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Extended Abstracts

پنجاه و دومین

کنفرانس ریاضی ایران

52nd Annual Iranian Mathematics Conference

Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

دانشکده ریاضی و کامپیوتر
دانشگاه شهید باهنر کرمان
۸ تا ۱۱ شهریور ۱۴۰۰

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52nd Annual Iranian Mathematics Conference

Faculty of Mathematics and Computer &
Mahani Mathematical Research Center
Shahid Bahonar University of Kerman, Kerman, Iran

30 August- 02 September, 2021

Extended Abstracts

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Preface

It is our great pleasure and honor to welcome you at the 52nd Annual Iranian Mathematics Conference (AIMC52). The AIMC52 will be held on 30-31 August & 1-2 September 2021, hosted by the Faculty of Mathematics and Computer & Mahani Mathematical Research Center, Shahid Bahonar University of Kerman (SBUK), Iran.

The Faculty of Mathematics and Computer of SBUK began its activities in 1995 and currently faculty has five departments: pure mathematics, applied mathematics, mathematics education, statistics and computer science. It has 56 faculty members, 59 doctoral, 143 master's degree and 618 undergraduate students. The members of the faculty are also active in the Mahani Mathematical Research Center located on campus. The mathematics department of the university was established in 1975 and is, in fact, one of the oldest mathematics departments in the country. It began offering a master's program in 1984 and doctoral program in 1988 and it is a matter of great pride that the first three doctorates in mathematics in Iran were awarded to graduates of this department. The faculty, and especially the mathematics department, were also instrumental in the award of the first honorary doctorate degree in mathematics in 2002 to the late Professor Parviz Shahriari, one of the pioneers of mathematics education in Iran.

The Iranian mathematics conferences are yearly events, the first conference being held in 1970. Each year a university in the country volunteers to host the conference together with the Iranian Mathematical Society (IMS). Mathematicians, researches and graduate students from around the world have attended these conferences where they have presented papers and participated in discussions. SBUK hosted the 13th conference in 1982 and has been the host and organizer of this conference every 13 years since then. This year (2021), the faculty of mathematics and computer of SBUK has the honor to organize the 52nd conference.

The AIMC52 received 548 submissions in that each submission was reviewed by some reviewers and one dedicated members of the Scientific Committee. Finally, 345 submissions were accepted for presentation: 251 as oral presentations and 94 as posters. We are proud to present a very interesting program. The conference program included 4 plenary talks, 12 invited talks with distinguished speakers, two panels, two workshops, and annual gathering of women.

Finally, we immensely thank the authors for submitting their research papers to the AIMC52, and are grateful to the members of the Scientific Committee for dedicating their attention and time to assessing the papers. We are also very thankful to the members of the Executive Committee for their efforts in the arrangement, promotion, and organization of the conference.

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Plenary Talks

1. Prof. David Eisenbud, University of California, Berkeley, USA

2. Prof. Esfandiar Eslami, Shahid Bahonar University of Kerman, Iran

3. Prof. Dusa McDuff, Barnard College, Columbia University, USA

4. Prof. Curtis Tracy McMullen, Harvard University, USA

Invited Talks

1. Prof. Rama Cont, University of Oxford, England
2. Prof. Hamid Reza Ebrahimi Vishki, Ferdowsi University of Mashhad, Iran
3. Prof. Behnam Hashemi, Shiraz University of Technology, Iran
4. Prof. Milan Hladik, Charles University, Prague, Czech Republic
5. Prof. Ramin Javadi, Isfahan University of Technology, Iran
6. Prof. Boris Mordukhovich, Wayne State University, USA
7. Prof. Witold Pedrycz, University of Alberta, Canada
8. Prof. Cheryl Elisabeth Praeger, University of Western Australia
9. Prof. Mohammad Safdari, Sharif University of Technology, Iran
10. Prof. Laure Saint-Raymond, University of Lyon, France
11. Prof. Siamak Yassemi, University of Tehran, Iran

Abstract of
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Linearity in the resolution of monomial ideals

David Eisenbud*

Department of Mathematics, University of California, Berkeley, US

Abstract. A well-known theorem of Froeberg describes the square-free quadratic ideals with linear resolutions, and this was extended in a paper of mine with Mark Green, Klaus Hulek and Sorin Popescu to tell when the resolution is linear for the first few steps; but no such result is known for monomials of degree greater than 2. I will recall these results, and discuss some new results of Hai Long Dao and myself on the "opposite" case of primary monomial ideals.

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Logic and its necessity

Esfandiar Eslami*

Department of Pure Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran

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Abstract. In this talk, we first review some general definitions of logic. Their common aspects together with a short history of logic is given. We show how human thinking paradigms give rise to different logics. Some logics with special domains of their applications are discussed. Each logic has a set of inference rules. Using these rules correctly in appropriate domains, we get true logical results. Otherwise we fall in the trap of fallacies. Some very common fallacies are mentioned. At the end, we introduce some modern non-classical logics which are used in Artificial Intelligence (AI). We will emphasize on the needs of future goals of AI to appropriate logics.

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New developments in the symplectic embedding problem

Dusa McDuff*

Department of Mathematics, Barnard College, Columbia University, New York, US

Abstract. I will discuss the question of when a four dimensional symplectic ellipse embeds into a target manifold such as a ball or a blow up of the complex projective plane. I will explain some recent work (due to Cristofaro-Gardiner, Holm, Magill and Weiler, among others) that exhibits the close connections between this question and properties of Pell equations and continued fractions. The talk will be elementary, and accessible to nonspecialists.

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Billiards and Moduli Spaces

Curtis Tracy McMullen*

Harvard University, USA

Abstract. The moduli space Mg of compact Riemann surface of genus g has been studied from diverse mathematical viewpoints for more than a century. In this talk, intended for a general audience, we will discuss moduli space from a dynamical perspective. We will present general rigidity results, provide a glimpse of the remarkable curves and surfaces in Mg discovered during the last two decades, and explain how these algebraic varieties are related to the dynamics of billiards in regular polygons, L-shaped tables and quadrilaterals. A variety of open problems will be mentioned along the way.

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Stochastic Calculus Without Probability: An Analytical Viewpoint

Rama Cont*

Mathematical Institute, University of Oxford, England

Abstract. Stochastic calculus was introduced by Kiyosi Ito and developed by Kunita, Watanabe, Meyer as a calculus for functions of stochastic processes with irregular trajectories, using a probabilistic definition for the stochastic integral. However, Ito's calculus may be alternatively seen as a calculus for causal functionals of systems with rough trajectories. We show that the main ingredients of the Ito calculus may be developed in a purely analytical framework, free of any probabilistic ingredients or assumptions, and sketch the foundations of a causal functional calculus which extends the Newton-Leibniz differential calculus to functionals of systems with rough trajectories of arbitrary irregularity [1, 2].

[1] H Chiu, R Cont (2020) Causal Functional Calculus, https://arxiv.org/abs/1912.07951.

[2] R Cont, N Perkowski (2019) Pathwise integration and change of variable formulas for continuous paths with arbitrary regularity, Transactions of the American Mathematical Society (Series B), Volume 6, 161-186.

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Pure and Applied Mathematics: Confrontation or Interaction?!

Hamid Reza Ebrahimi Vishki*

Ferdowsi University of Mashhad, Iran

Abstract. The subject of comparing pure and applied sciences has long been discussed, and there have been different views on their interaction. In particular, this comparison has been in the spotlight by those who work in mathematics, and there are different ideas for convergence or confrontation of these two tendencies. Indeed, the question may be raised as to which one is the other prerequisite? Should pure mathematics be at the forefront of the applied one, and its direction be determined by applied research and industry needs? Or the problematic nature of knowledge requires that pure mathematics be developed as a product of free thought beyond its immediate applications, leading to expanding the frontiers of knowledge? In this talk, we first give a brief introduction to these two trends and ideas in describing the duties of pure and applied mathematicians. Then we will discuss the existing challenges (especially in Iran) and focus on the more comprehensive question: "Pure or applied mathematics? Or both?".

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Least-squares spectral methods for solving operator eigenvalue problems

Behnam Hashemi*

Faculty of Mathematics, Shiraz University of Technology, Shiraz, Iran

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Abstract. We develop spectral methods for solving operator eigenvalue problems that are based on a least squares formulation of the problem. The key tool is a method for rectangular matrix pencils, which we extend to quasimatrices and objects combining quasimatrices and matrices. When applied to important eigenvalue problems like the standard Orr-Sommerfeld and Sturm-Liouville equations, the accuracy and speed of our methods are similar to typical spectral methods. The strength of the approach is its flexibility, allowing e.g. the basis functions to be chosen arbitrarily, and often giving high accuracy. It is particularly useful for solving challenging problems with boundary conditions depending affinely on the unknown spectral parameter. Such problems appear in a variety of applications e.g., in fluid and structural mechanics.

This talk is based on joint work with Yuji Nakatsukasa (University of Oxford)

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Absolute value programming

Milan Hladik*

Department of Applied Mathematics, Charles University, Prague, Czech Republic

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Abstract. Absolute value programming is quite recent and intensively developing discipline. It refers to systems of equations and inequalities and to mathematical programming problems involving absolute values. We focus primarily on absolute value equations $Ax - b = |x|$, which is the most frequently studied problem in this area, but we will also mention some extensions. Due to their relation to the linear complementarity problem, absolute value equations highly attracted the optimization community. We will discuss not only this relation, but also computational complexity issues, the structure of the solution set, and connections to other areas of mathematics. Important questions are those addressing solvability; we will present various conditions for (unique) solvability or unsolvability. Since the discipline is relatively new, there are many open and challenging problems; we pose some of them, too.

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Expanding Properties of Graphs and its Applications in Ramsey Theory

Ramin Javadi*

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Abstract. Given a positive number α, a graph G on n vertices is called an α-expander if the size of the external neighborhood of every vertex set U of size at most $n/2$ is at least $\alpha|U|$. Expander graphs have been studied widely in the literature and are proved to have significant applications in a wide range of fields such as computer science, computational complexity and coding theory. It is well-known that binomial random graphs are good expanders with high probability. Also, building regular expanders and regular bipartite expanders with an explicit construction is a central and well-studied problem in the context. Recently, some structural properties of expanders are studied and it is proved that expander graphs contain some families of sparse graphs such as trees and cycles of different lengths as subgraphs. In this talk, we explore some of these properties of expanders and then we present some important applications of these properties in Ramsey theory. Given a graph G and an integer $r \geq 2$, the multicolor size-Ramsey number of G, denoted by $\hat{R}(G,r)$, is the smallest integer m such that there is a graph H with m edges for which, in every edge

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coloring of H with r colors, H contains a monochromatic copy of G. The problem of finding the value of the size-Ramsey number of sparse graphs such as paths, trees and cycles, initiated by Paul Erdős, is a long-standing and well-known problem in Ramsey theory. Using expanding properties of random graphs, we give some results regarding the size-Ramsey numbers of paths and cycles.

Variational analysis: What is this about?

Boris Mordukhovich*

Wayne State University, USA

Abstract. Absolute value programming is quite recent and intensively developing discipline. It refers to systems of equations and inequalities and to mathematical programming problems involving absolute values. We focus primarily on absolute value equations $Ax - b = |x|$, which is the most frequently studied problem in this area, but we will also mention some extensions. Due to their relation to the linear complementarity problem, absolute value equations highly attracted the optimization community. We will discuss not only this relation, but also computational complexity issues, the structure of the solution set, and connections to other areas of mathematics. Important questions are those addressing solvability; we will present various conditions for (unique) solvability or unsolvability. Since the discipline is relatively new, there are many open and challenging problems; we pose some of them, too.

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Interpretability and Explainability in Data Analytics: From Data to Information Granules

Witold Pedrycz*

Department of Electrical & Computer Engineering, University of Alberta, Canada

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Abstract. In data analytics, system modeling, and decision-making models, the aspects of interpretability and explainability are of paramount relevance, just to refer here to explainable Artificial Intelligence (XAI). They are especially timely in light of the increasing complexity of systems one has to cope with. We advocate that there are two factors that immensely contribute to the realization of the above important features, namely, (i) a suitable level of abstraction along with its hierarchical aspects in describing the problem and (ii) a logic fabric of the resultant construct. It is shown that their conceptualization and the following realization can be conveniently carried out with the use of information granules (for example, fuzzy sets, sets, rough sets, and alike). Concepts are building blocks forming the interpretable environment capturing the essence of data and key relationships existing there. The emergence of concepts is supported by a systematic and focused analysis of data. At the same time, their initialization is specified by stakeholders or/and the owners and users of data. We present a comprehensive discussion of information granules-oriented design of concepts and their description by engaging an innovative mechanism of conditional

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(concept)-driven clustering. A detailed case study of enhancement of interpretability of functional rule-based models with the rules in the form "if x is A then $y = f(x)$". The interpretability mechanisms are focused on the elevation of interpretability of the conditions and conclusions of the rules. It is shown that augmenting interpretability of conditions is achieved by (i) decomposing a multivariable information granule into its one-dimensional components, (ii) their symbolic characterization, and (iii) linguistic approximation. A hierarchy of interpretation mechanisms is systematically established. We also discuss how this increased interpretability associates with the reduced accuracy of the rules and how sound trade-offs between these features are formed.

Codes and designs in Johnson graphs

Cheryl Elisabeth Praeger*

University of Western Australia

Abstract. The Johnson graph $J(v, k)$ has, as vertices, all k-subsets of a v-set V, with two k-subsets adjacent if and only if they share $k - 1$ common elements of V. Subsets of vertices of $J(v, k)$ can be interpreted as the block-set of an incidence structure, or as the set of codewords of a code, and automorphisms of $J(v, k)$ leaving the subset invariant are then automorphisms of the corresponding incidence structure or code. This approach leads to interesting new designs and codes. For example, numerous actions of the Mathieu sporadic simple groups give rise to examples of Delandtsheer designs (which are both ag-transitive and anti-ag transitive), and codes with large minimum distance (and hence strong error-correcting properties). In my talk I will explore links between designs and codes in Johnson graphs which have a high degree of symmetry, and I will mention several open questions.

This talk is based on joint work with R. A. Liebler, M. Neunhoeffer, and more recently J. Bamberg, A. C. Devillers and M. Ioppolo

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Local and nonlocal equations with gradient constraints

Mohammad Safdari*

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Abstract. We consider the questions of existence and regularity of fully nonlinear local or nonlocal equations with gradient constraints, which appear in singular stochastic control problems. We do not assume any regularity about the constraints, so in particular they need not be strictly convex. We will also consider local or nonlocal double obstacle problems which naturally arise in this study.

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Dynamics of perfect gases: a statistical approach

Laure Saint-Raymond*

Department of Mathematics, École Normale Supérieure, University of Lyon, France

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Abstract. The evolution of a gas can be described by different models depending on the observation scale. A natural question, raised by Hilbert in his sixth problem, is whether these models provide consistent predictions. In particular, for rarefied gases, it is expected that statistical models of kinetic theory can be obtained directly from molecular dynamics governed by the fundamental principles of mechanics. In the case of hard sphere gases, Lanford showed that the Boltzmann equation corresponds indeed to the law of large numbers in the low density limit, at least for very short times. The objective of this survey is to present recent progresses in the understanding of this limiting process, providing a complete statistical description of these dynamical systems.

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Cohen-Macaulayness in a Fixed Codimension

Siamak Yassemi*

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Abstract. A concept of Cohen-Macaulay in codimension t is defined and characterized for arbitrary finitely generated modules and coherent sheaves by Miller, Novik, and Swartz in 2011. Soon after, Haghighi, Yassemi, and Zaare-Nahandi defined and studied CMt simplicial complexes, which is the pure version of the above mentioned concept and naturally generalizes both Cohen- Macaulay and Buchsbaum properties. The purpose of this talk is to survey a number of recent studies of CMt simplicial complexes. We focus on the Stanley-Reisner ring of a simplicial complex, the shape of the Betti diagram of the Stanley-Reisner ideal of a simplicial complex in special cases and the independence simplicial complex of a simple graph. In the final step, we introduce the CMt property for an unmixed monomial ideal of a polynomial ring. This research program has produced many exciting results and, at the same time, opened many further interesting questions and conjectures.

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Papers

Part 1: Talks

weighted composition operators from S^p spaces into Bloch spaces

Ebrahim Abbasi[1,] and Mostafa Hassanlou[2]*

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Abstract. In this paper we give conditions for the boundedness and compactness of weighted composition operators between spaces of functions with derivative in Hardy spaces and Bloch spaces. As a result we find similar conditions for composition operators and multiplication operators.

Keywords: Hardy spaces, Bloch spaces, weighed composition operators

AMS Mathematics Subject Classification [2010]: Primary 47B38, 46E15; Secondary 30D55.

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1. Introduction

By $\mathbb{D}$ we mean the open unit disk in the complex plane and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. Let $S(\mathbb{D})$ be the space of analytic functions from $\mathbb{D}$ into itself. Given $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$, the weighted composition operator is defined by

$$uC_\varphi f = u(f \circ \varphi), \qquad f \in H(\mathbb{D}).$$

The class of weighted composition operators include multiplication operators $M_u f = uf$ and composition operators $C_\varphi f = f \circ \varphi$. The main subject concerning these operators on the spaces of analytic functions is investigating the operator properties such as boundedness, (weak) compactness, closed range and $\cdots$. Such a properties is related to function properties of u and φ. In this paper we are going to find some conditions for the boundedness and compactness of weighted composition operators between the spaces with derivative in Hardy spaces and Bloch spaces. For $1 \le p < \infty$, the Hardy space H^p consists of all analytic functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{H^p} = \left(\sup_{0<r<1} M_r(f,p) \right)^{1/p} = \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty.$$

These spaces are Banach with the norm $\| \cdot \|_{H^p}$.

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Let $1 \leq p < \infty$ and $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$. We denote by S^p, the space of analytic function on $\mathbb{D}$ such that derivative in Hardy spaces. So,

$$S^p = \{f \in H(\mathbb{D}) : f' \in H^p\}.$$

It can be see that S^p is a Banach space with the norm $||f||_{S^p} = |f(0)| + ||f'||_{H^p}$. Theorem 3.11 of [3] implies that if $f \in S^1$ then f extends continuously to $\overline{\mathbb{D}}$. Thus $S^p \subset A$, where A is the disc algebra consisting all analytic functions on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ with the norm $||f||_A = \sup_{z\in\mathbb{D}} |f(z)|$. Also the inclusion map $S^p \hookrightarrow A$ is bonded for $1 \leq p \leq \infty$, [2]. Another space which we use here is Bloch space which

$$\mathcal{B} = \{f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)|\} < \infty,$$

and the Banach norm for the space is

$$||f||_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)|.$$

Here we just use the weight function $(1 - |z|^2)$. But it can be generalized to the weights $(1 - |z|^2)^\alpha$ for $0 < \alpha < \infty$ or even more for general weight functions which are known in the literature.

If $f \in H^p$ then

$$|f(z)| \leq \frac{||f||_{H^p}}{(1 - |z|^2)^{1/p}}.$$

So if $f \in S^p$ then

$$|f'(z)| \leq \frac{||f'||_{H^p}}{(1 - |z|^2)^{1/p}} \leq \frac{||f||_{S^p}}{(1 - |z|^2)^{1/p}},$$

and $|f(z)| \leq c||f||_\infty$ where $||f||_\infty = \sup_{z\in\mathbb{D}} |f(z)|$ and c is a positive constant.

Weighted composition operators and the S^p spaces have been investigated in [2]. In [5] the authors studied Volterra type operators on these spaces. For complete characterization on some properties of composition and multiplication operators on S^2 we can refer to [4]. For more information about Hardy spaces see [3].

2. Main results

In this section we give necessary and sufficient conditions for the compactness and bondedness of weighted composition operators from S^p spaces into Bloch spaces.

Theorem 2.1. *Let $1 \leq p < \infty$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $uC_\varphi : S^p \to \mathcal{B}$ is bounded if and only if $u \in \mathcal{B}$ and*

$$\sup_{z\in\mathbb{D}} \frac{(1 - |z|^2)|u(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{1/p}} < \infty.$$

Corollary 2.2. *Let $1 \leq p < \infty$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the composition operators $C_\varphi : S^p \to \mathcal{B}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1 - |z|^2)|\varphi'(z)|}{(1 - |\varphi(z)|^2)^{1/p}} < \infty,$$

and the multiplication operators $M_u : S^p \to \mathcal{B}$ is bounded if and only if $u \in \mathcal{B}$ and

$$\sup_{z\in\mathbb{D}} \frac{(1 - |z|^2)|u(z)|}{(1 - |z|^2)^{1/p}} < \infty.$$

Theorem 2.3. *Let* $1 \leq p < \infty$, $u \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$. *Then* $uC_\varphi : S^p \to \mathcal{B}$ *is compact if and only if*

$$\limsup_{|z|\to 1}(1-|z|^2)|u'(z)| = 0$$

$$\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1/p}} = 0.$$

Corollary 2.4. *Let* $1 \leq p < \infty$, $u \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$. *Then the composition operators* $C_\varphi : S^p \to \mathcal{B}$ *is compact if and only if*

$$\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi'(z)|}{(1-|\varphi(z)|^2)^{1/p}} = 0,$$

and the multiplication operators $M_u : S^p \to \mathcal{B}$ is compact if and only if

$$\limsup_{|z|\to 1}(1-|z|^2)|u'(z)| = 0$$

$$\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|u(z)|}{(1-|z|^2)^{1/p}} = 0.$$

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Faculty of Mathematics and Computer
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30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

The use of artificial neural network (ANN) to simulate HIV infection model

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Abstract. In this work we implement an artificial neural network for the approximate solution of the mathematical model which describes the behavior of $CD4^+$ T-cells, infected CD4+ T-cells and free HIV virus particles after HIV infection. Also, the effect of constant and different variable source terms used for supplying the new CD4+ T-cells from thymus on the dynamics of $CD4^+$ T-cells, infected $CD4^+$ T-cells and free HIV virus are investigated.

Keywords: Artificial neural network, Ordinary differential equations, Numerical analysis.

AMS Mathematics Subject Classification [2010]: 65MX

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1. Introduction

The acquired immunodeficiency syndrome (AIDS) is a communicable disease and human immunodeficiency virus (HIV) is the causative agent for AIDS which damages ability of body to fight against diseases and leave it open to attack from usual innocuous infections. On entering the body HIV infects a large amount of $CD4^+$ T-cells and replicates quickly. The recent decade, researchers have proposed several models for human immune system to understand HIV dynamics, HIV infection, disease progression and interaction of the immune system with HIV. In the current paper, we consider the following model

$$
\begin{aligned}
\frac{dT(t)}{dt} &= s_0 - \mu_T T(t) + \alpha T(t)\left(1 - \frac{T(t)+I(t)}{T_{\max}}\right) - \beta V(t)T(t), \\
\frac{dI(t)}{dt} &= \beta V(t)T(t) - \mu_I I(t), \\
\frac{dT(t)}{dt} &= \gamma \mu_I I(t) - \mu_V V(t),
\end{aligned} \tag{1}
$$

with initial conditions

$$
T(0) = T_0, \qquad I(0) = I_0, \qquad V(0) = V_0. \tag{2}
$$

In model (1)-(2), we have

*Speaker

(1) The new supply rate of healthy T-cells from thymus, $s_0 = 0.1$,
(2) Growth rate of healthy T-cells population, $\alpha = 3$,
(3) Turnover rate of healthy T-cells, $\mu_T = 0.02$,
(4) Turnover rate of infected T-cells, $\mu_I = 0.3$,
(5) Turnover rate of free virus, $\mu_V = 2.4$,
(6) The infection rate, $\beta = 0.0027$,
(7) Maximum population level of healthy T-cells, $T_{max} = 1500$,
(8) Number of virus produced by infected T-cells, $\gamma = 10$.

We want to solve this model using the Artificial neural network [3]. Artificial neural network (ANN) is one of the popular areas of artificial intelligence (AI) research and also an abstract computational model based on the organizational structure of the human brain. ANN is a data modeling tool that depends to different parameters and learning methods. ANN acquires knowledge through learning, and this knowledge is stored within interneuron connections" strength, which is expressed by numerical values called "weights." These weights are used to compute output signal values for a new testing input signal value. Patterns are presented to the network via the "input layer," which communicates to one or more "hidden layers," where the actual processing is done via a system of weighted "connections." Figure 1 demonstrates the hidden layers then link to an output layer where the answer is output.

2. Proposed Method

We start with a system of differential equations with initial conditions which are introduced in equations (1.1) to (1.4). The goal is to learn THI model by using Artificial Neural Network (ANN) approach. We have chosen an Artificial Neural Network with an architecture of a fully connected network with one Hidden layer of neurons, three inputs and three outputs. see Figure 2 .The activation function chosen is a purelin function which is linear and is defined as $a = x$. The BFGS algorithm is used as an optimizer here and is proposed in the next section. The stopping criteria is the maximum iteration amount of 1000. It means that when the neural networks learning has reached the number of iterations, the learning process will stop, if the number of iterations is less than 1000, the learning will continue.

3. BFGS Algorithm

In this article, we are going to use a variant of gradient descent method known as Broyden-Fletcler-Goldfarb-Shanno (BFGS) optimization algorithm. The BFGS algorithm overcomes some of the limitations of plain gradient descent by seeking the second derivative of the cost function. Obiviously, BFGS method is one of the most effective methods for unconstrained optimization [1] . This method is one of the quasi-Newton methods which was successfully used for minimizing errors on Artificial Neural Networks and is useful when the calculation of the Hessian matrix is difficult or time-consuming. This method has a faster convergence in comparison with the method of gradient descent.
The following problem can be solved by BFGS algorithm which is introduced as below.

$$min f(x_k) \tag{3}$$

(1) Input x_0, ε (stopping criteria) and k_{max}.
(2) Set an initial iteration $k = 0$ and $B = I$, where I is an identity matrix.
(3) Calculate $f(x_k)$

(4) While ($\|\Delta(x_k)\| \geq \varepsilon$) or ($k \leq k_{max}$) do:
a. Calculate $d_k = -B_k f(x_k)$, where d_k is a generating direction.
b. Select α that is able to minimize $f(x_k + \alpha_k d_k)$.
c. $x_{k+1} = x_k + \alpha_k d_k$.
d. Calculate B_{k+1} by using the following equation.

$$B_{k+1} = B_k - \frac{B_k s_k (B_k s_k)^T}{s_k^T B_k S_k} + \frac{y_k y_k^T}{y_k^T s_k} + \Phi_k [s_k^T B_k s_k] v_k v_k^T \tag{4}$$

Where $\Phi_k \in [0, 1]$, $s_k = x_{k+1} - x_k$, $y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$ and $v_i = [\frac{y_k}{y_k^T s_k} - \frac{B_k s_k}{s_k^T B_k s_k}]$

(5) k=k+1
(6) End while.

4. Numerical Example

In this section, we use the proposed ANN to solve the main model based upon the mentioned parameters in [2]. Figures 3 and 4 show the graph of $T(t)$, $I(t)$ and $V(t)$ for one day in which they show a decaying oscillatory behavior. Figure **??** demonstrates the graph of $T(t)$, $I(t)$ and $V(t)$ for 70 days.

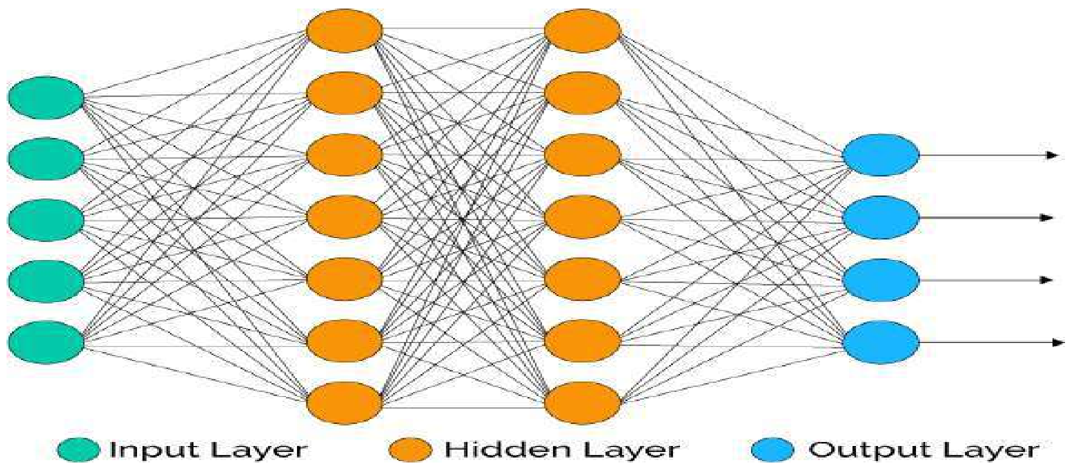

FIGURE 1. Structure of artificial neural network

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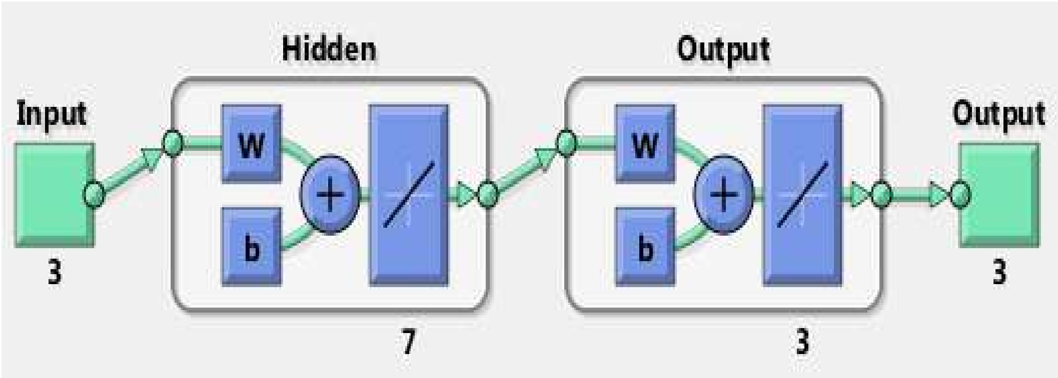

FIGURE 2. Structure of artificial neural network in THI model

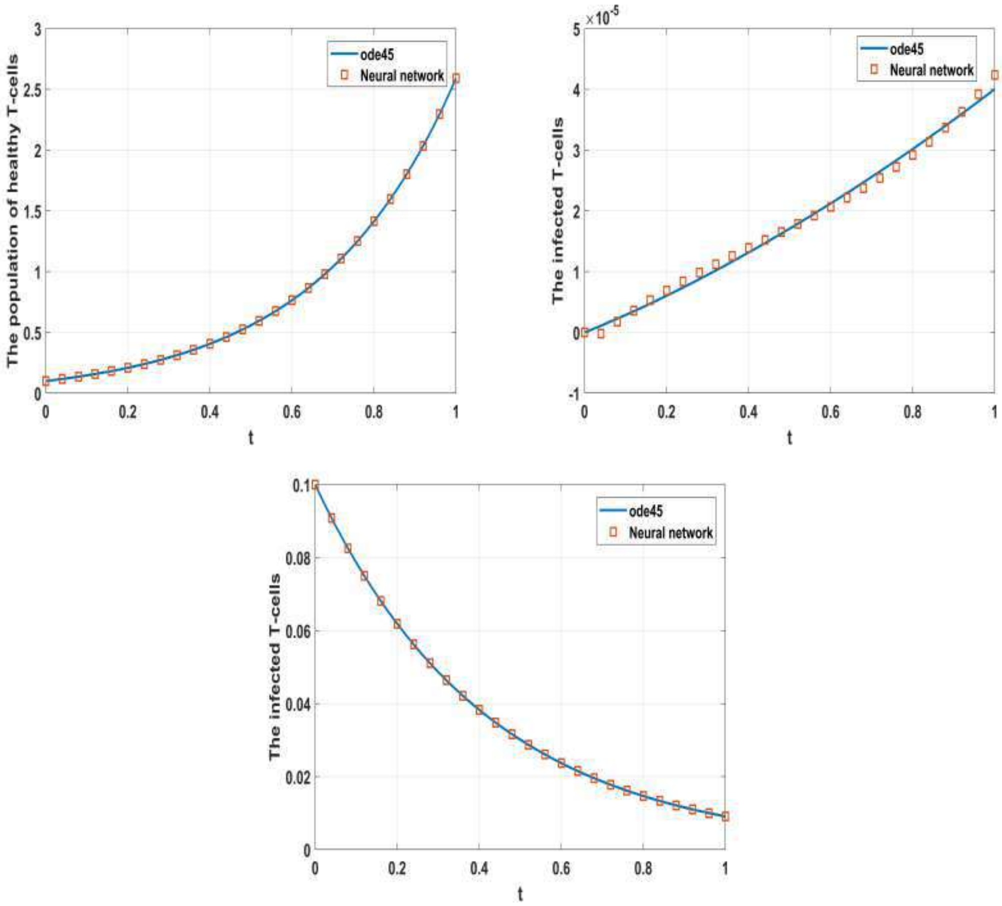

FIGURE 3. Approximate values of $T(t)$, $I(t)$ and $V(t)$ for one day.

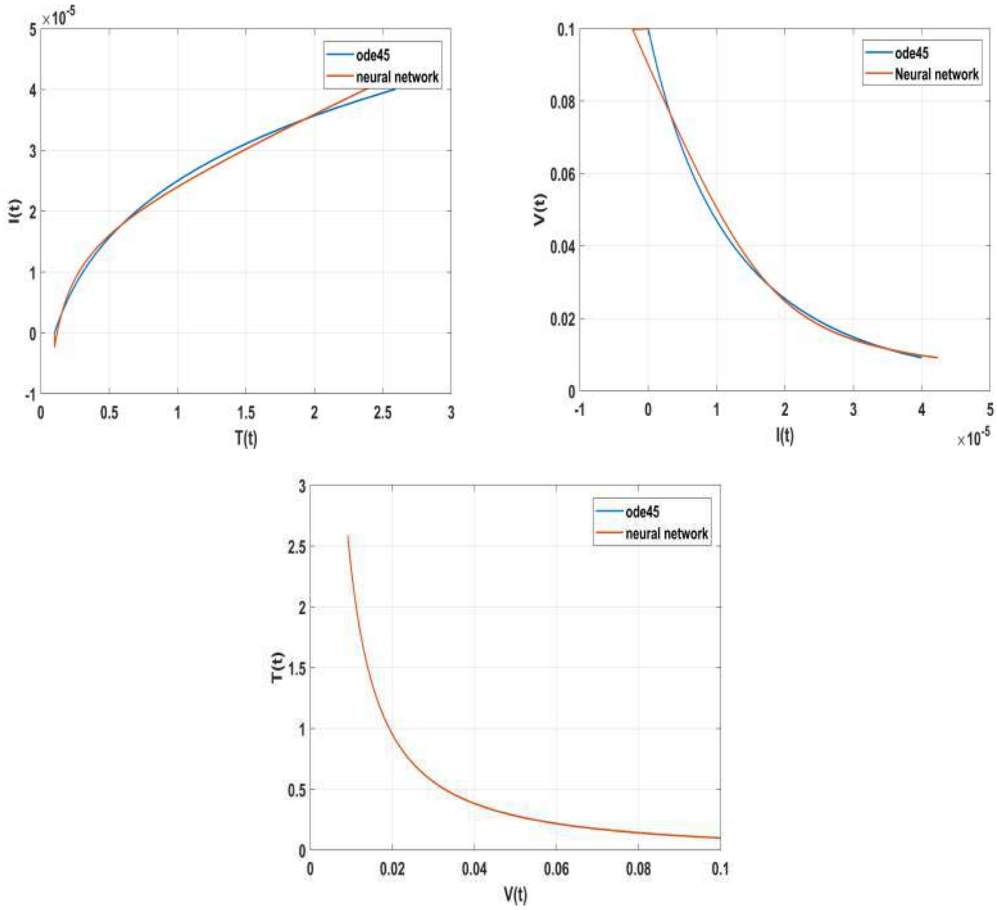

FIGURE 4. Approximate values of $T(t)$, $I(t)$ and $V(t)$ for one day.

Simultaneously Hard Thresholding Algorithms with Feedbacks and Partially Known Row Support for Multiple Measurement Vectors

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ABSTRACT. In this paper, we introduce simultaneously hard thresholding feedbacks with partially known row support (SNST+HT+FB+PKRS) for solving Multiple Measurement Vector Problem (MMV). This method has higher accuracy than solving the problem by breaking it apart into independent Single Measurement Vector (SMV) problems and applying the hard thresholding feedbacks with partially known support (NST+HT+FB+PKS). Furthermore, we compare it with MMV Orthogonal Matching Pursuit (M-OMP), MMV Basic Matching Pursuit (M-BMP) and MMV FOCal Underdetermined System Solver (M-FOCUSS).

Keywords: Multiple measurement vector, Compressed sensing, Restricted isometry principle, Fast thresholding algorithm.

AMS Mathematics Subject Classification [2010]: 65F10, 65F50, 15A29.

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1. Introduction

The Multiple Measurement Vector Problem (MMV) has wide applications in the neuromagnetic inverse problem, nonparametric spectrum analysis of time series and radar imaging. In MMV problem, we are looking for the unknown sparse signal $X(X \in \mathbf{R}^{m\times l}(l < n))$ of the linear equation $AX = B$, where $A \in \mathbf{R}^{n\times m}(n \ll m)$ and $B \in \mathbf{R}^{n\times l}$ are known. If l is equal to one, this model is called Single Measurement Vector (SMV) model, otherwise it is called Multiple Measurement vectors (MMV), where the solutions are assumed to have a common sparsity profile. One way to solve the MMV model which comes to mind is solving the following l underdetermined systems of equations:

$$Ax^{(j)} = b^{(j)},\ j = 1, \dots, l(l < n)$$

where the solution vectors $x^{(j)}$ have the same sparsity. Some researchers have worked on the case MMV. For instance, Chen and Huo have proved that OMP, which is a greedy algorithm, can find the sparset representations for the MMV model under certain conditions [2]. In addition, Cotter et al. offered a variant of FOCUSS algorithm to solve MMV model [3]. In this research we extend NST+HT+FB+PKS method in the SMV

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model [4] to the MMV model. The rest of this paper is organized as follows. At first, we express some important definitions. Seconly, we introduce SNST+H+FB+PKRS and the theoretical analysis of this algorithm. Finally, we study numerical tests and comparisons. The support of a vector $x \in \mathbf{R}^m$ is the index set of its nonzero entries, i.e.,

$$supp(x) = \{j \in \{1, \dots, m\} | x_j \neq 0\}.$$

The vector $x \in \mathbf{R}^m$ is called s-sparse if at most s of its entries are nonzero, i.e., if $\|x\|_0 = |supp(x)| \leq s$, where $|supp(x)|$ is the number of indices in $supp(x)$.

Definition 1.1. (see [5]) The row support set of a matrix $X = (x^{(1)}, \dots, x^{(l)}) \in \mathbf{R}^{m \times l}$ is,

$$L = supp(X) := \{i \in \{1, \dots, l\} | x^{(i)} \neq 0\}.$$

The matrix $X \in \mathbf{R}^{m \times l}$ is called s-row-sparse if at most s of its rows are nonzero.

Here A is assumed to have both full row rank and Gaussian entries. A_L is the submatrix consisting of columns of A indexed by L, X_L includes rows of X indexed by L and L^c is the set of $\{1, \dots, m\} \setminus L$. We assume that L is PKRS, i.e., $L = L_0 \cup L_1$, where L_0 is prior row support set of X and L_1 is the unknown part of the row support set. $RH_{l-t}^{L_0}(X)$ is a nonlinear operator which selects the $l-t$ largest rows of X not indexed in L_0 in terms of magnitude of norm and sets other ones to zero.

To state the convergence results, we first recall the definition of restricted isometry property (RIP) and preconditioned restricted isometry property (P-RIP) (see [1,6]).

Definition 1.2. For each integer $s = 1, 2, \dots,$ the restricted isometry constant δ_s of a matrix A is defined as the smallest number δ_s such that

$$(1 - \delta_s)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_s)\|x\|_2^2, \tag{1}$$

holds for all s-sparse vector x.

Definition 1.3. For each integer $s = 1, 2, \dots,$ the preconditioned restricted isometry constant γ_s of a matrix A is defined as the smallest number γ_s such that

$$(1 - \gamma_s)\|x\|_2^2 \leq \|(AA^*)^{\frac{-1}{2}} Ax\|_2^2, \tag{2}$$

holds for all s-sparse vector x.

2. Main results

In this section, we propose a new algorithm that is designed to recovery of s-row-sparse matrices X from measurements $Y = AX + E$. This algorithm is a natural extension of the NST+HT+FB+PKRS algorithm, which was introduced in [4]. We named this algorithm SNST+HT+FB+PKRS and is implemented as follows. In this algorithm, the $k-$th update row support set is L_k and the set L_0 is the prior row support set.

In the following theorm, which is an extension of Theorem 1 in [4], theoretical performance of the SNST+HT+FB+PKRS method is presented.

Theorem 2.1. *Let $X \in \mathbf{R}^{m \times l}$ be an arbitrary signal. Define $L^\sharp = supp(X^{[\sharp]})$, where $X^{[\sharp]}$ is a real s-row-sparse solution of $AX + \tilde{e} = Y$ and L_0 is the partially known row* *support set, which meets $L_0 \subset L^\sharp$ with $|L_0| = t$. If the P-RIP and RIP constants of A satisfy $\delta_{2s-t} + \sqrt{2}\gamma_{3s-2t} < 1$, then $U^{[k]}$ in $SNST + HT + FB + PKRS$ satisfies*

$$\|U^{[k]} - X^{[\sharp]}\|_F \leq \rho^k \|U^{[0]} - X^{[\sharp]}\|_F + \tau \frac{1}{1-\rho} \|e^{[\sharp]}\|_F$$

where $\rho = \frac{\sqrt{2}\gamma_{3s-2t}}{1-\delta_{2s-t}}$, $\tau = \frac{\sqrt{2}+\sqrt{1+\delta_s}}{1-\delta_{2s-t}}$ and $e^{[\sharp]} = A(X - X^{[\sharp]}) + \tilde{e}$.

Algorithm 1 SNST+HT+FB+PKRS Algorithm

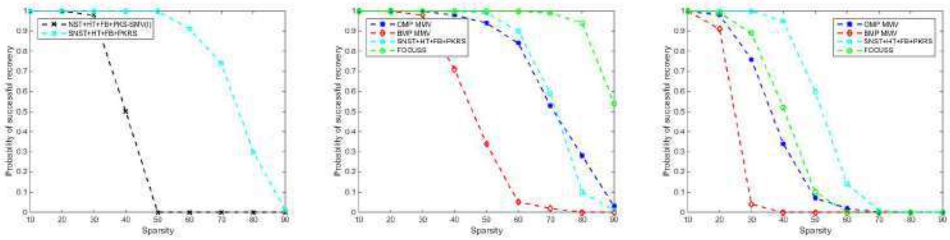
Input: A, Y, s, L_0;
Output: X;
Initialization: $k = 0$, $X^{[0]} = A^*(AA^*)^{-1}Y$, $U^{[-1]} = 0$, $U^{[0]} = RH_{l-t}^{L_0}(X^{[0]})$, $\epsilon_1 = 10^{-5}$, $\epsilon_2 = 10^{-6}$
 while $\left(\frac{\|AU^{[k]}-Y\|_F}{\|Y\|_F} \geq \epsilon_1 \text{ and } \frac{\|U^{[k]}-U^{[k-1]}\|_F}{\|U^{[k-1]}\|_F} \geq \epsilon_2\right)$ **do**
  $L_{k+1} = L_0 \cup supp(RH_{l-t}^{L_0}(X^{[k]} + \mathcal{P}(U^{[k]} - X^{[k]}))$.
  $U_{L_{k+1}}^{[k]} = X_{L_{k+1}}^{[k]} + (A_{L_{k+1}}^* A_{L_k})^{-1} A_{L_{k+1}}^* A_{L_{k+1}^c} X_{L_{k+1}^c}^{[k]}$.
  $U_{L_{k+1}^c}^{[k]} = 0$.
  $X^{[k+1]} = X^{[k]} + \mathcal{P}(U^{[k]} - X^{[k]})$.
  $k = k + 1$.
 end while

---

3. Numerical results

In this section, we compare SNST+HT+FB+PKRS method with NST+HT+FB+PKS-SMV(l), M-OMP, M-BMP and M-FOCUSS methods.

3.1. Table and Figure. In all the comparisons, the measurement matrix A has *i.i.d* entries drawn from a standard normal disturbtion with normalized columns. We have set $\epsilon_1 = 10^{-5}$ and $\epsilon_2 = 10^{-6}$ as stoping parameters and done 100 repetitions of each experiment. We have stated the noise-free case here, similar results are obtained for the noise case. In Test 1, 2 and 3, the signal matrix lenght is assumed as 256×10 and in each repetition its s-row-sparsity level varies. The size of the measurement matrix A is 128×256. In accordance with the probability of successful recovery diagram in Figure 1, in Test 1, using algorithm SNST+HT+FB+PKRS is better than solving ten linear equations with NST+HT+FB+PKS algorithm. In Test 2, we cosider X as a Gaussian matrix and algorithm SNST+HT+FB+PKRS performs close to the M-OMP method. The M-FOCUSS method works better than other methods and in Test 3, we set $X(i,j) = i^{1.1}, i = 1, \ldots, 256, j = 1, \ldots, 10$ and the algorithm SNST+HT+FB+PKRS performs better than the others. In Figure 2, the row of matrix A varies from 90 to 330 by step 30 and the signal matrix has s-row-sparsity 50. In addition, the signal matrix lenght is assumed as 1000×10. We conclude from Figure 2 that with increasing the percentage of partially known row support, the probability of successful recovery also increases and the number of repetitions of SNST+HT+FB+PKRS method decreases.



(A) Test1 (B) Test2 (C) Test3

FIGURE 1. Plots of the probability of successful recovery as a function of the s-row-sparsity.

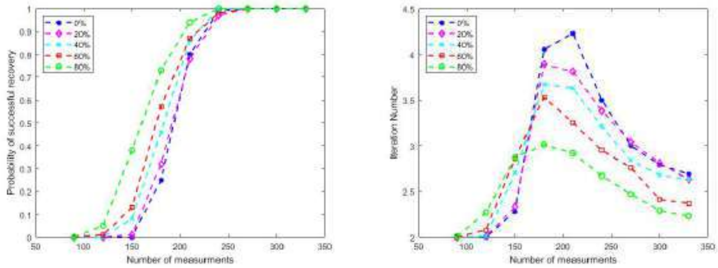


FIGURE 2. Plots of the probability of successful recovery and the iteration numbers as a function of the varying number of measurements with s-row-sparsity 50.

4. Conclusion

When we break the MMV problem apart into independent SMV problems and apply NST+HT+FB+PKS, the probability of successful signal reconstruction reduces. Moreover, we conclude that the reconstruction accuracy grows and the iteration number of SNST+HT+FB+PKRS reduces as the percenntage of known row support increases.

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Faculty of Mathematics and Computer
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30 August - 02 September 2021
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52nd Annual Iranian Mathematics Conference
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WENO-Z schemes based on the identification of extreme points for Hamilton-Jacobi equations

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Abstract. The aim of this work is to design a fifth-order WENO-Z scheme in the framework of finite difference for Hamilton-Jacobi (HJ) equations. By finding the extreme points of the reconstruction polynomial, the scheme (MWENO-Z) automatically adapts between the linear upwind scheme and a WENO-Z scheme. By comparing the numerical results of MWENO-Z and the classical WENO proposed by Jiang and Peng for HJ, the efficiency and robustness of MWENO-Z is appeared.

Keywords: WENO-Z scheme, finite difference framework, Hamilton-Jacobi equation, computational efficiency

AMS Mathematics Subject Classification [2010]: 65M06, 35F21

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1. Introduction

This research aims at obtaining numerical solutions for 1D HJ equations of the form

$$\phi_t(x,t) + H(x,t,\phi,\phi_x) = 0. \tag{1}$$

This type of equations are often appeared in many applications such as image processing, variational calculus, computer vision, material science and geometric optics. Although the solutions of HJ equations are usually continuous, their derivatives are discontinuous; even when the initial condition is smooth. Accordingly, it is better to study them in a suitable weak formulation. Such a weak formulation is presented by so-called *viscosity solutions*. Since the HJ equations are well known to be closely related to conservation laws, hence successful numerical methods for them can be adjusted to approximate the solutions of the HJ equations. In 2000, Jiang and Peng proposed the first version of WENO in the framework of finite difference for HJ equations [3]. The design of MWENO-Z has steps that are given in the following. First, in the big stencil, a fourth degree polynomial is constructed using the finite differences of ϕ. Second, the extreme points of the fourth degree polynomial, found in the previous step, are obtained. Third if in the big stencil the polynomial has at least one extreme point, then a new WENO-Z reconstruction is employed to approximate the numerical flux; otherwise, the numerical flux is approximated directly by the reconstruction polynomial. WENO schemes for HJ equations are concluded from a semi-discrete form. Accordingly, a uniform mesh with cells $I_x = [x - \Delta x, x]$ is supposed.

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Let $|I_x| = \Delta x$ to be the length of I_x. Also, the notations $\phi_j \equiv \phi_j(t) = \phi(x_j, t)$ and $\Delta^- \phi_j = \phi_j - \phi_{j-1}$ are considered. The semi-discretization formula is derived as

$$\frac{d\phi_j(t)}{dt} = -\hat{H}(x_j, t, \phi_j, \phi^-_{x,j}, \phi^+_{x,j}) = F(\phi_j(t)), \tag{2}$$

where the term $\hat{H} := \hat{H}(x_j, t, \phi_j, \phi^-_{x,j}, \phi^+_{x,j})$ is the numerical flux function. In this research the Lax-Friedrichs flux is used.

Details on how to form the MWENO-Z method are provided in Sect. 2. The numerical results of the new scheme are presented in Sect. 3 and concluding remarks are presented in Sect. 4.

2. Modified WENO-Z scheme

This section briefly describes how to design MWENO-Z scheme to solve Eq. (1).

step 1. By considering the big stencil $S = \{I_{x_{j-2}}, \ldots, I_{x_{j+2}}\}$, the fourth degree reconstruction polynomial p_0 can be easily obtained by making the following condition

$$\int_{I_i} p_0(\eta)\mathrm{d}\eta = \Delta^- \phi_{j+l}, \qquad l = -2, \cdots, 2. \tag{3}$$

step 2. Identify the extreme points of $p_0(x)$. Since the degree of $p_0'(x)$ is at most three, therefore, the real zero points of $p_0'(x)$ can be explicitly solved [2] and one is the extreme point of $p_0(x)$ if it is not a doubled zero point of $p_0'(x)$.

step 3. Now if the extreme points of the reconstruction polynomial $p_0(x)$ are outside the big stencil S or there is no extreme point at all, the approximations at the boundaries of each cell are directly given by $\phi^{-up}_{x,j} = p_0(x_j)$ and $\phi^{+up}_{x,j-1} = p_0(x_{j-1})$ and the procedure jumps to step 5.

step 4. Now if there is one or more extreme points in the big stencil S, a new WENO reconstruction is applied to approximate $\phi^-_{x,j}$ and $\phi^+_{x,j-1}$ as follows. The big stencil S is divided into three smaller stencils $S_1 = \{I_{x_{j-2}}, I_{x_{j-1}}, I_{x_j}\}$, $S_2 = \{I_{x_{j-1}}, I_{x_j}, I_{x_{j+1}}\}$ and $S_3 = \{I_{x_j}, I_{x_{j+1}}, I_{x_{j+2}}\}$ whose community is the same S. Now we need three reconstruction polynomials $p_r(x)$, $r = 1, 2, 3$ associated to these small stencils. First we find the polynomial $p_r(x)$ such that

$$\int_{I_{x_{j+l+r-1}}} p_r(x)dx = \Delta^- \phi_{j+l+r-1}, \qquad l = -2, -1, 0. \tag{4}$$

The new WENO-Z reconstruction on cell $I_{x_j} = [x_{j-1}, x_j]$ is considered as follow:

$$R(x) = \frac{w_0}{d_0}\big[p_0(x) - \sum_{l=1}^{3} d_l p_l(x)\big] + \sum_{l=1}^{3} w_l p_l(x). \tag{5}$$

Here, the set $\{d_0, d_1, d_2, d_3\}$ is the associated linear weights, and the set $\{w_0, w_1, w_2, w_3\}$ is the associated non-linear weights. The WENO reconstruction (5) is a non-linear convex combination of the polynomials $p_r(x)$, $r = 0, 1, 2, 3$, thus their ideal weights can be any positive constants with only condition that their sum equals to one. The smoothness indicators β_r are calculated to measure the smoothness of $p_r(x)$ on cell I_{x_j} and are computed by applying various order derivatives. The smaller β_r, the smoother $p_r(x)$ is in different target cells. The smoothness indicators are computed as follows [1]:

$$\beta_r = \sum_k \Delta x^{2k-1} \int_{I_{x_j}} \left(\frac{d^k p_r(x)}{dx^k}\right)^2 dx, \; r = 0, 1, 2, 3. \tag{6}$$

In order to complete the reconstruction of (5), we calculate the non-linear weights based on the associated linear weights and the smoothness indicators to obtain the fifth-order accuracy for smooth areas and non-oscillatory performance near singularities and discontinuities. Accordingly, we consider

$$w_r = \frac{\alpha_r}{\sum_{k=0}^{3} \alpha_k}, \ \alpha_r = d_r(1 + \frac{\tau}{\Delta x^2 + \beta_r}), \ r = 0, 1, 2, 3, \tag{7}$$

where $\tau = |\beta_1 - \beta_3|$. As mentioned, the linear weights of any convex combination whose sum is equal to one can be considered. Accordingly, the choice: $d_1 = d_3 = \frac{1}{8}, d_2 = \frac{1}{4}, d_0 = \frac{1}{2}$ is supposed. Therefore, the final approximations at the boundaries of each cell are given by

$$\phi^-_{x,j} = R(x_j), \ \phi^+_{x,j-1} = R(x_{j-1}). \tag{8}$$

step 5. The semi-discrete scheme (2) is discretized in time by using the third-order total variation diminishing (TVD) Runge-Kutta scheme [4], which is obtained by

$$\begin{aligned} \phi_j^{(1)} &= \phi_j^n + \Delta t F(\phi_j^n), \\ \phi_j^{(2)} &= \tfrac{3}{4}\phi_j^n + \tfrac{1}{4}\phi_j^{(1)} + \tfrac{1}{4}\Delta t F(\phi_j^{(1)}), \\ \phi_j^{n+1} &= \tfrac{1}{3}\phi_j^n + \tfrac{2}{3}\phi_j^{(2)} + \tfrac{2}{3}\Delta t F(\phi_j^{(2)}). \end{aligned} \tag{9}$$

3. Numerical results

In this section, the numerical results obtained from MWENO-Z are compared with WENO-JP [3]. Therefore, Eq. (1) with convex hamiltonian $H(x, t, \phi, \phi_x) = \frac{1}{2}(\phi_x + 1)^2$, known as Burgers equation, with initial condition $\phi(x, 0) = -\cos(\pi x)$ is considered for $x \in [-1, 1]$. In this test case periodic boundary conditions are applied. In Fig. 1, numerical results are shown. The MWENO-Z use less CPU time to reach certain error levels than that for the WENO-JP. The MWENO-Z is more efficient and robust than WENO-JP in this one dimensional non-linear test case. Now, we solve Eq. (1) with $H(x, t, \phi, \phi_x) = \phi_x$, known as linear advection equation, with the initial condition $\phi(x, 0) = g(x - 0.5)$ for $x \in [-1, 1]$ with periodic boundary conditions. Here,

$$g(x) = -\left(\frac{\sqrt{3}}{2} + \frac{9}{2} + \frac{2\pi}{3}\right)(x+1) + \begin{cases} 2\cos(\frac{3\pi x^2}{2}) - \sqrt{3}, & x \in [-1, -\frac{1}{3}), \\ \frac{3}{2} + 3\cos(2\pi x), & x \in [-\frac{1}{3}, 0), \\ \frac{15}{2} - 3\cos(2\pi x), & x \in [0, \frac{1}{3}), \\ \frac{1}{3}(28 + 4\pi + \cos(3\pi x)) + 6\pi x(x - 1), & x \in [\frac{1}{3}, 1]. \end{cases} \tag{10}$$

The results that were computed at $T = 32$ with 100 grid points are presented in Fig. 2. Clearly, in this example the MWENO-Z gives better resolution at the singularities.

4. Conclusion

In this research, we propose a new modified WENO-Z scheme for solving the non-linear 1D Hamilton-Jacobi equation in finite difference framework. By comparing the new scheme with WENO-JP, it can be realized that the new proposed scheme is more efficient and robust by considering a normal CFL constant and without the need for additional process.

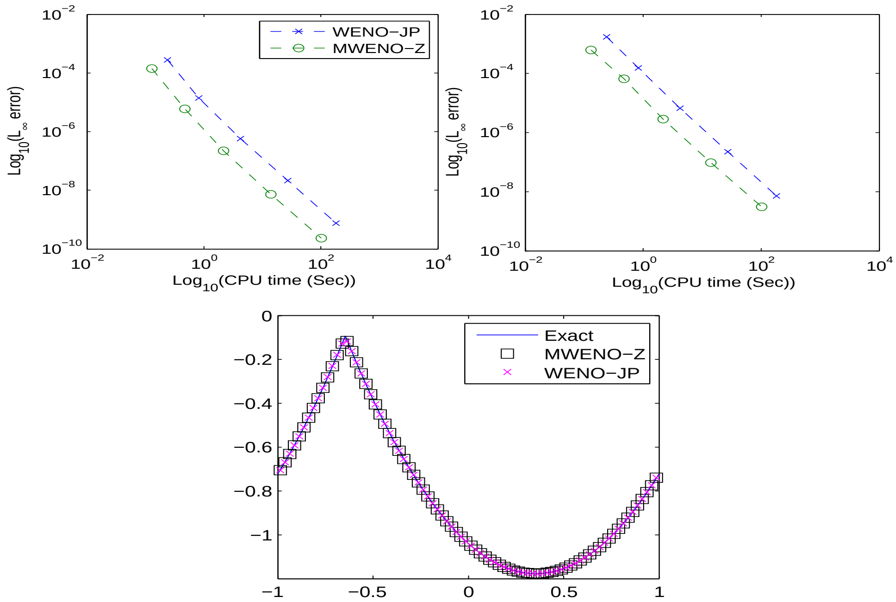

FIGURE 1. Top: Computing time and error. Bottom: 80 grid points and $t = 3.5/\pi^2$.

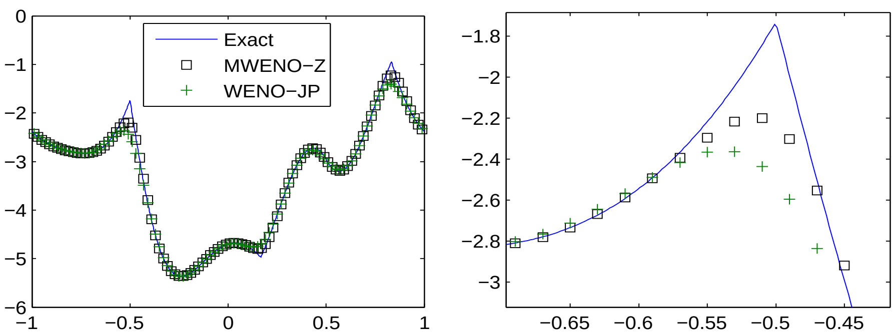

FIGURE 2. Left: linear advection at $T = 32$ with 100 grid points. Right: zoomed region of solution.

Acknowledgement

The author thanks all the organizers of this conference.

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L^B-valued Operators on L^B-valued GFA

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ABSTRACT. In this paper, at first, we introduce the concepts of L^B-valued general fuzzy automata and L^B-valued operators with t-norm and L^B-valued operators with t-conorm, where L stands for residuated lattice and $\mathbf{B}$ is a set of propositions about the GFA. Further, we study the relationships between the L^B-valued operators with t-norm and the L^B-valued operators with t-conorm.

Keywords: Operator, Norm, General fuzzy automata

AMS Mathematics Subject Classification [2010]: 03D05, 20M35

1. Introduction

It is well-known that the importance of the algebraic study is due to the fact that the algebraic properties play a vital role in the development of fundamentals of computer science [2]. In fuzzy set theory, a fuzzy implicator is a generalization of the classical one to fuzzy logic, much the same way as a t-norm and t-conorm are generalizations of the classical conjunction and disjunction, respectively; and in literature, there exist many families of fuzzy implicators (cf., [3, 4]).

2. Preliminaries

DEFINITION 2.1. [2] A lattice $(L, \leq, \wedge, \vee, 0, 1)$ is called a complete lattice with the greatest element 1 and the least element 0 if every subset (finite as well as infinite) has a supremum and infimum.Throughout this paper, we assume that L is a complete lattice $(L, \leq, \wedge, \vee, 0, 1)$ and for a nonempty set $X, \lambda : X \to L$ is an L-fuzzy set in X. Further, for a nonempty set X, L^X denotes the collection of all L-fuzzy sets in X. Furthermore, for $a \in L$, $\mathbf{a}$, $\mathbf{0}$, $\mathbf{1} : X \to L$ are the maps such that for all $x \in X$, $\mathbf{a}(x) = a, \mathbf{0}(x) = 0$, and $\mathbf{1}(x) = 1$. Now, let $f : X \to X'$ be a map. Then according to Zadeh's extension principle, f can be extended to the L-fuzzy operators $f^{\rightarrow} : L^X \to L^{X'}$ and $f^{\leftarrow} : L^{X'} \to L^X$ such that for all $\lambda \in L^X, \lambda' \in L^{X'}$, and $x' \in X'$,

$$f^{\rightarrow}(\lambda)(x') = \vee\{\lambda(x) : x \in X, f(x) = x'\} \text{ and } f^{\leftarrow}(\lambda') = \lambda' \circ f.$$

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Definition 2.2. [2] A triangular norm (t-norm, for short) $\mathcal{T}$ is a map $\mathcal{T} : L \times L \to L$ such that the following conditions hold:
(i) $\mathcal{T}(1, a) = a, \forall a \in L$;
(ii) $\mathcal{T}(a, b) = \mathcal{T}(b, a), \forall a, b \in L$;
(iii) $\mathcal{T}(a, \mathcal{T}(b, c)) = \mathcal{T}(\mathcal{T}(a, b), c), \forall a, b, c \in L$;
(iv) if $a \le c$ and $b \le d$, then $\mathcal{T}(a, b) \le \mathcal{T}(c, d)$.
In addition, $\mathcal{T}$ is called a left (respectively, right) continuous if for all $a, b \in L$ and $\{a_j : j \in J\}, \{b_j : j \in J\} \subseteq L, \mathcal{T}(\vee\{a_j : j \in J\}, b) = \vee\{\mathcal{T}(a_j, b) : j \in J\}$ (respectively, $\mathcal{T}(a, \wedge\{b_j : j \in J\}) = \wedge\{\mathcal{T}(a, b_j) : j \in J\}$).
For example, let $L = [0, 1]$. Then for all $a, b \in L$,
(i) Gödel t-norm $\mathcal{T}_G(a, b) = \min\{a, b\}$; and
(ii) Łukasiewicz t-norm $\mathcal{T}_L(a, b) = \max\{0, a + b - 1\}$.

Definition 2.3. [2] A triangular conorm (t-conorm, for short) $\mathcal{S}$ is a map $\mathcal{S} : L \times L \to L$ such that the following conditions hold:
(i) $\mathcal{S}(0, a) = a, \forall a \in L$;
(ii) $\mathcal{S}(a, b) = \mathcal{S}(b, a), \forall a, b \in L$;
(iii) $\mathcal{S}(a, \mathcal{S}(b, c)) = \mathcal{S}(\mathcal{S}(a, b), c), \forall a, b, c \in L$; and
(iv) if $a \le c$ and $b \le d$, then $\mathcal{S}(a, b) \le \mathcal{S}(c, d)$. In addition, $\mathcal{S}$ is called a left (respectively, right) continuous if for all $a, b \in L$ and $\{a_j : j \in J\}, \{b_j : j \in J\} \subseteq L, \mathcal{S}(\wedge\{a_j : j \in J\}, b) = \wedge\{S(a_j, b) : j \in J\}$ (respectively, $\mathcal{S}(a, \vee\{b_j : j \in J\}) = \vee\{\mathcal{S}(a, b_j) : j \in J\}$.
For example, let $L = [0, 1]$. Then for all $a, b \in L$,
(i) Gödel t-conorm $\mathcal{S}_G(a, b) = \max\{a, b\}$; and
(ii) Łukasiewicz t-conorm $\mathcal{S}_L(a, b) = \min\{1, a + b\}$.

Definition 2.4. [2] A negator $\mathcal{N}$ is a decreasing map $\mathcal{N} : L \to L$ such that $\mathcal{N}(0) = 1$ and $\mathcal{N}(1) = 0$. If $\mathcal{N}(\mathcal{N}(a)) = a, \forall a \in L$, then $\mathcal{N}$ is called a strong negator.
For a given negator $\mathcal{N}$, t-norm $\mathcal{T}$ and t-conorm $\mathcal{S}$ are called dual with respect to $\mathcal{N}$ if

$$\mathcal{S}(\mathcal{N}(a), \mathcal{N}(b)) = \mathcal{N}(\mathcal{T}(a, b)) \text{ and } \mathcal{T}(\mathcal{N}(a), \mathcal{N}(b)) = \mathcal{N}(\mathcal{S}(a, b)), \forall a, b \in L.$$

For example, let $L = [0, 1]$. Then for all $a \in L$,
(i) standard negator $\mathcal{N}_{\mathcal{S}}(a) = 1 - a$, which is strong; and
(ii) Gödel negators

$$\mathcal{N}_{G_1}(a) = \begin{cases} 1 & if\ a = 0 \\ 0 & otherwise; \end{cases} \quad \mathcal{N}_{G_2}(a) = \begin{cases} 0 & if\ a = 1 \\ 1 & otherwise, \end{cases}$$

which are non-strong least and greatest negators, respectively.

Definition 2.5. [1] A general fuzzy automaton (GFA) is considered as:

$$\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2),$$

where (i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$, (ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \ldots, a_m\}$, (iii) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$, (iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$, (v) $\omega : Q \to Z$ is the output function, (vi) $\tilde{\delta} : (Q \times [0, 1]) \times \Sigma \times Q \to [0, 1]$ is the augmented transition function. (vii) Function $F_1 : [0, 1] \times [0, 1] \to [0, 1]$ is called membership assignment function. Function $F_1(\mu, \delta)$, as is seen, is motivated by two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition.

3. Algebraic characterizations of L^B-valued general fuzzy automata via L^B-valued operators

Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, w, F_1, F_2)$ be a general fuzzy automaton. If we fix an input $a_k \in \Sigma$ at time t_i the proposition $\alpha|_{a_k}$ can be computed by $\mu^{t_i}(q_i)$ if the general fuzzy automaton $\tilde{F}$ is in the state q_i at time t_i otherwise $\alpha|_{a_k}$ is 0 if $\tilde{F}$ is not in the active state q_i. Accordingly, for each state $q_i \in Q$ we can assess the truth value of $\alpha|_{a_k}$, it is indicated by $\alpha|_{a_k}(q_i)$. As explained above $\alpha|_{a_k}(q_i) \in [0,1]$. In this section, we derive the logic B which is a set of propositions about the general fuzzy automaton $\tilde{F}$ formulated by the observer and constructing a complete infinitely distributive lattice $\mathbf{B} = (B, \leq, \wedge, \vee, 0, 1)$. We can establish the order $\leq$ on B as follows:
For $\alpha, \beta \in B, \alpha \leq \beta$ if and only if $\alpha(q_i) \leq \beta(q_i)$ for all $q_i \in Q$. One can instantly check that the contradiction, i.e., the proposition with constant truth value 0, is the least element and the tautology, i.e., the proposition with constant truth value 1, is the greatest component of the $\mathbf{B}$. Note that any component ith of 1 is the maximum membership values of active states at time t_i, for any $i \geq 0$.
We define L^B-valued subset of $Q \times \Sigma \times Q$, i.e., a map $\delta : Q \times \Sigma \times Q \to L^B$. The range set L^B allows to interpret L^B as a map assigning each (q, a_k, p) to $\delta(q, a_k, p) : B \to L$. This interpretation of transition map δ allows to represent it as the family $\{\delta^\alpha : \alpha \in B\}$ of L-valued sets $\delta^\alpha \in L^{Q\times\Sigma\times Q}$ of $Q \times \Sigma \times Q$ ordered by the elements of B, where the L-valued sets δ^α are defined by $\delta^\alpha(q, a_k, p) = \delta(q, a_k, p)(\alpha) = \begin{cases} 1 & \text{if } q = p \\ \alpha(q) \vee \alpha(p) & otherwise. \end{cases}$

Definition 3.1. An L^B-valued general fuzzy automaton is a 8-tuple $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$, where
(i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$,
(ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \ldots, a_m\}$,
(iii) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$,
(iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$,
(v) $\omega : Q \to Z$ is the output function,
(vi) $\tilde{\delta} : (Q \times L) \times \Sigma \times Q \to L^B$ is the L^Bvalued augmented transition function defined by

$$\tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j)(\alpha) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)(\alpha)).$$

(vii) Function $F_1 : [0,1] \times [0,1] \to [0,1]$ is called membership assignment function.
(viii) Function $F_2 : [0,1]^* \times [0,1] \to [0,1]$ is called multi-membership resolution function.

Definition 3.2. Let $\tilde{F}$ be an L^B-valued general fuzzy automaton and $\alpha \in B$. Then the L^B-valued operators with t-norm are maps $\mathcal{T}R, \mathcal{T}R^{-1} : B \to (L^B)^Q$ such that $\forall q \in Q_{act}(t_i)$ and $\forall \lambda \in B$,

$$\mathcal{T}R(\lambda)(q)(\alpha) = \vee\{\mathcal{T}(\lambda(p), \tilde{\delta}^*((p, \mu^{t_j}(p)), u, q)(\alpha)) | p \in Q_{pred}(q, u), u \in \Sigma^*\}; \text{ and}$$

$$\mathcal{T}R^{-1}(\lambda)(q)(\alpha) = \vee\{\mathcal{T}(\lambda(p), \tilde{\delta}^*((q, \mu^{t_i}(q)), u, p)(\alpha)) | p \in Q_{succ}(q, u), u \in \Sigma^*\}.$$

Proposition 3.3. *Let $\tilde{F}$ be an L^B-valued GFA and $\mathcal{T}R, \mathcal{T}R^{-1} : B \to (L^B)^Q$ be the L^B-valued operators with t-norm. Then for all $a \in L, \lambda, \mu \in B$ and $\{\lambda_j | j \in J\} \subseteq B$,*
(i) $\mathcal{T}R(\boldsymbol{a}) = \boldsymbol{a}$ and $\mathcal{T}R^{-1}(\boldsymbol{a}) = \boldsymbol{a}$, if $\mathcal{T}$ is a left continuous;
(ii) $\lambda \leq \mathcal{T}R(\lambda)$ and $\lambda \leq \mathcal{T}R^{-1}(\lambda)$;
(iii) $\mathcal{T}R(\vee\{\lambda_j | j \in J\}) = \vee\{\mathcal{T}R(\lambda_j) | j \in J\}$ and $\mathcal{T}R^{-1}(\vee\{\lambda_j | j \in J\}) = \vee\{\mathcal{T}R^{-1}(\lambda_j) | j \in J\}$, if $\mathcal{T}$ is a left continuous;

(iv) $\mathcal{T}R(\mathcal{T}R(\lambda)) = \mathcal{T}R(\lambda)$ *and* $\mathcal{T}R^{-1}(\mathcal{T}R^{-1}(\lambda)) = \mathcal{T}R^{-1}(\lambda)$, *if* $\mathcal{T}$ *is a left continuous;*
(v) if $\lambda \leq \mu$, *then* $\mathcal{T}R(\lambda) \leq \mathcal{T}R(\mu)$ *and* $\mathcal{T}R^{-1}(\lambda) \leq \mathcal{T}R^{-1}(\mu)$; *and*
(vi) $\mathcal{T}R(\wedge\{\lambda_j|j \in J\}) \leq \wedge\{\mathcal{T}R(\lambda_j)|j \in J\}$ *and* $\mathcal{T}R^{-1}(\wedge\{\lambda_j|j \in J\}) = \wedge\{\mathcal{T}R^{-1}(\lambda_j)|j \in J\}$,

Definition 3.4. Let $\tilde{F}$ be an L^B-valued general fuzzy automaton. Then the L^B-valued operators with t-conorm are maps $\mathcal{T}C, \mathcal{T}C^{-1} : B \to (L^B)^Q$ such that $\forall\lambda, \alpha \in B$ and $\forall q \in Q_{act}(t_i)$,

$$\mathcal{T}C(\lambda)(q)(\alpha) = \wedge\big\{\mathcal{S}(\lambda(p), \mathcal{N}(\tilde{\delta}^*((p, \mu^{t_j}(p)), u, q)(\alpha)))|p \in Q_{pred}(q, u), u \in \Sigma^*\big\};$$

and

$$\mathcal{T}C^{-1}(\lambda)(q)(\alpha) = \wedge\big\{\mathcal{S}(\lambda(p), \mathcal{N}(\tilde{\delta}^*((q, \mu^{t_i}(q)), u, p)(\alpha)))|p \in Q_{succ}(q, u), u \in \Sigma^*\big\}.$$

Proposition 3.5. *Let* $\tilde{F}$ *be an* L^B*-valued general fuzzy automaton and* $\mathcal{T}C, \mathcal{T}C^{-1} : B \to (L^B)^Q$ *be the* L^B*-valued operators with t-conorm. Then for all* $a \in L, \lambda, \mu \in B$ *and* $\{\lambda_j|j \in J\} \subseteq B$,
(i) $\mathcal{T}C(\boldsymbol{a}) = \boldsymbol{a}$ *and* $\mathcal{T}C^{-1}(\boldsymbol{a}) = \boldsymbol{a}$, *if* $\mathcal{S}$ *is a left continuous and* $\mathcal{N}$ *is strong;*
(ii) $\mathcal{T}C(\lambda) \leq \lambda$ *and* $\mathcal{T}C^{-1}(\lambda) \leq \lambda$;
(iii) $\mathcal{T}C(\wedge\{\lambda_j|j \in J\}) = \wedge\{\mathcal{T}C(\lambda_j)|j \in J\}$ *and* $\mathcal{T}C^{-1}(\wedge\{\lambda_j|j \in J\}) = \wedge\{\mathcal{T}C^{-1}(\lambda_j)|j \in J\}$, *if* $\mathcal{S}$ *is a left continuous;*
(iv) $\mathcal{T}C(\mathcal{T}C(\lambda)) = \mathcal{T}C(\lambda)$ *and* $\mathcal{T}C^{-1}(\mathcal{T}C^{-1}(\lambda)) = \mathcal{T}C^{-1}(\lambda)$, *if* $\mathcal{S}$ *is a left continuous,* $\mathcal{T}$ *and* $\mathcal{S}$ *are dual with respect to* $\mathcal{N}$;
(v) if $\lambda \leq \mu$, *then* $\mathcal{T}C(\lambda) \leq \mathcal{T}C(\mu)$ *and* $\mathcal{T}C^{-1}(\lambda) \leq \mathcal{T}C^{-1}(\mu)$; *and*
(vi) $\mathcal{T}C(\vee\{\lambda_j|j \in J\}) \leq \vee\{\mathcal{T}C(\lambda_j)|j \in J\}$ *and* $\mathcal{T}C^{-1}(\vee\{\lambda_j|j \in J\}) = \vee\{\mathcal{T}C^{-1}(\lambda_j)|j \in J\}$.

Proposition 3.6. *Let* $\tilde{F}$ *be an* L^B*-valued general fuzzy automaton,* $\alpha \in B, \mathcal{T}$ *and* $\mathcal{S}$ *be dual with respect to a strong negation* $\mathcal{N}$. *Then for all* $\lambda \in B$,
(i) $\mathcal{N}(\mathcal{T}R(\lambda)) = \mathcal{T}C(\mathcal{N}(\lambda))$, *i.e.,* $\mathcal{T}R(\lambda) = \mathcal{N}(\mathcal{T}C(\mathcal{N}(\lambda)))$; *and*
(ii) $\mathcal{N}(\mathcal{T}R^{-1}(\lambda)) = \mathcal{T}C^{-1}(\mathcal{N}(\lambda))$, *i.e.,* $\mathcal{T}R^{-1}(\lambda) = \mathcal{N}(\mathcal{T}C^{-1}(\mathcal{N}(\lambda)))$.

4. Conclusion

This study is an attempt to scrutinize the theory of L^B-valued general fuzzy automata with the help of some L^B-valued operators based on t-norm/t-conorm. These L^B-valued operators lead us to characterize some algebraic concepts associated with an L^B-valued general fuzzy automaton.

