an ordered vector space and it is easy to show that $\frac{X}{A}$ is a vector lattice space (for more details see [2]).

Let $A$ be a closed ideal of a Banach lattice space $X$. Then the vector lattice space $\frac{X}{A}$ under the quotient norm

$$\|\dot{x}\| = \inf\{\|y\| : y \in \dot{x}\},$$

is a Banach lattice space. In fact, the quotient vector space $\frac{X}{A}$ is itself a Banach lattice space.

The map $x \to \dot{x}$, from $X$ to $\frac{X}{A}$, is a linear operator called the canonical projection of $X$ onto $\frac{X}{A}$. The lattice homomorphisms are closely related to ideals. For every ideal $A$ of a Banach lattice space $X$, the canonical projection of $X$ onto the Banach lattice space $\frac{X}{A}$ is a lattice homomorphism (for more details see [2]).

Let $W$ be a nonempty subset of a normed linear space $X$. An element $w_0 \in W$ is called a best approximation to $x \in X$ from $W$ if for every $w \in W$,

$$\|x - w_0\| \leq \|x - w\|.$$

The set of all elements of best approximation to $x \in X$ from $W$ is denoted by $P_W(x)$. If each $x \in X$ has at least one best approximation $w_0 \in W$, then $W$ is called a proximinal subset of $X$.

A nonempty subset $W$ of an ordered vector space $X$ is called downward if

$$(w \in W \ , \ x \leq w) \Longrightarrow x \in W.$$

A simple example of a downward set is a set of the form $\{y \in X \ : \ y \leq g\}$ , where $g \in X$. For another example, let $f : X \longrightarrow \mathbb{R}$ be an increasing function, then its lower level sets $S_c(f) = \{x \in X : f(x) \leq c\}$ for all $c \in \mathbb{R}$, are downward.

## 2. Main results

In this section we discuss about proximality of downward sets and sublattices in quotient of Banach lattice spaces.

**Proposition 2.1.** *Let $W$ be a downward and proximinal subset of Banach lattice space $X$ and $w_1, w_2 \in P_W(x)$. Then $w_1 \wedge w_2 \in P_W(x)$.*

**Corollary 2.2.** *Let $W$ be a vector sublattice and $P_W(x)$ be a subspace of Banach lattice space $X$, then $P_W(x)$ is a vector sublattice of $X$.*

**Proposition 2.3.** [4] *Let $W$ be a closed downward subset of $X$. Then $W$ is proximinal in $X$. Moreover, $w_0 = x_0 - r\mathbf{1} \in P_W(x_0)$, where $r = d(x_0, W)$.*

**Theorem 2.4.** [4] *Let $W$ be a closed downward subset of $X$ and $x_0 \in X$ . Then there exists a least element $w_0 := \min P_W(x_0)$ of the set $P_W(x_0)$, namely, $w_0 = x_0 - r\mathbf{1}$ where $r := d(x_0, W)$.*

**Proposition 2.5.** *Let $A$ be an ideal of a vector lattice space $X$ and $W \subseteq X$. Then, $W$ is downward in $X$ if and only if $\frac{W}{A}$ is downward in $\frac{X}{A}$.*

**Theorem 2.6.** *Let $S$ be a sublattice and $A$ be an closed ideal in Banach lattice $X$ such that $A \subseteq S$. If $S$ is proximinal in $X$, then $\frac{S}{A}$ is a proximinal sublattice of $\frac{X}{A}$.*

**Corollary 2.7.** *Let $W$ be a closed and downward subset and $A$ be an ideal of $X$. Then $\frac{W}{A}$ is proximinal in $\frac{X}{A}$. Moreover, $\dot{w_0} \in P_{\frac{W}{A}}(\dot{x_0})$, where $w_0 \in P_W(x_0), \dot{w_0} = x_0 - r\mathbf{1} + A$ and $r = d(x_0, W)$.*

**Theorem 2.8.** *Let $S$ be a sublattice and $A$ be a closed and proximinal ideal in Banach lattice $X$ such that $A \subseteq S$. If $\frac{S}{A}$ is proximinal in $\frac{X}{A}$, then $S$ is a proximinal sublattice of $X$.*

**Theorem 2.9.** *Let $S$ be a sublattice and $A$ be an closed ideal in Banach lattice $X$ such that $A \subseteq S$. If $S$ is proximinal in $X$ , then we have*

$$P_S\dot{(x)} \subseteq P_{\frac{S}{A}}(\dot{x}),$$

*In particular if $A$ is proximinal in $X$, then*

$$P_S\dot{(x)} = P_{\frac{S}{A}}(\dot{x}).$$

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# NORM INEQUALITIES INVOLVING A SPECIAL CLASS OF FUNCTIONS FOR SECTOR MATRICES

DAVOOD AFRAZ*, RAHMATOLLAH LASHKARIPOUR, AND MOJTABA BAKHERAD

*Department of Mathematics, University of Sistan and Baluchestan, Tahedan, Iran*
*Davood.afraT@pgs.usb.ac.ir*

Abstract. In this paper, we present some unitarily invariant norm inequalities for sector matrices involving a special class of functions. In particular, if $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ is a $2n \times 2n$ matrix such that numerical range of $T$ is contained in a sector region $S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$, then for a submultiplicative function $h$ of the class $\mathcal{C}$ and every unitarily invariant norm, we have

$$\|h\left(|T_{ij}|^2\right)\| \leq \|h^r\left(\sec(\alpha)|T_{11}|\right)\|^{\frac{1}{r}} \|h^s\left(\sec(\alpha)|T_{22}|\right)\|^{\frac{1}{s}},$$

where $r$ and $s$ are positive real numbers with $\frac{1}{r} + \frac{1}{s} = 1$ and $i, j = 1, 2$.

## 1. Introduction

Let $\mathcal{M}_n$ be the algebra of all $n \times n$ complex matrices. For $T \in \mathcal{M}_n$, the conjugate transpose of $T$ is denoted by $T^*$. A complex matrix $T \in \mathcal{M}_{2n}$ can be partitioned as a $2 \times 2$ block matrix

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \tag{1.1}$$

where $T_{ij} \in \mathcal{M}_n$ $(i, j = 1, 2)$. For $T \in \mathcal{M}_n$, let $T = \mathcal{R}e(T) + i\mathcal{I}m(T)$ be the Cartesian decomposition of $T$, where the Hermitian matrices

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$\mathcal{R}e(T)=\frac{T+T^*}{2}$ and $\mathcal{I}m(T)=\frac{T-T^*}{2i}$ are called the real and imaginary parts of $T$, respectively. We say that a matrix $T\in\mathcal{M}_n$ is positive semidefinite if $z^*Tz\geq 0$ for all complex numbers $z$. For $T\in\mathcal{M}_n$, let $s_1(T)\geq s_2(T)\geq\cdots\geq s_n(T)$ denote the singular values of $T$, i.e. the eigenvalues of the positive semidefinite matrix $|T|=(T^*T)^{\frac{1}{2}}$ arranged in a decreasing order and repeated according to multiplicity. Note that, $s_j(T)=s_j(T^*)=s_j(|T|)$ for $j=1,2,\cdots,n$. A norm $\|\,.\,\|$ on $\mathcal{M}_n$ is said to be unitarily invariant if $\|UTV\|=\|T\|$ for every $T\in\mathcal{M}_n$ and for every unitary $U,V\in\mathcal{M}_n$. For $T\in\mathcal{M}_n$ and $p>0$, let $\|T\|_p=\left(\sum\limits_{j=1}^{n}s_j^p(T)\right)^{\frac{1}{p}}$. This defines the Schatten $p$-norm(quasinorm) for $p\geq 1(0<p<1)$. It is clear that, Schatten $p$-norm is an unitarily invariant norm. The $w$-norm of a matrix $T\in\mathcal{M}_n$ is defined by $\|T\|_w=\sum\limits_{j=1}^{n}w_js_j(T)$, where $w=(w_1,w_2,\cdots,w_n)$ is a decreasing sequence of nonnegative real numbers.

In this paper, we assume that all functions are continuous. It is known that if $T\in\mathcal{M}_n$ is positive semidefinite and $h$ is a nonnegative increasing function on $[0,\infty)$, then $h(s_j(T))=s_j\left(h(T)\right)$ for $j=1,2,\cdots,n$. For positive semidefinite $X,Y\in\mathcal{M}_n$ and a nonnegative increasing function $h$ on $[0,\infty)$, if $s_j(X)\leq s_j(Y)$ for $j=1,2,\cdots,n$, then $\|h(X)\|\leq\|h(Y)\|$, where $\|\,.\,\|$ is a unitarily invariant norm.

We say that a matrix $T$ is accretive(respectively dissipative) if in the Cartesian decomposition $T=X+iY$, the matrix $X$(respectively $Y$) is positive semidefinite. If both $X$ and $Y$ are positive semidefinite, $T$ is called accretive-dissipative.

Another important class of matrices, which is related to the class of accretive-dissipative matrices, is called sector matrices. To introduce this class, let $\alpha\in[0,\frac{\pi}{2})$ and $S_\alpha$ be a sector defined in the complex plane by

$$S_\alpha=\left\{z\in\mathbb{C}\ :\mathcal{R}e(z)\geq 0,\ |\mathcal{I}m(z)|\leq\tan(\alpha)\mathcal{R}e(z)\right\}.$$

For $T\in\mathcal{M}_n$, the numerical range of $T$ is defined by

$$W(A)=\{z^*Tz : z\in\mathbb{C},\ \|z\|=1\}.$$

A matrix whose its numerical range is contained in a sector region $S_\alpha$ for some $\alpha\in[0,\frac{\pi}{2})$, is called a sector matrix. It follows from the definition of sector matrices that $T$ is positive semidefinite if and only if $W(T)\subseteq S_0$ and also $T$ is accretive-dissipative if and only if $W(e^{\frac{-i\pi}{4}}T)\subseteq S_{\frac{\pi}{4}}$. Moreover, if $W(T)\subseteq S_\alpha$, then $T$ is invertible with $\mathcal{R}e(T)>0$ and therefore $T$ is accretive. A nonnegative function $h$ on

the interval $[0,\infty)$ is said to be submultiplicative if $h(ab) \le h(a)h(b)$ whenever $a, b \in [0,\infty)$.
Gumus et al. [3] introduced the special class $\mathcal{C}$ involving all nonnegative increasing functions $h$ on $[0,\infty)$ satisfying the following condition: If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are two decreasing sequences of nonnegative real numbers such that $\prod_{j=1}^{k} x_j \le \prod_{j=1}^{k} y_j$ $(k = 1, 2, \ldots, n)$, then $\prod_{j=1}^{k} h(x_j) \le \prod_{j=1}^{k} h(y_j)$ $(k = 1, 2, \ldots, n)$.
Note that, the power function $h(t) = t^p$ $(p > 0)$ belongs to class $\mathcal{C}$. For positive semidefinite matrix $\begin{pmatrix} X & Z \\ Z^* & Y \end{pmatrix} \in \mathcal{M}_{2n}$, they proved [3] that if $h \in \mathcal{C}$ is a submultiplicative function, then

$$\|h\left(|Z|^2\right)\| \le \|h^r\left(X\right)\|^{\frac{1}{r}} \|h^s\left(Y\right)\|^{\frac{1}{s}}, \tag{1.2}$$

where $r$ and $s$ are positive real numbers with $\frac{1}{r} + \frac{1}{s} = 1$. Moreover for a sector matrix $T \in \mathcal{M}_{2n}$ partitioned as in (1.1), Thang [6] proved the following inequality

$$\|T_{12}\|^2 \le \sec^2(\alpha)\|T_{11}\|\|T_{22}\| \tag{1.3}$$

for any unitarily invariant norm. Alakhrass [2] extended inequality (1.3) to

$$\||T_{12}|^p\| \le \sec^p(\alpha)\|T_{11}^{\frac{pr}{2}}\|^{\frac{1}{r}}\|T_{22}^{\frac{ps}{2}}\|^{\frac{1}{s}}, \tag{1.4}$$

where $r, s$ and $p$ are positive numbers and $\frac{1}{r} + \frac{1}{s} = 1$.
In the present paper, we establish some unitarily invariant norm inequalities for sector matrices involving the functions of class $\mathcal{C}$. For instance, we extend inequalities (1.2) and (1.4) to sector matrices and the class $\mathcal{C}$(Theorem 2.2).

## 2. Main results

Through the following, we give a lemma which are needed to prove our main statement.

**Lemma 2.1.** [2, Theorem 3.2] *Suppose that $T \in \mathcal{M}_{2n}$ partitioned as in (1.1) such that $W(T) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then*

$$\prod_{m=1}^{k} s_m(T_{ij}) \le \prod_{l=1}^{k} \sec(\alpha) s_m^{\frac{1}{2}}(\mathcal{R}e(T_{ii})) s_m^{\frac{1}{2}}(\mathcal{R}e(T_{jj})), \qquad (i, j = 1, 2),$$

*where $k = 1, 2, \ldots, n$.*

In the sequel, we give an unitarily invariant norm inequalities for sector matrices regarding of special class $\mathcal{C}$.

**Theorem 2.2.** *Let $T \in \mathcal{M}_{2n}$ partitioned as in (1.1) be a sector matrix and let $h \in \mathcal{C}$ be submultiplicative. If $r$ and $s$ are positive real numbers with $\frac{1}{r} + \frac{1}{s} = 1$, then*

$$\begin{aligned}\|h\left(|T_{ij}|^2\right)\| &\leq \|h^r\left(\sec(\alpha)\mathcal{R}e(T_{11})\right)\|^{\frac{1}{r}}\|h^s\left(\sec(\alpha)\mathcal{R}e(T_{22})\right)\|^{\frac{1}{s}} \\ &\leq \|h^r\left(\sec(\alpha)|T_{11}|\right)\|^{\frac{1}{r}}\|h^s\left(\sec(\alpha)|T_{22}|\right)\|^{\frac{1}{s}}\end{aligned}$$

*for every unitarily invariant norm $\| \,.\, \|$ on $\mathcal{M}_n$ and $i, j = 1, 2$.*

*Remark* 2.3. If $T \in \mathcal{M}_{2n}$ is positive semidefinite, i.e. $W(T) \subseteq S_0$, then Theorem 2.2 reduce to inequality (1.2). Applying Theorem 2.2 for $h(t) = t^{\frac{p}{2}}\,(p > 0)$, we get inequality (1.4). Therefore Theorem 2.2 is an extension of inequality (1.2) and inequality (1.4).

**Corollary 2.4.** *Suppose $T \in \mathcal{M}_{2n}$ partitioned as in (1.1) is accretive-dissipative and $h \in \mathcal{C}$ is submultiplicative. If $r$ and $s$ are positive real numbers with $\frac{1}{r} + \frac{1}{s} = 1$, then*

$$\|h\left(|T_{ij}|^2\right)\| \leq \|h^r\left(\sqrt{2}\mathcal{R}e(T_{11})\right)\|^{\frac{1}{r}}\|h^s\left(\sqrt{2}\mathcal{R}e(T_{22})\right)\|^{\frac{1}{s}}, \qquad (i, j = 1, 2),$$

*where $\| \,.\, \|$ is a unitarily invariant norm.*

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# IDEAL AMENABILITY OF MODULE EXTENSION BANACH ALGEBRAS

M. ALIKAHI*, M. RAMEZANPOUR

[1] *School of Mathematics and Computer Science, Damghan University, Damghan, Iran.*
*mahdiehalikahi91@gmail.com, ramezanpour@du.ac.ir*

Abstract. In this note, we extend and improve some results on ideal amenability of module extensions and triangular Banach algebras.

## 1. Introduction

Let $A$ be a Banach algebra and let $X$ be a Banach $A$-bimodule. Then $X^*$, the dual space of $X$ , is also a Banach $A$-bimodule with module multiplications defined by

$$(fa)(x) = f(ax), \quad (af)(x) = f(xa) \quad (a \in A, f \in X^*, x \in X).$$

In particular, $I$ and $I^*$ are Banach $A$-bimodule for every closed ideal $I$ of $A$.

A derivation from $A$ into $X$ is a continuous linear operator $D : A \to X$ such that

$$D(ab) = D(a)b + aD(b) \quad (a, b \in A).$$

For each $x \in X$, the map $d_x : A \to X$ defined by $d_x(a) = ax - xa$, is a derivation, which is called an inner derivation.

Recall that a Banach algebra $A$ is weakly amenable if every derivation from $A$ into $A^*$ is inner, in other words $H^1(A, A^*) = \{0\}$. This

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notion have introduced by Bade, Curtis and Dales for commutative Banach algebras in [3] and it was extended to the noncommutative case by Johnson in [6].

Let $A$ be a Banach algebra and let $I$ be a closed ideal of $A$. Following [5], $A$ is said to be $I$-weakly amenable if every derivation from $A$ into $I^*$ is inner. We call $A$ ideally amenable if $A$ is $I$-weakly amenable for every closed ideal $I$ of $A$. Obviously, an ideally amenable Banach algebra is weakly amenable. There are examples of weakly amenable Banach algebras which is not ideally amenable; (see [5]).

In this note, we give general necessary and sufficient conditions for ideal amenability of an especial kind of Banach algebras which are constructed from a Banach algebra $A$ and an algebraic Banach $A$-bimodule $X$, called generalized module extension Banach algebras $A \bowtie X$, [7]. As a consequence, we extend and improve some results on ideal amenability of module extensions and triangular Banach algebras.

## 2. Main results

Let $A$ and $X$ be Banach algebras and let $X$ be a Banach $A$-module. We say that $X$ is an algebraic Banach $A$-module if for every $x, y \in X$ and $a \in A$,

$$a(xy) = (ax)y, (xy)a = x(ya), (xa)y = x(ay),$$

$$\|ax\| \le \|a\|\|x\|, \ \|xa\| \le \|x\|\|a\|.$$

Then a direct verification shows that the direct sum $A \times X$ under the multiplication

$$(a, x)(b, y) = (ab, a.y + x.b + xy) \ (a, b \in A, x, y \in X)$$

and the norm $\|a + x\| = \|a\| + \|x\|$ is a Banach algebra which is called the genelarized module extension Banach algebra of $A$ and $X$ and is denoted by $A \bowtie X$. This Banach algebra, which contains some important classes of Banach algebras such as Lau product, module extension and semidirect product of Banach algebras, introduced by Ramezanpour and Barootkoob in [7].

Let $A$ be a Banach algebra and let $X$ be an algebraic Banach $A$-bimodule. Let $I$ be a closed ideal of $A$ and $Y$ be a closed ideal of $X$ which is also an $A$-submodule of $X$ with $IX \cup XI \subset Y$. Then $I \bowtie Y$ is a closed ideal of $A \bowtie X$.

In the next we give a general necessary and sufficient conditions for ideal amenability of $A \bowtie X$; [2].

**Theorem 2.1.** *Let $I \bowtie Y$ be a closed ideal of $A \bowtie X$. Then $A \bowtie X$ is $I \bowtie Y$-weakly amenable if and only if the following conditions hold:*

i. *$A$ is $I$-weakly amenable.*
ii. *If $D_X : A \longrightarrow Y^*$ is a derivation such that $x.D_X(a) = D_X(a).x = 0$, for all $a \in A; x \in X$; then there exists an element $g \in Y^*$ such that $D_X = ad_g$ and $x.g = g.x$, for all $x \in X$.*
iii. *If $S : X \longrightarrow Y^*$ is a derivation such that there exist a derivation $D_X : A \longrightarrow Y^*$ and a bounded linear map $T : X \longrightarrow I^*$ satisfying $T(a.x) = a.T(x) + D_X(a) \star x$; $T(x.a) = T(x).a + x \star D_X(a)$; $S(a.x) = D_X(a).x + a.S(x)$; $S(x.a) = x.D_X(a) + S(x).a$ and $S(x) \star y + x \star S(y) = T(xy)$; for all $a \in A$; $x \in X$; then $S$ is inner.*
iv. *If $T : X \longrightarrow I^*$ is a bounded $A$-module homomorphism such that $T(xy) = 0$ for all $x; y \in X$; then $T = 0$.*

Let the algebra multiplication on $X$ be trivial, that is, $xy = 0$ for each $x, y \in X$, then $A \bowtie X$ is same as the module extension of $A$ by $X$ which we denote by $A \oplus X$. In [4], ideal amenability for a module extension Banach algebra has been investigated. in the next we improve the main result of [4].

**Theorem 2.2.** *A module extension Banach algebra $A \oplus X$ is $I \oplus Y$-weakly amenable if and only if the following conditions hold:*

i. *$A$ is $I$-weakly amenable.*
ii. *$H^1(A, Y^*) = 0$.*
iii. *For every continuous $A$-bimodule morphism $T : X \longrightarrow I^*$, there exists $g \in Y^*$ such that $a.g - g.a = 0$ for $a \in A$ and $T(x) = x \star g - g \star x$ for $x \in X$.*
iv. *The only continuous $A$-bimodule morphism $S : X \longrightarrow Y^*$ for which $S(x) \star y + x \star S(y) = 0$ $(x, y \in X)$ in $I^*$ is $T = 0$.*

Let $A$ and $B$ be Banach algebras, $X$ be a Banach $(A, B)$-module. The algebra

$$\mathrm{Tri}(A, X, B) = \left\{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} : a \in A, x \in X, b \in B \right\}$$

with the usual $2 \times 2$ matrix operations and the norm $\| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \| = \|a\|_A + \|x\|_X + \|b\|_B$, is a Banach algebra wich is called triangular Banach algebra. The triangular Banach algebra $\mathrm{Tri}(A, X, B)$ is an example of module extension Banach algebra, in fact $\mathrm{Tri}(A, X, B) = (A \times B) \oplus X$ where $(a, b).x = a.x$ and $x.(a, b) = x.b$ for all $a \in A$, $b \in B$ and $x \in X$.

Let $I_1$ and $I_2$ be closed ideals in $A$ and $B$ respectively and $Y$ be a closed ideal in $X$ which is also an $(A, B)$-submodule of $X$ with $I_1X \cup XI_2 \subset Y$. Then $(I_1 \oplus I_2) \oplus Y$ is a closed ideal of $\mathrm{Tri}(A, X, B)$.

In [1], Abbaspour Tabadkan considered the case where $A$ and $B$ are unital Banach algebras and $X$ is a unital Banach $(A, B)$-module and showed that the corresponding triangular Banach algebra $\mathrm{Tri}(A, X, B)$ is ideally amenable if and only if $A$ and $B$ are ideally amenable. Now we apply Theorem 2.1 to give a general necessary and sufficient conditions for ideal amenability of triangular Banach algebras.

**Theorem 2.3.** *The triangular Banach algebra* $\mathrm{Tri}(A, X, B)$ *is* $(I_1 \oplus I_2) \oplus Y$*-weakly amenable if and only if*

1. $A \oplus B$ *is* $(I_1 \oplus I_2)$*-weakly amenable.*
2. $H^1(A \oplus B, Y^*) = 0$
3. $\overline{\langle I_1X + XI_2 \rangle} = Y$

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# CRITERIA FOR CONVEX INTEGRAL OPERATORS

E. AMINI

*Department of Mathematics, Payme Noor University, P. O. Box 19395-4697*
*Tehran, IRAN.*
*eb.amini.s@pnu.ac.ir*

Abstract. In this paper, the author used some integral operator and determine the subclass of $n$-valent analytic functions. Furthermore, we obtain result such that describes starlike and convex functions.

## 1. Introduction

Let $D$ be the open unit disk in the complex plan $\mathbb{C}$ and $\mathcal{H}(D)$ the class of all holomorphic functions on $D$. Let $\mathcal{A}_n$ be the class of all $n$-valent analytic functions of the form

$$f(z) = z + a_{n+1}z^{n+1} + \cdots, \qquad n \in \mathbb{N} = \{1, 2, \cdots\},$$

and let $\mathcal{A}_1 = \mathcal{A}$. The class of all functions in $\mathcal{A}$, such that those are univalent in the unit disk $D$ denote by $S$. The function $f \in \mathcal{A}$ is a starlike of order $\beta$, $0 \leqslant \beta < 1$, if

$$Re\Big(\frac{zf'(z)}{f(z)}\Big) > \beta, \qquad z \in D,$$

The class of function is denoted by $S^*(\beta)$ and the class of starlike functions is $S^* = S^*(0)$. Also, the function $f(z) \in \mathcal{A}$ is a convex of order $\alpha$, $(0 \leqslant \alpha < 1)$, if

$$Re\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > \alpha, \qquad z \in D.$$

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The class of function is denoted by $C(\alpha)$ and the class of convex functions is $C = C(0)$. Furthermore, $C \subseteq S^*$. All of these classes are subclasses of univalent functions (see [1]).

For $f_i \in \mathcal{A}$ and $\alpha_i > 0$, $i \in \{1, 2, \cdots, k\}$, D. Breaz and N. Breaz [2] introduced the following integral operator:

$$F_k(z) = \int_0^z \Big(\frac{f_1(\xi)}{\xi}\Big)^{\alpha_1} \cdots \Big(\frac{f_k(\xi)}{\xi}\Big)^{\alpha_k} d\xi. \tag{1.1}$$

Recently, Breaz et al. in [3] introduced the following integral operator:

$$F_{\alpha_1,\cdots,\alpha_k}(z) = \int_0^z \Big(f_1'(\xi)\Big)^{\alpha_1} \cdots \Big(f_k'(\xi)\Big)^{\alpha_k} d\xi \tag{1.2}$$

where $k \in \mathbb{N}$ and $z \in D$.

The most recent, Frasin [4] introduced the following integral operators, for $\alpha_i > 0$ and $f_i \in \mathcal{A}_n$:

$$F_n(z) = \int_0^z n\xi^{(n-1)} \left(\frac{f_1(\xi)}{\xi^n}\right)^{\alpha_1} \cdots \left(\frac{f_k(\xi)}{\xi^n}\right)^{\alpha_k} d\xi, \tag{1.3}$$

and

$$G_n(z) = \int_0^z n\xi^{(n-1)} \left(\frac{f_1'(\xi)}{n\xi^{n-1}}\right)^{\alpha_1} \cdots \left(\frac{f_k'(\xi)}{n\xi^{n-1}}\right)^{\alpha_k} d\xi, \tag{1.4}$$

*Remark* 1.1.

- **i:** If we put $n = 1$ in (1.3), then we obtain the integral operator $F_k(z)$ which is defined in (1.1),
- **ii:** If we put $n = 1$ in (1.4), then we obtain the integral operator $F_{\alpha_1,\alpha_2,\ldots,\alpha_k}(z)$ which is defined in (1.2).
- **iii:** For $n = k = 1$, $\alpha_1 = \alpha \in [0, 1]$ in (1.3), we obtain the integral operator

$$F_\alpha(z) = \int_0^z \Big(\frac{f(\xi)}{\xi}\Big)^{\alpha} d\xi, \tag{1.5}$$

  which is studied in [5].
- **iv:** For $n = k = 1$, $\alpha_1 = \alpha = 1$ in (1.3), we obtain the Alexander integral operator

$$G(z) = \int_0^z \frac{f(\xi)}{\xi} d\xi. \tag{1.6}$$

S.S. Miller and P. T. Mocanu (see [5]) has been studied some integral operator. In order to give our results, we have the following result.

**Lemma 1.2.** [5] *Let a function* $\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ *satisfies*

$$Re\{\Phi(ix, y)\} \leq 0$$

*for all real $x$ and all real $y$ with $y \leq -(1+x^2)/2$. If $p(z) = 1+p_1 z+\cdots$ is analytic in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ and*

$$Re\{\Phi\big(p(x), zp'(x)\big)\} > 0, \qquad z \in D,$$

*then*

$$Re\{p(z)\} > 0, \qquad z \in D.$$

## 2. MAIN RESULTS

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**Theorem 2.1.** *Let $f_i \in \mathcal{A}_n$ and $\alpha_i > 0$ for all $i \in \{1, 2, \cdots, k\}$. If*

$$Re\Big\{\sum_{i=1}^{k} \alpha_i \Big[\frac{z^2 f_i''(z)}{f_i'(z)} - (\frac{z f_i'(z)}{f_i(z)})^2\Big]\Big\} > 1, \tag{2.1}$$

*then $F_n$ is $n$-valent convex. Where, $F_n$ is the integral operator define as in (1.3).*

Making $n = 1$ in Theorem 2.1, we have the following:

**Corollary 2.2.** *Let $f_i \in \mathcal{A}$, $\alpha_i > 0$ for all $i \in \{1, 2, \cdots, k\}$ and $\sum_{i=1}^{k} \alpha_i \geq 1$. If (2.1) is hold, then the integral operator $F_k$ defined in (1.1) is convex of order each $\alpha$, $0 \leq \alpha < 1$.*

Making $k = n = 1$ in Theorem 2.1, we have the following:

**Corollary 2.3.** *Let $f \in \mathcal{A}$, $\alpha \in [0, 1]$. If*

$$Re\Big\{\alpha\Big[\frac{z^2 f''(z)}{f'(z)} - (\frac{z f'(z)}{f(z)})^2\Big]\Big\} > 1,$$

*then the integral operator $F_\alpha$ defined in (1.5), is convex of order each $\alpha$, $0 \leqslant \alpha < 1$.*

Making $k = n = 1$ and $\alpha = 1$ in Theorem 2.1, we obtain the following:

**Corollary 2.4.** *Let $f \in \mathcal{A}$. If*

$$Re\Big[\frac{z^2 f''(z)}{f'(z)} - (\frac{z f'(z)}{f(z)})^2\Big] > 1,$$

*then the integral operator $G(z)$ defined in (1.6), is convex of order each $\alpha$, $0 \leq \alpha < 1$.*

then $G(z)$ is convex. Here $G(z)$ is the integral operator define as in (1.6).

**Theorem 2.5.** *Let $f_i \in \mathcal{A}_n$ and $\alpha_i > 0$ for all $i \in \{1, 2, \cdots, k\}$. If*

$$Re\Big\{\sum_{i=1}^{k} \alpha_i \Big[\frac{z^2 f_i^{(3)}(z)}{f_i''(z)} - \big(\frac{z f_i''(z)}{f_i'(z)}\big)^2\Big]\Big\} > 1, \tag{2.2}$$

*then $G_n$ is $n$-valent convex. where $G_n$ is the integral operator define as in (1.4).*

Making $n = 1$ in Theorem 2.5, we have the following:

**Corollary 2.6.** *Let $f_i \in \mathcal{A}$, $\alpha_i > 0$ for all $i \in \{1, 2, \cdots, k\}$. If (2.2) is hold, then the integral operator $F_{\alpha_1,\cdots,\alpha_k}(z)$ defined in (1.2) is convex of order each $\alpha$, $0 \leq \alpha < 1$.*

Taking $k = n = 1$ and $\alpha = 1$ in Theorem 2.5, we obtain the following:

**Corollary 2.7.** *Let $f \in \mathcal{A}$. If*

$$Re\Big[\frac{z^2 f^{(3)}(z)}{f'(z)} - \big(\frac{z f''(z)}{f'(z)}\big)^2\Big] > 1,$$

*then $f(z)$ is convex function.*

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# UNIFORMLY CONVEX FUNCTIONS IN TERMS OF CONVOLUTION

E. AMINI

*Department of Mathematics, Payme Noor University, P. O. Box 19395-4697 Tehran, IRAN.*
*eb.amini.s@pnu.ac.ir*

Abstract. In this paper, we investigate $k$-uniformly convex functions and obtain some convolution results.

## 1. Introduction

In the theory of geometric function (see [1]), let $\mathcal{A}$ be the class of all analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $D = \{z \in \mathbb{C}, |z| < 1\}$. The class of univalent functions in $\mathcal{A}$ denoted by $S$

A function $f \in \mathcal{A}$ is said to be starlike if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0.$$

The class of starlike functions in $\mathcal{A}$ denoted by $ST$

A set $E$ is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining

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any two points of $E$ lies entirely in $E$. A function $f \in S$ maps $D$ onto a convex domain $E$ if and only if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0.$$

The class of convex functions in $\mathcal{A}$ denoted by $CV$.

**Definition 1.1.** [2] A function $f$ is said to be uniformly convex (starlike) in $D$ if $f$ is in $CV$ ($ST$) and has the property that for every circular arc $\gamma$ contained in $D$, with center $\zeta$, also in $D$, the arc $f(\gamma)$ is convex ( starlike w.r.t $(f(\gamma))$.

The class of uniformly starlike functions in $\mathcal{A}$ denoted by $UST$ and the class of uniformly convex functions in $\mathcal{A}$ denoted by $UCV$.

**Theorem 1.2.** [2] *Let $f \in \mathcal{A}$. Then*

**i:** *$f \in UCV$ if and only if*

$$Re\left\{1+(z-\zeta)\frac{f''(z)}{f'(z)}\right\}\geq 0,\ (z,\zeta)\in D\times D.$$

**ii:** *$f \in UST$ if and only if*

$$Re\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)}\geq 0,\ (z,\zeta)\in D\times D.$$

Note that by taking $\zeta = 0$ in theorem 1.2 we are back to the classes $CV$ and $ST$.

**Definition 1.3.** [2] $S_p = \{F \in ST|\ F(z) = zf'(z), f \in UCV\}$

**Definition 1.4.** [4] Let $0 \leq k < \infty$. A function $f \in S$ is said to be $k$-uniformly convex in $U$, if the image of every circular arc $\gamma$ contained in $D$, with center $\zeta$, where $|\zeta| \leq k$, is convex.

For fixed $k$, the class of all $k$-uniformly convex function will be denoted by $k-UCV$. Clearly, $0-UCV = CV$, and $1-UCV = UCV$.

**Theorem 1.5.** [4] *Let $f \in \mathcal{A}$ and $0 \leq k < \infty$. Then $f \in k-UCV$ if and only if*

$$Re\left\{1+(z-\zeta)\frac{f''(z)}{f'(z)}\right\}\geq 0,\ z\in D,\ |\zeta|\leq k.$$

**Definition 1.6.** $S_{k,p} = \{F \in ST|\ F(z) = zf'(z), f \in k-UCV\}$

Let $f(z) \in \mathcal{A}$ given by (2) and $g(z) = z+\sum_{n=2}^{\infty} b_n z^n$. The convolution $f$ and $g$, denoted by $f * g$, is a function in $\mathcal{A}$, given by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n$$

For more dentils to this theory refer the book by Ruscheweyh [3].

## 2. Convolution condition

**Theorem 2.1.** *Let $|z| < R \leq 1$, $0 < k < 1$, $|x| = 1$ and $x \neq -1$. The function $f$ is a $k$-unformly convex function if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z + ke^{-i\theta}(x+1) - x + 2)z^2}{(1-z)^3}\right] \neq 0. \tag{2.1}$$

*where $\theta = Arg\left[\frac{zf''(z)}{f'(z)}\right]$.*

*Proof.* The function $f$ is $k$-uniformly convex in $|z| < R$ if and only if

$$Re\left\{1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right\} > 0,\ z \in D,\ 0 < |\zeta| < k. \tag{2.2}$$

Since $\frac{(zf'(z))'}{f'(z)} = 1$ at $z = 0$, (2) is equivalent to

$$1 + (z-\zeta)\frac{f''(z)}{f'(z)} \neq \frac{x-1}{x+1},\ |x| = 1, x \neq -1, \tag{2.3}$$

Hence, we have:

$$1 + \frac{zf''(z)}{f'(z)} - \frac{\zeta}{z}(1 + \frac{zf''(z)}{f'(z)}) + \frac{\zeta}{z} \neq \frac{x-1}{x+1},$$

or equivalently

$$\frac{(zf'(z))' - \frac{\zeta}{z}(zf'(z))' + \frac{\zeta}{z}f'(z)}{f'(z)} - \frac{x-1}{x+1} \neq 0,\ z \in D, |\zeta| \leq k$$

which simplifies to

$$(zf'(z))'\left[(x+1)(1-\frac{\zeta}{z})\right] + f'(z)[\frac{\zeta}{z}(x+1) - x + 1] \neq 0. \tag{2.4}$$

The condition is fulfilled for every $z \in D$ and $0 \leq |\zeta| \leq k$. Choosing $\theta = Arg[\frac{zf''(z)}{f'(z)}]$ and $\zeta = kze^{-i\theta}$, from (2.4), we obtain that

$$(zf'(z))'\left((x+1)(1-ke^{-i\theta})\right) + f'(z)\left(ke^{-i\theta}(x+1) - x + 1\right) \neq 0. \tag{2.5}$$

Since the function $f$ given by (1.1), we obtain

$$(zf'(z))' = 1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1} = f'(z) * \left(\sum_{n=1}^{\infty} nz^{n-1}\right) = f'(z) * \frac{1}{(1-z)^2}$$

So that the left hand side of (2.5) may be expressed as

$$f'(z) * \left[\frac{(x+1)(1-ke^{-i\theta})}{1-z^2} + \frac{ke^{-i\theta}(x+1) - x + 1}{1-z}\right] \neq 0 \tag{2.6}$$

thus, the inequality (2.6) is equivalent to

$$f'(z) * \frac{1+\frac{ke^{-i\theta}(x+1)-x+1}{2}z}{(1-z)^2} \neq 0.$$

Hence, we have

$$\frac{1}{z}\left[zf'(z) * \frac{z+\frac{ke^{-i\theta}(x+1)-x+1}{2}z^2}{(1-z)^2}\right] \neq 0, \tag{2.7}$$

Since $zf'(z) * g(z) = f(z) * zg'(z)$, we can write (2.1) and this complete the proof of theorem. □

**Corollary 2.2.** *Let $|z| < R \leq 1$, $0 < k < 1$, $|x| = 1$ and $x \neq -1$. $f \in S_{k,p}$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z+\frac{ke^{-i\theta}(x+1)-x+1}{2}z^2}{(1-z)^2}\right] \neq 0.$$

*where $\theta = Arg\left[\frac{zf''(z)}{f'(z)}\right]$.*

**Corollary 2.3.** *Let $|z| < R \leq 1$, $|x| = 1$ and $x \neq -1$. The function $f$ is a uniformly convex function if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z+e^{-i\theta}(x+1)-x+2)z^2}{(1-z)^3}\right] \neq 0.$$

*where $\theta = Arg\left[\frac{zf''(z)}{f'(z)}\right]$.*

**Corollary 2.4.** *Let $|z| < R \leq 1$, $0 < k < 1$, $|x| = 1$ and $x \neq -1$. $f \in S_p$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z+\frac{e^{-i\theta}(x+1)-x+1}{2}z^2}{(1-z)^2}\right] \neq 0.$$

*where $\theta = Arg\left[\frac{zf''(z)}{f'(z)}\right]$.*

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# SOME OPTIMIZATION METHODS BASED ON BERNSTEIN POLYNOMIALS FOR APPROXIMATE SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS

M.R. ANSARI

*Department of Applied Mathematics, Imam Khomeini International University, Qazvin, Iran*
*mr.ansari@sci.ikiu.ac.ir*

Abstract. A main difficulty with the collocation method is that its coefficients matrices become ill-conditioned when the degree of approximations increases. This can cause numerical troublesomes and decreases the accuracy of the solutions. Here, three methods are presented based on the combination of Bernstein collocation and optimization methods for approximate solutions of linear integro-differential equations. Several numerical examples are presented which demonstrate the effectiveness of presented methods.

## 1. Introduction

A linear Fredholm-Volterra integro-differential equation (LFVIDE) of order $n$ is generally as follows:

$$\begin{aligned} L\big(y(x)\big) \equiv \sum_{i=0}^{n} p_k(x)y^{(k)}(x) + \lambda_1 \int_a^b K_f(x,t)y(t)dt \\ +\lambda_2 \int_a^x K_v(x,t)y(t)dt = g(x) \end{aligned} \tag{1.1}$$

along with the mixed conditions

$$S_l\big(y(x)\big) \equiv \sum_{k=0}^{n-1} \sum_{j=1}^{n_k} a_{jkl}y^{(k)}(d_{jkl}) = \mu_l, \quad l = 1,\dots,n-1, \tag{1.2}$$

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where, the known functions $p_k(x)$, $K_f(x,t)$, $K_v(x,t)$ and $g(x)$ are defined on $[a,b]$, $\lambda_1$, $\lambda_2$, $a_{jkl}$, $d_{jkl}$, $\mu_l$ are appropriate constants, and $n_k$ are positive integers. Clearly, $L$ and $S_l$ are linear operators.

The Bernstein polynomials are well known as the basis for polynomial spaces and have been used extensively in the literature [1, 2]. Also, collocation methods have been widely used to solve a variety of functional equations [1, 5]. But, a main difficulty with these methods is that its coefficients matrices become ill-conditioned as the degree of approximations increases.

In this paper, we present some optimization-based schemes to approximate solution of LFVIDE (1.1) and (1.2). Our methods guarantee that the obtained approximate solution satisfies exactly the mixed condition (1.2). The numerical results show that they are more accurate than the standard collocation method. In addition, they are numerically stable as degree of approximations increases.

## 2. Bernstein polynomials

Let denote the Bernstein polynomials of degree $m$ on the interval $[a,b]$ by $B_{i,m}(x)$. It is mathematicaly convenience to set $B_{i,m}=0$, if $i<0$ or $i>m$. Define

$$\phi_m(x) = [B_{0,m}(x)\ B_{1,m}(x)\ \ldots\ B_{m,m}(x)]^T, \tag{2.1}$$

we can write $\phi'_m(x) = D\phi_m(x)$ and $\phi_m^{(k)}(x) = D^k\phi_m(x)$ by extension, where, $D$ is operational matrix of derivative with elements [2]:

$$d_{i,j} = \begin{cases} -j, & i=j-1 \\ 2j-m, & i=j \\ m-j, & i=j+1 \\ 0, & \text{otherwise.} \end{cases}, \qquad i,j=1,\ldots,m+1$$

Next underlying theorem provides an appropriate framework of using Bernstein polynomials for approximating of functions [6].

**Theorem 2.1.** *If $y\in C^k[a,b]$, for some integer $k\geq 0$, then $B_m^{(i)}(y;x)$ converges uniformly to $y^{(i)}(x)$ for $i=0,\ldots,k$, as $m\to\infty$, where,*

$$B_m(y;x) = \sum_{k=0}^{m} y(a+\frac{(b-a)i}{m})B_{i,m}(x).$$

## 3. Standard collocation method

Let

$$y_m(x) = \sum_{i=0}^{m} c_i B_{i,m}(x) = \phi_m(x)^T C \tag{3.1}$$

approximate the solution of (1.1) and (1.2) where, $C = [c_0, c_1, \ldots, c_m]$ is the unknown coefficients vector and $\phi_m(x)$ is the basis functions vector is defined in (2.1). By substituting $y_m$ in (1.2), using the operational matrix of derivative $D$ and the linearity of $S_l$, we obtained:

$$\big(S_l(\phi_m(x))\big)^T C = \mu_l, \quad l = 1, \ldots, n-1, \tag{3.2}$$

Also, by substituting $y_m$ in (1.1), using the linearity of $L$ and applying the collocation points $x_j \in [a, b]$, $j = 1, \ldots, m-n+2$, we can write

$$L\big(\phi_m(x_j)\big)^T C = g(x_j) \;\; j = 1, \ldots, m-n+2. \tag{3.3}$$

Now, by solving the set of $m+1$ linear equations (3.2) and (3.3) we obtain the unknown vector $C$ and thus approximate solution $y_m$.

## 4. New optimization-based schemes

The residual function for integro-differential (1.1), is defined as:

$$R(y; x) = L\big(y(x)\big) - g(x).$$

Therefor, for approximate solution $y_m$, the residual function is

$$R_m(x) \equiv R(y_m; x) = L\big(\phi_m(x)\big)^T C - g(x).$$

We determine the unknown vector $C$ by combining the collocation method and minimizing the error corresponding to $R_m(x)$.

Obviously, $y_m$ must satisfy the mixed conditions (1.2), this gives us $n-1$ equations. Consider collocation points $x_1, \ldots x_{n_p} \in [a, b]$, where, $n_p \geq m-n+2$. We partition these points into two distinct subsests $A$ and $B$ and represent the point indices of these two sets with $I$ and $J$, respectively. For each $j \in J$, we set $R_m(x_j) = 0$ to mean that at these points (1.1) is set exactly. These points are chosen so that they are distributed throughout the interval $[a, b]$. Their number should be less than $m-n+2$ (say, $\frac{1}{2}(m-n+2)$). Otherwise, corresponding collocation equations precisely determine $y_m$. For points in $A$, we minimize the errors corresponding to the residual function $R_m(x)$. This explanation leads to the suggestion of the following three schemes: i) Least squares of errors, ii) Minimizing the sum of absolute errors, and iii) Minimizing the maximum of absolute errors. In each of these schemes if $n_p = m-n+2$, then the errors are zero at all collocation points and so the method is equivalent to the collocation method.

## 5. Numerical experiments

We considered eight experimental problems from literature. The performance profiles that introduced by Dolan and Moré [3] is used as a comparison measure of different methods for $m = 5, 10, 15, 25$ and 35. Here, $p_i(\tau)$ indicates the fraction of problems for which method $i$

is within a factor of $\tau \geq 1$ of the best method according to the statistic chosen [3]. Therefore, the algorithm whose efficiency function graph is higher has better performance in comparison. The performance profiles of the methods on all considered test problems could be seen in Figure 1. The curves denoted by "CM", "PM1", "PM2" and "PM3" respec-

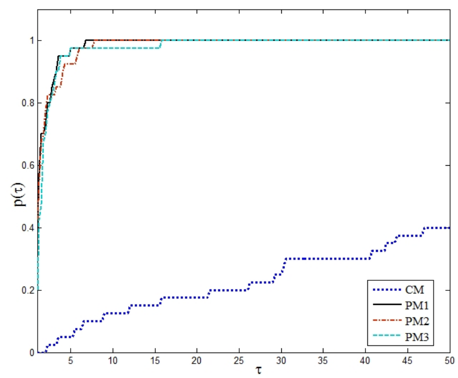


FIGURE 1. The performance profiles for collocation and our three proposed methods.

tively give mean errors of the collocation method, minimizing squares of errors, minimizing sum of absolute errors, and minimizing maximum of absolute errors. Clearly, Figure 1 shows that our optimization-based methods here significantly outperform the standard collocation method on the test problems.

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**Oral Presentation** **$*$ : Speaker**

# A NEW CLASS OF ALMOST L$^*$–LIMITED SETS IN BANACH LATTICES

M. SALIMI AND H. ARDAKANI*

[1] *Department of Mathematics, Farhangian University, Iran.*
*m_salimi52@yahoo.com*

[2] *Department of Mathematics, Payame Noor University,Tehran, Iran.*
*halimeh_ardakani@yahoo.com*

Abstract. We introduce the concepts of L$^*$–limited and almost L$^*$–limited sets in Banach lattices. We obtain some characterizations of them with respect to some well known geometric properties of Banach spaces, such as, weak DP$^*$ property, strong relatively compact Dunford-Pettis property and almost limited completely continuous operators on such Banach lattices.

## 1. Introduction

A subset $A$ of a Banach space $X$ is called limited, if every weak$^*$ null sequence $(x_n^*)$ in $X^*$ converges uniformly on $A$, that is

$$\lim_{n\to\infty} \sup_{a\in A} |\langle a, x_n^*\rangle| = 0.$$

Every relatively compact subset of $E$ is limited. If every limited subset of a Banach space $X$ is relatively compact, then $X$ has the Gelfand–Phillips (abb. GP) property. For example, $c_0$, $\ell_1$, reflexive spaces and Schur spaces (i.e., weak and norm convergence of sequences in $X$ coincide) have the GP property [3]. Also we recall that a Banach space

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$X$ has the GP property if and only if every limited and weakly null sequence $(x_n)$ in $X$ is norm null.
In this article, at first we define the concepts of L$^*$–limited and almost L$^*$–limited sets in Banach lattices. At first, we remember some definitions and terminologies from Banach lattice theory.
It is evident that if $E$ is a Banach lattice, then its dual $E^*$, endowed with the dual norm and pointwise order, is also a Banach lattice. The norm $\|.\|$ of a Banach lattice $E$ is order continuous if for each generalized net $(x_\alpha)$ such that $x_\alpha \downarrow 0$ in $E$, $(x_\alpha)$ converges to 0 for the norm $\|.\|$, where the notation $x_\alpha \downarrow 0$ means that the net $(x_\alpha)$ is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. $B_E$ is the closed unit ball of $E$. The lattice operations in the Banach lattice $E$ are weakly sequentially continuous if for every weakly null sequence $(x_n)$ in $E$, $|x_n| \to 0$ for $\sigma(E, E^*)$. We refer the reader for undefined terminologies, to the classical references [4].

## 2. Main results

Following the introducing of the concept limited sets in Banach spaces, we define L$^*$–limited sets and almost L$^*$–limited sets in Banach lattices.

**Definition 2.1.** Let $E$ be a Banach lattice. A norm bounded subset $B$ of a Banach lattice $E$ is said to be an L$^*$–limited set if every weakly null and limited sequence $(x_n)$ of $E^*$ converges uniformly to zero on the set $B$, that is $\sup_{f\in B} |f(x_n)| \to 0$.

It is clear that every limited set in $X$ is L$^*$–limited and every subset of an L$^*$–limited set is the same. Also, it is evident that every L$^*$–DP set is weak$^*$ bounded and so is bounded.
Recall from [2], that a subset $A$ of a Banach lattice $E$ is called almost limited if every disjoint weakl$^*$ null sequence $(x_n^*)$ in $E^*$ converges uniformly on $A$.

**Definition 2.2.** Let $E$ be a Banach lattice. A norm bounded subset $B$ of a Banach space $E$ is said to be an almost L$^*$–limited set if every weakly null and almost limited sequence $(x_n)$ of $E^*$ converges uniformly to zero on the set $B$, that is $\sup_{f\in B} |f(x_n)| \to 0$.

It is clear that every almost L$^*$–limited in $E$ is L$^*$–DP and every subset of an almost L$^*$–limited set is the same. Also, it is evident that every almost L$^*$–limited set is weak$^*$ bounded and so is bounded. The following theorem gives aditional properties of these sets.

**Proposition 2.3.** (a) *Absolutely closed convex hull of an almost L$^*$–limited is an almost L$^*$–DP set,*

(b) *relatively weakly compact subsets of Banach lattices are almost L$^*$–limited set.*

Note that the converse of assertion (b) in general, is false. In fact, the following theorem, shows that the closed unit ball of $c_0$ is an almost L$^*$–DP set, but it is not relatively weakly compact.
Recall from [**?**] that a Banach lattice $E$ has the strong GP property if all almost limited subsets of $E$ are relatively compact. It is clear that the strong GP property implies the GP property.
It is well known that every limited set is conditionally weakly compact. However, $B_{\ell_\infty}$ is indeed almost limited and by Rosenthal's $\ell_1$-theorem, $B_{\ell_\infty}$ is not conditionally weakly compact.
By [4], an element $e$ in a Banach lattice $E$ is called a *weak unit* if $B_e = E$, where $B_e$ is the band generated by $e$. For example, $C[0,1]$ is a Banach lattice with the weak unit $u(t) = t$. Banach lattices $M[0,1]$, of all signed Borel measures on $[0,1]$ of bounded variation and $(\ell_\infty)^*$, do not have any weak unit.

**Theorem 2.4.** [**?**] *Let $E$ be a Banach lattice such that $E^*$ has a weak unit or $E$ has order-continuous norm. Then every almost limited set $A$ in $E$ is conditionally weakly compact.*

**Theorem 2.5.** *Let $E$ be a Banach lattice such that $E^{**}$ has a weak unit or $E^*$ has order-continuous norm. Then $E^*$ has the strong GP property iff every bounded subset of $E$ is an almost L$^*$–limited set.*

**Definition 2.6.** A bounded linear operator $T$ from a Banach lattice $X$ into a Banach space $E$ is almost L$^*$–limited if $T(B_X)$ is an almost L$^*$–limited set in $E$. We denote this class of operators by $L_a^*li(E,Y)$.

**Definition 2.7.** A bounded linear operator $T$ from a Banach space $X$ into a Banach lattice $E$ is almost L$^*$–limited if $T(B_X)$ is an almost L$^*$–limited set. We denote this class of operators by $L_a^*li(X,E)$.

It is easy to see that the operator space $L_a^*li(X,E)$ is a norm-closed subspace of $L(X,E)$.
A Banach space $X$ has the GP proeprty if and only if for each Banach space $Y$, $Lcc(X,Y) = L(X,Y)$.

**Theorem 2.8.** *An operator $T$ is almost L$^*$–limited if and only if its adjoint $T^*$ is alcc. Also each weakly compact operator is almost L$^*$–limited.*

A Banach lattice $E$ has the weak DP$^*$ property if every relatively weakly compact set in $E$ is almost limited set.

**Theorem 2.9.** *If $E^*$ has the weak DP* property, then each almost $L^*$–limited set in $E$ is a limited set.*

**Definition 2.10.** A Banach lattice $E$ into a Banach space $E$ is almost L*–limited property, if all almost L*–limited sets in $E$ are relatively weakly compact.

**Corollary 2.11.** *Dual Banach lattice $E^*$ has the strong GP property and $E$ has the almost $L^*$–limited property, iff $E$ is reflexive.*

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# SOME PROPERTIES OF G-MATRICES

ALI ARMANDNEJAD

*Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran*
*armandnejad@vru.ac.ir*

Abstract. A nonsingular $n\times n$ real matrix $A$ is called a G-matrix if there exist nonsingular diagonal matrices $D_1$ and $D_2$ such that $A^{-T} = D_1AD_2$. Let $J = \text{diag}(\pm 1)$ be a signature matrix (a diagonal matrix that each of whose diagonal entries is $+1$ or $-1$). A nonsingular real matrix $Q$ is called $J$-orthogonal if $Q^TJQ = J$. In this note we investigate some properties of G-matrices and we find a relation between G-matrices and $J$-orthogonal matrices.

## 1. Introduction

The concept and properties of G-matrices were originally studied in [1] by Fiedler and Hall, see also [4]. For $n \le 3$, the set of all $n \times n$ G-matrices has been characterized but there is no characterization in case $n \ge 4$, see [2].

**Proposition 1.1.** *A $2 \times 2$ matrix is G-matrix if and only if it is nonsingular and has four or two nonzero entries. In other words every $2 \times 2$ G-matrix has one of the following forms:*
1$-$ $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ *or* $\begin{pmatrix} 0 & a \\ d & 0 \end{pmatrix}$ *where* $a \neq 0 \neq d$.
2$-$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *where* $a \neq 0 \neq d$, $b \neq 0 \neq c$ *and* $ad - bc \neq 0$.

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*Key words and phrases.* G-matrices, Connected components, J-orthogonal matrices.

**Proposition 1.2.** *A $3 \times 3$ matrix $A$ is G-matrix if and only if it is nonsingular and one of the following condition holds:*
1$-$ *There exist a $2\times 2$ G-matrix $B$ and a scalar $a$ such that $A = B \oplus [a]$, (up to multiplication by a permutation matrix).*
2$-$ *$A = [a_{ij}]$ has at most one zero entry and*

$$\det \begin{pmatrix} a_{12}a_{13} & a_{22}a_{23} & a_{32}a_{33} \\ a_{11}a_{13} & a_{21}a_{23} & a_{31}a_{33} \\ a_{11}a_{12} & a_{21}a_{22} & a_{31}a_{23} \end{pmatrix} = 0.$$

Some properties of G-matrices are collected in the following proposition:

**Proposition 1.3.** *The following statements are true:*
1$-$ *All orthogonal ($J$-orthogonal) matrices are G-matrices.*
2$-$ *All nonsingular diagonal matrices are G-matrices.*
3$-$ *Any $n$ positive real numbers are the singular values and eigenvalues of a diagonal G-matrix $D$.*
4$-$ *If $A$ is a G-matrix, then both $A^T$ and $A^{-1}$ are G-matrices.*
5$-$ *If $A$ is an $n \times n$ G-matrix and $D$ is an $n \times n$ nonsingular diagonal matrix, then both $AD$ and $DA$ are G-matrices.*
6$-$ *If $A$ is an $n \times n$ G-matrix and $P$ is an $n \times n$ permutation matrix, then both $AP$ and $PA$ are G-matrices.*

## 2. G-matrices and $J$-orthogonal matrices

Denote by $J = \mathrm{diag}(\pm 1)$ a diagonal (signature) matrix, each of whose diagonal entries is $+1$ or $-1$. As in [5], a nonsingular real matrix $Q$ is called $J$-orthogonal if $Q^T JQ = J$, or equivalently, if $Q^{-T} = JQJ$.

The following proposition gives a characterization of $J$-orthogonal matrices. For a fixed signature matrix $J$, let

$$\Gamma_n(J) = \{A \in \mathbf{M}_n : A^\top JA = J\}.$$

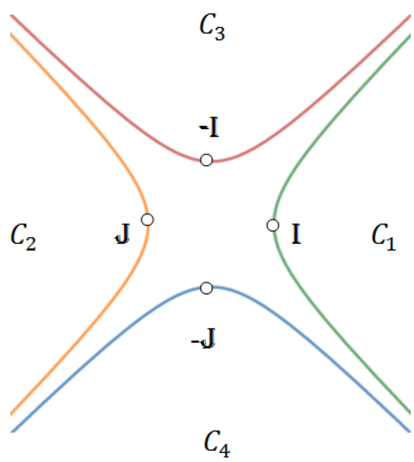
**Proposition 2.1.** *(hyperbolic CS decomposition) Let $q \geq p$ and $J = I_p \oplus (-I_q)$. Then every $A \in \Gamma_n(J)$ is of the form*

$$(U_1 \oplus U_2)(\begin{pmatrix} C & -S \\ -S & C \end{pmatrix} \oplus I_{q-p})(V_1 \oplus V_2), \tag{2.1}$$

*where $U_1, V_1 \in \mathcal{O}_p$, $U_2, V_2 \in \mathcal{O}_q$ and $C, S \in \mathbf{M}_p$ are nonnegative diagonal matrices such that $C^2 - S^2 = I$. Also, any matrix of the form (2.1) is $J$-orthogonal.*

The following proposition gives an interesting topological property of $J$-orthogonal matrices.

**Proposition 2.2.** *Let $J$ be an $n \times n$ signature matrix. If $J \neq \pm I$ then $\Gamma_n(J)$ has four connected components.*




For fixed nonsingular diagonal matrices $D_1$ and $D_2$, let

$$\mathbb{G}(D_1, D_2) = \{A \in \mathbf{M}_n : A^{-T} = D_1 A D_2\}.$$

Let $A$ be a nonsingular Hermitian matrix. The inertia matrix of $A$ is the diagonal matrix $i(A) = I_{i_+(A)} \oplus -I_{i_-(A)}$ in which $i_+(A)$ is the number of positive eigenvalues of $A$, and $i_-(A)$ is the number of negative eigenvalues of A, see [6].

The following known result from [1] shows that if $A^{-T} = D_1 A D_2$ then $D_1$ and $D_2$ have the same inertia matrix.

**Proposition 2.3.** *Suppose $A$ is a G-matrix and $A^{-T} = D_1 A D_2$, where $D_1$ and $D_2$ are nonsingular diagonal matrices. Then $i(D_1) = i(D_1)$.*

**Theorem 2.4.** *Let $D_1$ and $D_2$ be nonsingular diagonal matrices with the inertia matrix $J$. Then there exist permutation matrices $P$ and $Q$ such that*

$$\mathbb{G}(D_1, D_2) = \{|D_1|^{-1/2} P^T A Q |D_2|^{-1/2} : A \in \Gamma_n(J)\}.$$

**Corollary 2.5.** *Let $D_1$ and $D_2$ be nonsingular diagonal matrices with the inertia matrix $J$. If $J \neq \pm I$ then $\mathbb{G}(D_1, D_2)$ has four connected components.*

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# VECTOR BUNDLES AND HILBERT $C^*$-MODULES

MOHAMMAD B. ASADI

*School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran*
*mb.asadi@ut.ac.ir*

Abstract. In this talk, we review the categorical approach to Hilbert $C^*$-modules. We would also present some applications of vector bundles to study Hilbert $C^*$-modules.

## 1. Introduction and Preliminaries

One of the main approaches to study Hilbert $C^*$-modules is the categorical one. Indeed, a generalization of the Serre-Swan theorem states that the category of Hilbert $C_0(Z)$-modules is equivalent to the category of bundles of Hilbert spaces over locally compact Hausdorff space $Z$ [7], [9]. Moreover, the generalization of this result to a $C^*$-algebra over a non-commutative $C^*$-algebra holds. More precisely, the category of Hilbert C$^*$-modules over C$^*$-algebra $A$ is equivalent to some category consisting of Hilbert bundles over the pure state space of $A$ ([5]). We mention in section 4 some results in Hilbert $C^*$-module theory which has been obtained by regarding this equivalence of categories.

## 2. Continuous Fields of Hilbert Spaces

While studying category of Hilbert $A$-modules, where $A$ is a commutative C$^*$-algebra, the notion of continuous fields of Hilbert spaces appears.

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**Definition 2.1.** Let $Z$ be a locally compact Hausdorff space. Consider $((H_z)_{z\in Z},\Gamma)$,where $(H_z)_{z\in Z}$ is a family of Hilbert spaces and $\Gamma$ is a subset of $\prod_{z\in Z} H_z$. Also, we set

$$C_0-\prod_{z\in Z} H_z=\Big\{x\in\prod_{z\in Z} H_z : [x\mapsto\|x(z)\|]\in C_0(Z)\Big\}$$

The pair $((H_z)_{z\in Z},\Gamma)$ satisfying the following properties is said to be a continuous field of Hilbert spaces.

- $\Gamma$ is a subset of $C_0-\prod_{z\in Z} H_z$.
- For every $z\in Z$, the set $\{x(z)|x\in\Gamma\}$ is equal to $H_z$.
- For every $x\in\prod_{z\in Z} H_z$, if for every $z\in Z$ and every $\epsilon>0$, there is an $x'\in\Gamma$ such that $\|x(z')-x'(z')\|<\epsilon$, for all $z'$ in some neighborhood of $z$, then $x\in\Gamma$;

The space $\mathcal{H}=\prod_{z\in Z} H_z$ is called the total space.

Note that the function $z\mapsto\langle x(z),y(z)\rangle$ is an element of $C_0(Z)$, for every $x,y\in\Gamma$.

**Example 2.2.** Let $((H_z)_{z\in Z},\Gamma)$ be a continuous field of Hilbert spaces over a discrete space $Z$. One can easily conclude from definition of continuous field of Hilbert spaces that $\Gamma=\prod_{z\in Z} H_z$.

A morphism $\psi:((H_z)_{z\in Z},\Gamma)\longrightarrow((K_z)_{z\in Z},\Gamma')$ of continuous fields of Hilbert spaces is a family of linear maps $\{\psi_z:H_z\longrightarrow K_z : z\in Z\}$ such that the induced map $\psi:\mathcal{H}\longrightarrow\mathcal{K}$ on the total spaces satisfies $\{\psi\circ x : x\in\Gamma\}\subseteq\Gamma'$ and also the map $z\mapsto\|\psi_z\|$ is locally bounded. By [7, Proposition 4.7.], $\Gamma$ has a structure of Hilbert $C_0(Z)$-module with pointwise multiplication and inner product

$$\langle x,y\rangle(z)=\langle x(z),y(z)\rangle \qquad (x,y\in\Gamma, z\in Z).$$

Indeed, the category of Hilbert $C_0(Z)$-modules is equivalent to the category of continuous fields of Hilbert spaces [7, Proposition 4.8]. In particular, if $((H_z)_{z\in Z},\Gamma)$ and $((K_z)_{z\in Z},\Gamma')$ are the corresponding continuous fields of Hilbert spaces to Hilbert $C_0(Z)$-modules $X$ and $Y$, then for each $\Lambda$ in $End(X,Y)$, the map $\Lambda_z:H_z\longrightarrow K_z$ defined by $\Lambda_z(x(z))=(\Lambda(x))(z)$ is a well-defined bounded linear operator, for every $z\in Z$ [7].

**Example 2.3.** If we consider $A=C_0(Z)$ as a Hilbert $A$-module, in the natural way, then the corresponding continuous field of Hilbert spaces to Hilbert $A$-module $A$ is $((\mathbb{C}_z)_{z\in Z},\Gamma_A)$, where $\mathbb{C}_z=\mathbb{C}$, for every $z\in Z$ and $\Gamma_A=\{(f(z))_{z\in Z} : f\in C_0(Z)\}$. In particular, when $Z$ is discrete then $\Gamma_A=C_0-\prod_{z\in Z}\mathbb{C}_z$.

## 3. Holomorphic Hilbert bundles

Let $A$ be a C*-algebra, $\hat{A}$ the spectrum of $A$ and $P(A)$ be the set of pure states of $A$. In general, $P(A)$ is not compact, in this case we consider $P_0(A) = P(A) \cup \{0\}$. However, we set $P_0(A) = P(A)$, when $P(A)$ is compact.

We use the notations $\pi = [f]$ and $f = (\pi, e)$, whenever $\pi : A \longrightarrow B(H_\pi)$ is a member of $\hat{A}$ and $e = h \otimes h$ for some unit vector $h \in H_\pi$ and $f$ is the pure state $f(\cdot) = \langle \pi(\cdot)h, h\rangle$.

In this case, the unitary equivalence class of $f$ (as a set) is equal to

$$R_1(H_\pi) := \{e \in B(H_\pi) : \text{e is a rank one projection}\}.$$

The set $R_1(H_\pi)$ has a natural holomorphic manifold structure that is independent of the chosen representative element in each equivalence class in $P(A)$ [5]. Therefore, we can identify $P(A)$ as the disjoint union of projective spaces, i. e.,

$$P(A) = \bigcup_{\pi \in \hat{A}} \{\pi\} \times R_1(H_\pi).$$

Then $P_0(A)$ has a natural holomorphic manifold structure and it has a natural uniform structure determined by the seminorms arising from evaluation at the elements of $A$.

In [5], G. A. Elliott and K. Kawamura introduced the concept of uniformly continuous holomorphic Hilbert bundle of dual Hopf type over pure states of a C*-algebra.

## 4. Some Applications

In [8], the categorical approach to Hilbert $C_0(Z)$-modules used to determine the existence of frames in Hilbert $C^*$-modules [6, Question 8.1]. More precisely, the existence of a family $\{f_i\}_{i\in I}$ of elements of $X$, such that $\Sigma_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle$ is convergent in unltra weak operator topology to some element in universal enveloping von Neumann algebra of $A$. Also, there exist constants $0 < C \leq D < \infty$ such that for all $x \in X$,

$$C\langle x, x\rangle \leqslant \Sigma_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle \leq D\langle x, x\rangle$$

**Theorem 4.1.** *Let $(H, X(H))$ be a continuous field of Hilbert spaces over an infinite locally compact Hausdorff space $T$ with some accumulation points. There is a countable subset $W \subseteq T$ and a point $t_\infty \in \bar{W}/W$ that $H_t$ is separable for every $t \in W$ and $H_{t_\infty}$ is non-separable. Moreover, $X(H)$ as a Hilbert $C_0(T)$-module has no frames.*

Moreover, for a $C^*$-algebra of compact operators, i.e. a $C^*$-algebra that admits a non-degenerate representation in $K(H)$, for some Hilbert space $H$, this approach has been applied to determine the structure of a Hilbert $A \dot{+} \mathbb{C}$-module that admits no-frames [1] and [2].

In [3, Definition 1.3], algebra-valued G-frames in Hilbert $C^*$-modules has been defined as a family $\{\Lambda_i \in End(X, A) : i \in I\}$ such that there exist constants $0 < C \leq D < \infty$ that for every $x \in X$,

$$C\langle x, x\rangle \leq \sum_{i\in I}\langle \Lambda_i(x), \Lambda_i(x)\rangle \leq D\langle x, x\rangle, \tag{4.1}$$

where, by using the standard isometric embedding of $A$ into its universal enveloping von Neumann algebra $A^{**}$, the value $\sum_{i\in I}\langle \Lambda_i(x), \Lambda_i(x)\rangle$ is the limit of the increasingly ordered net of its finite partial sums with respect to the ultraweak topology on $A^{**}$. Considering a Hilbert $C_0(Z)$-module as a continuous field of Hilbert spaces, the following result holds.

**Theorem 4.2.** [3, Corollary 2.6] *Every Hilbert $C^*$-module over a commutative $C^*$-algebra $A$ admits an algebra-valued G-frame iff $A$ is $C^*$-algebra of compact operators.*

Also, in [4], orthogonality preserving pairs of operators on continuous fields of Hilbert spaces, which is one of the interesting problems in the study of linear preserving problems has been studied. By definition, pair of $\Psi, \Phi : E \to F$, where $E$ and $F$ are Hilbert $C_0(Z)$-modules, such that for every $x, y \in E$, $\langle x, y\rangle = 0$ implies $\langle \Psi(x), \Phi(y)\rangle = 0$.

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# SOME TOPOLOGICAL PROPERTIES OF $J_M(x)$

MEYSAM ASADIPOUR

*Department of Mathematics, College of Sciences, Yasouj University, Yasouj, 75918-74934, Iran.*
*Asadipour@yu.ac.ir*

Abstract. An operator $T \in \mathcal{L}(X)$ is called $M$-topologically transitive, if for each pair of nonempty open subsets $U$,$V$ of $M$ that are relatively open, there is a positive integer $n \geq 0$, such that $T^n(U) \cap V \neq \phi$ and $M$ is an invariant subspace under $T^n$. In this paper, we describe the local subspace transitivite operator and some topological properties of the set $\mathbb{J}_M(x)$ will be investigated.

## 1. Introduction

Assume that $X$ is an arbitrary Banach space with a closed subspace $M$ and $T \in \mathcal{L}(X)$ is a continuous linear map. If for every pair of nonempty open subset $(U, V)$ of $X$, there is an $n \in \mathbb{N}$, so that subset $T^n(U) \cap V \neq \phi$ and $M$ is an invariant subspace under $T^n$, then the operator $T$ is called $M$-topologically transitive. When $M = X$, then $T$ is called topologically transitive and for the first time, Birkhoff [3] proposed the topological transitivity operator with an example, which plays an important role in approximating any entire function belonging to $H(\mathbb{C})$. In [7], [8] and [9] one can see more information about $M$-topologically transitive.

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Authors in [4] and [5] tried to introduce the localized notion of topologically transitivity. In fact, they proposed $J(x)$ for a vector $x$ in Banach space $X$ as following:

$$J(x) = \{z \in X; \textit{there exist a sequence } \{z_n\} \subset X \textit{ and a strictly increasing sequence of positive integers } \{m_n\},\ \textit{such that } z_n \longrightarrow x \textit{ and } T^{m_n} z_n \longrightarrow z\}.$$

An equivalent definition for the set $J(x)$ through the open sets and a criterion for $J$-class operators were presented in [1]. Two good books for the study on transitivity and $J$-class operators are [2] and [6].

In this paper, we will define the definition of $M$-extended limit set for a vector $x$ and some topological properties of it will be investigated.

## 2. Preliminaries and Main Results

Although this section is devoted to the main results of this paper, but firstly we give some preliminaries.

**Definition 2.1.** $M$-extended limit set of vector $x$ under an operator $T \in \mathcal{L}(X)$ is the set of all $y \in M$ such that there is a sequence $\{z_n\}$ in subspace $M$ and a strictly increasing sequence $\{k_n\} \subset \mathbb{N}$ such that $z_n \longrightarrow x$ and $T^{k_n} z_n \longrightarrow y$, and for all n, $T^{k_n}(M) \subseteq M$.
The $M$-extended limit set of vector $x$ under an operator $T$ is denoted by $\mathbb{J}_M(x)$.

**Proposition 2.2.** *Let $T \in \mathcal{L}(X)$ be an $M$-topologically transitive operator. Then for every nonempty relatively open subsets $U$, $V$ of $M$, there is an infinite subset $P \subseteq \mathbb{N}$ such that for all $n \in P$,the set $V \cap T^{-n}(U)$ is nonempty and $M$ is an invariant subspace under $T^n$.*

*Proof.* If $T \in \mathcal{L}(X)$ is an $M$-transitive operator and $U$, $V$ are nonempty relatively open subsets of $M$ and if $n_0 \geq 0$, so that $T^{-n_0}(U) \cap V \neq \emptyset$ and $T^{n_0}(M) \subseteq M$, then two distinct vectors $x, y \in T^{-n_0}(U) \cap V$ and relatively open subsets $\mathcal{U}_x$ and $\mathcal{U}_y$ of $M$ are assumed, so that $\mathcal{U}_x, \mathcal{U}_y \subset T^{-n_0}(U) \cap V$ and $\mathcal{U}_x \cap \mathcal{U}_y = \emptyset$.
According to the assumptions, there is an integer $k \geq 1$, such that we have $T^k(M) \subseteq M$ and $T^{-k}(\mathcal{U}_x) \cap \mathcal{U}_y \neq \emptyset$. Thus, the set $V \cap T^{-(k+n_0)}(U)$ is nonempty and, clearly, $T^{(n_0+k)}(M) \subseteq M$. By repeating the above method, it can be ensured that there is an infinite subset $P$ of $\mathbb{N}$, such that for every $n \in P$, $V \cap T^{-n}(U) \neq \emptyset$ and $T^n(M) \subseteq M$. ∎

The recent proposition shows that one can find a strictly icreasing sequence $\{k_n\} \subset \mathbb{N}$ with desirable property in the following theorem.

**Theorem 2.3.** *Assume that $T \in \mathcal{L}(X)$ and $x \in M$. In this case, we have $\mathbb{J}_M^{tra}(x) = \mathbb{J}_M(x)$ where $\mathbb{J}_M^{tra}(x)$ is the set of all $y \in M$ such that for every relatively open neighborhoods $U_x$, $V_y$ of vectors $x, y$ in $M$ respectively, and every positive integer $n_0$, there exists an integer $n > n_0$ such that $T^n(U_x) \cap V_y \neq \emptyset$ and $T^n(M) \subseteq M$.*

*Proof.* Assume that $y \in \mathbb{J}_M^{tra}(x)$, $k_0 = 1$, $N = k_{n-1}$ and set $U_{(x,n)} = B(x, \dfrac{1}{n}) \cap M$, $V_{(y,n)} = B(y, \dfrac{1}{n}) \cap M$, for every $n \in \mathbb{N}$. By the assumption, integer $k_n > N$ and vector $x_n \in U_{(x,n)}$ exist such that $T^{k_n}(M) \subseteq M$ and $T^{k_n} x_n \in V_{(y,n)}$. Thus the sequences $\{k_n\}$ and $\{x_n\}$ are proposed by induction so that we have $x_n \longrightarrow x$ and $T^{k_n} x_n \longrightarrow y$. This completes the proof of nontrivial side. □

For an operator $T \in \mathcal{L}(X)$, if equality $\mathbb{J}_M(x) = M$ is established for a vector $x \in M$, then $x$ is a subspace $\mathbb{J}$-class vector for subspace $M$ under $T$. We then call $T$ a $\mathbb{J}_M$-class operator.

According to the earlier proposition which in fact defines a localized $M$-transitive operator, it is concluded that $T \in \mathcal{L}(X)$ is an $M$-transitive operator if and only if $\mathbb{J}_M(x) = M$ for all $x \in M$. Now, some topological properties of the set $\mathbb{J}_M(x)$ will be investigated.

**Theorem 2.4.** *Let $T \in \mathcal{L}(X)$. Then the set of all subspace $\mathbb{J}_M$-class vectors for subspace $M$ under $T$ is a closed, connected and $T$-invariant set.*

*Proof.* For convenience in the proof process, set:

$$\mathcal{J} = \{x \in M;\ \ \mathbb{J}_M(x) = M\}.$$

Clearly, the set $\mathcal{J}$ is a $T$-invariant set .

**step1.** the set $\mathcal{J}$ is connected. For the proof of this claim and to overlook the trivial state, assume that $x \in \mathcal{J}$, $y \in M$, $\lambda \in \mathbb{C}$ and $\lambda \neq 0$. In this case, there is a strictly increasing sequence $\{k_n\} \subset \mathbb{N}$ and a sequence $\{x_n\} \subset M$, such that $x_n \longrightarrow x$ and $T^{k_n} x_n \longrightarrow \dfrac{1}{\lambda} y$.
If $\{\lambda_n\} \subset \mathbb{C}$, $\lambda_n \longrightarrow \lambda$, then, $\lambda_n x_n \longrightarrow \lambda x$ and $T^{k_n}(\lambda_n x_n) \longrightarrow y$. Thus $y \in \mathbb{J}_M(\lambda x)$, or equivalently, $\mathbb{J}_M(\lambda x) = \mathbb{C}x$.
Note that fore every vector $x$, the set $\mathbb{C}x$ is connected and $J = \bigcup\limits_{x \in J} \mathbb{C}x$.
In the end of this step, if $\lambda_n \longrightarrow 0$, then $\mathbb{J}_M(0) = M$ or $0 \in \mathcal{J}$. Since $0 \in \bigcap\limits_{x \in J} \mathbb{C}x$, so $\mathcal{J}$ is a connected set.

**step2.** Suppose that an arbitrary sequence $\{y_n\} \subseteq \mathbb{J}_M(x)$ converges to $x$, then we claim that $y \in \mathbb{J}_M(x)$.
If $N \in \mathbb{N}$ and two arbitrary relatively open sets $U_x$ and $U_y$ are considered, then by the theorem 2.3, there exis integer $n_1 > N$, $y_{n_1} \in U_y$ and relatively open set $U_{y_{n_1}}$ such that $U_{y_{n_1}} \subset U_y$. Reuse of the theorem 2.3 shows that there is an integer $n > n_1$, such that;

$$\emptyset \neq T^n(U_x) \cap U_{y_{n_1}} \subseteq T^n(U_x) \cap U_y \quad and \quad T^n(M) \subseteq M.$$

Hence, the proof is completed.

$\square$

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# AN APPLICATION OF EQUIVALENT DEFINITION OF $J$-SET

MEYSAM ASADIPOUR

*Department of Mathematics, College of Sciences, Yasouj University, Yasouj, Iran*
*Asadipour@yu.ac.ir*

Abstract. For an operator $T \in L(X)$ and a vector $x \in X$, $J_T(x)$ denotes the set of all points $y \in X$ for which there is a sequence $\{z_n\} \subset X$ and a strictly increasing sequence $\{k_n\} \subset \mathbb{N}$ such that $z_n \longrightarrow x$ and $T^{k_n} z_n \longrightarrow y$. For some non-zero vector $x$, if $J_T(x) = X$, then $x$ is called a $J$-class vector for $J$-class operator $T$. In this paper, we show that, the set of all $J$-class operators on a Banach space $X$ is either empty or, it contains at least two operators of $L(X)$. Also, by providing a nontrivial example, we show that $J_T(0) = X$ however, $T$ is not a $J$-class operator, so $J$-class vector should be a non-zero vector.

## 1. Introduction

Assume that $X$ is a Banach space over the field $\mathbb{C}$ of complex numbers space and $T \in L(X)$. If for every pair of nonempty open subsets $(U, V)$ of $X$, there is $n \in \mathbb{N}$, so that subset $T^n(U) \cap V$ is nonempty; then, the map $T$ is transitive. For the first time, Birkhoff [3] proposed the topological transitivity operator with an example, which plays an important role in approximating any entire function belonging to $H(\mathbb{C})$. If the underlying space is considered as a separable Banach space, then transitivity is equivalent to hypercyclicity. To clarify the notion of hypercyclicity, note that, if for an $x \in X$ the orbit of $x$ under $T$, i.e.

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$orb(T,x) = \{T^n x;\ n = 0, 1, 2, \cdots\}$ is dense in $X$, then $x$ is called a hypercyclic vector for the hypercyclic operator $T$. Ansari [1] constructed a hypercyclic operator for an arbitrary infinite-dimensional separable Banach space. Therefore every non-separable Banach space is deleted in hypercyclicity.

Authors in [4] proposed $J_T(x)$ for an operator $T$ and a vector $x$ in Banach space $X$ as following:

$$\begin{aligned} J_T(x) = \{z \in X : & \textit{ there exist a sequence } \{z_n\} \subset X \textit{ and a strictly} \\ & \textit{increasing sequence of positive integers } \{m_n\},\ \textit{such} \\ & \textit{that } z_n \longrightarrow x \textit{ and } T^{m_n} z_n \longrightarrow z\}. \end{aligned}$$

Then they claimed that an equivalent definition for this set as follows;

$$\begin{aligned} J_T(x) = \{z \in X : & \textit{ for every pair of neighborhoods } U,\ V \textit{ of} \\ & x,\ z \textit{ respectively, there exists a positive integer} \\ & n \textit{ such that } T^n U \cap V \neq \phi\}. \end{aligned}$$

An operator $T$ is called a $J$-class operator if there exists a non-zero $x \in X$ such that $J_T(x) = X$. In fact, authors in [4] have tried to introduce the localized notion of hypercyclicity. Firstly with an example, we demonstrate the above localization is wrong. Indeed, consider the backward shift operator $B$ on $\ell^2(\mathbb{N})$ the space of square summable sequences, and set $T = \frac{1}{2}B$. Now consider a vector $z \in \ell^2(\mathbb{N})$ such that $Tz \neq 0$. Obviously the vector $Tz$ belongs to the recent set, but for every strictly increasing sequence of positive integers $\{k_n\}$ and every sequence $\{z_n\} \subset X$, if $z_n \longrightarrow z$ then $T^{k_n} z_n \longrightarrow 0$ and we get $J_T(x) = \{0\}$. Thus the above recent set is not equivalent to $J_T(x)$. It worth to mention that, an equivalent definition for the set $J_T(x)$ through the open sets was introduced by Asadipour and Yousefi, [2]. To be precise;

$$\begin{aligned} J_T(x) = \{z \in X : & \textit{ for every pair of neighborhoods } U,\ V \textit{ of} \\ & x,\ z \textit{ respectively, and every } N \in \mathbb{N} \textit{ there exists} \\ & \textit{an integer } n > N \textit{ such that } T^n U \cap V \neq \phi\}. \end{aligned}$$

In the following, we use this new definition to construct new $J$-class operators from a $J$-class operator. Also, we show that, why should the $J$-class vector be non-zero?

More information on the $J$-class operators and hypercyclicity can be seen in [5], [6] and [7].

## 2. Main results

As we mentioned in the previous section, the underlying Banach space in hypercyclicity should be separable, however, this restriction is not in the $J$-class. In other words, in addition to the separable spaces, which are considered in the hypercyclicity, some non-separable Banach spaces such as $\ell^{\infty}(\mathbb{N})$ support $J$-class operators, [4]. So we stress that in the following, $X$ denotes only a Banach space, unless emphasis on its separability or non-separability.

**Theorem 2.1.** *Let $T \in L(X)$ be a $J$-class operator. Then for every invertible operator, $S \in L(X)$, the operator $S^{-1}TS$ is a $J$-class operator.*

*Proof.* Consider $x \in X$ as a $J$-class vector for the operator $T$ and fix an $N \in \mathbb{N}$. If $y \in X$ is an arbitrary vector and $U_{S^{-1}x}, V_y$ are two open neighborhoods of $S^{-1}x$ and $y$, respectively, then by invertibility of $S$, there exists an integer $n > N$ such that;

$$T^n S(U_{S^{-1}x}) \cap S(V_y) \neq \phi.$$

Therefore $S^{-1}T^n S(U_{S^{-1}x}) \cap V_y \neq \phi$ or equivalently $y \in J_{S^{-1}TS}(S^{-1}x)$. Hence the vector $S^{-1}x$ is a $J$-class vector for the operator $S^{-1}TS$. □

So the set of all $J$-class operators on a Banach space $X$ is either empty, or contains many operators of $L(X)$. As you see, the proof was very easily expressed with the help of the equivalent definition for $J_T(x)$ through the open sets. Now we want to answer the following question;

Why should the $J$-class vector be non-zero in definition?
Contrary to the obvious example provided in [4], in the following, a nontrivial operator $T$ will be raised such that $J_T(0) = X$ however, is not a $J$-class operator. In fact, the unilateral weighted backward shift $T$ on $\ell^2(\mathbb{N})$ with the weight sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ is topological transitive, if, and only if $\limsup_n (\prod_{i=1}^{n} w_i) = +\infty$, [6], therefore the following operator $T$ is not local topological transitive anywhere except at zero.

**Example 2.2.** Consider weighted backward shift operator $T$ on $\ell^2(\mathbb{N})$ given by:

$$T(x^1, x^2, \cdots) = (2x^2, \frac{3}{2}x^3, \frac{4}{3}x^4, \cdots).$$

Also let $Y$ be the set of finite sequences with entries $z \in \mathbb{C}$ that $Re(z) \in \mathbb{Q}$, $Im(z) \in \mathbb{Q}$. Since $Y$ is dense in $\ell^2(\mathbb{N})$, so there are strictly increasing sequence $\{2k\}_k$, sequence $\{x_k\} \subset Y$ that

$$x_k = (x^1, 0, x^3, 0, \cdots, x^{2k-1}, 0, 0, \cdots),$$

$x_k \longrightarrow 0$ as $k \longrightarrow \infty$ and $T^{2k} x_k = 0$. Now, for the random member

$$y = (y^1, 0, y^3, 0, \cdots, y^{2m+1}, 0, 0, \cdots) \in Y$$

and $k \geq 1$, we set;

$$w_{2k}(y) = (\underbrace{0, \cdots, 0}_{2k-times}, \ \frac{y^1}{2k+1}, 0 \ , \frac{3y^3}{2k+3}, \ 0, \cdots, \frac{(2m+1)y^{2m+1}}{2(k+m)+1}, \ 0, \ 0, \cdots).$$

Clearly, for every $k \in \mathbb{N} \cup \{0\}$, $w_{2k}(y)$ belongs to $Y$ and the sequence $\{w_{2k}(y)\}$ is a sequence in $\ell^2(\mathbb{N})$. Since

$$||w_{2k}(y)||^2 = \sum_{j=1}^{2m} |\frac{j}{2k+j} y^j|^2 \leq \frac{4m^2}{(2k+1)^2} ||y||^2,$$

so $w_{2k}(y) \longrightarrow 0$, as $k \longrightarrow \infty$. Note that for $n \geq 1$:

$$T^n(x^1, x^2, x^3, \cdots) = \left( (n+1)x^{n+1}, \ \frac{1}{2}(n+2)x^{n+2}, \ \frac{1}{3}(n+3)x^{n+3}, \ \cdots \right),$$

thus

$$\begin{aligned} T^{2k} w_{2k}(y) = \Bigg( &(2k+1)\frac{1}{2k+1} y^1, 0, (\frac{2k+3}{3})(\frac{3}{2k+3}) y^3, 0, \cdots, \\ &(\frac{2(k+m)+1}{(2m+1)})(\frac{2m+1}{2(k+m)+1}) y^{2m+1}, 0, 0, \cdots \Bigg) = y. \end{aligned}$$

Hence all conditions of the $J$-class Criterion in [2] holds and $J_T(0) = \ell^2(\mathbb{N})$.

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# A NOTE ON GENERALIZED SHIFTS OVER ENTIRE FUNCTIONS FAMILY

F. AYATOLLAH ZADEH SHIRAZI*, E. HAKIMI

*Faculty of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran*
*f.a.z.shirazi@ut.ac.ir , fatemah@khayam.ut.ac.ir and elaheh.hakimi@gmail.com*

Abstract. In the following text for $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ we show that if $\sigma_\varphi(E) \subseteq E$, then $\varphi$ is finite fiber, however the inverse implication is not true (where $E$ denotes the collection of all entire functions and for $\sum\limits_{n\geq 0} a_n z^n \in \mathbb{C}[[z]]$ we have $\sigma_\varphi(\sum\limits_{n\geq 0} a_n z^n) = \sum\limits_{n\geq 0} a_{\varphi(n)} z^n$).

## 1. Introduction

Generalized shifts have been studied in different areas like: "Dynamical Systems", "Functional Analysis", "Group Theory", "Ergodic Theory", "Transformation groups", etc. (see e.g., [1, 3]), our main aim in this text is to have a glance at generalized shifts with taste of "Complex Analysis", let's mention that shift operators have been studied in analytic approaches by so many authors (see, e.g., [4, 5, 6]) although our point of view is different in some senses. It's common to consider an entire function as a member of all formal power series with coeffecients in complex domain $\mathbb{C}$, $\mathbb{C}[[z]] = \left\{ \sum\limits_{n=0}^{+\infty} a_n z^n : \forall n \geq 0 \, a_n \in \mathbb{C} \right\}$,

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where $\mathbb{C}[[z]]$ is a ring equipped with following sum and product (consider $\sum\limits_{n=0}^{+\infty} a_n z^n, \sum\limits_{n=0}^{+\infty} b_n z^n \in \mathbb{C}[[z]]$):

$$\sum_{n=0}^{+\infty} a_n z^n + \sum_{n=0}^{+\infty} b_n z^n = \sum_{n=0}^{+\infty} (a_n + b_n) z^n \ ,$$

$$(\sum_{n=0}^{+\infty} a_n z^n)\,(\sum_{n=0}^{+\infty} b_n z^n) = \sum_{n=0}^{+\infty} (\sum_{i+j=n} a_i b_j) z^n \ .$$

For nonempty sets $\Gamma, X$ and $\varphi : \Gamma \to \Gamma$ we call $\underset{(x_\alpha)_{\alpha\in\Gamma}\mapsto(x_{\varphi(\alpha)})_{\alpha\in\Gamma}}{\sigma_\varphi : X^\Gamma \to X^\Gamma}$ a generalized shift (generalized shift has been introduced for the first time in [2] as a generalization of one–sided and two–sided shifts).
For region $U$ in $\mathbb{C}$ suppose $H(U)$ denote the collections of all holomorphic functions on $U$, in particular $E := H(\mathbb{C})$ denotes the collection of all entire functions, so one may consider inclusions $E \subseteq H(\mathbb{U}) \subseteq \mathbb{C}[[z]](= \mathbb{C}^{\mathbb{N}_0}$, where $\mathbb{N}_0 = \{0, 1, 2, ...\}$ and $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\})$, so for $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ one may consider $\sigma_\varphi \restriction_E$ and $\sigma_\varphi \restriction_{\mathbb{U}}$, with $\sigma_\varphi(\sum\limits_{n\geq 0} a_n z^n) = \sum\limits_{n\geq 0} a_{\varphi(n)} z^n$. Note that $\sigma_\varphi : \mathbb{C}[[z]] \to \mathbb{C}[[z]]$ is a morphism of algebras.
Let's mention that:

$$E = \left\{ \sum_{n\geq 0} a_n z^n \in \mathbb{C}[[z]] : \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = 0 \right\} \ ,$$

$$H(\mathbb{U}) = \left\{ \sum_{n\geq 0} a_n z^n \in \mathbb{C}[[z]] : \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leq 1 \right\} \ .$$

## 2. Main results and counterexamples

In this section for $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ we show:

- $\sigma_\varphi(H(r\mathbb{U})) \subseteq H(\mathbb{U})$ for all $r > 1$,
- if $\sigma_\varphi(E) \subseteq E$, then $\varphi$ is finite fiber, i.e., for all $n \geq 0$, $\varphi^{-1}(n)$ is finite.

The text will be motivated by counterexamples.

**Theorem 2.1.** *For $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ if $\sigma_\varphi(E) \subseteq E$, then for all $n \geq 0$, $\varphi^{-1}(n)$ is finite.*

*Proof.* If there exists $m \geq 0$ such that $\varphi^{-1}(m) = \{n_1, n_2, \ldots\}$ with $n_1 < n_2 < \cdots$, and $z^m = \sum\limits_{n\geq 0} a_n z^n \in E$ then $\limsup\limits_{n\to\infty} |a_{\varphi(n)}|^{\frac{1}{n}} = \lim\limits_{k\to\infty} |a_{\varphi(n_k)}|^{\frac{1}{n_k}} = \lim\limits_{k\to\infty} |a_m|^{\frac{1}{n_k}} = 1$ and $\sigma_\varphi(z^m) \notin E$. $\square$

In the following example we show in Theorem 2.1 one can not replace $E$ with $\mathbb{U}$.

**Example 2.2.** For $p \geq 0$, consider constant map $\varphi_p : \mathbb{N}_0 \underset{n\mapsto p}{\to} \mathbb{N}_0$, then for all $\sum\limits_{n\geq 0} a_n z^n \in \mathbb{C}[[z]]$ we have:

$$\lim_{n\to\infty} |a_{\varphi_p(n)}|^{\frac{1}{n}} = \lim_{n\to\infty} |a_p|^{\frac{1}{n}} = \left\{ \begin{array}{ll} 1 & a_p \neq 0 \\ 0 & a_p = 0 \end{array} \right.$$

which leads to $\sigma_\varphi(H(\mathbb{U})) = \sigma_\varphi(E) = \sigma_\varphi(\mathbb{C}[[z]]) \subseteq H(\mathbb{U})$.

**Theorem 2.3.** *For $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ and $r > 1$ we have $\sigma_\varphi(E) \subseteq \sigma_\varphi(H(r\mathbb{U})) \subseteq H(\mathbb{U})$.*

*Proof.* Consider $\sum\limits_{n\geq 0} a_n z^n \in H(r\mathbb{U})$, then $\sum\limits_{n\geq 0} a_n$ is a(n absolutely) convergent serries and $\lim\limits_{n\to\infty} a_n = 0$. Suppose $\theta := \limsup\limits_{n\to\infty} |a_{\varphi(n)}|^{\frac{1}{n}}$. There exists $n_1 < n_2 < \cdots$ with $\theta = \lim\limits_{k\to\infty} |a_{\varphi(n_k)}|^{\frac{1}{n_k}}$. We have the following cases:
- *Case 1*: $\{\varphi(n_k)\}_{k\geq 1}$ has a constant subsequnce $\{\varphi(n_{k_j})\}_{j\geq 1}$: in this case suppose $\varphi(n_{k_j}) = p$ for all $j \geq 1$, then $\theta = \lim\limits_{k\to\infty} |a_{\varphi(n_k)}|^{\frac{1}{n_k}} = \lim\limits_{j\to\infty} |a_{\varphi(n_{k_j})}|^{\frac{1}{n_{k_j}}} = \lim\limits_{j\to\infty} |a_p|^{\frac{1}{n_{k_j}}} \in \{0, 1\}$.
- *Case 2*: $\{\varphi(n_k)\}_{k\geq 1}$ does not have any constant subsequnce: in this case using $\lim\limits_{n\to\infty} a_n = 0$ for all $\varepsilon > 0$ there exists $N \geq 1$ such that $|a_n| < \varepsilon$ for all $n \geq N$. Since $\{\varphi(n_k)\}_{k\geq 1}$ does not have any constant subsequnce there exists $L \geq 1$ such that $\varphi(n_k) \geq N$ for all $k \geq L$, hence $\theta = \lim\limits_{k\to\infty} |a_{\varphi(n_k)}|^{\frac{1}{n_k}} \leq \limsup\limits_{k\to\infty} \varepsilon^{\frac{1}{n_k}} = 1$
Using the above cases $\theta = \limsup\limits_{n\to\infty} |a_{\varphi(n)}|^{\frac{1}{n}} \leq 1$ and $\sigma_\varphi(\sum\limits_{n\geq 0} a_n z^n) \in H(\mathbb{U})$. $\square$

**Example 2.4.** Consider $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ with $\varphi(n!) = n$ $(n \geq 1)$, then for $e^z = \sum\limits_{n\geq 0} \frac{z^n}{n!} \in E$ and $a_n = \frac{1}{n!}$ we have $\limsup\limits_{n\to\infty} |a_{\varphi(n)}|^{\frac{1}{n}} \geq \limsup\limits_{n\to\infty} |a_{\varphi(n!)}|^{\frac{1}{n!}} = \limsup\limits_{n\to\infty} |a_n|^{\frac{1}{n!}} = 1$ hence by Theorem 2.3, $\sigma_\varphi(\sum\limits_{n\geq 0} \frac{z^n}{n!}) \in H(\mathbb{U}) \setminus E$.

Following example shows that Theorem 2.3 is not valid for $r = 1$.

**Example 2.5.** Consider $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ with $\varphi(n) = (n+1)^n$, then for $\sum\limits_{n\geq 0} nz^n \in H(\mathbb{U})$ and $a_n = n$ we have:

$$\limsup_{n\to\infty} |a_{\varphi(n)}|^{\frac{1}{n}} = \limsup_{n\to\infty} |a_{(n+1)^n}|^{\frac{1}{n}} = \limsup_{n\to\infty}(n+1) = +\infty \, .$$

Hence

$$\forall s > 0 \;\; \sigma_\varphi(\sum_{n\geq 0} nz^n) \notin H(s\mathbb{U}) \, .$$

## Acknowledgement

The authors wish to dedicate this text to the martyr of Ramadan Imam Ali (as).

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# SOME RESULTS ON OSTROWSKI'S INEQUALITY

H. BARSAM

*Department of Mathematics, Faculty of Science, University of Jiroft,*
*P.O. Box 78671-61167, Jiroft, Iran*
*hasanbarsam1360@gmail.com*

ABSTRACT. In this paper, we establish Ostrowski's type inequality for uniformly s-convex functions. Also, we obtain some new inequalities of Ostrowski's type for functions whose derivatives in absolute value are the class of uniformly s-convex.

## 1. INTRODUCTION

In 1928 Ostrowski proved the following result:
IF $f : I \to \mathbb{R}$ is continous on $(a,b)$ and $f' : I \to \mathbb{R}$ is bounded on $(a,b)$ such that $||f'||_\infty < \infty$ then

$$|f(x) - \frac{1}{b-a}\int_a^b f(t)dt| \leq [\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}](b-a)||f'||_\infty$$

for all $x \in (a,b)$. The constant $\frac{1}{4}$ in above inequality is the best. Because of the attractiveness of the inequality topic, in recent years, a lot of researchers have improved the Ostrowski and other inequality to other functions (see[1], [3], [4], [5]).
In this section, we consider the basic concepts and results, which are needed to obtain our main results.

In [[2], Definition 10.5], the class of uniformly convex functions is defined as follows and we generalize this definition to uniformly convex functions in the following.

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**Definition 1.1.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then $f$ is called uniformly s-convex function with modulus $\psi : [0, +\infty) \to [0, +\infty]$ if $\psi$ is increasing, $\psi$ vanishes only at 0, and

$$f(tx + (1-t)y) + t^s(1-t)\psi(|x-y|) \leq t^s f(x) + (1-t)^s f(y), \quad (1.1)$$

for each $x, y \in [0, +\infty)$ and $t \in [0, 1]$. Furthermore, if $s = 1$, then $f$ is called uniformly convex.

**Example 1.2.** ([2]) In view of the equality,

$$(tx + (1-t)y)^2 + t(1-t)(x-y)^2 = tx^2 + (1-t)y^2,$$

for all $t \in (0, 1)$ and $x, y \in \mathbb{R}$, the function $f(t) = t^2$ for $t \in \mathbb{R}$ is uniformly convex with $s = 1$ and modulus $\psi(t) = t^2$ for all $t \geq 0$.

In [1], Alomari et al. proved the following inequality of Ostrowski type for functions whose derivative in absolute value are s-convex in the second sense.

**Lemma 1.3.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^o$ and $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\begin{aligned} f(x) - \frac{1}{b-a}\int_a^b f(t)dt = & \frac{(x-a)^2}{b-a}\int_0^1 tf'(tx + (1-t)a)dt \\ & - \frac{(b-x)^2}{b-a}\int_0^1 tf'(tx + (1-t)b)dt \end{aligned}$$

*for each $x \in [a, b]$.*

## 2. Main results

### 2.1. Ostrowski type inequalities.

**Theorem 2.1.** *Let $f : I \subset [0, +\infty) \to [0, +\infty)$ be a differentiable mapping on $I^o$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is uniformly s-convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $|f'(x)| \leq M, x \in [a, b]$, then the following inequality holds:*

$$\begin{aligned} |f(x) - \frac{1}{b-a}\int_a^b f(t)dt| \leq & \frac{M}{b-a}[\frac{(x-a)^2 + (x-b)^2}{s+1}] \\ & - \frac{1}{b-a}[\frac{(x-a)^2\psi(|x-a|) + (x-b)^2\psi(|x-b|)}{(s+1)(s+2)}]. \end{aligned}$$

*Proof.* In view of Lemma 1.3 and uniformly s-convexity of $|f'|$, one has

$$|f(x)-\frac{1}{b-a}\int_a^b f(t)dt|\leq$$
$$\frac{(x-a)^2}{b-a}\int_0^1 t|f'(tx+(1-t)a)|dt+\frac{(b-x)^2}{b-a}\int_0^1 t|f'(tx+(1-t)b)|dt$$
$$\leq\frac{(x-a)^2}{b-a}\int_0^1 t[t^s|f'(x)|+(1-t)^s|f'(a)|-t^s(1-t)\psi(|x-a|)]dt$$
$$+\frac{(x-b)^2}{b-a}\int_0^1 t[t^s|f'(x)|+(1-t)^s|f'(b)|-t^s(1-t)\psi(|x-b|)]dt$$
$$\leq\frac{(x-a)^2}{b-a}[\frac{|f'(x)|}{s+2}+\frac{\Gamma(2)\Gamma(s+1)}{\Gamma(s+3)}|f'(a)|-\frac{\Gamma(s+2)\Gamma(2)}{\Gamma(s+4)}\psi(|x-a|)]$$
$$+\frac{(x-b)^2}{b-a}[\frac{|f'(x)|}{s+2}+\frac{\Gamma(2)\Gamma(s+1)}{\Gamma(s+3)}|f'(b)|-\frac{\Gamma(s+2)\Gamma(2)}{\Gamma(s+4)}\psi(|x-b|)]$$
$$\leq\frac{(x-a)^2}{b-a}[\frac{M}{s+1}-\frac{\psi(|x-a|)}{(s+3)(s+2)}]+\frac{(x-b)^2}{b-a}[\frac{M}{s+1}-\frac{\psi(|x-a|)}{(s+3)(s+2)}]$$
$$\leq\frac{M}{b-a}[\frac{(x-a)^2+(x-b)^2}{s+1}]-\frac{1}{b-a}[\frac{(x-a)^2\psi(|x-a|)+(x-b)^2\psi(|x-b|)}{(s+3)(s+2)}]$$

□

*Remark* 2.2. In Theorem 2.3, if $s=1$, then

$$|f(x)-\frac{1}{b-a}\int_a^b f(t)dt|\leq\frac{M}{b-a}[\frac{(x-a)^2+(x-b)^2}{2}]$$
$$-\frac{1}{b-a}[\frac{(x-a)^2\psi(|x-a|)+(x-b)^2\psi(|x-b|)}{12}].$$

New inequalities of Ostrowski's type for uniformly s-convex functions as follows:

**Theorem 2.3.** *Let $f: I\subset[0,+\infty)\to[0,+\infty)$ be a differentiable mapping on $I^o$ such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is uniformly s-convex on $[a,b]$ for some fixed $s\in(0,1]$, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $|f'(x)|\leq M, x\in[a,b]$, then the following inequality holds:*

$$|f(x)-\frac{1}{b-a}\int_a^b f(t)dt|\leq\frac{(x-a)^2}{b-a}(\frac{1}{p+1})^{\frac{1}{p}}[\frac{2(s+2)M^q-\psi(|x-a|)}{(s+1)(s+2)}]^{\frac{1}{q}}$$
$$+\frac{(x-b)^2}{b-a}(\frac{1}{p+1})^{\frac{1}{p}}[\frac{2(s+2)M^q-\psi(|x-b|)}{(s+1)(s+2)}]^{\frac{1}{q}}$$

*Proof.* By Lemma 1.3 and Hölder's inequality, we conclude

$$
\begin{aligned}
&|f(x) - \frac{1}{b-a}\int_a^b f(t)dt| \leq \\
&\frac{(x-a)^2}{b-a}[(\int_0^1 t^p dt)^{\frac{1}{p}}(\int_0^1 |f'(tx+(1-t)a)|^q dt)^{\frac{1}{q}} \\
&+ \frac{(x-b)^2}{b-a}[(\int_0^1 t^p dt)^{\frac{1}{p}}(\int_0^1 |f'(tx+(1-t)b)|^q dt)^{\frac{1}{q}} \\
&\leq \frac{(x-a)^2}{b-a}(\frac{1}{p+1})^{\frac{1}{p}}(\frac{|f'(x)|^q}{s+1} + \frac{|f'(a)|^q}{s+1} - \frac{\psi(|x-a|)}{(s+1)(s+2)})^{\frac{1}{q}} \\
&+ \frac{(x-b)^2}{b-a}(\frac{1}{p+1})^{\frac{1}{p}}(\frac{|f'(x)|^q}{s+1} + \frac{|f'(b)|^q}{s+1} - \frac{\psi(|x-b|)}{(s+1)(s+2)})^{\frac{1}{q}} \\
&\leq \frac{(x-a)^2}{b-a}(\frac{1}{p+1})^{\frac{1}{p}}[\frac{2(s+2)M^q - \psi(|x-a|)}{(s+1)(s+2)}]^{\frac{1}{q}} \\
&+ \frac{(x-b)^2}{b-a}(\frac{1}{p+1})^{\frac{1}{p}}[\frac{2(s+2)M^q - \psi(|x-b|)}{(s+1)(s+2)}]^{\frac{1}{q}}.
\end{aligned}
$$

□

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# NONSMOOTH OPTIMIZATION ON PREINVEX WEAKLY SUBDIFFERENTIABLE FUNCTIONS

A.P. FARAJZADEH AND P. CHERAGHI*

*Department of Mathematics, Razi University, Kermanshah, 67149, Iran*
*farajzadehali@gmail.com,a.sh58@yahoo.com*

Abstract. In this paper, some relationships between vector variational-like inequality problem and non-smooth vector optimization problem by using $\alpha$-preinvex non-smooth weak subdifferen- tiable functions are proved. The vector critical points, the weakly efficient points and the solutions of the weak vector variational- likeinequality, based on the weak subdifferentiablity and pseudo-$\alpha$-preinvexity assumptions are introduced.

## 1. Introduction

The notion of weak subdifferential which is a generalization of the classic subdifferential, is introduced by Azimov and Gasimov [7]. The main ingredient is the method of supporting the given nonconvex set. Subgradients plays an important role in deriving of optimality conditions and duality theorems. The first canonical generalized gradient introduced by Clarke. He applied this generalized gradient systematically to nonsmooth problems in a variety of problems.
The concept of vector variational inequality was introduced by [3]. Several authors have discussed relationships between vector variational inequalities and vector optimization problems under some convexity or generalized convexity assumptions. The variational-like inequality

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problems are closely related to the concept of the invex and pre-invex functions [2, 4],which generalize the notion of convexity of functions. Yangand Chen [6] and Noor [4] have shown that the minimum of invex functions on the invex sets can be characterized by variational-like inequalities. Ruiz-Garzon et al. [2] presented relationships between vectorvariational-likeinequality and optimization problems,under the assumptions of pseudo-preinvexity.

**Definition 1.1.** [7] Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $\bar{x} \in \mathbb{R}$ be a given point. A pair $(x^*, c) \in \mathbb{R} \times \mathbb{R}^+$ where $\mathbb{R}^+$, the set of nonnegative real numbers, is called the weak subgradient of $f$ at $\bar{x} \in X$ if the following inequality holds:

$$(\forall x \in \mathbb{R}) \qquad f(x) - f(\bar{x}) \geq x^*(x - \bar{x}) - c|x - \bar{x}|.$$

The set

$$\partial^w f(\bar{x}) = \Big\{(x^*, c) \in \mathbb{R}\times\mathbb{R}^+ : (\forall x \in \mathbb{R})\ f(x)-f(\bar{x}) \geq x^*(x-\bar{x})-c|x-\bar{x}|\ \Big\}$$

of all weak subgradients of $f$ at $\bar{x} \in \mathbb{R}$ is called the weak subdifferential of $f$ at $\bar{x} \in \mathbb{R}$. If $\partial^w f(\bar{x}) \neq \emptyset$ , then $f$ is called weakly subdifferentiable at $\bar{x}$.

Now we generalize the weak subdifferential for $f : \mathbb{R} \to \mathbb{R}^p$ as follows.

**Definition 1.2.** Let $f : \mathbb{R} \to \mathbb{R}^p$ be a function and $\bar{x} \in \mathbb{R}$ be a given point. A pair $(x^*, c) \in \mathbb{R}^p \times \mathbb{R}^{+p}$ where $\mathbb{R}^+$, the set of nonnegative real numbers, is called the weak subgradient of $f$ at $\bar{x} \in X$ if the following inequality for $i = 1, 2, ..., p$ holds:

$$(\forall x \in \mathbb{R})\ f_i(x) - f_i(\bar{x}) \geq {x^*}_i(x - \bar{x}) - c_i|x - \bar{x}|.$$

The set of all weak subgradients of $f$ at $\bar{x} \in \mathbb{R}$ is called the weak subdifferential of $f$ at $\bar{x} \in \mathbb{R}$. If $\partial^w f(\bar{x}) \neq \emptyset$ , then $f$ is called weakly subdifferentiable at $\bar{x}$.
Let $\mathbb{R}^p$ be the $p$-dimensional Euclidean space and $\mathbb{R}^+$, be its nonnegative orthant. The following stipulation for equalities and inequalities will be used throughout this paper. If $x, y \in \mathbb{R}^p$, then

(1) $x \leqslant y \Longleftrightarrow x_i \leq y_i, i = 1, 2, ..., p$ with strict inequality holding for atleast one i;
(2) $x \leq y \Longleftrightarrow x_i \leq y_i, i = 1, ..., p$;
(3) $x = y \Longleftrightarrow x_i = y_i, i = 1, ..., p$;
(4) $x < y \Longleftrightarrow x_i < y_i, i = 1, ..., p$.

## 2. Main results

In the sequel we first recall the definition of $\alpha$-invexity [1] and then we state the main results.

**Definition 2.1.** Let the function $\eta : S \times S \to \mathbb{R}$ and $\alpha : S \times S \to \mathbb{R}^+ - \{0\}$, $S \subset \mathbb{R}$. The set $S$ is called $\alpha - invex$, if for all $x, \bar{x} \in S$, $\lambda \in [0,1]$, $\bar{x} + \lambda\alpha(x,\bar{x})\eta(x,\bar{x}) \in S$

**Definition 2.2.** Let $S$ be a closed and $\alpha - invex$ non-empty set and $i = 1, 2, ...p$. The weak subdifferentiable function $f : S \to \mathbb{R}^p$ is called

(1) $\alpha$-preinvex if there exist the functions $\eta : S \times S \to \mathbb{R}$ and $\alpha : S \times S \to \mathbb{R}^+ - \{0\}$ satisfying
$$\forall x, \bar{x} \in S, f_i(x) - f_i(\bar{x}) \geq \alpha(x,\bar{x})(x_i^*\eta(x,\bar{x}) - c_i|\eta(x,\bar{x})|\ )$$
for all $(x_i^*, c_i) \in \partial^w f_i(\bar{x})$

(2) strictly $\alpha$-preinvex if there exist the functions $\eta : S \times S \to \mathbb{R}$ and $\alpha : S \times S \to R^+ - \{0\}$
satisfying
$$\forall x, \bar{x} \in S, f_i(x) - f_i(\bar{x}) > \alpha(x,\bar{x})(x_i^*\eta(x,\bar{x}) - c_i|\eta(x,\bar{x})|\ )$$
for all $(x_i^*, c_i) \in \partial^w f_i(\bar{x})$

(3) pseudo $\alpha$ -preinvex if there exist the functions $\eta : S \times S \to \mathbb{R}$ and $\alpha : S \times S \to \mathbb{R}^+ - \{0\}$ satisfying
$$\forall x, \bar{x} \in S, f_i(x) - f_i(\bar{x}) < 0 \Longrightarrow \alpha(x,\bar{x})(x_i^*\eta(x,\bar{x}) - c_i|\eta(x,\bar{x})|) < 0,$$
for all $(x_i^*, c_i) \in \partial^w f_i(\bar{x})$.

The following definitions will be utilized in the following.

**Definition 2.3.** [1] For a given open subset $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}^p$, a point $\bar{x} \in S$ is said to

(1) efficient(Pareto) solution to NVOP if there does not exist a $x \in S$ satisfying $f(x) \leqslant f(\bar{x})$;
(2) weakly efficient (Pareto) solution to NVOP if there does not exist a $x \in S$ satisfying $f(x) < f(\bar{x})$.

**Definition 2.4.** for a given open subset $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}^p$. For nonsmooth case,

(1) a vector variational-like inequality problem (VVLIP for short) is to find a point $\bar{x} \in S$, and for any $(x^*, c) \in \partial^w f(\bar{x})$, there is no $x \in S$ satisfying $x^*\eta(x,\bar{x}) - c|\eta(x,\bar{x})| \leq 0$
(2) a weak vector variational-like inequality problem (WVVLIP for short) is to find a point $x \in S$, and for any $(x^*, c) \in \partial^w f(\bar{x})$, there is no $x \in S$ satisfying $x^*\eta(x,\bar{x}) - c|\eta(x,\bar{x})| < 0$.

Now, we establish some results about VVLIP and NVOP under the condition of $\alpha$-preinvexity.

**Theorem 2.5.** *Let $S$ be a closed $\alpha$-invex non-empty set and $f : S \to \mathbb{R}^p$ weakly subdifferentiable at $\bar{x} \in S$ and $\alpha$-preinvex with respect to $\eta$ and $\alpha$ .If $\bar{x} \in S$ is a solution to the VVLIP with respect to the same $\eta$ and $\alpha$, then $\bar{x}$ is an efficient solution to the NVOP.*

The following result states relation between the solution sets of NVOP and WVVLIP.

**Theorem 2.6.** *Let $S$ be a closed $\alpha$-invex non-empty set and $f : S \to \mathbb{R}^p$ be weakly subdifferentiable at $\bar{x} \in S$ and pseudo-$\alpha$-preinvex with respect to $\eta$ and $\alpha$. If $\bar{x} \in S$ is a solution to the WVVLIP then $\bar{x}$ is a weak solution for NVOP.*

**Theorem 2.7.** *Let $S$ be a closed $\alpha$-invex non-empty set and $f : S \to \mathbb{R}^p$ and strictly $\alpha$-preinvex with respect to $\eta$ and $\alpha$ ,for $i = 1, 2, ...p,$. If $\bar{x} \in S$ is a weak efficient solution to NVOP, then $\bar{x}$ is an efficient solution to NVOP.*

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# ON EQUILIBRIUM PROBLEMS WITH GENERALIZED APPROXIMATE CONVEXITY

M. DARABI

*Department of Basic sciences, Golpayegan University of Technology, Isfahan, Iran*
*m.darabi@iut.ac.ir, marzi.darabi@yahoo.com*

Abstract. We extend the notion of approximate quas-iconvexity and we obtain relation between approximate solution existence of theses problems and differential of set-valued functions. Hence, we consider sufficient or necessary conditions of the existence of approximate solution for Stampacchia variational inequality.

## 1. Introduction

Optimization problems and variational inequalities have played a crucial role for solving engineering and economics problems. Set-valued optimization problem deals with the problem of finding efficient elements of a set-valued function. Hence, the pioneer work in the theory of vector variational inequalities in 1980 began by Giannessi that extended the classical variational inequality for vector-valued functions and proposed Stampacchia variational inequality. In the last decades, many problems with different constraint in engineering and economics have been considered that as mathematical modeling and these models can be considered as optimization problems and variational inequalities. Mishra and Laha in [3] and Gupta and Mishra in [2] considered approximate convexity assumption and obtained necessary or sufficient conditions for existence of solution of scalar optimization

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problems and Minty and Stampacchia variational inequalities. But we generalized approximate convexity concepts and focus on set-valued mappings and relation between set valued optimization problems and Minty and Stampacchia variational inequalities.
The outline of this paper is as follows: In this section, we define an optimization problem and some preliminary definitions and results which are utilized in the following. In section 2, we obtain necessary and sufficient condition relation between optimization problem and Clarke subdifferentials, hence we obtain some relations between existence of solution of vector parametric optimization problems and Stampacchia variational inequality.
We recall some definitions and preliminary results which are used in the next sections. Let $X$ and $Y$ be normed spaces and $P$ be a topological space. Let $A$ and $B$ be nonempty closed convex subsets of $X$ and $Y$, respectively, $\eta : A \times A \longrightarrow A$ is a continuous function such that $\eta(x, y) = -\eta(y, x)$ and $C : X \times P \longrightarrow 2^Y$ be a set-valued mapping such that for any $x \in X$ and for any $p \in P$, $C(x, p)$ is a closed, convex and pointed cone in $Y$ such that $\mathrm{int}C(x, p) \neq \emptyset$. Assume that $e : X \times P \longrightarrow Y$ is a continuous vector valued mapping satisfying $e(x, p) \in \mathrm{int}C(x, p)$. Hence, suppose that $K_1 : A \times P \longrightarrow 2^A$ and $K_2 : A \times P \longrightarrow 2^B$.
Let the machinery of the problems be expressed by $F : A \times P \longrightarrow 2^Y$. Consider the following parametric vector optimization problem, for given $p \in P$:

$$(VOP(p)) \text{ Find } \bar{x} \in clK_1(\bar{x}, p) \text{ such that, } \exists \bar{y} \in F(\bar{x}, p) \cap K_2(\bar{x}, p) :$$
$$\forall x \in K_1(\bar{x}, p) \ (F(x, p) - \bar{y}) \cap (-\mathrm{int}C(\bar{x}, p)) = \emptyset.$$

We denote the set of solutions of the above problems $(VOP(p))$ by $S(p)$. Special cases of the above problems are considered in [2] and [3]. We extended approximately convex concept for set-valued functions then by using of submonotonicity definition extend the following statements for set-valued functions that stablished for single-valued functions by Daniilidis and Georgiev in [5].

Function $f$ is approximately convex iff $\partial f$ is submonotone.

We introduce the following new classes of set valued functions, that generalize definitions [2, 4].

## 2. Main results

In this section, we obtain some sufficient conditions for the existence of solution of Problem $(VOP(p))$. Let us define

$$\Gamma(p) = \{\bar{x} \in A : \ \forall \bar{y} \in F(\bar{x}, p) \cap K_2(\bar{x}, p)$$
$$\exists x \in K_1(\bar{x}, p) : \ (F(x, p) - y) \cap (-\mathrm{int}C(x, p)) \neq \emptyset\},$$

$$\bar{E}(p) = \{x \in A : \ x \in clK_1(x, p)\}.$$

**Theorem 2.1.** *Let $A$ be a convex set and $x_0 \in A$ and $F$ be approximately pseudoconvex of type II at $x_0 \in X$. Then $(x_0, y_0)$ is locally weak minimal solution of*

$$\min_{x \in A} F(x),$$

*if and only if $0 \in \partial_0 F|_A(x_0, y_0)$.*

The following definitions generalize definitions of approximate efficient solutions for optimization problems that were introduced in [3].

**Definition 2.2.** (a) A vector $x_0 \in X$ is said to be an approximate efficient solution of type one of the Problem $(VOP(p))$ if and only if for all $\varepsilon > 0$, there isn't $\delta > 0$ such that for all $x \in k_1(x_0, p) \cap B_\delta(x_0) \setminus \{x_0\}$

$$\exists y_0 \in K_2(x_0, p) \cap F(x_0, p) : \ (F(x, p) - y_0 - \varepsilon e(x_0, p)\|x - x_0\|) \cap -\mathrm{int}C(x_0, p) = \emptyset.$$

(b) A vector $x_0 \in X$ is said to be an efficient solution of type two of the Problem $(VOP(p))$ if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in B_\delta(x_0) \cap K_1(x_0, p)$,

$$\exists y_0 \in K_2(x_0, p) \cap F(x_0, p) : \ F(x, p) - y_0 + \varepsilon e(x_0, p)\|x - x_0\| \subseteq Y \setminus -\mathrm{int}C(x_0, p).$$

(c) A vector $x_0 \in X$ is said to be an efficient solution of type three of the Problem $(VOP(p))$ if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in B_\delta(x_0) \cap K_1(x_0, p)$,

$$\exists y_0 \in K_2(x_0, p) \cap F(x_0, p) : \ (F(x, p) - y_0 - \varepsilon e(x_0, p)\|x - x_0\|) \cap (-\mathrm{int}C(x_0, p)) = \emptyset.$$

The following definitions is genaralization of Definition 2.4 [2].

**Definition 2.3.** Function $F$ is
(a) approximately quasiconvex of type I corresponding to $\eta$ at $x_0$ if for all $\varepsilon > 0$ there exists $\delta > 0$, that if $x, y \in k_1(x_0, p) \cap B_\delta(x_0)$ and

$$F(x, p) - F(y, p) \subseteq Y \setminus -\mathrm{int}C(x_0, p).$$

then

$$\langle x^*, \eta(x, y)\rangle - \varepsilon e(x_0, p)\|x_0 - x\| \in -C(x_0, p) \ \forall x^* \in \partial F(x, p).$$

(b) approximately quasiconvex of type II corresponding to $\eta$ at $x_0$ if for all $\varepsilon > 0$ there exists $\delta > 0$, that if $x, y \in k_1(x_0, p) \cap B_\delta(x_0)$ and

$$(F(x, p) - F(y, p) + \varepsilon e(x_0, p)\|x - x_0\|) \cap C(x_0, p) \neq \emptyset,$$

then

$$\langle x^*, \eta(x, y)\rangle \in -intC(x_0, p) \ \forall x^* \in \partial F(x, p).$$

**Definition 2.4.** (a) A vector $x_0 \in clK_1(x_0, p)$ is said to be an approximate efficient solution of type one of Stampacchia variational inequality if and only if for all $\varepsilon > 0$, there isn't $\delta > 0$ such that for all $x \in K_1(x_0, p) \cap B_\delta(x_0)$

$$\exists z \in \partial F(x, p) : < z, \eta(y, x_0)) > -\varepsilon e(x_0, p)\|x_0 - x\| \in Y \setminus -C(x_0, p).$$

(b) A vector $x_0 \in clK_1(x_0, p)$ is said to be an efficient solution of type two of Stampacchia variational inequality if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in K_1(x_0, p) \cap B_\delta(x_0)$,

$$\exists z \in \partial F(x, p) : < z, \eta(x_0, x) > +\varepsilon e(x_0, p)\|x - x_0\| \in Y \setminus -\text{int}C(x_0, p).$$

(c) A vector $x_0 \in X$ is said to be an efficient solution of type three of Stampacchia variational inequality if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in K_1(x_0, p) \cap B_\delta(x_0)$,

$$\exists z \in \partial F(x, p) : < z, \eta(x_0, x) > -\varepsilon e(x_0, p)\|x - x_0\| \in Y \setminus -\text{int}C(x_0, p).$$

**Theorem 2.5.** *Let $F : X \longrightarrow 2^Y$ be a function. Then*

- *(i) if $F$ is approximately quasiconvex of type II at $x_0 \in X$ and $x_0$ is an efficient solution of type one of Stampacchia variational inequality, then $x_0$ is also an efficient solution of type one of the Problem $(VOP(p))$ .*
- *(ii) if $F$ is approximately quasiconvex of type II at $x_0 \in X$ and $x_0$ is an efficient solution of type two of the Stampacchia variational inequality, then $x_0$ is also an efficient solution of type two of the Problem $(VOP(p))$.*
- *(iii) if $F$ is approximately quasiconvex of type II at $x_0 \in X$ and $x_0$ is an efficient solution of type three of Stampacchia variational inequality, then $x_0$ is also an efficient solution of type three of the Problem $(VOP(p))$ .*

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# THE STABILITY OF THE JENSEN FUNCTIONAL EQUATION IN QUASILINEAR SPACES

Z. DEHVARI* AND S. M. S. MODARRES MOSADDEGH

*Faculty of Mathematical, University of Yazd, Yazd, Iran*
*zrdehvari@gmail.com*
*Faculty of Mathematical, University of Yazd, Yazd, Iran*
*smodarres@yazd.ac.ir*

Abstract. In this paper we by using fixed point Theorem prove the stability of the Jensen functional equation in quasilinear spaces.

## 1. Introduction

Many authors have used the following fixed point theorem to proof the stability of functional equations ( see[2]-[4]).

**Theorem 1.1.** *Let $(X,d)$ be a complete generalized metric space and let $J : X \to X$ be a contraction map with a Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*

1. *$d(J^n x, J^{n+1}x) < \infty$ for all $n > n_0$;*
2. *the sequence $J^n x$ converges to a fixed point $x^*$ of $J$;*
3. *$x^*$ is the unique fixed point of $J$ in the set $Y := \{y \in X | d(J^{n_0}x, y) < \infty\}$;*
4. *$d(y, x^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

**Definition 1.2.** [1]A set $X$ is called a quasilinear space (qls, for short), if a partial order relation $\leqslant$ , an algebraic sum operation and an operation of multiplication by real numbers are defined in it in such a way

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that the following conditions hold for any elements $x, y, z, v \in X$ and any $a, b \in \mathbb{R}$:

(1) $x \leqslant x$;
(2) $x \leqslant z$ if $x \leqslant y$ and $y \leqslant z$;
(3) $x = y$ if $x \leqslant y$ and $y \leqslant x$;
(4) $x + y = y + x$;
(5) $x + (y + z) = (x + y) = z$;
(6) there exists an element $0_X \in X$ such that $x + 0_X = x$;
(7) $a.(b.x) = (a.b).x$;
(8) $a.(x + y) = a.x + a.y$;
(9) $1.x = x$;
(10) $0.x = 0_X$;
(11) $(a + b).x \leqslant a.x + b.x$;
(12) $x + z \leqslant y + v$ if $x \leqslant y$ and$z \leqslant v$;
(13) $a.x \leqslant a.y$ if $x \leqslant y$.

**Definition 1.3.** Let $X$ be a qls. A real function $\|.\|_X : X \to \mathbb{R}$ is called a norm if the following conditions hold:

(1) $\|x\|_X > 0$ if $x \neq 0_X$;
(2) $\|x + y\|_X \leq \|x\|_X + \|y\|_X$;
(3) $\|\alpha.x\|_X = |\alpha|.\|x\|_X$;
(4) if $x \leqslant y$, then $\|x\|_X \leq \|y\|_X$;
(5) if for any $\varepsilon > 0$ there exists an element $x_\varepsilon \in X$ such that $x \leqslant y + x_\varepsilon$ and $\|x\|_X \leq \varepsilon$ then $x \leqslant y$.

A qls $X$, with a norm defined on it, is called normed quasilinear space.

Hausdorff metric on normed quasilinear space $X$ is defined by

$$h_X(x, y) = inf\{r \geq 0 : x \leqslant y + a_1^r, y \leqslant x + a_2^r\}, \|a_i^r\| \leq r\}.$$

The function $h_X(x, y)$ satisfies all of metric axioms and $h_X(x, y) \leq \|x - y\|_X$.

## 2. Main results

Throughout this section, assume that $X$ is a linear space and $Y$ is a complete normed quasilinear space.

**Theorem 2.1.** *If function $f : X \to Y$ with $f(0) = 0_Y$ and symetric function $\varphi : X \times X \to [0, \infty)$ for all $x, y \in X$ satisfy the following conditions:*

(1) $h_Y(2f(\dfrac{x + y}{2}), f(x) + f(y)) \leq \varphi(x, y)$,

(2) $h_Y(2f(x), f(2x)) \leq h_Y(2f(x), f(x) + f(x))$,
(3) $\varphi(2x, 2x) \leq 2L\varphi(x, x)$,
(4) $\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0$,

*for some* $0 \leq L < 1$*. Then there exists an unique mapping* $g : X \to Y$ *such that for all* $x \in X$*,*

$$g(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n},$$

$$h_Y(f(x), g(x)) \leq \frac{1}{2-2L}\varphi(x, x),$$

$$2g(\frac{x+y}{2}) = g(x) + g(y),$$

*and* $g(nx) = ng(x)$ *for all* $n \in \mathbb{N}$*.*

*Proof.* Suppose $E = \{g | g : X \to Y, g(0) = 0_Y\}$ and define a generalized metric d on $E$ by

$$d(g_1, g_2) := \inf\{c \in [0, \infty] : h_Y(g_1(x), g_2(x)) \leq c\varphi(x, x)\}$$

Then, $d$ is a complete generalized metric on $E$. Now define the mapping $J : E \to E$ by $J(h(x)) := \frac{1}{2}h(2x)$. By (3), we have

$$h_Y(\frac{1}{2}g_1(2x), \frac{1}{2}g_2(2x)) \leq \frac{1}{2}d(g_1, g_2)\varphi(2x, 2x) \leq Ld(g_1, g_2)\varphi(x, x).$$

Therefore, $J$ is a contraction mapping with constant at most $L$.

letting $y = x$ in (1) and by condition (2) we get

$$h_Y(2f(x), f(2x)) \leq \varphi(x, x),$$

for all $x \in X$. Hence $d(f, Jf) \leq \frac{1}{2}$. By theorem 1.1, $J$ has a unique fixed point $g : X \to Y$ in $A = \{g \in E : d(f, g) < \infty\}$. Furthermore,

$$d(f, g) \leq \frac{1}{1-L}d(f, Jf) \leq \frac{1}{2-2L}.$$

This implies the following inequality,

$$h_Y(f(x), g(x)) \leq \frac{1}{2-2L}\varphi(x, x).$$

Since $d(J^n f, g) \to 0$, then $g(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$.

It follows from (1), (4),

$$
\begin{aligned}
h_Y(2g(\frac{x+y}{2}), g(x)+g(y)) &= \lim_{n\to\infty} \frac{1}{2^n} h_Y(2f(\frac{2^n x + 2^n y}{2}), f(2^n x) + f(2^n y)) \\
&\leq \frac{1}{2^n}\varphi(2^n x, 2^n y) = 0.
\end{aligned}
$$

So $2g(\frac{x+y}{2}) = g(x) + g(y)$.

If $I$ is an another mapping such that for all $n \in \mathbb{N}$, $I(nx) = nI(x)$ and for all $x, y \in X$, $2I(\frac{x+y}{2}) = I(x) + I(y)$, and

$$h_Y(f(x), I(x)) \leq \frac{1}{2-2L}\varphi(x,x),$$

then $d(f, I) < \infty$ and $I$ is a fixed point of $J$ in $A$. Since $g$ is an unique fixed point of $J$ in $A$, therefore $I = g$. □

**Corollary 2.2.** *Let $r < \frac{1}{2}$ and $\theta$ be nonnegative real numbers and $f : X \to Y$ be a mapping that $f(0) = 0_Y$ and for all $x, y \in X$,*

$$h_Y(2f(\frac{x+y}{2}), f(x)+f(y)) \leq \varphi(x,y),$$
$$h_Y(2f(x), f(2x)) \leq h_Y(2f(x), f(x)+f(x)),$$

*Then there exists a unique mapping $g : X \to Y$ such that,*

$$2g(\frac{x+y}{2}) = g(x) + g(y),$$
$$h_Y(f(x), g(x)) \leq \frac{\theta}{1-2^{r-1}}\|x\|^r.$$

*Proof.* By taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ in Theorem 2.1, we get the desired result. □

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# EXISTENCE AND UNIQUENESS OF FIXED POINTS F-CONTRACTION MAPPINGS

A. FARAJZADEH, M. DELFANI*

*Department of Mathematics, Razi University, Kermanshah 67149, Iran*
*farajzadehali@gmail.com, ma.delfani@gmail.com*

Abstract. A new generalization of the Banach contraction through the notions of the generalized $F$-contraction, simulation function and admissible function is introduced. The existence and uniqueness of fixed points for a self-mapping on complete metric spaces by the new constructed contraction are investigated.

## 1. Introduction and Preliminaries

In 1922, Banach proved the following famous and fundamental result in fixed-point theory [2]. Let $(X, d)$ be a complete metric space. Let $T$ be a contractive mapping on $X$, that is, there exists $q \in [0, 1)$ satisfying

$$d(Tx, Ty) \leq q.d(x, y), \forall x, y \in X.$$

This theorem, is called the Banach contraction principle. This principle has been generalized in different directions by various researchers.

**Theorem 1.1.** [6] *Let $(X, d)$ be a complete metric space, $q \in [0, 1)$ and $T : X \to X$ be a continuous mapping. If for each $x \in X$ there exists a positive integer $k = k(x)$ such that*

$$d(T^{k(x)}x, T^{k(x)}y) \leq qd(x, y),$$

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*for all $y \in X$, then $T$ has a unique fixed point $u \in X$. Moreover, for any $x \in X$, $u = \lim_{n\to\infty} T^n x$.*

**Definition 1.2.** [8] Let $(X, d)$ be a metric space. The mapping $T : X \to X$ is called an $F$-contraction, if there exist $F$ and $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $F : (0, \infty) \to \mathbb{R}$ is strictly increasing, $\lim_{n\to\infty} F(\alpha_n) = -\infty$ iff $\lim_{n\to\infty} \alpha_n = 0$ and there exists a number $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = -\infty$. We denote the set of all these functions by $\mathcal{F}$.

If we replace the third condition with $F$ is a continuous mapping, then the collection of all functions $F : (0, +\infty) \to \mathbb{R}$ is denoted by $\mathcal{G}$.

**Definition 1.3.** [4] Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping, then $\zeta$ is called a simulation function if satisfies the following conditions:
($\zeta$1) $\zeta(0, 0) = 0$;
($\zeta$2) $\zeta(t, s) < s - t$ for all $t, s > 0$;
($\zeta$3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

We denote the set of all simulation functions by $\mathcal{Z}$.

**Theorem 1.4.** [5] *Let $(X, d)$ be a complete metric space and $T : X \to X$ a mapping which satisfies the following condition: If there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for each $x \in X$ there is a positive integer $n(x)$ such that for all $y \in X$*

$$d(T^{n(x)}(x), T^{n(x)}(y)) > 0 \Rightarrow \zeta(F(d(x, y)), \tau + F(d(T^{n(x)}(x), T^{n(x)}(y)))) \geq 0.$$

*Then, $T$ has a unique fixed point $z \in X$ and $T^n(x_0) \to z$ for each $x_0 \in X$, as $n \to \infty$.*

**Definition 1.5.** [3] Let $\alpha : X \times X \to (0, +\infty)$ be a given mapping. The mapping $T : X \to X$ is said to be an $\alpha$-admissible, whenever $\alpha(Tx, Ty) \geq 1$ provided $\alpha(x, y) \geq 1$ and $x, y \in X$.

**Definition 1.6.** [1] An $\alpha$-admissible map $T$ is said to have the K-property, while for each sequence $\{x_n\} \subseteq X$ with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$, the nonnegative integer numbers, there exists a positive integer number $k$ such that $\alpha(Tx_n, Tx_m) \geq 1$, for all $m > n \geq k$.

## 2. Main results

In this section the main achievements of this article are presented.

**Theorem 2.1.** *Let $(X,d)$ be a complete metric space, $\alpha : X \times X \to (0,+\infty)$ be a symmetric function, where $\alpha(x,y) \geq 1$ and $T : X \to X$ be a continuous mapping which satisfies the condition: If there exist $F \in \mathcal{F}$, $\tau > 0$, $L \geq 0$ and simulation function $\zeta$ such that for all $x \in X$ there is a positive integer $n(x)$ such that for all $y \in X$ and $d(T^{n(x)}(x), T^{n(x)}(y)) > 0$,*

$$\zeta(\tau + \alpha(x,y)F(d(T^{n(x)}x, T^{n(x)}y)), F(m(x,y) + LN_1(x,y))) \geq 0, \quad (2.1)$$

*where*

$$m(x,y) = \max\left\{d(x,y), d(x,T^{n(x)}x), d(y,T^{n(x)}y), \frac{d(x,T^{n(x)}y) + d(y,T^{n(x)}x)}{2}\right\},$$

*and*

$$N_1(x,y) = \min\{d(x,T^{n(x)}x), d(x,T^{n(x)}y), d(y,T^{n(x)}x)\}.$$

*Then, $T$ has a unique fixed point.*

**Corollary 2.2.** *Theorem 3.3 of [7] reduces to Theorem 2.1 by taking $n(x) = 1$. Because in this case*

$$\zeta(\tau + \alpha(x,y)F(d(Tx,Ty)), F(m(x,y) + LN_1(x,y))) \geq 0.$$

*Now, by $(\zeta 2)$ we have*

$$\begin{aligned} 0 &\leq \zeta(\tau + \alpha(x,y)F(d(Tx,Ty)), F(m(x,y) + LN_1(x,y))) \\ &\leq F(m(x,y) + LN_1(x,y)) - (\tau + \alpha(x,y)F(d(Tx,Ty))). \end{aligned}$$

*Therefore*

$$\tau + \alpha(x,y)F(d(Tx,Ty)) \leq F(m(x,y) + LN_1(x,y)).$$

**Corollary 2.3.** *Theorem 1.4 is contained in Theorem 2.1 by taking $m(x,y) = d(x,y)$, $\alpha(x,y) = 1$ and $L = 0$.*

**Theorem 2.4.** *Let $(X,d)$ be a complete metric space, $\alpha : X \times X \to (0,+\infty)$ a symmetric function, where $\alpha(x,y) \geq 1$. Assume that $T : X \to X$ is a mapping which there exist $F \in \mathcal{G}$, $\tau > 0$ and the simulation function $\zeta$ such that for all $x, y \in X$ with $T^{n(x)}x \neq T^{n(x)}y$, where $n(x)$ is a positive integer and $\frac{1}{2}d(x,T^{n(x)}x) \leq d(x,y)$ implies*

$$\zeta(\tau + \alpha(x,y)F(d(T^{n(x)}x, T^{n(x)}y)), F(m(x,y))) \geq 0 \qquad (2.2)$$

*where $m(x,y)$ is defined as in Theorem 2.1, satisfying the following conditions:*

*(i)* $T$ *is* $\alpha$*-admissible,*

*(ii) there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*,*

*(iii) if* $\{x_n\}$ *is a sequence in* $X$ *such that* $x_n \to x$ *as* $n \to \infty$ *and* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}_0$*, then* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}_0$*,*

*(iv)*$T$ *has the K-property.*

*Then* $T$ *has a fixed point in* $X$*.*

**Corollary 2.5.** *If in Theorem 2.4, we put* $n(x) = 1$*, then*

$$\zeta(\tau + \alpha(x,y)F(d(Tx,Ty)), F(m(x,y))) \geq 0.$$

*Now, by* $(\zeta 2)$*, we have*

$$\begin{aligned} 0 &\leq \zeta(\tau + \alpha(x,y)F(d(Tx,Ty)), F(m(x,y))) \\ &\leq F(m(x,y)) - (\tau + \alpha(x,y)F(d(Tx,Ty))). \end{aligned}$$

*Therefore*

$$\tau + \alpha(x,y)F(d(Tx,Ty)) \leq F(m(x,y)).$$

*Hence we get Theorem 3.3 of* [7].

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**Oral Presentation** **∗ : Speaker**

# ON CERTAIN SUB-HILBERT SPACES IN THE FOCK SPACE

S. ESKANDARI∗, A. ABKAR

*Department of Pure Mathematics, Imam Khomeini International University, Qazvin 34194, Iran*
*S.Eskandari@edu.ikiu.ac.ir; abkar@sci.ikiu.ac.ir*

Abstract. Two specific sub-Hilbert spaces in the Fock spaces are studied. First the reproducing kernels of these spaces are identified and then the members of the $\mathcal{H}(\overline{\varphi})$ are shown. Finally, we will see that these sub-Fock Hilbert spaces are equal with equivalence of norms.

## 1. Introduction

Let $\mathbb{C}$ be the complex planee, $H(\mathbb{C})$ be the family of entire functions on $\mathbb{C}$ and for any positive parameter $\alpha$, we consider

$$d\lambda_\alpha(z) = \frac{\alpha}{\pi} e^{-|z^2|} dA(z).$$

where $dA(z)$ is the Euclidean area measure on the complex plane. Let $F^2_\alpha$ denote the Fock space consisting of all entire functions $f$ in $L^2(\mathbb{C}, d\lambda_\alpha)$, which means

$$F^2_\alpha = L^2_\alpha \cap H(\mathbb{C}).$$

$F^2_\alpha$ is a closed subspace of $L^2(\mathbb{C}, d\lambda_\alpha)$. consequently, $F^2_\alpha$ is a Hilbert space with the following inner product inherited from $L^2(\mathbb{C}, d\lambda_\alpha)$:

$$\langle f, g\rangle_\alpha = \int_{\mathbb{C}} f(z)\overline{g(z)} d\lambda_\alpha.$$

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It is well-known that for nonnegative integers $n$, the functions

$$e_n(z) = \sqrt{\alpha^n/n!}\, z^n$$

form an orthonormal basis for $F^2_\alpha$. This implies that for

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we have

$$\|f\|^2_\alpha = \sum_{n=0}^{\infty} \frac{n!}{\alpha^n} |a_n|^2.$$

The Fock space is a reproducing kernel Hilbert space; for every $f \in F^2_\alpha$ we have

$$f(w) = \langle f, k_w \rangle = \int_{\mathbb{C}} f(z) \overline{k^\alpha_w(z)} d\lambda_\alpha(z),$$

where

$$k^\alpha_w(z) = e^{\alpha z \bar{w}}$$

is the reproducing kernel for $F^2_\alpha$. More information on this topic can be found in [5].

Recall that for any fixed weight parameter $\alpha$, the orthogonal projection

$$P : L^2_\alpha \to F^2_\alpha$$

is an integral operator,

$$Pf(z) = \int_{\mathbb{C}} K(z, w) f(w) d\lambda_\alpha(w),$$

where $K(z, w) = \overline{k^\alpha_z(w)}$.

Given $\varphi \in L^\infty(\mathbb{C})$, we define the linear operator $T_\varphi : F^2_\alpha \to F^2_\alpha$ by

$$T_\varphi(f) = P(\varphi f), \quad f \in F^2_\alpha.$$

We call $T_\varphi$ the Toeplitz operator on $F^2_\alpha$ with symbol $\varphi$. It is clear that $T_\varphi$ is bounded with $||T_\varphi|| \leq ||\varphi||_\infty$. We have the following properties for any bounded functions $\varphi$ and $\psi$, for any complex numbers $a$ and $b$:

- $T_{a\varphi + b\psi} = aT_\varphi + bT_\psi$,
- $T_{\overline{\varphi}} = T^*_\varphi$,
- $T_\varphi \geq 0, if\, \varphi \geq 0$.

Suppose $T$ is a bounded operator on a Hilbert space $H$. We denote by $\mathcal{M}(T)$ the range of $T$ equipped with the following inner product:

$$< Tx, Ty >_{\mathcal{M}(T)} = < x, y >_H, \quad x, y \in H \ominus ker\, T.$$

If $T$ is a contraction on $H$, the Hilbert space

$$\mathcal{H}(T) = \mathcal{M}((I - TT^*)^{1/2})$$

is called the complemented space to $\mathcal{M}(T)$. The theory of sub-Hardy Hilbert spaces was studied by de Branges, Rovnyak, Sarason and some of their students. Kehe Zhu introduced sub-Bergman Hilbert spaces in [3] and generalized Sarason's work. He proved that both sub-Bergman Hilbert spaces contain the Banach space of all bounded analytic functions on the unit disk. In [1] Abkar and Jafarzade generalized the results obtained by Kehe Zhu to weighted Bergman spaces. Here we will see that some similar results hold in the case of Fock spaces.

## 2. Main results

We now identify the Sub-Fock Hilbert spaces, then we calculate the reproducing kernels of these spaces. let $\varphi \in L^\infty(\mathbb{C})$, we consider Toeplitz operators on $F^2_\alpha$, which are bounded operators on $F^2_\alpha$. For the sake of simplicity, we use $\mathcal{M}(\varphi)$ and$\mathcal{M}(\bar{\varphi})$ instead of $\mathcal{M}(T_\varphi)$ and $\mathcal{M}(T_{\bar{\varphi}})$, repectively. Soppuse $\varphi \in L^\infty(\mathbb{C})$ and $\varphi$ is a contractive multiplier, so $T_\varphi$ and $T_{\bar{\varphi}}$ are contractive operators on $F^2_\alpha$. We write $\mathcal{H}(\varphi)$ and $\mathcal{H}(\bar{\varphi})$ instead of $\mathcal{H}(T_\varphi)$ and $\mathcal{H}(T_{\bar{\varphi}})$, repectively and we call them sub-Fock Hilbert spaces.

First, the reroducing kernels of sub-Fock Hilbert spaces are computed.

**Proposition 2.1.** *Let $\varphi \in L^\infty(\mathbb{C})$ and $\varphi$ be a contractive multiplier on $F^2_\alpha$. For any positive parameter $\alpha$, the reproducing kernels of $\mathcal{H}_\alpha(\varphi)$ and $\mathcal{H}_\alpha(\overline{\varphi})$ are given, respectively, by*

(1) $K^\alpha_\varphi(z,w) = (1-\varphi(z)\overline{\varphi(w)})e^{\alpha z\overline{w}}$;
(2) $K^\alpha_{\overline{\varphi}}(z,w) = \int_{\mathbb{C}}(1-|\varphi(u)|^2)e^{\alpha z\overline{u}+\alpha u\overline{w}}d\lambda_\alpha(u)$.

Now for every $\varphi \in L^\infty(\mathbb{C})$, we find a representation for the elements of $\mathcal{H}_\alpha(\overline{\varphi})$.

**Proposition 2.2.** *Let $\varphi \in L^\infty(\mathbb{C})$ and $\alpha$ be any positive parameter. Then every element of $\mathcal{H}_\alpha(\overline{\varphi})$ has the representation*

$$f(z) = \int_{\mathbb{C}}(1-|\varphi(w)|^2)g(w)e^{\alpha z\overline{w}}d\lambda_\alpha(w)$$

*where $g$ is an entire function satisfying*

$$\int_{\mathbb{C}}|g(z)|^2(1-|\varphi(z)|^2)d\lambda_\alpha(z) < +\infty.$$

In the next proposition, we explore some relations between $\mathcal{H}_\alpha(\varphi)$ and $\mathcal{H}_\alpha(\overline{\varphi})$.

**Proposition 2.3.** *Let $\varphi \in L^\infty(\mathbb{C})$ be a contractive multiplier on $F^2_\alpha$, and let $\alpha$ be a positive parameter and $f \in F^2_\alpha$. Then*

(1) $f \in \mathcal{H}_\alpha(\varphi)$ *if and only if* $T^\alpha_{\overline{\varphi}} \in \mathcal{H}_\alpha(\overline{\varphi})$ *and in this case*

$$||f||^2_{\mathcal{H}_\alpha(\varphi)} = ||f||^2_{F^2_\alpha} + ||T^\alpha_{\overline{\varphi}} f||^2_{\mathcal{H}_\alpha(\overline{\varphi})}$$

(2) $f \in \mathcal{H}_\alpha(\overline{\varphi})$ *if and only if* $T^\alpha_{\varphi} \in \mathcal{H}_\alpha(\varphi)$ *and in this case*

$$||f||^2_{\mathcal{H}_\alpha(\overline{\varphi})} = ||f||^2_{F^2_\alpha} + ||T^\alpha_{\varphi} f||^2_{\mathcal{H}_\alpha(\varphi)}$$

(3) $\mathcal{M}(T^\alpha_\varphi) \cap \mathcal{H}_\alpha(\varphi) = \varphi \mathcal{H}_\alpha(\overline{\varphi})$.

**Proposition 2.4.** *Let* $\varphi \in L^\infty(\mathbb{C})$ *, be a contarctive muiltiplier on* $F^2_\alpha$ *for some positive parameter* $\alpha$. *Then every* $\psi \in L^\infty(\mathbb{C})$ *is a multiplier on both* $\mathcal{H}_\alpha(\varphi)$ *and* $\mathcal{H}_\alpha(\overline{\varphi})$, *moreover* $||T^\alpha_\psi|| \leq ||\psi||_\infty$.

The main theorem says that two sub-Fock spaces coincide.

**Theorem 2.5.** *Let* $\varphi \in L^\infty(\mathbb{C})$ *be a contractive multiplier on* $F^2_\alpha$ *for some positive parameter* $\alpha$. *Then* $\mathcal{H}_\alpha(\varphi) = \mathcal{H}_\alpha(\overline{\varphi})$ *with equivalence of norms.*

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# WEAKLY COMPACT COMPOSITION OPERATORS ON VECTOR-VALUED WEIGHTED BERGMAN SPACES

K. ESMAEILI

*Faculty of Engineering, Ardakan University, Ardakan, Iran*
*esmaeili@ardakan.ac.ir*

Abstract. Let $X$ be a Banach space. We prove that the composition operator on $X$-valued weighted Bergman spaces is weakly compact if and only if $X$ is reflexive and the corresponding composition operator on scalar-valued weighted Bergman space is weakly compact.

## 1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions on $\mathbb{D}$. Let $(X, \|\cdot\|_X)$ be a complex Banach space. A function $f : \mathbb{D} \to X$ is analytic if it is weakly analytic, i.e., if $x^* \circ f \in \mathcal{H}(\mathbb{D})$ for all functionals $x^* \in X^*$, where $X^*$ is the dual space of $X$. The space of analytic $X$-valued functions on $\mathbb{D}$ is denoted by $\mathcal{H}(\mathbb{D}, X)$.

For an analytic self-map $\varphi$ of $\mathbb{D}$, the composition operator $C_\varphi$ is defined by $C_\varphi(f) = f \circ \varphi$, for every $f \in \mathcal{H}(\mathbb{D}, X)$. An easy computation gives that $C_\varphi$ is a well defined linear operators as well as continuous with respect to the compact-open topology.

For $1 \leq p < \infty$, the vector-valued Hardy space $H^p(X)$ is the space of all analytic functions $f : \mathbb{D} \to X$ such that

$$\|f\|^p_{H^p(X)} = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|^p_X d\theta < \infty,$$

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and for $p=\infty$, $\|f\|_{H^\infty(X)}=\sup_{z\in\mathbb{D}}\|f(z)\|_X<\infty$. In the scalar-valued case where $X=\mathbb{C}$, the spaces $H^p(\mathbb{C})$ are just the classical Hardy spaces $H^p$.

Given a positive integrable function $\omega\in C^2[0,1)$, we extend it by $\omega(z)=\omega(|z|), z\in\mathbb{D}$, and call such $\omega$ a weight function.

**Definition 1.1.** [4] A weight function $\omega$ is called admissible if

($\mathcal{W}_1$) $\omega$ is non-increasing,
($\mathcal{W}_2$) $\omega(r)(1-r)^{-(1+\delta)}$ is non-decreasing for some $\delta>0$,
($\mathcal{W}_3$) $\lim_{r\to 1^-}\omega(r)=0$.

We define the weighted Bergman space $B^p_\omega(X)$ as the space of functions $f\in\mathcal{H}(\mathbb{D},X)$ with

$$\|f\|^p_{B^p_\omega(X)}=\frac{1}{\pi}\int_{\mathbb{D}}\|f(z)\|^p_X\omega(z)dA(z)<\infty,$$

where $dA$ denotes the Lebesgue area measure on the plane. If $\omega(z)=(1-|z|^2)^\alpha$ with $\alpha>-1$, then we write $B^p_\alpha(X)$ and if $\alpha=0$ we just omit $\alpha$. If $X=\mathbb{C}$ we omit $X$ in the notation.

Compactness and weak compactness of $C_\varphi$ have been studied on many classical Banach spaces such as Hardy spaces, Bergman and Bloch spaces, and BMOA (see [2, 6]). For fast and normal weights Kriete and MacCluer characterized the boundedness and compactness of composition operators on scalar-valued weighted Bergman spaces [5]. The boundedness and compactness of weighted composition operators between weighted Hilbert spaces, when the weight is admissible, has been characterized in [3]. In this work we investigate the boundedness and weak compactness of composition operators between vector-valued weighted Bergman spaces, in the case that the weight is admissible. The notations $A\lesssim B$ and $A\approx B$ mean that $A\le cB$ and $cB\le A\le CB$, respectively, for some positive constants $c$ and $C$.

## 2. Main Results

The generalized Nevanlinna counting function associated to the weight $\omega$ play a key role in our work and is defined as below.

**Definition 2.1.** Let $\varphi\in\mathcal{H}(\mathbb{D})$ such that $\varphi(\mathbb{D})\subseteq\mathbb{D}$ and $\omega$ be an admissible weight. For every $z\in\mathbb{D}\setminus\{\varphi(0)\}$ we define

$$N_{\varphi,\omega}(z)=\sum_{\varphi(a)=z,a\in\mathbb{D}}\omega(a).$$

Note that $N_{\varphi,\omega}(z) = 0$ when $z \notin \varphi(\mathbb{D})$ and when $z = \varphi(0)$, $N_{\varphi,\omega}(z) = 0$. When $\omega(z) = \log \frac{1}{|z|}$, $N_{\varphi,\omega}$ is the usual Nevanlinna counting function $N_\varphi$. For our purpose it is convenient to introduce the modified Nevanlinna counting function

$$\tilde{N}_{\varphi,\omega}(z) = \sum_{\varphi(a)=z, a\in\mathbb{D}} (1-|a|^2)^2\omega(a), \qquad z \in \mathbb{D} \setminus \{\varphi(0)\}.$$

The partial Nevanlinna counting function is defined for $0 < r < 1$ by

$$N_\varphi(r,z) = \sum_{\varphi(a)=z, |a|\le r} \log \frac{r}{|z|}, \qquad z \in \mathbb{D} \setminus \{\varphi(0)\}.$$

For a continuous subharmonic function $u$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} u(\varphi(re^{i\theta}))d\theta = u(0) + \frac{1}{2\pi}\int_{\mathbb{D}} N_\varphi(r,z)d[\Delta(u)](z),$$

where $0 < r < 1$ and $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ is analytic with $\varphi(0) = 0$. When $f \in \mathcal{H}(\mathbb{D}, X)$, $d[\Delta(\|f\|_X)](z)$ denotes integration with respect to the distributional Laplacian of $\|f\|_X$, which is a positive measure on $\mathbb{D}$ since the map $z \mapsto \|f(z)\|_X$ is subharmonic. This means that for every infinitely differentiable function $\tau$ on $\mathbb{C}$ with compact support we have

$$\int \tau(z)d[\Delta(\|f\|_X)](z) = \frac{1}{2\pi}\int_{\mathbb{D}} \|f(z)\|_X \Delta\tau(z)dA(z).$$

**Proposition 2.2.** *Let $X$ be a complex Banach space.*

(i) *If $C_\varphi : B^p_\omega(X) \to B^p_\omega(X)$ is bounded, then $C_\varphi : B^p_\omega \to B^p_\omega$ is bounded.*
(ii) *If $C_\varphi : B^p_\omega(X) \to B^p_\omega(X)$ is compact, then $C_\varphi : B^p_\omega \to B^p_\omega$ is compact and $X$ is finite-dimensional. In particular, if $X$ is infinite-dimensional, then $C_\varphi : B^p_\omega(X) \to B^p_\omega(X)$ is never compact.*
(iii) *If $C_\varphi : B^p_\omega(X) \to B^p_\omega(X)$ is weakly compact, then $X$ is reflexive and $C_\varphi : B^p_\omega \to B^p_\omega$ is weakly compact.*

For obtaining our results we need the following lemma.

**Lemma 2.3.** *Let $X$ be a complex Banach space and $\omega$ be an admissible weight. Then for each $z \in \mathbb{D}$ and $f \in B^1_\omega(X)$ there exists a positive constant $C$ independent of $f$ such that*

$$\|f(z)\|_X \le C\frac{\|f\|_{B^1_\omega(X)}}{\omega(z)(1-|z|^2)^2}.$$

**Theorem 2.4.** *If $f \in \mathcal{H}(\mathbb{D}, X)$, $\varphi(0) = 0$ and $\omega$ is an admissible weight, then*

$$\|C_\varphi(f)\|_{B^1_\omega(X)} \approx \|f(0)\|_X + \frac{1}{2\pi}\int_{\mathbb{D}} \tilde{N}_{\varphi,\omega}(z)d[\Delta(\|f\|_X)](z).$$

For the general case let $\varphi(0) = a$ and $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ denote the automorphism of the unit disc. Then $\psi = \varphi_a \circ \varphi$ is an analytic self-map with $\psi(0) = 0$ and $C_{\varphi_a}$ is bounded on $B^1_\omega(X)$. Thus $C_\varphi$ is bounded on $B^1_\omega(X)$ if and only if $C_\psi$ is bounded.

For the special case that $\varphi$ is the identity map, it follows from Theorem 2.4 that

$$\|f\|_{B^1_\omega(X)} \approx \|f(0)\|_X + \frac{1}{2\pi}\int_{\mathbb{D}} (1-|z|^2)^2\omega(z)d[\Delta(\|f\|_X)](z).$$

There is a precise connection between the weak compactness of $C_\varphi$ on $B^1_\omega(X)$ and the compactness of $C_\varphi$ on $B^1_\omega$. Note that Proposition 2.2(iii) implies that $X$ is reflexive, whenever of $C_\varphi$ is weakly compact on $B^p_\omega(X)$. For $1 < p < \infty$ the space $B^p_\omega(X)$ is reflexive whenever $X$ is reflexive, because $B^p_\omega(X)$ is then a closed subspace of the reflexive space $L^p(\mathbb{D}, X)$. Hence only $p = 1$ or $p = \infty$ are interesting for weak compactness.

**Theorem 2.5.** *Let $X$ be a complex Banach space and $\omega$ be a concave and admissible weight. Then $C_\varphi : B^1_\omega(X) \to B^1_\omega(X)$ is weakly compact if and only if $X$ is reflexive and* $\limsup_{|z|\to 1} \frac{N_{\varphi,\omega}(z)}{(1-|z|^2)^2\omega(z)} = 0$.

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# QUASI-MULTIPLIERS OF SECOND DUAL OF CERTAIN HYPERGROUP ALGEBRAS

M. ESSMAILI* AND A. ALINEJAD

*Department of Mathematics, Faculty of Mathematical and Computer Science, Kharazmi University, m.essmaili@khu.ac.ir*
*College of Farabi, University of Tehran, alinejad.ahmad@ut.ac.ir*

Abstract. At the present paper, we investigate quasi-multipliers on the second dual Banach algebra $L_c(\mathcal{X})^{**}$. Indeed, we show that $\mathfrak{QM}(L_c(\mathcal{X})^{**})$ is isomorphic with $M(\mathcal{X})$. As an application, we prove that $\mathfrak{QM}(L_c(\mathcal{X})^{**}) = L(\mathcal{X})$ if and only if $\mathcal{X}$ is discrete.

## 1. Introduction

Throughout the paper, $\mathcal{X}$ denotes a locally compact Hausdorff space, $M(\mathcal{X})$ is the Banach space of all bounded complex-valued regular Borel measures on $\mathcal{X}$ with the total variation norm and $M_p(\mathcal{X})$ is the set of all probability Borel Measures on $\mathcal{X}$. We also devote the symboles $C_b(\mathcal{X}), C_0(\mathcal{X})$, and $C_c(\mathcal{X})$ for the space of bounded continuous complex-valued functions on $\mathcal{X}$, those that vanish at infinity, and those that have compact support, respectively.

**Definition 1.1.** The space $\mathcal{X}$ is called a *hypergroup* if there is a map $\lambda\colon \mathcal{X} \times \mathcal{X} \longrightarrow M_p(\mathcal{X})$ with the following properties:

(i) the measures $\lambda_{(x,y)}$ have compact support for all $x, y \in \mathcal{X}$.

(ii) for each $f \in C_c(\mathcal{X})$, the mapping $(x, y) \mapsto \int_{\mathcal{X}} f(t)\, d\lambda_{(x,y)}(t)$ is in $C_b(\mathcal{X} \times \mathcal{X})$, and the mappings

$$x \mapsto \int_{\mathcal{X}} f(t)\, d\lambda_{(x,y)}(t), \quad x \mapsto \int_{\mathcal{X}} f(t)\, d\lambda_{(y,x)}(t)$$

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are in $C_c(\mathcal{X})$ for all $y \in \mathcal{X}$.

(iii) the convolution $(\mu,\nu) \mapsto \mu * \nu$ on $M(\mathcal{X})$ defined by

$$\int_{\mathcal{X}} f(t)\, d(\mu * \nu)(t) = \int_{\mathcal{X}} \int_{\mathcal{X}} \int_{\mathcal{X}} f(t)\, d\lambda_{(x,y)}(t)\, d\mu(t)\, d\nu(t)$$

is associative, where $f \in C_0(\mathcal{X}), \mu, \nu \in M(\mathcal{X})$.

(iv) there is a unique point $e \in \mathcal{X}$ (say the identity) such that

$$\lambda_{(x,e)} = \delta_x = \lambda_{(e,x)} \quad (x \in \mathcal{X}),$$

where $\delta_x$ denotes the Dirac measure at $x$.

With the above definition, $M(\mathcal{X})$ can be regarded as a Banach algebra. Furthermore, Ghahramani and Medghalchi in [1] defined $L(\mathcal{X})$ as a subalgebra $M(\mathcal{X})$, consisting of all measures $\mu$ for which the mappings $x \mapsto |\mu| * \delta_x$ and $x \mapsto \delta_x * |\mu|$ from $\mathcal{X}$ to $M(\mathcal{X})$ are norm continuous. They have shown that $L(\mathcal{X})$ is a closed ideal in $M(\mathcal{X})$ and also that if $\mathcal{X}$ admits a left invariant measure $m$, then $L(\mathcal{X}) = L^1(\mathcal{X}, m)$ [1, Remark 1]. The hypergroup $\mathcal{X}$ is called *foundation* if

$$\mathcal{X} = \overline{\bigcup \{(\mu) : \mu \in L(\mathcal{X})\}}.$$

In this case, the Banach algebra $L(\mathcal{X})$ has a bounded approximate identity, see [1] and [2]. We note that all hypergroups considered in this article are assumed to be foundation hypergroups without left invariant Haar measure. Let $L(\mathcal{X})^*$ and $L(\mathcal{X})^{**}$ be the first and second topological duals of $L(\mathcal{X})$, respectively.

**Definition 1.2.** A compact set $K \subseteq \mathcal{X}$ is called a *compact carrier* for $F \in L(\mathcal{X})^{**}$ if we have

$$\langle F, f\rangle = \langle F, f\chi_K\rangle \quad (f \in L(\mathcal{X})^*),$$

where $f\chi_K \in L(\mathcal{X})^*$ defined by $\langle f\chi_K, \mu\rangle = \langle f, \mu\chi_K\rangle$.

Now, we set

$$L_c(\mathcal{X})^{**} = \mathrm{cl}_{L(\mathcal{X})^{**}}\{F \in L(\mathcal{X})^{**} : F \text{ has compact carrier}\},$$

which is a Banach algebra with the first Arens Product $\Box$. We note that if $\mathcal{X} = G$ is a locally compact group, then $L_c(\mathcal{X})^{**} = L_0^\infty(G)^*$, where $L_0^\infty(G)$ is the introverted subspace of $L^\infty(G)$ consisting of all $f \in L^\infty(G)$ such that, for given $\varepsilon > 0$, there is a compact subset $K$ of $\mathcal{X}$ for which $\|g\|_{G\setminus K} \leq \varepsilon$; see also [3] and [4].

## 2. Main results

Quasi-multipliers were first studied within the framework of Banach algebras with a bounded approximate identity by K. McKennon [7]. In this paper, we concentrate on multiplier and quasi-multiplier algebra of the second dual of hypergroup algebras. Precisely, we show that the multiplier and quasi-multiplier algebra of $L_c(\mathcal{X})^{**}$ is isomorphic with $M(\mathcal{X})$, where $\mathcal{X}$ is a foundation hypergroup. As a consequence, we prove that $\mathfrak{QM}(L_c(\mathcal{X})^{**}) = L(\mathcal{X})$ if and only if $\mathcal{X}$ is discrete.

**Definition 2.1.** For a Banach algebra $\mathcal{A}$ a bilinear map $\mathfrak{m}\colon \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$ is called a *quasi-multiplier* if for all $a, b, c, d \in \mathcal{A}$,

$$\mathfrak{m}(ab, cd) = a\mathfrak{m}(b, c)d.$$

We note that the quasi-multiplier algebra of $\mathcal{A}$ is the set of all separately continuous quasi-multipliers on $\mathcal{A}$ which is denoted by $\mathfrak{QM}(\mathcal{A})$. In the case where $\mathcal{A}$ admits a bounded approximate identity, it is shown in [7, Theorem 2] that $\mathfrak{QM}(\mathcal{A})$ is a Banach space with the norm defined by

$$\|\mathfrak{m}\| = \sup\{\|\mathfrak{m}(a, b)\| : a, b \in \mathcal{A}, \|a\| = \|b\| = 1\}.$$

Recall that an approximate identity $(e_\alpha)$ in a Banach algebra $\mathcal{A}$ will be called an *ultra-approximate identity* for $\mathcal{A}$ if for all $\mathfrak{m} \in \mathfrak{QM}(\mathcal{A})$ and $a \in \mathcal{A}$, the nets $(\mathfrak{m}(e_\alpha, a))$ and $(\mathfrak{m}(a, e_\alpha))$ are $\|\cdot\|$-Cauchy.

**Proposition 2.2.** *Let $\mathcal{X}$ be a hypergroup. Then the multiplier algebra of $L_c(\mathcal{X})^{**}$ is isomorphic with $M(\mathcal{X})$.*

*Proof.* Let $(e_\alpha)$ be a bounded approximate identity of $L(\mathcal{X})$ with norm 1. So, if we assume that $E$ is a weak* cluster points of $(e_\alpha)$, then $E$ is a right identity in $L_c(\mathcal{X})^{**}$. Now, let $T\colon L_c(\mathcal{X})^{**} \longrightarrow L_c(\mathcal{X})^{**}$ be a right multiplier. We conclude that

$$T(m) = T(m\Box E) = m\Box T(E) \qquad (m \in L_c(\mathcal{X})^{**}).$$

Now, it follows that

$$T(m) = m\Box n = m\Box\pi(n).$$

Now, we know that the algebra consisting of elements $\pi(n)$ is exactly $M(\mathcal{X})$. On the other hand, since $M(\mathcal{X})$ has an identity we conclude that the multiplier algebra of $L_c(\mathcal{X})^{**}$ is $M(\mathcal{X})$. $\square$

Now, we are ready to characterize quasi-multipliers of the second dual Banach algebra $L_c(\mathcal{X})^{**}$.

**Theorem 2.3.** *Let $\mathcal{X}$ be a hypergroup. Then the quasi-multiplier algebra of $L_c(\mathcal{X})^{**}$ is isomorphic with $M(\mathcal{X})$.*

As a consequence, we obtain a necessary and sufficient condition for which the quasi-multiplier algebra $L_c(\mathcal{X})^{**}$ is equal to $L(\mathcal{X})$.

**Corollary 2.4.** *Let $\mathcal{X}$ be a hypergroup. Then the following assertions are equivalent.*

(i) $\mathfrak{QM}(L_c(\mathcal{X})^{**}) = L(\mathcal{X})$.
(ii) $\mathcal{X}$ *is discrete.*

*Proof.* Let $\mathcal{X}$ be discrete hypergroup. Thus by [6, Theorem 14(c)] we conclude that $M(\mathcal{X}) = L_c(\mathcal{X})^{**} = L(\mathcal{X})$. Now, it follows from Theorem 2.3 that

$$\mathfrak{QM}(L_c(\mathcal{X})^{**}) = M(\mathcal{X}) = L(\mathcal{X}).$$

Conversely, suppose that quasi-multiplier algebra of $L_c(\mathcal{X})^{**}$ is equal to $L(\mathcal{X})$. By Theorem 2.3, we deduce that $M(\mathcal{X}) = L(\mathcal{X})$ and so $\mathcal{X}$ is discrete. □

At the end, we obtain a result on the right annihilator of $L_c(\mathcal{X})^{**}$ which is denoted by $\mathrm{Ann}_r(L_c(\mathcal{X})^{**})$ and defined by

$$\mathrm{Ann}_r(L_c(\mathcal{X})^{**}) = \{R \in L_c(\mathcal{X})^{**} | \ \ L_c(\mathcal{X})^{**}\Box R = \{0\}\}.$$

**Theorem 2.5.** *Let $\mathcal{X}$ be a hypergroup and $\mathfrak{m}\colon L_c(\mathcal{X})^{**}\times L_c(\mathcal{X})^{**} L_c(\mathcal{X})^{**}$ be a quasi-multiplier. Then the following statements hold:*

(i) $\mathfrak{m}(L(\mathcal{X}) \times L(\mathcal{X})) \subseteq L(\mathcal{X})$.
(ii) $\mathfrak{m}(Ann_r(L_c(\mathcal{X})^{**}) \times Ann_r(L_c(\mathcal{X})^{**})) \subseteq Ann_r(L_c(\mathcal{X}))^{**})$.

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# A COINCIDENCE POINT THEOREM FOR TWO SELF-MAPPINGS IN UNIFORM SPACES

H. FARAJI

*Department of Mathematics, College of Technical and Engineering, Saveh Branch, Islamic Azad University, Saveh, Iran.*
*faraji@iau-saveh.ac.ir*

Abstract. In this work, we prove a the coincident point theorem of two self mappings in a uniform space generated by a family of $b$-pseudometrics. Our results are generalizations and extensions of the results Acharya (1974) and Faraji et al. (2019).

## 1. Introduction

In 2019, Faraji et al. [5] introduced the concept of the uniform spaces generated by a family of $b$-pseudometrics, in order to generalize one of the main results in [1]. For further details on fixed point theory in uniform space, see e.g. [2, 3, 4, 6, 7, 9, 10].

A uniformity $\mathcal{U}$ on $X$ is a family of subsets of $X \times X$, that the following conditions hold:

U1): $U \in \mathcal{U}$ implies $\Delta = \{(x,x) \in X \times X : x \in X\} \subset U$;
U2): $U_1, U_2 \in \mathcal{U}$ implies $U_1 \cap U_2 \in \mathcal{U}$;
U3): For each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$;
U4): $U \in \mathcal{U}$ implies that $V^{-1} = \{(x,y) \in X \times X : (y,x) \in V\} \subset U$ for some $V \in \mathcal{U}$;
U5): If $U \in \mathcal{U}$ and $U \subseteq V$ imply $V \in \mathcal{U}$.

Then, the pair $(X,\mathcal{U})$ is called a uniform space. A sequence $\{x_n\}$ in uniform space $(X,\mathcal{U})$ is convergent to a point $x \in X$, if for each $U \in \mathcal{U}$ there exsits $n_0 \in \mathbb{N}$ such that $(x_n, x) \in U$, for all $n \geq n_0$.

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Moreover, it is called a Cauchy sequence, if for any $U \in \mathcal{U}$, there exsits $n_0 \in \mathbb{N}$ such that $(x_n, x_m) \in U$, for all $n, m \geq n_0$. A uniform space $(X, \mathcal{U})$ is called sequentially complete, if every Cauchy sequence in $X$ is convergent. Let $\mathcal{U}$ be the uniformity generated by a nonempty family $\mathcal{F}$ of $b$-pseudometrics with the same parameter $s \geq 1$. Define

$$V_{(p,r)} = \{(x, y) \in X : p(x, y) < r\},$$

where $p \in \mathcal{F}$ and $r > 0$ and let $\mathcal{V}$ be the family of all sets of the form

$$\bigcap_{i=1}^{k} V_{(p_i, r_i)},$$

where $k$ is a positive integer, $p_i \in \mathcal{F}$ and $r_i > 0$ for $i = 1, \ldots, k$. We can easily check that $\mathcal{V}$ is a base for the uniformity $\mathcal{U}$ and for $V = \bigcap_{i=1}^{k} V_{(p_i, r_i)} \in \mathcal{V}$ and $\alpha > 0$, we have

$$\alpha V = \bigcap_{i=1}^{k} V_{(p_i, \alpha r_i)} \in \mathcal{V}.$$

Let $Y \subseteq X$, then

$$\mathcal{U}_Y = \{U \cap (Y \times Y) | U \in \mathcal{U}\},$$

is a uniformity on $Y$ and $\mathcal{V}_Y = \{V \cap (Y \times Y) | V \in \mathcal{V}\}$ is a base for $\mathcal{U}_Y$ (see, e.g., [8]).

## 2. Main results

Throughout this section, we assume that $(X, \mathcal{U})$ is a Hausdorff uniform space and uniformity $\mathcal{U}$ is generated by a family $\mathcal{F}$ of $b$-pseudometrics with the same parameter $s \geq 1$ on $X$. Let $\mathcal{V}$ be the family of all sets of the form

$$\bigcap_{i=1}^{k} \Big\{(x, y) \in X \times X : p_i(x, y) < r_i\Big\},$$

where $k$ is a positive integer, $p_i \in \mathcal{F}$ and $r_i > 0$ for $i = 1, \ldots, k$.

**Theorem 2.1.** *Let $f, g$ be self-mappings on $X$ such that*

$$(f(x), f(y)) \in \alpha V \qquad \text{if} \qquad (g(x), g(y)) \in V, \tag{2.1}$$

*for all $V \in \mathcal{V}$ and $x, y \in X$, where $\alpha > 0$ and $0 < s\alpha < 1$. If $f(X) \subseteq g(X)$ and $(g(X), \mathcal{U}_{g(X)})$ is sequentially complete, then $f$ and $g$ have a unique coincident point.*

*Proof.* Let $x_0$ be an arbitrary point in $X$. Choose a point $x_1$ in $X$ such that $f(x_0) = g(x_1)$. Continuing this process, we can define a sequence $\{x_n\}$ such that $y_n = f(x_n) = g(x_{n+1})$ for each $n \geq 0$. Let $V \in \mathcal{V}$ and $p$ be the Minkowski's $b$-pseudometric of $V$. Fix $x, y \in X$ and set $p(g(x), g(y)) = r$. Let $\varepsilon > 0$ be given. Then, we have

$$(g(x), g(y)) \in (r + \varepsilon)V.$$

Therefore by (2.1), we have

$$(f(x), f(y)) \in \alpha(r + \varepsilon)V.$$

Therefore,

$$p(f(x), f(y)) \leq \alpha(r + \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we get

$$p(f(x), f(y)) \leq \alpha p(g(x), g(y)). \tag{2.2}$$

Appling (2.2) for $y_n$ and $y_{n+1}$, we get

$$\begin{aligned} p(y_n, y_{n+1}) &= p(f(x_n), f(x_{n+1})) \\ &\leq \alpha p(g(x_n), g(x_{n+1})) \\ &= \alpha p(y_{n-1}, y_n), \end{aligned}$$

for all $n \geq 1$. Therefore,

$$p(y_n, y_{n+1}) \leq \alpha^n p(y_0, y_1), \quad (n = 1, 2, 3, \dots). \tag{2.3}$$

Let $m, n \in N$ and $m > n$. Since $s\alpha < 1$, by (2.3), we have

$$\begin{aligned} p(y_n, y_m) &\leq sp(y_n, y_{n+1}) + s^2 p(y_{n+1}, y_{n+2}) + \cdots + s^{m-n} p(y_{m-1}, y_m) \\ &\leq \big(s\alpha^n + s^2\alpha^{n+1} + \cdots + s^{m-n}\alpha^{m-1}\big) p(y_0, y_1) \\ &= s\alpha^n \big(1 + s\alpha + \cdots + (s\alpha)^{m-n-1}\big) p(y_0, y_1) \\ &\leq \frac{s\alpha^n}{1 - s\alpha} p(y_0, y_1). \end{aligned}$$

Now choose $N_0 \in \mathbb{N}$ such that

$$\frac{s\alpha^n}{1 - s\alpha} p(y_0, y_1) < 1,$$

for all $n \geq N_0$. Then, $p(y_n, y_m) < 1$ for all $n, m \geq N_0$ . This implies that $(y_n, y_m) \in V$, for all $n, m \geq N_0$. Since $y_i \in g(X)$, for all $i \in \mathbb{N}$, then $(y_n, y_m) \in V \cap (g(X) \times g(X))$, for all $n, m \geq N_0$. Since $V$ was arbitrary, $\{y_n\}$ is a Cuachy sequence in $g(X)$. Hence, there is $z$ in $X$ such that $y_n \to g(z)$ as $n \to +\infty$, that is

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_{n+1}) = g(z). \tag{2.4}$$

We show that $f(z) = g(z)$. Let $V \in \mathcal{V}$ and $p$ be the Minkowski's $b$-pseudometric of $V$. So, by (2.2), we have

$$p(f(x_n), f(z)) \;\; \leq \;\; \alpha p(g(x_n), g(z)), \quad (n = 1, 2, 3, \dots).$$

From (2.4), choose $N_1 \in \mathbb{N}$ such that

$$\alpha p(g(x_n), g(z)) < 1,$$

for all $n \geq N_1$. Then $(f(x_n), f(z)) \in V$, for all $n \geq N_1$. Since $V$ was arbitrary, $\lim_{n\to\infty} f(x_n) = f(z)$. Therefore, $f(z) = g(z) = t$, i.e., $f$ and $g$ have a coincident point $t$. The point $t$ is unique. To see this, let $f(z_1) = g(z_1) = t_1$. Let $V \in \mathcal{V}$ and $p$ is the Minkowskis $b$-pseudometric of $V$. Using (2.2), we have

$$p(t, t_1) = p(f(z), f(z_1)) \leq \alpha p(g(z), g(z_1)) = \alpha p(t, t_1) < p(t, t_1).$$

Then $p(t, t_1) = 0$ and $(t, t_1) \in V$. Since $V$ was arbitrary, it follows that $t = t_1$. $\square$

**Corollary 2.2.** *For $s = 1$ and $g = Id_X$, Theorem 2.1 reduces to Theorem 3.1 in* [1].

**Corollary 2.3.** *For $g = Id_X$, Theorem 2.1 is a generalization of the Acharya type Theorem* [5].

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# WOVEN FRAME IN HILBERT $C^*$- MODULES

A. KHOSRAVI AND M. R. FARMANI*

*Faculty of Mathematical Sciences and Computer, Kharazmi University, 599 Taleghani Ave., Tehran 15618, Iran*
*khosravi_amir@yahoo.com, khosravi@khu.ac.ir mr.farmanis@gmail.com*

Abstract. In this paper we introduce modular frame, woven modular frame in Hilbert $C^*$- Modules. And we show that under certain conditions the image of two woven modular frames is woven modular framed under linear operators.

## 1. Introduction

Hilbert space frames were originally introduced by Duffin and Schaeffer to deal with some problems in non harmonic Fourier analysis[5]. Frames can be viewed as redundant bases which are generalizations of Riesz bases [1, 2, 3, 4]. This redundancy property sometimes is extremely important in applications such as signal and image processing, data compression and sampling theory. Hilbert $C^*$-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a $C^*$-algebra rather than in the field of complex numbers. Frames for Hilbert spaces have natural analogues for Hilbert $C^*$-modules. These frames are called Hilbert $C^*$-modular frames or just simply modular frames. Modular frames are not trivial generalizations of Hilbert space frames due to the complex structure of $C^*$-algebras. It is well known that the theory of Hilbert $C^*$-modules is quite different from that of Hilbert spaces. For example, we know that, any closed linear subspace in a Hilbert space has an orthogonal complement. But this is no longer

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true in Hilbert $C^*$-module setting since not every closed submodule of a Hilbert $C^*$-module is complemented. Moreover, the Riesz representation theorem for continuous functionals on Hilbert spaces does not hold in Hilbert $C^*$-modules, and so there exist nonadjointable bounded linear operators on Hilbert $C^*$-modules [1, 2]. Therefore it is expected that problems about frames in Hilbert$C^*$-modules are more complicated than those in Hilbert spaces. While some of the results about frames in Hilbert spaces can be easily extended to Hilbert $C^*$-modular frames, many others cannot be obtained by simply modifying the approaches used in Hilbert spaces case.

## 2. WOVEN MODULAR FRAME

In this section, first we recall some definitions and basic properties of Hilbert $C^*$- Modules and p-woven frame and g-frame in Hilbert $C^*$-Modules [1, 2, 4]. Throughout this note $A$ is a unital $C^*$-algebra and $H, K_i$ are finitely or countably generated Hilbert A-modules. For each $i \in I$, $L(H, K_i)$ will denote the set of all adjointable A-linear maps from $H$ to $K_i$. We also define

$$\ell^2(A) := \{a = (a_i) \in A : \sum_{i\in I} a_i^* a_i \text{ is norm convergent in } A\}$$

**Definition 2.1.** A pre-Hilbert A-module is a left A-module $H$ equipped with an A-valued inner product $\langle .,. \rangle : H \times H \longrightarrow A$, such that
$(i)$ $\langle x, x\rangle \geq 0$ for all $x \in H$ and $\langle x, x\rangle = 0$ if and only if $x = 0$,
$(ii)\langle x, y\rangle = \langle y, x\rangle^*$ for all $x, y \in H$,
$(iii)$ $\langle ax + y, z\rangle = a\langle x, z\rangle + \langle y, z\rangle$ for all $a \in A$ and $x, y, z \in H$.

We assume that the linear operations of $A$ and $H$ are compatible.i.e. $\lambda(ax) = (\lambda a)x$ for every $\lambda \in \mathbb{C}$, $a \in A$ and $x \in H$. For every $x \in H$, we define

$$\| x \| = \| \langle x, x\rangle \|^{\frac{1}{2}} \qquad \text{and} \qquad |x| = \langle x, x\rangle^{\frac{1}{2}}.$$

If the pre-Hilbert A-module $(H, \langle .,. \rangle)$ is complete with respect to $\| . \|$, it is called a Hilbert A-module or a Hilbert $C^*$-modules over A. In this paper we focus on finitely and countably generated Hilbert $C^*$-modules over unital $C^*$-algebra A. A Hilbert A-module H is (algebraically) finitely generated if there exists a finite subset $\{x_1, x_2, ..., x_m\}$ of $H$ such that every element $x \in H$ can be expressed as an A-linear combination $x = \sum_{i=1}^{m} a_i x_i, a_i \in A$. A Hilbert $A$-module $H$ is countably generated if there exists a countable set of generators.
We now recall the definitions of frames and Riesz bases in Hilbert $C^*$-modules as follows.

**Definition 2.2.** Let $H$ be a Hilbert A-module. A family $\{x_i : i \in I\}$ of elements of $H$ is a (standard) frame for $H$, if there exits constants $0 < C \leq D < \infty$, such that for all $x \in H$,

$$C\langle x, x\rangle \leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq D\langle x, x\rangle. \qquad (1)$$

Where the sum in the middle of the inequality convergent in norm for $x \in H$.
The numbers $C$ and $D$ are called frame bounds, If $C = D = \lambda$, it is called a $\lambda$-tight frame and when $C = D = 1$, it is called a Parseval frame. $\{x_i : i \in I\}$ is said to be a Bessel sequence if only the right-hand side inequality is required.If the sum of (1) is convergent in norm, the frame is called standard.

According to what Arambasic and Khosravi proved, the above definition is equivalent to

$$C \parallel x \parallel^2 \leq \parallel \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \parallel^2 \leq D \parallel x \parallel^2, \qquad (1)$$

A sequence $\{x_i : i \in I\}$ is said to be a Riesz basis of $H$ if it is a frame and a generating set with the additional property that $A$-linear combinations $\sum_{i\in S} a_i x_i$ with coefficients $\{a_i : i \in S\} \subseteq A$ and $S \subseteq I$ are equal to zero if and only if in particular every summand $a_i x_i$ equal zero for $i \in S$.
Note that we can also define the analysis operator, synthesis operator and frame operator for modular frame as follows.
Suppose that $\{x_i : i \in I\}$ is a frame of a finitely or countably generated Hilbert $A$-module $H$ over a unital $C^*$-algebra $A$. The operator $T : H \to \ell^2(A)$ defined by $Tx = \{\langle x, x_i\rangle\}_{i\in I}$, is called the analysis operator. The adjoint operator $T^* : \ell^2(A) \to H$ is given by $T^*\{a_i\}_{i\in I} = \sum_{i\in I} a_i x_i$. $T^*$ is called pre-frame operator or the synthesis operator. By composing $T$ and $T^*$, we obtain the frame operator $S : H \to H$,

$$Sx = T^*Tx = \sum_{i\in I}\langle x, x_i\rangle x_i, \qquad (1)$$

is a frame operator for $H$. That is $S \in End^*_A(H)$, positive and invertible. Where $End^*_A(H)$ is the set of adjointable A-linear maps on $H$.
The frame $\{S^{-1}x_i : i \in I\}$ is said to be the canonical dual frame of $\{x_i : i \in I\}$.

*Remark* 2.3. If $A$ be a unital $C^*$-algebra, $H$ be a finitely or countably generated Hilbert $A$-module and $\{x_i : i \in I\}$ be Parseval frame(not necessarily standard) of $H$, then the reconstruction formula $x = \sum_{i\in I}\langle x, x_i\rangle x_i$, holds for every $x \in H$. Also from equation (1) we see that $x = \sum_{i\in I}\langle x, S^{-1}x_i\rangle x_i$, is vailid for every $x \in H$.
Moreover, if $\{x_i : i \in I\}$ be standard frame, then there exists a unique operator $S \in End^*_A(H)$ such that $x = \sum_{i\in I}\langle x, Sx_i\rangle x_i$.

**Theorem 2.4.** *Let $\{x_i : i \in I\}$, $\{y_i : i \in I\}$ be woven modular frame with frame bounds $C, D$ and $Q_1, Q_2 \in L(H)$, $Q_1$ be invertible and $\| Q_1^{-1} \| \| Q_1 - Q_2 \| < \sqrt{\frac{C}{D}}$. Then $\{Q_1 x_i : i \in I\}$, $\{Q_2 y_i : i \in I\}$ are woven modular frame.*

*Proof.* Since $\| I - Q_1^{-1} Q_2 \| < \sqrt{\frac{C}{D}} < 1$, then $Q_2$ is invertible , so $\{Q_2 y_i : i \in I\}$ is a modular frame. Now for every $\sigma \subseteq I$ and each $x \in H$ :

$$(\sum_{i\in\sigma}\langle x, Q_1x_i\rangle\langle Q_1x_i, x\rangle + \sum_{i\in\sigma^c}\langle x, Q_2y_i\rangle\langle Q_2y_i, x\rangle)^{\frac{1}{2}} =$$

$$(\sum_{i\in\sigma}\langle Q_1^*x, x_i\rangle\langle x_i, Q_1^*x\rangle + \sum_{i\in\sigma^c}\langle Q_1^*x + (Q_2^* - Q_1^*)x, y_i\rangle\langle y_i, Q_1^*x + (Q_2^* - Q_1^*)x\rangle)^{\frac{1}{2}}$$

$$\geq \sqrt{C}\langle Q_1^*x, Q_1^*x\rangle - \sqrt{D} \| Q_2^* - Q_1^* \| \| x \|$$

$$\geq (\frac{\sqrt{C}}{\| Q_1^{*^{-1}} \|} - \sqrt{D} \| Q_2^* - Q_1^* \|) \| x \| .$$

and

$$(\sum_{i\in\sigma}\langle x, Q_1x_i\rangle\langle Q_1x_i, x\rangle + \sum_{i\in\sigma^c}\langle x, Q_2y_i\rangle\langle Q_2y_i, x\rangle)^{\frac{1}{2}} \leq D(\| Q_1^* \|^2 + \| Q_2^* \|^2) \| x \|^2 .$$

□

**Corollary 2.5.** *Let $\{x_i : i \in I\}$, $\{y_i : i \in I\}$ be woven modular frame with bounds $C, D$ and with frame operator $S, S'$, respectively. Then $\{S^{-1}x_i : i \in I\}$ and $\{S'^{-1}y_i : i \in I\}$ are woven modular frames, when*

$$\| S - S' \| < \sqrt{\frac{C}{D}} \| S^{-1} \|^{-1} .$$

*In this case their cononical dual frame $\{S^{-1}x_i : i \in I\}$ and $\{S'^{-1}y_i : i \in I\}$ are woven modular frames.*

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# NORMS OF COMPOSITION–DIFFERENTIATION OPERATORS ON THE HARDY SPACE

M. FATEHI* AND C. N. B. HAMMOND

*Department of Mathematics*
*Shiraz Branch, Islamic Azad University*
*Shiraz, Iran*
*fatehimahsa@yahoo.com*
*Department of Mathematics and Statistics*
*Connecticut College*
*New London, CT 06320*
*cnham@conncoll.edu*

Abstract. Let $\varphi$ be a nonconstant analytic self-map of the open unit disk in $\mathbb{C}$, with $\|\varphi\|_\infty < 1$. Consider the operator $D_\varphi$, acting on the Hardy space $H^2$, given by differentiation followed by composition with $\varphi$. We obtain results relating to the norm of such an operator.

## 1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. The *Hardy space* $H^2$ is the Hilbert space consisting of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$ such that

$$\|f\| = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

We write $H^\infty$ to denote the space of all bounded analytic functions on $\mathbb{D}$, with $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}$.

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For an analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$, the *composition operator* $C_\varphi$ is defined by the rule $C_\varphi(f) = f \circ \varphi$. Every composition operator is bounded on $H^2$, with

$$\sqrt{\frac{1}{1-|\varphi(0)|^2}} \leq \|C_\varphi\| \leq \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

(See, for example, [2, Corollary 3.7].) For a function $\psi$ in $H^\infty$, the *Toeplitz operator* $T_\psi$ is defined $T_\psi(f) = \psi \cdot f$. Every such operator is bounded on $H^2$, with $\|T_\psi\| = \|\psi\|_\infty$ (see [7, Theorem 5]).

In the context of analytic functions on $\mathbb{D}$, it also seems reasonable to consider operators defined in terms of differentiation. It is easy to see that the differentiation operator $D(f) = f'$ is unbounded on the Hardy space: $\|D(z^n)\|/\|z^n\| = n$ for any natural number $n$. Nevertheless, for many analytic maps $\varphi\colon \mathbb{D} \to \mathbb{D}$, the operator

$$f(z) \mapsto f'(\varphi(z)) \tag{1.1}$$

is bounded on $H^2$. Many authors, following the example of [5] and [6], have used the notation $C_\varphi D$ to denote such an operator. Because of the unboundedness of $D$, it makes sense to write (1.1) as a single operator, particularly when that operator is bounded on $H^2$. We will write $D_\varphi$ to denote the operator on $H^2$ given by the rule

$$D_\varphi(f) = f' \circ \varphi.$$

We will refer to such an operator as a *composition–differentiation operator*. The Closed Graph Theorem shows that $D_\varphi$ is bounded on $H^2$ whenever $D_\varphi$ takes $H^2$ into itself.

Ohno [6] established a basic set of results relating to when the operators we are calling $D_\varphi$ are bounded or compact on $H^2$. We will only be considering $\varphi$ with $\|\varphi\|_\infty < 1$, in which case $D_\varphi$ is guaranteed to be Hilbert–Schmidt on $H^2$, and hence both bounded and compact (see [6, Theorem 3.3]). There are instances of bounded or compact $D_\varphi$ with $\|\varphi\|_\infty = 1$, but they are beyond the scope of our current investigation.

The purpose of this note is to explore the operators $D_\varphi$ in more detail. In particular, we find a representation for the adjoint $D_\varphi^*$ when $\varphi$ is linear fractional (Theorem 2.1). In the specific case where $\rho(z) = rz$ for $0 < |r| < 1$, we compute the norm $\|D_\rho\|$ explicitly (Theorem 2.2). Applying established results relating to composition operators, we also obtain estimates for the norm of $D_\varphi$ whenever $\|\varphi\|_\infty < 1$ (Theorem 2.3). In this paper, we state some results of [3]. Moreover, normality and self-adjointness of a slightly broader class of composition–differentiation operators were investigated in [4].

For any point $w$ in $\mathbb{D}$, define $K_w(z) = \frac{1}{1-\overline{w}z}$. It is well known that $K_w$ acts as the reproducing kernel function for point-evaluation:

$$\langle f, K_w \rangle = f(w)$$

for any $f$ in $H^2$. In a similar manner, define

$$K_w^{(1)}(z) = \frac{z}{(1-\overline{w}z)^2}.$$

Observe that $K_w^{(1)}$ acts as the reproducing kernel for point-evaluation of the first derivative:

$$\left\langle f, K_w^{(1)} \right\rangle = f'(w)$$

## 2. Main results

The goal of this section is to obtain information about the adjoint and norm of $D_\varphi$ in certain specific instances. If $\varphi(z) = \frac{az+b}{cz+d}$ is a nonconstant linear fractional self-map of $\mathbb{D}$, then the map $\sigma(z) = \frac{\overline{a}z-\overline{c}}{-\overline{b}z+\overline{d}}$. also takes $\mathbb{D}$ into itself (see [1, Lemma 1]). It is not difficult to show that $\|\sigma\|_\infty < 1$ whenever $\|\varphi\|_\infty < 1$. The relationship between these two maps has long been considered in reference to the adjoints of composition operators. In the context of composition–differentiation operators, we obtain the following formula.

**Theorem 2.1.** *For a pair of linear fractional maps $\varphi$ and $\sigma$, as described above, $D_\varphi^* T_{K_{\sigma(0)}^{(1)}}^* = T_{K_{\varphi(0)}^{(1)}} D_\sigma$.*

This result bears a close resemblance to Cowen's adjoint formula for composition operators (see [1, Theorem 2]), which can be rewritten $C_\varphi^* T_{K_{\sigma(0)}}^* = T_{K_{\varphi(0)}} C_\sigma$.

**Theorem 2.2.** *If $\rho(z) = rz$ for some real number $0 < r < 1$, then*

$$\|D_\rho\| = \left\lfloor \frac{1}{1-r} \right\rfloor r^{\lfloor 1/(1-r) \rfloor - 1}, \tag{2.1}$$

*where $\lfloor \cdot \rfloor$ denotes the greatest integer function.*

There are several interesting consequences of Theorem 2.2. First of all, $\|D_\rho\| = 1$ for $0 < r \le 1/2$ and $\|D_\rho\| > 1$ for $1/2 < r < 1$. Secondly, $\|D_\rho\|$ tends to $\infty$ as $r$ goes to 1. Since composition with a rotation is an isometry, (2.1) holds with $r$ replaced by $|r|$ for any complex number $r$ with $0 < |r| < 1$. Likewise, the same formula holds for $\|D_\varphi\|$ where $\varphi(z) = rz^k$ for any $k$ in $\mathbb{N}$.

Let $\|\varphi\|_\infty \le r < 1$ and define $\varphi_r = (1/r)\varphi$. Observe that

$$D_\varphi = C_{\varphi_r} D_\rho. \tag{2.2}$$

Since $\|D_\varphi\| \le \|C_{\varphi_r}\|\|D_\rho\|$, we obtain the following estimate for $\|D_\varphi\|$.

**Theorem 2.3.** *If $\varphi$ is a nonconstant analytic self-map of $\mathbb{D}$, with $\|\varphi\|_\infty < 1$, then*

$$\sqrt{\frac{1+|\varphi(0)|^2}{\big(1-|\varphi(0)|^2\big)^3}} \le \|D_\varphi\| \le \sqrt{\frac{r+|\varphi(0)|}{r-|\varphi(0)|}}\left\lfloor\frac{1}{1-r}\right\rfloor r^{\lfloor 1/(1-r)\rfloor - 1},$$

*whenever $\|\varphi\|_\infty \le r < 1$.*

**Example 2.4.** If $\|\varphi\|_\infty \le 1/2$, we may take $r = 1/2$ to see that

$$\sqrt{\frac{1+|\varphi(0)|^2}{\big(1-|\varphi(0)|^2\big)^3}} \le \|D_\varphi\| \le \sqrt{\frac{1+2|\varphi(0)|}{1-2|\varphi(0)|}}.$$

In particular, $\|D_\varphi\| = 1$ whenever both $\|\varphi\|_\infty \le 1/2$ and $\varphi(0) = 0$.

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# EXISTENCE OF THREE WEAK SOLUTIONS FOR A FOURTH-ORDER BOUNDARY VALUE PROBLEM

[1,2]HAIFFA MUHSAN B. ALRIKABI, [2]GHASEM A. AFROUZI AND [2]SAMANEH FIROUZI*

[1]*Department of Mathematics, Faculty of Mathematical Sciences, University of Thi-Qar, Iraq*
*alrfayalrfay@gmail.com*
[2]*Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran*
*afrouzi@umz.ac.ir and samanehfirouzi@gmail.com*

Abstract. In this article we consider the existence of three weak solutions to the fourth order boundary value problem (1.1) Our approach is based on variational methods and critical point theory.

## 1. Introduction

The aim of this paper is to establish the existence of at least three weak solutions for the following nonlinear fourth-order boundary value problem

$$\begin{cases} u^{iv} + \alpha u'' + \beta(x)u = \lambda f(x,u) + h(u), & x \in ]0,1[, \\ u(0) - u(1) = u''(0) - u''(1) = 0, \end{cases} \tag{1.1}$$

where $\alpha$ is a real constant, $\beta(x)$ is a continuous function on $[0,1]$ and $\lambda$ is a positive parameter, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an $L^2-$ Carathéodory function and $h : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with the Lipschitz constant $L \geq 0$; i.e.,

$$|h(t_1) - h(t_2)| \leq L|t_1 - t_2| \tag{1.2}$$

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*Key words and phrases.* Fourth-order equation; Three weak solutions; Variational methods .

for all $t_1, t_2 \in \mathbb{R}$, satisfying $h(0) = 0$.

## 2. PRELIMINARIES

**Theorem 2.1.** *([2, Theorem 5.1]) Let $X$ be a reflective real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on $X^*$, $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_{x \in X} \Phi(x) = \Phi(0) = \Psi(0) = 0.$$

*Assume that there exist $r > 0$ and $\bar{v} \in X$, with $r < \Phi(\bar{v})$, such that:*

$(a_1)$ $\frac{\sup_{\Phi(x) \le r} \Psi(x)}{r} < \frac{\Psi(\bar{v})}{\Phi(\bar{v})}$;

$(a_2)$ *for each $\lambda \in \Lambda_r := \left] \frac{\Phi(\bar{v})}{\Psi(\bar{v})}, \frac{r}{\sup_{\Phi(x) \le r} \Psi(x)} \right[$ the functional $\Phi - \lambda\Psi$ is coercive.*

*Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in $X$.*

**Lemma 2.2.** *([3, Lemma 2.3]). For every $u \in X$*

$$\| u \|_2 \le \frac{1}{\pi^2} \| u'' \|_2 . \tag{2.1}$$

*From which we have as a consequence*

$$\| u' \|_2 \le \frac{1}{\pi} \| u'' \|_2 . \tag{2.2}$$

*Now put*

(1) $\sigma_1 := 1 - \frac{\alpha}{\pi^2} + \frac{\beta_1}{\pi^4}$, $\sigma_2 := 1$ *when* $\beta_2 \le 0$ *and* $\alpha \ge 0$;
(2) $\sigma_1 := 1 + \frac{\beta_1}{\pi^4}$, $\sigma_2 := 1 - \frac{\alpha}{\pi^2}$ *when* $\beta_2 \le 0$ *and* $\alpha < 0$;
(3) $\sigma_1 := 1 - \frac{\alpha}{\pi^2}$, $\sigma_2 := 1 + \frac{\beta_2}{\pi^4}$ *when* $\beta_1 \ge 0$ *and* $\alpha \ge 0$;
(4) $\sigma_1 := 1$ *and* $\sigma_2 := 1 - \frac{\alpha}{\pi^2} + \frac{\beta_2}{\pi^4}$ *when* $\beta_1 \ge 0$ *and* $\alpha < 0$;
(5) $\sigma_1 := 1 - \frac{\alpha}{\pi^2} + \frac{\beta_1}{\pi^4}$ *and* $\sigma_2 := 1 + \frac{\beta_2}{\pi^4}$ *when* $\beta_1 < 0 < \beta_2$ *and* $\alpha \ge 0$.

*In each of these cases, if $\sigma_1 > 0$ and*

$$\theta_i := \sqrt{\sigma_i} \quad (i = 1, 2) \tag{2.3}$$

*then by (2.1) and (2.2)*

$$\theta_1 \| u'' \|_2 \le \| u \| \le \theta_2 \| u'' \|_2 \tag{2.4}$$

*where*

$$\| u \|= \left( \int_0^1 (|u''(x)|^2 - \alpha |u'(x)|^2 + \beta(x)|u(x)|^2) dx \right)^{1/2}$$

*and so,* $\| \, . \, \|$ *defines a norm on* $X$ *equivalent to usual norm of* $X$ *inherited from* $W^{2,2}([0,1])$.
*In the remainder, we suppose* $\theta_1$ *defined by* (2.3) *satisfies* $\theta_1 > 0$ *and therefore* (2.5) *holds. The following result is useful for proving our main result.*

**Proposition 2.3.** *For every* $u \in X$ *then*

$$\| u \|_\infty \leq \frac{1}{2\pi\theta_1} \| u \| \, .$$

*Proof.* Taking (2.1) and (2.2) into account, the conclusion follows from the well-known inequality $\| u \|_\infty \leq \frac{1}{2} \| u' \|_2$. □

Let us recall that a weak solution of (1.1) is a function $u \in X$ such that

$$\int_0^1 [u''(x)v''(x) - \alpha u'(x)v'(x) + \beta(x)u(x)v(x)] dx - \int_0^1 h(u(x))v(x) dx \tag{2.5}$$

$$- \lambda \int_0^1 f(x, u(x))v(x) dx \ \textit{for every} \ v \in X$$

Moreover, a function $u : [0,1] \to \mathbb{R}$ is said to be a generalized solution to problem (1.1) if $u \in C^3([0,1]), u''' \in AC([0,1]), u(0) = u(1) = 0, u''(0) = u''(1) = 0$, and $u^{iv} + \alpha u'' + \beta(x)u = \lambda f(x,u) + h(u)$ for almost every $t \in [0,1]$. If $f$ is continuous in $[0,1] \times \mathbb{R}$, therefore each generalized solution $u$ is a classical solution.
Corresponding to the functions $f$ and $H$, we introduce the functions $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$, defined as follow

$$F(x,\xi) = \int_0^\xi f(x,t) dt \quad \textit{and} \quad H(\xi) = \int_0^\xi h(x) dx \tag{2.6}$$

Put

$$\Phi(u) = \frac{1}{2} \| u \|^2 - \int_0^1 H(u(x)) dx \quad \textit{and} \quad \Psi(u) = \int_0^1 F(x, u(x)) dx \tag{2.7}$$

for every $u \in X$.
Let us introduce the energy functional $I_\lambda$ related to the problem (1.1):

$$I_\lambda(u) := \Phi(u) - \lambda \Psi(u)$$

and we observe that, for each $\lambda > 0$ the critical points $u$ of $I_\lambda$ are the weak solutions of (1.1), i.e.

$$\int_0^1 [u''(x)v''(x) - \alpha u'(x)v'(x) + \beta(x)u(x)v(x)]dx - \int_0^1 h(u(x))v(x)dx \\ - \lambda \int_0^1 f(x, u(x))v(x)dx = 0$$

for every $v \in X$. In order to obtain weak solutions for the problem (1.1), we make use of recent critical points results.

**Lemma 2.4.** *Suppose $h : \mathbb{R} \to \mathbb{R}$ satisfies (1.2) and $H(\xi)$ defined by (2.6) for every $\xi \in \mathbb{R}$. Then the functional $\Theta : X \to \mathbb{R}$ defined by*

$$\Theta(u) := \int_0^1 H(u(x))dx \tag{2.8}$$

*is a Gâteaux differentiable sequentially weakly continuous functional on $X$ with compact derivative*

$$\Theta'(u)[v] = \int_0^1 h(u(x))v(x)dx$$

*for every $v \in X$.*

## 3. Main results

**Theorem 3.1.** *Assume that there exist $M > 0$, $L < \pi^4\theta_1^2$, and*

(1) $F(x,t) \geq 0$, *for each* $(x,t) \in [0,1] \times \mathbb{R}$;

(2) $k(\xi) = \min\{K(a,b), a, b \in [\xi, 1-\xi], a \leq b\}$.

*which is defined for every $0 < \xi < \frac{1}{2}$. such that*

$$\frac{\pi^3 k(\xi) M\theta_1}{\pi^4\theta_1^2 - L} \leq \frac{\pi^2 k(\xi)\int_a^b F(x,\bar{v})dx}{2(\pi^4\theta_1^2 + L)}. \tag{3.1}$$

*Then, for every $\lambda = \left]\frac{2(\pi^4\theta_1^2+L)}{\pi^2 k(\xi)\int_a^b F(x,\bar{v})dx}, \frac{\pi^4\theta_1^2 - L}{\pi^3 k(\xi)M\theta_1}\right[$, the problem (1.1) admits at least three distinct weak solutions in $X$.*

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# FREE BOUNDARY PROBLEM FOR A SUBLINEAR ELLIPTIC EQUATION

M. FOTOUHI

*Department of Mathematical Sciences, Sharif University of Technology*
*fotouhi@sharif.edu*

ABSTRACT. We study solutions and the free boundary $\partial\{\pm u > 0\}$ of the sublinear equation

$$\Delta u = f(x, u),$$

from a regularity point of view. We assume that $f(x, u) \approx |u|^q$ for wome $0 < q < 1$.

## 1. Introduction

Let $B_1$ be the unit ball in $\mathbb{R}^n$ $(n \geq 2)$, $g \in W^{1,2}(B_1) \cap L^\infty(B_1)$, and consider a minimizer of the functional

$$J(u) = \int_{B_1} \left(|\nabla u|^2 + 2F(x, u)\right) dx, \quad u - g \in W_0^{1,2}(B_1).$$

This minimizer satisfies the semilinear problem

$$\Delta u = f(x, u), \qquad \text{in } B_1, \tag{1.1}$$

where $f$ is the derivative of $F$ with respect to the variable $u$, i.e. $f = F_u$. The main question concerns the analysis of local regularity of solutions and the free boundary $\partial\{u > 0\} \cup \partial\{u < 0\}$. We assume that $f(x, u) = \lambda(x, u)|u|^q$ for some $q \in (0, 1)$ and the function $\lambda(x, u)$ satisfies

$$0 < \lambda_0 \leq |\lambda(x, u)| \in C^{0,\beta}(B_1 \times \mathbb{R}), \qquad \lambda(x, u)u \geq 0.$$

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## 2. Regularity of solution

It was shown that in [7] the positive solutions have the optimal regularity $C^{[\kappa],\kappa-[\kappa]}$, where $\kappa = 2/(1-q)$. However, without the assumption $u \geq 0$, the result will be more complicated and we can not expect to obtain $C^{[\kappa],\kappa-[\kappa]}$-regularity in general case. Indeed, the ODE $y'' = y^q$, with initial condition $y(0) = 0 \neq y'(0)$, has a solution whose third derivative is unbounded, $y''' = qy'y^{q-1}$.

In order to study the optimal growth (regularity) of solution, we start with the following definition.

**Definition 2.1.** The vanishing order of $u$ at point $x_0$ is defined to be the largest value $\mathcal{V}(x_0)$ which satisfies

$$\limsup_{r\to 0^+} \frac{\|u\|_{L^\infty(B_r(x_0))}}{r^{\mathcal{V}(x_0)}} < +\infty.$$

The following proposition classifies the vanishing order of the solution at $x_0 \in \{u = 0\}$. This also shows the necessary condition for the regularity of the solution.

**Proposition 2.2.** *Let $u$ be a nontrivial solution of (1.1) and $u(x_0) = 0$, then*

$$\mathcal{V}(x_0) \in \{1, 2, 3, \cdots, \lfloor\kappa\rfloor, \kappa\}.$$

*Proof.* The classical regularity theory implies that $u \in C^{1,\alpha}$ and then we can assume that $|u(x)| \leq C_1|x-x_0|$ for some constant $C_1 > 0$. Thus $\mathcal{V}(x_0) \geq 1$ and also $|\Delta u| \leq C|u|^q \leq CC_1^q|x-x_0|^q$. Now apply Lemma 1.1 in [3], we can write $u(x) = P(x) + \Gamma(x)$, where $P$ is harmonic polynomial of degree $\lfloor 2+q\rfloor = 2$ and $|\Gamma(x)| \leq \tilde{C}C_1^q|x-x_0|^{2+q}$. Then if $\mathcal{V}(x_0) > 1$, we must have $\mathcal{V}(x_0) \geq 2$. And if $\mathcal{V}(x_0) > 2$, then $P \equiv 0$ and so $|u(x)| \leq CC_1^q|x-x_0|^{2+q}$. If we repeat this argument and apply again Lemma 1.1 in [3], we will see that $\mathcal{V}(x_0) \in \{1, 2, 3, \cdots, \lfloor\kappa\rfloor\}$ or $\mathcal{V}(x_0) \geq \kappa$. Now use Lemma 4.1 in [5], the nondegeneracy property, to find out that $\|u\|_{L^\infty(B_r(x_0))} \geq c_0 r^\kappa$, for some positive constant $c_0$. Therefore $\mathcal{V}(x_0) \leq \kappa$. □

A sufficient condition for the vanishing order of $\kappa$ is proved in the next theorem.

**Theorem 2.3** (Theorem 1.2 in [5])**.** *Let $u$ be a solution of (1.1) and $\|u\|_{L^\infty(B_r(x_0))} = o(r^s)$ for some $0 < s < \kappa$. Then $\mathcal{V}(x_0) > s$.*

Note that the definition of vanishing order obviously yields that $\mathcal{V}(x_0) \geq s$, but it is not easy to find out $\mathcal{V}(x_0)$ is strictly greater than $s$. In particular, when $s = \lfloor\kappa\rfloor$, it along with Proposition 2.2 proves that $\mathcal{V}(x_0) = \kappa$.

## 3. Regularity of free boundary

There are few results for the regularity of free boundary for problem (1.1). In [7], it has been shown that for $f(u) = (u^+)^q$ the free boundary $\partial\{u > 0\}$ has locally finite $\mathcal{H}^{n-1}$-Hausdorff dimension when $-1 < q < 1$. Then the non-coincident set $\{u > 0\}$ has locally finite perimeter and we are able to define the reduced part of free boundary, $\partial_{\text{red}}\{u > 0\}$, where a tangent plane exists in a weak sense. Alt and Phillips shows $\partial_{\text{red}}\{u > 0\}$ is a $C^{1,\alpha}$ surface, [1]. They, however, show that $\mathcal{H}^{n-1}\left(\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}\right) = 0$, when $-1 < q < 0$. They also can prove $C^{1,\alpha}$-regularity of the free boundary near the regular points for $-1 < q < 1$. Moreover, in [2] the regularity of free boundary around the singular points is studied and it is shown that the singular points lie in a Lipschitz surface. The critical assumption in these researches is positivity of the solution.

One of the main difficulties encountered in studying the regularity of the free boundary in problem (1.1) is classification of global solutions. In dimension two when $f(u) = \lambda_+(u^+)^q - \lambda_-(u^-)^q$, we are able to present a fairly good analysis of global homogeneous solutions, and hence a better understanding of the behavior of the free boundary, [4, 5, 6]. In higher dimensions the problem becomes quite complicated, but we are still able to obtain partial results; e.g. we prove that if a solution is close to one-dimensional solution in a small ball, then in an even smaller ball the free boundary can be represented locally as two $C^1$-regular graphs $\partial\{u > 0\}$ and $\partial\{u < 0\}$, tangential to each other. It is noteworthy that the above problem (in contrast to the case $q = 0$) introduces interesting and quite challenging features, that are not encountered in the case $q = 0$. For example one obtains homogeneous global solutions that are not one-dimensional. This complicates the analysis of the free boundary particularly in singular points of free boundary.

To investigate the regularity of free boundary, we consider "asymptotically one-phase-points" that is, a subset of $\{u = 0\}$ such that the blow-ups[1] belong to

$$\mathbb{H} := \{x \mapsto \alpha \max(x \cdot \nu, 0)^\kappa : \nu \in \mathbb{R}^n \text{ is a unit vector}\}.$$

Members of this class are $\kappa$-homogeneous global solutions of (1.1).

**Definition 3.1.** We denote by $\mathcal{R}_u$ the set of all (regular free boundary) points $x_0 \in \{u = 0\}$ such that at least one blow-up of $u$ at $x_0$ is in $\mathbb{H}$.

[1]Any limit of the sequence $u(x_0 + r_n x)/r_n^\kappa$ when $r_n \to 0$ is called a blow-up of solution $u$ at point $x_0$.

Our main result concerning the regularity of the free boundary is presented in the following theorem.

**Theorem 3.2** (Regularity of the free boundary). *The set of regular free boundary points $\mathcal{R}_u$ is locally in $B_1$ a $C^{1,\beta}$-manifold.*

We choose the epiperimetric inequality approach to prove this result.

**Theorem 3.3** (The epiperimetric inequality, Theorem 3.1 in [6]). *There exist $\epsilon \in (0,1)$ and $\delta > 0$ such that if $c \in W^{1,2}(B_1)$ is a homogeneous function of degree $\kappa$ and $\|c - h\|_{W^{1,2}(B_1)} \leq \delta$ for some $h \in \mathbb{H}$, then there exists a function $v \in W^{1,2}(B_1)$ such that $v = c$ on $\partial B_1$ and*

$$M(v) - M(h) \leq (1-\epsilon)\left(M(c) - M(h)\right),$$

*where $M$ is the boundary adjusted energy*

$$M(u) = \int_{B_1} \left(|\nabla u|^2 + 2F(x,u)\right) dx - \kappa \int_{\partial B_1} |u|^2 \, d\sigma.$$

*Proof of Theorem 3.2.* The epiperimetric inequality with a monotonicity formula, Proposition 2.3 in [6], provides an estimate for the energy decay. Indeed, one can control the rate of convergence $\|u(x_0 + \cdot) - h\|_{L^1(B_r)}$, where $h$ belongs to $\mathbb{H}$. We can show that the rate of convergence is $r^{n+\kappa+\beta}$ for some $\beta > 0$. For the detail of proof refer to [6]. $\square$

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# REMARKS ON FIXED POINTS IN PARTIAL SYMMETRIC SPACES

A.P. FARAJZADEH AND S.GHAHRAMANI*

*Department of Mathematics, Razi University, Kermanshah, 67149, Iran*
*farajzadehali@gmail.com,saeedgha68@gmail.com*

Abstract. In this note, a class of partial symmetric spaces (PSS) on a family of bounded and continuous functions is introduced. Some fixed points theorems on different contractions defined on a family are presented. An example to illustrate the main theorem is given.

## 1. Introduction

In 1922, Banach[1] introduced Banach contraction principle. This principle has been extended to different class of spaces, see, for instance [3, 4] . Matthews[2] originated the concept of partial metric spaces and generalized Banach contraction principle , Kannan-Ciric and Ciric quasi type fixed point results. The symmetric spaces are not continuous and the uniqueness of the limit of a sequence may fail because of the awol of triangular inequality. In this note we consider the combination of partial metric spaces and symmetric spaces which called partial symmetric spaces(PSS, for short) and we provide some fixed point theorems in the setting of PSS.

The following definitions are needed in the next section.

**Definition 1.1.** Let $\xi$ denote the set of all bounded and continuous functions defined on a non-empty set $S$. The mapping $\zeta : \xi \times \xi \longrightarrow R_+$

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is a partial symmetric space PSS if, for all $\alpha, \beta, \gamma \in \xi$,
(i) $\beta = \alpha$ iff $\zeta(\alpha, \alpha) = \zeta(\beta, \beta) = \zeta(\alpha, \beta)$;
(ii) $\zeta(\alpha, \alpha) \leq \zeta(\alpha, \beta)$;
(iii) $\zeta(\alpha, \beta) = \zeta(\beta, \alpha)$;

Remark that one can equip $\xi$ by a complete meter as follows

$$d(f, g) = \sup\{f(x) - g(x) : x \in S\}.$$

**Definition 1.2.** A sequence $\{\alpha_n\}$ in $(\xi, \zeta)$ is said to if

- convergent to $\alpha \in \xi$, if
$$\zeta(\alpha, \alpha) = \lim_{n \to \infty} \zeta(\alpha_n, \alpha).$$
- $\zeta$-caushy if $\lim_{m,n \to \infty} \zeta(\alpha_n, \alpha_m)$ exists and is finite

**Definition 1.3.** $(\xi, \zeta)$ is said to be $\zeta$-complete if every $\zeta$-cauchy sequence $\{\alpha_n\}$ in $\xi$ is $\zeta$-convergent to a point $\alpha \in \xi$, such that

$$\zeta(\alpha, \alpha) = \lim_{n \to \infty} \zeta(\alpha_n, \alpha) = \lim_{m,n \to \infty} \zeta(\alpha_n, \alpha_m).$$

**Definition 1.4.** Let $(\xi, \zeta)$ be a PSS. A mapping $F : \xi \longrightarrow \xi$ is said to be a k-contraction if for all $\alpha, \beta \in \xi$, $\zeta(F\alpha, F\beta) \leq k\zeta(\alpha, \beta)$, where $k \in [0, 1)$.

## 2. Main results

The first theorem is a version of Banach contraction principle in the setting of PSS.

**Theorem 2.1.** *Assume that $\xi$ denotes the set of all bounded continuous functions defined on a non-empty set $S$, $\zeta : \xi \times \xi \longrightarrow R_+$ is a partial symmetric space and $F$ is a function defined on the complete partial symmetric space $(\xi, d)$. If $F$ is a $k-$contraction mapping, for some $0 \leq k < 1$ then $F$ has a unique fixed function.*

The following theorem is PSS version of the main theorem presented in [2] .

**Theorem 2.2.** *Let $(\xi, \zeta)$ be a PSS and $F$ be a continuous mapping on $\xi$ such that*

$$d(F(f), F(g)) \leq k \max\{d(f, g), d(f, F(f)), (d(g, F(g)), d(f, F(g)), d(f, F(g))\}, \tag{2.1}$$

*where $k \in [0, 1/2)$. In that case, $F$ has a unique fixed function.*

*Proof.* Let $\mu_0 \in \xi$ and $\{\mu_n\}$ an iterative sequence as follow

$$\mu_1 = F\mu_0,\ \mu_2 = F^2\mu_0,\ \mu_3 = F^3\mu_0, ..., \mu_n = F^n\mu_0.$$

Then

$$\begin{aligned} d(\mu_{n+1}, \mu_n) =& d(F(\mu_n), F(\mu_{n-1})) \leq k \max\{d(\mu_n, \mu_{n-1}), d(\mu_n, \mu_{n+1}) \\ &, d(\mu_n, \mu_{n-1}), d(\mu_n, \mu_n), d(\mu_{n-1}, \mu_{n+1})\} \\ &= k \max\{d(\mu_n, \mu_{n-1}), d(\mu_n, \mu_{n+1}), d(\mu_{n-1}, \mu_{n+1})\}. \end{aligned}$$

Since $0 \leq k \ < 1/2$, so

$$k \max\{d(\mu_n, \mu_{n-1}), d(\mu_n, \mu_{n+1}), d(\mu_{n-1}, \mu_{n+1})\} \leq \\ k \max\{d(\mu_n, \mu_{n-1}), d(\mu_{n-1}, \mu_{n+1})\}.$$

Now, if $d(\mu_n, \mu_{n-1}) \leq d(\mu_{n-1}, \mu_{n+1})$, then

$$d(\mu_{n+1}, \mu_n) \leq kd(\mu_{n-1}, \mu_{n+1}) \leq k(d(\mu_{n-1}, \mu_n) + d(\mu_n, \mu_{n+1})),$$

and this results $d(\mu_n, \mu_{n+1}) \leq \dfrac{k}{(1-k)} d(\mu_{n-1}, \mu_n)$.

Since $0 \leq k < 1/2$, therefor $\lambda = \dfrac{k}{(1-k)} < 1$. In any case $d(\mu_{n+1}, \mu_n) \leq \lambda d(\mu_{n-1}, \mu_n)$. Since $0 \leq \lambda < 1$, it is easy to see that the sequence $\mu_n$ is a cauchy sequence and so $\lim_{n\to\infty} \mu_n = \mu$.

On the other hand, $(\xi, d)$ is a complete metric space, then $\mu \in \xi$. Since $F$ is continuous, so $\mu = \lim_{n\to\infty} \mu_{n+1} = F(\lim_{n\to\infty} \mu_n) = F(\mu)$.

Uniqueness, let there exist $\alpha, \beta \in \xi$ such that $F\alpha = \alpha,\ F\beta = \beta$. It follows from (2.1) that

$$\begin{aligned} d(F\alpha, F\beta) \leq& k \max\{d(\alpha, \beta), d(\alpha, F\alpha), d(\beta, F\beta), d(\alpha, F\beta), d(\beta, F\alpha)\} \\ =& k \max\{d(\alpha, \beta), d(\alpha, \alpha), d(\beta, \beta), d(\alpha, \beta), d(\beta, \alpha)\} \\ =& k \max\{d(\alpha, \beta), d(\alpha, \beta), d(\beta, \alpha)\} = kd(\alpha, \beta). \end{aligned}$$

Since $0 \leq k < 1/2$, then $d(\alpha, \beta) = 0$, and because $d(\alpha, \beta) = sup\,|\alpha - \beta|$. Therefore $\alpha = \beta$. □

**Example 2.3.** Suppose $F\colon \xi \longrightarrow \xi$ is defined by $F(\alpha(x)) = \dfrac{1}{4}\alpha(x)$, for all $\alpha \in \xi$ and $0 \leq x < 1$ where $\xi$ is bounded and continuous.
Then, for all $\alpha, \beta \in \xi$

$$\begin{aligned} d(F\alpha, F\beta) =& d(\frac{1}{4}\alpha, \frac{1}{4}\beta) = sup \left| \frac{1}{4}\alpha - \frac{1}{4}\beta \right| = \frac{1}{4} sup\,|\alpha - \beta| \\ \leq& \frac{1}{4} \max\{(d(\alpha, \beta), d(\alpha, F\alpha), d(\beta, F\beta), d(\alpha, F\beta), d(\beta, F\alpha)\}. \end{aligned}$$

Hence by Theorem 2.1, $F$ has a unique fixed function which is $\alpha = 0$.

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# QUANTUM TIME SCALE AND POISSON DISTRIBUTION

F. GHARARI* AND M. GANJI

*Department of Statistics and Computer Science, University of Mohaghegh Ardabili, Ardabil, IRAN*
*f.gharari@uma.ac.ir, mganji@uma.ac.ir*

Abstract. We obtained $q-$versions of Poisson and gamma probability distributions by using basic tools of quantum time scale. For these quantum nabla and delta distributions or $q-$distributions and $1/q-$distributions, some statistical properties are proved.

## 1. Introduction

Time scale calculus is a methodology which aims to unify and extend continuous and discrete calculus. By choosing $\mathbb{T} = \mathbb{R}$, the theory reduces to real analysis, while choosing $\mathbb{T} = \mathbb{Z}$, the theory reduces to discrete analysis. However, the concepts extend to other time scales. Probability is a discipline in which appears to be many applications of time scales. The approach of time scales gives to probability theory the unification of discrete and continuous. For further details about time scale theory, we refer readers to [4, 5]. Quantum calculus is obtained by choosing the time scale

$$\mathbb{T}_q := q^{\mathbb{Z}} = \{q^n;\, n \in \mathbb{Z}\} \cup \{0\},$$

where we have fixed $q \in (0,1) \cup (1,\infty)$. The choice of choosing $q > 1$ or $0 < q < 1$ is a matter of preference. One can switch between the two using the transformation $q \to q^{-1}$. Recently, we have presented a system of probability distributions in our works [6, 7, 8] on time scale

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calculus. In these works, some of probability distributions on discrete time sale, where equipment of delta and nabla calculus are introduced. Here, we discover Poisson and gamma distributions on quantum time scales equipment with delta and nabla calculus. The Poisson distribution on quantum time scale equipment of delta calculus is called the delta Poisson ( $1/q-$Poisson) distribution and ones on quantum time scale equipment with nabla calculus is called the nabla Poisson ( $q-$Poisson) distribution. CH. Charalambides in [1] has studied discrete $q-$distributions such as Euler and Heine distributions that are $q-$analogous of Poisson distribution. Here, we get them by defining Poisson distribution on quantum time scale equipment of delta and nabla calculus, respectively.

## 2. Main results

We say that the delta Poisson distribution or $1/q-$ Poisson distribution with $(\lambda, q)$ parameters is given by pmf

$$P([X]_{1/q} = [x]_{1/q}) = e_q(-\lambda)\frac{q^{\binom{x}{2}}\lambda^x}{[x]_q!} \qquad x = 0, 1, ... \tag{2.1}$$

where $0 < q < 1$, $0 < \lambda < \infty$ and which is denoted by $[X]_{1/q} \sim Po_{1/q}(\lambda)$. Since $[X]_{r,1/q} = q^{-xr+\binom{r+1}{2}}[X]_{r,q}$, for this distribution we have

$$\begin{aligned}
E([X]_{r,1/q}) &= e_q(-\lambda)\sum_{x=r}^{\infty}\frac{[X]_{r,1/q}q^{-rx+\binom{r+1}{2}}q^{x(x-1)/2}\lambda^x}{[x]_q!} \\
&= \lambda^r e_q(-\lambda)\sum_{x=r}^{\infty}\frac{q^{-rx+\binom{r+1}{2}}q^{r(x-r)}q^{r(r-1)/2}q^{(x-r)(x-r-1)}\lambda^{x-r}}{[x-r]_q!} \\
&= \lambda^r e_q(-\lambda)E_q(\lambda) = \lambda^r,
\end{aligned}$$

then $E([X]_{1/q}) = \lambda$ and $Var([X]_{1/q}) = \lambda(1 - \lambda(1 - 1/q))$, where $Var([X]_{1/q}) = 1/qE[X]_{2,1/q} + E[X]_{1/q} - E^2[X]_{1/q}$. A family of non-negative integer valued random variable $\{X_\lambda,\ 0 < \lambda < \rho \leq \infty\}$ is said to be a power series distribution if its pmf can be written in the form of

$$f(x;\lambda) = \frac{a(x)\lambda^x}{g(\lambda)}, \qquad x = 0, 1, ...$$

where $0 < \lambda < \rho$ and $g(\lambda) = \sum_{x=0}^{\infty} a(x)\lambda^x$. It is well known that mean-variance equality $E(X_\lambda) = Var(X_\lambda),\ 0 < \lambda < \rho$ characterizes the Poisson family of distributions (see [2] and [3]). This requirement that this equality holds for all $0 < \lambda < \rho$ has been overlooked by some authors (see [10]). In the following theorem is derived an analogues of this characterization for the $1/q-$Poisson distribution.

**Theorem 2.1.** *Assume that a family of non-negative integer valued random variables obeys a power series distribution. Then, $[X]_{1/q}$ has a $1/q-$Poisson distribution if and only if*

$$E[X]_{2,1/q} = E^2[X]_{1/q} \tag{2.2}$$

*for all $\lambda \in (0, \rho)$.*

To continue, first we provide a definition of $1/q-$gamma distribution, and then obtain a relationship between it and $1/q-$Poisson distribution.

The random variable $X_{1/q}$ has a $1/q-$gamma distribution with $(\alpha, \beta, q)$ parameters if

$$f_{X_{1/q}}(x_{1/q}) = \frac{e_q(-\beta x) q^{\binom{\alpha}{2}} x^{\alpha-1} \beta^\alpha}{\Gamma_q(\alpha)}, \qquad x \in \mathbb{R}_q^+, \tag{2.3}$$

where $\alpha > 0,\ \beta > 0,\ 0 < q < 1$ and $q^{\binom{\alpha}{2}} = \dfrac{\Gamma_q(\alpha+1)}{2!\Gamma_q(\alpha-2+1)}$. It denotes as $\Gamma_{1/q}(\alpha, \beta)$. We call the special case $\alpha = 1,\ f_{X_{1/q}}(x_{1/q}) = \beta e_q(-\beta x)$ as $1/q-$exponential distribution.

**Theorem 2.2.** *($1/q-$gamma$-$Poisson relationship) Suppose that $X_{1/q} \sim \Gamma_{1/q}(\alpha, 1)$ and $[Y]_{1/q} \sim Po_{1/q}(x, q)$, where $\alpha$ is a integer, then $P(X_{1/q} \leqslant x) = P([Y]_{1/q} \geqslant \alpha)$.*

Similarly, we can define $q-$Poisson and $q-$gamma distributions. We say that, the nabla Poisson distribution or $q-$Poisson distribution with $(\lambda, q)$ parameters is given by the pmf

$$P([X]_q = [x]_q) = E_q(-\lambda) \frac{\lambda^x}{[x]_q!}, \qquad x = 0, 1, ... \tag{2.4}$$

where $0 < q < 1,\ 0 < \lambda < 1/(1-q)$ and which is denoted by $[X]_q \sim Po_q(\lambda)$. By example 3.3.4 in [9], we saw that for this $q-$distribution $E[X]_{r,q} = \lambda$, then, we have $E([X]_q) = \lambda$ and $Var([X]_q) = \lambda(1 - \lambda(1 - q))$, where $Var([X]_q) = qE[X]_{2,q} + E[X]_q - E^2[X]_q$ (see [1]). Also, there is a similar theorem to theorem 2.1 in [9] for $q-$Poisson distribution, such that gives the following result for all $\lambda \in (0, \rho)$,

$$E[X]_{2,q} = E^2[X]_q. \tag{2.5}$$

It is said that the random variable $X_q$ has a $q-$gamma distribution with $(\alpha, \beta, q)$ parameters if its pdf is given by

$$f_{X_q}(x_q) = \frac{E_q(-q\beta x) x^{\alpha-1} \beta^\alpha}{\Gamma_q(\alpha)}, \qquad x \in \mathbb{R}_q^+, \tag{2.6}$$

where $\alpha > 0,\ \beta > 0,\ 0 < q < 1$ and it denotes as $\Gamma_q(\alpha, \beta)$. The special case $\alpha = 1,\ f_{X_q}(x_q) = \beta E_q(-q\beta x)$ is called $q-$exponential distribution.

**Theorem 2.3.** *($q-$gamma$-$Poisson relationship) Suppose that $X_q \sim \Gamma_q(\alpha, 1)$ and $[Y]_q \sim Po_q(x, q)$, where $\alpha$ is a integer, then $P(X_q \leqslant x) = P([Y]_q \geqslant \alpha)$.*

## 3. Conclusion

In this work, we studied some quantum distributions belong to the class of distributions on quantum time scales equipment of delta and nabla calculus. the class of $1/q-$ distributions generate by delta quantum calculus and $q-$ distributions by delta quantum calculus.

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# CONTINUOUS $*$-CONTROLLED FRAMES IN HILBERT $C^*$-MODULES

HADI GHASEMI* AND TAYEBE LAL SHATERI

[1] *Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, P.O. Box 397, IRAN*
*h.ghasemi@hsu.ac.ir and t.shateri@hsu.ac.ir; shateri@ualberta.ca*

Abstract. In this paper, we introduce the notion of continuous $*$-$C$-controlled frames in Hilbert $C^*$-modules. We present some results of these frames in Hilbert $C^*$-modules. Then we define $*$-$(C, C')$-frames and study multiplier operators for these frames.

## 1. Introduction

In 1952, the concept of discrete frames for Hilbert spaces were introduced by Duffin and Schaeffer [3] to study some problems in nonharmonic Fourier series. Frame theory has been used in many fields such as filter bank theory, image processing, etc. we refer to [1] for an introduction to frame theory in Hilbert spaces and its applications.

Frank and Larson [4] presented a general approach to the frame theory in Hilbert $C^*$-modules. Theory of frames have been extended from Hilbert spaces to Hilbert $C^*$-modules, see [5, 6, 7, 8].

In this paper, we introduce the notion of continuous $*$-$C$-controlled frames in Hilbert $C^*$-modules. We present some results of these frames in Hilbert $C^*$-modules. Then we define $*$-$(C, C')$-frames and study multiplier operators for these frames.

First, we recall some definitions and basic properties of Hilbert $C^*$-modules. Throughout this paper, $\mathcal{A}$ shows a $C^*$-algebra.

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**Definition 1.1.** A pre-Hilbert module over $C^*$-algebra $\mathcal{A}$ is a complex vector space $U$ which is also a left $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : U \times U \to \mathcal{A}$ which is $\mathbb{C}$-linear and $\mathcal{A}$-linear in its first variable and satisfies the following conditions:
$(i)$ $\langle f, f\rangle \geq 0$,
$(ii)$ $\langle f, f\rangle = 0$ iff $f = 0$,
$(iii)$ $\langle f, g\rangle^* = \langle g, f\rangle$,
$(iv)$ $\langle af, g\rangle = a\langle f, g\rangle$,
for all $f, g \in U$ and $a, b \in \mathcal{A}$.

A pre-Hilbert $\mathcal{A}$-module $U$ is called Hilbert $\mathcal{A}$-module if $U$ is complete with respect to the topology determined by the norm $\|f\| = \|\langle f, f\rangle\|^{\frac{1}{2}}$. We consider $GL(U)$ as the set of all bounded linear invertible operators with bounded inverse.

## 2. Main Results

In this section, we introduce continuous controlled frames in Hilbert $C^*$-modules with $C^*$-valued bounds, and then we give some results for these frames. We assume that $\mathcal{A}$ is a unital $C^*$-algebra and $U$ is a Hilbert $\mathcal{A}$-module.

Let $\mathcal{Y}$ be a Banach space, $(\mathcal{X}, \mu)$ a measure space, and $f : \mathcal{X} \to \mathcal{Y}$ a measurable function. Integral of the Banach-valued function $f$ has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions (see [2]). Since every $C^*$-algebra and Hilbert $C^*$-module is a Banach space, hence we can use this integral in these spaces.

**Definition 2.1.** Let $C \in GL(U)$. A mapping $F : \mathcal{X} \to U$ is called a continuous $*$-$C$-controlled frame for $U$ if

$(C1)$ $F$ is weakly-measurable, i.e, for any $f \in U$, the mapping $\mathcal{X} \longmapsto \langle f, F(\mathcal{X})\rangle$ is measurable on $\mathcal{X}$.

$(C2)$ There exist strictly nonzero elements $A, B$ in $\mathcal{A}$ such that

$$A\langle f, f\rangle A^* \leq \int_{\mathcal{X}} \langle f, F(x)\rangle\langle CF(x), f\rangle d\mu(x) \leq B\langle f, f\rangle B^*, \quad (f \in U). \tag{2.1}$$

**Example 2.2.** Let $U = \mathcal{A} = l^2(\mathbb{C})$ with the $\mathcal{A}$-inner product $\langle\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}\rangle = \{a_n\overline{b_n}\}_{n\in\mathbb{N}}$. Consider the linear operator $C : U \to U$ defined as $C\{a_n\}_{n\in\mathbb{N}} = \{\alpha a_n\}_{n\in\mathbb{N}}$, where $\alpha \in \mathbb{R}^+$. Let $(\mathcal{X}, \mu)$ be a measure space in which $\mathcal{X} = [0, 1]$ and $\mu$ is the Lebesgue measure. Suppose

$$\begin{aligned} F : \mathcal{X} &\to \quad U \\ x &\longmapsto \{\sqrt{3}(\tfrac{1}{2} + \tfrac{1}{n})x\}_{n\in\mathbb{N}}. \end{aligned}$$

If $f=\{a_n\}_{n\in\mathbb{N}}\in U$, then we see that

$$\begin{aligned}&\int_{\mathcal{X}}\langle f,F(x)\rangle\langle CF(x),f\rangle d\mu(x)\\&=\sqrt{\alpha}\{\frac{1}{2}+\frac{1}{n}\}_{n\in\mathbb{N}}\langle\{a_n\}_{n\in\mathbb{N}},\{a_n\}_{n\in\mathbb{N}}\rangle\sqrt{\alpha}\{\frac{1}{2}+\frac{1}{n}\}_{n\in\mathbb{N}}.\end{aligned}$$

Therefore $F$ is a continuous tight $*$-$C$-controlled frame with bound $\sqrt{\alpha}\{\frac{1}{2}+\frac{1}{n}\}_{n\in\mathbb{N}}$.

**Definition 2.3.** Let $C\in GL(U)$, and let the mapping $F:\mathcal{X}\to U$ be a continuous $*$-$C$-controlled Bessel map in Hilbert $C^*$-module $U$. The operator $S_Cf=\int_{x\in\mathcal{X}}\langle f,F(x)\rangle CF(x)d\mu(x)$ is called a continuous $*$-$C$-controlled frame operator.

**Theorem 2.4.** *Let $C\in GL^+(U)$, and let the mapping $F:\mathcal{X}\to U$ be a continuous $*$-$C$-controlled frame in Hilbert $C^*$-module $U$. Then continuous $*$-$C$-controlled frame operator $S_C$ is invertible, positive and $\|A^{-1}\|^{-2}\leq\|S_C\|\leq\|B\|^2$.*

In the sequel, we assume that $C,C'\in GL(U)$.

**Definition 2.5.** The mapping $F:\mathcal{X}\to U$ is called a continuous $*$-$(C,C')$-controlled frame in Hilbert $C^*$-module $U$, if
$(CC1)$ $C'F$ is weakly-measurable, i.e, for any $f\in U$, the mapping $\mathcal{X}\longmapsto\langle f,C'F(\mathcal{X})\rangle$ is measurable on $\mathcal{X}$.

$(CC2)$ There exists strictly nonzero elements $A,B$ in $\mathcal{A}$ such that

$$A\langle f,f\rangle A^*\leq\int_{\mathcal{X}}\langle f,C'F(x)\rangle\langle CF(x),f\rangle d\mu(x)\leq B\langle f,f\rangle B^*,\quad(f\in U).\tag{2.2}$$

**Theorem 2.6.** *Let $F$ and $G$ be continuous $*$-$(C,C)$ and $*$-$(C',C')$-controlled Bessel maps with $*$-frame bounds $B$ and $B'$, respectively. Let $m\in\mathcal{L}^\infty(\mathcal{X},\mu)$. The operator*

$$\begin{aligned}M_{m,CF,C'G}:U&\to U\\f&\longmapsto\int_{\mathcal{X}}m(x)\langle f,CF(x)\rangle C'G(x)d\mu(x)\end{aligned}$$

*is a well-defined bounded operator.*

Now, we give the concept of multipliers for continuous $*$-$(C,C')$-controlled Bessel maps.

**Definition 2.7.** Let $F$ and $G$ be continuous $*$-$(C,C)$ and $*$-$(C',C')$-controlled Bessel maps for $U$, respectively. Let $m\in\mathcal{L}^\infty(\mathcal{X},\mu)$. The operator $M_{m,CF,C'G}$ is called the continuous $*$-$(C,C')$-controlled Bessel multiplier of $F,G$ and $m$.

We give the following results.

**Corollary 2.8.** *Let $C$ and $C'$ be unitary operators in $End^*_{\mathcal{A}}(U)$. Suppose $F$ and $G$ be continuous $*$-$(C,C)$ and $*$-$(C',C')$-controlled Bessel maps with $*$-frame bounds $B$ and $B'$, respectively. Then*

$$M_{m,CF,C'G} = C'M_{m,F,G}C^*.$$

**Corollary 2.9.** *Let $C \in GL^+(U)$, and let $m(x) \geq \delta > 0$ a.e., then for any continuous $*$-$(C,C)$-controlled Bessel map $F$, the multiplier $M_{m,CF,CF}$ is a positive invertible operator.*

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# SOME NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

SH. GHASEMZADEHDIBAGI

*Department of Mathematics, Hamedan Branch, Islamic Azad University, Hamedan, Iran*
*sh.ghasemzadeh@iauh.ac.ir*

Abstract. We present some fixed point theorems for $(\alpha, p)$-nonexpansive mappings in $CAT(0)$ spaces. Moreover we study the existence of fixed points for $(\alpha, p)$-nonexpansive mappings.

## 1. Introduction and Preliminaries

A metric space $M$ is a $CAT(0)$ space if it is geodesically connected and if every geodesic triangle in $M$ is at least as thin as its comparison triangle in the Euclidean plane. $CAT(0)$ spaces are convex metric spaces, that is, for each $x, y \in M$ and $t \in [0,1]$, there exists $z \in [x, y]$ such that

$$d(x,z) = td(x,y), \qquad d(y,z) = (1-t)d(x,y). \tag{1}$$

For the sake of readerships, denote the unique element $(1-t)x \oplus ty$ in (1) by $z$. If $(M,d)$ is a $CAT(0)$ space, then

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z),$$

for all $x, y, z \in M$ and $t \in [0,1]$.

**Lemma 1.1.** *[1, Proposition 2.2] Let $(M,d)$ be a $CAT(0)$ space, $p, q, r, s \in M$ and $t \in [0,1]$. Then*

$$d(tp \oplus (1-t)q, tr \oplus (1-t)s) \leq td(p,r) + (1-t)d(q,s).$$

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We now adopt the notation $\alpha_1 x_1 \oplus \alpha_2 x_2 \oplus \cdots \oplus \alpha_n x_n = \sum_{i=1}^n [\alpha_i, x_i]$. Then for each convex metric space, when $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i \geq 0$ for all $i$, we have

$$d(\sum_{i=1}^n [\alpha_i, x_i], z) \leq \sum_{i=1}^n \alpha_i d(x_i, z),$$

for all $x_1, \cdots, x_n$ and $z \in X$. Therefore lem 1.1, yields the following inequality:

$$d(\sum_{i=1}^n [\alpha_i, x_i], \sum_{i=1}^n [\alpha_i, y_i] \leq \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j d(x_i, y_j), \tag{2}$$

for all $x_1, \cdots, x_n$ and $y_1, \cdots, y_n \in X$. Goebel and Pineda introduced the class of $(\alpha, p)$-nonexpansive mappings in Banach spaces [3]. Similarly in $CAT(0)$ spaces we define:

**Definition 1.2.** A function $T : C \to C$ is called $(\alpha, p)$-nonexpansive if, for some $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i \geq 0$ for all $1 \leq i \leq n$, $\alpha_1, \alpha_n > 0$, and for some $p \in [1, \infty)$,

$$\sum_{i=1}^n \alpha_i d(T^i x, T^i y)^p \leq d(x, y)^p,$$

for all $x, y \in C$.