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۸ تا ۱۱ شهریور ۱۴۰۰

On the normality of matrix polynomials

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Abstract. A matrix polynomial $P(\lambda)$ is called weakly normal if for every $\mu \in \mathbb{C}$, the matrix $P(\mu)$ is normal. It is said to be normal if all the eigenvalues of $P(\lambda)$ are semisimple. In this note, by using invertibility of the Vandermonde matrix, it is proved that $P(\lambda)$ is weakly normal if and only if all its coefficients are normal and mutually commuting. The relation between normal and weakly normal matrix polynomials are studied and some results about the polynomial numerical hulls of the companion linearization of the matrix polynomial $P(\lambda) = \lambda^m I - A$ are given.

Keywords: matrix polynomial, companion linearization, polynomial numerical hulls

AMS Mathematics Subject Classification [2010]: 15A18, 15A60, 15A22

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1. Introduction

The normality of matrices and operators arises in many problems in pure and applied linear algebra, and other branches of mathematics. Matrix polynomials have also some important applications; for example, their spectral analysis is useful in the study of differential equations, their numerical ranges are important in overdamped vibration systems with finite number of degrees of freedom. For more information, see [2].

To set some notation, $\mathbb{C}$ denotes the set of complex numbers, and $M_n(\mathbb{C})$ is algebra of all $n \times n$ complex matrices. We consider the matrix polynomial

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0, \tag{1}$$

where $A_j \in M_n(\mathbb{C})$, $A_m \neq 0$, and $\lambda \in \mathbb{C}$. The number m is the degree and n is the order of $P(\lambda)$, respectively. If $A_m = I_n$ the identity matrix, then $P(\lambda)$ is called monic. A scalar $\lambda_0 \in \mathbb{C}$ is called an eigenvalue of of $P(\lambda)$ if there exists a nonzero vector $x_0 \in \mathbb{C}^n$ such that $P(\lambda_0)x_0 = 0$. The vector x_0 is called an eigenvector of $P(\lambda)$ corresponding to the eigenvalue λ_0. The set of all eigenvalues of $P(\lambda)$ is the spectrum of $P(\lambda)$, i.e., $\sigma[P(\lambda)] = \{\mu \in \mathbb{C} : detP(\mu) = 0\}$. A multiple eigenvalue of $P(\lambda)$ is called semisimple if its algebraic multiplicity is equal to its geometric multiplicity. A matrix polynomial $P(\lambda)$ as in (1), is called weakly normal if for every $\mu \in \mathbb{C}$, the matrix $P(\mu)$ is normal. If all the eigenvalues of $P(\lambda)$ are semisimple, then $P(\lambda)$ is called a normal matrix polynomial.

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In Section 2 of this paper, using the invertibility of the Vandermonde matrix, we will prove that $P(\lambda)$ is weakly normal if and only if A_j's are normal and $A_iA_j = A_jA_i$ for all i, j. Moreover, we study the relation between normal and weakly normal matrix polynomials. In Section 3, we study the polynomial numerical hulls of the monic matrix polynomial $P(\lambda) = \lambda^m I - A$, where $A \in M_n(\mathbb{C})$. In this case, obviously, $P(\lambda)$ is weakly normal if and only if A is normal.

2. Weakly normal and normal matrix polynomials

In [3, Corollary 3.6], a characterization for weakly normal matrix polynomials with nonsingular leading coefficients is given. In the following theorem, we give a new proof according to the invertibility of the Vandermonde matrix for arbitrary matrix polynomials. We denote the Vandermond matrix by:

$$V(x_1, x_2, \dots, x_n) := \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^n & x_2^n & \cdots & x_n^n \end{bmatrix}.$$

It is known that $det(V(x_1, x_2, \dots, x_n) = (\Pi_{1\leq j\leq n} x_j)(\Pi_{1\leq i<j\leq n}(x_j - x_i))$.

Theorem 2.1. *Let $P(\lambda)$ be a matrix polynomial as in (1). Then, $P(\lambda)$ is weakly normal if and only if A_j's are normal and $A_iA_j = A_jA_i$.*

Proof. At first, we assume that $P(\lambda)$ is a weakly normal matrix polynomial. Then, for every $\lambda \in \mathbb{C}$, $P(\lambda)P(\lambda)^* - P(\lambda)^*P(\lambda) = 0$. So, we have $\sum_{i,j=0}^{m} \lambda^i \bar{\lambda}^j (A_iA_j^* - A_j^*A_i) = 0$ for all $\lambda \in \mathbb{C}$. Therefore,

$$V(\underbrace{1, \lambda, \dots, \lambda^m}, \underbrace{\bar{\lambda}, \bar{\lambda}\lambda, \dots, \bar{\lambda}\lambda^m}, \dots, \underbrace{\bar{\lambda}^m, \bar{\lambda}^m\lambda, \dots, \bar{\lambda}^m\lambda^m}) \begin{bmatrix} A_0A_0^* - A_0^*A_0 \\ A_1A_0^* - A_0^*A_1 \\ \vdots \\ A_mA_0^* - A_0^*A_m \\ A_0A_1^* - A_1^*A_0 \\ \vdots \\ A_mA_1^* - A_1^*A_m \\ \vdots \\ A_0A_m^* - A_m^*A_0 \\ \vdots \\ A_mA_m^* - A_m^*A_m \end{bmatrix} = 0.$$

In the above matrix multiplication, we choose the left matrix as a nonsingular Vandrmonde matrix, and so, we conclude that $A_iA_j^* = A_j^*A_i$ for all $i, j \in \{0, 1, \dots, m\}$. This shows that $A_iA_i^* = A_i^*A_i$ for all $i \in \{0, 1, \dots, m\}$, and so, A_i's are normal and mutually commuting.

The converse is trivial, and so, the proof is complete. □

By Theorem 2.1, $P(\lambda)$, as in (1), is weakly normal if and only if there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that $U^*P(\lambda)U$ is a diagonal matrix whose every entry of the main diagonal is a scalar polynomial of degree at most m. So, if each of this scalar polynomials has exactly m distinct zeros, or equivalently, all the eigenvalues of $P(\lambda)$ are semisimple, then $P(\lambda)$ is a normal matrix polynomial. Therefore, every normal matrix polynomial is a weakly normal matrix polynomial; but the converse is not true in general.

3. On polynomial numerical hulls of $P(\lambda) = \lambda^m I - A$

Let $P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ be a matrix polynomial as in (1). For a given positive integer k, the polynomial numerical hull of order k of $P(\lambda)$ is defined and denoted, see [1], by

$$V^k[P(\lambda)] = \{\mu \in \mathbb{C} : |q(0)| \leq \|q(P(\mu))\| \text{ for all } q \in \mathbb{P}_k\},$$

where, $\mathbb{P}_k$ is the set of all scalar polynomials of degree k or less, and $\|.\|$ is the spectral matrix norm. For the case $P(\lambda) = \lambda I - A$, where $A \in M_n(\mathbb{C})$, we see $V^k[P(\lambda)] = \{\mu \in \mathbb{C} : |q(\mu)| \leq \|q(A)\| \text{ for all } q \in \mathbb{P}_k\} =: V^k(A)$, which is the polynomial numerical hull of order k of the matrix A.

For the monic matrix polynomial $P(\lambda) = I_n\lambda^m + A_{m-1}\lambda^{m-1} + \ldots + A_1\lambda + A_0$, the companion linearization of $P(\lambda)$ is the following matrix:

$$\mathcal{C} = \begin{bmatrix} 0 & I_n & 0 & \ldots & 0 & 0 \\ 0 & 0 & I_n & \ldots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots & 0 & I_n \\ -A_0 & -A_1 & -A_2 & \ldots & -A_{m-2} & -A_{m-1} \end{bmatrix} \in M_{mn}(\mathbb{C}). \tag{2}$$

By [1, Theorem 3.3], $\mathcal{C}$, as in (2), is normal if and only if $A_1 = A_2 = \ldots = A_{m-1} = 0$, and A_0 is unitary. In this case, $\mathcal{C} = \pi_{-A_0}$, where

$$\pi_A = \begin{bmatrix} 0 & I_n & 0 & \ldots & 0 & 0 \\ 0 & 0 & I_n & \ldots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots & 0 & I_n \\ A & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}. \tag{3}$$

Note that π_A is the companion linearization of the matrix polynomial $P(\lambda) = \lambda^m I - A$, where $A \in M_n(\mathbb{C})$. In this case, obviously, $P(\lambda)$ is normal if and only if weakly normal, if and only if, A is a normal matrix. In the following theorem, we show that $V^k(\pi_A)$ has circular symmetric property. For some other properties, see [1, Section 3].

THEOREM 3.1. *Let $A \in M_n(\mathbb{C})$, and π_A be as in (3). Then, for every $\theta \in \mathbb{R}$, $e^{i\theta}V^k(\pi_A) = V^k(\pi_A)$.*

PROOF. Let $\mu \in V^k(\pi_A)$. Then, there exist $0 \leq t_1, \ldots, t_d \leq 1$ with $\sum_{j=1}^d t_j = 1$, and there are $x_1, \ldots, x_d \in \mathbb{C}^{nm}$ with $x_j^* x_j = 1$ such that $\mu^l = \sum_{j=1}^d t_j x_j^* \pi_A^l x_j$ for $l = 1, 2, \ldots, k$. Now, by setting $U_\theta = diag(e^{im\theta}, e^{i(m-1)\theta}, \ldots, e^{i\theta})$ and $y_j = (U_\theta \otimes I_n)x_j$, we see that

$$\sum_{j=1}^d t_j y_j^* \pi_A^l y_j = e^{-il\theta} \sum_{j=1}^d t_j x_j^* \pi_A^l x_j = e^{-il\theta}\mu^l = (e^{-i\theta})^l,$$

for $l = 1, 2, \ldots, k$. This shows that $e^{-i\theta}\mu \in V^k(\pi_A)$, and so, $V^k(\pi_A) \subseteq e^{i\theta}V^k(\pi_A)$.

By changing θ by $-\theta$, we conclude that $e^{i\theta}V^k(\pi_A) \subseteq V^k(\pi_A)$, and so, the result holds. □

Finally, we state the following result.

THEOREM 3.2. *Let $P(\lambda) = \lambda^m I - A$, where $A \in M_n(\mathbb{C})$, be a normal matrix polynomial. If A is unitary, then $V^2(\pi_A) = \sigma(\pi_A)$ if and only if $m = 2$ and A is a scalar unitary matrix.*

4. Conclusion

The normal or weakly normal matrix polynomials have been studied in [3] and its references. In the present paper, we gave a new proof to study the weak normality of arbitrary matrix polynomials according to the Vandermonde matrix. We also gave some results on the polynomial numerical hulls of the block companion linearization of the (normal) matrix polynomial $P(\lambda) = \lambda^m I_n - A$, where $A \in M_n(\mathbb{C})$.

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Iteration operator frames and their relation to their dual

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Abstract. The purpose of the paper is to analyze frames $\{f_k\}_{k\in\mathbb{Z}}$ having the form $\{T^k f_0\}_{k\in\mathbb{Z}}$ for some bounded linear operator T. We characterize all dual frames which are representable in terms of iterations of an operator. Moreover, we show that under some condition a Parseval iteration operator frame has a unique iteration operator dual frame by the same operator.

Keywords: iteration operator frames, iteration operator dual frames, Parseval iteration operator frames

AMS Mathematics Subject Classification [2010]: 42C15, 42C40

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1. Introduction

In this paper we consider frames $\{f_k\}_{k\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ arising via iterated action of a linear operator T. We say that the form $\{f_k\}_{k\in\mathbb{Z}}$ is represented via the operators T_j ($j = 1,\dots J$) that is given by the action of a class of bounded linear operators on a single element in the underlying Hilbert space. In particular, it could be given by iterated action of a fixed operator on a single element, i.e., as the form

$$\{f_k\}_{k\in\mathbb{Z}} = \{T^k f_0\}_{k\in\mathbb{Z}}. \tag{1}$$

Systems of vectors on this form play an important role in mathematical physics, operator theory and modern applied harmonic analysis [4]. Also this appears in the more recent context of dynamical sampling [1,2]. The Fourier orthonormal basis, single generator shift invariant systems and Gabor systems have the form of (1).

In the rest, we will collect some definitions and standard results from frame theory. A sequence $\{f_k\}_{k\in\mathbb{Z}}$ in Hilbert space $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \le \sum |\langle f, f_k\rangle|^2 \le B\|f\|^2, \qquad (f \in \mathcal{H}).$$

If just the right inequality in the above holds, then $\{f_k\}_{k\in\mathbb{Z}}$ is called a *Bessel* sequence. A sequence $F = \{f_k\}_{k\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ is called a *Riesz sequence* if there are constants $A, B > 0$ so that for all finite scalars c_k we have

$$A\sum |c_k|^2 \le \|\sum c_k f_k\|^2 \le B\sum |c_k|^2.$$

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In addition, if F is complete in $\mathcal{H}$, then it is a *Riesz basis* for $\mathcal{H}$. Furthermore, the class of Riesz bases is precisely the class of frame $\{f_k\}_{k=1}^{\infty}$ for which the equation $\sum_{k=1}^{\infty} c_k f_k = 0$, $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$, forces that $c_k = 0$ for all $k \in \mathbb{N}$. Frequently the latter condition is expressed by saying that $\{f_k\}_{k=1}^{\infty}$ is ω-independent. This is a much stronger condition that $\{f_k\}_{k=1}^{\infty}$ being linearly independent, which means that if a finite linear combination of vectors from $\{f_k\}_{k=1}^{\infty}$ is zero, all the coefficients must be zero. A frame which is not a Riesz basis is said to be *redundant* or *overcomplete*.

If $F = \{f_k\}_{k=1}^{\infty}$ is a Bessel sequence, its *synthesis operator* $T_F : \ell^2(\mathbb{N}) \to \mathcal{H}$ is defined by

$$T_F\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k.$$

It is well known that T_F is well-defined and bounded. A central role will be played by the kernel of the operator T_F, i.e., the subset of $\ell^2(\mathbb{Z})$ given by

$$N_T = \left\{ \{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{Z}) \quad ; \quad \sum_{k=1}^{\infty} c_k f_k = 0 \right\}.$$

The *excess* of a frame is the number of elements that can be removed yet leaving a frame. It is well known that the excess equals $dim(N_T)$; see [3]. The adjoint of T_F, $T_F^* : \mathcal{H} \to \ell^2(\mathbb{N})$, which is called the *analysis operator*, is given by $T_F^* f = \{\langle f, f_k\rangle\}_{k \in k \in \mathbb{Z}}$. Moreover, $S_F : \mathcal{H} \to \mathcal{H}$ the *frame operator* of F, is given by

$$S_F f = T_F T_F^* f = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k.$$

If F is a frame, then S_F is invertible and $A_F \leq S_F \leq B_F$. The sequence $\tilde{F} = \{S_F^{-1} f_k\}_{k\in\mathbb{Z}}$, which is also a frame, is called the *canonical dual frame*. A frame $\{g_k\}_{k=1}^{\infty}$ is called a *dual* of $\{f_k\}_{k=1}^{\infty}$ if

$$f = \sum_{k=1}^{\infty} \langle f, g_k\rangle f_k, \qquad (f \in \mathcal{H}).$$

Also, if $F = \{f_k\}_{k=1}^{\infty}$ is a frame, then every dual frame of F is of the form of $F^d = \{S_F^{-1} f_k + u_k\}_{k=1}^{\infty}$ where $\{u_k\}_{k=1}^{\infty}$ is a Bessel sequence such that

$$\sum_{k=1}^{\infty} \langle f, f_k\rangle u_k = 0, \qquad (f \in \mathcal{H}). \tag{2}$$

In the rest, we bring a few results which are used in the following [5]. The next theorem shows that any frame which is norm-bounded below is a linear union of iterated operator actions on certain elements.

THEOREM 1.1. *[5] Consider a frame $\{f_k\}_{k=1}^{\infty}$ which is norm-bounded below. Then the following hold:*

(1) *The frame $\{f_k\}_{k=1}^{\infty}$ can be decomposed as a finite union*

$$\{f_k\}_{k=1}^{\infty} = \bigcup_{j=1}^{J} \{f_k^{(j)}\}_{k\in I_j},$$

where each of the sequences $\{f_k^{(j)}\}_{k\in I_j}$ is an infinite Riesz sequences.

(2) *There is a finite collection of vectors from* $\{f_k\}_{k=1}^{\infty}$*, to be called* $\varphi_1, \ldots, \varphi_j$ *and the corresponding bounded operators* $T_j : \mathcal{H} \to \mathcal{H}$ *which closed range, such that*

$$\{f_k\}_{k=1}^{\infty} = \bigcup_{j=1}^{J} \{T_j^n \varphi_j\}_{n=0}^{\infty}.$$

In the next proposition it is introduced the necessary and sufficient condition for writing a frame as the iteration operator form and implies that it is independent of the ordering of the elements in $\{f_k\}_{k=1}^{\infty}$.

Proposition 1.2. *[5] Consider a frame* $\{f_k\}_{k=1}^{\infty}$ *which span*$\{f_k\}_{k=1}^{\infty}$ *is an infinite-dimensional subspace. Then the following are equivalent:*

(1) *The frame* $\{f_k\}_{k=1}^{\infty}$ *is linearly independent.*
(2) *There exists a linear operator* $T : \text{span}\{f_k\}_{k=1}^{\infty} \to \mathcal{H}$ *such that* $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$.

Proposition 1.3. *[5] Any Riesz sequence* $\{f_k\}_{k=1}^{\infty}$ *in* $\mathcal{H}$ *has the form* $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$ *for some operator* $T \in B(\mathcal{H})$ *with close range.*

The next proposition shows that if a frame $\{f_k\}_{k=1}^{\infty}$ with finite excess has a representation $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$, the operator T is forced to be unbounded.

Proposition 1.4. *[5] Assume that the frame* $\{f_k\}_{k=1}^{\infty}$ *is linearly independent and has finite excess. If* T *is a linear operator and* $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$*, then* T *is unbounded.*

Corollary 1.5. *[5] Assume that the frame* $\{f_k\}_{k=1}^{\infty}$ *is linearly independent, contains a Riesz basis and has infinite excess. Then,* $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$*, the operator* T *is unbounded.*

Note that for a frame containing a Riesz basis and having positive excess, a representation $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$ is not possible with a bounded operator T.

2. Main results

In this section we show that dual frames of an iteration operator frame have the same structure. In [6], it is supposed that a frame $\{f_k\}_{k\in\mathbb{Z}} = \{T^k f_0\}_{k\in\mathbb{Z}}$, where $T \in B(\mathcal{H})$ is invertible. Then the canonical dual $\{S^{-1} f_k\}_{k\in\mathbb{Z}}$ has the form of iteration operator frame as $\{S^{-1} f_k\}_{k\in\mathbb{Z}} = \{(T^*)^k S^{-1} f_0\}_{k\in\mathbb{Z}}$. In the next theorem we remove the invertibility of T and represent the canonical dual as iteration operator frame.

Lemma 2.1. *Let* $T_1 \in B(\mathcal{H})$ *and* $\{f_k\}_{k\in\mathbb{Z}} = \{T^k f_0\}_{k\in\mathbb{Z}}$ *be an iteration operator frame. Then* $\{g_k\}_{k\in\mathbb{Z}}$ *is the canonical dual of* $\{f_k\}_{k\in\mathbb{Z}}$ *if and only if* $\{g_k\}_{k\in\mathbb{Z}} = \{T_2^k g_0\}_{k\in\mathbb{Z}}$ *where* $T_2 = S_F^{-1} T S_F$ *and* $g_0 = S_F^{-1} f_0$.

In the next theorem we show that under some condition alternate dual of an iteration operator frame is also of this form.

Theorem 2.2. *Let* $T \in B(\mathcal{H})$ *and* $\{f_k\}_{k\in\mathbb{Z}} = \{T^k f_0\}_{k\in\mathbb{Z}}$ *be an iteration operator frame, also let* $\{g_k\}_{k\in\mathbb{Z}} = \{S_F^{-1} f_k + v_k\}_{k\in\mathbb{Z}}$ *be a dual of* $\{f_k\}_{k\in\mathbb{Z}}$ *where* $\{v_k\}_{k\in\mathbb{Z}}$ *is a Bessel sequence satisfies* (2) *and* $T S_F v_k = S_F v_{k+1}$*. Then* $\{g_k\}_{k\in\mathbb{Z}}$ *is also an iteration operator frame.*

In the next theorem we show the converse holds by some additional condition.

Theorem 2.3. *Let* $T, W \in B(\mathcal{H})$*,* T *be an injective operator and* $\{f_k\}_{k\in\mathbb{Z}} = \{T^k f_0\}_{k\in\mathbb{Z}}$ *be an iteration operator frame. Also, let* $\{g_k\}_{k\in\mathbb{Z}} = \{S_F^{-1} f_k + v_k\}_{k\in\mathbb{Z}} = \{W^k g_0\}$ *be an iteration operator dual of* $\{f_k\}_{k\in\mathbb{Z}}$ *where* $\{v_k\}_{k\in\mathbb{Z}}$ *is a Bessel sequence satisfies* (2) *. Then* $TS_F v_k = S_F v_{k+1}$*.*

In the last theorem we show that under some condition a Parseval iteration operator frame has a unique iteration operator dual frame by the same iteration operator.

Theorem 2.4. *Let* $T \in B(\mathcal{H})$ *be a self-adjoint operator,* $\{f_k\}_{k\in\mathbb{Z}} = \{T^k f_0\}_{k\in\mathbb{Z}}$ *a Parseval iteration operator frame and* $\{g_k\}_{k\in\mathbb{Z}} = \{T^k g_0\}_{k\in\mathbb{Z}}$ *be a dual of* $\{f_k\}_{k\in\mathbb{Z}}$ *such that* $TS_G = S_G T$ *and* $\langle g_0, T^k f_0\rangle$ *is real value for all* $k \in \mathbb{Z}$*. Then* $\{f_k\}_{k\in\mathbb{Z}}$ *has a unique iteration operator dual frame by the iteration operator* T*.*

In the next theorem we state the duality on iteration operator frames by a condition on their generator vectors.

Theorem 2.5. *Let* $\{f_k\}_{k\in\mathbb{Z}} = \{T^k f_0\}_{k\in\mathbb{Z}}$ *be an iteration operator frame and* $g_0 \in \mathcal{H}$ *such that* $\{(T^*)^k f_0\}_{k\in\mathbb{Z}}$ *and* $\{(T^*)^k g_0\}_{k\in\mathbb{Z}}$ *be a pair of dual frames. Then*

$$\langle S_F f_0, g_0\rangle = \langle f_0, f_0\rangle.$$

The converse is true if $T^2 = I$*.*

3. Conclusion

Iteration operator frames play a key role in frame theory. Here we study dual of such frames and characterize their dual by the same structure.

Acknowledgement

We thank all those involved in coordinating this conference.

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Some Results about Dedekind-finite Acts over Monoids

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Abstract. Dedekind-finite rings and modules are an interesting research in the theory of modules. In this talk, we introduce and study Dedekind-finiteness in the theory of acts over monoids. we will indicate when Dedekind-finiteness and cohopfian property are equivalent in the theory of acts over monoids. we present a powerful characterization of a quasi-injective act in terms of the endomorphisms of its injective envelope.

Keywords: S-act, Dedekind-finite, injective envelope, cohopfian

AMS Mathematics Subject Classification [2010]: 20M30, 20M50, 08B25

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1. Introduction

Dedekind [4] defined an infinite set as a set that can be placed in one-to-one correspondence with a proper subset of itself. He then defined a finite set as one that is not infinite. It follows that a set A is finite if and only if every injective function $f : A \to A$ is an isomorphism of sets. In the theory of modules, a module A (on a unitary ring R) is said to be cohopfian if every injective endomorphism of A is an automorphism and is said to be Dedekind-finite if A can never be isomorphic to a proper direct summand of itself. An easy argument shows that cohopfian property implies Dedekind-finiteness [5] and [7].

Let S be a monoid with identity 1. Recall that a (*right*) S-*act* is a non-empty set A equipped with a map $\mu : A \times S \to A$ called its action, such that, denoting $\mu(a, s)$ by as, we have $a1 = a$ and $a(st) = (as)t$, for all $a \in A$, and $s, t \in S$. An element $\theta \in A$ is called a *zero* of A if $\theta s = \theta$ for every $s \in S$. Let A be an S-act and $B \subseteq A$ a non-empty subset. Then B is called a *subact* of A if $bs \in B$ for all $s \in S$ and $b \in B$. Clearly S itself is an S-act with its operation as the action. An equivalence relation ρ on an S-act A is called a *congruence* on A if $a\rho a'$ implies $(as)\rho(a's)$ for $a, a' \in A$ and $s \in S$. Let $f : A \to B$ be an

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S-homomorphism. Then the kernel equivalence relation of f, $\ker f$, defined by $a(\ker f)a'$ if and only if $f(a) = f(a')$ for $a, a' \in A$, is an act congruence which is called the *kernel congruence* of f.

Throughout this paper, S is a monoid with 0, all S-acts will be right S-acts with unique zero θ and any subact of an S-act contains the zero θ. The category of all S-acts with unique zero θ and S-homomorphisms preserving zero (i.e., $f : A \to B$ with $f(as) = f(a)s$, for $s \in S$, $a \in A$ and $f(\theta) = \theta$), is denoted by $\mathbf{Act}_0$-S. Note that for each $A \in \mathbf{Act}_0$-S we have $A0 = \{\theta\}$.

Recall that the category $\mathbf{Act}_0$-S has coproducts of any non-empty families of S-acts. More precisely, if I is a non-empty set, $X_i \in \mathbf{Act}_0$-$S, i \in I$, and θ_i be the zero of X_i, then by [6, Proposition 2.1.15] the coproduct of $\{X_i : i \in I\}$ is $\coprod_{i\in I} X_i = (\dot{\bigcup_{i\in I}}(X_i \backslash \{\theta_i\}))\dot{\cup}\{\theta\}$ with $x_i s = \theta$, if $x_i s = \theta_i$ in X_i, $\theta s = \theta$ for $s \in S$. Likewise, if B and C are two subacts of an S-act A such that $A = B \cup C$ and $B \cap C = \{\theta\}$, then we write $A = B \oplus C$. In this case $A = B \oplus C$ is called a *decomposition* of A. Otherwise, A is called *indecomposable*. By [6, Theorem 1.5.10], every S-act A has a unique decomposition into indecomposable subacts. For more information about S-acts we encourage the reader to see [6].

In this talk, we are going to study Dedekind-finiteness on $\mathbf{Act}_0$-S. In Theorem 2.6, we will indicate when Dedekind-finiteness and cohopfian property are equivalent. In Theorem 2.8, we present a powerful characterization of a quasi-injective act in terms of the endomorphisms of its injective envelope. By this theorem we show that Dedekind-finiteness of A implies Dedekind-finiteness of $E(A)$ (injective envelope of A), if A is a quasi-injective torsion free act and its injective envelope is strongly faithful.

2. Main results

We begin with a definition.

Definition 2.1. By a *Dedekind-finite* S-act we mean an S-act A which is not isomorphic to any proper direct summand of itself. Equivalently, A is Dedekind-finite if and only if $B = \{\theta\}$ is the only S-act for which $A \coprod B \cong A$.

The next example shows how one can obtain a non-Dedekind-finite act from an indecomposable act.

Example 2.2. [1] Let S be a monoid. Take a non-zero indecomposable S-act A and an arbitrary infinite set I. Then $B = \coprod_{i\in I} A_i$, in which $A_i = A$ for any $i \in I$, is not Dedekind-finite, because $B \coprod A \cong B$ but $A \neq \{\theta\}$.

Definition 2.3. Let A be an S-act. Then A is called *cohopfian* if every injective endomorphism of A is an automorphism.

Lemma 2.4. [1] *If an S-act A is cohopfian, then A is Dedekind-finite.*

Definition 2.5. Let A be an S-act. Then A is called *quasi-injective* if, for any subact B of A, any $f \in \mathrm{Hom}_S(B, A)$ can be extended to an endomorphism of A, it means that the diagram

(1)
$$\begin{tikzcd} B \arrow[r, "\imath"] \arrow[d, "f"'] & A \arrow[dl, "\tilde{f}"] \\ A & \end{tikzcd}$$

commutes, where $\imath$ is the canonical inclusion.

In the following theorem we will show when a cohopfian act is Dedekind-finite and vice versa. But first let us recall [8] that an S-act A is *torsion free* if for any $x, y \in A$ and for any element $s \in S$ the equality $xs = ys$ implies $x = y$. Note that in [6] torsion free is considered in much weaker sense.

Theorem 2.6. [1] *Let A be a torsion free and quasi-injective S-act. Then the following statements are equivalent:*
i) *A is a Dedekind-finite S-act.*
ii) *A is a cohopfian S-act.*

Recall [6] that a subact A of an S-act B is *large* in B if any homomorphism $g : B \to C$ such that $g|_A$ is a monomorphism is itself a monomorphism. An extension B of A with the embedding $f : A \to B$ is an *essential extension* of A whenever Imf is large in B. Moreover, every minimal injective extension of an act A is called an *injective envelope* of A. Injective envelope of an S-act A will be denoted by $E(A)$. By [6, Proposition 3.1.24], every injective envelope of an S-act A is isomorphic to any maximal essential extension of A.

Next, we will consider the relationship between Dedekind-finiteness of an S-act A with its injective envelope.

Lemma 2.7. [1] *Let A and B be S-acts. Then every isomorphism $f : A \to B$ can extend to an isomorphism $\tilde{f} : E(A) \to E(B)$.*

Recall [6] that an S-act A is *strongly faithful* if for $s, t \in S$ the equality $as = at$ for some nonzero element $a \in A$ implies that $s = t$. The next result gives us a powerful characterization of a quasi-injective act in terms of the endomorphisms of its injective envelope.

Theorem 2.8. [1] *Let A be an S-act in which its injective envelope is strongly faithful. Then A is quasi-injective iff it is fully invariant in its injective envelope $E(A)$ (that is, iff any endomorphism of $E(A)$ takes A into A).*

THEOREM 2.9. [1] *Let A be a quasi-injective torsion free S-act which its injective envelope is strongly faithful. If A is Dedekind-finite, then so is $E(A)$.*

In the sequel we peruse connection between Dedekind-finiteness of an S-act A and the monoid $\mathrm{End}(A)$.

LEMMA 2.10. [1] *Let A be a torsion free S-act and let $e : A \to A$ be a homomorphism such that $e^2 = e$. Then $A \backslash e(A) \cup \{\theta\}$ is a subact of A.*

PROPOSITION 2.11. [1] *Let A be a torsion free S-act. Then the following statements are equivalent:*
i) *A is Dedekind-finite.*
ii) *for any homomorphism $e \in \mathrm{End}(A)$ in which $e^2 = e$, if $e(A) \cong A$ then $e = \mathrm{id}_A$.*

In the next theorem we present the connection between Dedekind-finite of an S-act A and its endomorphisms monoid. More explicitly:

THEOREM 2.12. [1] *Let A be an S-act. Then:*
i) *if A is Dedekind-finite and torsion free, then $fg = \mathrm{id}_A$ implies $gf = \mathrm{id}_A$ for any $f, g \in \mathrm{End}(A)$.*
ii) *if $fg = \mathrm{id}_A$ implies $gf = \mathrm{id}_A$ for any $f, g \in \mathrm{End}(A)$, then A is Dedekind-finite.*

3. Conclusion

We answer to this question that what relation there exists between Dedekind-finiteness of an S-act A and the monoid $\mathrm{End}(A)$ of endomorphisms of A.

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پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Group algebras satisfying some normalized Laurent polynomial identities

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Abstract. Let FG be the group algebra of a group G over a field F of characteristic $p \neq 2$. In this talk we discus on Laurent polynomial identity of $\mathcal{U}(FG)$, the unit group of FG. Particularly, we show that is G is torsion and $\mathcal{U}(FG)$ satisfies a normalized Laurent polynomial identity, then FG satisfies a polynomial identity. For a non-torsion group G, we also provide some necessary conditions for unit group of semiprime FG to satisfies a normalized Laurent polynomial identity.

Keywords: Laurent polynomial identity, group algebra, group identity, Engel group

AMS Mathematics Subject Classification [2010]: 16R50, 16S34, 16U60

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1. Introduction

Let $X = \{x_1, x_2, \ldots\}$ be a set of countably many non-commuting variables, F a field and R an F-algebra. Denote the units group of R by $\mathcal{U}(R)$. The free group generated by X is denoted by $\langle X \rangle$ and group algebra of the group $\langle X \rangle$ over the field F is denoted by $F\langle X \rangle$. Each element of $F\langle X \rangle$ is called a *Laurent polynomial*. Let $L(x_1, \ldots, x_n)$ be a nonzero Laurent polynomial. We say that L is *Laurent polynomial identity* (LPI, for short) of $\mathcal{U}(R)$ (or $U(R)$ satisfies an LPI, L) if $L(a_1, \ldots, a_n)$ for every $a_1, \ldots, a_n \in \mathcal{U}(R)$. In particular if $\mathcal{U}(R)$ Satisfies a group identity $w(x_1, \ldots, x_n)$ then $\mathcal{U}(R)$ satisfies the LPI $L(x_1, \ldots, x_n) = 1 - w$. Recall that if $L(x_1, \ldots, x_n)$ does not involve the inverses of all x_j's (i.e., $L \in F[X]$, the free algebra of X over F) and $L(a_1, \ldots, a_n) = 0$ for every $a_1, \ldots, a_n \in R$, then L is a polynomial identity of R (or R satisfies a PI, L).

Let G be a group and F be a field of characteristic $p \geq 0$. The condition under which the group algebra FG satisfies a polynomial identity were determined in classical results due to Isaacs and Passman. Recall that G is said to be *p-abelian* if its commutator subgroup G' is a finite p-group and that 0-abelian means abelian. Then FG satisfies a polynomial identity if and only if G has a p-abelian subgroup of finite index (see [4, Propositoin 1.1.4]).

In the 1980's, Brian Hartley conjectured that if G is torsion and $\mathcal{U}(FG)$ satisfies a group identity, then FG satisfies a polynomial identity. This Conjecture positively answered by many authors (see [4, Section 1.2]). However, there are group rings FG such

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that satisfies a polynomial identity but $\mathcal{U}(FG)$ does not satisfy a group identity. The necessary and sufficient conditions for the unit group FG (of torsion groups G) to satisfy a group identity were found by Liu and Passman in [6]. But for non-torsion groups G, Giambruno-Sehgal-Valenti [3] proved some necessary conditions for $\mathcal{U}(FG)$ to satisfies a group identity.

Some algebras whose unit groups satisfy an LPI were studied in [1, 2]. The main result of [2] states that if R is an algebraic algebra over an infinite field and $\mathcal{U}(R)$ satisfies an LPI, then R satisfies a PI [2, Theorem 5]. This yields that if G is a locally finite group and $\mathcal{U}(FG)$ satisfies an LPI then FG satisfies a PI. For torsion groups G and any field F, in [1] it was showed that if $\mathcal{U}(FG)$ satisfies an LPI which is not satisfied by the units of the relative Free algebra $\mathcal{F} = F\left[\alpha, \beta : \alpha^2 = \beta^2 = 0\right]$, then FG satisfies a PI; this generalizes Hartley's conjecture (because, as indicated in [1], $\mathcal{U}(\mathcal{F})$ does not satisfy any group identity).

In this talk we study some kinds of LPI on the units of group algebras. In the next section, we define the notions of LEI and normalized LPI as kinds of LPI. Particularly, as a generalization of Hartley's conjecture, we show that if $\mathcal{U}(FG)$ (of torsion groups G) satisfies an LEI or normalized LPI, then FG satisfies a PI.

2. Main results

Let G be a group. For any x, y in G, define $E_0(x, y) = x$, $E_1(x, y) = (x, y) = x^{-1}y^{-1}xy$ and inductively, $E_{n+1}(x, y) = (E_n(x, y), y)$. The group G is bounded Engel group if for each x, y in G there exists a positive integer n such that $E_n(x, y) = 1$. Let R be a ring. For any x_1, x_2 in R, define Lie product $[x_1, x_2] = x_1x_2 - x_2x_1$ and inductively, $[x_1, \ldots, x_{n+1}] = [[x_1, \ldots, x_n], x_{n+1}]$. The ring R is bounded Lie Engel if for each x, y in R there exists a positive integer n such that $[x, \underbrace{y, \ldots, y}_{\text{n times}}] = 0$.

Lemma 2.1. ([4, Corollary 5.2.13]) *Let F be a field and G a torsion group. Then $\mathcal{U}(FG)$ is bounded Engel if and only if FG is bounded Lie Engel.*

Lemma 2.2. ([4, Theorem 3.1.2]) *If char $F = 0$, then FG is bounded Lie Engel if and only if G is abelian. If char $F = p > 0$, then FG is bounded Lie Engel if and only if G is nilpotent and G has a p-abelian normal subgroup of finite p-power index.*

Let R be an F-algebra. By the *Laurent Engel identity* (LEI, for short) on $\mathcal{U}(R)$, we mean an LPI of the form $E(x, y) = \sum_{i=0}^{n} a_iE_i(x, y)$, where $a_i \in F$ and $n \in \mathbb{N}$. Clearly, if $\mathcal{U}(R)$ is a bounded Engel group, then it satisfies an LEI of the form $E(x, y) = 1 - E_n(x, y)$, for some natural numbers n. We now state one of our main results as

Theorem 2.3. *Let F be a field and G a torsion group. If $\mathcal{U}(FG)$ Satisfies an LEI, then FG satisfies a PI.*

Let R be an F-algebra whose $\mathcal{U}(R)$ satisfies an LPI, $p(x_1 \ldots, x_n)$. Substituting $x_i = x^iyx^{-i}$ and then multiplying by suitable power of x^{-1} from the left and suitable power of y from the right, if necessary, we may assume that $\mathcal{U}(R)$ Satisfies an LPI in 2 variable of the form $p_1(x, y) = a_1w_1 + \ldots + a_kw_k$ where each non-trivial w_i is of the form $w_i(x, y) = x^{a_{i_1}}y^{b_{i_1}} \ldots x^{a_{i_s}}y^{b_{i_s}}$, where each a_{i_j} and b_{i_j} is nonzero, $a_{i_1} < 0$ and $b_{i_s} > 0$. Once again, substituting $x = x_2x_1^{-1}$ and $y = x_1^{-1}x_2$, we may assume $\mathcal{U}(R)$ satisfies an LPI,

$$P_2(x_1, x_2) = a_1w_1 + \ldots + a_kw_k,$$

where each non-trivial w_i is of the form

$$(*)\ \ w_i\,(x_1,x_2) = x_1^{m_{i_1}} x_2^{n_{i_1}} \dots x_1^{m_{i_r}} x_2^{n_{i_r}},\ \ m_{i_1} = n_{i_r} = 1,\ \ m_{ij}, n_{ij} \in \{\pm 1, \pm 2\}.$$

Moreover, if w_i is as $(*)$, we say that the length of w_i is r and write $\ell(w_i) = r$. For example, $\ell(xy^{-2}x^3y) = 2$.

Definition 2.4. A *normalized LPI* is an LPI $P(x_1, x_2) = a_1w_1 + \ldots + a_kw_k$ where each non-trivial w_i is as $(*)$ and there exists some j such that $\ell(w_j) > \ell(w_i)$ for all $i \neq j$.

From the above argument, it is clear that if $\mathcal{U}(R)$ satisfies a group identity, then it also satisfies a normalized LPI.

Lemma 2.5. *Let F be a field of characteristic $p \neq 2$. There exists a non-zero polynomial $f(t) \in F[t]$ such that for every F-algebra R whose unit group satisfies a normalized LPI, we have $f(ab) = 0$ for every square zero elements $a, b \in R$.*

Now, by Lemma 2.5 and [1, Lemma 1.3 and Theorem 1.1], we deduce the following generalization of Hartley's Conjecture.

Theorem 2.6. *Let F be a field of characteristic $p \neq 2$ and G a torsion group. If $\mathcal{U}(FG)$ satisfies a normalized LPI, then FG satisfies a PI.*

Let us now discuss the semiprime group algebras FG of *non-torsion* groups G whose unit groups satisfy a normalized LPI. Note that if $p = 0$, then FG is semiprime and for $p \neq 0$, FG is semiprime if and only if G does not have a finite normal subgroup with order divisible by p (see [4, Proposition 1.2.9]).

Theorem 2.7. *Let F be a field of characteristic $p \neq 2$ and G a non-torsion group. Let FG be semiprime. If $\mathcal{U}(FG)$ satisfies a normalized LPI, then*

(1) *every idempotent in FG is central.*
(2) *the torsion elements of G form a (normal) subgroup T which is either abelian on Hamiltonian; and,*
(3) *if $p > 0$, G is a p'-group and T is abelian.*

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30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

On the Laplacian Spectrum of the comaximal graph of $\mathbb{Z}_n$

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Abstract. Assume that R is a commutative ring. The comaximal graph of R, denoted by $\Gamma(R)$, is a simple graph whose vertex set consists of all elements of R, and two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$. In this paper, we investigate the Laplacian spectrum of the comaximal graph of the ring $\mathbb{Z}_n$.

Keywords: Comaximl graph, Laplacian spectrum

AMS Mathematics Subject Classification [2010]: 05C50, 13A70

1. Introduction

Let G be a simple graph with n vertices, whose vertex set is $V(G) = \{v_1, v_2, \ldots, v_n\}$. For a vertex v of G, $N_G(v)$ denotes the set of vertices of G that are adjacent to v in G, and we denote $|N_G(v)|$ by $deg(v)$. For two distinct vertices v_i and v_j of G, we write $v_i \sim v_j$ if v_i and v_j are adjacent in G. The *adjacency matrix* of G is the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $v_i \sim v_j$, and otherwise $a_{ij} = 0$. Also, $D(G)$ is the diagonal matrix with the (i,i)-th entry having value $deg(v_i)$. The *Laplacian matrix* $L(G)$ of G is defined by $L(G) = D(G) - A(G)$, and the eigenvalues of $L(G)$ are called the *Laplacian eigenvalues* of G. The *spectrum* of a square matrix B, denoted by $\sigma(B)$, is the multiset of all the eigenvalues of B. Let $\mu_1, \mu_2, \ldots, \mu_r$ be distinct eigenvalues of B with multiplicities $m_1, m_2, \ldots, m_r$, respectively. Then we denote the spectrum of B by

$$\sigma(B) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_r \\ m_1 & m_2 & \ldots & m_r \end{pmatrix}.$$

For a graph G, the spectrum of $L(G)$, is called the *Laplacian spectrum* and it is denoted by $\sigma_L(G)$. The Laplacian spectrum of graphs have been widely studied in [5].

In a graph G, $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. Also $\overline{G}$ denotes the complement of G. We say that G is an *empty graph* if $E(G) = \emptyset$ and it is a *null graph* if $V(G) = \emptyset$. A vertex v is an isolated vertex if $deg(v) = 0$. Also, K_n denotes the complete graph on n vertices and P_n denotes the path with n vertices.

Let R be a commutative ring with nonzero identity. We denote the set of all unit elements and zero divisors of R by $U(R)$ and $Z(R)$, respectively. Also by $Z^*(R)$ we denote the set $Z(R) - \{0\}$. Sharma and Bhatwadekar [7] defined the comaximal graph of

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a commutative ring R, which is denoted by $\Gamma(R)$. The *comaximal graph* of R is a simple graph whose vertices consists of all elements of R, and two distinct vertices a and b are adjacent if and only if $aR + bR = R$, where cR is the ideal generated by c, for $c \in R$. Let $\Gamma_2(R)$ be an induced subgraph of $\Gamma(R)$ with nonunit elements of R as vertices. In this paper, we denote the graph $\Gamma_2(R) - \{0\}$ by $\Gamma_2^*(R)$.

Let $n > 1$ be an integer and let $\mathbb{Z}_n$ denote the ring of integers modulo n. Recently, in [2], the authors studied the Laplacian eigenvalues of the zero divisor graph of $\mathbb{Z}_n$.

In this paper, we first study the structure of the comaximal graph of $\mathbb{Z}_n$ and then we investigate and discuss the Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$.

2. Structure of $\Gamma_2^*(\mathbb{Z}_n)$ and $\Gamma(\mathbb{Z}_n)$

In this paper, for two integers r and s, the notation (r, s) stands for the greatest common divisor of r and s. Also we denote the elements of the ring $\mathbb{Z}_n$, where $n > 1$, by $0, 1, 2, \ldots, n-1$. For every nonzero element a in $\mathbb{Z}_n$, if $(a, n) = 1$, then a is a unit element; otherwise, $(a, n) \neq 1$, and so a is a zero divisor. Therefore, $|U(\mathbb{Z}_n)| = \phi(n)$ and $|Z(\mathbb{Z}_n)| = n - \phi(n)$, where ϕ is the Euler's totient function.

An integer d is said to be a *proper divisor* of n if $1 < d < n$ and $d \mid n$. Now let $d_1, d_2, \ldots, d_k$ be the distinct proper divisors of n. For $1 \leqslant i \leqslant k$, set

$$A_{d_i} := \{x \in \mathbb{Z}_n \mid (x, n) = d_i\}.$$

Clearly, the sets $A_{d_1}, A_{d_2}, \ldots, A_{d_k}$ are pairwise disjoint and we have

$$Z^*(\mathbb{Z}_n) = A_{d_1} \cup A_{d_2} \cup \cdots \cup A_{d_k}$$

and

$$V(\Gamma(\mathbb{Z}_n)) = \{0\} \cup A_{d_1} \cup A_{d_2} \cup \cdots \cup A_{d_k} \cup U(\mathbb{Z}_n).$$

In the rest of the paper, the induced subgraph of $\Gamma(\mathbb{Z}_n)$ on the set A_{d_i} is denoted by $\Gamma(A_{d_i})$, where $1 \leqslant i \leqslant k$.

In the following lemma, we investigate some adjacencies in $\Gamma(\mathbb{Z}_n)$.

LEMMA 2.1. *The following statements hold:*

(i) *Two distinct vertices x and y are adjacent in $\Gamma(\mathbb{Z}_n)$ if and only if $(x, y) \in U(\mathbb{Z}_n)$.*
(ii) *For $1 \leqslant i \leqslant k$, $\Gamma(A_{d_i})$ is isomorphic to $\overline{K}_{\phi(\frac{n}{d_i})}$.*
(iii) *For $1 \leqslant i \neq j \leqslant k$, a vertex of A_{d_i} is adjacent to a vertex of A_{d_j} if and only if $(d_i, d_j) = 1$.*

In the following, we introduce a simple graph G_n, which plays an important role in the rest of the paper. The graph G_n is the simple graph with vertex set $\{d_1, d_2, \ldots, d_k\}$, where d_i's, $1 \leqslant i \leqslant k$, are the proper divisors of n, and two distinct vertices d_i and d_j are adjacent if and only if $(d_i, d_j) = 1$.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$ be the factorization of n to its prime powers, where $t, \alpha_1, \ldots, \alpha_t$ are positive integers and $p_1, \ldots, p_t$ are distinct prime numbers. Every divisor of n is of the form $p_1^{\beta_1} p_2^{\beta_2} \ldots p_t^{\beta_t}$, for some integers $\beta_1, \ldots, \beta_t$, where $0 \leqslant \beta_i \leqslant \alpha_i$ for each $i \in \{1, 2, \ldots, t\}$. Hence the number of proper divisors of n is equal to $\prod_{i=1}^{t}(n_i + 1) - 2$. Therefore we have $k = |V(G_n)| = \prod_{i=1}^{t}(n_i + 1) - 2$.

Recall that for two graphs H_1 and H_2 with disjoint vertex sets, the *join* $H_1 \vee H_2$ of the graphs H_1 and H_2 is the graph obtained from the union of H_1 and H_2 by adding new edges from each vertex of H_1 to every vertex of H_2. The concept of join graph is generalized (in [6], it is called as a generalized composition graph). Assume that G is a graph on k vertices with $V(G) = \{v_1, v_2, \dots, v_k\}$, and let $H_1, H_2, \dots, H_k$ be k pairwise disjoint graphs. The *G-generalized join graph* $G[H_1, H_2, \dots, H_k]$ of $H_1, H_2, \dots, H_k$ is the graph formed by replacing each vertex v_i of G by the graph H_i and then joining each vertex of H_i to each vertex of H_j whenever $v_i \sim v_j$ in the graph G. Now, if the graph G consists of two adjacent vertices, then the G-generalized join graph $G[H_1, H_2]$ coincides with the join $H_1 \vee H_2$ of the graphs H_1 and H_2.

In the next lemma, we state that $\Gamma_2^*(\mathbb{Z}_n)$ is a generalized join of some certain non-empty graphs.

LEMMA 2.2. *In the comaximal graph of $\mathbb{Z}_n$, we have*

$$\Gamma_2^*(\mathbb{Z}_n) = G_n[\overline{K}_{\phi(\frac{n}{d_1})}, \overline{K}_{\phi(\frac{n}{d_2})}, \dots, \overline{K}_{\phi(\frac{n}{d_k})}]$$

and

$$\Gamma(\mathbb{Z}_n) = (K_1 \cup \Gamma_2^*(\mathbb{Z}_n)) \vee K_{\phi(n)}.$$

EXAMPLE 2.3. Consider the ring $\mathbb{Z}_{12}$. We have $d_1 = 2, d_2 = 3, d_3 = 4$, and $d_4 = 6$. Then G_{12} is the graph $2 \sim 3 \sim 4 \cup \{6\}$, which is isomorphic to $P_3 \cup K_1$. Now by Lemma 2.2, we have

$$\Gamma_2(\mathbb{Z}_{12}) = K_1 \cup G_{12}[\overline{K}_2, \overline{K}_2, \overline{K}_2, K_1].$$

In the following theorem, which was proved by Cardoso et al. in [1, Theorem 8], the Laplacian spectrum of a generalized join graph $G[H_1, \dots, H_k]$ is determined in terms of the Laplacian spectrum of the graphs H_i's and the spectrum of another $k \times k$ matrix $\mathcal{L}(G)$.

THEOREM 2.4. *[1] Let G be a graph with vertex set $\{v_1, v_2, \dots, v_k\}$ and let $H_1, H_2, \dots, H_k$ be k pairwise disjoint graphs with $m_1, m_2, \dots, m_k$ vertices, respectively. Then the Laplacian spectrum of $G[H_1, H_2, \dots, H_k]$ is given by*

$$\sigma_L(G[H_1, H_2, \dots, H_k]) = \left(\bigcup_{j=1}^{k} (M_j + (\sigma_L(H_j) \setminus \{0\}))\right) \bigcup \sigma(\mathcal{L}(G)),$$

where

$$M_j = \begin{cases} \sum_{v_i \sim v_j} m_i & \text{if } N_G(v_j) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases},$$

$$\mathcal{L}(G) = \begin{bmatrix} M_1 & -s_{1,2} & \dots & -s_{1,k} \\ -s_{1,2} & M_2 & \dots & -s_{2,k} \\ \dots & \dots & \ddots & \dots \\ -s_{1,k} & -s_{2,k} & \dots & M_k \end{bmatrix},$$

and

$$s_{i,j} = \begin{cases} \sqrt{m_i m_j} & \text{if } v_i \sim v_j \text{ in } G, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\sigma_L(H_j) \setminus \{0\}$ means that one copy of the eigenvalue 0 is removed from the multiset $\sigma_L(H_j)$, and $M_j + (\sigma_L(H_j) \setminus \{0\})$ means that M_j is added to each element of $\sigma_L(H_j) \setminus \{0\}$.

Let G be a vertex weighted graph by assigning the weight $m_i = |V(H_i)|$ to the vertex v_i of G, for $1 \leqslant i \leqslant k$. Let $W(G) = (w_{i,j})$ be the $k \times k$ matrix, where

$$w_{i,j} = \begin{cases} -m_j & \text{if } i \neq j \text{ and } v_i \sim v_j, \\ \sum_{v_i \sim v_r} m_r & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $W(G)$ is called a vertex weighted Laplacian matrix of G. In [3, p. 317], it is shown that the matrices $W(G)$ and $\mathcal{L}(G)$ are similar, and so we have $\sigma(\mathcal{L}(G)) = \sigma(W(G))$. Hence in Theorem 2.4, we can use $\sigma(W(G))$ instead of $\sigma(\mathcal{L}(G))$ for determining the Laplacian spectrum of $G[H_1, H_2, \ldots, H_k]$.

Suppose that $d_1, d_2, \ldots, d_k$ are the proper divisors of n. We assign the weight $|A_{d_j}| = \phi(\frac{n}{d_j})$, for $1 \leqslant j \leqslant k$, to the vertex d_j of the graph G_n. Now the $k \times k$ vertex weighted Laplacian matrix $W(G_n)$ of G_n is given by

$$W(G_n) = \begin{bmatrix} M_{d_1} & -t_{1,2} & \ldots & -t_{1,k} \\ -t_{2,1} & M_{d_2} & \ldots & -t_{2,k} \\ \ldots & \ldots & \ddots & \ldots \\ -t_{k,1} & -t_{k,2} & \ldots & M_{d_k} \end{bmatrix},$$

where $M_{d_j} = \sum_{d_i \in N_{G_n}(d_j)} \phi(\frac{n}{d_i})$, for $1 \leqslant j \leqslant k$, and

$$t_{i,j} = \begin{cases} \phi(\frac{n}{d_j}) & \text{if } d_i \sim d_j \text{ in } G_n, \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \leqslant i \neq j \leqslant k$.

In the following theorem, we determine the Laplacian spectrum of $\Gamma_2(\mathbb{Z}_n)$.

THEOREM 2.5. *Let $d_1, d_2, \ldots, d_k$ be the proper divisors of n. Then the Laplacian spectrum of $\Gamma_2(\mathbb{Z}_n)$ is given by*

$$\sigma_L(\Gamma_2(\mathbb{Z}_n)) = \begin{pmatrix} 0 & M_{d_1} & M_{d_2} & \ldots & M_{d_k} \\ 1 & \phi(\frac{n}{d_1}) - 1 & \phi(\frac{n}{d_2}) - 1 & \ldots & \phi(\frac{n}{d_k}) - 1 \end{pmatrix} \bigcup \sigma(W(G_n))$$

Note that since $\Gamma(\mathbb{Z}_n) = \Gamma_2(\mathbb{Z}_n) \vee K_{\phi(n)}$ and

$$\sigma_L(K_{\phi(n)}) = \begin{pmatrix} 0 & \phi(n) \\ 1 & \phi(n) - 1 \end{pmatrix},$$

by Theorem 2.5, in order to find the Laplacian spectrum $\Gamma(\mathbb{Z}_n)$, it is enough to determine $\sigma(W(G_n))$.

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Dynamics of a fractional SIR model with two different fractional derivatives

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Abstract. In this paper, an SIR model with fractional derivatives in the sense of Caputo and Caputo-Fabrizio definitions is introduced. The dynamical properties of the system analysed and some simulations are presented to verify the analytical results.

Keywords: SIR epidemic model, Caputo derivatives, Caputo-Fabrizio derivatives, Adams-Bashforth predictor method

AMS Mathematics Subject Classification [2010]: 26A33, 34K37

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1. Introduction

Today, the role of epidemic diseases is well observed in the society. For this reason, modelling and analyzing the infectious disease model has attracted the attention of many scientists. One of the well-known models that is widely used in literature for modelling infectious diseases is the SIR model [2, 8]. On the other hand, with the development and application of fractional derivatives, some researchers pay attention to the epidemic models as a fractional dynamical systems [1, 2]. In this paper, we study the following model

$$\begin{cases} D^q S(t) = \Lambda - \beta S I - (\mu + d) S, \\ D^q I(t) = \beta S I - (\alpha + \gamma + d) I, \\ D^q R(t) = \mu S + \gamma I - dR, \end{cases} \tag{1}$$

where D^q denotes the Caputo or Caputo-Fabrizio fractional derivative of order $0 < q \leq 1$. In this system, S, I and R denote the susceptible, infective and recovered individuals, respectively. Λ the recruitment rate, β is the disease transmission rate of susceptible to infected individuals, μ is the proportion of the susceptible that is vaccinated per unit time, d is the natural death rate, γ is the recovered rate and α is the disease-induced death rate.

2. Preliminaries

The popular definition of fractional derivatives is the Caputo type which is defined by

$$^{C}D_x^q f(x) = \frac{1}{\Gamma(1-q)} \int_0^x \frac{\dot{f}(x)}{(t-\tau)^q} d\tau, \tag{2}$$

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where $0 < q \leq 1$. In 2015 Caputo and Fabrizio defined the new definition of fractional derivative as follows.

Definition 2.1. [3] Let $0 < q < 1$. If $a \in (-\infty, x)$, the Caputo-Fabrizio fractional derivative of a function $f(x) \in H^1(a,b)$, $b > a$ is defined as

$$ {}^{CF}D_x^q f(x) = \frac{M(q)}{1-q}\int_a^x \dot{f}(s)\exp(-\frac{q(x-s)}{1-q})ds \tag{3}$$

where $M(a)$ is the normalization function such that $M(0) = M(1) = 1$.

Now suppose the fractional-order linear system

$$\mathcal{D}^q x(t) = Ax(t), \tag{4}$$

where $x(t) \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}, 0 < q < 1$ and $\mathcal{D} = {}^{C}D$ or $\mathcal{D} = {}^{CF}D$.

Suppose in (4), $\mathcal{D} = {}^{CF}D$ then the characteristic equation of (4) is

$$\det(\lambda(I - (1-q)A) - qA) = 0. \tag{5}$$

Theorem 2.2. *[5] If $(I-(1-q)A)$ is invertible, then system (4), described by Caputo-Fabrizio derivative, is asymptotically stable if and only if the real parts of the roots to the discriminant equation (5) are negative.*

Moreover, Li et al. [5] proved the next theorem.

Theorem 2.3. *The system (1) described by Caputo-Fabrizio derivative, is asymptotically stable if eigenvalues λ of matrix A satisfy one of the following conditions:*

(i) $\|\lambda\| \geq \frac{1}{1-q}, \quad \lambda \neq \frac{1}{1-q}$;
(ii) $Re(\lambda) > \frac{1}{1-q}$;
(iii) $Re(\lambda) < 0$;
(iv) $|Im(\lambda)| > \frac{1}{2(1-q)}$.

Now, consider system (4) described by Caputo derivative ($\mathcal{D} = {}^{C}D$), then we have the following theorem about the stability of the system.

Theorem 2.4. *[6] System (4), described by Caputo-Fabrizio derivative, with $x(t_0) = x_0$, is asymptotically stable if and only if $|\arg(spec(A))| > \frac{q\pi}{2}$, where $spec(A)$ is the spectrum (set of all eigenvalues) of A.*

Suppose the fractional-order nonlinear system

$$\mathcal{D}^q x(t) = f(x(t)), \tag{6}$$

where, $f \in C^1$ is a nonlinear function. We can determine the local dynamics of (6) by Jacobian linearization about the equilibrium points x^*. In the other words, It suffices to consider $\mathcal{D}^q x(t) = Ax(t)$ where $A = Df(x^*)$. Now we can use two above theorems to characterize the stability of the nonlinear system (6).

3. Equilibria and stability

By simple calculation, one can obtain that system (1) has always a disease-free equilibrium (DFE) $E_0 = (\frac{\Lambda}{d+\mu}, 0, \frac{\mu\Lambda}{d(\mu+d)})$ and the basic reproduction number for (1) is defined by

$$\mathfrak{R}_0 = \frac{\beta\Lambda}{(\mu+d)(\gamma+\alpha+d)}. \tag{7}$$

The unique endemic equilibrium $E_1 = (S_1, I_1, R_1)$ is obtained by

$$\begin{aligned} S_1 &= \frac{\gamma+\alpha+d}{\beta}, \\ I_1 &= \frac{\beta\Lambda - (\mu+d)(\gamma+\alpha+d)}{\beta(\gamma+\alpha+d)} = (\Re_0 - 1)\frac{\mu+d}{\beta}, \\ R_1 &= \frac{\mu(\gamma+\alpha+d) + \gamma(\mu+d)(\Re_0-1)}{\beta d}. \end{aligned}$$

It is clear that E_1 is positive if $\Re_0 > 1$.

THEOREM 3.1. *Let $0 < q \leq 1$, in the sense of Caputo and Caputo-Fabrizio derivatives,*

(i) *The disease free equilibrium E_0 is locally asymptotically stable if $\Re_0 < 1$ and it is a saddle $\Re_0 > 1$,*

(ii) *The endemic equilibrium E_1 is locally asymptotically stable if $\Re_0 > 1$.*

PROOF. (i) The eigenvalues of the Jacobian matrix evaluated at E_0 are $\lambda_1 = -d$, $\lambda_2 - (d+\mu)$ and $\lambda_3 = (\alpha+d+\mu)(\Re_0 - 1)$. Therefore, for $\Re_0 < 1$ all of eigenvalues are negative (or the $|\arg(\lambda_i)| > \frac{q\pi}{2}$ for $0 < q \leq 1$). It is obvious for $\Re_0 > 1$, the third eigenvalue is positive.
(ii) The Jacobian matrix evaluated at E_1 is

$$J(E_1) = \begin{pmatrix} -\Re_0(\mu+d) & -(\mu+d+\alpha) & 0 \\ (\Re_0-1)(\mu+d) & 0 & 0 \\ \mu & \gamma & -d \end{pmatrix}.$$

Therefore, the eigenvalues of $J(E_1)$ are

$$\lambda_{1,2} = \frac{-\Re_0(\mu+d) \pm \sqrt{\Re_0^2(\mu+d)^2 - 4(\Re_0-1)(\mu+d)}}{2}, \quad \lambda_3 = -d.$$

If $\Re_0 > 1$ then $\lambda_{1,2} \in \mathbb{R}^-$. Hence E_1 is is locally asymptotically stable. □

4. Numerical simulations and discussion

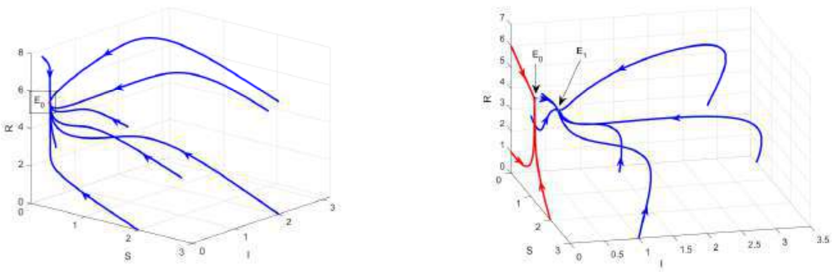
In this part, we simulate the system (1) numerically. For simulation of the system with the Caputo derivative, we use the predictor-corrector method of Adams-Bashforth-Moulton described in [4]. For the Caputo-Fabrizio fractional model we use three-step Adams-Bashforth predictor method described in [7]

Here we take following numerical values for parameters:

$$\Lambda = 0.3, \quad \alpha = 0.1, \quad d = 0.02, \quad \beta = 0.75, \quad \gamma = 0.03,$$

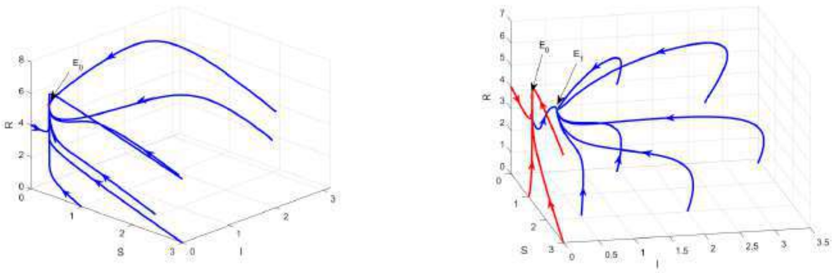
and also we take vaccination rate μ as a changing parameter. According to Theorem 3.1, we know if $\mu > \frac{\beta\Lambda - d(\gamma+\alpha+d)}{\gamma+\alpha+d} = 0.2833$ or equivalently $\Re_0 < 1$, the disease is eradicated in the society.

Now, we consider two cases. First, let $\mu = 0.8$, then we get $E_0 = (0.3529, 0, 5.6471)$ and $\Re_0 = 0.3922$. In this case the unique DFE is stable. In other word, trajectories of he system with the initial in the first octant converge to the DFE, see Figures 1 (a) and 2 (a).
In the second case, take $\mu = 0.2$, then we have $E_0 = (1.2, 0, 4.8)$, $E_1 = (0.9, 0.4167, 3.85)$ and $\Re_0 = 1.3333$. In this case the endemic equilibrium E_1 is stable and the DFE is a saddle, see Figures 1 (b) and 2 (b). The trajectories started out of the plane $I = 0$ converges to the E_1 and trajectories on the plane $I = 0$ converges to the E_0 (see the red



(a) $\mu = 0.8$ and $\Re_0 = 0.3922$. (b) $\mu = 0.2$ and $\Re_0 = 1.3333$.

FIGURE 1. Phase portrait of system (1) with Caputo derivatives of order $q = 0.95$. In (a) there is the stable DFE and in (b) DFE is a saddle and the endemic equilibrium is stable.



(a) $\mu = 0.8$ and $\Re_0 = 0.3922$. (b) $\mu = 0.2$ and $\Re_0 = 1.3333$.

FIGURE 2. Phase portrait of system (1) with Caputo-Fabrizio derivatives of order $q = 0.9$. In (a) there is the stable DFE and in (b) DFE is a saddle and the endemic equilibrium is stable.

trajectories in Figures 1 and 2). In the other word, plane $I = 0$ is a stable manifold for the saddle point E_0. Hence, numerical simulations verify the theorems of Section 3.

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Faculty of Mathematics and Computer
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Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Hardness Results of k-Efficient Domination in Chordal Graphs

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Abstract. A set $D \subseteq V$ of a graph $G = (V, E)$ is called an efficient dominating set of G if every vertex v has exactly one neighbor in D, in an alternative view, the vertex set V is partitioned to some circles with radius one such that the vertices in D are the centers of partitions. A generalization of this concept, introduced by Chellali et al. [1], is called k-efficient dominating set that briefly is partitioning the vertices of a graph with different radiuses. It leads to a partition set $\{U_1, U_2, \ldots, U_t\}$ such that each U_i consists a centre vertex u_i and all the vertices in distance d_i where $d_i \in \{0, 1, \ldots, k\}$. The problem of finding minimum set $\{u_1, u_2, \ldots, u_t\}$ is called the Minimum k-efficient domination problem. Given a positive integer S and a graph $G = (V, E)$, the k-efficient Domination Decision problem is to decide whether G has an k-efficient dominating set of cardinality at most S. The k-efficient Domination Decision problem is known to be NP-complete even for bipartite graphs [1]. Clearly, every graph has a k-efficient Dominating set, but it is not correct for efficient dominating set. In this paper, we study the NP-completeness of the k-efficient domination problem in Chordal graphs.

Keywords: Domination; Efficient domination; Computational complexity.

AMS Mathematics Subject Classification [2010]: 05C69, 11Y16

1. Introduction

Domination problem is one of the most fundamental types of problems that have been widely explored in computer science. There are many extensions of this problem, such as independent, total, efficient, mixed, paired, signed, and rainbow dominating sets. Many experts have investigated finding bounds and designing algorithms to compute the dominating sets of each kind in various classes of graphs. A discussion of some of these can be found in [2].

1.1. Definition and notation. Let $G = (V, E)$ be a simple graph. The neighbourhood of a vertex $v \in V$ is the set of vertices adjacent to v, denoted by $N(v)$, and the closed neighbourhood of vertex v is defined $N[v] = N(v) \cup \{v\}$. The neighbourhood of a vertex $v \in V$ up to radius i, denoted $N_i[v]$, is the set of all vertices in distance at most i of v. The notation $[t]$ is used for the set $\{1, 2, \ldots, t\}$.

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Domination A set $D \subseteq V$ is a dominating set if every vertex not in D is adjacent to at least one vertex in D. The domination number, denoted by $\gamma(G)$, is the cardinality of a smallest dominating set. Finding a minimum dominating set was one of the first problems that shown is NP-hard [3].

Efficient Domination A specific version of domination problems is efficient domination was introduced in [4] as an extension of the perfect error-correcting code in coding theory. An efficient dominating set in G is a subset of vertices $D \subseteq V$ such that every vertex $v \in V$ is dominated by precisely one vertex from D. Clearly, there exist graphs without any efficient dominating sets. So, if a graph consists of an efficient dominating set, it is efficiently dominatable. Vital applications of efficient domination in coding theory and other branches like parallel processing systems [5] have made it a noticeable domination. Therefore, the different notions and names of efficient dominating set appeared in the literature such as: perfect code, 1-perfect code, independent perfect dominating set, and perfect dominating set. The efficient domination problem is NP-complete even for restricted graph classes such as bipartite graphs and chordal graphs [6].

***k*-Efficient Domination** To relax the concept of efficient domination, Chellali, Haynes, and Hedetniemi in [1], introduce the k-efficient domination based on concept k-efficient partition.

A partition $\pi = \{N_{i_1}[v_1], N_{i_2}[v_2], \ldots, N_{i_t}[v_t]\}$ is called a k-efficient partition of V if for every $j \in [t]$, we have $i_j \in \{0, 1, \ldots, k\}$. The vertices $\{v_1, v_2, \ldots, v_t\}$ are the centres of partitions.

Definition 1.1. [1] For any integer $k \geq 0$, the k-efficient domination number of G, denoted $\epsilon_k(G)$, equals the smallest cardinality of a k-efficient partition of G.

In other words, k-efficient dominating set is the set $\{v_1, v_2, \ldots, v_t\}$ the centres of the circles with radius d where $d \in \{0, 1, 2, \ldots, k\}$ can partition the vertices of the graph. Clearly, typical efficient domination differs from 1-efficient domination due to in 1-efficient, we can select a singleton vertex that only dominates itself. It follows that, for instance, C_5 has a 1-efficient dominating set but does not have any efficient dominating set. It could be easily investigate that $\epsilon_j \leq \epsilon_i$ for $j > i$.

2. NP-complete for chordal graphs

This section shows that the k-efficient domination problem is NP-complete when restricted to chordal graphs. Note that, for $k = 1$, the problem of k-efficient domination differs from the typical efficient domination problem, shown is NP-complete for chordal graphs [6]. We prove the NP-completeness of k-efficient domination by using a reduction from Exact 3-Cover problem (EX3C), which is known to be NP-complete [3].

Exact 3-Cover problem (EX3C)

Instance: A finite set X with $|X| = 3q$, where q is a positive integer and a collection C of 3-element subsets of X.

Question: Is there a sub-collection C' of C such that every element of X appears in exactly one element of C'.

Theorem 2.1. *1-efficient Domination Problem is NP-complete for chordal graphs.*

Clearly, k-efficient Domination Problem for chordal graph is in NP. We describe a polynomial reduction from EX3C to 1-efficient Domination Problem for chordal graphs. Given any instance (X, C) of EX3C, we obtain a chordal graph $G = (V, E)$ and an integer

k such that EX3C has a solution if and only if G has a 1-efficient dominating set of cardinality at most N.

Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$ be an arbitrary instance of EX3C. The vertex set of the newly formed graph $G = (V, E)$ constructed of disjoint union three parts. For each $C_p \in C, p \in [t]$, we have a path with vertices c_p, v_p and u_p. This make the first part. The second part have a set of $3q$ vertices $\{x_1, x_2, \ldots, x_{3q}\}$, each corresponding to an element X. We add edges between every pair of vertices in this set to make a clique. For each vertex $x_i, i \in [3q]$, we have a claw, induced subgraph of $K_{1,3}$, centred at a vertex z_i and y_i is one of its the pendant. We connect x_i to y_i. Now we add the edges between x_i and c_p if the element corresponding to x_i is in c_p. Note that degree of each c_p is four for all $p \in [t]$. The construction of G from the instance (X, C) of EX3C is illustrated in Figure 1. Clearly, the graph G is a chordal graph.

Let $N = 4q + t$. Theorem 2.1 directly follows from the following result.

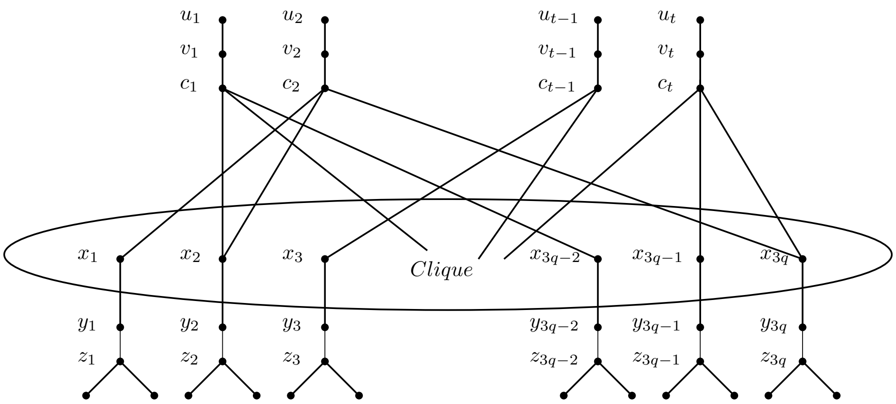

FIGURE 1. Reducing the EX3C problem to the k-efficient Domination Problem.

LEMMA 2.2. *EX3C has a solution if and only if G has a 1-efficient dominating set of cardinality at most $N = 4q + t$.*

PROOF. Suppose the instance (X, C) has a solution C'. Since each element of X is covered by exactly one element of C', $|C'| = q$. Let P be the index of the corresponding vertices in C'. We claim that the following set D is a 1-efficient dominating set:

$$D = \{u_i | i \in [P]\} \bigcup \{c_i | i \in [P]\} \bigcup \{v_i | i \in [t]/P\} \bigcup \{z_i | i \in [3q]\}.$$

The vertices u_i's only dominate themselves but other vertices in D dominate their adjacent vertices. One can easily check that D contains $4q + t$ and forms a 1-efficient domination.

Conversely, assume that G has a 1-efficient dominating set of cardinality at most $N = 4q + t$. Since D is a dominating set, in each 3-path $u_i v_i c_i$ where $i \in [t]$, at least one vertex have to appear in D. For a fixed $i \in [3q]$ consider the claw that is connected to x_i. To dominate the pendant vertices of it we need to select at least the centre vertex z_i or select all its pendent vertices. So summing up over all i, we get that D contains more than $3q$ vertices from selecting pendents. So, the cardinality of D is at least $t + 3q$. Now to complete the proof we shall show that $X \cap D = \emptyset$. Since if this is the case then each x_i has to be dominated by a $c_p \in C, p \in [t]$. We have to dominate the $3q$ vertices of X

using at most q vertices, as each $c_i \in C$ dominates exactly three x_is. This is possible only if there exist q vertices $\{c_{i_1}, \ldots, c_{i_q}\}$, which can dominate the $3q$ vertices X. Now define C' to be the sets corresponding to these vertices, i.e. $C' = \{c_{i_1}, \ldots, c_{i_q}\}$. Clearly C' is an exact cover of X, and has only q sets.

Till now we have only used the fact that D is a dominating set but for showing $D \cap X = \emptyset$, we will be crucially using the fact that D is a 1-efficient domination. To reach a contradiction let us suppose some $x_i \in D$. Then the connected vertex y_i is dominated. To dominate the pendent vertices in claw we have to select all pendents, one of them is 1-dominator and the other is self-dominator. So, we need two dominators to dominate this claw. To dominate other claws we need at least $3q-1$. This implies totally the $|D| > t+4q$ which contradicts the assumption that $|D| \leq N$. Therefore $D \cap X = \emptyset$. $\square$

3. Conclusion

In this work, the NP-completeness of the k-efficient domination problem, restricted to chordal graphs, is proven. The problem of k-efficient domination is a generalization of the typical efficient domination problem.

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On the epi-superfluous submodules

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Abstract. Let R be a ring. In this paper, we introduce epi-superfluous R-submodules and epimorphism radical of an R-module which are larger than that of superfluous submodules and radical of a module, respectively. Then we examine some characteristics of these submodules on epi-Notherian and epi-Artinian modules. Various examples are also given.

Keywords: epi-Superfluous submodule, epi-Artinian module, epi-Notherian module.

AMS Mathematics Subject Classification [2010]: 16D10; 16D99; 13C13.

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1. Introduction

Throughout this paper, all the rings we consider are associative rings with identity, and modules are unitary right modules. A submodule N of an R-module M is superfluous in M and denoted by $N \ll M$, in case for any submodule L of M, $L + N = M$ implies $L = M$. In 2015, Babak Amini and Afshin Amini in [2] introduced the notion of strongly superfluous submodule, and then investigated the basic properties of these submodules on max rings. A submodule K of an R-module M is said to be *strongly superfluous* in M and denoted by $K \leq_{ss} M$, if $\bigoplus_{i\in I} K \ll \bigoplus_{i\in I} M$ for any index set I. Recently, in 2018, Prakash and Chaturvedi in [6], introduced the notions of epi-Artinian and epi-Noetherian modules. A module M is *epi-Artinian* (*epi-Noetherian*) if for every descending (ascending) chain $M_1 \geq M_2 \geq M_3 \geq \cdots$ ($M_1 \leq M_2 \leq M_3 \leq \cdots$) of submodules of M, there exists an index $n \geq 1$ such that for every $i \geq n$, there exists an epimorphism $\phi_i : M_i \to M_{i+1}$ ($\phi_i : M_{i+1} \to M_i$), where in [3], Dastanpour and Ghorbani call these chain conditions epi-ACC (epi-DCC). Also, Ghorbani and Vedadi [5] defined the notion of epi-retractable modules. The R-module M is called *epi-simple (or;epi-retractable)* if every non-zero submodule of M is an epimorphism image of M.