In particular, their proofs rely heavily on the nonexpansive property of the following mapping

$$T_\alpha := \alpha_1 T \oplus \alpha_2 T^2 \oplus \cdots \oplus \alpha_n T^n.$$

Clearly $d(Tx, Ty)^p \leq \frac{1}{\alpha_1} d(x, y)^p$, that is $d(Tx, Ty) \leq \left(\frac{1}{\alpha_1^{1/p}}\right) d(x, y)$, and consequently $k(T) \leq \frac{1}{\alpha_1^{1/p}}$, hence $d(T^j x, T^j y) \leq \left(\frac{1}{\alpha_1^{1/p}}\right)^j d(x, y)$. When the multi-index $\alpha$ isn't specified, we say $T$ is mean nonexpansive. If $T$ is identity or constant mapping, then $T$ is $(\alpha, p)$-nonexpansive mapping. Also, let $C = [0, 1] \times [0, 1]$ and $T(x, y) = (x, 0)$ for all $(x, y) \in C$ be the projection. Therefore, $T$ is an $(\alpha, p)$-nonexpansive mapping.

## 2. Main Results

Let $(M, d)$ be a complete $CAT(0)$ space and $C$ be a nonempty convex subset of $M$.

**Definition 2.1.** We say that the $(\alpha, p)$-nonexpansive mapping $T : C \to C$ is $(\alpha, p)$-metrically invariant, if for any two points $x$ and $y$ in $C$ and for all $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$, we have

$$d(T^i x, T^i y) - d(T^i x, T^j y) - d(T^j x, T^i y) + d(T^j x, T^j y) \geq 0,$$

The following Lemma plays an important role in this article.

**Lemma 2.2.** *The mapping $T_\alpha$ is nonexpansive mapping, whenever $T$ is $(\alpha, p)$-metrically invariant.*

**Theorem 2.3.** *Let $(M, d)$ be a complete $CAT(0)$ space. Let $C$ be a nonempty bounded closed and convex subset of $M$. Then any $((\alpha_1, \alpha_2), 1)$-metrically invariant mapping $T : C \to C$ with $\alpha_1 \geq \dfrac{1}{2}$, has an approximate fixed point sequence.*

**Example 2.4.** Let $M = l^1$ with a metric inherited from the standard norm, $\|x\| = \|(x_1, x_2, \cdots)\| = \sum_{i=1}^{\infty} |x_i|$, $C := B_{l^1}$ the closed unit ball of $l^1$ and $\alpha = (1-k, k)$ for all $0 < k < 1$. Suppose $T : C \to C$ be a mapping defined for every $(x_1, x_2, \cdots) \in C$ by

$$T(x) = (s_2 x_2, s_3 x_3, \cdots, s_j x_j, \cdots)$$

where $s_j = \dfrac{1-(-k)^{j-1}}{1-(-k)^j}$ for $j \in \{2, 3, \cdots\}$. Then $d(T) = 0$. Clearly, $(a, \dfrac{a}{s_2}, \dfrac{a}{s_2 s_3}, \dfrac{a}{s_2 s_3 s_4}, \cdots, \dfrac{a}{s_2 s_3 \cdots s_j}, \cdots)$ is a fixed point of $T$ for all $a \in (0, 1)$.

Now, we generalize the results of Section 3 for $(\alpha, p)$-nonexpansive mappings, when $p \geq 1$.

**Theorem 2.5.** *Let $(M, d)$ be a complete $CAT(0)$ space. Let $C$ be a nonempty bounded closed and convex subset of $M$ with $diam(C) > 0$. If $T : C \to C$ is $(\alpha, p)$-nonexpansive with $n \geq 2$ for some $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $p \geq 1$, such that*

$$(1-\alpha_1)\Big(1-\alpha_1^{\frac{n-1}{p}}\Big) \leq \alpha_1^{\frac{n-1}{p}}\Big(1-\alpha_1^{\frac{1}{p}}\Big), \tag{3}$$

*then $d(T) = \inf\{d(x, Tx) : x \in C\} = 0$.*

**Theorem 2.6.** *Let $(M, d)$ be a complete locally compact $CAT(0)$ space. If $C$ be a nonempty bounded closed and convex subset of $M$, then $C$ has the fixed point property for $(\alpha, p)$-nonexpansive mappings with $n \geq 2$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $p \geq 1$ such that*

$$(1-\alpha_1)\Big(1-\alpha_1^{\frac{n-1}{p}}\Big) \leq \alpha_1^{\frac{n-1}{p}}\Big(1-\alpha_1^{\frac{1}{p}}\Big).$$

*Moreover, if the above inequality is strict then $Fix(T) = Fix(T_\alpha)$.*

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# THE NUMERICAL SOLUTION OF THE WAVE EQUATION BY THE FRAGILE POINTS METHOD

D. HAGHIGHI∗ AND S. ABBASBANDY

*Department of Applied Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, 34149-16818, Iran*
*haghighi.donya@edu.ikiu.ac.ir*
*abbasbandy@ikiu.ac.ir*

Abstract. In this paper, the numerical solution of the wave equation is considered by the fragile points method. To achieve the test and trial functions, the generalized finite difference method has been applied. Interior Penalty Numerical Fluxes (IPNF) have been proposed to establish the consistency of the method. Finally, numerical results are provided.

## 1. Introduction

In many problems, numerical analysis researchers use the finite element method (FEM) [2] , Finite Volume Method (FVM) and Boundary Element Method (BEM) to discretize the spatial dimension. Another group of numerical methods known as meshless methods that do not require mesh for discretization such as Element Free Galerkin (EFG) and Meshless Local Petrov-Galerkin (MLPG). Attaining trial functions by these methods cause a lot of complexity in trial functions and weak form integrations. Hence, Leiting Dong and colleagues to provide a general method for solving problems so that the test and trial functions are simple, local, and discontinuous polynomials introduced a new meshless method called the Fragile Points Method (FPM) [3].

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In this paper, wave equation

$$\frac{\partial^2 u}{\partial t^2}(\mathbf{x},t) = \alpha^2 \nabla^2 u(\mathbf{x},t) + f(\mathbf{x},t) \quad \mathbf{x} \in \Omega, \tag{1.1}$$

with initial conditions

$$u(\mathbf{x},0) = g_1(\mathbf{x}), \qquad \frac{\partial u}{\partial t}(\mathbf{x},0) = g_2, \tag{1.2}$$

and the boundary conditions

$$u(\mathbf{x},t) = h_1(\mathbf{x},t),\ \mathbf{x} \in \Gamma_D, \quad \nabla u.n(\mathbf{x},t) = h_2(\mathbf{x},t),\ \mathbf{x} \in \Gamma_N, \tag{1.3}$$

will be studied by Fragile Points Method (FPM).

## 2. Main results

2.1. **Polynomial discontinuous trial and test functions.** Inside the domain $\Omega$ and it's boundary $\partial\Omega$, several points are distributed sporadically. The Voronoi Diagram method has been selected for the partition of the domain. The trial function $u_h$ in the subdomain $E_0$ which includes the point $P_0$ can be written as

$$u_h(\mathbf{x},t) = u_0(\mathbf{x},t) + (\mathbf{x}-\mathbf{x}_0)\nabla u(\mathbf{x},t)|_{P_0}, \quad \mathbf{x} \in E_0. \tag{2.1}$$

In the above equation, $u_0$ is the value of $u_h$ at $P_0$ and $\mathbf{x}_0$ denotes the coordinate of the point $P_0$.

We employ the Generalized Finite Difference (GFD) method to calculate $\nabla u$ at $P_0$ in terms of the values of $u_h$ at several neighboring points of $P_0$ that named $q_1, q_2, ..., q_m$. In the following, to calculate the amount of the gradient of $\nabla u$ at $P_0$, we minimize a weighted discrete $L^2$ norm $\mathbf{J}$ so that

$$\mathbf{J} = \sum_{i=0}^{m} \Big( \nabla u|_{P_0}.\,(\mathbf{x}_i-\mathbf{x}_0)^T - (u_i-u_0)\Big)^2 w_i. \tag{2.2}$$

If we assume that weight function w is constant, due to the stationarity of $\mathbf{J}$ we have

$$\nabla u = \mathbf{B}\mathbf{u}_E, \tag{2.3}$$

where

$$A = \begin{bmatrix} x_1-x_0 & y_1-y_0 \\ x_2-x_0 & y_2-y_0 \\ \cdots & \cdots \\ x_m-x_0 & y_m-y_0 \end{bmatrix}, \quad \mathbf{B} = \left(A^T A\right)^{-1} A^T \begin{bmatrix} -1 & 1 & 0 & \ldots & 0 \\ -1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -1 & 0 & \cdots & 0 & 1 \end{bmatrix}_{m\times(m+1)}$$

and $\mathbf{u}_E = (u_0, u_1, ..., u_m)^T$. Also by substituting (2.3) into (2.1) the relation between $u_h$ and $\mathbf{u}_E$ will be obtained as

$$u_h = \mathbf{N}\mathbf{u}_E, \ \forall \mathbf{x} \in E_0, \quad \mathbf{N} = [\mathbf{x} - \mathbf{x}_0]\mathbf{B} + [1, 0, \ldots, 0]_{1\times(m+1)}. \quad (2.4)$$

2.2. **Implementation of numerical flux corrections.** We can rewrite wave equation (1.1) using mixed form as following,

$$\begin{cases} \sigma = \nabla u(\mathbf{x}, t), & in \ \Omega, \\ -\alpha^2 \nabla.\sigma = -\dfrac{\partial^2 u}{\partial t^2}(\mathbf{x}, t) + f(\mathbf{x}, t), & in \ \Omega. \end{cases} \quad (2.5)$$

By multiplying the first and second equations in (2.5) by test functions $\tau$ and $\nu$ respectively and integrating it on the subdomain $E$, using the Green formula and by summing these equations over all subdomains we have

$$\int_\Omega \sigma_h.\tau d\Omega = -\int_\Omega u_h \nabla.\tau d\Omega + \sum_{E\in\Omega} \int_{\partial E} \hat{u}_h n.\tau d\Gamma,$$

$$\alpha^2 \int_\Omega \sigma_h.\nabla\nu = \alpha^2 \sum_{E\in\Omega} \int_{\partial E} \hat{\sigma}_h.n\nu d\Gamma - \int_\Omega \frac{\partial^2 u}{\partial t^2}(\mathbf{x}, t)\nu d\Omega + \int_\Omega f(\mathbf{x}, t)\nu d\Omega.$$

In the above equations values $\hat{\sigma}_h$ and $\hat{u}_h$ represent approximations $\sigma_h$ and $u_h$ on $\partial E$. These values are named Numerical Fluxes [1]. Then we define operators the *average* and the *jump* which by these operators, we can manage the numerical fluxes. As regards $\Gamma = \Gamma_h + \Gamma_D + \Gamma_N$, Table 3.1 in [3] and by substituting the Interior Penalty Numerical Fluxes (IPNF), we have

$$\begin{aligned} & \alpha^2 \sum_{E\in\Omega} \int_E \nabla u_h.\nabla\nu d\Omega - \alpha^2 \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \left(\{\nabla u_h\}[\nu] + \{\nabla\nu\}[u_h]\right) d\Gamma \\ + \ & \alpha^2 \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{\eta}{h_e} \int_e [\nu][u_h] \, d\Gamma = \int_\Omega f(\mathbf{x}, t)\nu d\Omega - \int_\Omega \frac{\partial^2 u}{\partial t^2}(\mathbf{x}, t)\nu d\Omega \\ + \ & \alpha^2 \sum_{e\in\Gamma_D} \int_e \left(\frac{\eta}{h_e}\nu - \nabla\nu.\mathbf{n}\right) h_1(\mathbf{x}, t) d\Gamma + \alpha^2 \sum_{e\in\Gamma_N} \int_e \nu h_2(\mathbf{x}, t) d\Gamma. \end{aligned}$$

The above equation is the formula of FPM, which is called FPM-Primal method. If the matrix form of this method is expressed as follows:

$$\alpha^2 \mathbf{K}\mathbf{u} + \mathbf{C}\ddot{\mathbf{u}} = \mathbf{F}, \quad (2.6)$$

By Substituting values $\mathbf{B}$ instead of $\nabla\nu$ and $\nabla u$, $\mathbf{N}$ instead of $u_h$ and $\nu$ in equation (2.6), the point stiffness matrices will be achieved.

2.3. **Numerical results.** In this section, a example to verify the accuracy and efficiency of the present method will be evaluated.

**Example 2.1.** Consider the two-dimensional inhomogeneous wave equation equation as follows

$$u_{tt}(x, y, t) = \nabla^2 u(x, y, t) + cos(x), \;\; x \in (0, \pi), \;\; y \in (0, \pi), \;\; t > 0.$$

Dirichlet boundary conditions $u(0, y, t) = 1 + sin(y)sin(t)$, $u(\pi, y, t) = -1 + sin(y)sin(t)$,$u(x, 0, t) = cos(x)$, $u(x, \pi, t) = cos(x)$ and initial condition $u(x, y, 0) = cos(x)$ and $u_t(x, y, 0) = sin(y)$ and analytical solution can be expressed as $u(x, y, t) = cos(x) + sin(y)sin(t)$. Relative errors $r_0$ and $r_1$ have been shown in Table 1.

Table 1. The relative errors of the Example 2.1 at $T = 1$.

| points | Parameters | $r_0$ | $r_1$ | cpu time |
|---|---|---|---|---|
| $N = 121$ | $h_e = 1$ , $\eta = 6$ | $1.7263 \times 10^{-2}$ | $9.5218 \times 10^{-2}$ | $1.6s$ |
| $N = 676$ | $h_e = 0.1$ , $\eta = 3$ | $4.4831 \times 10^{-3}$ | $2.7873 \times 10^{-2}$ | $11s$ |
| $N = 2601$ | $h_e = 0.1$ , $\eta = 3$ | $2.7653 \times 10^{-3}$ | $1.5448 \times 10^{-2}$ | $91s$ |

According to the results of the Table by FPM, we can be seen that the method is stable and has good precision. Also, the method does not have much computational cost and depending on the number of points used, it will achieve numerical solutions with good accuracy in a short time that this is an advantage. Other advantages of this method over other numerical methods are described in detail in Table 1 in [4].

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# $\ell_p$-DIVERSITIES AND EMBEDDINGS

POUYA HAGHMARAM* AND KOUROSH NOUROUZI

*Faculty of Mathematics,*
*K. N. Toosi University of Technology,*
*Tehran, Iran*
*haghmaram@email.kntu.ac.ir and nourouzi@kntu.ac.ir*

Abstract. In this talk, we first introduce $\ell_p$-diversities as a generalization of $\ell_p$-metrics, for all $1 \leq p < \infty$. Then, we state the diversity setting of the well-known Bourgain theorem in metric embeddings. We show that every diversity $(X, \delta)$ on $n$ points embeds in diversities $\ell_p^{\mathcal{O}(\log n)}$ with distortion $\mathcal{O}(n \log^{\frac{1+p}{p}} n)$ as $1 \leq p \leq 2$, and $\ell_p^{\mathcal{O}(\log^2 n)}$ with distortion $\mathcal{O}(n \log^{\frac{2+p}{p}} n)$ as $2 < p < \infty$.

## 1. Introduction

Diversities were introduced in [1] as a generalization of metric spaces. Actually, a nonnegative real number is assigned to each finite subset of a given set. To be precise, let $X$ be a nonempty set and $\langle X \rangle$ denote the collection of all finite subsets of $X$. A diversity is a pair $(X, \delta)$ where $\delta : \langle X \rangle \to \mathbb{R}$ satisfies the following conditions for all $A, B, C \in \langle X \rangle$:

(D1) $\delta(A) \geq 0$, and $\delta(A) = 0$ if and only if $|A| \leq 1$;
(D2) If $B \neq \emptyset$, then $\delta(A \cup C) \leq \delta(A \cup B) + \delta(B \cup C)$.

**Example 1.1.** Let $(X, d)$ be a metric space. Then the mapping $\delta_{\text{diam}} : \langle X \rangle \to [0, \infty)$ defined as $\delta_{\text{diam}}(A) = \max\{d(a, b) : a, b \in A\}$, for all $A \in \langle X \rangle$ is a diversity on $X$.

Below, we introduce $\ell_p$-diversities for all $p \in [1, \infty)$. For $p = 1$, see [2].

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**Example 1.2.** Let $p \in [1, \infty)$ and $\ell_p$ denote the set of all real sequences $\{x_i\}_{i=1}^{\infty}$ with $\Sigma_{i=1}^{\infty} |x_i|^p < \infty$. Define the mapping $\delta_p : \langle \ell_p \rangle \to [0, \infty)$ as

$$\delta_p(A) = \left( \sum_{i=1}^{\infty} \operatorname{diam} \{a_i : a \in A\}^p \right)^{\frac{1}{p}} \qquad (A \in \langle L_p \rangle).$$

Then $\delta_p$ is a diversity on $\ell_p$.

More examples of diversities can be seen in [1, 2, 3, 4, 6].

Diversity embeddings were defined in [2] and then some results on metric embeddings and $L_1$-embeddable metrics were generalized to diversities, leading to some applications in combinatorial optimization and theoretical computer science (see [2, 3, 7]). More precisely, a mapping $f : (X, \delta_X) \to (Y, \delta_Y)$ of diversities is called a diversity embedding if $\delta_X(A) = \delta_Y(f(A))$, for all $A \in \langle X \rangle$. Also, a mapping $\phi : (X, \delta_X) \to (Y, \delta_Y)$ is said to have distortion $c$ if for some $\alpha, \beta > 0$ such that $c = \alpha\beta$ we have

$$\frac{1}{\alpha} \delta_X(A) \leq \delta_Y(\phi(A)) \leq \beta \delta(A), \tag{1.1}$$

for all $A \in \langle X \rangle$. In this case, it is said that $(X, \delta_X)$ embeds in $(Y, \delta_Y)$ with distortion $c$.

## 2. Main results

In this section, embedding of arbitrary finite diversities into $\ell_p$-diversities is given. The case $p = 1$, a special case, is given in [2, Theorem 13]. Let us first see the following lemma.

**Lemma 2.1.** *Let $\delta_{\text{diam}}^{(p)}$ be the diameter diversity derived from the metric space $(\mathbb{R}^m, d_p)$, as $1 \leq p < \infty$. Then, $\delta_{\text{diam}}^{(p)} \leq \delta_p \leq m^{\frac{1}{p}}\, \delta_{\text{diam}}^{(p)}$.*

*Proof.* The first inequality is trivial. For the second one, note that for all $k \in \{1, \cdots, m\}$, there exist $a^k, b^k \in A$ such that $\operatorname{diam}\{a_k : a \in A\} = a_k^k - b_k^k$. We have

$$\delta_p(A)^p = \sum_{k=1}^{m} (a_k^k - b_k^k)^p \leq \sum_{k=1}^{m} \|a^k - b^k\|_p^p \leq m\, \delta_{\text{diam}}^{(p)}(A)^p.$$

□

**Theorem 2.2.** *Every diversity $(X, \delta)$ on $n$ points embeds in the diversities $\ell_p^{\mathcal{O}(\log n)}$ with distortion $\mathcal{O}(n \log^{\frac{1+p}{p}} n)$ as $1 \leq p \leq 2$, and $\ell_p^{\mathcal{O}(\log^2 n)}$ with distortion $\mathcal{O}(n \log^{\frac{2+p}{p}} n)$ as $2 < p < \infty$.*

*Proof.* Let $d$ be the induced metric for $\delta$ and $\delta_{\text{diam}}$ denote the diameter diversity. By [2, Theorem 4], for each finite subset $A$ of $X$, we have

$$\delta_{\text{diam}}(A) \le \delta(A) \le (|A|-1)\delta_{\text{diam}}(A). \tag{2.1}$$

Let $1 \le p \le 2$. According to [5, Theorem 3.2, 3], there is a mapping $\phi : X \to \mathbb{R}^{\mathcal{O}(\log n)}$ such that

$$\frac{1}{\alpha} d(x,y) \le \|\phi(x)-\phi(y)\|_p \le \beta d(x,y), \tag{2.2}$$

where $\alpha\beta = \mathcal{O}(\log n)$. From (2.1), and (2.2), and Lemma 2.1, we have

$$\begin{aligned} \frac{1}{n\alpha}\delta(A) \le \frac{1}{\alpha}\delta_{\text{diam}}(A) &\le \delta^{(p)}_{\text{diam}}(\phi(A)) \le \delta_p(\phi(A)) \\ &\le m^{\frac{1}{p}}\delta^{(p)}_{\text{diam}}(\phi(A)) \le m^{\frac{1}{p}}\beta\delta_{\text{diam}}(A) \le m^{\frac{1}{p}}\beta\delta(A), \end{aligned} \tag{2.3}$$

where $m = \mathcal{O}(\log n)$. We then see that the distortion of the mapping $\phi : (X,\delta) \to \ell_p^{\mathcal{O}(\log n)}$ is $n\alpha m^{\frac{1}{p}}\beta = \mathcal{O}(n \log^{\frac{1+p}{p}} n)$.

Now let $2 < p < \infty$. According to [5, Theorem 3.2, 5], there is a mapping $\phi : X \to \mathbb{R}^{\mathcal{O}(\log^2 n)}$ satisfying (2.2). Thus (2.3) holds for $m = \mathcal{O}(\log^2 n)$. Observe that $n\alpha m^{\frac{1}{p}}\beta = \mathcal{O}(n \log^{\frac{2+p}{p}} n)$, which completes the proof. ☐

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# CLOSED RANGE WEIGHTED COMPOSITION OPERATORS ON THE HARDY AND WEIGHTED BERGMAN SPACES

M. HAJI SHAABANI

*Department of Mathematics, Shiraz University of Technology, P. O. Box 71555-313, Shiraz, Iran.*
*shaabani@sutech.ac.ir*

Abstract. In this paper, we investigate bounded below weighted composition operators $C_{\psi,\varphi}$ on a Hilbert space of analytic functions. Then for $\psi \in H^\infty$ and a univalent map $\varphi$, we characterize all closed range weighted composition operators $C_{\psi,\varphi}$ on $H^2$ and $A^2_\alpha$. Also we show that for $\psi \in H^\infty$ which is bounded away from zero near the unit circle, the weighted composition operator $C_{\psi,\varphi}$ is bounded below on $H^2$ or $A^2_\alpha$ if and only if $C_\varphi$ has closed range.

## 1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. The Hardy space $H^2$ is the set of all analytic functions $f$ on $\mathbb{D}$ such that

$$\|f\|_1^2 = \lim_{r\to 1}\int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

We recall that $H^\infty(\mathbb{D}) = H^\infty$ is the space of all bounded analytic functions defined on $\mathbb{D}$, with supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

Let $dA$ be the normalized area measure in $\mathbb{D}$. The weighted Bergman spaces $A^2_\alpha(\mathbb{D}) = A^2_\alpha$, for $\alpha > -1$, are defined by

$$A^2_\alpha(\mathbb{D}) = \{f \text{ analytic in } \mathbb{D} : ||f||^2_{\alpha+2} = \int_{\mathbb{D}} |f|^2 dA_\alpha < \infty\},$$

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where $dA_\alpha = (\alpha+1)(1-|z|^2)^\alpha dA$. We know that for $\alpha > -1$, $A^2_\alpha$ is a Hilbert space. The case when $\alpha = 0$ is known as the (unweighted) Bergman space, and is often denoted simply $A^2$.

Let $\varphi$ be an analytic map from the open unit disk $\mathbb{D}$ into itself. The operator that takes the analytic map $f$ to $f \circ \varphi$ is a composition operator and is denoted by $C_\varphi$. A natural generalization of a composition operator is an operator that takes $f$ to $\psi \cdot f \circ \varphi$, where $\psi$ is a fixed analytic map on $\mathbb{D}$. This operator is aptly named a weighted composition operator and is usually denoted by $C_{\psi,\varphi}$. More precisely, if $z$ is in the unit disk then $(C_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$. In this paper, we assume that $C_{\psi,\varphi}$ is a bounded operator.

The automorphisms of $\mathbb{D}$, that is, the one-to-one analytic maps of the disk onto itself, are just the functions $\varphi(z) = \lambda \frac{a-z}{1-\bar{a}z}$, where $|\lambda| = 1$ and $|a| < 1$. We denote the class of automorphisms of $\mathbb{D}$ by $\mathrm{Aut}(\mathbb{D})$.

Closed range composition operators were studied on the Hardy and weighted Bergman spaces in [1], [2], [4] and [5]. In the second section, we investigate closed range weighted composition operators. We show that if $C_{\psi,\varphi}$ is bounded below on a Hilbert space of analytic functions, then $C_\varphi$ has closed range. Next, we show that $C_{\psi,\varphi}$ is bounded below on $H^2$ or $A^2_\alpha$ if and only if $C_\varphi$ has closed range, when $\psi \in H^\infty$ is bounded away from zero near the unit circle. In Theorem 2.5, for $\psi \in H^\infty$ and $\varphi$ which is a univalent, holomorphic self-map of $\mathbb{D}$, we determine all closed range operators $C_{\psi,\varphi}$ on $H^2$ and $A^2_\alpha$. In this paper, we state some results of [3].

## 2. Main results

Let $H$ be a Hilbert space. The set of all bounded operators from $H$ into itself is denoted by $B(H)$. We say that an operator $A \in B(H)$ is bounded below if there is a constant $c > 0$ such that $c\|h\| \leq \|A(h)\|$ for all $h \in H$.

In the next lemma, we show that for operators $A$ and $B$ which have closed range, if $A$ is bounded below, then $AB$ has closed range.

**Lemma 2.1.** *Suppose that $A$ and $B$ belong to $B(H)$. Let $A$ and $B$ have closed range. If $A$ is bounded below, then $AB$ has closed range.*

If $f$ is defined on a set $V$ and if there is a positive constant $m$ so that $|f(z)| \geq m$, for all $z$ in $V$, we say $f$ is bounded away from zero on $V$. In particular, we say that $\psi$ is bounded away from zero near the unit circle, that is, there are $\delta > 0$ and $\epsilon > 0$ such that

$$|\psi(z)| > \epsilon \text{ for } \delta < |z| < 1.$$

In Propositions 2.2 and 2.3 and Theorem 2.5, we investigate bounded below weighted composition operators

**Proposition 2.2.** *Suppose that $H$ is a Hilbert space of analytic functions. If $C_{\psi,\varphi}$ is bounded below on $H$, then $C_\varphi$ has closed range.*

Suppose that $\varphi$ is a constant function. Then the range of $\varphi$ on $\mathbb{D}$ misses a neighborhood of the unit circle. [5, Corollary 4.2] and [5, Example 1] imply that $C_\varphi$ does not have a closed range on $H^2$ and $A^2_\alpha$. We use this fact in Theorem 2.3.

**Theorem 2.3.** *Let $\psi \in H^\infty$ be bounded away from zero near the unit circle. The composition operator $C_\varphi$ has closed range if and only if $C_{\psi,\varphi}$ is bounded below on $H^2$ or $A^2_\alpha$.*

In [2, Theorem 5.1], Akeroyd et al. found the following proposition. We use it in the next theorem in order to characterize all closed range weighted composition operators $C_{\psi,\varphi}$ on $H^2$ and $A^2_\alpha$, when $\varphi$ is univalent.

**Proposition 2.4.** *LSuppose that $\varphi$ is a univalent, holomorphic self-map of $\mathbb{D}$. Then $C_\varphi$ has closed range on $H^2$ or $A^2_\alpha$ if and only if $\varphi$ is an automorphism of $\mathbb{D}$.*

If $\psi \equiv 0$, then it is easy to see that $C_{\psi,\varphi}$ has closed range. Therefore, in the following theorem, we assume that $\psi \not\equiv 0$.

**Theorem 2.5.** *Assume that $\varphi$ is a univalent, holomorphic self-map of $\mathbb{D}$. Let $\psi \in H^\infty$ and $\psi \not\equiv 0$. The weighted composition operator $C_{\psi,\varphi}$ has closed range on $H^2$ if and only if $\varphi$ is an automorphism of $\mathbb{D}$ and there exits a constant $m > 0$ such that $|\psi| \geq m$ almost everywhere on $\partial\mathbb{D}$. Moreover, the weighted composition operator $C_{\psi,\varphi}$ has closed range on $A^2_\alpha$ if and only if $\varphi$ is an automorphism of $\mathbb{D}$ and $\psi = hb$, where $h \in H^\infty$ is invertible in $H^\infty$ and $b$ is a finite product of interpolating Blaschke products.*

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Poster Presentation

# WHEN THE ADJOINT OF A WEIGHTED COMPOSITION OPERATOR IS BOUNDED BELOW

M. HAJI SHAABANI

*Department of Mathematics, Shiraz University of Technology, P. O. Box 71555-313, Shiraz, Iran.*
*shaabani@sutech.ac.ir*

Abstract. In this paper, we obtain that $C^*_{\psi,\varphi}$ is bounded below on $H^2$ or $A^2_\alpha$ if and only if $C_{\psi,\varphi}$ is invertible.

## 1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. For $\alpha > -1$, the weighted Bergman space $A^2_\alpha(\mathbb{D}) = A^2_\alpha$ is the set of functions $f$ analytic in $\mathbb{D}$ with

$$\|f\|^2_{\alpha+2} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^\alpha dA(z) < \infty,$$

where $dA$ is the normalized area measure in $\mathbb{D}$. The case when $\alpha = 0$ is known as the (unweighted) Bergman space, and is often denoted simply $A^2$.

The Hardy space, denoted $H^2(\mathbb{D}) = H^2$, is the set of all analytic functions $f$ on $\mathbb{D}$, satisfying the norm condition

$$\|f\|^2_1 = \lim_{r\to 1}\int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

The space $H^\infty(\mathbb{D}) = H^\infty$ consists of all the functions that are analytic and bounded on $\mathbb{D}$, with supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

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Let $\varphi$ be an analytic map from the open unit disk $\mathbb{D}$ into itself. The operator that takes the analytic map $f$ to $f \circ \varphi$ is a composition operator and is denoted by $C_\varphi$. A natural generalization of a composition operator is an operator that takes $f$ to $\psi \cdot f \circ \varphi$, where $\psi$ is a fixed analytic map on $\mathbb{D}$. This operator is aptly named a weighted composition operator and is usually denoted by $C_{\psi,\varphi}$. More precisely, if $z$ is in the unit disk then $(C_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$.

Suppose that $H$ and $H'$ are Hilbert spaces and $A : H \to H'$ is a bounded operator. The operator $A$ is said to be left semi-Fredholm if there is a bounded operator $B : H' \to H$ and a compact operator $K$ on $H$ such that $BA = I + K$. Analogously, $A$ is right semi-Fredholm if there is a bounded operator $B' : H' \to H$ and a compact operator $K'$ on $H'$ such that $AB' = I + K'$. An operator $A$ is said to be Fredholm if it is both left and right semi-Fredholm. It is not hard to see that $A$ is left semi-Fredholm if and only if $A^*$ is right semi-Fredholm. Hence $A$ is Fredholm if and only if $A^*$ is Fredholm. Note that an invertible operator is Fredholm. By using the definition of Fredholm operator, it is not hard to see that if the operators $A$ and $B$ are Fredholm on a Hilbert space $H$, then $AB$ is also Fredholm on $H$.

The automorphisms of $\mathbb{D}$, that is, the one-to-one analytic maps of the disk onto itself, are just the functions $\varphi(z) = \lambda\frac{a-z}{1-\bar{a}z}$, where $|\lambda| = 1$ and $|a| < 1$. We denote the class of automorphisms of $\mathbb{D}$ by $\mathrm{Aut}(\mathbb{D})$. Automorphisms of $\mathbb{D}$ take $\partial\mathbb{D}$ onto $\partial\mathbb{D}$. It is known that $C_\varphi$ is Fredholm on the Hardy space if and only if $\varphi \in \mathrm{Aut}(\mathbb{D})$ (see [1]).

In the second section, we investigate Fredholm and invertible weighted composition operators. In Theorem 2.7, we show that the operator $C^*_{\psi,\varphi}$ is bounded below on $H^2$ or $A^2_\alpha$ if and only if $C_{\psi,\varphi}$ is invertible. In this paper, we state some results of [4].

## 2. Main results

Let $H$ be a Hilbert space. The set of all bounded operators from $H$ into itself is denoted by $B(H)$. We say that an operator $A \in B(H)$ is bounded below if there is a constant $c > 0$ such that $c\|h\| \leq \|A(h)\|$ for all $h \in H$.

If $f$ is defined on a set $V$ and if there is a positive constant $m$ so that $|f(z)| \geq m$, for all $z$ in $V$, we say $f$ is bounded away from zero on $V$. In particular, we say that $\psi$ is bounded away from zero near the unit circle, that is, there are $\delta > 0$ and $\epsilon > 0$ such that

$$|\psi(z)| > \epsilon \text{ for } \delta < |z| < 1.$$

Now we state the following simple and well-known lemma, and we frequently use it in this paper.

**Lemma 2.1.** *Let $C_{\psi,\varphi}$ be a bounded operator on $H^2$ or $A^2_\alpha$. Then, for each $w \in \mathbb{D}$, $C^*_{\psi,\varphi} K_w = \overline{\psi(w)} K_{\varphi(w)}$.*

**Lemma 2.2.** *Suppose that $A$ and $B$ are two bounded operators on a Hilbert space $H$. If $AB$ is a Fredholm operator, then $B$ is left semi-Fredholm.*

Zhao in [5] characterized Fredholm weighted composition operators on $H^2$. Also Zhao in [6] found necessary conditions of $\varphi$ and $\psi$ for a weighted composition operator $C_{\psi,\varphi}$ on $A^2_\alpha$ to be Fredholm. In the following proposition, we obtain a necessary and sufficient condition for $C_{\psi,\varphi}$ to be Fredholm on $H^2$ and $A^2_\alpha$. The idea of the proof of the next proposition is different from [5] and [6].

**Proposition 2.3.** *The operator $C^*_{\psi,\varphi}$ is left semi-Fredholm on $H^2$ or $A^2_\alpha$ if and only if $\varphi \in Aut(\mathbb{D})$ and $\psi \in H^\infty$ is bounded away from zero near the unit circle. Under this conditions $C_{\psi,\varphi}$ is a Fredholm operator.*

In the next proposition, we find a necessary condition of $\psi$ for an operator $C^*_{\psi,\varphi}$ to be bounded below on $H^2$ and $A^2_\alpha$. Then we use Proposition 2.4 in order to obtain all invertible weighted composition operators on $H^2$ and $A^2_\alpha$.

**Proposition 2.4.** *Let $\psi$ be an analytic map of $\mathbb{D}$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. If $C^*_{\psi,\varphi}$ is bounded below on $H^2$ or $A^2_\alpha$, then $\psi \in H^\infty$ is bounded away from zero on $\mathbb{D}$ and $\varphi \in Aut(\mathbb{D})$.*

Bourdon in [2, Theorem 3.4] obtained the following corollary; we give another proof (see also [3, Theorem 2.0.1]).

**Corollary 2.5.** *Let $\psi$ be an analytic map of $\mathbb{D}$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. The weighted composition operator $C_{\psi,\varphi}$ is invertible on $H^2$ or $A^2_\alpha$ if and only if $\varphi \in Aut(\mathbb{D})$ and $\psi \in H^\infty$ is bounded away from zero on $\mathbb{D}$.*

Note that if $C_{\psi,\varphi}$ is invertible, then $C^*_{\psi,\varphi}$ is bounded below. Hence by Proposition 2.4 and Corollary 2.5, we can see that $C^*_{\psi,\varphi}$ is bounded below if and only if $C_{\psi,\varphi}$ is invertible.

The algebra $A(\mathbb{D})$ consists of all continuous functions on the closure of $\mathbb{D}$ that are analytic on $\mathbb{D}$. In the next corollary, we find some Fredholm weighted composition operators which are not invertible.

**Corollary 2.6.** *Suppose that $\varphi \in Aut(\mathbb{D})$ and $\psi \in A(\mathbb{D})$. Assume that $\{z \in \mathbb{D} : \psi(z) = 0\}$ is a nonempty finite set and for each $z \in \partial\mathbb{D}$, $\psi(z) \neq 0$. Then $C_{\psi,\varphi}$ is Fredholm, but it is not invertible.*

**Theorem 2.7.** *Suppose that $\psi$ is an analytic map of $\mathbb{D}$ and $\varphi$ is an analytic self-map of $\mathbb{D}$. The operator $C^*_{\psi,\varphi}$ is bounded below on $H^2$ or $A^2_\alpha$ if and only if $\varphi \in Aut(\mathbb{D})$ and $\psi \in H^\infty$ is bounded away from zero on $\mathbb{D}$.*

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# NOTES ON THE PRODUCT OF COMPOSITION AND DIFFERENTIATION OPERATORS

S. HASANNEJAD* AND S. JAHEDI

*Department of Mathematics, Shiraz University of Technology, Shiraz 71557-13876, Iran*
*S.hasannejad@sutech.ac.ir*
*jahedi@sutech.ac.ir*

Abstract. In this paper, we study on the boundedness and compactness of a certain type of the product of composition and differentiation operators on the weighted Hardy spaces. We also determine its spectra and an expression to its adjoint.

## 1. Introduction

Let $\beta = \{\beta(n)\}_{n\in\mathbb{N}\cup\{0\}}$ be a sequence of positive real numbers with $\beta(0) = 1$ and $1 \leq p < \infty$. Denote the weighted Hardy space $H^p(\beta)$ by the space of formal power series

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$$

such that

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

Let $\hat{f}_k(n) = \delta_k(n)$. Then $f_k(z) = z^k$ and $\{f_k\}_{k\in\mathbb{N}\cup\{0\}}$ is a basis for $H^p(\beta)$. The weighted Hardy space $H^p(\beta)$, $1 \leq p < \infty$, is a reflexive

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Banach space with the $\|.\|_\beta$ and its dual is $H^q(\beta^{p/q})$ where $\frac{1}{p}+\frac{1}{q}=1$ and $\beta^{p/q}=\{\beta(n)^{p/q}\}_n$. The classical Hardy, Bergman and Dirichlet spaces are examples of weighted Hardy spaces, where $\beta(n)=1$, $\beta(n)=(n+1)^{-1/2}$ and $\beta(n)=(n+1)^{1/2}$, respectively.

If $p=2$, $H^2(\beta)$ is a Hilbert space with the inner product

$$\langle f\ ,\ g\rangle=\sum_{n=0}^{\infty}\hat{f}(n)\overline{\hat{g}(n)}\beta(n)^2$$

for all $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$ and $g(z)=\sum_{n=0}^{\infty}\hat{g}(n)z^n$ in $H^2(\beta)$. If the weighted sequence $\{\beta(n)\}$ satisfies

$$\lim_{n\to\infty}\beta(n)^{\frac{1}{n}}=\infty \qquad (1)$$

then the elements of $H^2(\beta)$ are entire functions.

Let $\varphi:\mathbb{C}\longrightarrow\mathbb{C}$ be any analytic map. The composition operator $C_\varphi$ on $H^2(\beta)$ is defined by

$$C_\varphi(f)=f\circ\varphi.$$

A number of researchers have studied on the relation between operator theoretic properties of $C_\varphi$ and the function theoretic of $\varphi$. For more information, the reader can refer to [1, 4, 5, 6].

Let $D$ be the differentiation operator defined by

$$Df(z)=f^{'}(z)$$

for any analytic function $f$ on the complex plane. Note that the operator $D$ is not necessarily bounded on any analytic function space. Since $H^2(\beta)$ is a functional Hilbert space, $D_\varphi$ is bounded on $H^2(\beta)$ if and only if $D_\varphi$ maps $H^2(\beta)$ into itself.

The product of composition operator $C_\varphi$ and differentiation operator $D$, $C_\varphi D$, have been studied by many authors, for example see [2, 3]. As in [2], we denote the product of composition operator and differentiation operator $C_\varphi D$ by $D_\varphi$ on $H^2(\beta)$ is given by

$$D_\varphi f=f^{'}\circ\varphi.$$

## 2. Main results

Let $\varphi(z)=az+b,\ a,\ b\in\mathbb{C}$. In the case $a=0,\ b\neq 0$ for $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n\in H^2(\beta)$ we have

$$D_\varphi f(z)=\sum_{n=1}^{\infty}n\hat{f}(n)b^{n-1}$$

and obviousely

$$\|D_\varphi f\|^2 \leq \|f\|^2 \sum_{n=1}^{\infty} \frac{n^2|b|^{2(n-1)}}{\beta(n)^2}.$$

Using root test and condition (1), the above right hand series is convergent and so the operator $D_\varphi$ is bounded on $H^2(\beta)$.

If $a,\ b \in \mathbb{C} \setminus \{0\},\ \varphi(z) = az + b$, then for $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$,

$$\begin{aligned} D_\varphi f(z) &= \sum_{n=1}^{\infty} n\hat{f}(n)(az+b)^{n-1} \\ &= \sum_{k=0}^{\infty} \sum_{n=k+1}^{\infty} \binom{n-1}{k} n\hat{f}(n) a^k b^{n-1-k} z^k. \end{aligned}$$

**Theorem 2.1.** *Let $\varphi(z) = az + b$ for some $a, b \in \mathbb{C} \setminus \{0\}$ with $|a| < 1$. Then the operator $D_\varphi$ is bounded on $H^2(\beta)$ if the sequence $\{n\frac{\beta(n-1)}{\beta(n)}\}_{n\in\mathbb{N}}$ is bounded.*

*In the special case, if $b = 0$, then the operator $D_\varphi$ is bounded on $H^2(\beta)$ if and only if the sequence $\{\frac{n|a|^{n-1}\beta(n-1)}{\beta(n)}\}_{n\in\mathbb{N}}$ is bounded.*

**Theorem 2.2.** *Let $\varphi(z) = az$ for some $a \in \mathbb{C} \setminus \{0\}$. Then the operator $D_\varphi$ is compact on $H^2(\beta)$ if and only if*

$$\lim_{n\to\infty} \frac{n|a|^{n-1}\beta(n-1)}{\beta(n)} = 0.$$

**Proposition 2.3.** *Let $\varphi(z) = az$ for some $a \in \mathbb{C} \setminus \{0\}$. Suppose that the sequence $\{(n+1)|a|^n \frac{\beta(n)}{\beta(n+1)}\}_{n\in\mathbb{N}\cup\{0\}}$ is monotonically decreasing and converges to a real number $k$. Then the spectrum of $D_\varphi$ is contained in the closed ball with the radius $k$.*

Assume that $\varphi(z) = az+b$, for some $a,\ b \in \mathbb{C}$ and $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ belongs to the domain of $D_\varphi$. So

$$f^{(k)}(b) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} \hat{f}(n) b^{n-k}.$$

By Cauchy-Schwarz's inequality and the root test one can conclude that the mapping $f \mapsto f^{(k)}(b)$ is a bounded linear functional of $H^2(\beta)$.

This shows that the domain of the adjoint of $D_\varphi$ contains all holomorphic polynomials. So we have the following result.

**Proposition 2.4.** *If $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ belongs to the domain of $D_\varphi^*$, then $D_\varphi^* f(z) = \sum_{n=0}^{\infty} \hat{f}_*(n)z^n$ where $\hat{f}_*(0) = 0$ and for $n \geq 1$*

$$\hat{f}_*(n) = \sum_{k=0}^{n-1} \frac{n!}{k!(n-k-1)!} \hat{f}(k)\bar{a}^k \bar{b}^{n-1-k} (\frac{\beta(k)}{\beta(n)})^2.$$

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# PRODUCTS OF DIFFERENTIATION AND COMPOSITION OPERATORS BETWEEN ZYGMUND TYPE SPACES

MOSTAFA HASSANLOU* AND AMIR H. SANATPOUR

*Engineering Faculty of Khoy, Urmia University, Urmia, Iran*
*m.hassanlou@urmia.ac.ir*
*Department of Mathematics, Kharazmi University, Tehran, Iran*
*a_sanatpour@khu.ac.ir; a.sanatpour@gmail.com*

Abstract. We give necessary and sufficient conditions for the compactness of products of differentiation and composition operators between Zygmund type spaces.

## 1. Introduction

Let $\mathbb{D}$ denote the open unit ball of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denote the space of all complex-valued analytic functions on $\mathbb{D}$. For $0 < \alpha < \infty$, a function $f \in H(\mathbb{D})$ belongs to *Bloch type space* $\mathcal{B}_\alpha$ if

$$\|f\|_{s\mathcal{B}_\alpha} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f'(z)| < \infty.$$

The space $\mathcal{B}_\alpha$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{B}_\alpha} = |f(0)| + \|f\|_{s\mathcal{B}_\alpha},$$

for each $f \in \mathcal{B}_\alpha$. The *little Bloch type space* $\mathcal{B}_{\alpha,0}$ is the closed subspace of $\mathcal{B}_\alpha$ consists of those functions $f \in \mathcal{B}_\alpha$ satisfying

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f'(z)| = 0.$$

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The classic *Zygmund space* $\mathcal{Z}$ consists of all functions $f \in H(\mathbb{D})$ which are continuous on the closed unit ball $\overline{\mathbb{D}}$ and

$$\sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty,$$

where the supremum is taken over all $\theta \in \mathbb{R}$ and $h > 0$. By [1, Theorem 5.3], an analytic function $f$ belongs to $\mathcal{Z}$ if and only if $\sup_{z\in\mathbb{D}}(1 - |z|^2)|f''(z)| < \infty$. Motivated by this, for each $0 < \alpha < \infty$, the *Zygmund type space* $\mathcal{Z}_\alpha$ is defined to be the space of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{s\mathcal{Z}_\alpha} = \sup_{z\in\mathbb{D}}(1 - |z|^2)^\alpha |f''(z)| < \infty.$$

The space $\mathcal{Z}_\alpha$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{Z}_\alpha} = |f(0)| + |f'(0)| + \|f\|_{s\mathcal{Z}_\alpha},$$

for each $f \in \mathcal{Z}_\alpha$. The *little Zygmund type space* $\mathcal{Z}_{\alpha,0}$ is the closed subspace of $\mathcal{Z}_\alpha$ consists of those functions $f \in \mathcal{Z}_\alpha$ satisfying

$$\lim_{|z|\to 1}(1 - |z|^2)^\alpha |f''(z)| = 0.$$

The *essential norm* of a bounded operator $T : X \to Y$ between Banach spaces $X$ and $Y$ is defined by

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\|_{X\to Y} : K \in \mathcal{K}(X, Y)\},$$

where $\mathcal{K}(X, Y)$ is the space of all compact operators $K : X \to Y$.

For each non-negative integer $k$, the *generalized weighted composition operator* $D^k_{\varphi,u}$ is defined by

$$D^k_{\varphi,u} f(z) = u(z)f^{(k)}(\varphi(z)),$$

for each $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. The class of generalized weighted composition operators include weighted composition operators $uC_\varphi = D^0_{\varphi,u}$, *composition operators followed by differentiation* $DC_\varphi = D^1_{\varphi,\varphi'}$ and *composition operators proceeded by differentiation* $C_\varphi D = D^1_{\varphi,1}$ [2]. Also, weighted types of operators $DC_\varphi$ and $C_\varphi D$ are of the form $D^k_{\varphi,u}$, that is $uDC_\varphi = D^1_{\varphi,u\varphi'}$ and $uC_\varphi D = D^1_{\varphi,u}$ [3]. The aim of this paper is to investigate compactness of products of differentiation and composition operators between Zygmund type spaces.

## 2. Main results

It is known that for each $n \geq 2$ and $0 < \alpha < \infty$ we have

$$|f^{(n)}(z)| \leq \frac{\|f\|_{\mathcal{B}_\alpha}}{(1 - |z|^2)^{\alpha+n-1}},$$

for all $f \in \mathcal{B}_\alpha$ and $z \in \mathbb{D}$ (see [5]). Therefore, for each $n \geq 2$ and $0 < \alpha < \infty$ we have

$$|f^{(n+1)}(z)| \leq \frac{\|f\|_{\mathcal{Z}_\alpha}}{(1-|z|^2)^{\alpha+n-1}}, \tag{2.1}$$

for all $f \in \mathcal{Z}_\alpha$ and $z \in \mathbb{D}$. Note that, by the definition of Zygmund type spaces, it is clear that (2.1) also holds in the case of $n = 1$. Before stating the main results of this section, we define the following test functions for each $0 < \alpha < \infty$ and $z, a \in \mathbb{D}$.

$$f_a(z) = \frac{(1-|a|^2)^2}{(1-\overline{a}z)^\alpha}, \quad g_a(z) = \frac{(1-|a|^2)^3}{(1-\overline{a}z)^{\alpha+1}}, \quad h_a(z) = \frac{(1-|a|^2)^4}{(1-\overline{a}z)^{\alpha+2}}.$$

**Theorem 2.1.** *Let $0 < \alpha, \beta < \infty$ and $n \geq 1$ with $(n, \alpha) \neq (1, 1)$. If $D^n_{\varphi,u} : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is a bounded operator, then the following statements are equivalent:*

(*i*) $D^n_{\varphi,u} : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ *is compact.*

(*ii*) $\limsup_{j\to\infty} j^{\alpha-2}\|D^n_{\varphi,u} I^{j+1}\|_{\mathcal{Z}_\beta} = 0$.

(*iii*) $\limsup_{|a|\to 1} \|D^n_{\varphi,u} f_a\|_{\mathcal{Z}_\beta} = 0$, $\quad \limsup_{|a|\to 1} \|D^n_{\varphi,u} g_a\|_{\mathcal{Z}_\beta} = 0$,
$\limsup_{|a|\to 1} \|D^n_{\varphi,u} h_a\|_{\mathcal{Z}_\beta} = 0$.

(*iv*) $\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+n-2}} |u''(z)| = 0$,
$\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+n-1}} |2u'(z)\varphi'(z) + u(z)\varphi''(z)| = 0$,
$\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+n}} |u(z)\varphi'^2(z)| = 0$.

By applying Theorem 2.1, we get the following results for the compactness of products of differentiation and composition operators $DC_\varphi : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ and $C_\varphi D : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ in the case of $\alpha \neq 1$.

**Corollary 2.2.** *Let $0 < \alpha, \beta < \infty$ with $\alpha \neq 1$. If $DC_\varphi : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is a bounded operator, then the following statements are equivalent:*

(*i*) $DC_\varphi : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ *is compact.*

(*ii*) $\limsup_{j\to\infty} j^{\alpha-2}\|\varphi'(I^{j+1})' \circ \varphi\|_{\mathcal{Z}_\beta} = 0$.

(*iii*) $\limsup_{|a|\to 1} \|\varphi' f'_a \circ \varphi\|_{\mathcal{Z}_\beta} = 0$, $\quad \limsup_{|a|\to 1} \|\varphi' g'_a \circ \varphi\|_{\mathcal{Z}_\beta} = 0$,
$\limsup_{|a|\to 1} \|\varphi' h'_a \circ \varphi\|_{\mathcal{Z}_\beta} = 0$.

(*iv*) $\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-2}} |\varphi'''(z)| = 0$,
$\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha}} |3\varphi''(z)\varphi'(z)| = 0$,
$\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} |\varphi'^3(z)| = 0$.

**Corollary 2.3.** *Let $0 < \alpha, \beta < \infty$ with $\alpha \neq 1$. If $C_\varphi D : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is a bounded operator, then the following statements are equivalent:*

(*i*) $C_\varphi D : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ *is compact.*

(*ii*) $\limsup_{j\to\infty} j^{\alpha-2}\|(I^{j+1})' \circ \varphi\|_{\mathcal{Z}_\beta} = 0$.

(*iii*) $\limsup_{|a|\to 1} \|f'_a \circ \varphi\|_{\mathcal{Z}_\beta} = 0, \quad \limsup_{|a|\to 1} \|g'_a \circ \varphi\|_{\mathcal{Z}_\beta} = 0,$
$\limsup_{|a|\to 1} \|h'_a \circ \varphi\|_{\mathcal{Z}_\beta} = 0.$

(*iv*) $\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |\varphi''(z)| = 0,$
$\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} |\varphi'^2(z)| = 0.$

We next give the result of Theorem 2.1 in the case of $(n, \alpha) = (1, 1)$.

**Theorem 2.4.** *Let* $0 < \beta < \infty$ *and* $D^1_{\varphi,u} : \mathcal{Z} \to \mathcal{Z}_\beta$ *be a bounded operator. Then,* $D^1_{\varphi,u} : \mathcal{Z} \to \mathcal{Z}_\beta$ *is compact if and only if*

(*i*) $\limsup_{|\varphi(z)|\to 1} (1-|z|^2)^\beta |u''(z)| \log \frac{2}{(1-|\varphi(z)|^2)^\alpha} = 0,$

(*ii*) $\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2} |2u'(z)\varphi'(z) + u(z)\varphi''(z)| = 0,$

(*iii*) $\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^2} |u(z)\varphi'^2(z)| = 0.$

Finally, it is worth mentioning that by applying Theorem 2.4 for the operators $DC_\varphi = D^1_{\varphi,\varphi'} : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ and $C_\varphi D = D^1_{\varphi,1} : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$, one can get criteria for the compactness of such operators as in Corollary 2.2 and Corollary 2.3.

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# INFINITELY MANY SOLUTIONS FOR A $(p_1, \cdots, p_n)$-BIHARMONIC SYSTEM

S. HEIDARI* AND A. RAZANI

*Department of Pure Mathematics, Faculty of Science,*
*Imam Khomeini International University,*
*P. O. Box 34149-16818, Qazvin, Iran.*
*S.Heidari@edu.ikiu.ac.ir and razani@sci.ikiu.ac.ir*

Abstract. In this paper, we prove the existence of infinitely many weak solutions for the $(p_1, \cdots, p_n)$-biharmonic type systems with Hardy potential. The main tool is an infinitely many critical points theorem.

## 1. Introduction

The purpose of the present paper is to establish the existence of infinitely many weak solutions for the following $(p_1, \cdots, p_n)$-biharmonic singular type systems

$$\begin{cases} \Delta^2_{p_i} u_i + a_i(x)\dfrac{|u_i|^{q_i-2}u_i}{|x|^{2q_i}} = \lambda F_{u_i}(x, u_1, \cdots, u_n) & \text{in} \quad \Omega, \\ u_i = \Delta u_i = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

for $1 \leq i \leq n$, where $\Delta^2_{p_i} u_i = \Delta(|\Delta u_i|^{p_i-2}\Delta u_i)$ is the $p_i$-biharmonic operator, $1 < q_i < \frac{N}{2} < p_i$ for $i = 1, \cdots, n$, $\Omega$ is a bounded domain in $\mathbb{R}^N (N \geq 3)$, containing the origin and with smooth boundary $\partial\Omega$, $a_i \in L^\infty(\Omega)$ such that $a_i := ess_\Omega \inf a_i(x) > 0$ for $i = 1, \cdots, n$. $\lambda \in (0, \infty)$ and $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a function such that the mapping $(t_1, t_2, \cdots, t_n) \to F(x, t_1, t_2, \cdots, t_n)$ is in $C^1$ in $\mathbb{R}^n$ for all $x \in \Omega$, $F_{t_i}$ is continuous in $\Omega \times \mathbb{R}^n$ for $i = 1, \cdots, n$, and $F_{t_i}$ denotes the partial

*Key words and phrases.* Infinitely many solutions, variational methods, Hardy potential, $(p_1, \cdots, p_n)$-biharmonic system.

derivative of $F$ with respect to $t_i$.
The investigation of singular elliptic problems has drawn the attention of many authors (for example [2, 3, 4]), because it applies in some parts of science such that boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, chemical catalyst kinetics, and theory of heat conduction in electrically conducting materials. These kinds of problems also appear in glacial advance, in transport of coal slurries down conveyor belts and in some other geophysical and industrial contents.
Let $X = \prod_{i=1}^{n} W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega)$ endow with the norm

$$\|(u_1, \cdots, u_n)\|_X = \sum_{i=1}^{n} \|u_i\|,$$

where $\|u_i\| = \big(\int_\Omega |\Delta u_i|^{p_i} dx\big)^{\frac{1}{p_i}}$ for $1 \le i \le n$.
We recall Rellich inequality [5], which says that

$$\int_\Omega \frac{|u(x)|^s}{|x|^{2s}} dx \le \frac{1}{H} \int_\Omega |\Delta u(x)|^s dx, \quad \forall u \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega) \tag{1.2}$$

where $1 < s < \frac{N}{2}$ and $H := (\dfrac{N(s-1)(N-2s)}{s^2})^s$ .

The embedding $W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ is compact whenever $p_i > \frac{N}{2}$, for $i = 1, 2, \cdots, n$. So we have the compact embeding $X \to C(\bar{\Omega}) \times \cdots \times C(\bar{\Omega})$ and there is a constant $\kappa > 0$ such that

$$\kappa = \max\{ \sup_{u_i \in (W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega)) \setminus \{0\}} \frac{\max_{x \in \bar{\Omega}} |u_i(x)|^{p_i}}{\|u_i\|^{p_i}} : \text{ for } 1 \le i \le n\} < +\infty.$$

Let $p_* = \min\{p_i \,; i = 1, 2, \cdots, n\}$ and for all $\nu > 0$ we define

$$\mathcal{Q}(\nu) := \{(t_1, \cdots, t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} |t_i| \le \nu\}.$$

We consider the functional $g_\lambda : X \to \mathbb{R}$ by $g_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ for each $u = (u_1, \cdots, u_n) \in X$, where the functionals $\Phi, \Psi : X \to \mathbb{R}$, as follows

$$\Phi(u) := \sum_{i=1}^{n} \frac{1}{p_i} \int_\Omega |\Delta u_i(x)|^{p_i} dx + \sum_{i=1}^{n} \frac{1}{q_i} \int_\Omega a_i(x) \frac{|u_i(x)|^{q_i}}{|x|^{2q_i}} dx,$$

$$\Psi(u) := \int_\Omega F(x, u_1(x), \cdots, u_n(x)) dx.$$

By (1.2) we have

$$\sum_{i=1}^{n}\frac{\|u_i\|^{p_i}}{p_i}\leq \Phi(u)\leq \sum_{i=1}^{n}\frac{\|u_i\|^{p_i}}{p_i}+\sum_{i=1}^{n}\frac{\|a_i\|_\infty}{q_iH}\int_\Omega |\Delta u_i(x)|^{q_i}dx,$$

for every $u\in X$, so $\Phi$ is well defined, coercive and sequentially weakly lower semicontinuous. Moreover $\Psi$ is sequentially weakly upper semicontinuous.

For fixed $x_0\in\Omega$, set $R_2>R_1>0$ such that $B(x_0,R_2)\subset\Omega$, where $B(x_0,R_2)$ denotes the ball with center at $x_0$ and radius $R_2$ and $\overline{B(x_0,R_2)}$ not containing the origin. Set

$$\begin{aligned}
\mathcal{L}_{p_i} &:= \frac{\Gamma(1+\frac{N}{2})}{\left(\sum_{i=1}^{n}(\kappa p_i)^{\frac{1}{p_i}}\right)^{p_*}\pi^{\frac{N}{2}}}\left(\frac{R_2^2-R_1^2}{2N}\right)^{p_i}\frac{1}{R_2^N-R_1^N}\\
\mathcal{L}_{q_i} &:= \frac{\Gamma(1+\frac{N}{2})}{\left(\sum_{i=1}^{n}(\kappa p_i)^{\frac{1}{p_i}}\right)^{p_*}\pi^{\frac{N}{2}}}\left(\frac{R_2^2-R_1^2}{2N}\right)^{q_i}\frac{1}{R_2^N-R_1^N}
\end{aligned}\tag{1.3}$$

for every $i=1,\cdots,n$.

## 2. Multiple solutions

Our goal is to prove the existence of infinitely many solutions for the problem (1.1). Due do this, we introduce the suitable hypothesis and establish an open interval of positive parameters such that the problem (1.1) admits infinitely many weak solutions.

The main tool used to prove our multiplicity result is the critical point theorem of Bonanno (see Bonanno and Molica Bisci [1]). We set

$$\varphi(r):=\inf_{u\in\Phi^{-1}(-\infty,r)}\frac{(\sup_{v\in\Phi^{-1}(-\infty,r)}\Psi(v))-\Psi(u)}{r-\Phi(u)}$$

$$\gamma:=\liminf_{r\to+\infty}\varphi(r),\qquad \delta:=\liminf_{r\to(\inf_X\Phi)^+}\varphi(r).$$

In the first case, we show that $\gamma<\infty$, so we obtain an unbounded sequence of solutions (Theorem 2.1), in the second case, we show that $\delta<\infty$, so we obtain a sequence of non-zero solutions strongly converging at zero (Theorem 2.2).

**Theorem 2.1.** *Assume that*

(A1) $F(x,t_1,\cdots,t_n)\geq 0$ *for each* $(x,t_1,\cdots,t_n)\in\Omega\times\mathbb{R}^n_+$.

(A2) *there exist* $x_0 \in \Omega$, $0 < R_1 < R_2$ *as considered in* (1.3) *such that* $\mathcal{A} < \mathcal{L}\mathcal{B}$, *where* $\mathcal{L} := \min\{\mathcal{L}_{p_i},\ , i = 1, 2, \cdots, n\}$,

$$\mathcal{A} := \liminf_{\xi \to +\infty} \frac{\int_\Omega \sup_{(t_1,\cdots,t_n)\in \mathcal{Q}(\xi)} F(x, t_1, \cdots, t_n)dx}{\xi^{p_*}},$$

$$\mathcal{B} := \limsup_{(t_1,\cdots,t_n)\to(+\infty,\cdots,+\infty)} \frac{\int_{B(x_0,R_1)} F(x, t_1, \cdots, t_n)dx}{\sum_{i=1}^n \frac{t_i^{p_i}}{q_i}}.$$

*Then for each* $\lambda \in \Lambda := \frac{1}{\left(\sum_{i=1}^n (\kappa p_i)^{\frac{1}{p_i}}\right)^{p_*}} \left]\frac{1}{\mathcal{L}\mathcal{B}}, \frac{1}{\mathcal{A}}\right[,$

*the system* (1.1) *has an unbounded sequence of weak solutions in* $X$.

**Theorem 2.2.** *Assume that* (*A*1) *holds and*

(A3) $F(x, 0, 0, \cdots, 0) = 0$ *for* $x \in \Omega$.

(A4) *There exist* $x_0 \in \Omega, 0 < R_1 < R_2$ *as considered in* (1.3) *such that,* $\mathcal{E} < \mathcal{L}'\mathcal{F}$, *where* $\mathcal{L}' := \min\{\mathcal{L}_{q_i},\ , i = 1, 2, \cdots, n\}$ *and*

$$\mathcal{E} := \liminf_{\xi \to 0^+} \frac{\int_\Omega \sup_{(t_1,\cdots,t_n)\in \mathcal{Q}(\xi)} F(x, t_1, \cdots, t_n)dx}{\xi^{p_*}},$$

$$\mathcal{F} := \limsup_{(t_1,\cdots,t_n)\to(0^+,\cdots,0^+)} \frac{\int_{B(x_0,R_1)} F(x, t_1, \cdots, t_n)dx}{\sum_{i=1}^n \frac{t_i^{q_i}}{q_i}}.$$

*Then for each* $\lambda \in \Lambda' := \frac{1}{\left(\sum_{i=1}^n (\kappa p_i)^{\frac{1}{p_i}}\right)^{p_*}} \left]\frac{1}{\mathcal{L}'\mathcal{F}}, \frac{1}{\mathcal{E}}\right[,$

*problem* (1.1) *admits a sequence* $(u_n)$ *of weak solutions which converges to zero.*

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Oral Presentation

# PYTHAGOREAN RELATION FOR OPERATOR

M. T. HEYDARI

*Department of Mathematics, College of Sciences, Yasouj University, Yasouj, 75918, Iran*
*heydari@yu.ac.ir*

Abstract. Let $T$ be a bounded linear operator on Hilbert space $H$. Then there exists a unique $z_0 \in \mathbb{C}$, such that $\|T-z_0\|^2+|\lambda|^2 \leq \|(T-z_0)+\lambda\|^2$ for all $\lambda \in \mathbb{C}$.

## 1. Introduction

Let $H$ be a complex Hilbert space with inner product $< .,. >$ and the norm $\|.\|$. Let $B(H)$ be the set of all bounded linear operators on $H$ and $T \in B(H)$. In [4], Stampfli introduced the concept of maximal numerical range. The maximal numerical range of $T$ is defined to be the set,

$$W_0(T) := \{\lambda \in \mathbb{C} : \langle Tx_n, x_n\rangle \to \lambda,\ \|x_n\| = 1 \ \ and \ \ \|Tx_n\| \to \|T\|\},$$

and the normalized maximal numerical range is given by

$$W_N(T) = \begin{cases} W_0(T/\|T\|) & if\ T \neq 0, \\ 0 & if\ T = 0. \end{cases} \tag{1.1}$$

When $H$ is finite dimensional, $W_0(T)$ corresponds to the numerical range produced by the maximal vectors (vectors $x$ such that $\|x\| = 1$ and $\|Tx\| = \|T\|$ ) . In [1], the author have shown that:

$$W_0(T) := \{\Lambda(T) : \Lambda \in B(H)^*, \Lambda(I) = 1 = \|\Lambda\| \ \ and \ \ \Lambda(T^*T) = \|T\|^2\},$$

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and the set is a non-empty, closed and convex set. Hamed and Spitkovsky in [2] proved that the maximal numerical range $W_0(A)$ of a matrix A is the (regular) numerical range $W(B)$ of its compression $B$ onto the eigenspace $L$ of $A^*A$ corresponding to its maximal eigenvalue. Some conditions under which $W_0(A)$ has a non-empty intersection with the boundary of W(A) were established, in particular, when $W_0(A) = W(A)$. The set $W_0(A)$ is also described explicitly for matrices unitarily similar to direct sums of 2-by-2 blocks, and some insight into the behavior of $W_0(A)$ was provided when $L$ has codimension one. Recently, Spitkovsky in [3] proved that the maximal numerical range of an operator has a non-empty intersection with the boundary of its numerical range if and only if the operator is normaloid (An operator $T$ is said to be normaloid if $\|T\|$ = spectral radius of $T$). Stampfli, though using it as a powerful tool in determining the norm of a derivation, seemed not to pay much interest to it, but instead treated it casually. It is nevertheless remarkable to note that Fong(l979) considered the essential maximal numerical range as an independent subject. The work of Sheth and Duggal(l984) is also worth being acknowledged. They initiated the study of maximal numerical range as a subject in itself. Other major contributors in this field are Khan(l988), who revisited the idea of essential maximal numerical range; and Cho(l988), whose result about the joint maximal numerical range is of particular importance in this area. One can clearly see from the foregoing that the maximal numerical range is still a virgin area of study as opposed to the usual numerical range. This project is therefore geared towards providing a curtain raiser for future research in the area of maximal numerical range.

## 2. Main results

Here we begin our investigation of when $0 \in W_0(T)$.

**Theorem 2.1.** *If $0 \in W_0(T)$, then $\|T\|^2 + |\lambda|^2 \leq \|T + \lambda\|^2$ for all $\lambda \in \mathbb{C}$. Conversely, if $\|T\| \leq \|T + \lambda\|$ for all $\lambda \in \mathbb{C}$, then $0 \in W_0(T)$.*

*Proof.* If $0 \in W_0(T)$, then there exist $\Lambda \in B(H)^*$ such that

$$\Lambda(I) = 1 = \|\Lambda\| \quad and \quad \Lambda(T^*T) = \|T\|^2$$

Therefore,

$$\begin{aligned} \|T\|^2 + |\lambda|^2 &= \Lambda(T^*T + |\lambda|^2) \\ &= \Lambda((T^* + \overline{\lambda})(T + \lambda)) \quad (since\ \Lambda(T) = 0) \\ &\leq \|(T^* + \overline{\lambda})(T + \lambda)\| \\ &= \|T + \lambda\|^2. \end{aligned}$$

Conversely, let $\|T\| \leq \|T+\lambda\|$ for all $\lambda \in \mathbb{C}$. Assume $0 \notin W_0(T)$. By rotating $T$, we may assume that $ReW_0(T) \geq \tau > 0$. Let $\Im = \{x \in H,\ Re < Tx, x > \leq \tau/2\}$. Let $\eta = \sup\{\|Tx\| : x \in \Im\}$. Thus $\eta < \|T\|$, since if $\eta = \|T\|$, then there exist $\{x_n\} \subset \Im$ such that $\|Tx_n\| \to \|T\|$. Put $\lambda = lim < Tx_n, x_n >$. Therefore $\lambda \in W_0(T)$ and so $Re\lambda \geq \tau$. On the other hand $\{x_n\} \subset \Im$, then $Re < Tx_n, x_n > \leq \tau/2$,which implies $Re\lambda \leq \tau/2$, contrary to hypothesis. Therefore $\|T\| - \eta > 0$. Put $\mu = min\{\tau/2, (\|T\| - \eta)/2\}$. If $x \in \Im$ then

$$\|(T-\mu)x\| \leq \|Tx\| + \mu \leq \eta + \mu < \|T\|$$

Let $Tx = (a+ib)x + y$ where $x \notin \Im$ and $\|x\| = 1$ and $< x, y >= 0$. Then $Ref(a) = x > \tau/2$ and

$$\|(T-\mu)x\|^2(a-\mu)^2 + b^2 + \|y\|^2 = \|Tx\|^2 + (\mu^2 - 2a\mu) < \|T\|^2$$

since $a > \mu > 0$. Thus, $\|T-\mu\| < \|T\|$, contrary to hypothesis. □

**Corollary 2.2.** *(Pythagorean relation for operators) Let $T$ be a bounded linear operator. Then there exists a unique $z_0 \in \mathbb{C}$, such that $\|T - z_0\|^2 + |\lambda|^2 \leq \|(T-z_0)+\lambda\|^2$ for all $\lambda \in \mathbb{C}$. Moreover, $0 \in W_0(T-\lambda)$ if and only if $\lambda = z_0$.*

*Proof.* Since $\|T-\lambda\|$ is large for $|\lambda|$ large, so $\inf_{\lambda\in\mathbb{C}} \|T-\lambda\|$ must be taken on at some point, say $z_0$. Then $\|T-z_0\| \leq \|(T-z_0)+\lambda\|$ for all $\lambda \in \mathbb{C}$ and so $0 \in W_0(T-z_0)$. Thus, by Theorem 2.1, $\|T-z_0\|^2+|\lambda|^2 \leq \|(T-z_0)+\lambda\|^2$ for all $\lambda \in \mathbb{C}$. If there exists another $z$, say $z_1$, such that $\|T-z_1\|^2+|\lambda|^2 \leq \|(T-z_1)+\lambda\|^2$ for all $\lambda \in \mathbb{C}$. Put $\lambda = z_0 - z_1$ in the first inequality, so we have $\|T-z_0\|^2 + |z_0-z_1|^2 \leq \|T-z_1\|^2$ and put $\lambda = z_1 - z_0$ in second inequality, then $\|T-z_1\|^2+|z_0-z_1|^2 \leq \|T-z_0\|^2$. Therefore

$$\|T-z_0\|^2 + |z_0-z_1|^2 \leq \|T-z_1\|^2 \leq \|T-z_0\|^2 - |z_0-z_1|^2$$

which implies $|z_0-z_1|^2 \leq -|z_0-z_1|^2 \leq 0$, and then $z_0 = z_1$. □

## 3. Further remarks

Given an element $T$, we define the center (or center of mass) of $T$ to be the point $z_0$ specified in the Pythagorean relation, and designate it by $c_T$. Given an element, how does one determine $c_T$? In general there is no simple answer. However, if $T$ is normal (or hyponormal) then $c_T$ is the center of the smallest circle containing the $\sigma(T)$=spectrum of $T$. In any event, $c_T \in$ *closure* $W(T)$, however, $c_T$ need not be contained in the convex hull of $\sigma(T)$. For more details see [4].

**Theorem 3.1.** *Let $\|S - T\| \leq \delta$. Then*

$$|c_S - c_T| \leq \frac{(\delta + [\delta^2 + 8\delta\|S - c_S\|]^{\frac{1}{2}})}{2}$$

.

*In particular, the map $T \to c_T$ is pointwise continuous.*

*Proof.* By the Pythagorean relation for $T$ and $S$ we have

$$\|T - c_T\|^2 + |\lambda|^2 \leq \|(T - c_T) + \lambda\|^2 \quad \text{for all } \lambda \in \mathbb{C}, \tag{3.1}$$

and

$$\|S - c_S\|^2 + |\lambda|^2 \leq \|(S - c_S) + \lambda\|^2 \quad \text{for all } \lambda \in \mathbb{C}. \tag{3.2}$$

We first assume that $c_S = 0$. Put $\lambda = c_T$ in inequality (3.1) and $\lambda = -c_T$ in inequality (3.2). Then we have

$$\begin{aligned}\|T\|^2 &\geq |c_T|^2 + \|T - c_T\|^2 \\ &\geq |c_T|^2 + (\|S - c_T\| - \|T - S\|)^2 \\ &\geq |c_T|^2 + (\|S - c_T\| - \delta)^2 \\ &= |c_T|^2 + \|S - c_T\|^2 - 2\delta\|S - c_T\| + \delta^2\end{aligned}$$

and

$$\|S - c_T\|^2 \geq \|S\|^2 + |c_T|^2.$$

Therefore

$$\|T\|^2 \geq \|T\|^2 - 2\delta\|S\| + 2|c_T|^2 - 2\delta\|S\| - 2\delta|c_T|$$

Thus

$$|c_T| \leq \frac{\delta + (\delta^2 + 8\delta\|S\|)^{1/2}}{2}.$$

To handle the case $c_S \neq 0$, we merely translate both $T$ and $S$ by $c_S I$. ∎

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# GENERALIZED FRAMES IN HILBERT MODULES OVER PRO-C*-ALGEBRAS

S. HOSSEINI

*Department of Mathematics, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran.*
*rsana7238@gmail.com*

Abstract. In this paper we study Hilbert pro-c*-modules and we introduce some properties of generalized frames with algebraic bounds in Hilbert pro-c*-modules.

## 1. Introduction

Frames that are a generalization of bases in Hilbert space, were introduced by [1] in 1952. In 1986, Daubechies et al., [3] reintroduced them. In other words, they replaced the sequence of bounded linear operators instead of the sequence of element in Hilbert space. Frames have many applications, such as: study and characterization of function spaces , signal and image processing, wireless communications, transceiver design, data compression and so on.

The rest of this section will recall some definitions and results which are needed in the next section.

We recall some basic definitions and properties of pro-$C^*$-algebras and Hilbert modules over pro-$C^*$-algebras.

**Definition 1.1.** A pro-$C^*$-algebra is a complete Hausdorff complex topological $*$-algebra $A$ whose topology is determined by its continuous

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$C^*$-seminorms in the sense that a net $(a_\lambda)$ converges to 0 if and only if $\rho(a_\lambda) \to 0$ for any continuous $C^*$-seminorm $\rho$ on $A$. For any $C^*$-seminorm $\rho$ on $A$ and each $a, b \in A$, we have

(i) $\rho(ab) \leq \rho(a)\rho(b)$;
(ii) $\rho(a^*a) = \rho(a)^2$.

For each pro-$C^*$-algebra $A$, the set of all positive elements in $A$ is denoted by $A^+$. Moreover, $a \geq 0$ denotes $a \in A^+$ and $a \leq b$ means that $b - a \geq 0$. We recall that every $C^*$-algebra is a pro-$C^*$-algebra.

The set of all continuous $C^*$-seminorms on $A$ is denoted by $S(A)$. An element $a \in A$ is bounded if $\|a\|_\infty = \sup\{\rho(a) : \rho \in S(A)\} < \infty$. The set of all bounded elements in $A$ is denoted by $b(A)$. Let $A$ be a unitary pro-$C^*$-algebra and $a \in A$. Then, non-zero element $a \in A$ is called strictly non-zero if zero does not belong to $\sigma(a)$. Here, we remember the following elementary result from [2].

**Proposition 1.2.** *Let $A$ be a unital pro- $C^*$-algebra with the identity $1_A$ and $\rho \in S(A)$. Then*

(1) *$\rho(a) = \rho(a^*)$ for all $a \in A$;*
(2) *$\rho(1_A) = 1$;*
(3) *if $a, b \in A^+$ and $a \leq b$, then $\rho(a) \leq \rho(b)$;*
(4) *if $1_A \leq b$, then $b$ is invertible and $b^{-1} \leq 1_A$;*
(5) *if $a, b \in A^+$ are invertible and $0 \leq a \leq b$, then $0 \leq b^{-1} \leq a^{-1}$;*
(6) *if $a, b, c \in A$ and $a \leq b$, then $c^*ac \leq c^*bc$;*
(7) *if $a, b \in A^+$ and $a^2 \leq b^2$, then $0 \leq a \leq b$.*

**Definition 1.3.** Let $A$ be a pro-$C^*$-algebra. A pre-Hilbert $A$-module is a complex vector space $E$ which is also a right $A$-module, compatible with the complex algebra structure, equipped with an $A$-valued inner product $\langle \cdot, \cdot \rangle : E \times E \longrightarrow A$ which is $\mathbb{C}$ and $A$-linear in second variable and satisfies the following conditions:

(i) $\langle x, y \rangle^* = \langle y, x \rangle$
(ii) $\langle x, x \rangle \geq 0$
(iii) $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

We say that $E$ is a *Hilbert $A$-module* or *Hilbert pro-$C^*$-module over $A$* if $E$ is complete with respect to the topology determined by the family of seminorms

$$\bar{\rho}_E(x) = \sqrt{\rho(\langle x, x \rangle)} \quad x \in E \rho \in S(A).$$

Let $E$ be a pre-Hilbert $A$-module. For every $\rho \in S(A)$ and for each $x, y \in E$, the following Cauchy-Schwartz inequality holds

$$\rho(\langle x, y \rangle)^2 \leq \rho(\langle x, x \rangle)\rho(\langle y, y \rangle).$$

Consequently, $\bar{\rho}_E(ax) \leq \rho(a)\bar{\rho}_E(x)$ for all $a \in A, x \in E$.

Let $A$ be a pro-$C^*$-algebra and $E$ be a pre-Hilbert $A$-module. We recall that an element $x$ in $E$ is bounded if $\|x\|_\infty = \sup\{\bar{\rho}_E(x) : \rho \in S(A)\} < \infty$. We denote by $b(E)$, the set of all bounded elements in $E$. (See [4, Proposition 1.11] and [5, Theorem 2.1] for more details).

**Definition 1.4.** Let $E$ and $F$ be two Hilbert modules over pro-$C^*$-algebra $A$. Then, the operator $T : E \longrightarrow F$ is called uniformly bounded (below), if there exists $C > 0$ such that $\bar{\rho}_F(Tx) \leq C\bar{\rho}_E(x)$ ($C\bar{\rho}_E(x) \leq \bar{\rho}_F(Tx)$) for all $\rho \in S(A)$ and $x \in E$. The number $C$ is called an upper bound for $T$ and hence we set

$$\|T\|_\infty = \inf\{C : C \,\text{is an upper bound for T}\}.$$

Clearly, in this case we have $\hat{\rho}(T) \leq \|T\|_\infty$, for all $\rho \in S(A)$.

## 2. Main results

This section is devoted to the main result of this paper.
Throughout this section, $A$ is a pro-$C^*$-algebra, $X$ and $Y$ are two Hilbert $A$-modules. Moreover, $\{Y_i\}_{i\in I}$ is a countable sequence of closed submodules of $Y$.

**Definition 2.1.** Let $X$ be a Hilbert pro-$C^*$-module. A sequence $\{x_i\}_{i\in I}$ in $X$ is said to be the *standard $*$-frame* for $X$ if for each $x \in X$, the series $\sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle$ is convergent in $A$ and there exist two strictly non-zero elements $C$ and $D$ in $A$ such that

$$C\langle x, x\rangle C^* \leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq D\langle x, x\rangle D^*$$

for all $x \in X$. The elements $C$ and $D$ are called *$*$-frame bounds* for $\{x_i\}_{i\in I}$. The $*$-frame is called *tight* if $C = D$ and called *Parseval* if $C = D = 1$. If in the above relation, we only need to have the upper bound, then $\{x_i\}_{i\in I}$ is called a *$*$-Bessel sequence*.

**Definition 2.2.** A sequence $\Lambda = \{\Lambda_i \in \mathrm{Hom}^*_A(X, Y_i)\}_{i\in I}$ is called a *$*$-g-frame* for $X$ with respect to $\{Y_i\}_{i\in I}$ if there exist two strictly non-zero elements $C$ and $D$ in $A$ such that for every $x \in X$,

$$C\langle x, x\rangle C^* \leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \leq D\langle x, x\rangle D^*.$$

The elements $C$ and $D$ are called *$*$-g-frame bounds* for $\Lambda$. The $*$-g-frame is called *tight* if $C = D$ and called *Parseval* if $C = D = 1$. If in the above we only need to have the upper bound, then $\Lambda$ is called a $*$-g-Bessel sequence. If $\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle$ is convergent in $A$ , the $*$-g-frame is called standard. Besides, if for each $i \in I$, $Y_i = Y$, we call it a $*$-g-frame for $X$ with respect to $Y$.

Let $\Lambda = \{\Lambda_i \in \mathrm{Hom}^*_A(X, Y_i)\}_{i\in I}$ be a $*$-g-frame for $X$ with respect to $\{Y_i\}_{i\in I}$ and bounds $C$ and $D$ in $A$. We define the corresponding $*$-g-frame transform

$$T_\Lambda : X \to \oplus_{i\in I} Y_i \quad T_\Lambda(x) = \{\Lambda_i x\}_{i\in I}.$$

**Theorem 2.3.** *Let $\Lambda = \{\Lambda_i \in Hom^*_A(X, Y_i)\}_{i\in I}$ be a $*$-g-frame for $X$ with respect to $\{Y_i\}_{i\in I}$ and bounds $C$ and $D$ in $A$. Then $T_\Lambda$ is well-defined, closed and injective. Also $T_\Lambda$ is an uniformly bounded below operator in $Hom_A(X, \oplus_{i\in I} Y_i)$.*

Now, we define the synthesis operator $T^*_\Lambda : \oplus_{i\in I} Y_i \longrightarrow X$ for $*$-g-frame $\Lambda$ through

$$T^*_\Lambda(\{y_i\}_i) := \sum_{i\in I} \Lambda^*_i y_i, \tag{2.1}$$

where $\Lambda^*_i$ is the adjoint operator of $\Lambda_i$.

**Proposition 2.4.** *The synthesis operator defined by (2.1) is well-defined, uniformly bounded and adjoint of the transform operator.*

Let $\Lambda = \{\Lambda_i : i \in I\}$ is a $*$-g-frame for $X$ with respect to $\{Y_i : i \in I\}$. Define the corresponding $*$-g-frame operator $S_\Lambda = T^*_\Lambda T_\Lambda : X \longrightarrow X$ via $S_\Lambda(x) = \sum_{i\in I} \Lambda^*_i \Lambda_i x$. Then, $S_\Lambda$ is a combination of two bounded operators and so it is a bounded operator.

**Theorem 2.5.** *Let $\Lambda = \{\Lambda_i : i \in I\}$ is a $*$-g-frame for $X$ with respect to $\{Y_i : i \in I\}$ with frame bounds $C$ and $D$. Then, $S_\Lambda$ is invertible positive operator. Moreover, it is a self-adjoint operator*

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# SOME COUPLED COINCIDENCE POINT RESULTS IN PARTIALLY ORDERED METRIC SPACESS

HASAN HOSSEINZADEH*,LAYA FADAKAR, SHEIRN TAVKOLIE , AREZOO MARDAZAD, ROBAB ABDEI AND ATEFEH SOJODEI

*Department of Mathematics, Ardabil Branch, Islamic Azad University, Ardabil, Iran*

Abstract. In this paper, we introduce the notion of partial-compatibility of mappings in an ordered partial metric space and use this notion to establish coupled coincidence point theorems for $g$-mixed monotone mappings satisfying a nonlinear contraction condition. Our consequences is an extention of results of Shatanawi et al [W. Shatanawi, B. Samet and M. Abbas, *Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces*,Mathematical and Computer Modelling, 55(3-4)(2012) 680-687]. We also provide an example to illustrate the results presented herein.

## 1. Introduction

The concepts of mixed monotone mapping and coupled fixed point have been introduced in [2] by Bhaskar and Lakshmikantham and they established some coupled fixed point theorems for a mixed monotone mapping in partially ordered metric spaces.

**Definition 1.1.** Let $X$ be a nonempty set and let $p : X \times X \to R^+$ satisfies:

(P1) $x = y \Longleftrightarrow p(x,x) = p(y,y) = p(x,y)$.

(P2) $p(x,x) \leq p(x,y)$.

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(P3) $p(x, y) = p(y, x)$.
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$, for all $x, y, z \in X$.
Then the pair $(X, p)$ is called a partial metric space and $p$ is called a partial metric on $X$.

The function $d_p : X \times X \to R^+$ defined by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

satisfies the conditions of a metric on $X$, therefore it is a (usual) metric on $X$. On a partial metric space the following concepts has been defined as follows:

**Definition 1.2.** (See e.g. [4, 5]).
(i) A sequence $x_n$ in a PMS $(X, p)$ converges to $x \in X$ iff $p(x, x) = \lim_n p(x, x_n)$.
(ii) A sequence $x_n$ in a PMS $(X, p)$ is called Cauchy if and only if $\lim_{n,m} p(x_n, x_m)$ exists (and is finite).
(iii) A PMS $(X, p)$ is said to be complete if every Cauchy sequence $x_n$ in $X$ converges, with respect to $\tau_p$, to a point $x \in X$ such that $p(x, x) = \lim_{n,m} p(x_n, x_m)$.
(iv) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon)$.

In this paper, we establish some coupled coincidence point results of nonlinear contraction mappings in the framework of ordered partial metric spaces. Our results extend and generalize the results of Shatanawi et al [6].

## 2. Main results

We recall three easy lemmas which have a essential role in the proof of the main results. This results can be derived easily (see e.g. [1, 4, 5]).

**Lemma 2.1.** *(1) A sequence $x_n$ is a Cauchy sequence in the PMS $(X, p)$ if and only if it is a Cauchy sequence in the metric space $(X, d_p)$.*
*(2) A PMS $(X, p)$ is complete if and only if the metric space $(X, d_p)$ is complete. Moreover,*

$$\lim_n d_p(x, x_n) = 0 \Longleftrightarrow p(x, x) = \lim_n p(x, x_n) = \lim_{n,m} p(x_n, x_m).$$

We define a notion of compatibility in the following:

**Definition 2.2.** The mappings $F$ and $g$ where $F : X \times X \to X$ and $g : X \to X$, are said to be partial-compatible if
1.

$$\lim_{n\to\infty} p(g(F(x_n, y_n)), F(g(x_n), g(y_n))) = 0,$$

and
$$\lim_{n\to\infty} p(g(F(y_n,x_n)),F(g(y_n),(gx_n)))=0,$$
whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $X$, such that $\{F(x_n,y_n)\}\to x$, $\{g(x_n)\}\to x$, $\{F(y_n,x_n)\}\to y$ and $\{g(y_n)\}\to y$, for some $x,y\in X$.

2. $p(x,x)=0$ implies that $p(gx,gx)=0$.

Note that the above Definition extends and generalizes the notion of compatibility introduced by Choudhury and Kundu [3].

Our main result is the following.

**Theorem 2.3.** *Let $(X,\preceq)$ be a partially ordered set and suppose that there is a partial metric $p$ on $X$ such that $(X,p)$ be a complete partial metric space. Let $F:X\times X\to X$ and $g:X\to X$ be two mappings such that*
$$\begin{aligned}&p(F(x,y),F(u,v))\\&\leq \alpha_1 p(gx,gu)+\alpha_2 p(gy,gv)+\alpha_3 p(F(x,y),gx)+\alpha_4 p(F(y,x),gy)\\&+\alpha_5 p(F(x,y),gu)+\alpha_6 p(F(y,x),gv)+\alpha_7 p(F(u,v),gx)\\&+\alpha_8 p(F(v,u),gy)+\alpha_9 p(F(u,v),gu)+\alpha_{10} p(F(v,u),gv)),\end{aligned}\tag{2.1}$$
*for every pairs $(x,y),(u,v)\in X\times X$ such that $gx\preceq gu$ and $gy\succeq gv$, where:*

*a.* $\alpha_7+\alpha_8+\sum_{i=1}^{10}\alpha_i<1$, *if* $\alpha_5-\alpha_7<0$ *and* $\alpha_6-\alpha_8<0$.

*b.* $\alpha_8+\sum_{i=1}^{10}\alpha_i<1$, *if* $\alpha_5-\alpha_7\geq 0$ *and* $\alpha_6-\alpha_8<0$.

*c.* $\alpha_7+\sum_{i=1}^{10}\alpha_i<1$, *if* $\alpha_5-\alpha_7<0$ *and* $\alpha_6-\alpha_8\geq 0$.

*d.* $\sum_{i=1}^{10}\alpha_i<1$, *if* $\alpha_5-\alpha_7\geq 0$ *and* $\alpha_6-\alpha_8\geq 0$.

*Suppose that*

*1. $F(X\times X)\subseteq g(X)$.*

*2. $F$ has the mixed $g$-monotone property.*

*3. $g$ is continuous and monotone increasing and $F$ and $g$ be partial-compatible mappings.*

*Also suppose*

*(a) $F$ is continuous, or,*

*(b) $X$ has the following properties:*

*(i) if $\{x_n\}$ is a non-decreasing sequence and $x\in X$ with $\lim_{n\to\infty} p(x_n,x)=p(x,x)=0$, then $x_n\preceq x$ for all $n$,*

*(ii) if $\{x_n\}$ is a non-increasing sequence and $x\in X$ with $\lim_{n\to\infty} p(x_n,x)=p(x,x)=0$, then $x\preceq x_n$ for all $n$.*

*If there exist $x_0,y_0\in X$ such that $gx_0\preceq F(x_0,y_0)$ and $F(y_0,x_0)\preceq gy_0$, then $F$ and $g$ have a coupled coincidence point in $X$.*

Many results can be deduced from the above theorem as follows.

**Corollary 2.4.** *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a partial metric $p$ on $X$ such that $(X, p)$ be a complete partial metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that*

$$p(F(x, y), F(u, v)) \leq \alpha_1 p(gx, gu) + \alpha_2 p(gy, gv), \tag{2.2}$$

*for every pairs $(x, y), (u, v) \in X \times X$ such that $gx \preceq gu$ and $gy \succeq gv$, where: $\alpha_1 + \alpha_2 < 1$.*

*Suppose that*

*1. $F(X \times X) \subseteq g(X)$.*

*2. $F$ has the mixed $g$-monotone property.*

*3. $g$ is continuous and monotone increasing and $F$ and $g$ be partial-compatible mappings.*

*Also suppose*

*(a) $F$ is continuous, or,*

*(b) $X$ has the following properties:*

*(i) if $\{x_n\}$ is a non-decreasing sequence and $x \in X$ with $\lim_{n\to\infty} p(x_n, x) = p(x, x) = 0$, then $x_n \preceq x$ for all $n$,*

*(ii) if $\{x_n\}$ is a non-increasing sequence and $x \in X$ with $\lim_{n\to\infty} p(x_n, x) = p(x, x) = 0$, then $x \preceq x_n$ for all $n$.*

*If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$, then $F$ and $g$ have a coupled coincidence point in $X$.*

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*Email address*: `hasan_hz2003@yahoo.com & hasan.hosseinzadeh@iauardabil.ac.ir`

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# THE COUPLED MEASURE OF NONCOMPACTNESS AND FUNCTIONAL INTEGRAL EQUATIONS VIA DARBO FIXED POINT THEOREMS

HASAN HOSSEINZADEH*,SAMEIRA HADEI BONABIE, SHEIRN TAVKOLIE AND AREZOO MARDAZAD

*Department of Mathematics, Ardabil Branch, Islamic Azad University, Ardabil, Iran*

ABSTRACT. The aim of this article is to study the results of the fixed point in coupled measure of noncompactness. We will use the technique of measure of noncompactness for coupled measure of noncompactness. Also, at the end of this paper, an application with an example is provided to illustrate the results.

## 1. INTRODUCTION

In the nonlinear analysis, one of the important tools is the concept of measure of noncompactness to address the problems in functional operator equations. This important concept in the mathematical sciences has been defined by many authors in various ways (see [1, 3, 4, 5, 8]). In [2] was introduced Some generalizations of Darbo fixed point theorem and they presented an application in functional integral equations.

In this paper, we investigate the results of the fixed point in coupled measure of noncompactness via Darbo fixed point theorem.

Throughout this article, we consider $E$ as the Banach space and briefly show a measure of noncompactness with MNC, $B(x,r)$ represents the closed ball in Banach space $E$ to centre $x$ and radius $r$. Also

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we use $B_r$ to represent $B(\theta, r)$, where $\theta$ is the zero elements, the family of all nonempty bounded subsets of Banach space $E$ is represented with $\mathcal{B}_E$. To begin, we have the following preliminaries from [7, 8, 9].

**Definition 1.1.** [8]. Let $\mu : \mathcal{B}_E \to \mathbb{R}_+$ be a mapping. The family $\mathcal{B}_E$ is called MNC on Banach space $E$ if the following conditions hold:

(1) for each $X^1 \in \mathcal{B}_E,\ \mu(X^1) = \theta$ if and only if $X^1$ is a precompact set;

(2) for each pair $(X^1, X^2) \in \mathcal{B}_E \times \mathcal{B}_E$; we have

$$X^1 \subseteq X^2 \Rightarrow \mu(X^1) \preceq \mu(X^2);$$

(3) for each $X^1 \in \mathcal{B}_E$, one has

$$\mu(X^1) = \mu(\overline{X^1}) = \mu(conv(X^1))$$

where $\overline{X^1}$ represents the closure of $X^1$ and $convX^1$ represents the convex hull of $X^1$;

(4) $\mu(\lambda X^1 + (1-\lambda)X^2) \leq \lambda\mu(X^1) + (1-\lambda)\mu(X^2)$ for $\lambda \in [0, 1]$;

(5) if $\{x_n\}_0^\infty \in \mathcal{B}_E$ is a decreasing sequence of closed sets and $\lim_{n\to\infty} \mu(x_n) = 0$, then $X^1_\infty = \bigcap_{n=0}^\infty X^1_n \neq \phi$.

## 2. Main results

We start this section with the following concept and then we turn to the main subject.

**Definition 2.1.** Let $E$ be a Banach space and $\mu : \mathcal{B}^2_E \to \mathbb{R}_+$ a mapping. We say that $\mu$ is a coupled MNC on $E$ If it has the following conditions:

(1) $\ker \mu = \{(X^1, X^2) \in \mathcal{B}^2_E : \mu(X^1, X^2) = \theta\}$ is nonempty

(2) for every $X^1, X^2 \in \mathcal{B}^2_E,\ \mu(X^1, X^2) = \theta \Leftrightarrow (X^1, X^2)$ is a precompact set;

(3) for each $((X^1, X^2), (X'^1, X'^2)) \in \mathcal{B}^2_E \times \mathcal{B}^2_E$ and $(X^1, X^2) \subseteq (X'^1, X'^2) \Rightarrow X^1 \subseteq X'^1, X^2 \subseteq X'^2$; we have

$$(X^1, X^2) \subseteq (X'^1, X'^2) \textit{ implies } \mu(X^1, X^2) \preceq \mu(X'^1, X'^2);$$

(4) for every $(X^1, X^2) \in \mathcal{B}^2_E$, one has

$$\mu(\overline{X^1}, \overline{X^2}) = \mu(X^1, X^2) = \mu(conv(X^1, X^2))$$

where $conv(X^1, X^2)$ denotes the convex hull of $(X^1, X^2)$;

(5) $\mu(\lambda(X^1, X^2) + (1-\lambda)(X'^1, X'^2)) \leq \lambda\mu(X^1, X^2) + (1-\lambda)\mu(X'^1, X'^2)$ for $\lambda \in [0, 1]$;

(6) if $\{X^1_n\}_0^\infty, \{X^2_n\}_0^\infty$ in $\mathcal{B}_E$ are decreasing sequences of closed sets and
$\lim_{n\to\infty} \mu\{(X^1_n, X^2_n)\}_0^\infty = 0$ , then $(X^1_\infty, X^2_\infty) = \bigcap_{n=0}^\infty (X^1_n, X^2_n) \neq \phi$.

**Theorem 2.2.** *Let $G$ be a nonempty, bounded, closed and convex subset of $E$ and $F : G \to G$ a continuous mapping such that*

$$\varphi_2(\mu(FX^1, FX^2)) \leq \varphi_2[\mu(X^1, X^2) - \varphi_1(\mu(X^1, X^2))] \tag{2.1}$$

*for each $\emptyset \neq X^1 \subseteq G, \emptyset \neq X^2 \subseteq G$, where $\mu$ is an arbitrary coupled MNC and functions $\varphi_1, \varphi_2 : \mathcal{R}_+^2 \to \mathcal{R}_+$, such that $\varphi_2$ is continuous and $\varphi_1$ is lower semicontinuous on $\mathcal{R}_+$. Furthermore, $\varphi_1(0,0) = 0$ and $\varphi_1(r,s) > 0$ for $r, s > 0$. Then $F$ has at least one fixed point in $G$.*

**Theorem 2.3.** *Let $G$ be a nonempty, bounded, closed and convex subset of $E$ and the mapping $F : G \to G$ be a continuous that in the following condition satisfying*

$$\mu(FX^1, FX^2) \leq \varphi_1(\mu(X^1, X^2)) \tag{2.2}$$

*for each $\emptyset \neq X^1 \subseteq G, \emptyset \neq X^2 \subseteq G$, where $\mu$ is an arbitrary coupled MNC and $\varphi_1 : \mathcal{R}_+^2 \to \mathcal{R}_+$ is a nondecreasing functions such that $\lim_{n\to\infty} \varphi_1^n(r,s) = 0$ for every $r, s \geq 0$. Then $F$ has at least one fixed point.*

**Lemma 2.4.** *Let $\varphi_1 : \mathcal{R}_+^2 \to \mathcal{R}_+$ be a upper semicontinuous and nondecreasing function. In this case, the following conditions are equivalent:*
*(1) $\lim_{n\to\infty} \varphi_1^n(r,s) = 0$ for each $r, s \geq 0$.*
*(2) $\varphi_1(r,s) < ts$ for any $r, s > 0$.*

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*Email address*: hasan_hz2003@yahoo.com, hasan.hosseinzadeh@iauardabil.ac.ir

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# MEAN ERGODICITY OF MULTIPLES OF COMPOSITION OPERATORS ON BLOCH TYPE SPACES

Z. KAMALI

*Department of Mathematics, Faculty of Sciences, Shiraz Branch, Islamic Azad University, Shiraz, Iran*
*zkamali@shirazu.ac.ir*

Abstract. In this paper we give necessary and sufficient conditions for the power boundedness and mean ergodicity of multiples of composition operators $\lambda C_\varphi$ on Bloch type spaces $\mathcal{B}^\alpha$ for $\alpha > 1$.

## 1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. For $\alpha > 0$, the Bloch type spaces, which is denoted by $\mathcal{B}^\alpha$, is the space of all functions in $H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty.$$

Consider that if $\alpha = 1$ then $\mathcal{B}^1 = \mathcal{B}$ is the classical Bloch space. It is easy to see that for each $\alpha > 0$ the space $\mathcal{B}^\alpha$ is a Banach space with the norm

$$||f|| = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|.$$

[5] is a perfect source for studying about these spaces.
Each $\varphi \in H(\mathbb{D})$ induces a linear composition operator $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ by $C_\varphi(f)(z) = f(\varphi(z))$ for every $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. For a

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positive integer $n$, the nth iterates of $\varphi$ is denoted by $\varphi_n$.
The author in [4] showed that if $0 < \alpha < \infty$, then $C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\alpha$ is bounded, if and only if, $\tau_{\varphi,\alpha} = \sup_{z\in\mathbb{D}}(\frac{1-|z|^2}{1-|\varphi(z)|^2})^\alpha|\varphi'(z)| < \infty$. In the case $\alpha > 1$, always $\tau_{\varphi,\alpha} < \infty$, so $C_\varphi$ is always bounded on $\mathcal{B}^\alpha$ for $\alpha > 1$.

The analytic self map of the unit disk are divided in two classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in $\mathbb{D}$. The non-elliptic one has a unique fixed point $p \in \overline{\mathbb{D}}$, such that $\{\varphi_n\}_n$ converges to $p$ uniformly on compact subset of $\mathbb{D}$. This point is called Denjoy-Wolff point. See [3] for more details.
Suppose $T$ is a bounded linear operator on a locally convex Hausdorff space, the Cesáro means of $T$ is defined by $T_{[n]} := \frac{1}{n}\sum_{m=1}^{\infty} T^m$, for all $n \in \mathbb{N}$. An operator $T$ is uniformly mean ergodic if the sequence of Cesáro means of $T$, $\{T_{[n]}\}_{n=1}^\infty$ converges in operator norm topology and it is mean ergodic if $\{T_{[n]}\}_n$ converges in the strong operator topology. Also it is called power bounded if $\sup_{n\in\mathbb{N}} ||T^n|| < \infty$.

## 2. Main results

This section is devoted to the main result of this paper. Here we give sketch of proof of all statements.

**Proposition 2.1.** *Let $\varphi$ be an analytic self map of $\mathbb{D}$, $\alpha > 0$ and $\lambda \in \mathbb{C}$. Suppose $\lambda C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\alpha$ is bounded. If $\lambda C_\varphi$ is power bounded, mean ergodic or uniformly mean ergodic, then $|\lambda| \leq 1$.*

*Proof.* Since $||\lambda^n C_{\varphi_n} 1|| = |\lambda|^n \leq ||\lambda^n C_{\varphi_n}||$ and if $\lambda C_\varphi$ is mean ergodic or uniformly mean ergodic, $\frac{||\lambda^n C_{\varphi_n} 1||}{n} = \frac{|\lambda^n|}{n} \to 0$ as $n \to \infty$, in three cases $\{|\lambda|^n\}_n$ must be a bounded sequence. ☐

**Theorem 2.2.** *Let $\varphi$ be an analytic self map of $\mathbb{D}$ and $\alpha > 1$. If $|\lambda| < 1$ and $\varphi$ has interior fixed point or $|\lambda| = 1$ and $\varphi$ has interior Denjoy-Wolff point, then $||\lambda^n C_{\varphi_n}|| \to 0$ and consequently, it is power bounded, mean ergodic and uniformly mean ergodic.*

*Proof.* Without loss of generality we may assume that $\varphi(0) = 0$. Since other wise, if $\varphi(a) = a$, for some $a \neq 0$, let $\Phi = \varphi_a \ o\ \varphi \ o\ \varphi_a$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. Then $\Phi(0) = 0$ and $C_\Phi = C_{\varphi_a} \ o\ C_\varphi \ o\ C_{\varphi_a}$ is similar to $C_\varphi$ and $||C_\varphi|| = ||C_\Phi||$. In the first case by Schwarz-Pick lemma for all $n \in \mathbb{N}$, $\tau_{\varphi_n,\alpha} \leq 1$ and $||\lambda^n C_{\varphi_n}|| \leq |\lambda|^n$. In the second case, we can show $||\lambda^n \varphi_n|| \to o$. By this fact the theorem follows. ☐

**Proposition 2.3.** *Let $\varphi$ be analytic self map of $\mathbb{D}$, $|\lambda| = 1$ and $\alpha > 0$. $\lambda C_\varphi$ is power bounded, if and only if, $C_\varphi$ is power bounded, if and only if, $\varphi$ has an interior fixed point.*

**Theorem 2.4.** *Let $\varphi$ be analytic self map of $\mathbb{D}$ with $z_0 \in \partial\mathbb{D}$ as boundary Denjoy-Wolff point, $\alpha > 1$ and $|\lambda| = 1$. If $\lambda C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\alpha$ is bounded, then $||\lambda^n C_{\varphi_n}|| \to \infty$, so it is not power bounded, nor mean ergodic, nor uniformly mean ergodic.*

*Proof.* Easy computation shows that $f(z) = \frac{1}{(z-z_0)^{\alpha-1}} - 1$ is in $\mathcal{B}^\alpha$ for $\alpha > 1$. Also

$$|\lambda^n C_{\varphi_n} f(0)| = |\frac{1}{(\varphi_n(0) - z_0)^{\alpha-1}} - 1| \leq ||\lambda^n C_{\varphi_n} f||.$$

By Denjoy-Wolff theorem $\varphi_n(0) \to z_0$ and the theorem follows. □

Recall that for an operator $T \in L(X)$ on a Banach space $X$, the *spectrum* $\sigma(T)$ is the set of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible. The approximate spectrum $\sigma_{ap}(T)$ is the set of $\beta \in \mathbb{C}$ for which there is $\{x_n\} \subseteq X$, with $||x_n|| = 1$ and $\lim_{n\to\infty} ||T(x_n) - \beta x_n|| = 0$.

**Theorem 2.5.** *Suppose $\varphi$ is an elliptic automorphism and $|\lambda| = 1$.*

(1) *If $\varphi$ is similar to a rational rotation, then $\lambda C_\varphi$ is uniformly mean ergodic.*
(2) *If $\varphi$ is similar to a irrational rotation, then $\lambda C_\varphi$ is not uniformly mean ergodic.*

*Proof.* We can assume $\varphi(z) = \beta z$, where $|\beta| = 1$.

(1) Since for all $k \in \mathbb{N}$, $\beta^k \neq 1$, we have $\overline{\{\lambda\beta^k : k \in \mathbb{N}\}} = \partial\mathbb{D}$. For $z_0 \in \partial\mathbb{D}$, there exists $\{n_k\}$ such that $\lambda\beta^{n_k} \to z_0$. Let $g_{n_k}(z) = \frac{z^{n_k}}{||z^{n_k}||}$, so $||\lambda C_\varphi g_{n_k} - z_0 g_{n_k}|| = |\lambda\beta^{n_k} - z_0| \to 0$, as $k \to \infty$. This means $\partial\mathbb{D} \subseteq \sigma_{ap}(\lambda C_\varphi) \subseteq \sigma(\lambda C_\varphi)$. By Dunford-Lin theorem (see [2]) $\lambda C_\varphi$ can not be uniformly mean ergodic.
(2) If $\beta^k = \lambda^k = 1$ for some $k \in \mathbb{N}$, then $(\lambda C_\varphi)_{[n]} \to \frac{1}{k}\sum_{m=1}^{k} \lambda^m C_{\varphi_m}$. If for all $k \in \mathbb{N}$, $\lambda^k \neq 1$, $(\lambda C_\varphi)_{[n]} \to 0$.

□

Bloch type spaces are Grothendieck Banach spaces which satisfy Dunford Pettis property (GDP spaces) which Lotz proved that mean ergodicity and uniform mean ergodicity are equivalent in these spaces. See [1].

**Theorem 2.6.** *Let $\varphi$ be analytic self map of $\mathbb{D}$ and $|\lambda| = 1$. If $\alpha > 1$, then $\lambda C_\varphi$ is uniformly mean ergodic if and only it is mean ergodic, if and only if, $\varphi$ has interior Denjoy-Wolff point or it is an elliptic automorphism which is similar to a rational rotation.*

## 3. Further remarks

Ergodic theory is one of the most important branches of mathematics which is related to dynamical system and especially used in "random process" topics.

The study of mean ergodicity of linear operators on Banach spaces goes back to 1931, when Von Numann proved that for a unitary operator $T$ on a Hilbert space $H$, there is a projection $P$ on $H$, such that $T_{[n]}$ converges to $P$ in the strong operator topology. In 1939 Lorch demonstrated that for reflexive Banach spaces, power bounded operators are mean ergodic and Lin showed that if $T$ is an operator such that $||T^n/n|| \to 0$, then $T$ is uniformly mean ergodic if and only if $\mathrm{Im}(I-T)$ is closed.Then Lotz extended this last result for Grothendieck Dunford-Pettis (GDP) spaces and established that for a bounded linear operator $T$ on Grothendieck Dunford-Pettis space $X$ satisfying $||T^n/n|| \to 0$, mean ergodicity is equivalent with uniformly mean ergodicity of $T$. See [2]. The study of ergodic properties of composition operators has received a special attention from many authors and this topic was investigated on various spaces of holomorphic functions. In [1] the authors completely characterized power bounded, mean ergodic and uniformly mean ergodic composition operators on Bloch type spaces. In this paper we consider the multiples of composition operators on these spaces. As we show the ergodic properties of $\lambda C_\varphi$ depend on both $\lambda$ and the fixed point configuration of $\varphi$.

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# SOME PROPERTIES OF F-HARMONIC MAPS WITH POTENTIAL

[1]S. M. KAZEMI TORBAGHAN* AND [2]K. SALEHI

[1]*Faculty of Basic Sciences, Univesity of Bojnord, Bojnord, Iran,*
*m.kazemi@ub.ac.ir*
[2] *Central of theoretical physic and chemistry (ctcp), Massey university, Auckland , Newzealand, K.salehi@massey.ac.nz*

Abstract. In this paper, $F-$harmonic maps with potential between Riemannian manifolds are studied. First, the variational formulas for these types of maps are obtained. Then, stability of $F-$harmonic maps from a Riemannian manifold into a standard unit sphere is studied.

## 1. Introduction

In 1964, Sampson and Eells investigated the properties of harmonic maps. They also proved the fundamental existence theorem for harmonic maps. From up to now, many scholars have done research on this topic,[3, 4]. These kind of maps have an important role in many branch of physics, mathematics and mechanics such as liquid crystal, ferromagnetic material, super conductor, etc., see [5, 6].
In [7], Ratto introduced the notion of harmonic maps with potential. Recently many research have done on this topic, Y. Chu [2]. Let $H$ be a smooth function on a smooth manifold $N$ and let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds, . Assume that

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$e(\phi) := \frac{1}{2} \mid d\phi \mid^2$. The function

$$E_H(\phi) = \int_M [e(\phi) - H(\phi)] dv_g, \tag{1.1}$$

is called the $H-$energy function of $\phi$. Moreover, any critical points of $E_H$ is said to be harmonic map with potential $H$.

$\mathcal{F}$-harmonic maps as an extension of geodesics , minimal surfaces and harmonic maps were first investigated by Ara in 1999, [1]. Consider a $C^2$-function $\mathcal{F} : [0, \infty) \longrightarrow [0, \infty)$ such that $F' > 0$ on $(0, \infty)$. The smooth map $\phi$ is called $\mathcal{F}$*-harmonic* if $\phi$ is a critical point of the $\mathcal{F}$-energy functional:

$$E_{\mathcal{F}}(\phi) = \int_M \mathcal{F}(\frac{\mid d\phi \mid^2}{2}) dv_g \tag{1.2}$$

$\mathcal{F}$-energy functional could be categorized as exponential energy, p-energy, or energy when $\mathcal{F}(t)$ is equal to $e^t$, $(2t)^{\frac{p}{2}}/p \quad (p \geq 4)$ or $t$, respectively. By calculating the first variation formula for $\mathcal{F}$-energy functional , it can be obtain that

$$\tau_{\mathcal{F}}(\phi) := \mathcal{F}'(\frac{\mid d\phi \mid^2}{2}) \tau(\phi) + d\phi(grad_g(\mathcal{F}'(\frac{\mid d\phi \mid^2}{2}))) = 0 \tag{1.3}$$

The operator $\tau_{\mathcal{F}}(\phi)$ is said to be the $\mathcal{F}$*-tension field* of the map $\phi$.

In view of physics, $\mathcal{F}$-harmonic maps have a key role in physical cosmology, physics and mechanics. For instance, they are studied to investigate the phenomenon of the quintessence,[3].

In this paper, $\mathcal{F}$-harmonic maps with potential is introduced. Then, the first and second variation formulas for these maps are derived. Finally, the stability of $\mathcal{F}$- harmonic maps with potential into the unit sphere equipped with induced metric is studied.

## 2. Main results

In this part, first, the notion of $\mathcal{F}-$energy functional with potential $H$ is studied. Then, the variation formulas are obtained. Finally, the stability of these maps are investigated.

Consider the $C^3$ map $\phi : M \longrightarrow N$ between Riemannian manifolds. Denote the Levi-Civita connection of $M, N$ and $\phi^{-1}TN$ by ${}^M\nabla, {}^N\nabla$ and $\hat{\nabla}$. Let $H$ be a smooth function on $N$ and let $F : [0, \infty) \longrightarrow [0, \infty)$ be a $C^3-$ strictly increasing function . F-bienergy functional with potential $H$ can be considered as follows:

$$E_{\mathcal{F},H}(\phi) = \int_M (\mathcal{F}(\frac{\mid d\phi \mid^2}{2}) + H \circ \phi) dv_g. \tag{2.1}$$

A map $\phi$ is said to be $\mathcal{F}$*-harmonic* with potential $H$ if $\phi$ is a critical point of the F-energy functional. F-harmonic maps with potential $H$ can be categorized as harmonic , p-harmonic or exponentially harmonic when $F(t)$ is equal to $t$, $(2t)^{\frac{p}{2}}/p$ ( $p \geq 4$) or $e^t$ respectively. By choosing a local orthonormal frame field $\{e_i\}$ on $M$, The $F-H-$tension field of $\phi$, $\tau_{F,H}(\phi)$, is defined by

$$\tau_{F,H}(\phi) = F'(\frac{\mid d\phi \mid^2}{2})\tau(\phi) + d\phi(gradF'(\frac{\mid d\phi \mid^2}{2})) +^N \nabla H \circ \phi, \quad (2.2)$$

here $\tau(\phi) = \sum_{i=1}^m \{\hat{\nabla}_{e_i} d\phi(e_i) - d\phi(^M\nabla_{e_i} e_i)\}$ is the tension field of $\phi$. According to the above notations we get

**Lemma 2.1.** *(The first variation formula) Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map. Then*

$$\frac{d}{dt} E_{F,H}(\phi_t) \mid_{t=0} = -\int_M h(\tau_{F,H}(\phi), V) dv_g, \quad (2.3)$$

*where* $V = \frac{d\phi_t}{dt} \mid_{t=0}$ .

By 2.1, the notion of $F-$harmonic map with potential $H$ for the functional $E_{F,H}$ can be defined as follows

**Definition 2.2.** A $C^2$ map $\phi$ is said to be $F-$harmonic with potential $H$ for the functional $E_{F,H}$ if $\tau_{F,H}(\phi) = 0$.

**Definition 2.3.** Let $\phi : (M, g) \longrightarrow (N, h)$ be an $F-$harmonic map with potential $H$, and let $\phi_t : M \longrightarrow N$ $(-\epsilon < t < \epsilon)$ be a smooth variation of $\phi_0 = \phi$ and $V = \dfrac{\partial \phi_t}{\partial t} \mid_{t=0}$. Setting

$$I(V) = \frac{d^2}{dt^2} E_{F,H}(\phi_t) \mid_{t=0}$$

The map $\phi$ is said to be stable if $I(V) \geq 0$ for any vector field $V$ along $\phi$.

By computing the second variation formula, it can be seen that

$$\begin{aligned} I(V) &= \int_M F''(\frac{\mid d\phi \mid^2}{2}) \langle \hat{\nabla} V, d\phi \rangle^2 dv_g \\ &+ \int_M F'(\frac{\mid d\phi \mid^2}{2}) \Big\{ \langle \mid \hat{\nabla} V \mid^2 - h(trace_g\, {}^N R(V, d\phi) d\phi \\ &- (\nabla_V^N grad^N H) \circ \phi, V) \Big\} dv_g \qquad (2.4) \end{aligned}$$

where $\mid \hat{\nabla} V \mid$ denotes the Hilbert-Schmidt norm of the $\hat{\nabla} V \in \Gamma(T^* M \times \phi^{-1} TN)$. By (2.4), we have

**Theorem 2.4.** *Let $\phi : (M, g) \longrightarrow \mathbb{S}^n$ be a stable $F-$ harmonic map with potential $H$ from a Riemannian manifold $(M, g)$ to $\mathbb{S}^n(n > 2)$, and let $\triangle^{\mathbb{S}^n} H \circ \phi \geq 0$. Suppose that $(\mathcal{F}(e(\phi)))'' < 0$ for $n < 2$. Then $\phi$ is constant.*

According to (2.4), we get

**Corollary 2.5.** *Let $\phi : (M, g) \longrightarrow \mathbb{S}^n$ be a stable $\mathcal{F}-$harmonic map with potential $H$ from a Riemannian manifold $(M, g)$ to $\mathbb{S}^n(n > 2)$. Suppose that $H$ is an affine function. Then $\phi$ is constant.*

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# SOLVING SINGULAR RICCATI DIFFERENTIAL EQUATIONS USING RKHS METHOD

N. KARAMIKABIR

*Department of Mathematics, Hamedan Branch, Islamic Azad University, Hamedan, Iran*
*n.karamikabir@iauh.ac.ir*

Abstract. This paper deals with the approximation of the solution of singular Riccati differential equations using the reproducing kernel Hilbert space scheme. In the meantime, the $n$-term approximate solution is obtained. Some numerical examples have also been studied to demonstrate the accuracy of the present method.

## 1. Introduction

In this work, we consider the following quadratic Riccati differential equation with singularity in reproducing kernel space

$$\mathcal{P}(r)u'(r) = \mathcal{Q}(r)u^2(r) + \mathcal{R}(r)u(r) + \mathcal{S}(r), \quad u(0) = \xi, \quad 0 \le r \le 1, \tag{1}$$

where $\mathcal{P}(r)$, $\mathcal{Q}(r)$, $\mathcal{R}(r)$ and $\mathcal{S}(r)$ coefficients are continuous real functions, perhaps $\mathcal{P}(0) = 0$ or $\mathcal{P}(1) = 0$ and $u(r) \in W_2^2[0,1]$.
In order to solution of (1), we construct a reproducing kernel functions.

**Definition 1.1.** ( [2]) Let $E$ be a nonempty abstract set and $\mathbb{C}$ be the set of complex numbers. A function $K : E \times E \to \mathbb{C}$ is a reproducing kernel of the Hilbert space $\mathcal{H}$ if

- for each $t \in E$, $K(.,t) \in \mathcal{H}$,
- for each $t \in E$ and $\psi \in \mathcal{H}$, $\langle \psi(.), K(.,t) \rangle = \psi(t)$.

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In Definition 1.1, second condition called the reproducing property, the value of the function $\psi$ at the point $t$ is reproducing by the inner product of $\psi(t)$ with $K(.,t)$.

**Remark 1.2.** A Hilbert space $\mathcal{H}$ of functions on a set $E$ is called an RKHS if there exists a reproducing kernel $K$ of $\mathcal{H}$. That is a Hilbert space which that possesses a reproducing kernel is called the RKHS.

**Definition 1.3.** The inner product space $W_2^2[0,1]$ is defined as $W_2^2[0,1] = \{x(t)|x(t), x'(t),$ is absolutely continuous, $x(0) = \alpha$ and $x''(t) \in L^2[0,1]\}$. The inner product $W_2^2[0,1]$ is defined by

$$\langle x(t), y(t)\rangle_{W_2^2[0,1]} = x(0)y(0) + x(1)y(1) + \int_0^1 x''(t)y''(t)\mathrm{d}t, \qquad (2)$$

and the norm $\|x\|_{W_2^2[0,1]}$ is denoted by $\|x\|_{W_2^2[0,1]} = \sqrt{\langle x, x\rangle_{W_2^2[0,1]}}$, where $x, y \in W_2^2[0,1]$.

**Theorem 1.4.** *( [4]) The space $W_2^2[0,1]$ is a reproducing kernel space. That is, for any $x(t) \in W_2^2[0,1]$ and each fixed $t \in [0,1]$ there exists $R_z(t) \in W_2^2[0,1]$, such that $\langle x(t), R_z(t)\rangle_{W_2^2[0,1]} = x(z)$. The reproducing kernel $R_z(t)$ can be denoted by*

$$\mathcal{R}_z(t) = \begin{cases} \frac{t}{6}((z-1)t^2 + z(z^2 - 3z + 8)), & t \leq z, \\ \frac{z}{6}(z^2(t-1) + t(t^2 - 3t + 8)), & t > z. \end{cases} \qquad (3)$$

**Definition 1.5.** The inner product space $W_2^1[0,1]$ is defined as $W_2^1[0,1] = \{x(t)|x(t),$ is absolutely continuous, $x(0) = \alpha$ and $x'(t) \in L^2[0,1]\}$. The inner product $W_2^1[0,1]$ is defined by

$$\langle x(t), y(t)\rangle_{W_2^1[0,1]} = x(0)y(0) + \int_0^1 x'(t)y'(t)\mathrm{d}t, \qquad (4)$$

and the norm $\|x\|_{W_2^1[0,1]}$ is denoted by $\|x\|_{W_2^1[0,1]} = \sqrt{\langle x, x\rangle_{W_2^1[0,1]}}$, where $x, y \in W_2^1[0,1]$.

In [1] proved that $W_2^1[0,1]$ is a complete reproducing kernel space and its reproducing kernel is

$$\overline{\mathcal{R}}_z(t) = \begin{cases} 1+t, & t \leq z, \\ 1+z, & t > z. \end{cases} \qquad (5)$$

## 2. Main Results

In this section, the solution of (1) is given in the reproducing kernel space $W_2^2[0,1]$. In (1), it is clear that $\mathcal{L} : W_2^2[0,1] \to W_2^1[0,1]$ is a bounded linear operator. Put $\chi_s(z) = \overline{\mathcal{R}}_{z_s}(z)$ and $\phi_s(z) = \mathcal{L}^*\chi_s(z)$,

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$. The orthonormal system $\{\overline{\phi}_s(z)\}_{s=1}^{\infty}$ of $W_2^2[0,1]$ can be derived from Gram Schmidt orthogonalization process of $\{\phi_s(z)\}_{s=1}^{\infty}$ such that
$\overline{\phi}_s(z) = \sum_{i=1}^{s} \alpha_{si}\phi_i(z), \quad (\alpha_{ss} > 0, s = 1, 2, \ldots).$

**Theorem 2.1.** *If $\{z_s\}_{s=1}^{\infty}$ is dense on $[0,1]$ and the solution of (1) is unique, then the solution of (1) satisfies the form*

$$v(z) = \sum_{s=1}^{\infty}\sum_{i=1}^{s} \alpha_{si} f(z_i, v(z_i))\overline{\phi}_s(z). \tag{6}$$

## 3. Numerical simulations

In this section, the scheme in the paper will be applied to two numerical examples.

**Example 3.1.** In equation (1), if $\mathcal{P}(r) = r$, $\mathcal{Q}(r) = 1$, $\mathcal{R}(r) = -\sqrt{r}$ and $\mathcal{S}(r) = -1$ then true solution is $u(r) = \frac{2\sqrt{r}-3}{-2r+4\sqrt{r}-3}$.
RKHS method, taking $u(0) = 1$, $r_s = \frac{s-1}{N-1}$, $s = 1, 2, \ldots, N$ with the reproducing kernel function $\overline{\mathcal{R}}_z(t)$ on $[0,1]$ . The numerical results at some selected gird points for $N = 51$ and $n = 7$ are given in Table 1.

Table 1. Numerical results for Example 3.1.

| $r$ | True solution | Approximate solution | Absolute error | Relative error |
|---|---|---|---|---|
| 0.1 | 1.223480959455358 | 1.223480953436179 | $6.0192 \times 10^{-9}$ | $4.9197 \times 10^{-9}$ |
| 0.2 | 1.306879269928204 | 1.306879257935626 | $1.1993 \times 10^{-8}$ | $9.1765 \times 10^{-9}$ |
| 0.3 | 1.351601504414804 | 1.351601534632646 | $3.0218 \times 10^{-8}$ | $2.2357 \times 10^{-9}$ |
| 0.4 | 1.366020440416440 | 1.366020491746745 | $5.1330 \times 10^{-8}$ | $3.7577 \times 10^{-8}$ |
| 0.5 | 1.353553390593274 | 1.353553470343467 | $7.9750 \times 10^{-8}$ | $5.8919 \times 10^{-8}$ |
| 0.6 | 1.316983583242455 | 1.316983693532165 | $1.1029 \times 10^{-7}$ | $8.3744 \times 10^{-8}$ |
| 0.7 | 1.259474520209441 | 1.259474421125751 | $9.9084 \times 10^{-8}$ | $7.8671 \times 10^{-8}$ |
| 0.8 | 1.184736379760737 | 1.184736316453742 | $6.3307 \times 10^{-8}$ | $5.3436 \times 10^{-8}$ |
| 0.9 | 1.096856471680698 | 1.096856438463759 | $3.3217 \times 10^{-8}$ | $3.0284 \times 10^{-8}$ |
| 1.0 | 1.000000000000000 | 1.000000025319537 | $2.5320 \times 10^{-8}$ | $2.5320 \times 10^{-8}$ |

**Example 3.2.** Consider the following singular equation

$$(1-r)u'(r) = u^2(r) + u(r), \quad u(0) = -\frac{1}{2}, \quad 0 \leq r < 1,$$

with true solution $u(r) = \frac{1}{r-2}$. Using our method, taking $r_s = \frac{s-1}{N-1}$, $s = 1, 2, \ldots, N$ and gird points $N = 51$ and $n = 9$, the numerical results are as given in Figure 1.

FIGURE 1. Comparisons of approximate solution with the exact solution (left) and the absolute errors with the relative errors of Example 3.2 (right).

## 4. Conclusions

Here, the reproducing kernel Hilbert space the scheme was implemented for solving a singular Riccati differential equations. This confirms the validity of the present method, and it is efficient, accurate and reliable for singular Riccati differential equations.

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Poster Presentation

# DEMI-CLOSEDNESS PRINCIPLE FOR $(\alpha, \beta)$-GENERALIZED NONEXPANSIVE MAPPINGS

N. KARAMIKABIR

*Department of Mathematics, Hamedan Branch, Islamic Azad University, Hamedan, Iran*
*n.karamikabir@iauh.ac.ir*

Abstract. Here, $(\alpha, \beta)$-nonexpansive mappings in Banach spaces are introduced, which are a generalization of $\alpha$-nonexpansive mappings. Then the demi-closedness principle is presented for these maps.

## 1. Introduction

In 1973, Hardy and Rogers [1] introduced the following concept of generalized nonexpansive mappings. A self-map $F$ on a nonempty subset $E$ of a Banach space is called generalized nonexpansive if for all $x, y \in E$, $\|Fx - Fy\| \leq$

$$a_1\|x-y\| + a_2\|x - Fx\| + a_3\|y - F(y)\| + a_4\|x - F(y)\| + a_5\|y - F(x)\|,$$

where $a_1, \cdots, a_5$ are nonnegative real numbers with $a_1 + a_2 + a_3 + a_4 + a_5 \leq 1$.
By interchanging $x$ and $y$, the above inequality is equivalent to the condition $\|Fx - Fy\| \leq$

$$a\|x-y\| + b\big\{\|x - Fx\| + \|y - F(y)\|\big\} + c\big\{\|x - F(y)\| + \|y - F(x)\|\big\},$$

for all $x, y \in E$, where $a, b, c > 0$ and $a + 2b + 2c \leq 1$, by putting $a = a_1$, $b = (a_2 + a_3)/2$ and $c = (a_4 + a_5)/2$. Here, we study the following case

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in which $a_2 = a_3 = 0$, $a_5 = \alpha$, $a_4 = \beta$ and $a_1 = \big(1-(\alpha+\beta)\big)$. Thus $b = 0$, $c = (\alpha+\beta)/2$ and $a = \big(1-(\alpha+\beta)\big)$.

**Definition 1.1.** A mapping $T$ with domain $D(T)$ and range $R(T)$ in Banach space $E$, is said to be $(\alpha,\beta)$-generalized nonexpansive mapping if there exists $0<\alpha<1$ and $0<\beta<1$ with $\alpha+\beta<1$, such that

$$\|Tx-Ty\| \leq \alpha\|Tx-y\| + \beta\|Ty-x\| + \big(1-(\alpha+\beta)\big)\|x-y\|,$$

for all $x,y \in D(T)$.

In fact, a mapping $T: E \to E$ is nonexpansive if and only if it is $(0,0)$-generalized nonexpansive. Every $(\alpha,\alpha)$-generalized nonexpansive is $\alpha$-nonexpansive which is introduced by Song et al. in 2018 [2]. The followings are direct consequence of Theorems $3,4$ and 6 in [3] and Theorem 2 of [4].

**Theorem 1.2.** *Let $E$ be a compact Banach space and $T$ be a continuous $(\alpha,\beta)$-generalized nonexpansive on $E$. If for any $T$-invariant closed subset $A$ of $E$ with $diam\,A>0$, there exist $y,z \in A$ such that*

$$\sup\{\|y-T^n(z)\| : n \geq 0\} < diam\,A,$$

*then for any $x \in E$, $\{T^n(x)\}$ has a subsequence which converges to a fixed point of $T$.*

The subset $E$ is normal with respect to $T$, if for any nonempty closed convex $T$-invariant subset $A$ of $E$, either $diam\,A = 0$ or there exists $z$ in $A$ such that

$$\sup\{\|y-z\| : y \in A\} < diam A.$$

**Theorem 1.3.** *Let $C$ be a weakly compact convex subset of a Banach space $E$ and $T$ be a continuous $(\alpha,\beta)$-generalized nonexpansive on $C$. If $C$ is normal with respect to $T$, then $T$ has a fixed point.*

Let $C$ be a bounded subset of Banach space $E$ and $y \in C$. We set

$$r_y = \sup\{\|x-y\| : x \in C\}, \text{ and } r = \inf\{r_y : y \in C\}.$$

The point $x$ is called the generalized center of $C$, if $\{x \in C : r_x = r\} = \{x\}$.

**Theorem 1.4.** *Let $C$ be a bounded convex subset of Banach space $E$. Suppose that $C$ has a generalized centre $x_0$. If $T: C \to C$ is $(\alpha,\beta)$-generalized nonexpansive mapping such that $C \subset \overline{coC}$, then $x_0$ is a fixed point of $T$.*

**Theorem 1.5.** *Let $C$ be a closed convex subset of a strictly convex Banach space $E$ and $T$ be a $(\alpha,\beta)$-generalized nonexpansive mapping on $C$. Then*

(a) *The fixed point set $Fix(T) = \{x \in C : T(x) = x\}$ of $T$ is convex.*
(b) *If $T$ is continuous, then $F$ is nonempty and compact.*
(c) *If $T$ is continuous, then for any $x_0 \in C, t \in (0,1)$, $\{T_t^n(x_0)\}$ converges to a fixed point of $T$, where $T_t(x) = (1-t)x + tT(x)$ for every $x \in X$.*

## 2. Main Results

Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ with the inner product $\langle ., . \rangle$ and the norm $\|.\|$. Then for each $u \in H$, the metric projection $P_C : H \to C$ is define by $P_C(u) = z$, $\|z - u\| = \inf_{x \in C} \|x - u\|$. The following property for metric projection $P_C$ is well known:

$$z = P_C(u) \iff \langle u - z, x - z \rangle \leq 0 \qquad \forall x \in C,$$

**Lemma 2.1.** *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and $T : C \to C$ be an $(\alpha, \beta)$-generalized nonexpansive mapping with $Fix(T) \neq \emptyset$. Then $Fix(T)$ is closed convex and*

$$\|Tx - p\| \leq \|x - p\|,$$

*for all $x \in C$ and $p \in Fix(T)$.*

**Lemma 2.2.** *Let $C$ be a nonempty subset of Hilbert space $H$ and $T : C \to C$ be an $(\alpha, \beta)$-generalized nonexpansive mapping. Then for all $x, y \in C$ we have*

$$\|Tx - Ty\| \leq \|x - y\| + \frac{\alpha + \beta}{1 - \beta} \|Tx - x\|. \tag{1}$$

**Theorem 2.3.** *(Demi-closedness principle) Let $C$ be a nonempty closed convex subset of Hilbert space $H$ and $T : C \to C$ be an $(\alpha, \beta)$-generalized nonexpansive mapping. If a sequence $\{x_n\}$ in $C$ converges weakly to $x \in C$ and*

$$\lim_{n \to \infty} \|x_n - Tx_n\| = 0,$$

*then $x \in Fix(T)$.*

**Theorem 2.4.** *Let $C$ be a bounded subset of Hilbert space $H$. If $T : C \to C$ is $(\alpha, \beta)$-generalized nonexpansive mapping and $\alpha \leq \beta$, then $T$ is asymptotically regular, that is, for any $x \in C$,*

$$\lim_{n \to \infty} \|T^{n+1}(x) - T^n(x)\| = 0.$$

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# APPROXIMATE CONVEXITY FOR SET-VALUED MAPS

ZOHREH KEFAYATI* AND MORTEZA OVEISIHA

*Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin 34149-16818, Iran*
*Zohrekefayati68@gmail.com and Oveisiha@sci.ikiu.ac.ir*

Abstract. In this paper, we consider the relationships between generalized approximate convexity for set-valued map, and generalized monotonicity for their subdifferentials. Also, some relations between variational inequalities and quasi efficient solutions under approximate convexity are given.

## 1. Introduction

In recent years, extending and characterizing generalized convex scalar-valued functions to set-valued maps have been studied by many authers. In 1997, Sach and Yen [7] gave some necessary and sufficient conditions for a set-valued map $F$ to be $K$-convex in terms of contingent derivative of the epigraphical multifunction for $F$ with respect to the ordering cone $K$. Also, Yang [9] introduced Dini directional derivative for set-valued mappings and used it to obtain some properties of $K$-convex set-valued maps. Luc et al. [5] introduced the concept of approximate convex functions using the $\varepsilon$-convex functions. The class of approximate convex functions contain the class of convex functions, strongly convex functions and weakly convex functions. The rest of this section is devoted to recall some definitions and results which are needed in the next section.

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Let $X$ and $Y$ be two Banach spaces and $X^*$ be topological dual space of $X$. The norm in $X$ and $X^*$ will be denoted by $\| \, . \, \|$. Also, suppose that $B_X$ is the closed unit ball of $X$, and $K \subset Y$ is a closed convex cone.

**Definition 1.1.** [1] Let $F : X \rightrightarrows Y$ be a set-valued mapping. Then, epigraphical multifunction $\mathcal{E}_F : X \rightrightarrows Y$ is defined by

$$\mathcal{E}_F(x) := \{y \in Y | y \in F(x) + K\}.$$

**Definition 1.2.** [4] Let $X$ be a Banach space, $\Omega$ be a nonempty subset of $X$, $x \in \Omega$ and $\varepsilon \geq 0$. The set of $\varepsilon$-normals to $\Omega$ at $x$ is

$$\hat{N}_\varepsilon(x;\Omega) := \{x^* \in X^* | \ \limsup_{u \to x} \frac{< x^*, u - x >}{\| \, u - x \, \|} \leq \varepsilon\}.$$

If $\varepsilon = 0$, the above set is denoted by $\hat{N}(x;\Omega)$ and called regular normal cone to $\Omega$ at $x$. Let $\bar{x} \in \Omega$, the basic normal cone to $\Omega$ at $\bar{x}$ is

$$N(\bar{x};\Omega) := \limsup_{x \to \bar{x}, \varepsilon \downarrow 0} \hat{N}_\varepsilon(x;\Omega).$$

**Definition 1.3.** [5] Let $f : \Omega \subset X \to \mathbb{R}$ be a real-valued function. $f$ is said to be approximately convex at $x_0 \in \Omega$, if for any $\alpha > 0$, there exists $\delta > 0$ such that for any $x_1, x_2 \in B(x_0, \delta) \cap \Omega$ and any $t \in [0,1]$, one has

$$f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2) + \alpha t(1-t) \, \| \, x_1 - x_2 \, \| \, .$$

**Definition 1.4.** [4] A set-valued mapping $F : \Omega \subset X \rightrightarrows Y$ is said to be

- Lipschitz around $\bar{x} \in domF$ iff there are a neighborhood $U$ of $\bar{x}$ and $l \geq 0$ such that

$$F(x) \subset F(u) + l \, \| \, x - u \, \| \, B_Y, \ \forall x, u \in \Omega \subset X.$$

- epi-Lipschitz around $\bar{x} \in domF$ iff $\mathcal{E}_F$ is Lipschitz around this point.

The set-valued mapping $F$ is locally Lipschitz on $X$, if it is Lipschitz around $\bar{x}$, for every $\bar{x} \in domF \cap X$.
Let $K$ be a closed, convex and pointed cone in $Y$ and denote the positive polar cone of $K$ by

$$K^+ := \{y^* \in Y^* | \ y^*(k) \geq 0, \ \forall k \in K\}.$$

Given $F : X \rightrightarrows Y$ and $y^* \in Y^*$. We associated to $F$ and $y^*$ a marginal function $f_{y^*} : X \to \mathbb{R} \cup \{\pm\infty\}$,

$$f_{y^*}(x) := inf\{y^*(y)| \ y \in F(x)\},$$

and the minimum set

$$M_{y^*} := \{y \in F(x) |\ f_{y^*}(x) = y^*(y)\}.$$

## 2. Main results

First, we present a generalization of approximate convexity for set-valued maps.

**Definition 2.1.** Let $\Omega \subset X$ be a convex set and $F : \Omega \subset X \rightrightarrows Y$.

- $F$ is said to be approximately $K$-convex at $x_0 \in domF$ if for every $\alpha \in \mathbb{R}$ there exists $\delta > 0$ (depending on $x_0$ and $\varepsilon$) such that for all $x_1, x_2 \in B(x_0, \delta)$ and $t \in [0, 1]$, one has

$$tF(x_1)+(1-t)F(x_2)+\alpha t(1-t) \parallel x_1-x_2 \parallel e \subseteq F(tx_1+(1-t)x_2)+K,$$

  for an $e \in intK$ with $\parallel e \parallel = 1$.
- $F$ is said to be approximately weakly $K$-convex type I at $x_0 \in domF$ if for every $\alpha \in \mathbb{R}$ there exist $\delta > 0$ for any $x_i \in B(x_0, \delta)$, $y^* \in K^+ \cap S_{Y^*}$ and $y_i \in M_{y^*}(x_i)$ $(i = 1, 2)$, one has

$$< \xi, x_2 - x_1 > -\alpha \parallel x_2 - x_1 \parallel \leq y^*(y_2) - y^*(y_1),$$

  for some $\xi \in \partial F(x_1, y_1)(y^*)$.
- $F$ is said to be approximately $K$-convex type II at $x_0 \in domF$, if for every $\alpha \in \mathbb{R}$, there exist $\delta > 0$ such that for any $x_1, x_2 \in B(x_0, \delta)$, $y^* \in K^+ \cap S_{Y^*}$ and $y_1 \in M_{y^*}(x_1)$ and $\xi \in \partial F(x_1, y_1)(y^*)$, one has

$$< \xi, x_2 - x_1 > -\alpha \parallel x_2 - x_1 \parallel \leq y^*(y_2) - y^*(y_1).$$

**Definition 2.2.** The set-valued mapping $\partial F_i : X \times Y \times Y^* \rightrightarrows X^*$ is said to be approximately $K$-monotone at $x_0 \in domF$ if for any $\alpha \in \mathbb{R}$, there exist $\delta > 0$ such that for any $x_i \in B(x_0, \delta)$, $y^* \in K^+ \cap S_{Y^*}$, $y_i \in M_{y^*}(x_i)$ and $\xi_i \in \partial F(x_i, y_i)(y^*), (i = 1, 2)$, one has

$$< \xi_2 - \xi_1, x_2 - x_1 > \geq -\alpha \parallel x_2 - x_1 \parallel .$$

Now, we consider the following optimization problem:

$$\min F(x), \text{ subject to } \quad x \in domF, \tag{2.1}$$

where $F : \Omega \subset X \rightrightarrows Y$.
The next two definitions are allocated solutions of Problem (2.1).

**Definition 2.3.** A point $(\bar{x}, \bar{y}) \in grF$ is said to be a scalarized locally quasi efficient solution (SLQE) of Problem (2.1) iff there exist $\alpha \in \mathbb{R}$ and $\delta > 0$, such that for any $y^* \in K^+ \backslash \{0\}$, $x \in B(\bar{x}, \delta) \cap \Omega$ and $y \in F(x)$, one has

$$y^*(\bar{y}) \leq y^*(y) + \alpha \parallel x - \bar{x} \parallel .$$

**Definition 2.4.** A point $(\bar{x}, \bar{y}) \in grF$ is said to be locally weak quasi efficient solution (LWQE) of Problem (2.1) iff there exist $\alpha \in \mathbb{R}$ and $\delta > 0$, such that

$$(F(x) - \bar{y}) \cap (-\alpha \parallel x - \bar{x} \parallel e - intK) = \emptyset, \ \forall x \in B(\bar{x}, \delta) \cap \Omega.$$

The next lemma gives a relation between (SLQE) and (LWQE).

**Lemma 2.5.** *Every solution of (SLQE) is a (LWQE) of Problem* (2.1)*.*

**Theorem 2.6.** *Suppose that $X$, $Y$ are Asplund spaces and $F : X \rightrightarrows Y$ is a locally epi-lipschitz map. If $F$ is approximately $K$-convex at $x_0 \in domF$, then $F$ is approximately weakly $K$-covex type I at this point.*

**Lemma 2.7.** *Let $F : \Omega \subset X \rightrightarrows Y$ be locally lipschitz and approximately weakly $K$-convex type I at $x_0$. Then $\partial F$ is approximately $K$-monotone at $x_0$.*

Now, we consider the following variational inequality:
(MVIP): Minty variational inequality problem consists of finding a vector $\bar{x}$ such that there exists $\delta > 0$ such that for any $x \in B(\bar{x}, \delta) \cap \Omega$ and $y^* \in K^+ \cap S_{Y^*}$, there exist $y \in M_{y^*}(x)$ and $\xi \in \partial F(x, y)(y^*)$ such that

$$< \xi, \bar{x} - x >\leq 0.$$

**Theorem 2.8.** *Let $F : \Omega \subseteq X \rightrightarrows Y$ be approximately weakly $K$-convex type II. If $\bar{x}$ is a solution of (SLQE), then it is a solution of (MVIP).*

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# TRIPLE JENSEN S-FUNCTIONAL EQUATIONS AND TRIPLE LIE HOM-DERIVATION

VAHID KESHAVARZ* AND SEDIGHEH JAHEDI

*Department of Mathematics, Shiraz University of Technology, Shiraz, Iran*
*v.keshavarz68@yahoo.com and jahedi@sutech.ac.ir*

Abstract. In this paper, we introduce the concept of triple Lie hom-derivation on the triple Lie Banach algebras. We prove the Hyers-Ulam stability of a special kind of triple Jensen s-functional equation on these spaces and give some related results.

## 1. Introduction

A ternary Banach algebra $A$ is a complex linear space, endowed with a ternary product $(a_1, a_2, a_3) \to [a_1, a_2, a_3]$ from $A^3$ into $A$ such that

$$[[a_1, a_2, a_3], b_1, b_2] = [a_1, [a_2, a_3, b_1], b_2] = [a_1, a_2, [a_3, b_1, b_2]]$$

and satisfies

$$\|[[a_1, a_2, a_3]\| \leq \|a_1\|.\|a_2\|.\|a_3\| \text{ } and \text{ } \|[a, a, a]\| = \|a\|^3 \text{ (see [8])}.$$

Recall that, a Lie algebra is a Banach algebra endowed with the Lie product $[x, y] := \frac{(xy-yx)}{2}$. Similarly, a triple Lie Banach algebra is a Banach algebra endowed with the product $\Big[[x, y], z\Big] := \frac{[x,y]z-z[x,y]}{2}$.

**Definition 1.1.** [4] Let $A$ be a triple Lie Banach algebra. A $\mathbb{C}$-linear mapping $h : A \to A$ is called triple Lie homomorphism if

$$h([[x, y], z]) = [[h(x), h(y)], h(z)] \qquad \forall\ x, y, z \in A.$$

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**Definition 1.2.** [4] Let $A$ be a triple Lie Banach algebra. A $\mathbb{C}$-linear mapping $D : A \to A$ is called triple Lie derivation if

$$D\Big([x,y,z]\Big) = [[D(x),y],z]+[[x,D(y)],z]+[[x,y],D(z)], \quad \forall\, x,y,z \in A.$$

**Definition 1.3.** Let $A$ be a triple Lie Banach algebras. Let $h : A \to A$ be a triple Lie homomorphism. A mapping $D : A \to A$ is called a triple Lie hom-derivation if $d$ is a $\mathbb{C}$-linear mapping and satisfies

$$\begin{aligned} D([x,y,z]) =& [[D(x),h(y)],h(z)] + [[h(x),D(y)],h(z)] \\ &+ [[h(x),h(y)],D(z)] \qquad \forall\, x,y,z \in A. \end{aligned}$$

The stability problem of functional equations has been first raised by Ulam [7]. In 1941, Hyers [3] gave a first affirmative answer to the question of Ulam for Banach spaces. Th. Rassias [6] then gave a positive answer for both additive mappings and linear mappings by using $\theta(\|x\|^p+\|y\|^p)$ where $p<1$. In 1994, Găvruta [2] generalized these theorems for approximate additive mappings controlled by a function $\varphi(x,y)$. In 2003, L. Cădariu and V. Radu [1] proved the Hyers–Ulam–Rassias stability of the additive Cauchy equation by using the fixed point method.

**Theorem 1.4.** [5] *Let $(A,d)$ be a complete generalized metric space and let $F : A \to A$ be a strictly contractive mapping with Lipschitz constant $0<L<1$. Then for each given element $a \in A$, either*

$$d(F^i(a), F^{i+1}(a)) = \infty$$

*for all nonnegative integers $i$ or there exists a positive integer $i_0$ such that*

$1 : d(F^i(a), F^{i+1}(a)) < \infty, \qquad \forall i \geq i_0$*;*

$2 :$ *the sequence $\{F^i(a)\}$ converges to a unique fixed point $b^*$ of $F$ in the set $B = \{b \in A \mid d(F^{i_0}a, b) < \infty\}$;*

$3 : d(b,b^*) \leq \frac{1}{1-L} d(b,F(b))$ *for all $b \in A$.*

Solution of the equation $2f(\frac{x+y}{2}) = f(x)+f(y)$ is called an Jensen mapping. Consider the triple Jensen type s-functional equations:

$$\begin{aligned} &3f\Big(\frac{x_1+x_2+x_3}{3}\Big) + 2f\Big(\frac{x_1-x_2}{2}\Big) - 2f\Big(\frac{x_1+x_3}{2}\Big) + 2f\Big(\frac{x_2+x_3}{2}\Big) \\ &- f(x_1) - f(x_2) - f(x_3) = s(f(x_1+x_2+x_3) - f(x_1) - f(x_2) - f(x_3)) \end{aligned} \tag{1.1}$$

where $s \neq 0, \pm 1$ is a complex number.

In this paper, we solve the triple Jensen s-functional equation (1.1) in triple Lie Banach algebras and we prove the Hyers–Ulam stability of triple Lie hom-derivation in triple Lie Banach algebras.

## 2. Main results

Throughout the paper, let $\mathbb{T}^1_{1/n_0}$ be the set of all complex numbers $e^{i\theta}$, where $0 \le \theta \le \frac{2\pi}{n_0}$ and $n_0$ is a fix positive integer number. Let $A$ be a triple Lie Banach algebra.

**Lemma 2.1.** *A mapping $f : A \to A$ which satisfies in (1.1) is additive.*

Throughout this section, assume that $\varphi, \psi : A^3 \to [0, \infty)$ are functions satisfy conditions

$$\varphi(\frac{x_1}{3}, \frac{x_2}{3}, \frac{x_3}{3}) \le \frac{L}{3}\varphi(x_1, x_2, x_3), \qquad \psi(\frac{x_1}{3}, \frac{x_2}{3}, \frac{x_3}{3}) \le \frac{L}{3^3}\psi(x_1, x_2, x_3) \tag{2.1}$$

for all $x_1, x_2, x_3 \in A$ and some $0 < L < 1$.

**Theorem 2.2.** *Let $f : A \to A$ be a function satisfies*

$$\begin{aligned} \Big\|3f\Big(\frac{\lambda(x_1+x_2+x_3)}{3}\Big) &+ 2f\Big(\frac{\lambda(x_1-x_2)}{2}\Big) - 2f\Big(\frac{\lambda(x_1+x_3)}{2}\Big) \\ &+ 2f\Big(\frac{\lambda(x_2+x_3)}{2}\Big) - \lambda f(x_1) - \lambda f(x_2) - \lambda f(x_3) \\ &- s\Big(f(\lambda(x_1+x_2+x_3)) - \lambda f(x_1) - \lambda f(x_2) - \lambda f(x_3)\Big)\Big\| \\ &\le \varphi(x_1, x_2, x_3) \end{aligned} \tag{2.2}$$

*where $\lambda \in \mathbb{T}^1_{1/n_0}$ and $\varphi : A^3 \to [0, \infty)$ fulfill (2.1). Then there exist a unique additive mapping $T : A \to A$ such that*

$$\|f(x) - T(x)\| \le \frac{L}{2(1-L)}\varphi(x, 0, 0)$$

In the next theorem, we prove the Hyer-Ulam stability of triple Lie hom-derivation on triple Lie Banach algebras.

**Theorem 2.3.** *Let $f, h : A \to A$ be functions satisfy in (2.2) and*

$$\|h([x_1, x_2, x_3]) - [[h(x_1), h(x_2)], h(x_3)]\| \le \psi(x_1, x_2, x_3) \tag{2.3}$$

$$\begin{aligned} \|f([x_1, x_2, x_3]) - [[f(x_1), h(x_2)], h(x_3)] &- [[h(x_1), f(x_2)], h(x_3)] \\ - [[h(x_1), h(x_2)], f(x_3)]\| &\le \psi(x_1, x_2, x_3) \end{aligned} \tag{2.4}$$

*where $\lambda \in \mathbb{T}^1_{1/n0}$ and $\varphi, \psi : A^3 \to [0, \infty)$ fulfill (2.1). Then there exists a unique homomorphism $H : A \to A$ and a unique hom-derivation*

$D : A \to A$ *such that*

$$\|h(x)-H(x)\| \le \frac{L}{2(1-L)}\varphi(x,0,0), \quad \|f(x)-D(x)\| \le \frac{L}{2(1-L)}\varphi(x,0,0)$$

In Theorem 2.2 and Theorem 2.3, by taking and $L = 2^{1-r}$ and

$$\varphi(x_1,x_2,x_3) = \psi(x_1,x_2,x_3) = \theta(\|x_1\|^r + \|x_2\|^r + \|x_3\|^r)$$

where $x, y, z \in X$, $r < 1$ and $\theta$ are nonnegative real numbers, we obtain the Hyers-Ulam-Rassias stability of Jensen s-functional and the Hyers-Ulam-Rassias stability of triple Lie hom-derivation on triple Banach algebras.

**Corollary 2.4.** *Let $r < 1$ and $\theta$ be two elements of $\mathbb{R}^+$. Suppose that $\varphi(x_1,x_2,x_3) = \theta(\|x_1\|^r + \|x_2\|^r + \|x_3\|^r)$. Assume $f : A \to A$, satisfies in (2.2). Then there exists a unique $\mathbb{C}$-linear $T : A \to A$ such that*

$$\|f(x) - T(x)\| \le \frac{2\theta}{2^r - 2}\|x\|^r$$

**Corollary 2.5.** *Let $r < 1$ and $\theta$ be two elements of $\mathbb{R}_+$. Suppose that $\varphi(x_1,x_2,x_3) = \psi(x_1,x_2,x_3) = \theta(\|x_1\|^r + \|x_2\|^r + \|x_3\|^r)$. Assume $f, h : A \to A$, are functions satisfying in (2.2), (2.3) and (2.4). Then there exist a unique triple homomorphism $H : A \to A$ and a unique triple Lie hom-derivation $D : A \to A$ such that*

$$\|h(x) - H(x)\| \le \frac{2\theta}{2^r - 2}\|x\|^r, \quad \|f(x) - D(x)\| \le \frac{2\theta}{2^r - 2}\|x\|^r$$

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# THE LEFT AND RIGHT NABLA DISCRETE FRACTIONAL DIFFERENCE IN MATRIX STRUCTURE

M. KHALEGHI MOGHADAM* AND Y. KHALILI

*Department of Basic Science, Agricultural and natural source university, Sari, Iran*
*m.khaleghi@sanru.ac.ir*

Abstract. In this paper, we establish the matrix structure for the following discrete fractional operator

$$({}_0\nabla_k^\alpha u)(k), \quad ({}_k\nabla_{T+1}^\alpha u)(k)$$

for $k \in [1,T]_{\mathbb{N}_0}$ where $0 \leq \alpha \leq 1$ and ${}_0\nabla_k^\alpha$ is the left nabla discrete fractional difference and ${}_k\nabla_{T+1}^\alpha$ is the right nabla discrete fractional difference and $\nabla$ is the backward difference operator. An example is included to illustrate the main results.

## 1. Introduction

This kind of problems play a fundamental role in different fields of research, such as mechanical engineering, control systems, economics, computer science, physics, artificial or biological neural networks, cybernetics, ecology and many others[3, 4].
The aim of this paper is to establish the matrix structure for the following discrete fractional operator

$$\left({}_k\nabla_{T+1}^\alpha u\right)(k), \qquad \left({}_0\nabla_k^\alpha u\right)(k)$$

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where $0 \leq \alpha \leq 1$ and ${}_0\nabla_k^\alpha$ is the nabla discrete fractional difference and ${}_k\nabla_{T+1}^\alpha$ is the nabla discrete fractional difference and $\nabla u(k) = u(k) - u(k-1)$ is the backward difference operator and $T \geq 2$ is fixed positive integer and $\mathbb{N}_1 = \{1, 2, 3, \cdots\}$ and ${}_T\mathbb{N} = \{\cdots T-2, T-1, T\}$ and $[1, T]_{\mathbb{N}_0}$ is the discrete set $\{1, 2, \cdots, T-1, T\} = \mathbb{N}_1 \bigcap {}_T\mathbb{N}$.

**Definition 1.1.** [2] **(i)** Let $m$ be a natural number, then the $m$ rising factorial of $t$ is written as $t^{\overline{m}} = \prod_{k=0}^{m-1}(t+k), \quad t^{\overline{0}} = 1$.
**(ii)** For any real number, the $\alpha$ rising function is increasing on $\mathbb{N}_0$ and $t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad$ such that $t \in \mathbb{R} \backslash \{\cdots, -2, -1, 0\}, 0^{\overline{\alpha}} = 0$.

**Definition 1.2.** let $f$ be defined on $\mathbb{N}_{a-1} \bigcap {}_{b+1}\mathbb{N}$, $a < b$, $\alpha \in (0,1)$, then the nabla discrete new left Caputo fractional difference and the right Caputo fractional difference are, respectively, defined by
$\left({}_k^C\nabla_{a-1}^\alpha f\right)(k) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=a}^{k} \nabla_s f(s)(k-\rho(s))^{\overline{-\alpha}}, \quad k \in \mathbb{N}_a$
$\left({}_{b+1}^C\nabla_k^\alpha f\right)(k) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=k}^{b} (-\Delta_s f)(s)(s-\rho(k))^{\overline{-\alpha}}, \quad k \in {}_b\mathbb{N}$,
and in the left Riemann sense by
$\left({}_k^R\nabla_{a-1}^\alpha f\right)(k) = \frac{1}{\Gamma(1-\alpha)} \nabla_k \sum_{s=a}^{k} f(s)(k-\rho(s))^{\overline{-\alpha}}, \quad k \in \mathbb{N}_a$,
and the right Riemann one by
$\left({}_{b+1}^R\nabla_k^\alpha f\right)(k) = \frac{1}{\Gamma(1-\alpha)} (-\Delta_k) \sum_{s=k}^{b} f(s)(s-\rho(k))^{\overline{-\alpha}}, \quad k \in {}_b\mathbb{N}$,
where $\rho(k) = k-1$ be the backward jump operator.

We note that the nabla Riemann fractional differences and the nabla Caputo fractional differences, for $0 < \alpha < 1$, coincide when $f$ vanishes at the end points that is $f(a-1) = 0 = f(b+1)$ [1]. So, for convenience, from now on we will use the symbol $\nabla^\alpha$ instead of ${}^R\nabla^\alpha$ or ${}^C\nabla^\alpha$.

## 2. Main results

Now, define the finite $T-$dimensional Hilbert space

$$W := \{u : [0, T+1]_{\mathbb{N}_0} \to \mathbb{R} : u(0) = u(T+1) = 0, u = (u(1), u(2), ..., u(T))^\dagger\},$$

which $u^\dagger$ denotes the transpose of $u$ and $W$ is equipped with the usual inner product and the norm $\langle u, v\rangle = \sum_{k=1}^{T} u(k)v(k)$, $\|u\| := \langle u, u\rangle = \left(\sum_{k=1}^{T} |u(k)|^2\right)$. So, we set $\ll u \gg := \langle {}_k\nabla_0^\alpha u, {}_k\nabla_0^\alpha u\rangle + \langle {}_{T+1}\nabla_k^\alpha u, {}_{T+1}\nabla_k^\alpha u\rangle$, hence $\ll u \gg = \left\{\sum_{k=1}^{T} |\left({}_k\nabla_0^\alpha u\right)(k)|^2 + |\left({}_{T+1}\nabla_k^\alpha u\right)(k)|^2\right\}$ is an equivalent norm in $W$ . A direct computation shows that,

$$\ll u \gg = u^\dagger \mathbb{A} u, \quad \forall u \in W. \tag{2.1}$$

Next, observe by Definition 1.2 that, for $k \in [1,T]_{\mathbb{N}_0}$

$$\left({}_k\nabla_0^{\alpha}u\right)(k) = \nabla_k \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{k} u(s)(k-\rho(s))^{\overline{-\alpha}},$$

$$\left({}_{T+1}\nabla_k^{\alpha}u\right)(k) = (-\Delta_k)\frac{1}{\Gamma(1-\alpha)} \sum_{s=k}^{T} u(s)(s-\rho(k))^{\overline{-\alpha}}$$

we let

$$z(k) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{k} u(s)(k-\rho(s))^{\overline{-\alpha}}, \quad w(k) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=k}^{T} u(s)(s-\rho(k))^{\overline{-\alpha}}$$

$\widetilde{z}(k) = \left(\nabla_k z\right)(k) = z(k)-z(k-1), \quad \widetilde{w}(k) = \left(-\Delta_k w\right)(k) = w(k)-w(k+1),$
thus, one have

$$\begin{aligned} \ll u \gg &= \sum_{k=1}^{T} \left|\left({}_k\nabla_0^{\alpha}u\right)(k)\right|^2 + \left|\left({}_{T+1}\nabla_k^{\alpha}u\right)(k)\right|^2 \\ &= \sum_{k=1}^{T} \left|\left(\nabla_k z\right)(k)\right|^2 + \left|\left(-\Delta_k z\right)(k)\right|^2 \\ &= \sum_{k=1}^{T} |\widetilde{z}(k)|^2 + |\widetilde{w}(k)|^2 = \|\widetilde{z}\|_2^2 + \|\widetilde{w}\|_2^2, \end{aligned} \tag{2.2}$$

$$z = (z(1), z(2), ..., z(T))^{\dagger}, \quad w = (w(1), w(2), ..., w(T))^{\dagger}$$
$$\widetilde{z} = (\widetilde{z}(1), \widetilde{z}(2), ..., \widetilde{z}(T))^{\dagger}, \quad \widetilde{w} = (\widetilde{w}(1), \widetilde{w}(2), ..., \widetilde{w}(T))^{\dagger}$$

hence, $z = Bu$, $\widetilde{z} = Dz$ and $w = B^{\dagger}u$, $\widetilde{w} = D^{\dagger}w$ where $\dagger$ denotes the transpose and

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ (1-\alpha) & 1 & 0 & \cdots & 0 \\ \frac{1}{2!}(2-\alpha)(1-\alpha) & (1-\alpha) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(T-\alpha-1)(T-\alpha-2)\cdots(1-\alpha)}{(T-1)!} & \frac{(T-\alpha-2)(T-\alpha-3)\cdots(1-\alpha)}{(T-2)!} & \cdots & \cdots & 1 \end{bmatrix}_{T\times T}.$$

$$D = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & -1 & 1 \end{bmatrix}_{T\times T}.$$

It is clear that $BD = DB$ and $B^\dagger D^\dagger = D^\dagger B^\dagger$, let $A = (DB)^\dagger DB$, $\widetilde{A} = DB(DB)^\dagger$. Hence, for all $u \in W$

$$u^\dagger A u = u^\dagger (DB)^\dagger DBu = z^\dagger D^\dagger Dz = \widetilde{z}^\dagger \widetilde{z} = \|\widetilde{z}\|_2^2, \tag{2.3}$$

$$u^\dagger \widetilde{A} u = u^\dagger DB(DB)^\dagger u = w^\dagger DD^\dagger w = \widetilde{w}^\dagger \widetilde{w} = \|\widetilde{w}\|_2^2, \tag{2.4}$$

Let $\mathbb{A} = A + \widetilde{A}$, thus

$$u^\dagger \mathbb{A} u = u^\dagger A u + u^\dagger \widetilde{A} u = \|\widetilde{z}\|_2^2 + \|\widetilde{w}\|_2^2,$$

therefore from (2.2) and (2.3) and (2.4), we have $\ll u \gg = u^\dagger \mathbb{A} u, \quad \forall u \in W$. Let $\lambda_{\min}$ and $\lambda_{\max}$ denote respectively the minimum and the maximum eigenvalues of $\mathbb{A}$, for any $u \in W$, we have

$$\lambda_{\min}\langle u, u\rangle < u^\dagger \mathbb{A} u < \lambda_{\max}\langle u, u\rangle, \tag{2.5}$$

Then from (2.1) and (2.5), $\ll u \gg \to +\infty$ if and only if $\langle u, u\rangle \to +\infty$. We note that, this matrix can be useful in solving problems of differential equations with fractional order in discrete case.

We now present an example to illustrate the result.

**Example 2.1.** Let, $T = 3$ and $\alpha = 0.5$. Then $\lambda_{\min} = \lambda_1 = 1 < \lambda_2 = 5 < 18 = \lambda_{\max}$ where

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ \frac{5}{8} & \frac{1}{2} & 1 \end{bmatrix}, \; D = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \; \mathbb{A} = \begin{bmatrix} 7 & -6 & 2 \\ -6 & 10 & -6 \\ 2 & -6 & 7 \end{bmatrix}.$$

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# DISCONTINUOUS INVERSE PROBLEMS FOR A DIFFERENTIAL PENCIL WITH THE TURNING POINT

Y. KHALILI* AND M. KHALEGHI MOGHADAM

*Department of Basic Sciences, Sari Agricultural Sciences and Natural Resources University, 578 Sari, Iran*
*y.khalili@sanru.ac.ir*

Abstract. In this work, a differential pencil with the turning point and jump condition on the half line is studied. We establish properties of the spectrum, give the formulation of the inverse problem and prove the uniqueness theorem.

## 1. Introduction

Consider the boundary value problem $B = B(q_1, q_0, \beta_1, \beta_0, a_1, a_2)$ for second order differential pencil

$$y''(x) + (\rho^2 r(x) + i\rho\, q_1(x) + q_0(x))y(x) = 0, \quad x \geq 0, \tag{1.1}$$

with the spectral boundary condition

$$U(y) := y'(0) + (\beta_1\rho + \beta_0)y(0) = 0, \tag{1.2}$$

and the discontinuous conditions

$$y(d+0,\rho) = y(d-0,\rho), \qquad y'(d+0,\rho) = a_1 y'(d-0,\rho) + a_2 y(d-0,\rho), \tag{1.3}$$

at $x = d$. Let $r(x) = -1,\ x < a$ and $r(x) = 1,\ x \geq a$. The weight-function $r(x)$ changes the sign in an interior point $x = a$ $(0 < a < d)$, which is called the turning point. The coefficients $\beta_0,\ \beta_1 (\beta_1 \neq \pm 1)$, $a_1 > 0$ and $a_2$ are complex numbers and $\rho$ is a spectral parameter.

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The complex-valued functions $q_0(x)$ and $q_1(x)$ satisfy $(1+x)q_l^{(\iota)}(x) \in L(0,\infty)$ as $0 \le \iota \le l \le 1$ and $q_1(x)$ is absolutely continuous.

Differential equations with a turning point and jump condition arise in various branches of natural sciences like physics, geophysics, etc. [3]. The inverse problem for differential pencils without turning point and discontinuities was studied in [1]. The presence of the turning point and discontinuity produces essential modifications in the problem. Some researchers have investigated the discontinuous inverse problem for differential pencils with the turning point. In this work, we study the inverse problem for the discontinuous differential pencil with the turning point and spectral boundary condition.

## 2. Preliminaries

Denote $\Pi_+ := \{\rho : \Im\rho > 0\}$, $\Pi_0 := \{\rho : \Im\rho = 0\}$ and $\Pi := \overline{\Pi_+} \setminus \{0\}$. There exists the Jost solution of Eq. (1.1) for sufficiently large $\rho \in \overline{\Pi_+}$ with the following formula for $m = 0, 1$,

$$e^{(m)}(x,\rho) = (i\rho)^m exp(i\rho x - Q(x))[1], \qquad x > d, \tag{2.1}$$

where $Q(x) = \frac{1}{2}\int_0^x q_1(t)dt$ and $[1] := 1 + O(\rho^{-1})$ (see [4]).
Also uniformly for $a \le x < d$ and sufficiently large $\rho \in \overline{\Pi_+}$, we have

$$\begin{aligned} e^{(m)}(x,\rho) = \frac{(i\rho)^m}{a_1}&(\exp(i\rho x - Q(x))[b_+] \\ &+(-1)^{m+1}\exp(i\rho(2d-x) - (2Q(d) - Q(x)))[b_-]), \end{aligned} \tag{2.2}$$

where $[b_\pm] := b_\pm + O(\rho^{-1})$ in which $b_\pm = \frac{1\pm a_1}{2}$.
For every fixed $x$, the functions $e^{(m)}(x,\rho)$, $m = 0, 1$, are holomorphic and continuous for $\rho \in \Pi_+$ and $\rho \in \overline{\Pi_+}$, respectively. Also these functions are continuously differentiable for $\rho \in \Pi$.
Taking the Birkhoff FSS [4], we will have for sufficiently large $\rho \in \overline{\Pi_+}$,

$$\begin{aligned} e^{(m)}(x,\rho) = \frac{\rho^m}{2a_1}&(E_+(\rho)\exp(\rho(x-a) - i(Q(x) - Q(a))) \\ &+E_-(\rho)(-1)^m\exp(-\rho(x-a) + i(Q(x) - Q(a)))),\ x \in [0,a), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} E_\pm(\rho) = (1\pm i)&\exp(i\rho a - Q(a))[b_+] \\ &-(1\mp i)\exp(i\rho(2d-a) - (2Q(d) - Q(a)))[b_-]. \end{aligned}$$

It follows from (2.1), (2.2) and (2.3) that for $m = 0, 1$ and $C = const$,

$$\begin{aligned} |e^{(m)}(x,\rho)| &\le C|\rho|^m\exp(-|\Im\rho|x), \qquad x \in [a,d) \cup (d,\infty), \\ |e^{(m)}(x,\rho)| &\le C|\rho|^m\exp(-|\Im\rho|a)\exp(|\Re\rho|(a-x)), \quad x \in [0,a] \end{aligned} \tag{2.4}$$

Let us consider $\Delta(\rho)=e'(0,\rho)+(\beta_1\rho+\beta_0)e(0,\rho)$ as the characteristic function of $B$ which is entire in $\rho\in\Pi_+$. Therefore by using (2.3), we can give for sufficiently large $\rho\in\Pi$,

$$\begin{aligned}\Delta(\rho)=\frac{\rho}{2a_1}&(N_+(\rho)(\beta_1-1)\exp(\rho a-iQ(a))\\&+N_-(\rho)(\beta_1+1)\exp(-\rho a+iQ(a))),\end{aligned}\tag{2.5}$$

where

$$\begin{aligned}N_\pm(\rho)=&(1\mp i)\exp(i\rho a-Q(a))[b_+]\\&-(1\pm i)\exp(i\rho(2d-a)-(2Q(d)-Q(a)))[b_-].\end{aligned}$$

Let $\delta>0$ be fixed. Put $G_\delta:=\{\rho\in\overline{\Pi_+};|\ \rho-\rho_n\ |\geq\delta,\ \rho_n\in\Lambda\}$, as $\Lambda=\{\rho\in\Pi_+\cup\mathbb{R};\Delta(\rho)=0\}$. Taking (2.5) one gets

$$|\Delta(\rho)|\geq C|\rho|\exp(-\Im\rho a)\exp(\Re\rho a),\quad \rho\in G_\delta.\tag{2.6}$$

By the Rouche's theorem [2] and the known technique [5], one can give that the roots of $\Delta(\rho)$ have the asymptotics

$$\rho_n=\frac{1}{a}\left(n\pi i+iQ(a)+\kappa_1+\kappa_2\right)+O\left(n^{-1}\right),$$

for large enough $n$, wherein $\kappa_1=\frac{1}{2}ln\frac{\beta_1+1}{\beta_1-1}$ and $\kappa_2=\frac{1}{2}ln\frac{i+1}{i-1}$.

We put

$$\phi(x,\rho)=\frac{e(x,\rho)}{\Delta(\rho)}.\tag{2.7}$$

The function $\phi(x,\rho)$ is a solution of Eq. (1.1) that is called the Weyl solution for BVP($B$). Denote $M(\rho)=\phi(0,\rho)$. We will call it the Weyl function for BVP($B$).

From (2.7) and properties of $e(x,\rho)$ and $\Delta(\rho)$, we will have

$$M(\rho)=\frac{1}{\rho(\beta_1\mp 1)}[1].\tag{2.8}$$

**Inverse Problem.** Given the Weyl function $M(\rho)$, find $q_1,q_0,\beta_1,\beta_0$.

## 3. Uniqueness theorem

A boundary value problem $\widetilde{B}=B(\widetilde{q}_1,\widetilde{q}_0,\widetilde{\beta}_1,\widetilde{\beta}_0,a_1,a_2)$ of the similar form (1.1)-(1.3) with tilde, alongside $B=B(q_1,q_0,\beta_1,\beta_0,a_1,a_2)$, is considered. We suppose that if $\alpha$ signifies an object relevant to $B$, then $\widetilde{\alpha}$ will signify the similar object relevant to $\widetilde{B}$.

**Theorem 3.1.** *Let $M(\rho)=\widetilde{M}(\rho)$. Then $\beta_1=\widetilde{\beta}_1$, $\beta_0=\widetilde{\beta}_0$, $q_1(x)=\widetilde{q}_1(x)$ and $q_0(x)=\widetilde{q}_0(x)$ a.e. on $x\geq 0$.*

**Proof.** By the assumption of theorem and the Weyl function (2.8), we infer that $\beta_1 = \widetilde{\beta}_1$.

Now we define the matrix $P(x,\rho) = [P_{j,k}(x,\rho)]_{j,k=1,2}$ , by the formula

$$P(x,\rho)\begin{bmatrix}\widetilde{\varphi}(x,\rho) & \widetilde{\phi}(x,\rho)\\ \widetilde{\varphi}'(x,\rho) & \widetilde{\phi}'(x,\rho)\end{bmatrix} = \begin{bmatrix}\varphi(x,\rho) & \phi(x,\rho)\\ \varphi'(x,\rho) & \phi'(x,\rho)\end{bmatrix},$$

where $\varphi(x,\rho) = \varphi_1(x,\rho) - (\beta_1\rho + \beta_0)\varphi_2(x,\rho)$ in which the entire functions $\varphi_j(x,\rho)$ are the discontinuous solutions of Eq. (1.1) under the jump condition (1.3) and the initial conditions $\varphi_j^{(m)}(0,\rho) = \delta_{j(m+1)}, m = 0,1$ ($\delta_{j(m+1)}$ is the Kronecker delta). From the assumption of the theorem, the functions $P_{j1}(x,\rho)$ and $P_{j2}(x,\rho)$ are entire in $\rho$. Also, from (2.4), (2.6), (2.7) and similar inequalities for $\varphi(x,\rho)$, we have

$$|P_{11}(x,\rho)| \le C, \qquad |P_{12}(x,\rho)| \le C|\rho|^{-1}. \tag{3.1}$$

Thus we will have for sufficiently large $\rho$,

$$P_{11}(x)\widetilde{\varphi}(x,\rho) = \varphi(x,\rho), \qquad P_{11}(x)\widetilde{\phi}(x,\rho) = \phi(x,\rho). \tag{3.2}$$

Taking the functions $\varphi(x,\rho)$ and $\phi(x,\rho)$ in $[0,a]$, one gets for sufficiently large $\rho$ and $\arg\rho \in (0,\frac{\pi}{2})$ :

$$\frac{\varphi(x,\rho)}{\widetilde{\varphi}(x,\rho)} = \exp(-i(Q(x) - \widetilde{Q}(x)))[1], \ \frac{\phi(x,\rho)}{\widetilde{\phi}(x,\rho)} = \exp(i(Q(x) - \widetilde{Q}(x)))[1].$$

Together with (3.2), this yeids that $Q(x) = \widetilde{Q}(x)$ and $P_{11}(x) = 1$ for $x \in [0,a]$. Similarly we can prove that $Q(x) = \widetilde{Q}(x)$ for $x \in [a,d) \cup (d,\infty)$ and $P_{11}(x) = 1$. So $q_1(x) = \widetilde{q}_1(x)$, $q_0(x) = \widetilde{q}_0(x)$ a.e. for $x \ge 0$, and $\beta_0 = \widetilde{\beta}_0$. The proof is completed.

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# SOME EXTENSIONS OF THE OPERATOR KANTOROVICH INEQUALITY IN HILBERT SPACES

YASER KHATIB

*Department of Mathematics, Mashhad Branch,*
*Islamic Azad University, Iran.*
*yaserkhatibam@yahoo.com*

Abstract. In this paper, we present some extensions of Kantorovich inequality for two operators on a Hilbert space. Moreover, the multiple version and a related inequality for positive linear maps are obtained. Also, we introduce the concept of Specht's ratio and improve some inequalities related to Specht's ratio.

## 1. Introduction

The Kantorovich inequality state that if $A$ is a positive operator on a Hilbert space $\mathcal{H}$ satisfying $MI_{\mathcal{H}} \geq A \geq mI_{\mathcal{H}}$ for positive scalars $M, m$, then for every unit vector $x$ in $\mathcal{H}$

$$\langle Ax, x\rangle\langle A^{-1}x, x\rangle \leq \frac{(M+m)^2}{4Mm}. \tag{1.1}$$

Also the Kantorovich type inequality for positive linear maps asserts that if $A$ is a positive operator on a Hilbert space $\mathcal{H}$ satisfying $MI_{\mathcal{H}} \geq A \geq mI_{\mathcal{H}}$ for positive scalars $M, m$ with $m < M$, and $\varphi$ is a normalized positive linear map from $B(\mathcal{H})$ to $B(\mathcal{K})$, where $\mathcal{H}, \mathcal{K}$ are Hilbert spaces, then

$$\varphi(A^{-1}) \leq \frac{(M+m)^2}{4Mm}\varphi(A)^{-1}. \tag{1.2}$$

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*Key words and phrases.* Kantorovich inequality, positive operators, normalized positive linear map, geometric mean of operators.

Several mathematicians have explored Kantorovich inequality for matrices and operators on a Hilbert space, see [1, 4, 6]. This note intends to present an extension of inequalities (1.1) and (1.2). At first, let us recall some definitions and concepts from [5].
The aim of this paper is to describe some new inequalities for operators in Hilbert space $\mathcal{H}$. We improve and generalize these inequalities by Specht's ratio concept.
Firstly, we state the Specht's ratio concept and some properties. The Specht's ratio [2] was defined by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}, (h \neq 1 \;\; h > 0.) \tag{1.3}$$

The Specht's ratio has some properties in the following:

(i) $S(1) = 1$ and $S(h) = S(\frac{1}{h}) > 1$, for $h > 0$.
(ii) $S(h)$ is a monotone increasing function on $(1, \infty)$.
(iii) $S(h)$ is a monotone decreasing function on $(0, 1)$.

**Lemma 1.1.** [3, Theorem 1] *For $a, b > 0$ and $\nu \in [0, 1]$,*
$(1-\nu)a + \nu b \geq S((\frac{b}{a})^r)a^{1-\nu}b^{\nu}$,
*where $r = \min\{\nu, 1-\nu\}$ and $S(.)$ is the Spech's ratio.*

**Theorem 1.2.** [3, Theorem 2] *Let $A$ and $B$ be two positive operators and $m, m', M, M'$ be positive real numbers satisfying one of the following conditions:*

(i) $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$.
(ii) $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$,

*with $h \equiv \frac{M}{m}$, then we have*

$$\begin{aligned} (1-\nu)A + \nu B &\geq S(h^r)A\natural_\nu B \\ &\geq A\natural_\nu B \\ &\geq S(h^r)(1-\nu)A^{-1} + \nu B^{-1} \\ &\geq (1-\nu)A^{-1} + \nu B^{-1}. \end{aligned}$$

*where $\nu \in [0, 1]$, $r = \min\{\nu, 1-\nu\}$, $S(.)$ is the Spech's ratio.*

*Remark* 1.3. Note that if $A = aI$, $B = bI$, $\nu = \frac{1}{2}$, and $r = \frac{1}{2}$ in Theorem 1.2, then

$$S(\sqrt{h})\sqrt{ab} \leq \tfrac{a+b}{2},$$

## 2. Main results

**Theorem 2.1.** *Let $A$ and $B$ be two positive operators on a Hilbert space $\mathcal{H}$, satisfying $A \leq aI_{\mathcal{H}}, B \leq bI_{\mathcal{H}}$, $AB = BA$ and $AB \leq I_{\mathcal{H}}$ for positive scalars $a, b$. If $\varphi$ is a normalized positive linear map from $B(\mathcal{H})$*

*to $B(\mathcal{K})$, and $m, m', M, M'$ are positive real numbers satisfying one of the following conditions:*

(i) $0 < m'I \leq bA \leq mI < MI \leq aB \leq M'I$.
(ii) $0 < m'I \leq aB \leq mI < MI \leq bA \leq M'I$,

*with $h \equiv \frac{M}{m}$, then for every unit vector $x$ in $\mathcal{H}$*

$$\langle \varphi(A)x, x\rangle \langle \varphi(B)x, x\rangle \leq \frac{(ab+1)^2}{4S^2(\sqrt{h})ab}. \tag{2.1}$$

*and*

$$\langle Ax, x\rangle \langle Bx, x\rangle \leq \frac{(ab+1)^2}{4S^2(\sqrt{h})ab}. \tag{2.2}$$

**Example 2.2.** Let

$$\text{A}=\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \quad \text{and} \quad \text{B}=\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}.$$

Therefore $A \leq 2I$, $B \leq 6I$, $AB = BA$ and $AB \leq I$. In this case the condition (ii) of theorem 2.1 satify. Then by equality 1.3, we have $S^2(\sqrt{h}) = 1/955967187$. Moreover, for unit vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we have $\langle Ax, x\rangle = \frac{1}{2}x_1^2 + \frac{2}{3}x_2^2$.
Then

$$\tfrac{1}{2} = \min\{\tfrac{1}{2}, \tfrac{2}{3}\} \leq \langle Ax, x\rangle \leq \max\{\tfrac{1}{2}, \tfrac{2}{3}\} = \tfrac{2}{3}.$$

Similarly, we can write

$$\tfrac{1}{3} = \min\{\tfrac{1}{3}, \tfrac{3}{4}\} \leq \langle Bx, x\rangle \leq \max\{\tfrac{1}{3}, \tfrac{3}{4}\} = \tfrac{3}{4}.$$

Hence $\langle Ax, x\rangle\langle Bx, x\rangle \leq \frac{1}{2}$. Also, $\frac{(ab+1)^2}{4S^2(\sqrt{h})ab} = 1/8000472384$. Then we get

$$\langle Ax, x\rangle\langle Bx, x\rangle \leq \tfrac{(ab+1)^2}{4S^2(\sqrt{h})ab}.$$

Therfore, we find two matrices that satisfy in conditions of inequality (2.2) in Theorem 2.1.

**Theorem 2.3.** *Let $A$ and $B$ be two positive operators on $\mathcal{H}$ satisfying $0 < A \leq aI_{\mathcal{H}}$, $0 < B \leq bI_{\mathcal{H}}$ ,$AB = BA$ and $AB \leq I_{\mathcal{H}}$ for positive scalars $a, b$. If $\varphi$ is a normalized positive linear map from $B(\mathcal{H})$ to $B(\mathcal{K})$, and $m, m', M, M'$ are positive real numbers satisfying one of the following conditions:*

(i) $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$.
(ii) $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$,

*with $h \equiv \frac{M}{m}$, then*

$$\varphi\left(A\right) \natural \varphi\left(B\right) \leq \tfrac{ab+1}{2S(\sqrt{h})(ab)^{\frac{1}{2}}}.$$

**Theorem 2.4.** *Let $A$ and $B$ be two positive operators on a Hilbert space $\mathcal{H}$, satisfying $A \leq aI_{\mathcal{H}}$, $B \leq bI_{\mathcal{H}}$, $AB = BA$ and $AB \leq I_{\mathcal{H}}$ for positive scalars $a, b$. If $\varphi$ is a normalized multiplicative positive linear map from $B(\mathcal{H})$ to $B(\mathcal{K})$, and $m, m', M, M'$ are positive real numbers satisfying one of the following conditions:*

(i) $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$.

(ii) $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$,

*with $h \equiv \frac{M}{m}$, then*

$$\langle\varphi(A^2)x, x\rangle\langle\varphi(A)\varphi(B)x, x\rangle \leq \frac{(ab+1)^2}{4S^2(\sqrt{h})ab}\langle\varphi(A)x, x\rangle^2, \tag{2.3}$$

*and*

$$\langle A^2x, x\rangle\langle ABx, x\rangle \leq \frac{(ab+1)^2}{4S^2(\sqrt{h})ab}\langle Ax, x\rangle^2. \tag{2.4}$$

**Theorem 2.5.** *Let $A$ and $B$ be two positive operators on a Hilbert space $\mathcal{H}$, satisfying $A \leq aI_{\mathcal{H}}$, $B \leq bI_{\mathcal{H}}$, $AB = BA$ and $AB \leq I_{\mathcal{H}}$ for positive scalars $a, b$ and $A$ is invertible. If $\varphi$ is a normalized positive linear map from $B(\mathcal{H})$ to $B(\mathcal{K})$, then for every unit vector $x$ in $\mathcal{H}$*

$$\varphi(B) - \varphi(A)^{-1} \leq (\sqrt{b} - \tfrac{1}{\sqrt{a}})^2 I_{\mathcal{K}}.$$

**Corollary 2.6.** *Let $A$ and $B$ be two positive operators on a Hilbert space $\mathcal{H}$, satisfying $A \leq aI_{\mathcal{H}}$, $B \leq bI_{\mathcal{H}}$, $AB = BA$ and $AB \leq I_{\mathcal{H}}$ for positive scalars $a, b$ and $A$ is invertible. Then for every unit vector $x$ in $\mathcal{H}$*

$$\langle Bx, x\rangle - \langle Ax, x\rangle^{-1} \leq (\sqrt{b} - \frac{1}{\sqrt{a}})^2.$$

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# REFINEMENTS OF NUMERICAL RADIUS INEQUALITIES IN HILBERT SPACES

YASER KHATIB* AND MAHMOUD HASSANI.

*Department of Mathematics, Mashhad Branch,*
*Islamic Azad University, Iran.*
*yaserkhatibam@yahoo.com*
*mhassanimath@gmail.com*

Abstract. We present some inequalities related to the powers of numerical radius inequalities of Hilbert space operators. Some results that employ the Hermite–Hadamard inequality for vectors in normed linear spaces are also obtained.

## 1. Introduction

Let $B(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle .,. \rangle$. We recall some definitions and concepts from [5]. The numerical radius satisfies

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|. \tag{1.1}$$

The second inequality in (1.1) has been improved in [4, Theorem 1] as follows:

$$\omega(A) \leq \frac{1}{2}\||A| + |A^*|\| \leq \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}}) \tag{1.2}$$

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*Corresponding author.

for every operator $A \in \mathcal{B}(\mathcal{H})$. The left hand of inequality (1.2) was extended in [3, Theorem 1] as follows:

$$\omega^r(A) \leq \frac{1}{2}\left\| |A|^{2r\nu} + |A^*|^{2r(1-\nu)} \right\|, \quad r \geq 1, 0 < \nu < 1, \tag{1.3}$$

which, this inequality will be improved in the end of this paper. Dragomir in [1, Theorem 1], proved the following inequality by the product of two operators:

$$\omega^r(B^*A) \leq \frac{1}{2}\left\| |A|^{2r} + |B|^{2r} \right\|, \quad r \geq 1. \tag{1.4}$$

By using of operator inequality, we improve the inequality (1.4). Recall that the Specht's ratio [2] was defined by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}} \quad (h \neq 1)$$

for a positive real number $h$, and it has some properties as follows:

(i) $S(1) = 1$ and $S(h) = S(\frac{1}{h}) > 1$ for $h > 0$.
(ii) $S(h)$ is a monotone increasing function on $(1, \infty)$.
(iii) $S(h)$ is a monotone decreasing function on $(0, 1)$.

**Lemma 1.1.** *For $a, b > 0$ and $\nu \in [0, 1]$, it follows that $(1-\nu)a + \nu b \geq S((\frac{b}{a})^r)a^{1-\nu}b^\nu$, where $r = \min\{\nu, 1-\nu\}$ and $S(.)$ is the Specht's ratio.*

**Theorem 1.2.** *Let $A$ and $B$ be two positive operators and let $m, m', M, M'$ be positive real numbers satisfying the following conditions (i) or (ii):*

(i) $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$,
(ii) $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$,

*with $h = \frac{M}{m}$. Then*

$$\begin{aligned}(1-\nu)A + \nu B \geq S(h^r)A\natural_\nu B \geq A\natural_\nu B &\geq S(h^r)\{(1-\nu)A^{-1} + \nu B^{-1}\}^{-1} \\ &\geq \{(1-\nu)A^{-1} + \nu B^{-1}\}^{-1},\end{aligned}$$

*where $\nu \in [0, 1]$, $r = \min\{\nu, 1-\nu\}$, and $S(.)$ is the Specht's ratio.*

*Remark* 1.3. Note that if $A = aI$, $B = bI$, $\nu = \frac{1}{2}$, and $r = \frac{1}{2}$ in Theorem 1.2, then

$$S(\sqrt{h})\sqrt{ab} \leq \tfrac{a+b}{2},$$

## 2. Main results

**Lemma 2.1.** *Let $f$ be a twice differentiable on $[a,b]$. If $f$ is convex such that $f'' \geq \lambda := \min_{x\in[a,b]} f(x) > 0$. Then*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{f(a)+f(b)}{2} - \frac{1}{8}\lambda(b-a)^2. \tag{2.1}$$

**Theorem 2.2.** *Let $A, B, X \in \mathcal{B}(\mathcal{H})$, let the continuous functions $f$ and $g$ be non-negative functions on $[0,\infty)$ satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$, and let $k$ be a non-negative increasing convex function on $[0,\infty)$ and twice differentiable such that $k'' \geq \lambda > 0$, with $k(0) = 0$. Also let the positive real numbers $m, m', M, M'$ and $h = \frac{M}{m}$ satisfy one of the following conditions:*

(i) $0 < m' \leq \langle B^* f^2(|X|)Bx, x\rangle \leq m < M \leq \langle A^* g^2(|X^*|)Ax, x\rangle \leq M'$

(ii) $0 < m' \leq \langle A^* f^2(|X|)Ax, x\rangle \leq m < M \leq \langle B^* g^2(|X^*|)Bx, x\rangle \leq M'$,

$$k\big(\omega(A^*XB)\big) \leq \frac{1}{2S(\sqrt{h})}\big\|k(B^*f^2(|X|)B)+k(A^*g^2(|X^*|)A)\big\| - \inf_{\|x\|=1}\xi(x), \tag{2.2}$$

*whenever*

$$\xi(x) = \frac{1}{8S(\sqrt{h})}\lambda\big(\big\langle\big(A^*g^2(|X^*|)A - B^*f^2(|X|)B\big)x, x\big\rangle\big)^2,$$

**Corollary 2.3.** *Let the assumptions of Theorem 2.2 hold. By taking $k(t) = t^2$ on $[0,\infty)$, thus the required $\lambda$ would be '2'. (i) If $0 < m'I < B^*|X|B \leq mI < MI \leq A^*|X^*|A < M'I$ or $0 < m'I < A^*|X^*|A \leq mI < MI \leq B^*|X|B < M'I$, for positive real numbers $m, m', M, M'$, then*

$$\omega^2(A^*XB) \leq \frac{1}{2S(\sqrt{h})}\big\|\big(A^*|X^*|A\big)^2 + \big(B^*|X|B\big)^2\big\| - \inf_{\|x\|=1}\frac{1}{4S(\sqrt{h})}\big(\big\langle\big(A^*|X^*|A - B^*|X|B\big)x, x\big\rangle\big)^2,$$

*(ii) If $X = I$ holds in conditions of (i), then*

$$\omega^2(A^*B) \leq \frac{1}{2S(\sqrt{h})}\Big\||A|^4+|B|^4\Big\| - \inf_{\|x\|=1}\frac{1}{4S(\sqrt{h})}\big(\big\langle\big(A^*A-B^*B\big)x, x\big\rangle\big)^2,$$

*which improves inequality (1.4) in especial conditions. (iii) If $A = B = I$ holds in conditions of (i), then*

$$\omega^2(X) \leq \frac{1}{2S(\sqrt{h})}\Big\||X^*|^2 + |X|^2\Big\| - \inf_{\|x\|=1}\frac{1}{4S(\sqrt{h})}\big(\big\langle\big(|X^*| - |X|\big)x, x\big\rangle\big)^2,$$

**Theorem 2.4.** *Suppose that $A, B, C, D$ in $B(\mathcal{H})$ are operators that $f$ is a positve increasing operator convex function on $\mathbb{R}$ and also that $f$ is twice differentiable such that $f'' \geq \lambda > 0$, with $f(0) = 0$. Let the positive real numbers $m, m', M, M'$ satisfy one of the following conditions:*

(i) $0 < m'I \leq A^*|B|^2A \leq mI \leq MI \leq D|C^*|^2D^* \leq M'I$
(ii) $0 < m'I \leq D|C^*|^2D^* \leq mI \leq MI \leq A^*|B|^2A \leq M'I,$

*with $h = \frac{M}{m}$. Then for every $x, y \in \mathcal{H}$, it follows that*

$$f(|\langle DCBAx, y\rangle|) \leq \frac{1}{2S(\sqrt{h})}\big[\langle f(A^*|B|^2A)x, x\rangle + \langle f(D|C^*|^2D^*)y, y\rangle \tag{2.3}$$

$$-\frac{1}{4}\lambda(\langle A^*|B|^2Ax, x\rangle - \langle D|C^*|^2D^*y, y\rangle)^2\big],$$

**Corollary 2.5.** *Suppose that $T$ in $B(\mathcal{H})$ that $f$ is a positve increasing operator convex function on $\mathbb{R}$ and also that $f$ is twice differentiable such that $f'' \geq \lambda > 0$. Let the positive real numbers $m, m', M, M'$ satisfy one of the following conditions:*

(i) $0 < m'I \leq |T|^{2\alpha} \leq mI \leq MI \leq |T^*|^{2\beta} \leq M'I$
(ii) $0 < m'I \leq |T^*|^{2\beta} \leq mI \leq MI \leq |T|^{2\alpha} \leq M'I,$

*with $h = \frac{M}{m}$. Then for every $x, y \in \mathcal{H}$ and $\alpha, \beta \in [0, 1]$ (with $\alpha+\beta \geq 1$), it follows that*

$$f\Big(\Big|\Big\langle T|T|^{\alpha+\beta-1}x, y\Big\rangle\Big|\Big) \leq \frac{1}{2S(\sqrt{h})}\Big[\Big\langle f(|T|^{2\alpha})x, x\Big\rangle + \Big\langle f\big(|T^*|^{2\beta}\big)y, y\Big\rangle \tag{2.4}$$

$$-\frac{1}{4}\lambda\Big(\langle |T|^{2\alpha}x, x\rangle - \langle |T^*|^{2\beta}y, y\rangle\Big)^2\Big],$$

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# COMMON BEST PROXIMITY POINTS THEOREM FOR TWO F-CONTRACTIVE NON-SELF MAPPINGS

P. LO′LO′* AND E. MOVAHEDNIA

*Department of Mathematics, Behbahan Khatam Alanbia University of Technology, Behbahan, Iran*
*lolo@bkatu.ac.ir*
*movahednia@bkatu.ac.ir*

Abstract. In this paper, we first prove the existence and uniqueness of a common best proximity point for a pair of non-self mappings satisfying in a F- contractive condition. Some interesting consequences including fixed point results are presented.

## 1. Introduction

In a recent paper, Wardowski [1] presented a new contraction, which called $F$-contraction and proved a fixed point results in complete metric spaces. Then Omidvari et al.[2] proved existence of a unique best proximity point for $F$-contractive non-self mappings. In this paper, we extend their results by introduce a new version of Wardowski's contraction for two mappings in a complete metric space and estabilish a new common best proximity point theorem. By some fixed point results, we support our main theorem and show some application of them .
Given two non-empty subsets $A$ and $B$ of a metric space $(X, d)$, the following notions and notations are used in the sequel .

$$d(A, B) = inf\{d(a, b) : a \in A, b \in B\}$$
$$A_0 = \{a \in A : d(a, b) = d(A, B) \text{for some } b \in B\}$$
$$B_0 = \{b \in B : d(a, b) = d(A, B) \text{for some } a \in A\}.$$

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**Definition 1.1.** If $A_0 \neq \varnothing$ then the pair $(A, B)$ is said to have P-property if and only if for any $a_1, a_2 \in A_0$ and $b_2, b_2 \in B_0$

$$\begin{cases} d(a_1, b_1) = d(A, B) \\ d(a_2, b_2) = d(A, B) \end{cases} \Longrightarrow d(a_1, a_2) = d(b_1, b_2)$$

## 2. Main result

We begin our study with following definitions.

**Definition 2.1.** The mappings $S : A \to B$ and $T : A \to B$ are said to have $K$-property if for all $x, y, u, v \in A$, they satisfy the condition that

$$\begin{cases} d(u, Sx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{cases} \Longrightarrow d(u, v) \leq d(x, y)$$

**Definition 2.2.** Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a mapping satisfying:

($F_1$) $F$ is strictly increasing, i.e
$\alpha < \beta \Longrightarrow F(\alpha) < F(\beta) \qquad \forall \alpha, \beta \in \mathbb{R}_+,$

($F_2$) For each sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of positive numbers
$\lim_{n\to\infty} \alpha_n = 0 \Longleftrightarrow \lim_{n\to\infty} F(\alpha_n) = -\infty,$

($F_3$) There exists $k \in (0, 1)$such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0,$

then self mappings $S, T : X \to X$ are said to satisfy an $F$-contractive condition if there exists $C > 0$ such that

$$\forall x \neq y \in X \quad \text{s.t} \quad d(Sx, Ty) > 0 \quad \Longrightarrow \quad C + F(d(Sx, Ty)) \leq F(d(x, y)).$$

**Theorem 2.3.** *Let $A$ and $B$ be non-empty subsets of a complete metric space $(X, d)$. Moreover, assume that $A_0$ is nonempty and closed. Let also the non-self mappings $S, T : A \to B$ satisfy the following conditions :*

i) *The pair $(S, T)$ has the $K$-property,*
ii) *The pair $(A, B)$ the $P$-property,*
iii) *$S$ and $T$ are continuous,*
iv) *$T(A_0), S(A_0) \subseteq B_0$,*
v) *$S$ and $T$ satisfy the $F$-contractive condition,*

*Then there exists a unique point $a \in A$ such that $d(a, Sa) = d(a, Ta) = d(A, B)$.*

*Proof.* Fix $a_0$ in $A_0$, since $S(A_0) \subseteq B_0$, then there exists an element $a_1$ in $A_0$ such that $d(a_1, Sa_0) = d(A, B)$.
Similarly, since $T(A_0) \subseteq B_0$, $a_2 \in A_0$ can be chosen such that $d(a_2, Ta_1) = d(A, B)$. Continuing this process, we achieve a sequence $\{a_n\} \in A_0$ such that

$$\begin{cases} d(a_{2n+1}, Sa_{2n}) = d(A, B) \\ d(a_{2n+2}, Ta_{2n+1}) = d(A, B) \end{cases} \tag{2.1}$$

We wil prove that the sequence $\{a_n\}$ is convergent in $A_0$. $(A, B)$ satisfies the P-property therefore from (2.1) we obtain

$$d(a_{2n+1}, a_{2n+2}) = d(Sa_{2n}, Ta_{2n+1}) \tag{2.2}$$

If there exists $m \in \mathbb{N}$ such that $d(a_m, a_{m+1}) = 0$, since $S, T$ have $K$-property, it can be clearly shown $\{a_n\} \to a_m$ in $A$ and $d(a_m, Sa_m) = d(a_m, Ta_m) = d(A, B)$.
Then we suppose that $d(a_n, a_{n+1}) > 0$ for all $n \in \mathbb{N}$ and by (2.2) we have $d(Sa_{2n}, Ta_{2n+1}) > 0$ for all $n \in \mathbb{N}$ .
$S$ and $T$ are the $F$-contraction and (2.2) holds, hence for any positive integer $n$ we have

$$C + F(d(a_{2n+1}, a_{2n+2})) = C + F(d(Sa_{2n}, Ta_{2n+1})) \leq F(d(a_{2n}, a_{2n+1})), \tag{2.3}$$

also similarly

$$C+F(d(a_{2n+2}, a_{2n+3})) = C+F(d(Sa_{2n+2}, Ta_{2n+1})) \leq F(d(a_{2n+1}, a_{2n+2})). \tag{2.4}$$

Therefor by 2.3 and 2.4 we have

$$\begin{aligned} F(d(a_n, a_n + 1)) &\leq F(d(a_{n-1}, a_n)) - C \qquad (2.5)\\ &. \\ &\leq F(d(a_0, a_1)) - nC. \end{aligned}$$

Put $\alpha_n =: d(a_n, a_{n+1})$. By (2.5), we obtain $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$ that together with $(F_2)$ gives

$$\lim_{n\to\infty} \alpha_n = 0 \tag{2.6}$$

Also From $(F_3)$ we have

$$\exists k \in (0, 1) \qquad \text{such that} \qquad \lim_{n\to\infty} \alpha_n^k F(\alpha_n) = 0 \tag{2.7}$$

On the Other hand, by (2.5) $F(\alpha_n) - F(\alpha_0) \leq -nC$. Therefor
$\alpha_n^k F(\alpha_n) - \alpha_n^k F(\alpha_0) \leq -n\alpha_n^k C \leq 0$.

Letting $n \longrightarrow \infty$ in the above inequality and using (2.6) and (2.7) , we obtain $\lim\limits_{n\to\infty} n\alpha_n^k = 0$. Hence there exists $N_1 \in \mathbb{N}$ such that $n\alpha_n^k \leq 1$ for all $n \geq N_1$ . Therefor for any $n \geq N_1$ $\alpha_n \leq \frac{1}{n^{\frac{1}{k}}}$. This means that series $\sum_{i=1}^{\infty} \alpha_i$ is convergent, then

$$\forall \epsilon > 0 \quad \exists N \geq 0 \quad \text{such that} \quad m \geq n \geq N, \quad \sum_{i=n}^{m} \alpha_i \leq \epsilon. \tag{2.8}$$

By the triangular inequality and(2.8)
$d(a_m, a_n) \leq \alpha_{m-1} + \alpha_{m-2} + ... + \alpha_n \leq \sum_{i=n}^{m} \alpha_i \leq \epsilon$.

Therefor $\{a_n\}$ is a cauchy seqvence in $A_0$.
Since$\{a_n\} \subseteq A_0$ and $A_0$ is a closed subset of the complete metrice space

$(X, d)$, we can find $a \in A_0$ such that $\lim_{n\to\infty} a_n = a$.
Now, from $d(a_{2n+1}, Sa_{2n}) = d(A, B)$ $d(a_{2n+2}, Ta_{2n+1}) = d(A, B)$, taking $n \to \infty$ and by continuity of $S$, $T$ and $d$, we have $d(a, Sa) = d(a, Ta) = d(A, B)$. So $a$ is a common best proximity point of the mappings $S$ and $T$. The uniqueness of the best proximity point follows from the condition that $S$ and $T$ are the $F$-contraction. That is, suppose $x$ is another common best proximity point of $S$ and $T$ such that $x \neq a$, i.e, $d(a, Sa) = d(a, Ta) = d(A, B)$ $d(x, Sx) = d(x, Tx) = d(A, B)$. Then by the $P$-property of $(A, B)$, we have $d(a, x) = d(Sa, Tx)$. Also $d(a, x) > 0 \Rightarrow d(Sa, Tx) > 0$. Therefore $F(d(a, x)) = F(d(Sa, Tx)) \leq F(d(a, x)) - C < F(d(a, x))$ which is a contradiction. Hence the common best proximity point of $S$ and $T$ is unique.
If in the definition (2.2) $S$ is equal to $T$, then $T$ is called $F$-contraction. The following result is a special case of Theorem (2.3) by setting $S = T$.

**Corollary 2.4.** *Let $A$ and $B$ be non-empty subsets of a complete metric space $(X, d)$ such that $A_0$ is nonempty and closed. Let $T : A \to B$ be a $F$-contraction non-self mapping such that $T(A_0) \subset B_0$. Assume that the pair $(A, B)$ has the $P$-property. Then there exists a unique $a \in A_0$ such that $d(a, Ta) = d(A, B)$.*

The following result is a special case of Theorem (2.3) by setting $S = T$ and $A = B$.

**Corollary 2.5.** *Let $A$ be non-empty closed subsets of a complete metric space $(X, d)$. Let $T : A \to A$ be a $F$-contractive self-mapp. Then $T$ has a unique fixed point $a \in A$.*

The next result ia an immediate consequence of Theorem (2.3) by taking $S = T$, $A = B$ and $F(\alpha) = ln\alpha$.

**Corollary 2.6.** *(Banach Contraction Principle)Let $A$ be non-empty closed subsets of a complete metric space $(X, d)$. Let $T : A \to A$ be a contractive self-map. Then $T$ has a unique fixed point $a \in A$.*

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# NEW INEQUALITIES FOR HIENZ AND HERON MEAN

S. MALEKINEJAD

*Department of Mathematics, Payame noor University, Tehran 12345, Iran*
*maleki60313@gmail.com*

ABSTRACT. In this paper, some refinements of Heron inequality for positive numbers are proved and using these inequalities for positive operator of these inequalities are obtained.

## 1. INTRODUCTION

Heron mean is defined by

$$F_\nu(a,b) = (1-\nu)\sqrt{ab} + \nu\frac{a+b}{2}.$$

It is easy to see that $F_\nu(a,b)$ is an increasing function in $\nu$ on $[0,1]$ and Heinz mean is defined by

$$H_\nu(a,b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu}b^\nu}{2}$$

$H_\nu(a,b)$ is a symmetric and convex function in $\nu$ on $[0,1]$ hence

$$\sqrt{ab} \leq F_\nu(a,b) \leq \frac{a+b}{2}, \sqrt{ab} \leq H_\nu(a,b) \leq \frac{a+b}{2} \tag{1.1}$$

Let H be a Hilbert space and let $B_h(H)$ be the semi space of all bounded linear self-adjoint operators on H. Further, let $B(H)$ and $B(H)^+$ , respectively, denote the set of all bounded linear operators on a complex Hilbert space H and set of all positive operators in $B_h(H)$.

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The set of all positive invertible operators is denoted by $B(H)^{++}$. For more details about this property, the reader is referred to [2].

Zhao, Wu, Cao and Liao [3] gave an inequality for the Hienz and Heron means as follows:
If $A$ and $B$ be two positive and invertible operators then

$$H_\nu(A,B) \leq F_{\alpha(\nu)}(A,B) \tag{1.2}$$

for $\nu \in [0,1]$, where $\alpha(\nu) = 1 - 4(\nu - \nu^2)$.

Refinements of Heron and Hienz means Kittaneh and manasrah [1] gave a different type of improvement of Youngs matrix inequalities :

$$2r(A\nabla B - A\sharp B) \leq A\nabla_\nu - A\sharp_\nu B \leq 2s(A\nabla B - A\sharp B) \tag{1.3}$$

for $A, B \in B(H), \nu \in [0,1], r = min\{\nu, 1-\nu\}$, and $s = max\{\nu, 1-\nu\}$. Young and Ren [4] obtained

**Theorem 1.1.** *Let $A, B \in B(H)^{++}$ be positive invertible operators, $I$ be indentity operator, and $\nu \in [0,1]$, then we have*

$$\nu(1-\nu)(A\nabla B - A\sharp B) + A\sharp B \leq F_\nu(A,B) \tag{1.4}$$

$$F_\nu(A,B) \leq A\nabla B - \nu(1-\nu)(A\nabla B - A\sharp B) \tag{1.5}$$

## 2. Main results

**Theorem 2.1.** *For $a, b \geq 0$ and $\nu \in [0,1)$, we have*

$$F_\nu(a,b) \geq \frac{1}{2}\nu(1-\nu^{n-1})(\sqrt{a}-\sqrt{b})^2 + \sqrt{ab} \tag{2.1}$$

*and*

$$(1 - \frac{2s}{1-\nu^{n-1}})(\frac{a+b}{2}) + \frac{2s}{1-\nu^{n-1}}\sqrt{ab} \leq H_\nu(a,b) \tag{2.2}$$

*where $s = max\{\nu, 1-\nu\}$ and $\forall n \in \mathbb{N}$.*

*Proof.*

$$\begin{aligned}
&F_\nu(a,b) - \frac{1}{2}\nu(1-\nu^{n-1})(\sqrt{a}-\sqrt{b})^2 \\
&= \frac{1}{2}(\nu a + \nu b + 2\sqrt{ab} - 2\nu\sqrt{ab}) \\
&- \frac{1}{2}(\nu a + \nu b - 2\nu\sqrt{ab} - \nu^n a - \nu^n b + 2\nu^n\sqrt{ab}) \\
&= \frac{1}{2}(\nu^n a + \nu^n b - 2\nu^n\sqrt{ab}) + \sqrt{ab} \\
&= \nu^n(\frac{a+b}{2} - \sqrt{ab}) + \sqrt{ab} \\
&\geq \sqrt{ab}.
\end{aligned}$$

then

$$F_\nu(a,b) - \frac{1}{2}\nu(1-\nu^{n-1})(\sqrt{a}-\sqrt{b})^2 \geq a\sharp b$$
$$F_\nu(a,b) - \frac{1}{2}\nu(1-\nu^{n-1})(2a\nabla b - 2a\sharp b) \geq a\sharp b$$
$$F_\nu(a,b) - \frac{\nu(1-\nu^{n-1})}{2s}(a\nabla_\nu b - a\sharp_\nu b) \geq a\sharp b \tag{2.3}$$

replacing $a$ by $b$ and $b$ by $a$, implies that

$$F_\nu(a,b) \geq \frac{\nu(1-\nu^{n-1})}{2s}(b\nabla_\nu a - a\sharp_{1-\nu} b) + a\sharp b \tag{2.4}$$

by the sum of the (2.3) and (2.4), we have

$$F_\nu(a,b) \geq \frac{\nu(1-\nu^{n-1})}{2s}(a\nabla b - H_\nu(a,b)) + a\sharp b$$

hence

$$\begin{aligned} H_\nu(a,b) &\geq \frac{a+b}{2} - \frac{2s}{\nu(1-\nu^{n-1})}(F_\nu(a,b) - a\sharp b) \\ &= \frac{a+b}{2} - \frac{2s}{\nu(1-\nu^{n-1})}(\nu(a\nabla b) - \nu a\sharp b) \\ &= \frac{a+b}{2} - \frac{s}{(1-\nu^{n-1})}(2a\nabla b - 2a\sharp b) \\ &= (1 - \frac{2s}{1-\nu^{n-1}})(a\nabla b) + \frac{2s}{1-\nu^{n-1}}a\sharp b, \end{aligned}$$

where $s = max\{\nu, 1-\nu\}$. □

**Theorem 2.2.** *For $a,b \geq 0$ and $\nu \in [0,1]$, we have*

$$F_{\nu^n}(a,b) = F_\nu(a,b) - \nu(1-\nu^{n-1})(a\nabla b - a\sharp b) \tag{2.5}$$

**Theorem 2.3.** *Let $A, B \in B(H)^{++}$ and $\nu \in [0,1)$. Then*

$$F_\nu(A,B) \geq \frac{\nu(1-\nu^{n-1})}{2s}(A\nabla B - H_\nu(A,B)) + A\sharp B \tag{2.6}$$

*and*

$$A\nabla B - \frac{2s}{\nu(1-\nu^{n-1})}(F_\nu(A,B) - A\sharp B) \leq H_\nu(A,B) \tag{2.7}$$

*for $\forall n \in N$. Where $s = max\{\nu, 1-\nu\}$.*

It is know that $\nu(1-\nu^{n-1})(A\nabla B - A\sharp B)$is a positive real number, this implies that the inequality (2.6) is a stronger than of the operator Heinz inequality (1.1) and also know that $1-\nu^{n-1} \geq 1-\nu$ hence inequality (2.6) is a stronger than of the inequality (1.4).

**Theorem 2.4.** *Let $A, B \in B(H)^{++}$ and $\nu \in [0,1]$. Then*

$$F_{\nu^n}(A,B) = F_\nu(A,B) - \nu(1-\nu^{n-1})(A\nabla B - A\sharp B) \tag{2.8}$$

*for $\forall n \in N$.*

**Theorem 2.5.** *Let $A, B \in B(H)^{++}$ and $\nu \in [0,1)$. Then*

$$(1 - \frac{2s}{1-\nu^{n-1}})A\nabla B + (\frac{2s}{1-\nu^{n-1}})A\sharp B \leq H_\nu(A,B) \tag{2.9}$$

*where $s = max\{\nu, 1-\nu\}$ and $\forall n \in \mathbb{N}$.*

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# DERIVATIONS ON $C^*-$ALGEBRAS

M. GHASEMI AND M. J. MEHDIPOUR*

*Department of Mathematics, Shiraz University of Technology, Shiraz, Iran*
*mi.ghasemi@sutech.ac.ir and mehdipour@sutech.ac.ir*

Abstract. Let $A$ and $B$ be $C^*-$algebras with $\sigma(B) \neq \emptyset$. In this paper, we investigate the Posner's first and second theorems for elements of $\text{Der}(A \times_\theta^{\phi,\gamma} B)$. We also characterize strong commutativity preserving elements of $\text{Der}(A \times_\theta^{\phi,\gamma} B)$.

## 1. Introduction

Let $A$ and $B$ be Banach algebras and $\theta \in \sigma(B)$, the spectrum of $B$. Let us recall from [8] that the $\theta-$Lau product $A$ and $B$ is the direct product $A \times B$ together with the component wise addition and the multiplication

$$(a, b) \cdot_\theta (x, y) = (ax + \theta(y)a + \theta(b)x, by).$$

Note that if we permit $\theta = 0$, the $\theta-$Lau product $A \times_\theta B$ is the usual direct product of Banach algebras. Hence we disregard the possibility that $\theta = 0$.

The $\theta-$Lau products $A \times_\theta B$ were first introduced by Lau [6], for Banach algebras that are pre-duals of von Neumann algebras, and for which the identity of the dual is a multiplicative linear functional. Sanjani Monfared [8] extended this product to arbitrary Banach algebras $A$ and $B$. In fact, he introduced a strongly splitting Banach algebra extension of $B$ by $A$ which present many properties that are not shared

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by arbitrary strongly splitting extension. He also gave characterizations of bounded approximate identity, spectrum, topological center and minimal idempotents of these products.

Let us recall that a Banach algebra $A$ is called *prime* if $aAb = (0)$ implies that either $a = 0$ or $b = 0$.

Let $T : A \to A$ be a linear map. Then $T$ is called *centralizing* if for every $a \in A$

$$[T(a), a] \in \mathrm{Z}(A),$$

where for each $a, x \in A$

$$[a, x] = ax - xa$$

and $\mathrm{Z}(A)$ denotes the center of $A$. Also, $T$ is called a *derivation* if for every $a, x \in A$

$$T(ax) = T(a)x + aT(x).$$

One can define the concept of derivation on rings similarly.

Derivations on rings studied by several authors [2, 3, 4, 5, 7]. For example, Posner [7] showed that the product of two nonzero derivations on prime rings with characteristic different from two is not a derivation. He also proved that the zero map is the only centralizing derivation on a noncommutative prime ring. These results are known as the Posner's first and second theorems, respectively. Starting from this work, a number of authors studied the relationship between the structure of a prime or semiprime ring $R$ and the behavior of additive mappings defined on $R$. For example, Bresar [2] proved that there is no nonzero additive mapping in a prime ring $R$ of characteristic different from 2 which is skew-commuting on $R$.

In this paper, let $\theta, \phi, \gamma \in \sigma(B)$ and $\mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$ be the set of all linear mappings $d : A \times B \to A \times B$ satisfying

$$d((a, b) \cdot_\theta (x, y)) = d(a, b) \cdot_\phi (x, y) + (a, b) \cdot_\gamma d(x, y)$$

for all $a, x \in A$ and $b, y \in B$. We show that if $\mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$ has a nonzero element, then $\phi = \gamma$. We also prove that if $d_1, d_2$ and $d_1d_2$ are elements of $\mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$, then $d_1d_2 = 0$. Finally, we investigate the concepts centralizing and strong commutativity preserving for elements of $\mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$.

## 2. Main results

Let $A$ and $B$ be $C^*-$algebras. In the sequel, let $A \times_\theta B$ be a $C^*-$algebra with following involution

$$(a, b)^* = (a^*, b^*).$$

**Theorem 2.1.** *Let $d \in Der(A \times_\theta^{\phi,\gamma} B) \neq 0$. Then $\phi = \gamma$.*

Now, we investigate analogues of Posner's first and second theorems.

**Theorem 2.2.** *Let $d_1, d_2 \in Der(A \times_\theta^{\phi,\gamma} B)$. If $d_1 d_2$ is an element of $Der(A \times_\theta^{\phi,\gamma} B)$, then $d_1 d_2 = 0$.*

Let $\eta_1, \eta_2 \in \sigma(B)$. A mapping $T : A \times B \to A \times B$ is called $(\eta_1, \eta_2)-$*centralizing* if for every $a \in A$ and $b \in B$,

$$[T(a,b),(a,b)]_{\eta_1,\eta_2} := T(a,b) \cdot_{\eta_1} (a,b) - (a,b) \cdot_{\eta_2} T(a,b) \in \mathrm{Z}(A) \times \mathrm{Z}(B).$$

**Theorem 2.3.** *Let $\eta_1, \eta_2 \in \sigma(B)$ and $d \in Der(A \times_\theta^{\phi,\gamma} B)$. If the mapping $(a,b) \mapsto [d(a,b),(a,b)]_{\eta_1,\eta_2}$ is $(\eta_1,\eta_2)-$centralizing, then either $\eta_1 = \eta_2$ or $d = 0$.*

As a consequent of Theorem 2.3 we present the following result.

**Corollary 2.4.** *Let $\eta_1, \eta_2 \in \sigma(B)$. Then the only $(\eta_1, \eta_2)$-centralizing element of $Der(A \times_\theta^{\phi,\gamma} B)$ is zero.*

A mapping $T : A \times B \to A \times B$ is called $(\eta_1, \eta_2)-$*skew commuting* if for every $a \in A$ and $b \in B$

$$\langle d(a,b),(a,b)\rangle_{\eta_1,\eta_2} := d(a,b) \cdot_{\eta_1} (a,b) + (a,b) \cdot_{\eta_2} d(a,b) = 0.$$

**Theorem 2.5.** *Let $\eta_1, \eta_2 \in \sigma(B)$ and $d \in Der(A \times_\theta^{\phi,\gamma} B)$. If $d$ is a $(\eta_1,\eta_2)-$skew commuting, then $d = 0$.*

**Definition 2.6.** A linear mapping $T : A \to A$ is called *strong commutativity preserving* if

$$[T(a), T(x)] = [a, x]$$

for all $a, x \in A$.

Derivation as well as strong commutativity preserving mappings have been studied by several authors; see for example [1, 2]. In the next result, we investigate this concept for elements of $\mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$.

**Theorem 2.7.** *Let $\eta_i, \rho_i \in \sigma(B)$ for $i = 1, 2$. If $d$ is an element of $Der(A \times_\theta^{\phi,\gamma} B)$ satisfying $[d(a,b), d(x,y)]_{\eta_1,\eta_2} = [(a,b),(x,y)]_{\rho_1,\rho_2}$ for all $a, x \in A$ and $b, y \in B$, then $d = 0$ and $\rho_1 = \rho_2$.*

**Proposition 2.8.** *Let $\eta \in \sigma(B)$. If $d$ is an element of $Der(A \times_\theta^{\phi,\gamma} B)$ satisfying $d((a,b) \cdot_\theta (x,y)) - (a,b) \cdot_\eta (x,y) \in Z(A) \times Z(B)$, then $d = 0$.*

We finish the paper with the following result.

**Theorem 2.9.** *Let $A \times_\theta B$ be a $C^*-$algebra. If $\theta$ is $*-$isometry, then $B$ is a prime $C^*-$algebra.*

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# MEASURES OF NONCOMPACTNESS IN THE SPACE OF REGULATED FUNCTIONS $R(J, \mathbb{R}^\infty)$ AND ITS APPLICATION TO SOME NONLINEAR INFINITE SYSTEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS

H. MEHRAVARAN*, H. AMIRI KAYVANLOO, AND R. ALLAHYARI

*Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran*
*hamid_mehravaran@mshdiau.ac.ir*
*Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran*
*amiri.hojjat93@mshdiau.ac.ir*
*Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran*
*rezaallahyari@mshdiau.ac.ir*

Abstract. In this paper, we prove the existence of solutions for the infinite systems of fractional boundary value problem. We also provide an illustrative example in verification of our existence theorem.

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## 1. Introduction

The technique of measures of noncompactness is very important tool for investigation of solvability of nonlinear integral-functional equations. In, (2018) Banas [2] formulate a criterion for relative compactness in the space of functions regulated on a bounded and closed interval $[a, b]$, so-called regulated functions, and proved that the mentioned criterion is equivalent to a known criterion obtained earlier by D. Frankova. In this paper, we shows the applicability of the mentioned measures of noncompactness to the existence result for some nonlinear infinite systems of fractional boundary value problem.

## 2. Main results

The symbol $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ will denote the set of all real numbers, set of all nonnegative real numbers, and the set of all positive integers, respectively. Assume that $\mathbb{E}$ be a Banach space with zero element $\theta$. If $X$ is subset of $\mathbb{E}$, then $\overline{X}$ and Conv $X$ denote closure and convex closure of $X$, respectively. Also, denote by $B(x, r)$ the closed ball centered at $x$ and with radius $r$. Moreover, let $\mathfrak{M}_{\mathbb{E}}$ denote the family of all nonempty and bounded subsets of $\mathbb{E}$ and $\mathfrak{N}_{\mathbb{E}}$ its subfamily consisting of all relatively compact sets.

**Definition 2.1.** [1] A mapping $\mu : \mathfrak{M}_{\mathbb{E}} \to [0, \infty)$, is said to be a measures of noncompactness in the Banach space $\mathbb{E}$ if it satisfies the following conditions:

1° The family $\ker \mu := \{X \in \mathfrak{M}_{\mathbb{E}} : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_{\mathbb{E}}$.
2° $\mu(X) \leq \mu(Y)$ for $X \subset Y$.
3° $\mu(ConvX) = \mu(X)$.
4° $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
5° If $(X_i)$ is a sequence of closed sets from $\mathfrak{M}_{\mathbb{E}}$ such that $X_{i+1} \subset X_i (i = 1, 2, \dots)$ and if $\lim_{i\to\infty} \mu(X_i) = 0$, then the intersection set $X_\infty := \bigcap_{i=1}^{\infty} X_i$ is nonempty.

In the sequel we will use measures of noncompactness having some additional properties. Namely, a measure $\mu$ is said to be sublinear if it satisfies the following two conditions:

6° $\mu(\lambda X) = \mu|\lambda|(X)$ for $\lambda \in \mathbb{R}$.
7° $\mu(X + Y) \leq \mu(X) + \mu(Y)$.

A sublinear measure of noncompactness $\mu$ satisfying the condition (weak maximum property)

8° $\mu(X \cup \{y\}) = \mu(X)$, $y \in \mathbb{E}$

and such that $\ker \mu = \mathfrak{N}_{\mathbb{E}}$ is said to be regular.

Now, we recall some facts concerning regulated functions.

**Definition 2.2.** [2] A function $x : [a, b] \to \mathbb{R}$ is said to be a regulated function if for every $t \in [a, b)$ the right-sided limit $x(t^+) := \lim_{s \to t^+} x(s)$ exists and for every $t \in (a, b]$ the left-sided limit $x(t^-) := \lim_{s \to t^-} x(s)$ exists.

Denote by $R(J, \mathbb{R}^\infty)$ the space consisting of all regulated functions defined on the interval $J = [a, b]$ with values in $\mathbb{R}^\infty$. Now, for $x = (x^i) \in R(J, \mathbb{R}^\infty)$ we put $\pi_i(x) = x^i$. This space equipped with the family of seminorms
$\|x\|_n := \sup\{|\pi_i(x)(t)| : t \in J, i \leq n\}$ becomes a Fréchet space. We are going to present the construction of the regular measures of noncompactness in the space $R(J, \mathbb{R}^\infty)$. Assume that $X \in \mathfrak{M}_{R(J,\mathbb{R}^\infty)}$. For $x \in X$ and $\varepsilon > 0$ and let $p^i : J \to (0, \infty)$ $(i = 1, 2, \ldots)$ is a sequence of functions. Put
$\overline{\omega}^+(X) = \sup\{p^i(T)\omega^+(\pi_i(X))\}$, $\overline{\omega}^-(X) = \sup\{p^i(T)\omega^-(\pi_i(X))\}$,
*and* $\mathcal{X}_{\mathbb{R}^\infty}(X(t)) = \sup\{p^i(T)\mathcal{X}_{\mathbb{R}}(\pi_i(X)(t))\}$, for $i = 1, 2, \ldots$.
Finally, let us define the quantity

$$\mu(X) := \max_{x \in X} \Big\{\overline{\omega}^-(X), \overline{\omega}^+(X)\Big\} + \sup_{t \in J} \mathcal{X}_{\mathbb{R}^\infty}\big(X(t)\big). \tag{2.1}$$

We can formulate our first results.

**Theorem 2.3.** [3] *The mapping $\mu : \mathfrak{M}_{R(J,\mathbb{R}^\infty)} \to \mathbb{R}_+$ given by (2.1), defines a regular measures of noncompactness on $R(J, \mathbb{R}^\infty)$.*

Next, we present an existence result for the infinite systems of boundary value problem of Caputo fractional differential equations of arbitrary order $q$ with finite many multistrip Riemann-Liouville type integral boundary conditions:

$$\begin{cases} {}^cD^q x_i(t) = f^i(t, x^1(t), x^2(t), \ldots), \ q \in (n-1, n], \ n \geq 2 \ t \in [0, T], \\ x_i(0) = 0, x_i^{'}(0) = 0, \ldots, x_i^{(n-2)}(0) = 0, \ x_i(T) = \displaystyle\sum_{\varsigma=1}^{m} \gamma_\varsigma [I^{\beta_\varsigma} x_i(\eta_\varsigma) - I^{\beta_\varsigma} x_i(\zeta_\varsigma)], \end{cases} \tag{2.2}$$

where ${}^cD^q$ is Caputo fractional derivative of order $q$ and $f^i : J \times \mathbb{R}^\infty \to \mathbb{R}$ $(i = 1, 2, \ldots)$ is given continuous function and $I^{\beta_j}$ is the Riemann-Liouville fractional integral of order $\beta_j > 0$, $\varsigma = 1, 2, \ldots m$, $0 < \zeta_1 < \eta_1 < \zeta_2 < \eta_2 < \ldots < \zeta_m < \eta_m < T$ and $\gamma_\varsigma \in \mathbb{R}$ are suitable chosen constants. we will write $f^i(t, x(t))$ instead of $f^i(t, \pi_1(x)(t), \pi_2(x)(t) \ldots)$ for $i = 1, 2, \ldots$.
We assume that the following conditions are satisfied.

$(A_1)$ The functions $f^i : J \times \mathbb{R}^\infty \to \mathbb{R}$ $(i \in \mathbb{N})$ are increasing for every $t \in J = [0, T]$ and equiregulated on $J$ for bounded variables $x = (x^i)$ *i.e.* for every $R^i > 0$ family of $\{f^i(., \pi_1(x)(t), \pi_2(x)(t) \ldots) : x \in \mathbb{B}_{\mathbb{R}^\infty}(\theta, R) = (\mathbb{B}_{\mathbb{R}}(\theta, R^i))$ and $R = \min\{R^1, R^2, \ldots\}\}$ is equiregulated on $J$ also, $f^i$ satisfies assumptions $(i)$, $(ii)$ of the Theorem **??**.

$(A_2)$ There exists the constant $k_{ij} \geq 0$ such that

$$|f^i(t, \pi_1(x)(t), \pi_2(x)(t) \ldots) - f^i(t, \pi_1(y)(t), \pi_2(y)(t) \ldots)| \leq \sum_{j=1}^{\infty} k_{ij} |\pi_j(x)(t) - \pi_j(y)(t)|,$$

for every $t \in J$ and $i = 1, 2, \ldots$.

$(A_3)$ Define the functions $g^i : J \times \mathbb{R}^\infty \to \mathbb{R}$ $(i \in \mathbb{N})$ such that $g^i$ are increasing and satisfies the following conditions:

$(a)$ $|f^i(t, \pi_1(x)(t), \pi_2(x)(t) \ldots)| \leq |f^i(\tau, \pi_1(x)(\tau), \pi_2(x)(\tau) \ldots)) - f^i(s, \pi_1(x)(s), \pi_2(x)(s) \ldots))| + |g^i(t, \pi_1(x)(t), \pi_2(x)(t) \ldots))|$, for every $t, s, \tau \in J$ such that $\tau \neq s$ and $i = 1, 2 \ldots$.

$(b)$ There exist continuous functions $a^i, b^i : J \to \mathbb{R}_+$ such that

$$|g^i(t, \pi_1(x)(t), \pi_2(x)(t) \ldots)| \leq \Big(a^i(t) \sum_{j=1}^{\infty} k_{ij} |\pi_j(x)(t)| + b^i(t)\Big)(|\tau - s|),$$

for every $t, s, \tau \in J$ such that $\tau \neq s$ and $i = 1, 2 \ldots$.

$(A_4)$ There exists a constant $l \in [0, 1)$ and increasing functions $m^i : J \to [1, \infty)$, such that $\Lambda \sum_{j=1}^{\infty} k_{ij} \frac{m^j(t)}{m^i(t)} \leq l$, *where* $2\Big\{\frac{T^q}{\Gamma(\mathbf{q+1})} + \frac{T^{q+n-1}}{|\lambda|\Gamma(\mathbf{q}+1)} + \frac{T^{n-1}}{|\lambda|} \sum_{\varsigma=1}^{m} \gamma_\varsigma \frac{\eta_\varsigma^{q+\beta_\varsigma} - \zeta_\varsigma^{q+\beta_\varsigma}}{\Gamma(q+\beta_\varsigma+1)}\Big\} = \Lambda > 0$, for $i = 1, 2, \ldots$ and $t \in J$.

**Theorem 2.4.** [3] *Suppose that the assumptions $(A_1) - (A_4)$ are satisfied. Then the nonlinear infinite System of Caputo fractional differential equation (2.2) has at least one solution in the space $R(J, \mathbb{R}^\infty)$.*

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# MULTIPLICITY RESULTS FOR FRACTIONAL $p(x,\cdot)$-KIRCHHOFF-TYPE PROBLEM

MARYAM MIRZAPOUR

*Faculty of Mathematical Sciences, Farhangian University, Tehran, Iran*
*m.mirzapour@cfu.ac.ir*

Abstract. This paper is concerned with the existence of infinitely many weak solutions for a fractional $p(x,\cdot)$-Kirchhoff-type problem. By means of the Fountain Theorem, we establish conditions ensuring the multiplicity of solutions for the problem.

## 1. Introduction

A very interesting area of nonlinear analysis lies in the study of elliptic equations involving fractional operators. This category of operators come up in a quite natural way in many different applications such as phase transition phenomena, continuum mechanics, population dynamics, minimal surface and game theory. For the basic properties of fractional Sobolev spaces, we refer the reader to [3].
In this paper we are interested in the following fractional equation

$$\begin{cases} M\big(\sigma_{p(x,y)}(u)\big)\mathcal{L}_K^{p(x,\cdot)}(u) = f(x,u) & \text{in } \Omega, \\ u(x) = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where

$$\sigma_{p(x,y)}(u) = \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)} K(x,y)\, dx\, dy, \tag{1.2}$$

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where $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary $\partial\Omega$, $\mathcal{Q} := \mathbb{R}^{2N} \backslash (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ with $\mathcal{C}\Omega = \mathbb{R}^N \backslash \Omega$, $p : \overline{\mathcal{Q}} \to (1, +\infty)$ is bounded continuous function, $N \geq 3$, $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $\mathcal{L}_K^{p(x,\cdot)}$ is the integro-differential operator that generalized $(-\Delta_{p(x,\cdot)})^s$, defined as follows

$$\begin{aligned}\mathcal{L}_K^{p(x,\cdot)}(u) &= p.v. \int_{\mathbb{R}^N} |u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))K(x, y)\, dy \\ &= \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \backslash B_\epsilon(x)} |u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))K(x, y)\, dy,\end{aligned}$$

for all $x \in \mathbb{R}^N$, where $p.v.$ is a commonly used abbreviation in the principal value sense. The energy functional associated with it is given by the functional

$$\mathcal{J}_K(u) = \widehat{M}\big(\sigma_{p(x,y)}(u)\big) - \int_\Omega F(x, t)\, dx, \tag{1.3}$$

where $\sigma_{p(x,y)}(u)$ is defined in (1.2), $\widehat{M}(t) = \int_0^t M(\tau)\, d\tau$ and $F(x, t) = \int_0^t f(x, s)\, ds$. Now, we can state our main result.

**Theorem 1.1.** *Let $\Omega$ be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0, 1)$, let $p : \overline{\mathcal{Q}} \to (1, +\infty)$ be a continuous function satisfying (2.1) and (2.2) with $sp^+ < N$. Assume that the Kirchhoff function $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying*

- $(\mathbf{H_0})$ *there exists $m_0 > 0$ such that $M(t) \geq m_0$ for all $t \geq 0$;*
- $(\mathbf{H_1})$ *there exists $\mu \in (0, 1)$ such that $\widehat{M}(t) \geq (1 - \mu)\, M(t)\, t$ for all $t \geq 0$;*
- $(\mathbf{H_2})$ *$M$ is differentiable and decreasing function on $\mathbb{R}^+$.*
- $(\mathbf{f_0})$ *there exist $c_1 > 0$ and $1 < q(x) < p_s^*(x) = \frac{N\overline{p}(x)}{N - s\overline{p}(x)}$ with $\overline{p}(x) = p(x, x)$ such that $|f(x, t) \leq c_1(1 + |t|^{q(x)-1})$ for all $(x, t) \in \Omega \times \mathbb{R}$;*
- $(\mathbf{f_1})$ *$\lim_{|t| \to +\infty} \frac{F(x,t)}{|t|^{\frac{p^+}{1-\mu}}} = +\infty$, uniformly for a.e. $x \in \Omega$;*
- $(\mathbf{f_2})$ *there exists $C_* > 0$ such that $G(x, t) \leq G(x, t') + C_*$ for each $x \in \Omega$, $0 < t < t'$ or $t' < t < 0$ where $G(x, t) = t f(x, t) - \frac{p^+}{1-\mu} F(x, t)$ and $p^+$ is defined in (2.1) ;*
- $(\mathbf{f_3})$ *$\lim_{t \to 0} \frac{F(x,t)}{|t|^{p^+}} = 0$, uniformly for a.e. $x \in \Omega$;*
- $(\mathbf{f_4})$ *$f(x, -t) = -f(x, t)$ for all $x \in \Omega$ and $t \in \mathbb{R}$;*

*If $q^- > p^+$, then problem (1.1) has a sequence of weak solutions $\{\pm u_k\}$ such that $\mathcal{J}_k(\pm u_k) \to +\infty$ as $k \to +\infty$.*

## 2. Variational framework and proof of main result

Let $\Omega$ be a Lipschitz open bounded set in $\mathbb{R}^N$. We assume that

$$1 < p^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) \le p(x,y) \le p^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) < +\infty \tag{2.1}$$

and

$$p \text{ is symmetric, that is } p(x,y) = p(y,x) \text{ for all } (x,y) \in \overline{\Omega}\times\overline{\Omega}. \tag{2.2}$$

We set $\overline{p}(x) = p(x,x)$, for any $x \in \overline{\Omega}$. Define the fractional Sobolev space with variable exponent via the *Gagliardo approach* as follows

$$X := \Big\{ u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable, such that } u|_\Omega \in L^{\overline{p}(x)}(\Omega) \text{ with}$$
$$\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}} K(x,y)\,dx\,dy < +\infty, \text{ for some } \lambda > 0 \Big\}.$$

Moreover, $X$ is endowed with the norm $\|u\|_X := \|u\|_{L^{\overline{p}(x)}(\Omega)} + [u]_{K,p(x,y)}$, where

$$[u]_{K,p(x,y)} = \inf\Big\{ \lambda > 0; \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}} K(x,y)\,dx\,dy \le 1 \Big\},$$

then we have $(X, \|\cdot\|_X)$ is a separable reflexive Banach space, see [1]. We shall work in the closed linear subspace $X_0 = \{u \in X;\ u(x) = 0 \text{ a.e. in } \mathbb{R}^N\backslash\Omega\}$. For any $u \in X_0$, we define the functional $\rho^0_{K,p(\cdot,\cdot)}(u) = \int_{\mathcal{Q}} |u(x)-u(y)|^{p(x,y)} K(x,y)\,dx\,dy$, then the norm associated with the *convex modular* $\rho^0_{K,p(\cdot,\cdot)}$ is given by $\|u\|_{X_0} = [u]_{K,p(x,y)} = \inf\Big\{\lambda > 0;\ \rho^0_{K,p(\cdot,\cdot)}(\frac{u}{\lambda}) \le 1\Big\}$. We know $(X_0, \|\cdot\|_{X_0})$ is a separable, reflexive and uniformly Banach space, see [1].

**Definition 2.1.** We say that $u \in X_0$ is a weak solution of problem if

$$M\left(\sigma_{p(x,y)}(u)\right) \int_{\mathcal{Q}} |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x,y)\,dx\,dy$$
$$- \int_\Omega f(x,u)\varphi(x)\,dx = 0,$$

for all $\varphi \in X_0$. By the assumptions on $M$, $f$, we have $\mathcal{J}_K \in C^1(X_0)$.

**Definition 2.2** (See [2])**.** Let $X_0$ be a Banach space and $\mathcal{J}_K \in C^1(X_0, \mathbb{R})$. given $c \in \mathbb{R}$, we say that $\mathcal{J}_K$ satisfies the Cerami condition (we denote condition $(C_c)$), if

(*i*) any bounded sequence $\{u_n\} \subset X_0$ such that $\mathcal{J}_K(u_n) \to c$ and $\mathcal{J}'_K(u_n) \to 0$ has a convergent subsequence;

$(ii)$ there exist constants $\delta,\ R,\ \beta>0$ such that

$$\|\mathcal{J}_K'(u)\|_{X_0^*}\|u\|_{X_0}\geq\beta \quad \text{for all} \quad u\in\mathcal{J}_K^{-1}([c-\delta,c+\delta]) \quad \text{with} \quad \|u\|_{X_0}\geq R.$$

If $\mathcal{J}_K\in C^1(X_0,\mathbb{R})$ satisfies condition $(C_c)$ for any $c\in\mathbb{R}$, we say that $\mathcal{J}_K$ satisfies condition $(C)$.

**Proposition 2.3.** *Under assumptions* $(\mathbf{H_0})$-$(\mathbf{H_2})$ *and* $(\mathbf{f_0})$-$(\mathbf{f_2})$, $\mathcal{J}_K$ *satisfies the Cerami condition.*

*Remark* 2.4. Since $X_0$ is a reflexive and separable Banach space, then $X_0^*$ is too. Then, there exist (see [5]) $\{e_j\}_{j\in\mathbb{N}}\subset X_0$ and $\{e_j^*\}_{j\in\mathbb{N}}\subset X_0^*$ such that $X_0=\overline{\text{ span }\{e_j\ :j=1,2,...\}}$, $X_0^*=\overline{\text{ span }\{e_j^*\ :j=1,2,...\}}$, and $\langle e_i,e_j^*\rangle=\begin{cases}1 & \text{if} \quad i=j,\\ 0 & \text{if} \quad i\neq j,\end{cases}$
where $\langle\cdot,\cdot\rangle$ denote the duality product between $X_0$ and $X_0^*$. We define $X_k=$ span $\{e_k\}$,$Y_k=\bigoplus_{j=1}^k X_j$ and $Z_k=\overline{\bigoplus_{j=k}^\infty X_j}$.

**Proposition 2.5** (Fountain Theorem, see [4])**.** *Let* $(X_0,\|\cdot\|_{X_0})$ *be a real reflexive Banach space,* $\mathcal{J}_K\in C^1(X_0,\mathbb{R})$ *is an even functional satisfying the Cerami condition. Moreover, for each* $k=1,2,\cdots$, *there exist* $\rho_k>r_k>0$ *such that*

$(\mathbf{A_1})$ $a_k:=\inf_{\{u\in Z_k,\ \|u\|=r_k\}}\mathcal{J}_K(u)\to+\infty$ *as* $k\to+\infty$,
$(\mathbf{A_2})$ $b_k:=\max_{\{u\in Y_k,\ \|u\|=\rho_k\}}\mathcal{J}_K(u)\leq 0$,

*then the functional* $\mathcal{J}_K$ *has a sequence of critical values which tends to* $+\infty$.

The functional $\mathcal{J}_K$ satisfies the Cerami condition by Proposition 2.3 and using $(\mathbf{f_4})$, we get $\mathcal{J}_K(-u)=\mathcal{J}_K(u)$ for any $u\in X_0$. As for the geometric features of $\mathcal{J}_K$, conditions $(\mathbf{A_1})$ and $(\mathbf{A_2})$ can be proved. Hence, the proof of Theorem 1.1 is complete.

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# EXISTENCE RESULT FOR A VARIABLE $s(\cdot)$-ORDER WITH $p(\cdot)$-FRACTIONAL LAPLACE PROBLEM

MARYAM MIRZAPOUR

*Faculty of Mathematical Sciences, Farhangian University, Tehran, Iran*
*m.mirzapour@cfu.ac.ir*

Abstract. In this paper we deal with the existence of weak solution for a variable $s(\cdot)$-order with $p(\cdot)$-fractional problem

$$\begin{cases} (-\Delta)^{s(\cdot)}_{p(\cdot)} u(x) = \lambda(x)|u|^{q(x)-2}u & \text{in } \Omega, \\ u(x) = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases}$$

By using the Mountain Pass Theorem, we establish conditions ensuring the existence result.

## 1. Introduction

Recently, a great attention has been focused on the study of the fractional and nonlocal operators of elliptic type, see for example [1, 3, 8] .. Indeed, the change in temperature can be better describe by using variable order derivatives of nonlocal integro-differential operators. From this, a great attention has been devoted to the study of fractional variable order spaces. We refer the reader to [5], [6] and [7]. In this paper we are interested in the following fractional equation

$$\begin{cases} (-\Delta)^{s(\cdot)}_{p(\cdot)} u(x) = \lambda(x)|u|^{q(x)-2}u & \text{in } \Omega, \\ u(x) = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded smooth domain with $N > s(x,y)\, p(x,y)$ for any $(x,y) \in \overline{\Omega} \times \overline{\Omega}$ where $s(\cdot) \in C(\mathbb{R}^{2N}, (0,1))$ and

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$p(\cdot) \in C(\mathbb{R}^{2N}, (1,\infty))$.Here, the main operator $(-\Delta)^{s(\cdot)}_{p(\cdot)}$ is the fractional variable $s(\cdot)$-order $p(\cdot)$-Laplacian given by

$$(-\Delta)^{s(\cdot)}_{p(\cdot)} u(x) = p.v. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+s(x,y)p(x,y)}}\, dy, \qquad x \in \mathbb{R}^N,$$

along any $u \in C_0^\infty(\mathbb{R}^N)$, where $p.v.$ denotes the Cauchy principle value. From now, we set some notations as follows. We denote

$$s^- = \min_{(x,y)\in\mathbb{R}^{2N}} s(x,y), \quad s^+ = \max_{(x,y)\in\mathbb{R}^{2N}} s(x,y), \quad p^- = \min_{(x,y)\in\mathbb{R}^{2N}} p(x,y)$$

$$p^+ = \max_{(x,y)\in\mathbb{R}^{2N}} p(x,y), \quad \overline{s}(x) = s(x,x), \quad \overline{p}(x) = p(x,x) \qquad \text{for} \quad x \in \overline{\Omega},$$

and the fractional critical exponent

$$p_s^*(x) = \frac{N\overline{p}(x)}{N - \overline{s}(x)\overline{p}(x)}, \qquad x \in \overline{\Omega}.$$

Now, we assume that $s(\cdot) : \mathbb{R}^{2N} \to (0,1)$ and $p(\cdot) : \mathbb{R}^{2N} \to (1,\infty)$ are continuous functions fulfilling

- (**H1**) $0 < s^- \leq s^+ < 1 < p^- \leq p^+$,
- (**H2**) $s(\cdot)$ and $p(\cdot)$ are symmetric, that is $s(x,y) = s(y,x)$ and $p(x,y) = p(y,x)$ for any $(x,y) \in \mathbb{R}^{2N}$.