In this paper, we introduce and study epi-superfluous submodules and then, we examine some properties modules with chain conditions on epi-superfluous submodules.

2. Results

We begin this section by the following definition.

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Definition 2.1. A submodule N of an R-module M is called epi-superfluous in M and denoted by $N \leq_{epi} M$, in case for any submodule L of M, $L + N = M$ implies that L is an epimorphism image of M.

Clearly, every superfluous submodule is epi-superfluous, but not conversely, see Example 2.2.

Example 2.2. (1) Every superfluous submodule is epi-superfluous.
(2) $2\mathbb{Z}$ is epi-superfluous in $\mathbb{Z}$, but $2\mathbb{Z}$ is not superfluous in $\mathbb{Z}$.

Lemma 2.3 ([6], Lemma 3.2). *A module M is epi-artinian (epi-noetherian) if and only if, for every non-empty set F of submodules of M, there exists $N \in F$ such that, for every submodule $K \leq N(N \leq K)$, if $K \in F$, then K is a homomorphic image of N (N is a homomorphic image of K).*

Consider $\mathcal{EM} = \{M \mid N$ *is not an epimorphism image of the R-module M for any maximal submodule N of M*$\}$. In the following example, we show that $\mathcal{EM}$ is not an empty set.

Example 2.4. Let R be a local ring with non cyclic maximal ideal M (for example, R can be considered as the localization of the ring $R = \mathbb{C}[x, y]$ in ideal $< x, y >$). Then, there is no R-epimorphism $f : R \to M$, and hence R-module R belongs to the $\mathcal{EM}$ set.

Lemma 2.5. *Let M be a right R-module such that $M \in \mathcal{EM}$. Then,* $\mathrm{rad}(M) = \{x \in M \mid xR \ll M\}$, *where* $\mathrm{rad}(M)$ *is the intersection of all maximal submodules of M.*

Proof. See, [1, Proposition 9.13]. □

Proposition 2.6. *Let M be a right R-module such that $M \in \mathcal{EM}$. If* $\mathrm{rad}(M)$ *is epi-Noetherian, then for every non-empty set F of epi-superfluous submodules of M, there exists $N \in F$ such that, for every $N \leq K$, if $K \in F$, then N is a homomorphic image of K.*

Proof. It suffices to show that $N \leq \mathrm{rad}(M)$ for every epi-superfluous submodules N of M, and then by Lemma 2.3, the result is proved. Let $N \leq_{\mathrm{epi}} M$ and $N \nleq \mathrm{rad}(M)$. Then, there is a maximal submodule K of M such that $N \nleq K$. So, $K + N = M$. On the other hand, N is an epi-superfluous submodules of M, and it follows that K is an epimorphism image of M, a contradiction. □

Recall that a submodule N of an R-module M is a fully invariant submodule if $f(N) \subseteq N$ for all endomorphisms f of M. A module M is said to be a duo module if every submodule of M is fully invariant, see [6].

Theorem 2.7. *Let M be a right R-module such that $M \in \mathcal{EM}$. If* $\mathrm{rad}(M)$ *is epi-Artinian, then:*

(1) *Every epi-superfluous submodule of M is epi-Artinian.*
(2) *If every cyclic submodule of* $\mathrm{rad}(M)$ *is duo, then* $\mathrm{Soc}(\frac{\mathrm{rad}(M)}{N}) \leq_e \frac{\mathrm{rad}(M)}{N}$.

Proof. (1) If $L \leq_{epi} M$, then we show that $L \leq \mathrm{rad}(M)$. If $L \nleq \mathrm{rad}(M)$, then by [1, Proposition 9.13], there exists maximal submodule K of M such that $L \nleq K$. So $K + L = M$ and hence K is an epimorphism image of M, a contradiction. Therefore by Lemma 2.3, L is epi-Artinian.

(2) Let $t \in \mathrm{rad}(M)$. Then by (1), tR is epi-Artinian, since $tR \leq_{epi} M$. So by [6, Propositions 3.3], for every non-zero proper submodule N of $\mathrm{rad}(M)$ such that $t \notin N$, it

follows that $\frac{tR+N}{N} = \frac{tR}{tR\cap N}$ is epi-Artinian. Hence by [6, Theorem 3.11], every non-zero proper submodule of $\frac{\mathrm{rad}(M)}{N}$ contains an essential submodule that is a direct sum of epi-simple modules. Hence, epi-Soc$(\frac{\mathrm{rad}(M)}{N}) \leq_e \frac{\mathrm{rad}(M)}{N}$ and consequently by [4, Remark 4.3], $\mathrm{Soc}(\frac{\mathrm{rad}(M)}{N}) \leq_e \frac{\mathrm{rad}(M)}{N}$. □

THEOREM 2.8. *Let M be a right R-module such that $M \in \mathcal{EM}$. If every cyclic submodule and non-zero proper submodule of* $\mathrm{rad}(M)$ *is duo and finitely generated, respectively, then* $\mathrm{rad}(M)$ *is epi-Artinian if and only if every epi-superfluous submodule of M is epi-Artinian.*

PROOF. ($\Longrightarrow$) is clear by Theorem 2.7(1).

($\Longleftarrow$) First, we claim that $\mathrm{Soc}(\frac{\mathrm{rad}(M)}{N}) \leq_e \frac{\mathrm{rad}(M)}{N}$ for every proper submodule N of $\mathrm{rad}(M)$. Let $t \in \mathrm{rad}(M)$. Then by hypothesis and Lemma 2.5, tR is epi-Artinian, since $tR \leq_{epi} M$. So by [6, Proposition 3.3], for every non-zero proper submodule N of $\mathrm{rad}(M)$ such that $t \notin N$, it follows that $\frac{tR+N}{N} = \frac{tR}{tR\cap N}$ is epi-Artinian. Hence by [6, Theorem 3.11], every non-zero proper submodule of $\frac{\mathrm{rad}(M)}{N}$ contains an essential submodule that is a direct sum of epi-simple modules. Hence, epi-Soc$(\frac{\mathrm{rad}(M)}{N}) \leq_e \frac{\mathrm{rad}(M)}{N}$ and consequently by [4, Remark 4.3], $\mathrm{Soc}(\frac{\mathrm{rad}(M)}{N}) \leq_e \frac{\mathrm{rad}(M)}{N}$.

Now, suppose that $\mathrm{rad}(M)$ is not epi-Artinian. Then $\mathrm{rad}(M)$ is not Artinian. So by [1, Proposition 10.10], the set γ of submodules B of $\mathrm{rad}(M)$ such that $\frac{\mathrm{rad}(M)}{B}$ is not finitely cogenerated, is non-empty. Let $\{B_i : i \in I\}$ be any chain of submodules in γ. If $B = \cap_{i\in I} B_i$ and $B \notin \gamma$, then $\frac{\mathrm{rad}(M)}{B}$ is finitely cogenerated and hence $B = B_i$ for some $i \in I$. So, $B \in \gamma$ and by Zorn's Lemma, γ has a minimal member Z. Let U denote the submodule of $\mathrm{rad}(M)$, containing Z , such that $\mathrm{Soc}(\frac{\mathrm{rad}(M)}{Z}) = \frac{U}{Z}$. Then, $\frac{U}{Z} \leq_e (\frac{\mathrm{rad}(M)}{Z})$ and hence by [1, Proposition 10.7], $\frac{U}{Z}$ is not finitely generated, which a contradiction by hypothesis. □

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option pricing in the fractional stochastic volatility models using malliavin calculus

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Abstract. We study the fractional stochastic volatility model in which the volatility is driven by a fractional Brownian motion and the price is driven by an independent simple Brownian motion. We relate the option price to a quadratic average of the exponential fractional Brownian motion and We prove the existence of the Implied Volatilities distribution density function by using malliavin calculus and we derive the asymptotics of the mentioned average as t tends to infinity.

Keywords: Fractional stochastic volatility model, volatility smile, call option pricing, asymptotics of the distribution density, malliavin calculus

AMS Mathematics Subject Classification [2010]: 60G22, 91G20, 60H07

1. Introduction

Implied volatility surface is the plot of the implied volatility (σ) as a two variable function of moneness (strike price K) and expiration time (T). This surface is obtained from empirical data of option prices traded in the options markets. Figure 1 shows the volatility surface.

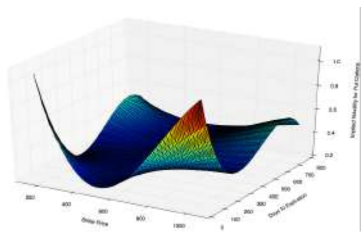

Figure 1. Implied volatility surface

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Should Black-Scholes model be the ground truth of the market, the volatility surface would be flat (which is not the case). Hence the curvature of this surface is an indicator of how much the Black-Scholes model fits to the market.

So far, one of the challenges of mathematical finance has been to build more sophisticated models that illustrate the same implied volatility as observed in real data.

One of the efforts in this direction has been made using the so called stochastic volatility models (SVM). These models assume that the volatility is not constant but instead is a stochastic process in itself. Hence $\sigma(t)$ also follows a stochastic differential equation alongside the price process $S(t)$. One of the most famous such models is the following due to Hull and White (1987).

$$\frac{\mathrm{d}S(t)}{\mathrm{S(t)}} = \mu(t, S(t))dt + \sigma(t)dw^1(t) \tag{1}$$

$$d(\ln\sigma(t)) = k(\theta - \ln\sigma(t))dt + \gamma dw^2(t) \tag{2}$$

where w^1 and w^2 are independent Wiener processes. A simple argument shows that when conditioned on the trajectory of $\sigma(t)$, the price at time t of a European option of exercise date T is indeed the Black-Scholes price where the constant volatility σ is replaced by its quadratic average over the period $\sigma_t^2 = \frac{1}{T-t}\int_t^T \sigma^2(u)du$. Hence the option price can be obtained by taking expectation of this Black-Scholes price.

Although the SVM models fit better to the volatility surface, they are still far from a good fit. In recent years a new family of models have been introduced which is a generalization of SVM models in the sense that they use the fractional Brownian motion as the noise in the volatility process.

1.1. A fractional Brownian motion (fBm) with Hurst parameter $0 \leq H \leq 1$ is a zero-mean Gaussian process $(W_t^H)_{t\in R}$ with the covariance

$$E\left[W_t^H W_s^H\right] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$$

The fractional stochastic volatility model (FSVM) is given by the following system:

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu dt + \sqrt{V_t}dW(t) \tag{3}$$

$$d(V_t) = \alpha V_t dt + \eta . V_t dW_t^H \tag{4}$$

The same argument as in the SVM models implies:

1.2 (see [2]). *The European call option price is given by*

$$C(t) = S(t)E_t^Q\left[\Phi\left(\frac{m_t}{U_t} + \frac{U_t}{2}\right)|F_t\right] - e^{-m_t}E_t^Q\left[\Phi\left(\frac{m_t}{U_t} - \frac{U_t}{2}\right)|F_t\right]$$

where

$$m_t = \ln\left(\frac{e^{-r(T-t)}S(t)}{K}\right), \quad U_t = \sqrt{\int_0^t \sigma^2(u)du}$$

and Φ is the standard Gaussian distribution function.

2. Main results

Our ultimate goal is to study the properties of the option prices in the fractional Hull-White model of the previous section. Theorem 1.2 shows that any information on the distribution of the variable $U(t)$ could be employed to obtain information on the distribution of the option price as well. Hence one can study the distributional properties of $U(t)$.

Article [4] does the same thing in the classical Hull White model and then uses it to study the asymptotic behaviour of the distribution density of the stock price process. Following the framework of [4] we define

$$\alpha_t = \int_0^t e^{\mu s + \sigma W_s^H} ds$$

And notice that by the time reversing property of the fBm we have,

$$\alpha_t \overset{d}{=} e^{\mu t + \sigma W_t^H} \int_0^t e^{-(\mu s + \sigma W_s^H)} ds := V_t$$

Now we want to use the Ito formula. The Ito formula for fBm exists for $0 < H < 1$ (except $H = \frac{1}{2}$ which is indeed the Bm itself)[see [1]]. By applying Ito formula we find,

$$dV_t = \big((\mu + Ht^{2H-1}\sigma^2)V_t + 1\big)\, dt + \sigma V_t dW_t^H$$

2.1. *V has the density function called $f(v,t)$.*

Proof. Suppose

$$G = \int_0^T e^{B_s^H} ds$$

It is enough to prove that G has a density function. So for this purpose, we prove $G \in \mathbb{D}^{1,2}$. And for this purpose, we prove[see [5]]

$$\mathbb{E}(\|DG\|_H^2) = \textstyle\sum_{n=1}^{\infty} nn!\|g_n\|^2_{L^2(\mathbb{R}_+^n)} < \infty.$$

That

$$g_n = \frac{1}{n!}\mathbb{E}(D^n G).$$

So we must find n order Malliavin derivative of G. According to[[6]]:

$$B_s^H = \textstyle\int_0^s K_H(s,u)\, dB_u,$$

That

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \textstyle\int_s^t \mid u - s \mid^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\, du,$$

$$c_H = [\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}]^{\frac{1}{2}},$$

According to Malliavin derivative

$$D_{t_1,\dots,t_n}(e^{B_s^H}) = e^{B_s^H} K_H(s,t_1)\dots K_H(s,t_n),$$

$$\Rightarrow D_{t_1,\dots,t_n}(\textstyle\int_0^T e^{B_s^H}\, ds) = \int_0^T e^{B_s^H} K_H(s,t_1)\dots K_H(s,t_n)\, ds,$$

$$\Rightarrow \mathbb{E}(D_{t_1,\dots,t_n}(\textstyle\int_0^T e^{B_s^H}\, ds)) = \int_0^T e^{\frac{1}{2}s^{2H}} K_H(s,t_1)\dots K_H(s,t_n)\, ds = n! g_n.$$

And we have

$$\int_s^t |u-s|^{H-\frac{3}{2}}\,du = \int_0^{t-s} u^{H-\frac{3}{2}}\,du = \frac{1}{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} < \frac{1}{H-\frac{1}{2}}T^{H-\frac{1}{2}},$$

$$\Rightarrow K_H(t,s) < c.s^{\frac{1}{2}-H},$$

That c is constant. so, as a result:

$$\mid n!g_n \mid \leqslant c^n.t_1^{\frac{1}{2}-H}...t_n^{\frac{1}{2}-H}.$$

and finally

$$\int_0^T ... \int_0^T \mid n!g_n \mid^2 = \int_0^T ... \int_0^T t_1^{1-2H}...t_n^{1-2H}\,dt = \big(\int_0^T t_1^{1-2H}\,dt_1\big)^n = \frac{1}{2-2H}T^{2-2H} < (c')^n.$$

So the proposition is proved. □

By using the fractional version of the Kolmogorovs forward equation we the following partial differential equation governing f:

$$\partial_t f = Ht^{2H-1}\sigma^2 v^2 \partial_v^2 f - \big((\mu - Ht^{2H-1}\sigma^2)v + 1\big)\partial_v f - \big(\mu - Ht^{2H-1}\sigma^2\big) f$$

$$f(t,0) = f(t,\infty) = 0, \quad f(0,x) = \delta_{x_0}$$

Questions of interest, regarding the function $f(t,x)$ are its asymptotic behaviour when $t \to 0, \infty$ and also $x \to 0, \infty$.

In this article we provide an asymptotic bound for f when $t \to \infty$ and show that under certain assumptions, it decays exponentially and obtain the rate.

2.2. *If for some $t_0 > 0$ and a positive constant M, we have $|f(t_0,v)| < M$, then*

$$f(t,v) \le Me^{\frac{1}{2}\sigma^2 t^{2H} - \mu t}$$

For proof we use the result of [3] on asymptotic of the solution of hyperbolic PDEs.

3. Conclusion

We relate the option price to a quadratic average of the exponential fractional Brownian motion and We prove the existence of the Implied Volatilities distribution density function by using malliavin calculus and we derive the asymptotics of the mentioned average as t tends to infinity.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Mean-square stability analysis of a stochastic Runge-Kutta scheme for stiff Itô SDE systems

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Abstract. In this paper, we investigate the mean-square stability analysis of a stochastic Runge-Kutta (SRK) schemes for stiff SDE systems of Itô types. In this class the schemes which are not fully implicit, while appropriate for stiff SDEs. For a subclass of these schemes with stochastic weak second order, the mean-square stability (MS-stability) is analysed.

Keywords: stiff stochastic differential equations, mean-square stability, stochastic Runge-Kutta.

AMS Mathematics Subject Classification [2010]: 65L04, 60H10 ,65C30 ,37H30.

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1. Introduction

Most of the problems in modeling of many chemical, physical and economic systems are formulated in the forms of stiff SDEs [2–4, 7]. The authors in [5] worked on weak second order SRK methods for Itô SDEs and calculated the coefficients of the schemes with minimized error constant, but these methods are not A-stable. Also, their SRK methods are developed as diagonally drift-implicit Runge-Kutta methods for Itô SDEs in [4] and the stability analysis of them are analysed but all of them are not A-stable and they may fail for stiff SDEs with high stiffness. So, in this paper, after introducing a general coefficient matrices of the SRK schemes, we obtain the Butcher table of a fully specified stochastic weak second order scheme with appropriate MS-stability properties when applied to the stiff SDEs.

2. Some definitions and preliminary results

Suppose $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$. We assume that the $(X_t)_{t\in\mathcal{I}}$ be solution of the Itô SDE

$$dX_t = a(X_t)dt + b(X_t)dW_t, \qquad X_{t_0} = x_0, t \in \mathcal{I}, \tag{1}$$

where $\mathcal{I} = [t_0, T]$ for some $0 \leq t_0 < T < \infty$ and $a : \mathbb{R}^d \to \mathbb{R}^d$ is the drift coefficient, $d \times m$-matrix function $b = (b^{i,j})$ such that $b : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is the diffusion coefficient and $\{W(t) = (W_t^1, ..., W_t^m)\}_{t\geq 0}$ is an m-dimensional Wiener process. Then we split $a(X_t)$

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into two parts f, g such that $a(X_t) \equiv f(X_t) + g(X_t)$. If we choose f and g as the stiff and nonstiff parts of the drift function a, respectively, then the numerical method will be implicit only in terms of the stiff part f and will be explicit with respect to the nonstiff part g, and the computational cost of the method will be reduced.

3. A class of stochastic weak second order SRK method

In this section, for the weak approximation of the solution of the Itô SDE (1), we consider a class of stochastic weak second order SRK schemes [1] which are more general than the class of SRK methods introduced by [4], as follows:

$$\begin{aligned} Y_{n+1} = Y_n \quad &+ \quad h\sum_{i=1}^{s}\alpha_i f(H_i^{(0)}) + h\sum_{i=1}^{s}\beta_i g(\widehat{H}_i^{(0)}) + \sum_{i=1}^{s}\sum_{k=1}^{m}\left(\gamma_i^{(1)}\hat{I}_{(k)} + \gamma_i^{(2)}\frac{\hat{I}_{(k,k)}}{\sqrt{h}}\right)b^k(H_i^{(k)}) \\ &+ \quad \sum_{i=1}^{s}\sum_{k=1}^{m}\left(\gamma_i^{(3)}\hat{I}_{(k)} + \gamma_i^{(4)}\sqrt{h}\right)b^k(\widehat{H}_i^{(k)}), \end{aligned} \tag{2}$$

for $n = 0, 1, ...N-1$ with the stage values,

$$\begin{aligned} H_i^{(0)} &= Y_n + h\sum_{j=1}^{s}A_{ij}^{(0)}f(H_j^{(0)}) + h\sum_{j=1}^{s}B_{ij}^{(0)}g(\widehat{H}_j^{(0)}) + \sum_{j=1}^{s}\sum_{l=1}^{m}\hat{I}_{(l)}C_{ij}^{(0)}b^l(H_j^{(l)}), \\ \widehat{H}_i^{(0)} &= Y_n + h\sum_{j=1}^{s}A_{ij}^{(1)}f(H_j^{(0)}) + h\sum_{j=1}^{s}B_{ij}^{(1)}g(\widehat{H}_j^{(0)}) + \sum_{j=1}^{s}\sum_{l=1}^{m}\hat{I}_{(l)}C_{ij}^{(1)}b^l(H_j^{(l)}), \\ H_i^{(k)} &= Y_n + h\sum_{j=1}^{s}A_{ij}^{(2)}f(H_j^{(0)}) + h\sum_{j=1}^{s}B_{ij}^{(2)}g(\widehat{H}_j^{(0)}) + \sqrt{h}\sum_{j=1}^{s}C_{ij}^{(2)}b^k(H_j^{(k)}), \\ \widehat{H}_i^{(k)} &= Y_n + h\sum_{j=1}^{s}A_{ij}^{(3)}f(H_j^{(0)}) + h\sum_{j=1}^{s}B_{ij}^{(3)}g(\widehat{H}_j^{(0)}) + \sum_{j=1}^{s}\sum_{\substack{l=1 \\ l\neq k}}^{m}\frac{\hat{I}_{(k,l)}}{\sqrt{h}}C_{ij}^{(3)}b^l(H_j^{(l)}). \end{aligned} \tag{3}$$

for $i = 1, \dots, s$ and $k = 1, \dots, m$. Also the random variables $\hat{I}_{(j)}$ and $\hat{I}_{(i,j)}$ are defined in [1, 6]. Thus, with the help of the B-series analysis, we can derive the stochastic weak second order conditions for the coefficients of the SRK schemes (2)-(3). For more details see [1, 4].

4. An efficient stochastic weak second order SRK scheme with appropriate asymptotic MS-stability properties

In this section to obtain a SRK scheme with suitable asymptotic MS-stability properties, we suppose that some diagonal elements of the matrices $A^{(0)}, C^{(0)}, C^{(1)}, C^{(2)}$ and $C^{(3)}$ are nonzero and in order to reduce the computational cost of the method, we take the coefficients as lower-triangular matrix. Therefore, the corresponding coefficient matrices can be given as follows:

$$A^{(i)} = \begin{bmatrix} A_{11}^{(i)} & 0 & 0 \\ A_{21}^{(i)} & A_{22}^{(i)} & 0 \\ A_{31}^{(i)} & A_{32}^{(i)} & A_{33}^{(i)} \end{bmatrix}, B^{(j)} = \begin{bmatrix} 0 & 0 & 0 \\ B_{21}^{(j)} & 0 & 0 \\ B_{31}^{(j)} & B_{32}^{(j)} & 0 \end{bmatrix}, C^{(k)} = \begin{bmatrix} 0 & 0 & 0 \\ C_{21}^{(k)} & C_{22}^{(k)} & 0 \\ C_{31}^{(k)} & C_{32}^{(k)} & C_{33}^{(k)} \end{bmatrix}, A^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ A_{21}^{(2)} & 0 & 0 \\ A_{31}^{(2)} & 0 & 0 \end{bmatrix},$$

in which $i = 0, 1, j = 0, 1, 2, k = 0, 1, 2, 3$. For more reduction in computational cost, we set $A^{(3)} = B^{(3)} = 0$. It should be mentioned that by these settings, the scheme is explicit w.r.t nonlinear part g, because the matrices $B^{(k)}, k = 1, 2, 3$ are considered as to be strictly lower triangular ones. Now, we consider the scalar SDE test,

$$dX_t = \lambda X_t dt + \mu X_t dW_t, \quad t \geq t_0 \geq 0, \quad X_{t_0} = x_0, \tag{4}$$

then we apply schemes (2)-(3) to SDE (4). We can obtain the stability function $\Pi_{SRK}(x,y)$ of SRK schemes as follows:

$\Pi_{SRK}(x,y) =$

$$
\begin{aligned}
&\Bigg(x\Bigg(\frac{\alpha_1}{1-A_{11}^{(0)}x}+\alpha_2\Bigg(\frac{A_{21}^{(0)}x}{1-A_{11}^{(0)}x}+1\Bigg)+\frac{\alpha_3}{\big(1-A_{33}^{(0)}x\big)}\Bigg(\frac{A_{31}^{(0)}x}{1-A_{11}^{(0)}x}+\frac{A_{32}^{(0)}x}{\big(1-A_{22}^{(0)}x\big)}\Bigg(\frac{A_{21}^{(0)}x}{1-A_{11}^{(0)}x}+1\Bigg)+1\Bigg)\Bigg)+1\Bigg)^2+\\
&\Bigg(x\Bigg(\frac{\alpha_2}{\big(1-A_{22}^{(0)}x\big)}\Bigg(\frac{C_{22}^{(0)}y}{\big(1-C_{22}^{(2)}y\big)}\Bigg(\frac{C_{21}^{(2)}y}{1-C_{11}^{(2)}y}+1+\frac{A_{21}^{(2)}x}{1-A_{11}^{(0)}x}\Bigg)+\frac{C_{21}^{(0)}y}{1-C_{11}^{(2)}y}\Bigg)+\\
&\frac{1}{1-A_{33}^{(0)}x}\Bigg(\alpha_3\Bigg(\frac{A_{32}^{(0)}x}{\big(1-A_{22}^{(0)}x\big)}\Bigg(\frac{C_{22}^{(0)}y}{\big(1-C_{22}^{(2)}y\big)}\Bigg(\frac{C_{21}^{(2)}y}{1-C_{11}^{(2)}y}+1+\frac{A_{21}^{(2)}x}{1-A_{11}^{(0)}x}\Bigg)+\frac{C_{21}^{(0)}y}{1-C_{11}^{(2)}y}\Bigg)+\\
&\frac{C_{32}^{(0)}y}{\big(1-C_{22}^{(2)}y\big)}\Bigg(\frac{C_{21}^{(2)}y}{1-C_{11}^{(2)}y}+1+\frac{A_{21}^{(2)}x}{1-A_{11}^{(0)}x}\Bigg)+\\
&\frac{C_{33}^{(0)}y}{\big(1-C_{33}^{(2)}y\big)}\Bigg(\frac{C_{32}^{(2)}y}{\big(1-C_{22}^{(2)}y\big)}\Bigg(\frac{C_{21}^{(2)}y}{1-C_{11}^{(2)}y}+1+\frac{A_{21}^{(2)}x}{1-A_{11}^{(0)}x}\Bigg)+\frac{C_{31}^{(2)}y}{1-C_{11}^{(2)}y}+1+\frac{A_{31}^{(2)}x}{1-A_{11}^{(0)}x}\Bigg)+\frac{C_{31}^{(0)}y}{1-C_{11}^{(2)}y}\Bigg)\Bigg)\Bigg)+\\
&+\frac{\gamma_1^{(2)}y}{\big(1-C_{22}^{(2)}y\big)}\Bigg(\frac{C_{21}^{(2)}y}{1-C_{11}^{(2)}y}+1+\frac{A_{21}^{(2)}x}{1-A_{11}^{(0)}x}\Bigg)+\\
&\frac{\gamma_1^{(3)}y}{\big(1-C_{33}^{(2)}y\big)}\Bigg(\frac{C_{32}^{(2)}y}{\big(1-C_{22}^{(2)}y\big)}\Bigg(\frac{C_{21}^{(2)}y}{1-C_{11}^{(2)}y}+1+\frac{A_{21}^{(2)}x}{1-A_{11}^{(0)}x}\Bigg)+\frac{C_{31}^{(2)}y}{1-C_{11}^{(2)}y}+1+\frac{A_{31}^{(2)}x}{1-A_{11}^{(0)}x}\Bigg)+\frac{\gamma_1^{(1)}y}{1-C_{11}^{(2)}y}\Bigg)^2+\\
&+\frac{1}{2}\Bigg(\frac{\gamma_2^{(2)}y}{\big(1-C_{22}^{(2)}y\big)}\Bigg(\frac{C_{21}^{(2)}y}{1-C_{11}^{(2)}y}+1+\frac{A_{21}^{(2)}x}{1-A_{11}^{(0)}x}\Bigg)+\\
&\frac{\gamma_2^{(3)}y}{\big(1-C_{33}^{(2)}y\big)}\Bigg(\frac{C_{32}^{(2)}y}{\big(1-C_{22}^{(2)}y\big)}\Bigg(\frac{C_{21}^{(2)}y}{1-C_{11}^{(2)}y}+1+\frac{A_{21}^{(2)}x}{1-A_{11}^{(0)}x}\Bigg)+\frac{C_{31}^{(2)}y}{1-C_{11}^{(2)}y}+1+\frac{A_{31}^{(2)}x}{1-A_{11}^{(0)}x}\Bigg)+\frac{\gamma_2^{(1)}y}{1-C_{11}^{(2)}y}\Bigg)^2,
\end{aligned}
$$

in which $x = h\lambda, y = \sqrt{h}\mu$. Now, we investigate the behavior of the above mentioned MS-stability function for large amounts of λ and μ, i.e., for $x \to -\infty, y \to \pm\infty$. So, to gain the effective SRK schemes, we must choose suitable parameters such that the obtained scheme becomes more suitable for stiff SDEs. Therefore, we choose the parameters such that the terms of $\Pi_{SRK}(x,y)$ vanish for large values of λ and μ. Therefore, for obtaining the coefficients of an effective scheme of stochastic weak and deterministic order 2 with appropriate MS-stability properties, we solve the nonlinear system of order conditions along with this stability condition, by MATLAB numerically. The corresponding method is denoted by SRK4. The Butcher array of the proposed scheme is then presented in Table 1. By using the parameters of this scheme and also for DDIRDI4 and DDIRDI5 methods [4] we have

$$\lim_{(x,y)\to(-\infty,\pm\infty)} \Pi_{SRK4}(x,y) = 0, \lim_{(x,y)\to(-\infty,\pm\infty)} \Pi_{DDIRDI4}(x,y) = \lim_{(x,y)\to(-\infty,\pm\infty)} \Pi_{DDIRDI5}(x,y) = \infty.$$

Therefore, the DDIRDI4 and DDIRDI5 methods may fail for stiff SDEs with high stiffness, while we can apply the proposed SRK4 scheme to stiff SDEs more appropriately. The MS-stability regions of the above mentioned methods and the analytical region of SDE are illustrated in Fig 1. It should be mentioned that as the region of MS-stability of DDIRDI4 is contained in MS-stability region of DDIRDI5, we do not plot its region. Fig 1 shows that the MS-stability region of the proposed SRK4 scheme is larger than those of DDIRDI4 and DDIRDI5. Clearly one should expect a better performance of the proposed SRK4 scheme, compared with DDIRDI4 and DDIRDI5 methods, when applied to the stiff SDEs.

0.5000	0	0	0	0	0	0	0	0			
1.0549	−7.8207	0	0.1481	0	0	0.4652	−.3692	0			
−1.6813	−1.4753	6.8786	0.5070	0.5070	0	0.3241e-1	−0.6970e-2	1.0117			
0.1764	0	0	0	0	0	0	0	0			
0.2265e-1	0.2265e-1	0	0.6978e-1	0	0	$-0.6311e-2$	0.5009e-2	0			
0.2705	0.2705	0.2705	0.5034	0.5034	0	0.9108e-1	−0.5443e-1	0.9635			
0	0	0	0	0	0	6.1780	0	0			
−0.2979	0	0	−0.2979	0	0	−0.1854	7.9690	0			
0.5064	0	0	0.5064	0	0	−0.3939	0.2501	−0.7551e-3			
0	0	0	0	0	0	0	0	0			
0	0	0	0	0	0	−1.6609	1.3183	0			
0	0	0	0	0	0	1.6610	−1.3183	0			
0.3315	0.2054	0.4631	0.54653	−0.4641e-1	0.4999	0.6089e-1	−0.3044e-1	0.9695	0.7525	−0.473963	−0.2785
						−4.2588	2.1294	2.1293	0.1662e-3	−1.4593	1.4592

TABLE 1. Butcher tableau for proposed SRK4 scheme

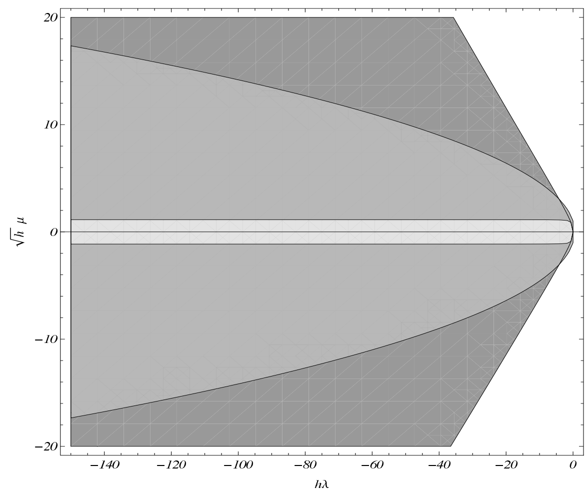

FIGURE 1. MS-stability regions: DDIRDI5 method (light-gray), proposed SRK4 scheme (dark-gray) and SDE (4) (gray: analytical region)

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دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

SOME RESULTS ON TORSION AND EXTENSION FUNCTORS

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Abstract. In this paper we generalize the Zero Divisor Conjecture and Rigidity Theorem for k-regular sequence. For this purpose for any k-regular M-sequence $x_1, ..., x_n$ we prove that if $\dim \operatorname{Tor}_2^R(\frac{R}{(x_1,...,x_n)}, M) \leq k$, then $\dim \operatorname{Tor}_i^R(\frac{R}{(x_1,...,x_n)}, M) \leq k$, for all $i \geq 1$. Also we show that if $\dim \operatorname{Ext}_R^{n+2}(\frac{R}{(x_1,...,x_n)}, M) \leq k$, then $\dim \operatorname{Ext}_R^i(\frac{R}{(x_1,...,x_n)}, M) \leq k$, for all integers $i \geq 0$ $(i \neq n)$.

Keywords: k-regular sequence, extension functor, zero divisor conjecture

AMS Mathematics Subject Classification [2010]: 13D45, 13D07

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1. Introduction

Throughout this paper, R denotes a commutative and Noetherian ring with non zero identity, I denotes an arbitrary ideal, M denotes a finitely generated R-module, and $k \geq -1$ an arbitrary integer. In 1961, M. Auslander proposed the zero divisor conjecture in [5] as follows.

Remark 1.1. (Zero divisor conjecture) Let R be a Noetherian local ring and M be a finitely generated R module of finite projective dimension over R. If $x \in R$ is a non-zerodivisor on M, then x is a non-zerodivisor of R.

This conjecture was proved by M. Hochster, L. Szpiro, C. Peskin, and P. Robert, in special cases. Also M. Auslander introduced rigidity concept as a generalization of Zero Divisor Conjecture.

Definition 1.2. Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. An R-module M is called rigid if $\operatorname{Tor}_i^R(M, N) = 0$ for some finitely generated R-module N, then $\operatorname{Tor}_j^R(M, N) = 0$ for any $j \geq i$ (see [5]).

He also stated the following theorem.

Theorem 1.3. *(Rigidity Theorem.) Let R be a Noetherian regular local ring and M be a finitely generated R-module. Then M is rigid.*

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The Rigidity Theorem was proved by M, Auslander in unramified case. S. Lichtenbaum proved the theorem for arbitrary regular local rings in 1966, see [6]. The notion of k-regular sequence was introduced by Chinh and Nhan [3] which is an extension of the well-known notion of filter regular sequence introduced by Schenzel, Trung and Cuong [4]. It is known that for $k = -1$ any k-regular M-sequence is an M-regular sequence. In this paper we generalize the Zero Divisor Conjecture and Rigidity Theorem for k-regular sequences.

2. Main results

THEOREM 2.1. *[1, Theorem $2 \cdot 3$] Let R be a Noetherian ring and $x_1, ..., x_n$ be a sequence of elements of R. The following statements are equivalent:*

(1) *$x_1, ..., x_n$ is a poor k-regular M-sequence;*
(2) *$\frac{x_1}{1}, ..., \frac{x_n}{1}$ is a poor regular $M_\mathfrak{p}$-sequence in $R_\mathfrak{p}$ for all $\mathfrak{p} \in (\operatorname{Supp} M)_{>k}$ and all $i = 1, ..., n$;*
(3) *$x_1^{m_1}, ..., x_n^{m_n}$ is a poor k-regular M-sequence for all $m_1, ..., m_n \in \mathbb{N}$.*

THEOREM 2.2. *Let R be a Noetherian (not necessary local) ring and M be a non zero finitely generated R-module. Let $n \geq 1$ be an integer and $x_1, ..., x_n$ be a poor k-regular M-sequence. If $\dim \operatorname{Tor}_2^R(\frac{R}{(x_1,...,x_n)}, M) \leq k$, then $\dim \operatorname{Tor}_i^R(\frac{R}{(x_1,...,x_n)}, M) \leq k$, for any $i \geq 1$.*

PROOF. To prove the assertion, we show that $\operatorname{Tor}_i^{R_\mathfrak{p}}(\frac{R_\mathfrak{p}}{(\frac{x_1}{1},...,\frac{x_n}{1})}, M_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in (\operatorname{Supp} M)_{>k}$ and all $i \geq 0$. Therefore, without loss of generality, we may assume that $(\operatorname{Supp} M)_{>k} \subseteq V(x_1, ..., x_n)$ and $\dim M > k$. The rest of proof will be done by induction. □

THEOREM 2.3. *Let R be a Noetherian (not necessary local) ring, M be a non zero finitely generated R-module, and I be an ideal of R with k-depth $(I, M) = n$. Let $x_1, ...x_n$ be a maximal k-regular M-sequence in I. If $\dim \operatorname{Ext}_R^{n+2}(\frac{R}{(x_1,...,x_n)}, M) \leq k$, then $x_1, ..., x_n \in I$ is a k-regular R-sequence.*

PROOF. It is proved by induction on n.

□

3. Conclusion

THEOREM 3.1. *Let R be a Noetherian (not necessary local) ring, M be a non zero finitely generated R-module, and I be an ideal of R with k-depth $(I, M) = n \geq 1$. Assume that $x_1, x_2, ..., x_n \in I$ is a maximal k-regular M-sequence. If $\dim \operatorname{Ext}_R^{n+2}(\frac{R}{(x_1,...,x_n)}, M) \leq k$, then $\dim\ \operatorname{Ext}_R^i(\frac{R}{(x_1,...,x_n)}, M) \leq k$, for all integers $i \geq 0$ ($i \neq n$).*

PROOF. Let $\mathfrak{p} \in (\operatorname{Spec} R)_{>k}$. By Theorem 2.3 and 2.1 $\frac{x_1}{1}, ..., \frac{x_n}{1}$ is a poor regular $R_\mathfrak{p}$-sequence. We show that $\operatorname{Ext}_{R_\mathfrak{p}}^i(\frac{R_\mathfrak{p}}{(\frac{x_1}{1},...,\frac{x_n}{1})}, M_\mathfrak{p}) = 0$ for all $i \geq n+1$. For this purpose, we may assume that $\mathfrak{p} \in V(x_1, ..., x_n)$. Since

$$\operatorname{pd}_{R_p}(\frac{R_p}{(\frac{x_1}{1}, ..., \frac{x_n}{1})}) = n,$$

clearly

$$\operatorname{Ext}_{R_\mathfrak{p}}^i(\frac{R_\mathfrak{p}}{(\frac{x_1}{1}, ..., \frac{x_n}{1})}, M_\mathfrak{p}) = 0$$

for all $i \geq n+1$, and so
$$\dim \operatorname{Ext}^i_R(\frac{R}{(x_1,...,x_n)}, M) \leq k.$$
for all $i \geq n+1$.

□

COROLLARY 3.2. *Let R be a Noetherian (not necessary local) ring, M be a non zero finitely generated R-module, and I be an ideal of R with k-depth $(I, M) = n \geq 1$, assume that $x_1, ..., x_n \in I$ is a maximal k-regular M-sequence. Then the following statements are equivalent:*

(1) *$x_1, ..., x_n$ is an k-regular R-sequence;*
(2) *$\dim \operatorname{Ext}^i_R(\frac{R}{(x_1,...,x_n)}, M) \leq k$ for all $i > n$;*
(3) *$\dim \operatorname{Ext}^{n+2}_R(\frac{R}{(x_1,...,x_n)}, M) \leq k$;*
(4) *$\dim \operatorname{Ext}^2_R(\frac{R}{(x_1,...,x_n)}, \operatorname{H}^n_{(x_1,...,x_n)}(M)) \leq k$;*
(5) *$\dim \operatorname{Ext}^i_R(\frac{R}{(x_1,...,x_n)}, \operatorname{H}^n_{(x_1,...,x_n)}(M)) \leq k$ for all integers $i \geq 1$.*

PROOF. This is an immediate consequence of [2, Corollary $2 \cdot 5$].

□

COROLLARY 3.3. *Let R be a Noetherian ring and M be a non zero finitely generated R-module and $x_1, x_2, ..., x_n$ be a poor k-regular M-sequence. Then* $\dim \operatorname{Tor}^R_i(\frac{R}{(x_1,x_2,...,x_n)}, M) \leq k$ *if and only if* $\dim \operatorname{Tor}^R_{n+i}(\frac{R}{(x_1,x_2,...,x_n)}, \operatorname{H}^n_{(x_1,...,x_n)}(M)) \leq k$*, for all $i \geq 0$.*

THEOREM 3.4. *Let R be a Noetherian ring and M be a non zero finitely generated R-module. Let $n \geq 1$ be an integer and $x_1, ..., x_n$ be a poor k-regular M-sequence. Then the following statements are equivalent:*

(1) *$\dim \operatorname{Tor}^R_i(\frac{R}{(x_1,...,x_n)}, M) \leq k$ for every $i \geq 1$;*
(2) *$\dim \operatorname{Tor}^R_2(\frac{R}{(x_1,...,x_n)}, M) \leq k$;*
(3) *$\dim \operatorname{Tor}^R_i(\frac{R}{(x_1,...,x_n)}, \operatorname{H}^n_{(x_1,...,x_n)}(M)) \leq k$ for all integers $i \geq n+1$;*
(4) *$\dim \operatorname{Tor}^R_{n+2}(\frac{R}{(x_1,...,x_n)}, \operatorname{H}^n_{(x_1,...,x_n)}(M)) \leq k$.*

PROOF. It is easy consequence Theorem 2.2. □

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A class of almost L*–Dunford–Pettis sets in Banach lattices

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Abstract. We introduce the concepts of L*–Dunford–Pettis and almost L*–Dunford–Pettis sets in Banach lattices. We obtain some characterizations of them with respect to some well known geometric properties of Banach spaces, such as, weak Dunford-Pettis property, strong relatively compact Dunford-Pettis property and almost Dunford-Pettis completely continuous operators on such Banach lattices.

Keywords: Dunford–Pettis set, relatively compact Dunford–Pettis property (DP$_{rc}$P), almost Dunford-Pettis set, strong DP$_{rc}$P

AMS Mathematics Subject Classification [2010]: 46A40, 47L20, 46B28

1. Introduction

A subset A of a Banach space X is called Dunford–Pettis (DP), if every weak null sequence (x_n^*) in X^* converges uniformly on A, that is

$$\lim_{n\to\infty} \sup_{a\in A} |\langle a, x_n^*\rangle| = 0.$$

Every relatively compact subset of E is DP. If every DP subset of a Banach space X is relatively compact, then X has the relatively compact DP property (abb. DP$_{rc}$P). For example, dual Banach spaces with the weak Radon-Nikodym property and Schur spaces (i.e., weak and norm convergence of sequences in X coincide) have the DP$_{rc}$P [3]. Also we recall that a Banach space X has the DP$_{rc}$P if and only if every DP and weakly null sequence (x_n) in X is norm null.
In this article, at first we define the concepts of L*–DP and almost L*–DP sets in Banach lattices. At first, we remember some definitions and terminologies from Banach lattice theory.
It is evident that if E is a Banach lattice, then its dual E^*, endowed with the dual norm and pointwise order, is also a Banach lattice. The norm $\|.\|$ of a Banach lattice E is order continuous if for each generalized net (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|.\|$, where the notation $x_\alpha \downarrow 0$ means that the net (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. B_E is the closed unit ball of E. The lattice operations in the Banach lattice E are weakly sequentially continuous if for every weakly null sequence

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(x_n) in E, $|x_n| \to 0$ for $\sigma(E, E^*)$. We refer the reader for undefined terminologies, to the classical references [4].

2. Main results

Following the introducing of the concept DP sets in Banach spaces, we define L*–DP and almost L*–DP sets in Banach lattices.

Definition 2.1. Let E be a Banach lattice. A norm bounded subset B of a Banach lattice E is said to be an L*–DP set if every weakly null and DP sequence (x_n) of E^* converges uniformly to zero on the set B, that is $\sup_{f\in B} |f(x_n)| \to 0$.

It is clear that every DP set set in X is L*–DP and every subset of an L*–DP set is the same.
Recall from [2], that a subset A of a Banach lattice E is called almost DP if every disjoint weakly null sequence (x_n^*) in E^* converges uniformly on A.

Definition 2.2. Let E be a Banach lattice. A norm bounded subset B of a Banach space E is said to be an almost L*–DP set if every weakly null and almost DP sequence (x_n^*) of E^* converges uniformly to zero on the set B, that is $\sup_{f\in B} |f(x_n)| \to 0$.

It is clear that every almost L*–DP in E is L*–DP and every subset of an almost L*–DP set is the same. The following theorem gives aditional properties of these sets.

Proposition 2.3. (a) *Absolutely closed convex hull of an almost L^*–DP set is an almost L^*–DP set,*
(b) *relatively weakly compact subsets of a Banach lattice are almost L^*–DP set.*

Note that the converse of assertion (b) in general, is false. In fact, the following theorem, shows that the closed unit ball of c_0 is an almost L*–DP set, but it is not relatively weakly compact.
A Banach lattice E has the strong DP$_{rc}$P if all almost DP subsets of E are relatively compact. It is clear that the strong DP$_{rc}$P implies the DP$_{rc}$P.
It is well known that every DP set is conditionally weakly compact. However, B_{ℓ_∞} is indeed almost DP and by Rosenthal's ℓ_1-theorem, B_{ℓ_∞} is not conditionally weakly compact.
By [4], an element e in a Banach lattice E is called a *weak unit* if $B_e = E$, where B_e is the band generated by e. For example, $C[0, 1]$ is a Banach lattice with the weak unit $u(t) = t$. Banach lattices $M[0, 1]$, of all signed Borel measures on $[0, 1]$ of bounded variation and $(\ell_\infty)^*$, do not have any weak unit.

Theorem 2.4. *[1] Let E be a Banach lattice such that E^* has a weak unit or E has order-continuous norm. Then every almost DP set A in E is conditionally weakly compact.*

Theorem 2.5. *Let E be a Banach lattice such that E^{**} has a weak unit or E^* has order-continuous norm. Then E^* has the strong $DP_{rc}P$ iff every bounded subset of E is an almost L^*–DP set.*

Definition 2.6. A bounded linear operator T from a Banach space X into a Banach space Y is L*–DP if $T(B_X)$ is an L*–DP set in Y. We denote this class of operators by L$^*DP(X, Y)$.

Definition 2.7. A bounded linear operator T from a Banach space X into a Banach lattice E is almost L*–DP if $T(B_X)$ is an almost L*–DP set. We denote this class of operators by $L_a^*DP(X, E)$.

It is easy to see that the operator space $L_a^*DP(X,E)$ is a norm-closed subspace of $L(X,E)$. A Banach space X has the DP$_{rc}$P if and only if for each Banach space Y, DP$cc(X,Y) = L(X,Y)$.

Theorem 2.8. *An operator T is almost L^*–DP if and only if its adjoint T^* is DP^{a}cc. Also each weakly compact operator is almost L^*–DP.*

A Banach lattice E has the weak DP property if every relatively weakly compact set in E is almost DP set.

Theorem 2.9. *If E^* has the weak DP property, then each almost L^*–DP set in E is a DP set.*

Proof. Let A be an almost L*–DP set in E, and let (x_n^*) be a weakly null sequence in E^*. Since E^* has the weak DP property, then (x_n^*) is almost DP and by hypothesis, it converges uniformly on A. Hence A is a DP set. □

Definition 2.10. A Banach lattice E into a Banach space E is almost L*–DP property, if all almost L*–DP sets in E are relatively weakly compact.

Corollary 2.11. *Dual Banach lattice E^* has the strong DP$_{rc}$P and E has the almost L^*–DP property, iff E is reflexive.*

Proof. If E is reflexive, then E^* is reflexive and so it has the strong DP$_{rc}$P. Clearly, E has the almost L*–DP property, since the closed unit ball B_E is relatively weakly compact.
For the converse, if E^* has the strong DP$_{rc}$P, then every weakly null and almost DP sequence in E^* is norm null; that is, B_E is an almost L*–DP set, and so it is relatively weakly compact, by the almost L\$ˆ*\$–DP property of E. Hence E is reflexive. □

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۸ تا ۱۱ شهریور ۱۴۰۰

Finite one-step plus unitary rings are commutative

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Abstract. As a generalization of one-step rings we define "one-step plus rings" as one-step rings such that all of their subrings (of the same structure or not) are commutative. In this paper we show that finite one-step unitary rings are commutative.

Keywords: One-step ring, Unitary ring, Division ring

AMS Mathematics Subject Classification [2010]: 16P10, 17C60.

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1. Introduction

An structure is called one-step if all of its proper substructures (of the same kind) are commutative. Aside commutative structures which are obvious examples of one-step structures, non-commutative structures have only been classified for groups, rings and semigroups. Note that Wedderburn's little theorem [4] implies that "Finite one-step skew fields are commutative". One should note that finite one-step structures are of great importance because every finite structure contains at least one one-step substructure, for in finite states we usually are confronted with a miniaml condition which is one-step [5]. As a generalization of one-step rings we define "one-step plus rings" as the class of one-step rings such that all of their sub-structures (of the same kind or not) are commutative. In this paper we show that finite one-step plus unitary rings are commutative. So first we define:

Definition 1.1. One-step plus rings are the class of all rings such all of their subrings (of the same kind or not) are commutative.

In this paper, we characterize the structure of finite one-step plus unitary rings and as our main result show that these rings exclusively are commutative. There is no full describtion of one-step rings. Real quaternions are an example of infinite one-step non-commutative rings of finite dimensional over its center. It is proved that if R is one-step non-commutative ring and finite dimensional module over its center then R is finite [9]. A result due to M. Ikeda [3] states that if N is the Jacobson radical of a one-step non-commutative ring R such that R/N is finite over its center, then R is a finite ring. The structure of nilpotent and non-nilpotent finite one-step non-commuatative rings are characterized in [5] [13, p. 753]. For notations and preliminaries we refer to [4]. In the

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*Speaker. Email address: arian@znu.ac.ir following example we show that no all one-step rings are one step rings. This show that this class of rings may deserve distict study:

EXAMPLE 1.2. Consider the ring $R = \{\begin{pmatrix} a & b \\ 0 & a^2 \end{pmatrix}; a, b \in GF(4)\}$. This ring is a one-step unitary ring but it has the following non-commutative subring (which is not the same kind since it has no unity element) $\{\begin{pmatrix} 0 & b \\ 0 & a^2 \end{pmatrix}; a, b \in GF(4)\}$. So there are one-step unitary rings in which some of their subrings (without unity) are not commutative. This show that R is an one-step ring but is not an one-step plus ring!

2. Main results

To prepare our main results first we remind some theorems and give some lemmas:

LEMMA 2.1. *[2] Let R be a finite unitary ring of order p^n, where p is a prime and n is a positive integer. If $n < 3$, then R is a commutative ring.*

LEMMA 2.2. *Let R be a ring and I an ideal of R such that $I \subseteq Z(R)$. If $\frac{R}{I}$ is a finite field, then R is a commutative ring.*

PROOF. Since $\frac{R}{I}$ is a finite field, there exists an element $a \in R$ such that $\frac{R}{I} = \{I, a + I, ..., a^k + I\}$. Let $x, y \in R$. There are $r, s \in I$ and two integers m, n such that $x = s + a^n$ and $y = r + a^m$. Since $I \subseteq Z(R)$, we have $xy = (s + a^n)(r + a^m) = (r + a^m)(s + a^n) = yx$. This show that R is commutative. □

Now, we give our main result but first note that, an additive subgroup S of a ring R is a subring if and only if S is closed under multiplication. Therefore all left or right ideals of R are considered as subrings of R.

THEOREM 2.3. *Finite one-step plus unitary rings are commutative.*

PROOF. The proof is by induction on the order of R or $|R|$. Let $|R| = p_1^{\alpha_1} ... p_k^{\alpha_k}$ be the regular decomposition of $|R|$ into distinct prime numbers p_i, where α_i are positive integers $(i = 1, \cdots, k)$. Then

$$R = R_1 \bigoplus R_2 \bigoplus ... \bigoplus R_k,$$

where each R_i is an ideal of order $p_i^{\alpha_i}$ [2]. If $k > 1$, then by assumption each R_i is a commutative ring and hence $R = R_1 \bigoplus R_2 \bigoplus ... \bigoplus R_k$ is commutative by its componentwise structure. So we may assume that $|R| = p^\alpha$, where p is a prime number and α is a positive integer. By Lemma 2.1 every unitary ring of order p^α with $\alpha < 3$ is commutative, so the base step in our induction process is valid. Now we have $|R| = p^\alpha$, where $\alpha \geq 3$ and we assume that the claim is true for all unitary rings of order p^γ, where $\gamma < \alpha$. Now first consider R is a simple ring. Then by Wedderburn-Artin Theorem and Wedderburn's little Theorem [4] $R \cong M_r(F)$, where F is a finite field and r is a positive integer. If $r > 1$, then the set of upper-triangular matrices in $M_r(F)$ is a non-commutative subring of R, which contradicts our main assumption, hence $r = 1$ and R is a field, as desired. Now we may suppose that R is not a simple ring and let I be a nontrivial minimal ideal of R. By induction hypothesis $\frac{R}{I}$ is a commutative ring and we have the following two cases:

Case 1. First suppose that R is a local ring. Let M be the maximal ideal of R. Clearly M as a subring of R is commutative but is not central, otherwise by Lemma 2.2, R is commutative. Let $x \in M \setminus Z(R)$, then if R is not commutative there is an element

$w \in R \setminus M = R^*$ such that $xw \neq wx$. Since $\frac{R}{I}$ is a commutative ring, we have $[R, R] \subseteq I$, so $0 \neq [w, x] \in I$ and the non-commutative subring generated by $\{x, w\}$ is not a proper subring, in other words $R = \langle x, w\rangle$. Since $\frac{R}{M}$ is a finite commutative simple ring it is a field and we may assume that $(\frac{R}{M})^* = \langle w + M\rangle$, where $o(w + M) = n = p^\beta - 1$, for we may consider $|\frac{R}{M}| = p^\beta$, where $\beta < \alpha$. To follow the proof we separating the argument in two subcases regarding $Z(R) \cap I \neq 0$ or $= 0$, where both lead to contradiction.

Case2. Suppose that R is not a local ring. Suppose that M and L are two non-central maximal ideals of R. Since M and L are commutative rings and $R = M + L$, we have $M \cap L \subseteq Z(R)$. If R is not commutative there are $y \in L \setminus M$ and $x \in M \setminus L$ such that $xy \neq yx$, so $R = \langle x, y\rangle$. Since $0 \neq xy - yx = [x, y] \in Z(R)$, the ideal $I = ann_R([x, y])$ is a two sided proper ideal, for $1 \notin I$. We have $0 = (yx)y - y(yx) = y(xy - yx)$, therefore $y \in I$ and by the same way $x \in I$. Therefore $R = \langle x, y\rangle \subseteq I$, which is a contradiction. [1] □

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Stability and Hopf bifurcation in a diffusive predator-prey system with Holling type II response

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Abstract. In this paper, stability and Hopf bifurcation in a diffusive predator-prey system are discussed. The interaction term is Holling type II. The local behavior is first discussed for the corresponding homogeneous system. Then, the diffusive system's linear stability is discussed around a homogeneous equilibrium state followed by bifurcations in the infinite-dimensional system. By choosing a proper bifurcation parameter, we prove that a Hopf bifurcation occurs in both the homogeneous and nonhomogeneous systems. We compute the normal form of this bifurcation up to the third order and obtain the direction of the Hopf bifurcation. Finally, we provide numerical simulations to illustrate our analytical conclusions.

Keywords: Stability, Hopf bifurcation, Reaction-diffusion system, Predator-prey model

AMS Mathematics Subject Classification [2010]: 35K57, 92D25, 70K50

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1. Introduction

One of the crucial topics in theoretical biology/ecology and applied mathematics is the dynamic relationship between predators and their prey due to its importance in population dynamics. Non-linear differential systems are usually used to describe predator-prey interactions. The interaction between the two population can be divided into several types, as Holling I-III types, Crowley-Martin Type, Bedington-Deangelis type, and Hasell-Varley type. In this paper, we consider a well-motivated model in population dynamics. The model is a reaction-diffusion system with a Holling type II functional response in the reaction term, motivated by applications in spatial ecology. From the biological point of view, it is more interesting to study the dynamical behavior of the positive equilibrium points. The main result is to show that there is a Hopf bifurcation at an internal steady state. We construct a center manifold to study the bifurcation and establish that the Hopf bifurcation is generic. The periodic solution arising in the Hopf bifurcation corresponds to an oscillating biological interaction between two species, prey and predator, in an isolated environment. From a biological/ecological perspective, the stable periodic solution keeps the population densities in balance. When the direction of the Hopf bifurcation is supercritical, the bifurcating periodic solution is stable, and when the direction of the Hopf bifurcation is subcritical, the bifurcating periodic solution is unstable. The biological

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motivation for studying the Hopf bifurcation in the model is to maintain balance in the ecosystem. Almost all mathematical models constructed for predator-prey systems have either a stable equilibrium point or a stable limit cycle. The observable oscillations in biological systems are usually modeled by the limit cycles of the corresponding mathematical model. The basic predator-prey model with logistic growth in the prey and a Holling type II response function is given by

$$\begin{cases} \frac{dX}{dt} = rX(1 - \frac{X}{N}) - \frac{\alpha XY}{1+t_h\alpha X}, \\ \frac{dY}{dt} = -sY + \frac{c\alpha XY}{1+t_h\alpha X}, \end{cases} \tag{1}$$

where $X(t)$ denotes the prey and $Y(t)$ the predator. The parameter r is the prey's growth rate, N is its carrying capacity, and s is the predator's natural death rate in the absence of prey. The parameter α is the search efficiency of Y for X, c is the consumption rate, and t_h is the average handling time for each prey. Upon rescaling the variables and the parameters of (1) as

$$x = \frac{X}{N},\ y = \frac{\alpha}{r}Y,\ t = \frac{\tau}{r},\ \bar{s} = \frac{s}{r},\ \bar{\alpha} = t_h\alpha N,\ \bar{c} = c\alpha\frac{N}{r},$$

it becomes

$$\begin{cases} x' = \frac{dx}{d\tau} = x(1-x) - \frac{xy}{1+\bar{\alpha}x}, \\ y' = \frac{dy}{d\tau} = -\bar{s}y + \bar{c}\frac{xy}{1+\bar{\alpha}x}. \end{cases} \tag{2}$$

To study the impact of spatial diffusion, we consider, in this paper, the following system

$$\begin{cases} \frac{\partial u}{\partial t} = d_1\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + u(1-u) - \frac{uv}{1+\alpha u}, \\ \frac{\partial v}{\partial t} = d_2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - v(1 - \rho\frac{u}{1+\alpha u}), \end{cases} \tag{3}$$

in which d_1, d_2, ρ, α are positive constants, and $W = (u, v)$ is a function of $(x, y, t) \in (0, 1) \times (0, 1) \times \mathbb{R}$, together with the Neumann boundary conditions

$$\frac{\partial W}{\partial x}(0, y, t) = \frac{\partial W}{\partial x}(1, y, t) = \frac{\partial W}{\partial y}(x, 0, t) = \frac{\partial W}{\partial y}(x, 1, t) = (0, 0). \tag{4}$$

2. Main results

Our main results are stated in the following theorems.

2.1. *Consider system*

$$\begin{cases} \dot{u} = u(1-u) - \frac{uv}{1+\alpha u}, \\ \dot{v} = -v(1 - \rho\frac{u}{1+\alpha u}), \end{cases} \tag{5}$$

on the first quadrant, with the positive parameters ρ, α. Then

(i) *The boundary equilibrium point at the origin $(u, v) = (0, 0)$ is always a hyperbolic saddle.*

(ii) *The boundary equilibrium point at $(u, v) = (1, 0)$ is a stable node when $\frac{\rho}{\alpha+1} < 1$, a hyperbolic saddle when $\frac{\rho}{\alpha+1} > 1$, and a saddle-node point when $\frac{\rho}{\alpha+1} = 1$.*

(iii) *When $\rho > \alpha + 1$, then there exists a nontrivial (positive) equilibrium point at $(u, v) = (\bar{u}, \bar{v})$, with*

$$\bar{u} = \frac{1}{\rho - \alpha},\ \bar{v} = \frac{\rho(\rho - (\alpha + 1))}{(\rho - \alpha)^2}.$$

(iv) *If* $\rho > \frac{\alpha(\alpha+1)}{\alpha-1}$*, then the positive equilibrium point* $(\bar{u}, \bar{v})$ *is an unstable strong focus, and in this case, there exists an asymptotically stable limit cycle. If* $1+\alpha < \rho < \frac{\alpha(\alpha+1)}{\alpha-1}$*, then the previous limit cycle disappears and the fixed point* $(\bar{u}, \bar{v})$ *becomes a stable strong focus.*

(v) *When* $\rho = \frac{\alpha(\alpha+1)}{\alpha-1}$*, the positive equilibrium point* $(\bar{u}, \bar{v})$ *is a stable weak focus of order one.*

(vi) *The point* $\rho = \frac{\alpha(\alpha+1)}{\alpha-1}$*, with* $\alpha > 1$*, is the Hopf bifurcation value for system* (5)*.*

2.2. *Consider system* (3)*, with the positive parameters* d_1, d_2, ρ, α*. Then, the following holds:*

(i) *When* $\rho > \alpha+1$*, then there exists a nontrivial steady-state solution* $(u, v) = (\bar{u}, \bar{v})$*, with*

$$\bar{u} = \frac{1}{\rho - \alpha}, \quad \bar{v} = \frac{\rho(\rho - (\alpha+1))}{(\rho-\alpha)^2}.$$

(ii) *The point* $\rho = \frac{\alpha(\alpha+1)}{\alpha-1}$*, with* $\alpha > 1$*, is the Hopf bifurcation value for system* (3)*.*

(iii) *When* $\rho > \frac{\alpha(\alpha+1)}{\alpha-1}$*, then there exists a unique periodic solution which is stable, and the nontrivial steady-state solution* $(u, v) = (\bar{u}, \bar{v})$ *is unstable.*

(iv) *When* $1+\alpha < \rho \leq \frac{\alpha(\alpha+1)}{\alpha-1}$*, then the steady-state solution* $(u, v) = (\bar{u}, \bar{v})$ *is stable.*

3. Numerical results

In this section, we provide numerical simulations to validate the main results given in Theorems 2.1 and 2.2.

3.1. Let us consider system (5) with $\alpha = 2$ and $\rho = 8$. Then, its phase portrait on the first quadrant is as in Figure 1. In this figure, the positive equilibrium point $E^* = (\frac{1}{6}, \frac{10}{9})$ is an unstable strong focus, and there exists a stable limit cycle around it arising from the Hopf bifurcation.

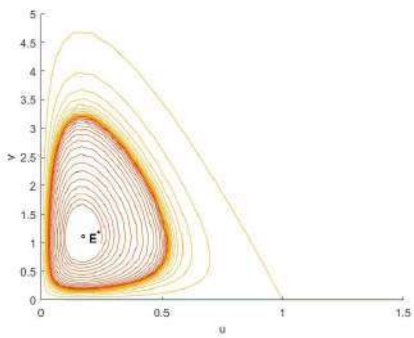


Figure 1. The phase portrait of system (5) in the first quadrant for $(\alpha, \rho) = (2, 8)$.

3.2. Let us consider system (5) with $\alpha = 2$, and $\rho = 6$. Then, its phase portrait on the first quadrant is as in Figure 2. In this figure, the positive equilibrium point $E^* = (\frac{1}{4}, \frac{9}{8})$ is a stable weak focus of order one.

3.3. Let us consider system (3) with $d_1 = d_2 = 1$, $\alpha = 2$ and $\rho = 8$. Then, the solution of system (3) with the initial values $(u_0, v_0) = (0.17, 1)$ is shown in Figure 3 for $0 \leq t \leq 1500$. In this figure, the steady state $(u, v) = (\frac{1}{6}, \frac{10}{9})$ is unstable. Moreover, there exists a stable periodic solution emerging from the Hopf bifurcation.

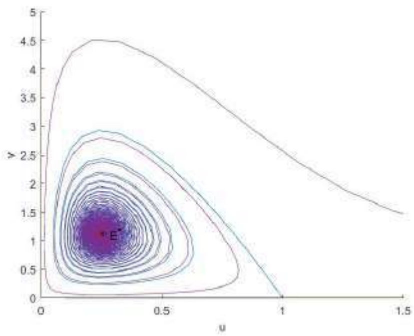
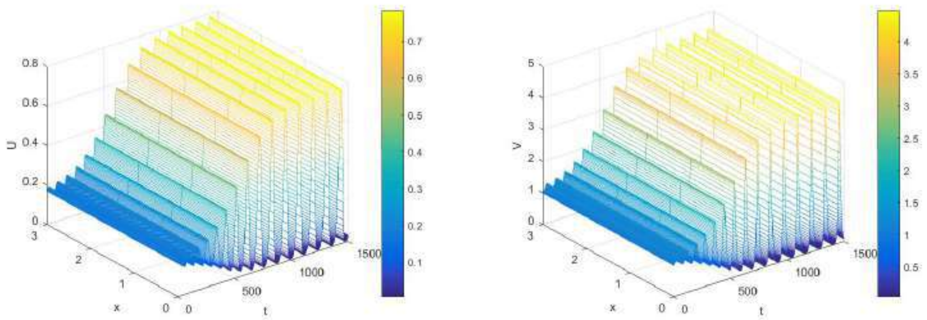


Figure 2. The phase portrait of system (5) in the first quadrant for $(\alpha, \rho) = (2, 6)$.



(a) Plots of $(t, x, u(x, t))$ for $0 \leq x \leq 3$ and $0 \leq t \leq 1500$

(b) Plots of $(t, x, v(x, t))$ for $0 \leq x \leq 3$ and $0 \leq t \leq 1500$

Figure 3. The graph of the functions $u = u(x, t)$ and $v = v(x, t)$ as a solution of (3) for $(d_1, d_2, \alpha, \rho) = (1, 1, 2, 8)$, and with the initial values $(t_0, x_0, u_0, v_0) = (0, 0, 0.17, 1)$.

3.4. The periodic solution of system (3) with the initial values

$$(u_0, v_0) = (0.664764172873410, 0.140596846060007)$$

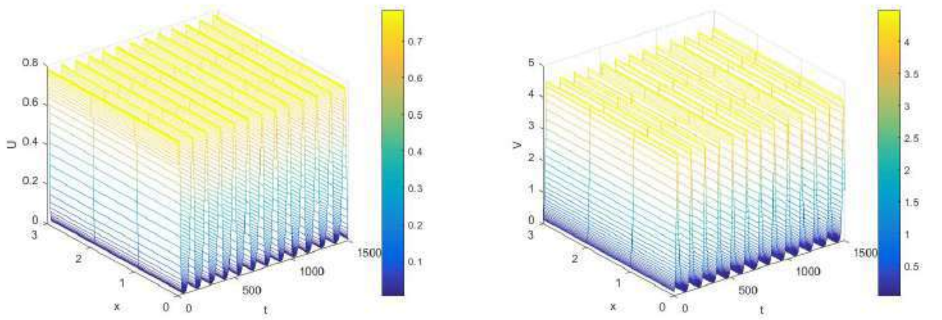
is shown in Figure 4 for $0 \leq t \leq 1500$.

4. Conclusion

In this paper, we considered a spatially extended prey-predator model with Holling type II functional response and investigated the local behaviors of spatially homogeneous steady states. By employing the linearization technique, the existence of Hopf bifurcation for both the non-spatial and spatial systems has been established. Further, we determined the stability and direction of the Hopf-bifurcating periodic solutions by using the center manifold and normal form theories. Finally, some numerical simulations have been provided to corroborate the obtained theoretical results.

Acknowledgement

Isfahan University of Technology supports this paper.



(a) Plots of $(t, x, u(x,t))$ for $0 \le x \le 3$ and $0 \le t \le 1500$

(b) Plots of $(t, x, v(x,t))$ for $0 \le x \le 3$ and $0 \le t \le 1500$

Figure 4. The graph of the functions $u = u(x,t)$ and $v = v(x,t)$ as a solution of (3) for $(d_1, d_2, \alpha, \rho) = (1, 1, 2, 8)$, and with the initial values $(t_0, x_0, u_0, v_0) = (0, 0, 0.664764172873410, 0.140596846060007)$.

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52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

On symmetries of the Riemannian manifold $\mathbb{H}^2 \times \mathbb{R}$

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Abstract. In this article, we study the existence of Killing and affine vector fields on the Riemannian manifold $\mathbb{H}^2 \times \mathbb{R}$. We also classify the matter collineations of this manifold.

Keywords: $\mathbb{H}^2 \times \mathbb{R}$ Riemannian metric, Affine vector field, Matter collineation

AMS Mathematics Subject Classification [2010]: 58D17, 53B20

1. Introduction

The study of symmetries in general relativity has long been considered due to they are interesting both from the mathematical and the physical point of view (see for example [7]). The term "symmetry" here refers to a one-parameter group of diffeomorphisms of Riemannian manifold (M, g), which leaves a special mathematical or physical quantity invariant. This statement is equivalent to the Lie derivative of the geometry quantity under the vector field X vanishes, i.e., one has the field equation $\mathcal{L}_X T = 0$. If T has geometrical or physical significance, then those special vector fields under which T is invariant will also be of significance. Isometries are a well known example of symmetries, for which $T = g$ is the metric tensor, the corresponding vector field X is then a Killing vector field. Homotheties and conformal motions on (M, g) are also examples of symmetries. A vector field X on (M, g) preserving Levi-civita connection ∇ is known as an affine vector field. It is obvious that if X preserves g, then it also preserves ∇, but the opposite is not always ture. Recently, other types of symmetries including curvature collineations ($T = \mathcal{R}$ being the curvature tensor), Ricci collineations ($T = \rho$ being the Ricci tensor) and etc., have been studied. Some examples may be found in [2, 3].

A matter collineation of a Riemannian manifold (M, g) is a vector field X, corresponding to a symmetry of the energy-momentum tensor $T = \rho - \frac{\tau}{2} g$, where τ displays the scalar curvature. Since the Ricci tensor is constructed from the connection of the metric tensor, Ricci collineations have geometrical importance [8]. However, matter collineations are more related to a physical viewpoint [4, 5]. These physical and geometric concepts give a single meaning in a particular case, for example, when the meter tensor has a zero scalar curvature.

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Due to the physical significance of symmetries, they have been investigated in several classes of Lorentziam manifold. Furthermore, the three-dimensional case has also been considerd as an interesting source of examples and different behaviours. Some examples may be found in [1, 6] and references therein.

Our goal in this article is to present a whole classification of Killing and affine vector fields of the Riemannian manifold $\mathbb{H}^2 \times \mathbb{R}$. We also classify the matter collineation of this manifold. All calculations have also been checked using $Maple16^{©}$.

2. Connection and curvature of the Riemannian manifold $\mathbb{H}^2 \times \mathbb{R}$

Assume $\mathbb{H}^2$ be expressed by the upper half-plane model $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the metric $g_{\mathbb{H}^2} = \frac{1}{y^2}(dx^2 + dy^2)$. Therefore, the left invariant product metric on the Riemannian manifold $\mathbb{H}^2 \times \mathbb{R}$ is given by

$$g = \frac{1}{y^2}(dx^2 + dy^2) + dz^2. \tag{1}$$

We will denote by ∇ the Levi-civita connection of $(\mathbb{H}^2 \times \mathbb{R}, g)$, by $\mathcal{R}$ its curvature tensor, taken with the sign contract $\mathcal{R}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and by ρ the Ricci tensor of $(\mathbb{H}^2 \times \mathbb{R}, g)$, which is defined by $\rho(X, Y) = \mathrm{tr}\{Z \mapsto \mathcal{R}(Z, X)Y\}$. The Ricci operator Ric is given by $\rho(X, Y) = g(Ric(X), Y)$ and the scalar curvature $\tau = \mathrm{tr}_g\, \rho$ is the metric trace of the Ricci tensor.

The non-zero components of the Levi-civita connection ∇ of the Riemannian manifold $\mathbb{H}^2 \times \mathbb{R}$ are given by

$$\nabla_{\partial_x}\partial_x = \tfrac{1}{y}\partial_y, \quad \nabla_{\partial_x}\partial_y = -\tfrac{1}{y}\partial_x, \quad \nabla_{\partial_y}\partial_y = -\tfrac{1}{y}\partial_y. \tag{2}$$

The non-zero component of the curvature tensor $\mathcal{R}$ is given by

$$\mathcal{R}(\partial_x, \partial_y)\partial_y = -\frac{1}{y^2}\partial_x,$$

and the non-zero components of the Ricci tensor are $\rho_{11} = \rho_{22} = -\frac{1}{y^2}$.

3. Killing and affine vector fields of the Riemannian manifold $\mathbb{H}^2 \times \mathbb{R}$

In this section, we want to classify the Killing and affine vector fields of $(\mathbb{H}^2 \times \mathbb{R})$. In general, we have the following theorem.

THEOREM 3.1. Assume $X = X^1\partial_x + X^2\partial_y + X^3\partial_z$ be an arbitrary vector field on the Riemannian manifold $(\mathbb{H}^2 \times \mathbb{R})$. Then

(i) X is a Killing vector field if and only if

$$X^1 = \frac{1}{2}c_1(x^2 - y^2) + c_2x + c_3, \; X^2 = (c_1x + c_2)y, \; X^3 = c_4,$$

(ii) X is an affine, non-Killing vector field if and only if

$$X^1 = c_1(x^2 - y^2) + c_2x + c_3, \; X^2 = (-2c_1x + c_2)y, \; X^3 = c_4z + c_5,$$

In the above expressions, c_i is arbitrary real number, for any indices i.

PROOF. A straightforward computations displays that Lie derivative of g is given by

$$\mathcal{L}_X g = \frac{2}{y^3}(y\partial_x X^1 - X^2)dxdx + \frac{2}{y^2}(\partial_x X^2 + \partial_y X^1)dxdy$$

$$
\begin{aligned}
&+ \frac{2}{y^2}(y^2\partial_x X^3 + \partial_z X^1)dxdz + \frac{2}{y^3}(y\partial_y X^2 - X^2)dydy \\
&+ \frac{2}{y^2}(y^2\partial_y X^3 + \partial_z X^2)dydz + 2\partial_z X^3 dzdz.
\end{aligned}
$$

By seeting all the cofficients in the $\mathcal{L}_X g$ equivalent to zero and solving the system of PDEs, the killing vector fields is obtained which gives the case (i).

To specify the affine vector fields, we require to compute the Lie derivative of the Levi-civita connection ∇. By (2) we prove that the possibly non-zero components of the $\mathcal{L}_X\nabla$ are assumed with

$$
\begin{aligned}
\mathcal{L}_X\nabla(\partial_x, \partial_x) &= -\frac{1}{y}(2\partial_x X^2 - y\partial^2_{xx}X^1 + \partial_y X^1)\partial_x + \frac{1}{y^2}(2y\partial_x X^1 + y^2\partial^2_{xx}X^2 - y\partial_y X^2 - X^2)\partial_y \\
&+ \frac{1}{y}(y\partial^2_{xx}X^3 - \partial_y X^3)\partial_z, \\
\mathcal{L}_X\nabla(\partial_x, \partial_y) &= \frac{1}{y^2}(-y\partial_y X^2 + y^2\partial^2_{yx}X^1 + X^2)\partial_x + \frac{1}{y}(\partial_y X^1 + y\partial^2_{yx}X^2)\partial_y \\
&+ \frac{1}{y}(\partial_x X^3 + y\partial^2_{yx}X^3)\partial_z, \\
\mathcal{L}_X\nabla(\partial_x, \partial_z) &= -\frac{1}{y}(\partial_z X^2 - y\partial^2_{yx}X^1)\partial_x + \frac{1}{y}(\partial_z X^1 + y\partial^2_{zx}X^2)\partial_y + \partial^2_{zx}X^3\partial_z, \\
\mathcal{L}_X\nabla(\partial_y, \partial_y) &= \frac{1}{y}(-\partial_y X^1 + y\partial^2_{yy}X^1)\partial_x + \frac{1}{y^2}(-y\partial_y X^2 + y^2\partial^2_{yy}X^2 + X^2)\partial_y \\
&+ \frac{1}{y}(y\partial^2_{yy}X^3 + \partial_y X^3)\partial_z, \\
\mathcal{L}_X\nabla(\partial_y, \partial_z) &= \frac{1}{y}(-\partial_z X^1 + y\partial^2_{zy}X^1)\partial_x - \frac{1}{y}(\partial_y X^2 - y\partial^2_{zy}X^2)\partial_y + \partial^2_{zy}X^3\partial_z, \\
&+ \frac{1}{y}(y\partial^2_{yy}X^3 + \partial_y X^3)\partial_z, \\
\mathcal{L}_X\nabla(\partial_z, \partial_z) &= \partial^2_{zz}X^1\partial_x + \partial^2_{zz}X^2\partial_y + \partial^2_{zz}X^3\partial_z.
\end{aligned}
$$

Affine vector fields are specified by solving the system of PDEs, obtained from the vanishing of the cofficients of the $\mathcal{L}_X\nabla$. This yelds the case (ii) and ends the proof. □

4. Matter collineations

We formerly remembered that an arbitrary vector field X on a Riemannian manifold (M, g) is nameed a matter collineation when $T = \rho - \frac{\tau}{2}g$. Now, We classify matter collineations on Riemannian manifold $(\mathbb{H}^2 \times \mathbb{R})$.

Theorem 4.1. Assume $X = X^1\partial_x + X^2\partial_y + X^3\partial_z$ be an arbitrary smooth vector field on the Riemannian manifold $(\mathbb{H}^2 \times \mathbb{R}, g)$. Then, X is a matter collineation if and only if X^1, X^2 are arbitrary and $X^3 = c$, where c is a real constant.

Proof. A straightforward computation displays that only the non-zero component of the tensor field T is $T(\partial_z, \partial_z) = 1$. Now, we compute the Lie derivative of the tensor field T. We have

$$\mathcal{L}_X T = 2\partial_x X^3 dxdy + 2\partial_y X^3 dydz + 2\partial_z X^3 dzdz.$$

Requiring that $\mathcal{L}_X T = 0$. So, we attain the system of PDEs, which solutions specify the matter collineations of $(\mathbb{H}^2 \times \mathbb{R}, g)$. Thus, X is a matter collineation if and only if X^1, X^2 are arbitrary and X^3 is a real constant and this completes the proof. $\square$

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
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A numerical scheme to solve the nonlinear time-fractional stochastic beam equation

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Abstract. In this paper, we employ a spectral collocation method based on Legendre polynomials (LPs) to solve the nonlinear time-fractional stochastic beam equation (NTFSBE). This method is applied to convert the solution of NTFSBE to the solution of a nonlinear system of algebraic equations. The numerical approach is completely described. Finally, a test example is implemented to validate the robustness of the proposed scheme.

Keywords: Fractional calculus, Stochastic beam equation, Legendre collocation scheme

AMS Mathematics Subject Classification [2010]: 35R60, 60H35, 35R11

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1. Introduction

There are many phenomena in physics, chemistry, and engineering that appear randomly and are explained by stochastic processes [1–3]. Stochastic behaviour arises naturally in many different phenomena where the effects of random "noise" perturbations to a system are being considered. For this reason, in recent years, the theory of stochastic partial differential equations has attracted more attention of scholars.

In this paper, we focus on the following NTFSBE

$$\rho\partial_t^2 u(x,t)+\partial_t^\alpha u(x,t)+\partial_x^4 u(x,t)-\mu\partial_x^2 u(x,t)=\mathbf{F}(x,t,u)\dot{\mathrm{B}}(t)+\mathtt{g}(x,t), \quad \text{in} \quad \Omega\times(0,\mathtt{T}), \tag{1}$$

with the initial and boundary conditions

$$u(x,t)=0, \qquad \text{in} \quad \partial\Omega\times(0,\mathtt{T}), \tag{2}$$

$$u(x,0)=\zeta_0(x), \qquad \text{in} \quad \Omega, \tag{3}$$

where $\rho,\mu\in\mathbb{R}^+$, $\dot{\mathrm{B}}(t):=\frac{\mathrm{dB}(t)}{\mathrm{d}t}$ denotes a time white noise and $\mathbf{F}\in \mathrm{C}^1(\Omega\times(0,\mathtt{T})\times\mathbb{R})$ satisfies the Lipschitz condition with respect to u. Moreover, $\mathtt{L},\mathtt{T}\in\mathbb{R}^+$, $\Omega:=[0,\mathtt{L}]$, $\zeta_0(x)$ is the continuous function and the operator $\partial_t^\alpha[\cdot]$ denotes Caputo fractional derivative of order $\alpha\in(0,1)$ [4],

$$\partial_t^\alpha u(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial u}{\partial s}(x,s)\mathrm{d}s, \tag{4}$$

where $\Gamma(\cdot)$ shows the Gamma function.

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2. The Shifted LPs and Their Properties

Definition 2.1. ([5]) The shifted LPs on $[a,b]$ are defined by explicit analytic form

$$\theta_i^{a,b}(t)=\sum_{r=0}^{[\frac{i}{2}]}\sum_{m=0}^{i-2r}\sum_{j=0}^{m}\varpi_{r,m,j}^{a,b}\ t^j, \tag{5}$$

where

$$\varpi_{r,m,j}^{a,b}=\frac{(-1)^{r-j+m}2^{m-i}b^{m-j}(2i-2r)!}{(b-a)^m(i-r)!r!(i-2r-m)!j!(m-j)!}.$$

Theorem 2.2. *([5]) Suppose that $\tilde{\Omega}:=[0,\mathtt{L}]\times[0,\mathtt{T}]$, $C^{n+1,m+1}(\tilde{\Omega})$ is the space of functions with continuous partial derivatives and let $f(x,t)\in C^{n+1,m+1}(\tilde{\Omega})$ satisfies the conditions*

$$\max_{(x,t)\in\tilde{\Omega}}\left|\frac{\partial^{n+1}f}{\partial x^{n+1}}(x,t)\right|\leq\beta_1,\ \max_{(x,t)\in\tilde{\Omega}}\left|\frac{\partial^{m+1}f}{\partial t^{m+1}}(x,t)\right|\leq\beta_2,\ \max_{(x,t)\in\tilde{\Omega}}\left|\frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial t^{m+1}}(x,t)\right|\leq\beta_3,$$

where β_1, β_2 and β_3 are positive constants. Let $f_{n,m}(x,t)$ is an approximation of $f(x,t)$ defined by $f_{n,m}(x,t)=\sum_{i=0}^{n}\sum_{j=0}^{m}c_{i,j}\theta_i^{0,\mathtt{L}}(x)\theta_j^{0,\mathtt{T}}(t)$, then

$$\|f-f_{n,m}\|_2\leq\frac{\beta_1\mathtt{L}^{n+1}}{(n+1)!2^{2n+1}}+\frac{\beta_2\mathtt{T}^{m+1}}{(m+1)!2^{2m+1}}+\frac{\beta_3\mathtt{L}^{n+1}\mathtt{T}^{m+1}}{(n+1)!(m+1)!2^{2n+2m+2}}.$$

3. Description of the Collocation Approach

To find a numerical solution of Eq.(1), assume

$$u_{n,m}(x,t)=\sum_{i=0}^{n}\sum_{j=0}^{m}c_{i,j}\theta_i^{0,\mathtt{L}}(x)\theta_j^{0,\mathtt{T}}(t)=\Theta(x)^{\mathrm{T}}\mathbf{C}\tilde{\Theta}(t), \tag{6}$$

where $\Theta(x)=[\theta_0^{0,\mathtt{L}}(x),\cdots,\theta_n^{0,\mathtt{L}}(x)]^{\mathrm{T}}$ and $\tilde{\Theta}(t)=[\theta_0^{0,\mathtt{T}}(t),\cdots,\theta_m^{0,\mathtt{T}}(t)]^{\mathrm{T}}$. Also

$$\mathbf{C}:=\Big[c_{i,j}\Big]_{(n+1)\times(m+1)},\qquad i=0,1,...,n,\ j=0,1,...,m,$$

is an $(n+1)\times(m+1)$ unknown coefficients matrix that must be determined. According to Eqs. (1) and (6), we have

$$\begin{aligned}\mathbf{R}(x,t)&\triangleq\rho\Theta(x)^{\mathrm{T}}\mathbf{C}\tilde{\Psi}^2(t)+\Theta(x)^{\mathrm{T}}\mathbf{C}\tilde{\Psi}^{\alpha}(t)+\Psi^4(x)^{\mathrm{T}}\mathbf{C}\tilde{\Theta}(t)\\&-\mu\Psi^2(x)^{\mathrm{T}}\mathbf{C}\tilde{\Theta}(t)-\mathbf{F}(x,t,\Theta(x)^{\mathrm{T}}\mathbf{C}\tilde{\Theta}(t))\frac{\mathrm{dB}(t)}{\mathrm{d}t}-\mathtt{g}(x,t)\simeq0,\end{aligned}\tag{7}$$

where $\tilde{\Psi}^{\alpha}(t)$ is the Caputo fractional derivative of the vector $\tilde{\Theta}(t)$ and is obtained by

$$\tilde{\Psi}^{\alpha}(t)=\partial_t^{\alpha}\tilde{\Theta}(t)=[0,\vartheta_1^{\alpha}(t),...,\vartheta_m^{\alpha}(t)]^T, \tag{8}$$

in which for $j=1,...,m$, we have

$$\vartheta_j^{\alpha}(t):=\partial_t^{\alpha}\left(\theta_j^{0,\mathtt{T}}(t)\right)=\sum_{r=0}^{[\frac{j}{2}]}\sum_{k=0}^{j-2r}\sum_{j=0}^{k}\frac{\Gamma(j+1)\varpi_{r,k,j}^{0,\mathtt{T}}}{\Gamma(j+1-\alpha)}\ t^{j-\alpha}.$$

Also

$$\tilde{\Psi}^2(t)=[\frac{d^2}{dt^2}\theta_0^{0,\mathtt{T}}(t),\cdots,\frac{d^2}{dt^2}\theta_m^{0,\mathtt{T}}(t)]^{\mathrm{T}}, \tag{9}$$

$$\Psi^r(x) = [\frac{d^r}{dx^r}\theta_0^{0,\mathrm{L}}(x), \cdots, \frac{d^r}{dx^r}\theta_n^{0,\mathrm{L}}(x)]^{\mathrm{T}}, \qquad r = 2, 4. \tag{10}$$

Let $x_0 = 0$, $x_n = \mathtt{L}$ and $\{x_i;\ i = 1, ..., n-1\}$ are the roots of $\theta_{n-1}^{0,\mathrm{L}}(x)$. Also, suppose $\{t_j;\ j = 1, ..., m\}$ are the roots of $\theta_m^{0,\mathrm{T}}(t)$. By evaluating Eq. (7) at collocation points (x_i, t_j), $i = 1, ..., n-1$, $j = 1, ..., m$, we have

$$\begin{aligned}\mathbf{R}(x_i, t_j) &= \rho\Theta(x_i)^{\mathrm{T}}\mathbf{C}\tilde{\Psi}^2(t_j) + \Theta(x_i)^{\mathrm{T}}\mathbf{C}\tilde{\Psi}^{\alpha}(t_j) + \Psi^4(x_i)^{\mathrm{T}}\mathbf{C}\tilde{\Theta}(t_j)\\ &- \mu\Psi^2(x_i)^{\mathrm{T}}\mathbf{C}\tilde{\Theta}(t_j) - \mathbf{F}(x_i, t_j, \Theta(x_i)^{\mathrm{T}}\mathbf{C}\tilde{\Theta}(t_j))\frac{\mathrm{B}(t_j) - \mathrm{B}(t_{j-1})}{t_j - t_{j-1}} - \mathtt{g}(x_i, t_j).\end{aligned} \tag{11}$$

Also, from Eqs.(2)-(3) and (6), we have

$$\Lambda_j^1 \triangleq \Theta(x_0)^{\mathrm{T}}\mathbf{C}\tilde{\Theta}(t_j) \simeq 0, \qquad j = 1, ..., m, \tag{12}$$

$$\Lambda_j^2 \triangleq \Theta(x_n)^{\mathrm{T}}\mathbf{C}\tilde{\Theta}(t_j) \simeq 0, \qquad j = 1, ..., m, \tag{13}$$

$$\Pi_i \triangleq \Theta(x_i)^{\mathrm{T}}\mathbf{C}\tilde{\Theta}(0) - \zeta_0(x_i) \simeq 0, \quad i = 0, ..., n. \tag{14}$$

Hence, we solve the following system of $(n+1) \times (m+1)$ nonlinear algebraic equations

$$\begin{cases}(t_j - t_{j-1})\,\mathbf{R}(x_i, t_j) = 0, & i = 1, ..., n-1,\ j = 1, ..., m,\\ \Lambda_j^r = 0, & r = 1, 2,\ j = 1, ..., m,\\ \Pi_i = 0, & i = 0, ..., n,\end{cases} \tag{15}$$

by using the Newton's iterative technique [6]. As a result, an approximate solution $u_{n,m}(x, t)$ can be attained from (6).

4. Numerical test example

In this section, we investigate our proposed approach for solving the TFSACE. We evaluate the numerical solution $u(x, t)$ along $\mathtt{p}$ discretized Brownian paths. Also, the arithmetic mean of $u(x, t)$ over these paths is considered. The codes are written in Matlab software and the computations are performed on a machine using a 1.70 GHz processor.

Consider the Eq.(1) with $\rho = 2$, $\mu = 1$ and $\mathbf{F}(x, t, u) = \sigma\sin(u)\dot{\mathrm{B}}(t)$. Note that the exact solution of this example is $u(x, t) = t^2x^4$. Figure 1 shows the exact and numerical solution of $u(x, t)$ with $\mathtt{p} = 80$, $\sigma = 0.05$, $\alpha = 0.5$ and $n = m = 8$. Figure 2 indicates the numerical solution of $u(x, \mathtt{T})$ along $\mathtt{p} = 100$ different discretized Brownian paths (Blue) and their arithmetic mean (Red) and Figure 3 displays the absolute error and contour plot of $u(x, t)$ with $\sigma = 0.02$, when $\alpha = 0.45$, $\mathtt{p} = 100$ and $n = m = 9$.

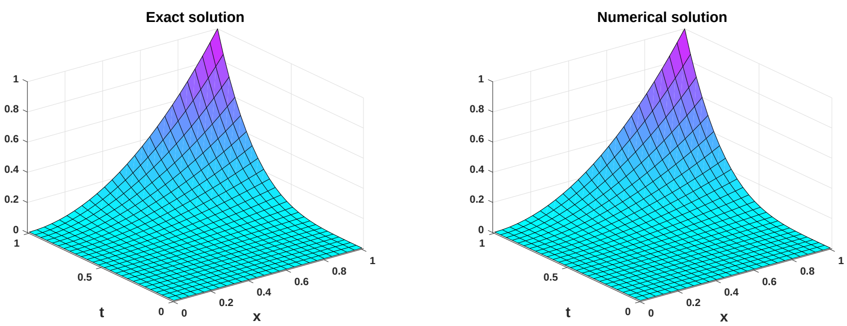

FIGURE 1. The exact and numerical solution of $u(x, t)$ with $\alpha = 0.5$.

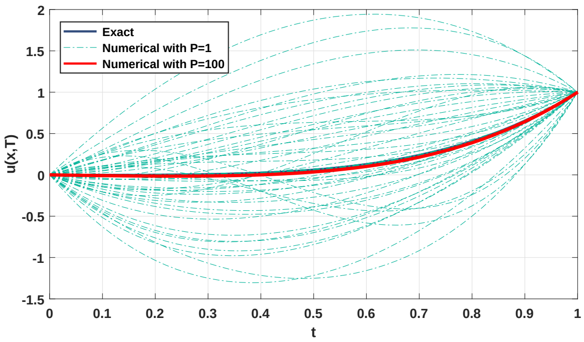

FIGURE 2. The numerical solution of $u(x, \mathtt{T})$ along $\mathtt{p} = 100$ different discretized Brownian paths (Blue) and their arithmetic mean (Red).

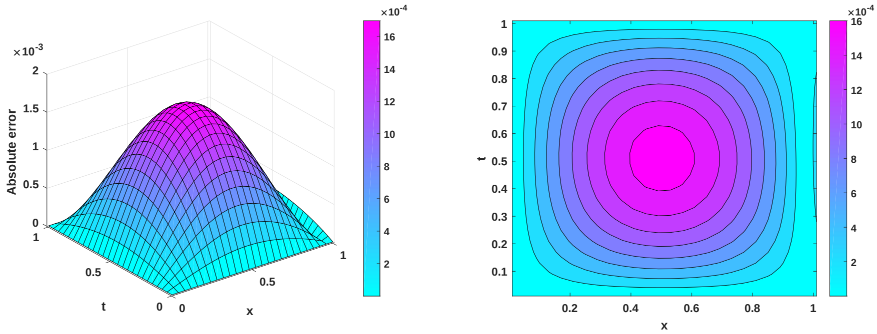

FIGURE 3. The absolute error and contour plot of $u(x,t)$ with $\sigma = 0.02$.

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Numerical solution of an inverse diffusion-convection problem based on the Chebyshev-collocation method

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Abstract. In this work, we consider an inverse diffusion-convection problem with an unknown function in the boundary condition. Since, in the sense of stability, this inverse problem is generally ill-posed, a mollification regularization technique is utilized. Then, a sixth-kind Chebyshev-collocation method will be introduced to solve the resulted mollified problem. At the end, to validate the accuracy of the proposed method a numerical example is investigated.

Keywords: Inverse problem. Diffusion-convection equation. Collocation method. Mollification technique.

AMS Mathematics Subject Classification [2010]: 35R30, 65M70, 41A50

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1. Introduction

In this article, we investigate the diffusion-convection equation in the form

$$u_t(x,t) = u_{xx}(x,t) + \eta(x)u_x(x,t) + f(x,t), \qquad (x,t) \in \Omega \times \mathtt{T}, \tag{1}$$

with the conditions

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{2}$$

$$u(0,t) = \varphi_0(t), \quad u(1,t) = \varphi_1(t), \qquad t \in \mathtt{T}, \tag{3}$$

such that $\eta(x)$ is a known coefficient, $f(x,t)$ is the source term, and $u_0(x)$ and $\varphi_0(t)$ show two known continuous functions. Also, let $\Omega := (0,1)$, $\mathtt{T} := (0,1)$. In this problem $\varphi_1(t)$ is an unknown boundary function. To find the solution of this inverse problem, we need an additional condition. Here, the condition

$$u(\hat{x},t) = \beta(t), \qquad t \in \mathtt{T}, \tag{4}$$

is used at an interior point $\hat{x} \in \Omega$.

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1.1. Mollification method. The inverse problem with unknown boundary condition is sensitive to the noisy input data and is generally ill-posed [1,2]. In practice, we have only a perturbed approximation of the input function $\beta(t)$ in the condition (4). Thus, using an appropriate regularization method is necessary to find a stable numerical solution. Here, we employ the mollification regularization technique. This method utilizes a convolution of the input data and a smooth function, to filter the high-frequency components of the noisy data. So, the noise will be controlled and the resulted problem becomes well-posed. Let $\delta > 0$ and $p > 0$ such that $p\delta < 0.5$ and $A_p = \Big(\int_{-p}^{p} exp(-s^2)ds\Big)^{-1}$. Suppose $\varphi(t)$ is a locally integrable function over $[0,1]$ and $t \in I_\delta = [p\delta, 1-p\delta]$. Then

$$\mathcal{J}_\delta\varphi(t) = (\rho_{\delta,p} * \varphi)(t) = \int_{t-p\delta}^{t+p\delta} \rho_{\delta,p}(t-s)\varphi(s)ds,$$

is the δ-mollification [3] of φ and in which the Gaussian kernel

$$\rho_{\delta,p}(t) = \begin{cases} A_p\delta^{-1}exp(-\frac{t^2}{\delta^2}), & |t| \leqslant p\delta, \\ 0, & |t| > p\delta. \end{cases}$$

is a non-negative $C^\infty(-p\delta, p\delta)$ function. Now, suppose that we have a discrete version of $\varphi(t)$. Then, to obtain the mollification of this function, let $K = \{t_j : j \in M\} \subset T$ and $\Delta t = sup\{(t_{j+1} - t_j) :\ j \in M,\ t_{j+1} - t_j > d > 0\}$ where M is a set of integers, and d is a positive constant. Let $\mathrm{B} = \{\varphi(\mathrm{t_j}) = \varphi_\mathrm{j} : \mathrm{j} \in \mathrm{M}\}$ be a discrete function defined on K. We set $s_j = (t_j + t_{j+1})/2,\ j \in M$. The discrete mollification of B [4] is defined as follows:

$$\mathcal{J}_\delta\mathrm{B(t)} = \sum_{\mathrm{j}=-\infty}^{\infty} \Big(\int_{\mathrm{s_{j-1}}}^{\mathrm{s_j}} \rho_\delta(\mathrm{t-s})\mathrm{ds}\Big)\varphi_\mathrm{j}.$$

Assume that in (4), instead of $\beta(t)$, some noisy function $\beta^\varepsilon \in \mathrm{C}^0(\mathtt{T})$ is in hand, such that $\|\beta - \beta^\varepsilon\|_{\infty,\mathtt{T}} \leq \varepsilon$. Consider a set of sample points $\{\tilde{\mathbf{t}}_r := rh;\ r = 0,...,\tilde{\mathtt{m}}\}$, where $\tilde{\mathtt{m}}$ is a positive integer and $h = 1/\tilde{\mathtt{m}}$ is the mesh size. So, by applying the mentioned discrete mollification approach on $\{\beta^\varepsilon(\mathbf{t}_r);\ r = 0,...,\tilde{\mathtt{m}}\}$, a mollified function $\{\mathbf{J}_\delta\,[\beta^\varepsilon]\,(\mathbf{t}_r);\ r = 0,...,\tilde{\mathtt{m}}\}$ will be resulted where the smoothing parameter δ is determined automatically by the generalized cross-validation (GCV) criterion [3]. At the end, interpolating these values results a stabilized approximation $\mathbf{B}(t)$ of the additional noisy function β^ε.

2. Numerical procedure

Definition 2.1. The shifted sixth-kind Chebyshev polynomials are defined by [5]

$$\theta_0(t) = 1, \quad \theta_1(t) = 2t - 1, \qquad \theta_{i+1}(t) = t\theta_i(t) + \Pi_i\ \theta_{i-1}(t), \quad i = 2, 3, ...,$$

on $[0,1]$, where

$$\Pi_i := \frac{-i^2 + 2i(-3 - (-1)^i) - 3(1 - (-1)^i)}{(-2i-4)(-2i-2)}.$$

To obtain a numerical approximation of the solution of (1), assume

$$u(x,t) \simeq \sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j}\theta_i(x)\theta_j(t) = \Theta(x)^{\mathrm{T}}\mathbf{C}\Theta(t), \tag{5}$$

where $\Theta(x) := [\theta_0(x), ..., \theta_n(x)]^{\mathrm{T}}$, $\Theta(t) := [\theta_0(t), ..., \theta_m(t)]^{\mathrm{T}}$ and $\mathbf{C} := \Big[c_{i,j}\Big]_{(n+1)\times(m+1)}$, $i = 0, ..., n,\ j = 0, ..., m$ is an unknown coefficients matrix that should be determined. So, according to (5) and (1), we have

$$\Theta(x)^{\mathrm{T}}\mathbf{C}\Theta_t(t) = \Theta_{xx}(x)^{\mathrm{T}}\mathbf{C}\Theta(t) + \eta(x)\Theta_x(x)^{\mathrm{T}}\mathbf{C}\Theta(t) + f(x,t), \tag{6}$$

where $\Theta_t(t) := [\theta_0'(t), ..., \theta_m'(t)]^{\mathrm{T}}$ and $\Theta_{xx}(x) := [\theta_0''(x), ..., \theta_n''(x)]^{\mathrm{T}}$. Let $x_0 = 0$, $x_n = \hat{x}$ and $\{x_i : i = 1, ..., n-1\}$ be the roots of $\theta_{n-1}(x)$ and $\{t_j : j = 1, ..., m\}$ be the roots of $\theta_m(t)$. Now, by evaluating (6) at $(n-1)\times m$ collocation points $\{(x_i, t_j) : i = 1, ..., n-1, j = 1, ..., m\}$, we have

$$\Theta(x_i)^{\mathrm{T}}\mathbf{C}\Theta_t(t_j) = \Theta_{xx}(x_i)^{\mathrm{T}}\mathbf{C}\Theta(t_j) + \eta(x_i)\Theta_x(x_i)^{\mathrm{T}}\mathbf{C}\Theta(t_j) + f(x_i,t_j), \tag{7}$$

Also, with collocating (2) at $\{x_i : i = 0, ..., n\}$, and collocating (3) and (4) at $\{t_j : j = 1, ..., m\}$, we have

$$\Theta(x_i)^{\mathrm{T}}\mathbf{C}\Theta(0) = u_0(x_i), \tag{8}$$

$$\Theta(0)^{\mathrm{T}}\mathbf{C}\Theta(t_j) = \varphi_0(t_j), \tag{9}$$

$$\Theta(\hat{x})^{\mathrm{T}}\mathbf{C}\Theta(t_j) = \mathbf{B}(t_j). \tag{10}$$

Let

$$\begin{aligned}
&\Psi_1 = [\Theta(x_1), ..., \Theta(x_{n-1})]^{\mathrm{T}}_{(n-1)\times(n+1)}, \qquad \Psi_2 = [\Theta_{xx}(x_1), ..., \Theta_{xx}(x_{n-1})]^{\mathrm{T}}_{(n-1)\times(n+1)},\\
&\Psi_3 = [\Theta_x(x_1), ..., \Theta_x(x_{n-1})]^{\mathrm{T}}_{(n-1)\times(n+1)}, \quad \Phi_1 = [\Theta(t_1), ..., \Theta(t_m)]_{(m+1)\times m},\\
&\Phi_2 = [\Theta_t(t_1), ..., \Theta_t(t_m)]_{(m+1)\times m}, \qquad \mathcal{D}_\eta = diag\,[\eta(x_1), ..., \eta(x_n)],\\
&\mathcal{F} = [f_{i,j}]_{(n-1)\times m}, \qquad f_{i,j} := f(x_i,t_j), \quad i = 1,2,...,n-1,\ j = 1,2,...,m.
\end{aligned}$$

Using the Kronecker product, Eq. (7) is equivalent to

$$\boldsymbol{\lambda}_1\mathbf{X} = \mathbf{Z}, \tag{11}$$

where $\boldsymbol{\lambda_1} = \Phi_2^{\mathrm{T}} \otimes \Psi_1 - \Phi_1^{\mathrm{T}} \otimes \Big(\Psi_2 + \mathcal{D}_\eta\Psi_3\Big)$, $\mathbf{X} = \mathrm{vec}(\mathbf{C})$ and $\mathbf{Z} = \mathrm{vec}(\mathcal{F})$. From Eqs. (8)-(10), we have

$$\boldsymbol{\lambda}_2\mathbf{X} = \mathbf{S}, \qquad \boldsymbol{\lambda}_3\mathbf{X} = \mathbf{H}_0, \qquad \boldsymbol{\lambda}_4\mathbf{X} = \mathbf{H}_{\hat{x}}, \tag{12}$$

where $\boldsymbol{\lambda_2} = \Theta(t_0)^{\mathrm{T}} \otimes \Psi_4$, $\boldsymbol{\lambda_3} = \Phi_1^{\mathrm{T}} \otimes \Theta(x_0)^{\mathrm{T}}$, $\boldsymbol{\lambda_4} = \Phi_1^{\mathrm{T}} \otimes \Theta(\hat{x})^{\mathrm{T}}$, $\Psi_4 = [\Theta(x_0), ..., \Theta(x_n)]^{\mathrm{T}}$, $\mathbf{S} = [u_0(x_1), ..., u_0(x_n)]^{\mathrm{T}}$, $\mathbf{H}_0 = [\varphi_0(t_1), ..., \varphi_0(t_m)]^{\mathrm{T}}$ and $\mathbf{H}_{\hat{x}} = [\mathbf{B}(t_1), ..., \mathbf{B}(t_m)]^{\mathrm{T}}$. Thus, from (11)-(12), we obtain

$$\boldsymbol{A}\mathbf{X} = \mathbf{F}, \tag{13}$$

where $\boldsymbol{A} = [\boldsymbol{\lambda}_1^{\mathrm{T}}, \boldsymbol{\lambda}_2^{\mathrm{T}}, \boldsymbol{\lambda}_3^{\mathrm{T}}, \boldsymbol{\lambda}_4^{\mathrm{T}}]^{\mathrm{T}}$ and $\mathbf{F} = [\mathbf{Z}^{\mathrm{T}}, \mathbf{S}^{\mathrm{T}}, \mathbf{H}_0^{\mathrm{T}}, \mathbf{H}_{\hat{x}}^{\mathrm{T}}]^{\mathrm{T}}$. The relation (13) gives a system of $(n+1)(m+1)$ linear algebraic equations which can be solved. Solving this system leads to an approximate solution of $u(x,t)$ in the form (5). Finally, to obtain an estimation of the unknown boundary function $\varphi_1(t)$, we have

$$\varphi_1(t) = u(1,t) \simeq \sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j}\theta_i(1)\theta_j(t). \tag{14}$$

3. Numerical test Example

In this section, we consider Eqs.(1)-(4) with $\eta(t) = 2$, $u_0(x) = x^2$ and $\varphi_0(t) = 0$. The exact solution to this problem is $u(x,t) = x^2 exp(-t)$. Let $\hat{x} = 0.7$, Figure 1 shows the exact and numerical solution and Figure 2 shows the absolute error of $u(x,t)$, when $N = M = 9$, $\varepsilon = 0.01$ and $\varepsilon = 0.05$. Table 1 displays the l_2-norm error for several values of ε and $n = m$.

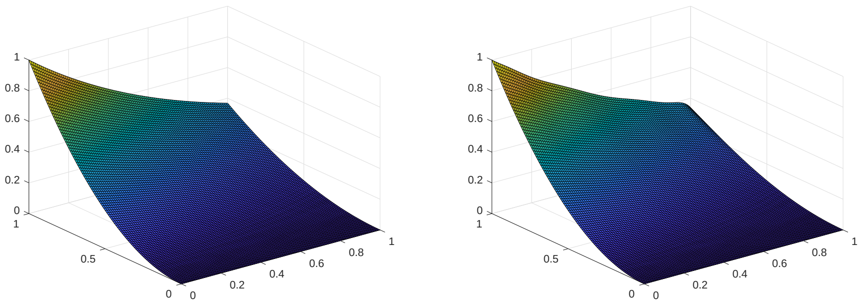

FIGURE 1. Exact solution (left) and numerical solution (right).

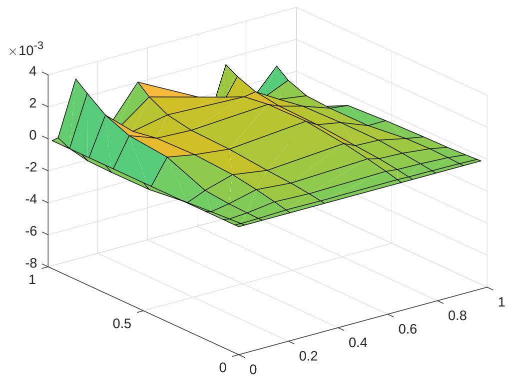

FIGURE 2. Absolute error.

TABLE 1. The l_2-norm errors.

N=M	5	7	9
$\varepsilon = 0.01$	0.003	0.001	0.001
$\varepsilon = 0.05$	0.008	0.005	0.005

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Groups with few p-parts of co-degrees of irreducible characters

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Abstract. For a character χ of a finite group G, the number $\chi^c(1) = \frac{[G:\mathrm{Ker}\chi]}{\chi(1)}$ is called the codegree of χ. Let p be a prime and let e be a positive integer. In this talk, we first show that the p-parts of co-degrees of non-principal irreducible characters of G are same if and only if G is an elementary abelian p-group. Next, we show that if G is a p-solvable group such that $p^{e+1} \nmid \chi^c(1)$, for every irreducible character χ of G, then the p-length of G is not greater than e. Finally, we study the finite groups satisfying the condition that p^2 does not divide the co-degrees of their irreducible characters.

Keywords: co-degree of a character, p-length, p-solvable group.

AMS Mathematics Subject Classification [2010]: 20C15, 20D10, 20D05

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1. Introduction

Throughout this paper, G is a finite group and p is a prime number. For a positive integer a, $|a|_p$ denotes the p-part of a. For a p-solvable group G, the p-length of G, denoted by $\ell_p(G)$, is the minimum possible number of factors that are p-groups in any normal series of G which every factor is either a p-group or a p'-group. Let $\mathrm{Irr}(G)$ denote the set of the (complex) irreducible characters of G. For a character χ of G, the number $\chi^c(1) = \frac{[G:\mathrm{Ker}\chi]}{\chi(1)}$ is called the co-degree of χ (see [6]). Set $\mathrm{Codeg}(G) = \{\chi^c(1) : \chi \in \mathrm{Irr}(G)\}$. Some properties of $\chi^c(1)$ have been studied in [1–4, 6].

2. Main results

J.G. Thompson proved that if the degree of every nonlinear irreducible character of G is divisible by p, then G has a normal p-complement. In [3, Proposition 2.3 and Theorem 1.1], it has been shown that:

Theorem 2.1. *Let G be a finite p-solvable group.*

(i) *If $p \mid \chi^c(1)$ for every non-principal character $\chi \in \mathrm{Irr}(G)$, then G is solvable.*
(ii) *Suppose that p is neither 2 nor a Mersenne prime. Then $p \mid \chi^c(1)$ for every non-principal character $\chi \in \mathrm{Irr}(G)$ if and only if G is a p-group.*

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Alizadeh et al. proved that G is an elementary abelian group if and only if $|\mathrm{Codeg}(G)| = 2$ (see [2] and the references cited in it.). In [3, Theorem 1.2], we have considered the analogues of this result for p-parts of co-degrees of irreducible characters and we prove that:

Theorem 2.2. *For every non-principal irreducible character χ of G, $\chi^c(1)_p = p^e$ if and only if $e = 1$ and G is an elementary abelian p-group.*

In [1], it has been proven that the p-length of a p-solvable group is not greater than the number of the distinct co-degrees of its irreducible characters which are divisible by p. In [4, Theorem 1], we have found the other bound for the p-length of a p-solvable group in terms of the co-degrees of its irreducible characters, as follows:

Theorem 2.3. *If G is a p-solvable group and $p^{e+1} \nmid \chi^c(1)$, for every $\chi \in \mathrm{Irr}(G)$, then $\ell_p(G) \leqslant e$.*

In [5] and the references cited in it, it has been shown that if $p^2 \nmid \chi(1)$, for every $\chi \in \mathrm{Irr}(G)$, then $[G : O(G)]_p \leqslant p^3$. In [4, Theorems 1,2], we investigate the same problem for irreducible character co-degrees and we prove that:

Theorem 2.4. *If $\chi^c(1)_p \leqslant p$, for every irreducible character χ of G, then either $|G|_p = p$ or G is a p-solvable group of p-length one.*

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Relative N-Weight codes over direct product of finite chain rings

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Abstract. In this paper, we study a relative N-weight linear codes over $R_1^\alpha \times R_2^\beta$, where R_1 and R_2 are finite chain rings. We introduce the concept of relative N-weight code over$R_1^\alpha \times R_2^\beta$ as a generalization of one- weight and two- weights codes. It is shown that the Gray image of N-distance of relative N-weight is a N-distance code and that the Gray image of a relative N-weight code is a linear relative N-weight code.

Keywords: Codes over chain rings, Linear code, Gray image, Relative N-weight code

AMS Mathematics Subject Classification [2010]: 94B60,94B05,94B25

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1. Preliminaries

A relative one-weight code was first introduced in [2] in order to study the relative generalized Hamming weight [5]. Let $\mathcal{C}$ be a linear q-ary code and $\mathcal{C}_1$ be a linear subcode of $\mathcal{C}$. If all the codewords $\mathcal{C} \setminus \mathcal{C}_1 = \{c : c \in \mathcal{C} \quad and \quad c \notin \mathcal{C}_1\}$ have the same weight, then $\mathcal{C}$ is called a relative one-weight code with respect to the subcode $\mathcal{C}_1$. The other special classes are two-weight codes and three-weight codes introduced in [3], [4], and these two classes of codes are useful in the wire-tap channel of type II with multiple users [5].
We introduce the concept of relative N-weight codes over$R_1^\alpha \times R_2^\beta$.
In [1] codes over $R_1^\alpha \times R_2^\beta$ are defined. In general case we let R_1 and R_2 be chain rings with maximal ideal $\langle\gamma_1\rangle$,$\langle\gamma_2\rangle$ and nilpotency indices e_1 and e_2, respectively. Moreover, we will suppose that R_1 and R_2 have the same residue field $\mathbb{F}_q = \frac{R_1}{\langle\gamma_1\rangle} = \frac{R_2}{\langle\gamma_2\rangle}$, and $e_1 \le e_2$. Also, suppose that $T_1 = \{r_0, \dots, r_{q-1}\}$ and $T_2 = \{r'_0, \dots, r'_{q-1}\}$ are the Teichmüller sets of representatives of R_1 and R_2, respectively. One can see that there exists the natural surjective ring homomorphism π from R_2 to R_1 such that $\pi(\gamma_2) = \gamma_1$ and $\pi(r'_j) = r_j$. Using this map, the scalar multiplication $*$ can be defined as follows:

$$a * \mathbf{u} = (\pi(a)u_0, \pi(a)u_1, \dots, \pi(a)u_{\alpha-1}|au'_0, \dots, au'_{\beta-1}), \tag{1}$$

for all $a \in R$ and $\mathbf{u} = (u|u') = (u_0, \dots, u_{\alpha-1}|u'_0, \dots, u'_{\beta-1}) \in R_1^\alpha \times R_2^\beta$. Consider injective map $\iota : R_1 \to R_2$ by definition $\iota(\gamma_1) = \gamma_2$ and $\iota(r_j) = r'_j$. It is obvious that $\pi\iota = Id$.

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It is asserted that $R_1^\alpha \times R_2^\beta$ is an R_2-module with respect to the scalar multiplication defined in (1); see [1].

Definition 1.1. A subset $\mathcal{C}$ over $R_1^\alpha \times R_2^\beta$ is a linear code if it is a submodule of $R_1^\alpha \times R_2^\beta$.

Now we define the weight of $\mathbf{u} = (u|v)$ as $w(u|v) = wt(u) + wt(v)$.

Definition 1.2. The inner product of vectors $\mathbf{u}$ and $\mathbf{v}$ in $R_1^\alpha \times R_2^\beta$ is defined by

$$< \mathbf{u}, \mathbf{v} >= \gamma_2^{e_2-e_1}\iota(u.v) + u'.v' \in R_2,$$

where $u.v$ and $u'.v'$ are standard inner product.

Then we define the dual code of a linear code $\mathcal{C}$ over $R_1^\alpha \times R_2^\beta$ as

$$\mathcal{C}^\perp = \{\mathbf{v} \in R_1^\alpha \times R_2^\beta \ : < \mathbf{u}, \mathbf{v} >= 0, \ \text{ for all } \mathbf{u} \in \mathcal{C} \ \}.$$

2. Relative N-weight codes over $R_1^\alpha \times R_2^\beta$

Throughout in this paper we denote the calligraphic $\mathcal{C}$ as a code in $R_1^\alpha \times R_2^\beta$ and the standard C as a code over the $\mathbb{F}_q^{q^{e_1-1}\alpha+q^{e_2-1}\beta}$, where residue field of chain rings R_1 and R_2 is $\mathbb{F}_q$.

Definition 2.1. Let $\mathcal{C}$ A nonzero code over $R_1^\alpha \times R_2^\beta$ and $\mathcal{C}_1$ be a k_1 dimensional subcode of $\mathcal{C}$, $\mathcal{C}_2$ be a k_2 dimensional subcode of $\mathcal{C}$,..., $\mathcal{C}_N$ be a k_N dimensional subcode of $\mathcal{C}$ satisfying in relation

$$\mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_{N-1} \subset \mathcal{C}_N = \mathcal{C}.$$

Then $\mathcal{C}$ is called a relative N-weight with respect to $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_N$, where $\mathcal{C}_1, \mathcal{C}_2 \setminus \mathcal{C}_1, \mathcal{C}_3 \setminus \mathcal{C}_2, \cdots, \mathcal{C}_N \setminus \mathcal{C}_{N-1}$ are all constant weights codes.
If these N-constant weight codes have weights $m_1, \cdots m_N$,respectively, then the relative N- weight code $\mathcal{C}$ is denoted by $\mathcal{C}(m_1, m_2, \cdots, m_N).(m_i > 0)$

Example 2.2. $\mathcal{C}$ is called a relative two-weight code with respect to a subcode $\mathcal{C}_1$ and $\mathcal{C} \setminus \mathcal{C}_1$ are both constant weight codes. If these two constant weight codes have weights m_1 and m_2, then the relative two-weight code $\mathcal{C}$ is denoted by $C(m_1, m)$.
If $\mathcal{C}_1$ be a k_1 dimensional subcode of $\mathcal{C}$, and $\mathcal{C}_2$ be a k_2-dimensional subcode of $\mathcal{C}$, satisfying $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}$, then $\mathcal{C}$ is called a relative three-weight code with respect to $\mathcal{C}_1$ and $\mathcal{C}_2$,provided that $\mathcal{C}_1$,$\mathcal{C}_2 \setminus \mathcal{C}_1$,$\mathcal{C} \setminus \mathcal{C}_2$ are all constant weight codes.If these three constant weight codes have weights m_1, m_2 and m respectively, then the relative three- weight code $\mathcal{C}$ is denoted by $\mathcal{C}(m, m_1, m_2)$.

Definition 2.3. A non zero linear code $\mathcal{C}$ in $R_1^\alpha \times R_2^\beta$ is called N-distance if there exist N distinct positive integers $m_1, \cdots, m_N$, such that for any two distinct codwords $c_i, c_j \in \mathcal{C}$,$d(c_i, c_j) \in \{m_1, \cdots, m_t\}$.

Lemma 2.4. *The linear code $\mathcal{C}$ in $R_1^\alpha \times R_2^\beta$ is a relative N-distance code if and only if $\mathcal{C}$ is a relative N- weight code.*

Proof. Let the linear code $\mathcal{C}$ is relative N-weight code with respect to $\mathcal{C}_1$,$\cdots$,$\mathcal{C}_N$ with wights $m_1, m_2, \cdots, m_N$. Then for two distinct codwords $c_i, c_j \in \mathcal{C}$, we are different cases:

First case:
If $c_i, c_j \in \mathcal{C}_1$ then, $w(c_i = w(c)_j = m_1$ therefore
$$d(c_i, c_j) = w(c_i - c_j) = m_1 \in \{m_1, \cdots, m_t\}.$$
Two case:
If $c_i, c_j \in \mathcal{C}_i \setminus c_{i-1}$,$(1 \leq i \leq n)$ then
$$d(c_i, c_j) = m_i \in \{m_1, \cdots, m_N\}.$$
Three case:
If $c_i \in \mathcal{C}_i \setminus \mathcal{C}_{i-1}$, $c_2 \in \mathcal{C}_2 \setminus C_{j-1}$ where $i \neq j$, then
$$d(c_i, c_j) = w(c_i - c_j) \in \{\{m_1, \cdots, m_N\}.$$
Therefore in any case $\mathcal{C}$ is N-distance code.
Vice versa If $\mathcal{C}$ is N-distance code ,then for two distinct codwords we have:

$$\exists m_i, m_j \in \{m_1, \cdots, m_N\}, \text{such that}, \quad d(c_i, 0) = w(c_i) = m_i, \quad d(c_j, 0) = w(cj) = m_j.$$

So there exist subcode $\mathcal{C}_\rangle$ of $\mathcal{C}$, such that for $c_i \in \mathcal{C}_i$, $w(c_i) = m_i$. And for $c_j \in \mathcal{C}_\rangle \in \mathcal{C}_|$, $w(c_j) = m_j$ $(1 \leq i, j \leq n)$. There fore $\mathcal{C}$ is relative N-weight code. □

THEOREM 2.5. *If $\mathcal{C}$ is a N-distance linear code over $R_1^\alpha \times R_2^\beta$ with distance $\{m_1, \cdots, m_N\}$, then $\varphi(\mathcal{C})$is a N-distance code with the same distance $\{m_1, \cdots, m_N\}$, where φ is arbitrary isometry gray over $R_1^\alpha \times R_2^\beta$.*

PROOF. Let $c_1, c_2 \in \mathcal{C}$ are two distinct codwords. then $\varphi(c_1), \varphi(c_2) \in \varphi(\mathcal{C})$. It is clear that $\varphi(c_1) \neq \varphi(c_2)$. By Lemma 2.4 $\mathcal{C}$ is N-distance code so $d(c_1, c_2 \in \{m_1, \cdots, m_N\}$ but φ is isometric then
$$d(\varphi((c_1), \varphi(c_2)) = d(c_1, c_2) \in \{m_1, \cdots, m_N\}.$$
Therefore $\varphi(\mathcal{C})$ is N-distance with weights $\{m_1, \cdots, m_N\}$. □

If $\mathcal{C}$ is a N-relative weight linear code, then $\mathcal{C}^\perp$ need not be so.

EXAMPLE 2.6. Let $\mathcal{C} = \langle(0|1)\rangle$ be a linear code in $\mathbb{Z}_2 \times \mathbb{Z}_2[u]$. Then $\mathcal{C}$ is a $\mathcal{C}(2, 1)$ relative two-weight code and its dual code $\mathcal{C}^\perp = \langle(1, 0)\rangle$ is not a relative two-weight linear code.

THEOREM 2.7. *Let $\mathcal{C}$ be a $\mathcal{C}(m_1, \cdots, m_N)$ code in $R_1^\alpha \times R_2^\beta$, then the Gray image $\mathcal{C}$ is a relative N-weight code $\mathcal{C}(m_1, \cdots m_N)$ in $\mathbb{F}_q^{q^{e_1-1}\alpha+q^{e_2-1}\beta}$.*

PROOF. Let $x \in (C_i) \setminus (C_{i-1})$,$(1 \leq i \leq n)$, then there exists $c \in \mathcal{C}_i \setminus \mathcal{C}_{i-1}$ such that $x = \varphi(c)$. Since $wt(\varphi(c)) = wt(c)$ for all $c \in \mathcal{C}$. Therefore, $wt(x) = wt(\varphi(x)) = wt(c) = m_i$.Let $x \in \varphi(\mathcal{C}_i$ with $x \neq 0$, then there exists$0 \neq c_i \in \mathcal{C}$ such that $x = \varphi(c_i)$. Therefore, $wt(x) = wt(\varphi(c_i) = wt(c_i) = m_i$. Hence $\varphi((C))(m_1, \cdots, m_N)$ relative N-weight code in $\mathbb{F}_q^{q^{e_1-1}\alpha+q^{e_2-1}\beta}$. □

THEOREM 2.8. *Let $\mathcal{C}$ be a $\mathcal{C}(m_1, \cdots, m_N)$ code in $R_1^\alpha \times R_2^\beta$. Then for any positive integer t, there exists a relative N-weight code $\mathcal{D}(tm_1, \cdots, tm_N)$ in $R_1^{t\alpha} \times R_2^{t\beta}$.*

PROOF. Let $\mathcal{C}$ be a relative N-weight code $\mathcal{C}(m_1, \cdots, m_N)$ with subcodes $\mathcal{C}_1, \cdots, \mathcal{C}_N$. Define
$$\mathcal{D} = \{\underbrace{(u, \cdots, u}_{t\ times} | \underbrace{v, \cdots, v)}_{t\ times} | (u, v) \in \mathcal{C}\} \subset R_1^{t\alpha} \times R_2^{t\beta},$$

and

$$\mathcal{D}_i = \{((\underbrace{u, \cdots, u}_{t\ times} | \underbrace{v, \cdots, v}_{t\ times}) | (u, v) \in \mathcal{C}_i\},$$

it is clear that, $\mathcal{D}_i \subset \mathcal{D}$ is a linear code in $R_1^{t\alpha} \times R_2^{t\beta}$ and also $\mathcal{D}_1 \subset \mathcal{D}_2 \subset \cdots \subset \mathcal{D}_N$. Now Let

$$(x|y) = (\underbrace{x, \cdots, x}_{t\ times} | \underbrace{y, \cdots, y}_{t\ times}) \in \mathcal{D}_i \setminus \mathcal{D}_{i-1}, \qquad 1 \leq i \leq n.$$

Then $(x|y) \in \mathcal{C}_i \setminus \mathcal{D}_{i-1}$, $wt(x|y) = m_i$ and hence $wt(u|v) = tm_i$.
Therefore ,$\mathcal{D}(tm_1, tm_2, \cdots, tm)$ is a relative N-weight code with subcodes $\mathcal{D}_1, \cdots, \mathcal{D}_N$. $\square$

THEOREM 2.9. *Let $\mathcal{C}$ be a linear code over $\mathbb{Z}_{p^r}^{\alpha} \times \mathbb{Z}_{p^s}^{\beta}$. If $\mathcal{C}(m_1, \cdots, m_N)$ is a relative N-weight code, then $p^r | m_1$.*

PROOF. Let $c = (u|v) \in \mathcal{C}$, and $\mathcal{C}_1$ is a subcode of $\mathcal{C}$.Since $\mathcal{C}_1$ is a linear one weight subcode with weight m_1

$$\underbrace{c + c + \cdots + c}_{p^r} = (p^r u | p^r v) = (0|p^r v) \in \mathcal{C}_1.$$

On other hand $p^r v \in \mathbb{Z}_{p^s}^{\beta}$, so $p^r | wt(0|p^r v)$. Therefore $wt(c) = wt(p^r c)$ and $p^r | w(c)$ then $p^r | m_1$. $\square$

THEOREM 2.10. *Let $\mathcal{C}$ be a linear code over $\mathbb{Z}_{p^r}^{\alpha} \times \mathbb{Z}_{p^s}^{\beta}$. Then the weight of all codwords of $\mathcal{C}$ are multiple p^r if and only if $\left(p_\alpha^{r-1} | p_\beta^{s-1}\right) \in \mathcal{C}^\perp$, where $p_\alpha^{r-1} = (p^{r-1}, \cdots, p^{r-1}) \in \mathbb{Z}_{p^r}^{\alpha}$ and $p_\beta^{s-1} = (p^{s-1}, \cdots, p^{s-1}) \in \mathbb{Z}_{p^s}^{\beta}$.*

PROOF. Let $(u|v) \in \mathcal{C}$, where $u \in \mathbb{Z}_{p^r}^{\alpha}$ and $v \in \mathbb{Z}_{p^s}^{\beta}$. Let take $u = (u_1, \cdots, u_\alpha)$ and $v = (v_1, \cdots, v_\beta)$, then we have

$$\langle (p_\alpha^{r-1} | p_\beta^{s-1}), (u|v) \rangle = p^{s-r} (\sum_{i=1}^{\alpha} p^{r-1} u_i) + \sum_{j=1}^{\beta} p^{s-1} v_j \in \mathbb{Z}_{p^s}^{\beta}.$$

It is clear that $\langle (p_\alpha^{r-1} | p_\beta^{s-1}), (u|v) \rangle = 0$ if and only if $p^r | wt(u|v)$, hence, the weight of all codewords in $\mathcal{C}$ are multiple p^r iff $\left(p_\alpha^{r-1} | p_\beta^{s-1}\right) \in \mathcal{C}^\perp$ $\square$

COROLLARY 2.11. *Let $\mathcal{C}(m_1, \cdots, m_N)$ be a relative N-weight code over $\mathbb{Z}_{p^r}^{\alpha} \times \mathbb{Z}_{p^s}^{\beta}$. Then the weights of all codewords of $\mathcal{C}$ are multiple p^r if and only if $\left(p_\alpha^{r-1} | p_\beta^{s-1}\right) \in \mathcal{C}^\perp$*

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On the lattice of fuzzy filters of quantales

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Abstract. The notion of an $\mathcal{L}$-fuzzy filter in a quantale is introduced. After that some properties are given, using the notion of a closure operator, the lattice structure of these substructures is studied. Particularly, it is shown that this lattice is a complete Brouwerian lattice and so is a complete Heyting lattice.

Keywords: Quantale, $\mathcal{L}$-fuzzy filter, lattice

AMS Mathematics Subject Classification [2010]: 03G12; 08A72, 03E72

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1. Introduction and Preliminaries

Quantales were introduced by Mulvey [3] to provide a lattice-theoretic setting for studying non-commutative C^*-algebras as well as a constructive foundations for quantum mechanics. Some familiar examples of quantales are frames, complete Boolean algebras, ideal lattices of rings, the power set of a semigroup and also C^*-algebras. Now, we give some difinitions and results from [2, 4, 5, 9].

Definition 1.1. A Quantale is a relational structure $Q = \langle Q, *, \leq\rangle$, where $\langle Q, \leq\rangle$ is a complete lattice (with top element $\top$ and bottom element $\bot$) and $\langle Q, *\rangle$ is a semigroup satisfying

$$a * \left(\bigvee_{i\in I} b_i\right) = \bigvee_{i\in I} a * b_i, \qquad \left(\bigvee_{i\in I} b_i\right) * a = \bigvee_{i\in I} b_i * a \tag{1}$$

for all $a, b, a_i, b_i \in Q$ with $(i \in I)$.

Quantal Q is called unital if it has a unit with respect to the operation $*$; i.e., there exists an element $1 \in Q$ such that $a * 1 = 1 * a = a$, for all $a \in Q$. Q is called strictly two-sided if $a * \top = \top * a = a$. Obviously, any strictly two-sided quantal is unital with $\top = 1$. Because of (1), the operation $*$ has two adjoints $\rightsquigarrow$ and $\to$ which satisfy

$$a * c \leq b \Leftrightarrow c \leq a \to b \ \Leftrightarrow a \leq c \rightsquigarrow b. \tag{2}$$

For the simplicity of reference, let's denote $a \to 0$ and $a \rightsquigarrow 0$ by $\neg a$ and $\sim a$, respectively. Some basic properties of quantales are as follows:

(1) $a * \bot = \bot * a = \bot$, $a \to \top = a \rightsquigarrow \top = \top$.
(3) $a \leq b \to c$ if and only if $b \leq a \rightsquigarrow c$.

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(4) $a * (a \to b) \leq b$ and $(a \rightsquigarrow b) * a \leq b$. Particularly, $a * \neg a = 0 = \sim a * a = 0$.
(5) $b \leq a \to (a * b)$ and $b \leq a \rightsquigarrow (b * a)$.
(6) $a \leq b$ implies that $c * a \leq c * b$ and $a * c \leq b * c$.
(7) $(x \to y) * (y \to z) \leq x \to z$, $(y \rightsquigarrow z) * (x \rightsquigarrow y) \leq x \rightsquigarrow z$.
(8) if Q is two-sided, $x * y \leq x$ and $x * y \leq y$ and so $x \leq y \rightsquigarrow x$ and $y \leq x \to y$.
(9) if Q is two-sided, $x \to x = x \rightsquigarrow x = \top$.
(10) If Q is two-sided, $a \leq b$ if and only if $a \to b = a \rightsquigarrow b = \top$.
(11) $(a * b) \to c = b \to (a \to c)$ and $(a * b) \rightsquigarrow c = a \rightsquigarrow (b \rightsquigarrow c)$.
(12) $b \to c \leq (a \to b) \to (a \to c)$ and $b \rightsquigarrow c \leq (a \rightsquigarrow b) \rightsquigarrow (a \rightsquigarrow c)$. Particularly, if Q is strictly two-sided, $a \leq b$ implies that $c \to a \leq c \to b$ and $c \rightsquigarrow a \leq c \rightsquigarrow b$.
(13) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$ and $(a \vee b) \rightsquigarrow c = (a \rightsquigarrow c) \wedge (b \rightsquigarrow c)$.
(14) $b \to c \leq (a * b) \to (a * c)$, $b \rightsquigarrow c \leq (b * a) \rightsquigarrow (c * a)$.

Definition 1.2. [4] A nonempty subset F of quantale Q is said to be a filter if it is closed with respect to the operation $*$ and it is an upper set; i.e. $a \leq b$ and $a \in F$ imply that $b \in F$.

For more details about lattices, fuzzy sets and L-fuzzy sets , we refer to references [6], [8] and [1].

In this paper, $\mathcal{L}$ will denote a complete lattice.

2. Main results

Definition 2.1. [9] A non-zero $\mathcal{L}$-fuzzy set μ of quantale Q is called an $\mathcal{L}$-fuzzy filter if for all $a, b \in Q$ it satisfies

(LF1) $\mu(a) \wedge \mu(b) \leq \mu(a * b)$.
(LF2) $a \leq b$ implies that $\mu(a) \leq \mu(b)$.

Let **LS**(Q) and **FF**(Q) be the set of all $\mathcal{L}$-fuzzy sets and $\mathcal{L}$-fuzzy filters of Q, respectively.

Example 2.2. [7] Consider the quantale $\langle Q, *, \leq \rangle$ in which $Q = \{\perp, a, b, \top\}$ which is a complete lattice with the ordering $\perp < a, b < \top$ and the operation $*$ is defined as in Table 1. We define $\mathcal{L}$-fuzzy sets μ and ν by $\mu(\perp) = \mu(b) = s, \mu(a) = \mu(\top) = t$ and

Table 1. Cayley table of $*$

$*$	$\perp$	a	b	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
a	$\perp$	a	b	$\top$
b	$\perp$	b	b	b
$\top$	$\perp$	$\top$	b	$\top$

$\nu(\perp) = \nu(a) = s, \nu(b) = \nu(\top) = t$, where $s, t \in L$ with $s > t$. It is easily verified that μ and ν are $\mathcal{L}$-fuzzy filters of Q.

Proposition 2.3. *An $\mathcal{L}$-fuzzy set μ of strictly two-sided Q is an $\mathcal{L}$-fuzzy filter if and only if for all $a, b \in Q$*

(LF3) $\mu(1) \geq \mu(a)$.
(LF4) $\mu(b) \geq \mu(a) \wedge \mu(a \to b)$ *(or* $\mu(b) \geq \mu(a) \wedge \mu(a \rightsquigarrow b)$*).*

Proposition 2.4. *An $\mathcal{L}$-fuzzy subset μ of Q is an $\mathcal{L}$-fuzzy filter if and only if any nonempty level subset μ_t (with $t \in [0, 1]$) is a filter of Q.*

It is easy to verify that the intersection of any family of $\mathcal{L}$-fuzzy filters of Q is again an $\mathcal{L}$-fuzzy filter of Q. It follows that for an $\mathcal{L}$-fuzzy subset μ of Q, the $\mathcal{L}$-fuzzy filter of Q generated by μ; i.e., the intersection of all $\mathcal{L}$-fuzzy filters of Q which contain μ, exists. We show it by $\langle\mu\rangle$. We observe that $\langle\mathbf{0}\rangle = \chi_Q$. Generally, $\langle\cdot\rangle$ satisfies the following properties: (1) $\mu \subseteq \langle\mu\rangle$, (2) $\mu \subseteq \nu$ implies that $\langle\mu\rangle \subseteq \langle\nu\rangle$, (3) $\langle\mu\rangle = \mu$ if and only if μ is an $\mathcal{L}$-fuzzy filter of Q. Thus, the mapping $\mu \mapsto \langle\mu\rangle$ is a closure operator on $\mathbf{LS}(Q)$, where closed subsets of $\mathbf{LS}(Q)$ are $\mathcal{L}$-fuzzy filters of Q. For more details about closure operators and their properties, we refer the reader to [6]. Hence, by [6, Theorem I.5.2], $\mathbf{LS}(Q)_C = \mathbf{FF}(Q)$ is a complete lattice in which infimum and supremum are defined as

$$\sqcap_{i\in I}\ \mu_i = \cap_{i\in I}\mu_i,\ \sqcup_{i\in I}\mu_i = \langle\cup_{i\in I}\mu_i\rangle. \tag{3}$$

From the above observations, it follows that the intersection of any family of filters of Q is again a filter. So, observing the previous explanations, for subset X of Q, the mapping $X \mapsto \langle X\rangle$ is a closure operator on 2^Q, the power set of Q. Hence

Corollary 2.5. *$(\mathbf{F}(Q), \sqcup, \sqcap)$ is a complete lattice, where $\mathbf{F}(Q)$ denotes the set of all filters of Q.*

Now, we give a characterization of $\mathcal{L}$-fuzzy filter generated by an $\mathcal{L}$-fuzzy subset of Q.

Theorem 2.6. *Let μ be an $\mathcal{L}$-fuzzy subset of Q. Then the $\mathcal{L}$-fuzzy filter η of Q generated by μ is as follows: for all $x \in Q$,*

$$\langle\mu\rangle(x) = \bigvee\{\mu(a_1) \wedge \mu(a_2) \wedge \cdots \wedge \mu(a_n) : x \geq a_1 * a_2 * \cdots * a_n, n \in \mathbf{N}, a_1, \ldots, a_n \in Q\}$$

Corollary 2.7. *The filter of Q generated by a subset X of Q is characterized as*

$$\langle X\rangle = \{x \in Q : x \geq a_1 * a_2 * \cdots * a_n, n \in \mathbf{N}, a_1, a_2, \ldots, a_n \in X\}$$

Proposition 2.8. *For subset X of Q we have*

$$\langle X\rangle = \bigcup\{\langle A\rangle : A \subseteq X \text{ is finite }\}.$$

That is $\langle\cdot\rangle$ is an algebraic closure operator.

Theorem 2.9. *If Q is strictly two-sided, then $\mathbf{FF}(Q) = (\mathbf{FF}FF(Q), \sqcup, \sqcap)$ is a complete Brouwerian lattice.*

Proof. Let μ and η_i $(i \in I)$ be $\mathcal{L}$-fuzzy filters of Q. All we need is to prove that $\mu \sqcap (\sqcup_{i\in I}\eta_i) \subseteq \sqcup_{i\in I}(\mu \sqcap \eta_i)$. Let $x \in Q$ and $\epsilon > 0$ be arbitrary. Then there exist $n \in \mathbf{N}$ and $a_1, a_2, \ldots, a_n \in Q$ such that $x \geq a_1 * a_2 * \cdots * a_n$ and

$$(\sqcup\eta_i)(x) < \epsilon + (\sqcup\eta_i)(a_1) \sqcap (\sqcup\eta_i)(a_2) \sqcap \cdots \sqcap (\sqcup\eta_i)(a_n).$$

By the definition of $\sqcup\eta_i$ we get $(\sqcup\eta_i)(a_i) < \eta_j(a_j) + \epsilon$, for at least one j. Without loss of generality, we can assume that

$(\sqcup\eta_i)(a_1) < \eta_1(a_1) + \epsilon, (\sqcup\eta_i)(a_2) < \eta_2(a_2) + \epsilon, \ldots, (\sqcup\eta_i)(a_n) < \eta_n(a_n) + \epsilon.$

Hence, $(\sqcup\eta_i)(x) < n\epsilon + \eta_1(a_1) \sqcap \eta_2(a_2) \sqcap \cdots \sqcap \eta_n(a_n)$ and so

$$\mu \sqcap (\sqcup\eta_i)(x) \quad < \quad n\epsilon + (\mu(x) \sqcap \eta_1(a_1)) \sqcap \cdots \sqcap (\mu(x) \sqcap \eta_n(a_n)).$$

On the other hand $(a_1 * a_2 * \cdots * a_n) \to x = (a_1 * a_2 * \cdots * a_n) \rightsquigarrow x = \top$.

Now, let $b_n = (a_1 * a_2 * \cdots * a_{n-1}) \to x$, $b_{n-1} = (b_n * a_1 * a_2 * \cdots * a_{n-2}) \to x$, ..., $b_2 = (b_3 * \cdots * b_{n-1} * b_n * a_1) \to x$ and $b_1 = (b_2 * b_3 * \cdots * b_n) \to x$.

We observe that $(b_1 * b_2 * \cdots * b_n) \to x = b_1 \rightsquigarrow ((b_2 * \cdots * b_n) \to x) = b_1 \rightsquigarrow b_1 = \top$, whence $x \geq b_1 * b_2 * \cdots * b_n$.

On the other hand,

$$a_n \to b_n = a_n \to ((a_1 * a_2 * \cdots * a_{n-1}) \to x) = (a_1 * a_2 * \cdots * a_n) \to x = \top,$$

which implies that $a_n \leq b_n$. Similarly,

$$\begin{aligned} a_{n-1} \to b_{n-1} &= a_{n-1} \to ((b_n * a_1 * a_2 * \cdots * a_{n-2}) \to x) = (b_n * a_1 * a_2 * \cdots * a_{n-1}) \to x \\ &= b_n \rightsquigarrow ((a_1 * a_2 * \cdots * a_{n-1}) \to x) = b_n \rightsquigarrow b_n = \top, \end{aligned}$$

which implies that $a_{n-1} \leq b_{n-1}$. Also,

$$\begin{aligned} a_1 \to b_1 &= a_1 \to ((b_2 * b_3 * \cdots * b_n) \to x) = (b_2 * b_3 \cdots * b_n * a_1) \to x \\ &= b_2 \rightsquigarrow ((b_3 \cdots * b_n * a_1) \to x) = b_2 \rightsquigarrow b_2 = \top. \end{aligned}$$

Hence, for $i = 2, \ldots, n-1$ we have

$$\begin{aligned} a_i \to b_i &= a_i \to ((b_{i+1} * b_{i+2} * \cdots * b_n) \to x) = (b_{i+1} * b_{i+2} * \cdots * b_n * a_i) \to x \\ &= b_{i+1} \rightsquigarrow (b_{i+2} * \cdots * b_n * a_i) \to x) = b_{i+1} \rightsquigarrow b_{i+1} = \top, \end{aligned}$$

whence $b_i \leq a_i$. Hence, for all $i \in \{1, 2, \ldots, n\}$ we have $b_i \leq a_i$ and so $\eta_i(b_i) \leq \eta_i(a_i)$. Obviously,

$$\begin{aligned} x \to b_i &= x \to (b_{i+1} * b_{i+2} * \cdots * b_n) \to x = (b_{i+1} * b_{i+2} * \cdots * b_n) \rightsquigarrow (x \to x) \\ &= (b_{i+1} * b_{i+2} * \cdots * b_n) \rightsquigarrow \top = \top, \end{aligned}$$

whence $x \leq b_i$ and so $\mu(x) \leq \mu(b_i)$, for all $i \in \{1, 2, \ldots, n\}$. Thus for all $i \in \{1, 2, \ldots, n\}$, $\mu(x) \sqcap \eta_i(a_i) \leq (\mu \sqcap \eta_i)(b_i)$ and so

$$\mu \sqcap (\sqcup \eta_\alpha)(x) < n\epsilon + (\mu \sqcap \eta_{\alpha_1})(b_1) \sqcap \cdots \sqcap (\mu \sqcap \eta_{\alpha_n})(b_n).$$

Obviously, $(\mu \sqcap \eta_i)(b_i) \leq [\sqcup(\mu \sqcap \eta_i)](b_i), \quad \forall i \in \{1, 2, \ldots, r\}$. Thus

$$\mu \sqcap (\sqcup \eta_\alpha)(x) < n\epsilon + \sqcap_{i=1}^{n} [\sqcup(\mu \sqcap \eta_\alpha)](b_i) \leq n\epsilon + [\sqcup(\mu \sqcap \eta_\alpha)](x).$$

Since ϵ is arbitrary, we have $\mu \sqcap (\sqcup \eta_i)(x) \leq \sqcup(\mu \sqcap \eta_i)(x)$. □

Corollary 2.10. *If Q is strictly two-sided, then $\mathbf{FF}(Q) = (\mathbf{FF}(Q), \sqcup, \sqcap)$ is a complete Heyting lattice.*

3. Conclusions

In this paper, we conclude that the set of all $\mathcal{L}$-fuzzy filters of a quantale under the set-theoretic inclusion forms a lattice, and if Q is strictly two-sided, this lattice is a complete Brouwerian lattice and so a complete Heyting lattice.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Preconditioning methods for weighted Toeplitz regularized least-squares problems

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Abstract. We consider two types of block preconditioners and the corresponding iterative methods for the solution of the weighted Toeplitz least-squares problems. We show that the proposed iterative methods are convergent unconditionally. These two preconditioners can be used to accelerate the convergence rate of the Krylov subspace methods. Numerical results are given for GMRES.

Keywords: Least-squares problems, Weighted Toeplitz matrix, preconditioner.

AMS Mathematics Subject Classification [2010]: 65F10, 65F08.

1. Introduction

This paper is devoted to the study of the numerical solution of the weighted Toeplitz least-squares problem is described by:

$$\min_{x\in\mathbb{R}^n} ||Bx-b||_2^2, \quad B=\begin{pmatrix}\Xi K\\ \sqrt{v}I_n\end{pmatrix}, \quad b=\begin{pmatrix}\Xi f\\ 0\end{pmatrix}, \tag{1}$$

where $\Xi\in\mathbb{R}^{m\times m}$ is a symmetric positive definite (SPD) matrix (as a weighting matrix), $K\in\mathbb{R}^{m\times n}$ $(m\geq n)$ is a full-rank Toeplitz matrix, $I_n\in\mathbb{R}^{n\times n}$ is the identity matrix, $v>0$ is a regularization parameter and $f\in\mathbb{R}^m$ is a given vector. Because of the spatially variant property of the weighted Toeplitz matrix ΞK, its displacement rank is very large. Therefore, reasonable preconditioning techniques need to be considered for solving such weighted Toeplitz regularized least-squares problems.

Let $M=(\Xi^T\Xi)^{-1}$ and $y=\Xi^T\Xi\,(f-Kx)$, then the system (1) are expressed as-

$$\begin{pmatrix}M & K\\ K^T & -vI_n\end{pmatrix}\begin{pmatrix}y\\ x\end{pmatrix}=\begin{pmatrix}f\\ 0\end{pmatrix}, \tag{2}$$

where $M\in\mathbb{R}^{m\times m}$ is an SPD matrix. The linear system (2) can also be reformulated as

$$\mathscr{A}z\equiv\begin{pmatrix}M & K\\ -K^T & vI_n\end{pmatrix}\begin{pmatrix}y\\ x\end{pmatrix}=\begin{pmatrix}f\\ 0\end{pmatrix}\equiv b. \tag{3}$$

There are many efficient iterative methods devoted to solving the linear system (3) over the past few years, such as the Hermitian and skew-Hermitian splitting (HSS) iterative method

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[1], the Krylov subspace methods [5] and so on. The HSS method and its corresponding HSS preconditioner [2] have received further attention. For solving linear system (3), several preconditioners have been introduced in recent years. For instance, according to special properties of weighted Toeplitz matrix, Ng and Pan [4] established a new HSS (NHSS) iterative method and constructed corresponding preconditioner of the form

$$\mathscr{P}_{NHSS} = \begin{pmatrix} \frac{1}{2\alpha}(\alpha I_m + M) & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} \alpha I_m & K \\ -K^T & vI_n \end{pmatrix}, \qquad \alpha > 0. \tag{4}$$

By introducing the other parameter in the NHSS scheme, a generalized NHSS (GNHSS) method and the corresponding GNHSS preconditioner were developed in [3]. More precisely, the following preconditioner has been suggested

$$\mathscr{P}_{GNHSS} = \begin{pmatrix} \frac{1}{\alpha+\beta}(\alpha I_m + M) & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} \beta I_m & K \\ -K^T & vI_n \end{pmatrix}, \qquad \alpha,\ \beta > 0. \tag{5}$$

In the sequel we present two preconditioners for solving weighted Toeplitz regularized least-squares problems. Consider the following splittings for the coefficient matrix $\mathscr{A}$ in (3):

$$\mathscr{A} = \mathscr{P}_1 - \mathscr{R}_1 = \begin{pmatrix} M & 2K \\ -K^T & 2vI_n \end{pmatrix} - \begin{pmatrix} 0 & K \\ 0 & vI_n \end{pmatrix}, \tag{6a}$$

$$\mathscr{A} = \mathscr{P}_2 - \mathscr{R}_2 = \begin{pmatrix} M & (1+\frac{v}{\alpha})K \\ -K^T & vI_n \end{pmatrix} - \begin{pmatrix} 0 & \frac{v}{\alpha}K \\ 0 & 0 \end{pmatrix}, \qquad \alpha > 0. \tag{6b}$$

Then, the iterative methods based on the splittings (6a) and (6b) can be derived as follows:

Method: Let $z^{(0)} = (y^{(0)}; x^{(0)}) \in \mathbb{C}^{m+n}$ be an initial guess. By using the following iterative scheme, for $k = 0, 1, 2, \ldots$, compute $z^{(k+1)} = (y^{(k+1)}; x^{(k+1)})$ until $z^{(k)} = (y^{(k)}; x^{(k)})$ converges:

$$\begin{pmatrix} M & 2K \\ -K^T & 2vI_n \end{pmatrix} \begin{pmatrix} y^{(k+1)} \\ x^{(k+1)} \end{pmatrix} = \begin{pmatrix} 0 & K \\ 0 & vI_n \end{pmatrix} \begin{pmatrix} y^{(k)} \\ x^{(k)} \end{pmatrix} + \begin{pmatrix} f \\ 0 \end{pmatrix}, \tag{7a}$$

$$\begin{pmatrix} M & (1+\frac{v}{\alpha})K \\ -K^T & vI_n \end{pmatrix} \begin{pmatrix} y^{(k+1)} \\ x^{(k+1)} \end{pmatrix} = \begin{pmatrix} 0 & \frac{v}{\alpha}K \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y^{(k)} \\ x^{(k)} \end{pmatrix} + \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{7b}$$

We define the iterative schemes (7a) and (7b) to solve linear system (3) as follows:

$$z^{(k+1)} = \Gamma z^{(k)} + c, \quad k = 0, 1, 2, \ldots, \tag{8}$$

where Γ is the iteration matrix and $c = \Gamma^{-1}b$.

2. Main results

It is well-known that iterative methods (8) are convergent for any initial guess $z^{(0)}$ if and only if $\rho(\Gamma) < 1$, where $\rho(\Gamma)$ is the spectral radius of the iteration matrix. For the splitting in (6a), the iteration matrix is given by

$$\Gamma_1 = \begin{pmatrix} M & 2K \\ -K^T & 2vI_n \end{pmatrix}^{-1} \begin{pmatrix} 0 & K \\ 0 & vI_n \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}I_n \end{pmatrix}. \tag{9}$$

THEOREM 2.1. *let $M \in \mathbb{R}^{m\times m}$ be an SPD matrix, $K \in \mathbb{R}^{m\times n}$ be a full-rank Toeplitz matrix, and Γ_1 be defined as in (9). Then $\rho(\Gamma_1) < 1$, that is, the iterative scheme in (7a) converges to the solution of (3) unconditionally.*

PROOF. Let λ be an arbitrary eigenvalue of $\Gamma_1 = \mathscr{P}_1^{-1}\mathscr{R}_1$. From (9), it is easy to check that the matrix Γ_1 has eigenvalues $\lambda = 0, \frac{1}{2}$ with multiplicity m and n, respectively. The result follows immediately from the fact that $\max|\lambda| < 1$. □

REMARK 2.2. Under the same assumptions of Theorem 2.1, the eignvaluse of $\mathscr{P}_1^{-1}\mathscr{A}$ are $\theta = 1, \frac{1}{2}$.

The preconditioner $\mathscr{P}_2$ in (6b) can be factorized as

$$\mathscr{P}_2 = \begin{pmatrix} M & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_m & 0 \\ -K^T & I_n \end{pmatrix} \begin{pmatrix} I_m & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I_m & \left(1+\frac{v}{\alpha}\right) M^{-1}K \\ 0 & I_n \end{pmatrix}, \tag{10}$$

where $S = vI_n + (1+\frac{v}{\alpha})K^TM^{-1}K$. Then the iteration matrix for this preconditioner is obtained as

$$\Gamma_2 = \mathscr{P}_2^{-1}\mathscr{R}_2 = \begin{pmatrix} 0 & \Omega \\ 0 & \Phi \end{pmatrix}, \tag{11}$$

where $\Phi = \frac{v}{\alpha} S^{-1}K^TM^{-1}K$. As the form Ω is not the focus in argument, we do not write it here.

THEOREM 2.3. *Let the conditions in Theorem (2.1) be satisfied. Then it holds that $\rho(\Gamma_2) < 1$, i.e., the iterative scheme in (7b) converges to the solution of (3) unconditionally.*

PROOF. From (11), it is clear that Γ_2 has an eignvalue 0 with algebraic multiplicity m and its remaining eigenvalues are given by those of Φ. Let (ξ, x) be the eigenpair of the matrix Φ. Without loss of generality, we can assume that $||x||_2 = 1$. Then, we have

$$vK^TM^{-1}Kx = \alpha\xi Sx. \tag{12}$$

Multiplying both sides of the relation (12) on the left by x^T, we obtain

$$v\, x^TK^TM^{-1}Kx = \alpha\xi\Big(vx^Tx + (1+\frac{v}{\alpha})\, x^TK^TM^{-1}Kx\Big). \tag{13}$$

We set quantity $\eta = x^TK^TM^{-1}Kx$. It then follows from (13) that $\xi = \frac{v\eta}{v\eta+\alpha(v+\eta)}$. In view of the positive definiteness of $K^TM^{-1}K$, we can see that $|\xi| < 1$, which completes the proof. □

REMARK 2.4. From (11), we can obtain

$$\mathscr{P}_2^{-1}\mathscr{A} = I - \Gamma_2 = \begin{pmatrix} I & -\Omega \\ 0 & I-\Phi \end{pmatrix}, \tag{14}$$

then the m eigenvalues of matrix $\mathscr{P}_2^{-1}\mathscr{A}$ are equal to 1 and other eigenvalues are of the form $\lambda = \frac{\alpha(v+\eta)}{v\eta+\alpha(v+\eta)}$.