Let us assume that function $\lambda(x)$ satisfies the following conditions:

- ($\mathbf{\Lambda 1}$) $\lambda(x) \in L^\infty(\Omega)$,
- ($\mathbf{\Lambda 2}$) there exists an $x_0 \in \Omega$ and two positive constants $r$ and $R$ with $0 < r < R$ such that $\overline{B_R(x_0)} \subset \Omega$ and $\lambda(x) = 0$ for $x \in \overline{B_R(x_0)\backslash B_r(x_0)}$ while $\lambda(x) > 0$ for $x \in \Omega\backslash\overline{B_R(x_0) \setminus B_r(x_0)}$.

Furthermore the function $q(x)$ fulfilling:

- (**Q1**) $q \in C_+(\overline{\Omega})$ and $1 \leq q(x) < p_s^*(x)$ for any $x \in \overline{\Omega}$,
- (**Q2**) either $\max_{\overline{B_r(x_0)}} q(x) < p^- < p^+ < \min_{\overline{\Omega\backslash B_R(x_0)}} q(x)$, or $\max_{\overline{\Omega\backslash B_R(x_0)}} q(x) < p^- < p^+ < \min_{\overline{B_r(x_0)}} q(x)$.

Our main result concerning problem (1.1) is given by the following theorem.

**Theorem 1.1.** *Let $\Omega$ be a bounded smooth domain of $\mathbb{R}^N$, with $N > s(x,y)p(x,y)$ for any $(x,y) \in \overline{\Omega}\times\overline{\Omega}$. Assume that* (**H1**) − (**H2**), ($\mathbf{\Lambda 1}$) − ($\mathbf{\Lambda 2}$) *and* (**Q1**) − (**Q2**) *hold. Then there exists a positive constant $\lambda^* > 0$ such that problem (1.1) has a positive non-trivial weak solution, provided that $\|\lambda\|_{L^\infty(\Omega)} < \lambda^*$.*

## 2. Variational framework

In this section, we give some definitions and results of variable order fractional Sobolev spaces with variable exponent. We first define the new space

$$X := \Big\{ u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable, such that } u|_\Omega \in L^{\overline{p}(x)}(\Omega) \text{ with} \\ \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\nu^{p(x,y)}} K(x,y)\,dx\,dy < +\infty, \text{ for some } \nu > 0 \Big\}.$$

where $\mathcal{Q} := \mathbb{R}^{2N} \backslash (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ with $\mathcal{C}\Omega := \mathbb{R}^N \backslash \Omega$. We endow $X$ with the norm $\|u\|_X = \|u\|_{p(\cdot)} + [u]_X$, where

$$[u]_X = \inf \Big\{ \nu > 0; \iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\nu^{p(x,y)}|x-y|^{N+s(x,y)p(x,y)}}\,dx\,dy < 1 \Big\}.$$

Now, let $X_0 = \{u \in X;\ u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$, with the norm

$$\begin{aligned} \|u\|_{X_0} = \inf \Big\{ \nu > 0; & \iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\nu^{p(x,y)}|x-y|^{N+s(x,y)p(x,y)}}dx\,dy \\ &= \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{\nu^{p(x,y)}|x-y|^{N+s(x,y)p(x,y)}}dx\,dy \le 1 \Big\}, \end{aligned}$$

where the equality is a consequence of the fact that $u = 0$ a.e. in $\mathbb{R}^N$. Then $(X_0, \|\cdot\|_{X_0})$ is a separable reflexive Banach space, see [4].

**Definition 2.1.** Let $\mathcal{J} \in C^1(X_0, \mathbb{R})$ and $c \in \mathbb{R}$. The functional $\mathcal{J}$ satisfies the $(PS)_c$ condition, if any sequence $(u_n) \subset X_0$ such that $\mathcal{J}(u_n) \to c$ and $\mathcal{J}'(u_n) \to 0$ as $n \to +\infty$ in $X_0^*$, where $X_0^*$ is the dual space of $X_0$, has a convergent subsequence in $X_0$.

Next, we state the Sobolev-type embedding theorem for $X_0$.

**Theorem 2.2.** [4, Lemma 3.4] *Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, $N \ge 2$ and $s(\cdot)$ and $p(\cdot)$ satisfy* $(\mathbf{H1}) - (\mathbf{H2})$ *such that $N > s(x,y)p(x,y)$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$ and $q(x) \in C_+(\overline{\Omega})$ such that $q(x) < p_s^*(x)$ for any $x \in \overline{\Omega}$. Then there exists a constant $C = C(N,s,p,q,\Omega)$ such that for every $u \in X_0$*

$$\|u\|_{q(\cdot)} \le C\|u\|_{X_0}.$$

*Moreover, this embedding is compact.*

## 3. Proof of the main result

Clearly, the weak solutions of (1.1) are exactly the critical points of the Euler-Lagrange functional $\mathcal{J} : X_0 \to \mathbb{R}$, defined by

$$\mathcal{J}(u) = \iint_{\mathbb{R}^{2N}} \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\, dx\, dy - \int_\Omega \frac{\lambda(x)}{q(x)}|u|^{q(x)}\, dx.$$

In what follows, in order to study the existence of solution for problem (1.1), we prove that the functional $\mathcal{J}$ possesses the Mountain Pass geometry [2].

**Lemma 3.1.** *Assume that* $(\mathbf{H1})-(\mathbf{H2})$, $(\mathbf{\Lambda1})-(\mathbf{\Lambda2})$ *and* $(\mathbf{Q1})-(\mathbf{Q2})$ *hold. Then there exists* $\lambda^* > 0$ *such that provided* $\|\lambda\|_{L^\infty(\Omega)} < \lambda^*$, *there exist* $\rho_1 > 0$ *and* $\delta_1 > 0$ *such taht* $\mathcal{J}(u) \geq \delta_1 > 0$ *for any* $u \in X_0$ *with* $\|u\|_{X_0} = \rho_1$.

**Lemma 3.2.** *Assume that* $(\mathbf{H1})-(\mathbf{H2})$, $(\mathbf{\Lambda1})-(\mathbf{\Lambda2})$ *and* $(\mathbf{Q1})-(\mathbf{Q2})$ *hold. Then there exists* $\psi \in X_0$, $\psi \neq 0$ *such that* $\lim_{t\to+\infty} \mathcal{J}(t\,\psi) \to -\infty$.

**Lemma 3.3.** *Assume that* $(\mathbf{H1})-(\mathbf{H2})$, $(\mathbf{\Lambda1})-(\mathbf{\Lambda2})$ *and* $(\mathbf{Q1})-(\mathbf{Q2})$ *hold. Then* $\mathcal{J}$ *satisfies the* $(PS)_c$ *condition in* $X_0$ *for any* $c \in \mathbb{R}$.

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# REFINEMENT OF JENSEN-MERCER INEQUALITY

A. MORASSAEI

*Department of Mathematics, Faculty of Sciences, University of Zanjan, University Blvd., Zanjan 45371-38791, Iran.*
*morassaei@znu.ac.ir*

Abstract. In this paper we present the refinement of Jensen's inequality and then we state Jensen-Mercer inequality. In continuation, we give refinement of Jensen-Mercer inequality.

## 1. Introduction

Throughout this paper, suppose that $I$ and $J$ are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and functions $h$ and $f$ are real non-negative functions defined on $J$ and $I$, respectively.

In [4], Varošanec defined the $h$ -convex function as follows:
Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. We say that $f : I \to \mathbb{R}$ is a $h$-convex function, or that $f$ belongs to the class $SX(h, I)$, if $f$ is non-negative and for all $x, y \in I$, $t \in (0,1)$ we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y)\,. \tag{1.1}$$

If inequality (1.1) is reversed, then $f$ is said to be $h$-concave, that is $f \in SV(h, I)$.

If $h(t) = t$, then all non-negative convex functions belong to $SX(h, I)$ and all non-negative concave functions belong to $SV(h, I)$.

A function $h : J \to \mathbb{R}$ is said to be a *super-additive function* if

$$h(x+y) \geq h(x) + h(y)\,, \tag{1.2}$$

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for all $x, y \in J$. If inequality (1.2) is reversed, then $h$ is said to be a *sub-additive function.* If the equality holds in (1.2), then $h$ is said to be a *additive function.*

Function $h$ is called a *super-multiplicative function* if

$$h(xy) \geq h(x)h(y)\,, \tag{1.3}$$

for all $x, y \in J$ [4]. If inequality (1.3) is reversed, then $h$ is called a *sub-multiplicative function.* If the equality holds in (1.3), then $h$ is called a *multiplicative function.*

**Example 1.1.** [4] Consider the function $h : [0, +\infty) \to \mathbb{R}$ by $h(x) = (c+x)^{p-1}$. If $c = 0$, then the function $h$ is multiplicative. If $c \geq 1$, then for $p \in (0,1)$ the function $h$ is super-multiplicative and for $p > 1$ the function $h$ is sub-multiplicative.

## 2. Jensen's and Jensen-Mercer type inequality

In [1], Mercer proved that

$$f\left(x_1 + x_n - \sum_{j=1}^{n} t_j x_j\right) \leq f(x_1) + f(x_n) - \sum_{j=1}^{n} t_j f(x_j)\,. \tag{2.1}$$

where $x_j$'s also satisfy in the condition $0 < x_1 \leq x_2 \leq \cdots \leq x_n$, $t_j \geq 0$ with $\sum_{j=1}^{n} t_j = 1$ and $f$ is a convex function on an interval containing the $x_j$.

In this section, we present the Jensen-Mercer inequality for $h$-convex functions and then we give a refinement of Jensen-Mercer inequality for convex functions and a refinement of Jensen's inequality for $h$-convex functions.

**Theorem 2.1.** [4, Theorem 19] *Let $t_1, \cdots, t_n$ be positive real numbers ($n \geq 2$). If $h$ is a non-negative super-multiplicative function, $f$ is a $h$-convex function on $I$ and $x_1, \cdots, x_n \in I$, then*

$$f\left(\frac{1}{T_n}\sum_{j=1}^{n} t_j x_j\right) \leq \sum_{j=1}^{n} h\left(\frac{t_j}{T_n}\right) f(x_j)\,, \tag{2.2}$$

*where $T_n = \sum_{j=1}^{n} t_j$.*

**Theorem 2.2.** *Let $f$ be a $h$-convex function on an interval containing the $x_j$ $(j = 1, \cdots, n)$ such that $0 < x_1 \leq \cdots \leq x_n$, then*

$$f\left(x_1 + x_n - \sum_{j=1}^{n} t_j x_j\right) \leq \left(\sum_{j=1}^{n} h(t_j)[h(\lambda_j) + h(1-\lambda_j)]\right)\big(f(x_1) + f(x_n)\big) - \sum_{j=1}^{n} h(t_j) f(x_j)\,,$$

*where for every $j = 1, \cdots, n$, there exists $\lambda_j \in [0,1]$ such that $x_j = \lambda_j x_1 + (1-\lambda_j)x_n$.*

**Corollary 2.3.** *With the assumptions of previous theorem, if $h$ is a super-additive function such that for every probability vector $(t_1, \cdots, t_n)$, $\sum_{j=1}^{n} h(t_j) \leq 1$, then*

$$f\left(x_1 + x_n - \sum_{j=1}^{n} t_j x_j\right) \leq h(1)\big(f(x_1) + f(x_n)\big) - \sum_{j=1}^{n} h(t_j) f(x_j)\,.$$

*Moreover, if $h$ is multiplicative, then*

$$f\left(x_1 + x_n - \sum_{j=1}^{n} t_j x_j\right) \leq f(x_1) + f(x_n) - \sum_{j=1}^{n} h(t_j) f(x_j)\,.$$

Let $r = (r_1, \cdots, r_m)$ and $t = (t_1, \cdots, t_n)$ be two probability tuples i.e. $r_i,\ t_j \geq 0$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$, $\sum_{i=1}^{m} r_i = 1$ and $\sum_{j=1}^{n} t_j = 1$. By a (discrete) weight function (with respect to $r$ and $t$), we mean a mapping $\omega : \{(i,j) \ :\ 1 \leq i \leq m,\ 1 \leq j \leq n\} \to [0, \infty)$, such that $\sum_{i=1}^{m} \omega(i,j) r_i = 1$ $(j = 1, \cdots, n)$, and $\sum_{j=1}^{n} \omega(i,j) t_j = 1$ $(i = 1, \cdots, m)$.
In [3] Rooin proved that if $C$ is a convex subset of a real linear space, $x_1, \cdots, x_n \in C$ and $\varphi : C \to \mathbb{R}$ is a convex mapping, then

$$\varphi\left(\sum_{j=1}^{n} t_j x_j\right) \leq \sum_{i=1}^{m} r_i \varphi\left(\sum_{j=1}^{n} \omega(i,j) t_j x_j\right) \leq \sum_{j=1}^{n} t_j \varphi(x_j)\,.$$

The following example shows that above inequalities can be strict [2, Example 4.1].

**Example 2.4.** If $u = (u_1, \cdots, u_m)$ and $v = (v_1, \cdots, v_n)$ with $\|u\| = \left(\sum_{i=1}^{m} u_i^2\right)^{1/2} \leq 1$ and $\|v\| = \left(\sum_{j=1}^{n} v_j^2\right)^{1/2} \leq 1$ belong to $r^{\perp}$ and $t^{\perp}$, respectively, where $\perp$ denotes the usual orthogonality, then the function $\omega$ with $\omega(i,j) = 1 + u_i v_j \quad (1 \leq i \leq m,\ 1 \leq j \leq n)$ is a weight function. For instant, set $t = (t_1, t_2) = (\frac{2}{5}, \frac{3}{5})$, $r = (r_1, r_2, r_3) =$

$(\frac{1}{5}, \frac{3}{10}, \frac{1}{2})$, $v = (\frac{1}{2}, -\frac{1}{3})$ and $u = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$. Also, assume that $\varphi : \mathbb{R} \to \mathbb{R}$ is the convex function $\varphi(t) = t^2$. Hence

$$\varphi\left(\sum_{j=1}^{2} t_j x_j\right) < \sum_{i=1}^{3} r_i \varphi\left(\sum_{j=1}^{2} \omega(i,j) t_j x_j\right) < \sum_{j=1}^{2} t_j \varphi(x_j).$$

**Theorem 2.5** (Refinement of Jensen-Mercer inequality)**.** *Assume that* $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ *and* $t_j \geq 0$ *with* $\sum_{j=1}^{n} t_j = 1$ *and* $f$ *is a convex function on an interval containing the* $x_j$*. Then*

$$\begin{aligned} f\left(x_1 + x_n - \sum_{j=1}^{n} t_j x_j\right) &\leq \sum_{i=1}^{m} r_i f\left(\sum_{j=1}^{n} \omega(i,j) t_j (x_1 + x_n - x_j)\right) \\ &\leq \sum_{i=1}^{m} r_i \left(\sum_{j=1}^{n} \omega(i,j) t_j f(x_1 + x_2 - x_n)\right) \qquad (2.3) \\ &\leq f(x_1) + f(x_n) - \sum_{j=1}^{n} t_j f(x_j)\,, \end{aligned}$$

*where* $r = (r_1, \cdots, r_m)$ *and* $t = (t_1, \cdots, t_n)$ *are two probability tuples and* $\omega(i,j)$ *is a weight function.*

**Theorem 2.6** (Refinement of Jensen's inequality for $h$-convex function)**.** *Assume that* $h : J \to \mathbb{R}^+$ *is multiplicative function such that* $h(t) \leq t$ *and* $f$ *is a* $h$*-convex function on an interval containing the* $x_j$ $(j = 1, \cdots, n)$*. Then*

$$f\left(\sum_{j=1}^{n} t_j x_j\right) \leq \sum_{i=1}^{m} h(r_i) f\left(\sum_{j=1}^{n} \omega(i,j) t_j x_j\right) \leq \sum_{j=1}^{n} h(t_j) f(x_j)\,, \quad (2.4)$$

*where* $t = (t_1, \cdots, t_n)$ *and* $r = (r_1, \cdots, r_m)$ *are two probability tuples and* $\omega(i,j)$ *is weight function (with respect to* $r$ *and* $t$*).*

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Poster Presentation

# OPERATOR ALGEBRAS IN TWO PARAMETER $\sigma$-$C^*$-DYNAMICS

M. MOSADEQ

*Department of Mathematics, Behbahan Branch, Islamic Azad University, Behbahan, Iran*
*mosadeq@mshdiau.ac.ir*

Abstract. In this talk, we introduce the concept of $\sigma$-two parameter groups of bounded linear operators as a generalization of ($\sigma$-) one parameter groups and analyze their basic properties. We also, describe a two parameter $\sigma$-$C^*$-dynamics as a uniformly continuous $\sigma$-two parameter group of $*$-linear automorphisms on a $C^*$-algebra and associate with each so-called $\sigma$-$C^*$-dynamics a pair of $\sigma$-derivation, named as its generator. Finally, as an application, we characterize each two parameter $\sigma$-$C^*$-dynamics on the concrete $C^*$-algebra $\mathcal{A} := \mathbf{B}(H)$, where $H$ is a Hilbert space.

## 1. Introduction

Let $\mathcal{A}$ be a Banach space and $\sigma$ be a bijective bounded linear operator on $\mathcal{A}$. A one parameter family $\{\alpha_t\}_{t\in\mathbb{R}}$ of bounded linear operators on $\mathcal{A}$ is called a $\sigma$-one parameter group if $\alpha_0 = \sigma$ and $\sigma\alpha_{t+s} = \alpha_t\alpha_s$ $(t, s \in \mathbb{R})$. The $\sigma$-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ is called uniformly continuous if $\lim_{t\to 0} \|\alpha_t - \sigma\| = 0$. The generator $\delta$ of the $\sigma$-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ as a mapping $\delta : D(\delta) \subseteq \mathcal{A} \to \mathcal{A}$ such that $\delta(a) = \lim_{t\to 0} \frac{\alpha_t(a) - \sigma(a)}{t}$ where $D(\delta) = \{a \in A \textit{ such that } \lim_{t\to 0} \frac{\alpha_t(a) - \sigma(a)}{t} \textit{exists}\}$. If $\{\alpha_t\}_{t\in\mathbb{R}}$ is

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a $\sigma$-one parameter group with the generator $\delta$, then one can easily see that

(i) $\sigma\alpha_t = \alpha_t\sigma$ and $\sigma^{-1}\alpha_t = \alpha_t\sigma^{-1}$ for each $t \in \mathbb{R}$.
(ii) $\sigma(\delta(a)) = \delta(\sigma(a))$ and $\sigma^{-1}(\delta(a)) = \delta(\sigma^{-1}(a))$ for each $a \in D(\delta)$.

The reader is referred to [4] for more details.

By a $\sigma$-two parameter group of bounded linear operators on $\mathcal{A}$, we mean a mapping $\alpha : \mathbb{R} \times \mathbb{R} \to \mathbf{B}(\mathcal{A})$ which fulfills $\alpha_{0,0} = \sigma$ and $\sigma\alpha_{s+s',t+t'} = \alpha_{s,t}\alpha_{s',t'}$, for each $s, s', t, t' \in \mathbb{R}$. As in $\sigma$-one parameter case, the $\sigma$-two parameter group $\{\alpha_{s,t}\}_{s,t\in\mathbb{R}}$ is called uniformly continuous if $\lim\limits_{(s,t)\to(0,0)} \| \alpha_{s,t} - \sigma \| = 0$.

To any $\sigma$-two parameter group $\{\alpha_{s,t}\}_{s,t\in\mathbb{R}}$ on $\mathcal{A}$, we associate two $\sigma$-one parameter groups $\{u_s\}_{s\in\mathbb{R}}$ and $\{v_t\}_{t\in\mathbb{R}}$ defined by $u_s := \alpha_{s,0}$ and $v_t := \alpha_{0,t}$. One can see that the $\sigma$-two parameter group $\{\alpha_{s,t}\}_{s,t\in\mathbb{R}}$ is uniformly continuous if and only if so are $\{u_s\}_{s\in\mathbb{R}}$ and $\{v_t\}_{t\in\mathbb{R}}$. The generators of $\{u_s\}_{s\in\mathbb{R}}$ and $\{v_t\}_{t\in\mathbb{R}}$ are denoted by $\delta_1$ and $\delta_2$, respectively. We denote the pair $(\delta_1, \delta_2)$ as the generator of $\{\alpha_{s,t}\}_{s,t\in\mathbb{R}}$. In the case that $\sigma := I_{\mathcal{A}}$, then the concept of uniformly continuous $\sigma$-two parameter group is nothing more than a uniformly continuous two parameter group in the usual sense. The concept of two parameter semigroups and their relative concept was appeared in the frameworks of Janfada and Niknam in 2004. We refer the reader to [2] for more details. The $\sigma$-two parameter group property of $\{\alpha_{s,t}\}_{s,t\in\mathbb{R}}$ implies that $u_s v_t = \alpha_{s,0}\alpha_{0,t} = \sigma\alpha_{s,t} = \alpha_{0+s,t+0} = \alpha_{0,t}\alpha_{s,0} = v_t u_s$.

On the other hand, it is known from [2] that if $\{u_s\}_{s\in\mathbb{R}}$ and $\{v_t\}_{t\in\mathbb{R}}$ are two uniformly continuous $\sigma$-one parameter groups on $\mathcal{A}$ with the generators $\delta_1$ and $\delta_2$, respectively, then the two parameter family $\{\alpha_{s,t}\}_{s,t\in\mathbb{R}}$ defined by $\alpha_{s,t} := u_s v_t$ forms a $\sigma$-two parameter group if and only if $\delta_1\delta_2 = \delta_2\delta_1$.

One parameter groups of bounded linear operators are of highly considerable magnitude because of their applications in the theory of dynamical systems. The classical $C^*$-dynamical systems are expressed by means of uniformly continuous one parameter groups of $*$-automorphisms on $C^*$-algebras. On the other hand, the generator $d$ of a $C^*$-dynamics is a $*$-derivation. Recently, various generalized notions of derivations have been investigated in the context of Banach algebras. For instance, let $\mathcal{A}$ be a $*$-Banach algebra, and $\sigma$ be a $*$-linear operator. A $*-$linear map $\delta$ from a $*$-subalgebra $D(\delta)$ of $\mathcal{A}$ into $\mathcal{A}$ is called a $\sigma$-derivation if $\delta(ab) = \delta(a)\sigma(b) + \sigma(a)\delta(b)$ for all $a, b \in D(\delta)$. For instance, let $\sigma$ be a linear $*$-endomorphism and $h$ be an arbitrary self-adjoint element of $\mathcal{A}$. Then the mapping $\delta : \mathcal{A} \to \mathcal{A}$ defined by $\delta = i[h, \sigma(a)]$ is a $\sigma$-derivation which is called *inner* (see [3, 5]).

## 2. Two parameter $\sigma$-$C^*$-dynamics of operator algebras

Throughout this section, let $\mathcal{A}$ be a $C^*$-algebra and $\sigma$ is a $*$-linear automorphism on $\mathcal{A}$.

**Definition 2.1.** A *two parameter $\sigma$-$C^*$-dynamics* is a uniformly continuous $\sigma$-two parameter group $\{\alpha_{s,t}\}_{s,t\in\mathbb{R}}$ of $*$-linear automorphisms on the $C^*$-algebra $\mathcal{A}$.

To each two parameter $\sigma$-$C^*$-dynamics $\{\alpha_{s,t}\}_{s,t\in\mathbb{R}}$, on can associate $C^*$-dynamics $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ on $\mathcal{A}$ defined by $\varphi_{s,t}(a) := \alpha_{s,t}\left(\sigma^{-1}(a)\right)$. In this case, the pair $(d_1, d_2)$ defined by $d_j(\sigma(a)) = \delta_j(a)$, $a \in D(\delta_j)$ is its generator, $j = 1, 2$. On the other hand, the following lemma provides a method to construct a two parameter $\sigma$-$C^*$-dynamics from a two parameter $C^*$-dynamics.

**Lemma 2.2.** *Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a $*$-linear automorphism and $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ be a two parameter $C^*$-dynamics on $\mathcal{A}$ such that $\varphi_{s,t}\sigma = \sigma\varphi_{s,t}$. Then, $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ induces the two parameter $\sigma$-$C^*$-dynamics $\{\alpha_{s,t}\}_{s,t\in\mathbb{R}}$ on $\mathcal{A}$ defined by $\alpha_{s,t}(a) := \varphi_{s,t}\left(\sigma(a)\right)$.*

**Example 2.3.** Let $\mathcal{B}$ be a $C^*$-algebra and take $\mathcal{A} := \mathcal{B} \times \mathcal{B}$. Suppose that $\{\phi_{s,t}\}_{s,t\in\mathbb{R}}$ is a two parameter $C^*$-dynamics on $\mathcal{B}$ and consider the associated two parameter $C^*$-dynamics $\{\phi_{s,t} \oplus \phi_{s,t}\}_{s,t\in\mathbb{R}}$ on $\mathcal{A}$. Define $\sigma : \mathcal{A} \to \mathcal{A}$ by $\sigma(a, b) := (b, a)$. Then, $\sigma$ is a $*$-linear automorphism on $\mathcal{A}$ and the two parameter family $\{\alpha_{s,t}\}_{s,t\in\mathbb{R}}$ defined by $\alpha_{s,t} := (\phi_{s,t} \oplus \phi_{s,t})\sigma$ forms a two parameter $\sigma$-$C^*$-dynamics on $\mathcal{A}$ with the same continuity of $\{\phi_{s,t} \oplus \phi_{s,t}\}_{s,t\in\mathbb{R}}$.

**Theorem 2.4.** *Let $\{\alpha_{s,t}\}_{s,t\in\mathbb{R}}$ be a two parameter $\sigma$-$C^*$-dynamics on $\mathcal{A}$ with the generator $(\delta_1, \delta_2)$. Then, $\delta_j$ is a $\sigma$-derivation, for $j = 1, 2$.*

**Definition 2.5.** A $\sigma$-inner automorphism implemented by a unitary element $u$ of $\mathcal{A}$ is a $*$-linear automorphism $\alpha : \mathcal{A} \to \mathcal{A}$ such that $\alpha(a) = u\sigma(a)u^*$ for every $a \in \mathcal{A}$.

In the rest of the paper, we investigate this construction for a two parameter $\sigma$-$C^*$-dynamics on the concrete $C^*$-algebra $\mathcal{A} := \mathbf{B}(H)$.

**Theorem 2.6.** *Let $\{U_{s,t}\}_{s,t\in\mathbb{R}}$ be a uniformly continuous two parameter group of unitary operators on $\mathbf{B}(H)$, and $\{\alpha_{s,t}\}_{s,t\in\mathbb{R}}$ be the two parameter $\sigma$-$C^*$-dynamics implemented by the unitary operators group $\{U_{s,t}\}_{s,t\in\mathbb{R}}$ of $\sigma$-inner automorphisms with the generator $(\delta_1, \delta_2)$. Then, $\delta_j$ is an inner $\sigma$-derivation ($j = 1, 2$).*

It is now a pleasant surprise that each two parameter $\sigma$-$C^*$-dynamics on $\mathbf{B}(H)$ is of this form, i.e., it is implemented by a unitary operators two parameter group on $H$. To achieve this nontrivial result, first note that each bounded derivation on $\mathbf{B}(H)$ is inner see [1, Lemma 1.3.16.2]). So, we can characterize uniformly continuous two parameter inner $*$-automorphisms groups on the $C^*$-algebra $\mathbf{B}(H)$ as follows.

**Theorem 2.7.** *Let $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ be a two parameter group on $\mathbf{B}(H)$. Then, the following properties are equivalent.*

(i) *$\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ is a uniformly continuous two parameter $*$-automorphisms group on $\mathbf{B}(H)$.*

(ii) *There are self-adjoint operators $A, B$ in $B(\mathcal{H})$ satisfying such that $\varphi_{s,t}(T) = e^{it(A+B)}Te^{-it(A+B)}$.*

Applying the previous theorem, one can obtain the following main result.

**Theorem 2.8.** *Let $\{\alpha_{s,t}\}_{s,t\in\mathbb{R}}$ be a $\sigma$-two parameter group on $\mathbf{B}(H)$. Then, the following properties are equivalent.*

(i) *$\{\alpha_{s,t}\}_{s,t\in\mathbb{R}}$ is a two parameter $\sigma$-$C^*$-dynamics on $\mathbf{B}(H)$.*

(ii) *There is a uniformly continuous two parameter group $\{U_{s,t}\}_{s,t\in\mathbb{R}}$ of unitary operators on $\mathbf{B}(H)$ such that $\alpha_{s,t}(T) = U_{s,t}\sigma(T)U^*_{s,t}$.*

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**Oral Presentation**

# THE IMPLEMENTED $\sigma$-INNER AUTOMORPHISM GROUPS OF OPERATOR ALGEBRAS

M. MOSADEQ

*Department of Mathematics, Behbahan Branch, Islamic Azad University, Behbahan, Iran*
*mosadeq@mshdiau.ac.ir*

Abstract. Let $A$ be a $C^*$-algebra acting on a Hilbert space $H$ which includes $\mathcal{K}(H)$, $\sigma$ be a $*$-linear automorphism on $A$ and $\{\alpha_t\}_{t\in\mathbb{R}}$ be a $\sigma$-$C^*$-dynamics on $A$ with the generator $\delta$. In this paper, we demonstrate some conditions under which $\{\alpha_t\}_{t\in\mathbb{R}}$ can be implemented by a $C_0$-groups of unitaries on $H$.

## 1. Introduction

Let $A$ be a Banach space and $\sigma$ be a bijective bounded linear operator on $A$. A one parameter family $\{\alpha_t\}_{t\in\mathbb{R}}$ of bounded linear operators on $A$ is called a $\sigma$-one parameter group if $\alpha_0 = \sigma$ and $\sigma\alpha_{t+s} = \alpha_t\alpha_s$ $(t, s \in \mathbb{R})$. The $\sigma$-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ is called strongly continuous if $\lim\limits_{t\to 0}\alpha_t(a) = \sigma(a)$ for all $a \in A$. The generator $\delta$ of the $\sigma$-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ as a mapping $\delta : D(\delta) \subseteq A \to A$ such that $\delta(a) = \lim\limits_{t\to 0}\dfrac{\alpha_t(a) - \sigma(a)}{t}$ where $D(\delta) = \{a \in A \; such \; that \; \lim\limits_{t\to 0}\dfrac{\alpha_t(a) - \sigma(a)}{t} exists\}$. If $\{\alpha_t\}_{t\in\mathbb{R}}$ is a $\sigma$-one parameter group with the generator $\delta$, then one can easily see that

(i) $\sigma\alpha_t = \alpha_t\sigma$ and $\sigma^{-1}\alpha_t = \alpha_t\sigma^{-1}$ for each $t \in \mathbb{R}$.

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(ii) $\sigma(\delta(a)) = \delta(\sigma(a))$ and $\sigma^{-1}(\delta(a)) = \delta(\sigma^{-1}(a))$ for each $a \in D(\delta)$.

As an example of $\sigma$-one parameter group, let Let $B$ be a Banach space and take $A := B \times B$. Suppose that $\{\phi_t\}_{t\in\mathbb{R}}$ is a one parameter group on $B$ and consider the associated one parameter group $\{\phi_t \oplus \phi_t\}_{t\in\mathbb{R}}$ on $A$. Define $\sigma : A \to A$ by $\sigma(a, b) := (b, a)$. Then, $\sigma$ is a bijective bounded linear operator on $A$ and the one parameter family $\{\alpha_t\}_{t\in\mathbb{R}}$ defined by $\alpha_t := (\phi_t \oplus \phi_t)\sigma$ is a $\sigma$-one parameter group on $A$ with the same continuity of $\{\phi_t \oplus \phi_t\}_{t\in\mathbb{R}}$. The reader is referred to [3] for more details.

One parameter groups of bounded linear operators are of highly considerable magnitude because of their applications in the theory of dynamical systems. The classical $C^*$-dynamical systems are expressed by means of strongly continuous one parameter groups of $*$-automorphisms on $C^*$-algebras. On the other hand, the infinitesimal generator $d$ of a $C^*$-dynamical system is a $*$-derivation.

Recently, various generalized notions of derivations have been investigated in the context of Banach algebras. For instance, it can be pointed to "$\sigma$-derivations" as follows.

Let $A$ be a $*$-Banach algebra, and $\sigma$ be a $*$-linear operator. A $*-$linear map $\delta$ from a $*$-subalgebra $D(\delta)$ of $A$ into $A$ is called a $\sigma$-derivation if $\delta(ab) = \delta(a)\sigma(b) + \sigma(a)\delta(b)$ for all $a, b \in D(\delta)$. For instance, let $\sigma$ be a linear $*$-endomorphism and $h$ be an arbitrary self-adjoint element of $A$. Then the mapping $\delta : A \to A$ defined by $\delta = i[h, \sigma(a)]$ is a $\sigma$-derivation which is called *inner* (see [1, 6] for more information on $\sigma$-derivations).

In each case of generalization of derivation, a noted point which draws the attention of analysts is trying to represent a suitable dynamical system whose infinitesimal generator is exactly the desired extended derivation as well as being an extension of a $C^*$-dynamical system. Such dynamical system is usually provided by adjoining a suitable property to (an extension of) a one parameter group of bounded linear operators. Some approaches to preparing new dynamical systems and their applications have been explained in [2, 4, 5].

Let $\sigma$ be a $*$-linear automorphism on a $C^*$-algebra $A$. By a $\sigma$-$C^*$-dynamics, we mean a strongly continuous $\sigma$-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ of $*$-linear automorphisms on $A$. It has been proved in [2] that, the generator $\delta$ of the $\sigma$-$C^*$-dynamics $\{\alpha_t\}_{t\in\mathbb{R}}$ is a $*$-$\sigma$-derivation.

Let $\sigma$ be a $*$-linear automorphism on the $C^*$-algebra $A$. A $\sigma$-inner automorphism implemented by a unitary element $u$ of $A$ is a $*$-linear automorphism $\alpha : A \to A$ such that $\alpha(a) = u\sigma(a)u^*$ for every $a \in A$.

Suppose that $h$ is a self-adjoint element in $A$, $\sigma : A \to A$ is a $*$-linear automorphism such that $\sigma(h) = h$. Inspiring the method as stated in ([2], Theorem 3.7) one can prove that, the inner $\sigma$-derivation $\delta(a) = i[h, \sigma(a)]$ induces the $\sigma$-$C^*$-dynamical system $\alpha_t(a) = e^{ith}\sigma(a)e^{-ith}$ of $\sigma$-inner automorphisms.

Let $H$ be a Hilbert space and $\mathbf{B}(H)$ be the set of all bounded linear operators on $H$. Due to the Gelgand-Naimark-Segal representation, each non-commutative $C^*$-algebra can be regarded as a $C^*$-subalgebra of $\mathbf{B}(H)$, for some Hilbert space $H$. It is one of the key ideas of quantum mechanics to use $C_0$-one parameter groups of unitary operators on a Hilbert space $H$ to implement new dynamical systems on the operator algebra $\mathbf{B}(H)$ and its $C^*$-subalgebras.

We recall that $\mathcal{K}(H)$, the set of all compact operators on $H$, is a $C^*$-subalgebra of $\mathbf{B}(H)$ which contains $\mathcal{F}(H)$, the set of all finite rank operators on $H$. Especially, $\mathcal{K}(H) = \overline{\mathcal{F}(H)}$ (see [7, Theorem 2.4.5]).

Let $A$ be a $C^*$-algebra acting on a Hilbert space $H$ which includes $\mathcal{K}(H)$, $\sigma$ be a $*$-linear automorphism on $A$ and $\{\alpha_t\}_{t\in\mathbb{R}}$ be a $\sigma$-$C^*$-dynamics on $A$ with the generator $\delta$. In this talk, we are going to characterize $\{\alpha_t\}_{t\in\mathbb{R}}$ with respect to a $C_0$-group of unitaries on $H$.

## 2. $\sigma$-$C^*$-DYNAMICS OF OPERATOR ALGEBRAS

**Theorem 2.1.** *Let $\{u_t\}_{t\in\mathbb{R}}$ be a $C_0$-group of unitary operators on a Hilbert space $H$ and $\sigma : \mathbf{B}(H) \to \mathbf{B}(H)$ be a $*$-linear automorphism satisfying $\sigma(u_t) = u_t$. Then, $\alpha_t(a) = u_t\sigma(a)u_t^*$ is a $\sigma$-$C^*$-dynamics on $\mathcal{K}(H)$.*

The following theorem manifests some conditions under which a $\sigma$-$C^*$-dynamics on $A$ can be implemented by a $C_0$-groups of unitaries on $H$.

**Theorem 2.2.** *Let $A$ be a $C^*$-algebra acting on a Hilbert space $H$ which includes $\mathcal{K}(H)$, $\sigma$ be a $*$-linear automorphism on $A$ and $\{\alpha_t\}_{t\in\mathbb{R}}$ be a $\sigma$-$C^*$-dynamics on $A$ with the generator $\delta$. If there exists a rank one projection $p \in A$ such that $\alpha_t(p) = \sigma(p)$ for each $t \in \mathbb{R}$, then there is a $C_0$-group $\{u_t\}_{t\in\mathbb{R}}$ of unitaries in $\mathbf{B}(H)$ such that $\alpha_t(a) = u_t\sigma(a)u_t^*$.*

The following result gives us a version of perturbation theorem in the setting of $\sigma$-$C^*$-dynamical systems.

**Theorem 2.3.** *Let $\delta_1$ be the generator of a $\sigma$-$C^*$-dynamics $\{\alpha_t\}_{t\in\mathbb{R}}$ on $A$ and $\delta_2$ be a bounded $*$-$\sigma$-derivation on $A$ such that $\delta_2\sigma = \sigma\delta_2$. Then, $\delta_1 + \delta_2$ generates a $\sigma$-$C^*$-dynamical system on $A$.*

**Theorem 2.4.** *Let $A$ be a $C^*$-algebra acting on a Hilbert space $H$ which includes $\mathcal{K}(H)$, $\sigma$ be a $*$-linear automorphism on $A$ and $\{\alpha_t\}_{t\in\mathbb{R}}$ be a $\sigma$-$C^*$-dynamics on $A$ with the generator $\delta$ satisfying $\sigma\delta = \delta$. If there is a rank-one projection $p \in D(\delta)$, then there exists a bounded $*$-$\sigma$-derivation $\delta^p$ on $A$ such that $(\delta + \delta^p)(p) = 0$ and $\delta + \delta^p$ generates a $\sigma$-$C^*$-dynamical system on $A$.*

**Theorem 2.5.** *Let $A$ be a $C^*$-algebra acting on a Hilbert space $H$ which includes $\mathcal{K}(H)$, $\sigma$ be a $*$-linear automorphism on $A$ and $\{\alpha_t\}_{t\in\mathbb{R}}$ be a $\sigma$-$C^*$-dynamics on $A$ with the generator $\delta$ satisfying $\sigma\delta = \delta$. Then, for a rank-one projection $p \in D(\delta)$, there exist a a $\sigma$-$C^*$-dynamical system $\{\alpha_{t,p}\}_{t\in\mathbb{R}}$ and a self-adjoint operator $h^p$ on $H$ and such that $\alpha_{t,p}(a) = e^{ith^p}\sigma(a)e^{-ith^p}$ on $A$. Furthermore, if $\sigma(h^p) = h^p$, then there is a self-adjoint operator $h$ on $H$ such that for each $a \in D(\delta) \cap \mathcal{K}(H)$, $\delta(a) = i[h, \sigma(a)]$ and $\alpha_t(a) = e^{ith}\sigma(a)e^{-ith}$ on $\mathcal{K}(H)$.*

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# A STUDY ON BEHAVIOR OF THE NORM OF THE MATRIX EXPONENTIAL

M. H. MOSLEHI* AND E. ZANGOOIEZADEH

*Department of Mathematics, Payame Noor University, P. O. Box 19395-3697, Tehran, Iran.*
*mh_moslehi@pnu.ac.ir and e.zangoie@pnu.ac.ir*

Abstract. In this paper, we study on the possible behavior of the norm of the Matrix exponential that plays an important role in linear control systems and ordinary differential equations. For this Purpose, we consider the relation between the concepts of the weighted logarithmic norm, the Lyapunov equation and the norm of the matrix exponential.

## 1. Introduction

The logarithmic norm of a matrix $A$ is defined by

$$\mu[A] = \lim_{h \longrightarrow 0^+} \frac{\|I + \Delta A\| - 1}{\Delta}$$

for matrix norm $\|.\|$ induced by a vector norm in $\mathbb{R}^n$ [2]. The formula $\mu_2[A] = \lambda_{\max}(\frac{\hat{A}+\hat{A}^T}{2})$ is well-known [5] where $\lambda_{\max}(F)$ stands for the maximal eigenvalue of a symmetric matrix $F$.

**Lemma 1.1.** [2, 3]. *For any inner product on $\mathbb{R}^n$, and the corresponding inner product norm $\|.\|$, we have*

$$\mu[A] = \max_{x \neq 0} \frac{(Ax, x)}{\|x\|^2}$$

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*Key words and phrases.* Weighted logarithmic norm, Lyapunov equation, stable matrix, Matrix exponential .

*and*

$$\|\exp(At)\| \le \exp(\mu[A]t) \tag{1.1}$$

**Lemma 1.2.** [1]. *A real matrix $A$ is stable if and only if, for any given real symmetric positive definite matrix $W$, there is a symmetric positive definite matrix $H$ which is the unique solution of the following Lyapunov equation:*

$$A^T H + HA = -2W$$

**Definition 1.3.** Assume that a symmetric matrix $H$ is positive definite. For any vector $x$ and any matrix $A$, the vector norm with weight $H$, the matrix norm with weight $H$, and the logarithmic norm with weight $H$ defined, respectively, by

$$\|x\|_H = \sqrt{x^T Hx},\ \|A\|_H = \max_{x\neq 0} \frac{\|Ax\|_H}{\|x\|_H},\ \mu_H[A] = \max_{x\neq 0} \frac{(Ax, x)_H}{\|x\|_H^2}$$

## 2. Main results

Throughout this section, we assume that the symmetric positive definite matrix $H$ satisfies the Lyapunov equation

$$A^T H + HA = -2I \tag{2.1}$$

**Theorem 2.1.** [4] *If a real matrix $A$ is stable, then there is a weight $H$ logarithmic norm of $A$ such that*

$$\mu_H[A] = -\frac{1}{\lambda_{\max}(H)} \tag{2.2}$$

**Theorem 2.2.** [5] *For any real matrix $A$,*

$$\mu_{(H)}[A] = \lambda_{\max}\left(\frac{\hat{A} + \hat{A}^T}{2}\right),\ \|A\|_{(H)} = \sqrt{\lambda_{\max}(\hat{A}^T\hat{A})} \tag{2.3}$$

*where $H_0 = \sqrt{H}$ and $\hat{A} = H_0 A H_0^{-1}$.*

**Theorem 2.3.** *If a real matrix $A$ is stable, we have*

$$\|\exp(At)\|_2 = \beta \exp\left(-\frac{t}{\lambda_{\max}(H)}\right) \tag{2.4}$$

*where,*

$$\beta = \sqrt{\frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}} \tag{2.5}$$

.

*Proof.* Here, with a different approach from [5] , the proof of the theorem is stated. Let $H_0 = \sqrt{H}$ and $\hat{A} = H_0 A H_0^{-1}$ , we have

$$\exp(At) = H_0^{-1} \exp(\hat{A}t) H_0.$$

and

$$\|\exp(At)\|_2 = \|H_0^{-1} \exp(\hat{A}t) H_0\|_2 = \|H_0^{-1}\|_2 \|H_0\|_2 \|\exp(\hat{A}t)\|_2 \quad (2.6)$$

According to Lemma 1.1, Theorem 2.2 and Theorem 2.2, we have

$$\begin{aligned}
\|\exp(\hat{A}t)\|_2 &= \exp(\mu_2[\hat{A}]t) \\
&= \exp\left(t\lambda_{\max}\left(\frac{\hat{A} + \hat{A}^T}{2}\right)\right) \\
&= \exp\left(\mu_H[A]t\right) \\
&= \exp\left(-\frac{t}{\lambda_{\max}(H)}\right) \qquad (2.7)
\end{aligned}$$

On the other hand, since $H_0$ is a symmetric positive definite matrix,

$$\|H_0^{-1}\|_2 \|H_0\|_2 = \sqrt{\frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}} = \beta \quad (2.8)$$

From (2.6)–(2.8), the inequality (2.4) holds. $\square$

**Corollary 2.4.** *Consider a stable linear time-invariant system of the form*

$$\dot{y}(t) = Ay(t). \quad (2.9)$$

*system* (2.9) *is asymptotically stable if and only if the weight $H$ logarithmic norm of matrix $A$ is negative.*

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# BIFURCATION ANALYSIS IN A CLASS OF DELAYED SYSTEM

M. H. MOSLEHI* AND E. ZANGOOIEZADEH

*Department of Mathematics, Payame Noor University, P. O. Box 19395-3697, Tehran, Iran.*
*mh_moslehi@pnu.ac.ir and e.zangoie@pnu.ac.ir*

Abstract. In this paper we consider a delayed Hopfield neural networks model with three neurons. This system will analyze by proving the local asymptotic stability, bifurcation and existence of a Hopf bifurcating periodic solution. This purpose is achieved by analyzing the associated characteristic transcendental equation.

## 1. Introduction

Analysis of neural networks from the viewpoint of non-linear dynamics is helpful in solving problems of theoretical and practical importance. The vast applications of Hopfield neural networks such as classification, associative memory, pattern recognition and optimization [1, 3], have draw the attentions of researchers. Marcus and Westervelt [2] first found out that the delay can destabilize the network as a whole and create oscillatory behavior. We consider here a bidirectional three-neuron network with discrete delaya described by the following system of delay differential equations:

$$\dot{u}_i\left(t\right) = -u_i\left(t\right) + \sum_{j=1}^{3} a_{ij} f\left(u_j\left(t - \ \tau_j\right)\right), \ i = 1, 2, 3, i \neq j \tag{1.1}$$

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where $u_i\left(t\right)$ represents the activation state of $i$-th neuron $i = 1, 2, 3$ at time $t$ , $a_{ij}$ is the weight of synaptic connections from $i$-th neuron to $j$-th neuron and $\tau_j \geq 0$ is the time delay. In (1.1), each neuron not only is connected to itselfe but also it is connected to the other neuron too via a non linear sigmoidal function $f$, which is a typical transmitting function, among neurons. The initial value is assumed to be $u_i\left(\theta\right) = \varphi_i\left(\theta\right),\ \theta \in [-k, 0]$ where $\varphi_i\left(\theta\right) \in C\left([-k,\ 0], \mathbb{R}\right), i = 1, 2, 3$ and $k = max_{1\leq j\leq 3}\,\tau_j$. The natural phase space for (1.1) is the space $C$= $C([-k, 0]$,$\mathbb{R}^3)$ of continuous functions defined on $[-k, 0]$ equipped with the supremum norm $\|\varphi\| = sup_{-k\leq s\leq 0}\,\|\varphi(s)\|$ . Suppose $f$ :$\mathbb{R} \to \mathbb{R}$ is continuous. Then the solutions of equation (1.1) define the continuous semiflow

$$\begin{aligned} \Phi : \mathbb{R}^+ \times C &\longmapsto C \\ (t, \varphi) &\longmapsto\ x_t^{\varphi} \end{aligned}$$

A function $\widehat{\xi} \in C$ is an equilibrium point (or stationary point) of $\Phi$ if $\widehat{\xi}\left(s\right) = (\xi_1, \xi_2, \xi_3)$ for all $-k \leq s \leq 0$ , satisfying $\sum_{j=1}^{3} a_{ij} f\left(\xi_j\right) = -\xi_i$, $i = 1, 2, 3$. Suppose that $f \in C^1(\mathbb{R})$ , $f\left(0\right) = 0$ and $uf\left(u\right) > 0$ for $u \neq 0$ . Therefore $(0, 0, 0)$ is a stationary point of system (1.1).

## 2. Main results

For stability analysis, the system (1.1) has been linearized about $(0, 0, 0)$ and the following system of linearized equations obtained:

$$\dot{u}_i\left(t\right) = -u_i\left(t\right) + \sum_{j=1}^{3} \alpha_{ij} u_j\left(t - \tau_j\right),\ i = 1, 2, 3,\ i \neq j \tag{2.1}$$

Where $\alpha_{ij} = a_{ij} f^{'}\left(0\right), i, j = 1, 2, 3, i \neq j$. Let $a_{11} + a_{22} + a_{33} = 0$ and $\tau_1 = \tau_2 = \tau_3 = \tau$.The associated characteristic equation of system (2.1) is as follow

$$\begin{vmatrix} -\lambda - 1 + a_{11}e^{-\lambda\tau} & -\alpha_{12}e^{-\lambda\tau} & \alpha_{13}e^{-\lambda\tau} \\ \alpha_{21}e^{-\lambda\tau} & -\lambda - 1 + a_{22}e^{-\lambda\tau} & \alpha_{23}e^{-\lambda\tau} \\ \alpha_{31}e^{-\lambda\tau} & \alpha_{32}e^{-\lambda\tau} & -\lambda - 1 + a_{33}e^{-\lambda\tau} \end{vmatrix} = 0 \tag{2.2}$$

The zero solution of system (1.1) is stable if and only if all roots $\lambda$ of characteristic equation (2.2) have negative real parts. If $A = (\alpha_{ij})_{n\times n}, B = e^{-\lambda\tau} I_{n\times n}, \Lambda = \lambda + 1$, Then the characteristic equation (2.2) can be written as the following equation

$$\mathcal{P}\left(\lambda, \tau\right) = \det\left(\Lambda I - AB\right) = 0 \tag{2.3}$$

Formula (2.3) can be rewritten as follows

$$\mathcal{P}\left(\lambda,\tau\right)=\prod_{k=1}^{3}\mathcal{P}_{k}\left(\lambda,\tau\right)=0 \tag{2.4}$$

where $\mathcal{P}_k\left(\lambda,\tau\right)=\lambda+1-\mu_j e^{-\lambda\tau}$ and $\mu_1$, $\mu_2$ and $\mu_3$ are the eigenvalues of the matrix $A$, i.e. the roots of polynomial

$$F(\mu)=\mu^3-p\mu-q=0 \tag{2.5}$$

where $p=\alpha_{12}\alpha_{21}+\alpha_{23}\alpha_{32}+\alpha_{13}\alpha_{31}-\alpha_{11}\alpha_{22}-\alpha_{11}\alpha_{33}-\alpha_{22}\alpha_{33}$ and $q=det(A)$. To solve $F\left(\mu\right)=0$, the following two cases are considered.

Case 1: If $p\leq 3(q^2/4)^{\frac{1}{3}}$, solutions of $F\left(\mu\right)=0$ are given by

$$z=2\delta,-\delta\pm i\xi$$

where $\delta=\frac{\alpha+\gamma}{2}$ , $\xi=\frac{\sqrt{3}(\alpha-\gamma)}{2}$,
$\alpha$ is the real value of $[(q+\sqrt{q^2-(4p^3/27)})/2]^{\frac{1}{3}}$ and
$\gamma$ is the real value of $[(q-\sqrt{q^2-(4p^3/27)})/2]^{\frac{1}{3}}$.

Case 2: If $p>3(q^2/4)^{\frac{1}{3}}$, solutions of $F\left(z\right)=0$ are given by $z=(2\sqrt{p}cos(2m\pi+\theta)/3)/\sqrt{3}\;(m=0,1,2)$ where $tan\theta=\sqrt{-(q^2-4p^3/27)}/q$.

**Theorem 2.1.** *If*

(a) $p\leq 3(q^2/4)^{\frac{1}{3}}$, $max[|2\delta|,|\delta|+|\xi|]<1$ *and* $\tau\geq 0$ *or*
(b) $p>(q^2/4)^{\frac{1}{3}}$, $2|\sqrt{p}|/\sqrt{3}\leq 1$ *and* $\tau\geq 0$.

*then all the roots of characteristic equation* (2.4) *have negative real part and hence the trivial steady state* $(0,0,0)$ *of* (1.1) *is locally asymptotically stable.*

**Theorem 2.2.** *If* $p\leq 3(q^2/4)^{\frac{1}{3}}$, $\delta\neq\frac{1}{2}$, $|2\delta|<1$, $|\delta^2+\xi^2|>1$ *then there exist some* $\tau^*>0$ *such that the trivial steady state* $(0,0,0)$ *of* (1.1) *is locally asymptotically stable when* $\tau<\tau^*$. *Moreover, a Hopf bifurcation occurs at the trivial steady state* $(0,0,0)$ *of* (1.1) *when* $\tau=\tau^*$.

*Proof.* Clearly $\lambda=0$ is not a solution of characteristic equation (2.4). $\lambda=i\omega$ is a root of (2.4) if and only if $\mathcal{P}\left(i\omega,\tau\right)=0$.
Now $\mathcal{P}_1\left(i\omega,\tau\right)=i\omega+1-(2\delta)e^{-(i\omega\;)\tau}\neq 0$, since $|2\delta|<1$. If $\mathcal{P}_2(i\omega,\tau)=i\omega+1-\left(\delta+i\xi\right)e^{-(i\omega)\tau}=0$ then

$$cos\left(\omega\tau\right)=\frac{-\delta-\xi\omega}{\delta^2+\xi^2}\;,sin(\omega\tau)=\frac{-\xi+\delta\omega}{\delta^2+\xi^2}$$

and we have $\omega^2=\delta^2+\xi^2-1$. By $\delta^2+\xi^2>1$, We denote the positive root by $\omega^*=\sqrt{\delta^2+\xi^2-1}$. Therefore, for the imaginary root $\lambda= i\omega$ of $\mathcal{P}_2(i\omega,\tau)=0$, we have two sequences $\{\tau^{1,j}\}_0^{\infty}$ , $\{\tau^{2,j}\}_0^{\infty}$ $(j=0,1,2,3,\ \ldots)$, $(m=1,2)$ as follow

$$\tau^{m,j}=\frac{1}{\omega^*}\left[2[\frac{m}{2}]\pi-\arccos\left(\frac{-\delta-\xi\omega^*}{\delta^2+\xi^2}\right)\ +2j\pi\right],\ -\xi+\delta\omega^*<0$$

We assume that $\tau^*=min_{\ j=0,1,2,\ldots}\{\tau^{1,j},\tau^{2,j}\}$ and $\lambda\left(\tau\right)=\mu\left(\tau\right)+\omega\left(\tau\right)$ is a solution of $\mathcal{P}_2(\lambda,\tau)=0$ i.e. $\mu\left(\tau^*\right)=0,\omega\left(\tau^*\right)=\omega^*$. Thus we have

$$\frac{d\lambda}{d\tau}=-\frac{\lambda(-\delta+i\xi)e^{-\lambda\tau}}{1+\tau(-\delta+i\xi)e^{-\lambda\tau}}$$

From $\mathcal{P}_2(i\omega,\tau)=0$, we have $e^{-\lambda\tau}=\frac{1+\lambda}{-\delta+i\xi}$

Thus, $\left(\frac{d\lambda}{d\tau}\right)^{-1}=-\frac{1}{\lambda(\delta+i\xi)e^{-\lambda\tau}}-\frac{\tau}{\lambda}$ . Evaluating $\left(\frac{d\lambda}{d\tau}\right)^{-1}$ at $\tau=\tau^*$ (i.e., $\lambda=i\omega^*$ ) and taking the real part, we have

$$Re\ \left[\left(\frac{d\lambda}{d\tau}\right)^{-1}|_{\ \tau=\tau^*}\right]=\frac{1}{1+\left(\omega^*\right)^2}=\frac{1}{\delta^2+\xi^2}>0$$

So $\frac{dRe(\lambda)}{d\tau}$ is positive at $\tau=\tau^*$. Thus, the solution curve of the characteristic Eq. $\mathcal{P}_2(i\omega,\tau)=0$ crosses the imaginary axis. This shows that a Hopf bifurcation occurs at $\tau=\tau^*>0$. When $\tau<\tau^*$, he origin of state space of system (1) is locally asymptotically stable by continuity. By the same argument for $\mathcal{P}_3(i\omega,\tau)=i\omega+1-\left(-\delta-i\xi\right)e^{-(i\omega)\tau}=0$, the proof of the theorem is completes. $\square$

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# FABER POLYNOMIAL COEFFICIENT ESTIMATES OF COMPREHENSIVE SUBCLASS OF MEROMORPHIC BI-UNIVALENT FUNCTIONS

A. MOTAMEDNEZHAD

*Faculty of Mathematical Sciences, Shahrood University of Technology, P. O. Box 316-36155, Shahrood, Iran*
*a.motamedne@gmail.com*

Abstract. In this paper, we introduce a new comprehensive subclass of meromorphic bi-univalent functions. We also find the upper bounds for the initial Taylor-Maclaurin coefficients $|b_0|$ and$|b_1|$ for functions in this comprehensive subclass. The results presented in this paper would generalize and improve several recent works on the subject.

## 1. Introduction

Let $\Sigma$ denote the family of meromorphic univalent functions $f$ of the form

$$f(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}, \tag{1.1}$$

which defined on the domain $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. Since $f \in \Sigma$ is univalent, it has an inverse $f^{-1}$, that satisfy

$$f^{-1}(f(z)) = z \ (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \ \ (M < |w| < \infty, \ M > 0)\,.$$

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Furthermore, the coefficients of $g$, the inverse map of $f$, are given by the Faber polynomial ([6]):

$$g(w) = f^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} = w - b_0 - \sum_{n=1}^{\infty} \frac{1}{n} K_{n+1}^n \frac{1}{w^n}, \qquad (1.2)$$

where $M < |w| < \infty$,

$$K_{n+1}^n = nb_0^{n-1}b_1 + n(n-1)b_0^{n-2}b_2 + \frac{1}{2}n(n-1)(n-2)b_0^{n-3}(b_3 + b_1^2) + \frac{n(n-1)(n-2)(n-3)}{3!}b_0^{n-4}(b_4 + 3b_1b_2) + \sum_{j\geq 5} b_0^{n-j}V_j$$

and $V_j$ with $5 \leq j \leq n$ is a homogeneous polynomial of degree $j$ in the variables $b_1, b_2, ..., b_n$. (See [1, 2] or [12]).

A function $f \in \Sigma$ is said to be meromorphic bi-univalent if $f^{-1} \in \Sigma$. The family of all meromorphic bi-univalent functions is denoted by $\Sigma_{\mathfrak{B}}$.

Estimates on the coefficient of meromorphic univalent functions were widely studied in the literature; for instance, the estimate $|b_2| \leq 2/3$ for meromorphic univalent functions $f \in \Sigma$ with $b_0 = 0$ was obtained by Schiffer [10] and the inequality $|b_n| \leq 2/(n+1)$ for $f \in \Sigma$ with $b_k = 0$, $1 \leq k \leq n/2$ was proven by Duren [5].

For the coefficients of the inverse of meromorphic univalent functions, Springer [11] proved that

$$|B_3| \leq 1 \textit{ and } |B_3 + \frac{1}{2}B_1^2| \leq \frac{1}{2}$$

and conjectured that

$$|B_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!} \ (n = 1, 2, 3, ...).$$

In 1977, Kubota [7] has proved that the Springer's conjecture is correct for $n = 3, 4, 5$ and afterwards sharp bounds for the coefficients $B_{2n-1}$, $1 \leq n \leq 7$ were obtained by Schober [8].

Recently, Bulut [3] introduced the following subclass of meromorphic bi-univalent function and obtained non sharp estimates on the initial coefficients $|b_0|$ and $|b_1|$ for functions in this subclass. In this paper, we use the Faber polynomial expansion [6] to obtain not only improvement of estimates of coefficients $|b_0|$ and $|b_1|$ which obtained by Bulut [3], but also we find estimates of coefficients $|b_n|$ where $n \geq 1$.

**Definition 1.1** ([3]). A function $f(z) \in \Sigma_{\mathfrak{B}}$ given by (1.1) is said to be in the class $\mathcal{M}_\sigma(\alpha, \lambda)$ $(0 \leq \alpha < 1,\ 0 \leq \lambda < 1)$, if the following conditions are satisfied:

$$f \in \Sigma_{\mathfrak{B}}, \ Re\left\{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right\} > \alpha$$

and

$$Re\left\{\frac{wg'(w)}{(1-\lambda)g(w)+\lambda wg'(w)}\right\} > \alpha,$$

where $z, w \in \Delta$ and the function $g$ is the inverse of $f$ given by (1.2).

**Theorem 1.2.** [3]] *Let the function $f(z)$ given by (1.1) be in the class be in the class $\mathcal{M}_\sigma(\alpha, \lambda)$. Then*

$$|b_0| \leq \frac{\sqrt{2(1-\alpha)}}{1-\lambda} \ \textit{and} \ |b_1| \leq \frac{1-\alpha}{1-\lambda}.$$

In the present paper by using the Faber polynomial expansions we obtain estimates of coefficients $|b_n|$, of functions in the subclasses $\mathcal{N}_{\Sigma_{\mathfrak{B}}}(\lambda, \beta, \alpha)$. The Faber polynomials introduced by Faber [6] play an important role in various areas of mathematical sciences, especially in geometric function theory. Several authors worked on using Faber polynomial expansions to find coefficient estimates for classes meromorphic bi-univalent functions, see for example [3, 4, 9, 13].

## 2. Main results

In this section, we introduce and investigate the subclass $\mathcal{N}_{\Sigma_{\mathfrak{B}}}(\lambda, \beta, \alpha)$ of meromorphic bi-univalent functions defined on $\Delta$.

**Definition 2.1.** A function $f(z) \in \Sigma_{\mathfrak{B}}$ given by (1.1) is said to be in the class $\mathcal{N}_{\Sigma_{\mathfrak{B}}}(\lambda, \beta, \alpha)$ $(0 \leq \lambda < 1, \ 0 \leq \beta \leq 1, \ 0 \leq \alpha < 1)$, if the following conditions are satisfied:

$$f \in \Sigma_{\mathfrak{B}}, \ Re\left\{\frac{zf'(z)+\beta z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right\} > \alpha$$

and

$$Re\left\{\frac{wg'(w)+\beta w^2 g''(w)}{(1-\lambda)g(w)+\lambda wg'(w)}\right\} > \alpha,$$

where $z, w \in \Delta$ and the function $g$ is the inverse of $f$ given by (1.2).

In the following theorem we find the upper bounds for the initial Taylor-Maclaurin coefficients $|b_0|$, $|b_1|$ and $|b_2|$ for functions in subclass $\mathcal{N}_{\Sigma_{\mathfrak{B}}}(\lambda, \beta, \alpha)$ $(0 \leq \lambda < 1, \ 0 \leq \beta \leq 1, \ 0 \leq \alpha < 1)$.

**Theorem 2.2.** *Let $f(z) \in \Sigma_{\mathfrak{B}}$ given by (1.1) be in the class $\mathcal{N}_{\Sigma_{\mathfrak{B}}}(\lambda, \beta, \alpha)$ $(0 \leq \lambda < 1,\ 0 \leq \beta \leq 1,\ 0 \leq \alpha < 1)$. Then*

$$|b_0| \leq \begin{cases} \frac{\sqrt{2(1-\alpha)}}{1-\lambda};\ 0 \leq \alpha \leq \frac{1}{2} \\ \\ \frac{2(1-\alpha)}{1-\lambda};\ \frac{1}{2} \leq \alpha < 1 \end{cases} \quad \text{and } |b_1| \leq \frac{1-\alpha}{|\beta+\lambda-1|}.$$

By putting $\beta = 0$ in Theorem 2.2, we conclude the following corollary.

**Corollary 2.3.** *Let $f(z) \in \Sigma_{\mathfrak{B}}$ given by (1.1) be in the class $\mathcal{M}_{\sigma}(\alpha, \lambda)$ $(0 \leq \lambda < 1,\ 0 \leq \alpha < 1)$. Then*

$$|b_0| \leq \begin{cases} \frac{\sqrt{2(1-\alpha)}}{1-\lambda};\ 0 \leq \alpha \leq \frac{1}{2} \\ \\ \frac{2(1-\alpha)}{1-\lambda};\ \frac{1}{2} \leq \alpha < 1 \end{cases} \quad \text{and } |b_1| \leq \frac{1-\alpha}{1-\lambda}.$$

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# EXTENSION OF SOME INEQUALITIES FOR THE POLAR DERIVATIVE OF POLYNOMIALS IN COMPLEX DOMAIN

A. MOTAMEDNEZHAD

*Faculty of Mathematical Sciences, Shahrood University of Technology, P. O. Box 316-36155, Shahrood, Iran*
*a.motamedne@gmail.com*

ABSTRACT. Let $p(z)$ be a polynomial of degree $n$. In this paper we prove results concerning maximum modulus of the polar derivative of $p(z)$ with restricted zeros. Our results refine and generalize certain well-known polynomial inequalities.

## 1. Introduction

Let $p$ be a polynomial of degree at most $n$, according to Bernstein theorem, we have

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1.1}$$

For the class of polynomials which does not vanish in $|z| < 1$, Erdös conjectured and Lax [8] proved that

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

This result was improved by Aziz and Dawood [2] who, under the same hypothesis, proved that

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \{\max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)|\}. \tag{1.3}$$

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*Key words and phrases.* Polar derivative, Inequalities, Maximum modulus, Polynomials.

Bernstein [4] proved the following result from which inequality (1.1) can be obtained for $Q(z) = z^n$.

**Theorem A.** *Let $p(z)$ and $Q(z)$ be two polynomials with degree of $p(z)$ not exceeding that of $Q(z)$. If $Q(z)$ has all its zeros in $|z| \le 1$ and*

$$|p(z)| \le |Q(z)|, \quad for \quad |z| = 1$$

*then*

$$|p'(z)| \le |Q'(z)|, \quad for \quad |z| = 1. \tag{1.4}$$

More generally, it was proved by Malik and Vong [10] that for any $\beta$ with $|\beta| \le 1$, inequality (1.4) can be replaced by

$$|zp'(z) + \frac{n\beta}{2}p(z)| \le |zQ'(z) + \frac{n\beta}{2}Q(z)| \qquad for \ \ |z| = 1. \tag{1.5}$$

Concerning maximum of polynomial $p(z)$ Dewan and Hans [6] proved the following theorem.

**Theorem B.** *If $p(z)$ is a polynomial of degree $n$ which does not vanish in $|z| < 1$, then for $\beta$ with $|\beta| \le 1$*

$$|zp'(z) + \frac{n\beta}{2}p(z)| \le \frac{n}{2}\{(|1 + \frac{\beta}{2}| + |\frac{\beta}{2}|) \max_{|z|=1} |p(z)| - (|1 + \frac{\beta}{2}| - |\frac{\beta}{2}|) \min_{|z|=1} |p(z)|\},$$

Let $D_\alpha p(z)$ be an operator which carries $p(z)$ to the polynomial $D_\alpha p(z) = np(z) + (\alpha - z)p'(z), \alpha \in \mathbb{C}$, which is a polynomial of degree at most $(n-1)$. $D_\alpha p(z)$ generalizes the ordinary derivative $p'(z)$ in the sense that

$$\lim_{\alpha \longrightarrow \infty} \frac{D_\alpha p(z)}{\alpha} = p'(z).$$

Here we construct a sequence of polar derivatives

$$D_{\alpha_1} p(z) = np(z) + (\alpha_1 - z)p'(z)$$

$$\begin{aligned} D_{\alpha_k} D_{\alpha_{k-1}} ... D_{\alpha_1} p(z) =& (n - k + 1) D_{\alpha_{k-1}} ... D_{\alpha_1} p(z) + \\ & (\alpha_k - z)(D_{\alpha_{k-1}} ... D_{\alpha_1} p(z))' \quad for \ \ k = 2, 3, ..., n. \end{aligned}$$

The $k^{th}$ polar derivative $D_{\alpha_k} D_{\alpha_{k-1}} ... D_{\alpha_1} p(z)$ of $p(z)$ is a polynomial of degree at most $n - k$. For $p_j(z) = D_{\alpha_j} D_{\alpha_{j-1}} ... D_{\alpha_1} p(z)$, we have

$$\begin{aligned} & p_j(z) = (n - j + 1)p_{j-1}(z) + (\alpha_j - z)p'_{j-1}(z), \qquad j = 1, 2, ..., t, \\ & p_0(z) = p(z) \end{aligned} \tag{1.6}$$

As an extension of (1.1) for the polar derivative Aziz and Shah [3] proved

$$|D_\alpha p(z)| \le n|\alpha z^{n-1}| \max_{|z|=1} |p(z)|, \qquad for \quad |z| \geqslant 1. \tag{1.7}$$

Aziz [1] extended the inequality (1.7) to the $j^{th}$ polar derivative and proved the following theorem.

**Theorem C.** *If $p(z)$ is a polynomial of degree $n$ such that $\alpha_1, \alpha_2, ..., \alpha_t$, $(t \leq n-1)$, are complex numbers with $|\alpha_i| \geqslant 1$ for all $i = 1, 2, ..., t$, then for $|z| \geqslant 1$*

$$|p_t(z)| \leq n(n-1)(n-2)...(n-t+1)|\alpha_1\alpha_2...\alpha_t||z|^{n-t} \max_{|z|=1} |p(z)| \quad (1.8)$$

For the class of polynomials having no zeros in $|z| < 1$, Dewan, Singh and Mir ([7],Theorem 1 for $\mu = k = 1$) proved

**Theorem D.** If $p(z)$ is a polynomial of degree $n$ which does not vanish in $|z| < 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant 1$ and $|z| = 1$

$$|D_\alpha p(z)| \leq \frac{n}{2}\left\{(|\alpha|+1)\max_{|z|=1}|p(z)| - (|\alpha|-1)\min_{|z|=1}|p(z)|\right\}. \quad (1.9)$$

Recently Liman et al [9] obtained the following generalization of inequality (1.9).

**Theorem E.** *If $p(z)$ is a polynomial of degree $n$ which does not vanish in $|z| < 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geqslant 1$, $|\beta| \leq 1$ and $|z| = 1$*

$$\begin{aligned} &|zD_\alpha p(z) + n\beta\frac{\alpha-1}{2}p(z)| \leq \\ &\frac{n}{2}[\{|\alpha+\beta\frac{\alpha-1}{2}| + |z+\beta\frac{\alpha-1}{2}|\}\max_{|z|=1}|p(z)| - \\ &\{|\alpha+\beta\frac{\alpha-1}{2}| - |z+\beta\frac{\alpha-1}{2}|\}\min_{|z|=1}|p(z)|], \end{aligned} \quad (1.10)$$

## 2. Main results

In this paper we generalize the inequality (1.10) to the class of polynomials of degree $n$ which not vanishing in $|z| < k$ where $k \leq 1$. For this reason, we shall extend inequality (1.5) to the polar derivative of a polynomial. For proof of the main result, the following lemma is needed. This lemma is due to Zireh [12, 13].

**Lemma 2.1.** *If $p(z)$ is a polynomial of degree $n$ having all zeros in $|z| \leq k$, $k \leq 1$, then for all $\alpha_1, \cdots \alpha_t \in \mathbb{C}$ with $|\alpha_1| \geq k$, $|\alpha_2| \geq k, \cdots, |\alpha_t| \geq k$, $(1 \leq t < n)$, and $|z| = 1$ we have*

$$|p_t(z)| \geqslant \frac{n_t}{(1+k)^t} \times A_{\alpha_t}|p(z)|. \quad (2.1)$$

*where $n_t = n(n-1)(n-2)...(n-t+1)$ and $A_{\alpha_t} = (|\alpha_1|-k)(|\alpha_2|-k)\cdots(|\alpha_t|-k)$.*

**Theorem 2.2.** *Let $Q(z)$ be a polynomial of degree $n$ having all its zeros in $|z| \leq k$, $(k \leq 1)$ and $p(z)$ be a polynomial of degree at most $n$. If $|p(z)| \leq |Q(z)|$ for $|z| = k$, then for all $\beta$, $\alpha_i$ with $|\beta| \leq 1$, $|\alpha_i| \geqslant k$ for $i = 1, 2, ..., t$, $(t < n)$,*

$$|z^t p_t(z) + \beta \frac{n_t A_{\alpha_t}}{(1+k)^t} p(z)| \leq |z^t Q_t(z) + \beta \frac{n_t A_{\alpha_t}}{(1+k)^t} Q(z)| \qquad for \quad |z| \geqslant 1. \tag{2.2}$$

*where $n_t = n(n-1)(n-2)...(n-t+1)$ and $A_{\alpha_t} = (|\alpha_1| - k)(|\alpha_2| - k)...(|\alpha_t| - k)$.*

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# INTEGRAL MEANS OF ANALYTIC FUNCTIONS IN THE UNIT DISK

A. ABKAR AND E. NAJAFI*

*Department of Pure Mathematics, Imam Khomeini International University, Qazvin 34194, Iran*
*abkar@sci.ikiu.ac.ir and elanajafi@yahoo.com*

Abstract. We review variable exponent Hardy spaces in the unit disk. In particular, we study the behavior of integral means of analytic functions on small disks $\{z \in \mathbb{C} : |z| < r < 1\}$.

## 1. Introduction

Variable Lebesgue spaces are generalizations of classical Lebesgue spaces $L^p$ where we allow the exponent $p$ to be a measurable function. Let $\mathbb{D}$ denote the unit disk in the complex plane. For a function $f$ analytic in the unit disk $\mathbb{D}$ and $0 < p < \infty$, the classical integral means of $f$ are defined by

$$M_p(f, r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{\imath\theta})|^p d\theta, \quad 0 \leq r < 1.$$

For fixed $f$ and $p$, it is well-known that $M_p(f, r)$ is a nondecreasing function of $r$. For a given $0 < p < \infty$, the Hardy space $H^p(\mathbb{D})$ is defined as the space of all analytic functions on $\mathbb{D}$ for which $\lim_{r\to 1^-} M_p(f, r)$ is finite. For details on the theory of Hardy spaces, one refer to [4].
The well-known Hardy convexity theorem assert that $M_p(f, r)$, as a function of $r$ on $[0, 1)$, is nondecreasing and logarithmically convex.

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Recall that the logarithmic convexity of a given function $M$ means that the function $r \mapsto \log M(r)$ is a convex function of $\log r$; which means that if

$$\log r = \lambda \log r_1 + (1-\lambda) \log r_2, \quad 0 < r_1 < r_2 < 1,\ 0 < \lambda < 1,$$

then

$$\log M(r) \leq \lambda \log M(r_1) + (1-\lambda) \log M(r_2).$$

Recently, Kehe Zhu and his colleagues proved that a similar result is true for the mean area of functions in the weighted Bergman spaces (see [1] and [8]). Suppose that $0 < p < \infty$, $-1 \leq \alpha \leq 0$, and $f$ is a function in the weighted Bergman space $A^p_\alpha(\mathbb{D})$ consisting of analytic functions in the unit disk for which the integral

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z)$$

is finite (here $dA$ denotes the area measure on $\mathbb{D}$). Then the function

$$r \mapsto M_{p,\alpha}(f,r) = \frac{\int_{|z|<r} |f(z)|^p (1-|z|^2)^\alpha dA(z)}{\int_{|z|<r} (1-|z|^2)^\alpha dA(z)}$$

is logarithmically convex. In particular, if $\alpha = 0$, then

$$r \mapsto M_{p,0}(f,r) = \frac{1}{\pi r^2} \int_{|z|<r} |f(z)|^p dA(z)$$

is logarithmically convex.

## 2. Variable exponent spaces of analytic functions

Let $\Omega = [0, 2\pi]$ and put

$$p^+ = p^+_\Omega := ess \sup_{\theta \in \Omega} p(\theta), \ and \ p^- = p^-_\Omega := ess \inf_{\theta \in \Omega} p(\theta)$$

where $p : \Omega \to [1, \infty)$ is a measurable function. The measurable function $p$ is called a variable exponent. We denote by $\mathcal{P}(\Omega)$ the set of all variable exponents $p$ for which $p^+ < \infty$.

For a complex-valued function $f : \partial\mathbb{D} \to \mathbb{C}$ we define

$$\rho_{p(.)}(f) := \int_0^{2\pi} |f(e^{i\theta})|^{p(\theta)} d\theta.$$

Let $p \in \mathcal{P}(\Omega)$. The variable exponent Lebesgue spaces $L^{p(\cdot)}(\partial\mathbb{D})$ is the set of all complex-valued measurable functions $f : \partial\mathbb{D} \to \mathbb{C}$ for which $\rho_{p(.)}(f) < \infty$. Equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\partial\mathbb{D})} := \inf \left\{ \lambda > 0 : \ \rho_{p(.)}(\frac{f}{\lambda}) \leq 1 \right\}.$$

$L^{p(.)}(\partial\mathbb{D})$ is a Banach space.

**Definition 2.1.** A function $p : [0, 2\pi] \to [1, \infty)$ is said to be locally long-Holder continuous on $[0, 2\pi)$ if there exists a positive constant $C$ such that

$$|p(x) - p(y)| \leq \frac{C}{\log(\frac{1}{|x-y|})}$$

for all $x, y \in [0, 2\pi]$.

**Definition 2.2.** ([2]) Let $p : [0, 2\pi] \to [1, \infty)$ be a measurable function such that $p(0) = p(2\pi)$. The variable exponent Hardy space $H^{p(.)}(\mathbb{D})$ is defined as the space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\sup_{0\leq r<1} \|f_r\|_{L^{p(.)}(\partial\mathbb{D})} < \infty,$$

where $f_r : \partial\mathbb{D} \to \mathbb{C}$ is defined by $f_r(e^{i\theta}) := f(re^{i\theta})$.

Each $H^{p(.)}(\mathbb{D})$ is a linear space in which the norm of $f \in H^{p(.)}(\mathbb{D})$ is defined by

$$\|f\|_{H^{p(.)}(\mathbb{D})} := \sup_{0\leq r<1} \|f_r\|_{L^{p(.)}(\partial\mathbb{D})}.$$

Also it is shown that $H^{p(.)}(\mathbb{D})$ can be identified with the subspace of functions in $L^{p(.)}(\partial\mathbb{D})$ whose negative Fourier coefficients are zero, and thus $H^{p(.)}(\mathbb{D})$ is a Banach space.

## 3. Variable exponent version of the integral means

First we define a variable exponent version of the integral means for analytic functions in the unit disk $\mathbb{D}$.

**Definition 3.1.** For a function $f$ analytic in the unit disk $\mathbb{D}$ and for a measurable function $p : [0, 2\pi] \to [1, \infty)$, the variable exponent version of the integral means are defined by

$$M_{p(.)}(f, r) := \frac{1}{2\pi}\rho_{p(.)}(f_r),\ \ 0 \leq r < 1$$

where $f_r : \partial\mathbb{D} \to \mathbb{C}$ is defined by $f_r(e^{i\theta}) := f(re^{i\theta})$.

It is now tempting to ask the following question.

**Question 3.2.** Let $p : [0, 2\pi] \to [1, \infty)$ be a log-Holder continuous function such that $p(0) = p(2\pi)$, and let $f \in H^{p(.)}(\mathbb{D})$ and $0 < r < 1$. Is the function $r \mapsto M_{p(.)}(f, r)$ a convex function of $\log r$?

We now shift from functions in the variable exponent Hardy spaces to functions in variable exponent Bergman spaces. From now on we assume that $\Omega = \mathbb{D}$. Let $p : \mathbb{D} \to [1, \infty)$ be a log-Holder continuous function on the unit disk with $p^+ < \infty$. For an analytic function $f : \mathbb{D} \to \mathbb{C}$ we define the modular $\rho_{p(\cdot)}$ by

$$\rho_{p(\cdot)}(f) := \int_{\mathbb{D}} |f(z)|^{p(z)}\, dA(z),$$

where $dA(z)$ is the Lebesgue area measure on the unit disk. The norm induced by this modular is given by

$$\|f\|_{L^{p(.)}} := \inf \left\{ \lambda > 0 : \rho_{p(.)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

The variable exponent Bergman space $A^{p(\cdot)}(\mathbb{D})$ consists of all analytic functions in the unit disk for which

$$\int_{\mathbb{D}} |f(z)|^{p(z)}\, dA(z) < \infty.$$

**Question 3.3.** Is there any $\alpha$ for which the function

$$r \mapsto M_{p(.),\alpha}(f, r) = \frac{\int_{|z|<r} |f(z)|^{p(z)}(1 - |z|^2)^{\alpha} dA(z)}{\int_{|z|<r}(1 - |z|^2)^{\alpha} dA(z)}$$

is a convex function of $\log r$?

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# A NONSTANDARD FINITE DIFFERENCE SCHEME FOR STOCHASTIC PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

[1]M. J. MOHAMMAD TAGHIZADEH, [2]M. AMINIAN, [3]M. NAMJOO*,
[4]M. KARAMI AND [5]A. MOHEBBIAN

[1]*Atherosclerosis Research Center, Ahvaz Jundishapur University of Medical Sciences, Ahvaz, Iran*
[2,3,4,5]*Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran*
*dr.taghizadeh87@gmail.com, mehran.aminian@vru.ac.ir, namjoo@vru.ac.ir,*
*m.karami@vru.ac.ir, a.mohebbiyan@stu.vru.ac.ir*

Abstract. The aim of this manuscript is to introduce and analyze a nonstandard finite difference (NSFD) scheme for Itô stochastic partial differential equations (SPDEs). We also discuss on consistency, stability and convergence the NSFD scheme. The numerical simulations obtained from the NSFD scheme show the efficiency of the proposed NSFD scheme.

## 1. Introduction

Over the past few decades, stochastic partial differential equations have been considered for the modelling of realistic phenomena due to their possession real behavior. The footsteps of SPDEs are found in various branches of sciences, technologies and medical problems including cardiovascular disease such as myocardial infraction, myocardial ischemia, hypertension, heart failure, etc., and many scientists have devoted growing attention to the analysis SPDEs. Hence, it is noteworthy to study the problem of modelling related to that phenomenon. Finding the exact solutions of SPDEs in many cases is difficult. So, it is necessary to obtain their numerical solutions by using some numerical methods. In this manuscript, a new NSFD scheme constructed for solving SPDE in the following form:

$$u_t(x,t) + a\gamma u_{xx}(x,t) + bu(x,t) + cu(x,t)\dot{W}(t) = 0,\ x \in [0,1],\ t \in [0,1], \tag{1.1}$$

with the initial and boundary conditions $u(x,0) = u_0(x)$, $u(0,t) = f(t)$ and $u(1,t) = g(t)$, where the parameters $b$ and $c$ are positive constants. Here $\dot{W}(t)$ is called white noise in which $\dot{W}(t)$ is a Gaussian distribution with zero mean [1].

## 2. A NSFD scheme for parabolic SPDEs

In this part, we construct a NSFD scheme for SPDEs (1.1). In order to the concept of NSFD scheme, consider an ordinary differential equation (ODE) in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(x(t),t), \quad x(0) = x_0, \quad t \in [0,t_f]. \tag{2.1}$$

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*Key words and phrases.* Nonstandard finite difference scheme, cardiovascular disease, consistency, stability, convergence, stochastic.

Let $x_n$ be the approximation of $x(t_n)$, then the discrete approximation $\frac{\mathrm{d}x}{\mathrm{d}t}|_{t=t_n}\approx \frac{x_{n+1}-x_n}{\phi(h)}$, is used for the differentiation in (2.1), where $\phi(h) = h+O(h^2)$ [3]. Moreover, we will require linear and nonlinear terms must be modeled by the nonlocal discrete representation. For example, linear and nonlinear terms in the SPDE can be considered as $u_k^n \approx \frac{1}{2}(u_{k+1}^n + u_k^n)$ and $(u_k^n)^2 \approx u_k^n u_{k+1}^n$, where the value of the numerical solution SPDE at the nodal point $(x_k, t_n) = (k\Delta x, n\Delta t)$ will be denoted by $u_k^n$. Now we extend the above mentioned approach for the SPDE (1.1). For this purpose, the time and space derivatives in the SPDE and also the function $u(x,t)$ in deterministic part can be approximated in the form $u_t(k\Delta x, n\Delta t) \approx \frac{u_k^{n+1}-u_k^n}{\Delta t}$, $u_{xx} \approx \frac{u_{k-1}^n-2u_k^n+u_{k+1}^n}{\Delta x^2}$ and $u_k^n \approx \frac{u_{k+1}^n+u_k^n}{2}$. Therefore, the NSFD scheme of SPDE (1.1) is given by

$$u_{k+1}^n = -ru_{k-1}^n + (1-2r-s)u_k^n - (r+s)u_{k+1}^n + cu_k^n\Delta W_n, \tag{2.2}$$

where $r = \frac{a\Delta t}{\Delta x^2}$, $s = \frac{b\Delta t}{2}$ and $\Delta W_n = W((n+1)\Delta t) - W(n\Delta t) \sim N(0,\Delta t)$. In the reminder of this manuscript we assume that the increments of Wiener process are independent of the state $u_k^n$.

## 3. Consistency and stability analysis

Consider a SPDE of the form $Lv = G$ where $L$ denotes the differential operator and $G$ is source term. Let $L_k^n u_k^n = G_k^n$ denote the NSFD scheme. In order to analysis consistency and stability the NSFD scheme (2.2), it is very important to consider a norm. Hence, for a sequence $\{u_k^n\}_{0\le k\le M}$, we define $\|u^n\| = \sqrt{\sup\limits_{0\le k\le M} |u_k^n|^2}$. In order to get consistency, stability and convergence results, we give the following definitions [2].

**Definition 3.1.** A NSFD scheme $L_k^n u_k^n = G_k^n$ is consistent with the SPDE $Lv = G$ at point $(x,t)$, if for any continuously differentiable function $\Phi(x,t)$ in the sense of mean square we have $\mathbb{E}\|u^{n+1} - v^{n+1}\| \to 0$, for $t = (n+1)\Delta t$ and $\Delta t, \Delta x \to 0$, where the vectors $u^{n+1}$ and $v^{n+1}$ are defined by $u^{n+1} = \big(u_0^{n+1}, u_1^{n+1}, \ldots, u_M^{n+1}\big)$ and $v^{n+1} = \big(v_0^{n+1}, v_1^{n+1}, \ldots, v_M^{n+1}\big)$.

**Theorem 3.2.** *The NSFD scheme (2.2) is consistent in the sense of mean square.*

*Proof.* Let $\Phi(x,t)$ be a smooth function, thus

$$\begin{aligned} L(\Phi)|_k^n = {}& \Phi(k\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) \\ & + a\int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s)\,\mathrm{d}s + b\int_{n\Delta t}^{(n+1)\Delta t} \Phi(k\Delta x, s)\,\mathrm{d}s \\ & + c\int_{n\Delta t}^{(n+1)\Delta t} \Phi(k\Delta x, s)\,\mathrm{d}W(s), \end{aligned}$$

and

$$\begin{aligned} L_k^n\Phi = {}& \Phi(k\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) \\ & + \frac{a\Delta t}{\Delta x^2}\Big(\Phi((k+1)\Delta x, n\Delta t) - 2\Phi(k\Delta x, n\Delta t) + \Phi((k-1)\Delta x, n\Delta t)\Big) \\ & + \frac{b\Delta t}{2}\Big(\Phi((k+1)\Delta x, n\Delta t) + \Phi(k\Delta x, n\Delta t)\Big) + c\,\Phi(k\Delta x, n\Delta t)\Delta W_n. \end{aligned}$$

Now by using the square property of Itô integral, one can obtain the following inequality

$$
\begin{aligned}
&\mathbb{E}\Big|L(\Phi)|_k^n - L_k^n\Phi\Big|^2 \\
&= \mathbb{E}\Big|a\int_{n\Delta t}^{(n+1)\Delta t}\Big[\Phi_{xx}(k\Delta x, s) - \frac{1}{\Delta x^2}\Big(\Phi((k+1)\Delta x, n\Delta t) \\
&\quad - 2\Phi(k\Delta x, n\Delta t) + \Phi((k-1)\Delta x, n\Delta t)\Big)\Big]\mathrm{d}s \\
&+ b\int_{n\Delta t}^{(n+1)\Delta t}\Big[\Phi(k\Delta x, s) - \frac{1}{2}\Big(\Phi((k+1)\Delta x, n\Delta t) + \Phi(k\Delta x, n\Delta t)\Big)\Big]\mathrm{d}s \\
&+ c\int_{n\Delta t}^{(n+1)\Delta t}\left(\Phi(k\Delta x, s) - \Phi(k\Delta x, n\Delta t)\right)\mathrm{d}W(s)\Big|^2 \\
&\le 4a^2\mathbb{E}\Big|\int_{n\Delta t}^{(n+1)\Delta t}\Big[\Phi_{xx}(k\Delta x, s) - \frac{1}{\Delta x^2}\Big(\Phi((k+1)\Delta x, n\Delta t) \\
&\quad - 2\Phi(k\Delta x, n\Delta t) + \Phi((k-1)\Delta x, n\Delta t)\Big)\Big]\mathrm{d}s\Big|^2 \\
&+ 4b^2\mathbb{E}\Big|\int_{n\Delta t}^{(n+1)\Delta t}\Big[\Phi(k\Delta x, s) - \frac{1}{2}\Big(\Phi((k+1)\Delta x, n\Delta t) + \Phi(k\Delta x, n\Delta t)\Big)\Big]\mathrm{d}s\Big|^2 \\
&+ 4c^2\int_{n\Delta t}^{(n+1)\Delta t}\mathbb{E}\Big|\Phi(k\Delta x, s) - \Phi(k\Delta x, n\Delta t)\Big|^2\mathrm{d}s.
\end{aligned}
$$

Since $\Phi(x,t)$ is a deterministic function, hence $\mathbb{E}|L(\Phi)|_k^n - L_k^n\Phi|^2 \to 0$, as $n$, $k \to \infty$. ☐

**Definition 3.3.** A stochastic NSFD scheme is said to be stable in the sense of mean square, if there exist some positive constants $\overline{\Delta x_0}$ and $\overline{\Delta t_0}$ and nonnegative constants $K$ and $\beta$ such that $\mathbb{E}\|u^{n+1}\|^2 \le Ke^{\beta t}\mathbb{E}\|u^0\|^2$, for all $t = (n+1)\Delta t$, $0 \le \Delta x \le \overline{\Delta x_0}$, and $0 \le \Delta t \le \overline{\Delta t_0}$.

**Theorem 3.4.** *The NSFD scheme (2.2) with $t = (n+1)\Delta t$ and $\frac{s-1}{2} \le r \le -s$ is stable with respect to sup–norm.*

*Proof.* By applying $\mathbb{E}|\cdot|^2$ in (2.2) and using the independence of the Wiener process increments, we get

$$
\begin{aligned}
\mathbb{E}\big|u_k^{n+1}\big|^2 &= \mathbb{E}\big| - ru_{k-1}^n + (1+2r-s)u_k^n - (r+s)u_{k+1}^n\big|^2 + c^2\Delta t\mathbb{E}\big|u_k^n\big|^2 \\
&\le (|r| + |1+2r-s| + |r+s|^2 + c^2\Delta t)\sup_k \mathbb{E}\big|u_k^n\big|^2.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\sup_k \mathbb{E}|u_k^{n+1}|^2 &\le (1+c^2\Delta t)\sup_k \mathbb{E}|u_k^n|^2 \le \ldots \le (1+c^2\Delta t)^{n+1}\sup_k \mathbb{E}|u_k^0|^2 \\
&\le e^{c^2\Delta t(n+1)}\sup_k \mathbb{E}|u_k^0|^2 = e^{c^2 t}\sup_k \mathbb{E}|u_k^0|^2.
\end{aligned}
$$

So, the NSFD scheme (2.2) is stable, with $K = 1$ and $\beta = c^2$. ☐

**Definition 3.5.** A stochastic difference scheme $L_k^n u_k^n = G_k^n$ approximating the SPDE $Lv = G$ is convergent in the sense of mean square at time $t$, if $\mathbb{E}\|u^{n+1} - v^{n+1}\|^2 \to 0$, for $t = (n+1)\Delta t$, $\Delta x \to 0$ and $\Delta t \to 0$.

**Theorem 3.6.** *The NSFD scheme* (2.2) *for SPDE* (1.1) *is convergent in the sense of mean square with respect to sup–norm.*

*Proof.* From the stochastic Lax–Richtmyer Theorem, one can conclude that the NSFD scheme (2.2) is convergent. □

## 4. Numerical simulation

In this section, we apply the proposed NSFD scheme in Section 2 to illustrate the results of the previous section.

**Example 4.1.** Consider the SPDE as follows

$$u_t(x,t) - u_{xx}(x,t) + u(x,t) + c\, u(x,t)\dot{W}(t) = 0, \quad x \in [0,1], \quad t \in [0,1],$$

subject to the following initial condition $u(x,0) = \sin \pi x$ for $x \in [0,1]$, with the boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$ for $t \in [0,1]$. It is easy to check that in the absence of the stochastic term, the exact solution is $u(x,t) = \sin \pi x\; e^{-(\pi^2+1)t}$.

Suppose $M$ and $N$ are the total number for the space and time discretizations, respectively. The NSFD scheme is conditionally stable for $\frac{s-1}{2} < r < -s$. In Figure 1 we have presented the result of NSFD scheme and the exact solution for $c = 1$, $\Delta x = 0.04$ and $N = 4500$. The numerical results NSFD scheme and the exact solution by taking $c = 0.5$, $\Delta x = 0.04$ and $N = 2500$ have displayed in Figure 2. In Figure 3 the numerical solutions of NSFD scheme compared with the exact solution with $c = 2$, $\Delta x = 0.04$ and $N = 5000$.

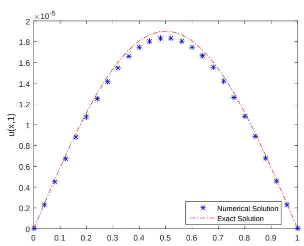


Figure 1. The curve of the solutions for $c = 1$, $\Delta x = 0.04$ and $N = 4500$.

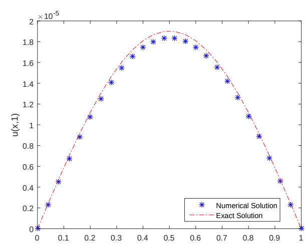


Figure 2. The curve of the solutions for $c = 0.5$, $\Delta x = 0.04$ and $N = 2500$.

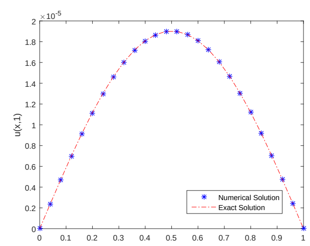


Figure 3. The curve of the solutions for $c = 2$, $\Delta x = 0.04$ and $N = 5000$.

## 5. Conclusions

In this paper to solve stochastic parabolic partial differential equation, we propose a NSFD scheme. Consistency, stability and convergence of the proposed NSFD scheme is established. The study shows that the NSFD scheme is effective to solve SPDEs.

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# KANNAN FIXED-POINT THEOREM ON BANACH GROUPS

[1]A. R. POURMOSLEMI, [2]S. REZAEI AND [3]T. NAZARI∗

[1,3]*Department of Mathematics, Payame Noor University, Tehran, Iran.*
*a_pourmoslemy@pnu.ac.ir and taheree.nazari2222@gmail.com.*
[2]*Department of Mathematics, Aligudarz Branch, Islamic Azad University, Aligudarz, Iran*
*sh.rezaei@iau-aligudarz.ac.ir*

Abstract. In this paper we present a fixed point theorem for Kannan type mapping in a Banach group.

## 1. Introduction

The fixed point theory is one of the most useful and essential tools of nonlinear analysis. The first important result in the theory of fixed point about contractive mapping is Banach theorem [1]. In 1968 Kannan [5] introduced a new type of contraction. Subrahmanyam [6] showed that a metric spaces is complete if and only if, every Kannan mapping has a fixed point. In 2018 Karapinar, by using the interpolation notion, introduced a new Kannan type contraction to maximize the rate of convergence [4].

On the other hand, group-norms have also played a role in the theory of topological groups [2, 3]. Some results on the existence and uniqueness of fixed points for Kannan mappings on normed groups and Banach group are proved in this paper.

We begin with some basic notions which will be needed in this paper.

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**Definition 1.1.** [2] Let $\mathcal{L}$ be a group. A norm on a group $\mathcal{L}$ is a function $\|.\| : \mathcal{L} \to \mathbb{R}$ with the following properties:

(1) $\|w\| \geq 0,\ for\ all\ w \in \mathcal{L};$
(2) $\|w\| = \|w^{-1}\|,\ for\ all\ w \in \mathcal{L};$
(3) $\|wk\| \leq \|w\| + \|k\|,\ for\ all\ w, k \in \mathcal{L};$
(4) $\|w\| = 0\ implies\ that\ w = e.$

A normed group $(\mathcal{L}, \|.\|)$ is a group $\mathcal{L}$ equipped with a norm $\|.\|$. By setting $d(w,k) := \|w^{-1}k\|$, it is easy to see that norms are in bijection with left-invariant metrics on $\mathcal{L}$.

**Definition 1.2.** A Banach group is a normed group $(\mathcal{L}, \|.\|)$, which is complete with respect to the metric

$$d(w,k) = \|wk^{-1}\|, \quad (w, k \in \mathcal{L}).$$

**Definition 1.3.** Let $(\mathcal{L}, \|.\|)$ be a normed group and $\vartheta : \mathcal{L} \to \mathcal{L}$ be a mapping. Then $\vartheta$ is called Kannan contraction if there exists $\eta \in [0, \frac{1}{2})$ such that

$$\|\vartheta(w)\vartheta(k)^{-1}\| \leq \eta \left[\|w\vartheta(w)^{-1}\| + \|k\vartheta(k)^{-1}\|\right], \quad (w, k \in \mathcal{L}).$$

## 2. Main results

In this section we extend the kannan's theorem and we continue by a generalization of the definition of Kannan type contraction via interpolation notion.

**Lemma 2.1.** *Let $(\mathcal{L}, \|.\|)$ be a Banach group and $A$ be a nonempty closed subset of $\mathcal{L}$ and let $\psi : A \to A$ be a mapping such that satisfying*

$$\|\psi(w)\psi(k)^{-1}\| \leq \eta \left[\|w\psi(w)^{-1}\| + \|k\psi(k)^{-1}\|\right],$$

*for all $w, k \in A$ and $0 \leq \eta < 1$. If for arbitrary point $a \in A$ there exists $b \in A$ such that $\|b\psi(b)^{-1}\| \leq r_1\|a\psi(a)^{-1}\|$ and $\|ba^{-1}\| \leq r_2\|a\psi(a)^{-1}\|$, when there exist constants $r_1, r_2 \in \mathbb{R}$ such that $0 \leq r_1 < 1$ and $r_2 > 0$, Then $\psi$ has at least one fixed point.*

*Proof.* For an arbitrary element $a_0 \in A$ define a sequence $(a_n)_{n=1}^{\infty} \subset A$ such that

$$\|\psi(a_{n+1})a_{n+1}^{-1}\| \leq r_1\|\psi(a_n)a_n^{-1}\|,$$

and

$$\|a_{n+1}a_n^{-1}\| \leq r_2\|\psi(a_n)a_n^{-1}\|,$$

for $n = 1, 2, ....$ It is easy to see that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. Because $A$ is complete, there exists $c \in A$ such that $\lim_{n\to\infty} a_n = c$. Then

$$\begin{aligned}\|\psi(c)c^{-1}\| &\leq \|\psi(c)\psi(a_n)^{-1}\| + \|\psi(a_n)a_n^{-1}\| + \|a_nc^{-1}\| \\ &\leq \eta \left[\|c\psi(c)^{-1}\| + \|a_n\psi(a_n)^{-1}\|\right] + \|\psi(a_n)a_n^{-1}\| + \|a_nc^{-1}\|,\end{aligned}$$

and

$$\begin{aligned}\|\psi(c)c^{-1}\| &\leq \frac{\eta+1}{1-\eta}\|\psi(a_n)a_n^{-1}\| + \frac{1}{1-\eta}\|a_nc^{-1}\| \\ &\leq \frac{\eta+1}{1-\eta}r_1^n\|\psi(a_0)a_0^{-1}\| + \frac{1}{1-\eta}\|a_nc^{-1}\| \to 0,\end{aligned}$$

as $n \to \infty$. So, $\psi(c) = c$. □

**Theorem 2.2.** *Let $(\mathcal{L}, \|.\|)$ be a Banach group and $\vartheta$ be a Kannan contraction. Then $\vartheta$ has a unique fixed point in $\mathcal{L}$.*

*Proof.* Let $w \in \mathcal{L}$ be an arbitrary element. Consider $k = \vartheta(w)$. Then

$$\|k\vartheta(k)^{-1}\| = \|\vartheta(w)\vartheta(k)^{-1}\| \leq \eta\left[w\vartheta(w)^{-1}\| + \|k\vartheta(k)^{-1}\|\right],$$

which implies

$$\|k\vartheta(k)^{-1}\| \leq \frac{\eta}{1-\eta}\|w\vartheta(w)^{-1}\|.$$

For arbitrary $w_0 \in \mathcal{L}$ define a sequence $(w_{n+1} = \vartheta(w_n))_{n=1}^{\infty}$. By the Lemma (2.1), this sequence is converges to $z$ and $\vartheta(z) = z$. If $w$ be an another fixed point of $\vartheta$, we have

$$0 < \|zw^{-1}\| = \|\vartheta(z)\vartheta(w)^{-1}\| \leq \eta\left[\|z\vartheta(z)^{-1}\| + \|w\vartheta(w)^{-1}\|\right] = 0.$$

Therefore, $\vartheta$ has a unique fixed point.

□

**Theorem 2.3.** *Let $(\mathcal{L}, \|.\|)$ be a compact normed group and let $\vartheta : \mathcal{L} \to \mathcal{L}$ be a continuous Kannan nonexpansive mapping. Then $\vartheta$ has a unique fixed point.*

*Proof.* The function $\alpha : \mathcal{L} \to [0, \infty)$ defined by $\alpha(w) = \|w\vartheta(w)^{-1}\|$ is continuous. Since $\mathcal{L}$ is compact, there exists an element $z \in \mathcal{L}$ such that $\vartheta(z) = \inf\{\vartheta(w) : \; w \in \mathcal{L}\}$. If $\vartheta(z) \neq z$, then

$$\|\vartheta(z)\vartheta(\vartheta(z))^{-1}\| < \frac{1}{2}\left[\|z\vartheta(z)^{-1}\| + \|\vartheta(z)\vartheta(\vartheta(z))^{-1}\|\right],$$

and

$$\alpha(\vartheta(z)) = \|\vartheta(z)\vartheta(\vartheta(z))^{-1}\| < \|z\vartheta(z)^{-1}\| = \alpha(z).$$

This is a contradiction and hence, $\vartheta(z) = z$. It is obvious that $z$ is a unique fixed point. □

**Definition 2.4.** Let $(\mathcal{L}, \|.\|)$ be a normed group. We say that the self-mapping $\vartheta : \mathcal{L} \to \mathcal{L}$ is an interpolative Kannan type contraction, if there exist a constant $\eta \in [0, 1)$ and $\mu \in (0, 1)$ such that

$$\|\vartheta(w)\vartheta(k)^{-1}\| \leq \eta[\|w\vartheta(w)^{-1}\|]^{\mu}.[\|k\vartheta(k)^{-1}\|]^{1-\mu}, \tag{2.1}$$

for all $w, k \in \mathcal{L}$ with $w \neq \vartheta(w)$.

**Theorem 2.5.** *Let $(\mathcal{L}, \|.\|)$ be a Banach group and $\vartheta$ be an interpolative Kannan type contraction.Then $\vartheta$ has a unique fixed point in $\mathcal{L}$.*

*Proof.* For arbitrary element $w_0 \in \mathcal{L}$ a sequence $(w_n)_{n=1}^{\infty} \subset \mathcal{L}$ be defined by $w_{n+1} = \vartheta^n(w_0)$. Without loss of generality, we assume that $w_n \neq w_{n+1}$ for each non-negative integer $n$. Then $\|w_n\vartheta(w_n)^{-1}\| = \|w_n w_{n+1}^{-1}\| > 0$. By taking $w = w_n$ and $k = w_{n-1}$ in (2.1) we get

$$\|w_{n+1}w_n^{-1}\| = \|\vartheta(w_n)\vartheta(w_{n-1})^{-1}\| \leq \eta[\|w_n\vartheta(w_n)^{-1}\|]^{\mu}$$

$$[\|w_{n-1}\vartheta(w_{n-1})^{-1}\|]^{1-\mu} = \eta[\|w_{n-1}w_n^{-1}\|]^{1-\mu}[\|w_n w_{n+1}\|]^{\mu},$$

so

$$\|w_n w_{n+1}^{-1}\|^{1-\mu} \leq \eta[\|w_{n-1}w_n^{-1}\|]^{1-\mu}.$$

So, the sequence $\|w_{n-1}w_n^{-1}\|$ is non-increasing and non-negative. As a result, there is a non-negative constant $z$ such that $\lim_{n\to\infty} \|w_{n-1}w_n^{-1}\| = z$. Then

$$\|w_n w_{n+1}^{-1}\| \leq \eta\|w_{n-1}w_n^{-1}\| \leq \eta^n\|w_0 w_1^{-1}\|.$$

By letting $n \to \infty$ in the inequality above, we observe that $z = 0$. By using a standard arguments based on the triangle inequality, we conclude that the sequence $(w_n)_{n=1}^{\infty}$ is a Cauchy sequence.Since $\mathcal{L}$ is a complete group, there exists $v \in \mathcal{L}$ such that $\lim_{n\to\infty} \|w_n v^{-1}\| = 0$. By substituting $w = w_n$ and $k = v$ in (2.1), we have

$$\|\vartheta(w_n)\vartheta(v)^{-1}\| \leq \eta[\|w_n\vartheta(w_n)^{-1}\|]^{\mu}.[\|v\vartheta(v)^{-1}\|]^{1-\mu}.$$

Taking $n \to \infty$ in the inequality above, we thus get $\|v\vartheta(v)^{-1}\| = 0$ and hence $v = \vartheta(v)$. It is obvious that $v$ is a unique fixed point. □

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# A CHARACTERISATION OF TERNARY DERIVABLE MAPS AT ZERO PRODUCT

MOHSEN NIAZI

*Department of Mathematics, University of Birjand, Birjand, Iran*
*niazi@birjand.ac.ir*

Abstract. In this note, concerning continuous conjugate linear maps from unital C*-algebras into their dual spaces, we show that it can be reached from ternary derivability at zero to derivability and ternary derivability at every point of the space.

## 1. Introduction

A linear map $D$ from a Banach algebra $A$ into a Banach $A$-bimodule $X$ is said to be *derivable at a point* $z \in A$ if the identity

$$D(z) = D(a)b + aD(b)$$

holds for every $a, b \in A$ with $ab = z$. It is obvious that $D$ is a derivation if and only if it is derivable at every point $z \in A$.

In what follows we illustrate the interest of the mathematical community in these kind of problems by surveying some of the achievements in this line. In 2002, W. Jing, S.J. Lu and P.T. Li [3] proved that if $D$ is a continuous linear map on a von Neumann algebra and also derivable at zero, then $D$ is a generalised derivation. Moreover, if $D(1) = 0$, then $D$ is a derivation. A.B.A. Essaleh and A.M. Peralta in [1, Corollary 2.16] proved the same result without assuming the continuity of $D$.

Linear maps which are generalised derivable at an element can be defined in a similar way. W. Jing proved in [2, Theorems 2.2 and

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2.6] that for each infinite dimensional Hilbert space $H$, a linear map $\delta : B(H) \to B(H)$ which is generalised Jordan derivable at zero, or at 1, is a generalised derivation. Another related results appear in [1], [5] and [7].

Every C*-algebra $A$ can be regarded as an element in the bigger class of JB*-triples when equipped with the triple product $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$, $(a, b, c \in A)$. A *ternary derivation* on a C*-algebra $A$ is a linear map $T : A \to A$ satisfying the generalised Leibnitz's rule:

$$T\{a, b, c\} = \{T(a), b, c\} + \{a, T(b), c\} + \{a, b, T(c)\},$$

for every $a, b, c \in A$. Further, a linear map $T : A \to A$ is *ternary derivable at a point* $z \in A$ if the identity

$$T(z) = \{T(a), b, c\} + \{a, T(b), c\} + \{a, b, T(c)\}$$

holds for every $a, b, c \in A$ with $\{a, b, c\} = z$.

Linear maps on C*-algebras which are ternary derivable at zero are studied in [1]. It is shown that if a continuous linear map on a C*-algebra is derivable or ternary derivable at zero, then it is a generalised derivation. Furthermore, if a continuous linear map $T$ on a C*-algebra is ternary derivable at zero with $T(1) = 0$, then $T$ is a ternary derivation, and if the hypothesis $T(1) = 0$ is replaced with $T(1)^* = -T(1)$ the same conclusion remains true. Thus being ternary derivable at zero implies that the mapping is ternary derivable on the whole domain. If the domain C*-algebra is replaced with a von Neumann algebra $W$, then every linear map $T : W \to W$ which is derivable or ternary derivable at zero is continuous.

The main result of our paper proves that every continuous conjugate linear map $T : A \to A^*$, where $A$ is a unital C*-algebra, is a ternary derivation whenever $T$ is a ternary derivable at the zero element of $A$ and $T(1) = 0$ (see Theorem 3.5).

## 2. Preliminary definitions

In this section, we recall some definitions and basic facts in triple structures.

**Definition 2.1.** A *triple product* on a complex vector space $E$ is a mapping

$$\{.,.,.\} : E \times E \times E \to E,$$

which is bilinear and symmetric in the outer variables and conjugate linear in the middle one and satisfying the so-called "Jordan Identity":

$$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\},$$

for all $a, b, c, d, e$ in $E$. A complex vector space $E$ is called a *Jordan triple* when it is endowed a non-trivial triple product. A Jordan triple $E$ is called a *Jordan Banach triple* when it is a Banach space and the triple product of $E$ is continuous.

Ternary modules over Jordan triples are defined in [6, Subsection 2.2]. We need this structure in the following definition.

**Definition 2.2.** Let $T$ be a conjugate linear mapping from a Jordan triple $E$ into a ternary $E$-module $X$. $T$ is called a *ternary derivable at* $z \in E$ whenever it satisfies the identity

$$T(z) = \{T(a), b, c\}_1 + \{a, T(b), c\}_2 + \{a, b, T(c)\}_3$$

for any $a, b, c \in E$ with $\{a, b, c\} = z$. $T$ is called a *ternary derivation* whenever it is ternary derivable at every point of $E$.

Let $A$ be a C$^*$-algebra. According to [4, Sec. 5] the following identity

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a), \quad (a, b, c \in A)$$

defines a triple product on $A$ which makes it a JB$^*$-triple. Let $A^*$ denotes the dual Banach space of $A$. The mappings:

$$\begin{aligned}
&\{\cdot,\cdot,\cdot\}_1 : A^* \times A \times A \to A^*, \quad \langle c, \{x, a, b\}_1\rangle = \langle \{a, b, c\}, x\rangle,\\
&\{\cdot,\cdot,\cdot\}_2 : A \times A^* \times A \to A^*, \quad \langle c, \{a, x, b\}_2\rangle = \overline{\langle \{a, c, b\}, x\rangle},\\
&\{\cdot,\cdot,\cdot\}_3 : A \times A \times A^* \to A^*, \quad \langle c, \{a, b, x\}_3\rangle = \langle \{c, a, b\}, x\rangle,
\end{aligned}$$

where $a, b, c \in A$ and $x \in A^*$, define a Banach ternary $A$-module structure on $A^*$.

We also consider $A^*$ as an (associative) Banach $A$-bimodule with actions:

$$\langle b, ax\rangle = \langle ba, x\rangle, \quad \langle b, xa\rangle = \langle ab, x\rangle, \quad (x \in A^*, a, b \in A), \tag{2.1}$$

and define an involution $^*$ on $A^*$ by the following identity:

$$\langle a, x^*\rangle = \overline{\langle a^*, x\rangle}, \quad (x \in A^*, a \in A).$$

Thanks to these structures we can consider associative and Jordan derivations from $A$ into $A^*$.

## 3. Main result

In what follows we extensively apply weak$^*$ continuity and weak$^*$ convergence, so the following theorem would has a pivotal role.

**Theorem 3.1.** *Let $A$ be a $C^*$-algebra, and let $D : A \to A^*$ be a continuous linear mapping. Then $D^{**} : A^{**} \to A^{***}$, the bitranspose of $D$, is $A^*$-valued.*

Given a C*-algebra $A$, $A^{***}$ is regarded as the dual space of $A^{**}$, and enjoys the $A^{**}$-bimodule structure, by actions (2.1).

**Proposition 3.2.** *Let $A$ be a $C^*$-algebra. Then $A^*$ is a $A^{**}$-submodule of $A^{***}$.*

The following two results prepare necessary background to reach the main result.

**Proposition 3.3.** *Let $T$ be a continuous conjugate linear mapping from a unital $C^*$-algebra $A$ into $A^*$ with $T(1) = 0$. The linear mapping $T \circ *$ is a derivation whenever $T$ is ternary derivable at zero.*

**Proposition 3.4.** *Let $T$ be a continuous conjugate linear mapping from a unital $C^*$-algebra $A$ into $A^*$ with $T(1) = 0$. If $T$ is ternary derivable at zero, then $T(a^*) = T(a)^*$, for every $a$ in $A$.*

We can now obtain the main result of the paper.

**Theorem 3.5.** *Let $T$ be a continuous conjugate linear mapping from a unital $C^*$-algebra $A$ into $A^*$ with $T(1) = 0$. Then $T$ is a ternary derivation whenever it is ternary derivable at zero.*

The restricting condition of the preceding result which emphasised that $T$ ought to vanish at the unit element, can not be removed. For example if we take $A$ to be a commutative unital C*-algebra and $T$ be defined by $T(x) = x^* b_0$, where $b_0 \in A^*$ with $b_0^* \neq -b_0$, we see that $T$ is not a ternary derivation, however, it is a ternary derivable at zero.

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Poster Presentation

# ITERATED DUAL VALUED TERNARY DERIVABLE MAPS AT UNIT

MOHSEN NIAZI

*Department of Mathematics, University of Birjand, Birjand, Iran*
*niazi@birjand.ac.ir*

Abstract. In this note we prove that a continuous (conjugate) linear map from a unital C*-algebra $A$ into its iterated dual space, $A^{(n)}$, which is ternary derivable at the unit element of $A$, is a ternary derivation. This result is a ternary module valued variant of a similar result on unital C*-algebras.

## 1. Introduction

One of the interesting problems in derivation theory is to find optimal conditions on a map which guaranty its derivability. In this paper we pursue this problem in the context of ternary derivations. In [1] it has been shown that a continuous linear mapping $T$ on a unital C*-algebra $A$ which is ternary derivable at the unit element of $A$, is a ternary derivation. Trying to replace the codomain of $T$ by a ternary module in the sense of M.R. Miri, H.R. Ebrahimi Vishki and the author of this note in [2], we prove that the codomain of the mapping $T$ in [1, Corollary 2.5] could be replaced by the iterated dual spaces, $A^{(n)}$, which we regard them as Banach ternary $A$-modules.

We begin by reviewing some basic facts in the category of Jordan triples. By a *triple product* on a complex vector space $E$, we mean a mapping

$$\{.,.,.\} : E \times E \times E \to E,$$

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which is bilinear and symmetric in the outer variables and conjugate linear in the middle one and satisfying the so-called “Jordan Identity”:

$$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\},$$

for all $a, b, c, d, e$ in $E$. A complex vector space $E$ is called a *Jordan triple* when it is endowed with a non-trivial triple product. A Jordan triple $E$ is called a *Jordan Banach triple* when it is a Banach space and the triple product of $E$ is continuous.

**Definition 1.1.** Let $E$ be a Jordan triple and $X$ be a complex vector space. Consider the following mappings and axioms:

$$\{\cdot,\cdot,\cdot\}_1 : X \times E \times E \to X, \qquad \{\cdot,\cdot,\cdot\}_2 : E \times X \times E \to X,$$
$$\{\cdot,\cdot,\cdot\}_3 : E \times E \times X \to X,$$

(1) $\{\cdot,\cdot,\cdot\}_1$ is linear in the first and second variables and conjugate linear in the third one. $\{\cdot,\cdot,\cdot\}_2$ is conjugate linear in each variable.

(1)$'$ Each of the mappings $\{\cdot,\cdot,\cdot\}_1, \{\cdot,\cdot,\cdot\}_2$ and $\{\cdot,\cdot,\cdot\}_3$ is linear in the first and third variables and conjugate linear in the second variable.

(2) $\{x, b, a\}_1 = \{a, b, x\}_3$, and $\{a, x, b\}_2 = \{b, x, a\}_2$ for every $a, b \in E$ and $x \in X$.

(3) Let $\{\cdot,\cdot,\cdot\}$ denote any of the mappings $\{\cdot,\cdot,\cdot\}_1, \{\cdot,\cdot,\cdot\}_2, \{\cdot,\cdot,\cdot\}_3$ or the triple product of $E$. Then the identity

$$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\},$$

holds for every $a, b, c, d, e$ where one of them is in $X$ and the other ones are in $E$.

When the mappings $\{\cdot,\cdot,\cdot\}_1, \{\cdot,\cdot,\cdot\}_2$ and $\{\cdot,\cdot,\cdot\}_3$ satisfy the axioms (1), (2) and (3), $X$ is called *a ternary $E$-module of type* $(I)$ and when they satisfy the axioms (1)$'$, (2) and (3), $X$ is called *a ternary $E$-module of type* $(II)$.

We usually write the expression “ternary $E$-module”, without declaring the type, whenever a statement is true for both types or the type is clear from the context. The following theorem connect the two types of the ternary modules.

**Theorem 1.2.** *Let $E$ be a Jordan triple. If $X$ is a ternary $E$-module of type* $(I)$ *(resp.* $(II)$*) then its dual space, $X^*$, with the following actions*

$$\{\cdot,\cdot,\cdot\}_1 : X^* \times A \times A \to X^*, \quad \langle \varphi, \{\theta, a, b\}_1 \rangle = \langle \{a, b, \varphi\}_3, \theta \rangle,$$
$$\{\cdot,\cdot,\cdot\}_2 : A \times X^* \times A \to X^*, \quad \langle \varphi, \{a, \theta, b\}_2 \rangle = \overline{\langle \{a, \varphi, b\}_2, \theta \rangle},$$
$$\{\cdot,\cdot,\cdot\}_3 : A \times A \times X^* \to X^*, \quad \langle \varphi, \{a, b, \theta\}_3 \rangle = \langle \{\varphi, a, b\}_1, \theta \rangle,$$

*where $a, b \in E$, $\varphi \in X$ and $\theta \in X^*$, is a ternary $E$-module of type* $(II)$ *(resp.* $(I)$*).*

It is easy to see that every Jordan Banach triple $E$ is a Banach ternary $E$-module of type (II) under its own ternary product as module actions. Now, Theorem 1.2 confirms that $E^*$ is a Banach ternary $E$-module of type (I), $E^{**}$ is a Banach ternary $E$-module of type (II), … . This means that the $n$th iterated dual space, $E^{(n)}$, is a Banach ternary $E$-module of type (I) whenever the integer $n$ is *odd* and is a Banach ternary $E$-module of type (II) whenever $n$ is *even*.

**Definition 1.3.** Let $E$ be a Jordan triple, and let $X$ be a ternary $E$-module of type (I) (*resp.* type (II)). A conjugate linear mapping (*resp.* linear mapping) $T : E \to X$ is called *ternary derivable at* $z \in E$ whenever it satisfies the identity

$$T(z) = \{T(a), b, c\}_1 + \{a, T(b), c\}_2 + \{a, b, T(c)\}_3$$

for any $a, b, c \in E$ with $\{a, b, c\} = z$. $T$ is called a *ternary derivation* whenever it is ternary derivable at every point of $E$.

Let $A$ be a Banach $^*$-algebra. It is easy to see that

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a), \quad (a, b, c \in A)$$

defines a triple product on $A$ which makes it a Jordan Banach triple. The iterated dual spaces of $A$, as the arguments before Definition 1.3 shows, are supplied with ternary $A$-module structures. We also consider $A^{(n)}$ as an (associative) Banach $A$-bimodule with the following recursively defined actions:

$$\langle b, \varphi\theta\rangle = \langle b\varphi, \theta\rangle, \quad \langle b, \theta\varphi\rangle = \langle \varphi b, \theta\rangle, \tag{1.1}$$

where $a \in A, \varphi \in A^{(n-1)}$ and $\theta \in A^{(n)}$, and define an involution $^*$ on $A^{(n)}$, recursively, by the following identity:

$$\langle \varphi, \theta^*\rangle = \overline{\langle \varphi^*, \theta\rangle}, \quad (\theta \in A^{(n)}, \varphi \in A^{(n-1)}). \tag{1.2}$$

## 2. Main results

To prove the main result of this paper we need the following two propositions.