Now we will give the implementation of these two preconditioners within a Krylov subspace method. Let $r = (r_1; r_2)$ and $z = (z_1; z_2)$ with r_1, $z_1 \in \mathbb{R}^m$ and r_2, $z_2 \in \mathbb{R}^n$. We have the following algorithms for these two iterative methods.

Algorithm 1: Computation of $z = \mathscr{P}_1^{-1}r$
1. Compute $d = r_1 - \frac{1}{v}Kr_2$;
2. Solve $(M + \frac{1}{v}KK^T)z_1 = d$;
3. Compute $z_2 = \frac{1}{2v}(K^Tz_1 + r_2)$.

Algorithm 2: Computation of $z = \mathscr{P}_2^{-1}r$
1. Compute $d = r_1 - \omega Kr_2$; with $\omega = \frac{\alpha+v}{\alpha v}$,
2. Solve $(M + \omega KK^T)z_1 = d$;
3. Compute $z_2 = \frac{1}{v}(K^Tz_1 + r_2)$.

3. Numerical results

Let us now consider the one-dimensional examples which are tested in [3,4], where K is a square Toeplitz matrix defined by

$$\text{(a)} \qquad K = (t_{ij}) \in \mathbb{R}^{n\times n} \quad \text{with } t_{ij} = \frac{1}{\sqrt{|i-j|+1}},$$

$$\text{(b)} \qquad K = (t_{ij}) \in \mathbb{R}^{n\times n} \quad \text{with } t_{ij} = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-|i-j|^2}{2\sigma^2}}.$$

The matrix K is well-conditioned in case (a), and it is highly ill-conditioned for case (b) with $\sigma = 2$. In our tests, we choose Ξ as a positive diagonal random matrix and scale its diagonal entries so that its condition number is around 10^3. The initial guess is the zero vector and the right-hand side vector is $b = (f; 0)$ with f being the vector of all entries equal to one. The iteration will be stopped whenever $||b - \mathscr{A} z^{(k)}||_2/||b||_2 < 10^{-6}$. Two methods are applied as preconditioners with GMRES method. The linear sub-systems in preconditioning step are solved by the Cholesky factorization. Computations were done on a 64-bit 1.80 GHz core i7 processor and 12.00GB RAM using Matlab version 2017a.

n	Pre	α	IT	CPU
2^{10}	$\mathscr{P}_1$	–	1	0.0952
	$\mathscr{P}_2$	0.1	1	0.0910
2^{11}	$\mathscr{P}_1$	–	1	0.5306
	$\mathscr{P}_2$	0.1	1	0.5410

TABLE 1. Numerical results for case (a)

n	Pre	α	IT	CPU
2^{10}	$\mathscr{P}_1$	–	2	0.1530
	$\mathscr{P}_2$	0.1	3	0.1721
2^{11}	$\mathscr{P}_1$	–	1	0.5448
	$\mathscr{P}_2$	0.1	3	0.7412

TABLE 2. Numerical results for case (b)

n	$\mathscr{A}$	$\mathscr{P}_1^{-1}\mathscr{A}$	$\mathscr{P}_2^{-1}\mathscr{A}$
2^{10}	4.6627e+16	2.0660	1.0104
2^{11}	1.2713e+15	2.000	1.6189

TABLE 3. Condition number for case (b)

In Tables 1 and 2, we report the number of iterations (IT) and the CPU time (CPU) for preconditioned GMRES Method with values of $n = 2^{10}, 2^{11}$ and $v = 0.001$. Table 3 shows the estimated condition number, as computed by MATLAB's condest function, for the related systems in case (b). Numerical results confirmed the effciency of the propsed preconditioners.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Pretty clean monomial ideals

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Abstract. Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. In this paper, we give some results for sum, product and colon of clean (pretty clean) monomial ideals. We also generalize Soleyman Jahan's result from monomial ideals with at most 3 variables to monomial ideals with number of arbitrary variables. Indeed, we prove that if $I = uJ$ is a monomial ideal of S, where u is a monomial in S, and J is a monomial ideal of height ≥ 2, then I is pretty clean if and only if J is pretty clean.

Keywords: pretty clean, clean, monomial ideals, simplicial complex, shellable

AMS Mathematics Subject Classification [2010]: 13F20, 13F55, 05E40

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1. Introduction

Throughout this paper, K is a field and I is a monomial ideal of the polynomial ring $S = K[x_1, \ldots, x_n]$.

A chain

$$\mathcal{F} : I = I_0 \subset I_1 \subset \cdots \subset I_r = S$$

of monomial ideals is called a monomial prime filtration of S/I, if for all $i = 1, \ldots, r$ there exists a monomial prime ideal p_i such that $I_i/I_{i-1} \cong S/p_i$. The set of prime ideals $p_1, \ldots, p_r$ is denoted by $\operatorname{Supp}(\mathcal{F})$. Let $\operatorname{Min}(I)$ denotes the set of minimal prime ideals of $\operatorname{Supp}(S/I)$. Then, we have

$$\operatorname{Min}(I) \subseteq \operatorname{Ass}(S/I) \subseteq \operatorname{Supp}(\mathcal{F}) \subseteq \operatorname{Supp}(S/I).$$

A prime filtration $\mathcal{F}$ of S/I is called clean (cf. [1]) if $\operatorname{Supp}(\mathcal{F}) = \operatorname{Min}(I)$. It is pretty clean (cf. [2]) if for all $i < j$ for which $p_i \subseteq p_j$ it follows that $p_i = p_j$. Furthermore I is called pretty clean (clean) if S/I admits a pretty clean (clean) prime filtration. It is clear that if I is clean, then it is pretty clean. For a squarefree monomial ideal I, Since $\operatorname{Ass}(S/I) = \operatorname{Min}(I)$, it follows by [2, Corollary 3.5], I is clean if and only if I is pretty clean.

Our aim of this paper is to investigate clean and pretty clean monomial ideals. We study sum, product and colon of clean (pretty clean) monomial ideals. We also generalize Soleyman Jahan's result ([3, Lemma 1.9]) from monomial ideals with at most 3 variables to monomial ideals with number of arbitrary variables. Indeed, we prove that if $I = uJ$

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is a monomial ideal of S, where u is a monomial in S, and J is a monomial ideal of height ≥ 2, then I is pretty clean if and only if J is pretty clean.

2. Main results

A simplicial complex Δ on the vertex set $[n] = \{1, \dots, n\}$ is a collection of subsets of $[n]$ with the property that if $F \in \Delta$, then all subsets of F are also in Δ. An element of Δ is called a face of Δ, and the maximal faces of Δ, under inclusion, are called facets. We denote by $\mathcal{F}(\Delta)$ the set of all facets of Δ. When $\mathcal{F}(\Delta) = \{F_1, \dots, F_r\}$, we write $\Delta =< F_1, \dots, F_r >$. For each $F \in \Delta$, we set $\dim F = |F| - 1$. A simplicial complex Δ is called pure if all facets of Δ have the same dimension.

The Stanley-Reisner ideal I_Δ is a monomial ideal of $S = K[x_1, \dots, x_n]$ generated by all squarefree monomials $x_{i_1} x_{i_2} \cdots x_{i_k}$ such that $\{i_1, \dots, i_k\}$ is not a face of Δ. For $F \subseteq [n]$, we set $X_F = \prod_{i \in F} x_i$.

Definition 2.1. Let Δ be a simplicial complex on $[n] = \{1, \dots, n\}$. We say that Δ is (non-pure) shellable if there exists an order $F_1, \dots, F_r$ of Δ such that for $i = 2, \dots, r$ the simplicial complex $\langle F_1, \dots, F_{i-1} \rangle \cap \langle F_i \rangle$ is a pure $(\dim F_i - 1)$-dimensional simplicial complex.

Theorem 2.2. *([1, Theorem on page 53]) A simplicial complex Δ is (non-pure) shellable if and only if I_Δ is a clean ideal.*

Lemma 2.3. *Let I and J be monomial ideals of S. Then we have the following:*

a) *I and J are clean (pretty clean) $\nRightarrow$ IJ is clean (pretty clean).*

b) *I and J are clean (pretty clean) $\nRightarrow$ $I \cap J$ is clean (pretty clean).*

Let $I = (x_1, x_2)$ and $J = (x_3, x_4)$. Then I and J are clean (pretty clean). Since $\Delta =< \{1, 2\}, \{3, 4\} >$ is not shellable, so by Theorem 2.2, $IJ = I \cap J = I_\Delta$ is not clean (pretty clean).

c) *I and J are clean (pretty clean) $\nRightarrow$ $I + J$ is clean (pretty clean).*

Let $I = I_{\Delta_1}$ and $J = I_{\Delta_2}$, where $\Delta_1 =< \{1, 2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{4, 6\} >$ and $\Delta_2 =< \{1, 2\}, \{1, 5\}, \{5, 6\}, \{2, 6\} >$. Since Δ_1 and Δ_2 are shellable, it follows by Theorem 2.2 that I and J are clean (pretty clean). But $\Delta_1 \cap \Delta_2 =< \{1, 2\}, \{5, 6\} >$ is not shellable. So $I + J$ is not clean (pretty clean), because,

$$\begin{aligned} I + J = I_{\Delta_1} + I_{\Delta_2} &= (X_F \mid F \notin \Delta_1) + (X_F \mid F \notin \Delta_2) \\ &= (X_F \mid F \notin \Delta_1 \cap \Delta_2) = I_{\Delta_1 \cap \Delta_2}. \end{aligned}$$

The saturation $\widetilde{I}$ of I is defined to be $\widetilde{I} = I : \mathfrak{m}^\infty = \bigcup_k I : \mathfrak{m}^k$, where $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of S.

Lemma 2.4. *([3, Lemma 1.1]) Let I be a monomial ideal of S. The ideal I is pretty clean if and only if $\widetilde{I}$ is pretty clean.*

Lemma 2.5. *Let I be a monomial ideal of S. Then $\widetilde{I} = \widetilde{I\mathfrak{m}^i}$ for all $i \in \mathbb{N}$.*

Proof. It is enough to show that $\widetilde{I} = \widetilde{I\mathfrak{m}}$. Since $I\mathfrak{m} \subseteq I$, it implies that $\widetilde{I\mathfrak{m}} \subseteq \widetilde{I}$. Now let $u \in \widetilde{I}$, so $u\mathfrak{m}^k \subseteq I$ for some k. Then $u\mathfrak{m}^{k+1} \subseteq I\mathfrak{m}$, which implies that $u \in \widetilde{I\mathfrak{m}}$. □

Proposition 2.6. *Let I be a monomial ideal of S. Then I is pretty clean if and only if $I\mathfrak{m}^i$ is pretty clean for all $i \in \mathbb{N}$.*

PROOF. Let $i \in \mathbb{N}$. By Lemma 2.4 and Lemma 2.5, we have that

$$\begin{aligned} I \text{ is pretty clean} &\Leftrightarrow \widetilde{I} \text{ is pretty clean} \\ &\Leftrightarrow \widetilde{I\mathfrak{m}^i} \text{ is pretty clean} \\ &\Leftrightarrow I\mathfrak{m}^i \text{ is pretty clean.} \end{aligned}$$

□

Lemma 2.7. *Let I be a monomial ideal of S. Then $\widetilde{I} = \widetilde{I : \mathfrak{m}^i}$ for all $i \in \mathbb{N}$.*

PROOF. Let $i \in \mathbb{N}$, so $I \subseteq I : \mathfrak{m}^i \subseteq \widetilde{I}$. Hence $\widetilde{I} \subseteq \widetilde{I : \mathfrak{m}^i} \subseteq \widetilde{\widetilde{I}} = \widetilde{I}$. Then $\widetilde{I} = \widetilde{I : \mathfrak{m}^i}$. □

Proposition 2.8. *Let I be a monomial ideal of S. Then I is pretty clean if and only if $I : \mathfrak{m}^i$ is pretty clean for all $i \in \mathbb{N}$.*

PROOF. Let $i \in \mathbb{N}$. By Lemma 2.4 and Lemma 2.7, we have that

$$\begin{aligned} I \text{ is pretty clean} &\Leftrightarrow \widetilde{I} \text{ is pretty clean} \\ &\Leftrightarrow \widetilde{I : \mathfrak{m}^i} \text{ is pretty clean} \\ &\Leftrightarrow I : \mathfrak{m}^i \text{ is pretty clean.} \end{aligned}$$

□

Remark 2.9. Let the chain of monomial ideals

$$\mathcal{F} : I = I_0 \subset I_1 \subset \cdots \subset I_r = S,$$

be a monomial prime filtration of S/I. Then it is satisfies in the following conditions:

a) $I_i = (I_{i-1}, u_i)$, where u_i is a monomial of S for $i = 1, \ldots, r$,
b) $I_i/I_{i-1} \cong S/p_i$, where $p_i = (I_{i-1} : u_i)$ is a monomial prime ideal of S for $i = 1, \ldots, r$.

In the next, we generalize Soleyman Jahan's result ([3, Lemma 1.9]) from monomial ideals with at most 3 variables to monomial ideals with number of arbitrary variables.

Proposition 2.10. *Let I be a monomial ideal of $S = K[x_1, \ldots, x_n]$. If $I = uJ$, where u is a monomial in S, and J is a monomial ideal of* height ≥ 2, *then I is pretty clean if and only if J is pretty clean.*

PROOF. Let I be a pretty clean ideal. So by Remark 2.9, there exists a pretty clean filtration

$$\mathcal{F} : I = I_0 = uJ \subset I_1 \subset \cdots \subset I_r = S$$

of S/I such that

a) $I_i = (I_{i-1}, u_i)$, where u_i is a monomial of S for $i = 1, \ldots, r$,
b) $I_i/I_{i-1} \cong S/p_i$, where $p_i = (I_{i-1} : u_i)$ is a monomial prime ideal of S for $i = 1, \ldots, r$.

If $I_i : u \neq I_{i-1} : u$, then

$$\frac{I_i : u}{I_{i-1} : u} = \frac{(I_{i-1} : u) + ((u_i) : u)}{I_{i-1} : u} \cong \frac{S}{(I_{i-1} : uv)} = \frac{S}{p_i},$$

where $v = u_i/\gcd(u, u_i)$.

The last equality follows from the fact that $u/gcd(u,u_i) \notin p_i$. If $u/gcd(u,u_i) \in p_i$, then $uv = uu_i/gcd(u,u_i) \in I_{i-1}$, so $v \in (I_{i-1} : u)$. Therefore

$$I_i : u = (I_{i-1}, u_i) : u = (I_{i-1} : u) + ((u_i) : u) = (I_{i-1} : u) + (v) = I_{i-1} : u,$$

which is a contradiction.

Therefore

$$\mathcal{F}' : J = I : u \subseteq I_1 : u \subseteq \cdots \subseteq I_r : u = S,$$

is a monomial prime filtration of S/J with $\operatorname{Supp}(\mathcal{F}') \subseteq \operatorname{Supp}(\mathcal{F})$. Hence J is a pretty clean ideal .

Converse follows by [3, Lemma 1.9]. For convenience, we bring the proof. Let J be a pretty clean ideal. By [2, Corolarry 3.4], there is a prime filtration

$$\mathcal{F} : J = J_0 \subset J_1 \subset \cdots \subset J_r = S,$$

such that $J_i/J_{i-1} \cong S/p_i$, where $p_i \in \operatorname{Supp}(S/I) = \operatorname{Ass}(S/J)$, so height $p_i \geq 2$. By the prime filtration $\mathcal{F}$, there exists a chain of monomials

$$\mathcal{F}_1 : I = uJ \subset uJ_1 \subset \cdots \subset uJ_r = (u),$$

such that $uJ_i/uJ_{i-1} \cong S/p_i$.

On the other hand, there exists a prime filtration

$$\mathcal{F}_2 : (u) = uJ_r \subset J_{r+1} \subset \cdots \subset J_{r+l} = S,$$

of $S/(u)$, where J_{r+k} are principal monomial ideals with $J_{r+k}/J_{r+k-1} \cong S/q_k$ and where $q_k \in \operatorname{Ass}(S/(u))$ has height 1 for $k = 1, \ldots, l$. In fact, if $u = u_0 = \prod_{t=1}^{s} x_{i_t}^{a_t}$ and $u_j = \prod_{r=j+1}^{s} x_{i_r}^{a_r}$ for $j = 1, \ldots, s-1$, then the prime filtration $\mathcal{F}_2$ is the following:

$$\mathcal{F}_2 : J_r = (u) \subset (x_{i_1}^{a_i-1}u_1) \subset (x_{i_1}^{a_i-2}u_1) \subset \cdots \subset (u_1) \subset (x_{i_2}^{a_2-1}u_2) \subset \cdots \subset (u_2) \subset \cdots$$
$$\subset (x_{i_s}) \subset S.$$

Therefore $\mathcal{F}_2$ is a pretty clean filtration. Now composing the above filtration $\mathcal{F}_1$ and $\mathcal{F}_2$ we obtain a pretty clean filtration of S/I. □

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Characterization of generalized matrix Banach algebras

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Abstract. In this paper, the Banach algebras which can be assumed as generalized matrix Banach algebras will be characterized. Then we show that there is a Banach algebra A which can not be assumed as a triangular Banach algebra, but $H^1(A, A) = 0$. This example gives a negative answer to the open question raised by Bennis and Fahid "Does the condition $H^1(A \oplus X, A \oplus X) = 0$ imply that $A \oplus X$ has a triangular matrix representation?"

Keywords: Banach algebra, idempotent, generalized matrix Banach algebra

AMS Mathematics Subject Classification [2010]: 46H25, 46M18

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1. Introduction

Let A and B be two Banach algebras, M be an (A, B)-module (i.e. a left A-module and a right B-module with compatible actions) and N be a (B, A)-module (i.e. a left B-module and a right A-module with compatible actions). Also, let $\Phi : M \times N \to A$ and $\Psi : N \times M \to B$ be two bounded bilinear mappings which are bimodule morphisms on each of their coordinates and satisfying the following equalities.

$$m(\Psi(n, m')) = (\Phi(m, n))m' \quad and \quad n(\Phi(m, n')) = (\Psi(n, m))n' \qquad (n, n' \in N, m, m' \in M).$$

Then $G = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ is a Banach algebra with the norm

$$\left\| \begin{matrix} a & m \\ n & b \end{matrix} \right\|_G = ||a|| + ||m|| + ||b|| + ||n||,$$

which is called a generalized matrix Banach algebra.

The generalized matrix algebras were introduced by Sands in [4]. Obviously, when $M = 0$ or $N = 0$, G exactly degenerates to the so-called triangular algebra. When $\Phi = 0$ and $\Psi = 0$ such kinds of generalized matrix algebras are called *trivial generalized matrix algebras*. For example, the module extension Banach algebra $(A \dot{+} B) \oplus (M \dot{+} N)$, which is defined in [5], where $(A \dot{+} B)$ is the direct l_1-sum of Banach algebras A and B, and $(M \dot{+} N)$ as an l_1-direct sum of modules is an $(A \dot{+} B)-$module with the module actions

$$(a, b).(m, n) = (am, bn), \quad (m, n).(a, b) = (mb, na) \quad (a \in A, b \in B, m \in M, n \in N),$$

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is a trivially generalized matrix algebra.

Let A be a Banach algebra and let $\mathcal{X}$ be a Banach A-module. The dual $\mathcal{X}^*$ of $\mathcal{X}$ with the module operations

$$(fa)(x) = f(ax), \quad (af)(x) = f(xa), \quad (a \in A, x \in \mathcal{X}, f \in \mathcal{X}^*);$$

is a Banach A-module. Also, the second dual $\mathcal{X}^{**}$ of $\mathcal{X}$ under the module operations

$$(aF)(f) = F(fa), \quad (Fa)(f) = F(af), \quad (a \in A, f \in \mathcal{X}^*, F \in \mathcal{X}^{**});$$

is a Banach A-module. Similarly, the n-th dual $\mathcal{X}^{(n)}$ of $\mathcal{X}$ may be considered as a Banach A-module. In particular, for $\mathcal{X} = A$, $A^{(n)}$ is a Banach A-module.

A derivation D is a bounded linear operator $D : A \to \mathcal{X}$ such that

$$D(ab) = D(a)b + aD(b), \quad (a, b \in A).$$

A derivation D is called an inner derivation if there exists $x \in \mathcal{X}$ such that

$$D(a) = \delta_x(a) = ax - xa, \quad (a \in A).$$

Let $Z^1(A, \mathcal{X})$ be the linear space of all derivations from A into $\mathcal{X}$, and $N^1(A, \mathcal{X})$ be the linear subspace of all inner derivations from A into $\mathcal{X}$. Then the cohomology group of A with coefficients in $\mathcal{X}$ is denoted by $H^1(A, \mathcal{X}) = \frac{Z^1(A,\mathcal{X})}{N^1(A,\mathcal{X})}$.
The concept of n-weak amenability and permanent weak amenability of Banach algebras were introduced by Dales et al [3]. A Banach algebra A is called n-weakly amenable if $H^1(A, A^{(n)}) = 0$, and A is said to be permanently weakly amenable if it is n-weakly amenable for all $n \geq 1$.

In this paper, we try to investigate the Banach algebras which can be considered as generalized matrix Banach algebras. That is the Banach algebras such as $\mathfrak{A}$, that there exist the algebras A, B, (A, B)-module M and (B, A)-module N such that $\mathfrak{A}$ is isometric isomorphic to $\begin{bmatrix} A & M \\ N & B \end{bmatrix}$.

Note that we can consider every Banach algebra as a (trivial) generalized matrix Banach algebra with $M = 0, N = 0$ and $B = 0$. But the purpose of this paper is to investigate the Banach algebras which have the (non-trivial) generalized matrix representations.

2. Main results

The following theorem characterizes all unital Banach algebras which may be considered as a generalized Matrix Banach algebra.

THEOREM 2.1. *A unital Banach algebra $\mathfrak{A}$ can be considered as a generalized Matrix Banach algebra if and only if it has a non-trivial idempotent.*

Note that if $eae = a$ for each $a \in \mathfrak{A}$, and $e\mathfrak{A}(1-e), (1-e)\mathfrak{A}e$, and $(1-e)\mathfrak{A}(1-e)$ are zero, then $\mathfrak{A}$ has the trivial generalized Matrix representation and $e = 1$.

COROLLARY 2.2. *[2, Theorem 5.1.4] A unital Banach algebra $\mathfrak{A}$ has a triangular representation if and only if it has a non-trivial idempotent e such that $(1-e)\mathfrak{A}e = 0$.*

THEOREM 2.3. *Let A be a unital Banach algebra and let X be a unital Banach A-module. Then the module extension Banach algebra $A \oplus X$ can be considered as a generalized Matrix Banach algebra if and only if A can be considered as a generalized Matrix Banach algebra.*

If we denote the nontrivial idempotents of the algebra A by $I(A)$, then the Theorems 2.1 and 2.3 imply the following corollary.

Corollary 2.4. *$I(A \oplus X) = \emptyset$ if and only if $I(A) = \emptyset$.*

Theorem 2.5. *Let G be a locally compact group. Then $\ell^1(G)$ doesn't have any matrix representation and in addition $H^1(\ell^1(G), \ell^(G)^{(n)}) = \{0\}$, for each $n \in \mathbb{N} \cup \{0\}$.*

Now Theorems 2.3 and 2.5 imply the following theorem, which gives a negative answer to the question raised in [1].

Theorem 2.6. *There is a permanently weakly amenable module extension Banach algebra $A \oplus X$ without any (triangular) matrix representation.*

3. Conclusion

The only unital Banach algebras which have generalized Matrix representations are those which have at least one non-trivial idempotent. matrix representation of the module extension Banach algebra $A \oplus X$ depends only on the matrix representation of A. Even permanently weakly amenable module extension Banach algebras may not have a matrix representation.

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۸ تا ۱۱ شهریور ۱۴۰۰

$M-$ideals in $MV-$algebras

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Abstract. In this paper $M-$ideals has been introduced and their properties have been studied. We examined their relationship to the minimal prime ideals. Also, We studied their behavior under homomorphisms on $MV-$algebra.

Keywords: $MV-$algebra, Maximal ideal, $M-$ideal.

AMS Mathematics Subject Classification [2010]: 06D35, 06B10

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1. Introduction

$MV-$algebras are introduced by C. C. Chang in 1958 [1] as an algebraic counterpart of the Łukasiewicz infinite valued propositional logic. In order to keep the paper brief, we refer the reader to [1, 6] for results on $MV-$algebras. In particular, emphasis has been put the ideal theory of $MV-$ algebra [3]. F. Forouzesh et al, introduced obstinate ideals of an $MV-$ algebra [5]. In this paper, we introduce $M-$ideals and provide equivalent conditions for it. In addition, it has been investigated some relationships between such ideals and Minimal prime ideals. It is also shown that every prime ideal of an $MV-$ algebra contains a minimal prime ideal We recollect some definitions and results which will be used in the sequel:

Definition 1.1. [1] An $MV-$algebra is a structure $(A, \oplus, ^*, 0)$ where $\oplus$ is a binary operation, *, is a unary operation, and 0 is a constant such that the following axioms are satisfied for any $x, y \in A$:
$(MV1)$ $(A, \oplus, 0)$ is an abelian monoid;
$(MV2)$ $(x^*)^* = x$;
$(MV3)$ $0^* \oplus x = 0^*$;
$(MV4)$ $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Note that we have $1 = 0^*$ and the auxiliary operation $\odot$ which are as follows:

$$x \odot y = (x^* \oplus y^*)^*.$$

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We recall that the natural order determines a bounded distributive lattice structure such that

$$x \vee y = x \oplus (x^* \odot y) = y \oplus (x \odot y^*) \quad \text{and} \quad x \wedge y = x \odot (x^* \oplus y) = y \odot (y^* \oplus x).$$

Also for any two elements $x, y \in A$, $x \leq y$ if and only if $x^* \oplus y = 1$ if and only if $x \odot y^* = 0$. Obviously, $\leq$ is a partial order on A which is called the natural order on A.

In this paper, A is an $MV-$algebra.

Definition 1.2. [2] An ideal of A is a nonempty subset I of A satisfying the following conditions:
$(I1)$ If $x \in I,\ y \in A$ and $y \leq x$, then $y \in I$,
$(I2)$ If $x, y \in I$, then $x \oplus y \in I$.
We denote by $Id(A)$ the set of all ideals of A.

Definition 1.3. [2] Let I be an ideal of A. If $I \neq A$, then I is a proper ideal of A.
• [2] A proper ideal I of A is called prime ideal if for all $x, y \in A, x \wedge y \in I$, then $x \in I$ or $y \in I$.
We denote by $Spec(A)$ the set of all prime ideals of an $MV-$algebra A.
• [2] An ideal P of A is called a minimal prime ideal of A when:
1)$P \in Spec(A)$;
2) $If\ there\ exists\ Q \in Spec(A)\ such\ that\ Q \subseteq P,\ then\ P = Q.$
We denote by $Min(A)$ the set of all minimal prime ideals of A.
• [6] A proper ideal I of A is called maximal if and only if for each ideal $J \neq I$, if $I \subseteq J$, then $J = A$.
We denote by $Max(A)$ the set of all maximal ideals of A.

Note: Minimal prime ideal P of A is called minimal prime ideal over ideal I, if
1) $I \subseteq P$;
2) $If\ there\ exists\ Q \in Min(A)\ such\ that\ I \subseteq Q \subseteq P,\ then\ P = Q.$
We denote by $Min(I)$ the set of all minimal prime ideals over ideal I.

Corollary 1.4. *[6] Every prime ideal of A is contained in a unique maximal ideal of A.*

Theorem 1.5. *[6] Let S be a $\wedge-$closed system of A and $I \in Id(A)$ such that $I \cap S = \emptyset$. Then there exists a prime ideal P of A such that $I \subseteq P$ and $P \cap S = \emptyset$.*

Note: Let $a \in A$. Define $P_a = \bigcap\{P : P \in Min(A), a \in P\}$.

$$M_a = \bigcap\{M : M \in Max(A), a \in M\} \qquad M(a) = \{M : M \in Max(A), a \in M\}$$

2. $M-$ideals

Definition 2.1. A proper ideal I of A is called an $M-$ideal if $M_a \subseteq I$, for each $a \in I$.

Example 2.2. (1) Let $A = \{0, a, b, c, d, 1\}$. where $0 < a, c < d < 1$ and $0 < a < b < 1$. Define $\oplus$ and $*$ as follows:

$*$	0	a	b	c	d	e	f	1
	1	f	e	d	c	b	a	0

$\oplus$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	a	c	c	e	e	1	1
b	b	c	b	c	f	1	f	1
c	c	c	c	c	1	1	1	1
d	d	e	f	1	f	1	f	1
e	e	e	1	1	1	1	1	1
f	f	1	f	1	f	1	f	1
1	1	1	1	1	1	1	1	1

Then $(A, \oplus, *, 0, 1)$ is an MV-algebra [6]. It has four ideals: $I_0 = \{0\}, I_1 = \{0, c\}, I_2 = \{0, a, b\}, I_3 = A$. Obviously, I_0, I_1 and I_2 are $M-$ideal.
(2) Let $\mathbb{R}^*$ be a nonstandard model of real numbers with natural order and ε be a positive infinitesimal element of $\mathbb{R}^*$. Let $\varepsilon^2 = \varepsilon.\varepsilon, ..., \varepsilon^n = \varepsilon.\varepsilon...\varepsilon(n - times)$, where is the usual product in the field $\mathbb{R}^*$; then $\varepsilon^i > 0$, for any $i \in \mathbb{N}$ and $\varepsilon^i \leqslant \varepsilon^j$, for $i > j$.
The unit interval $[0, 1]^* \subseteq \mathbb{R}^*$ is an $MV-$algebra under the operations: $x \oplus y = min\{1, x + y\}, x^* = 1 - x$. Let $\mathbb{N}$ be the ordered set of positive natural numbers. For every $n \in \mathbb{N}$, let E_n be the subalgebra of $[0, 1]^*$ generated by $\{\varepsilon, \varepsilon^2, ..., \varepsilon^n\}$ and E be the subalgebra $\bigcup\limits_{n \in \mathbb{N}} E_n$ we recall that [4], $E =< (\varepsilon^i) \mid i \in \mathbb{N} >$. The set of all ideals of E are $\{0\}, < \varepsilon >, ..., < \varepsilon^i > ...$,where $i \in \mathbb{N}$ and $< \varepsilon^i > \subseteq < \varepsilon^j >$, for any $i > j$. Obviously, $< \varepsilon^2 >$ is not an $M-$ideal.

Theorem 2.3. *Let I be a proper ideal of A. Then the following statements are equivalent:*
(1) I is a $M-$ideal in A,
(2) $M(a) \subseteq M(b)$ and $a \in I$, imply that $b \in I$,
(3) $M(a) = M(b)$ and $a \in I$, imply that $b \in I$.

Proof. $1 \Rightarrow 2)$ Let $M(a) \subseteq M(b)$ and $a \in I$. Then $M_b \subseteq M_a$, so $b \in I$.
$2 \Rightarrow 1)$ Let there exists $a \in I$ such that $M_a \nsubseteq I$. Then there exists $b \in M_a$ such that $b \notin I$. Obviously, $M(a) \subseteq M(b)$ so $b \in I$, which is a contradiction.
$2 \Rightarrow 3)$ It is clear.
$3 \Rightarrow 2)$ Let $M(a) \subseteq M(b)$ and $a \in I$. Obviously, $a \wedge b \in I$ and $M(b) = M(a \wedge b)$, hence $b \in I$. □

Proposition 2.4. *Let $f : A \to B$ be a $MV-$homomorphism. Then every $M-$ideal of B contracts to an $M-$ideal of A if and only if every maximal ideal of B contracts to an $M-$ideal.*

Proof. Let J be an $M-$ideal of $B, M_A(a) = M_A(b)$ and $a \in J^c$. Then $f(a) \in J$. We show that $M_B(f(a)) = M_B(f(b))$. Let $M \in Max(B)$ and $f(a) \in M$. Then $a \in M^c$, hence $b \in M^c$ so $f(b) \in M$ imply that $M_B(f(a)) \subseteq M_B(f(b))$. Similarly, $M_B(f(b)) \subseteq M_B(f(a))$. Hence $M_B(f(a)) = M_B(f(b))$, so $f(b) \in J$ then $b \in J^c$. The converse is clear. □

Theorem 2.5. *Let I be a proper ideal of A and $P \in Spec(A)$ such that $I \subseteq P$. Then there exists $P^* \in Min(I)$ such that $P^* \subseteq P$.*

Proof. Set $\Gamma = \{P^{'} \in Spec(A) : I \subseteq P^{'} \subseteq P\}$. Since $P \in \Gamma$, then $\Gamma \neq \emptyset$. We define $\leq$ on Γ by $\forall P^{'}, P^{''} \in \Gamma \quad ; \quad P^{'} \leq P^{''} \Longleftrightarrow P^{''} \subseteq P^{'}$. Obviously, $\leq$ is a partial order on Γ.

Let $\beta = \{P_i\}_{i \in I}$ be a nonempty chain of elements of Γ. Put $Q = \bigcap_{i \in I} P_i$. It is claimed that $Q \in Spec(A)$. Since $\beta \neq \emptyset$, then $Q \neq A$. Let $r \wedge r' \in Q$ and $r \notin Q$. Thus there exists $i \in I$ such that $r \notin P_i$. If P_j such that $j \in I$ is an arbitrary element of β, then we consider two cases:
Case 1. if $P_i \subseteq P_j$, then $r \wedge r' \in P_i$ and $r \notin P_i$ so $r' \in P_i \subseteq P_j$. Thus $r' \in P_j$.
Case 2. if $P_j \subseteq P_i$, then $r \notin P_j$ and $r \wedge r' \in P_j$, so $r' \in P_j$.
So $r' \in P_j$, for all $j \in I$, implies that $r' \in Q$. Obviously, $I \subseteq Q \subseteq P$. Hence Q is an upper bound of chain β in Γ, so by $Zorn$'Lemma Γ has a maximal element P^* such that $P^* \in \Gamma$. Now it is shown that $P^* \in Min(A)$ such that $I \subseteq P^*$. Let $Q^* \in Spec(A)$ be such that $I \subseteq Q^* \subsetneq P^*$. Then $Q^* \in \Gamma$ and $P^* \leq Q^*$, which is a contradiction. □

THEOREM 2.6. *Every $M-$ideal of A is the intersection of the minimal prime ideals containing it.*

PROOF. It is clear that $I \subseteq P_I$. Since I is an $M-$ideal, so $M_x \subseteq I$, for each $x \in I$. Obviously, $M_I \subseteq I$. On the other hand by Corollary 1.4 and Theorem 2.5, $P_I \subseteq M_I$, thus $P_I \subseteq I$. This implies that $I = P_I$ □

THEOREM 2.7. *Let I be an $M-$ideal of A. Then every minimal prime ideal over I is an $M-$ideal.*

PROOF. Let Q be a minimal prime ideal over I and there exists $b \in Q$ such that $M(b) \subseteq M(a)$ and $a \in A \setminus Q$. Put $S = (A \setminus Q) \cup \{c \wedge b | c \in A \setminus Q\}$. Obviously, S is a $\wedge-$closed system of A. It is claimed that $I \cap S = \emptyset$. Let $c \wedge b \in I$ such that $c \notin Q$. We show that $M(c \wedge b) \subseteq M(c \wedge a)$. If $M \in Max(A)$ such that $c \wedge b \in M$, then $c \in M$ or $b \in M$ hence $M(c \wedge b) \subseteq M(c \wedge a)$, which implies that $c \wedge a \in I$. Thus $c \wedge a \in Q$, so $c \in Q$ or $a \in Q$, which is a contradiction. Hence $I \cap S = \emptyset$. By Theorem 1.5, there exists $Q' \in Spec(A)$ such that $Q' \cap S = \emptyset$ and $I \subseteq Q'$. Obviously, $Q' \subsetneq Q$ which is a contradiction. □

Obviously, in Example 2.2(1), I_0 is an $M-$ideal but is not a minimal prime ideal of A.

3. Conclusion

We concluded if I is an $M-$ideal of A, then every minimal prime ideal over I is an $M-$ideal. By giving counter example, it is showed that the posite is not true. Also, It is proved every $M-$ideal is the intersection of the minimal prime ideals containing it.

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A classification of 2-designs admitting flag-transitive automorphism groups

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Abstract. In this talk, we give a classification of 2-designs admitting flag-transitive automorphism groups. Here, we present a recent achievement on a classification of 2-designs with $\gcd(r, \lambda) = 1$ admitting flag-transitive automorphism groups which states that all such 2-designs are known except for those admitting one dimensional affine type automorphism groups.

Keywords: 2-design, flag-transitive automorphism group, primitive group

AMS Mathematics Subject Classification [2010]: 05B05, 20B25, 05B25

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1. Introduction

A 2-(v, k, λ) design $\mathcal{D}$ is a pair $(\mathcal{P}, \mathcal{B})$ with a set $\mathcal{P}$ of v points and a set $\mathcal{B}$ of b blocks such that each block is a k-subset of $\mathcal{P}$ and each two distinct points are contained in λ blocks. The *replication number* r of $\mathcal{D}$ is the number of blocks incident with a given point. We always assume that $\mathcal{D}$ is nontrivial, that is to say, $2 < k < v$. An *automorphism* of $\mathcal{D}$ is a permutation on $\mathcal{P}$ which maps blocks to blocks and preserving the incidence. The *full automorphism* group $\mathrm{Aut}(\mathcal{D})$ of $\mathcal{D}$ is the group consisting of all automorphisms of $\mathcal{D}$. A *flag* of $\mathcal{D}$ is a point-block pair (α, B) such that $\alpha \in B$. For $G \leqslant \mathrm{Aut}(\mathcal{D})$, G is called *flag-transitive* if G acts transitively on the set of flags. The group G is said to be *point-primitive* if G acts primitively on $\mathcal{P}$.

2. Main results

In this talk, we present a recent achievement on a classification of flag-transitive 2-designs with $\gcd(r, \lambda) = 1$. Zieschang [10] proved that a flag-transitive automorphism group of a 2-designs with $\gcd(r, \lambda) = 1$ is point-primitive of almost simple or affine type. Such designs admitting an affine automorphism group have been determined in [4–6]. The case where a 2-design with $\gcd(r, \lambda) = 1$ admits an almost simple automorphism group has

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†The main results presented in this talk are part of joint papers with my colleague Seyed Hassan Alavi and my PhD students Mohsen Bayat and Fatemeh Mouseli at Bu-Ali Sina University.

been studied in [1–3, 7–9]. In conclusion, all 2-designs with $\gcd(r, \lambda) = 1$ admitting flag-transitive automorphism groups are known except for those admitting one dimensional affine type automorphism groups.

Theorem 2.1. *Suppose that G is a flag-transitive automorphism group of the non-trivial 2-designs $\mathcal{D}$ with $\gcd(r, \lambda) = 1$. Then either $(\mathcal{D}, G)$ is known, or $\mathcal{D}$ has $q = p^a$ points with p prime and G is a subgroup of the group $\mathrm{A\Gamma L}_1(q)$ of 1-dimensional semilinear affine transformations.*

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On Minty Variational Inequality with Generalized Approximate Convexity

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Abstract. We extend the notion of approximate convexity for set-valued mappings and approximate efficient solution for perturbed vector optimization problems. We obtain relation between approximate solution for optimization problem and approximate solution for Minty variational inequality.

Keywords: Approximate efficient solution, approximately convexity, Minty inequality

AMS Mathematics Subject Classification [2010]: 47J30, 30H05, 46A18.

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1. Introduction

Optimization problems and variational inequalities have played a crucial role for solving engineering and economics problems. The standard optimality concept in multiobjective optimization goes back to 1881 and 1906. Set-valued optimization problem deals with the problem of finding efficient elements of a set-valued function. Actually, set-valued optimization problem generalizes the concept of scalar optimization and vector optimization. In scalar optimization, there is a solution which generates the smallest function value. But, due to the lack of a total order in $\mathbb{R}^n$ or a topological vector space, there are several elements which cannot be compared. Hence, the pioneer work in the theory of vector variational inequalities in 1980 began by Giannessi [4] that extended the classical variational inequality for vector-valued functions in addition, Giannessi introduced Minty variational inequality in 1998. In the last decades, many problems with different constraint in engineering and economics have been considered that as mathematical modeling and these models can be considered as optimization problems and variational inequalities.

On the other hand, set-valued maps are of interest both theoretically and in practice and in recent years, as generalization of mathematical problems such as the vector variational inequalities and optimization problems, different types of problems for set-valued maps were intensively studied and many results on the existence of solutions for these problems were obtained, see [1]. Ngai et al in 2000, were introduced the concept of approximately convexity, Bhatia et al. [6] introduced new classes of approximate pseudoconvex functions, then Mishra and Laha in [3] and Gupta and Mishra in [2] considered

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the approximate pseudoconvexity assumption and by Stampacchia and Minty variational inequalities characterized an approximate efficient solution of the scalar and vector optimization problems. Here, we generalized the approximate convexity concepts and focus on set-valued mappings and relation between set valued optimization problems and Minty variational inequalities.
In this section, we define an optimization problem and some preliminary definitions and results which are utilized in the following. In section 2, we obtain some relations between solution of parametric optimization problems for set-valued maps and Minty variational inequalities. Let X and Y be normed spaces and P be a topological space. Let A and B be nonempty closed convex subsets of X and Y, respectively, $\eta : A \times A \longrightarrow A$ is a continuous function such that $\eta(x,y) = -\eta(y,x)$ and $C : X \times P \longrightarrow 2^Y$ be a set-valued mapping such that for any $x \in X$ and for any $p \in P$, $C(x,p)$ is a closed, convex and pointed cone in Y such that $\text{int}C(x,p) \neq \emptyset$. Assume that $e : X \times P \longrightarrow Y$ is a continuous vector valued mapping satisfying $e(x,p) \in \text{int}C(x,p)$. Hence, suppose that $K_1 : A \times P \longrightarrow 2^A$ and $K_2 : A \times P \longrightarrow 2^B$. Let the machinery of the problems be expressed by $F : A \times P \longrightarrow 2^Y$. Consider the following parametric vector optimization problem, for given $p \in P$:

$$(VOP(p)) \text{ Find } \bar{x} \in clK_1(\bar{x},p) \text{ such that, } \exists \bar{y} \in F(\bar{x},p) \cap K_2(\bar{x},p) :$$
$$\forall x \in K_1(\bar{x},p) \ (F(x,p) - \bar{y}) \cap (-\text{int}C(\bar{x},p)) = \emptyset.$$

We denote the set of solutions of the above problems $(VOP(p))$ by $S(p)$. Special cases of the above problems are considered in [2] and [3]. For existence results of the above problems in topological vector spaces and its special cases, we refer to [1]. In the sequal, we recall some concepts of approximately convexity.

Definition 1.1. [5] f on A, is said to be approximately convexity if

$$\forall a \in A, \ \forall \varepsilon > 0, \ \exists \delta > 0 : \ \forall x, y \in B_\delta(a), \ \forall \lambda \in (0,1)$$
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + \varepsilon t(1-t)\|x-y\|.$$

Bhatia et al. [6] extended approximately convexity concept and defined new versions of this concept. We extended approximately convex concept for set-valued functions, that generalize definitions [2, 6].

Definition 1.2. Let $F : A \longrightarrow 2^Y$, function F is said to be

- approximately pseudoconvex (strictly approximately pseudoconvex) of type I corresponding to η at x_0 if for all $\varepsilon > 0$ there exists $\delta > 0$, that if $x, y \in B_\delta(x_0)$ and
$$\langle x^*, \eta(x,y) \rangle \subseteq C(x_0,p)(intC(x_0,p)) \ \exists x^* \in \partial F(x,p),$$
then
$$\exists y_0 \in K_2(x_0,p) \cap F(x,p) : \ \forall y \in K_1(x_0,p) \ F(y,p) - y_0 + \varepsilon e(x_0,p)\|x-y\| \subseteq C(x_0,p).$$
- approximately pseudoconvex (strictly approximately pseudoconvex) of type II corresponding to η at x_0 if for all $\varepsilon > 0$ there exists $\delta > 0$, that if $x, y \in B_\delta(x_0)$ and
$$\langle x^*, \eta(x,y) \rangle + \varepsilon e(x_0,p)\|x-y\| \subseteq C(x_0,p)(intC(x_0,p)) \ \exists x^* \in \partial F(x,p),$$
then
$$\exists y_0 \in K_2(x_0,p) \cap F(x_0,p) : \ \forall x \in K_1(x_0,p) \ y_0 - F(x,p) \subseteq C(x_0,p).$$

Remark 1.3. (i) For the case when $f : X \longrightarrow Y$, approximate pseudoconvex of type I is the same as the approximate pseudoconvex defined in [5]. Obviously, one can deduce the concept of approximate pseudoconvex of type I from approximate pseudoconvex of type II.
(ii) Bhatia et al. [6] showed that Converse of the above statement was not true.

The following definition provided by Song in 2003, Aubin and Frankowska in 1990 and Chen and Jahn in 1998.

Definition 1.4.
- Let $\bar{x} \in clA$ (closure of A) a given element. The contingent cone $T_A(\bar{x})$ is defined by

$$T_A(\bar{x}) = \{u \in X : \lim_{h \searrow 0} inf \frac{d_A(\bar{x} + hu)}{h} = 0\},$$

where $d_A(u) = \inf_{v \in A} \|u - v\|$.
- Let $F : X \longrightarrow 2^Y$ and $(\bar{x}, \bar{y}) \in gr(F)$. A set-valued map $DF(\bar{x}, \bar{y}) : X \longrightarrow 2^Y$ whose graph equals the contingent cone to the graph of F at $(\bar{x}, \bar{y})$, i.e.

$$gr(DF(\bar{x}, \bar{y})) = T_{gr(F)}(\bar{x}, \bar{y})$$

- Let $(x, y) \in gr(F)$. We say that the set-valued map $D_{\uparrow}F(x, y) : X \longrightarrow Y$ defined by

$$D_{\uparrow}F(x, y)(u) := MinD(F + C)(x, y)(u)$$

is the contingent epiderivative of F at (x, y).
- Let $F : X \longrightarrow 2^Y$ be a set-valued mapping, $x_0 \in domF$ and $y_0 \in F(x_0)$. Assume that F is contingently epidifferentiable at (x_0, y_0). The set

$$\partial F(x_0, y_0) = \{T \in L(X, Y) : \; Tu \notin D_{\uparrow}F(x_0, y_0)(u) + intC(x_0, p), \forall u \in X\}$$

is called weak contingent generalized gradient of F at (x_0, y_0).

2. Main results

In this section, we obtain some relations between existence of solution of vector parametric optimization problems for set-valued maps and Minty variational inequalities. The following definitions generalize definitions of approximate efficient solutions for optimization problems that were introduced in [3].

Definition 2.1. (a) A vector $x_0 \in X$ is said to be an approximate efficient solution of type one of the Problem $(VOP(p))$ if and only if for all $\varepsilon > 0$, there isn't $\delta > 0$ such that for all $x \in k_1(x_0, p) \cap B_\delta(x_0) \setminus \{x_0\}$

$$\exists y_0 \in K_2(x_0, p) \cap F(x_0, p) : \; (F(x, p) - y_0 - \varepsilon e(x_0, p)\|x - x_0\|) \cap -\text{int}C(x_0, p) = \emptyset.$$

(b) A vector $x_0 \in X$ is said to be an efficient solution of type two of the Problem $(VOP(p))$ if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in B_\delta(x_0) \cap K_1(x_0, p)$,

$$\exists y_0 \in K_2(x_0, p) \cap F(x_0, p) : \; F(x, p) - y_0 + \varepsilon e(x_0, p)\|x - x_0\| \subseteq Y \setminus -\text{int}C(x_0, p).$$

(c) A vector $x_0 \in X$ is said to be an efficient solution of type three of the Problem $(VOP(p))$ if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in B_\delta(x_0) \cap K_1(x_0, p)$,

$$\exists y_0 \in K_2(x_0, p) \cap F(x_0, p) : \; (F(x, p) - y_0 - \varepsilon e(x_0, p)\|x - x_0\|) \cap (-\text{int}C(x_0, p)) = \emptyset.$$

In the sequel, we define approximate efficient solutions for Minty variational inequalities that extend concepts approximate efficient solutions for Minty variational inequalities were given by Mishra and Laha in [3].

Definition 2.2. (a) A vector $x_0 \in clK_1(x_0,p)$ is said to be an approximate efficient solution of type one of Minty variational inequality if and only if for all $\varepsilon > 0$, there isn't $\delta > 0$ and for all $x \in K_1(x_0,p) \cap B_\delta(x_0) \setminus \{x_0\}$

$$\forall z \in \partial F(x,p) \ < z, \eta(x_0,x) > -\varepsilon e(x_0,p)\|x - x_0\| \in -\mathrm{int}C(x_0,p).$$

(b) A vector $x_0 \in clK_1(x_0,p)$ is said to be an efficient solution of type two of Minty variational inequality if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in K_1(x_0,p) \cap B_\delta(x_0)$,

$$\forall z \in \partial F(x,p) \ < z, \eta(x_0,x) > +\varepsilon e(x_0,p)\|x - x_0\| \in Y \setminus -\mathrm{int}C(x_0,p).$$

(c) A vector $x_0 \in X$ is said to be an efficient solution of type three of Minty variational inequality if and only if for all $\varepsilon > 0$, $\delta > 0$ and for any $x \in k_1(x_0,p) \cap B_\delta(x_0)$,

$$\forall z \in \partial F(x,p) \ < z, \eta(x_0,x) > -\varepsilon e(x_0,p)\|x - x_0\| \in Y \setminus -\mathrm{int}C(x_0,p).$$

Theorem 2.3. *Let $F : X \longrightarrow 2^Y$ be a function. Then*

- *(i) if F is approximately pseudoconvex of type II at $x_0 \in X$ and x_0 is an efficient solution of type one of the Problem $(VOP(p))$, then x_0 is also an efficient solution of type one of Minty variational inequality.*
- *(ii) if F is approximately pseudoconvex of type II at $x_0 \in X$ and x_0 is an efficient solution of type two of the Problem $(VOP(p))$, then x_0 is also an efficient solution of type two of Minty variational inequality.*
- *(iii) if F is approximately pseudoconvex of type II at $x_0 \in X$ and x_0 is an efficient solution of type three of the Problem $(VOP(p))$, then x_0 is also an efficient solution of type three of Minty variational inequality.*

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

A generalized Wilcoxon test for multivariate central symmetry

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Abstract. We propose a new class of tests for central symmetry around a known point based on the center-outward depth ranking. The asymptotic distribution of the proposed tests under the null distribution is derived. This class includes the celebrated Wilcoxon signed-rank test as a special case in the univariate setting. For illustration, we apply the tests to a well-known data set to illustrate the method developed in this paper.

Keywords: Depth function, Central symmetry, Wilcoxon signed-rank test

AMS Mathematics Subject Classification [2010]: 62H15, 62G10

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1. Introduction

Testing the departures from multivariate symmetry is important subject in statistics. For the multivariate case, there are several notions of symmetry including spherical, elliptical, central or diagonal and angular symmetry, in increasing order of generality. Specifically, we consider testing for central symmetry about a known point. The random vector $\mathbf{X}$ is centrally symmetric around $\boldsymbol{\mu}$ provided $\mathbf{X}-\boldsymbol{\mu}$ and $\boldsymbol{\mu}-\mathbf{X}$ have the same distribution. Let $\mathbf{X}_1, ..., \mathbf{X}_n$ denote independent copies of the random vector $\mathbf{X} = (\mathbf{X}_1, ..., \mathbf{X}_d)^T$ from a continuous $d-$variate population. The problem is to test the hypothesis

$$H_0 : \mathbf{X} - \boldsymbol{\mu}_0 \overset{d}{=} \boldsymbol{\mu}_0 - \mathbf{X},$$

against general alternatives. Without loss of generality, we can take $\boldsymbol{\mu}_0 = 0$, if it is not, consider $\mathbf{X}_i - \boldsymbol{\mu}_0$ instead of $\mathbf{X}_i$ $(i = 1, ..., n)$.

For testing central symmetry about known center, most tests have been developed by employing the empirical characteristic functions, or the empirical distribution function. Recently, [1] introduced an affine invariant class of tests for central symmetry based on depth function.

In this paper, we aim to propose a class of tests based on the comparison of center-outward ranks of points in opposite regions. The proposed class of tests is asymptotically distribution-free under the null hypothesis and do not require any moment assumption.

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2. The proposed test statistics

According to the signed-rank test in univariate cases, extending such test in multivariate problem requires the multidimentional version of rank and sign. Indeed, the univariate signed-rank test is defined based on the rank of absolute value of observations and the sign of the orginal data. Then, dealing with the multivariate case, we need to obtain an ordering which simultaneously captures the centrality and the relative magnitude of deviation and an appropriate sign vector. Employing the depth function provides the requirements of such ordering.

Associated with a given distribution F on R^p, a depth function is designed to provide a center-outward ordering of points $\mathbf{x}$ in R^p. Indeed, a notion of data depth is used to measure centrality/outlyingness of a point with respect to given data cloud or distribution. Let $\mathbf{X} = (X_1, ..., X_p)^T$ be a random vector on a probability space $(\Omega, \mathcal{F}, P)$ and F denote a distribution function corresponding to P. Zuo and Serfling [4] provided a formal definition of statistical depth function as a function $D(., F) : \mathbb{R}^p \to \mathbb{R}$ satisfying the four properties including affine invariance, maximised somewhere in the center of the distribution F, quasi-concavity and vanishing at infinity. Various depth functions have been proposed for ranking multivariate data, among which the more popular are Tukey, Mahalanobis, spatial and projection depth functions.

The *Tukey depth* of $\mathbf{x} \in \mathbb{R}^p$ with respect to F is defined as

$$HD\left(\mathbf{x}, F\right) = \inf_{H} \left\{P\left(H\right) : \ H \ is \ a \ closed \ halfspace \ in \ \mathbb{R}^p \ and \ \mathbf{x} \in H\right\}.$$

The *Mahalanobis* depth of $\mathbf{x}$ with respect to F is given by

$$MD\left(\mathbf{x}, F\right) = \frac{1}{1 + \left(\mathbf{x} - \boldsymbol{\mu}\right)^T \boldsymbol{\Sigma}^{-1} \left(\mathbf{x} - \boldsymbol{\mu}\right)}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are the mean vector and dispersion matrix of F distribution, respectively.

The *spatial depth* of $\mathbf{x}$ with respect to F is given by

$$SD(\mathbf{x}, F) = 1 - ||E(S(\mathbf{x} - \mathbf{X}))||$$

where

$$S\left(\mathbf{x}\right) = \begin{cases} \frac{\mathbf{x}}{||\mathbf{x}||} & \mathbf{x} \neq 0 \\ 0 & \mathbf{x} = 0 \end{cases}$$

and $\mathbf{X} \sim F$.

The *projection depth* of $\mathbf{x}$ with respect to F is given by

$$PD\left(\mathbf{x}, F\right) = \frac{1}{1 + OP\left(\mathbf{x}, F\right)},$$

where $OP\left(\mathbf{x}, F\right) = \sup\limits_{\|\mathbf{u}\|=1} \left|\frac{\mathbf{u}^T\mathbf{x} - \mu(F_\mathbf{u})}{\sigma(F_\mathbf{u})}\right|$ with $\mu(.)$ and $\sigma(.)$ as univariate location and scale measures, respectively and $F_\mathbf{u}$ is the distribution of $\boldsymbol{u}^T\mathbf{X}$.

Now, we present the definition of center-outward ranking of data points.

DEFINITION 2.1. Assume that $\mathbf{X}_1, ..., \mathbf{X}_n$ is a random sample from distribution function F in $\mathbb{R}^p$. The center-outward rank $\mathbf{X}_i$ within the sample $\mathbf{X}_1, ..., \mathbf{X}_n$ is

$$\#\left\{\mathbf{X}_j \in \left\{\mathbf{X}_1, ..., \mathbf{X}_n\right\} : \quad D\left(\mathbf{X}_j, F_n\right) \geq D\left(\mathbf{X}_i, F_n\right)\right\}$$

where F_n is the sample distribution function.

Let the random sample $X_1, ..., X_n$, with sample distribution function F_n comes from p-dimensional distribution function F. The sample depth function will be obtained by replacing F with the sample distribution F_n. Based on the sample depth function $D(., F_n)$, the center-outward rank of $\mathbf{X}_i$, $i = 1, ..., n$ is defined as

$$\# \{\mathbf{X}_j \in \{\mathbf{X}_1, ..., \mathbf{X}_n\} : \quad D(\mathbf{X}_j, F_n) \leq D(\mathbf{X}_i, F_n)\} .$$

Let $D(., F)$ be a depth function on $\mathbb{R}^p$ associated with a distribution function F. The proposed test compares the sum of the depth-based ranks in the opposite regions with respect to the origin in space $\mathbb{R}^p$. We need to order the multivariate points in such a way that the difference between the center-outward ranks in the opposite regions detects depurtures from the null symmetry point. More precisely, the center-outward rank of points relative to the null symmetry center should be obtained instead of relative to the median of $\mathbf{X}_1, ..., \mathbf{X}_n$. Let F_n and F_n^s denote the sample distribution function of random sample $\mathbf{X}_1, ..., \mathbf{X}_n$ and the symmetrized sample $(\pm\mathbf{X}_1, ..., \pm\ \mathbf{X}_n)$, respectively. Then, we have ordered $\mathbf{X}_1, ..., \mathbf{X}_n$ based on $D(., F_n^s)$ rather than $D(., F_n)$. More precisely, define

$$(1) \qquad R_i = \# \{\mathbf{X}_j \in \{\mathbf{X}_1, ..., \mathbf{X}_n\} : \quad D(\mathbf{X}_j, F_n^s) \geq D(\mathbf{X}_i, F_n^s)\} , \quad i = 1, ..., n.$$

For each data point $\mathbf{X}_i$, the coresponding componentwise sign vector is defined as $\boldsymbol{\delta}_i = (\delta_i^1, ..., \delta_i^p)^T$ where

$$\delta_i^j = \begin{cases} 1 & if\, X_{ij} \geq 0 \\ -1 & otherwise \end{cases}$$

with X_{ij} is jth component of the vector $\mathbf{X}_i$, $i = 1, ..., n$, $j = 1, ..., p$. The signed-rank vector is proposed as

$$\mathbf{V}_i = \frac{R_i}{n+1}\boldsymbol{\delta}_i, \qquad i = 1, ..., n,$$

Now, the peoposed class of test statistics is introduced as

$$W_{n,D} = n\bar{\boldsymbol{V}}'_n \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{V}_i\mathbf{V}'_i\right)^{-1} \bar{\boldsymbol{V}}_n$$

where $\bar{\boldsymbol{V}}_n = \frac{1}{n}\sum\limits_{i=1}^{n} \mathbf{V}_i$. It is clear that, the large values of the test statistic $W_{n,D}$ reject H_0 in favor of alternative hypothesis.

To complete the procedure of test, we need to obtain the exact or asymptotics distribution of $W_{n,D}$ under the null distribution. If the sample depth function satisfies affine invariance property, then under the null hypothesis of centrally symmetric about 0, $W_{n,D}$ converges in distribution to a chi-square random variable with p degree of freedom. Therefore, for a sufficiently large sample size n, the null hypothesis will be rejected at level α when

$$W_{n,D} \geq \chi^2_{p,1-\alpha}$$

where $\chi^2_{p,1-\alpha}$ denotes the $1-\alpha$ quantile of the chi-square distribution with p degree of freedom.

It is worth to mentioning that in the special case $p = 1$ and defining the proposed test based on the halfspace depth, $W_{n,D}$ is equivalent to the two-sided Wilcoxon signed-rank test which is a popular nonparametric test in the univariate case.

TABLE 1. Analysis of the real data example

	$W_{n,MD}$	$W_{n,SD}$	$W_{n,PD}$	$W_{n,HD}$
Test statistics	3.657	3.5514	7.3142	5.8349
P-value	0.744	0.756	0.422	0.648

3. A real data example

In this section, the proposed class of tests is illustrated with a well-known real data set on the health survey of paint sprayers in a car assembly plant presented by [2]. This data set consists of six measurements including haemoglobin concentration, PCV packed cell volume, white blood cell count, lymphocyte count, neutrophil count and serum lead concentration for 103 black worker. The data were analyzed earlier by some authors. Specifically, the test of [2] confirms that the data are not following from a multivariate normal population such that the p-value of the test is approximately equal to 0. In the sequal, [3] designed to test the elliptical symmetry and obtained the p-value of 0.11 where results that it is not unreasonable to assume that the data are from an elliptical distribution. We determined four versions of $W_{n,D}$, derived through the Mahalanobis, spatial, projection and halfspace depths as $W_{n,MD}$, $W_{n,SD}$, $W_{n,PD}$ and $W_{n,HD}$, respectively. The value of test statistics and p-values are presented in Table 1. All tests show no statistical evidence against the null hypothesis of central symmetry of data.

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Some new characterizations of inner product spaces in terms of HH-I-orthogonality

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ABSTRACT. In this study, we consider Hermite-Hadamard type of isosceles orthogonality (HH-I-orthogonality) in normed linear spaces. We prove that the existence property of HH-I-orthogonality in the sense of Alonso and Benítez. In conclusion, some new characterizations of real inner product spaces in terms of HH-I-orthogonality and its relationship with Birkhoff-James orthogonality are presented.

Keywords: Birkhoff-James orthogonality, Hermite-Hadamard type of isosceles orthogonality, inner product space, strictly convex normed linear space

AMS Mathematics Subject Classification [2010]: 46B20, 46C05

1. Introduction

Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space. We say that a vector $x \in X$ is orthogonal to a vector $y \in X$, denoted by $x \perp y$, if their inner product is zero, that is $\langle x, y\rangle = 0$. However, there are various notions of orthogonality in a normed linear space, if the norm is not induced by an inner product. One of the most well-known concept of orthogonality in normed linear spaces is Birkhoff-James orthogonality [6]. Let $(X, \|\cdot\|)$ be a real normed linear space. A vector $x \in X$ is said to be orthogonal to a vector $y \in X$ in the sense of Birkhoff-James, written as $x \perp_B y$, if

$$\|x + \lambda y\| \geq \|x\| \quad (\forall \lambda \in \mathbb{R}).$$

Also, James introduced isosceles orthogonality in normed linear spaces [5]. A vector $x \in X$ is said to be isosceles orthogonal to a vector $y \in X$, written as $x \perp_I y$, if

$$\|x - y\| = \|x + y\|.$$

Some main properties of the orthogonality in inner product spaces do not always carry over to generalized orthogonalities in normed linear spaces, such as Birkhoff-James orthogonality and isosceles orthogonality. Taking this into account different types of orthogonalities provide a good tools for studying the geometric properties of normed linear spaces. Also, there are interesting characterizations of inner product spaces connected with the notions

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of orthogonality in normed linear spaces; see e.g., [2]. For example, it is known that isosceles orthogonality is not homogeneous and James [5] proved that isosceles orthogonality in a real normed linear space X is homogeneous if and only if X is an inner product space. Moreover, it is well known that Birkhoff-James orthogonality is not symmetric, and so the following characterization of inner product spaces has been proved in [6]:

Theorem 1.1. *[6] A normed linear space X, whose dimension is at least three, is an inner product space if and only if Birkhoff-James orthogonality is symmetric in X.*

In 2010, Kikianty and Dragomir introduced the Hermite-Hadamard type of isosceles orthogonality by utilizing the 2-HH-norm in [4]. Precisely, a vector $x \in X$ is called HH-I-orthogonal to $y \in X$, denoted by $x \perp_{HH-I} y$, if and only if

$$\int_0^1 \|(1-t)x - ty\|^2 dt = \int_0^1 \|(1-t)x + ty\|^2 dt.$$

The main properties of HH-I-orthogonality were determined in [4]. In particular, it was proved by [4] that HH-I-orthogonality in a real normed linear space X is homogeneous if and only if X is an inner product space. Some characterizations of the real inner product spaces using the notion of HH-I-orthogonality and by considering its relationship with Birkhoff–James orthogonality have been obtained in [3].

In this paper, first we prove that HH-I-orthogonality is existent in the Alonso and Benítez sense (Theorem 2.1). Next, some new characterizations of real inner product spaces in terms of HH-I-orthogonality and its relation with Birkhoff-James orthogonality are given. In particular, a characterization of inner product spaces by using some weakened hypothesis of the homogeneity of HH-I-orthogonality are presented.

2. Main results

James [5] proved that isosceles orthogonality has the α-existence property, i.e., for every linearly independent vectors $x, y \in X$, there is $\alpha \in \mathbb{R}$ such that $x \perp_I (\alpha x + y)$. Another existence property for isosceles orthogonality was stated by Alonso and Benítez in [1]. Alonso proved that if $(X, \|\cdot\|)$ be normed linear space and Y is a two-dimensional subspace of X, then for every $x \in Y$ and every $\gamma > 0$ there exists $y \in Y$ such that $\|y\| = \gamma$ and $x \perp_I y$. Furthermore, it was proved in [4] that HH-I-orthogonality has the α-existence property, but Alonso and Benítez noticed that since HH-I-orthogonality is nonhomogeneous, the α-existence of HH-I-orthogonality is not equivalent to their definition. In the next result we prove that HH-I-orthogonality is existence in the Alonso and Benítez sense.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a normed linear space and let Y be a two-dimensional subspace of X. Then for every $x \in Y$ and every $\gamma > 0$, there exists $y \in Y$ such that $\|y\| = \gamma$ and $x \perp_{HH-I} y$.*

Before stating our next result, we recall the following characterization of strictly convex normed linear spaces with respect to left uniqueness of Birkhoff-James orthogonality.

Theorem 2.2. *[6, Theorem 4] Let X be a normed linear space. Then the following statements are equivalent:*

(i) *X is strictly convex.*
(ii) *The Birkhoff–James orthogonality is unique at left, i.e., for every $x, y \in X$, with $x \neq 0$ there exists a unique $\alpha \in \mathbb{R}$ such that $(\alpha x + y) \perp_B x$.*

Also, note that the α-uniqueness of Birkhoff-James orthogonality is equivalent to its uniqueness in the sense of Alonso and Benítez in strictly convex normed linear space $(X, \|\cdot\|)$. Indeed, if Y is a two dimensional subspace of X, $x \in Y$ and $\gamma > 0$, then for $y \in Y$ which is linearly independent of x, there is a unique $\alpha \in \mathbb{R}$ such that $(\alpha x + y) \perp_B x$. Now, let $z = \gamma \frac{\alpha x + y}{\|\alpha x + y\|}$. Then z is a unique vector in Y such that $\|z\| = \gamma$ and $z \perp_B x$, by homogeneity of Birkhoff-James orthogonality. Conversely, assume that $x, y \in X$ are linearly independent vectors and $Y = \operatorname{span}\{x, y\}$. Then there is a unique $z \in Y$ such that $\|z\| = \|y\|$ such that $z \perp_B x$. Now, let $z = \lambda x + \mu y$ $(\lambda, \mu \in \mathbb{R})$. Then $(\alpha x + y) \perp_B x$ for which $\alpha = \frac{\lambda}{\mu}$ (if $\mu = 0$, then $x = 0$, which is impossible).

THEOREM 2.3. *Let $(X, \|\cdot\|)$ be a strictly convex normed linear space whose dimension is at least 3. If*

$$x \perp_{HH-I} y \Rightarrow x \perp_B y \quad (\forall x, y \in \mathbb{S}_X),$$

then Birkhoff-James orthogonality is symmetric in X, and therefore X is an inner product space.

PROOF. Let $x, y \in \mathbb{S}_X$ such that $x \perp_B y$ and let $Y = \operatorname{span}\{x, y\}$ be the two dimensional subspace of X generated by x and y. Then there is a vector $z \in \mathbb{S}_Y$ such that $y \perp_{HH-I} z$, by Theorem 2.1, and so $z \perp_B y$. Then remarks preceding this theorem and Theorem 2.2 imply that $z = x$. Thus $y \perp_B x$. Therefore Birkhoff-James orthogonality is symmetric in X, and so Theorem 1.1 concludes that X is an inner product space. □

COROLLARY 2.4. *Let $(X, \|\cdot\|)$ be a strictly convex space whose dimension is at least 3. If for all $x, y \in \mathbb{S}_X$ there is $\delta > 0$ such that*

$$x \perp_{HH-I} y \Rightarrow x \perp_{HH-I} \lambda y \quad (\forall |\lambda| < \delta),$$

then X is an inner product space.

PROOF. Let $x, y \in \mathbb{S}_X$ such that $x \perp_{HH-I} y$. Then there is $\delta > 0$ such that $x \perp_{HH-I} \lambda y$ for all λ with $|\lambda| < \delta$. Define the mapping $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(\lambda) = \int_0^1 \|(1-t)x + \lambda t y\|^2 dt.$$

Since $x \perp_{HH-I} \lambda y$ for all λ such that $|\lambda| < \delta$, we have $\phi(\lambda) = \varphi(-\lambda)$ for all $\lambda \in (-\delta, \delta)$. On the other hand, φ is a continuous and convex function. Then φ attains its minimum at zero, that is

$$\varphi(\lambda) \geq \varphi(0) = \int_0^1 \|(1-t)x\|^2 dt.$$

It follows that $\|(1-t)x + \lambda t y\| \geq \|(1-t)x\|$ for almost all $t \in [0, 1]$. It follows from the continuity property of Birkhoff-James orthogonality that $(1-t)x \perp_B ty$ for all $t \in [0, 1]$. Thus $x \perp_B y$, since Birkhoff-James orthogonality is homogeneous. Therefore we have proved that $\perp_{HH-I} \subseteq \perp_B$ on $\mathbb{S}_X$. Consequently, Theorem 2.3 implies that X is an inner product space. □

3. Conclusion

There are interesting characterizations of inner product spaces connected with the notions of orthogonality in normed linear spaces. In this paper the Hermite-Hadamard type of isosceles orthogonality (HH-I-orthogonality) is investigated. Considering HH-I-orthogonality and Birkhoff-James orthogonality, some new characterizations of real inner product spaces are presented.

Acknowledgement

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
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دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

A two-step cubic regularization algorithm to solve unconstrained optimization problems

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Abstract. The current paper studies an improved adaptive cubic regularized method to solve the unconstrained minimization problems. This work is focused on the adaptive regularization algorithm using cubics. we present a two-step version of adaptive cubic regularization algorithm for unconstrained optimization problems. The global convergence analysis is investigated under appropriate conditions. Several numerical results are given to illustrate the efficiency and robustness of the suggested method.

Keywords: Unconstrained optimization, Cubic regularization, Trust region method.

AMS Mathematics Subject Classification [2010]: 90C30, 49M37, 65K05.

1. Introduction

Let us consider the unconstrained optimization problem:

$$\min_{x\in\mathbb{R}^n} f(x), \tag{1}$$

where $f:\mathbb{R}^n\to\mathbb{R}$ is a twice continuously differentiable function. We assume that the solution set of (1) is nonempty. Classical iterative methods for optimization can be classified into two categories: line search and trust region methods [5]. In line search methods (LS), for a given x_k, we have $x_{k+1}=x_k+\alpha_k d_k$ where $d_k\in\mathbb{R}^n$ is a descent search direction and α_k is called the step size. The standard trust region (TR) method is a prominent class of iterative methods to solve the optimization problems. Similar to line search method, iterative formula to solve (1) is generally in the following form:

$$x_0\in\mathbb{R}^n,\ x_{k+1}=x_k+d_k,\ k\geq 0,$$

in which d_k satisfies the following minimization problem:

$$\begin{aligned} \min_{d\in\mathbb{R}^n} \quad & m_k(d)=f_k+g_k^Td+\frac{1}{2}d^TB_kd, \\ s.t. \quad & \|d\|\leq\sigma_k, \end{aligned} \tag{2}$$

where $g(x)=\nabla f(x)$, $H(x)=\nabla^2 f(x)$ and $\sigma_k>0$ is the TR radius. Also, B is an approximation of H. Here, we adopt the notations $f(x_k):=f_k$, $g(x_k):=g_k$, $H(x_k):=$

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$H_k,\ B(x_k) := B_k$. Using the ratio

$$\rho_k = \frac{f_k - f(x_k + d_k)}{m_k(0) - m_k(d_k)},$$

the classical TR algorithms evaluate an agreement between the model m_k and the function f_k. The step d_k is accepted whenever $\rho_k \geq \mu$ (constant $\mu > 0$). This leads us to the new point $x_{k+1} = x_k + d_k$. Otherwise, the step d_k is rejected and the problem (2) must be solved again [5]. In recent years, adaptive regularized methods have been investigated as an alternative to classical globalization methods for constrained and unconstrained minimization problems [1, 2]. Nesterov and Polyack [4] were the first researchers who analysed the Newton method by a cubic regularization approach to theoretically solve the unconstrained minimization problems and prove the convergence results. They showed a worst-case iteration count of order $O(\epsilon^{-3/2})$ to get $\|\nabla f_k\| \leq \epsilon$. Cartis et al. [2] introduced an adaptive regularization method (ARC) with a cubic model m_k for the unconstrained minimization problems.

2. The proposed algorithm

This section presents a two-step nonmonotone type of the ARC algorithm. For this purpose, first, we give a brief review of the basic ARC algorithm as mentioned in [2]. The key feature of the ARC method is the calculation of the step d by minimizing a cubic overestimator of the function f. The cubic model is as follows:

$$m_k(d) := f_k + d^T g_k + \frac{1}{2} d^T B_k d + \frac{1}{3}\sigma_k \|d\|^3, \tag{3}$$

which was used by Cartis et al. in [2]. The step d is computed as an approximate minimizer of $m_k(d)$. The cubic model $m_k(d)$ is used as an approximation of $f(x)$ and the subproblem

$$\min_{d\in\mathbb{R}^n} m_k(d),$$

is solved. Zhang and Hager [6] proposed a nonmonotone line search, such that the step size α_k satisfying the following inequality:

$$f(x_x + \alpha_k d_k) \leq C_k + \beta \alpha_k g(x_k)^T d_k, \tag{4}$$

where $0 < \beta < 1$,

$$C_k = \begin{cases} f_k & k = 0, \\ (\eta_{k-1} Q_{k-1} C_{k-1} + f_k)/Q_k & k \geq 1, \end{cases} \tag{5}$$

and

$$Q_k = \begin{cases} 1 & k = 0, \\ \eta_{k-1} Q_{k-1} + 1 & k \geq 1. \end{cases} \tag{6}$$

Here, we use a variable η_k in definition of Q_k :

$$\eta_k = \begin{cases} \frac{\eta_0}{2}, & k = 1, \\ \frac{\eta_{k-1}+\eta_{k-2}}{2}, & k \geq 2, \end{cases} \tag{7}$$

where $0 \leq \eta_{\min} \leq \eta_{k-1} \leq \eta_{\max} < 1$. Here, we present a two-step nonmonotone ARC algorithm as follows:

Algorithm 1: A two-step nonmonotone ARC algorithm (two-step-NARC).

Input: Let $x_0 \in \mathbb{R}^n$, $\sigma_0 > 0$, $0 < \eta_0 < 1$, $\gamma_2 \geq \gamma_1 > 1$, $0 < \epsilon < 1$ and $\mu_2 \geq \mu_1 > 0$, $k = 0$.

Step 1. If $\|g_k\| \leq \epsilon$ stop.

Step 2. Compute an approximate minimizer d_k of m_k such that

$$m_k(d_k) \leq m_k(d_k^c),$$

where d_k^c is the Cauchy point defined by the equation as follow:

$$d_k^c = -\alpha_k^c g_k \quad \text{and} \quad \alpha_k^c = \arg\min_{\alpha\in\mathbb{R}_+} m_k(-\alpha g_k).$$

Set $y_k = x_k + d_k$

Step 3. Compute an approximate minimizer $\widehat{d}_k$ of m_k such that

$$m_k(\widehat{d}_k) \leq m_k(d_k^c),$$

where d_k^c is the Cauchy point due to y_k by (3).

Step 4. Compute $\rho_k = \frac{C_k - f(x_k+d_k+\widehat{d}_k)}{Pred_k}$.

Step 5. Set
$$x_{k+1} = \begin{cases} x_k + d_k + \widehat{d}_k, & \rho_k \geq \mu_1, \\ x_k, & \text{otherwise.} \end{cases}$$

Step 6. Update σ_{k+1} as

$$\sigma_{k+1} = \begin{cases} [0, c_1\sigma_k], & \rho_k > \mu_2, \\ [\sigma_k, \gamma_1\sigma_k], & \mu_1 \leq \rho_k \leq \mu_2, \\ [\gamma_1\sigma_k, \gamma_2\sigma_k], & \text{otherwise.} \end{cases}$$

Step 7. Set $k = k + 1$ and go to Step 1.

3. Global convergence

We first give the following assumptions.

(A1). The mapping f is twice continuously differentiable, below bounded and the level set $L_0 = \{x \in \mathbb{R}^n | f(x) \leq f(x_0)\}$ is bounded.

(A2). $g(x)$ is Lipschitz continuous, that is, there exists positive constant L such that

$$\|g(y) - g(x)\| \leq L\|y - x\|, \; x, y \in \mathbb{R}^n. \tag{8}$$

(A3). The matrix sequence $\{B_k\}$ is uniformly bounded.

Lemma 3.1. *[2] Suppose that the step d_k satisfies $m_k(d_k) \leq m_k(d_k^c)$. Then for all $k \geq 0$ we have*

$$f(x_k) - m_k(d_k) \geq \frac{\|g_k\|}{6\sqrt{2}} \min\left\{\frac{\|g_k\|}{1 + \|B_k\|}, \frac{1}{2}\sqrt{\frac{\|g_k\|}{\sigma_k}}\right\}.$$

For simplicity, we define two index sets $I = \{k \mid \rho_k \geq \mu_1\}$ and $J = \{k \mid \rho_k < \mu_1\}$.

Lemma 3.2. *Let $\{x_k\}$ be the sequence generated by Algorithm 1. Then $f_{k+1} \leq C_{k+1} \leq C_k$ holds for all $k \in I \cup J$.*

Lemma 3.3. *Suppose that* **(A3)** *holds and the step d_k satisfies $m_k(d_k) \leq m_k(d_k^c)$, for all $k \geq 0$. Then*

$$\|d_k\| \leq \frac{3}{\sigma_k} \max(M, \sqrt{\sigma_k\|g_k\|}).$$

THEOREM 3.4. *Suppose that* **(A1)**, **(A2)** *and* **(A3)** *hold. Then the sequence* $\{x_k\}$ *generated by Algorithm 1 satisfies*

$$\liminf_{k\to\infty} \|g_k\| = 0.$$

4. Numerical results

In this part, we report some results on the following numerical results for the proposed algorithm. We also compare the effectiveness of the proposed method with the classical monotone ARC algorithm [2] and NARC [3]. We start with $\sigma_0 = 0.1, \mu_1 = 0.1, \mu_2 = 0.9, \gamma_1 = 1.1, \gamma_2 = 2, \eta_0 = 0.75$. The number of evaluations of the objective function N_f and the number of evaluations of its gradient N_g. As regards the CUTEr collection, we selected a set of 20 CUTEr medium-sized unconstrained problems. All algorithms have been compared by means of the performance profiles. The results are plotted in Fig. 1. Obviously, the proposed algorithm has a better performance than the ARC algorithm [2] and NARC [3] based on $N_f + N_g$ and the number of iterations. Generally the proposed algorithm has better numerical results in comparison with the other two algorithms.

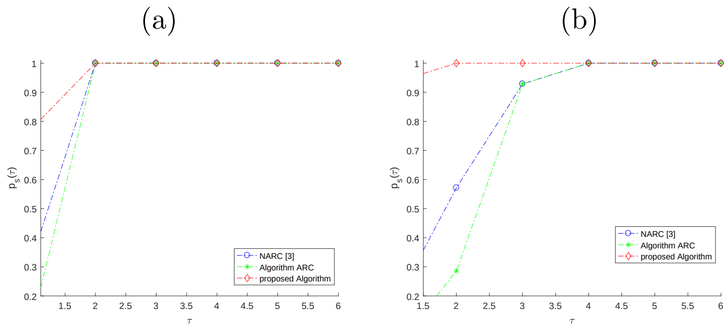

FIGURE 1. (a) Performance profiles for $N_f + N_g$ (b) The number of iterations.

5. Conclusions

In this paper, combing with nonmonotone line search, we present a two-step ARC method for unconstrained optimization problem. Convergence of the method is analyzed under some suitable assumptions. The numerical results have shown the effectiveness of the presented algorithm.

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۸ تا ۱۱ شهریور ۱۴۰۰

Partially projective modules and locally partially free sheaves

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Abstract. In this paper, the notions of partially projective modules and locally partially free sheaves are introduced. These notions are generalizations of projective modules and locally free sheaves respectively, and have some interesting properties in common with projective modules and locally free sheaves. As in the Serre-Swan theorem, the relationship between partially projective modules and locally partially free sheaves is obtained.

Keywords: faithful surjection, partially projective module, partially free module, locally partially free sheaf.

AMS Mathematics Subject Classification [2010]: 13C10, 13C13, 14A15 (at least 1 and at most 3)

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1. Introduction

Since introduced in 1956 in the influential book [1], projective modules have played an important role in both ring and module theory. In the characterization of rings, the celebrated theorem of Auslander-Buchsbaum and Serre which states that a commutative Noetherian local ring is regular if and only if it has finite global dimension (see e.g. [2, Theorem 19.12]) is an outstanding example. This article concerns a generalization of projective modules. (See for example [7] for complete discussion on projective modules). In section 2, we extend the notion of projective modules as follows: We call a surjection of R-modules $g : A \to B$ faithful, if for any maximal ideal P of R and for any $b \in B$, if $P = \mathrm{Ann}_R(b)$ then there exists some $a \in A$ such that $P = \mathrm{Ann}_R(a)$ and $g(a) = b$. The idea is to replace the surjective map, in the definition of projective module, by a faithful surjection. We call the resulting modules "partially projective" (p-projective for short), as in some situations they are not projective, but they have a projective submodule as a direct summand (see Theorem 2.9). p-projective modules have some interesting properties in common with projective modules. For example, the class of p-projective R-modules is closed under direct sums and summands and every p-projective module is a direct summand of a "partially free module" (see Definition 2.6). In section 3, we define locally

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partially free modules and, study the relationship between partially projective modules and locally partially free sheaves.

Throughout this paper all rings are commutative with unity and all modules are unital. We denote the set of all maximal ideals of a ring R by $\mathrm{Max}(R)$ and the set of all associated prime ideals of an R-module G by $Ass_R(G)$. By a regular element $t \in R$ we mean a nonzero-divisor, i.e. $tr = 0$ implies $r = 0$. We define the torsion submodule of an R-module G as $\mathrm{T}(G) = \{x \in G : tx = 0, \text{ for some regular element } t \in R\}$.

2. Definition and basic properties of p-projective modules

In this section, we illustrate some properties of p-projective modules. First, some definitions:

Definition 2.1. Let P be a maximal ideal of R. An R-module epimorphism $g : A \to B$ is called *P-surjective* whenever for any $b \in B$, if $P = \mathrm{Ann}_R(b)$ then there exists some $a \in A$ such that $P = \mathrm{Ann}_R(a)$ and $g(a) = b$.
An R-module epimorphism $g : A \to B$ is called a *faithful surjection* if for all maximal ideals P of R, g is P-surjective.

Definition 2.2. An R-module G is called *partially projective* (*p-projective* for short), if for every faithful surjection $g : A \to B$ and any R-module homomorphism $f : G \to B$, there exists an R-module homomorphism $\bar{f} : G \to A$ such that $g\bar{f} = f$.

Proposition 2.3. *every direct sum of p-projective modules is p-projective.*

Remark 2.4. Let P be a maximal ideal of R. We show that R/P is a p-projective R-module. Suppose $g : A \to B$ is a faithful surjection and $f : R/P \to B$ is a nonzero R-homomorphism. Thus $b := f(1+P)$ is annihilated by P and since $b \neq 0$ and P is maximal, hence $\mathrm{Ann}_R(b) = P$. The P-surjectivity of g implies that there exists some $a \in A$ such that $P = \mathrm{Ann}_R(a)$ and $g(a) = b$. Now we define $\bar{f} : R/P \to A$ by $\bar{f}(r + P) = ra$ for every $r \in R$. Obviously $\bar{f}$ is an R-homomorphism and $g\bar{f} = f$. Therefore R/P is a p-projective R-module. Since every R/P-vector space $V(P)$ is a direct sum of copies of R/P, hence Proposition 2.3 implies that $V(P)$ is a p-projective R-module. So we have the following.

Proposition 2.5. *Let F be a free (projective or p-projective) module and let $\{P_\lambda\}_{\lambda\in\Lambda}$ be a family of maximal ideals of R. For any $\lambda \in \Lambda$, let $V(P_\lambda)$ be an R/P_λ-vector space. Then $F \oplus \bigoplus_{\lambda\in\Lambda} V(P_\lambda)$ is a p-projective module.*

Definition 2.6. Let $\{P_\lambda\}_{\lambda\in\Lambda}$ be a family of maximal ideals of R. An R-module G is called *partially free*, if $G \cong F \oplus \bigoplus_{\lambda\in\Lambda} V(P_\lambda)$ where F is a free R-module and the $V(P_\lambda)$ are R/P_λ-vector spaces.

Proposition 2.7. *every R-module M is a homomorphic image of a partially free module G such that the surjection is faithful. Furthermore, if $G = F \oplus \bigoplus_{\lambda\in\Lambda} V(P_\lambda)$, then the maximal ideals P_λ (if any) may be considered in $Ass_R(M)$.*

Theorem 2.8. *Let G be an R-module. Then the following conditions are equivalent:*

(1) *G is p-projective;*
(2) *every exact sequence $0 \longrightarrow N \longrightarrow M \xrightarrow{g} G \longrightarrow 0$ with g a faithful surjection, splits;*
(3) *G is a direct summand of a partially free module;*
(4) *G is a direct summand of a p-projective module.*

THEOREM 2.9. *Let G be an R-module such that $\Sigma = Max(R) \cap Ass_R(G)$ is a finite set and $I = \bigcap_{P\in\Sigma} P$ contains a regular element. Then G is p-projective if and only if $G \cong H \oplus K$ where H is a projective R-module and K is a direct summand of a free R/I-module.*

PROOF. Let $\Sigma = \{P_1, \dots, P_n\}$. We have $R/I = R/\cap_{i=1}^n P_i \cong R/P_1 \times \cdots \times R/P_n$ and so by Remark 2.4 and Proposition 2.3, every free R/I-module is a p-projective R-module. Thus by Theorem 2.8, (4)→(1), every projective R/I-module is a p-projective R-module. Now let $G \cong H \oplus K$ where H and K are R-modules such that H is projective and K is a direct summand of a free R/I-module. As mentioned above, K is a p-projective R-module and then by Proposition 2.3, G is p-projective.
Conversely, assume that G is a p-projective R-module. By Theorem 2.8, there exist a free R-module F_0 and for each $i \in \{1, \dots, n\}$, an R/P_i-vector space V_i such that

$$F_0 \oplus \bigoplus_{i=1}^{n} V_i \cong G \oplus M \tag{1}$$

where M is an R-module. by hypothesis, there exists a regular element $t \in \cap_{i=1}^n P_i$. Thus by (1), we have $tF_0 \cong tG \oplus tM$. Since t is regular, tF_0 is a free R-module and so $H := tG$ is projective. Again from (1) we have

$$\bigoplus_{i=1}^{n} V_i = \mathrm{T}(F_0 \oplus \bigoplus_{i=1}^{n} V_i) \cong \mathrm{T}(G) \oplus \mathrm{T}(M).$$

Therefore for every $x \in G$, $tx = 0$ if and only if $x \in \mathrm{T}(G)$. Thus the sequence
$0 \longrightarrow \mathrm{T}(G) \longrightarrow G \xrightarrow{t} tG \longrightarrow 0$ is exact. As $H = tG$ is projective, we have $G \cong H \oplus \mathrm{T}(G)$. On the other hand, $K := \mathrm{T}(G)$ is a direct summand of $\bigoplus_{i=1}^n V_i$ and it is easily seen that $\bigoplus_{i=1}^n V_i$ is a direct summand of a free R/I-module. So K is a direct summand of a free R/I-module and $G \cong H \oplus K$ as desired. □

3. Locally partially free module

In this section, we define locally partially free modules and study the relationship between p-projectives and locally partially free sheaves.

DEFINITION 3.1. Let R be a ring and M an R-module. We say that M is locally partially free if we can cover spec(R) by standard opens $D(f_i)$, $i \in I$ such that M_{f_i} is a partially free R_{f_i}-module for all $i \in I$

LEMMA 3.2. *Let M be a locally partially free R_m-module .Then M_p is a partially free R_p-module for all $p \in Spec(R)$.*

PROPOSITION 3.3. *Let R be a Noetherian ring and M be a finitely generated locally partially free R-module. Then M be a partially projective R-module.*

LEMMA 3.4. *Let M be a partially free R-module and $p \in specR$.Then M_p is a partially free R_p-module. If $p \in SpecR \setminus MaxR$ then M_p is free R_p-module.*

LEMMA 3.5. *Let M be a partially projective R-module and $p \in specR$.Then M_p be a partially projective R_p-module. If $p \in specR \setminus MaxR$ then M_p is projective R_p-module.*

DEFINITION 3.6. Let $X = Spec(R)$ and (X, O_X) be affine scheme. Let G be a sheaf of O_X-modules. We say G is locally partially free sheaf if we can cover Spec(R) by standard opens $D(f_i)$, $i \in I$ such that $G|_{D(f_i)} \simeq \bigoplus \mathrm{O}_X|_{D(f_i)} \oplus (\bigoplus O_X/\tilde{J})|_{D(f_i)}$, for some $J \in Max(R)$

In the following result, we give a scheme analog of the Serre-Swan Theorem[3.7]. For more information on sheaves and scheme see [3].

THEOREM 3.7. *Let $X = SpecR$ be an affine scheme and Let G be an R-module such that $\Sigma = Max(R) \cap Ass_R(G)$ is a finite set and $I = \bigcap_{P \in \Sigma} P$ contains a regular element. G is a p-projective module if and only if $\tilde{G}$ is a locally partially free sheaf.*

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A Numerical Method for The Two-Asset Black-Scholes PDE

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Abstract. In this paper, an efficient hybrid numerical method for solving two-asset option pricing problem is presented based on the Crank-Nicolson and the radial basis function methods. For this purpose, the two-asset Black-Scholes partial differential equation is considered. Also, the convergence of the proposed method are proved and implementation of the proposed hybrid method is specifically studied on Call on maximum Rainbow options. In addition, this method is compared to the explicit finite difference method as the benchmark and the results show that the proposed method can achieve a noticeably higher accuracy than the benchmark method.

Keywords: Two-asset option pricing, Black-Scholes equation, Radial basis functions

AMS Mathematics Subject Classification [2010]: 91G80, 65M06, 65M12

1. Introduction

The financial markets are becoming more complex with trading many types of financial derivatives. A financial derivative is a contract with a value dependant on one or several underlying assets. Black-Scholes and Merton introduced a parabolic partial differential equation (PDE) that the price of the European option satisfies under certain assumption. During the last decades, researchers have been presenting some numerical methods in order to solve Black-Scholes equation [2–5]. RBFs method is known as a powerful tool for interpolation of scattered data. The main advantage of radial basis functions method, is its meshless characteristic. In this paper we applied RBFs in order to solve the two-asset Black-Scholes PDE for Call on maximum model. In order to this purpose we use Multiquadratic radial basis functions. The rest of this paper is organized as follows: In section 2, a proposed method based on θ method and RBFs method for solving two-assets Black-Scholes PDE is presented. In addition, convergence of the proposed method are proved in section 3. Call on maximum Rainbow option are introduced in section 4. The proposed method is applied to solve these problems and their obtained numerical results are presented.

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2. Proposed method for two-asset option pricing

In this paper we consider two-asset Black-Scholes PDE

$$\begin{aligned}&\frac{\partial U}{\partial t}(x,y,t)-\frac{1}{2}\sigma_1^2x^2\frac{\partial^2 U}{\partial x^2}(x,y,t)-\sigma_1\sigma_2\rho xy\frac{\partial^2 U}{\partial x\partial y}(x,y,t)-\frac{1}{2}\sigma_2^2y^2\frac{\partial^2 U}{\partial y^2}(x,y,t)\\ &-rx\frac{\partial U}{\partial x}(x,y,t)-ry\frac{\partial U}{\partial y}(x,y,t)+rU(x,y,t)=0.\end{aligned}\tag{1}$$

where σ_1 and σ_2 are the volatility of assets x and y, respectively, ρ is the correlation coefficient between x and y and r is the risk-free rate. E is the strike price. The domain for each asset price is $[0,+\infty)$, but in a numerical method, we usually truncate the domain to $[0,L_1]$ and $[0,L_2]$ respectively. The choice of L_1 and L_2 usually depends on the evaluation area we are interested in. We consider (1) with initial condition $U(x,y,0)=payoff(x,y)$, and boundary conditions are defined according to the exact solution of (1). We discretize the domain with N division in x-axis and y-axis, not necessarily uniform as $\{x_i\}_{i=1}^N and \{y_i\}_{i=1}^N$, and M time steps, so interval $[0,T]$ is discretized with $\Delta t=\dfrac{T}{M}$, that T denotes the expiration time. Now, we approximate function U with RBF method according to $U(x,y,t)=\sum_{i=1}^{N^2}\lambda_i(t)\phi_i(x,y)$. where ϕ is a radial basis function. By defining

$$D=-\frac{1}{2}\sigma_1^2x^2\frac{\partial^2}{\partial x^2}-\sigma_1\sigma_2\rho xy\frac{\partial^2}{\partial x\partial y}-\frac{1}{2}\sigma_2^2y^2\frac{\partial^2}{\partial y^2}-rx\frac{\partial}{\partial x}-ry\frac{\partial}{\partial y}+r,\tag{2}$$

we can rewrite (1) to $\dfrac{\partial U}{\partial t}(x,y,t)+DU(x,y,t)=0$. Using the θ method

$$\left(\frac{U(x,y,t+\Delta t)-U(x,y,t)}{\Delta t}+O(\Delta t)\right)+(1-\theta)DU(x,y,t+\Delta t)+\theta DU(x,y,t)=0,\tag{3}$$

where the parameter θ is chosen in interval $[0,1]$. By rearranging (3) we have

$$[1+(1-\theta)\Delta tD]U^{n+1}=[1-\theta\Delta tD]U^n,\tag{4}$$

where $U^n=U(x,y,t^n)$. Defining $A=1+(1-\theta)\Delta tD \quad and \quad B=1-\theta\Delta tD$, we obtain

$$AU^{n+1}=BU^n.\tag{5}$$

By using RBF approximation, we find

$$U^{n+1}=\sum_{i=1}^{N^2}\lambda_i^{n+1}\phi_i(x,y),\tag{6}$$

$$U^{n}=\sum_{i=1}^{N^2}\lambda_i^{n}\phi_i(x,y).\tag{7}$$

Substituting values from (6) and (7) into (5) for all interial and boundary points of collocation points, we get the scheme in matrix form:

$$A\Phi\lambda^{n+1}=B\Phi\lambda^n+g^{n+1},\tag{8}$$

where $\Phi=[\phi(r_{i,j})]_{i,j=1}^N$ and g^{n+1} is a $N^2\times 1$ vector, such that according to interial points its components are equal to zero and its other componets are obtained by substituting boundary points into their boundary conditions. Subsequently (8) can be written as

$\lambda^{n+1} = (A\Phi)^{-1}(B\Phi)\lambda^n + (A\Phi)^{-1}g^{n+1}$. So, $\lambda^{n+1} = H\lambda^n + G$, where $H = (A\Phi)^{-1}(B\Phi)$ and $G = (A\Phi)^{-1}g^{n+1}$, it follows that

$$\mathcal{U}^{n+1} = \Phi H \Phi^{-1}\mathcal{U}^n + \Phi G^{n+1}. \tag{9}$$

In above relation $\mathcal{U}^0$ vector is obtained using initial condition.

3. The convergence of the proposed method

In this section, we prove the convergence of the scheme (9). We define matrix $E = \Phi H \Phi^{-1}$. The components of the matrix E depends on the constant $\gamma = \dfrac{\Delta t}{h^s}$, where h is the distance between any two nodes, and s is the highest order of partial differential operator, where s is equal to 2 for mentioned problem (1). We know that $|U_t(x,y) - u_t(x,y)| \leq \beta_l h^{l-1}|u|_{\mathcal{N}_\varphi(\Omega)}$, where $l \in \mathbb{N}$, $\mathcal{N}_\varphi(\Omega)$ is a native space of RBF φ and $u^n(x,y)$ is the exact solution of (1) at time $n\Delta t$. [6] We assume that (9) is accurate of order p, it yields that

$$\mathrm{u}^{n+1} = \Phi H \Phi^{-1}\mathrm{u}^n + \Phi G^{n+1} + O((\Delta t) + h^p), \quad \Delta t \to 0, h \to 0, \forall n \tag{10}$$

Now we define $e^n(x,y) = u^n(x,y) - U^n(x,y)$. By subtracting (9) from (10) we get:

$$\mathsf{e}^{n+1} = E\mathsf{e}^n + O((\Delta t) + h^p), \quad \Delta t \to 0, h \to 0 \tag{11}$$

By Lax-Richtmyer definiton of convergency the scheme in (9) is convergent if $\|E\| \leq 1$, hence, there exist a constant η such that $\|\mathsf{e}^{n+1}\| \leq \|E\|\|\mathsf{e}^n\| + \eta((\Delta t) + h^p)$. It is seen that $e^0 = 0$, using the initial condition. So we have $\|\mathsf{e}^{n+1}\| \leq \left(1 + \|E\|^2 + ... + \|E\|^{n+1}\right)\eta((\Delta t) + h^p)$. By considering the convergency condition, we obtain $\|\mathsf{e}^{n+1}\| \leq n\eta((\Delta t) + h^p)$. So convergence of the scheme is proved.

4. Implementation of the proposed method

In this section we introduce Rainbow options. The numerical solutions for them are further considered using the proposed method.

4.1. Rainbow option. Rainbow option is based on a combination of various assets like a rainbow is a combination of various colors. There are different forms of Rainbow option. In this paper we consider Call on maximum option. The exact solution is:

$$\begin{aligned} C(S_1,S_2,t) &= S_1[N(\delta_1) - N'(-d_1,\delta_1,\rho_1)] + S_2[N(\delta_2) - N'(-d_2,\delta_2,\rho_2)] \\ &+ Ee^{-r(T-t)}N'(-d_1 + \sigma_1\sqrt{T-t}, -d_2 + \sigma_2\sqrt{T-t}, \rho) - Ee^{-r(T-t)}, \end{aligned}$$

where

$$d_i = \frac{ln(\frac{S_i}{E}) + (r + \frac{1}{2}\sigma_i^2)(T-t)}{\sigma_i\sqrt{T-t}} \quad i = 1,2, \quad \delta_1 = \frac{ln(\frac{S_1}{S_2}) + (\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$\delta_2 = \frac{ln(\frac{S_2}{S_1}) + (\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad \rho_1 = \frac{\rho\sigma_2 - \sigma_1}{\sigma}, \quad \rho_2 = \frac{\rho\sigma_1 - \sigma_2}{\sigma}$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}, \quad N(\delta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\delta} e^{-\frac{z^2}{2}}\,dz.$$

$$N'(d,\delta,\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{d}\int_{-\infty}^{\delta} e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}}\,dxdy.$$

4.1.1. *An example of Call on maximum option.* Here we consider PDE (1) with initial and boundary conditions consistent with Call on maximum Rainbow option when, $E = 10, \sigma_1 = \sigma_2 = 0.2, T = 0.5, \rho = 0.1, r = 0.1, L_1 = L_2 = 40$. In order to use the proposed method we suppose $N = 10, M = 30$ and $\theta = 0.5$. This problem is solved by the proposed method by MQ RBF and appropriate shape parameter. The results of the proposed method and the explicit finite difference method with $\Delta S = 0.4, \Delta t = 0.005$ as benchmark, are shown in table 1, which is shown the high accuracy of the proposed method. In case

Table 1. Results for a Call on maximum option example

S_1	S_2	Approx by proposed method	Approx by explicite FDM	Exact solution
16	16	7.696995177509	7.694959078947	7.696995177078
20	8	10.487706098473	10.487772099135	10.487706094291
20	16	10.687059187097	10.686378201908	10.687059187049

of $S_1 = 20, S_2 = 8$ the effect of the time step size (Δt) to the computational accuracy is shown in figure 1 .

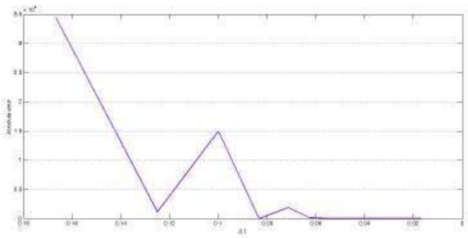

Figure 1. Variation of the absolute error with Δt

5. Conclusion

Two-asset options whose payoff depends on two underlying assets. In this paper, an efficient hybrid numerical method for solving PDE (1) was introduced based on the Crank-Nicolson and the radial basis functions methods. Furthermore, the convergence of the proposed method were proved. The proposed method were used for pricing of Rainbow options. The merit of the proposed hybrid method is its ability to achieve high accuracy without the need to use high computational cost.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Non- total imprisonment and Lorentzian length spaces

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Abstract. A sufficient and a necessary condition for non- total imprisonment on Lorentzian length spaces is given. Some properties of these kind of length spaces are investigated. In addition it is proved that any non-totally imprisoning locally causally closed and d-compatible Lorentzian length space which contains a lightlike line is causally disconnected.

Keywords: Length space, non- total imprisonment, ligthlike line

AMS Mathematics Subject Classification [2010]: 53C23, 53C50, 53B30

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1. Introduction

Non- imprisonment is an important causality condition in general relativity. Non- total imprisoning and non- partial imprisoning space-times are studied widely in [3]. In this paper we investigate these conditions on length spaces as a generalization of space-times. Let us recall some definitions and theorems which are needed in the second section.
Suppose that X is a set with reflexive and transitive relation $\leq$ and $\ll$ a transitive relation contained in $\leq$. Then $(X, \leq, \ll)$ is called a causal space [1].
$I^+(x) = \{y \in X : x \ll y\}$ and $I^-(x) = \{y \in X : y \ll x\}$
$J^+(x) = \{y \in X : x \leq y\}$ and $J^-(x) = \{y \in X : y \leq x\}$
We write $x < y$ if $x \leq y$ and $x \neq y$.

Definition 1.1. [1] Let $(X, \leq, \ll)$ be a causal space and d a metric on X. Let $\tau : X \times X \to [0, \infty]$ be a lower semi-continuous map that satisfies:

$$\tau(x, z) \geq \tau(x, y) + \tau(y, z)$$

For all $x, y, z \in X$ with $x \leq y \leq z$. Moreover, suppose that $\tau(x, y) = 0$ if $x \nleq y$ and $\tau(x, y) > 0 \Leftrightarrow x \ll y$. Then $(X, d, \ll, \leq, \tau)$ is called a Lorentzian pre-Length space and τ is called a time separation function.

Definition 1.2. [1] Let $I \subseteq R$ be an interval. A non-constant curve $\beta : I \to X$ is called future directed causal (timelike) if β is locally Lipschitz continuous and for all $t_1, t_2 \in I$, $t_1 < t_2$ we have $\beta(t_1) \leq \beta(t_2)$ $(\beta(t_1) \ll \beta(t_2))$. Furthermore, a future directed causal curve is called null if no two points on the curve are related with respect to $\ll$.

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Before defining a length space we need the following two definitions.

Definition 1.3. [1] A Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is called causally path connected if for every $x, y \in X$ with $x \ll y$ there is a future directed timelike curve from x to y and for $x < y$ there is a future directed causal curve from x to y.

Definition 1.4. [1] A Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is called localizable if $\forall x \in X$ there is an open neighborhood Ω_x of x in X with the following properties:

- There is $C > 0$ such that $L^d(\beta) \leq 0$ for all causal curve β contained in Ω_x,
- There is continuous map $\omega_x : \Omega_x \times \Omega_x \to [0, \infty)$ such that $(\Omega_x, d|_{\Omega_x \times \Omega_x}, \ll |_{\Omega_x \times \Omega_x}, \leq |_{\Omega_x \times \Omega_x}, \omega_x)$ is a Lorentzian pre-length space with the following condition: For every $y \in \Omega_x$ we have $I^{\pm}(y) \cap \Omega_x \neq \emptyset$,
- For all $p, q \in \Omega_x$ with $p < q$ there is a future directed causal curve $\beta_{p,q}$ from p to q that is maximal in Ω_x and satisfies
$$L_\tau(\beta_{p,q}) = \omega_x(p, q) \leq \tau(p, q),$$
(That the curve $\beta_{p,q}$ is maximal in Ω_x means that for every other future directed causal curve λ connecting p and q with image contained in Ω_x we have that $L_\tau(\beta_{p,q}) \geq l_\tau(\lambda)$)
We call such a neighborhood Ω_x a localizing neighborhood of x. If in addition, the neighborhood Ω_x can be chosen such that
- Whenever $p, q \in \Omega_x$ satisfy $p \ll q$ then $\beta_{p,q}$ is timelike and strictly longer than any future directed causal curve in Ω_x from p to q that contains a null segment.

Definition 1.5. [1] Let $(X, d, \ll, \leq, \tau)$ be a locally causally closed, causally path connected and localizable Lorentzian pre-length space and let $x, y \in X$. Then set
$\rho(x, y) = sup\{L_\tau(\beta) : \beta$ future directed causal from x to $y\}$, if the set of future -directed causal curves from x to y is not empty. Otherwise $\rho(x, y) = 0$. We call X a Lorentzian length space if $\rho = \tau$.

Definition 1.6. [1] Let $-\infty \leqslant a < b \leqslant \infty$ and let $\beta : [a, b) \to X$ be a future directed causal curve. It is called extendible if there exists a future directed causal curve $\widehat{\beta} : [a, b] \to X$ such that $\widehat{\beta}|_{[a,b)} = \beta$. The curve β is called inextendible if it is not extendible.

The following two theorems are used in section 2, to find a new characterization for non- total imprisonment.

Theorem 1.7. *[1] Let $(X, d, \ll, \leq, \tau)$ be a locally closed Lorentzian pre-length space. Let $-\infty < a < b \leqslant \infty$ and let $\beta : [a, b) \to X$ be a future directed causal curve parameterized with respect to d- arc length. If (X, d) is a proper metric space or the image of β is contained in a compact set then β is inextendible if and only if $b = \infty$. In this case, $L^d(\beta) = \infty$. Moreover β is inextendible if and only if $lim_{\nearrow b}\beta(t)$ does not exists.*

Theorem 1.8. *[1] Let $(X, d, \ll, , \tau)$ be a locally causally closed and d-compatible Lorentzian pre-length space. Let β_n be a sequence of future directed causal curves $\beta_n : [0, L_n) \to X$ that are parameterized with respect to d- arc length with $L_n = L^d(\beta_n) \to \infty$. If there is a compact set that contains every $\beta_n([0, L_n])$ or if d is proper and $\beta_n(0) \to x$ for some $x \in X$, then there is a subsequence (β_{n_k}) of β_n and a future directed causal curve $\beta : [0, \infty) \to X$ such that $\beta_{n_k} \to \beta$ locally uniformly. Moreover, β is inextendible.*

Definition 1.9. [1] Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space. A future directed causal curve $\beta : [a, b] \to X$ is maximal if $L_\tau(\beta) = \tau(\beta(a), \beta(b))$.

Null curves are always maximal on any compact interval.

2. non- imprisonment condition

Definition 2.1. [1] A Lorentzian length space $(X, d, \ll, \leq, \tau)$ is called:
non- total imprisoning if for every compact subset K of X there is a $C > 0$ such that the d- arc length of all causal curves contained in K is bounded by C.

Theorem 2.2. *Let $(X, d, \ll, \leq, \tau)$ be a locally causally closed and d-compatible Lorentzian length space.*

- *If for every compact set K and every future inextendible causal curve $\beta : [a, b) \to X$ there is $t_0 \in [a, b)$ such that for every $t \geq t_0$, $\beta(t) \notin K$ then X is non- total imprisoning,*
- *If X is non- total imprisoning then for every compact set K, every future inextendible causal curve $\beta : [a, b) \to X$ and every $t_0 \in [a, b)$, there is $t \geq t_0$ such that $\beta(t) \notin K$.*

Proof. For the first part suppose by contradiction that there is no $C > 0$ as in the definition of non- total imprisoning Lorentzian length spaces for a compact set K. Using theorem 1.8 there is an inextendible limit curve β in K and this is a contradiction.
To prove the second part let X be non- total imprisoning. Suppose by contradiction that there is a compact set K, an inextendible causal curve $\beta : [a, b) \to X$ and $t_0 \in [a, b)$ such that for every $t \geq t_0$, $\beta(t) \in K$. By using of theorem 1.7 this is a contradiction since the d-arc length of β is not finite. □

Definition 2.3. A lightlike line is an achronal inextendible causal curve.

The following definition is similar to what we have for Lorentzian space times [2].

Definition 2.4. A Lorentzian length space $(X, d, \ll, \leq, \tau)$ is causally disconnected by a compact set K if there are sequences p_n and q_n, $p_n \leq q_n$ going to infinity such that for each n every continuous causal curve connecting p_n to q_n intersects K.

Remark 2.5. Suppose that $(X, d, \ll, \leq, \tau)$ be a pre-length space. $x \ll y$ and $y \leq z$ implies that $x \ll z$, for $x, y, z \in X$. [1]

Theorem 2.6. *Any non-totally imprisoning locally causally closed and d-compatible Lorentzian length space which contains a lightlike line is causally disconnected.*

Proof. Let $\beta : (a, b) \to M$ be a lightlike line. Theorem 2.2 implies that there are sequences s_n and t_n, $s_n \to a$ and $t_n \to b$ such that $\beta(s_n) = p_n$ and $\beta(t_n) = q_n$ scapes every compact set. There is no causal curves from p_n to q_n without intersecting $K = \{\beta(0)\}$. Suppose by contradiction that there is such a curve, η. Then there is timelike curve from p_{n+1} to q_{n+1} by following β from p_{n+1} to p_n then following η from p_n to q_n and finally the curve β from q_n to q_{n+1}, which is a contradiction by using of remark 2.5, since β is ligthlike. □

The following two sets were used for characterization of non total imprisoning space-times in general relativity [3].
$\Omega_f(\beta) = \bigcap \overline{\beta(t, \infty]}$
$\Omega_p(\beta) = \bigcap \overline{\beta((-\infty, t])}$

Theorem 2.7. *Let* $(X, d, \ll, \leq, \tau)$ *be a Lorentzian length space. If* $\beta : R \to X$ *be an inextendible causal curve then the set* $\Omega_f(\beta)$ *is closed. For every causal curve* β*, the closure of its image is given by* $\overline{\beta} = \Omega_f(\beta) \cup \beta \cup \Omega_p(\beta)$.

Proof. Since the intersection of closed sets is close, $\Omega_f(\beta)$ is close. Suppose that the domain of β is (a, b) and $x \in \overline{\beta}$. There is a sequence $s_n \in (a, b)$ such that $\beta(s_n) \to x$. There are three cases. Either s_n admits a subsequence which converges to $s_0 \in (a, b)$, in this case by using of continuity $x = \beta(s_0) \in \beta$, or there is a subsequence which converges to b, in this case $x \in \Omega_f(\beta)$ or finally, there is a subsequence which converges to a which implies that $x \in \Omega_p(\beta)$. On the other hand it is obvious that $\Omega_f(\beta) \cup \beta \cup \Omega_p(\beta) \subset \overline{\beta}$. $\square$

Remark 2.8. If $(X, d, \ll, \leq, \tau)$ be a length space then $I^{\pm}(.)$ is open since:

- $x \ll y \Leftrightarrow \tau(x, y) > 0$,
- τ is lower semi continuous.

Theorem 2.9. *Let* $(X, d, \ll, \leq, \tau)$ *be a causal length space and* β *be a future inextendible causal curve then:*

- $\Omega_f(\beta)$ *is an achronal set,*
- *If* α *be a lightlike line that* $\alpha \subset \Omega_f(\beta)$ *then* $\Omega_f(\alpha) \cup \Omega_p(\alpha) \subset \Omega_f(\beta)$.

Proof. Suppose by contradiction that $\Omega_f(\beta)$ is not achronal then there are $x_1, x_2 \in \Omega_f(\beta)$ such that $x_2 \in I^+(x_1)$. Since X is length space remark 2.8 implies that I^+ is open. Hence there are neighborhoods U_1 and U_2 of x_1 and x_2 such that $U_1 \times U_2 \subset I^+$. There are, t_1, t_2 such that $t_2 < t_1$, $\beta(t_2) \in U_2$ and $\beta(t_1) \in U_1$, thus $\beta(t_2) \in J^+(\beta(t_1))$ and $\beta(t_1) \in I^+(\beta(t_2))$. This is a contradiction with the fact that X is causal.
To prove the second part suppose that $x \in \Omega_f(\alpha) \cup \Omega_p(\alpha)$ and U is a neighbourhood of x. There is s such that $\alpha(s) \in U$. But $\alpha(s) \in \Omega_f(\beta)$ and U is a neighborhood for $\alpha(s)$. Hence for an arbitrary T, there is $t > T$, such that $\beta(t) \in U$. T and U are arbitrary and consequently $x \in \Omega_f(\beta)$. $\square$

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30 August - 02 September 2021
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Adaptive integrated radial basis function method for time dependent partial differential equations

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ABSTRACT. The integrated radial basis function (RBF) method is a universal mesh-free method for the numerical solution of partial differential equations. Both global and local forms of this method achieve a higher order of accuracy. In this paper, we take advantage of the mesh-free property of the methods and use an adaptive algorithm to choose the location of the collocation points. An adaptive algorithm is used for Burgers equation and it is shown that it leads to high accuracy with fewer collocation points.

Keywords: Integrated radial basis function, Adaptive technique, Burgers equation

AMS Mathematics Subject Classification [2010]: 65M50, 65L50

1. Introduction

There are many methods for solving the partial differential equation. Known main methods can be categorized to finite difference method, finite element method, and finite volume method. The common feature of these methods is that they depend on grid and this limits their implementation for irregular areas. Meshless methods based on radial basis functions for solving partial differential equations have better features than the mentioned methods [1,2]. Some problems have high degree of space or time, steep gradients, corners, and physical changes resulting from nonlinearity, adaptive methods may be preferable to fixed grid methods. The purpose of adaptive methods is to find the PDE solution as well with a small number of the basis functions. Whereas integrated radial basis function (IRBF) method is completely meshfree, requiring just interpolation nodes and a set of points called center defining the basis functions, implementing adaptivity in terms of refining and coarsening nodes is very straightforward [3].

1.1. A brief review of the integrated radial basis functions method. First, a few primitive definitions are represented.

DEFINITION 1.1. Radial function [4]: A function : $\mathbb{R}^d \to \mathbb{R}$ is declared to be radial if there exists a function $\phi : [0, \infty] \to \mathbb{R}$ such that $\Phi(x) = \phi(\|x\|_2)$ for all $x \in \mathbb{R}^d$.

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The approximation of a distribution $S(\mathrm{x})$, using radial basis functions, may be impressed as a linear combination of N radial functions; commonly it takes the next form:

$$S(\mathrm{x}) \approx \sum_{l=1}^{N} \ell_l \phi_\epsilon(\|\mathrm{x}-\mathrm{x}_l\|) + P(\mathrm{x}), for\ \mathrm{x} \in \Omega \subset \mathbb{R}^d, \tag{1}$$

where N is the number of data points, $\mathrm{x} = (x_1, x_2, \ldots, x_d)$, d is the dimension of the problem, ℓ's are unknown coefficients to be determined, $\|.\|$ is Euclidean norm and ϕ_ϵ is the radial basis function and depends on ϵ in which ϵ is shape parameter. The IRBFs method employs integrating idea instead of derivating approach [5, 6].

2. Implementation of IRBF method for Burgers equation

Consider Burgers equation as follows

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \quad x \in (a,b),\ t > 0, \tag{2}$$

with the boundary and initial conditions

$$u(x,0) = f_0(x),\ x \in (a,b), \quad u(a,t) = g_1(t),\ u(b,t) = g_2(t),\ t > 0, \tag{3}$$

where f_0, g_1, g_2 are given functions. As seen from Eq. (2) , the order of PDE is two. According to the IRBF, the second- and first- derivatives and itself function are approximated as follows

(4)

$$\frac{\partial^2 u(x)}{\partial x^2} = \sum_{l=1}^{N} \beta_l(t)\psi(x),\ \frac{\partial u(x)}{\partial x} = \sum_{l=1}^{N} \beta_l(t)\phi_l(x) + \mu_1(t),\ u(x) = \sum_{l=1}^{N} \beta_l(t)\varphi_l(x) + \mu_1(t)x + \mu_2(t),$$

where

$$\phi(x) = \int \psi(x)dx, \varphi(x) = \int \phi(x)dx,$$

and $\mu_1(t)$ and $\mu_2(t)$ are the constants of integration at per time step. Collocating grid nodes in the associated stencil gives the following matrix

$$U = \Phi^{[0]} \begin{pmatrix} \beta(t) \\ \mu(t) \end{pmatrix} = \mathcal{C} \begin{pmatrix} \beta(t) \\ \mu(t) \end{pmatrix}, \quad \begin{pmatrix} \beta(t) \\ \mu(t) \end{pmatrix} = \mathcal{C}^{-1} U, \tag{5}$$

where $\Phi^{[0]}$ is a $N \times (N+2)$ matrix, $U = (u(x_1), u(x_2), \ldots, u(x_N))^T$, $\beta(t) = (\beta_1(t), \beta_2(t), \ldots, \beta_N(t))^T$, and $\mu(t) = (\mu_1(t), \mu_2(t))^T$ and $\mathcal{C}$ denotes conversion matrix. Substituting (5) into the following relation

$$U_{xx} = \Phi^{[2]} \begin{pmatrix} \beta(t) \\ \mu(t) \end{pmatrix} = \Phi^{[2]}\mathcal{C}^{-1}U = D_{xx}U,\ U_x = \Phi^{[1]} \begin{pmatrix} \beta(t) \\ \mu(t) \end{pmatrix} = \Phi^{[1]}\mathcal{C}^{-1}U = D_xU, \tag{6}$$

where $U_{xx} = (\frac{\partial^2 u(x_1)}{\partial x^2}, \ldots, \frac{\partial^2 u(x_N)}{\partial x^2}), U_x = (\frac{\partial u(x_1)}{\partial x}, \ldots, \frac{\partial u(x_N)}{\partial x})$ and D_{xx}, D_x are $N \times N$ matrices and putting (6) into (2) gives

$$\frac{dU}{dt} + U.*D_xU = \nu D_{xx}U, \text{ or more explicitly } \frac{dU}{dt} = \nu D_{xx}U - U.*D_xU,$$

where $.*$ is dot product.

2.1. Descripition of the adaptive method. For time-dependent PDEs, the adaptive algorithm is explained in the following algorithm.

INPUT: x, N, θ_{refine} and $\theta_{coarsen}$ and initial shape parameter;
for $i = 1 : T_{final}$ **do**
  Calculate the IRBF derivatives of u for x;
  Compute approximation of u by nemurical method for solving ODE in i-th time step ;
  while *All residual are less than* θ_{refine} **do**
    Calculate the global IRBF interpolant matrix for x;
    $y = x(1 : N - 1) + 0.5(\text{diff}(x))$ % "diff" is a MATLAB command that calculates differences between adjacent elements of x ;
    Calculate the global IRBF interpolant s_u for y;
    Compute residual $Res := |u - s_u|$;
    if $Res < \theta_{coarsen}$ **then**
      Find points and remove them;
    end
    if $Res > \theta_{refine}$ **then**
      Find points and add them to x;
    end
    Adapt the shape parameters;
  end
  Apply the boundary conditions;
  Update x, approximation of u and shape paramters;
end
OUTPUT: Approximation solution of u at final time;

Algorithm 1: Adaptive IRBF for time-dependent PDE

3. Numerical results

EXAMPLE 3.1. Investigate Eq. (4) with the following initial and boundary condition

$$u(0, t) = u(1, t) = 0, \; u(x, 0) = \sin(2\pi x) + \frac{1}{2}\sin(\pi x).$$

This method is solved by the presented technique and the obtained results will be compared with method suggested in [3]. Fig. 1 depicts numerical solution of Burgers equation with adaptive-IRBF method (right panel) and technique of [3] (left panel). According to this figure, it can be seen that the number of points used for the proposed method is less than the method of [3]. For sample, the method of [3] with $\nu = 0.001$ at $T = 0.3, 1$, the number of points created in this technique is $N = 153, N = 99$. Whereas the proposed method with $N = 84$ and $N = 59$ gives the same result. Also for $\nu = 0.0005$, the number of points created by the presented technique and technique of [3] are $N = 91, 76$ and $N = 179, 114$, respectively at times $T = 0.3$ and $T = 1$.

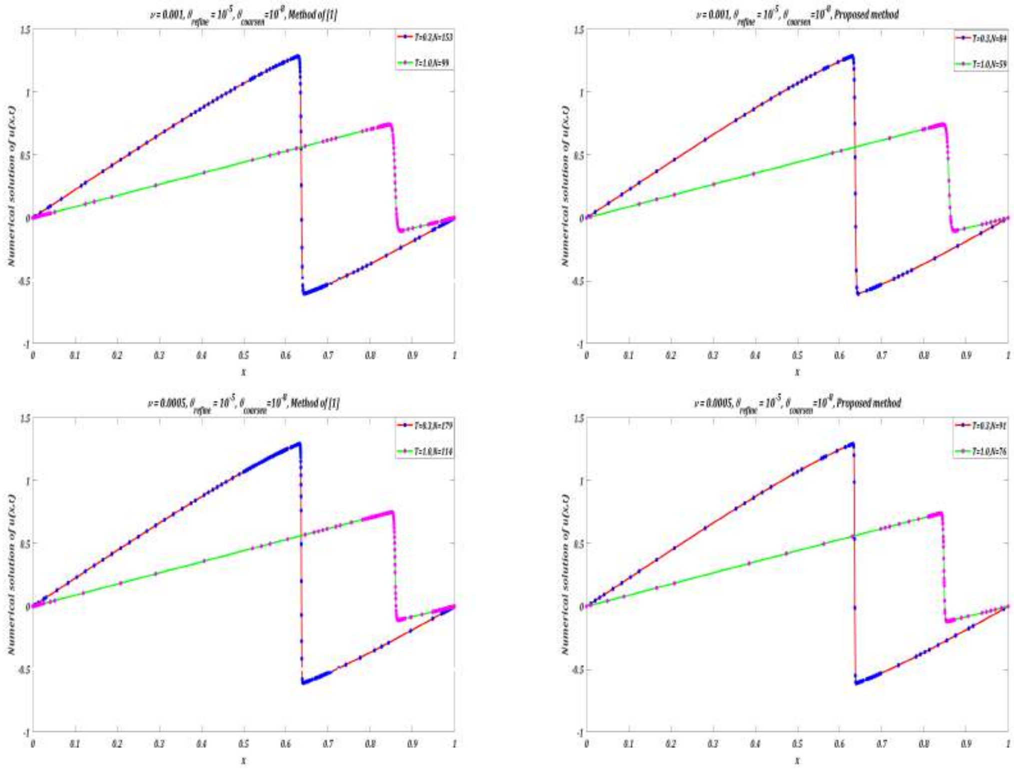

FIGURE 1. Numerical solution of Burgers equation with adaptive-IRBF method (right panel) and technique of [3] (left panel).

4. Conclusion

Integrated radial basis function method combined with an adaptive algorithm is used to solve Burgers equation. The inherent flexibility of the IRBF method allowed the node location to be chosen adaptively in a way that retained the desired accuracy but used significantly fewer centers. We compared the obtained results by the presented technique with technique in [3]. Our future research will concern the use of adaptive grid IRBF method in two and three space dimensions.

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پنجاه و دومین کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
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Nonlinear stochastic differential equations and novel operational matrix method

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Abstract. A novel operational matrix of the integral operator with respect to the variable-order fractional Brownian motion is applied to solve nonlinear stochastic differential equations. In addition, the convergence of the new method is investigated. Finally, the accuracy and efficiency of the new method are confirmed by solving a well-known model.

Keywords: Variable-order fractional Brownian motion, Stochastic differential equations, Generalized hat functions, Stochastic operational matrix

AMS Mathematics Subject Classification [2010]: 60H10, 60G22, 65G99

1. Introduction

In this research, we consider a class of nonlinear stochastic differential equations (NSDEs) given by:

$$\begin{cases} dy(t) = F(t, y(t))dt + G(t, y(t))dB_{H(t)}(t), \quad t \in [0, \ell], \\ y(0) = y_0, \end{cases} \tag{1}$$

where

$$dB_{H(t)}(t) = W_{H(t)}(t), \tag{2}$$

is a multi-fractional Gaussian noise. Here, y_0 is a given initial value; and the unknown function $y(t)$ is a stochastic process on probability space $(\Omega, \mathcal{F}, \mathbf{P})$; F, G are given functions; and $B_{H(t)}(t)$ is a variable-order fractional Brownian motion (fBm) of the Riemann-Liouville type with the time-dependent Hölder exponent $H(t) \in (\frac{1}{2}, 1)$ defined by Sheng et al. [4]:

$$B_{H(t)}(t) = \frac{1}{\Gamma\left(H(t) + \frac{1}{2}\right)} \int_0^t (t - \varsigma)^{H(t) - \frac{1}{2}} dB(\varsigma),$$

where $B(\varsigma)$ is a Brownian motion (Bm) process.

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The analytic solution of the above stochastic problem can not be obtained usually. Therefore, we present a new approach to provide approximate solution for the above problem.

2. Background

2.1. The GHFs are defined on the interval $[0,\ell]$ by [5]:

$$\psi_0(t)=\begin{cases}\dfrac{h-t}{h}, & 0\le t<h,\\ 0, & \text{otherwise},\end{cases}$$

$$\psi_j(t)=\begin{cases}\dfrac{t-(j-1)h}{h}, & (j-1)h\le t<jh,\\ \dfrac{(j+1)h-t}{h}, & jh\le t<(j+1)h,\\ 0, & \text{otherwise},\end{cases}\qquad j=1,2,\dots,n-2,$$

$$\psi_{n-1}(t)=\begin{cases}\dfrac{t-(\ell-h)}{h}, & \ell-h\le t\le \ell,\\ 0, & \text{otherwise},\end{cases}$$

where $h=\frac{\ell}{n-1}$.

2.2. We can approximate an arbitrary function $y(t)$ by the GHFs as follows:

$$y(t)\simeq y_n(t)=\sum_{j=0}^{n-1} y(jh)\psi_j(t)\triangleq \hat{Y}^T\Psi(t), \tag{3}$$

$$\hat{Y}=[\ y(0)\quad y(h)\quad y(2h)\quad \cdots\quad y(\ell)\ ]^T,$$
$$\Psi(t)=[\ \psi_0(t)\quad \psi_1(t)\quad \psi_2(t)\quad \cdots\quad \psi_{n-1}(t)\ ]^T.$$

3. The presented method

In this section, by converting Eq. (1) in integral form and using (3) we obtain:

$$\hat{Y}^T-y_0e^T-F(\hat{T}^T,\hat{Y}^T)\mathcal{P}-G(\hat{T}^T,\hat{Y}^T)\mathcal{P}_s\simeq 0, \tag{4}$$

where $\mathcal{P}$ and $\mathcal{P}_s$ are operational matrices of integration and integration with respect to the variable-order fBm [1], respectively, which for a sake of briefness we will not able to explain them in detail.

To determine unknown vector Y, existing in Eq. (4), we solve the above system. Substituting this vector in Eq. (3), an approximate solution can be obtained for Eq. (1).

4. Error bound

Here, we theoretically investigate an error bound for the proposed method. We recall that the definition of sup-norm for any continuous function y on the interval $[0,\ell]$ is as follows:

$$\|y\|=\sup_{t\in[0,\ell]}|y(t)|.$$

4.1. *Let $y(t)$ be an exact solution of Eq. (1) and $y_n(t)$ be the GHFs expansion of $y(t)$. Assume that F and G are Lipschitz continuous such that*

(C1) $|F(\varsigma, y(\varsigma)) - F(\varsigma, y_n(\varsigma))| \leq \eta_1 |y(\varsigma) - y_n(\varsigma)|$,
(C2) $|G(\varsigma, y(\varsigma)) - G(\varsigma, y_n(\varsigma))| \leq \eta_2 |y(\varsigma) - y_n(\varsigma)|$,

where η_1, η_2 are positive constants. Let $\alpha = \ell(\eta_1 + \eta_2 \|W_H\|)$ where

$$\|W_H\| = \sup_{t\in[0,\ell]} \left|W_{H(t)}(t)\right|.$$

Then $\lim_{n\to\infty} \|y - y_n\| = 0$ provided that $\alpha < 1$.

Proof. Suppose that $y_n(t)$, defined in (3), is an approximate solution of (1), then $y_n(t)$ satisfies the equation

$$\begin{aligned} y_n(t) =& y_0 + \int_0^t F(\varsigma, y_n(\varsigma))d\varsigma + \int_0^t G(\varsigma, y_n(\varsigma))dB_{H(\varsigma)}(\varsigma) \\ =& y_0 + \int_0^t F(\varsigma, y_n(\varsigma))d\varsigma + \int_0^t G(\varsigma, y_n(\varsigma))W_{H(\varsigma)}(\varsigma)d\varsigma. \end{aligned}$$

From (C1) and (C2), we obtain

$$\begin{aligned} |y(t) - y_n(t)| \leq& \left| \int_0^t (F(\varsigma, y(\varsigma)) - F(\varsigma, y_n(\varsigma)))\, d\varsigma \right. \\ &\left. + \int_0^t (G(\varsigma, y(\varsigma)) - G(\varsigma, y_n(\varsigma)))\, W_{H(\varsigma)}(\varsigma)d\varsigma \right| \\ \leq& \int_0^t |F(\varsigma, y(\varsigma)) - F(\varsigma, y_n(\varsigma))|\, d\varsigma \\ &+ \int_0^t |G(\varsigma, y(\varsigma)) - G(\varsigma, y_n(\varsigma))| \left|W_{H(\varsigma)}(\varsigma)\right| d\varsigma \\ \leq& \eta_1 \int_0^t |y(\varsigma) - y_n(\varsigma)|\, d\varsigma \\ &+ \eta_2 \int_0^t |y(\varsigma) - y_n(\varsigma)| \left|W_{H(\varsigma)}(\varsigma)\right| d\varsigma. \end{aligned} \tag{5}$$

Taking sup-norm of both sides of inequality (5) yields

$$\|y - y_n\| \leq \ell\eta_1 \|y - y_n\| + \ell\eta_2 \|y - y_n\| \|W_H\|, \tag{6}$$

therefore,

$$(1 - \ell(\eta_1 + \eta_2 \|W_H\|)) \lim_{n\to\infty} \|y - y_n\| \leq 0.$$

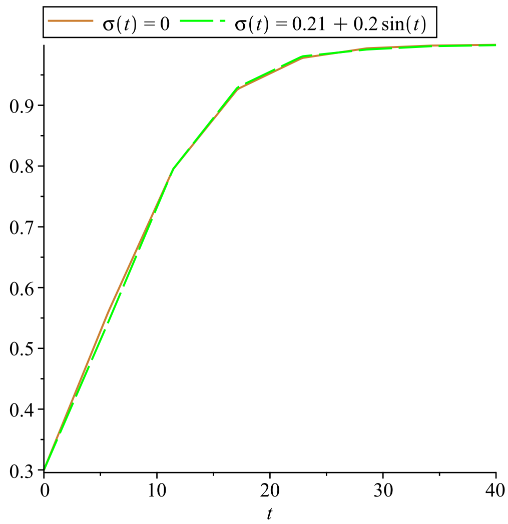
Let $\alpha = \ell(\eta_1 + \eta_2 \|W_H\|) < 1$, therefore, $\lim_{n\to\infty} \|y - y_n\| = 0$ and this completes the proof. □

5. Mathematical application

Stochastic logistic equation. We consider the stochastic logistic equation driven with the variable-order fBm and introduced in [2] as follows:

$$\begin{aligned} dy(t) &= \mu(t)y(t)\left(1 - \frac{y(t)}{\kappa}\right)dt + \sigma(t)y(t)\left(1 - \frac{y(t)}{\kappa}\right)dB_{H(t)}(t), \\ y(0) &= y_0. \end{aligned} \tag{7}$$

We apply our proposed method to solve Eq. (7), where $y_0 = 0.3$, $\kappa = 1$, $\mu(t) = 0.2$, $\sigma(t) = 0,\ 0.21 + 0.2\sin(t)$, and $t \in [0, 40]$. Also, we assume that $N = 600$, $n = 8$. The graphs of the approximate solutions are plotted in Figure 1 for $H(t) = 0.5 + 0.3\cos(1000t)$ and $H(t) = 0.3 + 0.3e^{-t}$, which illustrate the efficiency of our method.



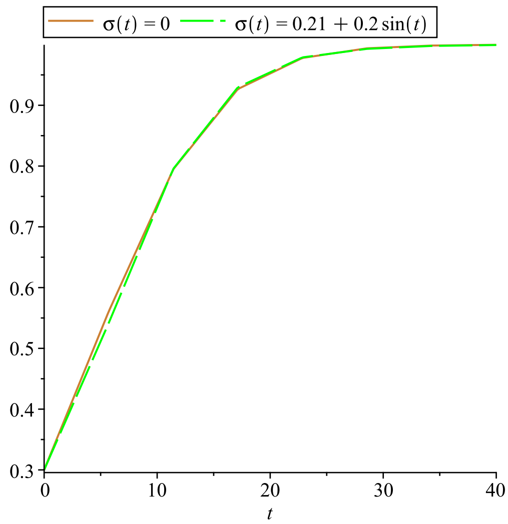


FIGURE 1. Plots of the approximate solutions with $H(t) = 0.5 + 0.3\cos(1000t)$ (left), $H(t) = 0.3 + 0.3e^{-t}$ (right) for the logistic equation.

6. Conclusion

In this research, a novel method was presented to determine numerical solutions of NSDEs. By using the stochastic operational matrix, these equations were reduced to systems of equations and solved by the Newton method. Convergence of the approach has been analyzed. Also, the method was evaluated by solving a well-known model. We can deduce that our method is an efficient numerical tool for solving NSDEs.

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On the existence of strictly positive doubly stochastic operators

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Abstract. Let $p \in [1, \infty)$ and I be a non-empty set. We denote by $\ell^p(I)$ the Banach space of all functions $f : I \to \mathbb{R}$ with $\|f\|_p := \left(\sum_{i\in I} |f(i)|^p\right)^{\frac{1}{p}} < \infty$. In this work, we investigate the existance of the strictly positive doubly stochastic operators in $\ell^p(I)$ for finite and infinite I. We prove that there is a doubly stochastic operator in $\ell^p(I)$ which is strictly positive if and only if I is countable. Also, some properties of such operators are considered.

Keywords: Strictly positive operator, Doubly stochastic operator, Matrix form

AMS Mathematics Subject Classification [2010]: 47A56, 47B60

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1. Introduction

Let $p \in [1, \infty)$ and I be a non-empty set. We denote by $\ell^p(I)$ the Banach space of all functions $f : I \to \mathbb{R}$ with $\|f\|_p := \left(\sum_{i\in I} |f(i)|^p\right)^{\frac{1}{p}} < \infty$. Let $\mathbf{e}_i : I \to \mathbb{R}$ be defined by $\mathbf{e}_i(j) = \delta_{ij}$, the Kronecker delta. It can be represented $f \in \ell^p(I)$ by $\sum_{i\in I} f(i)\mathbf{e}_i$.

Suppose that $p \in [1, \infty)$ and $D : \ell^p(I) \to \ell^p(I)$ be a bounded linear operator. Then D can be represented by a matrix $[d_{ij}]$, where $d_{ij} = (D\mathbf{e}_j)(i)$ and for all $f \in \ell^p(I)$ and $i \in I$, we have

$$(Df)(i) = \sum_{j\in I} d_{ij} f(j).$$

From now on we incorporate D to its matrix form $[d_{ij}]$.

A bounded linear operator $D : \ell^p(I) \to \ell^p(I)$ is said to be positive if $f \geq 0$ implies $Df \geq 0$, for $f \in \ell^p(I)$. It is known that D is positive if and only if $d_{ij} \geq 0$ for all $i, j \in I$. Similarly, D is said to be strictly positive if and only if for all $i, j \in I$, we have $d_{ij} > 0$.

Let us recall the definitions of a doubly stochastic operator.

Definition 1.1. Let $p \in [1, \infty)$ and I be a non-empty set. Then a positive bounded linear operator $D = [d_{ij}] : \ell^p(I) \to \ell^p(I)$ is called doubly stochastic operator if $\sum_{i\in I} d_{ij} = 1$ for all $j \in I$ and $\sum_{j\in I} d_{ij} = 1$ for all $i \in I$.

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We denote by $\Omega(\ell^p(I))$, the set of all doubly stochastic operators in $\ell^p(I)$ and $\Omega^s(\ell^p(I))$ denotes the set of all strictly positive doubly stochastic operators in $\ell^p(I)$.

In many areas of mathematics, physics and stochastic analysis, the finite and infinite matrices appear in functional equation, linear algebra, quantum mechanic and Markov chain. The doubly stochastic matrices play main role in the study of the doubly stochastic operators in the Banach space $\ell^p(I)$. On this subject we refer the reader to the books [5,6] and papers [1–4]. In this work, we prove that $\Omega^s(\ell^p(I))$ is a non-empty set if and only if I is a countable set. Also, some properties of such operators are considered.

2. The existence of strictly positive doubly stochastic operator

In this section, we discuss on the existence of a doubly stochastic operator in $\ell^p(I)$ that all of its entries are strictly positive. In fact, we know that the answer is yes for finite I. The simplest example which has this property, is $D := \frac{1}{n} J_{n\times n}$, where $J_{n\times n}$ is the $n \times n$ matrix of ones. Finding such example for infinite I is rather difficult.

Theorem 2.1. *If I is an uncountable set and $D = [d_{ij}] : \ell^p(I) \to \ell^p(I)$ is a doubly stochastic operator, then*

(i) *for all $i \in I$, there is $j \in I$ such that $d_{ij} = 0$,*
(ii) *for all $j \in I$, there is $i \in I$ such that $d_{ij} = 0$.*

Proof. Assume that $i \in I$ is arbitrary. Then we have $\sum_{j\in I} d_{ij} = 1$. Since I is an uncountable set and $d_{ij} \geq 0$, then there exists $j \in I$ such that $d_{ij} = 0$. The other case is similar. □

Remark 2.2. As in Theorem 2.1, for uncountable set I, there is no strictly positive doubly stochastic operator in $\ell^p(I)$ i.e., $\Omega^s(\ell^p(I)) = \emptyset$.

Theorem 2.3. *There is a strictly positive doubly stochastic operator $D : \ell^p(I) \to \ell^p(I)$ if and only if I is countable.*

Proof. Suppose that I is a countable set. If I is a finite set, then we have seen that all entries of the operator $D := \frac{1}{n} J_{n\times n}$ are positive and D is a strictly positive doubly stochastic operator. Now suppose that I is infinitely countable. For simplicity of notation, we assume that $I = \mathbb{N}$. Suppose that $D : \ell^p \to \ell^p$ is defined by the matrix form

$$D = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \cdots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \cdots \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then it is easily verified that $\frac{1}{2}(D + D^t) : \ell^p \to \ell^p$, where D^t is the transpose of D, is the doubly stochastic operator that all entries are positive. The converse is obtained by using Theorem 2.1. □

In the next example, we obtain that how we can constract a strictly positive doubly stochastic operator from $D \in \Omega^s(\ell^p)$.

Example 2.4. Let $n \in \mathbb{N}$ and $D = [d_{ij}] \in \Omega^s(\ell^p)$. Put $D_{ij} = \frac{d_{ij}}{n} J_{n\times n}$, where $J_{n\times n}$ is the $n \times n$ matrix of ones. Clearly, $[D_{ij}] \in \Omega^s(\ell^p)$.

Remark 2.5. In the following we obtain some properties of strictly positive doubly stochastic operators in $\ell^p(I)$.

(i) $\Omega^s(\ell^p(I))$ is a convex subset of $\Omega(\ell^p(I))$.
(ii) If $D_1,\ D_2 \in \Omega^s(\ell^p(I))$, then $D_1 D_2 \in \Omega^s(\ell^p(I))$.
(iii) $D \in \Omega^s(\ell^p(I))$ if and only if $D^t \in \Omega^s(\ell^p(I))$.
(iv) If card $I \geq 2$, then $\Omega^s(\ell^p(I)) \subsetneqq \Omega(\ell^p(I))$.
(v) If I is an uncountable set, then $\Omega^s(\ell^p(I)) = \emptyset$.
(vi) For $I = \mathbb{N}$, we have card $\Omega^s(\ell^p) = \mathcal{N}_1$.

3. Conclusion

In Theorem 2.1, we show that for uncountable set I, there is no strictly positive doubly stochastic operator in $\ell^p(I)$. In Theorem 2.3, we prove that there is a strictly positive doubly stochastic operator $D : \ell^p(I) \to \ell^p(I)$ if and only if I is countable. In Example 2.4, we show that how one can constract a strictly positive doubly stochastic operator from $D \in \Omega^s(\ell^p)$. We consider that card $\Omega^s(\ell^p) = \mathcal{N}_1$. Also, $\Omega^s(\ell^p(I))$ is a convex subset of $\Omega(\ell^p(I))$ which is closed under composition and transposition. If I is a non-empty set with at least two elements, then there exists a doubly stochastic operator in $\ell^p(I)$ which is not strictly positive.

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Faculty of Mathematics and Computer
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Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Best proximity point of weak contraction

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Abstract. In this article we extend the notion of orthogonal metric space to strongly orthogonal metric space. Also, the aim of this research is to define $\perp$-proximally increasing mapping and obtain several best proximity point results concerning this mapping in the framework of new spaces, which is called strongly orthogonal metric space.

Keywords: Best proximity point, O-set, strongly orthogonal

AMS Mathematics Subject Classification [2010]: 47H10; 54H25

1. Introduction

Recently, Eshaghi and et. [2] introduced the notion of orthogonal sets and orthogonal metric spaces. They also prove an existence of Banach fixed point theorem in orthogonal metric spaces [2] and generalizations of this theorem has been obtained in [1].
In 1969, Ky Fan [3] expressed fixed point problem for non-self mapping. In fact, the main idea of the theory the best proximity point. Clearly, a best proximity point theorem is a natural generalization of a fixed point theorem.

Definition 1.1. ([2]) Let X be a non-empty set and $\perp$ be a binary relation defined on $X \times X$. If $\perp$ satisfies the following condition

$$\exists x_0; \ ((\forall y; y \perp x_0) \quad or \quad (\forall y; x_0 \perp y)),$$

it is called an orthogonal set. The element x_0 is called an orthogonal element.

Let d be a usual metric on X. Then $(X, \perp, d)$ is called an orthogonal metric space.

Example 1.2. ([4]) Let $X = [2, \infty)$, we define $x \perp y$ if $x \leq y$ then by putting $x_0 = 2$, $(X, \perp)$ is an O-set.

Definition 1.3. ([2]) Let $(X, \perp)$ be an orthogonal set (O-set). Then a sequence $\{x_n\}_{n \in \mathbb{N}}$ is called orthogonal sequence (briefly O-sequence) if

$$((\forall n \in \mathbb{N}; x_n \perp x_{n+1}) \quad or \quad (\forall n \in \mathbb{N}; x_{n+1} \perp x_n)).$$

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Similarly, a Cauchy sequence $\{x_n\}$ is said to be a Cauchy O-sequence if

$$((\forall n \in \mathbb{N}; x_n \perp x_{n+1}) \quad or \quad (\forall n \in \mathbb{N}; x_{n+1} \perp x_n)).$$

Definition 1.4. [2] A mapping $T : X \longrightarrow X$ is called orthogonal preserving ($\perp$-preserving) if $x \perp y$ implies $T(x) \perp T(y)$ for all $x, y \in X$.

Definition 1.5. ([5]) Let $(X, \perp)$ be an orthogonal set (O-set). Then a sequence $\{x_n\}_{n \in \mathbb{N}}$ is called strongly orthogonal sequence (briefly SO-sequence) if

$$((\forall n, k \in \mathbb{N}; x_n \perp x_{n+k}) \quad or \quad (\forall n, k \in \mathbb{N}; x_{n+k} \perp x_n)).$$

Definition 1.6. ([5]) Let $(X, \perp)$ be an orthogonal set , then X is strongly orthogonal complete (briefly SO-complete) if every Cauchy SO-sequence is convergent.

It is easy to see that every complete metric space is SO-complete and the converse is not true. ([5])

Definition 1.7. ([5]) A mapping $T : X \longrightarrow X$ is strongly orthogonal continuous (SO-continuous) in $x \in X$ if for each SO-sequence $\{x_n\}_{n \in \mathbb{N}}$ in X such that $x_n \longrightarrow x$ then $T(x_n) \longrightarrow T(x)$. Also T is SO-continuous on X if T is SO-continuous in each $x \in X$.

It easy to see that every continuous mapping is SO-continuous, but [5] shows that the converse is not true.
Now, let A and B be two non-empty subsets of a metric space (X, d) and $T : A \to B$ be a non-self mapping. The point $x \in A$ is the best proximity point of T if $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(x, y); x \in A,\ y \in B\}$. Clearly, a best proximity point theorem is a natural generalization of a fixed point theorem.
For given two non-empty subsets A and B, we consider A_0 and B_0 by the following sets $A_0 = \{x \in A : d(x, y) = d(A, B)$ for some $y \in B\}$ and $B_0 = \{y \in B : d(x, y) = d(A, B)$ for some $x \in A\}$.

Definition 1.8. ([6]) Let (A, B) be a pair of non-empty subsets of (X, d) such that $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if and only if

$$\begin{cases} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B)\} \end{cases} \Rightarrow d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

2. Main results

We start with the following definition.

Definition 2.1. A mapping $T : A \longrightarrow B$ is a said to be $\perp$-proximally increasing if it satisfies the condition that

$$\begin{cases} y_1 \perp y_2 \\ d(x_1, Ty_1) = d(A, B) \\ d(x_2, Ty_2) = d(A, B) \end{cases} \Rightarrow x_1 \perp x_2,$$

where $x_1, x_2, y_1, y_2 \in A$.

Theorem 2.2. *Let $(X, \perp, d)$ is strongly orthogonal complete metric space and (A, B) be a pair of non-empty closed subset of the space $(X, \perp, d)$ with $A_0 \neq$. Let $T : A \longrightarrow B$ be a map such that* *i)* *T is a $\perp$-preserving and $\perp$-proximally increasing such that $T(A_0) \subseteq B_0$, (A, B) satisfies the P-property;*

ii) *There exist orthogonal elements x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$;*

iii) *T is a SO-continuous function on A such that*

$$d(Tx, Ty) \leq \varphi(d(x, y)) \tag{1}$$

for any point x and y in A such that $x \perp y$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ non-decreasing function with $\lim_{n\to\infty} \varphi^n(t) = 0$, for each $t > 0$.

Then T has a best proximity point $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$.

Proof. By (ii), there exist x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \perp x_1$. Since $Tx_1 \in T(A_0) \subseteq B_0$, there exist element x_2 in A_0 such that $d(x_2, Tx_1) = d(A, B)$. Since T is a $\perp$-preserving and $\perp$-proximally increasing, we get $x_1 \perp x_2$. Continuing this process, we can construct a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = d(A, B) \quad , \quad for\ all\ n \in \mathbb{N}, \tag{2}$$
$$with \quad x_0 \perp x_1,\ x_1 \perp x_2,\ x_2 \perp x_3,\ \ldots,\ x_n \perp x_{n+1},\ \ldots.$$

Thus $\{x_n\}$ is an O-sequence and consequently SO-sequence. Since (A, B) satisfies P-property, for any $n \in \mathbb{N}$, we have

$$\begin{cases} d(x_{n+1}, Tx_n) = d(A, B), \\ d(x_n, Tx_{n-1}) = d(A, B) \end{cases} \Longrightarrow d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}). \tag{3}$$

We claim that $\{x_n\}$ is a Cauchy SO-sequence. Now, since $x_0 \perp x_1$, we have $d(x_2, x_1) = d(Tx_1, Tx_0) \leq \varphi(d(x_1, x_0))$ and since $x_1 \perp x_2$, we have

$$d(x_3, x_2) = d(Tx_2, Tx_1) \leq \varphi(d(x_2, x_1)) \leq \varphi^2(d(x_1, x_0)).$$

By induction $d(x_{n+1}, x_n) \leq \varphi^n(d(x_1, x_0)) \to 0$, as $n \to \infty$. Let $\varepsilon > 0$ be fixed. Choose $N \in \mathbb{N}$ so that $d(x_{n+1}, x_n) < \varepsilon - \varphi(\varepsilon)$, for all $n \geq N$.
We denote a ball with center x and radius ε by $B[x, \varepsilon]$. Since $x_{N+1} \in B[x_N, \varepsilon]$, so $d(x_{N+1}, x_N) < \varepsilon$. Thus, from (1) and (3), we have

$$\begin{aligned} d(Tx_{N+1}, Tx_{N-1}) &\leq d(Tx_{N+1}, Tx_N) + d(Tx_N, Tx_{N-1}) \\ &\leq \varphi(d(x_{N+1}, x_N)) + d(x_{N+1}, x_N) < \varphi(\varepsilon) + (\varepsilon - \varphi(\varepsilon)) = \varepsilon. \end{aligned} \tag{4}$$

Therefore, $Tx_{N+1} \in B[Tx_{N-1}, \varepsilon]$. From (2), $d(x_{N+2}, Tx_{N+1}) = d(A, B)$ with $x_{N+2} \in A_0$ and $d(x_N, Tx_{N-1}) = d(A, B)$. From (3), we have $d(x_{N+2}, x_N) = d(Tx_{N+1}, Tx_{N-1})$. By (4), $d(x_{N+2}, x_N) < \varepsilon$, so $x_{N+2} \in B[x_N, \varepsilon]$ with $x_{N+2} \in A_0$, therefore

$$x_{N+2} \in B[x_N, \varepsilon] \cap A. \tag{5}$$

Again, from (1), (3) and (5), we get

$$d(Tx_{N+2}, Tx_{N-1}) \leq \varepsilon. \tag{6}$$

Therefore, $Tx_{N+2} \in B[Tx_{N-1}, \varepsilon]$. From (2), $d(x_{N+3}, Tx_{N+2}) = d(A, B)$ with $x_{N+3} \in A_0$ and $d(x_N, Tx_{N-1}) = d(A, B)$. From (3), we get $d(x_{N+3}, x_N) = d(Tx_{N+2}, Tx_{N-1})$. By (6), $d(x_{N+3}, x_N) < \varepsilon$. So $x_{N+3} \in B[x_N, \varepsilon]$ with $x_{N+3} \in A_0$, therefore $x_{N+3} \in B[x_N, \varepsilon] \cap A$. Continuing this process, we have $d(Tx_{N+n+1}, Tx_{N-1}) \leq \varepsilon$. So, we can conclude that $x_{N+m} \in B[x_N, \varepsilon] \cap A$, for all $m \in \mathbb{N}$. Then we get $\{x_n\}$ is a Cauchy SO-sequence in A. Since $\{x_n\}$ is a Cauchy SO-sequence in A, X is a SO-complete and A is a closed subset of X, the SO-sequence $\{x_n\}$ convergence to $x^* \in A$. Since T is a SO-continuous map on A, we have $Tx_n \to Tx^*$. By the continuity of the mapping d, we get $d(x_{n+1}, Tx_n) \to d(x^*, Tx^*)$.

But (2) shows that sequence is a constant sequence converges to $d(A,B)$. Therefore, $d(x^*,Tx^*) = d(A,B)$; that is, $x^* \in A$ is a best proximity point for T. □

THEOREM 2.3. *Suppose condition* (i) *and* (ii) *are true in Theorem 2.2, and we substitute the following condition instead of condition* (iii)*, and the rest of the conditions are true. Then* T *has a best proximity point* $x^* \in A$ *such that* $d(x^*,Tx^*) = d(A,B)$.

iii′) T is a SO-continuous function on A such that

$$d(Tx,Ty) \le \varphi(Q(x,y)) \tag{7}$$

where

$$Q(x,y) = \max\{d(x,y), d(x,Tx) - d(A,B), d(y,Ty) - d(A,B), \frac{1}{2}[d(x,Ty) + d(y,Tx) - 2d(A,B)], \frac{1}{2}[d(x,Tx) + d(y,Ty) - 2d(A,B)]\}$$

for any point x *and* y *in* A *such that* $x \perp y$ *and* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *non-decreasing function with* $\lim_{n\to\infty} \varphi^n(t) = 0$*, for each* $t > 0$.