**Proposition 2.1.** *Let $T$ be a continuous* (*conjugate*) *linear mapping from a unital $C^*$-algebra $A$ into its dual space, $A^{(n)}$. If $T$ is ternary derivable at the unit element of $A$, then $T \circ *$ is a derivation whenever $n$ is odd and $T$ is a derivation whenever $n$ is even.*

**Proposition 2.2.** *Let $T$ be a continuous* (*conjugate*) *linear mapping from a unital $C^*$-algebra $A$ into its dual space, $A^{(n)}$. If $T$ is ternary derivable at the unit element of $A$, then $T(a^*) = T(a)^*$ for every $a \in A$.*

**Theorem 2.3.** *Let $T$ be a continuous* (*conjugate*) *linear mapping from a unital $C^*$-algebra $A$ into its dual space, $A^{(n)}$. $T$ is a ternary derivation whenever it is ternary derivable at the unit element of $A$.*

*Proof.* Let $n$ be an odd integer. Since, by Proposition 2.1, $T\circ *$ is a derivation when $n$ is odd, for any $a,b,c\in A$, we have

$$\begin{aligned} T(ab^*c) &= (T\circ *)((ab^*c)^*) = (T\circ *)(c^*ba^*) \\ &= (T\circ *)(c^*)ba^* + c^*(T\circ *)(b)a^* + c^*b(T\circ *)(a^*) \\ &= T(c)ba^* + c^*T(b^*)a^* + c^*bT(a). \end{aligned}$$

Also, by Proposition 2.2, we have $T(b^*) = T(b)^*$. Therefore

$$T(ab^*c) = T(c)ba^* + c^*T(b)^*a^* + c^*bT(a).$$

Now, by definitions (1.1) and (1.2), for any $\varphi\in A^{(n-1)}$, we have

$$\begin{aligned} \langle\varphi, T(ab^*c)\rangle &= \langle\varphi, T(c)ba^*\rangle + \langle\varphi, c^*T(b)^*a^*\rangle + \langle\varphi, c^*bT(a)\rangle \\ &= \langle ba^*\varphi, T(c)\rangle + \langle a^*\varphi c^*, T(b)^*\rangle + \langle\varphi c^*b, T(a)\rangle \qquad (2.1) \\ &= \langle ba^*\varphi, T(c)\rangle + \overline{\langle c\varphi^*a, T(b)\rangle} + \langle\varphi c^*b, T(a)\rangle. \end{aligned}$$

Interchanging the role of $a$ and $c$ in the above identity, we obtain

$$\langle\varphi, T(cb^*a)\rangle = \langle bc^*\varphi, T(a)\rangle + \overline{\langle a\varphi^*c, T(b)\rangle} + \langle\varphi a^*b, T(c)\rangle. \qquad (2.2)$$

Applying the identities (2.1) and (2.2), we conclude that

$$\begin{aligned} \langle\varphi, T\{a,b,c\}\rangle &= \frac{1}{2}\big[\langle\varphi, T(ab^*c)\rangle + \langle\varphi, T(cb^*a)\rangle\big] \\ &= \langle ba^*\varphi, T(c)\rangle + \overline{\langle c\varphi^*a, T(b)\rangle} + \langle\varphi c^*b, T(a)\rangle \\ &\quad + \langle bc^*\varphi, T(a)\rangle + \overline{\langle a\varphi^*c, T(b)\rangle} + \langle\varphi a^*b, T(c)\rangle \\ &= \langle\{b,c,\varphi\}_3, T(a)\rangle + \overline{\langle\{a,\varphi,c\}_2, T(b)\rangle} + \langle\{\varphi,a,b\}_1, T(c)\rangle \\ &= \langle\varphi, \{T(a),b,c\}_1\rangle + \langle\varphi, \{a,T(b),c\}_2\rangle + \langle\varphi, \{a,b,T(c)\}_3\rangle. \end{aligned}$$

This shows that $T\{a,b,c\} = \{T(a),b,c\}_1 + \{a,T(b),c\}_2 + \{a,b,T(c)\}_3$.

A similar arguments, except for the position of the involution $^*$, can be applied to prove the above identity in the case of even integers. $\square$

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# GENERALISED LEIBNITZ'S RULE AND MAPPINGS ON VON NEUMANN ALGEBRAS

MOHSEN NIAZI

*Department of Mathematics, University of Birjand, Birjand, Iran*
*niazi@birjand.ac.ir*

Abstract. In this note by assuming a mapping $d$ on a von Neumann algebra $\mathcal{M}$, which satisfies the generalised Leibnitz's rule, we prove that $d$ is real linear and therefore is a real-homogeneous ternary derivation.

## 1. Introduction

Derivations are of the most well-behaved mappings in analysis in the sense that all their characters could be extracted from optimised sets of conditions. There are a large number of results in the field which explore these set of optimised conditions that guaranty the derivability. Among all these efforts are local derivability, derivability at one point and automatic continuity results. Maybe it could be said that the work of R.V. Kadison [3] in 1990 is of the most earliest results in this way by introducing the notion of local derivations and proving that local derivations are derivations. P. Šemrl [8] introduced the notion of 2-local derivations and without assuming the linearity in priori, he deduced the linearity and the derivability. This way continued by the author of this note and A.M. Peralta in [6] and [7] by introducing the notion of weak-2-local derivations. In a different direction the notion of derivability at one point of the space was introduced. To the best of

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our knowledge the work of W. Jing, S.J. Lu, and P.T. Li [2] is of the most earliest results in this line.

A same direction was followed up in the context of triple structures. M. Mackey [5] introduced the notion of local ternary derivations on Jordan triples. Weak-local ternary derivations are also studied in [1].

A *triple derivation* on a Jordan triple $E$ is a linear map $T$ on $E$ satisfying the generalised Leibnitz's rule:

$$T\{a,b,c\} = \{T(a),b,c\} + \{a,T(b),c\} + \{a,b,T(c)\} \qquad (1.1)$$

for all $a,b,c \in E$.

In this paper we obtain linearity of a mapping which satisfies the identity (1.1). Precisely speaking, we prove the following theorem:

**Theorem 1.1.** *Let $d$ be a mapping (no linearity is assumed in priori) on a von Neumann algebra $\mathcal{M}$ which satisfies the identity (1.1), and continuous at the origin with $d(1) = 0$. Then $d$ is a real-homogeneous ternary derivation.*

## 2. Preliminary definitions

In this section, we recall some definitions in the context of triple structures.

**Definition 2.1.** A *triple product* on a complex vector space $E$ is a mapping

$$\{.,.,.\} : E \times E \times E \to E,$$

which is bilinear and symmetric in the outer variables and conjugate linear in the middle one and satisfying the so-called "Jordan Identity":

$$\{a,b,\{c,d,e\}\} = \{\{a,b,c\},d,e\} - \{c,\{b,a,d\},e\} + \{c,d,\{a,b,e\}\},$$

for all $a,b,c,d,e$ in $E$. A complex vector space $E$ is called a *Jordan triple* when it is endowed a non-trivial triple product. A Jordan triple $E$ is called a *Jordan Banach triple* when it is a Banach space and the triple product of $E$ is continuous.

A *JB$^*$-triple* is a Jordan Banach triple $E$ which satisfies the following two axioms:

(1) For any $a$ in $E$ the mapping $x \mapsto \{a,a,x\}$ is a hermitian operator on $E$ with non-negative spectrum;
(2) $\|\{a,a,a\}\| = \|a\|^3$ for all $a$ in $A$.

A JB$^*$-triple which has a predual as a Banach space is called a *JBW$^*$-triple*.

Let $A$ be a C*-algebra. According to [4, Sec. 5] the following identity

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a), \quad (a, b, c \in A) \tag{2.1}$$

defines a triple product on $A$ which makes it a JB*-triple. Also, a von Neumann algebra with triple product (2.1) is a JBW*-triple.

## 3. Main result

Proof of the main result of this paper, i.e. Theorem 1.1, will be done through presenting a series of lemmata. Throughout this section $\mathcal{M}$ and $d$ will denote the same objects as in the statement of Theorem 1.1.

**Lemma 3.1.** *Let $p$ be a projection in $\mathcal{M}$. Then $d(1-p) = -\,d(p)$.*

**Lemma 3.2.** *Let $p$ be a projection in $\mathcal{M}$. Then, there exists an anti-symmetric element $\mu_p \in \mathcal{M}$, depending on $p$, such that*

$$d(p) = \mu_p p - p\mu_p \quad \textit{and} \quad d(1-p) = \mu_p(1-p) - (1-p)\mu_p.$$

In what follows, concerning any projection $p \in M$, we consider the auxiliary mapping $\phi_p$ on $\mathcal{M}$, defined by $\phi_p(a) = d(a) - \mu_p a + a\mu_p$, where $a \in \mathcal{M}$. Note that, by definition, $\phi_p(p) = 0$. In the other hand, by Lemma 3.1, we have

$$\begin{aligned}\phi_p(1-p) &= d(1-p) - \mu_p(1-p) + (1-p)\mu_p \\ &= -d(p) + \mu_p p - p\mu_p = -\phi_p(p),\end{aligned}$$

hence $\phi_p(1-p) = 0$.

Also, the element $\mu_{1-p}$, as Lemma 3.2 suggests, could be taken equal to $\mu_p$. We do so, and by this choice, we have $\phi_{1-p} = \phi_p$.

**Lemma 3.3.** *Let $p$ be a projection in $\mathcal{M}$. Then, for any $a \in \mathcal{M}$, we have*

$$\phi_p(a) = \phi_p(pap) + \phi_p(pa(1-p)) + \phi_p((1-p)ap) + \phi_p((1-p)a(1-p)).$$

**Lemma 3.4.** *Let $p$ be a projection in $\mathcal{M}$. Then, for any $a, b \in \mathcal{M}$, we have $\phi_p(p(a+b)(1-p)) = \phi_p(pa(1-p)) + \phi_p(pb(1-p))$.*

**Lemma 3.5.** *Let $p$ be a projection in $\mathcal{M}$. Then, for any $a, b \in \mathcal{M}$, we have $\phi_p(p(a+b)p) = \phi_p(pap) + \phi_p(pbp)$.*

We can now prove the main result of the paper.

*Proof of Theorem 1.1.* Let $a, b \in \mathcal{M}$, and $p$ be a projections in $\mathcal{M}$. By Lemma 3.3, we have

$$\phi_p(a) = \phi_p(pap) + \phi_p(pa(1-p)) + \phi_p((1-p)ap) + \phi_p((1-p)a(1-p)).$$

The same identity is true when $a$ is replaced by $b$ or $a+b$. Now, applying Lemmata 3.4 and 3.5, and considering the argument before Lemma 3.3, we obtain

$$\begin{aligned}
\phi_p(a+b) &= \phi_p(p(a+b)p) + \phi_p(p(a+b)(1-p)) + \phi_p((1-p)(a+b)p) \\
&\quad + \phi_p((1-p)(a+b)(1-p)) \\
&= [\phi_p(pap) + \phi_p(pbp)] + [\phi_p(pa(1-p)) + \phi_p(pb(1-p))] \\
&\quad + [\phi_p((1-p)ap) + \phi_p((1-p)bp)] \\
&\quad + [\phi_p((1-p)a(1-p)) + \phi_p((1-p)b(1-p))] \\
&= [\phi_p(pap) + \phi_p(pa(1-p)) + \phi_p((1-p)ap) \\
&\quad + \phi_p((1-p)a(1-p))] \\
&\quad + [\phi_p(pbp) + \phi_p(pb(1-p)) + \phi_p((1-p)bp) \\
&\quad + \phi_p((1-p)b(1-p))] \\
&= \phi_p(a) + \phi_p(b).
\end{aligned}$$

From this identity we conclude that

$$\begin{aligned}
d(a+b) &= \phi_p(a+b) - \mu_p(a+b) + (a+b)\mu_p \\
&= \phi_p(a) + \phi_p(b) - \mu_p a - \mu_p b + a\mu_p + b\mu_p \\
&= [\phi_p(a) - \mu_p a + a\mu_p] + [\phi(b) - \mu_p b + b\mu_p] = d(a) + d(b).
\end{aligned}$$

This proves that $d$ is additive. Obviously, rational homogeneity of $d$ is deduced from its additivity. In the other hand, since the topology of $\mathcal{M}$ is translation invariant and $d$ is additive, continuity of $d$ at the origin implies its continuity at every point. Now, real homogeneity of $d$ is deduced from its rational homogeneity. □

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**Oral Presentation**

# ON THE RANGE AND KERNEL OF JORDAN $*$-DERIVATIONS

ABOLFAZL NIAZI MOTLAGH

*Department of Mathematics,University of Bojnord,Bojnord, Iran*
*a.niazi@ub.ac.ir and niazimotlagh@gmail.com*

Abstract. In this talk,relation between kernel and Range of generalized Jordan $*$-derivation is investigated.

## 1. Introduction and preliminaries

Let $\mathfrak{A}$ be a $*$-algebra. A Jordan $*$-derivation on $\mathfrak{A}$ is a linear mapping $E : \mathfrak{A} \longrightarrow \mathfrak{A}$ which satisfies

$$E(a^2) = aE(a) + E(a)a^*$$

for all $a \in \mathfrak{A}$. Note that for a fixed $a \in \mathfrak{A}$ the mapping $\Delta_a(x) = ax - x^*a$ is a Jordan $*$-derivation; such Jordan $*$-derivation are said to be inner.

In [1] and [6] we can see the following results

(1) Every Jordan $*$-derivation on complex $*$-algebra with identity is inner.
(2) Every Jordan $*$-drivation on algebra of all bounded linear operators on a real Hilbert space $\mathcal{H}(dim\mathcal{H} > 1)$, is inner.
(3) Every Jordan $*$-drivation on the quaternion algebra is inner.

Let $\mathcal{H}$ be a complex infinite dimentional Hilbert space and let $\mathbb{B}(\mathcal{H})$ be the Banach algebra of all bounded linear operators on $\mathcal{H}$. For operators $\mathrm{A}, \mathrm{B} \in \mathbb{B}(\mathcal{H})$ we define generalized Jordan $*$-derivation $\Delta_{\mathrm{A,B}}$ by

$$\Delta_{\mathrm{A,B}}(X) = \mathrm{A}X - X^*\mathrm{B}$$

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for $X \in \mathbb{B}(\mathcal{H})$. Note that if $\mathrm{A} = \mathrm{B}$ then $\Delta_{\mathrm{A},\mathrm{A}} = \Delta_{\mathrm{A}}$ is a Joradan $*$-derivation.

## 2. Main Results

**Theorem 2.1.** *Let* $\mathrm{A}, \mathrm{B} \in \mathbb{B}(\mathcal{H})$. *Set* $T = \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{bmatrix}$ *on* $\mathcal{H} \oplus \mathcal{H}$. *Then we have the following results:*

1. *If* $\overline{R(\Delta_T)}^{\tau} \cap ker(\delta_{T^*}) = \{0\}$ *then* $\overline{R(\Delta_{\mathrm{A},\mathrm{B}})}^{\tau} \cap ker(\Delta_{\mathrm{A}^*,\mathrm{B}^*}) = \{0\}$, *where* $\overline{R(\Delta_T)}^{\tau}$ *denotes closure of* $R(\delta_T)$ *respect to the norme topoplogy or the weak operator topology.*
2. *If* $R(\Delta_T) \cap ker(\Delta_{T^*}) = \{0\}$ *then* $R(\Delta_{\mathrm{A},\mathrm{B}}) \cap ker(\Delta_{\mathrm{A}^*,\mathrm{B}^*}) = \{0\}$.

*Proof.* Let $\mathrm{C} \in \overline{R(\Delta_{\mathrm{A},\mathrm{B}})}^{\tau} \cap ker(\Delta_{\mathrm{A}^*,\mathrm{B}^*})$. Then there exists a sequence $\{X_n\} \subseteq \mathbb{B}(\mathcal{H})$ such that $\lim_\tau \Delta_{\mathrm{A},\mathrm{B}}(X_n) = \mathrm{C}$. Therefore we have $\lim_\tau \mathrm{A}X_n - X_n^*\mathrm{B} = \mathrm{C}$. Also $ker(\Delta_{\mathrm{A}^*,\mathrm{B}^*})(\mathrm{C}) = 0$ implies that $\mathrm{A}^*\mathrm{C} = \mathrm{C}^*\mathrm{B}^*$.

Now set $\mathrm{S} = \begin{bmatrix} 0 & \mathrm{C} \\ 0 & 0 \end{bmatrix}$ and $Y_n = \begin{bmatrix} 0 & X_n \\ 0 & 0 \end{bmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$. So

$$\begin{aligned} \lim_\tau (TY_n - Y_n^*T) &= \lim_\tau \left( \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{bmatrix} \begin{bmatrix} 0 & X_n \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ X_n^* & 0 \end{bmatrix} \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{bmatrix} \right) \\ &= \lim_\tau \begin{bmatrix} 0 & \mathrm{A}X_n \\ -X_n^*\mathrm{A} & 0 \end{bmatrix} \end{aligned}$$

Now suppose that $\lim_w \begin{bmatrix} 0 & \mathrm{A}X_n - X_n^*\mathrm{B} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$. Then for each $x, y \in \mathcal{H}$ we have

$$\lim \left| \left\langle \begin{bmatrix} T_{11} & T_{12} - (\mathrm{A}X_n - X_n^*\mathrm{B}) \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix}, \begin{bmatrix} y \\ 0 \end{bmatrix} \right\rangle \right| = 0.$$

Therefore $\lim \left| \langle T_{12} - (\mathrm{A}X_n - X_n^*\mathrm{B})x, y \rangle \right| = 0$ for all $x, y \in \mathcal{H}$, which implies that

$$\lim_w (\mathrm{A}X_n - X_n^*\mathrm{B}) = T_{12}.$$

Using same argument we prove that

$$T_{11} = T_{21} = T_{22} = 0.$$

So this implies that

$$\begin{aligned}
\lim_w (TY_n - Y_n^* T) &= \lim_w \begin{bmatrix} 0 & \mathrm{A}X_n - X_n^*\mathrm{B} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \lim_w(\mathrm{A}X_n - X_n^*\mathrm{B}) \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \mathrm{C} \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

In the following, we show the above results for the norm topology. Let

$$\lim_{\|\cdot\|} \begin{bmatrix} 0 & \mathrm{A}X_n - X_n^*\mathrm{B} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

on $\mathcal{H} \oplus \mathcal{H}$, then $\lim_n \left\| \begin{bmatrix} T_{11} & T_{12} - (\mathrm{A}X_n - X_n^*\mathrm{B}) \\ T_{21} & T_{22} \end{bmatrix} \right\| = 0.$

Hence $\lim_n \|T_{12} - (\mathrm{A}X_n - X_n^*\mathrm{B})\| = 0$ and $T_{11} = T_{21} = T_{22} = 0$.

Now we can conclude

$$\begin{aligned}
\lim_{\|\cdot\|} \begin{bmatrix} 0 & \mathrm{A}X_n - X_n^*\mathrm{B} \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & \lim_{\|\cdot\|}(\mathrm{A}X_n - X_n^*\mathrm{B}) \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \mathrm{C} \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

Hence, $S = \begin{bmatrix} 0 & \mathrm{C} \\ 0 & 0 \end{bmatrix} \in \overline{R(\Delta_T)}^{\tau}$. Since

$$S^*T^* = \begin{bmatrix} 0 & \mathrm{C}^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{A}^* & 0 \\ 0 & \mathrm{B}^* \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{C}^*\mathrm{B}^* \\ 0 & 0 \end{bmatrix}$$

and

$$T^*S = \begin{bmatrix} \mathrm{A}^* & 0 \\ 0 & \mathrm{B}^* \end{bmatrix} \begin{bmatrix} 0 & \mathrm{C} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{A}^*\mathrm{C} \\ 0 & 0 \end{bmatrix}$$

using equality $\mathrm{A}^*C = \mathrm{C}^*\mathrm{B}^*$ we can conclude that $S^*T^* = T^*S$. On the other hand $S \in \overline{R(\Delta_T)}^{\tau} \cap ker(\Delta_{T^*}) = \{0\}$, hence $\mathrm{C} = 0$.

This complete the proof of (1). The proof (2) follows immediately from replacing $\overline{R(\Delta_T)}^{\tau}$ with $R(\Delta_T)$. $\square$

**Theorem 2.2.** *Let* $\mathrm{A}_1, \mathrm{A}_2 \in \mathbb{B}(\mathcal{H})$. *Suppose that* $\overline{R(\Delta_{\mathrm{A}_i,\mathrm{A}_j})} \cap ker(\Delta_{\mathrm{A}_i,\mathrm{A}_j}) = \{0\}$ *for* $i, j = 1, 2$. *Then* $\overline{R(\Delta_T)} \cap ker(\Delta_T) = \{0\}$ *in which* $T = \mathrm{A}_1 \oplus \mathrm{A}_2 = \begin{bmatrix} \mathrm{A}_1 & 0 \\ 0 & \mathrm{A}_2 \end{bmatrix}$.

**Theorem 2.3.** *Let* $\mathrm{A}_1$ *and* $\mathrm{A}_2$ *denote unitary equivalent bounded operators on Hilbert space* $\mathcal{H}$*, so that* $\mathrm{A}_2 = \mathrm{U}\mathrm{A}_1\mathrm{U}^*$ *hold for some unitary operator* $\mathrm{U}$*. If* $\overline{R(\Delta_{\mathrm{A}_1})} \cap Ker(\Delta_{\mathrm{A}_1}) = \{0\}$ *then* $\overline{R(\Delta_{\mathrm{A}_2})} \cap Ker(\Delta_{\mathrm{A}_2}) = \{0\}$

Before stating the next corollary, we need the following theorem of Putnam[5].

**Theorem 2.4.** *Let* $\mathrm{A}, \mathrm{B} \in \mathbf{B}(\mathcal{H})$ *be normal operators and* $\mathrm{B}$ *similar to* $\mathrm{A}$*, so that* $\mathrm{B} = \mathrm{C}^{-1}\mathrm{A}\mathrm{C}$ *hold for some bounded invertable operator* $\mathrm{C}$*. If* $\mathrm{C} = P\mathrm{U}$ *is the polar factorization of* $\mathrm{C}$*, where* $P$ *is positive definite and* $\mathrm{U}$ *is unitary then* $\mathrm{B} = \mathrm{U}^*\mathrm{A}\mathrm{U}$*. Thus similarity equivalence of normal operators implies their unitary equivalence.*

**Corollary 2.5.** *Let* $\mathrm{A}, \mathrm{B} \in \mathbf{B}(\mathcal{H})$ *be normal operators and* $\mathrm{B}$ *similar to* $\mathrm{A}$*. If* $\overline{R(\Delta_{\mathrm{A}})} \cap Ker(\Delta_{\mathrm{A}}) = \{0\}$ *then* $\overline{R(\Delta_{\mathrm{B}})} \cap Ker(\Delta_{\mathrm{B}}) = \{0\}$*.*

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# THE GENERALIZED HYERS-ULAM STABILITY OF DERIVATIONS IN NON-ARCHIMEDEAN BANACH ALGEBRAS

ABOLFAZL NIAZI MOTLAGH

*Department of Mathematics,University of Bojnord,Bojnord, Iran*
*a.niazi@ub.ac.ir and niazimotlagh@gmail.com*

Abstract. In this paper, the generalized Hyers-Ulam stability of the functional inequality

$$\|f(a)+f(b)+cf(d)+f(c)d\| \leq \left\|kf\Big(\frac{a+b+cd}{k}\Big)\right\|, \quad |k|<2,$$

in non-Archimedean Banach algebras is established.

sectionIntroduction

Let $\mathbb{K}$ be a field. A non-archimedean absolute value on $\mathbb{K}$ is a function $|.| : \mathbb{K} \longrightarrow [0,\infty)$ such that for any $a, b \in \mathbb{K}$ we have
(i) $|a| \geq 0$ and equality holds if and only if $a = 0$,
(ii) $|ab| = |a||b|$,
(iii) $|a+b| \leq max\{|a|,|b|\}$.
Condition (iii) is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, is concluded from (iii) that $|n| \leq 1$ for each integer $n$. In all, we always assume that $|.|$ is non-trivial, i.e., that there is an $a_0 \in \mathbb{K}$ such that $|a_0| \notin \{0,1\}$.
Throughout this paper, we assume that that the base field is a non-Archimedean valution field.

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*Key words and phrases.* Banach algebra, Derivation, Generalized Hyers-Ulam stability, Non-Archimedean Banach algebra.

**Definition 0.1.** Let $\mathcal{X}$ be a linear space over a scalar field $\mathbb{K}$ with a non-archimedean non-trivial valuation $|.|$. A function $\|.\| : \mathcal{X} \longrightarrow \mathbb{R}$ is a non-archimedean norm (valuation) if it satisfies the following conditions:

(I)$\|x\| = 0$ if and only if $x = 0$;
(II) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in \mathcal{X}$;
(III) the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leq max\{\|x\|, \|y\| \quad (x, y \in \mathcal{X}).$$

Then $(\mathcal{X}, \|.\|)$ is called a non-archimedean normed space.

Stanislaw M. Ulam introduced a number of important unsolved problems during his talk at the university of Wisconsin in 1940 [11, 12].One of his open problems was the first stability problems:

Let $\mathcal{G}_1$ be a group and let $(\mathcal{G}_2, d)$ be a metric group. Given $\varepsilon > 0$ does there exists a $\delta > 0$ such that if a function $f : \mathcal{G}_1 \to \mathcal{G}_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in \mathcal{G}_1$, then there exists a homomorphism $T : \mathcal{G}_1 \to \mathcal{G}_2$ such that $d(f(x), T(x)) < \varepsilon$ for all $x \in \mathcal{G}_1$?

Hyers [3] gave a first affirmative partial answer to the quastion of ulam for Banach spaces. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Rassias [9] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [9] has had a lot of influence in the development of what we call generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations[2, 10].

Beginning around the year 1980, the topic of approximate homomorphisms and derivations and their stability theory in the field of functional equation and inequalities was taken up by several mathematicians. Also, the stability problems in non-Archimedean Banach spaces(algebras) are studied by Moslehian and Rssias [4], Moslehian and Sadeghi[5, 6], Mirmostafaee[7] and Najati and Moradlou[8].

In this paper, we prove that if $f$ satisfies the functional inequality

$$\|f(a) + f(b) + cf(d) + f(c)d\| \leq \left\|kf\Big(\frac{a + b + cd}{k}\Big)\right\|, \qquad |k| < |2|, \tag{0.1}$$

then $f$ is a derivation, and prove the generalized Hyers-ulam stability of functional inequlity (0.1) in non-Archimedean Babanach algebras.

## 1. The generalized Hyers–Ulam stability in non-Archimedean Banach algebras

Throughout this section, $\mathcal{A}$ is a non-archimedean Banach algebra on non-archimedean filed $\mathbb{K}$ that the characteristic of $\mathbb{K}$ is not 2 and $\mathcal{X}$ is a non-Archimedean Banach $\mathcal{A}$-bimodule.

**Proposition 1.1.** *Suppose that $k$ is a fixed integer greater than 2 and $|k| < |2|$. Let $\mathcal{A}$ be an unital non-archimedean Banach algebra,$\mathcal{X}$ is a non-Archimedean Banach $\mathcal{A}$-bimodule and $f : \mathcal{A} \longrightarrow \mathcal{X}$ be a mapping such that*

$$\|f(a) + f(b) + cf(d) + f(c)d\| \leq \left\| kf\left(\frac{a+b+cd}{k}\right)\right\| \tag{1.1}$$

*for all $a, b, c, d, \in \mathcal{A}$. Then $f$ is a derivation.*

**Theorem 1.2.** *Suppose that $k$ is a fixed integer greater than 2 and $|k| < |2|$. Let $r < 1, \theta$ be nonnegative real numbers and $f : \mathcal{A} \longrightarrow \mathcal{X}$ be an odd mapping such that $f(1) = 0$ and*

$$\begin{aligned}\|f(a) + f(b) + cf(d) + f(c)d\| \leq& \left\| kf\left(\frac{a+b+cd}{k}\right)\right\| \\ &+ \theta(\|a\|^r + \|b\|^r + \|cd\|^r)\end{aligned} \tag{1.2}$$

*for all $a, b, c, d \in \mathcal{A}$. Then there exists a unique derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ such that*

$$\|f(a) - D(a)\| \leq \frac{2 + |2|^r}{|2|^r}\theta\|a\|^r \quad (a \in \mathcal{A}). \tag{1.3}$$

**Theorem 1.3.** *Suppose that $k$ is a fixed integer greater than 2 and $|k| < |2|$. Let $r > 1, \theta$ be nonnegative real numbers and $f : \mathcal{A} \longrightarrow \mathcal{X}$ be an odd mapping such that $f(1) = 0$ and*

$$\begin{aligned}\|f(a) + f(b) + cf(d) + f(c)d\| \leq& \left\| kf\left(\frac{a+b+cd}{k}\right)\right\| \\ &+ \theta(\|a\|^r + \|b\|^r + \|cd\|^r)\end{aligned} \tag{1.4}$$

*for all $a, b, c, d \in \mathcal{A}$. Then there exists a unique derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ such that*

$$\|f(a) - D(a)\| \leq \frac{2 + |2|^r}{|2|}\theta\|a\|^r \quad (a \in \mathcal{A}). \tag{1.5}$$

**Theorem 1.4.** *Suppose that $k$ is a fixed integer greater than 2 and $|k| < |2|$. Let $r < \frac{1}{3}, \theta$ be nonnegative real numbers and $f : \mathcal{A} \longrightarrow \mathcal{X}$ be an odd mapping such that $f(1) = 0$ and*

$$\begin{aligned}\|f(a) + f(b) + cf(d) + f(c)d\| \leq& \left\| kf\left(\frac{a+b+cd}{k}\right)\right\| \\ &+ \theta\|a\|^r.\|b\|^r.\|cd\|^r\end{aligned} \tag{1.6}$$

*for all $a, b, c, d \in \mathcal{A}$. Then there exists a unique derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ such that*

$$\|f(a) - D(a)\| \leq \frac{\theta|2|^r}{|2|^{3r}}\|a\|^{3r} \quad (a \in \mathcal{A}). \tag{1.7}$$

**Theorem 1.5.** *Suppose that $k$ is a fixed integer greater than 2 and $|k| < |2|$. Let $r > \frac{1}{3}, \theta$ be nonnegative real numbers and $f : \mathcal{A} \longrightarrow \mathcal{X}$ be an odd mapping such that $f(1) = 0$ and*

$$\|f(a) + f(b) + cf(d) + f(c)d\| \leq \left\| kf\Big(\frac{a+b+cd}{k}\Big) \right\| + \theta \|a\|^r.\|b\|^r.\|cd\|^r) \tag{1.8}$$

*for all $a, b, c, d \in \mathcal{A}$. Then there exists a unique derivation $D : \mathcal{A} \longrightarrow \mathcal{A}$ such that*

$$\|f(a) - D(a)\| \leq \frac{\theta |2|^r}{|2|} \|a\|^{3r} \qquad (a \in \mathcal{A}). \tag{1.9}$$

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# THE EQUIVALENCY OF P-BESSEL SEQUENCES WITH CONTROLLED P-BESSEL SEQUENCES

ELNAZ. OSGOOEI

*Faculty of Science, Urmia University of Technology, Urmia, Iran*
*e.osgooei@uut.ac.ir*

Abstract. In this paper inspiring the concept of p-Bessel sequences, p-controlled Bessel sequences are presented and showed that in the case that $1 < p \leq 2$ these two concept could replace instead of each other.

## 1. Introduction

Controlled frames for spherical wavelets were first introduced in [2] and the relation between controlled frames and standard frames were developed in [1, 4, 5].
In this paper, motivated the concept of p-frames, we introduce p-controlled frames on Banach spaces. In Section 2, we show that under some strong condition the concept of p-Bessel sequences and controlled p-Bessel sequences are equivalent. In other words, the equivalency of these two concepts presented when $1 < p \leq 2$, however the general case $1 < p < \infty$ is more desirable that we did not reach.
Throughout this paper $GL(X)$ is denoted as the set of all bounded and invertible operators on the space $X$.
A sequence $\{f_i\}_{i=1}^{\infty} \subseteq H$ is a frame for $H$ if there exist $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad f \in H, \tag{1.1}$$

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The constants $A$ and $B$ are called lower and upper frame bounds, respectively. The sequence $\{f_i\}_{i=1}^{\infty} \subseteq H$ is a Bessel sequence for $H$, if only the right hand inequality in (1.1) holds for all $f \in H$.

**Definition 1.1.** Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of vectors in a Hilbert space $\mathcal{H}$ and $C, D \in GL(\mathcal{H})$. Then $\{f_i\}_{i=1}^{\infty}$ is called a frame controlled by $C$ and $D$ or $(C, D)$-controlled frame if there exist two constants $0 < A \leq B < \infty$, such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} \langle f, Cf_i\rangle\langle Df_i, f\rangle \leq B\|f\|^2, \quad f \in \mathcal{H}. \tag{1.2}$$

If only the right inequality in (1.2) holds, then $\{f_i\}_{i=1}^{\infty}$ is called a $(C, D)$-controlled Bessel sequence. If $A = B$ then $\{f_i\}_{i=1}^{\infty}$ is called a $(C, D)$-controlled tight frame.

**Definition 1.2.** A sequence $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p-frame for $X$ $(1 < p < \infty)$ if there exist constants, $0 < A \leq B < \infty$ such that

$$A\|f\| \leq (\sum_{i=1}^{\infty} |g_i(f)|^p)^{\frac{1}{p}} \leq B\|f\|. \tag{1.3}$$

The sequence $\{g_i\}_{i=1}^{\infty}$ is called a p-Bessel sequence if only the right inequality in (1.3) holds.

If $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p-frame for $X$. Then two operators

$$U : X \to \ell^p, \quad Uf = \{g_i(f)\}_{i=1}^{\infty},$$

and

$$T : \ell^q \to X^*, \quad T\{d_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} d_i g_i,$$

is defined. The operator $U$ is called the analysis operator and $T$ is called the synthesis operator of $\{g_i\}_{i=1}^{\infty}$. If $\{g_i\}_{i=1}^{\infty}$ is a p-frame (or just a p-Bessel sequence) then $U$ is a bounded operator and also $T$.

## 2. Main results

In this section first the concept of $(C, D)$-controlled p-Bessel sequence is introduced and then it is proved that under some suprising condition $1 < p \leq 2$ and $\overline{sgn(\langle f, Cg_i\rangle)}sgn(\langle f, Dg_i\rangle) > 0$ for each $f \in X$ and $i \in \mathbb{N}$, the concept of $(C, D)$-controlled p-Bessel sequence and p-Bessel sequence is equivalent.

**Definition 2.1.** Let $\{g_i\}_{i=1}^{\infty}$ be a family of vectors in $X^*$. Suppose that $C, D \in GL(X^*)$. The sequence $\{g_i\}_{i=1}^{\infty}$ is called a p-frame controlled by $C$ and $D$ or $(C,D)$-controlled p-frame if there exist constants $0 < A \leq B < \infty$ such that for each $f \in X$,

$$\begin{aligned} A^2\|f\|^2 &\leq \|\{\langle f, Cg_i\rangle\}_{i=1}^{\infty}\|_p^{2-p} \sum_{i=1}^{\infty} |\langle f, Dg_i\rangle||\langle Cg_i, f\rangle|^{p-1}\overline{sgn\langle f, Cg_i\rangle} sgn\langle f, Dg_i\rangle \\ &\leq B^2\|f\|^2. \end{aligned} \tag{2.1}$$

If only the right inequality in (2.1) satisfied, then $\{g_i\}_{i=1}^{\infty}$ is called a $(C,D)$-controlled p-Bessel sequence.

**Proposition 2.2.** *Let $1 < p \leq 2$. Suppose that $\overline{sgn(\langle f, Cg_i\rangle)} sgn(\langle f, Dg_i\rangle) > 0$ for each $f \in X$ and $i \in \mathbb{N}$. Then the sequence $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a $(C,D)$-controlled p-Bessel sequence if and only if it is a p-Bessel sequence.*

*Proof.* Suppose that $\{g_i\}_{i=1}^{\infty}$ is a $(C,D)$-controlled p-Bessel sequence. Then by (2.1)

$$(\sum_{i=1}^{\infty} |\langle Cg_i, f\rangle|^p)^{\frac{1}{p}} \infty, \quad f \in X.$$

Suppose that there exists $0 \neq f_0 \in X$ such that for each $M > 0$

$$(\sum_{i=1}^{\infty} |\langle Cg_i, f_0\rangle|^p)^{\frac{1}{p}} > M. \tag{2.2}$$

Consider $\sum_{i=1}^{\infty} |\langle f_0, Dg_i|\rangle|\langle Cg_i, f_0\rangle|^{p-1}\overline{sgn\langle f_0, Cg_i\rangle} sgn\langle f_0, Dg_i\rangle = K$. Then three cases may happen:

(i) $K = \infty$.
(ii) $0 < K < \infty$.
(iii) $K = 0$.

Since $1 < p \leq 2$, the cases (i) and (ii) is a contradiction with (2.1) and (2.2).
(iii) If $K = 0$, then $\overline{sgn\langle f, Cg_i\rangle} sgn\langle f, Dg_i\rangle = 0$, which is a contradiction.
Therefore $(\sum_{i=1}^{\infty} |\langle Cg_i, f\rangle|^p)^{\frac{1}{p}} < \infty, \quad f \in X$. So similar to the proof of Lemma 3.1.1 in [3], there exists $B' > 0$ such that $(\sum_{i=1}^{\infty} |\langle g_i, f\rangle|^p)^{\frac{1}{p}} < B'\|f\|, \quad f \in X$. Now suppose that $\{g_i\}_{i=1}^{\infty}$ is a p-Bessel sequence with bound $B$ for $X$. Since $1 < p \leq 2$, we have

$$\|\{\langle f, Cg_i\rangle\}_{i=1}^{\infty}\|_p^{2-p} \leq B^{2-p}\|C^*f\|^{2-p}, \quad f \in X. \tag{2.3}$$

Also since $\frac{1}{p}+\frac{1}{q}=1$ and $\overline{sgn(\langle f, Cg_i\rangle)}sgn(\langle f, Dg_i\rangle)>0$ for each $f\in X$ and $i\in\mathbb{N}$, we have

$$\begin{aligned}
\sum_{i=1}^{\infty}|\langle f, Dg_i\rangle||\langle Cg_i, f\rangle|^{p-1}&\overline{sgn\langle f, Cg_i\rangle}sgn\langle f, Dg_i\rangle\\
&=\sum_{i=1}^{\infty}|\langle f, Dg_i\rangle\langle f, Cg_i\rangle||\langle Cg_i, f\rangle|^{p-2}\\
&\leq\sum_{i=1}^{\infty}|\langle f, Dg_i\rangle||\langle Cg_i, f\rangle|^{p-1}\\
&\leq(\sum_{i=1}^{\infty}|\langle f, Dg_i\rangle|^{p})^{\frac{1}{p}}(\sum_{i=1}^{\infty}|\langle Cg_i, f\rangle|^{p})^{\frac{1}{q}}\\
&\leq B^p\|D^*f\|\|C^*f\|^{\frac{p}{q}},\quad f\in X.
\end{aligned}$$

Therefore by (2.3) and above equations for each $f\in X$ we have

$$\begin{aligned}
\|\{\langle f, Cg_i\rangle\}_{i=1}^{\infty}\|_p^{2-p}\sum_{i=1}^{\infty}|\langle f, Dg_i\rangle||\langle Cg_i, f\rangle|^{p-1}&\overline{sgn(\langle f, Cg_i\rangle}sgn(\langle f, Dg_i\rangle)\\
&\leq B^2\|D\|\|C\|\|f\|^2.
\end{aligned}$$

□

## 3. Further remarks

The equivalency of p-Bessel sequences with $(C,D)$-controlled p-Bessel sequences is our aim in this paper that we just obtain it in the case of $1<p\leq 2$, however the general case $1<p<\infty$ is more desirable that we did not reach and it remains as an open question.

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# ON THE BEST PROXIMITY POINT THEOREMS IN PARTIALLY ORDERED NON-ARCHIMEDEAN FUZZY METRIC SPACES

M. PAKNAZAR

*Department of Mathematics Education, Farhangian University.*
*m.paknazar@cfu.ac.ir*

Abstract. In this paper we establish some new best proximity theorems in partially ordered non-Archimedean fuzzy metric spaces. Moreover, we deduce some new fixed point results.

## 1. Introduction

Ran and Reurings [1] initiated the study of the existence of fixed points in partially ordered sets with applications to matrix equations. Recently many researchers presented some new results for nonlinear contractions in partially ordered metric spaces (see for instance [2] and the references therein). Paknazar in [3], introduced some new classes of proximal contraction mappings and established best proximity point theorems for such kinds of mappings in a non-Archimedean fuzzy metric space. In this paper, as an application of that results we present some best proximity and fixed point results in such spaces.

Here we recall some basic consepts used in the following.
If $(X, \preceq)$ is a partially ordered set and $T : X \to X$ is such that, for all $x, y \in X$ with $x \preceq y$ implies $Tx \preceq Ty$, then the mapping $T$ is said to be *non-decreasing*.

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**Definition 1.1.** (Schweizer and Sklar [4]) A binary operation $\star : [0,1] \times [0,1] \to [0,1]$ is called a *continuous t-norm* if it satisfies the following assertions:

(T1) $\star$ is commutative and associative;
(T2) $\star$ is continuous;
(T3) $a \star 1 = a$ for all $a \in [0,1]$;
(T4) $a \star b \leq c \star d$ when $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0,1]$.

**Definition 1.2.** (George and Veeramani [5]) A *fuzzy metric space* is an ordered triple $(X, M, \star)$ such that $X$ is a nonempty set, $\star$ is a continuous $t$-norm and $M$ is a fuzzy set on $X \times X \times (0, +\infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

(F1) $M(x, y, t) > 0$;
(F2) $M(x, y, t) = 1$ if and only if $x = y$;
(F3) $M(x, y, t) = M(y, x, t)$;
(F4) $M(x, y, t) \star M(y, z, s) \leq M(x, z, t + s)$;
(F5) $M(x, y, \cdot) : (0, +\infty) \to (0, 1]$ is left continuous.

If we replace (F4) by $M(x, y, t) \star M(y, z, s) \leq M(x, z, \max\{t, s\})$, then the triple $(X, M, \star)$ is called a *non-Archimedean fuzzy metric space.* Note that, each non-Archimedean fuzzy metric space is a fuzzy metric space.

## 2. Main results

In this section we are going to present some best proximity and fixed point results in partially ordered non-Archimedean fuzzy metric spaces.

**Definition 2.1.** Let $A$ and $B$ be nonempty subsets of a partially ordered fuzzy metric space $(X, M, \star, \preceq)$. A mapping $T : A \to B$ is said to be *proximally ordered-preserving* if, for all $x_1, x_2, u_1, u_2 \in A$ and $t > 0$,

$$\begin{cases} x_1 \preceq x_2, \\ M(u_1, Tx_1, t) = M(A, B, t), \\ M(u_2, Tx_2, t) = M(A, B, t) \end{cases} \Longrightarrow u_1 \preceq u_2.$$

Clearly, letting $A = B$, then, if $T : A \to A$ is proximally order-preserving, then $T$ is a non-decreasing mapping.

**Definition 2.2.** Let $A$ and $B$ be nonempty subsets of a partially ordered fuzzy metric space $(X, M, \star, \preceq)$. Let $T : A \to B$ be a non-self mapping. We say that $T$ is an *ordered $\varphi$-$\omega$-proximal contractive mapping* if, for all $x, y, u, v \in A$ and $t > 0$,

$$\begin{cases} x \preceq y, \\ M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t) \end{cases} \Rightarrow \varphi(M(u, v, t)) \leq \omega(t)\varphi(M(x, y, t)),$$

where $\varphi \in \Phi$ and $\omega : (0, +\infty) \to (0, 1)$ is a function.

**Definition 2.3.** Let $A$ and $B$ be nonempty subsets of a partially ordered fuzzy metric space $(X, M, \star, \preceq)$. Let $T : A \to B$ be a non-self mapping. We say that $T$ is an *ordered $\phi$-proximal contractive mapping* if, for all $x, y, u, v \in A$ and $t > 0$,

$$\begin{cases} x \preceq y, \\ M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t) \end{cases} \Rightarrow M(u, v, t) \geq M(x, y, t) + \phi(M(x, y, t)),$$

where $\phi : [0, 1] \to [0, 1]$ is continuous and $\phi(s) > 0$ for all $s \in (0, 1)$.

**Theorem 2.4.** *Let $A$ and $B$ be nonempty closed subsets of a complete partially ordered non-Archimedean fuzzy metric space $(X, M, \star, \preceq)$ such that $A_0(t)$ is nonempty for all $t > 0$. Let $T : A \to B$ be a non-self mapping satisfying the following assertions:*

(i) *$T$ is proximally order-preserving and $T(A_0(t)) \subseteq B_0(t)$, $\forall t > 0$;*
(ii) *$T$ is an ordered $\varphi$-$\omega$-proximal contractive mapping;*
(iii) *if $\{y_n\}$ is a sequence in $B_0(t)$ and $x \in A$, such that $M(x, y_n, t) \to M(A, B, t)$ as $n \to +\infty$, then $x \in A_0(t)$, $\forall t > 0$;*
(iv) *there exist $x_0, x_1 \in A_0(t)$ such that $\forall t > 0$ when $x_0 \preceq x_1$, $M(x_1, Tx_0, t) = M(A, B, t)$;*
(v) *if $\{x_n\}$ is an increasing sequence in $X$ such that $x_n \to x$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \geq 1$.*

*Then there exists $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$, $\forall t > 0$.*

*Proof.* All of the conditions of Theorem 2.6 in [3] hold. So $T$ has a best proximity point and this completes the proof. $\square$

**Corollary 2.5.** *Let $A$ and $B$ be nonempty closed subsets of a complete partially ordered non-Archimedean fuzzy metric space $(X, M, \star, \preceq)$ such that $A_0(t)$ is nonempty for all $t > 0$. Let $T : A \to B$ be a non-self mapping satisfying the following assertions:*

(i) *$T$ is proximally order-preserving and $T(A_0(t)) \subseteq B_0(t)$, $\forall t > 0$;*
(ii) *$T$ is an ordered $\varphi$-$\omega$-proximal contractive mapping;*
(iii) *if $\{y_n\}$ is a sequence in $B_0(t)$ and $x \in A$, such that $M(x, y_n, t) \to M(A, B, t)$ as $n \to +\infty$, then $x \in A_0(t)$, $\forall t > 0$;*
(iv) *there exist $x_0, x_1 \in A_0(t)$ such that such that $\forall t > 0$ when $x_0 \preceq x_1$, $M(x_1, Tx_0, t) = M(A, B, t)$;*
(v) *if $\{x_n\}$ is an increasing sequence in $X$ such that $x_n \to x$ as $n \to +\infty$, then there is a subsequence $\{x_{k_n}\}$ with $x_{k_n} \preceq x$ for all $n \geq 1$.*

*Then there exists $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$, $\forall t > 0$.*

Here we deduce the following fixed point results from the above best proximity results.

**Theorem 2.6.** *Let $(X, M, \star, \preceq)$ be a complete partially ordered non-Archimedean fuzzy metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) *$T$ is a non-decreasing mapping;*
(ii) *$T$ is an ordered $\varphi$-$\omega$-contractive mapping;*
(iii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
(iv) *if $\{x_n\}$ is an increasing sequence in $X$ such that $x_n \to x$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \geq 1$.*

*Then $T$ has a fixed point in $X$.*

**Theorem 2.7.** *Let $(X, M, \star, \preceq)$ be a complete partially ordered non-Archimedean fuzzy metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) *$T$ is a non-decreasing mapping;*
(ii) *$T$ is an ordered $\phi$-contractive mapping;*
(iii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
(iv) *if $\{x_n\}$ is an increasing sequence in $X$ such that $x_n \to x$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \geq 1$.*

*Then $T$ has a fixed point in $X$.*

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# A CLASS OF THREE-STAGE RUNGE-KUTTA METHODS FOR STOCHASTIC DIFFERENTIAL EQUATIONS

M. PARSAMANESH

*Faculty of Mohajer, Technical and Vocational University, Isfahan, Iran*
*mparsamanesh@tvu.ac.ir*

Abstract. A class of explicit Runge-Kutta methods for numerical solution of SDEs is described. These schemes are derivative-free. The order conditions that this class of stochastic Runge-Kutta methods must satisfy to have weak order two is obtained. Examples of the second-order explicit Runge-Kutta methods of this class are shown. Numerical examples are presented to support the theoretical results.

## 1. Introduction

Stochastic Differential Equations can describe realer models because of inserting the noise term in Ordinary Differential Equations. Since many of physical systems have no analytic solution, numerical schemes must be developed to obtain exact and efficient solutions.

A scalar Ito SDE is of the form

$$dX_t = a(t, x_t)dt + b(t, X_t)dW_t,\ X_{t_0} = x_0 \tag{1.1}$$

where $a, b : [t_0, T] \times R \to R$ are the drift and diffusion coefficients, $\{W_t\}_{t_0 \le t \le T}$ represents a one-dimensional standard Wiener process and the initial value $x_0 \in R$ is nonrandom. Suppose that SDE (1.1) satisfies conditions that is required for existence and uniqueness of solution [2].

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## 2. Runge-Kutta methods for SDEs

In this section we generalize the methods that are given in [4] and we propose the following form to obtain stochastic Runge-Kutta methods.

$$X_{n+1} = X_n + (\alpha_1 K_0 + \alpha_2 K_1 + \alpha_3 K_2)\Delta + S_0\Gamma + S_1\Lambda + S_2\Omega, \quad (2.1)$$

with

$$\begin{array}{l} K_0 = a(t_n, X_n), \\ S_0 = b(t_n, X_n), \\ K_1 = a(t_n + \mu_0\Delta, X_n + \lambda_0 K_0\Delta + S_0 L), \\ S_1 = b(t_n + \bar{\mu}_0\Delta, X_n + \bar{\lambda}_0 K_0\Delta + S_0 M), \\ K_2 = a(t_n + \rho_0\Delta, X_n + \varphi_0 K_0\Delta + S_0 N), \\ S_2 = b(t_n + \bar{\rho}_0\Delta, X_n + \bar{\varphi}_0 K_0\Delta + S_0 Q), \end{array} \quad (2.2)$$

where $\Gamma, \Lambda, \Omega$, L, M, N and Q are random variables of mean-square order $\frac{1}{2}$. We seek values for the constants and conditions on these random variables in order to the scheme has order two in the weak sense. For this task we write weak second-order expansion of scheme first and then we compare it with simplified order two Taylor scheme [3]. With this comparison, we have following theorem:

**Theorem 2.1.** *Suppose the coefficients a and b of the SDE (1.1) are continuous, satisfy both Lipschitz and linear growth bound conditions in x, and belong to $\varphi_p^{2\beta+2}$. The Runge-Kutta scheme (2.1)-(2.2) has order two in the weak sense if the following conditions hold:*

$$\begin{array}{l} \alpha_1 + \alpha_2 + \alpha_3 = 1, \\ \alpha_2\mu_0 + \alpha_3\rho_0 = \frac{1}{2}, \\ \alpha_2\lambda_0 + \alpha_3\varphi_0 = \frac{1}{2}, \end{array} \quad (2.3)$$

$$\begin{array}{l} \Delta(\alpha_2 L^2 + \alpha_3 N^2) \overset{(2)}{\simeq} \frac{1}{2}\Delta^2, \\ \Delta(\alpha_2 L + \alpha_3 N) \overset{(2)}{\simeq} \frac{1}{2}\Delta\Delta W_n, \end{array} \quad (2.4)$$

$$\begin{array}{l} \Gamma + \Lambda + \Omega \overset{(2)}{\simeq} \Delta W_n, \\ (\bar{\mu}_0\Lambda + \bar{\rho}_0\Omega)\Delta \overset{(2)}{\simeq} \frac{1}{2}\Delta\Delta W_n, \\ (\bar{\lambda}_0\Lambda + \bar{\varphi}_0\Omega)\Delta \overset{(2)}{\simeq} \frac{1}{2}\Delta\Delta W_n, \\ M\Lambda + Q\Omega \overset{(2)}{\simeq} \frac{1}{2}((\Delta W_n)^2 - \Delta), \\ M^2\Lambda + Q^2\Omega \overset{(2)}{\simeq} \frac{1}{2}\Delta\Delta W_n, \end{array} \quad (2.5)$$

$$\begin{array}{l} (\bar{\mu}_0 M\Lambda + \bar{\rho}_0 Q\Omega)\Delta \overset{(2)}{\simeq} 0, \\ (\bar{\lambda}_0 M\Lambda + \bar{\varphi}_0 Q\Omega)\Delta \overset{(2)}{\simeq} 0. \end{array} \quad (2.6)$$

Now, we can find two follwing schemes that satisfy above order conditions:

**Method SRK1**

$$X_{n+1} = X_n + \frac{1}{6}(K_0 + 2K_1 + K_2)\Delta + \frac{1}{4}(2S_0 + S_1 + S_2)\Delta W_n$$
$$+\frac{1}{4}(S_2 - S_1)(\sqrt{\Delta} - \frac{(\Delta W_n)^2}{\sqrt{\Delta}}),$$

with

$$\begin{aligned}
&K_0 = a(t_n, X_n),\\
&S_0 = b(t_n, X_n),\\
&K_1 = a(t_n + \tfrac{1}{2}\Delta, X_n + \tfrac{1}{2}K_0\Delta + (\tfrac{3}{5} \mp \tfrac{\sqrt{6}}{10})\Delta W_n s_0),\\
&S_1 = b(t_n + \Delta, X_n + K_0\Delta + \sqrt{\Delta}S_0),\\
&K_2 = a(t_n + \Delta, X_n + K_0\Delta + \tfrac{3}{5} \pm \tfrac{2\sqrt{6}}{5}\Delta W_n S_0),\\
&S_2 = b(t_n + \Delta, X_n + K_0\Delta - \sqrt{\Delta}S_0).
\end{aligned}$$

**Method SRK2**

$$X_{n+1} = X_n + \frac{1}{4}(K_0 + K_1 + 2K_2)\Delta + \frac{1}{4}(2S_0 + S_1 + S_2)\Delta W_n$$
$$+\frac{1}{4}(S_2 - S_1)(\sqrt{\Delta} - \frac{(\Delta W_n)^2}{\sqrt{\Delta}}),$$

with

$$\begin{aligned}
&K_0 = a(t_n, X_n),\\
&S_0 = b(t_n, X_n),\\
&K_1 = a(t_n + \tfrac{1}{2}\Delta, X_n + \tfrac{1}{2}K_0\Delta),\\
&S_1 = b(t_n + \Delta, X_n + K_0\Delta + \sqrt{\Delta}S_0),\\
&K_2 = a(t_n + \tfrac{3}{4}\Delta, X_n + \tfrac{3}{4}K_0\Delta + \Delta W_n S_0),\\
&S_2 = b(t_n + \Delta, X_n + K_0\Delta - \sqrt{\Delta}S_0).
\end{aligned}$$

## 3. Numerical examples

We compare methods SRK1, SRK2 with method that proposed by Platen [1]. We consider the nonlinear SDE

$$dX_t = (\frac{1}{3}X_t^{1/3} + 6X_t^{2/3})dt + X_t^{2/3}dW_t, \; X_0 = 1, \tag{3.1}$$

with exact solution $X_t = (2t + 1 + \frac{W_t}{3})^3$.
We approximated the first moment of the solution at point t = 1 via 5000 independent trials. The exact value is $E[X_1] = 28$. The results, summarized in Table 1. We consider also speed of methods and summarize time of evaluations as CPU-time in Table 2.

Table 1. Errors and standard deviations in the approximation of $E[X_1]$ in (3.1).

| $\Delta$ | PLATEN | | SRK1 | | SRK2 | |
|---|---|---|---|---|---|---|
| | error | st. d | error | st. d | error | st. d |
| $2^{-1}$ | 5.874 | 6.11 | 5.197 | 6.26 | 5.291 | 6.24 |
| $2^{-2}$ | 2.182 | 7.99 | 1.845 | 8.09 | 1.888 | 8.08 |
| $2^{-3}$ | 0.772 | 8.71 | 0.649 | 8.75 | 0.664 | 8.75 |
| $2^{-4}$ | 0.229 | 9.23 | 0.191 | 9.24 | 0.196 | 9.24 |
| $2^{-5}$ | 0.029 | 9.17 | 0.019 | 9.17 | 0.020 | 9.17 |

Table 2. CPU-time of evaluations in Table 1.

| $\Delta$ | PLATEN | SRK1 | SRK2 |
|---|---|---|---|
| $2^{-1}$ | 0.078125 | 0.062500 | 0.062500 |
| $2^{-2}$ | 0.109375 | 0.140625 | 0.125000 |
| $2^{-3}$ | 0.171875 | 0.296875 | 0.281250 |
| $2^{-4}$ | 0.359375 | 0.453125 | 0.453125 |
| $2^{-5}$ | 0.718750 | 0.921875 | 0.921875 |

## 4. Conclusions

In this paper we obtained the conditions that proposed stochastic Runge-Kutta method must satisfy to have order two in the weak sense. Results shows that errors obtained by methods SRK1 and SRK2 are less than those by Platen scheme for each step size. Also, CPU-time in large step size for these schemes is less than those by Platen scheme but not in smaller step size because of an extra stage.

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# A NOTE ON MULTIPLICATIVELY LOCAL SPECTRAL SUBSPACE PRESERVING MAPS

R. PARVINIANZADEH

*Department of Mathematics, University of Yasouj, Yasouj 75918, Iran*
*r.parvinian@yu.ac.ir*

Abstract. Let $B(X)$ be the algebra of all bounded linear operators on infinite-dimensional complex Banach space $X$. For $T \in B(X)$ and $\lambda \in \mathbb{C}$, let $X_T(\{\lambda\})$ denotes the local spectral subspace of $T$ associated with $\{\lambda\}$. We investigate the form of all maps $\varphi_1$ and $\varphi_2$ on $B(X)$ such that, for every $T$ and $S$ in $B(X)$, the local spectral subspace of $TS$ and $\varphi_1(T)\varphi_2(S)$ are the same associated with singleton set $\{\lambda\}$. Also, we obtain some interesting results in direction when $X = \mathbb{C}^n$.

## 1. Introduction

Throughout this paper, Let $B(X)$ be the algebra of all bounded linear operators on infinite-dimensional complex Banach space $X$ and its unit will be denoted by $I$. For any vector $x_0 \in X$, let $B_{x_0}(X)$ be the collection of all operators in $B(X)$ vanishing at $x_0$. The local resolvent set, $\rho_T(x)$, of an operator $T \in B(X)$ at some point $x \in X$ is the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood $U$ of $\lambda$ in $\mathbb{C}$ and a $X$-valued analytic function $f : U \longrightarrow X$ such that $(\mu I - T)f(\mu) = x$ for all $\mu \in U$. The complement of local resolvent set is called the local spectrum of $T$ at $x$, denoted by $\sigma_T(x)$, and is obviously a closed subset (possibly empty) of $\sigma(T)$, the spectrum of $T$. For every subset $F \subseteq \mathbb{C}$

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the local spectral subspace $X_T(F)$ is defined by

$$X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}.$$

Clearly, if $F_1 \subseteq F_2$ then $X_T(F_1) \subseteq X_T(F_2)$. For more information about these notions one can see the book [1].

The problem of describing linear or additive maps on $B(X)$ preserving the local spectra has been initiated by Bourhim and Ransford in [5], and continued by several authors; see for instance [3] and the references therein. Motivated by the result from the theory of linear preservers proved by Jafarian and Sourour [7], Dolinar et al. [6], characterised the form of maps preserving the lattice of sum of operators, they showed that maps (not necessarily linear) $\varphi : B(X) \to B(X)$ satisfy $Lat(\varphi(A) + \varphi(B)) = Lat(A + B)$ for all $A, B \in B(X)$, if and only if there is a non zero scalar $\alpha$ and a map $\phi : B(X) \to K$ such that $\varphi(A) = \alpha A + \phi(A)I$ for all $A \in B(X)$. Recall that $X_T(\Omega)$, the local spectral subspace of $T$ associated with a subset $\Omega$ of $\mathbb{C}$, is an element of $Lat(T)$, so one can replace the lattice preserving property by the local spectral subspace preserving property.
In [2], Benbouziane et al. characterized the forms of all maps preserving the local spectral subspace of sum, difference, product and triple product of operators associated with a singleton.
For a vector $x \in X$ and a linear functional $f$ in the dual space $X^*$ of $X$, let $x \otimes f$ stands for the operator of rank at most one defined by

$$(x \otimes f)y = f(y)x, \qquad \forall\ y \in X.$$

We denote $F_1(X)$ the set of all rank-one operators on $X$ and $N_1(X)$ be the set of nilpotent operators in $F_1(X)$. Note that $x \otimes f \in N_1(X)$ if and only if $f(x) = 0$.

The following lemma gives an explicit identification of local spectral subspace in the case of rank-one operator.

**Lemma 1.1.** [5] *Let $R \in F_1(X)$ be a non-nilpotent operator, and let $\lambda$ be a nonzero eigenvalue of $R$. Then $X_R(0) = ker(R)$ and $X_R(\{\lambda\}) = Im(R)$.*

The nonzero local spectrum of $T \in B(X)$ at any $x_0 \in X$ is defined by

$$\sigma_T^*(x_0) := \begin{cases} \{0\} & if\ \sigma_T(x_0) = \{0\}, \\ \sigma_T(T) \setminus \{0\} & if\ \sigma_T(x_0) \neq \{0\}. \end{cases}$$

**Lemma 1.2.** [4] *For a nonzero vector $x_0 \in X$ and a nonzero operator $R \in B(X)$, the following statements are equivalent.*

*(a) $R$ has rank one.*
*(b) $\sigma^*_{RT}(x_0)$ contains at most one element for all $T \in B(X)$.*

In this paper, we investigate the form of all maps $\varphi_1$ and $\varphi_2$ on $B(X)$ such that, for every $T$ and $S$ in $B(X)$, the local spectral subspace of $TS$ and $\varphi_1(T)\varphi_2(S)$ are the same associated with the singleton set $\{\lambda\}$.

## 2. Main results

The following Lemma is a key of the proofs coming after.

**Lemma 2.1.** [2] *Let $x$ be a nonzero vector in $X$ and $T, S \in B(X)$. If $X_T(\{\lambda\}) = X_S(\{\lambda\})$ for all $\lambda \in \mathbb{C}$. Then, $\sigma_T(x) = \{\mu\}$ if and only if $\sigma_S(x) = \{\mu\}$ for all $\mu \in \mathbb{C}$.*

This theorem will be useful in the proofs of the main results.

**Theorem 2.2.** [2] *Let $T, S \in B(X)$. The following statements are equivalent.*
(1) $T = S$
(2) $X_{TR}(\{\lambda\}) = X_{SR}(\{\lambda\})$ *for all* $\lambda \in \mathbb{C}$ *and* $R \in F_1(X)$.

**Theorem 2.3.** *If two surjective linear maps $\varphi_1$ and $\varphi_2$ from $B(X)$ onto $B(X)$ satisfy*

$$X_{\varphi_1(T)\varphi_2(S)}(\{\lambda\}) = X_{TS}(\{\lambda\}), \quad \forall\ T, S \in B(X),\ \forall\ \lambda \in \mathbb{C}$$

*then $\varphi_2$ maps $B_{x_0}(X)$ onto $B_{x_0}(X)$ and there exist two bijective linear mappings $A : X \to X$ and $B : X \to X$ such that*

$$\varphi_1(T) = ATB, \quad (T \in B(X)),$$

*and*

$$\varphi_2(T) = B^{-1}TA^{-1}, \quad (T \notin B_{x_0}(X)).$$

*Proof.* We break down the proof of Theorem into several steps.
Step 1. $\varphi_1$ is bijective.
Step 2. $\varphi_1$ preserves rank one operators in both directions.
step 3. $\varphi_2(B_{x_0}(X)) = B_{x_0}(X)$.
Step 4. There are bijective linear mappings $P : X \to X$ and $Q : X^\star \to X^\star$ such that $\varphi_1(x \otimes f) = Px \otimes Qf$ for all $x \in X$ and $f \in X^\star$.
Step 5. For any $x \in X$ and $f \in X^\star$, we have $f(x) = (Qf)(\varphi_2(I)Px)$
Step 6. $P$ is continuous and $\varphi_2(I)$ is invertible.
Step 7. $\varphi_2(T) = B^{-1}TA^{-1}$ for all $T \notin B_{x_0}(X))$, where $A = \alpha^{-1}P$ for some nonzero scalar $\alpha \in \mathbb{C}$ and $B = (\varphi_2(I)A)^{-1}$.
Step 8. $\varphi_1(T) = ATB$ for every $T \in B(X)$. ∎

In the case $X$ is a finite dimensional space, we have a good description of the concepts involved in local spectral theory; see for instance [8]. Let $M_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices, and for any vector $x_0 \in \mathbb{C}^n$, let $M_{n,x_0}(\mathbb{C})$ be the collection of all matrices in $M_n(\mathbb{C})$ vanishing at $x_0$.

**Remark.** [8]. Let $T \in M_n(\mathbb{C})$ and $\lambda_1, \lambda_2, ..., \lambda_r$ be the distinct eigenvalues of $T$ and denote by $E_1, E_2, ..., E_r$ the corresponding root spaces. We have $\mathbb{C}^n = E_1 \oplus E_2 \oplus ... \oplus E_r$ and $T = T_1 \oplus T_2 \oplus ... \oplus T_r$ where $T_i$ is the restriction of $T$ to $E_i$. It follows that for every $x \in \mathbb{C}^n$,

$$\sigma_T(x) = \bigcup \{\sigma_{T_i}(P_i x) : \ 1 \leq i \leq r\} = \{\lambda_i : \ 1 \leq i \leq r, P_i(x) \neq 0\}$$

where $P_i : \mathbb{C}^n \to E_i$ is the canonical projection.

However, if $X = \mathbb{C}^n$, then the surjectivity $\varphi_1$ and $\varphi_2$ in Theorem 2.3 is redundant, as is shown by our next result.

**Theorem 2.4.** *Two maps $\varphi_1$ and $\varphi_2$ on $M_n(\mathbb{C})$ satisfy*

$$X_{\varphi_1(T)\varphi_2(S)}(\{\lambda\}) = X_{TS}(\{\lambda\}) \quad \forall \ T, S \in M_n(\mathbb{C}), \quad \forall \ \lambda \in \mathbb{C}$$

*if and only if $\varphi_2$ maps $M_{n,x_0}(\mathbb{C})$ into itself and there are two invertible matrices $A$ and $B$ in $\mathbb{C}^n$ such that*

$$\varphi_1(T) = ATB, \quad (T \in M_n(\mathbb{C})),$$

*and*

$$\varphi_2(T) = B^{-1}TA^{-1}, \quad (T \notin M_{n,x_0}(\mathbb{C})).$$

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Poster Presentation

# ON MAPS PRESERVING OPERATORS OF LOCAL SPECTRAL SUBSPACE ASSOCIATED WITH ZERO

R. PARVINIANZADEH

*Department of Mathematics, University of Yasouj, Yasouj 75918, Iran*
*r.parvinian@yu.ac.ir*

ABSTRACT. Let $B(X)$ be the algebra of all bounded linear operators on Banach space $X$. For $T \in B(X)$, let $X_T(\{0\})$ denotes the local spectral subspace of $T$ associated with $\{0\}$. We describe surjective linear maps $\varphi$ on $B(X)$ that satisfy

$$X_{\varphi(T)}(\{0\}) = X_T(\{0\}), \quad \forall\, T \in Q(X).$$

Furthermore, we characterize maps $\varphi$ (not necessarily linear nor surjective) on $B(X)$ which satisfy $X_{\varphi(T)-\varphi(S)}(\{0\}) = X_{T-S}(\{0\})$ for every $T, S \in B(X)$ which $T - S \in Q(X)$.

## 1. INTRODUCTION

Linear preserver problems, in the most general setting, demand the characterization of linear maps between algebras that leave a certain property, a particular relation, or even a subset invariant. This subject is very old and goes back well over a century to the so-called first linear preserver problem, due to Frobenius [6], that determines linear maps preserving the determinant of matrices. Given a Banach space $X$ over the complex field $\mathbb{C}$, we shall denote by $B(X)$ the algebra of all linear bounded operators on $X$ and its unit will be denoted by $I$. The local resolvent set, $\rho_T(x)$, of an operator $T \in B(X)$ at some point $x \in X$ is the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood

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$U$ of $\lambda$ in $\mathbb{C}$ and a $X$-valued analytic function $f : U \longrightarrow X$ such that $(\mu I - T)f(\mu) = x$ for all $\mu \in U$. The complement of local resolvent set is called the local spectrum of $T$ at $x$, denoted by $\sigma_T(x)$, and is obviously a closed subset (possibly empty) of $\sigma(T)$, the spectrum of $T$. The local spectral radius of $T$ at $x$ is given by $r_T(x) := \limsup_{n\longrightarrow\infty} \|T^n(x)\|^{\frac{1}{n}}$, and coincides with the maximum modulus of $\sigma_T(x)$ provided that $T$ has the single-valued extension property. We recall an operator $T \in B(X)$ is said to have the single-valued extension property (henceforth abbreviated to SVEP) provided that for every open subset $U$ of $\mathbb{C}$, the equation

$$(\mu I - T)f(\mu) = 0, \quad \forall \mu \in U,$$

has no nontrivial analytic solution $f$. The set of all operators in $B(X)$ which have the single-valued extension property will be denoted by $Q(X)$. It is easily verified that $T \in B(X)$ has the single-valued extension property if no nonempty open subset of $\mathbb{C}$ is contained in the point spectrum of $T$. In particular, $T \in Q(X)$ if $T$ has no eigenvalues or if the spectrum of $T$ is nowhere dense in $\mathbb{C}$. The notion of SVEP at a point dates back to Finch [5]. For every subset $F \subseteq \mathbb{C}$ the local spectral subspace $X_T(\Omega)$ is defined by

$$X_T(\Omega) = \{x \in X : \sigma_T(x) \subseteq \Omega\}.$$

Clearly, if $\Omega_1 \subseteq \Omega_2$ then $X_T(\Omega_1) \subseteq X_T(\Omega_2)$. For more information about these notions one can see the books [1], [7].

Recently, there has been an upsurge of interest in linear and nonlinear local spectra preserver problems, which demand the characterization of maps on matrices or Banach space operators that leave the local spectra invariant. Bourhim and Ransford were the first ones to consider this type of preserver problem, characterizing in [4] additive maps on the algebra of all linear bounded operators on a complex Banach space $X$ that preserve the local spectrum of operators at each vector of $X$. Their results cleared the way for several authors to describe maps on matrices or operators that preserve local spectrum and local spectral radius; see, for instance, the last section of the survey article [3] and the references therein.

For a vector $x \in X$ and a linear functional $f$ in the dual space $X^*$ of $X$, let $x \otimes f$ stands for the operator of rank at most one defined by

$$(x \otimes f)y = f(y)x, \qquad \forall\ y \in X.$$

We denote $F_1(X)$ the set of all rank-one operators on $X$ and $N_1(X)$ be the set of nilpotent operators in $F_1(X)$. Note that $x \otimes f \in N_1(X)$

if and only if $f(x) = 0$.

The first lemma summarizes some known basic properties of the local spectrum.

**Lemma 1.1.** [1], [7] *Let $X$ be a Banach space and $T \in B(X)$. For every $x, y \in X$ and a scalar $\alpha \in \mathbb{C}$ the following statements hold.*
*(a) If $T$ has SVEP, then $\sigma_T(x) \neq \emptyset$ provided that $x \neq 0$.*
*(b) $\sigma_T(\alpha x) = \sigma_T(x)$ if $\alpha \neq 0$, and $\sigma_{\alpha T}(x) = \alpha\sigma_T(x)$.*
*(c) If $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then $\sigma_T(x) \subseteq \{\lambda\}$. If, further, $x \neq 0$ and $T$ has SVEP, then $\sigma_T(x) = \{\lambda\}$.*
*(d) If $S \in B(X)$ commutes with $T$, then $\sigma_T(Sx) \subseteq \sigma_T(x)$.*
*(e) $\sigma_{T^n}(x) = \{\sigma_T(x)\}^n$ for all $x \in X$ and $n \in \mathbb{N}$.*

We require the following elementary properties of local spectral subspace.

**Lemma 1.2.** [1] *Let $T \in B(X)$ and $F \subseteq \mathbb{C}$, then $X_T(F)$ is a $T$-hyperinvariant subspace of $X$, and*

$$(T - \lambda I)X_T(F) = X_T(F) \qquad \forall\ \lambda \in \mathbb{C}\backslash F.$$

The following Lemma is a key of the proofs coming after.

**Lemma 1.3.** [2] *Let $x$ be a nonzero vector in $X$ and $T, S \in B(X)$. If $X_T(\{\lambda\}) = X_S(\{\lambda\})$ for all $\lambda \in \mathbb{C}$. Then, $\sigma_T(x) = \{\mu\}$ if and only if $\sigma_S(x) = \{\mu\}$ for all $\mu \in \mathbb{C}$.*

In this paper, we describe surjective linear maps $\varphi$ on $B(X)$ that satisfy

$$X_{\varphi(T)}(\{0\}) = X_T(\{0\}), \quad \forall\ T \in Q(X).$$

Furthermore, we characterize maps $\varphi$ (not necessarily linear nor surjective) on $B(X)$ which satisfy $X_{\varphi(T)-\varphi(S)}(\{0\}) = X_{T-S}(\{0\})$ for every $T, S \in B(X)$ which $T - S \in Q(X)$.

## 2. Main results

The following lemma gives an explicit identification of local spectral subspace in the case of rank-one operator.

**Lemma 2.1.** [4] *Let $R \in F_1(X)$ be a non-nilpotent operator, and let $\lambda$ be a nonzero eigenvalue of $R$. Then*
*$X_R(\{0\}) = ker(R)$ and $X_R(\{\lambda\}) = Im(R)$.*

The following Lemma is a key of the proofs next theorem.

**Lemma 2.2.** *Let $\varphi : B(X) \longrightarrow B(X)$ be a surjective linear map. If satisfies one of the following assertion:*
*(a)* $X_{\varphi(T)}(\{0\}) \subseteq X_T(\{0\}), \quad \forall\, T \in Q(X)$
*(b)* $X_T(\{0\}) \subseteq X_{\varphi(T)}(\{0\}), \quad \forall\, T \in Q(X)$
*then there exists a nonzero scalar $\lambda \in \mathbb{C}$ such that $\varphi(I) = \lambda I$, where $I$ stands for the identity operator on $X$.*

**Theorem 2.3.** *Let $\varphi : B(X) \longrightarrow B(X)$ be a surjective linear map. Then the following assertions are equivalent.*
*(a)* $X_{\varphi(T)}(\{0\}) \subseteq X_T(\{0\}), \quad \forall\, T \in Q(X)$
*(b)* $X_T(\{0\}) \subseteq X_{\varphi(T)}(\{0\}), \quad \forall\, T \in Q(X)$
*(c) there exists a nonzero scalar $\lambda \in \mathbb{C}$ such that $\varphi(T) = \lambda T$, for all $T \in Q(X)$.*

In the following lemma and theorem we give a concrete form of maps that preserve the local subspace of the difference of two operators associated with $\{0\}$.

**Lemma 2.4.** *Let $\varphi : B(X) \longrightarrow B(X)$ be a map which satisfies $X_{\varphi(T)-\varphi(S)}(\{0\}) = X_{T-S}(\{0\})$ for every $T, S \in B(X)$ which $T - S \in Q(X)$. Then for every nonzero scalar $\lambda \in \mathbb{C}$, there exists a nonzero scalar $\mu_\lambda \in \mathbb{C}$ such that $\varphi(\lambda I) = \mu_\lambda I + \varphi(0)$.*

**Theorem 2.5.** *Let $\varphi : B(X) \longrightarrow B(X)$ be a map. Then the following assertions are equivalent.*
*(a) $X_{\varphi(T)-\varphi(S)}(\{0\}) = X_{T-S}(\{0\})$ for every $T, S \in B(X)$ which $T-S \in Q(X)$*
*(b) there exists a nonzero scalar $\lambda \in \mathbb{C}$ such that $\varphi(T) = \lambda T + \varphi(0)$, for all $T \in Q(X)$.*

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# ALMOST CONVERGENCE OF ALMOST PERIODIC SEQUENCES IN HADAMARD SPACES

H. KHATIBZADEH AND H. POULADI*

*Department of Mathematics, University of Zanjan, Zanjan, Iran*
*h23pouladi@gmail.com*

Abstract. In this paper, we recall the notion of Karcher mean in Hadamard spaces and prove every almost periodic sequence in a Hadamard space is almost convergent.

## 1. INTRODUCTION

Lorentz in [?] characterized almost convergent scalar sequences in terms of the concept of uniform convergence of the *Vallee-Poussin means*, i.e., the uniform convergence of $\frac{1}{n}\sum_{i=0}^{n-1} x_{k+i}$ respect to $k$, as $n \to \infty$, for the scalar sequence $\{x_n\}$. In [?, ?] we see that every almost periodic scalar sequence is almost convergent. Kurtz in [?] extended this result to the almost periodic vector sequence in Banach spaces. In this paper, we prove almost convergence (respect to the defined mean) of the almost periodic sequences in Hadamard spaces.

## 2. PRELIMINARIES

Let $(X, d)$ be a metric space, a geodesic segment (or geodesic) between two points $x_0, x_1 \in X$, is the image of isometric mapping $\gamma : [0, d(x_0, x_1)] \longrightarrow X$, with $\gamma(0) = x_0, \gamma(d(x_0, x_1)) = x_1$ and $d\big(\gamma(t), \gamma(t')\big) = |t - t'|$ for all $t, t' \in [0, d(x_0, x_1)]$. A metric space $(X, d)$ is said to be a geodesic metric space if every two points of $X$ are joined by a geodesic

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and it is said to be uniquely geodesic if between any two points there is exactly one geodesic that for two arbitrarily points $x_0, x_1$ denoted with $[x_0, x_1]$. All points in $[x_0, x_1]$ are denoted by $x_t = (1-t)x_0 \oplus tx_1$ for all $t \in [0,1]$, where $d(x_t, x_0) = td(x_0, x_1)$ and $d(x_t, x_1) = (1-t)d(x_0, x_1)$. A function $f : X \longrightarrow \mathbb{R}$ is said to be convex on the uniquely geodesic metric space $X$ if for all $x, y \in X$ and all $\lambda \in [0,1]$,

$$f\big((1-\lambda)x \oplus \lambda y\big) \leqslant (1-\lambda)f(x) + \lambda f(y),$$

also $f$ is said to be strongly convex with parameter $\gamma > 0$ if for all $x, y \in X$ and all $\lambda \in [0,1]$,

$$f\big((1-\lambda)x \oplus \lambda y\big) \leqslant (1-\lambda)f(x) + \lambda f(y) - \lambda(1-\lambda)\gamma d^2(x,y).$$

A uniquely geodesic metric space $(X, d)$ is a $CAT(0)$ space if and only if the function $d^2(x, \cdot)$, for all $x$, is strongly convex with $\gamma = 1$, i.e., for every three points $x_0, x_1, y \in X$ and all $0 \leqslant t \leqslant 1$,

$$d^2(y, x_t) \leqslant (1-t)d^2(y, x_0) + td^2(y, x_1) - t(1-t)d^2(x_0, x_1),$$

where $x_t = (1-t)x_0 \oplus tx_1$ for every $t \in [0,1]$.
A $CAT(0)$ space is uniquely geodesic. A complete $CAT(0)$ space is said to be Hadamard space. From now, we denote every Hadamard space by $\mathscr{H}$. Any lower semicontinuous (shortly, lsc), strongly convex function in a Hadamard space has a unique minimizer.

**Definition 2.1.** Given a finite number of points $x_0, \ldots, x_{n-1}$ in a Hadamard space, we define the function

$$\mathcal{F}_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} d^2(x_i, x), \tag{2.1}$$

and for the points $x_k, \cdots, x_{k+n-1}$, we define the function

$$\mathcal{F}_n^k(x) = \frac{1}{n}\sum_{i=0}^{n-1} d^2(x_{k+i}, x). \tag{2.2}$$

From [**?**, Proposition 2.2.17] we know that these functions have unique minimizers. For $\mathcal{F}_n(x)$ (resp. $\mathcal{F}_n^k(x)$) the unique minimizer is denoted by $\sigma_n(x_0, \ldots, x_{n-1})$ or shortly, $\sigma_n$, (resp. $\sigma_n^k(x_k, \ldots, x_{k+n-1})$ or shortly, $\sigma_n^k$) and it is called the mean of $x_0, \ldots, x_{n-1}$ (resp. $x_k, \ldots, x_{k+n-1}$). These mean is known as the Karcher mean of $x_0, \ldots, x_{n-1}$ (resp. $x_k, \ldots, x_{k+n-1}$) (see [**?**]).

The following lemma is a consequence of [**?**, Lemma 2.7] that we need it to prove the main results.

**Lemma 2.2.** *Let $\{x_n\}$ be a sequence in Hadamard space $\mathscr{H}$. Then for $\sigma_n^k$ defined as the above, for each $y \in \mathscr{H}$ and $k \geqslant 1$, we have:*

$$d^2\big(\sigma_n^k, y\big) \leqslant \frac{1}{n}\sum_{i=0}^{n-1} d^2(x_{k+i}, y) - \frac{1}{n}\sum_{i=0}^{n-1} d^2(x_{k+i}, \sigma_n^k).$$

## 3. MAIN RESULTS

**Definition 3.1** (Periodic and Almost Periodic Sequences)**.** Let $\{x_n\}$ be a sequence in metric space $(X, d)$, we call this sequence is periodic with the period $p$ (or $p-$periodic) if there exists a positive integer $p$ such that $x_{n+p} = x_n$ for all $n$. Also, we call it almost periodic if for each $\epsilon > 0$ there are natural numbers $L = L(\epsilon)$ and $N = N(\epsilon)$ such that any interval $(k, k + L)$ where $k \geqslant 0$ contains at least one integer $p$ satisfying

$$d(x_{n+p}, x_n) < \epsilon, \qquad \forall n \geqslant N.$$

We need the following lemma for proving almost convergence of almost periodic sequences in Hadamard spaces.

**Lemma 3.2.** *(see [**?**]) Let $(\mathscr{H}, d)$ be a Hadamard space and $\{f_n^k\}_{k,n}$ be a sequence of convex functions on $\mathscr{H}$. If $\{x_n^k\}_{k,n}$ is a sequence of minimum points of $\{f_n^k\}_{k,n}$ and $x$ is the unique minimizer of the strongly convex function $f$, satisfying:*

(I) *the sequence $\{f_n^k\}$ is pointwise convergent to $f$ as $n$ tends to infinity uniformly in $k \geqslant 0$,*
(II) $\limsup\limits_{n\to\infty} \sup\limits_{k\geqslant 0} \big(f(x_n^k) - f_n^k(x_n^k)\big) \leqslant 0$,

*then $x_n^k$ converges to $x$ uniformly in $k \geqslant 0$ as $n \to \infty$.*

**Theorem 3.3.** *Let $\{x_n\}$ be an almost periodic sequence in Hadamard space $\mathscr{H}$. Then the sequence $\{x_n\}$ is almost convergent.*

***Sketch of the Proof.*** Since $\{x_n\}$ is almost periodic, it is easy to check that for each $x$, $\{d(x_n, x)\}$ is almost periodic, also clearly for each $x$, $\{d^2(x_n, x)\}$ is almost periodic. We know that the scalar sequence $\{d^2(x_n, x)\}$ is almost convergent for all $x \in \mathscr{H}$ [**?**]. Define:

$$\mathcal{F}_n^k(x) := \frac{1}{n}\sum_{i=0}^{n-1} d^2(x_{k+i}, x), \tag{3.1}$$

and

$$\mathcal{F}(x) := \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d^2(x_{k+i}, x) \quad \text{uniformly in } k \geqslant 0. \tag{3.2}$$

Almost convergence of $\{d^2(x_n, x)\}$ for any $x \in \mathscr{H}$ shows that (3.2) is well defined. By the strong convexity of $d^2(\cdot, x)$, the functions $\mathcal{F}_n^k$ and $\mathcal{F}$ are strongly convex and therefore have unique minimizers $\sigma_n^k$ and $\sigma$ respectively. We can show that assumption (II) of Lemma 3.2 holds, then $\mathcal{F}_n^k$ and $\mathcal{F}$ satisfy all assumptions of Lemma 3.2 and hence, $\{x_n\}$ is almost convergent to $\sigma$. □

Every $N$-periodic sequence is almost periodic and by the previous theorem, it is almost convergent. We prove that it almost converges to the mean of its $N$ first points.

**Theorem 3.4.** *Let $\{x_n\}$ be a $N-$periodic sequence in Hadamard space $\mathscr{H}$. Then the sequence $\{x_n\}$ is almost convergent to $\sigma_N$.*

***Sketch of the Proof.*** In Definition 2.1, we see that $\sigma_n$ or the Karcher mean of $x_0, \cdots, x_{n-1}$ is the unique minimizer of the function (2.1) and $\sigma_n^k$ or the karcher mean of $x_k, \cdots, x_{k+n-1}$ is the unique minimizer of the function (2.2). By Lemma 2.2, we have:

$$d^2\left(\sigma_n^k, \sigma_N\right) \leqslant \frac{1}{n} \sum_{i=0}^{n-1} d^2(x_{k+i}, \sigma_N) - \frac{1}{n} \sum_{i=0}^{n-1} d^2(x_{k+i}, \sigma_n^k).$$

Without loss of generality, for all $n$ we can suppose that $n = tN + r$, $0 \leqslant r < N$. Now by $N-$periodicity of the sequence $\{x_n\}$ and the definition of $\sigma_N$ we obtain:

$$\sup_{k \geqslant 0} d^2\left(\sigma_n^k, \sigma_N\right) \leqslant \frac{r}{n} \sup_{k \geqslant 0} \sup_{tN \leqslant i \leqslant tN+r-1} d^2(x_{k+i}, \sigma_N).$$

Now letting $n \to \infty$ completes the proof. □

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# WEIGHTED RIESZ BASES IN G-FUSION FRAMES

GH. RAHIMLOU

*Department of Mathematics, Faculty of Tabriz Branch, Technical and Vocational University (TVU), East Azarbaijan , Iran*
*grahimlou@gmail.com*

Abstract. In this paper, we introduce orthonoramal and Riesz bases for g-fusion frames and show that the weights have basic roles. Next, we prove an effective theorem between frames and g-fusion frames by using an operator.

## 1. Introduction

The Riesz basis has been defined in [2] by the image of orthonormal bases with a bounded bijective operator, but for fusion and generalized frames, there are different strategies [1, 4]. In this paper, we transfer some common properties of g-frames to g-fusion frames which have been defined by authors. Throughout this paper, $H$ and $K$ are separable Hilbert spaces and $\mathcal{B}(H,K)$ is the collection of all bounded linear operators of $H$ into $K$. If $K=H$, then $\mathcal{B}(H,H)$ will be denoted by $\mathcal{B}(H)$. Also, $\pi_V$ is the orthogonal projection from $H$ onto a closed subspace $V\subset H$ and $\{H_j\}_{j\in\mathbb{J}}$ is a sequence of Hilbert spaces, where $\mathbb{J}$ is a subset of $\mathbb{Z}$. It is easy to check that if $u\in\mathcal{B}(H)$ and $V\subset H$ is a closed subspace, then ([3])

$$\pi_V u^*\pi_{\overline{uV}}=\pi_V u^*.$$

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*Key words and phrases.* g-fusion frame, Dual g-fusion frame, gf-complete, gf-orthonormal basis, gf-Riesz basis.

We define the space $\mathscr{H}_2 := (\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell_2}$ by

$$\mathscr{H}_2 = \big\{\{f_j\}_{j\in\mathbb{J}} \ : \ f_j \in H_j, \ \sum_{j\in\mathbb{J}} \|f_j\|^2 < \infty\big\}, \tag{1.1}$$

with the inner product defined by

$$\langle \{f_j\}, \{g_j\}\rangle = \sum_{j\in\mathbb{J}} \langle f_j, g_j\rangle.$$

It is clear that $\mathscr{H}_2$ is a Hilbert space with pointwise operations.

**Definition 1.1.** Let $W = \{W_j\}_{j\in\mathbb{J}}$ be a collection of closed subspaces of $H$, $\{v_j\}_{j\in\mathbb{J}}$ be a family of weights, i.e. $v_j > 0$ and $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in \mathbb{J}$. We say $\Lambda := (W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ is a generalized fusion frame (or g-fusion frame) for $H$ if there exists $0 < A \leq B < \infty$ such that for each $f \in H$

$$A\|f\|^2 \leq \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 \leq B\|f\|^2. \tag{1.2}$$

With the same method of Theorem 3.1.3 and 5.4.1 in [2], we can prove the following results.

**Theorem 1.2.** *$\Lambda$ is a g-fusion Bessel sequence for $H$ with bound $B$ if and only if the operator $T_\Lambda$ is well-defined and bounded operator with $\|T_\lambda\| \leq \sqrt{B}$.*

**Theorem 1.3.** *$\Lambda$ is a g-fusion frame for $H$ if and only if*

$$T_\Lambda : \mathscr{H}_2 \longrightarrow H,$$
$$T_\Lambda(\{f_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}} v_j \pi_{W_j} \Lambda_j^* f_j$$

*is a well-defined, bounded and surjective.*

## 2. Main results

**Definition 2.1.** Let $W = \{W_j\}_{j\in\mathbb{J}}$ be a collection of closed subspaces of $H$ and $j \in \mathbb{J}$. We say that $(W_j, \Lambda_j)_{j\in\mathbb{J}}$ is a g-f-orthonormal bases for $H$ with respect to $\{v_j\}_{j\in\mathbb{J}}$, if

$$\langle v_i \pi_{W_i} \Lambda_i^* g_i, v_j \pi_{W_j} \Lambda_j^* g_j\rangle = \delta_{i,j} \langle g_i, g_j\rangle \ , \quad i, j \in \mathbb{J} \ , \quad g_i \in H_i \ , \quad g_j \in H_j \tag{2.1}$$

$$\sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 = \|f\|^2 \ , \qquad f \in H. \tag{2.2}$$

**Definition 2.2.** $\Lambda = (W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ is called a g-f-Riesz basis for $H$ if

(1) $\Lambda$ is g-f-complete,
(2) There exist $0 < A \leq B < \infty$ such that for each finite subset $\mathbb{I} \subseteq \mathbb{J}$ and $g_j \in H_j$, $j \in \mathbb{I}$,

$$A \sum_{j\in\mathbb{I}} \|g_j\|^2 \leq \Big\| \sum_{j\in\mathbb{I}} v_j \pi_{W_j} \Lambda_j^* g_j \Big\|^2 \leq B \sum_{j\in\mathbb{I}} \|g_j\|^2. \tag{2.3}$$

It is easy to check that if $\Lambda$ is a g-f-Riesz bases for $H$, then the operator $T_\Lambda$ which is defined by

$$T_\Lambda : \mathscr{H}_2 \longrightarrow H$$
$$T_\Lambda(\{g_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}} v_j \pi_{W_j} \Lambda_j^* g_j,$$

is injective.

**Theorem 2.3.** [5] *Let $\Lambda = (W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ be a g-fusion frame for $H$ and suppose that (2.1) holds. Then $\Lambda$ is a g-f-orthonormal basis for $H$.*

**Theorem 2.4.** [5] *$\Lambda$ is a g-f-orthonormal bases for $H$ if and only if*

*(I) $v_j \pi_{W_j} \Lambda_j^*$ is isometric for any $j \in \mathbb{J}$;*
*(II) $\bigoplus_{j\in\mathbb{J}} v_j \pi_{W_j} \Lambda_j^*(H_j) = H$.*

**Corollary 2.5.** *Every g-f-orthonormal bases for $H$ is a g-f-Riesz bases for $H$ with bounds $A = B = 1$.*

**Theorem 2.6.** *Let $\Theta = (W_j, \Theta_j)_{j\in\mathbb{J}}$ be a g-f-orthonormal bases with respect to $\{v_j\}_{j\in\mathbb{J}}$ and $\Lambda = (W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ be a g-fusion frame for $H$ with same weights. Then, there exists a surjective operator $V \in \mathcal{B}(H)$ such that $\Lambda_j \pi_{W_j} = \Theta_j \pi_{W_j} V^*$ for all $j \in \mathbb{J}$.*

**Corollary 2.7.** *If $\Lambda$ is a Parseval g-fusion frame for $H$, then $V^*$ is isometric.*

**Corollary 2.8.** *If $\Lambda$ is a g-f-Riesz bases for $H$, then $V$ is invertible.*

*Proof.* Let $Vf = 0$ and $f \in H$. Since $T_\Lambda$ is injective and

$$Vf = \sum_{j\in\mathbb{J}} v_j^2 \pi_{W_j} \Lambda_j^* \Theta_j \pi_{W_j} f = T_\Lambda T_\Theta^* f,$$

therefore, $T_\Theta^* f = 0$. So, $\|f\|^2 = \|T_\Theta f\|^2 = 0$, hence, $f = 0$. □

**Corollary 2.9.** *If $(W_j, \Lambda_j)_{j\in\mathbb{J}}$ is a g-f-orthonormal bases for $H$ with respect to $\{v_j\}_{j\in\mathbb{J}}$, then $V$ is unitary.*

*Proof.* By Corollaries 2.5 and 2.8, the operator $V$ is invertible. Let $f \in H$, we obtain

$$\|f\|^2 = \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 = \sum_{j\in\mathbb{J}} \|\Theta_j \pi_{W_j} V^* f\|^2 = \|V^* f\|^2.$$

Thus, $VV^* = id_H$ and this means that $V$ is unitary. $\square$

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Oral Presentation

# BOUNDS FOR THE SECOND TOEPLITZ DETERMINANTS OF CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

H. RAHMATAN

*Department of Mathematics, Faculty of Mathematical Sciences, Payame noor University, P.O.Box 19395-3697, Tehran, Iran*
*h.rahmatan@gmail.com*

Abstract. In this paper, we consider the Toeplitz determinants $T_2(1) = 1 - a_2^2$ and $T_2(2) = a_2^2 - a_3^2$ defined for the coefficients of a functions $f$ which belongs to the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$, for some real $\alpha$ $(\alpha > 1)$.

## 1. Introduction

Let $\mathcal{A}$ the class of univalent functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}; |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are normalized univalent in $\mathbb{U}$.
SH. Owa and J. Nishivaki introduced the two subclass $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ of analytic functions $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ in $\mathbb{U}$.
Let $\mathcal{M}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$Re\{\frac{zf'(z)}{f(z)}\} < \alpha \qquad (z \in \mathbb{U}), \tag{1.2}$$

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for some real $\alpha$ $(\alpha > 1)$.
And let $\mathcal{N}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$Re\{1 + \frac{zf''(z)}{f'(z)}\} < \alpha \qquad (z \in \mathbb{U}), \tag{1.3}$$

for some real $\alpha$ $(\alpha > 1)$[3].
Toeplitz determinants are closely related to Hankel determinants [2]. That, Toelitz matrices have constant entries along the diagonal. Toeplitz matrices have some applications in pure and applied mathematics[5]. Thomas and Halim in [4] introduced the symmetric determinant $T_q(n)$ for analytic functions $f$ of the form 1.1 defined by,

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_n & \dots & a_{n+q-2} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n+q-1} & a_{n+q-2} & \dots & a_n \end{vmatrix} \qquad q \in \mathbb{N} \setminus 1, \quad n \in \mathbb{N}.$$

In the particular cases
$q = 2, n = 1, a_1 = 1$ and $q = 2, n = 2$,
the Toeplitz determinant simplifies respectively to

$$T_2(1) = 1 - a_2^2, \; and \; T_2(2) = a_2^2 - a_3^2.$$

In this paper, we seek the bounds for the functional $|1-a_2^2|$ and $|a_2^2-a_3^2|$ for functions belonging to the two classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$, for some real $\alpha$ $(\alpha > 1)$.

## 2. Main results

To prove our main resuts, we need follow lemma and theorems.

**Lemma 2.1** ([1]). *If the function $p \in \mathcal{P}$ is given by the following series,*

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots$$

*then*

$$|p_k| \leq 2, \quad (k = 1, 2, \dots)$$

*where $\mathcal{P}$ be the class of functions with positive real part consisting of analytic functions $p : \mathbb{U} \to \mathcal{C}$ satisfying $p(0) = 1$ and $Re(p(z)) > 0$[1].*

**Theorem 2.2** ([3]). *If $f(z) \in \mathcal{M}(\alpha)$, for some real $\alpha$ $(\alpha > 1)$, then*

$$|a_n| \leq \frac{\prod_{j=2}^{n}(j + 2\alpha - 4)}{(n-1)!}, \qquad (n \geq 2).$$

**Theorem 2.3** ([3]). *If $f(z) \in \mathcal{N}(\alpha)$, for some real $\alpha$ $(\alpha > 1)$, then*

$$|a_n| \leq \frac{\prod_{j=2}^{n}(j + 2\alpha - 4)}{n!}, \qquad (n \geq 2).$$

Our results for the bounds of $|T_2(1)|$ and $|T_2(2)|$ of functions $f$ belonging to the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ are contained in:

**Theorem 2.4.** *If $f(z) \in \mathcal{M}(\alpha)$, then*

$$|T_2(1)| = |1 - a_2^2| \leq 1 + 4(\alpha - 1)^2, \tag{2.1}$$

*and*

$$|T_2(2)| = |a_2^2 - a_3^2| \leq \frac{1}{4}(\alpha - 1)^2(16 + (4\alpha - 2)^2) \tag{2.2}$$

*Proof.* With results of theorem2.2 and lemma2.1, we have,

$$p(z) = \frac{\alpha - \frac{zf'(z)}{f(z)}}{\alpha - 1}$$

for $f(z) \in \mathcal{M}(\alpha)$. Since,

$$\alpha f(z) - zf'(z) = (\alpha - 1)p(z)f(z),$$

which yields the following relations,

$$-a_2 = (\alpha - 1)p_1, \tag{2.3}$$

$$-2a_3 = (\alpha - 1)(p_2 + a_2 p_1). \tag{2.4}$$

Therefore,

$$1 - a_2^2 = 1 - (\alpha - 1)^2 p_1^2,$$

by using triangle inequality and lemma2.1, we got desired estimate on $|T_2(1)|$ as asserted in 2.1.
From 2.3 and 2.4, we obtain,

$$a_2^2 - a_3^2 = \frac{1}{4}(\alpha - 1)^2(4p_1^2 - (p_2 + a_2 p_1)^2), \tag{2.5}$$

since,

$$|p_2 + a_2 p_1| \leq 4\alpha - 2,$$

from 2.5, by using triangle inequality and lemma2.1, we have,

$$|a_2^2 - a_3^2| \leq \frac{1}{4}(\alpha - 1)^2(16 + (4\alpha - 2)^2),$$

this completes the proof of theorem. ∎

**Theorem 2.5.** *If* $f(z)$ *in*$\mathcal{N}(\alpha)$. *then,*

$$|T_2(1)| \leq 1 + (\alpha - 1)^2, \tag{2.6}$$

*and*

$$|T_2(2)| \leq \frac{1}{36}(\alpha - 1)^2(36 + (2\alpha - 1)^2). \tag{2.7}$$

*Proof.* Similar to the proof of Theorem2.4, we obtain desired estimate on $|T_2(1)|$ and $|T_2(2)|$ as asserted in 2.6 and 2.7. □

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# MODIFIED ROPER-SUFFRIDGE EXTENSION OPERATOR FOR SOME SUBCLASSES OF THE UNIT BALL $B^n$

S. RAHROVI* AND H. PIRI

*Department of Mathematics, Basic Science Faculty, University of Bonab, Bonab, Iran.*
*sarahrovi@gmail.com and hossein_piri1979@yahoo.com*

Abstract. Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogenous polynomail of degree 2. We consider the normalized extension of $f$ to the Euclidean unit ball $B^n \subseteq \mathbb{C}^n$ given by $[\Phi_Q(f)](z) = (f(z_1) + f'(z_1)Q(\hat{z}), \sqrt{f'(z_1)}\hat{z})$, where $z = (z_1, \hat{z}) \in B^n$. In this paper, we show that under certain conditions, $\Phi_Q(f)$ preserve almost starlike mappings of complex order $\lambda$.

## 1. Introduction

Let $\mathbb{C}^n$ be the vector space of $n$-complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{k=1}^{n} z_k \overline{w}_k$ and Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. The open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by $B_r^n$ and the unit ball $B_1^n$ by $B^n$. In the case of one complex variable, $B^1$ is denoted by $U$. Let $H(B^n, \mathbb{C}^n)$ denote the topological vector space of all holomorphic mappings $F : B^n \to \mathbb{C}^n$, and let $S_n \subseteq H(B^n, \mathbb{C}^n)$ denotes the family of normalized univalent (one-to-one) mappings. The normalization is $F(0) = 0$, $DF(0) = I_n$, where $DF$ is the Fréchet differential of $F$ and $I_n$ is the identity operator on $\mathbb{C}^n$. Of course, $S_1 = S$ is the classical family of schlicht mappings of $U$. Let $\mathcal{Q}_n$ denote

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the set of all homogenous polynomials $Q : \mathbb{C}^n \to \mathbb{C}$ of degree 2. That is, $Q(\lambda z) = \lambda^2 Q(z)$ for all $z \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. It is well known that $Q_n$ is a Banach space with the norm $\|Q\| = \sup_{z\in\mathbb{C}^n\setminus\{0\}} \frac{|Q(z)|}{\|z\|^2}$, $Q \in \mathcal{Q}_n$. A map $f \in S_n$ is said to be starlike if the image is a starlike domain with respect to 0. We denote the classes of normalized starlike mappings on $B^n$ by $S^*(B^n)$. The following notion of starlikeness of order $\alpha$ was introduced in [1, 2].

**Definition 1.1.** Let $f : B^n \to \mathbb{C}^n$ be a normalized locally biholomorphic mapping and let $\alpha \in [0, 1)$. The mapping $f$ is said to be starlike of order $\alpha$ if $Re\left(\dfrac{\|z\|^2}{\langle [Df(z)]^{-1} f(z), z\rangle}\right) > \alpha$, $z \in B^n\setminus\{0\}$.

*Remark* 1.2. $(i)$ In the case of one complex variable, the above relation is equivalent to $Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha$ for $z \in U$. Let $S^*_\alpha(B^n)$ be the set of starlike mappings of order $\alpha$ on $B^n$. In the case $n = 1$, $S^*_\alpha(B^1)$ is denoted by $S^*_\alpha$. Note that if $f \in S^*_\alpha(B^n)$, then

$$Re\langle [Df(z)]^{-1} f(z), z\rangle > 0, \qquad z \in B^n\setminus\{0\},$$

and thus $f \in S^*(B^n)$ (see [3]).

The following notion of almost starlike mapping of complex order $\lambda$ was introduced by Bălăei and Nechita [4].

**Definition 1.3.** Let $f$ be a normalized locally biholomorphic mapping on $B^n$, and let $\lambda \in \mathbb{C}$, with $Re\lambda \leq 0$. The function $f$ is said to be an almost starlike mapping of complex order $\lambda$ if

$$Re\left((1-\lambda)\langle [Df(z)]^{-1} f(z), z\rangle\right) > -Re\lambda\|z\|^2, \qquad z \in B^n\setminus\{0\}.$$

In 1995, Roper and Suffridge [5] introduced an extension operator. This operator is defined for normalized locally biholomorphic function $f$ on the unit disc $U$ in $\mathbb{C}$ by

$$[\Phi_n(f)](z) = (f(z_1), \sqrt{f'(z_1)}\hat{z}), \qquad f \in S, z \in B^n,$$

where the branch of the power function is chosen so that $\sqrt{f'(z_1)}|_{z_1=0} = 1$.

It is well known that Roper-Suffridge extension operator has the following remarkable properties

$(i)$ if $f$ is a normalized convex function on $U$, then $\Phi_n(f)$ is a normalized convex mapping on $B^n$;

$(ii)$ if $f$ is a normalized starlike function on $U$, then $\Phi_n(f)$ is a normalized starlike mapping on $B^n$.

The above results $(i)$ was proved by Roper and Suffridge when they introduced their operator [5]. However, theirs proof were very complex, while the second result was given by Graham and Kohr [6]. Until now, it is difficult to construct the concrete convex mappings and starlike mappings on $B^n$. By making use of Roper-Suffridge extension operator, we may easily give many concrete examples about these mappings. This is one important reason why people are interested in this extension operator. In 2005, Muir [8] modified the Roper- Suffridge extension operator as follows:

**Definition 1.4.** Let $Q \in \mathcal{Q}_{n-1}$. For any $f \in S$, define the operator $\Phi_Q(f) : B^n \to \mathbb{C}^n$ by

$$[\Phi_Q(f)](z) = \Big(f(z_1) + f'(z_1)Q(\hat{z}), \sqrt{f'(z_1)}\hat{z}\Big), \qquad z = (z_1, \hat{z}) \in B^n, \tag{1.1}$$

we choose the branch of the power function such that $\sqrt{f'(z_1)}|_{z_1=0} = 1$.

Note that when $Q = 0$, the above operator is the well-kown Roper-Suffridge extension operator. It takes a simple calculation to verify that $\Phi_Q(f) \in S_n$ for each $f \in S$, and therefore $\Phi_Q : S \to S_n$ for all $Q \in \mathcal{Q}_{n-1}$. Muir [8] proved that this operator preserves starlikeness and convexity if and only if $\|Q\| \le 1/4$ and $\|Q\| \le 1/2$, respectively. For a function $f \in S$, we introduce the quantity

$$\Lambda_f(z) = \frac{1-|z|^2}{2}\frac{f''(z)}{f'(z)} - \bar{z}, \qquad z \in U. \tag{1.2}$$

The disk automorphism transform is denoted by $\psi$. In other word,

$$\psi(w) = \frac{z-w}{1-\bar{z}w}.$$

Consider the Koebe transform of $f$ with respect to disk automorphism $\psi$ by the form

$$g(w) = \frac{f(\psi(w)) - f(\psi(0))}{f'(\psi(0))\psi'(0)}, \qquad w \in U.$$

Clearly, $g \in S$ and a simple calculation shows that $g''(0) = -2\Lambda_f(z)$. It then follows that $g$ has a power series expansion of the form

$$g(w) = w - \Lambda_f(z)w^2 + \mathcal{O}(|w|^3), \qquad w \in U.$$

The well-known coefficient bound for the second coefficient of a function in $S$ gives $|\Lambda_f(z)| \le 2$, $f \in S$, and $z \in U$. In this paper, we show that on the special conditions the operator $\Phi_Q(f)$ preserves almost starlike mappings of complex order $\lambda$.

In order to prove the main results, we need the following lemmas.

**Lemma 1.5.** [9] *Let $f$ be a holomorphic function on the unit disc $U$. Then $Reh(z) \geq 0$, for every $z \in U$, if and only if there exists an increasing function $\mu$ on $[0, 2\pi]$, which satisfies $\mu(2\pi) - \mu(0) = Reh(0)$, such that $h(z) = \int_0^{2\pi} \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} d\mu(\theta) + iImh(0)$.*

**Lemma 1.6.** [8] *For all $w \in U$, $Re\left(\dfrac{1-|w|^2w^2}{(1-w)^2}\right) \geq \dfrac{1-|w|^4}{(1+|w|)^2}$. In particular, $Re\left(\dfrac{1-|w|^2w^2}{(1-w)^2}\right) \geq \dfrac{1-|w|^2}{2}$.*

**Lemma 1.7.** [7]*(Schwarz-Pick Lemma) Suppose that $g \in H(U)$ satisfies $g(U) \subset U$, then $|g'(\xi)| \leq \frac{1-|g(\xi)|^2}{1-|\xi|^2}$, for each $\xi \in U$.*

## 2. Main Results

We start our main results by the following theorem.

**Theorem 2.1.** *Let $\lambda \in \mathbb{C}$ with $Re\lambda \leq 0$, and $n \geq 2$ also $Q \in \mathcal{Q}_{n-1}$. If $f$ is an almost starlike function of complex order $\lambda$ on the unit disc $U$ and $\|Q\| \leq \frac{1}{4|1-\lambda|}$, then $\Phi_Q(f)$ is an almost starlike mapping of complex order $\lambda$ on $B^n$, where*

$$[\Phi_Q(f)](z) = \left(f(z_1) + f'(z_1)Q(\hat{z}), \sqrt{f'(z_1)}\hat{z}\right), \qquad z = (z_1, \hat{z}) \in B^n.$$

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# MODULE BIPROJECTIVITY AND MODULE BIFLATNESS OF THE FOURIER ALGEBRA OF AN INVERSE SEMIGROUP

R. REZAVAND

*School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Enghelab Ave., Tehran, Iran*
*rezavand@ut.ac.ir*

Abstract. For an inverse semigroup $S$ with the set of idempotents $E$, we find necessary and sufficient conditions for the Fourier algebra $A(S)$ to be module biflat, or module biprojective (as $\ell^1(E)$-module). As a result, when $S$ is either a bicyclic inverse semigroup or a Brandt inverse semigroup, $A(S)$ is module biflat and module biprojective.

## 1. Introduction

The concept of module amenability for a class of Banach algebras has been developed by Amini in [1] and he showed that for every inverse semigroup $S$ with subsemigroup $E$ of idempotents, the $l^1(E)$-module amenability of $^1(S)$ is equivalent to the amenability of $S$. Amini and the author defined the Fourier algebra $A(S)$ of an inverse semigroup $S$ as the predual of semigroup von Neumann algebra $L(S)$ in [3]. The co-algebra structure of $L(S)$ induces a canonical algebra structure on $A(S)$, making it a completely contractive Banach algebra. $A(S)$ has an extra structure of a Banach module on the semigroup algebra $l^1(E)$. It is shown in [3] that if $S$ is an amenable inverse semigroup $S$ with the set of idempotents $E$ and a minimal idempotent, then the Fourier

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algebra $A(S)$ is module operator amenable, as a completely contractive Banach algebra and an operator module over $l^1(E)$.

There are some other concepts related to the notions of module amenability such as module biprojectivity and module biflatness, introduced in [4]. In other words, these concepts are module versions of biprojectivity and biflatness for Banach algebras which are introduced by Helemskii [7]. In this talk which is based on [5], we give necessary and sufficient conditions for the Fourier algebra $A(S)$ to be module biflat, or module biprojective.

1.1. **Module structure.** Let $\mathfrak{A}$ be a Banach algebra and $\mathcal{A}$ be a Banach algebra and a Banach $\mathfrak{A}$-module with compatible actions,

$$\alpha \cdot (ab) = (\alpha \cdot a)b \ \ (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}),$$

and the same for the right action, then we say that $\mathcal{A}$ is an *Banach* $\mathfrak{A}$*-module*.

We know that $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A} \cong (\mathcal{A}\hat{\otimes}\mathcal{A})/I$ where $I$ is the closed ideal generated by elements of the form $a \cdot \alpha \otimes b - a \otimes \alpha \cdot b$, for $\alpha \in \mathfrak{A}$, $a, b \in \mathcal{A}$. We define $\omega : \mathcal{A}\hat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ by $\omega(a \otimes b) = ab$, and $\tilde{\omega} : \mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A} \cong (\mathcal{A}\hat{\otimes}\mathcal{A})/I \longrightarrow \mathcal{A}/J$ by

$$\tilde{\omega}(a \otimes b + I) = ab + J \quad (a, b \in \mathcal{A}),$$

both extended by linearity and continuity where $J = \overline{\langle \omega(I) \rangle}$ is the closed ideal of $\mathcal{A}$ generated by $\omega(I)$. Then $\tilde{\omega}, \tilde{\omega}^{**}$ are $\mathcal{A}$-$\mathfrak{A}$-module homomorphisms [1].

## 2. Main results

Here, we indicate two definitions from [4].

**Definition 2.1.** A Banach algebra $\mathcal{A}$ is called *module biprojective* (as an $\mathfrak{A}$-module) if $\widetilde{\omega}$ has a bounded right inverse which is an $\mathcal{A}/J$-$\mathfrak{A}$-module homomorphism.

**Definition 2.2.** A Banach algebra $\mathcal{A}$ is called *module biflat* (as an $\mathfrak{A}$-module) if $\widetilde{\omega}^*$ has a bounded left inverse which is an $\mathcal{A}/J$-$\mathfrak{A}$-module homomorphism.

**Proposition 2.3.** [5] *Let $\mathfrak{A}$ act trivially on $\mathcal{A}$ from the left and $\mathcal{A}/J$ be a commutative $\mathfrak{A}$-module such that $\mathcal{A}$ be a left (right) Banach essential $\mathfrak{A}$-module. If $\mathcal{A}$ is module biprojective, then $\mathcal{A}/J$ is biprojective.*

**Theorem 2.4.** [5] *Let $\mathcal{A}$ be a Banach algebra with $\mathcal{A}^2 = \mathcal{A}$ and $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from the left. If $\mathcal{A}/J$ is biprojective (biflat), then it is module biprojective (biflat). The converse is true if $\mathcal{A}$ is a left (right) Banach essential $\mathfrak{A}$-module.*

For an inverse semigroup $S$, the equivalence relation defined by: $s \sim t$ if and only if there is $e \in E$ with $es = et$, gives the maximal group homomorphic image $G_S := \{[s] : s \in S\}$ [6]. If $T \subseteq S$ is an inverse subsemigroup of $S$, then $G_{T,S} = \{[t] : t \in T\}$ is a subgroup of $G_S$. We say that two elements $x, y \in S$ are $T$-equivalent, and write $x \sim_T y$ if there is an element $t \in T$ with $xy^* \sim t$. $\sim_T$ is an equivalence relation [5]. We denote the set of all such classes by $S/T$, and define the $E$-index of $T$ in $S$ by $[S : T]_E := \big|S/T\big|$, where the right hand side is the cardinality of $S/T$.

**Lemma 2.5.** [5] $[S : T]_E = [G_S : G_{T,S}]$.

We say that $T$ is $E$-*a*belian if $st \sim ts$, for each $s, t \in T$. In this case, the subgroup $G_{T,S} \leq G_S$ is abelian. We say that $S$ is *a*lmost $E$-abelian, if it has a subsemigroup $T$ of finite $E$-index which is $E$-abelian.

**Proposition 2.6.** [5] *$S$ is almost $E$-abelian if and only if $G_S$ is almost abelian.*

It is shown in [10] that for a locally compact group $G$, $A(G)$ is biprojective if and only if $G$ is discrete and almost abelian. Also, it is proved in [10] that the module biflatness of $A(G)$ implies that $G$ is almost abelian. In the following theorem, which is our main result, we find the necessary and sufficient conditions for $A(S)$ to be module biprojective and module biflat.

**Theorem 2.7.** [5] *Let $S$ be an inverse semigroup with the set of idempotents $E$.*

(i) *$A(S)$ is module biprojective if and only if $S$ is almost $E$-abelian.*
(ii) *When $S$ is amenable, then $A(S)$ is module biflat if and only if $S$ is almost $E$-abelian.*

**Example 2.8.** [5] Let $\mathcal{S}$ be the bicyclic inverse semigroup generated by $p$ and $q$, that is

$$\mathcal{S} = \{p^a q^b : a, b \geq 0\}, \;\; (p^a q^b)^* = p^b q^a.$$

The multiplication operation is defined by

$$(p^a q^b)(p^c q^d) = p^{a-b+max\{b,c\}} q^{d-c+max\{b,c\}}.$$

The set of idempotents of $\mathcal{S}$ is $E_{\mathcal{S}} = \{p^a q^a : a = 0, 1, ...\}$ which is also totally ordered with the following order

$$p^a q^a \leq p^b q^b \Longleftrightarrow a \leq b.$$

Since $ts$ and $st$ are idempotents, $ts \sim st \sim e$ and hence $S$ is $E_{\mathcal{S}}$-abelian. On the other hand $[\mathcal{S} : \mathcal{S}]_E = 1$. Thus, $\mathcal{S}$ is almost $E_{\mathcal{S}}$-abelian. Also,

$G_{\mathcal{S}}$ is isomorphic to the group of integers $\mathbb{Z}$ [2]. Since $\mathcal{S}$ is amenable [6], $A(\mathcal{S})$ is module biprojective and module biflat by Theorem 2.7.

**Example 2.9.** [5] Let $G$ be a group with identity $e$, and let $I$ be a non-empty set (finite or infinite). Then, the Brandt inverse semigroup corresponding to $G$ and $I$, denoted by $S = \mathcal{M}(G, I)$, is the collection of all $I \times I$ matrices $(g)_{ij}$ with $g \in G$ in the $(i, j)^{\text{th}}$ place and 0 (zero) elsewhere and the $I \times I$ zero matrix 0. Multiplication in $S$ is given by the formula

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \qquad (g, h \in G,\, i, j, k, l \in I),$$

and $(g)^*_{ij} = (g^{-1})_{ji}$ and $0^* = 0$. The set of all idempotents is $E_S = \{(e)_{ii} : i \in I\} \bigcup \{0\}$. It is shown in [9, Example 3.2] that $(g)_{ij} \sim 0$ for all $g \in G$ and $i, j \in J$ and hence $G_S$ is the trivial group. This means that $S$ is almost $E_S$-abelian. Besides, $[S : S]_E = 1$. Due to the amenability of $S$ [6], $A(S)$ is module biprojective and module biflat by Theorem 2.7.

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# A BIHARMONIC INCLUSION PROBLEM ON THE HEISENBERG GROUP

F. SAFARI* AND A. RAZANI

*Department of Pure Mathematics, Faculty of Science,*
*Imam Khomeini International University, Qazvin, Iran.*
*f.safari@edu.ikiu.ac.ir and razani@sci.ikiu.ac.ir*

Abstract. The aim of this note is to prove the solvability of a biharmonic problem of a partial differential inclusion with Dirichlet boundary conditions on the Heisenberg group applying variational technique.

## 1. Introduction

We consider the following inclusion problem:

$$\begin{cases} -\Delta^2_{\mathbb{H}^n} u \in \mathcal{F}(\xi, u) & in\ \Omega, \\ \frac{\partial u}{\partial n} = 0 & on\ \partial\Omega, \\ u = 0 & on\ \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{H}^n$ is the unit Korányi ball and $\mathcal{F} : \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a real multifunction (set value mapping) with the following assertions:

(†) $\mathcal{F}$ is upper Carathéodory multifunction;

(††) For every $\tau \in \mathbb{R}$, there exist $w_1 \in L^p(\Omega), w_2 \in L^{\frac{p}{1-\sigma}}(\Omega)$ such that

$$|f| \leq w_1(\xi) + w_2(\xi)|\tau|^\sigma, \tag{1.2}$$

for a.e. $\xi \in \Omega$ and $f \in \mathcal{F}(\xi, \tau)$, where $2 \leq p < Q^*$, $0 \leq \sigma < 1$,

$\mathcal{P}(\mathbb{R}) := \{E \subset \mathbb{R} : E$ is nonempty, compact, and convex subset of $\mathbb{R}\}$.

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The notations that we use are standard but we bring some of them which are essetial tools in the next section.

Through this note $\Omega$ is the unit Korányi ball centered at the origin and by $\nabla_{\mathbb{H}^n}$ and $\Delta^2_{\mathbb{H}^n}$ we denote the Heisenberg gradient and the Heisenberg biharmonic operator on $\mathbb{H}^n$, respectively. We consider the Heisenberg Sobolev space $H^1(\Omega) := HW^{1,2}(\Omega)$ and $H^1_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the norm $\|u\|_* = (\int_\Omega |\nabla_{\mathbb{H}^n} u|^2 d\xi)^{\frac{1}{2}}$. Also we set $H^2(\Omega) := HW^{2,2}(\Omega)$ and we denote by $H^2_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ with respect to the norm $\|u\| = (\int_\Omega |\Delta_{\mathbb{H}^n} u|^2 d\xi)^{\frac{1}{2}}$. Actually, $H^{-2}(\Omega)$ is the dual space to $H^2_0(\Omega)$. The first theorem consists of compact embeddings established in [2]:

**Theorem 1.1.** *The following embeddings are compact:*

(i) *If $Q = 4$, then $H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow L^p(\Omega)$, $1 \le p < +\infty$.*
(ii) *If $Q > 4$, then $H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow L^p(\Omega)$, $1 \le p < Q^*$. where $Q^* = \frac{2Q}{Q-2}$ is the critical Sobolev exponent of $Q = 2n + 2$.*

*See more details about Heisenberg groups in* [5, 6, 7, 8].

**Definition 1.2.** (Upper Carathéodory) $F : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ is called an upper Carathéodory multifunction if $F(., \tau) : \Omega \to 2^{\mathbb{R}}$ is measurable for each $\tau \in \mathbb{R}$ and $F(\xi, .) : \mathbb{R} \to 2^{\mathbb{R}}$ is upper semi-continuous for (almost) each $\xi \in \Omega$.