PROOF. Similar to proof of Theorem 2.2, since (A,B) satisfies P-property and by (7), we have $d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1}) \le \varphi(Q(x_n,x_{n-1}))$, where

$$\begin{aligned} Q(x_n,x_{n-1}) = \max\{&d(x_n,x_{n-1}), d(x_n,Tx_n) - d(A,B), d(x_{n-1},Tx_{n-1}) - d(A,B),\\ &\frac{1}{2}[d(x_n,Tx_{n-1}) + d(x_{n-1},Tx_n) - 2d(A,B)],\\ &\frac{1}{2}[d(x_n,Tx_n) + d(x_{n-1},Tx_{n-1}) - 2d(A,B)]\}.\end{aligned}$$

We now claim

$$d(x_{n+1},x_n) \le \varphi(d(x_n,x_{n-1})). \tag{8}$$

Now, by using (2) and (3), we have $Q(x_n,x_{n-1}) \le \max\{d(x_n,x_{n-1}), d(x_n,x_{n+1})\}$. Thus, using the above inequality, (2), (3) and (7), we get

$$d(x_{n+1},x_n) \le \varphi(\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1})\}), \quad \forall n \in \mathbb{N}.$$

Suppose that $\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$ and since $\varphi(t) < t$ for all $t > 0$, we have $d(x_{n+1},x_n) \le \varphi(d(x_n,x_{n+1})) < d(x_n,x_{n+1})$, which is a contraction. Thus, for all $n \in \mathbb{N}$, we have $\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1})\} = d(x_n,x_{n-1})$, Thus $d(x_{n+1},x_n) \le \varphi(d(x_n,x_{n-1}))$, $\forall n \in \mathbb{N}$. So, (8) holds. By induction, we have

$$d(x_{n+1},x_n) \le \varphi^n(d(x_1,x_0)), \quad \forall n \in \mathbb{N}.$$

So $\lim_{n\to\infty} d(x_{n+1},x_n) = 0$. The continue of the proof is similar to the Theorem 2.2. □

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Well-posedness of the ZKB equation in the weighted spaces

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Abstract. In this work we consider a generalized dissipative ZK equation. The associated linear part produces both semigroup and group. As the dissipation is directional, we use a regularization method to study the associated initial value problem in Sobolev spaces $H^s(\mathbb{R}^n)$ and some weighted spaces $\mathcal{F}_r^{s,p}$. We also prove an ill-posedness result in the two-dimensional case.

Keywords: Well-posedness, Initial Value Problem, Semigroup

AMS Mathematics Subject Classification [2010]: 35Q35, 35K55, 35Q53

1. Introduction

In this paper, we study of the following evolution equation

$$u_t + (L_\alpha + f(u))_x = 0, \qquad t \in \mathbb{R}^+, \tag{1}$$

where $L_\alpha = \Delta u - \alpha u_x$ is the ZK operator with a directional dissipation. Here, $\alpha \in \mathbb{R}$, f is a differentiable real-valued function on $\mathbb{R}$ such that $f(0) = 0$ and $f'(0) = 0$ and we consider $u = u(x, y, t)$ such that $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $n \geq 2$. We also assume that $f(x) = O(x^{p+1})$, for $p \in \mathbb{N}$. The evolution equation (1) is known as the ZKB equation when $f(u) = u^2/2$, because of appearing the ZK operator and the Burgers-type dissipation. The ZKB equation (1) describes asymptotically the propagations of nonlinear dust acoustic waves in a nonuniform magnetized dusty plasma [2,4]. By neglecting the dissipative term in (1), we will get the so-called ZK equation.

$$u_t + (\Delta u + f(u))_x = 0. \tag{2}$$

In this work, we are going to study the Cauchy problem associated to (1) in Sobolev spaces. Our strategy is to use a regularization by applying more dissipative terms to the equation; in fact, we will consider the following regularized ZKB (rZKB) problem:

$$u_t + (\Delta u + f(u) - \alpha u_x)_x - \beta \Delta_\perp u = 0, \tag{3}$$

where $\beta \in \mathbb{R}^+$ and $\triangle = \partial_x^2 + \triangle_\perp$. Next, by using the semigroup properties of (3), we endeavor to prove a well-posedness result in $H^s(\mathbb{R}^2)$ spaces for $s > 2$ and show our results hold in weak topology as the parameter β tends to zero. In dimension two, we will also show that (1) is well-posed in the weighted spaces $H^s(\mathcal{W})$ (see Definition 2.1) for some

*Speaker. Email address: saesfahani@gmail.com suitable weight functions $\mathcal{W}$. Regarding on the ill-posedness issue, we are not able to derive a criterion to anticipate the minimum index of local well-posedness due to the directional dissipation; but we establish that the flow-map of equation (1) fails to be C^2 in $H^{s,0}(\mathbb{R}^2)$ for $s<-\frac{3}{4}$.

2. Main results

Now we summarize our main results skipping several propositions and technical lemmas.

Definition 2.1. We denote by $L^2(\mathcal{W})$ the space of all real-valued functions f such that $\|f\|^2_{L^2(\mathcal{W}dx)}=\int f^2(x)\mathcal{W}(x)\,dx<\infty$, where $H^s=H^s(\mathbb{R}^n)$ is the nonhomogeneous Sobolev space. Especially, for $\mathcal{W}(x)=1+\sum\limits_{i=1}^{n}x_i^{2r_i}$ and $r=(r_1,\cdots,r_n)\in\mathbb{R}^n$, we denote $\mathcal{F}^s_r$ the space of all real-valued measurable functions f such that $\|f\|_{\mathcal{F}^s_r}=\|f\|_{H^s}+\|f\|_{L^2(\mathcal{W})}<\infty$. Similarly for any $p\geq 1$, one can define $\mathcal{F}^{s,p}_r=H^s\cap L^p(\mathcal{W})$. For $r\in\mathbb{R}$, we denote $\mathcal{F}^s_r$ as $\mathcal{F}^s_{r,\cdots,r}$ and $H^s(\mathcal{W})=H^s\cap L^2(\mathcal{W})$.

Using properties of the semigroup associated to the linear problem, we can obtain the our main local well-posedness theorem.

Theorem 2.2. *Let $s>2$. Then for any initial data $u_0\in H^s$, there exist $T^s_{\alpha,\beta}=T\left(\alpha,\beta,\|u_0\|_{H^s}\right)$ and a unique solution of the initial value problem (3), $u_{\alpha,\beta}(\cdot)$, defined in the interval $\left[0,T^s_{\alpha,\beta}\right]$ satisfying*

$$u_{\alpha,\beta}\in C\left(\left[0,T^s_{\alpha,\beta}\right];H^s\right)\cap C^1\left(\left[0,T^s_{\alpha,\beta}\right];H^{s-2}\right).$$

Moreover,

$$u_{\alpha,\beta}\in C\left(\left(0,T^s_{\alpha,\beta}\right];H^\infty\right).$$

Furthermore, the theorem is true for $\beta=0$ (in the weak topology sense) and $\alpha=\beta=0$.

To prove Theorem 2.2, we use the following estimates. The semigroup associated with (1) is denoted by $U_{\alpha,\beta}$.

Lemma 2.3. *Let $\alpha,\beta>0$ and $s\in\mathbb{R}$, then for any $\delta\geq 0$ and all $t>0$, $U_{\alpha,\beta}(t)\in\mathcal{L}(H^s,H^{s+\delta})$. Moreover there exists $C_s>0$ such that*

$$\|u_{\alpha,\beta}(t)\|_{H^{s+\delta}}\leq C_s\sqrt{1+t^{-s}\max\{\alpha^{-s},\beta^{-s}\}}\;\|u_0\|_{H^s},\tag{4}$$

for any $u_0\in H^s$.

Lemma 2.4. *Let $U^0_{\alpha,\beta}(t)=U_{\alpha,\beta}(t)\delta_0$, $m=(m_1,m_2),k=(k_1,k_2)\in\left(\mathbb{Z}^+\right)^2$, $x\in\mathbb{R}^2$ and $t>0$, where δ_0 is Dirac delta.*

i) If $2\leq p\leq\infty$, then there exists $C(\alpha,\beta)>0$ such that

$$\left\|x^kD^mU^0_{\alpha,\beta}(t)\right\|_{L^p}\leq C(\alpha,\beta)\langle t\rangle^{\frac{1}{2}|k|}\,t^{-\frac{1}{2}|m|-2\left(1-\frac{1}{p}\right)}.\tag{5}$$

ii) If $1\leq p\leq 2$, then there exists $C(\alpha,\beta)>0$ such that

$$\left\|x^kD^mU^0_{\alpha,\beta}(t)\right\|_{L^p}\leq C(\alpha,\beta)\langle t\rangle^{\frac{1}{2}(|k|-1)}\,t^{-2\left(1-\frac{1}{p}\right)-\frac{|m|}{2}},\tag{6}$$

where $|k|=k_1+k_2$, $|m|=m_1+m_2$ and $\langle t\rangle=\left(1+t^2\right)^{1/2}$.

iii) $u_{\alpha,\beta}(t) \in L^p$ *for any* $2 \le p \le \infty$*, if* $u_0 \in L^2$*. Moreover,*

$$\|u_{\alpha,\beta}(t)\|_{L^p} \lesssim t^{-\theta}\|u_0\|_{L^p},$$

where $\theta = \theta(p) = 1 - \frac{2}{p}$.

Now, we use the properties of the Kato-Ponce commutator [3]. Let J^s be is the Bessel potential of order $-s$ and $\mathcal{S}(\mathbb{R}^2)$ is the Schwartz class.

Lemma 2.5. *If* $f, g \in \mathcal{S}(\mathbb{R}^2)$*,* $s > 0$ *and* $p \in (1, +\infty)$*, then*

$$\|[J^s, M_f]g\|_{L^p} \lesssim \left(\|\nabla f\|_{L^{p_1}} \left\|J^{s-1}g\right\|_{L^{p_2}} + \|J^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}\right), \tag{7}$$

$$\|fg\|_{L^p} \lesssim \left(\|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}} + \|J^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}\right), \tag{8}$$

where $p_2, p_3 \in (1, +\infty)$ *such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Theorem 2.6 (Continuous Dependence). *There exists a metric space* $E^s_{\alpha,\beta}$ *such that for* $R > 0$*, the correspondence* $u_0 \to u_{\alpha,\beta}$ *that associates to* $u_0 \in \mathcal{B}_R$ *the solution* $u_{\alpha,\beta}$ *of (3) with initial data* u_0 *is continuous mapping of* $\mathcal{B}_R$ *to* $E^s_{\alpha,\beta}$*, where* $\mathcal{B}_R$ *is the ball of radius* R *centered at the origin of* H^s*.*

To study the well-posedness in the weighted spaces, we need to understand the behavior of our semigroup in such spaces.

Remark 2.7. One can obtain the explicit form of $\mathcal{X}(t)$, T^s and $\mathcal{A}_T$. Indeed,

$$\mathcal{X}(t) = \frac{2^{\frac{2}{p}}\|u_0\|^2_{H^s}}{\left(2 - c_s tp\|u_0\|^p_{H^s}\right)^{\frac{2}{p}}},$$

$$T^s = \frac{2}{\left(c_s p\|u_0\|^2_{H^s}\right)} \text{ and } \mathcal{A}_T = \frac{2^{\frac{1}{p}}\|u_0\|_{H^s}}{\left(2 - c_s pT\|u_0\|^p_{H^s}\right)^{\frac{1}{p}}}, \text{ for any } T \in (0, T^s).$$

Lemma 2.8. *Let* $p, m \in \mathbb{N}$*,* $\beta > 0$*,* $t > 0$ *and* $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$*. Then there exists* $C(m, \beta, |\omega|) > 0$ *such that for any* $f \in \mathcal{F}^{0,p}_{0,m}$*,*

$$\|D^\omega U_{\alpha,\beta}(t)f\|_{\mathcal{F}^{0,p}_{0,m}} \le C(m, \beta, |\omega|)\left(1 + t^{-\frac{|\omega|}{2}} + t^{\frac{m-|\omega|}{2}}\right)\|f\|_{\mathcal{F}^{0,p}_{0,m}}$$

Theorem 2.9 (Well-posedness Result in Weighted Spaces). *Let* $\mathcal{W}$ *be a weight with all its first and second derivatives bounded and such that* $|\mathcal{W}(x,y)| \le C_\varepsilon e^{\varepsilon\left(x^2+y^2\right)}$*, for all* $(x,y) \in \mathbb{R}^2$ *and any* $\varepsilon \in (0, \widetilde{\varepsilon})$*, for some* $\widetilde{\varepsilon} > 0$ *and* $C_\varepsilon > 0$*. Let also* $u_0 \in H^s(\mathcal{W})$*,* $s > 2$*. Then the solution* $u_{\alpha,\beta}$ *of the equation (3) corresponding to the initial data* u_0 *is in* $C\left(\left[0, T^s_{\alpha,\beta}\right); H^s(\mathcal{W})\right)$*. Moreover, the continuous dependence of solutions of the equation (1) holds in* $H^s(\mathcal{W})$*.*

Theorem 2.10 (Persistence of Solutions). *Let* $s \in \mathbb{N}$*,* $s \ge 3$ *and* $\beta \ge 0$*. Also suppose that* $u_{\alpha,\beta} \in C\left(\left[0, T^s_{\alpha,\beta}\right); H^s\right)$ *is the maximal solution of the rZKB equation corresponding to the initial data* $u_0 \in \mathcal{F}^{s,2}_{1,s}$*. Then* $u_{\alpha,\beta} \in C\left(\left[0, T^s_{\alpha,\beta}\right); \mathcal{F}^{s,2}_{1,s}\right)$*.*

Next, we show that the Picard iteration method cannot be used to obtain a solution of (1). Indeed, we construct a sequence of initial data that will ensure the irregularity of the flow map for $s < -3/4$.

Theorem 2.11 (Ill-posedness Result). *Let $s < -\frac{3}{4}$ and $H^{s,0}(\mathbb{R}^2)$ be the x-directional Sobolev space. Then there is no $T > 0$ such that the ZKB equation (1), with $f(u) = u^2/2$, admits a unique solution u in $C([0,T]; H^{s,0}(\mathbb{R}^2))$ for any initial data in the same ball of $H^{s,0}(\mathbb{R}^2)$ centered at the origin and the map $\phi \to u$ is C^2-differentiable at the origin from $H^{s,0}(\mathbb{R}^2)$ to $C([0,T]; H^{s,0}(\mathbb{R}^2))$.*

To prove the above theorem, we notice that contrary to the ZKB equation, there is a the minimum index of local well-posedness for the gZK with power law nonlinearity

$$u_t + \Delta u_x + u^p u_x = 0, \qquad u = u(x,y,t), \quad (x,y) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{9}$$

More precisely, we establish that one cannot obtain local well-posedness of the gZK equation (9) for data in $H^s(\mathbb{R}^n)$ for $s \le \frac{n}{2} - \frac{2}{p}$ and $p \ge 4/n$. To proceed with this result, we use the ideas of Birnir et al. [1].

In the light of this comment, we show an nonexistence of traveling wave of the form $u(x,y,t) = \varphi(x - ct, y)$ of the gZKB equation in the plane.

Theorem 2.12. *There is no solution φ of gZKB satisfying*

$$\begin{cases} D^\beta \varphi \to 0 \quad \textit{as} \quad |(x,y)| \to +\infty \quad \textit{such that} \quad \beta \in \mathbb{N}^2 \quad \textit{and} \quad |\beta| \ge 1, \\ \varphi \to a \quad \textit{as} \quad x \to +\infty, \\ \varphi \to b \quad \textit{as} \quad x \to -\infty, \\ \varphi \to d \quad \textit{as} \quad |y| \to +\infty, \end{cases}$$

for any a, b and $d \in \mathbb{R}$.

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Embedding theorems on Bergman spaces with admissible weights

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Abstract. Let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions on the open unit disk $\mathbb{D}$ and ω be a radial weight defined on $\mathbb{D}$. For $0 < p < \infty$, the weighted Bergman space $\mathcal{A}^p_\omega$ consists of functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{\mathcal{A}^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) < \infty,$$

where $dA(z) = dxdy/\pi$ stands for the normalized area measure in $\mathbb{D}$. We describe those positive Borel measures μ in the unit disc $\mathbb{D}$ such that the Bergman space $\mathcal{A}^p_\omega \subset L^q(\mu)$, $0 < p \leq q < \infty$, where ω belongs to a large class of weights which includes the standard weights and the exponential type weights.

Keywords: Carleson measures; weighted Bergman spaces; compact operators

AMS Mathematics Subject Classification [2010]: Primary 47B33; Secondary 47B35

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1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions on $\mathbb{D}$. Given a positive integrable function ω on $[0, 1)$, we extend it by $\omega(z) = \omega(|z|), z \in \mathbb{D}$, and call such ω a weight function. Our results are stated for weights satisfying the following conditions.:

($\mathcal{W}_1$) ω is non-increasing,

($\mathcal{W}_2$) $\omega(r)(1 - r^2)^{-(1+\delta)}$ is non-decreasing for some $\delta > 0$.

A weight ω is called admissible if ω satisfies ($\mathcal{W}_1$) and ($\mathcal{W}_2$). For example, the standard weights $\omega(r) = (1-r^2)^\alpha$, $\alpha > -1$ and the exponential type weights $\omega(r) = (1-r^2)^\alpha \exp(1-r^2)^\beta$, $\alpha, \beta > 0$ are admissible weights.

For $0 < p < \infty$, the weighted Bergman space $\mathcal{A}^p_\omega$ consists of functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{\mathcal{A}^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) < \infty,$$

where $dA(z) = dxdy/\pi$ stands for the normalized area measure in $\mathbb{D}$.

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Note that $\mathcal{A}^p_\omega$ is the closed subspace of $L^p(\mathbb{D}, \omega dA)$ consisting of analytic functions and for $1 < p < \infty$, $(\mathcal{A}^p_\omega, \|\cdot\|_{\mathcal{A}^p_\omega})$ is a Banach space. If $\omega(r) = (1-r^2)^\alpha$, $\alpha > -1$, the standard Bergman spaces $\mathcal{A}^p_\alpha$ are obtained.

For $a \in \mathbb{D}$, by φ_a we mean the automorphism of the unit disc given by

$$\varphi_a(z) = \frac{a-z}{1-\overline{a}z}, z \in \mathbb{D},$$

and $E(a,r) := \{z \in \mathbb{D} : |\varphi_a(z)| < r\}$ denotes the pseudohyperbolic disc of radius r centered at a. It is well-known that for all $a \in \mathbb{D}$ and $z \in E(a,r)$ with $r \in (0,1)$, $1-|a|^2 \approx 1-|z|^2$, where the constants depend only on r. Moreover,

$$|\varphi_a'(z)| = \frac{1-|a|^2}{|1-\overline{a}z|^2} = \frac{1-|\varphi_a(z)|^2}{1-|z|^2}.$$

It is also well-known that for each $z \in E(a,r)$,

$$\frac{1}{|1-\overline{a}z|^2} \geq \frac{1-r^2}{(1-|a|^2)(1-|z|^2)}. \tag{1}$$

Let X be a space of analytic functions on the unit disc $\mathbb{D}$. A positive Borel measure μ in $\mathbb{D}$ is said to be a q-Carleson measure for X if the embedding $X \subset L^q(\mu)$, $0 < q < \infty$ is continuous. It means for some positive constant M

$$\left(\int_{\mathbb{D}} |f(z)|^q d\mu(z) \right)^{1/q} \leq M\|f\|_X.$$

A description of (vanishing) Carleson measures have been obtained for several spaces of analytic functions (see e.g. [1, 3–6]). Here we obtain a complete description of the q-(vanishing) Carleson measures for $\mathcal{A}^p_\omega$ when $0 < p \leq q < \infty$. Our results can be used in order to study several related questions such as the characterization of boundedness and compactness of composition operators and integral operators on weighted Bergman spaces.

Recall that the norm of the bounded operator $T : X \to Y$ is denoted by $||T||_{X\to Y}$ and the notation $A \approx B$ means that $cB \leq A \leq CB$ for some constants c and C.

2. Main results

We use the following lemmas frequently for obtaining our main results.

Lemma 2.1. *Let ω be a weight satisfying $(\mathcal{W}_1)$ and $(\mathcal{W}_2)$. Then for each $r \in (0,1)$, $a \in \mathbb{D}$ and $z \in E(a,r)$ we have $\omega(a) \approx \omega(z)$ and*

$$\int_{E(a,r)} \omega(z) dA(z) \approx \omega(a) \int_{E(a,r)} dA(z) \approx \omega(a)(1-|a|^2)^2.$$

Lemma 2.2. *[7] Let ω be a weight satisfying $(\mathcal{W}_1)$ and $(\mathcal{W}_2)$. Then*

$$\int_{\mathbb{D}} \frac{\omega(z)dA(z)}{|1-\overline{a}z|^{p(1+\delta)+2}} \approx \frac{\omega(a)}{(1-|a|^2)^{p(1+\delta)}}.$$

For obtaining our main results we need some auxiliary results.

Proposition 2.3. *If ω satisfies conditions $(\mathcal{W}_1)$ and $(\mathcal{W}_2)$, then there exists a positive constant C (independent of f and z) such that for each $f \in \mathcal{A}^p_\omega$ and $z \in \mathbb{D}$,*

$$|f(z)|^p \leq C \frac{\|f\|^p_{\mathcal{A}^p_\omega}}{(1-|z|^2)^2\omega(z)}.$$

We need the following family of test functions in $\mathcal{A}^p_\omega$ in order to characterize the q-Carleson measures.

For $a \in \mathbb{D}$, let $f_a(z) = \frac{(1-|a|^2)^{1+\delta}}{\omega(a)^{1/p}(1-\bar{a}z)^{1+\delta+2/p}}$. It can be shown that $\{f_a : \ a \in \mathbb{D}\}$ is a bounded subset of $\mathcal{A}^p_\omega$.

We will need the following decomposition of $\mathbb{D}$ into equal-sized squares in the pseudo-hyperbolic metric.

Lemma 2.4. *[2, Lemma 12] For each pseudohyperbolic radius r, there exist a sequence $\{z_n\}$ of points in $\mathbb{D}$ satisfying the following two conditions:*

(i) $\mathbb{D} = \cup_{n=1}^{\infty} E(z_n, r)$;

(ii) *There is a positive integer N such that each point $z \in \mathbb{D}$ belongs to at most N of the sets $E(z_n, 2r)$.*

Theorem 2.5. *Let μ be a positive Borel measure on $\mathbb{D}$, ω satisfies $(\mathcal{W}_1)$ and $(\mathcal{W}_2)$, $0 < p \leq q < \infty$ and $r \in (0,1)$. Then the following statements are equivalent:*

(i) $\|\mu\|_\omega = \sup_{a\in\mathbb{D}} \frac{\mu(E(a,r))}{(1-|a|^2)^{2q/p}\omega(a)^{q/p}} < \infty$,

(ii) *μ is a q-Carleson measure for $\mathcal{A}^p_\omega$ and $I_d : \mathcal{A}^p_\omega \to L^q(\mu)$ is bounded with norm equivalent to $\|\mu\|_\omega$.*

As an immediate consequence of Theorem 2.5 we have the following corollary.

Corollary 2.6. *Let μ be a positive Borel measure on $\mathbb{D}$ and ω satisfies $(\mathcal{W}_1)$ and $(\mathcal{W}_2)$. Then μ is q-Carleson if and only if for each pseudohyperbolic radius r, there exist a sequence $\{z_n\}$ of points in $\mathbb{D}$ such that $\mathbb{D} = \cup_{n=1}^{\infty} E(z_n, r)$ and $M = \sup_{n\in\mathbb{N}} \frac{\mu(E(z_n,r))}{(1-|z_n|^2)^{2q/p}\omega(z_n)^{q/p}} < \infty$.*

For the standard Bergman spaces A^p_α , the statement of Theorem 2.5 also holds, and one can see [4, Theorem 2.2]. Now we characterize the q-vanishing Carleson measures for $\mathcal{A}^p_\omega$. Before stating our main theorem, we need the following lemma.

Lemma 2.7. *For every bounded sequence in $\mathcal{A}^p_\omega$, there is a subsequence which converges uniformly on compact subsets of $\mathbb{D}$ to an element of $\mathcal{A}^p_\omega$.*

Theorem 2.8. *Let μ be a positive Borel measure on $\mathbb{D}$, ω satisfies $(\mathcal{W}_1)$ and $(\mathcal{W}_2)$, $0 < p \leq q < \infty$ and $r \in (0,1)$. Then the following statements are equivalent:*

(i) *μ is a q-vanishing Carleson measure for $\mathcal{A}^p_\omega$ (i. e. $I_d : \mathcal{A}^p_\omega \to L^q(\mu)$ is compact),*

(ii) $\limsup_{|a|\to 1^-} \frac{\mu(E(a,r))}{(1-|a|^2)^{2q/p}\omega(a)^{q/p}} = 0$.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

A novel fractional Legendre collocation method for a class of non-linear systems of fractional differential equations

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Abstract. In this paper, an interpolation operator based on the orthogonal fractional Legendre functions is introduced and employed to develop a high-order collocation approach for the numerical solution of a class of non-linear systems of fractional differential equations. The applicability and validity of the method are justified by a prototype example.

Keywords: Non-linear system of fractional differential equations, Fractional Legendre functions, Collocation method

AMS Mathematics Subject Classification [2010]: 34A09, 65L05, 65L20

1. Introduction

We intend to provide a highly accurate numerical approach for solving the following non-linear system of fractional differential equations (FDEs)

$$\begin{cases} D_C^{\gamma} y_j(t) = \sum_{i=1}^{n} p_{j,i}(t) y_i^{k_i}(t) + p_{j,n+1}(t), \quad j \in \aleph_n = \{1, 2, ..., n\}, \\ y_j(0) = y_j^0, \; k_i \in \mathbb{N}, \; t \in \Lambda = [0, 1], \end{cases} \tag{1}$$

where $\gamma = \frac{\eta}{\lambda} \in (0, 1)$ is a positive rational number described by the co-prime integers $\eta \geq 1$ and $\lambda \geq 2$. D_C^{γ} is known as Caputo fractional derivative of order γ defined by [1]

$$D_C^{\gamma}(.) = I^{1-\gamma} \partial_t(.),$$

in which $I^{1-\gamma}$ denotes the Riemann-Liouville fractional integral operator of order $(1 - \gamma)$ [1].

The rest of this paper is organized as follows. In the next section, we introduce the fractional Legendre functions and design a fractional interpolation operator. A novel fractional Legendre collocation method is developed to approximate the solution of (1) in Section 3. In Section 4, a prototype example is conducted to show the efficiency of the proposed method.

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2. The fractional Legendre functions

The fractional Legendre functions $L_n^{(\tau)}(t)$ with $\tau \in (0,1]$ and $t \in \Lambda$ are defined from the Legendre polynomials $L_n(x)$ through the coordinate transform $t = 2x^\tau - 1$ as follows [4]

$$L_n^{(\tau)}(t) = L_n(2x^\tau - 1).$$

These functions are mutually orthogonal concerning the weight function $w^{(\tau)}(t) = t^{\tau-1}$, i.e.,

$$\int_\Lambda L_m^{(\tau)}(t) L_n^{(\tau)}(t) w^{(\tau)}(t) dt = \frac{1}{\tau(2n+1)} \delta_{mn}, \quad m, n \geq 0.$$

We have the following result on fractional Legendre-Gauss quadrature.

Lemma 2.1. *[4] Suppose that $\{x_j, w_j\}_{j=0}^N$ are the Legendre-Gauss quadrature nodes and corresponding weights respectively [3]. Considering $t_j^{(\tau)} = \left(\frac{x_j+1}{2}\right)^{1/\tau}$ and $w_j^{(\tau)} = w_j$, we have*

$$\int_\Lambda u(t) w^{(\tau)}(t) dt = \sum_{j=0}^N u(t_j^{(\tau)}) w_j^{(\tau)}, \quad \forall u \in \phi_{2N+1}^{(\tau)},$$

where

$$\phi_{2N+1}^{(\tau)} = Span\{L_n^{(\tau)}(t) : \; 0 \leq n \leq 2N+1\}.$$

Further properties of Legendre polynomials and the fractional Legendre functions can be found in [3] and [4], repectively.

The fractional Legendre interpolation operator $I_N^{(\tau)} : L^2_{w^{(\tau)}}(\Lambda) \to \phi_N^{(\tau)}$ is introduced as follows

$$\left(I_N^{(\tau)} u\right)(t_j^{(\tau)}) = u(t_j^{(\tau)}), \quad 0 \leq j \leq N.$$

Since $I_N^{(\tau)} u \in \phi_N^{(\tau)}$, we can write

$$\left(I_N^{(\tau)} u\right)(t) = \sum_{j=0}^N \tilde{u}_j L_j^{(\tau)}(t),$$

where the unknown coefficients $\tilde{u}_j$ are obtained from $\tilde{u}_j = \tau(2n+1)\langle u, L_j^{(\tau)}\rangle_{N,w^{(\tau)}}, \quad 0 \leq j \leq N$. Here, $\langle u, v\rangle_{N,w^{(\tau)}}$ is the fractional Legendre-Gauss discrete inner product formula.

Clearly, from Lemma 2.1, we have

$$\langle u, v\rangle_{N,w^{(\tau)}} = (u, v)_{w^{(\tau)}}, \quad \forall uv \in \phi_{2N+1}^{(\tau)}.$$

3. Fractional Legendre collocation method

We set $\tau = \frac{1}{\lambda}$ and consider the fractional Legendre collocation solution as

(2)

$$y_{j,N}(t) = \sum_{l=0}^N a_{j,i} L_i^{(\tau)}(t) = \underline{a}_j \underline{L}^{(\tau)} = \underline{a}_j L^{(\tau)} \underline{T}_t, \qquad \underline{a}_j = [a_{j,0}, a_{j,1}, \dots, a_{j,N}], \qquad j \in \aleph_n,$$

where $\underline{L}^{(\tau)} = [L_0^{(\tau)}(t), L_1^{(\tau)}(t), \ldots, L_N^{(\tau)}(t)]^T$ is the vector of fractional Legendre functions, $L^{(\tau)}$ is an invertible lower triangular matrix and $\underline{T}_t = [1, t^\tau, \ldots, t^{N\tau}]^T$. Consider

(3)

$$p_{j,i}(t) \simeq p_{j,i,N}(t) = \sum_{l=0}^{N} \bar{p}_{j,i,l}\ t^{\tau l} = \underline{\bar{p}}_{j,i}\underline{T}_t,\ \underline{\bar{p}}_{j,i} = [\bar{p}_{j,i,0}, \bar{p}_{j,i,1}, \ldots, \bar{p}_{j,i,N}],\ j \in \aleph_n,\ i \in \aleph_{n+1},$$

Now, we give the following lemma which transforms $y_{i,N}^{k_i}(t),\ i \in \aleph_n$ into a suitable matrix form.

Lemma 3.1. *The following relation holds*

$$y_{i,N}^{k_i}(t) = \underline{a}_i L^{(\tau)} \mathcal{M}_i^{k_i - 1} \underline{T}_t, \quad i \in \aleph_n, \tag{4}$$

where $\mathcal{M}_i$ is the following infinite upper-triangular matrix

$$\mathcal{M}_i = \begin{bmatrix} \underline{a}_i L_0^{(\tau)} & \underline{a}_i L_1^{(\tau)} & \underline{a}_i L_2^{(\tau)} & \ldots \\ 0 & \underline{a}_i L_0^{(\tau)} & \underline{a}_i L_1^{(\tau)} & \ldots \\ 0 & 0 & \underline{a}_i L_0^{(\tau)} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

with $L_s^{(\tau)} = \{L_{m,s}^{(\tau)}\}_{m=0}^{\infty},\ s = 0, 1, \ldots$.

Employing the relations (2), (4) and (3) into the equivalence system of Volterra integral equations of (1) yields

$$\underline{a}_j L^{(\tau)} \underline{T}_t = y_j^0 + \sum_{i=1}^{n} \underline{a}_i L^{(\tau)} \mathcal{M}_i^{k_i - 1} \left(\sum_{l=0}^{N} \bar{p}_{j,i,l}\ I^\gamma \left(t^{\tau l} \underline{T}_t \right) \right) + \underline{\bar{p}}_{j,n+1} I^\gamma \left(\underline{T}_t \right),\ j \in \aleph_n. \tag{5}$$

Therefore, Computing $I^\gamma \left(t^{\tau l} \underline{T}_t \right)$ and $I^\gamma \left(\underline{T}_t \right)$ the relation (5) can be written as

$$\underline{a}_j L^{(\tau)} \underline{T}_t = y_j^0 + \sum_{i=1}^{n} \underline{a}_i L^{(\tau)} \mathcal{M}_i^{k_i - 1} \underline{T}_{j,i,t} + \underline{\bar{p}}_{j,n+1} \underline{T}_{j,t},\ j \in \aleph_n,$$

where

$$\underline{T}_{j,i,t} := \sum_{l=0}^{N} \bar{p}_{j,i,l} \left[\frac{\Gamma(\tau(l+k)+1)}{\Gamma(\tau(l+k)+\gamma+1)}\ t^{\gamma + \tau(l+k)} \right]_{k=0}^{N},$$

$$\underline{T}_{j,t} := \left[\frac{\Gamma(\tau k + 1)}{\Gamma(\tau k + \gamma + 1)}\ t^{\gamma + \tau k} \right]_{k=0}^{N},\ \ i, j \in \aleph_n.$$

are the vectors of order $N+1$. Defining $\underline{\underline{a}}_j = \underline{a}_j L^{(\tau)} = [\underline{a}_{j,0}, \underline{a}_{j,1}, \ldots, \underline{a}_{j,N}]$, we have

$$\underline{\underline{a}}_j (\underline{T}_t - \mathcal{M}_j^{k_j - 1} \underline{T}_{j,j,t}) = y_j^0 + \sum_{i=1, i \neq j}^{n} \underline{\underline{a}}_i \mathcal{M}_i^{k_i - 1} \underline{T}_{j,i,t} + \underline{\bar{p}}_{j,n+1} \underline{T}_{j,t},\ j \in \aleph_n, \tag{6}$$

such that the matrix $\mathcal{M}_i^{k_i-1}$, $i \in \aleph_n$ has the following upper-triangular Toeplitz structure [2]

$$\begin{bmatrix} \mathcal{M}_{i,0,0}^{k_i-1} & \mathcal{M}_{i,0,1}^{k_i-1} & \dots & \mathcal{M}_{i,0,N}^{k_i-1} \\ 0 & \mathcal{M}_{i,0,0}^{k_i-1} & \dots & \mathcal{M}_{i,0,N-1}^{k_i-1} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \mathcal{M}_{i,0,0}^{k_i-1} \end{bmatrix} \tag{7}$$

where $\{\mathcal{M}_{i,0,s}^{k_i-1}\}_{s=0}^{N}$ are non-linear functions of the elements $\underline{a}_{i,0}$, $\underline{a}_{i,1}, \dots,$ $\underline{a}_{i,s}$. In this step, setting the fractional Legendre-Gauss nodes $\{t_r^{(\tau)}\}_{r=0}^{N}$ introduced in Lemma 2.1 into (6), a non-linear system of algebraic equations of order $n(N+1)$ is achieved which is solved by the well known iterative quasi Newton method. Consequently, obtaining $\underline{\underline{a}}_j$, the approximate solutions (2) can be characterized by solving $\underline{\underline{a}}_j = \underline{a}_j L^{(\tau)}$.

4. Numerical example

Consider the following example.

Example 4.1.

$$\begin{cases} D_C^{\frac{1}{2}} y_1(t) = 2t\ y_1^4(t) + \frac{1}{2}\sin(t)\ y_2(t) + p_{1,3}(t), \\ D_C^{\frac{1}{2}} y_2(t) = E_{\frac{1}{2}}(-t^{\frac{1}{2}})\ y_1(t) - t^{\frac{3}{2}}\ y_2^3(t) + p_{2,3}(t), \\ y_1(0) = y_2(0) = 0, \end{cases}$$

where the forcing functions $p_{j,3}(t)$, $j = 1,2$ are derived such that the exact solutions are $y_1(t) = \sin(t^{\frac{1}{2}})$ and $y_2(t) = t^{\frac{5}{2}}$. Here, $E_\sigma(t)$ is the well-known Mittag-Leffler function.

This problem is solved via suggested approach, and the numerical consequences are depicted in Table 1. To derive the desired numerical errors, 200-terms of the Mittag-Leffler functions are considered.

Table 1. Obtained errors for Example 4.1 for different values of N.

N	$\Vert\epsilon_{1,N}\Vert_{L^2_{w^\tau}}$	$\Vert\epsilon_{2,N}\Vert_{L^2_{w^\tau}}$
2	8.01×10^{-2}	4.08×10^{-1}
4	3.33×10^{-3}	4.08×10^{-1}
8	8.64×10^{-7}	9.56×10^{-16}
16	6.18×10^{-16}	1.42×10^{-16}

5. Conclusion

In this paper, a suitable fractional collocation approach based on the fractional Legendre functions was implemented, and the high accuracy of the method was emphasized through the numerical solution of an example.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Some properties of superconvex-cyclic operators

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Abstract. A convex polynomial is a convex combination of the monomials $\{1, z, z^2, ...\}$. A bounded linear operator T on a Banach space X is called superconvex-cyclic if there is a vector $x \in X$ such that $\{\lambda p(T)x | \ \lambda \in \mathbb{C}, p \ is \ a \ convex \ polynomial\}$ is dense in X. Some spectral properties of superconvex-cyclic operators are obtained. It is proved that positive multiples of superconvex-cyclic operators are superconvex-cyclic. It is shown that the convex-cyclicity of an operator T is equivalent to the superconvex-cyclicity of $I_{\mathbb{C}} \oplus T$. Also, we discuss superconvex-cylicity of matrices.

Keywords: convex polynomial, convex-cyclic operator, superconvex-cyclic operator.

Mathematics subject classification [2010]: $47A16$, $46A25$.

1. Introduction and Preliminaries

Let $\mathcal{B}(X)$ denote the algebra of all bounded linear operators on a separable Banach space X. Also, $\mathbb{N}$ is referred to $\{0, 1, 2, \ldots\}$, and $\mathbb{D}$ is the open unit disc in the complex plane $\mathbb{C}$.

For an operator $T \in \mathcal{B}(X)$ and a vector $x \in X$, by the orbit of x under T we mean $orb(T, x) := \{T^n x | n \in \mathbb{N}\}$.

Let $\mathcal{CP}$ denote the convex hull of monomials $\{1, z, z^2, \ldots\}$. That is,

$$\mathcal{CP} := \{\sum_{i=0}^{m} a_i z^i : a_i \geqslant 0 \ for \ all \ 0 \leqslant i \leqslant m, \ and \sum_{i=0}^{m} a_i = 1\}.$$

The elements of $\mathcal{CP}$ are called convex polynomials.

An operator $T \in \mathcal{B}(X)$ is called

(1) cyclic with a cyclic vector x, if the linear span of $orb(T, x)$, i.e., $\{p(T)x| \ \ p \in \mathcal{P}\}$, is dense in X for some $x \in X$;
(2) hypercyclic with a hypercyclic vector x, if $orb(T, x)$ is dense in X for some $x \in X$;
(3) supercyclic with a supercyclic vector x, if $\mathbb{C}.orb(T, x)$ is dense in X for some $x \in X$;

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(4) convex-cyclic with a convex-cyclic vector x, if $co(orb(T,x)) = \{p(T)x| \ p \in \mathcal{CP}\}$ is dense in X for some $x \in X$;
(5) superconvex-cyclic with a superconvex-cyclic vector x, if $\mathbb{C}.co(orb(T,x))$ is dense in X for some $x \in X$.

In this presentation, we describe some known results about hypercyclic, supercyclic, and convex-cyclic operators and then present similar new results for superconvex-cyclic operators. Then we focus on superconvex-cyclicity of matrices.
The concept of convex-cyclicity is introduced by Rezaei in [6] and then it has been studied more by authors in [2], [3], and [5]. The concept of superconvex-cyclicity, introduced by authors in [4] in the weak sense, lies between cyclicity and convex-cyclicity or supercyclicity. So every convex-cyclic or supercyclic operator is superconvex-cyclic and every superconvex-cyclic operator is cyclic. Indeed, we have shown that the inclusions are strict. We denote the set of all superconvex-cyclic vectors and the set of all convex-cyclic vectors for T, as $\mathcal{SCC}(T)$ and $\mathcal{CC}(T)$, respectively.

2. Superconvex-cyclicity versus other kinds of cyclicity

The connection between dynamics of a linear operator and the point spectrum of its adjoint, denoted by $\sigma_p(T^*)$, is studied a lot in literature.

PROPOSITION 2.1. *Suppose that $T \in \mathcal{B}(X)$.*
(i) *If T is hypercyclic then $\sigma_p(T^*) = \emptyset$ [1].*
(ii) *If T is supercyclic then $\sigma_p(T^*)$ is empty or consists of exactly one nonzero number λ. In the latter case, dim ker $(T^* - \lambda) = 1$ [1].*
(iii) *If T is convex-cyclic, then $\sigma_p(T^*) \cap (\overline{\mathbb{D}} \cup \mathbb{R}) = \emptyset$. Besides, if $\lambda_1, \lambda_2 \in \sigma_p(T^*)$, then $\overline{\lambda_2} \neq \lambda_1$. ([3])*

We have obtained a related result for superconvex-cyclic operators that runs as follows.

THEOREM 2.2. *Let $T \in \mathcal{B}(X)$. If T is superconvex-cyclic, then $\sigma_p(T^*)$ contains at most one real number, and if it contains $\lambda \geq 1$, then $\sigma_p(T^*) \cap \overline{\mathbb{D}} = \emptyset$. Besides, if $\lambda_1, \lambda_2 \in \sigma_p(T^*)$, then $\overline{\lambda_2} \neq \lambda_1$.*

It has been proved that if T is hypercyclic and $|c| = 1$, then so is cT. Besides every nonzero multiple of a supercyclic operator is supercyclic. The next result gives a similar statement for a convex-cyclic operator.

PROPOSITION 2.3. *(Proposition 2.4 of [2]) If T is a convex-cyclic operator on a locally convex space X, and if $c > 1$, then cT is also convex-cyclic. Furthermore, $\mathcal{CC}(T) = \mathcal{CC}(cT)$.*

We have proved a similar result for superconvex-cyclic operators.

THEOREM 2.4. *If T is a superconvex-cyclic operator on a Banach space X, then so is cT for every scalar $c > 0$. Furthermore, $\mathcal{SCC}(T) = \mathcal{SCC}(cT)$.*

A connection between convex-cyclic operators and superconvex-cyclic ones is obtained in the next result.

THEOREM 2.5. *Let $\mathcal{H}$ be a Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is convex-cyclic if and only if $I_{\mathbb{C}} \oplus T$ is superconvex-cyclic on $\mathbb{C} \oplus \mathcal{H}$.*

We remark that if T is a convex-cyclic operator then $I_{\mathbb{C}} \oplus T$ is not convex-cyclic, because $1 \in \sigma_p((I_{\mathbb{C}} \oplus T)^*)$. This along with the preceding theorem shows that convex-cyclic operators form a strict subclass of superconvex-cyclic operators.

3. Superconvex-cyclicity of matrices

It is known that for any matrix A on $\mathbb{C}^n$, there are unique commuting matrices B and C such that B is diagonalizable, C is nilpotent, and $A = B + C$. So a method for investigating the superconvex-cyclicity of matrices, is to discuss the problem for sum of two such matrices. In this connection, we first consider the superconvex-cyclicity of nilpotent perturbation of multiples of the identity on $\mathbb{C}^n$ for $n > 1$. Note that, in light of Proposition 1(iii), since the spectrum of a nilpotent matrix consists of zero, it is not convex-cyclic. Also, λI is not superconvex-cyclic on $\mathbb{C}^n$. For their sum, we have obtained the following result.

THEOREM 3.1. *Suppose that Q is a nonzero nilpotent matrix on $\mathbb{C}^n$ for $n > 1$.*
(a) If $\lambda \in \mathbb{R}$, then $\lambda I + Q$ is not superconvex-cyclic.
(b) If $\lambda \notin \mathbb{R}$, then $\lambda I + Q$ is superconvex-cyclic if and only if $dimker(Q) = 1$.

Next, we consider the sum of a diagonal matrix and a nilpotent one. The following result is obtained.

THEOREM 3.2. *Suppose that $diag(\lambda_1, \cdots, \lambda_n)$ is a diagonal matrix on $\mathbb{C}^n$ and Q is a nilpotent matrix commuting with D. If at least two λ_j's are distinct real numbers, then $D + Q$ is not superconvex-cyclic.*

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Lie algebras of generalized Lie groups based on right-invariant vector fields

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Abstract. In this paper, we are going to specify a relation between the Lie algebras of a class of generalized Lie groups and the Lie algebras of a class of Lie groups, by using of the right-invariant vector fields of generalized Lie groups and their one-parameter subgroups.

Keywords: generalized Lie group, Lie algebra, right-invariant vector field

AMS Mathematics Subject Classification [2010]: 22E60, 17B60, 17B66

1. Introduction

Lie theory is a section of mathematics that relates to various branches of mathematics and physics namely algebra, non-associative algebras, quasi-groups, geometry, mathematical physics, combinatorics, H-spaces and related systems, operad theoretical and cohomological methods, jet theory, number theory and quantum theory. The notion of Lie algebra is one of the fundamental concepts of modern mathematics and mathematical physics. The general theory of Lie algebras leads to a rich assortment of important explicit examples of geometric objects. So importance of Lie algebras leads to importance of their generalizations. There are different kinds of generalizations of Lie algebras, such as hom-Lie algebras, hom-Lie superalgebras, color hom-Lie algebras, etc. [1, 4, 7]. A Lie group is, roughly speaking, an analytic manifold with a group structure such that the group operations are analytic. Lie groups provide a way to express the concept of a continuous family of symmetries for geometric objects. Some generalized structures of Lie groups such as Lie groupoids and generalized Lie groups have been defined through the recent century. The notion of a generalized Lie group or a top space was first introduced by Molaei [5]. In this generalized setting, several authors studied various aspects and concepts of generalized groups and top spaces [2, 3, 6].

First, we recall the definition of a generalized group and a generalized Lie group.

Definition 1.1. [5] A generalized group is a nonempty set $\mathcal{G}$ admitting an operation called multiplication which satisfies the following conditions:

- $(g_1.g_2).g_3 = g_1.(g_2.g_3)$ for all $g_1, g_2, g_3 \in \mathcal{G}$.

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- For any $g \in \mathcal{G}$ there exists a unique $e(g)$ in $\mathcal{G}$ such that
$$g.e(g) = e(g).g = g.$$
- For any $g \in \mathcal{G}$ there exists $h \in \mathcal{G}$ such that
$$g.h = h.g = e(g).$$

Remark 1.2. For any generalized group $\mathcal{G}$ and any $g \in \mathcal{G}$,
$$e^{-1}(e(g)) = \{h \in \mathcal{G} | e(h) = e(g)\},$$
has a canonical group structure.

A generalized Lie group (a top space) is a smooth manifold whose points can be (smoothly) multiplied together by a generalized group operation and generally its identity is a semigroup morphism, i.e.,

Definition 1.3. [2] A top space T is a Hausdorff d-dimensional differentiable manifold which is endowed with a generalized group structure such that the generalized group operations:

- $. = T \times T \to T$ by $(t_1, t_2) \mapsto t_1.t_2$ which is called the multiplication map;
- ${}^{-1} : T \to T$ by $t \mapsto t^{-1}$ which is called the inverse map are differentiable and it holds that
- $e(t_1.t_2) = e(t_1).e(t_2)$ for all $t_1, t_2 \in T$.

Now we state a theorem that plays an important role in our problem.

Theorem 1.4. *Let T be a top space and let the cardinality of e(T) be a natural number. Then the set of right-invariant vector fields on T, is a Lie algebra under the Lie bracket operation.*

Let T is a top space and e(T) is a finite set, then Theorem 1.4 implies that there exists a Lie algebra corresponding to T. According to this Lie algebra there is a Lie group.

Definition 1.5. [6] Let T is a top space. A curve $\phi : \mathbb{R} \to T$ is called one-parameter subgroup of top space T if it is satisfies the condition $\phi(t_1 + t_2) = \phi(t_1) + \phi(t_2)$; for all $t_1, t_2 \in \mathbb{R}$.

Lemma 1.6. *Assume $\phi : \mathbb{R} \to T$ is a one-parameter subgroup of T, then $\phi(0) \in e(T)$. Moreover $\phi(s)\phi(-s) \in e(T)$; for all $s \in \mathbb{R}$.*

Proposition 1.7. *Let $\phi : \mathbb{R} \to T$ is a one-parameter subgroup of T and X is a vector field such that $\frac{d\phi^\mu(t)}{dt} = X^\mu(\phi(t))$, where X^μ denotes a component of X in a coordinate system. Then vector field X is a right-invariant vector field.*

Proposition 1.8. *Let X be a right-invariant vector field on top space T. Then there exist one-parameter subgroups on T corresponding to X.*

Remark 1.9. Note that the correspondence between one-parameter subgroups of T and right-invariant vector fields on T is not one-to-one and we can find for every right-invariant vector field X, $|e(T)|$ one-parameter subgroup of T.

Example 1.10. Let $T = \mathbb{R}$, with the product $(a, b) \mapsto a$. We know that $card(e(T)) = \infty$. Then by the previous assertion there exists infinite right-invariant vector fields on T. Note that the vector field X on T is a right-invariant vector field if and only if $X : T \to \mathbb{R}$ is defined by $X(u) = cu$, for some constant number $c \in \mathbb{R}$. It is obvious that for every one-parameter subgroup ϕ, $\phi(\mathbb{R})$ is a commutative subgroup of T. By setting $\phi(0) \in e(T)$, we have a commutative subgroup of T. Therefore we can find a correspondence between right-invariant vector field and free commutative group $\prod^{*}_{\phi(0)\in e(T)} \phi(\mathbb{R})$.

Definition 1.11. Let T be a top space and let G be a topological group. Then a covering projection $P : T \to G$ is called a top space covering projection if P satisfies the following conditions:

(1) $P(t) = e$, for all $t \in e(T)$, where e is identity element;
(2) $P(t_1 t_2) = P(t_1)P(t_2)$, for all $t_1, t_2 \in T$.

Theorem 1.12. *Let P is a top space covering projection for a top space T and a topological group G and let $|e(T)| < \infty$. Then there exists a correspondence (but not necessarily one-to-one) between one-parameter subgroups G and one parameter subgroups of T.*

Corollary 1.13. *Let T is a top space with $|e(T)| < \infty$, and G is a Lie group and $P : T \to G$ a top space covering projection for G, then there exists a one-to-one correspondence between right-invariant vector field G and right invariant vector fields of T. Moreover the Lie algebra created by the right invariant vector fields of T is isomorphic to the Lie algebra of G.*

Proof. Let X be a right-invariant vector field, then there exist $|e(T)|$ one-parameter subgroups of T correspondence to X, and all of these one-parameter subgroups of T correspond to some one-parameter subgroups of G. Since G is a Lie group then there exists only one right-invariant vector field correspondence with that one-parameter subgroups. ☐

Corollary 1.14. *Let T and G be connected sets and $e(t_0) \in T$ be fixed. Moreover let $P : T \to G$ be a top space covering projection for G. Then there exists a unique Lie group structure on T such that $e(t_0)$ is identity element and Lie algebra of T (as a Lie group) is equal to the Lie algebra of right invariant vector fields of T (as a top space).*

Acknowledgement

This research is supported by Grant No. 99GRC1M82582 Shiraz University, Shiraz, Iran.

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Artinian and Noetherian PMV-algebras

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Abstract. In this paper, we introduce the notions of Noetherian and Artinian PMV-algebras and we state an equivalent definition of Noetherian PMV-algebras. We show that if A is a Noetherian (Artinian) PMV -algebra and $I \in Id(A)$, then A/I is a Noetherian (Artinian) PMV -algebra.

Keywords: PMV-algebra, $\cdot$-ideal, Noetherian , Artinian.

AMS Mathematics Subject Classification [2010]: $03B50$, $06D35$.

1. Introduction

A. Dvurecenskij and A. Di Nola in [1] introduced the notion of PMV-algebras, that is MV-algebras whose product operation $(\cdot)$ is defined on the whole MV-algebra. This operation is associative and left/right distributive with respect to partially defined addition. They showed that the category of product MV-algebras is categorically equivalent to the category of associative unital l-rings. In addition, they introduced and studied MVF-algebras [1]. They also introduced $\cdot$-ideals in PMV-algebras. Then they showed that: Any MVF-algebra is a subdirect product of subdirectly irreducible MVF-algebras [[1], Corollary 5.6]. Thus they concluded that a product MV-algebra is an MVF-ring if and only if it is a subdirect product of linearly ordered product MV-algebras [[1], Theorem 5.8]. By the notion of MVF-algebra, they can introduce PMV-algebras.
Also, in [4], F. Forouzesh introduce the notions of Noetherian and Artinian MV-modules and they study some equivalent definition of Noetherian MV-modules.

2. preliminaries

In this section, we summarize the basic concepts PMV-algebras. For more details on these concepts, we refer the reader to [3] and [7].

Definition 2.1. [1] A product MV-algebra (or PMV-algebra, for short) is a structure $(A, \oplus, ^*, \cdot, 0)$, where $(A, \oplus, ^*, 0)$ is an MV-algebra and $\cdot$ is a binary associative operation on A such that the following property is satisfied:
if $x + y$ is defined, then $x \cdot z + y \cdot z$ and $z \cdot x + z \cdot y$ are defined and

$$(x + y) \cdot z = x \cdot z + y \cdot z, \quad z \cdot (x + y) = z \cdot x + z \cdot y,$$

*Speaker. Email address: frouzesh@bam.ac.ir where, $+$ is a partial addition on an MV-algebra A as follows: For any $x, y \in M$, $x + y$ is defined if and only if $x \leq y^*$ and in this case,

$$x + y := x \oplus y.$$

The partial addition was defined [2].

If A is PMV-algebra, then a unity for the product is an element $e \in A$ such that $e \cdot x = x \cdot e = x$ for any $x \in A$. A PMV-algebra that has unity for the product will be called unital.

In the sequel, an lu-ring will be a pair (R, u) where $(R, \oplus, \cdot, \leq)$ is an l-ring and u is a strong unit of R such that $u \cdot u \leq u$. The last conditions imply that the intrval $[0, u]$ of an lu-ring (R, u) is closed under the product of R. Thus, if we consider the restriction of $\cdot$ to $[0, u] \times [0, u]$, then the interval $[0, u]$ has a canonical PMV-algebra structure:

$$x \oplus y := (x + y) \wedge u, \quad x^* := u - x, \quad x \cdot y := x \cdot y,$$

for any $0 \leq x, y \leq u$. We shall denote this structure $[0, u]_R$.

If $\mathcal{UR}$ is the category of lu-rings, whose objects are pairs (R, u) as above and whose morphisms are l-rings homomorphisms which preserve the strong unit, then we get a functor

$$\Gamma : \mathcal{UR} \to \mathcal{PMV},$$
$$\Gamma(R, u) := [0, u]_R, \text{ for any lu-ring } (R, u),$$
$$\Gamma(h) := h \mid_{[0,u]} \text{ for any lu-rings homomorphism h.}$$

In [1] is proved that Γ establishes a categorical equivalence between $\mathcal{UR}$ and $\mathcal{PMV}$. We recall that in an MV-algebra M, the Chang distance function is
$d : M \times M \longrightarrow M, \quad d(a, b) := (a \odot b^*) \oplus (b \odot a^*)$.

Lemma 2.2. *If A is a PMV-algebra, then the following properties hold for any $x, y, \alpha \in A$,*
(a) $(nx) \cdot y = x \cdot (ny)$, for any $n \in \mathbb{N}$, where $nx = x + \cdots + x$,
(b) $x \cdot y^* \leq (x \cdot y)^*$,
(c) $(x \cdot y)^* = x^* \cdot y + (1 \cdot y)^*$,
(d) $(\alpha \cdot x) \odot (\alpha \cdot y)^* \leq \alpha \cdot (x \odot y^*)$,
(e) $\alpha \cdot (x \oplus y) \leq \alpha \cdot x \oplus \alpha \cdot y$,
(f) $d(\alpha \cdot x, \alpha \cdot y) \leq \alpha \cdot d(x, y)$,
Moreover, if A is a unital PMV-algebra, then
$(x \cdot y)^* = x^* \cdot y + y^*$.

Definition 2.3. [1] A $\cdot$-ideal of PMV-algebra A is an ideal I of MV-algebra A such that if $a \in I$ and $b \in A$ entail $a \cdot b \in I$ and $b \cdot a \in I$.
We denote by $Id_p(A)$ the set of $\cdot$-ideals of a PMV-algebra A. The set of all maximal ideals of an MV-algebra A is denoted by $Max(A)$.

Definition 2.4. [5] Let P be a $\cdot$-ideal of of a PMV-algebra A. P is called a $\cdot$-prime ideal, if (i) $P \neq A$ and (ii) for every $a, b \in A$, if $a \cdot b \in P$, then $a \in P$ or $b \in P$.

Lemma 2.5. *[1] If A is PMV-algebra, then for any $a, b \in A$,*
(i) $a \cdot 0 = 0 = 0 \cdot a$,
(ii) if $a \leq b$, then for any $c \in A$, $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.

Lemma 2.6. *[4] If A is PMV-algebra and $x, y \in A$, then $(x] \vee (y] = (x \oplus y]$.*

3. Noetherian and Artinian PMV-algebras

Definition 3.1. A PMV-algebra A is called Noetherian (Artinian), if for every increasing (decreasing) chain of its $\cdot$-ideals like $I_1 \subseteq I_2 \subseteq \dots$ ($I_1 \supseteq I_2 \cdots$), there exists $n \in \mathbb{N}$ such that $I_i = I_n$, for all $i \geq n$.

The following examples show that Noetherian (Artinian) PMV-algebras exists and any PMV-algebra may not be a Noetherian (Artinian) PMV-algebra.

Example 3.2. Let $M_2(\mathbb{R})$ be the ring of square matrices of order 2 with real elements and 0 be the matrix with all element 0. If we define the order relation on components $A = (a_{ij})_{i,j=1,2} \geq 0$ iff $a_{ij} \geq 0$ for any i, j, such that $v = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, then $A = \Gamma(M_2(\mathbb{R}), v)$ is a PMV-algebra. Then $Id_p(A) = \{(0,0), A\}$. Hence A is a Noetherian and Artinian PMV-algebra.

Example 3.3. Let $G = \oplus\{Z_i/i \in \mathbb{N}\}$ be the lexicographic product of denumerable infinite copies of the abelian l-group $\mathbb{Z}$ of the relative integers and $e^i \in G$ such that $e^i_k = 0$ if $k \neq i$ and $e^i_k = 1$ if $k = i$.

Also, G with the usual product is an lu-ring. It follows from [1] that $A = \Gamma(G, u) = [0, u]$ is a PMV-algebra, where Γ is a functor from the category of abelian lu-ring to the category PMV-algebras and $u = (1, 0, 0, 0, \dots)$ is the strong unit of A, where $\leq$ is the lexicographic order on G.
If we set $P_i =< (0, e^i) >$, then $P_i \subseteq P_j$, for $i > j$. Since $P_1 \supseteq P_2 \supseteq P_3 \supseteq \cdots \supseteq P_n \supseteq \cdots$, hence A is not a Artinian PMV-algebra.

Theorem 3.4. *Let A be a PMV-algebra. The following conditions are equivalent:*
(i) Any non-empty collection of $\cdot$-ideals of A has a maximal element.
(ii) A is a Noetherian.
(iii) Every $\cdot$-ideal of A is principal.

Proof. (i) $\Rightarrow$ (ii) Let $\{I_i\}_{i \in J}$ be any increasing sequence of $\cdot$-ideals of A, and $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be given. Consider the collection $\Sigma = \{I_i\}_{i \in N}$. This collection has a maximal element I_m. Then $I_k = I_m$, for all $k \geq m$.
(ii) $\Rightarrow$ (iii) Let I be a $\cdot$-ideal of A. Choose any $x_1 \in I$. If $(x_1] \neq I$. Choose $x_2 \in I$, $x_2 \notin (x_1]$. If $(x_1] \vee (x_2] \neq I$, continue this process. By assumption (ii), after a finite number of steps, we have

$$(x_1] \vee (x_2] \vee \cdots \vee (x_n] = I.$$

Thus by Lemma 2.6, It follows that

$$I = (x_1 \oplus x_2 \oplus \cdots \oplus x_n].$$

Therefore every $\cdot$-ideal of A is principal.
(iii) $\Rightarrow$ (ii) Consider any increasing sequence of $\cdot$-ideals of A like $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m \subseteq \cdots$. Then Their union $I = \cup_{i=1}^n I_i$ is a $\cdot$-ideal of A which is a principal by (iii). Hence there exists $a \in A$ such that $I = (a]$. Choose m such that $I = (a] \subseteq I_m$. Then $I_k = I_m$, for all $k \geq m$.
(ii) $\Rightarrow$ (i) Let Σ be a non-empty collection of $\cdot$-ideals of I. Choose any $I_1 \in \Sigma$. If I_1 is not maximal, choose $I_2 \in \Sigma$, $I_1 \subseteq I_2$. If I_2 is not maximal continue this process. After a finite number of steps there exists some $I_m \in \Sigma$, which is maximal. $\square$

Theorem 3.5. *A is a Noetherian PMV-algebra if and only if every $\cdot$- ideal of A is finitely generated.*

Lemma 3.6. *Let A be a Noetherian (Artinian) PMV-algebra and $I \in Id(A)$. Then A/I is a Noetherian (Artinian) PMV-algebra.*

Theorem 3.7. *Let A be a Noetherian Boolean PMV-algebra. Then every $\cdot$-ideal of A is intersection of finitely many $\cdot$-prime ideals of A.*

Theorem 3.8. *Let A be a Boolean PMV-algebra. Then A is a Noetherian PMV-algebra if and only if every $\cdot$-prime ideal of A is finitely generated.*

We recall that a proper ideal of A is called an obstinate ideal of A if $x, y \notin I$ implies $x \odot y^* \in I$ and $y \odot x^* \in I$, for all $x, y \in A$ [6, Definition 2.1]. Also, it is proved that if I is a proper ideal of A, then I is an obstinate ideal if and only if $x \in I$ or $x^* \in I$, for all $x \in A$ [6].

Theorem 3.9. *The set of obstinate ideals of an MV-algebra A, satisfies both of the ascending chain condition and descending chain condition.*

Proof. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an increasing chain of obstinate ideals of A. Put $S = \{I_i\}_{i \in \mathbb{N}}$. Since S is a non-empty set, we claim that S has maximal element, like I_n, that is $I_i = I_n$, for all $i \geq n$.
Assume that there is $j \geq n$ such that $I_n \subsetneqq I_j$, hence there exists $a \in I_j$ and $a \notin I_n$. Since I_n is an obstinate ideal, $a^* \in I_n$, so $a^* \in I_j$. Hence $1 = a \oplus a^* \in I_j$, which is a contradiction. Thus $I_n = I_i$, for all $i \geq n$. Therefore the set of obstinate ideals of A satisfy the ascending chain condition. Similarly, the set of obstinate ideals of A satisfy the descending chain condition. □

Like Cohen's theorem on rings (Hungerford 1974), we have to the following theorem in MV-algebras:

Theorem 3.10. *Let A be an PMV-algebra. Then A is a Noetherian PMV-algebra if and only if every prime $\cdot$-ideal of A is finitely generated.*

4. Conclusion

In this paper, we introduced the notions of Noetherian and Artinian PMV-algebras and we stated some equivalent definitions of Noetherian PMV-algebras.

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On conjugacy class graph of a Frobenious group of order pq

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Abstract. Let G be the Frobenius group of order pq, where p and q are two distinct primes and $\Gamma(G)$ be the conjugacy class graph of G. In this paper, we show that $\Gamma(G)$ is a disconnected graph with two connected components. Also, we compute the characteristic polynomial, the energy and the Laplacian energy of $\Gamma(G)$.

Keywords: Frobenius group, conjugacy class graph, characteristic polynomial, Laplacian energy, eigenvalue

AMS Mathematics Subject Classification [2010]: 20E45, 05C50, 20D60

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1. Introduction

A graph Γ is a finite nonempty set of objects called vertices together with a set of unordered pairs of distinct vertices of Γ called the edges. The vertex-set of Γ is denoted by $V(\Gamma)$, while the edge-set is denoted by $E(\Gamma)$. Let Γ be a graph with set of vertices $V(\Gamma) = \{1, \ldots, n\}$ and the set of edges $E(\Gamma) = \{e_1, \ldots, e_n\}$. The adjacency matrix of Γ denoted by $A(\Gamma)$, is an $n \times n$ matrix defined as follows: the rows and the columns of $A(\Gamma)$ are indexed by $V(\Gamma)$. If $i \neq j$, then the $(i,\, j)$-entry of $A(\Gamma)$ is 0 for nonadjacent and 1 for adjacent vertices i and j. The $(i,\, i)$-entry of $A(\Gamma)$ is 0 for $i = \{1, \ldots, n\}$. The degree of vertex i is denoted by $d_\Gamma(i)$ and the degree matrix on the other hand denoted by $\Delta(\Gamma)$ is defined as $\Delta(\Gamma) = diag(d_\Gamma(1), d_\Gamma(2), \ldots, d_\Gamma(n))$ which is the diagonal matrix of vertex degrees. Then, the Laplacian matrix of Γ is denoted by $L(\Gamma)$ which satisfies $L(\Gamma) = \Delta(\Gamma) - A(\Gamma)$.

Let $\lambda_1, \lambda_2, \ldots , \lambda_n$ be the eigenvalues of the adjacency matrix of Γ, and let $\mu_1, \mu_2, \ldots , \mu_n$ be the eigenvalues of the Laplacian matrix of Γ. The energy of the graph Γ is defined as the sum of the absolute values of the eigenvalues of the adjacency matrix of Γ, i.e., $E(\Gamma) = \sum_{i=1}^{n} |\lambda_i|$. Also, the Laplacian energy of the graph Γ is defined as the sum of the absolute values of the difference between the Laplacian matrix eigenvalues and the ratio of twice the edges number divided by the vertices number, i.e., $LE(\Gamma) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$, where n is the vertices number and m is the edges number of the graph Γ.

Let G be a finite group and $V(G)$ be the set of all non-central conjugacy classes of G. From orders of representatives of conjugacy classes, the following conjugacy class graph $\Gamma(G)$ was introduced in [5]: its vertex set is the set $V(G)$ and two distinct vertices x^G and y^G are connected with an edge if $(o(x), o(y)) > 1$.

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In this paper, we compute the characteristic polynomial, the energy and the Laplacian energy of the conjugacy class graph of a Frobenius group of order pq, where p and q are two distinct primes. Recall that the Frobenius group with kernel K and complement H is denoted by $K \rtimes_f H$. Also, K_n is the complete graph with n vertices.

2. Preliminaries

In this section, we give some preliminary results that will be used in the proof of our main results.

Definition 2.1. The spectrum of a graph Γ is the set of numbers which are eigenvalues of $A(\Gamma)$, together with their multiplicities. If the distinct eigenvalues of $A(\Gamma)$ are $\lambda_0 > \lambda_1 > \ldots > \lambda_{s-1}$, and their multiplicities are $m(\lambda_0), m(\lambda_1), \ldots, m(\lambda_{s-1})$, then we shall write

$$Spec\ \Gamma = \begin{pmatrix} \lambda_0 & \lambda_1 & \ldots & \lambda_{s-1} \\ m(\lambda_0) & m(\lambda_1) & \ldots & m(\lambda_{s-1}) \end{pmatrix}$$

As a straightforward result of Proposition 3.5 of [3], we have the following remark.

Remark 2.2. Let Γ be a complete graph K_n. Then

$$Spec\ \Gamma = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$$

Lemma 2.3. *(Lemma 4.3 of [4]) Let G be a Frobenius group with Abelian Frobenius kernel N, and H be its Frobenius complement. Let N be the disjoint union of $t+1$ G-conjugacy classes. Then*
(i) All non-central conjugacy classes contained in N are of same length $|H|$, and hence $|N| = 1 + t|H|$.
(ii) If in addition H is also Abelian with $|H| = s+1$, then any non-central conjugacy class of G is of length either $s+1$ or $(s+1)t+1$. Moreover, the numbers of all different G-conjugacy classes with length $s+1$, $(s+1)t+1$ are respectively t, s.

Proposition 2.4. *([2]) The multiplicity of 0 as eigenvalue of $L(\Gamma)$ is equal to the number of connected components of the graph.*

Proposition 2.5. *([1]) The Laplacian matrix of the complete graph K_n has eigenvalues 0 with multiplicity 1 and n with multiplicity $n-1$.*

3. Main results

Let G be a Frobenius group with Frobenius kernel $\mathbb{Z}_q$ and Frobenius complement $\mathbb{Z}_p$, where p and q are two distinct primes. Also let t be the number of non-central conjugacy classes of G contained in $\mathbb{Z}_q$ and s be the number of non-central conjugacy classes of G contained in $\mathbb{Z}_p$. Therefore $t = \frac{q-1}{p}$ and $s = p-1$. In this section, we present our main results.

Proposition 3.1. *Let $G = \mathbb{Z}_q \rtimes_f \mathbb{Z}_p$ and $\Gamma(G)$ be the conjugacy class graph of G. Then $\Gamma(G)$ is a graph with two connected components K_t and K_s, the number of the vertices of $\Gamma(G)$ is $n = t+s$ and the number of edges of $\Gamma(G)$ is $m = \frac{t(t-1)}{2} + \frac{s(s-1)}{2}$.*

Proposition 3.2. *Let $G = \mathbb{Z}_q \rtimes_f \mathbb{Z}_p$ and $\Gamma(G)$ be the conjugacy class graph of G. Then we have:*

(i) The adjacency matrix eigenvalues of $\Gamma(G)$ *are* $\lambda = -1$ *with multiplicity* $s+t-2$, $\lambda = t-1$ *with multiplicity 1 and* $\lambda = s-1$ *with multiplicity 1.*
(ii) The Laplacian matrix eigenvalues of $\Gamma(G)$ *are* $\mu = 0$ *with multiplicity 2,* $\mu = t$ *with multiplicity* $t-1$ *and* $\mu = s$ *with multiplicity* $s-1$.

THEOREM 3.3. *Let* $G = \mathbb{Z}_q \rtimes_f \mathbb{Z}_p$ *and* $\Gamma(G)$ *be the conjugacy class graph of* G. *Then we have*
(i) The characteristic polynomial of $\Gamma(G)$ *is* $\chi(\Gamma(G);\lambda) = (\lambda+1)^{t+s-2}(\lambda-t+1)(\lambda-s+1)$.
(ii) The energy of $\Gamma(G)$ *is* $E(\Gamma(G)) = 2(t+s-2)$.
(iii) The Laplacian energy of $\Gamma(G)$ *is* $LE(\Gamma(G)) = 2[\frac{t(t-1)+s(s-1)}{t+s}]+(t-1)|t-\frac{t(t-1)+s(s-1)}{t+s}|+ (s-1)|s-\frac{t(t-1)+s(s-1)}{t+s}|$.

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The Epiperimetric Inequality Approach for the Regularity of a Free Boundary Problem

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Abstract. We apply the epiperimetric inequality approach and show $C^{1,\beta}$-regularity for the free boundary $\partial\{|u|>0\}$ at asymptotically flat points of the problem $\Delta u = |u|^{q-1}u + g(x,u)$, where g is Hölder continuous and vanishes faster than $|u|^q$ as $u \to 0$.

Keywords: Epiperimetric inequality, Semilinear elliptic equation, Free boundary, Regulairty.

AMS Mathematics Subject Classification [2010]: 35B65, 35J61, 35R35.

1. Introduction

We study the regularity of the free boundary of local minimzers u of the functional

$$J(u) = \int_{B_1} \left(|\nabla u|^2 + 2F(x,u)\right) dx, \quad u - u_0 \in W_0^{1,2}(B_1),$$

where B_1 is the unit ball in $\mathbb{R}^n$ $(n \geq 2)$, $u_0 \in W^{1,2}(B_1) \cap L^\infty(B_1)$. This minimizer satisfies the semilinear problem

$$\Delta u = f(x,u), \qquad \text{in } B_1, \tag{1}$$

where f is the derivative of F with respect to the variable u, i.e. $f = F_u$. Here, we assume that $f(x,u) = |u|^{q-1}u + g(x,u)$ for some $q \in (0,1)$ and g satisfies

(H1) $g(x,u)$ is Hölder continuous with respect to the both variables x and u,

(H2) $\lim_{u\to 0} \frac{g(x,u)}{|u|^q} = 0$ for every fixed $x \in B_1$.

We also use the notation $\Gamma = \partial\{|u| > 0\}$ which is referred to as the free boundary. We can consider the equation (1) as a perturbation of the problem

$$\Delta u = f_0(u) = |u|^{q-1}u, \qquad \text{in } B_1. \tag{2}$$

As a especial case, the classical form of the obstacle problem consider the semilinear elliptic equation

$$\Delta u = \chi_{\{u>0\}}.$$

The optimal $C^{1,1}_{\text{loc}}$ regularity for the solution is obtained in [2]. Caffarelli in his seminal papers [3, 4] shows that points of the free boundary are divided into two different classes:

*Speaker. Email address: fotouhi@sharif.edu regular points and singular points. Near the regular points, the solution behaves like half-space solutions and satisfies

$$\limsup_{r\to 0} \frac{|B_r(x_0)\cap\{u=0\}|}{|B_r|} > 0.$$

It is well-known that free boundary is an analytic hypersurface at regular points, [3]. The singular class contains points at which the solution behaves like quadratic polynomials and

$$\lim_{r\to 0} \frac{|B_r(x_0)\cap\{u=0\}|}{|B_r|} = 0.$$

Already, there is an example in dimension two in which the singular part is a cantor set. In general the singular part of free boundary cannot be a smooth manifold but is covered by manifolds with some regularity. In [4], it has shown that the singular part is locally contained in a C^1 manifold. This result has been improved in dimension two by Weiss [10] using the epiperimetric inequality. He proves the singular part is contained in a $C^{1,\alpha}$ curve. In [5], they show a logarithmic epiperimetric inequality to obtain $C^{1,\log}$ regularity of singular part in higher dimension. Finally, Figalli and Serra improved the result to an optimal $C^{1,1}$ regularity up to the presence of some "anomalous" points of higher codimension, [6].

2. Main results

The regularity of solutions of (1) and (2) is well studied and we expect to have the optimal regularity $C^{[\kappa],\kappa-[\kappa]}$, where $\kappa = 2/(1-q)$ (for the details of result refer to [7–9]). However, there are few results for the regularity of free boundary $\Gamma = \partial\{|u|>0\}$. In [9], it has been shown that for $f(u) = (u^+)^q$ the free boundary $\partial\{u>0\}$ has locally finite $\mathcal{H}^{n-1}$-Hausdorff dimension. Then the non-coincident set $\{u>0\}$ has locally finite perimeter and we are able to define the reduced part of free boundary, $\partial_{\mathrm{red}}\{u>0\}$, where a tangent plane exists in a weak sense. Alt and Phillips shows $\partial_{\mathrm{red}}\{u>0\}$ is a $C^{1,\alpha}$ surface, [1]. They also prove $C^{1,\alpha}$-regularity of the free boundary near the regular points.

One of the main difficulties encountered in studying the regularity of the free boundary in problem (1) is classification of global solutions. In dimension two, we are able to present a fairly good analysis of global homogeneous solutions, and hence a better understanding of the behavior of the free boundary, [7, 8]. In higher dimensions the problem becomes quite complicated, but we are still able to obtain partial results; e.g. we prove that if a solution is close to one-dimensional solution in a small ball, then in an even smaller ball the free boundary can be represented locally as two C^1-regular graphs $\partial\{u>0\}$ and $\partial\{u<0\}$, tangential to each other. It is noteworthy that the above problem (in contrast to the case $q=0$) introduces interesting and quite challenging features, that are not encountered in the case $q=0$. For example one obtains homogeneous global solutions that are not one-dimensional. This complicates the analysis of the free boundary particularly in singular points of free boundary.

Definition 2.1. To investigate the regularity of free boundary, we consider "asymptotically one-phase-points" that is, a subset of Γ such that the blow-ups[1] belong to

$$\mathbb{H} := \{x \mapsto \alpha\max(x\cdot\nu, 0)^\kappa : \nu\in\mathbb{R}^n \text{ is a unit vector}\}.$$

[1]Any limit of the sequence $u(x_0+r_nx)/r_n^\kappa$ when $r_n\to 0$ is called a blow-up of solution u at point x_0.

Members of this class are κ-homogeneous global solutions of (2), i.e. $u(rx) = r^\kappa u(x)$, and are called *half-plane* solutions. We denote by $\mathcal{R}_u$ the set of all (regular free boundary) points $x_0 \in \Gamma$ such that at least one blow-up limit of u at x_0 is in $\mathbb{H}$.

Our main result concerning the regularity of the free boundary is presented in the following theorem.

THEOREM 2.2. *The set of regular free boundary points $\mathcal{R}_u$ is locally in B_1 a $C^{1,\beta}$-manifold.*

3. Epiperimetric inequality

We choose the epiperimetric inequality approach to show the regularity of free boundary.

THEOREM 3.1 (The epiperimetric inequality, Theorem 3.1 in [8]). *There exist $\epsilon \in (0,1)$ and $\delta > 0$ such that if $c \in W^{1,2}(B_1)$ is a homogeneous function of degree κ and $\|c-h\|_{W^{1,2}(B_1)} \leq \delta$ for some $h \in \mathbb{H}$, then there exists a function $v \in W^{1,2}(B_1)$ such that $v = c$ on ∂B_1 and*

$$M(v) - M(h) \leq (1-\epsilon)\left(M(c) - M(h)\right),$$

where M is the boundary adjusted energy

$$M(u) = \int_{B_1} \left(|\nabla u|^2 + \frac{2}{1+q}|u|^{1+q}\right) dx - \kappa \int_{\partial B_1} |u|^2 \, d\sigma.$$

This inequality is a powerful tool in the regularity theory of free boundary and minimal surfaces. The concept of an epiperimetric inequality was first introduced by Reifenberg (1964) in the context of minimal surfaces. In [10], Weiss used this approach to study the free boundary of the obstacle problem in dimension two. Recently, using a direct argument, a logarithmic version of epiperimetric inequality has been introduced in [5] to study the regularity of the singular set of free boundary in the obstacle problem.