By $S = S(\Omega)$ we denote the set of all (classes of) Lebesgue-measurable real functions on $\Omega$ and by $\mathcal{N}_F(u)(\xi)$ the Nemytskii operator. Next lemma is a fact in Section 8 of [4].

**Lemma 1.3.** *If $F : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ is an upper Carathéodory multifunction, then $S \cap \mathcal{N}_F(u) \neq \emptyset$.*

For the last theorem of this section we need some assumptions:
Let $\psi : \Omega \times X \to \mathbb{R}$ be a function, such that

(i) for every $x \in X$ , the function $\Omega \ni \omega \to \psi(\omega, x)$ is measurable.
(ii) for any bounded subset $B \subset X$, there exists $k_B \in L^1(\Omega)$, such that for almost all $\omega \in \Omega$ and all $x, y \in B$ we have

$$|\psi(\omega, x) - \psi(\omega, y)| \le k_B(\omega)\|x - y\|_X.$$

Consider the integral functional $\Psi : X \to \mathbb{R}$ defined by $\Psi(x) := \int_\Omega \psi(\omega, x) d\mu$. The next is Theorem 1.3.9 of [3].

**Theorem 1.4.** *If $\Psi$ satisfies in above hypotheses and $\Psi$ is finite at some point $x \in X$, then $\Psi$ is finite, it is Lipschitz on every bounded subset of $X$ and $\partial\Psi(y) \subseteq \int_\Omega \partial\psi(\omega, y) d\mu$, for all $y \in X$.*

## 2. Proof of the main result

Let us start this section by defining what we mean by a solution to the problem (1.1). Assume that $\mathcal{N}_{\mathcal{F}}^{p} : H_0^2(\Omega) \to H^{-2}(\Omega)$ is the multivalued Nemytskii operator corresponding to $\mathcal{F}$ defined by

$$\mathcal{N}_{\mathcal{F}}^{p}(u) := \{v \in L^p(\Omega) : v \in \mathcal{F}(\xi, u) \text{ a.e. on } \Omega\}, \text{ for } p \geq 2.$$

**Definition 2.1.** (Weak Solution) We say that $u \in H_0^2(\Omega)$ is a (weak) solution of problem (1.1), if there exists $v \in \mathcal{N}_{\mathcal{F}}^{p}(u)$ such that

$$\int_{\Omega} \Delta_{\mathbb{H}^n} u \Delta_{\mathbb{H}^n} w d\xi = \langle v, w \rangle,$$

for all $w \in H_0^2(\Omega)$, where $\langle ., . \rangle$ denotes the duality pairing between $H_0^{-2}(\Omega)$ and $H_0^2(\Omega)$.

Notice that Definition 2.1 is quite natural, although it does not lead to a regularity theory, as in general the function $v \in L^p(\Omega)$ remain undetermined.
**step1:** We show that for $u \in H_0^2(\Omega)$, we have $\mathcal{N}_{\mathcal{F}}^{p}(u) \neq \emptyset$.
Applying Lemma 1.3, there exists a measurable selection $v : \Omega \to \mathbb{R}$ of $\mathcal{F}$ such that $v(\xi) \in \mathcal{F}(\xi, u)$. By hypothesis $(\dagger\dagger)$, $v \in L^p(\Omega)$;
But $L^p(\Omega) \subset H_0^{-2}(\Omega)$, and with similar method as applied in [1, Theorem 2.12], for each $f \in H_0^{-2}(\Omega)$ the problem

$$\begin{cases} -\Delta_{\mathbb{H}^n}^2 u = f(\xi) & in\ \Omega, \\ \frac{\partial u}{\partial n} = 0 & on\ \partial\Omega \\ u = 0 & on\ \partial\Omega. \end{cases} \tag{2.1}$$

has only one solution $u \in H_0^2(\Omega)$. Therefore, there exists one to one correspodence between $u \in H_0^2(\Omega)$ and $v \in \mathcal{N}_{\mathcal{F}}^{p}(u)$ defined as above.
**step2:** We define appropriate energy fuctional to the problem.
For $u \in H_0^2(\Omega)$, we introduce the function $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ by $\phi(\xi, u(\xi)) := v(\xi)$, where $v \in \mathcal{N}_{\mathcal{F}}^{p}(u)$ is the measurable selection associated with $u$ as mentioned above. Also we can define locally Lipschitz continuous potential $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ by setting $\Phi(\xi, \tau) = \int_0^\tau \phi(\xi, \eta) d\eta$, for all $\tau \in \mathbb{R}$.
We can now define an energy functional for the problem by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\Omega} \Phi(\xi, u(\xi)) d\xi. \tag{2.2}$$

for all $u \in \mathbb{X} := H_0^2(\Omega) \cap H_0^1(\Omega)$.
**step3:** To show that $I$ has a critical point ($K(I) \neq \emptyset$) and its critical points are weak solutions of the problem, we check the conditions of the non-smooth version of mountain pass theorem (MPT) ([3, Theorem 2.1.3]) in the following two lemmas:

**Lemma 2.2.** *The functional $I$ defined in (2.2) is locally Lipschitz continuous in $\mathbb{X}$, and if $u \in K(I)$, then $u$ is a solution of problem (1.1). Moreover, we have*

$$\partial I(u) \subseteq \Delta^2_{\mathbb{H}^n} u - \int_\Omega \partial \Phi(\xi, u(\xi)) d\xi.$$

Also, $I$ satisfies a **(PS)**-type pre-compactness property:

**Lemma 2.3.** *Let $(u_n)$ be a bounded sequence in $H^2_0(\Omega)$ such that $(I(u_n))$ is bounded and $m_I(u_n) \to 0$. Then, $(u_n)$ has a convergent subsequence.*

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# ON A GENERALIZED NOTION OF PSEUDO-AMENABILITY FOR CERTAIN BANACH ALGEBRAS

AMIR SAHAMI* AND SHAHAB KALANTARI

*Department of Mathematics Faculty of Basic Science, Ilam University, P.O. Box 69315-516 Ilam, Iran.*
*a.sahami@ilam.ac.ir*
*Department of Basic Sience, Babol Noshirvani University of Technology, Shariati Ave., Babol 47148-71167, Iran.*
*shahab.kalantari@nit.ac.ir*

Abstract. Strong pseudo-amenability is introduced and the properties of this generalized notion of pseudo-amenability is discussed. As an application, we investigate strong pseeudo-amenability of some semigroup algebras. For instance, we show that for a Brandt semigroup $S = M^0(G, I)$, $\ell^1(S)$ is strong pseudo-amenable if and only if $G$ is amenable and $I$ is finite. Also strong pseudo-amenability of some Munn algebras is studied here.

## 1. Introduction

Johnson introduced the class of amenable Banach algebras. A Banach algebra $A$ is called amenable, if there exists a bounded net $(m_\alpha)$ in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\pi_A(m_\alpha)a \to a$ for every $a \in A$. For further information about the history of amenability see [7].

By removing the boundedness condition in the definition of amenability, Ghahramani and Zhang in [5] introduced and studied two generalized notions of amenability, named pseudo-amenability and pseudo-contractibility. A Banach algebra $A$ is called pseudo-amenable (pseudo-contractible) if there exists a not necessarily bounded net $(m_\alpha)$ in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \to$

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0 $(a\cdot m_\alpha = m_\alpha\cdot a)$ and $\pi_A(m_\alpha)a\to a$ for every $a\in A$, respectively. Motivated by these considerations, we introduce a notion of amenability (say strong pseudo-amenability) that on $\ell^1(S)$ implies that $G$ is amenable and $I$ is finite, where $S=M^0(G,I)$ is the Brandt semigroup over an index set $I$. In fact we show that the strong pseudo-amenability stands between two notions pseudo-contractibility and pseudo-amenability on some semigroup algebras.

Here is the definition of our new notion:

**Definition 1.1.** A Banach algebra $A$ is called *strong pseudo-amenable*, if there exists a (not necessarily bounded) net $(m_\alpha)_\alpha$ in $(A\otimes_p A)^{**}$ such that

$$a\cdot m_\alpha - m_\alpha\cdot a\to 0,\quad a\pi_A^{**}(m_\alpha)=\pi_A^{**}(m_\alpha)a\to a\qquad (a\in A).$$

Note that every commutative pseudo-amenable Banach algebra is strong pseudo-amenable. Then the class of strong pseudo-amenable Banach algebra is wide enough.

We present some standard notations and definitions that we shall need in this paper. Let $A$ be a Banach algebra. If $X$ is a Banach $A$-bimodule, then $X^*$ is also a Banach $A$-bimodule via the following actions

$$(a\cdot f)(x)=f(x\cdot a),\quad (f\cdot a)(x)=f(a\cdot x)\qquad (a\in A, x\in X, f\in X^*).$$

Let $A$ and $B$ be Banach algebras. The projective tensor product $A\otimes_p B$ with the following multiplication is a Banach algebra

$$(a_1\otimes b_1)(a_2\otimes b_2)=a_1a_2\otimes b_1b_2\quad (a_1,a_2\in A, b_1b_2\in B).$$

Also $A\otimes_p A$ with the following action becomes a Banach $A-$bimodule:

$$a_1\cdot a_2\otimes a_3=a_1a_2\otimes a_3,\quad a_2\otimes a_3\cdot a_1=a_2\otimes a_3a_1,\quad (a_1,a_2,a_3\in A).$$

The product morphism $\pi_A: A\otimes_p A\to A$ is specified by $\pi_A(a\otimes b)=ab$ for every $a,b\in A$.

## 2. Some properties of strong-pseudo amenability

This section is devoted to the general properties of strong-pseudo amenability and its applications.

**Proposition 2.1.** *Let $A$ be a strong pseudo-amenable Banach algebra. Then $A$ is pseudo-amenable.*

Let $A$ be a Banach algebra and $\phi\in\Delta(A)$. A Banach algebra $A$ is called approximately left $\phi-$amenable, if there exists a (not necessarily bounded) net $(m_\alpha)$ in $A$ such that

$$am_\alpha-\phi(a)m_\alpha\to 0,\quad \phi(m_\alpha)\to 1,\qquad (a\in A).$$

For further information see [1].

**Corollary 2.2.** *Let $A$ be a Banach algebra and $\phi \in \Delta(A)$. If $A$ is pseudo-amenable, then $A$ is approximately left $\phi$-amenable.*

A Banach algebra $A$ is called biflat if there exists a bounded $A$-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ such that $\pi_A^{**} \circ \rho(a) = a$ for each $a \in A$ [7].

**Lemma 2.3.** *Let $A$ be a biflat Banach algebra with a central approximate identity. Then $A$ is strong pseudo-amenable.*

**Lemma 2.4.** *Let $A$ and $B$ be Banach algebras. Suppose that $B$ has a non-zero idempotent. If $A \otimes_p B$ is strong pseudo-amenable, then $A$ is strong pseudo-amenable.*

Let $A$ be a Banach algebra and $I$ be a totally ordered set. The set of $I \times I$ upper triangular matrices, with entries from $A$ and the usual matrix operations and also finite $\ell^1$-norm, is a Banach algebra and it denotes with $UP(I, A)$.

**Theorem 2.5.** *Let $I$ be a totally ordered set with smallest element and let $A$ be a Banach algebra with $\phi \in \Delta(A)$. Then $UP(I, A)$ is pseudo-amenable if and only if $A$ is pseudo-amenable and $|I| = 1$.*

Suppose that $A$ is a Banach algebra and $I$ is a non-empty set. We denote $M_I(A)$ for the Banach algebra of $I \times I$-matrices over $A$, with the finite $\ell^1$-norm and the matrix multiplication. This class of Banach algebras belongs to $\ell^1$-Munn algebras, see [2]. We also denote $\varepsilon_{i,j}$ for a matrix belongs to $M_I(\mathbb{C})$ which $(i, j)$-entry is 1 and 0 elsewhere. The map $\theta : M_I(A) \to A \otimes_p M_I(\mathbb{C})$ is defined by $\theta((a_{i,j})) = \sum_{i,j} a_{i,j} \otimes \varepsilon_{i,j}$ is an isometric algebra isomorphism.

**Theorem 2.6.** *Let $I$ be a non-empty set. Then $M_I(\mathbb{C})$ is strong pseudo-amenable if and only if $I$ is finite.*

*Remark* 2.7. We give a pseudo-amenable Banach algebra which is not strong pseudo-amenable.

Let $I$ be an infinite set. Using [6, Proposition 2.7], $M_I(\mathbb{C})$ is biflat. By [4, Proposition 3.6], $M_I(\mathbb{C})$ has an approximate identity. Then [4, Proposition 3.5] implies that $M_I(\mathbb{C})$ is pseudo-amenable. But by previous theorem $M_I(\mathbb{C})$ is not strong pseudo-amenable.

For a locally compact group $G$ and a non-empty set $I$, set

$$M^0(G, I) = \{(g)_{i,j} : g \in G, i, j \in I\} \cup \{0\},$$

where $(g)_{i,j}$ denotes the $I \times I$ matrix with $g$ in $(i, j)$-position and zero elsewhere. With the following multiplication $M^0(G, I)$ becomes a semigroup

$$(g)_{i,j} * (h)_{k,l} = \begin{cases} (gh)_{il} & j = k \\ 0 & j \neq k, \end{cases}$$

It is well known that $M^0(G,I)$ is an inverse semigroup with $(g)^*_{i,j} = (g^{-1})_{j,i}$. This semigroup is called Brandt semigroup over $G$ with index set $I$, which by the arguments as in [4, Corollary 3.8], $M^0(G,I)$ becomes a uniformly locally finite inverse semigroup.

**Corollary 2.8.** *Let $S = M^0(G,I)$ be a Brandt semigroup. Then the following are equivalent:*

(i) *$\ell^1(S)$ is strong pseudo-amenable;*
(ii) *$G$ is amenable and $I$ is finite.*

*Remark* 2.9. There exists a pseudo-amenable semigroup algebra which is not strong pseudo-amenable.

To see this, let $G$ be an amenable locally compact group. Suppose that $I$ is an infinite set. By [4, Corollary 3.8] $\ell^1(S)$ is pseudo-amenable but using Corollary 2.8 implies that $\ell^1(S)$ is not strong pseudo-amenable, whenever $S = M^0(G,I)$ is a Brandt semigroup.

Also there exists a strong pseudo-amenable semigroup algebra which is not pseudo-contractible.

To see this, let $G$ be an infinite amenable group. Suppose that $I$ is a finite set. By Corollary 2.8 $\ell^1(S)$ is strong pseudo-amenable but [3, Corollary 2.5] implies that $\ell^1(S)$ is not pseudo-contractible, whenever $S = M^0(G,I)$ is the Brandt semigroup.

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# *BSE*-PROPERTIES OF DALES-DAVIE ALGEBRAS

A.H. SANATPOUR* AND Z.S. HOSSEINI

*Department of Mathematics, Kharazmi University, Tehran, Iran*
*a_sanatpour@khu.ac.ir; a.sanatpour@gmail.com*
*Department of Mathematics, Bu-Ali Sina University, Hamedan, Iran*
*z.hoseini@basu.ac.ir; zahrahoseinizh3@gmail.com*

Abstract. In this paper we investigate conditions under which Dales-Davie algebras are $BSE$-algebras. We also study $BSE$-properties of infinitely differentiable Lipschitz algebras.

## 1. Introduction

Let $X$ be a perfect compact plane set. A complex-valued function $f : X \to \mathbb{C}$ is called *differentiable* on $X$ if at each point $z_0 \in X$ the following limit exists

$$f'(z_0) = \lim_{\substack{z \to z_0 \\ z \in X}} \frac{f(z) - f(z_0)}{z - z_0}.$$

For each $n \in \mathbb{N}$, we denote the set of all $n$-times continuously differentiable complex functions on $X$ by $D^n(X)$, and the set of all infinitely complex-differentiable functions on $X$ by $D^\infty(X)$.

**Definition 1.1.** Suppose that $M = \{M_n\}_{n=0}^\infty$ is a sequence of positive numbers such that $M_0 = 1$.

($i$) The sequence $M$ is called an *algebra sequence* if for all $m, n \in \mathbb{N}$

$$\frac{(m+n)!}{M_{m+n}} \leq \frac{n!}{M_n} \frac{m!}{M_m}.$$

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$(ii)$ The algebra sequence $M$ is called *non-analytic* if

$$d(M) = \lim_{n\to\infty} \left(\frac{n!}{M_n}\right)^{\frac{1}{n}} = 0.$$

**Definition 1.2.** Let $X$ be a perfect compact plane set and $M$ be an algebra sequence. The *Dales-Davie* algebra associated with $X$ and $M$ is defined by

$$D(X,M) = \left\{ f \in D^{\infty}(X) \ : \ \|f\|_{D(X,M)} = \sum_{k=0}^{\infty} \frac{\|f^{(k)}\|_X}{M_k} < \infty \right\},$$

where $\|\cdot\|_X$ denotes the uniform sup-norm on $X$ [1, 2].

Let $\mathcal{A}$ be a commutative Banach algebra with maximal ideal space $\Phi_{\mathcal{A}}$ and $C_0(\Phi_{\mathcal{A}})$ denote the space of all continuous functions on $\Phi_{\mathcal{A}}$ vanishing at infinity. The algebra $\mathcal{A}$ is embedded in $C_0(\Phi_{\mathcal{A}})$ by considering the Gelfand transform $a \mapsto \widehat{a}$, where $\widehat{a}(\varphi) = \varphi(a)$ for each $\varphi \in \Phi_{\mathcal{A}}$. A commutative Banach algebra $\mathcal{A}$ is called *without order* if $a \in \mathcal{A}$ and $a\mathcal{A} = \{0\}$ implies that $a = 0$. Given a without order commutative Banach algebra $\mathcal{A}$, a bounded linear operator $T : \mathcal{A} \to \mathcal{A}$ is called a *multiplier* if $a(Tb) = T(ab)$ for all $a, b \in \mathcal{A}$. The set of all multipliers on $\mathcal{A}$ is denoted by $M(\mathcal{A})$ which is a commutative unital Banach subalgebra of $\mathcal{B}(\mathcal{A})$, the space of all bounded linear operators on $\mathcal{A}$ [5]. Larsen in [5] proved that for every $T \in M(\mathcal{A})$ there exists a unique bounded continuous function $\widehat{T}$ on $\Phi_{\mathcal{A}}$ such that $\widehat{(Tx)} = \widehat{T}\widehat{x}$ for all $x \in \mathcal{A}$.

A bounded continuous function $\sigma$ on $\Phi_{\mathcal{A}}$ is called a $BSE$-function if the following condition satisfies: there exists a constant $\beta > 0$ such that for any finite numbers of $\varphi_1, \varphi_2, \ldots, \varphi_n$ in $\Phi_{\mathcal{A}}$ and complex numbers $c_1, c_2, \ldots, c_n$, the inequality

$$\left| \sum_{i=1}^{n} c_i \sigma(\varphi_i) \right| \leq \beta \left\| \sum_{i=1}^{n} c_i \varphi_i \right\|_{\mathcal{A}^*}$$

holds. The $BSE$-norm of $\sigma$ is defined to be the infimum of all such $\beta$ in the above inequality and $C_{BSE}(\Phi_{\mathcal{A}})$ denotes the set of all $BSE$-functions. The next definition is given by Takahasi and Hatori in [6].

**Definition 1.3.** A without order commutative Banach algebra $\mathcal{A}$ is called a $BSE$-*algebra* if $\widehat{M(\mathcal{A})} = C_{BSE}(\Phi_{\mathcal{A}})$.

Bochner and Schoenberg in 1934 studied these algebras on the real line and then Eberlein in 1955 gave the extension for locally compact abelian groups $G$. Takahasi, Hatori, Kaniuth, Ulger and some other mathematicians studied this topic for the commutative Banach algebras, Banach function algebras and some other well-known algebras [4, 6]. In this paper we study $BSE$-properties of Dales-Davie algebras and Lipschitz version of Dales-Davie algebras.

## 2. Main results

Dales-Davie algebras are often incomplete even for fairly nice plane sets $X$. It is known that completeness of $D^1(X)$ is a sufficient condition for the completeness of $D^n(X)$ $(n \in \mathbb{N})$ and $D(X, M)$. We next introduce the type of compact plane sets $X$ for which $D^1(X)$ is complete and we shall consider in this paper.

**Definition 2.1.** Let $X$ be a compact plane set which is connected by rectifiable arcs and let $\delta(z, w)$ be the geodesic metric on $X$, the infimum of the lengths of arcs joining $z$ and $w$.

- $(i)$ $X$ is *pointwise regular* if for each $z \in X$ there exists a constant $c_z > 0$ such that for every $w \in X$, $\delta(z, w) \leq c_z|z - w|$.
- $(ii)$ $X$ is *uniformly regular* if there exists a constant $c > 0$ such that for every $z, w \in X$, $\delta(z, w) \leq c|z - w|$.

Dales and Davie in [1] proved that if $X$ is a finite union of uniformly regular sets, then $D^1(X)$ is complete. The proof given in [1] is also valid when $X$ is a finite union of pointwise regular sets [3].

Before stating our first result, we recall the following definition.

**Definition 2.2.** Let $m, n \in \mathbb{N}$ with $n \geq m$ and $S(m, n)$ denote the set of all $n$-tuples of non-negative integers $(a_1, a_2, \cdots, a_n)$ for which $a_1 + a_2 + \cdots + a_n = m$ and $a_1 + 2a_2 + \cdots + na_n = n$. For an algebra sequence $M$, the sequence $\{A_m\}_{m=1}^{\infty}$ is defined as follows:

$$A_m = \sup \left\{ \frac{1}{P_n} \prod_{k=1}^{n} (P_k)^{a_k} \;:\; n \geq m, (a_1, a_2, \cdots, a_n) \in S(m, n) \right\},$$

where $P_n = \frac{M_n}{n!}$ for each non-negative integer $n$.

**Lemma 2.3.** *Let $X$ be a uniformly regular compact plane set and $M$ be a non-analytic algebra sequence. Then, the closed unit ball of the Banach function algebra $D(X, M)$ is pointwise closed in $C(X)$.*

**Theorem 2.4.** *Let $X$ be a uniformly regular compact plane set and $M$ be a non-analytic algebra sequence such that $\lim_{m\to\infty}(A_m)^{\frac{1}{m}} = 0$. Then, the natural Banach function algebra $D(X, M)$ is a $BSE$-algebra.*

Let $X$ be a compact plane set and $0 < \alpha \leq 1$. The *Lipschitz algebra of order* $\alpha$, denoted by $Lip(X, \alpha)$, is the algebra of all complex-valued functions $f$ on $X$ for which

$$p_\alpha(f) = \sup\left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} : x, y \in X \text{ and } x \neq y \right\} < \infty.$$

Similar to the definition of $D^n(X)$, for a perfect compact plane set $X$, $0 < \alpha \leq 1$ and $n \in \mathbb{N}$, the algebra of all complex-valued functions $f$ on $X$ whose derivatives up to order $n$ exist and $f^{(k)} \in Lip(X, \alpha)$ for each $k$ $(0 \leq k \leq n)$, is denoted by $Lip^n(X, \alpha)$ [3].

**Definition 2.5.** For an algebra sequence $M$ and $0 < \alpha \leq 1$, the Lipschitz version of Dales-Davie algebra $Lip(X, M, \alpha)$ [3] is defined to consists of those $f \in \bigcap_{n=1}^\infty Lip^n(X, \alpha)$ for which

$$\|f\|_{Lip(X,M,\alpha)} = \sum_{k=0}^\infty \frac{\|f^{(k)}\|_X + p_\alpha(f^{(k)})}{M_k} < \infty.$$

By applying a similar method as in the case of Dales-Davie algebras $D(X, M)$, we get the following results for the $BSE$-properties of the algebras $Lip(X, M, \alpha)$.

**Lemma 2.6.** *Let $X$ be a uniformly regular compact plane set and $M$ be a non-analytic algebra sequence. Then, the closed unit ball of the Banach function algebra $Lip(X, M, \alpha)$ is pointwise closed in $C(X)$.*

**Theorem 2.7.** *Let $X$ be a uniformly regular compact plane set and $M$ be a non-analytic algebra sequence such that $\lim_{m\to\infty}(A_m)^{\frac{1}{m}} = 0$. Then, the natural Banach function algebra $Lip(X, M, \alpha)$ is a $BSE$-algebra.*

It is worth mentioning that Theorem 2.4 and Theorem 2.7 are generalizations of [4, Theorem 2.5] and [4, Theorem 2.12], respectively.

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# ON THE EXISTENCE OF POSITIVE SOLUTIONS FOR A NONLINEAR SEMIPOSITONE PROBLEM INVOLVING NONLOCAL OPERATOR

S. SHAKERI

*Department of Mathematics, Ayatollah Amoli Branch, Islamic Azad University, Amol, P. O. Box 678, Iran*
*s.shakeri@iauamol.ac.ir*

Abstract. This paper is concerned with the existence of positive solutions for a class of kirchhoff type systems.Our aim is to establish the existence of positive solution for certain range of $\lambda$ using the method of sub-supersolutions.

## 1. Introduction

In this paper, we are interested in the existence of positive solutions for a class of Kirchhoff type problems of the form

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p\,dx\right)\Delta_p u = \lambda a(x) f(u) - \frac{1}{u^\alpha} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

W here $\Omega$ is a bounded domain of $R^N$ with smooth boundary $\partial\Omega, 0 < \alpha < 1, 1 < p < N$, $M : R^+ \to R^+$ is a continuous and increasing function , $\lambda$ is positive parameter, $f : [0,\infty] \longrightarrow R$ is continuous,nondecreasing function which are asymptotically $p$-linear at $\infty$, the functions $M, a, f$ satisfy the following conditions:

$(H_1)$ $M : R_0^+ \to R^+$ is a continuous and increasing function and $m_0 \leq M(t) \leq m_\infty$ for all $t \in R_0^+$, where $R_0^+ := [0, +\infty)$;

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$(H_2)$ There exist $\sigma_1 > 0, k > 0$ and $s_0 > 1$ such that $f(s) \geq \sigma_1 s^{p-1} - k$ for every $s \in [0, s_0]$.
$(H_3)$ $\lim_{s\to+\infty} \frac{f(s)}{s^{p-1}} = \sigma$ for some $\sigma > 0$.
$(H_4)$ $a : \overline{\Omega} \to (0, \infty)$ is a continuous function such that $a_0 = \min_{x\in\overline{\Omega}} a(x)$, $a_1 = \max_{x\in\overline{\Omega}} a(x)$.

Since the first equation in (1.1) contains an integral over $\Omega$, it is no longer a pointwise identity; therefore it is often called nonlocal problem. This problem models several physical and biological systems, where $u$ describes a process which depends on the average of itself, such as the population density, see [5].

In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [2, 3, 6, 8, 9], in which the authors have used variational method and topological method to get the existence of solutions for (1.1). In this paper, motivated by the ideas introduced in [4] and the properties of Kirchhoff type operators in [1, 7], we study problem (1.1) in the infinite semipositone case. Using the sub- and supersolutions techniques, we establish the existence of a positive solution directly for certain range of $\lambda$. To our best knowledge, this is a new research topic for nonlocal problems, see [1, 7].

In order to precisely state our main result we first consider the following eigenvalue problem for the $p$-Laplace operator $-\Delta_p u$:

$$\begin{cases} -\Delta_p u = \mu|u|^{p-2}u & \text{in } \Omega, \\ u = 0 \quad \text{on } x \in \partial\Omega. \end{cases} \tag{1.2}$$

Let $\phi_1 \in C^1(\overline{\Omega})$ be the eigenfunction corresponding to the first eigenvalue $\mu_1$ of (1.2) such that $\phi_1 > 0$ in $\Omega$ and $\|\phi_1\|_\infty = 1$. Consider the boundary-value problem

$$\begin{cases} -\Delta_p z - \mu|z|^{p-2}z = -1 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega \end{cases}$$

By Anti-maximum principle ([10]), there exists $\xi = \xi(\Omega) > 0$ such that the solution $z_\mu$ of for $\mu \in (\mu_1, \mu_1 + \xi)$ is positive in $\Omega$ and is such that $\frac{\partial z_\mu}{\partial \nu} < 0$ on $\partial\Omega$, where $\nu$ is outward normal vector at $\partial\Omega$.

Since $z_\mu > 0$ in $\Omega$ and $\frac{\partial z_\mu}{\partial \nu} < 0$ there exist $m > 0$, $A > 0$, and $\delta > 0$ be such that $|\nabla z_\mu| \geq m$ in $\overline{\Omega}_\delta$ and $z_\mu \geq A$ in $\Omega \setminus \overline{\Omega}_\delta$ where $\overline{\Omega}_\delta := \{x \in \Omega : \ d(x, \partial\Omega) \leq \delta\}$.

We will also consider the unique solution $e_p \in C^1(\overline{\Omega})$ of the boundary value problem

$$\begin{cases} -\Delta_p e_p = 1 & \text{in } \Omega, \\ e = 0 \quad \text{on } x \in \partial\Omega \end{cases} \tag{1.3}$$

to discuss our result. It is known that $e_p > 0$ in $\Omega$ and $\frac{\partial e_p}{\partial \eta} < 0$ on $\partial\Omega$.

Our main result is given by the following theorem.

**Theorem 1.1.** *If the conditions* $(H_1)$-$(H_4)$ *hold, then there exist positive constants* $s_0^*(\sigma, \Omega)$, $J(\Omega)$, $\underline{\lambda}$, *and* $\hat{\lambda}(> \underline{\lambda})$ *such that if* $s_0 \geq s_0^*$ *and* $\frac{\sigma_1}{\sigma} \geq J$, *(1.1) has a positive solution for* $\lambda \in [\underline{\lambda}, \hat{\lambda}]$.

## 2. Preliminaries

We will prove our result by using the method of sub- and supersolutions, we refer the readers to a recent paper [1, 7] on the topic. A function $\psi$ is said to be a subsolution of problem (1.1) if it is in $W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ such that $\psi = 0$ on $\partial\Omega$ and satisfies

$$M\left(\int_\Omega |\nabla\psi|^p\,dx\right)\int_\Omega |\nabla\psi|^{p-2}\nabla\psi\cdot\nabla w\,dx \leq \int_\Omega [\lambda a(x) f(\psi) - \frac{1}{\psi^\alpha}]w\,dx, \tag{2.1}$$

A function $\phi \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ is said to be a supersolution if $\phi = 0$ on $\partial\Omega$ and satisfies

$$M\left(\int_\Omega |\nabla\phi|^p\,dx\right)\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\cdot\nabla w\,dx \geq \int_\Omega [\lambda a(x) f(\phi) - \frac{1}{\phi^\alpha}]w\,dx, \tag{2.2}$$

Where $W := \{w \in C_0^\infty(\Omega) :\ w \geq 0 \text{ in } \Omega\}$

The following result plays an important role in our arguments. The readers may consult the papers [1, 7] for details.

**Lemma 2.1.** *Assume that* $M : R_0^+ \to R^+$ *satisfies the condition* $(H_1)$. *If the functions* $u, v \in W_0^{1,p}(\Omega)$ *satisfies*

$$\begin{aligned} &M\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx \\ &\leq M\left(\int_\Omega |\nabla v|^p\,dx\right)\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\,dx \end{aligned} \tag{2.3}$$

*for all* $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \geq 0$, *then* $u \leq v$ *in* $\Omega$.

From Lemma 2.1 we obtain the following basic principle of the sub- and supersolutions method.

**Proposition 2.2** (See [1, 7])**.** *Let* $M : R_0^+ \to R^+$ *be a function satisfying the condition* $(H_1)$. *Assume that* $f$ *satisfies the subcritical growth condition*

$$|f(x,t)| \leq C(1 + |t|^{q-1}), \quad llx \in \Omega, \quad \forall t \in R,$$

*where* $1 < q < p^*$, *and the function* $f(x,t)$ *is nondecreasing in* $t \in R$. *If there exist a subsolution* $\underline{u} \in W^{1,p}(\Omega)$ *and a supersolution* $\overline{u} \in W^{1,p}(\Omega)$

*of problem (1.1), then (1.1) has a minimal solution $u_*$ and a maximal solution $u^*$ in the order interval $[u_*, u^*]$, i.e., $\underline{u} \leq u_* \leq u^* \leq \overline{u}$ and if $u$ is any solution of (1.1) such that $\underline{u} \leq u \leq \overline{u}$, then $u_* \leq u \leq u^*$.*

In the practice problems, it is often known that the subsolution $\underline{u}$ and the supersolution $\overline{u}$ are of $L^\infty(\Omega)$, so the restriction on the growth condition of $f$ is needless. Hence, the following theorem is more suitable for our framework.

**Proposition 2.3** (See [1, 7]). *Let $M : R_0^+ \to R^+$ be a function satisfying the condition $(H_1)$. Assume that $\underline{u}, \overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ are a subsolution and a supersolution of problem (1.1) such that $\underline{u} \leq \overline{u}$ in $\Omega$. If $f \in C(\overline{\Omega} \times R, R)$ is nondecreasing in $t \in [\inf_\Omega \underline{u}, \sup_\Omega \overline{u}]$ then the conclusion of Proposition 2.2 is valid.*

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# A CORRESPONDENCE FOR UNBOUNDED REGULAR OPERATORS ON HILBERT C*-MODULES

K. SHARIFI

*Faculty of Mathematical Sciences, Shahrood University of Technology,*
*P. O. Box 3619995161, Shahrood, Iran.*
*sharifi.kamran@gmail.com*

Abstract. Let $E$ be a Hilbert C*-module over a fix C*-algebra and let $B$ be a non-degenerate C*-subalgebra of $\mathcal{L}(E)$. In this paper we study the embedding of $\mathcal{R}(B)$ into $\mathcal{R}(E)$. The inclusion map $i : \mathcal{K}(E) \to \mathcal{L}(E)$ is a non-degenerate $*$-homomorphism which induces the $*$-bijection $\mathcal{R}(\mathcal{K}(E)) \to \mathcal{R}(E); t \mapsto \tilde{t} = \pi(t)$.

## 1. Introduction

Hilbert C*-modules are essentially objects like Hilbert spaces, except that the inner product, instead of being complex-valued, takes its values in a C*-algebra. Although Hilbert C*-modules behave like Hilbert spaces in some ways, some fundamental Hilbert space properties like Pythagoras' equality, self-duality, and even decomposition into orthogonal complements do not hold. A (right) *pre-Hilbert C*-module* over a C*-algebra $A$ is a right $A$-module $E$ equipped with an $A$-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A\,,\ (x, y) \mapsto \langle x, y \rangle$, which is $A$-linear in the second variable $y$ and has the properties:

$$\langle x, y \rangle = \langle y, x \rangle^*, \quad \langle x, x \rangle \geq 0 \ \text{ with equality only when } x = 0.$$

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A pre-Hilbert $A$-module $E$ is called a *Hilbert $A$-module* if $E$ is a Banach space with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$. A Hilbert $A$-submodule $W$ of a Hilbert $A$-module $E$ is an orthogonal summand if $W \oplus W^\perp = E$, where $W^\perp$ denotes the orthogonal complement of $W$ in $X$. We denote by $\mathcal{L}(E)$ the C*-algebra of all adjointable operators on $E$, i.e., all $A$-linear maps $t : E \to E$ such that there exists $t^* : E \to E$ with the property $\langle tx, y\rangle = \langle x, t^*y\rangle$ for all $x, y \in X$. A bounded adjointable operator $v \in \mathcal{L}(E)$ is called a *partial isometry* if $vv^*v = v$, see [**?**] for some equivalent conditions. For the basic theory of Hilbert C*-modules we refer to the books [3].

An unbounded regular operator on a Hilbert C*-module is an analogue of a closed operator on a Hilbert space. Let us quickly recall the definition. A densely defined closed $A$-linear map $t : Dom(t) \subseteq E \to E$ is called *regular* if it is adjointable and the operator $1 + t^*t$ has a dense range. Indeed, a densely defined operator $t$ with a densely defined adjoint operator $t^*$ is regular if and only if its graph is orthogonally complemented in $E \oplus E$ (see e.g. [1, 3]). We denote the set of all regular operators on $E$ by $\mathcal{R}(E)$. If $t$ is regular then $t^*$ is regular and $t = t^{**}$, moreover $t^*t$ is regular and selfadjoint. Define $q_t = (1+t^*t)^{-1/2}$ and $f_t = tq_t$, then $Ran(q_t) = Dom(t)$, $0 \leq q_t = (1 - f_t^* f_t)^{1/2} \leq 1$ in $\mathcal{L}(E)$ and $f_t \in \mathcal{L}(E)$ [3, (10.4)]. The bounded operator $f_t$ is called the bounded transform of regular operator $t$. According to [3, Theorem 10.4], the map $t \to f_t$ defines an adjoint-preserving bijection

$$\mathcal{R}(E) \to \{f \in \mathcal{L}(E) : \|f\| \leq 1 \text{ and } Ran(1 - f^*f) \text{ is dense in } E\}.$$

Consider $t \in \mathcal{L}(E)$, then $t$ is regular and $\|f_t\| < 1$. Consider $t \in \mathcal{R}(E)$. Then $t$ belongs to $\mathcal{L}(E) \Leftrightarrow D(t) = E \Leftrightarrow t$ is bounded $\Leftrightarrow \|f_t\| < 1$. The space $\mathcal{R}(E)$ from a topological point of view is studied in [4]. Very often there are interesting relationships between regular operators and their bounded transforms. In fact, for a regular operator $t$, some properties transfer to its bounded transform $F_t$, and vice versa. Suppose $t \in \mathcal{R}(E)$ is a regular operator, then $t$ is called *normal* iff $Dom(t) = Dom(t^*)$ and $\langle tx, tx\rangle = \langle t^*x, t^*x\rangle$ for all $x \in Dom(t)$. The operator $t$ is called *selfadjoint* iff $t^* = t$ and $t$ is called *positive* iff $t$ is normal and $\langle tx, x\rangle \geq 0$ for all $x \in Dom(t)$. In particular, a regular operator $t$ is normal (resp., selfadjoint, positive) iff its bounded transform $f_t$ is normal (resp., selfadjoint, positive). Moreover, both $t$ and $f_t$ have the same range and the same kernel.

## 2. Main results

Let $A$, $B$, $C$ be C*-algebras, such that $A$ is an ideal in $B$, and let $E$ be a Hilbert C*-module. Suppose that $\alpha : A \to \mathcal{L}(E)$ is a non-degenerate $*$-homomorphism. It is well known that $\alpha$ can be extended uniquely to a $*$-homomorphism $\tilde{\alpha} : B \to \mathcal{L}(E)$. If $\alpha$ is injective and $A$ is essential in $B$ then $\tilde{\alpha}$ is injective [3, Proposition 2.1]. In particular, the inclusion map $i : \mathcal{K}(E) \to \mathcal{L}(E)$ is non-degenerate, and the idealiser of $\mathcal{K}(E)$ is $\mathcal{L}(E)$, so $i$ extends to a $*$-isomorphism between $M(\mathcal{K}(E))$, the multiplier of $\mathcal{K}(E)$, and $\mathcal{L}(E)$. The later fact motivates us for the following results.

Consider a C*-algebra $A$ and define $E$ to be the Hilbert C*-module over $A$ such that $E = A$ as a right $A$-module and $\langle a, b\rangle = b^* a$ for every $a, b \in A$. Then the elements of $\mathcal{R}(E)$ are called elements affiliated with $A$. We write also $t\eta A$ instead of $t \in \mathcal{R}(E)$.

We fix a Hilbert C*-module $E$ over a C*-algebra. At the same time, we will consider a non-degenerate C*-subalgebra $B$ of $\mathcal{L}(E)$. We will look at an embedding of $\mathcal{R}(B)$ into $\mathcal{R}(E)$. Concerning the multiplier algebra, we have:

$$M(B) = \{\, x \in \mathcal{L}(E) \mid \text{for every } b \in B \text{ that } xb,\, bx \in B \,\}$$

As pointed out in [5, 6] for Hilbert spaces, we can also embed $\mathcal{R}(B)$ in $\mathcal{R}(E)$. Following the argument of Woronowicz [6], we state that a non-degenerate $*$-homomorphism can be extended to the set of affiliated elements.

**Theorem 2.1.** *Consider a Hilbert C*-module $E$ over a C*-algebra $A$. Let $B$ be a C*-algebra and $\pi$ be a non-degenerate $*$-homomorphism from $B$ into $\mathcal{L}(E)$. Consider an element $t$ affiliated with $B$. Then there exists a unique element $s \in \mathcal{R}(E)$ such that $f_s = \pi(f_t)$ and we define $s = \pi(t)$. We have moreover that $\pi(D(t))\, E$ is a core for $\pi(t)$ and $\pi(t)(\pi(b)v) = \pi(t(b))\, v$ for every $b \in D(t)$ and $v \in E$.*

The last part of this theorem implies that $\pi(D)\, K$ is a core for $\pi(t)$ if $D$ is a core for $t$ and $K$ is a dense subspace of $E$.

*Remark* 2.2. Suppose moreover that $\pi$ is injective. Then the canonical extension of $\pi$ to $M(B)$ is also injective. Let $s$ and $t$ be two elements affiliated with $B$. Utilizing the bounded transform $f_t$, then $s = t$ if and only if $\pi(s) = \pi(t)$.

The following result can be proven as [6, Theorem 1.2]. It follows easily using the bounded transform $f_t$.

**Proposition 2.3.** *Consider a Hilbert C*-module $E$ over a C*-algebra $A$. Let $B$,$C$ be two C*-algebras. Consider a non-degenerate $*$-homomorphism*

*$\pi$ from $B$ into $M(C)$ and a non-degenerate $*$-homomorphism $\theta$ from $C$ into $\mathcal{L}(E)$. Then $(\theta\pi)(t) = \theta(\pi(t))$ for every $t \,\eta\, B$.*

**Definition 2.4.** Call $\pi$ the inclusion of $B$ into $\mathcal{L}(E)$, then $\pi$ is a non-degenerate $*$-homomorphism from $B$ into $\mathcal{L}(E)$. Let $t$ be an element affiliated to $B$. Then we define $\tilde{t} = \pi(t)$, so $\tilde{t}$ is a regular operator on $E$.

Because $\pi$ is injective, we know immediately that the mapping

$$\mathcal{R}(B) \to \mathcal{R}(E) : t \mapsto \tilde{t} = \pi(t)$$

is injective. We have also immediately that $\tilde{x} = x$ for every $x \in M(B)$. Looking at example 4 of [6], we have also the following result. Consider a regular operator $t$ on $E$. Then there exists an element $s$ affiliated with $B$ such that $\tilde{s} = t \quad \Leftrightarrow$

(1) $f_t$ belongs to $M(B)$,
(2) $(1 - f_t^* f_t)^{\frac{1}{2}} B$ is dense in $B$.

If there exists such an $s$, we have immediately that $f_t = f_s$, so $f_t$ will certainly satisfy the two mentioned conditions. If $f_t$ satisfies these two conditions, there exists an element $s$ affiliated with $B$ such that $f_s = f_t$. So we have that $f_{\tilde{s}} = f_s = f_t$ which implies that $\tilde{s} = t$. This implies immediately the following result.

**Theorem 2.5.** *Consider a Hilbert $C^*$-module over $E$ over a $C^*$-algebra $A$. Then the mapping $\mathcal{R}(\mathcal{K}(E)) \to \mathcal{R}(E) : t \mapsto \tilde{t} = \pi(t)$ is a $*$-bijection.*

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# MODULE COHOMOLOGY OF SEMIGROUP ALGEBRAS

A. SHIRINKALAM

*Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran*
*a.shirinkalam-science@iauctb.ac.ir*

Abstract. In this paper we introduce a new concept of module cohomology for Banach algebras. We study the relation between this type of cohomology with Hochschild cohomology of Banach algebras.

## 1. Introduction and Preliminaries

Let $\mathcal{L}^n(X,Y)$ (resp. $\mathcal{L}^n_{\mathfrak{A}}(X,Y)$) denote the space of all bounded $n$-linear (resp. module) maps from $X$ into $Y$ (when $X$ and $Y$ are $\mathfrak{A}$-bimodules). We denote by $X^{(n)}$ the $n^{\text{th}}$ dual of a Banach space $X$. Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule. We say that the action of $\mathfrak{A}$ on $\mathcal{A}$ is compatible, if for each $\alpha \in \mathfrak{A}$ and $a, b \in \mathcal{A}$, we have $\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha)$. Throughout the rest of the paper, we assume that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions. Let $X$ be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule such that for every $\alpha \in \mathfrak{A}, a \in \mathcal{A}$ and $x \in X$, $\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \quad a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x, \quad (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a)$, and the same for the right and two-sided actions. Then we say that $X$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule. If moreover, for each $\alpha \in \mathfrak{A}$ and $x \in X$, $\alpha \cdot x = x \cdot \alpha$, then $X$ is called a

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commutative Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule. In this case $X^*$, the dual of $X$ is also a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule with the canonical action.

Let $I$ be the closed ideal of the projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$ generated by the elements of the form $a\cdot\alpha\otimes b - a\otimes\alpha\cdot b$, for $\alpha\in\mathfrak{A}, a, b\in\mathcal{A}$. Then, the module projective tensor product $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is the quotient $\dfrac{\mathcal{A}\hat{\otimes}\mathcal{A}}{I}$ [4]. Let $J$ be the closed ideal of $\mathcal{A}$ generated by $\pi(I)$, where $\pi:\mathcal{A}\hat{\otimes}\mathcal{A}\to\mathcal{A}$ is the multiplication map.

Let $X$ be a Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule. The space of all $\mathfrak{A}$-module derivations from $\mathcal{A}$ to $X$ is denoted by $\mathcal{Z}^1_{\mathfrak{A}}(\mathcal{A},X)$ and the subspace of inner module derivations is denoted by $\mathcal{B}^1_{\mathfrak{A}}(\mathcal{A},X)$. The first $\mathfrak{A}$-module cohomology group of $\mathcal{A}$ with coefficients in $X$ is defined as the quotient seminormed space $\mathcal{H}^1_{\mathfrak{A}}(\mathcal{A},X) := \dfrac{\mathcal{Z}^1_{\mathfrak{A}}(\mathcal{A},X)}{\mathcal{B}^1_{\mathfrak{A}}(\mathcal{A},X)}$.

## 2. Relation between module cohomology and Hochschild cohomology

Let $\mathcal{A}$ and $\mathfrak{A}$ and the closed ideal $J$ be as in the previous section, and $X$ be a Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule. We say that the action of $\mathfrak{A}$ on $X$ is trivial from left, if for every $\alpha\in\mathfrak{A}$ and $x\in X, \alpha\cdot x = f(\alpha)x$, where $f$ is a character on $\mathfrak{A}$.

For the rest of this section, we assume that $X$ is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule, both with a left trivial action of $\mathfrak{A}$, via the same character $f$ on $\mathfrak{A}$. We also assume that the Banach algebra $A/J$ is unital. In the next section, we provide examples satisfying all these conditions.

**Proposition 2.1.** *The Banach spaces $\mathcal{Z}^1_{\mathfrak{A}}(\mathcal{A},X)$ and $\mathcal{Z}^1(\frac{\mathcal{A}}{J},X)$ are isometrically isomorphic.*

*Proof.* The map $\rho:\mathcal{Z}^1_{\mathfrak{A}}(\mathcal{A},X)\to\mathcal{Z}^1(\frac{\mathcal{A}}{J},X)$ defined by $\rho(D)(a+J) = D(a)$, for $D\in\mathcal{Z}^1_{\mathfrak{A}}(\mathcal{A},X)$ is a surjective isometry. □

**Theorem 2.2.** *The seminormed spaces $\mathcal{H}^1_{\mathfrak{A}}(\mathcal{A},X)$ and $\mathcal{H}^1(\frac{\mathcal{A}}{J},X)$ are isomorphic.*

*Proof.* The isomorphism $\rho$ in the previous proposition induces a surjective map $\Phi:\dfrac{\mathcal{Z}^1_{\mathfrak{A}}(\mathcal{A},X)}{\mathcal{B}^1_{\mathfrak{A}}(\mathcal{A},X)}\to\dfrac{\mathcal{Z}^1(\frac{\mathcal{A}}{J},X)}{\mathcal{B}^1(\frac{\mathcal{A}}{J},X)}$, given by $\Phi(D+\mathcal{B}^1_{\mathfrak{A}}(\mathcal{A},X)) := \rho(D)+\mathcal{B}^1(\frac{\mathcal{A}}{J},X)$ that is an isomorphism of seminormed spaces. □

For the definition of an $\mathfrak{L}_1$-space we refer the reader to [10]. Examples of such spaces are $L^1(\mu)$ for a measure $\mu$ and $C(K)^*$, for a compact space $K$.

Recall that $J$ is a closed ideal of $\mathcal{A}$ and is equal to the closed linear span of the set of elements of the form $(a\cdot\alpha)b - a(\alpha\cdot b)$ with $\alpha\in\mathfrak{A}, a, b\in\mathcal{A}$. Let $J_0$ be the closed linear span of the set of elements of the form $a\cdot\alpha - f(\alpha)a$ with $a\in\mathcal{A}, \alpha\in\mathfrak{A}$. Since $\frac{\mathcal{A}}{J}$ is assumed to be unital, $J_0\subseteq J$. In practice, for most interesting examples (see the next section), one has $J=J_0$.

**Lemma 2.3.** *With the above notation, assume that $J=J_0$ and $X$ is an $\mathfrak{L}_1$-space. Then the Banach spaces $\frac{\mathcal{A}}{J}\hat{\otimes}X$ and $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}X$ are isometrically isomorphic.*

**Proposition 2.4.** *(Reduction of dimension) Let $\frac{\mathcal{A}}{J}$ is an $\mathfrak{L}_1$-space. Under the assumptions of Lemma 2.3,*

(i) $\mathcal{L}^k_{\mathfrak{A}}(\mathcal{A},X^*)\simeq(\underbrace{\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\cdots\hat{\otimes}_{\mathfrak{A}}\mathcal{A}}_{k-times}\hat{\otimes}_{\mathfrak{A}}X)^*$ *is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$- bimodule, for every $k\in\mathbb{N}$,*
(ii) *We have the isomorphism $\mathcal{H}^{n+k}_{\mathfrak{A}}(\mathcal{A},X)\simeq\mathcal{H}^n_{\mathfrak{A}}(\mathcal{A},\mathcal{L}^k_{\mathfrak{A}}(\mathcal{A},X))$, of seminormed spaces, for every $k,n\in\mathbb{N}$.*

The isometric isomorphism in (i) follows from [9, Exercise 5.3.1]. Replacing cochains with module cochains, a similar argument in the proof of [9, Theorem 2.4.6] shows (ii).

**Corollary 2.5.** *Under the above assumptions, the spaces $\mathcal{H}^k_{\mathfrak{A}}(\mathcal{A},X^*)$ and $\mathcal{H}^k(\frac{\mathcal{A}}{J},X^*)$ are isometrically isomorphic.*

*Proof.* By Proposition 2.4 and Theorem 2.2 we have

$$\mathcal{H}^k_{\mathfrak{A}}(\mathcal{A},X^*)\simeq\mathcal{H}^1(\frac{\mathcal{A}}{J},\mathcal{L}^{k-1}_{\mathfrak{A}}(\mathcal{A},X^*))\simeq\mathcal{H}^1(\frac{\mathcal{A}}{J},\mathcal{L}^{k-1}(\frac{\mathcal{A}}{J},X^*))\simeq\mathcal{H}^k(\frac{\mathcal{A}}{J},X^*).$$

□

## 3. Applications to semigroup algebras

Let $S$ be an inverse semigroup with the set of idempotents $E$. Let $\ell^1(E)$ act on $\ell^1(S)$ by multiplication from right and trivially from left, that is, $\delta_e\cdot\delta_s=\delta_s,\quad \delta_s\cdot\delta_e=\delta_{se}=\delta_s*\delta_e\quad(e\in E, s\in S)$, where $\delta_s$ is the point mass at $s$. Here the closed ideal $J$ (see section 1) is the closed linear span of the set $\{\delta_{set}-\delta_{st}: s,t\in S, e\in E\}$. We consider an equivalence relation on $S$ defined by

$$s\sim t\Longleftrightarrow\delta_s-\delta_t\in J\quad(s,t\in S).\tag{3.1}$$

The discussion before [1, Theorem 2.4] shows that $S/\sim$ is a discrete group. In this case, by the proof of [8, Theorem 3.3], we observe that $\frac{\ell^1(S)}{J}\simeq\ell^1(S/\sim)$ as (commutative) $\ell^1(E)$-bimodules. The discrete group $S/\sim$ is the same as the maximal group homomorphic image $G_S$ of $S$.

**Theorem 3.1.** *Let $S$ be an inverse semigroup with the set of idempotents $E$. Let $\ell^1(E)$ act on $\ell^1(S)$ by multiplication from right and trivially from left. Then, $\mathcal{H}^1_{\ell^1(E)}(\ell^1(S), \ell^1(G_S)^{(2n+1)}) = 0$, for each $n \in \mathbb{N} \cup \{0\}$.*

Next we generalize [6, Corollary 3.5] from Clifford semigroups to arbitrary inverse semigroups.

**Theorem 3.2.** *Let $S$ be an inverse semigroup with the set of idempotents $E$. Let $\ell^1(E)$ act on $\ell^1(S)$ by multiplication from right and trivially from left. Then, for each $n \in \mathbb{N} \cup \{0\}$, $\mathcal{H}^2_{\ell^1(E)}(\ell^1(S), \ell^1(G_S)^{(2n+1)})$ is a Banach space.*

*Proof.* Note that for each locally compact group $G$ and every $n \in \mathbb{N}$, there is a compact Hausdorff space $K_n$ such that $L^1(G)^{(2n)} \simeq M(K_n)$ is an $\mathfrak{L}_1$-space. Now for $n = 0$, $\frac{\ell^1(S)}{J} \simeq \ell^1(G_S)$ is a unital Banach algebra and an $\mathfrak{L}_1$-space. Also $\frac{\ell^1(S)}{J} \hat{\otimes} \ell^1(G_S) \simeq \ell^1(G_S \times G_S)$ is an $\mathfrak{L}_1$-space. Therefore, by Corollary 2.5,$\mathcal{H}^2_{\ell^1(E)}(\ell^1(S), \ell^\infty(G_S)) \simeq \mathcal{H}^2(\frac{\ell^1(S)}{J}, \ell^\infty(G_S)) \simeq \mathcal{H}^2(\ell^1(G_S), \ell^\infty(G_S))$. By [7, Theorem 3.3], the last space is a Banach space. For $n \geq 1$, since $\ell^1(G_S)^{(2n)}$ is an $\mathfrak{L}_1$-space, by Corollary 2.5, $\mathcal{H}^2_{\ell^1(E)}(\ell^1(S), \ell^1(G_S))^{(2n+1)}) \simeq \mathcal{H}^2(\ell^1(G_S), (\ell^1(G_S))^{(2n+1)})$. Again, by [7, Theorem 3.3], the last space is a Banach space. $\square$

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# FRAMENESS BOUND FOR FUSION FRAMES IN FINITE-DIMENTIONAL HILBERT SPACES

AZAM SHEKARI* AND MOHAMMAD REZA ABDOLLAHPOUR

*Department of Mathematics, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran*
*azamshekari67@yahoo.com*
*mrabdollahpour@yahoo.com*

Abstract. In this paper, we show that in each finite dimensional Hilbert space, a frame of subspaces is an ultra Bessel sequence of subspaces. We also show that every frame of subspaces in a finite dimensional Hilbert space has frameness bound.

## 1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space. We say that a sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a frame for $\mathcal{H}$, if there exist constants $0 < A, B < \infty$ such that

$$A\|f\|^2 \leqslant \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2, \quad f \in \mathcal{H}. \tag{1.1}$$

If $A = B$ then we call $\{f_i\}_{i=1}^{\infty}$ is tight frame and if $A = B = 1$ it is called a Parseval frame. If the right hand inequality of (1.1) holds for all $f \in \mathcal{H}$, then we call $\{f_i\}_{i=1}^{\infty}$ a Bessel sequence for $\mathcal{H}$. In 2008, the concept of ultra Bessel sequences in Hilbert spaces introduced and investigated by Faroughi and Najati [5].

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**Definition 1.1.** Let $\mathcal{H}_0$ be an inner product space. Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of members of $\mathcal{H}_0$. Then $\{f_i\}_{i=1}^{\infty}$ is called an ultra Bessel sequence in $\mathcal{H}_0$, if

$$\sup_{\|f\|=1} \sum_{i=n}^{\infty} |\langle f, f_i\rangle|^2 \to 0 \tag{1.2}$$

as $n \to \infty$, i.e., the series $\sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2$ converges uniformly in unit sphere of $\mathcal{H}_0$.

As a generalization of ordinary frame, frame of subspaces introduced by Casazza and Kutyniok in [3].

**Definition 1.2.** Let $\{v_i\}_{i=1}^{\infty}$ be a family of weights, i.e., $v_i > 0$, for all $i \geq 1$. A family of closed subspaces $\{W_i\}_{i=1}^{\infty}$ of a Hilbert space $\mathcal{H}$ is a frame of subspaces or fusion frame with respect to $\{v_i\}_{i=1}^{\infty}$ for $\mathcal{H}$, if there exist constants $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \leq D\|f\|^2, \qquad f \in \mathcal{H}. \tag{1.3}$$

If the right hand inequality in (1.3) holds for all $f \in \mathcal{H}$, we call $\{W_i\}_{i=1}^{\infty}$ a Bessel sequence of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$ with Bessel bound $D$.

**Definition 1.3.** For each family of subspaces $\{W_i\}_{i=1}^{\infty}$ of $\mathcal{H}$, we define the set

$$\Big(\sum_{i=1}^{\infty} \oplus W_i\Big)_{\ell^2} = \Big\{\{f_i\}_{i=1}^{\infty} | f_i \in W_i, \quad \sum_{i=1}^{\infty} \|f_i\|^2 < \infty\Big\}.$$

It clear that $\Big(\sum_{i=1}^{\infty} \oplus W_i\Big)_{\ell^2}$ is a Hilbert space with the point wise operations and with the inner product given by

$$\langle \{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty}\rangle = \sum_{i=1}^{\infty} \langle f_i, g_i\rangle.$$

It is proved in [3], if $\{W_i\}_{i=1}^{\infty}$ is a frame of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$ for $\mathcal{H}$ then the operator

$$T_{W,v} : \Big(\sum_{i=1}^{\infty} \oplus W_i\Big)_{\ell^2} \to \mathcal{H}, \quad T_{W,v}(f) = \sum_{i=1}^{\infty} v_i f_i$$

is bounded and onto and its adjoint is

$$T^*_{W,v} : \mathcal{H} \to \Big(\sum_{i=1}^{\infty} \oplus W_i\Big)_{\ell^2}, \quad T^*_{W,v}(f) = \{v_i \pi_{W_i}(f)\}_{i=1}^{\infty}.$$

The operators $T_{W,v}$ and $T^*_{W,v}$ are called the synthesis and analysis operators for $\{W_i\}_{i=1}^\infty$ and $\{v_i\}_{i=1}^\infty$, respectively.

Also, it is proved in [3], if $\{W_i\}_{i=1}^\infty$ is a frame of subspaces with respect to $\{v_i\}_{i=1}^\infty$, the operator

$$S_{W,v} : \mathcal{H} \to \mathcal{H}, \quad S_{W,v}(f) = TT^*(f)$$

is a positive, self-adjoint and invertible operator on $\mathcal{H}$ and we have the reconstruction formula

$$S_{W,v}(f) = \sum_{i=1}^\infty v_i^2 S^{-1} \pi_{W_i}(f), \qquad f \in \mathcal{H}.$$

The operator $S_{W,v}$ is called the frame operator for $\{W_i\}_{i=1}^\infty$ and $\{v_i\}_{i=1}^\infty$.

The ultra Bessel sequence of subspaces were introduced in [2] by the authors of this paper.

**Definition 1.4.** Let $\mathcal{H}_0$ be an inner product space. Let $\{W_i\}_{i=1}^\infty$ be a family of closed subspaces of $\mathcal{H}_0$. Then $\{W_i\}_{i=1}^\infty$ is called an ultra Bessel sequence of subspaces in $\mathcal{H}_0$, if

$$\sup_{\|f\|=1} \sum_{i=n}^\infty v_i^2 \|\pi_{W_i}(f)\|^2 \to 0, \tag{1.4}$$

as $n \to \infty$, i.e., the series $\sum_{i=1}^\infty v_i^2 \|\pi_{W_i}(f)\|^2$ converges uniformly in the unit sphere of $\mathcal{H}_0$.

Following proposition has been proved in [2] and we use it in the rest of this paper.

**Proposition 1.5.** *Let $\{W_i\}_{i=1}^\infty$ be a family of closed subspaces in Hilbert space $\mathcal{H}$ and $\{v_i\}_{i=1}^\infty$ be a family of weights such that $\sum_{i=1}^\infty v_i^2 < \infty$. Then $\{W_i\}_{i=1}^\infty$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$.*

## 2. Main results

In this section, we prove that in a finite dimensional Hilbert space, each frame of subspaces is an ultra Bessel sequence of subspaces. Also we can divide a frame of subspaces $\{W_i\}_{i=1}^\infty$ in two sets $\{W_i\}_{i=1}^{N-1}$ and $\{W_i\}_{i=N}^\infty$, for which $\{W_i\}_{i=1}^{N-1}$ is not a frame of subspaces, but $\{W_i\}_{i=1}^N$ is a frame of subspaces. We refere to [1] for the proof of the following results.

**Theorem 2.1.** *Let $\{W_i\}_{i=1}^\infty$ be a frame of subspaces for Hilbert space $\mathcal{H}$ such that for all $i \geq 1$, $\dim W_i < \infty$. Then*

(i) *if $\mathcal{H}$ is an infinite dimensional Hilbert space, then $\{W_i\}_{i=1}^\infty$ is not an ultra Bessel sequence of subspaces.*

(ii) *if* $\mathcal{H}$ *is a finite dimensional Hilbert space, then* $\{W_i\}_{i=1}^{\infty}$ *is an ultra Bessel sequence of subspaces, and there exists* $N_0 \geqslant 1$ *such that for each* $1 \leqslant n < N_0$, $\{W_i\}_{i=1}^{n}$ *is not a frame of subspaces of* $\mathcal{H}$*, but* $\{W_i\}_{i=1}^{n}$ *is a frame of subspaces of* $\mathcal{H}$ *for each* $n \geqslant N_0$.

**Definition 2.2.** Let $\mathcal{H}$ be a finite dimensional Hilbert space and $\{W_i\}_{i=1}^{\infty}$ be a frame of subspaces of $\mathcal{H}$. Then we call the number $N_0$ in the Theorem 2.1, the frameness bound of the frame of subspaces $\{W_i\}_{i=1}^{\infty}$.

**Lemma 2.3.** [4] *Let* $\mathcal{H}$, $\mathcal{K}$ *be Hilbert spaces, and suppose that* $U : \mathcal{K} \to \mathcal{H}$ *is a bounded operator with closed range* $\mathcal{R}_U$*. Then there exists a bounded operator* $U^{\dagger} : \mathcal{H} \to \mathcal{K}$ *for which*

$$UU^{\dagger}f = f, \quad \forall f \in \mathcal{R}_U. \tag{2.1}$$

If $\{W_i\}_{i=1}^{\infty}$ is a frame of subspaces for $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$. Then

$$T_{W,v}(T^{*}_{W,v}S^{-1}_{W,v}f) = f, \quad f \in \mathcal{H},$$

so $T^{\dagger}_{W,v} = T^{*}_{W,v}S^{-1}_{W,v}$.

**Theorem 2.4.** *Let* $\mathcal{H}$ *be a finite dimensional Hilbert space and* $\{W_i\}_{i=1}^{\infty}$ *be a frame of subspaces of* $\mathcal{H}$ *with respect to* $\{v_i\}_{i=1}^{\infty}$ *an let* $N_0$ *be frameness bound of* $\{W_i\}_{i=1}^{\infty}$*. Let* $n \geqslant N_0$ *and* $S_n$ *and* $T_n$ *be the frame frame operator and synthesis operator of* $\{W_i\}_{i=1}^{n}$*, respectively. Then*

(i) $S_n \to S_{W,v}$ *in* $B(\mathcal{H})$,

(ii) $T_n \to T_{W,v}$ *in* $B\Big((\sum_{i=1}^{\infty} \oplus W_i)_{\ell^2}, \mathcal{H}\Big)$ *and* $T^{\dagger}_n \to T^{\dagger}_{W,v}$ *in* $B\Big(\mathcal{H}, (\sum_{i=1}^{\infty} \oplus W_i)_{\ell^2}\Big)$.

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**Oral Presentation**

# ON $\varphi \otimes \psi$ -CONNES AMENABILITY OF DUAL BANACH ALGEBRAS

E. TAMIMI

*Department of Mathematics, Velayat University, Iranshahr 9911131311, Iran*
*tamimi_ebrahim@yahoo.com*

ABSTRACT. In this paper, we investigate the relation between kernel of a homomorphism defined on a Banach algebra and the left identity of that Banach algebra. Then we generalize this cocept to tensor product of homomorphisms and also, we characterize Connes amenability of projective tensor product of dual Banach algebras with certain preduals. For this purpose, we focus on projective tensor product of homomorphisms. Some results are also given.

## 1. Introduction

The concept of amenability for Banach algebras was first introduced by Johnson [2]. A generalization of amenability which depends on homomorphisms was introduced by Kaniuth et al. in [3, 4]. This concept was also studied independently, by Monfared in [6]. Let $\mathcal{A}$ be a Banach algebra and E be a Banach $\mathcal{A}$-bimodule. A bounded Linear map $D : \mathcal{A} \to E$ is a derivation if it satisfies $D(ab) = D(a).b + a.D(b)$ for all $a, b \in \mathcal{A}$. Given $x \in E$, the inner derivation $ad_x : \mathcal{A} \to E$ is defined by $ad_x(a) = a.x - x.a$. A Banach algebra $\mathcal{A}$ is amenable if for every Banach $\mathcal{A}$-bimodule E, every derivation from $\mathcal{A}$ into $E^*$, the dual of E, is inner. A Banach $\mathcal{A}$-bimodule E is called dual if there is a closed submodule $E_*$ of $E^*$ such that $E = (E_*)^*$, we say $E_*$ predual of E. The

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Banach algebra $\mathcal{A}$ is called dual if it is dual as a Banach $\mathcal{A}$-bimodule. It is easily checked that a Banach algebra which is also a dual space is a dual Banach algebra if and only if the multiplication map is separately $weak^*$-continuous [7]. Every von Neumann algebra is a dual Banach algebra [9, Corollary III.3.9]. We write $\mathcal{A} = (\mathcal{A}_*)^*$ if we wish to stress that $\mathcal{A}$ is a dual Banach algebra with predual $\mathcal{A}_*$. A proper concept of amenability for dual Banach algebras is the Connes amenability. This notion under different name, for the first time was introduced by Johnson, Kadison, and Ringrose for von Neumann algebras [2]. Also the concept of Connes amenability for the larger class of dual Banach algebras, which seems to be a natural variant of amenability for dual Banach algebras, systematically was introduced and later extended by Runde [7]. A dual Banach $\mathcal{A}$-bimodule E is normal, if for each $x \in E$ the module maps $\mathcal{A} \to E$; $a \to x.a$ and $a \to a.x$ are $weak^*$-$weak^*$ continuous. A dual Banach algebra $\mathcal{A}$ is Connes amenable if every $weak^*$-continuous derivation from $\mathcal{A}$ into a normal, dual Banach $\mathcal{A}$-bimodule is inner. For a given dual Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule E, $\sigma wc(E)$ denote the set of all elements $x \in E$ such that the module maps $\mathcal{A} \to E$; $a \to a.x$ *and* $a \to x.a$ are $weak^*$-$weak$ continuous, one can see that, it is a closed submodule of E. In [8, Proposition 4.4], the author showed that $E = \sigma wc(E)$ if and only if $E^*$ is a normal dual Banach $\mathcal{A}$-bimodule. Also, it has been showed that a dual Banach algebra $\mathcal{A}$ is Connes amenable if and only if there exists a $\sigma wc$-virtual diagonal for $\mathcal{A}$ [8, Theorem 4.8]. Also, $\varphi$-Connes amenability and $\varphi$-Connes mean which seem to be natural variants of Connes amenability and Connes mean for dual Banach algebras, systematically were introduced by Ghaffari and Javadi [1] and Mahmoodi [5], although this concept is much older.

Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras and $\varphi \in \Delta_{\omega^*}(\mathcal{A})$, the set of all $weak^*$-continuous homomorphisms from $\mathcal{A}$ onto $\mathbb{C}$ and $\psi \in \Delta_{\omega^*}(\mathcal{B})$, and let E be a Banach $\mathcal{A}$-bimodule. In this note we study $\varphi \otimes \psi$-Connes amenability for projective tensor product of mentioned dual Banach algebras. We write $Z^1(\mathcal{A}, E)$ and $N^1(\mathcal{A}, E)$, for the space of all derivations and inner derivations from $\mathcal{A}$ onto E, respectively and we recall that the quotient space $\mathcal{H}^1(\mathcal{A}, E) = Z^1(\mathcal{A}, E)/N(\mathcal{A}, E)$ is the first cohomology group of $\mathcal{A}$ with coefficients in E. A Banach algebra $\mathcal{A}$ is called amenable if $\mathcal{H}^1(\mathcal{A}, E^*) = 0$, for every dual Banach $\mathcal{A}$-bimodule $E^*$ with a canonical action. We investigate the Connes amenability through vanishing of $\mathcal{H}^1_{\omega^*}(\mathcal{A}, E^{**})$ for Banach $\varphi$-bimodule E and through the existence of a left identity for $ker(\varphi \otimes \psi)$.

## 2. Main results

The next theorem improves the corresponding result given in [1].

**Theorem 2.1.** *Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a Banach algebra and let $\varphi \in \Delta(\mathcal{A}) \bigcap \mathcal{A}_*$. Let $\varphi^\sharp$, be the unique extention of $\varphi$ to an element of $\Delta(\mathcal{A}^\sharp)$. Then $ker\varphi^\sharp$ has a left identity if and only if $ker\varphi$ has a left identity.*

**Lemma 2.2.** *Suppose that $\mathcal{A} = (\mathcal{A}_*)^*$, $\mathcal{B} = (\mathcal{B}_*)^*$ and $\mathcal{A}\widehat{\otimes}\mathcal{B} = (\mathcal{A}_* \otimes_\omega \mathcal{B}_*)^*$ be unital dual Banach algebras, and let $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$, $\psi \in \Delta(\mathcal{B}) \cap \mathcal{B}_*$. Then the kernel of $\varphi \otimes \psi$ has a left identity if and only if both $ker\varphi$ and $ker\psi$ have left identity.*

As an immediate consequence of Lemma 2.2 and Theorem 2.1 we obtain

**Corollary 2.3.** *Let $\mathcal{A} = (\mathcal{A}_*)^*$, $\mathcal{B} = (\mathcal{B}_*)^*$ be unital, dual Banach algebras and let $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$, $\psi \in \Delta(\mathcal{B}) \cap \mathcal{B}_*$. Then $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is $\varphi \otimes \psi$-Connes amenable if and only if $ker(\varphi \otimes \psi)$ has a left identity.*

**Proposition 2.4.** *Let $\mathcal{A}$ be a Banach algebra and $\varphi \in \Delta_{\omega^*}(\mathcal{A})$. Then for each Banach $\varphi$-bimodule $E$,*
*(i) the following three conditions are equivalent:*

1. *$\mathcal{A}$ has a $\varphi$-Connes mean;*
2. *$\mathcal{H}^1_{\omega^*}(\mathcal{A}, E^{**}) = 0$ ;*
3. *for each weak$^*$-continuous derivation $D : \mathcal{A} \to E$, there exists a bounded net $(x_\alpha)$ in $E$ such that $Da = lim_\alpha(a.x_\alpha - x_\alpha.a)$ $(a \in \mathcal{A})$.*

*(ii) $(\mathcal{A}^{**}, \Box)$ is $\widetilde{\varphi}$-Connes amenable if and only if $\mathcal{A}$ is $\varphi$-Conns amenable.*

The purpose of following theorem is to investigate the relation between the $\varphi$-amenability and $\varphi$-Connes amenability of dual Banach algebra under certain condition.

**Theorem 2.5.** *Let $\mathcal{A}$ be a Banach algebra, $E$ is a Banach $\mathcal{A}$-bimodule and $\varphi \in \Delta_{\omega^*}(\mathcal{A})$. Then the following three conditions are equivalent.*
*(i) $\mathcal{A}$ has a $\varphi$-Connes mean.*
*(ii) If the module action of $\mathcal{A}$ on $E$ given by $a.x = \varphi(a)x$ for all $x \in E$ and $a \in \mathcal{A}$, then $\mathcal{H}^1_{\omega^*}(\mathcal{A}, E^*) = \{0\}$.*
*(iii) For $\mathcal{A}$-bimodule $\sigma wc(ker\varphi)^{**}$, by the left module action to be $a.F = \varphi(a)F$ for $F \in \sigma wc(\mathcal{A}^{**})$ and $a \in \mathcal{A}$, each $D \in \mathcal{Z}^1_{\omega^*}(\mathcal{A}, \sigma wc(ker\varphi)^{**})$ is inner.*

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# A ONE-STEP WITH MEMORY METHOD IS MORE EFFICIENT THAN THE TRAUB'S METHOD

†VALI TORKASHVAND∗ AND ‡MOHAMMAD JAVAD LALEHCHINI

† *Young Researchers and Elite Club, Shahr-e-Qods Branch, Islamic Azad University, Tehran, IRAN.*
*torkashvand1978@gmail.com*
‡*Department of Mathematics, Farhangian University Hamedan, IRAN.*

Abstract. In this work, a with memory method is developed. This technique enables us to achieve the high efficiency theoretically and practically. The improvement of the order of convergence this family is obtained by using one self-accelerating parameter, in which the order of convergence is increased from 2 to 3 without any new function evaluation. It means that, without any new function calculations, the order of convergence can be improved untill 50%. Numerical examples and the comparison with existing one till four-point methods are included to demonstrate exceptional convergence speed of the proposed method and confirm theoretical results. Aanother advantage is the convenient fact that this method does not use derivative.

## 1. Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis. In this paper, we consider iterative methods to find a simple root of a non-linear equation $f(x) = 0$, where $f : D \to \mathbb{R}$ for an open interval $D$ is a scalar function. Newton's method for a single

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non-linear equation is defined by:

$$x_{k+1} = x_k + \frac{f(x_k)}{f'(x_k)},\ k = 0, 1, 2, \ldots \tag{1.1}$$

This is an important and basic method, which converges quadratically. Traub in his book classified iterative methods for solving such equations as one or multi point [7]. He proved the best one point iterative method should achieve order of convergence $n$ using $n$ function evaluations. Also, he and his Phd student in 1974 conjectured that any multipoint method should achieve optimal order convergence $2^n$ using $n+1$ evaluations [4]. It should be noted that these criteria are about methods without memory, i.e., methods that use the information of the current iteration. On the other hand, Traub investigated that it is possible to increase the convergence order of without memory methods by reusing the obtained information in the previous iteration. He called such scheme as "with memory method". Following Traub's idea of developing method with memory, many authors have attempted to construct methods with memory.[2, 4, 6, 7]. We recall the so-called efficiency index defined by Ostrowski [5], as $EI = p^{1/n}$, where $p$ is the order of convergence and $n$ is the total number function evaluations per iteration.

## 2. Main Results

In section, we deal with modifying four-point without memory method by Geum-Kim [3]. Geum-Kim's method has the iterative expression

$$\begin{cases} y_k = x_k + \theta f(x_k),\ g(x_k) = \frac{\theta f(x_k) f(y_k)}{f(x_k) - f(y_k)}, z_k = y_k + g(x_k),\ \ k = 0, 1, 2, \ldots, \\ K(x_k) = g(x_k) \frac{f(x_k) f(z_k)}{(f(x_k) - f(z_k))(f(y_k) - f(z_k))},\ s_k = z_k + H(x_k) + K(x_k), \\ q_k = z_k + K(x_k),\ T(x_k) = K(x_k) \frac{f(q_k)}{(f(x_k) - f(q_k))(f(y_k) - f(q_k))(f(z_k) - f(q_k))}, \\ H(x_k) = T(x_k)(f(x_k) f(y_k) + f(z_k)^2 - f(z_k) f(q_k)), \\ h1_k = (f(z_k)(f(z_k) - f(s_k)) + f(x_k) f(y_k)) f(z_k), \\ h2_k = f(q_k)(f(q_k) - f(s_k))(-f(x_k) - f(y_k) + f(q_k) + f(s_k)), \\ t1_k = f(x_k) f(y_k)(h1 - h2) + f(z_k) f(q_k)(f(z_k) - f(q_k))(f(z_k) - f(s_k))(f(q_k - f(s_k)), \\ t2_k = (f(x_k) - f(s_k))(f(y_k) - f(s_k))(f(z_k) - f(s_k))(f(q_k) - f(s_k)), \\ W(x_k) = T(x_k) f(s_k) \frac{t1_k}{t2_k},\ x_{k+1} = s_k + W_k \end{cases} \tag{2.1}$$

Considering just one step, we will have the following one-step method without memory:

$$\begin{cases} w_k = x_k + \beta f(x_k),\ k = 0, 1, 2, \ldots, \\ x_{k+1} = w_k + \frac{\beta f(w_k) f(x_k)}{f(x_k) - f(w_k)}. \end{cases} \tag{2.2}$$

Denoted by TM, where $0 \neq \beta \in \mathbb{R}$. Let us note that in each iteration we only evaluate $f(x_k)$, and $f(w_k)$ so that the method will be optimal in the sense of Kung-Traub's conjecture. We show that its convergence order is 2. In what follows, we present the error equation of (2.2).

**Theorem 2.1.** *Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ be a scalar function which has a simple root $\alpha$ in the open interval $I$, and also the initail approximation $x_0$ is sufficiently close the simple zero, then, the one-step iteration method (2.2) has optimal convergence order 2.*

Now, we choose the parameter $\beta$ in without memory method (2.2) to make it with memory method. We propose the following new methods with memory, $x_0, \beta_0$ are given then $w_0 = x_0 + \beta_0 f(x_0)$

$$\begin{cases} \beta_k = \frac{-1}{N_2'(x_k)}, \; k = 1, 2, \cdots, \\ w_k = x_k + \beta_k f(x_k), \; x_{k+1} = w_k + \frac{\beta_k f(w_k) f(x_k)}{f(x_k) - f(w_k)}, \; k = 0, 1, 2, \cdots. \end{cases} \tag{2.3}$$

**Theorem 2.2.** *If an initial guess $x_0$ is sufficiently close to the zero $\alpha$ of $f(x)$ and the parameter $\beta_k$ in the iterative scheme (2.3) is recursively calculated by the form given in (2.3), then the convergence order of the with memory methods (2.3) is at least 3.*

In order to check the effectiveness of the proposed iterative methods we used the same test functions as Torkashvand et al. [8] in numerical comparison:(the approximation $x_0$ to $\alpha$, where $\alpha$ is the exact root.)

$$\begin{cases} f_1(t) = t\log(1 + t\sin(t)) + e^{-1+t^2+t\cos(t)}\sin(\pi t), \; \alpha = 0, \; x_0 = 0.6, \\ f_2(t) = 1 + \frac{1}{t^4} - \frac{1}{t} - t^2, \; \alpha = 1, \; x_0 = 1.4, \\ f_3(t) = e^{t^3-t} - \cos(t^2 - 1) + t^3 + 1, \; \alpha = -1, \; x_0 = -1.4. \end{cases}$$

By theoretical analysis and numerical experiments, we confirm that the proposed method which is a derivative-free one-point method has high computational efficiency. Its convergence order is 3 and its efficiency index is 1.73205. Computational results and comparison with the existing well known methods confirm robust and efficient of our methods.

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TABLE 1. Numerical results for the test functions $f_1(t), f_2(t), f_3(t)$

| Methods | function | $\|x_1-\alpha\|$ | $\|x_2-\alpha\|$ | $\|x_3-\alpha\|$ | $\|x_4-\alpha\|$ | COC | EI |
|---|---|---|---|---|---|---|---|
| *Abbasbandy* [1] | $f_1(t)$ | 0.83336(0) | 0.36700(2) | 0.36687(2) | 0.36674(2) | 3.00000 | 1.44225 |
| *Abbasbandy* [1] | $f_2(t)$ | 0.14069(−1) | 0.14716(−4) | 0.16318(−13) | 0.22249(−40) | 3.00000 | 1.44225 |
| *Abbasbandy* [1] | $f_3(t)$ | 0.45878(−1) | 0.19008(−4) | 0.21971(−14) | 0.33940(−44) | 3.00000 | 1.44225 |
| *Newton*(1.1) | $f_1(t)$ | 0.15910(0) | 0.24476(−1) | 0.58950(−3) | 0.34739(−6) | 2.00000 | 1.41421 |
| *Newton*(1.1) | $f_2(t)$ | 0.66116(−1) | 0.75908(−2) | 0.93257(−4) | 0.13917(−7) | 2.00000 | 1.41421 |
| *Newton*(1.1) | $f_3(t)$ | 0.19629(0) | 0.71856(−1) | 0.14357(−1) | 0.74750(−3) | 2.00000 | 1.41421 |
| *Chebyshev*[6] | $f_1(t)$ | 0.12676(1) | 0.11282(1) | 0.11213(1) | 0.11213(1) | 3.00000 | 1.44225 |
| *Chebyshev*[6] | $f_2(t)$ | 0.68718(−1) | 0.75645(−3) | 0.57650(−9) | 0.25292(−27) | 3.00000 | 1.44225 |
| *Chebyshev* [6] | $f_3(t)$ | 0.90711(−2) | 0.90621(−6) | 0.93272(−18) | 0.10170(−53) | 3.00000 | 1.44225 |
| *Halley* [6] | $f_1(t)$ | 0.33637(0) | 0.40072(−1) | 0.28364(−4) | 0.63850(−14) | 3.00000 | 1.44225 |
| *Halley* [6] | $f_2(t)$ | 0.68732(−1) | 0.29397(−3) | 0.31528(−10) | 0.38862(−31) | 3.00000 | 1.44225 |
| *Halley* [6] | $f_3(t)$ | 0.67737(−2) | 0.33595(−6) | 0.41455(−19) | 0.77888(−58) | 3.00000 | 1.44225 |
| *Torkashvand*(2.3) | $f_1(t)$ | 0.47811(0) | 0.69702(−1) | 0.30072(−3) | 0.10244(−10) | 3.00000 | 1.73205 |
| *Torkashvand*(2.3) | $f_2(t)$ | 0.60801(−1) | 0.88768(−3) | 0.78586(−9) | 0.60277(−27) | 3.00000 | 1.73205 |
| *Torkashvand*(2.3) | $f_3(t)$ | 0.24592(−1) | 0.18316(−4) | 0.63525(−14) | 0.28029(−42) | 3.00000 | 1.73205 |


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# ON DUALITY OF THE UNBOUNDED CONTINUOUS OPERATORS BETWEEN BANACH LATTICES

O. ZABETI

*Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran*
*o.zabeti@gmail.com*

Abstract. In this talk, we consider unbounded continuous operators between Banach lattices by replacing weak convergence with the unbounded absolute weak convergence. We shall investigate the duality notion for these classes of continuous operators.

Suppose $E$ is a Banach lattice. A net $(x_\alpha)$ in $E$ is said to be **unbounded absolute weak convergent** ($uaw$-convergent, for short) to $x \in E$ if for each $u \in E_+$, $|x_\alpha - x| \wedge u \xrightarrow{w} 0$. $(x_\alpha)$ is **unbounded norm convergent** ($un$-convergent, in brief) if $\||x_\alpha - x| \wedge u\| \to 0$. Both convergences are topological. For ample information on these concepts, see [2, 4, 5].

Now, we consider the following observations as unbounded versions of continuous operators.

Suppose $E$ is a Banach lattice and $X$ is a Banach space. A continuous operator $T : E \to X$ is called **unbounded continuous** if for each bounded sequence $(x_n) \subseteq E$, $x_n \xrightarrow{uaw} 0$ implies that $T(x_n) \xrightarrow{w} 0$.

Observe that a continuous operator $T : E \to F$, where $E$ and $F$ are Banach lattices, is said to be $uaw$**-continuous** if $T$ maps every norm bounded $uaw$-null sequence into a $uaw$-null sequence. Consider this point that sequentially $uaw$-continuous operators were introduced in [3] at first as a beside note. Unbounded continuous operators as well as $uaw$-continuous operators have been studied in [6], extensively.

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In this talk, we shall obtain some conditions under which, the dual of an either unbounded continuous operator or a $uaw$-continuous operator has a similar property. Furthermore, we investigate this duality with weakly compact operators, as well. For undefined terminology and related notions, see [1]

*Remark* 0.1. Observe that in general, there are no relations between unbounded continuous operators and weakly compact ones. Consider [3, Example 2.21]; the operator $T:\ell_1\to L_2[0,1]$ defined by $T(x_n)=(\sum_{n=1}^{\infty}x_n)\chi_{[0,1]}$ for all $(x_n)\in\ell_2$ where $\chi_{[0,1]}$ denotes the characteristic function of $[0,1]$. It is weakly compact but not unbounded continuous. Indeed, the standard basis $(e_n)$ in $\ell_1$ is $uaw$-null but $T(e_n)$ is not weakly null since $\int_0^1\chi_{[0,1]}dt=1$. Moreover, the identity operator on $\ell_\infty$ is not weakly compact yet it is unbounded continuous using [5, Theorem 7].

**Theorem 0.2.** *Suppose $E$ is a Banach lattice and $F$ is an order continuous Banach lattice. Then every weakly compact operator $T:E\to F$ has an unbounded continuous adjoint.*

*Proof.* Assume that $(x_n{}')$ is a norm bounded sequence in $F'$ which is $uaw$-null. By [5, Proposition 5] $x_n{}'\xrightarrow{w^*}0$. By the Gantmacher theorem [1, Theorem 5.23], $T'(x_n{}')\xrightarrow{w}0$, as desired. ☐

*Remark* 0.3. Weakly compactness of operator $T$ and also order continuity of $F$ are essential in Theorem 0.2 and can not be dropped. Consider the identity operator $I:c_0\to c_0$. $I$ is not weakly compact but $F$ is order continuous. Furthermore, $I$ is also unbounded continuous. Its adjoint, $I:\ell_1\to\ell_1$ is not unbounded continuous; assume $(e_n)$ is the standard basis of $\ell_1$. It is $uaw$-null by [5, Lemma 2]. But, certainly, it is not weakly null in $\ell_1$.

Also, consider the operator $T:L_2[0,1]\to\ell_\infty$ defined via $T(f)=(\int_0^1 f(t)dt,\int_0^1 f(t)dt,\ldots)$. It is weakly compact but $F$ is not order continuous. Consider the operator $T':(\ell_\infty)'\to L_2[0,1]$. It is not unbounded continuous. Consider the standard basis $(e_n)$ which is $uaw$-null in $(\ell_\infty)'$. But $<T'(e_n),1>=<\mathsf{e_n},\mathsf{T}(1)>=1$.

**Theorem 0.4.** *Suppose $E$ is a Grothendieck space and $F$ is an order continuous Banach lattice. Moreover, assume that $T:E\to F$ is an unbounded continuous operator. Then $T':F'\to E'$ is also unbounded continuous.*

*Proof.* Suppose $(x_n{}')$ is a norm bounded $uaw$-null sequence in $F'$. By [5, Proposition 5], $x_n{}'\xrightarrow{w^*}0$. So, $T'(x_n{}')\xrightarrow{w^*}0$ in $E'$. By the Grothendieck property, we have $T'(x_n{}')\xrightarrow{w}0$. ☐

Observe that the Grothendieck property of $E$ is essential in Theorem 0.4 and can not be removed. Consider again the identity operator $I$ on $c_0$. By using [5, Theorem 7], it is easy to see that $I$ is unbounded continuous but its adjoint is not. Note that $c_0$ dose not have the Grothendieck property.

**Theorem 0.5.** *Suppose $E$ is a Banach lattice whose dual is order continuous and atomic and $F$ is an order continuous Banach lattice. Then every positive operator $T : E \to F$ has a uaw-continuous adjoint.*

*Proof.* Consider the positive operator $T' : F' \to E'$. Suppose $({x_n}')$ is a positive bounded sequence in $F'$ such that ${x_n}' \xrightarrow{uaw} 0$. By [5, Proposition 5], ${x_n}' \xrightarrow{w^*} 0$ so that $T'{x_n}' \xrightarrow{w^*} 0$. By [4, Proposition 8.5] and [5, Theorem 7], we see that $T'{x_n}' \xrightarrow{uaw} 0$, as claimed. □

Furthermore, when the operator $T$ is not positive, we may consider the following.

**Proposition 0.6.** *Suppose $E$ is an order continuous Banach lattice whose dual is also order continuous and atomic and $F$ is an order continuous Banach lattice. Then every continuos operator $T : E \to F$ has a uaw-continuous adjoint.*

*Proof.* Consider the operator $T' : F' \to E'$. Suppose $({x_n}')$ is a positive bounded sequence in $F'$ such that ${x_n}' \xrightarrow{uaw} 0$. By [5, Proposition 5], ${x_n}' \xrightarrow{w^*} 0$ so that $T'{x_n}' \xrightarrow{w^*} 0$. By [4, Theorem 8.4] and [5, Theorem 7], we see that $T'{x_n}' \xrightarrow{uaw} 0$, as claimed. □

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# CHARACTERIZATION OF $n$-JORDAN MULTIPLIERS

ABBAS ZIVARI-KAZEMPOUR

*Department of Mathematics, Ayatollah Borujerdi University, Borujerd, Iran*
*zivari@abru.ac.ir*

Abstract. Let $A$ be a Banach algebra and $X$ be an $A$-bimodule. A linear map $T : A \longrightarrow X$ is called an $n$-Jordan multiplier if $T(a^n) = aT(a^{n-1})$, for all $a \in A$. In this paper, under special hypotheses we show that every $(n+1)$-Jordan multiplier is an $n$-Jordan multiplier and vice versa.

## 1. Introduction

Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. A map $T : A \longrightarrow X$ is called *left multiplier* [*right multiplier*] if for all $a, b \in A$,

$$T(ab) = T(a)b, \quad [T(ab) = aT(b)],$$

and $T$ is called a *multiplier* if it is both left and right multiplier. Also, $T$ is called *left Jordan multiplier* [*right Jordan multiplier*] if for all $a \in A$,

$$T(a^2) = T(a)a, \quad [T(a^2) = aT(a)],$$

and $T$ is called a *Jordan multiplier* if $T$ is a left and a right Jordan multiplier.

The general theory of multipliers on Banach algebras has been developed by Johnson in [3]. He proved that each multiplier $T : A \longrightarrow A$ on without order Banach algebra $A$ is linear and continuous.

Recall that the Banach algebra $A$ is called *without order*, if for all $x \in A$, $xA = \{0\}$ $[Ax = \{0\}]$ implies $x = 0$.

Clearly, every left (right) multiplier is a left (right) Jordan multiplier, but the converse is not true in general, see [2, Example 2.6]. One may

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refer to the monograph [5] for the additional fundamental results in the theory of multipliers.

There exists another related concept, called (two-sided) multiplier. A map $T : A \longrightarrow X$ is said to be multiplier if for every $a, b \in A$,

$$aT(b) = T(a)b. \tag{1.1}$$

If $T$ is both left and right multiplier, then $T$ is a multiplier, according to (1.1), but the converse is, in general, false. The following example obtained by the author in [6].

**Example 1.1.** *Let*

$$A = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : \quad a, b, c \in \mathbb{R} \right\},$$

*and define* $T : A \longrightarrow A$ *by*

$$T\left( \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}.$$

*Then, for all* $x, y \in A$, $T(x)y = xT(y)$, *hence* $T$ *is a multiplier as in (1.1), but it is not left (right) multiplier, because* $T(x)y \neq T(xy) = 0$, *in general.*

Let $A$ be a unital Banach algebra with unit $e_A$. An $A$-bimodule $X$ is called *unitary* if $e_A x = x e_A = x$, for all $x \in X$. For example, $A^*$ is an unitary $A$-bimodule with the following actions.

$$a \cdot f(x) := f(xa), \quad f \cdot a(x) := f(ax), \quad a, x \in A, \quad f \in A^*.$$

**Definition 1.2.** Let $A$ be a Banach algebra, $X$ be a left $A$-module and let $T : A \longrightarrow X$ be a linear map. Then $T$ is called *right $n$-multiplier* if

$$T(a_1 a_2 ... a_n) = a_1 T(a_2 ... a_n),$$

for all $a_1, a_2, ..., a_n \in A$. Moreover, $T$ is called *right $n$-Jordan multiplier* if for all $a \in A$,

$$T(a^n) = aT(a^{n-1}).$$

The left version of *$n$-multiplier* and *$n$-Jordan multiplier* can be defined analogously.

The concept of $n$-multiplier was introduced and studied by Laali and Fozouni in [4], where some interesting results related to these maps were obtained. The notion of $n$-Jordan multiplier was introduced in [1].

It is clear that every $n$-multiplier is an $(n+1)$-multiplier, while on the other hand it was shown in [4, Theorem 2], that in the general case $Mul_n(A,X) \subsetneq Mul_{n+1}(A,X)$, where $Mul_n(A,X)$ is the set of all $n$-multipliers from a Banach algebra $A$ into its module $X$.

Moreover, by [4, Theorem 3] if $A$ is an essential Banach algebra, then $Mul_n(A,X) = Mul_{n+1}(A,X)$.

Every $n$-multiplier is an $n$-Jordan multiplier but the converse is false by [1, Proposition 6]. See also [2, Example 2.6] for $n = 2$.

On the other hand some $(n+1)$-Jordan multipliers fail to be $n$-Jordan multipliers. Hence neither $(n+1)$-Jordan multipliers are neccessarily $n$-Jordan multipliers nor $n$-Jordan multipliers are automatically $(n+1)$-Jordan multipliers. Now the following questions can be raised.

*Under which conditions for a Banach algebra $A$ is any $(n+1)$-Jordan multiplier $T : A \longrightarrow X$ automatically an $n$-Jordan multiplier and vice versa? Moreover, when is any $n$-Jordan multiplier automatically an $n$-multiplier?*

In this paper we investigate this question and prove that under suitable conditions the concepts of $(n+1)$-Jordan multiplier, $n$-Jordan multiplier and Jordan multiplier are equivalent.

## 2. Main results

Since all results which are true for right multipliers have obvious analogue statements for left multipliers, we will focus in the sequel just on the right versions.

**Lemma 2.1.** *Let $A$ be a Banach algebra, $X$ be a left $A$-module and let $T : A \longrightarrow X$ be a linear and right Jordan multiplier. Then $T$ is a right $n$-Jordan multiplier, for $n \geq 2$.*

**Theorem 2.2.** *Let $A$ be a unital Banach algebra, and $X$ be a unitary Banach left $A$-module. Suppose that $T : A \longrightarrow X$ is a continuous linear map. If $T(ab) = aT(b)$ for all $a, b \in A$ with $ab = e_A$, then $T$ is a right $n$-Jordan multiplier.*

As a consequence of Theorem 2.2, we have the next result.

**Corollary 2.3.** *Let $A$ be a unital Banach algebra, $X$ be a unitary Banach left $A$-module and let $T : A \longrightarrow X$ be a continuous linear map. If $a \in Inv(A)$ and $T(aa^{-1}) = aT(a^{-1})$, then $T$ is a right $n$-Jordan multiplier.*

**Theorem 2.4.** *Let $A$ be a unital Banach algebra, and $X$ be a unitary Banach lef $A$-module. Let $T : A \longrightarrow X$ be a continuous linear map. If for an idempotent $p \in A$, $T(ab) = aT(b)$ for all $a, b \in A$ with $ab = p$, then $T$ is a right $n$-Jordan multiplier on $pAp$.*

**Theorem 2.5.** *Let $n \in \{2, 3\}$ be fixed, $A$ be a unital Banach algebra and $X$ be a unitary left $A$-module. Then every right $(n+1)$-Jordan multiplier $T : A \longrightarrow X$ is a right $n$-Jordan multiplier.*

Next we generalize Theorem 2.5, for all $n \in \mathbb{N}$.

**Theorem 2.6.** *Let $A$ be a unital Banach algebra, $X$ be a unitary Banach left $A$-module. Then every right $(n+1)$-Jordan multiplier $T : A \longrightarrow X$ is a right $n$-Jordan multiplier.*

From Lemma 2.1 and Theorem 2.6 we get the following result.

**Corollary 2.7.** *Let $A$ be a unital Banach algebra and $X$ be a unitary Banach left $A$-module. Suppose that $T : A \longrightarrow X$ is a linear map. Then the following conditions are equivalent.*

(1) *$T(a) = aT(e_A)$, for all $a \in A$.*
(2) *$T$ is a right Jordan multiplier.*
(3) *$T$ is a right $n$-Jordan multiplier.*
(4) *$T$ is a right $(n+1)$-Jordan multiplier.*

Combining Corollary 2.7 and Theorem 2.2, we get the next result.

**Corollary 2.8.** *Let $A$ be a unital Banach algebra and $X$ be a unitary Banach left $A$-module. Suppose that $T : A \longrightarrow X$ is a continuous linear map. If $T(ab) = aT(b)$ for all $a, b \in A$ with $ab = e_A$, then $T$ is a right $n$-multiplier.*

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# مجموعه مقاله های فارسی

سخنرانی

# روش‌های بهینه‌سازی مبتنی بر روش هم‌مکانی چندجمله‌ای برنشتاین برای حل تقریبی معادلات دیفرانسیل معمولی

محمدرضا انصاری

گروه ریاضی کاربردی، دانشگاه بین‌المللی امام خمینی(ره)، قزوین، ایران
*mr.ansari@sci.ikiu.ac.ir*

چکیده. یک مشکل اساسی در روش هم‌مکانی بدشرط شدن ماتریس ضرایب با افزایش درجه تقریب است. این پدیده باعث مشکلات عددی و کاهش در دقت جواب می‌شود. در این مقاله، روش‌هایی بر اساس ترکیب روش هم‌مکانی برنشتاین و روش‌های بهینه‌سازی برای جواب تقریبی معادلات دیفرانسیل خطی با شرایط اولیه و یا مرزی پیشنهاد شده است. در این‌جا، جواب تقریبی با استفاده از جواب یک مساله بهینه‌ازی مقید به دست می‌آید. برای بررسی کارایی روش‌ها، مسایل آزمایشی از مرتبه‌های مختلف در نظر گرفته شده و نتایج به دست آمده با نتایج روش‌های دیگر مقایسه می‌شوند. بررسی‌ها نشان می‌دهند که روش‌های پیشنهاد شده دقیق، کارا و دارای پایداری عددی خوبی هستند.

## ۱. پیش‌گفتار

در این مقاله، هدف ما ارایه روش‌هایی با دقت بالا و کارا برای حل تقریبی معادله دیفرانسیل خطی مرتبه $n$ با ضرایب متغیر به شکل کلی زیر است:

$$L(y) \equiv \sum_{k=0}^{n} p_k(x) y^{(k)}(x) = f(x), \qquad x \in [a, b] \tag{۱.۱}$$

که در آن، $L$ عملگر دیفرانسیل خطی مرتبه $n$، $f(x)$ و $p_i(x)$ ها توابعی معلوم و $[a, b]$ دامنه مساله می‌باشد. در اینجا و در ادامه، مشتق $k$ام تابع $y$ نسبت به متغیر $x$ با $y^{(k)}(x)$ نشان داده می‌شود. تعداد $n-1$ شرط اولیه یا شرط مرزی لازم برای حل (۱.۱) عبارتند از:

$$S_l(y) = q_l, \qquad l = 1, \ldots, n-1 \tag{۲.۱}$$

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**واژگان کلیدی.** معادلات مسایل مقدار اولیه و مرزی، معادله دیفرانسیل خطی، روش هم‌مکانی برنشتاین، بهینه‌سازی .

که در آن، $S_l$ها عملگرهای خطی شرایط اولیه و یا شرایط مرزی بوده و $q_l$ها حقیقی و معلوم هستند.

چندجمله‌ای‌های برنشتاین به دلیل ویژگی‌های خوب خود در نظریه تقریب و طراحی هندسی بسیار مورد توجه قرار داشته‌اند و از این چندجمله‌ای‌ها در حل انواع معادلات دیفرانسیل خطی و غیرخطی استفاده شده است [۱، ۲، ۳، ۵، ۷]. علاوه براین، در سال‌های اخیر روش‌های هم‌مکانی به صورت گسترده برای حل انواع معادلات دیفرانسیل استفاده شده‌اند [۱، ۲، ۶]. در مسایل خطی، روش‌های هم‌مکانی اغلب حل مساله را به حل یک دستگاه معادلات خطی تبدیل می‌کنند. مشکلی که در اینجا می‌تواند بروز کند، بدشرط بودن ماتریس ضرایب است که باعث بروز مشکلات عددی وکاهش دقت جواب می‌شود. این مشکل اغلب زمانی پدید می‌آید که نقاط هم‌مکانی خیلی به هم نزدیک باشند.

در گذشته، روش‌های مبتنی بر بهینه‌سازی در مدل‌سازی و حل بسیاری از مسایل با موفقیت به کار گرفته شده‌اند [۶، ۸]. در این مقاله، ما سه روش براساس کمینه‌سازی باقیمانده‌های متناظر با معادلات هم‌مکانی چندجمله‌ای‌های برنشتاین برای حل معادله (۱.۱) و (۲.۱) پیشنهاد می‌کنیم. این روش‌ها نه تنها نسبت به روش هم‌مکانی جواب‌های دقیق‌تری می‌دهند، بلکه از نظر عددی با افزایش درجه تقریب پایدار می‌باشند.

## ۲. چندجمله‌ای‌های برنشتاین

چندجمله‌ای‌های پایه برنشتاین از مرتبه $m$ بر بازه $[a,b]$ را $B_{i,m}(x)$ها می‌گیریم و برای سادگی، متناظر با $i<0$ و $i>m$ فرض می‌شود $B_{i,m}=0$. از مزایای این چندجمله‌ای‌ها، سادگی محاسبه مقادیر تابع و مشتقات آن در نقاط مرزی است [۴]. با تعریف

$$\phi_m(x)=[B_{0,m}(x)\ B_{1,m}(x)\ \dots\ B_{m,m}(x)]^T \tag{۱.۲}$$

داریم $\phi_m^{(k)}(x)=D^k\phi_m(x)$ که در آن، $D$ ماتریس عملیاتی مشتق با درایه‌های زیر است [۴]:

$$d_{i,j}=\begin{cases}-j, & i=j-1\\ 2j-m, & i=j\\ m-j, & i=j+1\\ 0, & \text{در غیر اینصورت}\end{cases}\qquad i,j=1,\dots,m+1$$

فرض کنید که $y\in C^k[a,b]$. چندجمله‌ای برنشتاین متناظر با $y$ در بازه $[a,b]$ به صورت زیر تعریف می‌شود:

$$B_m(y;x)=\sum_{k=0}^{m}y(a+\frac{(b-a)i}{m})B_{i,m}(x).$$

قضیه زیر بستر مناسب برای استفاده از این چندجمله‌ای‌های در تقریب توابع را فراهم می‌کند [۲].

**قضیه ۱.۲.** فرض کنید که $K>0$ عدد صحیح مثبتی باشد و $y\in C^k[a,b]$ . در این صورت،

$$\lim_{m\to\infty}B_m^{(k)}(y;x)=y^{(k)}(x),\qquad k=0,1,\dots,K.$$

## ۳. روش هم‌مکانی

یک جواب هم‌مکانی $y_m$ برای یک معادله تابعی روی بازه $\Omega$، عضوی از یک فضای تابعی با بعد متناهی به نام فضای هم‌مکانی است که معادله را در زیر مجموعه‌ای متناهی و خاص از نقاط به نام نقاط هم‌مکانی برقرار

می‌کند. اگر شرایط اولیه یا مرزی نیز وجود داشته باشند، آنگاه بدیهی است که $y_m$ باید این شرایط را نیز برقرار کند. معادله (۱.۱) را در نظر بگیرید و فرض کنید

$$y_m(x) = \sum_{i=0}^{m} c_i B_{i,m}(x) = \phi_m(x)^T C \tag{۱.۳}$$

تقریب جواب باشد که در آن، $C = [c_0,\ c_1,\ \ldots,\ c_m]$ بردار ضرایب مجهول و $\phi_m(x)$ بردار توابع پایه است که در (۱.۲) تعریف شده است. با جای‌گذاری (۱.۳) در (۱.۱) و استفاده از نقاط هم‌مکانی $x_j \in [a, b]$، برای $j = 1, \ldots, m - n + 2$ به دست می‌آید:

$$\Big(\sum_{k=0}^{n} p_k(x_j) D^k \phi_m(x_j)\Big)^T C = f(x_j), \qquad j = 1, \ldots, m-n+2 \tag{۲.۳}$$

هم‌چنین، با جای‌گذاری (۱.۳) در شرایط اولیه و یا مرزی (۲.۱) و استفاده از خطی بودن $S_l$ها به دست می‌آید:

$$\big(S_l(\phi_m)\big)^T C = q_l, \qquad l = 1,\ \ldots,\ n-1 \tag{۳.۳}$$

مجموع $m - n + 2$ معادله (۲.۳) و $n - 1$ معادله (۳.۳) دستگاهی متشکل از $m + 1$ معادله و $m + 1$ مجهول می‌سازد که با حل آن $c_i$ها و در نتیجه $y_m$ به دست می‌آید.

## ۴. روش‌های پیشنهادی مبتنی بر بهینه‌سازی

متناظر با معادله (۱.۱)، تابع باقیمانده به صورت زیر تعریف می‌شود:

$$R(y; x) = L\big(y(x)\big) - f(x).$$

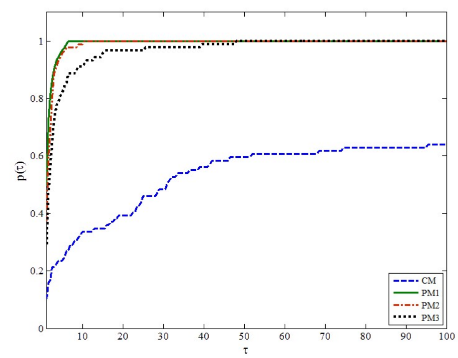
برای جواب تقریبی $y_m$ تابع باقیمانده برابر است با

$$R_m(x) \equiv R(y_m; x) = L\big(\phi_m(x)\big)^T C - f(x).$$

بردار $C$ با استفاده از ترکیب روش هم‌مکانی و کمینه‌سازی خطای متناظر با تابع $R_m(x)$ تعیین خواهد شد. $y_m$ باید طوری تعیین شود که $n - 1$ شرط اولیه یا مرزی به طور دقیق برآورده شوند. این $n - 1$ معادله به دست می‌دهد و در نتیجه برای تعیین $y_m$ تنها $m - n + 2$ درجه آزادی باقی می‌ماند. نقاط هم‌مکانی را $x_1, \ldots, x_{n_p}$ در نظر می‌گیریم که $n_p \geq m - n + 2$. این نقاط را به دو زیرمجموعه متمایز $A$ و $B$ افراز کرده و اندیس نقاط دو مجموعه را به ترتیب با $I$ و $J$ نشان می‌دهیم. برای هر $j \in J$ قرار می‌دهیم $R_m(x_j) = 0$، تا در این نقاط $y_m$ به طور دقیق در معادله دیفرانسیل صدق کند. بهتر است این نقاط تا حد ممکن به طور یکنواخت در بازه $[a, b]$ توضیع شده باشند و تعداد آن‌ها باید کمتر از $m - n + 2$ (برای مثال، $0.5(m - n + 2)$) باشد. در غیر این صورت، این معادلات $y_m$ را به طور دقیق معلوم می‌کنند. برای نقاط که اندیس آن‌ها را $I$ در نظر می‌گیریم، خطاهای متناظر با تابع باقیمانده را کمینه خواهیم کرد. توضیح بالا منجر به پیشنهاد سه رویکرد می‌شود: الف) کمینه‌سازی مجموع مربعات خطا ب) کمینه‌سازی مجموع خطا و پ) کمینه‌سازی مجموع خطا. در هر سه رویکرد اگر $n_p = m - n + 2$، خطا در همه نقاط هم‌مکانی صفر شده و روش هم‌ارز روش هم‌مکانی خواهد شد. در غیر این صورت، چون کمینه‌سازی خطا در نقاط بیشتری مد نظر قرار می‌گیرد (به عبارتی، فضای امکان‌پذیر بزرگ‌تر است)، باید حداکثر خطای تقریب دست‌کم روی نقاط مورد استفاده کم‌تر از خطای تقریب در روش هم‌مکانی باشد.

## ۵. نتایج عددی

ما مسایل آزمون گوناگون با شرایط اولیه و یا مرزی از مرتبه‌های مختلف در نظر گرفتیم که شامل ۷ مساله مرتبه ۲ و ۶ مساله از مرتبه بزرگ‌تر بودند. نمودارهای شکل ۱ سطح کارایی الگوریتم‌های پیشنهادی ما (کمینه‌سازی مجموع مربعات خطا PM1، مجموع خطا PM2 و بیشینه خطا PM3) و روش هم‌مکانی CM را روی تمامی مسایل آزمون و برای $m$ های مختلف را نشان می‌دهد. برای مفایسه از معیار دولان و موره [۴] استفاده شده است. با توجه به این معیار در مقایسه عملکرد دو الگوریتم، الگوریتمی کارایی بهتر دارد که نمودار آن بالاتر از نمودار متناظر با الگوریتم دیگر قرار بگیرد و البته هر چه اختلاف بین دو نمودار بیش‌تر باشد، تفاوت بیشتري بین کارایی دو الگوریتم روی مسایل مورد آزمون وجود داشته است. نمودارهای شکل ۱ به خوبی نشان می‌دهند که




شکل ۱: نمودارهای عملکرد الگوریتم‌های CM، PM۱، PM۲ و PM۳ با معیار خطای میانگین

الگوریتم‌های پیشنهادی مبتنی بر بهینه‌سازی ما نسبت به روش هم‌مکانی عملکرد بهتری داشته‌اند. در این میان، به ترتیب الگوریتم‌های PM۱، PM۲ و سپس PM۳ بهترین عملکردها را نشان داده‌اند.

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بیست و چهارمین سمینار آنالیز ریاضی و کاربردهای آن
۵-۶ خرداد ۱۴۰۰، دانشگاه بین المللی امام خمینی (ره)

پوستر

# پایداری مجانبی یک مدل اپیدمی زمان-گسسته با نرخ شیوع دو خطی

محمود پارسامنش [1]

[1] گروه ریاضی، دانشکده شهید مهاجر اصفهان، دانشگاه فنی و حرفه ای
*mparsamanesh@tvu.ac.ir*

چکیده. در این مقاله ابتدا یک مدل اپیدمی از نوع SIS با واکسیناسیون توصیف و در قالب دستگاه معادلات تفاضلی بیان می‌شود. سپس پایداری مجانبی نقاط تعادل مطالعه می‌گردد و دینامیک مدل بر حسب یک مقدار آستانه‌ای مشخص می‌شود. نتایج تئوری بدست آمده، توسط نتایج عددی شبیه سازی شده نیز بررسی می‌شود.

## ۱. پیش‌گفتار

بسیاری از مدلهای اپیدمی توسط معادلات دیفرانسیل فرمول بندی شده اند[۳]. با این وجود معادلات تفاضلی با توجه روز افزون مواجه هستند[۲]. چرا که معمولاً داده ها به صورت گسسته در دسترس هستند و پیاده سازی معادلات تفاضلی ساده‌تر از معادلات دیفرانسیل است. همچنین دستگاه معادلات تفاضلی دینامیک غنی‌تری به نمایش می‌گذارد.

## ۲. معرفی مدل

فرض کنیم افراد یک جامعه بسته به وضعیتشان نسبت به بیماری به سه گروه در معرض ابتلا، عفونی شده، و واکسینه تقسیم شوند و تعداد افراد هر یک از این گروه ها در زمان به ترتیب با $I_t$، $S_t$ و $V_t$ نمایش داده شوند. نرخ مرگ در همه گروه ها $\mu$ است و فرض می کنیم که مرگ در اثر بیماری نداریم. به علاوه فرض می کنیم هر مرگ با یک زاد همراه است و جمعیتی معادل $\mu N$ در هر واحد زمانی به جمعیت افزوده می شود که یک کسر $q$ از آن ها واکسینه شده و مابقی در معرض ابتلا خواهند بود. به علاوه واکسیناسیون بر روی افراد گروه $S$ نیز با نرخ $\varphi$ انجام

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**واژگان کلیدی.** مدل اپیدمی، واکسیناسیون، پایداری، انشعاب، ماتریس ژاکوبی .

می‌شود. از طرفی واکسیناسیون موقتی بوده و افراد گروه $V$ با نرخ $\theta$ مصونیت خود را از دست می‌دهند. $\beta$ نرخ تماس افراد عفونی شده با افراد سالم و $\gamma$ نرخ بهبود افراد از بیماری است. تمامی پارامترها نامنفی و کمتر از یک فرض شده‌اند و بعلاوه $\mu$ مثبت است. بنابراین دستگاه معادلات تفاضلی برای مدل به صورت زیر خواهد بود:

$$\begin{aligned} S_{t+1} &= (1-q)\mu N - \beta S_t I_t + [1-(\mu+\varphi)]S_t + \gamma I_t + \theta V_t \\ I_{t+1} &= \beta S_t I_t + [1-(\mu+\gamma)]I_t \\ V_{t+1} &= q\mu N + \varphi S_t + [1-(\mu+\theta)]V_t \end{aligned} \tag{۱.۲}$$

برای جمعیت کلی جامعه داریم :

$$N_{t+1} = S_{t+1} + I_{t+1} + V_{t+1} = S_t + I_t + V_t = S_0 + I_0 + V_0 = N_0 = N$$

لذا جمعیت در مدل همواره ثابت است. در این صورت $V_t = N - (S_t + I_t)$ و لذا با حذف آن از معادلات روی دستگاه زیر تمرکز می‌کنیم:

$$\begin{aligned} S_{t+1} &= [(1-q)\mu + \theta]N - \beta S_t I_t + [1-(\mu+\varphi+\theta)]S_t + (\gamma-\theta)I_t \\ I_{t+1} &= \beta S_t I_t + [1-(\mu+\gamma)]I_t \end{aligned} \tag{۲.۲}$$

شرایط $\mu+\varphi+\theta+\beta < 1$ و $\mu+\gamma < 1$ تضمین می‌کنند که جواب‌ها نامنفی می‌مانند.

## ۳. پایداری مدل

نقاط تعادل مدل (۲.۲) از جواب های دستگاه زیر بدست می‌آیند:

$$\begin{aligned} (\mu+\varphi+\theta)\bar{S} &= [(1-q)\mu+\theta]N - \beta\bar{S}\bar{I} + (\gamma-\theta)\bar{I} \\ (\mu+\gamma)\bar{I} &= \beta\bar{S}\bar{I} \end{aligned} \tag{۱.۳}$$

دو نقطه تعادل برای مدل (۲) وجود دارد. نقطه تعادل عاری از بیماری $Q^0 = (I^0, S^0) = (0, \frac{[(1-q)\mu+\theta]N}{\mu+\varphi+\theta})$ و نقطه تعادل همه‌گیر $Q^* = (I^*, S^*) = (\frac{[(1-q)\mu+\theta]\beta N - (\mu+\varphi+\theta)(\mu+\gamma)}{\beta(\mu+\theta)}, \frac{\mu+\gamma}{\beta})$. این نقطه تعادل وجود دارد اگر $\mathcal{R}_0 = \frac{[(1-q)\mu+\theta]\beta N}{(\mu+\varphi+\theta)(\mu+\gamma)} > 1$ که در آن $\mathcal{R}_0$ عدد مولد پایه برای مدل (۲.۲) است. ماتریس ژاکوبی این مدل در $X = (I, S)$ به صورت زیر است:

$$J(X) = \begin{bmatrix} 1-(\mu+\gamma)+\beta S & \beta I \\ -\beta S + (\gamma-\theta) & 1-(\mu+\varphi+\theta)-\beta I \end{bmatrix} \tag{۲.۳}$$

لذا ماتریس ژاکوبی در $Q^0$ به صورت زیر است:

$$J^0 = J(Q^0) = \begin{bmatrix} 1-(\mu+\gamma)+(\mu+\gamma)\mathcal{R}_0 & 0 \\ (\mu+\gamma)\mathcal{R}_0 + (\gamma-\theta) & 1-(\mu+\varphi+\theta) \end{bmatrix}$$

مقادیر ویژه ماتریس ژاکوبی عبارتند از $\lambda_1 = 1-(\mu+\gamma)+(\mu+\gamma)\mathcal{R}_0$ و $\lambda_2 = 1-(\mu+\varphi+\theta)$ با توجه به مفروضات مدل داریم $|\lambda_2| < 1$ و به علاوه، $|\lambda_1| < 1$ اگر

$\mathcal{R}_0 < 1$. همچنین ماتریس ژاکوبی در $Q^*$ به صورت زیر داده می‌شود:

$$J^* = J(Q^*) = \begin{bmatrix} 1 & \frac{(\mu+\varphi+\theta)(\mu+\gamma)}{(\mu+\theta)}(\mathcal{R}_0 - 1) \\ -(\mu+\theta) & 1-(\mu+\varphi+\theta) - \frac{(\mu+\varphi+\theta)(\mu+\gamma)}{(\mu+\theta)}(\mathcal{R}_0 - 1) \end{bmatrix}$$

معادله مشخصه $J^*$ به صورت زیر است:

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2$$

که در آن $a_1 = -Tr(J^*) = 2 + b_1$ و $a_2 = \det(J^*) = 1 + b_1 + b_2$ و $b_1$ و $b_2$ در آن به صورت زیر فرض شده‌اند: $b_1 = -[(\mu+\varphi+\theta)(1+(\frac{\mu+\gamma}{\mu+\theta})(\mathcal{R}_0 - 1))]$ و $b_2 = (\mu+\varphi+\theta)(\mu+\gamma)(\mathcal{R}_0 - 1)$. شرایط Jury برای پایداری مجانبی موضعی ماتریس ژاکوبی $J^*$ به صورت زیر است [۱]:

$$|Tr(J^*)| < 1 + \det(J^*) < 2$$

اولاً، با جایگذاری مقادیر قبلی عبارت $Tr(J^*) < 1 + \det(J^*) < 2$ معادل است با

$$0 < (\mu+\varphi+\theta)(\mu+\gamma)(\mathcal{R}_0 - 1) < (\mu+\varphi+\theta)[1 + (\frac{\mu+\gamma}{\mu+\theta})(R_0 - 1)]$$

که این رابطه خودبه‌خود برقرار است. ثانیاً، می‌توان نشان داد $-(1+\det(J^*)) < Tr(J^*)$ اگر وفقط اگر

$$2[(\mu+\varphi+\theta) + \beta I^*] < 4 + (\mu+\theta)\beta I^* \tag{۳.۳}$$

نتایج بدست آمده را در قضیه زیر بیان می‌کنیم:

**قضیه ۱.۳.** با مفروضات مدل (۲) داریم:

الف) اگر $\mathcal{R}_0 \leq 1$ فقط یک نقطه تعادل عاری از بیماری $Q^0$ برای مدل (۲.۲) وجود دارد. همچنین، مدل دو نقطه تعادل $Q^0$ و $Q^*$ دارد، اگر $\mathcal{R}_0 > 1$.

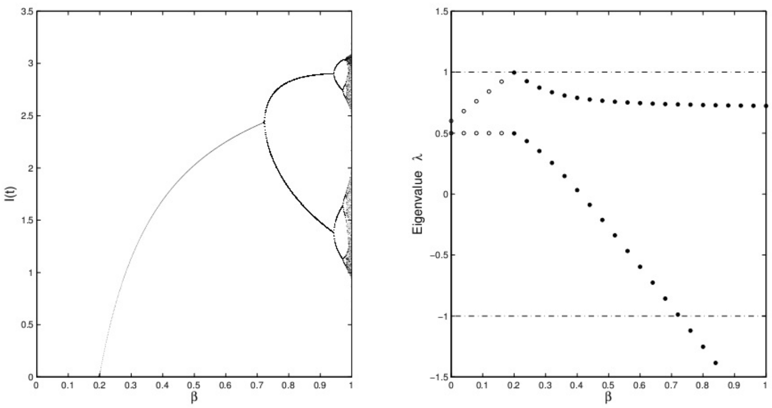
ب) نقطه تعادل عاری از بیماری $Q^0$ پایدار مجانبی است اگر $\mathcal{R}_0 < 1$ و ناپایدار است اگر $\mathcal{R}_0 > 1$.

ج) نقطه تعادل همه گیر $Q^*$ پایدار مجانبی است اگر و فقط اگر رابطه (۳.۳) برقرار باشد.

## ۴. مثال و بحث

فرض کنیم پارامترهای مدل را به صورت زیر است: $q = 0.6$، $\gamma = 0.3$، $\varphi = 0.2$، $\theta = 0.2$ و $\mu = 0.1$. به علاوه فرض کنیم واحد جمعیت یک میلیون باشد و جمعیت ثابت $N = 4.2$ با مقادیر اولیه $I_0 = 0.8$ و $S_0 = 2.4$ باشد. شکل ۱ یک نمودار انشعاب برای گروه جمعیتی افراد مبتلا و مقادیر ویژه ماتریس ژاکوبی (۲.۳) به عنوان تابعی از $\beta \in (0,1)$ را نمایش می‌دهد. وقتی $\beta < 0.2$ نقطه تعادل عاری از بیماری $Q^0$ پایدار است. در $\beta = 0.2$ بزرگترین مقدار ویژه برابر یک است و یک انشعاب شکستگی (fold bifurcation) رخ می‌دهد. برای $0.2 < \beta < 0.72$ نقطه تعادل همه گیر $Q^*$ پایدار است و در $\beta \approx 0.72$ یک انشعاب دوره-مضاعف (period-doubling bifurcation) اتفاق می‌افتد چراکه شعاع طیفی ماتریس ژاکوبی در این نقطه $\rho = -1$ می‌باشد. البته نقاط انشعاب توسط قضیه ۱.۳ نیز قابل استخراج است. همچنین، جواب‌های دستگاه (۱.۲) برای مقادیر مختلف پارامتر انشعاب $\beta$ در شکل ۲

آورده شده‌اند. رفتار جواب‌ها با مباحث فوق برای پایداری بر حسب مقادیر مختلف $\beta$ مطابقت دارد.




شکل ۱: نمودار انشعاب برای تعداد افراد مبتلا و مقادیر ویژه ماتریس ژاکوبی برحسب مقادیر مختلف $\beta$

شکل ۲: نمودار جوابهای تعداد افراد سالم (آبی)، مبتلا (سبز) و واکسینه (قرمز) در مدل برای مقادیر مختلف $\beta$

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سخنرانی

# شبیه سازی و حل معادله لاپلاس با مرزهای فرکتالی برفدانه کوخ با استفاده از نرم افزار کامسول

امیر پیشکو [1]

[1] گروه پژوهشی فیزیک نظری و محاسباتی، پژوهشکده فیزیک و شتابگرها، پژوهشگاه علوم و فنون هسته‌ای
*apishkoo@gmail.com*

چکیده. این مقاله گزارشی از مدل‌سازی معادله لاپلاس برای ناحیه‌ای با مرزهای فرکتالی (برفدانه کوخ) و شبیه‌سازی مربوط به آن با ارائه توزیع پتانسیل الکتریکی در تمامی نقاط ناحیه برای تکرار صفرم تا پنجم و شرایط مرزی دیریکله ارائه می‌کند.

## ۱. پیش‌گفتار

مدل‌سازی، روشی کمی و بنیادی برای درک و تحلیل سیستمهای پیچیده و پدیده‌ها می‌باشد. مدل‌سازی ریاضی می‌کوشد با استفاده از مدل‌های ریاضی در رایانه‌ها درک درستی از علم به‌دست آورد. مدل‌سازی ریاضی مکمل فعالیت گروهی است اما نمی‌تواند جایگزین کار نظری و تجربی در تحقیقات علمی باشد. وقتی انجام آزمایش‌ها خیلی گسترده، خیلی گران، خیلی خطرناک یا خیلی وقت گیر باشد، مدل‌سازی اغلب به‌جای آزمایش استفاده می‌شود. مدل‌سازی ریاضی به عنوان ابزاری قدرتمند و ضروری برای مطالعه انواع مسائل در تحقیقات علمی، توسعه فرآیند، و تولید ظهور کرده است. موفقیت یک مدل ریاضی بستگی به این دارد که تا چه اندازه استفاده از آن ''آسان'' است و دقت پیش‌بینی آن چقدر است. برای اینکه یک مسئله در دنیای واقعی شبیه‌سازی شود مراحل نشان داده شده در شکل ۱ می‌بایست انجام شود.

2010 Mathematics Subject Classification. Primary 28A80; Secondary 35J05, 65N30.

**واژگان کلیدی.** فرکتال، برفدانه کوخ، مدل‌سازی رایانه‌ای، شبیه‌سازی رایانه‌ای .

شکل ۱: مراحل شبیه‌سازی یک مسئله در دنیای واقعی

## ۲. حل معادله لاپلاس

برای حل معادله‌لاپلاس بعنوان یک معادله‌دیفرانسیل با مشتقات‌جزئی می‌توان از روشهای متنوع تحلیلی، نیمه تحلیلی، و عددی استفاده کرد. می‌توان روش تحلیلی جداسازی متغیرها، روش‌های عددی اویلر، تفاضل محدود، عناصر محدود، و همچنین روشهای نیمه تحلیلی HPM، DJM ADM، را نام برد[۴]. حل معادلات خطی از معادلات غیرخطی آسان‌تر است. اما حتی حل تحلیلی معادلات خطی نیز در بعضی شرایط خاص امکان‌پذیر نیست. ما معمولا به‌دنبال حل معادلات دیفرانسیل با مشتقات جزئی همراه با شرایط اولیه و شرایط مرزی هستیم. ما معمولا به‌دنبال حل معادلات دیفرانسیل با مشتقات جزئی همراه با شرایط اولیه و شرایط مرزی هستیم. شرایط مرزی میتوانند از نوع دیریکله (شکل ۲) یا نیومن باشند. در این مقاله شرایط مرزی ما از نوع دیریکله می‌باشدکه در شکل زیر مشخص شده است.

شکل ۲: شرایط مرزی دیریکله برای اضلاع ناحیه مثلث متساوی الاضلاع.

۱.۲. **هندسه فرکتال.** هندسه فرکتال یکی از ساختارهای ریاضی جذاب است. اگر سطحی فرکتال باشد هندسه آن بطور متناوب خودش را در مقیاس‌های مختلف تکرار می‌کند. فرکتالها با چهار ویژگی بیان می‌شوند که عبارتند از خودسانی، ناوردایی مقیاسی، خاصیت فضا پرکنی و بعد فرکتالی[۱، ۲، ۳]. این خواص را می توان در فرکتالها مثل خم مینکوفسکی و خم هیلبرت و برفدانه کوخ مشاهده کرد. در این مقاله مرز اولیه ما که مثلث متساوی الاضلاع است در تکرارهای اول تا پنجم تبدیل به برفدانه کوخ می‌شود و با استفاده از نرم افزار کامسول جواب عددی آن بدست می‌آید.

**۲.۲. سیستم عملکرد تکراری.** (Iterated Function System) روش سیستم عملکرد تکراری (IFS) روش فوق العاده انعطاف پذیری برای ایجاد ساختارهای فرکتالی مفید می باشد. این روش بر اساس یک مجموعه ای از تبدیلات آفین $w$، بصورت زیر می باشد.

$$w(x,y) = (ax+by+e, cx+dy+f),$$

طوری که ضرایب $(a,b,c,d,e,f)$ حرکت عنصر فرکتال را در فضا تعیین می کنند. $a$ و $d$ تغییر مقیاس، $b$ و $c$ چرخش و $e$ و $f$ هم انتقال خطی عنصر فرکتال را مشخص می نمایند [۵].

**تعریف ۱.۲.** فرض کنید $w_1, ..., w_n$ انقباض های متوالی باشند، آنگاه می گوییم مجموعه $S$ یک تبدیل ناوردا یا مجموعه ناوردا برای سیستم عملکرد تکراری است اگر و تنها اگر داشته باشیم

$$w_S = \cup_{i=1}^{n} w_i(S),$$

که در آن $w_S$ عملگر هوچینسون نامیده می شود.
به‌عنوان مثال سیستم عملکرد تکراری برای برفدانه کوخ [۵]در زیر آمده است.

**مثال ۲.۲.**

$$w_1(x,y) = [\tfrac{1}{2}x - \tfrac{\sqrt{3}}{6}y; \tfrac{\sqrt{3}}{6}x + \tfrac{1}{2}y] \quad (۱)$$
$$w_2(x,y) = [\tfrac{1}{3}x + \tfrac{1}{\sqrt{3}}; \tfrac{1}{3}y + \tfrac{1}{3}] \quad (۲)$$
$$w_3(x,y) = [\tfrac{1}{3}x; \tfrac{1}{3}y + \tfrac{2}{3}] \quad (۳)$$
$$w_4(x,y) = [\tfrac{1}{3}x - \tfrac{1}{\sqrt{3}}; \tfrac{1}{3}y + \tfrac{1}{3}] \quad (۴)$$
$$w_5(x,y) = [\tfrac{1}{3}x - \tfrac{1}{\sqrt{3}}; \tfrac{1}{3}y - \tfrac{1}{3}] \quad (۵)$$
$$w_6(x,y) = [\tfrac{1}{3}x; \tfrac{1}{3}y - \tfrac{2}{3}] \quad (۶)$$
$$w_7(x,y) = [\tfrac{1}{3}x + \tfrac{1}{\sqrt{3}}; \tfrac{1}{3}y - \tfrac{1}{3}] \quad (۷)$$

## ۳. محیط‌های سازنده مدل و سازنده کاربرد

در کامسول سه محیط متفاوت وجود دارد که در حالت عادی تنها دوتا از آنها فعال است: سازنده مدل، سازنده کاربرد، و سازنده فیزیک. اولی و دومی در حالت عادی فعال هستند اما برای فعال سازی محیط سازنده فیزیک به قسمت Prereferences باید رفت و آن‌را فعال نمود. اگر بر روی کلید ''سازنده کاربرد'' در منوی ابزار کلیک کنیم می توانیم به محیط ''سازنده کاربرد'' برویم و برعکس در آن محیط نیز اگر روی کلید ''سازنده مدل'' کلیک کنیم می توانیم به محیط اول بازگردیم (شکل ۲).

## ۴. نتیجه‌گیری

این مقاله ریاضیدانان را به کار با نرم‌افزار کامسول ترغیب می‌کند که معمولا مهندسین و فیزیکدانان از آن استفاده می‌کنند. کسانی که در زمینه آنالیز عددی و محاسبات عددی کار پژوهشی انجام می‌دهند. روند مدلسازی و حل معادله لاپلاس با مرزهای فرکتالی با شکل برفدانه کوخ و شرایط مرزی دیریکله نمونه کار تحقیقی نویسنده مقاله است که در انتهای مقاله شبیه‌سازی‌های مربوطه بصورت فایل MP۴ ویدئویی قرار داده شده‌است.

شکل ۳: در محیط سازنده مدل (چپ)، مدل‌سازی و در محیط سازنده کاربرد (راست)، فرم و روش طراحی کنیم

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سخنرانی

*:سخنران

# مقایسه حجم با شرط تقریباً سولیتون ریچی

سکینه حاجی‌آقاسی* و شاهرود اعظمی

گروه ریاضی، دانشکده علوم پایه، دانشگاه بین المللی امام خمینی(ره)
*azami@sci.ikiu.ac.ir* ،*s.hajiagasi@gmail.com*

چکیده. در این مقاله شرطی روی انحنای ریچی که شامل میدان‌های برداری است در نظر گرفته می‌شود. تحت این شرط مقایسه حجم و مقایسه لاپلاسین روی منیفلد بررسی می‌شود. در حالت خاص با شرط انحنای ریچی بکری-امری با به‌کار بردن یک روش کوتاه روی منیفلد اصلی تحت شرایط ضعیف‌تر از فرضیات معمولی می‌توان نتایج فوق را برای منیفلد اصلی بدست آورد.

## ۱. پیشگفتار

قضیه مقایسه‌ی حجم بیشاب-گرومو یک ابزار ضروری در هندسه دیفرانسیل و آنالیز روی منیفلدها است. این قضیه بیان می‌کند که بافرض کران پایین $-(n-1)K$، نسبت حجم گوی‌های ژئودزیک منیفلد وفضا یک تابع صعودی از شعاع است. در بسیاری از مواقع کران پایین نقطه‌ای روی انحنای ریچی در دسترس نیست. از این رو تلاش‌های بسیاری برای کم کردن این فرض انجام می‌شود.
در این مقاله، شرط انحنای ریچی شامل میدان‌های برداری $\mathrm{Ric}+\mathrm{Hess}L \geq -\lambda g$ را در نظر می‌گیریم که از شرط ریچی بکری-امری گسترده‌تر است.
در اینجا $\lambda$ یک تابع حقیقی‌مقدار غیرثابت و $L$ یک تابع هموار روی منیفلد است که $L$ یا $|\nabla L|$ کراندار باشد. به طور مشخص شرط کران پایین را روی یک انحنای ریچی تغییر یافته‌ی $\mathrm{Ric}+\frac{1}{2}L_V g$ اعمال می‌کنیم که $V$ یک میدان برداری هموار و $L_V g$ مشتق لی $g$ در جهت $V$

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**واژگان کلیدی.** متر ریمانی، مقایسه حجم، انحنای ریچی بکری-امری. .

است. مطالعه سولیتون ریچی و معادله میدان اینشتینی انگیزه قوی‌تری برای در نظر گرفتن این نوع از شرط ریچی را فراهم می‌کند. برای مطالعه بیشتر به [۱]، [۲]، [۳] ، [۴] و [۵] مراجعه کنید.

## ۲. دست‌آوردهای پژوهش

در این مقاله، منیفلد ریمانی $(M, g)$ با بعد $n$ را به همراه نقطه‌ی پایه‌ی ثابت $O \in M$ در نظر می‌گیریم. شرط‌های اساسی روی تانسور انحنای ریچی را به صورت

$$\mathrm{Ric} + \frac{1}{2}L_V g \geq -\lambda g, \tag{۱.۲}$$

و

$$|V|(y) \leq \frac{K}{d(y, O)^\alpha}, \tag{۲.۲}$$

در نظر بگیرید. در اینجا $\lambda$ یک تابع حقیقی مقدار غیر ثابت و $V$ یک میدان برداری هموار روی $M$ و $d(y, O)$ فاصله $O$ تا $y$ است و $K \geq 0$ و $0 \leq \alpha < 1$ ثابت هستند. در ادامه از نماد $C(a_1, a_2, ..., a_n)$ برای ثابت‌های وابسته به پارامترهای $a_1, a_2, ..., a_n$ استفاده می‌شود. در ادامه قضایا را به همراه طرح کلی اثبات آن‌ها می‌آوریم. به علت طولانی بودن جزئیات محاسبات از ارائه دقیق آن‌ها صرف نظر کرده‌ایم.

**قضیه ۱.۲.** (مقایسه حجم۱): فرض کنید روابط (۱.۲) و (۲.۲) برقرار باشند. به ازای $\forall r > 0$ بدست می‌آید:

$$Vol(B(x, r)) \leq e^{C(\alpha)Kr^{1-\alpha} + \int_0^r \lambda t^2 dt} \frac{Vol(S^{n-1})}{n} r^n \tag{۳.۲}$$

که در اینجا، $S^{n-1}$ کره‌ی واحد در $\mathbb{R}^n$ است.

برهان. اگر فرض کنیم که $s = d(x, y)$ فاصله تمام نقاط $y$ از نقطه ثابت $x$ باشد، آنگاه در مفهوم توزیع رابطه زیر را خواهیم داشت.

$$\Delta s - \frac{n-1}{s} \leq \frac{1}{s^2} \int_0^s \lambda t^2 dt + < V, \nabla s > + \frac{C(\alpha)K}{s^\alpha}. \tag{۴.۲}$$

با توجه به (۴.۲) داریم:

$$\frac{\partial}{\partial s} ln(\omega(s, .)) \leq \frac{n-1}{s} + \frac{1}{s^2} \int_0^s \lambda t^2 dt + \frac{K}{(d_0 - s)^\alpha} + \frac{C(\alpha)K}{s^\alpha},$$

که $\omega(s,.)$ عنصر حجمی متر $g$ روی منیفلد $M$ در مختصات قطبی ژئودزیک است. با انتگرال‌گیری از رابطه فوق برای حالت $s_1 \leq s_2 \leq d_0 = d(x,O)$ داریم:

$$ln\frac{\omega(d_0,.)}{\omega(s_1,.)} \leq ln(\frac{d_0}{s_1})^{n-1} + \frac{1}{s^2}\int_{s_1}^{d_0}\int_0^s \lambda t^2 dtds + C(\alpha)Kd_0^{1-\alpha}. \quad (۵.۲)$$

مشابهاً برای حالت‌های $d_0 \leq s_1 < s_2$ و $s_1 \leq d_0 \leq s_2$ بدست می‌آید و با میل دادن $s_1 \longrightarrow 0$ نتیجه می‌شود:

$$\omega(s,.) \leq e^{C(\alpha)Ks^{1-\alpha}+\int_0^s \lambda t^2 dt} s^{n-1}, \quad \forall s > 0. \quad (۶.۲)$$

با توجه به رابطه‌ی

$$\frac{d}{dr}Vol(B(x,r)) = \frac{d}{dr}\int_0^r \int_{S^{n-1}} \omega(t,\theta)d\theta dt = \int_{S^{n-1}} \omega(r,.)d\theta,$$

نتیجه نهایی به دست آمده و اثبات به پایان می‌رسد.

□

در قضایای بعدی قضیه ۱.۲ را برای تابع مثبت و کران‌دار $\lambda$ با کران بالای $K_1$ در نظر می‌گیریم.

**قضیه ۲.۲.** (قضیه مقایسه حجم ۲): الف) فرض کنید علاوه بر شرایط قضیه اول مقایسه حجم، شرط نافروپاشی حجم به ازای $\forall x \in M$ و بعضی $\rho > 0$ ثابت به صورت $Vol(B(x,r)) \geq \rho$ برقرار باشد. در این صورت برای هر $0 < r_1 < r_2 \leq 1$ کران نسبت حجم به صورت زیر بدست می‌آید.

$$\frac{Vol(B(x,r_2))}{r_2^n} \leq e^{C(n,K,K_1,\alpha,\rho)[K_1(r_2^3-r_1^3)+K(r_2-r_1)^{1-\alpha}]}.\frac{Vol(B(x,r_1))}{r_1^n}. \quad (۷.۲)$$

ب) در حالت خاص تانسور انحنای ریچی بکری-امری $\mathrm{Ric} + \mathrm{Hess}L$ را در نظر بگیرید و فرض کنید روابط $\mathrm{Ric} + \mathrm{Hess}L \geq -\lambda g$ و $|\nabla L(y)| \leq \dfrac{K}{d(y,O)^\alpha}$ به ازای هر $y$ متعلق به $M$ و نقطه‌ی ثابت $O \in M$ و ثابت‌های $K \geq 0$ و $\alpha \in [0,1]$ برقرار باشند. در این صورت نتایج قضیه مقایسه حجم ۱.۲ و قسمت (الف) همین قضیه برقرار هستند.

می‌توان نشان داد که تحت فرض قوی‌تر که $|V|$ کران دار باشد، نتایج بالا قابل بهبود است. به ویژه قضیه مقایسه حجم ۲.۲ بدون فرض شرط نافروپاشی حجم بدست می‌آید.
فرض کنید $(M,g)$ یک گرادیان سولیتون ریچی با نرمال‌سازی $R + |\nabla L|^2 - 2\lambda L = 0$ برای $\lambda \neq 0$ و یا $R + |\nabla L|^2 = 1$ برای $\lambda = 0$ باشد. بنابراین $R \geq 0$ است وقتی که $\lambda \geq 0$ باشد و هنگامی که $\lambda < 0$ باشد، $R \geq -C_1(n)$ است. اگر علاوه بر آن فرض کنیم $|L| \leq K$ باشد، آنگاه داریم:

$$|\nabla L| \leq \Lambda(n,\lambda,K),$$

به طوری که

$$\Lambda(n,\lambda,K)=\begin{cases}\sqrt{2\lambda K}, & \lambda>0;\\ 1, & \lambda=0;\\ \sqrt{-2\lambda K+C_1(n)}, & \lambda<0.\end{cases}$$

**نتیجه ۳.۲.** (حالت گرادیان تقریباً سولیتون ریچی): فرض کنید که $(M,g_{ij})$ گرادیان سولیتون ریچی به صورت $R_{ij}+\nabla_i\nabla_j L=\lambda g_{ij}$ باشد و رابطه $R+|\nabla L|^2-2\lambda L=0$ برقرار باشد. علاوه بر این فرض کنید در $B(O,2\delta)$ برای بعضی نقطه ثابت $O$ و شعاع $\delta$ داشته باشیم:$|L|\leq K$. بنابراین از آنجا که $\lambda$ مثبت است، لذا $R\geq 0$ است پس داریم: $|\nabla L|\leq\sqrt{2\lambda K}$ و با توجه به کران $\lambda$ نابرابری $|\nabla L|\leq\sqrt{2K_1K}$ بدست می‌آید. بنابراین اظهارات زیر برقرار هستند.

الف) برای هر $0<r<d(x,\partial B(O,\delta))$ نابرابری زیر را داریم.

$$Vol(B(x,r))\leq C(n)e^{2\sqrt{2K_1K}r+K_1r^3}r^n. \tag{۸.۲}$$

ب) برای هر $x\in B(O,\delta)$ و $0<r_1<r_2<d(x,\partial B(O,\delta))$ نابرابری زیر برقرار است.

$$\frac{Vol(B(x,r_2))}{r_2^n}\leq e^{[K_1(r_2^3-r_1^3)+2\sqrt{2K_1K}(r_2-r_1)]}.\frac{Vol(B(x,r_1))}{r_1^n}. \tag{۹.۲}$$

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پوستر

# ریشه های جواب معادلات دیفرانسیل خطی مرتبه دوم خودالحاقی

الهام خداپرست

دانشگاه دریانوردی و علوم دریایی چابهار.دانشکده علوم پایه. گروه ریاضیات و کاربردها
*el-kh-85@yahoo.com*

چکیده. در معادله دیفرانسیل $y''(x) + S(x)y(x) = 0$ محاسبه تعداد ریشه های جواب های غیر بدیهی در یک بازه به طور مستقیم ممکن نیست.در این مقاله در مورد وجود و موقعیت ریشه های جواب های این معادلات دیفرانسیل بحث می کنیم.

## ۱. پیش‌گفتار

بسیاری از مسائل فیزیکی و مکانیکی به صورت یک معادله دیفرانسیل معمولی ویا با مشتقات جزئی با شرایط اولیه و مرزی مدل سازی می شوند. در معادله دیفرانسیل $y''(x)+S(x)y(x) = 0$ محاسبه تعداد ریشه های جواب های غیر بدیهی در یک بازه به طور مستقیم ممکن نیست. از طرفی وجود و موقعیت ریشه های جواب های معادلات دیفرانسیل معمولی از مهم ترین مسائل در تئوری مسایل مقدار مرزی می باشند. به طوری که در طول قرن گذشته پژوهش های بسیاری در این زمینه صورت گرفته است. اگر معادله دیفرانسیل خطی مرتبه دوم

$$a_0(t)\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_2(t)y(t) = 0 \ (1.1)$$

خود الحاقی باشند ان گاه ان را به صورت یکی از مسائل مهم ریاضی می توان نوشت که به مسئله مرزی اشتورم لیوویل معروف است. در این مقاله در مورد وجود و موقعیت ریشه های جواب های این معادلات دیفرانسیل بحث می کنیم.

**تعریف ۱.۱.** فرض کنید $I \subset R$ برای معادله دیفرانسیل خطی همگن مرتبه دوم

$$a_0(t)y''(t) + a_1(t)y'(t) + a_2(t)y(t) = 0 \ (1.2)$$

**واژگان کلیدی.** معادلات دیفرانسیل خطی مرتبه دوم، معادلات الحاقی ، خودالحاقی .

که در ان $a_0(t) \neq 0$ دارای مشتق مرتبه دوم پیوسته و $a_1(t)$ مشتق مرتبه اول پیوسته و $a_2(t)$ روی I پیوسته باشد معادله الحاقی را به صورت زیر تعریف می کنیم

$$\frac{d^2}{dt^2}(a_0(t)y(t)) - \frac{d}{dt}(a_1(t)y(t)) + a_2(t)y(t) = 0 \ (1.3)$$

با محاسه مشتقات معادله بالا خواهیم داشت

$$a_0(t)\frac{d^2y}{dt^2} + (2a_0^{'}(t) - a_1(t))\frac{dy}{dt} + (a_0^{''}(t) - a_1^{'}(t) + a_2(t))y = 0 \ (1.4)$$

**تعریف ۲.۱.** معادله دیفرانسیل خطی مرتبه دوم را خود الحاقی گوییم هرگاه الحاق ان با خودش برابر باشد

**قضیه ۳.۱.** معادله دیفرانسیل خطی همگن مرتبه دوم (1.2) که در ان $a_0(t) \neq 0$ دارای مشتق مرتبه دوم پیوسته و $a_1(t)$ مشتق مرتبه اول پیوسته و $a_2(t)$ روی I پیوسته باشد در نظر می گیریم شرط لازم و کافی برای خودالحاقی (1.2) عبارت است از $a_0^{'}(t) = a_1(t)$

**نتیجه ۴.۱.** فرض کنیم معادله دیفرانسیل خطی همگن مرتبه دوم (1.2) خود الحاقی باشد در این صورت معادله (1.2) به صورت زیر بیان می شود

$$\frac{d}{dt}(a_0(t)\frac{dy}{dt}) + a_2(t)y(t) = 0 \ (1.5)$$

که به مسئله مقدار مرزی اشتورم–لیوویل معروف است

نتیجه مهم در مورد وجود و موقعیت ریشه های جواب های معادلات دیفرانسیل معمولی قضیه اشتورم است که برای معادلات دیفرانسیل خطی زیر در دامنه های کران دار و بی کران R در فضای n بعدی $E^n$ بدست می اید.

$$lu = \sum_{i,j=1}^{n} D_i(a_{ij}D_ju + 2\sum_{i=1}^{n} b_iD_iu + cu = 0, \ (1.6)$$

$$Lv = \sum_{i,j=1}^{n} D_i(A_{ij}D_jv + 2\sum_{i=1}^{n} B_iD_iv + Cv = 0 \ (1.7)$$

اگر $b_i = B_i = 0$ باشند ان گاه معادلات الحاقی به صورت زیر خواهد بود.

$$lu = \frac{d}{dx}(a(x)\frac{du}{dx}) + c(x)u = 0 \ (1.8)$$

$$Lv = \frac{d}{dx}(A(x)\frac{dv}{dx}) + C(x)v = 0 \ (1.9)$$

در معادلات الحاقی R دامنه کران دار و $\xi$ نقاط ابتدا و انتهای بازه باشند فرض کنید مرز $\xi$ از R تا بردار یکه نرمال n در هر نقطه به طور قطعه قطعه پیوسته باشند. نقاط $E^n$ به صورت $x = (x^1, x^2, ..., x^n)$ تعریف می شوند. مشتق ان ها را نسبت به $x^i$ با $D_i$ نشان می دهیم معادلات الحاقی مرتبه دوم روی بازه $\alpha < x < \beta$ مورد بررسی قرار می گیرند که ضرایب $a_{ij}, A_{ij}, c, C, b_i, B_i$ در معادلات$(1.6), (1.7)$ روی $R \cup \xi$ حقیقی مقدار پیوسته فرض می

شوند و ماتریس های $a_{ij}, A_{ij}$ در R متقارن و مثبت تعریف شده اند.و همچنین در معادلات الحاقی که$b_i = B_i = 0$ باشد تغییر از $lu$ را به این صورت تعریف می کنیم.

$$V[u] = \int_R [\sum_{i,j=1}^{n} (a_{ij} - A_{ij}) D_i u \, D_j u + (C - c) u^2] dx \ (1.10)$$

**قضیه ۵.۱.** فرض کنیدR یک دامنه محدود در $E^n$ باشد که مرز $\xi$ پیوستگی های قطعه قطعه نرمال یکه دارد . هم چنین فرض کنیم در دو رابطه$(1.6), (1.7)$ $b_i = B_i = 0$ باشد اگر جواب$u$
غیر بدیهی از $lu = 0$ در R وجود داشته باشد به طوری که روی $\xi$، $u = 0$ باشد و $V[u] \geq 0$
پس هر جواب از $Lv = 0$ یکی از خصوصیات زیر را دارد:

۱. $v$ حداقل یک صفر در $(\alpha, \beta)$ دارد.

۲. $v$ یک مضرب ثابت از $u$ است.

در مورد $n = 1$ جواب های معادلات دیفرانسیل خطی مرتبه دوم فقط صفرهای ساده دارند و هر جواب $v$ از $Lv = 0$ باید یک صفر در $[\alpha, \beta]$ داشته باشد که به یک بازه باز کاهش پیدا می کند یعنی حداقل یک صفر در $(\alpha, \beta)$ دارد مگر اینکه $v$ مضرب ثابتی از $u$ باشد.

## ۲. نتایج

**گزاره ۱.۲.** فرض کنید در بازه $\alpha < x < \beta$،$c(x) < C(x)$ و $a(x) = A(x)$ باشد. اگر جواب حقیقی و غیر بدیهی $u$ از $Lu = 0$ وجود داشته باشد به طوری که $u(\alpha) = u(\beta) = 0$، ان گاه هر جواب حقیقی غیر بدیهی از $Lv = 0$ حداقل یک ریشه در $(\alpha, \beta)$ دارد.

**گزاره ۲.۲.** فرض کنید $c(x) < C(x)$ و $a(x) > A(x)$ در بازه $\alpha < x < \beta$ باشد نتیجه قبلی معتبر است.

**گزاره ۳.۲.** فرض کنید $\phi_1, \phi_2$ جواب های مستقل خطی معادله دیفرانسیل $y'' + S(x)y(x) = 0$ باشند و ریشه های $\phi_1, \phi_2$ متمایز باشند در این صورت

۱. ریشه های جواب های غیر بدیهی معادله فوق منفرد هستند.

۲. بین هر دو ریشه متوالی$x_1, x_2$ از $\phi_1$ به طور دقیق یک ریشه از $\phi_2$ وجود دارد.

رفتار نوسانی معادله دیفرانسیل $y'' + S(x)y(x) = 0$ به علامت $S(x)$ بستگی دارد. بنابر این گزاره زیر را بیان می کنیم.

**گزاره ۴.۲.** اگر روی بازه I، $S(x) \leq 0$ باشد ان گاه جواب غیر بدیهی معادله $y'' + S(x)y(x) = 0$ نمی تواند روی این بازه دو ریشه داشته باشد.

برهان. بنا بر برهان خلف فرض کنیم یک جواب غیر بدیهی $y(x)$ روی بازه I حداقل دو ریشه داشته باشد $x_1, x_2$ دو ریشه متوالی $y(x)$ باشند که رو $(x_1, x_2)$ تابع y مثبت است. چون $S(x)$ پیوسته است لذا داریم $y'(x_1) > 0$ و $y'(x_2) < 0$ حال ثابت می کنیم که $y'(x_1)$ نمی تواند نامثبت باشد زیرا در غیر این صورت

$$y'(x_1) = 0 \Longrightarrow y(x_1) = 0$$

در نتیجه $y(x)=0$ که با غیر بدیهی بودن ان در تناقض است. اگر $y^{'}(x_1)<0$ پس $y(x)$ برای هر $x>x_1$ منفی خواهد بود که با فرضیه مثبت بودن $y(x)$ در تناقض است. بنابراین

$$y^{''}(x)=-S(x)y(x)\geq 0$$

چون $y^{''}(x)\geq 0$ می باشد لذا $y^{'}(x)$ روی بازه **I** یک تابع صعودی است پس

$$0>y^{'}(x_2)\geq y^{'}(x_1)>0$$

که این نیز یک تناقض است. بنابراین جواب غیربدیهی $y(x)$ نمی تواند دو ریشه داشته باشد.
$\square$

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پوستر

*:ارائه دهنده

# سفتی تابعی یک فضای حاصل‌ضربی

پریوش درخشان [1] * و مهرداد کاروان جهرمی [2]

[1] گروه ریاضی، دانشکده مهندسی سیستمهای هوشمند و علوم داده، دانشگاه خلیج فارس
*parivashdo@gmail.com*
[2] گروه ریاضی، دانشکده مهندسی سیستمهای هوشمند و علوم داده، دانشگاه خلیج فارس
*Mkaravanj@gmail.com*

چکیده. سفتی تابعی و سفتی تابعک ضعیف به عنوان یک تقریب برای دوگان یک کاردینال پایدار فضای تمام توابع پیوسته حقیقی مقدار روی تابعی مانند $X$ تحت توپولوژی همگرایی نقطه وار معرفی شده است. هدف ما از این مقاله ارائه برهان دیگری برای اثبات مسئله اولگ اوکانوز که در مورد سفتی تابعی روی فضای حاصل ضربی است، می‌باشد.

## ۱. پیش‌گفتار

سفتی تابعی و نیم‌سفتی (سفتی تابعک ضعیف) از یک فضا توسط آرخنگلسکی[۱] به عنوان یک تقریب برای دوگان یک کاردینال پایدار $C_p(X)$ (فضای تمام توابع پیوسته حقیقی مقدار روی $X$ تحت توپولوژی همگرایی نقطه وار ) معرفی شده است. سفتی تابعی و نیم‌سفتی نسبتاً یکسان به نظر می‌رسند و در حالت‌های طبیعی زیادی، منطبق هستند. در واقع، برای چند سال این سؤال که آیا سفتی تابعی و نیم‌سفتی یکی هستند، باز مانده بود. تا این که رزنیچنکو[۲] در ۱۹۸۷ یک مثال از یک فضا ساخت که نیم‌سفتی شماراست و سفتی تابعی نیست. در این مقاله همه‌ی فضاهای فرض شده تیخونف است. برای فضای $X$ و $Y$، نماد $Y^X$، $C^d(X,Y)$ و

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واژگان کلیدی. سفتی تابعی، مسئله اولگ اوکانوز، توپولوژی .

* سخنران

[1]A. Arhangel'skii
[2]Reznichenko

$C(X)$ به ترتیب منظور تمام نگاشت‌های از $X$ به $Y$، مجموعه‌ی تمام توابع پیوسته خوشه‌ای از $X$ به $Y$ و مجموعه تمام توابع حقیقی مقدار روی $X$ می‌باشند. برای دنباله‌ی $\{z_n\}_{n\in\mathbb{N}}$ از فضای $Z$ ما نماد $\{z_n\}^d$ را برای نمایش نقاط خوشه‌ای دنباله‌ی $\{z_n\}$ استفاده می‌کنیم. در این مقاله راه‌حل دیگری برای سوال اولگ اوکانوز " آیا روی فضای فشرده $X$، سفتی تابعی $X$ با سفتی تابعی $X^\omega$ برابر است؟"ارائه می‌دهیم.

از مهم‌ترین کاردینال‌های وابسته به فضای توپولوژی $X$، سفتی[3] و سفتی تابعی[4] آن است که به ترتیب با $t(X)$ و $t_0(X)$ نشان می‌دهیم. در این بخش منظور از $\kappa$، عدد اصلی نامتناهی است و همه‌ی فضاهای توپولوژی فرض شده تیخونف (کاملاً منظم و هاسدورف) می‌باشند.

**تعریف ۱.۱.** سفتی تابعی در نقطه $x \in X$ را تعریف می‌کنیم:

$$t_0(x,X) = \min\left\{ \kappa : \begin{array}{l} \text{هر تابع حقیقی مقدار } \kappa - \text{پیوسته} \\ \text{روی } X\text{، در نقطه‌ی } x \text{ پیوسته باشد.} \end{array} \right\}$$

و سفتی تابعی فضای $X$ برابر است با:

$$t_0(X) = \min\{\ \kappa : \text{هر تابع حقیقی مقدار } \kappa - \text{پیوسته روی } X\text{، پیوسته باشد.}\ \}$$

**گزاره ۲.۱.** اگر $p : X \longrightarrow Y$ نگاشت $\mathbb{R}-$خارج‌قسمتی باشد، آن‌گاه $t_0(Y) \leqslant t_0(X)$ است.

برهان. فرض کنیم $\kappa = t_0(X)$ و $\phi : Y \longrightarrow \mathbb{R}$ تابع $\kappa-$پیوسته باشد. تابع $\phi \circ p$، $\kappa-$پیوسته روی $X$ است و چون $t_0(X) = \kappa$ پس $\phi \circ p$ پیوسته و چون $p$، $\mathbb{R}-$خارج‌قسمتی است پس $\phi$ پیوسته است. $\square$

**گزاره ۳.۱.** فرض کنیم $X$ فضای موضعاً فشرده باشد، آن‌گاه $t_0(X \times Y) \leqslant t_0(X)t_0(Y)$ است.

برهان. به [۱] مراجعه کنید. $\square$

**تعریف ۴.۱.** نگاشت $f \in C^d(X,Y)$ را پیوسته نقطه خوشه‌ای نامیم هرگاه برای هر دنباله‌ی $f(x) \in f(\{x_j\}^d)$، آن‌گاه $f(x) \in \{f_{n_j}(x_j)\}^d$ باشد.

**تعریف ۵.۱.** دنباله‌ی $\{f_n\} \subseteq Y^X$ را همگرایی نقطه خوشه‌ای[5] به $f \in C^d(X,Y)$ ($cp-$همگرا) نامیم اگر برای هر دنباله‌ی $\{x_n\} \subseteq X$، $f(\{x_n\}^d) \subseteq \{f_n(x_n)\}^d$ باشد و هم‌چنین $\{f_n\}$ همگرایی نقطه خوشه‌ای مورثی[6] به $f$ ($hcp-$همگرا) نامیم اگر هر یک از زیردنباله‌هایش $cp-$همگرا باشد.

**قضیه ۶.۱.** فرض کنیم $X$ فضای فشرده‌ی شمارا باشد و $f_n \xrightarrow{hcp} f$ باشد، آن‌گاه $f_n$ به $f$ به طور یکنواخت همگراست.

[3] Tightness
[4] Functional Tightness
[5] cluster point convergent
[6] hereditarily cluster point convergent

برهان. فرض کنیم $f_n \xrightarrow{hcp} f$ باشد یعنی برای هر زیر دنباله‌ی $f_{n_j}$ از $f_n$، $x \in \{x_j\}^d$، آن‌گاه $f(x) \in \{f_{n_j}(x_j)\}^d$ است می‌خواهیم نشان دهیم $f_n \xrightarrow{uni} f$. به فرض خلف فرض کنیم $f_n$ به $f$ به طور یکنواخت همگرا نباشد.

$$\exists \varepsilon > 0,\ \exists n_1 < n_2 < ...\ \exists x_j,\ |f_{n_j}(x_j) - f(x_j)| \geqslant \varepsilon$$

فرض کنیم $x \in \{x_j\}^d$ باشد در این صورت

$$\exists j_0,\ \forall j > j_0\ |f(x) - f(x_{j_0})| < \varepsilon/2$$

و

$$\forall k > j_0 \ni\ |f(x) - f(x_k)| < \varepsilon/2\ \&\ |f_{n_k}(x_k) - f(x)| < \varepsilon/2$$

در نتیجه

$$|f_{n_k}(x_k) - f(x_k)| < \varepsilon$$

که تناقض است با فرض مسئله لذا فرض خلف باطل و حکم ثابت است. □

**لم ۷.۱.** برای هر $\kappa$، اگر $f : X \longrightarrow Y$، $\kappa$-پیوسته باشد، آن‌گاه برای هر دنباله $\{x_n\}_{n\in\mathbb{N}}$ در $X$، $f(\{x_n\}^d) \subseteq \{f(x_n)\}^d$ است.

برهان. فرض کنیم $x \in \{x_n\}^d$ باشد. چون $f$، $\kappa$-پیوسته است لذا $f$ روی $z = \{x_n\}\cup\{x\}$ پیوسته است. برای هر همسایگی $W$ از $f(x)$، $(f|_z)^{-1}(W)$ همسایگی بازی از $x$ در $z$ است بنابراین اشتراک نامتناهی زیرمجموعه از $\{x_n\}$ است. □

**تعریف ۸.۱.** الف) $x^0 = \{x^0_n\}_{n\in\omega} \in X^\omega$ را ثابت در نظر می‌گیریم. فرض کنیم

$$x_{\leq n} = \{y_k\}_{k\in\omega},$$

که اگر $k \leq n$، آن‌گاه $y_k = x_k$ و در غیر این صورت $y_k = x^0_k$ است. قرار دهید

$$S_n = \{x_{\leq n} : x \in \Pi_n X_n\},$$

و نگاشت $p_n : \Pi_n X_n \longrightarrow S_n$ را با قاعده‌ی $p_n(x) = x_{\leq n}$ تعریف می‌کنیم.

ب) فرض کنیم $f : \Pi_n X_n \longrightarrow \mathbb{R}$ است. برای هر $n$ نگاشت $f_n : \Pi_n X_n \longrightarrow \mathbb{R}$ را با قاعده‌ی $f_n(x) = f(x_{\leq n})$ تعریف می‌کنیم.

**لم ۹.۱.** فرض کنیم $f_n$ بنا به تعریف ۸.۱، پیوسته است اگر $f|_{S_n}$ پیوسته باشد.

برهان. بنا به قضیه خارج‌قسمتی به آسانی دیده می‌شود که نگاشت $p_n$ خارج‌قسمتی است. هم‌چنین اگر $x, y \in \prod_n X_n$ ، $x_{\leq n} = y_{\leq n}$، آن‌گاه $f_n(x) = f_n(y)$ است. لذا هر فیبر $p_n$ شامل فیبر $f_n$ است. هم‌چنین $f|_{S_n} o p_n = f_n$ است. این اصل نتیجه می‌گیرد که $f_n$ پیوسته است اگر $f|_{S_n}$ پیوسته باشد.

□

**نتیجه ۱۰.۱.** فرض کنیم $X$ فشرده، $t_0(X) = \kappa$ و $f : \Pi_n X_n \longrightarrow \mathbb{R}$، $\kappa$-پیوسته باشد. اگر $p_n$ و $f_n$ بنا به تعریف ۸.۱ باشند، آن‌گاه هر $f_n$ پیوسته است.

برهان. بنا به قضیه ۳.۱ $t_0(S_n)=\kappa$ و تحدید $f$ به $S_n$، $\kappa$-پیوسته است پس $f|_{S_n}$ پیوسته است. لذا بنا به لم ۹.۱، $f_n$ پیوسته است. $\square$

**لم ۱۱.۱.** فرض کنیم $f_n$ و $f$ بنا به ۸.۱ تعریف می‌کنیم. هم‌چنین فرض کنیم $\{x^n\}_{n\in\omega}\subseteq X^\omega$ و $x\in\{x^n\}^d$، آن‌گاه $f(x)\in\{f_n(x^n)\}^d$ است.

برهان. می‌خواهیم نشان دهیم که برای یک همسایگی دلخواه $V$ از $f(x)$، $V$ شامل $f_n(x^n)$ است. قرار دهید

$$Y=\{x^n_{\leq m}:n,m\in\omega\}\cup\{x_{\leq n}:n\in\omega\}$$

از آن‌جایی که $f$، $\kappa$-پیوسته پس $f$ پیوسته روی $Y$ است. لذا $(f|_Y)^{-1}(V)$ باز در زیرفضای $Y$ از $X^\omega$ است. همسایگی باز پایه‌ای $U$ از $x$ بیابید که

$$U=<U_{i_1},U_{i_2},...,U_{i_k}>\cap Y\subseteq(f|_Y)^{-1}(V)$$

است.

$x_{\leq n}\longrightarrow x$ همگراست پس وجود دارد $n_0$ به طوری که برای هر $n>n_0$، $x_{\leq n}\in U$ هم‌چنین $x\in\{x^n\}^d$ که وجود دارد $m\geq n_0$ به طوری که $x_{\leq m},x^m\in U$ است.

فرض کنیم $x^m_{\leq m}=\{x^m_t\}_{t\leq m}\times\{x^0_t\}_{t>m}$ است. می‌خواهیم نشان دهیم که $x^m_{\leq m}\in U$ که برای این منظور اگر نشان دهیم برای هر $i_j,j=1,2,...,k$ اگر $i_j\leq m$، $x^m_{i_j}\in U_{i_j}$ و اگر $i_j>m$، آن‌گاه $x^0_{i_j}\in U_{i_j}$ که هر دو حالت $x^m,x_{\leq m}\in U$ به دست می‌آید در نتیجه $f_m(x^m)=f(x^m_{\leq m})\in V$ پس $f$ پیوسته است. $\square$

**گزاره ۱۲.۱.** $t_0(X)\leqslant t_0(X^\omega)$

برهان. می‌دانیم که نگاشت تصویر $p:X^\omega\longrightarrow X$ یک نگاشت $\mathbb{R}$-خارج‌قسمتی است. پس طبق گزاره ۲.۱ حکم برقرار است. $\square$

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۵-۶ خرداد ۱۴۰۰، دانشگاه بین المللی امام خمینی (ره)

پوستر

*:ارائه دهنده

# کنترل بهینه برای مدل دینامیکی بیماری کووید همراه با قرنطینه و مهاجرت

زهرا دیهیم [1] * و امید سلیمانی فرد [2]

[1]گروه ریاضی کاربردی، دانشکده علوم ریاضی، دانشگاه فردوسی مشهد
*Z.deihim97@gmail.com*
[2] گروه ریاضی کابردی، دانشکده علوم ریاضی، دانشگاه فردوسی مشهد
*Soleimani@um.ac.ir*

چکیده. در این مطالعه، به نحوه مدل سازی دینامیکی انتقال بیماری $Covid - 19$ از طریق مهاجرت پرداخته ایم.وجود جواب برای سیستم و پایداری سیستم را بررسی کرده ایم و در نهایت با اعمال کنترل های مناسب روی سیستم به نحوه پیشگیری از انتقال بیماری پرداخته ایم.

## ۱. مقدمات

بیماری که امروزه تحت عنوان $Covid - 19$ شناخته می شود،نوعی از ویروس کرونا است که باعث بیماری های شدید تنفسی و گوارشی می شود. دراین پژوهش ما با اعمال دو کنترل روی سیستم و قرنطینه افراد مشکوک به بیماری و بستری افراد آلوده،از انتقال بیش از حد بیماری جلوگیری کرده ایم.

2010 Mathematics Subject Classification. Primary 47J30; Secondary 47H10, 47H05.

واژگان کلیدی. کووید-۱۹، کنترل بهینه، قرنطینه، سیستم های دینامیکی .

* سخنران

### ۱.۱. مدل دینامیکی انتقال بیماری $Covid-19$.

سیستم دینامیکی انتقال بیماری $C0vid-19$ با در نظر گرفتن قرنطینه و مهاجرت به صورت زیر بیان می شود:[۱]

$$\begin{aligned}
\frac{dS}{dt} &= \pi + \alpha q + \sigma(1-p)Q - (\lambda+\mu)S \\
\frac{dL}{dt} &= \alpha(1-q) + \lambda S - (\tau_1+\tau_2+\mu)L \\
\frac{dI}{dt} &= \tau_1 L - (\gamma+\nu_1+\delta_1+\mu)I \\
\frac{dQ}{dt} &= \tau_2 L - (\sigma+\mu)Q \\
\frac{dJ}{dt} &= \gamma I - (\nu_2+\delta_2+\mu)J \\
\frac{dR}{dt} &= \delta_1 I + \delta_2 J + \sigma p Q - \mu R
\end{aligned} \tag{۱.۱}$$

که در آن، افراد مستعد بیماری $= S$ ، کمون $= L$ ، افراد آلوده $= I$ ، افراد قرنطینه $= Q$ ، افراد بستری در بیمارسان $= J$ ، افراد بهبود یافته$= R$ ، سرعت انتقال بیماری $= \lambda$
کسری از انسان های مهاجر که در معرض بیماری هستند $= q$

### ۲.۱. تحلیل وجود جواب وبررسی پایداری مدل.

برای اثبات وجود جواب و ناحیه جذب از دو لم زیر استفاده میکنیم:

**لم ۱.۱.** جواب منحصر به فرد $(S(t), L(t), I(t), Q(t), J(t), R(t))$ برای مدل (۱.۱) با شرایط اولیه مثبت به ازای هر $t \geq 0$ وجود دارد.[۱]

**لم ۲.۱.** ناحیه زیستی شدنی برای مدل(۱.۱) به این صورت است

$$\Gamma = (S(t), L(t), I(t), Q(t), J(t), R(t)) \in R^6_+$$

که در آن

$$R^6_+ : \ S(t) + L(t) + I(t) + Q(t) + J(t) + R(t) \leq \frac{\pi+\alpha}{\mu}$$

ناحیه ای مثبت است و ناحیه جذب محسوب می شود .[۱]

این سیستم در صورتی در حالت تعادل قرار دارد که $q = 1$ یا $\alpha = 0$ یعنی تمام انسان های مهاجر در معرض بیماری قرار دارند و یا نرخ آن افرادی که در معرض بیماری قرار گرفتند ناچیز است. حال نقطه تعادل را در این حالت به صورت زیر در نظر میگیریم:

$$\varepsilon_0^q = (S^*, L^*, I^*, Q^*, J^*) = (\frac{\pi+\alpha}{\mu}, 0, 0, 0, 0).$$

**قضیه ۳.۱.** نقطه تعادل عاری از بیماری $\varepsilon_0^q$ از مدل(۱.۱) همواره با شرط $q = 1$ منجر به پایداری سراسری سیستم می شود $(GAS)$ هرگاه $R_i^* \leq 1$ .[۱]

**۳.۱. کنترل بهینه.**
در این بخش دو کنترل ( آزمایش موثر و قرنطینه هنگام باز شدن مرز ها) $= u_1$ و (استفاده از ماسک و شستن دست ها و محافظت از خود از طریق آموزش) $= u_2$ را روی سیستم (۱.۱) اعمال میکنیم و با استفاده از اصل ماکسیمم پونتریاگین کنترل بهینه را برای سیستم (۱.۱) به دست می آوریم. برای این کار باید تابعی $F$ را با شرایط زیر مینیم کنیم:

$$F(u_1,u_2)=min\int_{t_0}^{t_f}[L+I+Q+J+\frac{b_1}{2}u_1^2+\frac{b_2}{2}u_2^2]dt$$

به طوری که:

$$\begin{aligned}
\frac{dS}{dt}&=(1-u_1)[\pi+\alpha q]+\sigma(1-p)Q-(1-u_2)(\lambda S)-\mu S\\
\frac{dL}{dt}&=\alpha(1-q)+(1-u_2)\lambda S-(\tau_1+\tau_2+\mu)L\\
\frac{dI}{dt}&=\tau_1 L-(\gamma+\nu_1+\delta_1+\mu)I\\
\frac{dQ}{dt}&=\tau_2 L-(\sigma+\mu)Q\\
\frac{dJ}{dt}&=\gamma I-(\nu_2+\delta_2+\mu)J\\
\frac{dR}{dt}&=\delta_1 I+\delta_2 J+\sigma pQ-\mu R
\end{aligned}$$

(۲.۱)

که $S>0, L\geq 0, I\geq 0, Q\geq 0, J\geq 0, R\geq 0$ .
با توجه به اینکه اساسی ترین شرط برای وجود جواب کنترل بهینه،محدب بودن $F$ است.پس ابتدا ماتریس هسین $F(u_1,u_2)$ را به فرم زیر به دست می آوریم:

$$H=\begin{bmatrix} b_1 & 0\\ 0 & b_2\end{bmatrix}$$

با توجه به اینکه $b_1, b_2$ مثبت هستند، $H$ یک ماتریس معین مثبت است. پس $F(u_1,u_2)$ اکیدا محدب است. حال با توجه به اصل ماکسیمم پونتریاگین هامیلتونی را به فرم زیر داریم:

$$\begin{aligned}
H&=L^*+I^*+Q^*+J^*+\frac{b_1}{2}u_1^2+\frac{b_2}{2}u_2^2+\psi_1[(1-u_1)(\pi+\alpha q)\\
&+\sigma(1-q)Q^*-(1-u_2)(\lambda S^*)-\mu S^*]\\
&+\psi_2[\alpha(1-q)+(1-u_2)\lambda S^*-(\tau_1+\tau_2+\mu)L^*]\\
&+\psi_3[\tau_1 L^*-(\gamma+\nu_1+\delta_1+\mu)I^*]\\
&+\psi_4[\tau_2 L^*-(\sigma+\mu)Q^*]\\
&+\psi_5[\gamma I^*-(\nu_2+\delta_2+\mu)J^*]\\
&+\psi_6[\delta_1 I^*+\delta_2 J+\sigma pQ^*-\mu R^*]
\end{aligned}$$

(۳.۱)

حال متغیر های هم حالت $i=1,...,6, \psi_i$ را به این صورت در نظر می گیریم که در معادلات زیر با شرایط انتهایی $\psi_i(t_f)=0$ صدق کند:

$$\begin{aligned}
\frac{d\psi_1}{dt} &= -\frac{\partial H}{\partial S^*} = -\psi_1[-\mu-(1-u_2)\lambda]-\psi_2[(1-u_2)\lambda]\\
\frac{d\psi_2}{dt} &= -\frac{\partial H}{\partial L^*} = -1+\psi_2(\tau_1+\tau_2+\mu)-\psi_3\tau_1-\psi_4\tau_2\\
\frac{d\psi_3}{dt} &= -\frac{\partial H}{\partial I^*} = -1+\psi_3(\gamma+\nu_1+\delta_1+\mu)-\psi_6\delta_1\\
\frac{d\psi_4}{dt} &= -\frac{\partial H}{\partial Q^*} = -1-\psi_1\sigma(1-p)+\psi_4(\sigma+\mu)-\psi_6\sigma p\\
\frac{d\psi_5}{dt} &= -\frac{\partial H}{\partial J^*} = -1+\psi_5(\nu_2+\delta_2+\mu)-\psi_6\delta_2\\
\frac{d\psi_6}{dt} &= -\frac{\partial H}{\partial R^*} = \psi_6\mu
\end{aligned}$$

(۴.۱)

با توجه به شرایط لازم بهینگی $\frac{\partial H}{\partial u_i}=0$ $(i=1,2),$ و(۳.۱)و(۴.۱) کنترل بهینه برای سیستم(۱.۱) به صورت زیر است:

$$\begin{aligned}
u_1^* &= min[max(0,\psi_1(\frac{\pi+\alpha q}{b_1}),u_{1max}]\\
u_2^* &= min[max(0,(\frac{(\psi_1-\psi_2)\lambda S^*}{b_2}),u_{2max}]
\end{aligned}$$

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پوستر

# آماره پیوسته هموردای بسنده در فضای توپولوژیکی هاسدروف موضعاً همبند

مهدی شمس [1]

[1] گروه آمار، دانشکده علوم ریاضی، دانشگاه کاشان
*mehdishams@kashanu.ac.ir*

چکیده. در این مقاله با توجه به ساختار توپولوژیکی روی فضاها و به‌کمک گروه‌های توپولوژیکی و مفاهیم آنالیز ریاضی ثابت می‌شود هر خانواده از اندازه‌های احتمال با چگالی‌های پیوسته توسط یک گروه تبدیلات جابه‌جایی تولید شده و یک آماره بسنده پیوسته هموردا با مقادیر حقیقی را اختیار می‌کند.

## ۱. پیش‌گفتار

فرض کنید $(X,\mathcal{U})$ یک فضای توپولوژیکی هاسدورف موضعاً همبند و مسیری همبند با پایه شمارا باشد (در این مقاله تمام فضاهای توپولوژیکی، هاسدورف نیز هستند) $(\Theta,\mathcal{W})$ گروه تبدیل جابه‌جایی پیوسته موضعاً همبند و همبند روی $X$ باشد. گروه $(\Theta,\mathcal{W})$ روی $X$ پیوسته است، یعنی توابع $(\theta,\tau)\mapsto\theta\tau$ و $\theta\mapsto\theta^{-1}$ و $(\theta,x)\mapsto\theta x$ پیوسته هستند. فرض کنید $\mathcal{A}\subset\mathcal{P}(X)$، $\sigma$-میدان تولید شده توسط $\mathcal{U}$ باشد و برای هر $\theta\in\Theta$، $P_\theta$ اندازه احتمال تولید شده توسط تبدیل $\theta$ هست، یعنی برای هر $\theta\in\Theta$ اندازه احتمال $P_\theta|\mathcal{A}$، را توسط $P_\theta(A):=P(\theta^{-1}A)$، $A\in\mathcal{A}$ تعریف می‌کنیم که در حالت خاص اگر $\epsilon$ یک عضو خنثی از $\Theta$ باشد، $P_\epsilon=P$. خانواده $P_\theta$، $\theta\in\Theta$ توسط گروه تبدیل $\Theta$ تولید می‌شود. همچنین فرض کنید برای هر $\theta\in\Theta$، $P_\theta$ یک چگالی پیوسته وابسته به $P$ است و برای یک اندازه نمونه $n>1$ یک آماره بسنده پیوسته با مقادیر حقیقی مثل $T_n$ که هموورداست وجود داشته باشد. (تابع $T:X^n\to Y$ هموورداست اگر $T(x''_1,\ldots,x''_n)=T(x'_1,\ldots,x'_n)$، آن‌گاه برای

2010 Mathematics Subject Classification. Primary 43-XX; Secondary 62F10.

**واژگان کلیدی.** فضای توپولوژیکی هاسدروف موضعاً همبند، اندازه احتمال، اندازه لبگ، آماره بسنده هموردا .

هر $\theta \in \Theta$، $T(\theta x'_1, ..., \theta x'_n) = T(\theta x''_1, ..., \theta x''_n)$ ([۱]). تحت این فرضیات یک آماره پیوسته با مقادیر حقیقی $S$ روی $X$ وجود دارد که $P_\theta$، $\theta \in \Theta$ بسنده است به طوری که توزیع $S$ یا خانواده پارامتر مکان توزیع‌های نرمال با واریانس ۱ یا خانواده پارامتر مقیاس توزیع‌های گاما هست. به عبارت دیگر، میان خانواده‌های توزیع‌هایی که توسط گروه‌های تبدیل جابه‌جایی با شرایط نظم تولید می‌شوند، آماره بسنده در خانواده پارامتر مکان توزیع‌های نرمال و خانواده پارامتر مقیاس توزیع‌های گاما تنها برای برخی اندازه‌های نمونه بزرگتر از ۱، پیوسته و هم‌وردا هست. در مورد ارتباط بین ناوردایی و بسندگی تحقیقات زیادی از جمله [۲، ۴] انجام شده است که در این مقاله از دیدگاه مفاهیم آنالیز ریاضی در مورد خانواده مکان و مقیاس این ارتباط مورد بررسی قرار خواهد گرفت.

## ۲. گروه‌های تبدیل و آماره‌های بسنده

اگر $\Theta$ یک گروه تبدیلات روی $X$ و $T$ یک تابع هم‌وردا روی $X$ باشد، تبدیلات $\theta : X \to X$ به طور طبیعی تبدیلات $\bar{\theta} : T(X) \to T(X)$ تعریف شده به صورت $\theta t := T\theta T^{-1}\{t\}$ را القا می‌کند (براحتی دیده می‌شود برای $t \in T(X)$، $T\theta T^{-1}\{t\}$ تنها از یک عضو ثابت تشکیل شده و از این رو خوش تعریف است).

فرض کنید $\mathcal{O}$ توپولوژی $\mathbb{R}$ و $\mathcal{B}$ میدان بورل باشد. برای هر اندازه احتمال $Q|\mathcal{A}$ اندازه احتمال القایی $Q * T|\mathcal{B}$ را به صورت $Q * T(B) := Q(T^{-1}B)$، $B \in \mathcal{B}$ تعریف کنید.

**لم ۱.۲.** فرض کنید $(X,\mathcal{U})$ یک فضای هاسدروف و $(\Theta, \mathcal{W})$ یک گروه تبدیلات پیوسته روی $(X,\mathcal{U})$ باشد. اگر $T : X \to \mathbb{R}$ پیوسته و هم‌وردا باشد، آن گاه برای هر $t \in T(X)$، $\theta \to \theta t$ پیوسته است.

**لم ۲.۲.** فرض کنید $(X,\mathcal{U})$ همبند و موضعاً همبند و $T : X \to \mathbb{R}$ پیوسته باشد. اگر $A \subset T(X)$ و $T^{-1}A \in \mathcal{U}$، آن گاه $A \in \mathcal{O} \cap T(X)$.

**لم ۳.۲.** فرض کنید $(X,\mathcal{U})$ همبند و موضعاً همبند و $(\Theta, \mathcal{W})$ یک گروه تبدیل پیوسته روی $(X,\mathcal{U})$ باشد. اگر $T : X \to \mathbb{R}$ پیوسته و هم‌وردا باشد، آن گاه $t \to \theta t$ برای هر $\theta \in \Theta$، $\mathcal{O} \cap T(X)$-پیوسته است.

**لم ۴.۲.** فرض کنید $(X,\mathcal{U})$ یک گروه توپولوژیکی و $(\Theta, \mathcal{W})$ یک گروه تبدیلات پیوسته و همبند روی $(X,\mathcal{U})$ و $T : X \to \mathbb{R}$ پیوسته و هم‌وردا باشد. آن گاه:

(i) $S = \{t \in T(X) : \Theta t = \{t\}\}$ ناوردا و $\mathcal{O} \cap T(X)$-بسته است.

(ii) اگر برای یک $W \in \mathcal{W}$ و $t \in T(X)$، مجموعه $Wt$ از یک نقطه تشکیل شده باشد، آن گاه $t \in S$.

با توجه به تعریف بسندگی برای هر $A \in \mathcal{A}$ یک امید شرطی از $1_A$ به شرط $T$ وابسته به $P_\theta$ و مستقل از $\theta \in \Theta$ وجود دارد که آن را $p(A, .)$ می‌نامیم. داریم:

$$\int p(A,t)1_B(t)P_\theta * T(dt) = P_\theta(A \cap T^{-1}B),\ \ \forall B \in \mathcal{B}, \theta \in \Theta$$

**لم ۵.۲.** فرض کنید:

(i) $(X,\mathcal{U})$ همبند و موضعاً همبند باشد.

(ii) $(\Theta,\mathcal{W})$ گروه تبدیلات پیوسته که موضعاً همبند، همبند و جابه‌جایی باشد.

(iii) برای هر $\varnothing \neq U \in \mathcal{U}$، $P(U) > 0$.

(iv) یک آماره هموردای پیوسته با مقادیر حقیقی که برای خانواده تولید شده اندازه‌های احتمال $P_\theta \in \mathcal{A}$، $\theta \in \Theta$ بسنده باشد وجود داشته باشد.

مجموعه $S$ تعریف شده در لم ۴.۲ و $t_0 \in S$ و $U \in \mathcal{U}$ با $t_0 \in T(U)$ را در نظر بگیرید. برای هر امید شرطی $p(U,.)$، یک $P*T$-مجموعه پوچ $M$ و یک فاصله غیرتباهیده $I$ شامل $t_0$ وجود دارد به طوری که برای هر $t \in I \cap \overline{M}$، $p(U,t) > 0$.

**تعریف ۶.۲.** تابع $f : X \to Y$ را یک انقباض از تابع $g : X \to Y$ گویند، هرگاه $g(x') = g(x'')$ آن گاه $f(x') = f(x'')$. در این صورت یک تابع $\psi : g(X) \to Y$ وجود دارد که $f = \psi \circ g$.

**لم ۷.۲.** فرض کنید

(i) $(X,\mathcal{U})$ همبند و موضعاً همبند باشد.

(ii) $(\Theta,\mathcal{W})$ گروه تبدیلات همبند، موضعاً همبند، پیوسته و جابه‌جایی باشد.

(iii) $P$ یک اندازه احتمال روی $\sigma$-میدان $\mathcal{A}$ تولید شده توسط $U$ است به طوری که برای هر $\varnothing \neq U \in \mathcal{U}$، $P(U) > 0$.

(iv) هر اندازه احتمال خانواده تولید شده $P_\theta|\mathcal{A}$، $\theta \in \Theta$ یک چگالی پیوسته وابسته به $P|\mathcal{A}$ را می‌پذیرد که آن را $h(.,\theta)$ می‌نامیم.

(v) یک آماره پیوسته هموردا با مقادیر حقیقی که برای $P_\theta|\mathcal{A}$، $\theta \in \Theta$ بسنده است وجود دارد.

در این صورت برای هر $\theta \in \Theta$، $h(.,\theta)$ یک انقباض از $T$ است.

اکنون قضیه اصلی مقاله بیان می‌شود که در اثبات آن از لم‌های ۱.۲ تا ۷.۲ استفاده شده است.

**قضیه ۸.۲.** فرض کنید:

(i) $(X,\mathcal{U})$ همبند مسیری و موضعاً همبند با پایه شمارا باشد.

(ii) $(\Theta,\mathcal{W})$ گروه تبدیل پیوسته، همبند، موضعاً همبند و جابه‌جایی روی $X$ باشد.

(iii) $P$ یک اندازه احتمال روی $\sigma$-میدان $\mathcal{A}$ است به طوری که برای هر $\varnothing \neq U \in \mathcal{U}$، $P(U) > 0$.

(iv) خانواده تولید شده $P_\theta|\mathcal{A}$، $\theta \in \Theta$ شامل بیشتر از ۲ عمل باشند.

(v) برای هر $\theta \in \Theta$، اندازه احتمال $P_\theta|\mathcal{A}$ یک چگالی پیوسته وابسته به $P|\mathcal{A}$ به نام $h(.,\theta)$ را بپذیرد.

(vi) برای یک اندازه $n > 1$ یک آماره پیوسته هموردا با مقادیر حقیقی وجود دارد که برای $P_{\boldsymbol{\theta}}|\boldsymbol{\mathcal{A}}$، $\boldsymbol{\theta} \in \Theta$ بسنده است.

در این صورت یک آماره پیوسته و هموردا با مقادیر حقیقی $S$ روی $X$ وجود دارد که برای $P_{\boldsymbol{\theta}}|\boldsymbol{\mathcal{A}}$، $\boldsymbol{\theta} \in \Theta$ بسنده است به طوری که خانواده توزیع‌های القایی $P_\theta * S$، $\theta \in \Theta$ یا خانواده پارامتر مکان توزیع‌های نرمال با واریانس ۱ یا خانواده پارامتر مقیاس توزیع‌های گاما هست.

به طور دقیق‌تر، یک تابع پیوسته $w:\Theta\to\mathbb{R}$ وجود دارد به طوری که $P_\theta * S$ دارای یکی از چگالی‌های زیر وابسته به اندازه لبگ است:

$$a) r\to(2\pi)^{-\frac{1}{2}}e^{-\frac{(r-w(\theta))^2}{2}}, r\in\mathbb{R}; w(\theta\tau)=w(\theta)+w(\tau)$$

$$b) r\to\frac{1}{\Gamma(p)}w(\theta)^p r^{p-1}e^{-w(\theta)r}, r>0; p>0, w(\theta)>0, w(\theta\tau)=w(\theta)w(\tau).$$

این نتیجه می‌دهد در حالت خاص یک آماره بسنده و هموردا با مقادیر حقیقی که برای $P_{\boldsymbol{\theta}}|\boldsymbol{\mathcal{A}}$، $\boldsymbol{\theta}\in\Theta$ بسنده است وجود دارد که برای هر $n\in\mathbb{N}$، $(x_1,...,x_n)\longrightarrow\sum_{i=1}^n S(x_i)$.

**مثال ۹.۲.** فرض کنید $\Theta$ یک گروه از تمام تبدیلات روی $X=\mathbb{R}$ و $P|\mathcal{B}$ توزیع کوشی باشد:

(i) برای هر $n\in\mathbb{N}$ یک تابع یک به یک دواندازه‌پذیر $T_n:\mathbb{R}^n\to\mathbb{R}$ وجود دارد [۵]. از این که $T_n$ یک به یک است به وضوح هم‌ورداست. از این که دواندازه‌پذیر است، برای $P_{\boldsymbol{\theta}}|\boldsymbol{\mathcal{B}}$، $\boldsymbol{\theta}\in\Theta$ بسنده است. لذا شرط پیوستگی حذف نمی‌شود.

(ii) برای هر $n\in\mathbb{N}$ یک تابع پیوسته $T_n:\mathbb{R}^n\to\mathbb{R}$ وجود دارد که روی مکمل یک مجموعه صفر لبگ مناسب در $\mathcal{B}$، یک به یک است [۳]. $T_n$ برای تمام اندازه‌های احتمال که توسط اندازه لبگ $n$−بعدی روی $\boldsymbol{\mathcal{B}}^n$ مغلوب می‌شوند پیوسته و لذا در حالت خاص برای $P_{\boldsymbol{\theta}}|\boldsymbol{\mathcal{B}}$، $\boldsymbol{\theta}\in\Theta$ بسنده است و از این رو شرط همورداییی حذف نمی‌شود.

**گزاره ۱۰.۲.** فرض کنید $\Theta$ یک گروه از تبدیلات روی $\mathbb{R}$ و $P|\mathcal{B}$ هر اندازه احتمال با چگالی پیوسته نسبت به اندازه لبگ روی $\mathcal{B}$ باشد. در این صورت خانواده $P_\theta$، $\theta\in\Theta$ یک آماره بسنده هموردای پیوسته با مقادیر $\mathbb{R}^2$ برای هر اندازه نمونه $n$ را می‌پذیرد.

**مثال ۱۱.۲.** اگر $X=(0,\infty)$، $\Theta$ گروه تبدیلات همسانی و $P|\mathcal{B}$ توسط داشتن چگالی $1_{(0,1)}$ وابسته به اندازه لبگ تعریف شود، خانواده وابسته $P_\theta|\mathcal{B}$، $\theta\in\Theta$ یک آماره بسنده پیوسته هموردا با مقادیر حقیقی $T_n(x_1,...,x_n)=max\{x_1,...,x_n\}$ را برای هر نمونه می‌پذیرد.

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پوستر

# میدان‌های سیگمایی ناوردا، تقریباً ناوردا و بسنده

مهدی شمس [1]

[1] گروه آمار، دانشکده علوم ریاضی، دانشگاه کاشان
*mehdishams@kashanu.ac.ir*

چکیده. در این مقاله نشان داده می‌شود که $\sigma$-میدان‌های ناوردا و تقریباً ناوردا معادل هستند اگر و تنها اگر $\sigma$-میدان ناوردا مستقل از یک $\sigma$-میدان بسنده مناسب باشد.

## ۱. پیشگفتار

یکی از محدودیت‌ها روی توابع تصمیم، استفاده از اصول ناوردایی است. در این مقاله ارتباط بین ناوردایی و بسندگی مورد بررسی قرار می‌گیرد. در یک مسأله تصمیم آماری اغلب داده‌ها توسط بسندگی یا ناوردایی فشرده می‌شوند و هر دو در کاهش بعد مسأله موفق هستند. حال این سوال مطرح می‌شود که آیا امکان به‌کارگیری هر دو روش وجود دارد یا خیر؟ هال و همکاران [۴] نشان دادند که تحت شرایط خاص این تقلیل می‌تواند توسط به‌کارگیری هر دو اصل به‌طور همزمان و این که کدام بی‌اهمیت است انجام شود. به هر حال می‌توان از بررسی این شرایط اجتناب کرد و فضای نمونه را با فضای آماره بسنده جابه‌جا کرد و برای دوری از این مشکلات قبل از جستجوی یک گروه مناسب از تبدیلات برای ناوردایی مدل، در ابتدا داده‌ها توسط بسندگی فشرده شوند و سپس ناوردایی گروه را روی فضای آماره بسنده به‌کار برد. در برخی مسائل تصمیم اصول بسندگی و ناوردایی منجر به فشردگی یکسان در داده‌ها می‌شوند. به عنوان مثال اگر $X \sim N(0, \sigma^2)$، مسأله برآورد $\sigma^2$ تحت تابع زیان $SEL$ و تبدیل $g(x) = -x$، $\bar{g}(\sigma^2) = \sigma^2$ و $\tilde{g}(a) = a$ ناوردا خواهد بود و بنابراین قواعد تصمیم ناوردا (هموردا) به صورت $\delta(x) = \delta(-x)$ هستند

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**واژگان کلیدی.** گروه تبدیلات دواندازه‌پذیر، $\sigma$-میدان ناوردا، آماره بسنده مینیمال، ناوردای ماکسیمال، تقریباً هموردا .

(یعنی $\delta(x)$ تابعی از $x^2$ است). از این که $X^2$ آماره بسنده کامل است می‌توان نتیجه گرفت که اصول بسندگی و ناوردایی هر دو منجر به محدود کردن روی رده قواعدی که توابعی از $X^2$ هستند، می‌شوند. بنابراین آماردانان اغلب برای خلاصه کردن داده‌ها تا آنجا که ممکن است قبل از به‌کارگیری ناوردایی از بسندگی استفاده می‌کنند که این رهیافت تنها زمانی معقول است که دو شرط برقرار باشند. مسأله کاهش یافته نیز ناوردا باشد و برای هر قاعده هم‌وردا براساس نمونه اصلی، یک قاعده هم‌وردا بر اساس آمار بسنده وجود داشته باشد که به خوبی قاعده اولی باشد. این دو شرط در بیشتر مسائل آماری برقرارند. یک قانون هم ارز با شرط اول، این است که آماره بسنده هم‌وردا باشد که مثال زیر نشان می‌دهد این قاعده همیشه برقرار نیست.

**مثال ۱.۱.** فرض کنید $X_1, ..., X_n \overset{\text{iid}}{\sim} N(\theta, 1)$. این توزیع تحت تبدیلات $g(\mathbf{X}) = \mathbf{X} + a$ ناورداست. آماره بسنده مینیمال برای $\theta$ یعنی $T_1(\mathbf{X}) = \bar{X}$ هم‌ورداست، در صورتی که آماره بسنده $T_2(\mathbf{X}) = (\bar{X}, \frac{X_1}{X_2})$ هم‌وردا و تقریباً هم‌وردا نیست، زیرا $T_2(\mathbf{X} + a) = (\bar{X} + a, \frac{X_1+a}{X_2+a})$ تابعی (حتی تقریباً همه جا) از $\bar{X}$ و $\frac{X_1}{X_2}$ نیست.

نشان داده می‌شود که آماره بسنده مینیمال، هم‌ورداست و لذا اگر تا آنجا که ممکن است با استفاده از بسندگی فشرده سازی شود، شرط اول برقرار خواهد بود.

در مقالاتی نظیر [۵، ۳] به ارتباط مفاهیم بسندگی و ناوردایی پرداخته شده است. در این مقاله نشان داده می‌شود که $\sigma$−میدان‌های ناوردا و تقریباً ناوردا معادل هستند اگر و تنها اگر $\sigma$−میدان ناوردا مستقل از یک $\sigma$−میدان بسنده مناسب باشد.

## ۲. بسندگی و ناوردایی

فرض کنید $(\mathcal{X}, \mathcal{A})$ یک فضای نمونه اندازه‌پذیر، $\mathcal{P}$ یک خانواده از توزیع‌های احتمال روی $\mathcal{A}$ و $G$ یک گروه از تبدیلات یک به یک، پوشا و دواندازه‌پذیر از $\mathcal{X}$ به خودش باشد. فرض کنید که $\mathcal{P}$ تحت $G$ ناوردا باشد، یعنی برای هر $P \in \mathcal{P}$ و $g \in G$، $Pg^{-1} \in \mathcal{P}$. فرض کنید $\mathcal{A}_S \subset \mathcal{A}$ یک $\sigma$−میدان بسنده باشد. $\sigma$−میدان همه مجموعه‌های $\mathcal{P}$−تقریباً $G$−ناوردا را توسط $\mathcal{A}_A \subset \mathcal{A}$ و $\sigma$−میدان مجموعه‌های $G$−ناوردا در $\mathcal{A}$ را توسط $\mathcal{A}_I \subset \mathcal{A}$ نمایش می‌دهیم. داریم $\mathcal{A}_A \supset \mathcal{A}_I$. برای $A, B \in \mathcal{A}$ می‌نویسیم $A \sim B$ اگر $A$ و $B$، $\mathcal{P}$−معادل باشند. قرار می‌دهیم $\mathcal{A}_{SI} = \{A \in \mathcal{A}_I : \exists B \in \mathcal{A}_S, B \sim A\}$. به وضوح $\mathcal{A}_{SI} \subset \mathcal{A}_I$ ولی $\mathcal{A}_{SI} \subset \mathcal{A}_S$ لزوماً برقرار نیست. در بیشتر مثال‌های مورد علاقه، $\mathcal{A}_S$ طوری انتخاب می‌شود که $\mathcal{A}_{SI} = \mathcal{A}_S \cap \mathcal{A}_I$. اشتراک اساسی $\mathcal{A}_{SA} \subset \mathcal{A}$ به طور مشابه تعریف می‌شود. یک آماره (تابع اندازه‌پذیر از $\mathcal{X}$) $S$ هم‌وردا برای $G$ است اگر برای هر $x, y \in \mathcal{X}$ و $g \in G$ و $Sx = Sy$ نتیجه دهد $Sgx = Sgy$. ( در مرجع [۱] گفته می‌شود $S$ با $G$ جابه‌جا می‌شود). یک تابع هم‌وردا روی برد خودش یک گروه از تبدیلات یک به یک $\bar{G}$ تعریف شده توسط $\bar{g}S = Sg$ القا می‌کند و متناظر یک همریختی است. از این به بعد فرض می‌شود $S$ برای یک $\sigma$−میدان داده شده $\mathcal{B}$ روی برد $S$ اندازه‌پذیر است و باید اعضای $\bar{G}$، $(\mathcal{B}, \mathcal{B})$ اندازه‌پذیر و در پی آن دواندازه‌پذیر باشند. $S$ را تقریباً هم‌وردا گوییم اگر برای هر $g \in G$ یک تبدیل دواندازه‌پذیر $\bar{g}$ روی برد $S$ وجود داشته باشد و از این رو $Sg = \bar{g}S$.

**لم ۱.۲.** اگر $f$ یک آماره کران‌دار با مقادیر حقیقی باشد، آن‌گاه برای $g \in G$ و $P \in \mathcal{P}$، $E_P(f|g\mathcal{A}_S) \sim E(fg|\mathcal{A}_S)g^{-1}$.

**لم ۲.۲.** اگر $\mathcal{A}_S$ آماره بسنده مینیمال یا معادل با $\sigma$-میدان القایی توسط یک آماره تقریباً هموردای $S$ باشد، آن‌گاه برای هر $g \in G$، $g\mathcal{A}_S \sim \mathcal{A}_S$.

برهان. فرض می‌کنیم $\mathcal{A}_S$ بسنده مینیمال و $f$ آماره کران‌دار با مقادیر حقیقی باشد. با توجه به لم ۱.۲، $E_P(f|g\mathcal{A}_S)$ بستگی به $P$ ندارد. پس $g\mathcal{A}_S$ و لذا $g\mathcal{A}_S \supset \mathcal{A}_S$ که چون برای هر $g \in G$ درست است، داریم $g\mathcal{A}_S \sim \mathcal{A}_S$. آماره تقریباً هموردای $S$ را در نظر گرفته و فرض می‌کنیم $\mathcal{A}_S \sim S^{-1}\mathcal{B}$. لذا $gS^{-1}\mathcal{B} \sim S^{-1}\bar{g}\mathcal{B} = S^{-1}\mathcal{B} \sim \mathcal{A}_S$. هم‌ارزی اول به خاطر تقریباً هموردایی $S$ و تساوی یکی مانده به آخر به جهت دواندازه‌پذیر بودن $\bar{g}$ است. $\square$

**لم ۳.۲.** فرض کنید برای هر $g \in G$، $g\mathcal{A}_S \sim \mathcal{A}_S$. در این صورت گزاره‌های زیر هم‌ارزاند:

(i) اگر $f$ آماره کران‌دار با مقادیر حقیقی تقریباً ناوردا باشد، $E(f|\mathcal{A}_S)$ تقریباً ناورداست.
(ii) $\mathcal{A}_S$ و $\mathcal{A}_A$ به طور شرطی به شرط $\mathcal{A}_{SA}$ مستقل‌اند.

برهان. داریم $E(f|\mathcal{A}_S)g \sim E(fg|\mathcal{A}_S)g \sim E(f|g\mathcal{A}_S) \sim E(f|\mathcal{A}_S)$. اولین هم‌ارزی به خاطر $f \sim fg$ و دومی به جهت لم ۱.۲ و سومی به واسطه $g\mathcal{A}_S \sim \mathcal{A}_S$ برقرار است. هم‌ارزی گزاره‌های $(i)$ و $(ii)$ با توجه به لم ۳.۳ از [۴] است. $\square$

یک نتیجه این لم این است که اگر برای هر $g \in G$، $g\mathcal{A}_S \sim \mathcal{A}_S$، آن‌گاه $\mathcal{A}_S$ برای $\mathcal{A}_A$ بسنده است. تحت فرض اضافی $\mathcal{A}_{SA} \sim \mathcal{A}_{SI}$، داریم $\mathcal{A}_{SI}$ برای $\mathcal{A}_A$ بسنده است.

اکنون برای جمع‌بندی موضوع، شرایط و گزاره‌های زیر را در نظر بگیرید:

1) $\mathcal{A}_{SA} \sim \mathcal{A}_{SI}$;
2) $\mathcal{A}_{SI} \sim \{\varnothing, \mathcal{X}\}$;
3) $\mathcal{A}_{SA} \sim \{\varnothing, \mathcal{X}\}$;
4) $\mathcal{A}_S \perp \mathcal{A}_A$;
5) $\mathcal{A}_A \perp \mathcal{A}_I$;
6) $\mathcal{A}_{SA}$ برای $\mathcal{A}_A$ بسنده است;
7) $\mathcal{A}_S$ و $\mathcal{A}_A$ مستقل شرطی به شرط $\mathcal{A}_{SA}$ هستند;
8) $\forall g \in G, g\mathcal{A}_S \sim \mathcal{A}_S$;
9) $\mathcal{A}_S$ به طور کامل کران‌دار است;
10) $\mathcal{P}$ متلاشی می‌شود روی $\mathcal{A}_A$ به طور معادل هر زیر$\sigma$-میدان از $\mathcal{A}_A$ برای $\mathcal{A}_A$ بسنده است;
11) $G$ تولید می‌کند $P$;
12) $\mathcal{A}_S \vee \mathcal{A}_I \sim \mathcal{A}$.

در انتها می‌خواهیم روابط بین این گزاره‌ها را بررسی کنیم.

طبق لم ۳.۲ داریم $1 \Rightarrow 7 \Rightarrow 8$ و $1,2 \Leftrightarrow 3$.

اگر $\mathcal{A}_S$ توسط یک آماره هم‌وردا مثل $S$ تولید شود، آنگاه 2 اساساً لازم است که $\bar{G}$ روی برد $S$ انتقالی باشد.

شرط 10 یعنی محدودیت‌های اندازه‌ها در $\mathcal{P}$ به $\mathcal{A}_A$ منطبق می‌شود. به وضوح $10 \Rightarrow 11$ اما عکس آن برقرار نیست. (اختیار کنید $X \sim N(\mu, \sigma^2)$ و $Y \sim N(\mu, 1-\sigma^2)$ که $\mu \in \mathbb{R}$ و $0 < \sigma < 1$ و $X \perp Y$ گروه تبدیلات $\mathcal{P}$ را ناوردا نگه می‌دارد و آماره ناوردای ماکسیمال برابر با $X - Y$ است. $\mathcal{P}$ روی $\mathcal{A}_I$ متلاشی می‌شود. از این که $\mathcal{A}_A \sim \mathcal{A}_I$ برای این گروه، $\mathcal{P}$ روی $\mathcal{A}_A$ متلاشی می‌شود. با این حال $G$، $\mathcal{P}$ را تولید نمی‌کند.

**قضیه ۴.۲.** داریم $3,7 \Leftrightarrow 4$. لذا اگر 7 برقرار باشد $3 \Leftrightarrow 4$. همچنین $7,12,3,5 \Leftrightarrow 4$. لذا اگر $7,12$ برقرار باشد، $3,5 \Leftrightarrow 4 \Leftrightarrow 3$.

اگر $\mathcal{A}_S$ طوری انتخاب شود که 7 برقرار باشد (به طور مثال $\mathcal{A}_S$ آماره بسنده مینیمال باشد)، آنگاه برای $\mathcal{A}_{SA}$ بدیهی داریم $\mathcal{A}_S \perp \mathcal{A}_A$ و به عبارت دیگر $\mathcal{A}_{SA} \sim \mathcal{A}_{SI}$ و $\mathcal{A}_{SI} \sim \{\varnothing, \mathcal{X}\}$. همچنین داریم $4 \Rightarrow 1,2,8,11$.

با توجه به قضیه ۴.۲ به نظر می‌رسد شرط 11 اضافی باشد. به هر حال $10 \Rightarrow 6,3$.

همچنین $3,8 \Leftrightarrow 4 \Leftrightarrow 3,7$ و قضیه باسو به صورت $4 \Rightarrow 9,10$ بیان می‌شود.

در [۲] ثابت شده $5 \Rightarrow 12,4$ که از ۴.۲ نیز به دست می‌آید.

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پوستر

# وجود جواب غیربدیهی برای معادله $(p,q)$-لاپلاسین با نمای بحرانی سوبولف

مهدی لطیفی [1]

[1] گروه ریاضی، دانشکده علوم پایه، دانشگاه پدافند هوایی خاتم الانبیاء(ص)
*latifi.mmali@gmail.com*

چکیده. وجود جواب ضعیف نابدیهی برای معادله بیضوی غیرخطی

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda f(x)|u|^{k-2}u + g(x)|u|^{p^*-2}u \quad \text{in} \quad \mathbb{R}^N \\ u(x) \geq 0; \quad x \in \mathbb{R}^N \end{cases}$$

تحت شرایطی روی توابع و پارامترهای موجود، را بررسی می کنیم.

## ۱. پیش‌گفتار

هدف ما مطالعه وجود جواب ضعیف نامنفی، برای مسئله بیضوی غیرخطی شامل $(p,q)$-لاپلاسین زیر، که شامل توان بحرانی سوبولف است، می باشد:

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda f(x)|u|^{k-2}u + g(x)|u|^{p^*-2}u \quad \text{در} \quad \mathbb{R}^N \\ u(x) \geq 0; \quad x \in \mathbb{R}^N \end{cases} \tag{۱.۱}$$

که $p < N, 2 \leq q \leq p < k < p^* := \frac{Np}{N-p}$ و $0 < \lambda$ یک پارامتر است. همچنین $0 \leq f(x) \in C(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ با $r = \frac{p^*}{p^*-k}$ که روی زیرمجموعه باز $\Omega \subset \mathbb{R}^N$ با $|\Omega| > 0$، کراندار است، $0 \leq g(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ و $\Delta_m u$ عملگر $m$-لاپلاسین است.

**قضیه ۱.۱.** با فرضیات بالا بر روی توابع و پارامترهای موجود در ۱.۱، $\lambda^* > 0$ موجود است که برای هر $\lambda > \lambda^*$ مسئله حداقل یک جواب ضعیف غیربدیهی در فضای $X$ دارد.

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**واژگان کلیدی.** $(p,q)$-لاپلاسین، نمای بحرانی سوبولف، جواب نامنفی.

## ۲. تعاریف و فضایای اولیه

**گزاره ۱.۲.** (قضیه مسیر کوهی)[۲] $X$ یک فضای باناخ حقیقی و $\varphi \in C^1(X,\mathbb{R})$ . فرض کنیم $\varphi(0)=0$ و $\alpha,\rho>0$ و $x_1 \in X \setminus \overline{B_\rho}(0)$ ای موجود باشند که:

(۱) $\varphi(u) \geq \alpha$ برای هر $u \in X$ با $\|u\|_X = \rho$؛

(۲) $\varphi(x_1) < \alpha$؛

در آن صورت دنباله ی $\{u_n\} \subset X$ موجود است که: $\varphi'(u_n) \to 0$ و $\varphi(u_n) \to c$,

$$c := \inf\left\{\max_{t\in[0,1]} \varphi(\gamma(t)) : \gamma \in C([0,1],X),\gamma(0)=0 \ and \ \gamma(1)=x_1\right\}.$$

برای $1<m<N$ ، $\mathcal{D}^{1,m}(\mathbb{R}^N)$ را بستار فضای $C_0^\infty(\mathbb{R}^N)$ نسبت به نرم $W^{1,m}(\mathbb{R}^N)$ در نظر می گیریم. $\mathcal{D}^{1,m}(\mathbb{R}^N)$ یک فضای باناخ انعکاسی است که به صورت

$$\mathcal{D}^{1,m}(\mathbb{R}^N) := \left\{u \in L^{m^*}(\mathbb{R}^N) : \frac{\partial u}{\partial x_j} \in L^m(\mathbb{R}^N), j=1,\ldots,N\right\},$$

نیز مشخص می شود که $m^* := \frac{Nm}{N-m}$، با نرم $\|u\|_{\mathcal{D}^{1,m}} = \left(\int\limits_{\mathbb{R}^N} |\nabla u|^m dx\right)^{\frac{1}{m}}$ که با نرم $W^{1,m}(\mathbb{R}^N)$ معادل است.[۱].

**لم ۲.۲.** ( principle concentration-compactness ) $v_n \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ را دنباله ای کراندار بگیرید که $v_n \rightharpoonup v$ در $L^{p^*}(\mathbb{R}^N)$ . اگر زیردنباله ای از آن موجود باشد که، $|v_n|^{p^*}dx \rightharpoonup v$ برای اندازه $v$ ای، در آن صورت $x_j \in \mathbb{R}^N$ و $v_i>0$ ، برای $i=1,2,3,\cdots$ وجود دارد که

$$v_n^{p^*} \rightharpoonup |v|^{p^*} + \sum_{i=1}^{\infty} v_i \delta_{x_i} \quad \text{و} \quad \sum_{i=1}^{\infty} v_i^{\frac{p}{p^*}} < \infty$$

که $\delta_{x_i}$ اندازه دیراک در $x_i$ است.

ثابت سوبولف را به صورت زیر داریم: $S = \inf\left\{\frac{\|\nabla u\|_p^p}{\|u\|_{p^*}^p} : u \in \mathcal{D}^{1,p}(\mathbb{R}^N), u \neq 0\right\}$ .

## ۳. نتایج

فضای باناخ انعکاسی $X = \mathcal{D}^{1,p}(\mathbb{R}^N) \cap \mathcal{D}^{1,q}(\mathbb{R}^N)$ را با نرم $\|u\|_X = \|u\|_{\mathcal{D}^{1,p}} + \|u\|_{\mathcal{D}^{1,q}}$ در نظر می گیریم. قرار می دهیم $u_+ = \max\{0,u\}$ و $u_- = u - u_+$.

تابعک اویلر-لاگرانژ متناظر با ۱.۱ به صورت زیر است:

(۱.۳)
$$E_\lambda(u) = \frac{1}{p}\int\limits_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{q}\int\limits_{\mathbb{R}^N} |\nabla u|^q dx - \frac{\lambda}{k}\int\limits_{\mathbb{R}^N} f(x)u_+^k dx - \frac{1}{p^*}\int\limits_{\mathbb{R}^N} g(x)u_+^{p^*} dx$$

تابعک $E_\lambda$ در $X$ خوش تعریف و از کلاس $C^1$ است و نقاط بحرانی آن جوابهای ۱.۱ هستند.

**تعریف ۱.۳.** $X$ فضای باناخ و $E : X \to \mathbb{R}$ تابعکی دیفرانسیل پذیر:

(۱) دنباله $\{u_n\} \subset X$ یک $(PS)_c$-دنباله برای $E$ گفته می شود اگر $E(u_n) \to c$ و $E'(u_n) \to 0$ وقتی $n \to \infty$

(۲) اگر هر$(PS)_c$-دنباله دارای زیر دنباله ای همگرا ( در $X$ ) باشد، گوئیم $E$ در شرط $(PS)_c$ صدق می کند.

**لم ۲.۳.** اگر $\{u_n\}$ یک $(PS)_c$-دنباله برای $E_\lambda(u)$ باشد، آنگاه $\{u_n\}$ کراندار است.

**لم ۳.۳.** اگر $\{u_n\} \subset X$ یک $(PS)_c$-دنباله برای $E_\lambda$ باشد، در آن صورت تابع نامنفی $u \in X$ وجود دارد که $\nabla u_n \to \nabla u$ تقریبا همه جا در $\mathbb{R}^N$.

**لم ۴.۳.** برای هر $\lambda > 0$ تابعک $E_\lambda$ در شرایط زیر صدق می کند:

i : $R, \rho > 0$ وجود دارد که: $E_\lambda(u) \geq \rho$ با $\|u\| = R$.

ii : $u_0 \in X$ وجود دارد که $\|u_0\| > R$ و $E_\lambda(u_0) < 0$.

برای هر $\lambda > 0$ قرار می دهیم: $\tilde{C}_\lambda := \inf\limits_{u \in X\setminus\{0\}} \max\limits_{t\geq 0} E_\lambda(tu)$.

**لم ۵.۳.** $\lambda^* > 0$ ای وجود دارد که : $\forall \lambda > \lambda^* \qquad 0 < \tilde{C}_\lambda < \frac{S^{\frac{N}{p}}}{N}$.

برهان. از لم ۴.۳ داریم که $E_\lambda(u) \geq \rho > 0$ برای $\|u\| = R$ ، لذا $\tilde{C}_\lambda \geq \rho > 0$ (مقدار $\rho$ به $\lambda$ وابسته اما همیشه مثبت است). اگر $u_0 \in X \setminus \{0\}$ با ساپورت داخل $\Omega$ بگونه ای باشد که $u_0 \geq 0$ و $\|u_0\|_{p^*} = 1$ ، از رابطه

$$E_\lambda(tu_0) = \frac{t^p}{p}\|u_0\|^p_{D^{1,p}} + \frac{t^q}{q}\|u_0\|^q_{D^{1,q}} - \frac{\lambda t^k}{k}\int\limits_{\mathbb{R}^N} f(x)u_0^k dx - \frac{t^{p^*}}{p^*}\int\limits_{\mathbb{R}^N} g(x)u_0^{p^*}dx$$

می بینیم که $E_\lambda(tu_0) \to -\infty$ وقتی $t \to \infty$ و $E_\lambda(tu_0) \to 0^+$ ، هرگاه $t \to 0^+$ . لذا $t_\lambda > 0$ ای وجود دارد که $\max\limits_{t\geq 0} E_\lambda(tu_0) = E_\lambda(t_\lambda u_0)$ و بنابراین برای هر $\lambda > 0$ :

$$\lambda \int\limits_{\mathbb{R}^N} f(x)u_0^k dx = \frac{\|u_0\|^p_{D^{1,p}}}{{t_\lambda}^{k-p}} + \frac{\|u_0\|^q_{D^{1,q}}}{{t_\lambda}^{k-q}} - {t_\lambda}^{p^*-k}\int\limits_{\mathbb{R}^N} g(x)u_0^{p^*}dx. \qquad (۲.۳)$$

سمت چپ عبارت بالا مثبت است لذا نتیجه می گیریم که $t_\lambda \to 0$ وقتی $\lambda \to \infty$ . از طرفی چون $E_\lambda(t_\lambda u_0) \to 0^+$ وقتی $t_\lambda \to 0^+$ ، پس $\lambda^* > 0$ وجود دارد که

$$\lambda > \lambda^* \quad \text{برای هر} \quad \max_{t\geq 0} E_\lambda(tu_0) = E_\lambda(t_\lambda u_0) < \frac{S^{\frac{N}{p}}}{N}.$$

چون $\tilde{C}_\lambda \leq \max\limits_{t\geq 0} E_\lambda(tu_0)$ نتیجه می شود که $\tilde{C}_\lambda < \frac{S^{\frac{N}{p}}}{N}$ برای هر $\lambda > \lambda^*$. □

اثبات قضیه ۱.۱. از لم ۴.۳ و ۲.۲، داریم که دنباله $\{u_n\}$ موجود است که $E_\lambda(u_n) \to c_\lambda$ و $E'_\lambda(u_n) \to 0$. ($\{u_n\}$ یک $(PS)_{c_\lambda}$ - دنباله برای $E_\lambda$ است.) بر طبق لم ۲.۳ و ۳.۳ تابع نامنفی $u \in X$ موجود است که $u_n \to u$ و $\nabla u_n \to \nabla u$ تقریبا همه جا در $\mathbb{R}^N$ . بنابراین برای هر $\varphi \in X$:

$$\int |\nabla u_n|^{m-2} \nabla u_n \nabla \varphi dx \to \int |\nabla u|^{m-2} \nabla u \nabla \varphi dx \quad \text{برای} \quad m = p, q$$

$$\int g(x) |u_n|^{p^*-2} u_n \varphi dx \to \int g(x) |u|^{p^*-2} u \varphi dx$$

$$\int f(x) |u_n|^{k-2} u_n \varphi dx \to \int f(x) |u|^{k-2} u \varphi dx.$$

به عبارت دیگر $E'(u_n)(\varphi) \to E'(u)(\varphi)$ و نتیجتا $E'(u)\varphi = 0$ برای $\varphi \in X$ . حال اگر $u = 0$ از آنجا که $u_n$ در $X$ کراندار است، $\ell \geq 0$ وجود دارد که

$$\int |\nabla u_n|^p dx + \int |\nabla u_n|^q dx \to \ell.$$

از طرفی چون $E'_\lambda(u_n) \to 0$ ، داریم که

(۳.۳)
$$\int |\nabla u_n|^p dx + \int |\nabla u_n|^q dx = \lambda \int f(x) u_{n+}^k dx + \int g(x) u_{n+}^{p^*} dx + o(1)$$

و چون $u_{n+}^k$ در $L^{\frac{p^*}{k}}$ کراندار است و $u_{n+}^k \to 0$ تقریبا همه جا در $\mathbb{R}^N$ . پس داریم: $u_{n+}^k \rightharpoonup 0$ در $L^{\frac{p^*}{k}}$ و چون $f \in L^r = L^{\frac{p^*}{p^*-k}} = \left(L^{\frac{p^*}{k}}\right)'$ پس $\int g(x) u_{n+}^{p^*} dx \to \ell$.

اما برای هر $\lambda > \lambda_*$ ، $E_\lambda(u_n) \to c_\lambda \leq \tilde{C}_\lambda$ لذا

$$\frac{1}{p}\int |\nabla u_n|^p dx + \frac{1}{q}\int |\nabla u_n|^q dx - \frac{\lambda}{k}\int f u_{n+}^k dx - \frac{1}{p^*}\int g u_{n+}^{p^*} dx \to c_\lambda \leq \tilde{C}_\lambda$$

بنابراین $\left(\frac{1}{p} - \frac{1}{p^*}\right)\ell \leq \tilde{C}_\lambda$ یا $\frac{\ell}{N} \leq \tilde{C}_\lambda$. اکنون از

$$\int |\nabla u_n|^p dx + \int |\nabla u_n|^q dx \geq S\left(\int |u_n|^{p^*} dx\right)^{\frac{p}{p^*}} \geq S\left(\int u_{n+}^{p^*} dx\right)^{\frac{p}{p^*}}$$

پس $\ell \geq S(\ell)^{\frac{p}{p^*}}$ یا $\ell^{\frac{p}{N}} = \ell^{1-\frac{p}{p^*}} \geq S$ و بنابراین $\tilde{C}_\lambda > \frac{\ell}{N} \geq \frac{S^{\frac{N}{p}}}{N}$ . که تناقض است. $\square$

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پوستر

# وجود حداقل سه جواب برای یک معادله دیفرانسیل ضربه‌ای مرتبه ششم

عثمان هلاکو [1] *

[1]گروه ریاضی، واحد علی‌آباد کتول، دانشگاه آزاد اسلامی، علی آباد کتول، ایران
*halakoo@aliabadiau.ac.ir*

چکیده. در این مقاله با کمک قضایای بونانو وجود حداقل سه جواب را برای یک نوع معادله دیفرانسیل ضربه‌ای مرتبه ششم ثابت می‌کنیم. یک حالت خاص و یک مثال نیز در ادامه ارائه شده است.

## ۱. پیش‌گفتار

در این مقاله مسئله مقدار مرزی ضربه‌ای مرتبه ششم زیر را درنظر می‌گیریم:

$$\begin{cases} -u^{(6)}(t)+u^{(4)}(t)-(p(t)u'(t))'+q(t)u(t)=\lambda f(t,u(t)), & t\neq t_j,\ t\in[0,1] \\ u(0)=u(1)=u'''(0)=u'''(1)=0 & \\ \Delta\left(u^{(3)}(t_j)\right)=\mu I_{1j}(u(t_j)), & j=1,2,\dots,m, \\ -\Delta\left(u^{(4)}(t_j)\right)=\mu I_{2j}(u'(t_j)), & j=1,2,\dots,m, \\ \Delta\left(u^{(5)}(t_j)\right)=\mu I_{3j}(u''(t_j)), & j=1,2,\dots,m, \end{cases} \tag{۱.۱}$$

که در آن $p\in C^1([0,1]\times[0,+\infty))$ و $q\in L^\infty([0,1])$ و $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ یک تابع $L^1$-کاراتئودوری است و برای هر $1\leq j\leq m$، $I_{1j},I_{2j},I_{3j}\in C(\mathbb{R},\mathbb{R})$ و $0=t_0<t_1<t_2<\cdots<t_m<t_{m+1}=1$ بوده و عملگر $\Delta$ به‌صورت

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**واژگان کلیدی.** معادله دیفرانسیل ضربه‌ای، جواب کلاسیک، جواب ضعیف، نقطه بحرانی. .

* سخنران

$\Delta(u(t_j)) = u(t_j^+) - u(t_j^-) = \lim_{t\to t_j^+} u(t) - \lim_{t\to t_j^-} u(t)$ تعریف می‌شود و $\lambda$ و $\mu$ اعداد مثبت ثابتی هستند.

**قضیه ۱.۱** ([۱، نتیجه 3.1]). فرض کنیم $X$ یک فضای باناخ حقیقی انعکاسی باشد، $\Phi : X \to \mathbb{R}$ یک تابعک به‌طور پیوسته مشتق‌پذیر گاتو، محدب و اجباری باشد به‌طوری‌که مشتق گاتوی آن دارای یک وارون پیوسته روی $X^*$ است، و $\Psi : X \to \mathbb{R}$ یک تابعک به‌طور پیوسته مشتق‌پذیر گاتو باشد که مشتق گاتوی آن فشرده است و داشته باشیم:

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

فرض کنیم دو عدد مثبت $r_1$ و $r_2$ و $\bar{x} \in X$ موجود باشند به‌طوری‌که $2r_1 < \Phi(\bar{x}) < \frac{r_2}{2}$ و

$(B_1)$ $\frac{\sup_{x\in\Phi^{-1}((-\infty,r_1))} \Psi(x)}{r_1} < \frac{2}{3}\frac{\Psi(\bar{x})}{\Phi(\bar{x})}$؛

$(B_2)$ $\frac{\sup_{x\in\Phi^{-1}((-\infty,r_2))} \Psi(x)}{r_2} < \frac{1}{3}\frac{\Psi(\bar{x})}{\Phi(\bar{x})}$؛

$(B_3)$ برای هر

$$\lambda \in \Lambda_{r_1,r_2} := \left(\frac{3}{2}\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \min\left\{\frac{r_1}{\sup_{x\in\Phi^{-1}((-\infty,r_1))} \Psi(x)}, \frac{\frac{r_2}{2}}{\sup_{x\in\Phi^{-1}((-\infty,r_2))} \Psi(x)}\right\}\right)$$

و برای هر $x_1, x_2 \in X$ که کمینه‌های موضعی تابعک $\Phi - \lambda\Psi$ می‌باشند و نیز $\Psi(x_1) \geq 0$ و $\Psi(x_2) \geq 0$، داشته باشیم:

$$\inf_{s\in[0,1]} \Psi(sx_1 + (1-s)x_2) \geq 0.$$

در این‌صورت برای هر $\lambda \in \Lambda_{r_1,r_2}$، تابعک $\Phi - \lambda\Psi$ دارای حداقل سه نقطه بحرانی مجزا می‌باشد که در $\Phi^{-1}((-\infty, r_2))$ قرار می‌گیرند.

**قضیه ۲.۱** ([۳، قضیه 3.6]). فرض کنیم $X$ یک فضای باناخ حقیقی انعکاسی باشد، $\Phi : X \to \mathbb{R}$ یک تابعک به‌طور پیوسته مشتق‌پذیر گاتو، نیم‌پیوسته پایینی ضعیف دنباله‌ای و اجباری باشد به‌طوری‌که مشتق گاتوی آن دارای یک وارون پیوسته روی $X^*$ است، و $\Psi : X \to \mathbb{R}$ یک تابعک به‌طور پیوسته مشتق‌پذیر گاتو و نیم‌پیوسته بالایی ضعیف دنباله‌ای باشد که مشتق گاتوی آن فشرده است به‌طوری‌که $\Phi(0) = \Psi(0) = 0$. فرض کنیم $r \in \mathbb{R}$ و $u_1 \in X$ با شرط $0 < r < \Phi(u_1)$ موجود باشند به‌طوری‌که

$(A_1)$ $\sup_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) < r\frac{\Psi(u)}{\Phi(u_1)}$؛

$(A_2)$ برای هر $\lambda \in \Lambda_r := \left(\frac{\Phi(u_1)}{\Psi(u_1)}, \frac{r}{\sup_{u\in\Phi^{-1}((-\infty,r])} \Psi(u)}\right)$، تابعک $\Phi - \lambda\Psi$ اجباری باشد.

در این‌صورت برای هر $\lambda \in \Lambda_r$، تابعک $\Phi - \lambda\Psi$ دارای حداقل سه نقطه بحرانی مجزا در $X$ است.

**لم ۳.۱** ([۲]). $u \in X$ یک جواب ضعیف مسئله (۱.۱) می‌باشد اگر و تنها اگر $u$ یک جواب کلاسیک مسئله (۱.۱) باشد.

**لم ۴.۱**. فرض کنیم شرط زیر برقرار باشد:

$(H_1)$ اعداد $\alpha_i, \beta_i > 0$ و $\sigma_i \in [0,1)$، که $i = 1,2,3$ موجود باشند به‌طوری‌که برای $x \in \mathbb{R}$، $i = 1,2,3$ و $j = 1,2,\dots,m$ داشته باشیم:

$$|I_{ij}(x)| \leq \alpha_i + \beta_i |x|^{\sigma_i}$$

در این‌صورت برای $u \in X$ داریم:

$$\left| \sum_{j=1}^{m} \int_0^{u^{(i-1)}(t_j)} I_{ij}(x)dx \right| \leq m \left( \alpha_i \|u\|_\infty + \frac{\beta_i}{\sigma_i + 1} \|u\|_\infty^{\sigma_i+1} \right),\ \forall i = 1,2,3$$

## ۲. دست‌آوردهای پژوهش

قرار می‌دهیم:

$$k = \frac{\delta^2 \pi^2}{288 \left(1 + \frac{\|p\|_\infty}{\pi^2} + \frac{\|q\|_\infty}{\pi^4}\right) \left[\frac{1}{\alpha^5} + \frac{1}{(1-\beta)^5}\right]}$$

$$\Gamma_{i,c} = \frac{\alpha_i}{c} + \left(\frac{\beta_i}{\sigma_i + 1}\right) c^{\sigma_i - 1}$$

که در آن $0 < \alpha < \beta < 1$ و $c$ یک عدد مثبت بوده و برای $i = 1,2,3$، $\alpha_i$، $\beta_i$ و $\sigma_i$ در فرض $(H_1)$ لم ۴.۱ داده شده‌اند.

**قضیه ۱.۲.** فرض کنیم $(H_1)$ برقرار بوده و دو عدد مثبت $c$ و $d$ با شرط $c < \frac{2d}{\pi} \left[\frac{1}{\alpha^3} + \frac{1}{(1-\beta)^3}\right]^{\frac{1}{2}}$ موجود باشند به‌طوری‌که:

$(A_1)$ برای هر $(t, \varepsilon) \in ([0,\alpha] \cup [\beta, 1]) \times [0, d]$ داشته باشیم: $F(t,\varepsilon) \geq 0$

$(A_2)$ $\frac{\int_0^1 \max_{|\varepsilon| \leq c} F(t,\varepsilon)dt}{c^2} < \frac{k \int_\alpha^\beta F(t,d)dt}{d^2}$

$(A_3)$ $\limsup_{|\varepsilon| \to +\infty} \frac{\sup_{t \in [0,1]} F(t,\varepsilon)}{\varepsilon^2} \leq \frac{\pi^2 \int_0^1 \max_{|\varepsilon| \leq c} F(t,\varepsilon)dt}{4c^2}$

در این‌صورت برای هر

$$\lambda \in \Lambda = \left( \frac{2\delta^2\pi^2 d^2}{k \int_\alpha^\beta F(t,d)dt}, \frac{2\delta^2\pi^2 c^2}{\int_0^1 \max_{|\varepsilon| \leq c} F(t,\varepsilon)dt} \right)$$

عدد

$$\rho = \frac{1}{2m} \min_{1 \leq i \leq 3} \left\{ \frac{2\delta^2\pi^2c^2 - \lambda \int_0^1 \max_{|\varepsilon| \leq c} dt}{c^2 \Gamma_{i,c}}, \frac{k\lambda \int_\alpha^\beta F(t,d)dt - 2\delta^2\pi^2 d^2}{d^2 \Gamma_{i,\left(\frac{d}{\sqrt{k}}\right)}} \right\}$$

وجود دارد به‌طوری‌که برای هر $\mu \in [0, \rho)$، مسئله (۱.۱) حداقل سه جواب کلاسیک دارد.

**لم ۲.۲.** فرض کنیم به ازای هر $(t,x) \in [0,1] \times \mathbb{R}$، $f(t,x) \geq 0$ و برای هر $x \in \mathbb{R}$، $j = 1,2,\dots,m$ و $i = 1,2,3$، داشته باشیم: $I_{ij}(x) \leq 0$. اگر $u$ جواب کلاسیک مسئله (۱.۱) باشد آن‌گاه به ازای هر $t \in [0,1]$ داریم: $u(t) \geq 0$.

قرار می‌دهیم:

$$G_{i,c}=\sum_{j=1}^{m}\min_{|\varepsilon|\le c}\int_0^{\varepsilon} I_{ij}(x)dx,\quad \forall c>0,\ i=1,2,3$$

نتیجه دیگری را به‌صورت زیر بیان می‌کنیم:

**قضیه ۳.۲.** فرض کنیم سه عدد مثبت $c_1$، $c_2$ و $d$ با شرط

$$\frac{\pi c_1}{2\left[\frac{1}{\alpha^3}+\frac{1}{(1-\beta)^3}\right]^{\frac{1}{2}}}<d<\sqrt{\frac{k}{2}}c_2$$

موجود باشند به‌طوری‌که:

$(B_1)$ برای هر $(t,x)\in[0,1]\times[0,c_2]$ داشته باشیم: $f(t,x)\ge 0$؛

$(B_2)$ $\frac{\int_0^1 F(t,c_1)dt}{c_1^2}<\frac{2}{3}k\frac{\int_\alpha^\beta F(t,d)dt}{d^2}$؛

$(B_3)$ $\frac{\int_0^1 F(t,c_2)dt}{c_2^2}<\frac{k}{3}\frac{\int_\alpha^\beta F(t,d)dt}{d^2}$؛

در این‌صورت برای هر

$$\lambda\in\Lambda'=\left(\frac{3\delta^2\pi^2d^2}{k\int_\alpha^\beta F(t,d)dt},\delta^2\pi^2\min\left\{\frac{2c_1^2}{\int_0^1 F(t,c_1)dt},\frac{c_2^2}{\int_0^1 F(t,c_2)dt}\right\}\right)$$

و برای هر تابع نامثبت $I_{ij}$ که $j=1,2,\dots,m$ و $i=1,2,3$، عدد

$$\rho^*=\frac{1}{2}\min_{1\le i\le 3}\left\{\frac{\lambda\int_0^1 F(t,c_1)dt-2\delta^2\pi^2c_1^2}{G_{i,c_1}},\frac{\lambda\int_0^1 F(t,c_2)dt-\delta^2\pi^2c_2^2}{G_{i,c_2}}\right\}$$

وجود دارد به‌طوری‌که برای هر $\mu\in[0,\rho)$، مسئله (۱.۱) حداقل سه جواب کلاسیک $u_i$، $i=1,2,3$، در $X$ دارد که به ازای هر $i=1,2,3$، $0<\|u_i\|_\infty\le c_2$.

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پوستر

# حساب کوانتش غیرخطی عملگرهای شبه‌دیفرانسیل و بررسی مثالی از آن

سپیده یزدانی اندبیلی [1]

[1] گروه ریاضی، دانشکده علوم ریاضی، دانشگاه علم و صنعت ایران
*sepidehyazdani1394@gmail.com*

چکیده. مقاله به معرفی عملگر شبه‌دیفرانسیل با کوانتش مرتبط $\tau$ و حساب مربوطه می‌پردازد. حالت غیرخطی کوانتش‌های کوهن-نیرنبرگ، پادکوهن-نیرنبرگ و ویل معمولاً روی گروه‌های لی پوچتوان برای توابع چندجمله‌ای $\tau$، به ویژه جهت وجود تشابه مناسب برای کوانتش ویل در مجموعه غیرجابه‌جایی قابل مشاهده هستند. در این مقاله حساب مربوطه را در حالت $\mathbb{R}^n$ مورد بررسی قرار داده و در انتها مثال‌هایی برای $\tau$ غیرخطی روی گروه هایزنبرگ قطبی و هایزنبرگ بیان شده است که می‌توان مثال‌های دیگری از گروه لی پوچتوان بیان نمود.

## ۰. پیش‌گفتار

اهمیت و ضرورت استفاده از عملگرهای شبه‌دیفرانسیل در زمینه‌های متعددی از جمله معادلات دیفرانسیل با مشتقات جزئی، مکانیک کوانتوم و پردازش تصویر مشهود است. از ویژگی‌ها و خواص این عملگرها در [۶]، [۷]، [۵]، و [۱] یافت می‌شود. کوانتش‌های مورد اشاره در مقاله در چگونگی پیدایش تشابه مناسب برای کوانتش ویل در مجموعه غیرجابه‌جایی قابل مشاهده هستند به طوری که کلاس کوانتش‌های متقارن خاصیت اساسی ماشین کوانتوم را دارند و در آن سیمبل‌های خود-الحاق به عملگرهای خود-الحاق کوانتیده می‌شوند که مطالب مربوط در [۲] و نمونه‌ای از حساب عملگرهای شبه دیفرانسیل در این زمینه در [۴] قابل مشاهده هستند.

عملگر شبه دیفرانسیل مورد مطالعه در این مقاله به صورت زیر است:

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma(x+\tau(y-x),\xi)u(y)dyd\xi \qquad (۱.۰)$$

واژگان کلیدی. عملگر شبه دیفرانسیل، کوانتش (کوانتیزاسیون)ویل، گروه هایزنبرگ .

که در آن $\tau:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ یک تابع است به خصوص برای انتخاب‌های خطی $\tau(x)=0$، $\tau(x)=x$ و $\tau(x)=\dfrac{x}{2}$، به ترتیب کوانتش های کوهن–نیرنبرگ، پادکوهن–نیرنبرگ و ویل را پوشش می‌دهد.

روش‌های (کوانتش) متعددی برای ارتباط دادن عملگر به تابعی از متغیر $(x,\xi)$ روی فضای فاز وجود دارد که در اینجا کوانتش های زیر را که به نام‌های ویل، کوهن–نیرنبرگ و پادکوهن–نیرنبرگ معروف هستند، مطالعه می‌کنیم:

$$\begin{aligned}
&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma\left(\frac{x+y}{2},\xi\right)u(y)dyd\xi,\\
&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma(x,\xi)u(y)dyd\xi,\\
&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma(y,\xi)u(y)dyd\xi.
\end{aligned}$$

## ۲. دست‌آوردهای پژوهش

مقاله به بیان مطالعات و نتایجی پیرامون حساب خواص عملگرهای کلی توسط معادل قرار دادن آن با عملگر مورد نظر می‌پردازد و همچنین الحاقی، ترانهاده و فرمول ترکیب برای سیمبل‌های $\tau$–کوانتیده برای عملگر مرتبط با کوانتش $\tau$ ارائه داده می‌شود. سپس عملگرهای $\tau$–کوانتیده بیضوی و وارون تقریبی (پاراوارون) آنها بیان می‌شود.

همچنین در پی یافتن روش دیگری برای حرکت از یک $\tau$–کوانتش به کوانتش بعدی با تغییر متغیر و با استفاده از قضیه هادامارد نتیجه‌ای حاصل شده و برای $L^2$–کرانداری عملگر مورد نظر در کلاس هورماندر یک سؤال مهم این است که چه تعداد از مشتقات سیمبل باید کراندار باشد تا شرایط $L^2$–کرانداری عملگرهای شبه دیفرانسیلی در قضیه کالدرون–ویلانکورت تضمین شود و ضمناً یک نتیجه قضیه کالدرون–ویلانکورت برای عملگرهای $\tau$–کوانتیده به دست آمده است.

در انتها نشان داده می‌شود که چگونه $\tau$–کوانتش به طور طبیعی در آنالیز عملگرها در مجموعه‌های غیرجابه‌جایی ظاهر می‌شود و چون برای گروه‌های نمایی توابع تقارن همیشه وجود دارد منجر به کوانتش‌های با خاصیت در کوانتش ویل می‌شوند. بدین صورت که از توصیف ماتریسی گروه هایزنبرگ و قطبی استفاده کرده و نمونه‌های ساده‌ای از توابع تقارن در مجموعه‌های غیرجابه‌جایی هایزنبرگ نرمال و قطبی آورده شده و همچنین در گروه هایزنبرگ شکل ساده تابع ”نقطه میانی“ $m(x,y)$ را نشان می‌دهیم. در گروه هایزنبرگ قطبی $\tau$–کوانتش نقش ویل را ایفا می‌کند بدین معنا که عملگرهای $\tau$–کوانتیده با سیمبل‌های خود–الحاقی، خود–الحاقی هستند و تاب جالبی در آخرین متغیر روی $\mathbb{H}_{pol}$ حاصل می‌شود. در نهایت مثال‌های دیگری از گروه لی پوچتوان برای حالت غیرخطی در این زمینه قابل بیان هستند. برای حساب این عملگرها به قضیه زیر اشاره می‌کنیم که به طور کامل در [۲] بیان شده است.

**قضیه ۱.۲.** فرض کنید $a\in A_d^m$ و عملگر مربوطه $A_a$ را به صورت زیر تعریف می‌کنیم:

$$A_a u(x):=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)dyd\xi,\qquad u\in\mathcal{S}(\mathbb{R}^n)$$

آنگاه برای هر تابع قابل قبول $\tau$ از مرتبه $\mu \geq 0$، وجود دارد سیمبل $\sigma$ به طوری که:

$$A_a u(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma(x+\tau(y-x),\xi)u(y)dyd\xi,$$

و $\sigma$ دارای بسط مجانبی ضعیف زیر است:

$$\sigma(x+\tau(y-x),\xi) \sim \sum_{\alpha,\beta\geq 0}\ \sum_{|\alpha|+|\beta|\leq|\delta|\leq N(|\alpha|+|\beta|)} k_\delta(\tau,\alpha,\beta,x-y)\partial_x^\alpha\partial_y^\beta\partial_\xi^\delta a(v,v,\xi)|_{v=x+\tau(y-x)}, \tag{۱.۲}$$

که می‌توانیم برای هر $N \geqslant 1$ در نظر بگیریم و هر یک از موارد

$$k_\delta(\tau,\alpha,\beta,x-y)\partial_x^a\partial_y^\beta\partial_\xi^\delta a(v,v,\xi)|_{v=x+\tau(y-x)}\ ,$$

یک سیمبل در $S^{m-|\delta|}_{\mu(|\alpha|+|\beta|),\tau}$ است. علاوه بر این داریم:

$$k_0(\tau,0,0,x-y) \equiv 1\ .$$

بسط مجانبی ضعیف، در (۱.۲) و در ادامه، به این معنی خواهد بود که یک بسط مجانبی در $\xi$ در مفهوم کلاسیک موضعی در متغیرهای فضا است و به طور کلی برای هر $M \in \mathbb{N}_0$ داریم:

$$\sigma(x+\tau(y-x),\xi) - \sum_{\alpha,\beta\geq 0}\ \sum_{|\alpha|+|\beta|\leq|\delta|\leq N(|\alpha|+|\beta|)} k_\delta(\tau,\alpha,\beta,x-y)\times \times\ \partial_x^\alpha\partial_y^\beta\partial_\xi^\delta a(v,v,\xi)|_{v=x+\tau(y-x)} \in A^{m-M}_{(\mu+d)M}. \tag{۲.۲}$$

در چنین مواردی می‌گوییم $\sigma$ یک جمع مجانبی با شرایط در $S^{m-(|\alpha|+|\beta|)}_{\mu(|\alpha|+|\beta|),\tau}$ و باقیمانده در $A^{m-(|\alpha|+|\beta|)}_{(\mu+d)(|\alpha|+|\beta|)}$ است.

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# آنالیز ریاضی و کاربردهای آن

۵ و ۶ خرداد ۱۴۰۰

# دانشگاه بین المللی امام خمینی (ره)

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