In order to prove Theorem 2.2, we need to show that the monotonicity formula (which is established by Weiss in [10] for the classical obstacle problem), holds in the present setting. See Proposition 5.1 in [8] for a similar formula and the proof.

PROPOSITION 3.2. *Let u be a solution of (1) in $B_{r_0}(x_0)$ and let*

$$W_s(u,y,r) = \frac{1}{r^{n+2\kappa-2}} \int_{B_r(y)} \left(|\nabla u|^2 + 2F_{s,y}(x,u)\right) dx - \frac{\kappa}{r^{n+2\kappa-1}} \int_{\partial B_r(y)} |u|^2 \, d\mathcal{H}^{n-1},$$

where $F_{s,y}(x,u) := F(x_0 + s(x-y), u)$. The there exists constants $C < \infty$ and $\mu > 0$ such that $W_1(u,x_0,r) + Cr^\mu$ is increasing for $r > 0$.

The epiperimetric inequality with the monotonicity formula, Proposition 3.2, provides an estimate for the rate of convergence $\|u(x_0+\cdot) - h\|_{L^1(\partial B_r)}$, where h belongs to $\mathbb{H}$.

PROPOSITION 3.3. *Let $x_0 \in B_1 \cap \partial\{|u| > 0\}$, and suppose that the epiperimetric inequality holds with $\epsilon \in (0,1)$ for each*

$$c_r(x) := |x|^\kappa u_r(\frac{x}{|x|}) = \frac{|x|^\kappa}{r^\kappa} u(x_0 + \frac{r}{|x|}x)$$

and for all $r \leq r_0 < 1$. Finally let u_0 denote an arbitrary blow-up limit of u at x_0. Then there exists constants C and Λ depending only on n and ϵ such that

$$\int_{\partial B_1} |u_r(x) - u_0(x)| d\mathcal{H}^{n-1} \leq C|W_1(u,x_0,r_0) - W_1(u,x_0,0+)|^{1/2} (\frac{r}{r_0})^\Lambda.$$

Proposition 3.3 proves the uniqueness of blow-ups provided u_r remains in a δ-neighborhood of $\mathbb{H}$, where δ is the constant introduced in the epiperimetric inequality. The definition of $\mathcal{R}_u$ assures that this condition is satisfied during the blow-up process. Moreover, we imply that $\mathcal{R}_u$ is open relative to Γ.

Proof of Theorem 2.2. For every $x_0 \in \mathcal{R}_u$, assume that $h_{x_0}(x) = \alpha \max(x\cdot\nu(x_0), 0)^\kappa$ is the blow-up limit of u at x_0 which is a half-plane solution, for some $\nu(x_0) \in \partial B_1(0) \subset \mathbb{R}^n$. Recall that the uniqueness of blowup to see $x_0 \mapsto \nu(x_0)$ is well-defined. $\nu(x_0)$ is the orthogonal vector on free boundary and we show that it is Hölder continuous with exponent $\beta = \Lambda/(2\kappa+\Lambda)$. Here, Λ is the exponent defined in Proposition 3.3.

We can show easily by an indirect argument that there exists the constant $c(n)$ such that

$$c(n)|\nu(x_1)-\nu(x_2)| \leq \int_{\partial B_1} \left| \max(x\cdot\nu(x_1),0)^\kappa - \max(x\cdot\nu(x_2),0)^\kappa \right| d\mathcal{H}^{n-1},$$

for every $x_1, x_2 \in \mathcal{R}_u$. Now choose $\gamma := (\kappa+\Lambda)^{-1}$, $r := |x_1 - x_2|^\gamma$ and apply Proposition 3.3 to see that

$$\begin{aligned}\alpha c(n)|\nu(x_1)-\nu(x_2)| \leq & \int_{\partial B_1} |u(x_1+rx)/r^\kappa - h_{x_1}(x)| + |u(x_1+rx) - u(x_2+rx)|/r^\kappa \\ & \qquad + |u(x_2+rx)/r^\kappa - h_{x_2}(x)|\, d\mathcal{H}^{n-1} \\ \leq & 2Cr^{\Lambda/2} + \int_{\partial B_1}\int_0^1 \left|\nabla u\left(x_1+rx+t(x_2-x_1)\right)\right| \frac{|x_1-x_2|}{r^\kappa}\, dt d\mathcal{H}^{n-1} \\ \leq & 2Cr^{\Lambda} + C_1\frac{|x_1-x_2|}{r^\kappa} \leq (2C+C_1)|x_1-x_2|^{\gamma\Lambda},\end{aligned}$$

For the detail of proof refer to [8]. □

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Using Helly's theorem and BSE-functions to give a new vector space

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Abstract. Let $(X, \|\cdot\|)$ be a normed space and X^* be its dual. In this paper we introduce the space $\overline{C}_{\mathrm{BSE}}(X^*)$ consisting of all functions $\sigma : X^* \to \mathbb{C}$ which satisfy in a certain relation like the Bochner-Schoenberg-Eberlein property. Using the Helly theorem, we characterize this space and as an application we give some results on the real line $\mathbb{R}$. Indeed, we give a characterization of (continuous) linear functions on $\mathbb{R}$.

Keywords: Banach algebra, BSE-function, character space

AMS Mathematics Subject Classification [2010]: 46H05, 46J10

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1. Introduction

Suppose that A is a semi-simple commutative Banach algebra and $\Delta(A)$ is the character space of A, i.e., the space of all non-zero homomorphisms from A into $\mathbb{C}$. A bounded continuous function σ on $\Delta(A)$ is called a BSE-function if there exists a constant $M > 0$ such that for each $\varphi_1, \ldots, \varphi_n \in \Delta(A)$ and complex numbers $c_1, \ldots, c_n$ the inequality

$$\left| \sum_{i=1}^{n} c_i \sigma(\varphi_i) \right| \leq M \left\| \sum_{i=1}^{n} c_i \varphi_i \right\|_{A^*}$$

holds. Let $C_{\mathrm{BSE}}(\Delta(A))$ be the set of all BSE-functions. BSE-functions for the first time introduced and investigated by Takahasi and Hatori; see [4] and two notable works [1, 2].

In this paper for a normed space X with dual X^*, we study the space of all functions $\sigma : X^* \longrightarrow \mathbb{C}$ such that are w^*-continuous and satisfy in the following relation:

There exists an $M_\sigma > 0$ such that for all $n \in \mathbb{N}$ we have

$$\left| \sum_{i=1}^{n} c_i \sigma(x_i^*) \right| \leq M_\sigma \left\| \sum_{i=1}^{n} c_i x_i^* \right\|_{X^*} \qquad (x_1^*, \ldots, x_n^* \in X^*, c_1, \ldots, c_n \in \mathbb{C}).$$

We give a characterization of functions which belong to this space. Then using this characterization, for a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ we give a condition that is equivalent to the linearity of f.

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2. Main Results

Suppose that A is a semi-simple commutative Banach algebra. Clearly, $\Delta(A) \subseteq A^*$ and A^* is a normed space. We know that each $\sigma \in C_{\mathrm{BSE}}(\Delta(A))$ defined on $\Delta(A)$. One may ask this question:

Whether σ can be extended to A^* and satisfies

$$\left|\sum_{i=1}^{n} c_i\sigma(a_i^*)\right| \leq M \left\|\sum_{i=1}^{n} c_i a_i^*\right\|_{A^*} \qquad (a_1^*, a_2^*, a_3^*, \ldots, a_n^* \in A^*), \tag{1}$$

where $M > 0$ is a constant?

In this section we study the complex-valued functions which defined on the dual of a normed space and satisfy in a similar relation as 1.

Let $(X, \|\cdot\|)$ be a normed space and X^* be its dual. Let $C_{w^*}(X^*)$ show the space of all complex-valued functions on X^* which are continuous respect to the w^*-topology of X^*. By $\overline{C}_{\mathrm{BSE}}(X^*)$ we show the space consisting of all the complex-valued functions σ in $C_{w^*}(X^*)$ with this property:

There exists an $M_\sigma > 0$ such that for each $x_1^*, x_2^*, x_3^*, \ldots x_n^* \in X^*$ and complex numbers $c_1, c_2, c_3, \ldots, c_n$ the following relation holds:

$$\left|\sum_{i=1}^{n} c_i\sigma(x_i^*)\right| \leq M_\sigma \left\|\sum_{i=1}^{n} c_i x_i^*\right\|_{X^*}. \tag{2}$$

Clearly $\overline{C}_{\mathrm{BSE}}(X^*)$ is a vector space.

REMARK 2.1. Using relation 2, one can see that each $\sigma \in \overline{C}_{\mathrm{BSE}}(X^*)$ satisfies $\sigma(0_{X^*}) = 0$ where 0_{X^*} is the zero element of X^*. Therefore, the constant function 1 is not in $\overline{C}_{\mathrm{BSE}}(X^*)$. So, $\overline{C}_{\mathrm{BSE}}(X^*) \subsetneq C_{w^*}(X^*)$.

REMARK 2.2. Obviously, for each bounded subset of X^* like Δ, one can see that the restriction of $\sigma \in \overline{C}_{\mathrm{BSE}}(X^*)$ to Δ is in $C_b(\Delta)$. Also, in the case that $X = A$ is a commutative Banach algebra, we have

$$\overline{C}_{\mathrm{BSE}}(A^*)_{|\Delta(A)} = \{\sigma_{|\Delta(A)} : \sigma \in \overline{C}_{\mathrm{BSE}}(A^*)\} \subseteq C_{\mathrm{BSE}}(\Delta(A)).$$

To proceed further, we recall the Helly theorem.

THEOREM 2.3. (Helly) Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{C}$. Suppose that $M > 0$, $x_1^*, \ldots, x_n^*$ are in X^* and $c_1, \ldots, c_n$ are in $\mathbb{C}$. Then the following are equivalent:

(1) for all $\epsilon > 0$, there exists $x_\epsilon \in X$ such that $\|x_\epsilon\| \leq M + \epsilon$ and $x_k^*(x_\epsilon) = c_k$ for $k = 1, \ldots, n$.

(2) for all $a_1, \ldots, a_n \in \mathbb{C}$,

$$\left|\sum_{i=1}^{n} a_i c_i\right| \leq M \left\|\sum_{i=1}^{n} a_i x_i^*\right\|_{X^*}.$$

PROOF. See [3, Theorem 4.10.1]. □

As an application of Helly's theorem, we give the following characterization.

THEOREM 2.4. *$\overline{C}_{\mathrm{BSE}}(X^*)$ is equal to the set of all $\sigma \in C_{w^*}(X^*)$ for which there exists a bounded net $\{x_\alpha\}$ in X with $\sigma = w^* - \lim_\alpha \widehat{x_\alpha}$.*

PROOF. Suppose that $\sigma \in C_{w^*}(X^*)$ is such that there exists $\beta < \infty$ and a net $\{x_\alpha\} \subseteq X$ with $\|x_\alpha\| < \beta$ for all α and $\lim_\alpha x^*(x_\alpha) = \sigma(x^*)$ for all $x^* \in X^*$. Let $x_1^*, \ldots, x_n^*$ be in X^* and $c_1, \ldots, c_n$ be complex numbers. Then we have

$$\begin{aligned} \left|\sum_{i=1}^n c_i \sigma(x_i^*)\right| \leq & \left|\sum_{i=1}^n c_i x_i^*(x_\alpha)\right| + \left|\sum_{i=1}^n c_i(x_i^*(x_\alpha) - \sigma(x_i^*))\right| \\ \leq & \beta \left\|\sum_{i=1}^n c_i x_i^*\right\| + \sum_{i=1}^n |c_i||x_i^*(x_\alpha) - \sigma(x_i^*)| \end{aligned}$$

Taking the limit with respect to α, we conclude that $\sigma \in \overline{C}_{\mathrm{BSE}}(X^*)$.

Conversely, let $\sigma \in \overline{C}_{\mathrm{BSE}}(X^*)$. Suppose that Λ is the net consisting of all finite subsets of X^*. By Helly's theorem, for each $\epsilon > 0$ and $\lambda \in \Lambda$, there exists $x_{(\lambda,\epsilon)} \in X$ with $\|x_{(\lambda,\epsilon)}\| \leq M_\sigma + \epsilon$ and $x^*(x_{(\lambda,\epsilon)}) = \sigma(x^*)$ for all $x^* \in \lambda$. Clearly, $\{(\lambda, \epsilon) : \lambda \in \Lambda, \epsilon > 0\}$ is a directed set with $(\lambda_1, \epsilon_1) \preceq (\lambda_2, \epsilon_2)$ iff $\lambda_1 \subseteq \lambda_2$ and $\epsilon_1 \leq \epsilon_2$. Therefore, we have

$$\lim_{(\lambda,\epsilon)} x^*(x_{(\lambda,\epsilon)}) = \sigma(x^*) \qquad (x^* \in X^*).$$

□

We continue with the following question:

Is there any continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that satisfies in the following relation:

There exists an $M > 0$ such that for all $n \in \mathbb{N}, r_1, r_2, r_3, \ldots, r_n \in \mathbb{R}$ and $x_1, x_2, x_3, \ldots,\ x_n \in \mathbb{R}$ we have

$$\left|\sum_{i=1}^n r_i f(x_i)\right| \leq M \left|\sum_{i=1}^n r_i x_i\right| \tag{3}$$

Obviously, each linear function on $\mathbb{R}$ satisfies in the above condition. In the following corollary we show that if a function f satisfies in the above relation, it should be of the form $f(x) = \alpha x$ where α is in $\mathbb{R}$, i.e., f is a linear function.

COROLLARY 2.5. *The only functions which satisfy in 3 are of the form $f(x) = \alpha x$ where α is in $\mathbb{R}$.*

PROOF. Let a function f satisfy in relation 3. So, for each $c_1, \ldots, c_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in \mathbb{R}$ we have

$$\left|\sum_{i=1}^n c_i f(x_i)\right| \leq M \left|\sum_{i=1}^n c_i x_i\right|.$$

On the other hand, we know that $\mathbb{R} \cong \mathbb{R}^*$ (isometrically isomorphic) with $t \to g_t$ where $g_t(x) = tx$. Therefore the function σ defined by $\sigma(g_t) := f(t)$ is in $\overline{C}_{\mathrm{BSE}}(\mathbb{R}^*)$, because for each $c_1, \ldots, c_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in \mathbb{R}$ we have

$$\left|\sum_{i=1}^n c_i \sigma(g_{x_i})\right| = \left|\sum_{i=1}^n c_i f(x_i)\right| \leq M \left|\sum_{i=1}^n c_i x_i\right| \leq M \left\|\sum_{i=1}^n c_i g_{x_i}\right\|_{R^*}.$$

Using Theorem 2.4, there exists a bounded sequence $\{s_n\} \subseteq \mathbb{R}$ such that for each $t \in \mathbb{R}$ we have

$$f(t) = \sigma(g_t) = \lim_n g_t(s_n) = \lim_n t s_n = t \lim_n s_n = t\sigma(g_1) = tf(1).$$

Taking $\alpha = f(1)$, the proof is complete. □

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۸ تا ۱۱ شهریور ۱۴۰۰

On polygroup actions

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Abstract. In this paper we introduce the notion of polygroup action on polygroup from which we are able to build a regular hypergroup and among other results it is proved that actions of polygroups are associated with hyperrepresentations. Hypermatrix representations of multivalued structures were studied by T. Vougiouklis. Representations of polygroups were studied by R. Ameri and et. This study is more general.

Keywords: polygroup, action, hyperrepresentation

AMS Mathematics Subject Classification [2010]: 18F20

1. Introduction

The theory of algebraic hyperstructures is a well-established branch of classical algebraic theory. Hyperstructure theory was first proposed in 1934 by Marty, who defined hypergroups and began to investigate their properties with applications to groups, rational fractions and algebraic functions [4]. It was later observed that the theory of hyperstructures has many applications in both pure and applied sciences; for example, semi hypergroups are the simplest algebraic hyperstructures that possess the properties of closure and associativity. The theory of hyperstructures has been widely reviewed ([2], [3] and [6]).

In this paper we introduce the notion of polygroup action on polygroup from which we are able to build a regular hypergroup and among other results it is proved that actions of polygroups are associated with hyperrepresentations. Hypermatrix representations of multivalued structures were studied by T. Vougiouklis in [6]. Representations of polygroups were studied by R. Ameri and et. in [1]. This study is more general.

In this section we give some notions and results of hypergroupoids, which we need to develop our paper.

Definition 1.1. Let H be a set. A map $\cdot : H \times H \longrightarrow P^*(H)$ is called hyperoperation or join operation, where $P^*(H)$ is the set of all nonempty subsets of H. The join operation is extended to subsets of H in natural way, so that $A \cdot B$ is given by

$$A \cdot B = \bigcup \{a \cdot b : a \in A \text{ } and \text{ } b \in B\}.$$

the notations $a \cdot A$ and $A \cdot a$ are used for $\{a\} \cdot A$ and $A \cdot \{a\}$ respectively. Generally, the singleton $\{a\}$ is identified by its element a.

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Definition 1.2. A hypergroupoid is a set H endowed with a hyperoperation $\cdot : H\times H \longrightarrow P^*(H)$. A quasihypergroup is a hypergroupoid such that $x\cdot H = H\cdot x = H$, for all $x\in H$, (the reproduction axiom), where $H\cdot x = \bigcup_{h\in H} h\cdot x$.

Definition 1.3. [2] A hypergroup is a set H equipped with an associative hyperoperation $\cdot : H\times H \longrightarrow P^*(H)$ which satisfies the property $x\cdot H = H\cdot x = H$, for all $x\in H$. If the hyperoperation $\cdot$ is associative then H is called a semihypergroup. In the above definition if $A, B\subseteq H$ and $x\in H$ then we define

$$A\cdot B = \bigcup_{a\in A, b\in B} a\cdot b, \quad x\cdot B = \{x\}\cdot B \quad and \quad A\cdot x = A\cdot\{x\}.$$

Definition 1.4. [2] A hypergroup $(H,\cdot)$ is called a regular hypergroup, if it has at least an identity element and all element of H has at least an inverse. In other words, there exists $e\in H$, such that for all $x\in H$, we have $x\in (x\cdot e)\cap(e\cdot x)$ and for all $x\in H$ there exists $x^{-1}\in H$ such that $e\in (x\cdot x^{-1})\cap(x^{-1}\cdot x)$.

Definition 1.5. [3] A polygroup is a special case of a hypergroup. A polygroup is a system $P=\langle P,\cdot,e,^{-1}\rangle$, where $e\in P$, $^{-1}$ is a unary operation on P, $\cdot$ maps $P\times P$ into nonempty subsets of P, and the following axioms hold for all $x,y,z\in P$:
(P_1) $(x\cdot y)\cdot z = x\cdot(y\cdot z)$,
(P_2) $x\cdot e = e\cdot x = x$,
(P_3) $x\in y\cdot z$ implies $y\in x\cdot z^{-1}$ and $z\in y^{-1}\cdot x$.

The following elementary facts about polygroups follow easily from the axioms: $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$, $e^{-1}=e$, $(x^{-1})^{-1}=x$, and $(x\cdot y)^{-1}=y^{-1}\cdot x^{-1}$, where $A^{-1}=\{a^{-1} \ : \ a\in A\}$.
A polygroup in which every element has order 2 (i.e., $x^{-1}=x$ for all x) is called symmetric. As in group theory it can be shown that a symmetric polygroup is commutative.

Definition 1.6. [5] If $(H,\otimes)$ and $(H',\odot)$ are two hypergroupoids, then a function $\varphi : H \longrightarrow H'$ is called a good homomorphism if and only if $\varphi(x\otimes y)=\varphi(x)\odot\varphi(y)$, $\forall(x,y)\in H^2$.

Definition 1.7. [5] Let $(G,\odot)$ be a hypergroupoid. The action of $(G,\odot)$ on a nonempty set A is a map $\bullet : G\times A \longrightarrow P^*(A)$ such that for all $(g_1,g_2)\in G\times G, a\in A$:
(i) $\bigcup_{t\in g_1\cdot g_2} t\bullet a = \bigcup_{s\in g_2\bullet a} g_1\bullet s$,
(ii) there exists $e\in G$ such that $a\in e\bullet a$.

2. Main results

Definition 2.1. If $\langle P_1,\cdot,e_1,^{-1}\rangle$ and $\langle P_2,*,e_2,^{-I}\rangle$ are two polygroups, then a function $\varphi : P_1 \longrightarrow P_2$ is called a good homomorphism if and only if $\varphi(x\cdot y)=\varphi(x)*\varphi(y)$, $\forall(x,y)\in P_1^2$.

Definition 2.2. Let $\langle P,\cdot,e,^{-1}\rangle$ be a polygroup. The action of $\langle P,\cdot,e,^{-1}\rangle$ on a nonempty set A is a map $\bullet : P\times A \longrightarrow P^*(A)$ such that for all $(g_1,g_2)\in P\times P$, $a\in A$:
(i) $\bigcup_{t\in g_1\cdot g_2} t\bullet a = \bigcup_{s\in g_2\bullet a} g_1\bullet s$,
(ii) $a\in e\bullet a$.

Proposition 2.3. *Let $\langle P,\cdot,e,^{-1}\rangle$ be a polygroup and $A^{P^*(A)}$ be the set of all functions from A to $P^*(A)$, endowed with the composition operation $\circ$, then $\varphi : P \longrightarrow A^{P^*(A)}$ defined by $\varphi(g)(a)=g\bullet a$ is a homomorphism.*

The homomorphism $\varphi : P \longrightarrow A^{P^*(A)}$ is called a hyperrepresentation associated with the polygroup action. this process is reversible in the sense that if $\varphi : P \longrightarrow A^{P^*(A)}$ is any homomorphism then the map from $P \times A \longrightarrow P^*(A)$ defined by $g \bullet a = \varphi(g)(a)$ satisfies the properties of a polygroup action of P on A.

Definition 2.4. Let $\langle H, \cdot, e_1, ^{-1} \rangle$ and $\langle K, \otimes, e_2, ^{-I} \rangle$ be two polygroups and $\varphi : K \longrightarrow H^{P^*(H)}$ be a hyperrepresentation determined by the polygroup action $\bullet$ of K on H. Let G be the set of ordered pairs (h, k) where $(h, k) \in H \times K$ and define the following hyperoperation on G by

$$(h_1, k_1) * (h_2, k_2) = (h_1 \cdot \varphi(k_1)(h_2), k_1 \otimes k_2).$$

Clearly this hyperoperation makes G into hypergroupoid which is denoted by $H \int_\varphi K = (H \times K, *)_\varphi$.

Remark 2.5. $h_1 \cdot \varphi(k_1)(h_2) = \bigcup_{t \in \varphi(k_1)(h_2)} h_1 \cdot t$ where $\varphi(k_1)(h_2) = k_1 \bullet h_2$.

Definition 2.6. In the above definition φ is closed if $h \in \varphi(k)(h)$, for all $h \in H$.

Proposition 2.7. *Let $\langle H, \cdot, e_1, ^{-1} \rangle$ and $\langle K, \otimes, e_2, ^{-I} \rangle$ be two polygroups and $\varphi : K \longrightarrow H^{P^*(H)}$ be a hyperrepresentation determined by the polygroup action $\bullet$ of K on H. Then $H \int_\varphi K$ is a regular hypergroup if φ is closed.*

Proposition 2.8. *Let $\langle H, \cdot, e_1, ^{-1} \rangle$ and $\langle K, \otimes, e_2, ^{-I} \rangle$ be two polygroups and K is acting trivially on H, that is $k \bullet h = e_2 \bullet h$. Then $H \int_\varphi K$ is a regular hypergroup.*

Definition 2.9. Let $\langle K, \otimes, e_2, ^{-I} \rangle$ be a polygroup acting on the polygroup $\langle H, \cdot, e_1, ^{-1} \rangle$. K is called acting reversibly on H if the following implication holds:

$$a \in \varphi(k)(b) \Rightarrow \exists k' \in K \ ; \ b \in \varphi(k')(a).$$

Proposition 2.10. *Let $\langle H, \cdot, e_1, ^{-1} \rangle$ and $\langle K, \otimes, e_2, ^{-I} \rangle$ be two polygroups. If K is acting reversibly on H. Then the relation on H defined by aRb if and only if $a \in \varphi(k)(b)$ for some $k \in K$, is an equivalence relation.*

Remark 2.11. Let $C_a = \{\varphi(k)(a) \ : \ k \in K\}$ denote the class of the element a and let $H_a = \{k \in K \ : \ a \in \varphi(k)(a)\}$ denote the stabilizer of a in H, then we have

Corollary 2.12. *Let $\langle H, \cdot, e_1, ^{-1} \rangle$ and $\langle K, \otimes, e_2, ^{-I} \rangle$ be two polygroups. The action of K on H is transitive if and only if $C_a = H_a$.*

3. Conclusion

In this paper, we have considered the notion of hyperrepresentation associated with the polygroup action as a new concept. As concerning future works, we will generalize these notions.

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Homological properties of Banach algebras

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Abstract. Let G be a locally compact group, $L^1(G)$ be group algebra and $M(G)$ be measure algebra of G. In this paper, we investigate the homological properties of Banach left module $L^1_\mu(G)$ over algebras $L^1(G)$ and $M(G)$. We show that $L^1_\mu(G)$ is injective Banach left $L^1(G)-$module if and only if G is discrete and amenable.

Keywords: Locally compact group, Banach module, projectivity, injectivity, flatness

AMS Mathematics Subject Classification [2010]: 43A15; 43A20, 46H25

1. Introduction

Throughout this paper, G denotes a locally compact group with a fixed left Haar measure λ. As usual, let $L^1(G)$ denote the group algebra of G as defined in [4] equipped with the norm $\|.\|_1$ and the convolution product "$*$" of functions on G defined by

$$(\phi * \psi)(x) = \int \phi(y)\psi(y^{-1}x)d\lambda(y)$$

for all $\phi, \psi \in L^1(G)$ and locally almost all $x \in G$. Let $M(G)$ denote the measure algebra of G as defined in [4] endowed with the convolution product "$*$" and the total norm $\|.\|$. Then $M(G)$ is a Banach algebra with the identity element δ_e, the Dirac measure at the identity element e of G. Also for $\mu \in M(G)$, let $L^1_\mu(G)$ denote the deformed L^1- space $L^1_\mu(G)$ given by

$$L^1_\mu(G) = \mu * L^1(G)$$

with the norm $\|\mu * f\|_\mu = \|f\|_{L^1(G)}$, see [5].

Let E and F be two Banach spaces and denote by $B(E,F)$ the Banach space of all bounded operators from E into F. An operator $T \in B(E,F)$ is called *admissible* if $T \circ S \circ T = T$ for some $S \in B(F,E)$. In the case where, $\mathfrak{A}$ is a Banach algebra and E and F are Banach left $\mathfrak{A}-$modules, $\mathfrak{A}_B(E,F)$ denotes the closed linear subspace of $B(E,F)$ of all left $\mathfrak{A}-$module morphisms. An operator $T \in \mathfrak{A}_B(E,F)$ is a *retraction* if there exists $S \in \mathfrak{A}_B(F,E)$ with $T \circ S = I_F$, the identity operator on F; in this case, F is called a *retract* of E.

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Let us recall that a Banach left $\mathfrak{A}-$module P is called *projective* if, for Banach left $\mathfrak{A}-$modules E and F, each admissible epimorphism $T \in \mathfrak{A}_B(E,F)$ and each $S \in \mathfrak{A}_B(P,F)$, there exists $R \in \mathfrak{A}_B(P,E)$ such that $T \circ R = S$. Let also recall that a Banach left $\mathfrak{A}-$module I is called *injective* if for Banach left $\mathfrak{A}-$modules E and F, each admissible monomorphism $T \in \mathfrak{A}_B(E,F)$ and each $S \in \mathfrak{A}_B(E,I)$, there exists $R \in \mathfrak{A}_B(F,I)$ such that $R \circ T = S$. A Banach left $\mathfrak{A}-$module E is called *flat*, if its dual E^* be an injective Banach right $\mathfrak{A}-$module.

Homological properties of Banach modules have been studied by several authors [1–3, 6]. For example, Dales and Polyakov [1] studied homological properties of modules over group algebras. They gave necessary and sufficient conditions for some Banach left $L^1(G)-$modules to have homological properties such as projectivity, injectivity and flatness.

In this paper, we investigate the homological properties of the Banach left $L^1_\mu(G)$ module over $L^1(G)$ and $M(G)$. We show that $L^1_\mu(G)$ is projective and flat Banach left $L^1(G)-$ module. We also, prove that $L^1_\mu(G)$ is injective Banach left $L^1(G)-$module if and only if G is discrete and amenable.

2. Main results

The deformed L^1-space $L^1_\mu(G)$ is a left $L^1(G)-$ module (left $M(G)-$ module) with the module actions defined by

$$g \cdot (\mu * f) = \mu * (g * f) \quad (\nu \cdot (\mu * f) = \mu * (\nu * f))$$

for all $f, g \in L^1(G)$ and $\nu \in M(G)$.

A Banach left $\mathfrak{A}-$module E is called *essential* if $\mathfrak{A}E = E$. Also, E is called *faithful* if $\mathfrak{A} \cdot x = 0$ for all nonzero elements $x \in E$.

Proposition 2.1. *The following statements are fulfilled.*

(1) *$L^1_\mu(G)$ is essential.*
(2) *$L^1_\mu(G)$ is faithful.*

Theorem 2.2. *Let G be a locally compact group. Then $L^1_\mu(G)$ is a projective Banach left $L^1(G)-$ module.*

Proof. It is well know, each retraction of a projective Banach left $L^1(G)-$module is projective [2]. We therefore only need to prove that $L^1_\mu(G)$ is a retraction of $L^1(G)$. To this end, define $\rho : L^1(G) \to L^1_\mu(G)$ by

$$f \mapsto \mu * f$$

for $f \in L^1(G)$. So it is clear that ρ is a left $L^1(G)-$module morphism. Now, we define $Q : L^1_\mu(G) \to L^1(G)$ by

$$\mu * f \mapsto f.$$

Then Q is a left $L^1(G)-$module morphism and a right inverse for ρ; indeed, for $\phi \in L^1(G)$ and $\mu * f \in L^1_\mu(G)$, we have

$$Q(\phi \cdot \mu * f) = Q(\mu * \phi * f) = \phi * f = \phi \cdot Q(\mu * f)$$

and

$$\rho \circ Q(\mu * f) = \rho(f) = \mu * f$$

for all $\mu * f \in L^1_\mu(G)$. Thus $L^1_\mu(G)$ is projective □

Let us recall that a locally compact group G is called *amenable* if there is a left invariant mean on $L^\infty(G)$.

THEOREM 2.3. *Let G be a locally compact group. Then $L^1_\mu(G)$ is an injective Banach left $L^1(G)-$ module if and only if G is discrete and amenable.*

THEOREM 2.4. *Let G be a locally compact group. Then $L^1_\mu(G)$ is a flat Banach left $L^1(G)-$ module.*

Let E be a Banach left $L^1(G)-$ module. A functional $\Lambda \in E^*$ is called *augmentation invariant* if every $x \in E$ and $\phi \in L^1(G)$, we have

$$\langle \Lambda, \phi \cdot x \rangle = \varphi_G(\phi)\langle \Lambda, x \rangle,$$

where $\varphi_G : M(G) \to \mathbb{C}$ is defined by $\varphi_G(\mu) = \mu(G)$. Note that the restriction φ_G to $L^1(G)$ has the form

$$\varphi_G(\phi) = \int \phi(x) d\lambda(x)$$

for all $\phi \in L^1(G)$. In the case where, Λ is a non-zero augmentation invariant functional in E^*, then E is said to be *augmentation invariant.*

THEOREM 2.5. *Let G be a locally compact group. Then $L^1_\mu(G)$ is augmentation invariant.*

THEOREM 2.6. *Let G be a locally compact group. Then the following statements are fulfilled.*

(1) *$L^1_\mu(G)$ is projective in $M(G)-$ module if and only if $L^1_\mu(G)$ is projective in $L^1(G)-$ module.*
(2) *$L^1_\mu(G)$ is injective in $M(G)-$ module if and only if $L^1_\mu(G)$ is injective in $L^1(G)-$ module.*

THEOREM 2.7. *Let G be a locally compact group. If $L^1_\mu(G)$ is flat in $M(G)-$ module, then $L^1_\mu(G)$ is flat in $L^1(G)-$ module.*

3. Conclusion

We summarize our results in the following Table.

	Projective	*Injective*	Flat
$L^1_\mu(G)$	all G 2.2	G discrete and amenable 2.3	all G 2.4

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Semi-analytical solution of novel coupled partial integral differential equations (PIDEs) with application in Aeroelasticity

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Abstract. One way to derive the governing equations in complex dynamic systems is to use several intermediate coordinate systems. When the systems also include concentrated discrete nodes, can utilize Delta Dirac function in order to apply the properties of nodes in the equations. In this work, a semi-analytical solution of novel coupled partial integral differential equations is developed. The extracted equations include parameter-dependent and time-dependent integral parts which Dirac Delta function is multiplied by itself several times in the parameter-dependent terms. The validation of results for the flutter speed shows that there is a proper precision in the presented semi-analytical solution.

Keywords: Semi-analytical, PIDE, Dirac Delta function, Aeroelasticity

AMS Mathematics Subject Classification [2010]: 74F10, 45K05, 37M15

1. Introduction

BWA configuration is one of innovative configurations commonly used by airplane designers to reduce the aircraft emissions [1]. The structure is included front wing, rear wing and winglet that connects to the wings tip as Figure 1 [2]. Where $(XYZ)_f$, $(XYZ)_r$ and $(XYZ)_0$ are un-swept coordinate systems of front and rear wings and coordinate system on fuselage C.G., respectively. Because of the BWA complicated dynamic, several coordinate systems are used to extract the governing equations [3]. The equations are obtained via Hamilton's variational principle and are as coupled partial integral differential equations (PIDEs). The equations include parameter-dependent and time-dependent integral parts which Dirac Delta function is multiplied by itself several times in the parameter-dependent terms. Furthermore, the time-dependent terms are presented in Wagner unsteady model that is utilized to apply the aerodynamic forces and moments [4]. In order to simulate the torsional and longitudinal behavior of the winglet, is used two springs [5]. Therefore in this article, a semi-analytical solution of the novel equations is presented and in order to

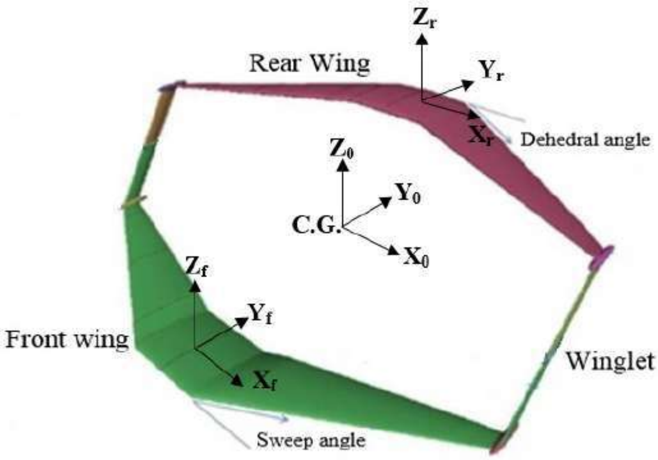

FIGURE 1. A schematic of BWA configurations and used intermediate coordinate systems [2]

validate the solution, the obtained results are compared with professional software results.

2. Main results

In this work, plunge and pitch motions are considered for each wing. So, four governing equations are as follows. It should be noted that w, θ , l, $\phi(t)$, δ_D and σ indicate plunge and pitch motions of the wings, wing length, Wagner function, Dirac Delta function and parameter of Duhamel integral form, respectively. Furthermore, f and r indexes denote front and rear wings and A_i to V_i constants are functions of BWA geometry and properties, springs stiffness and airstream properties.

$$\begin{aligned}(1)\quad & A_1\ddot{w}_f+B_1\ddot{\theta}_f+C_1w^{(4)}{}_f+\int_0^{l_r}\{D_l\,w_f[\delta_D\left(x_f-l_f\right)]^3\delta_D\left(x_r-l_r\right)-E_1w_r[\delta_D\left(x_f-l_f\right)]^2\\ & [\delta_D\left(x_r-l_r\right)]^2\}dx_r+\{F_1\ddot{w}_f-G_1\ddot{w}''_f-H_1\ddot{\theta}'_f\}\delta_D\left(x_f-l_f\right)=I_1\dot{\theta}_f+J_1\dot{w}'_f-K_1\dot{\theta}'_f\\ & +[L_1\dot{w}_f\left(0\right)+M_1\theta_f\left(0\right)+N_1w'{}_f\left(0\right)\ +O_1\dot{\theta}_f\left(0\right)+P_1\theta'{}_f\left(0\right)]\varphi_f\left(t\right)\\ & +\int_0^t\varphi_f\left(t-\sigma\right)[Q_1\ddot{w}_f+R_1\dot{\theta}_f+S_1\dot{w}'_f+T_1\ddot{\theta}_f+U_1\dot{\theta}'_f]d\sigma\end{aligned}$$

$$\begin{aligned}(2)\quad & A_2\ddot{\theta}_f+B_2\ddot{w}_f-C_2\theta''{}_f+\int_0^{l_r}\{D_2\,\theta_f[\delta_D\left(x_f-l_f\right)]^3\delta_D\left(x_r-l_r\right)-E_2\theta_r[\delta_D\left(x_f-l_f\right)]^2\\ & [\delta_D\left(x_r-l_r\right)]^2\}dx_r+\{F_2\ddot{\theta}_f+G_2\ddot{w}''_f\}\delta_D\left(x_f-l_f\right)=H_2\dot{w}'_f+I_2\dot{\theta}_f-J_2\theta'{}_f-K_2\ddot{\theta}_f\\ & -L_2\dot{\theta}'_f+M_2\dot{w}_f(0)+N_2\theta_f(0)\ +O_2w'{}_f(0)+P_2\dot{\theta}_f(0)+Q_2\theta'{}_f(0)\\ & +\int_0^t\phi_f\left(t-\sigma\right)\left[R_2\ddot{w}_f+S_2\dot{\theta}_f\ +\ T_2\dot{w}'_f+U_2\ddot{\theta}_f+V_2\dot{\theta}'_f\right]d\sigma\end{aligned}$$

$$\begin{aligned}(3)\quad & A_3\ddot{w}_r+B_3\ddot{\theta}_r+C_3w^{(4)}{}_r+\int_0^{l_f}\{D_3w_r[\delta_D\left(x_r-l_r\right)]^3\delta_D\left(x_f-l_f\right)+E_3w_r[\delta_D\left(x_r-l_r\right)]^3\\ & \delta_D\left(x_f-l_f\right)-F_3w_f[\delta_D\left(x_f-l_f\right)]^2[\delta_D\left(x_r-l_r\right)]^2\}dx_f+\{G_3\ddot{w}_r-H_3\ddot{w}''_r\\ & -I_3\ddot{\theta}'_r\}\delta_D\left(x_r-l_r\right)=J_3\dot{\theta}_r+K_3\dot{w}'_r-L_3\dot{\theta}'_r\ +[M_3\dot{w}_r\left(0\right)+N_3\theta_r\left(0\right)+O_3w'{}_r\left(0\right)\ +\end{aligned}$$

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$$P_3\dot{\theta}_r\,(0)$$
$$+\,Q_3\theta'_r\,(0)]\varphi_r\,(t)\;+\int\limits_0^t \varphi_r\,(t-\sigma)[R_3\ddot{w}_r + S_3\dot{\theta}_r + T_3\dot{w}'_r + U_3\ddot{\theta}_r + V_3\dot{\theta}'_r]d\sigma$$

(4) $$A_4\ddot{\theta}_r + B_4\ddot{w}_r - C_4\theta''_r + \int\limits_0^{l_f}\{-D_4\theta_f[\delta_D\,(x_f-l_f)]^2[\delta_D\,(x_r-l_r)]^2 - E_4\theta_r[\delta_D\,(x_r-l_r)]^3$$
$$\delta_D\,(x_f-l_f)\;\}dx_f + \{F_4\ddot{\theta}_r + G_4\ddot{w}''_r\}\delta_D\,(x_r-l_r) = H_4\dot{w}'_r + I_4\dot{\theta}_r - J_4\theta'_r - K_4\ddot{\theta}_r - L_4\dot{\theta}'_r$$
$$+\,M_4\dot{w}_r(0) + N_4\theta_r(0)\;+O_4w'_r(0) + P_4\dot{\theta}_r(0) + Q_4\theta'_r(0)$$
$$+\int\limits_0^t \phi_r\,(t-\sigma)\,[R_4\ddot{w}_r + S_4\dot{\theta}_r\;+\;T_4\dot{w}'_r + U_4\ddot{\theta}_r + V_4\dot{\theta}'_r]\,d\sigma$$

2.1. Develop the parameter-dependent terms. A new class of generalized functions was developed by NASA report in 1967 [6]. The functions can derive from hyperbolic tangent and Gaussian families which based on the hyperbolic tangent family, the Heaviside function can be stated as follows:

(5) $$H_n\,(x) = \tfrac{1}{2}\,[1 + \tanh\,(nx)]$$

where n is the sequence index and the derivative of Eq. (5) will be as follows:

(6) $$\tfrac{d}{dx}H_n\,(x) = \delta_n\,(x) = 2n\left[H_n\,(x) - H^2{}_n\,(x)\right]$$

Using further differentiation and reapplication of Eq. (6), can be obtained the developed relations. The Gaussian representation for delta function is as follows:

(7) $$\delta_n\,(x) = \tfrac{n}{\sqrt{\pi}}e^{-n^2x^2}$$

The derivatives of above relation are given by

(8) $$\delta_n{}^{(\alpha)}\,(x) = (-1)^\alpha n^\alpha H_\alpha\,(nx)\,\delta_n\,(x)$$

where $H_\alpha(nx)$ are Hermite polynomials of argument nx. The following developed relations are obtained based on the hyperbolic tangent family.

(9) $$\delta^2{}_n\,(x) = \tfrac{n\,\delta_n(x)}{3} - \tfrac{\delta''{}_n(x)}{12n}$$

(10) $$\delta^3{}_n\,(x) = \tfrac{2n^2\,\delta_n(x)}{15} - \tfrac{\delta''{}_n(x)}{24} + \tfrac{\delta^{(4)}{}_n(x)}{480n^2}$$

(11) $$\delta_n\,(x)\,\delta'{}_n\,(x) = \tfrac{n\,\delta'{}_n(x)}{6} - \tfrac{\delta^{(3)}{}_n(x)}{24n}$$

(12) $$\delta_n\,(x)\,\delta''{}_n\,(x) = -\tfrac{4n^3\,\delta_n(x)}{15} + \tfrac{n\,\delta''{}_n(x)}{6} - \tfrac{\delta^{(4)}{}_n(x)}{40n}$$

(13) $$\delta'{}_n\,(x)\,\delta''{}_n\,(x) = \tfrac{4n^3\,\delta_n(x)}{15} - \tfrac{\delta^{(5)}{}_n(x)}{120n}$$

Therefore, the parameter-dependent terms can be developed using the mentioned relations and by part integral method.

2.2. Eliminate the time-dependent terms. Using by part integral method and some mathematical techniques as Ref. [4], the time-dependent terms are eliminated from the equations developed in the previous step.

2.3. Transform PDEs to ODEs. The expanded equations from the previous steps, will be as partial differential equations (PDEs) and transformed to ordinary differential equations (ODEs) using the assumed modes method [3]. The bending and torsion deflections $(w_f, \theta_f, w_r, \theta_r)$ are expanded by means of series of trial functions which only must satisfy geometric boundary conditions. The final equations can be written in the state-space form.

3. Numerical results

There is no numerical solution or experimental data for the studied model so far. Therefore in this work, the flutter analyses of the BWA is performed in a professional software and the results are compared as in Table 1. As can be seen, a good agreement is reported.

Table 1. Validation of flutter speed for BWA configuration

Component	Computational Method	Flutter Speed (m/s)	Mach No.
Front wing	Semi-analytical solution	287	0.84
Front wing	Professional software solution	289	0.85
Rear wing	Semi-analytical solution	274	0.81
Rear wing	Professional software solution	271	0.80
BWA	Semi-analytical solution	269	0.79
BWA	Professional software solution	270	0.79

4. Conclusion

The extracted equations included several parameter/time-dependent integral parts. Furthermore, Dirac Delta function was multiplied by itself several times. The validation revealed that can utilize the procedure for the solution of PIDEs which include generalized functions, parameter-dependent and time-dependent integral parts.

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Some operator inequalities involving operator monotone functions for sector matrices

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Abstract. Recently, important inequalities for some operator mean inequalities via operator monotone and operator monotone decreasing functions have been proved. In this paper, we extend these inequalities to inequalities for sector matrices which involving the mean of sector matrices, the positive linear maps and operator monotone functions or more precisely, operator monotone increasing (decreasing).

Keywords: Sector matrices, Heinz mean, Heron mean.

AMS Mathematics Subject Classification [2010]: 15A60, 15B48, 47A64

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1. Introduction

An operator $A \in B(H)$ is called accretive if in its Cartesian (or Toeplitz) decomposition, $A = \mathcal{R}z + i\mathcal{I}z$, $\mathcal{R}z$ is positive, where $\mathcal{R}z = \frac{A+A^*}{2}$, $\mathcal{I}z = \frac{A-A^*}{2}$. A linear map $\Phi : B(H) \to B(H)$ is called positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. If $\Phi(I) = I$, where I denoted the identity operator, then we say that ϕ is unital. A continuous real valued function f(resp. g) defined on interval J is said to be operator monotone or more precisely, operator monotone increasing (decreasing) if for every two positive operators A and B with spectra in J, the inequality $A \leq B$ implies $f(A) \leq f(B)(g(A) \geq g(B))$.
At the end of this section, we present the Lemmas we need to prove the main theorems.

Lemma 1.1. *[1] If $f \in m$ and $A, B \in \mathbb{M}_n^+$, then*

$$f(A\nabla_t B) \geq f(A)\nabla_t f(B), \ 0 \leq t \leq 1.$$

Lemma 1.2. *[5] Let $\alpha, \beta > 0$, $\nu \in (0,1)$, $A, B \in B(H)$ and accretive. Then*

$$(\alpha A)\sharp_\nu(\beta B) = \alpha^{1-\nu}\beta^\nu(A\sharp_\nu B)$$

Lemma 1.3. *([5], [6]) Let $A, B \in B(H)$ be accretive and let $\nu \in [0,1]$. Then*

$$RA\sharp_\nu RB \leq R(A\sharp_\nu B) \leq sec^2(\alpha)((RA)\sharp_\nu(RB)).$$

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Lemma 1.4. *[4] Let $A, B \in \mathbb{M}$, be such that $W(A), W(B) \subset S_\alpha$ and $\nu \in (0,1)$. Then*
$$H_\nu(RA, RB) \leq RH_\nu(A,B) \leq sec^2(\alpha) H_\nu(RA, RB),$$

Lemma 1.5. *[1] Let $f \in m$ and $A, B \in S_\alpha$ for some $0 \leq \alpha < \frac{\pi}{2}$. Then*
$$\mathcal{R}A\sigma_f\mathcal{R}B \leq \mathcal{R}(A\sigma_f B) \leq \sec^2\alpha(\mathcal{R}A\sigma_f\mathcal{R}B).$$

Lemma 1.6. *[3] Let $A \in \mathbb{M}_n$ are positive , Φ is a unital positive linear mapping and $f \in m$. Then*
$$\Phi(A\sigma_f B) \leq \Phi(A)\sigma_f\Phi(B),$$
and
$$\Phi(f(A)) \leq f(\Phi(A)).$$

Lemma 1.7. *[1] Let $f \in m$ and $A, B \in S_\alpha$ for some $0 \leq \alpha < \frac{\pi}{2}$. Then*
$$\mathcal{R}\Phi(f(A)) \leq \sec^2\alpha\mathcal{R}f(\Phi(A)).$$

Lemma 1.8. *[1] Let $f \in m$ and $A, B \in S_\alpha$ for some $0 \leq \alpha < \frac{\pi}{2}$. Then*
$$f(\mathcal{R}A) \leq \mathcal{R}(f(A)) \leq \sec^2\alpha f(\mathcal{R}A)$$

Lemma 1.9. *[1, Theorem 6.4] Let $A, B \in \mathbb{M}_n$ be accretive such that $\omega(A), \omega(B) \subset S_\alpha$, for some $0 \leq \alpha < \frac{\pi}{2}$. Then for any $f \in m$ and $\nu \in (0,1)$,*
$$\mathcal{R}(f(A)\nabla_\nu f(B)) \leq \sec^2\alpha\mathcal{R}f(A\nabla_\nu B).$$

Lemma 1.10. *[2, Theorem 2.5] Let $0 < mI \leq A, B \leq MI$, $0 \leq \nu \leq 1$, $!_\nu \leq \tau_\nu$, $\sigma_\nu \leq \nabla_\nu$ and Φ be a positive unital linear map. If f is an operator monotone function on $(0,\infty)$, then*
$$f(\Phi(A\tau_\nu B)) \leq \mathcal{K}\left(f(\Phi(A))\sigma_\nu f(\Phi(B))\right).$$

Lemma 1.11. *[Theorem 2.8] [2] Let $0 < mI \leq A, B \leq MI$, $0 \leq \nu \leq 1$, $!_\nu \leq \tau_\nu$, $\sigma_\nu \leq \nabla_\nu$ and Φ be a positive unital linear map. If f is an operator monotone function on $(0,\infty)$, then*
$$\Phi(f(A))\tau_\nu\Phi(f(B)) \leq \mathcal{K}\Phi(f(A\sigma_\nu B)).$$
where $f : (0,\infty) \to (0,\infty)$ is an operator monotone function.

2. Main results

Theorem 2.1. *Let $A, B \in S_\alpha$ such that $0 < mI \leq A, B \leq MI$, $0 \leq \nu \leq 1$ and ϕ be a positive unital linear map. If f is an operator monotone function on $(0,\infty)$, then*
$$\mathcal{R}(\Phi(f(A)\sharp_\nu f(B))) \leq \mathcal{K}\sec^4\alpha\mathcal{R}\left(\Phi(f(A\sharp_\nu B))\right).$$

Proof. If in Lemma 1.11 let $\tau_\nu = \sigma_\nu = \sharp_\nu$ we have
$$\Phi(f(\mathcal{R}A))\sharp_\nu\Phi(f(\mathcal{R}B)) \leq \mathcal{K}\Phi\left(f(\mathcal{R}A\sharp_\nu\mathcal{R}B)\right),$$
$$\begin{aligned}\mathcal{K}\Phi\left(f(\mathcal{R}A\sharp_\nu\mathcal{R}B)\right) &\leq \mathcal{K}\Phi(f(\mathcal{R}(A\sharp_\nu B)) && \text{(by Lemma 1.3)}\\ &\leq \mathcal{K}\Phi(\mathcal{R}(f(A\sharp_\nu B)) && \text{(by Lemma 1.8)}\\ &= \mathcal{K}\mathcal{R}\left(\Phi(f(A\sharp_\nu B))\right),\end{aligned}$$
and
$$\begin{aligned}\Phi(f(\mathcal{R}A))\sharp_\nu\Phi(f(\mathcal{R}B)) &\geq \Phi(f(\mathcal{R}A)\sharp_\nu f(\mathcal{R}B)) && \text{(by Lemma 1.6)}\\ &\geq \Phi\left(\cos^2\alpha\mathcal{R}f(A)\sharp_\nu\cos^2\alpha\mathcal{R}f(B)\right) && \text{(by Lemma 1.8)}\end{aligned}$$

$$
\begin{aligned}
&= \cos^2\alpha\Phi(\mathcal{R}f(A)\sharp_\nu\mathcal{R}f(B)) \\
&\geq \cos^2\alpha\Phi\left(\cos^2\alpha\mathcal{R}(f(A)\sharp_\nu f(B))\right) && \text{(by Lemma 1.3)} \\
&= \cos^4\alpha\mathcal{R}(\Phi(f(A)\sharp_\nu f(B))).
\end{aligned}
$$

then

$$
\begin{aligned}
\cos^4\alpha\mathcal{R}(\Phi(f(A)\sharp_\nu f(B))) &\leq \Phi(f(\mathcal{R}A))\sharp_\nu\Phi(f(\mathcal{R}B)) \\
&\leq \mathcal{K}\Phi\left(f(\mathcal{R}A\sharp_\nu\mathcal{R}B)\right) \\
&\leq \mathcal{K}\sec^2\alpha\mathcal{R}\left(\Phi(f(A\sharp_\nu B))\right).
\end{aligned}
$$

□

THEOREM 2.2. *Let $A, B \in S_\alpha$ such that $0 < mI \leq A, B \leq MI$, $0 \leq \nu \leq 1$ and Φ be a positive unital linear map. If f is an operator monotone function on $(0, \infty)$, then*

(1)
$$f(\Phi(\mathcal{R}A\sharp_\nu\mathcal{R}B)) \leq \mathcal{K}\sec^2\alpha\left(\mathcal{R}(f(\Phi(A\nabla_\nu B))\right).$$

PROOF.

$$
\begin{aligned}
f(\Phi(\mathcal{R}A\sharp_\nu\mathcal{R}B)) &\leq \mathcal{K}(f(\Phi(\mathcal{R}A))\nabla_\nu f(\Phi(\mathcal{R}B)) && \text{(by Lemma1.10)} \\
&= \mathcal{K}(f(\mathcal{R}(\Phi(A)))\nabla_\nu f(\mathcal{R}(\Phi(B))) \\
&\leq \mathcal{K}\left(\mathcal{R}(f(\Phi(A)))\nabla_\nu\mathcal{R}(f(\Phi(B)))\right) && \text{(by Lemma1.8)} \\
&= \mathcal{K}\left(\mathcal{R}(f(\Phi(A))\nabla_\nu f(\Phi(B))\right) \\
&\leq \mathcal{K}\sec^2\alpha\left(\mathcal{R}(f(\Phi(A)\nabla_\nu\Phi(B)))\right) && \text{(by Lemma1.9)} \\
&= \mathcal{K}\sec^2\alpha\mathcal{R}(f(\Phi(A\nabla_\nu B))).
\end{aligned}
$$

□

THEOREM 2.3. *Let $A, B \in S_\alpha$ such that $0 < mI \leq A, B \leq MI$, $0 \leq \nu \leq 1$ and Φ be a positive unital linear map. If f is an operator monotone function on $(0, \infty)$, then*

$$f\left(\Phi(\mathcal{R}\mathcal{H}_\nu(A, B)\right) \leq \sec^2\alpha\mathcal{K}f(\Phi(A\nabla_\nu B)).$$

PROOF. If in Lemma 1.10 let $\tau_\nu = \mathcal{H}_\nu$ and $\sigma_\nu = \nabla_\nu$ we have

$$
\begin{aligned}
f\left(\Phi(\mathcal{R}\mathcal{H}_\nu(A, B))\right) &\leq \sec^2\alpha f(\Phi(\mathcal{H}_\nu(\mathcal{R}A, \mathcal{R}B))) && \text{(by Lemma 1.4)} \\
&\leq \sec^2\alpha\mathcal{K}\left(f(\Phi(\mathcal{R}A))\nabla_\nu f(\Phi(\mathcal{R}B))\right) && \text{(by Lemma 1.10)} \\
&\leq \sec^2\alpha\mathcal{K}\mathcal{R}\left(f(\Phi(A\nabla_\nu B)\right) && \text{(by Lemma 1.1).}
\end{aligned}
$$

□

THEOREM 2.4. *Let $0 < mI \leq A, B \leq MI$, $0 \leq \nu \leq 1$, $!_\nu \leq \tau_\nu, \sigma_\nu \leq \nabla_\nu$ and Φ be a positive unital linear map. If f is an operator monotone function on $(0, \infty)$, then*

(2)
$$\mathcal{R}(\Phi(f(A)\tau_\nu\Phi(f(B)) \leq \mathcal{K}\sec^6\alpha\mathcal{R}(f(\Phi(A\sigma_\nu B))).$$

PROOF.

$$
\begin{aligned}
\mathcal{R}(\Phi(f(A)\tau_\nu\Phi(f(B)) &\leq \mathcal{R}(\sec^2\alpha\Phi f(A))\tau_\nu\mathcal{R}(\sec^2\alpha\Phi f(B)) && \text{(by Lemma 1.5)} \\
&= \Phi(\sec^2\alpha\mathcal{R}f(A))\tau_\nu\Phi(\sec^2\alpha\mathcal{R}f(B)) && \text{(by Lemma 1.2)} \\
&\leq \sec^4\alpha\Phi(f(\mathcal{R}A))\tau_\nu\Phi(f(\mathcal{R}B)) && \text{(by Lemma 1.8)} \\
&\leq \sec^4\alpha\mathcal{K}\Phi(f(\mathcal{R}A\sigma_\nu\mathcal{R}B)) && \text{(by Lemma 1.11)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sec^4 \alpha \mathcal{K} \Phi(f(\mathcal{R} A \sigma_\nu B))) && \text{(by Lemma 1.5)}\\
&\leq \sec^4 \alpha \mathcal{K} \Phi(\mathcal{R}(f(A \sigma_\nu B))) && \text{(by Lemma 1.8)}\\
&= \sec^4 \alpha \mathcal{K} \mathcal{R}(\Phi(f(A \sigma_\nu B))) && \\
&\leq \mathcal{K} \sec^6 \alpha \mathcal{R}(f(\Phi(A \sigma_\nu B))) && \text{(by Lemma 1.7).}
\end{aligned}
$$

□

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On the complexity function of leading digits of powers of one-digit primes

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Abstract. In this paper we investigate a class of words with low complexity function and accomplish it to the following problem. For a prime number $\mathbf{p}$, consider a sequence of digits $\mathbf{w}_n$, where $\mathbf{w}_n$ is the first digit in the decimal representation of $\mathbf{p}^n$. How to find the number of subwords of $\mathbf{w}_n$ of a given length?

Keywords: Complexity functions, digital problems, combinatorics on words, symbolic dynamics, Sturmian words

AMS Mathematics Subject Classification [2010]: 37B10, 68R15, 11A63

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1. Introduction

The following Olympiad problem is widely known, it was designed by Alexei Kanel-Belov:

We consider a sequence whose n^{th} term is the first digit of 2^n. Show that the number of different words with length of 13 sets of thirteen consecutive digits is 57.

Consider the sequence of $\mathbf{w} = \{\omega_k\}_{k>0}$, where ω_k is the first digit in the decimal representation of $\mathbf{p}^k$. For $\mathbf{p} = 2$, we study the sequence $124813612512481361251248136\cdots$. In particular, we are interested in the number of factors of length n that may occur in such a sequence (i.e., the subwords made of n consecutive digits).

Digital problems of this type in Number theory are well-known to be difficult, e.g., in the literature, least non-zero digit of $n!$ in base 12 (Deshouillers et al. [1]) or digits of n^n have been investigated.

A *word* over an alphabet $\mathcal{A}$ is a sequence taking values in the finite set $\mathcal{A}$. A *finite word* of length n is thus a map from $\{1, \cdots, n\}$ to $\mathcal{A}$ and a (one-sided) *infinite word* is just a map from $\mathbb{N}$ to $\mathcal{A}$. Words have a strong representational power: they can encode elements

*Speaker. Email address: m.golafshan@phystech.edu of an infinite set using finitely many symbols, e.g., the characteristic sequence of a subset of integers or the base-d expansion of an irrational number in $[0, 1]$. They naturally appear in a variety of contexts: computability theory, symbolic dynamics, algebra, number theory and numeration systems or theoretical computer science and text algorithms.

It is natural to associate some measures of complexity with infinite words. The most studied one is the *factor complexity* studied in 1975 by Ehrenfeucht, Lee and Rozenberg (see Fig. 1). It counts the number $c_{\mathbf{w}}(n)$ of distinct factors of length n (blocks of n consecutive letters) occurring in an infinite word $\mathbf{w}$. For instance, ultimately periodic words are characterized, thanks to the Morse-Hedlund theorem, by the fact that $c_{\mathbf{w}}(n)$ is bounded.

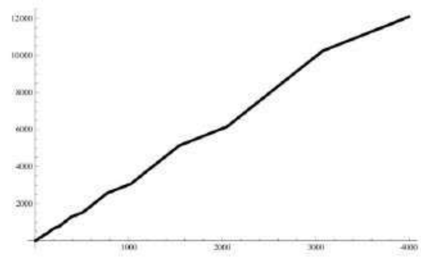

FIGURE 1. The first values of the factor complexity of the Thue-Morse word [2] (showing fluctuations and a $O(n)$ behavior).

The famous theorem of Morse-Hedlund can also be stated as follows. An infinite word is *ultimately periodic* if and only if there exists N such that $c_{\mathbf{w}}(N)$ is less than or equal to N. Therefore, a word is non-periodic if and only if, for all n, $c_{\mathbf{w}}(n) > n$. Remarkably, *Sturmian words* can be defined by the fact that $c_{\mathbf{w}}(n) = n + 1$ for all n. They are in particular over a binary alphabet. In that respect, they are non-periodic words of minimal complexity.

THEOREM 1.1. *(Kronecker's theorem) [3]. Let α be an irrational number. The set*

$$\{\{n\alpha\}_{n>0} | n \in \mathbb{N}\}$$

is dense in $[0, 1]$.

1.1. Strategy. Moving to the unipotent dynamics of the torus and counting the number of regions into which the torus is divided by a family of parallel hyperplanes [4]. Here, we consider that one dimensional torus, i.e., circle.

2. Main Results

In this paper, we study a generalization of this problem and prove the following results.

2.1. Linearity.

THEOREM 2.1. *The factor complexity of leading digits of sequence $\{\mathbf{p}^k\}_{k>0}$, where $\mathbf{p}$ be a one-digit prime number is a linear function; i.e., $c_{\mathbf{w}}(n) = an + b$.*

PROOF. Without loss of generality, we prove it for $\mathbf{p} = 2$; For other primes will be proved similar.

Let d_k be the first digit of 2^k. Then,

$$d_k \cdot 10^m \leqslant 2^k < (d_k + 1)10^m,$$

and by taking the logarithm from all sides,

$$\log(d_k) + m \leqslant k\log(2) < \log(d_k + 1) + m.$$

In the other words,

$$\{k\log(2)\} \in [\log(d_k), \log(d_k + 1)),$$

where $\{r\}$ denotes the fractional part of the real number r.

The intervals $I_{d_k} := [\log(d_k), \log(d_k + 1))$ form a partition of the unit circle $\mathcal{S} := \mathbb{R}/\mathbb{Z}$, hence $\omega_k = d_k$ if and only if $\{n\log(2)\} \in I_{d_k}$. For example $[0, 0.301) \subset [0, \log(2)) = I_1$.

Let T be the rotation of the unit circle $\mathcal{S}$ by $\phi := \log(2)$. Then $\omega_k = d_k$ if and only if, $T^k(0) \in I_{d_k}$. A word obtained this way is said to have the Interval Coding Property (ICP), for obvious reasons.

If $x_1 \cdots x_s$ is a factor of $\mathbf{w}$, then there exists $x \in \mathcal{S}$ such that $x + i\phi \in I_{(x_i)}$ for all $i = 1, \cdots, s$. By B_i denote the set $T^{-1}(I_i)$. We have $x \in \bigcap_{i=1}^{s} B_i =: V_x$, this is an intersection of s intervals and $x_1 \cdots x_s$ is a factor if and only if, V_x is non-empty. Therefore to calculate $c_{\mathbf{w}}(s)$ we need to consider not only the sets I_{d_k}, yet also the inverse rotations $T^{-i}(I_{d_k})$ for $0 \leqslant i < s$ and count the size of the obtained partition of $\mathcal{S}$.

Let $\theta_i := \log(i)$ and denote by $\Phi := \{\theta_1, \cdots, \theta_9\}$ the set 9 points. Now let us add $\Phi - \phi$ to the set Φ and consider how many new points we get. In principle, we get 9 extra points when we rotate the boundary points of the I_{d_k} over $-\log(2)$, yet we have some dependence. A priori 9 new points, but we double-count points that are expressed both as $\theta_t - \phi$ and as θ_l for some t, l. This happens for $t, l : t = 2l \cdot 10^{\sigma}$, where σ is an integer, i.e., for

$$(t, l) \in \{(1, 5), (2, 1), (4, 2), (6, 3), (8, 4)\}.$$

Hence, we get only 4 new points with each rotation. The same reasoning works when we proceed with adding sets $\Phi - 2\phi, \Phi - 3\phi, \cdots$ to the set Φ.

Analogously, for primes $3, 5$ and 7, respectively we have 6, 4 and 8.

In the final stage of the proof we have to show why the results make the complexity function a linear function. According to the concept of finite difference method, since after finite rotation we reach a constant number of new points in each period, then it can be claimed that the complexity function of this word is in the form of a polynomial. In particular, because for all prime numbers, after first rotation, we reach a fixed value at each stage, these polynomials must be of the first degree, i.e., $c_{\mathbf{w}}(n) = an + b$. □

2.2. Close formulas.

Corollary 2.2. *For* $\mathbf{w}$ *derived from the leading digits of the powers of* $2, 3, 5,$ *and* 7 *the complexity function is* $c_{\mathbf{w}}(n) = 4n + 5$, $c_{\mathbf{w}}(n) = 6n + 3$, $c_{\mathbf{w}}(n) = 4n + 5$ *and* $c_{\mathbf{w}}(n) = 8n + 1$, *respectively.*

Proof. As with the previous theorem, we know that the complexity functions of all of them is in the form of $c_{\mathbf{w}}(n) = an + b$. So it is enough to determine the two parameters of a and b.

1) We first examine the value of a because we have almost done it in the previous theorem. If we put the numbers 1, 2, 3, etc. in this sequence, then we get the following sequence:

$$a + b, 2a + b, 3a + b, \cdots.$$

With the method of finite differences, we reach this sequence after one step: $a, a, a, \cdots$. Values of a was the same constant that we obtained after each first rotation in each rotation. Yields, a for primes $2, 3, 5$, and 7, respectively be equal to 4, 6, 4, and 8.

2)Now we look for value of b. We know the value of a, so it is enough to find the value of $c_{\mathbf{w}}(1)$; That is, the number of words in length 1. Obviously, $c_{\mathbf{w}}(1) = 9$ since $T^k(0)$ lies dense, hits every interval I_{d_k}. The density of this sequence is based on Kronker's theorem, as the angle of rotation $\log(\mathbf{p})$ for prime number $\mathbf{p}$ is an irrational.

$\square$

3. Conclusion

As a result, all the complexity functions of the first digit of one-digit prime numbers were found. For instance, 2^k, 3^k and etc. These problems in the field of word combinations fall into the realm of words with a function of low factor complexity. As the ultimate goal is to generalize this problem in different ways. As an example, factor complexity of 2^{n^3}.

Acknowledgement

We thank Andrei Raigorodsky and Ivan Mitrofanov for their valuable comments and suggestions.

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A nonlinear chattering-free sliding mode control for stabilization of fractional-order complex systems

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Abstract. In this paper, a nonlinear chattering-free sliding mode control method is designed to stabilize uncertain fractional chaotic systems. The main feature of this controller is rapid convergence to the point of equilibrium and minimize chattering and resistance against uncertainties. Moreover, in order to prove the stability of the controlled system based on direct method of Lyapunov theory is used. It is worth noticing that the proposed fractional-order sliding mode controller can be applied to control a broad range of fractional-order dynamical systems.

Keywords: fractional-order, sliding mode control, stabilization

AMS Mathematics Subject Classification [2010]: 26A33, 93D20

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1. Introduction

Historically, the emergence of fractional calculations has been consistent with the invention of the theory of integer calculus; But its applications and development are relate to recent decades [1]. In recent years, differential equations of fractional order (FO) have been used to represent and model more accurate real-world nonlinear systems. The control of these type of systems is one of the most important issues in the design of control systems. In addition to the high accuracy of modeling, the fractional equations, also have an interesting feature that linear fractional-order systems, in contrast to integer-order systems, can still remain stable despite the presence of poles on the right side of the complex plane [2].

Sliding mode control (SMC) is a nonlinear control strategy expressing considerable properties such as robustness, accuracy, simple implementation and immutability to uncertainties. Generally, SMC includes two steps

1. Designing an appropriate sliding surface.
2. Designing control input for the closed-loop system by the sliding surface.

In this paper, we introduce a suitable sliding surface and using a suitable controller, we prove the convergency of all the system state trajectories to the sliding surface. Finally

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illustrated examples and numerical simulations are presented to demonstrate the validity of the proposed method.

The rest of this paper is organized as follows. In Section 2, some preliminaries and problem formulation are given. In Section 3, the design procedure of the proposed FO sliding mode approach is presented. Section 4 gives illustrative example. Finally, concluding remarks are included in Section 5.

2. Preliminaries

Definition 2.1. The Riemann-Liouville fractional integration of order α is defined as follows [3]:

$$ {}_{t_0}I_t^{\alpha} f(t) = {}_{t_0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau \tag{1}$$

In which $\Gamma(.)$ is the Euler's gamma function.

Definition 2.2. The αth order Caputo fractional derivative of a continuous function $f(t) : R^+ \to R$ is defined as [3]:

$$ {t_0}^{C} D_t^{\alpha} f(t) = {}_{t_0}D_t^{-(m-\alpha)} \frac{d^m}{dt^m} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} f^{(m)}(\tau)(t-\tau)^{m-\alpha-1} d\tau \tag{2}$$

Because in practical applications, system dynamics is often influenced by model uncertainties and external disturbances [4], in this paper, we consider the following nonlinear fractional order system with model uncertainties and external disturbances:

$$\begin{cases} D^{\alpha}x_1(t) = f_1(t, x(t)) + \Delta f_1(t, x(t)) + d_1(t) + u_1(t) \\ D^{\alpha}x_2(t) = f_2(t, x(t)) + \Delta f_2(t, x(t)) + d_2(t) + u_2(t) \\ \vdots \\ D^{\alpha}x_n(t) = f_n(t, x(t)) + \Delta f_n(t, x(t)) + d_n(t) + u_n(t) \end{cases} \tag{3}$$

Where $\alpha \in (0, 1)$ is the order of system, $x(t) = [x_1, x_2, .., x_n]^T \in R^n$ denote the state vector, $f(t, x(t)) = [f_1(t, x(t)), f_2(t, x(t)), ..., f_n(t, x(t))]^T \in R^n$ is the given nonlinear function, $\Delta f(t, x(t)) = [\Delta f_1(t, x(t)), \Delta f_2(t, x(t)), ..., \Delta f_n(t, x(t))] \in R^n$ is the uncertainty term of system,
$d(t) = [d_1(t), d_2(t), ..., d_n(t)] \in R^n$ is the external disturbance term of the system, and $u(t) = [u_1(t), u_2(t), ..., u_n(t)] \in R^n$ is the control input.
Assumption 1: In this paper, the uncertainty term and external disturbance are considered bounded as follows :

$$ |\Delta f(t, x(t))| + |d(t)| \leq \rho \tag{4}$$

Where $\rho = [\rho_1, \rho_2, ..., \rho_n]$ is vector of positive constants.

3. Controller design

For the fractional system (3), we define the sliding surface as follows:

$$ S_i(t) = |x_i(t)| + \gamma_i x_i(t) + k_i D^{-\alpha} |x_i(t)|^{\mu} \tanh(x_i(t)) \tag{5}$$

Where $k_i > 0$, $\gamma_i > 1$, $0 < \mu < 1$.
Now, we design the control input as shown below and show that the system (3) with the designed control input converges to the sliding surface (5).

$$u_i(t) = -\left(f_i(t, x_i(t)) + \lambda_i \tanh(x_i(t)) + \frac{\tanh(x_i(t))(k_i |x_i(t)|^\mu + \beta_i |S_i(t)|^q)}{\gamma_i + sgn(x_i(t))} + \rho_i \right). \tag{6}$$

THEOREM 3.1. *Consider the fractional system (3) with the condition (4) and the sliding surface (5). If the system is controlled by the controller (6), then the system state trajectories converge to the zero asymptotically.*

Proof: We define the Lyapunov function as

$$V(t, x(t)) = \|S(t)\|_1 = \sum_{i=1}^{n} |S_i(t)| \tag{7}$$

Now by taking fractional derivation from both side of above equality, we have

$$D^\alpha V(t, x(t)) = D^\alpha \sum_{i=1}^{n} |S_i(t)| = \sum_{i=1}^{n} D^\alpha |S_i(t)| \tag{8}$$

Using Lyapunov stability theorem

(9)

$$D^\alpha V(t, x(t)) \leq \sum_{i=1}^{n} [sgn(S_i(t)) (sgn(x_i(t)) D^\alpha x_i(t) + \gamma_i D^\alpha x_i(t) + k_i |x_i(t)|^\mu \tanh(x_i(t)))]$$

$$\begin{aligned}
&= sgn(S_i(t)) (D^\alpha x_i(t) (\gamma_i + sgn(x_i(t))) + k_i |x_i(t)|^\mu \tanh(x_i(t))) && (10)\\
&= sgn(S_i(t)) [(f_i(t, x_i(t)) + \Delta f_i(t, x_i(t)) + d_i(t) + u_i(t) + k_i |x_i(t)|^\mu \tanh(x_i(t)))] && (11)\\
&\leq sgn(S_i(t))[(f_i(t, x_i(t)) + \rho_i - f_i(t, x_i(t)) - \lambda_i \tanh(x_i(t))b \\
&\quad - \frac{\tanh(x_i(t))(k_i |x_i(t)|^\mu + \beta_i |S_i(t)|^q)}{\gamma_i + sgn(x_i(t))} - \rho_i) + k_i |x_i(t)|^\mu \tanh(x_i(t))] && (12)\\
&= -\lambda_i \tanh(x_i(t)) (\gamma_i + sgn(x_i(t))) - \beta_i |S_i(t)|^q \tanh(S_i(t)) \\
&\quad - \lambda_i \tanh(x_i(t)) - \gamma_i . \lambda_i \tanh(x_i(t)) sgn(S_i(t)) && (13)\\
&= -\lambda_i \tanh(x_i(t)) (1 + \gamma_i sgn(S_i(t))) < 0 && (14)
\end{aligned}$$

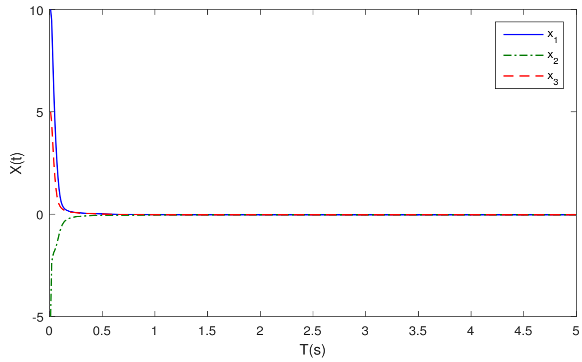
Thus the proof is finished.

4. Numerical simulation

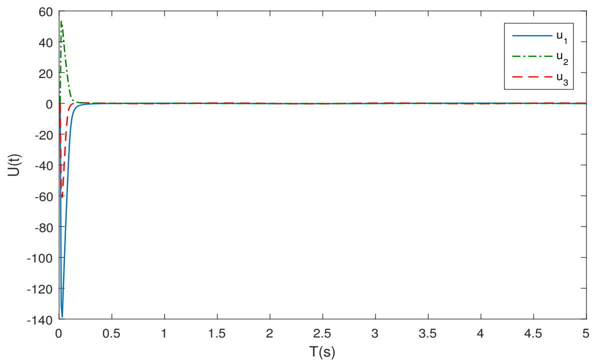
In this section, control of FO Brushless DC Motor system (BLDCM)system is presented to illustrate the effectiveness of the proposed control method.
Differential equations of the chaotic Brushless DC Motor system (BLDCM) are given as follows [5]:

$$\begin{cases} D^\alpha x_1(t) &= -0.875 x_1(t) + x_2(t) x_3(t) + u_1 \\ D^\alpha x_2(t) &= -x_2(t) + 55 x_3(t) - x_1(t) x_3(t) + u_2 \\ D^\alpha x_3(t) &= 4(x_2(t) - x_3(t)) + u_3 \end{cases} \tag{15}$$

This system for $0.96 < \alpha \leq 1$ is chaotic. So, we consider $\alpha = 0.98$ and the initial states $x_1(0)$, $x_2(0)$ and $x_3(0)$ to be 10, -5 and 5, respectively.



In the design of the sliding surface, the parameters are selected as $\mu = [0.7, 0.7, 0.7]$, $\gamma = [2, 2, 2]$ and $k = [0.1, 0.1, 0.1]$. Also, control parameters are selected as $\rho = [0.2\,,\, 0.35,\, 0.35]^T$, $\lambda = [1.5, 1.5, 1.5\,]^{\,T}$, $q = [0.6, 0.6, 0.6]$. The stability of the FOBLDCM system (15), using the proposed control input is shown in Fig.1. Also, Fig.2 shows the control signals. Clearly, the control efforts are feasible in practice, without damaging chattering actions.



5. Conclusion

In this paper, a nonlinear controller method is proposed to control of chaotic FOSs. The theoretical and analytical results in this paper are based on the direct method of the Lyapunov stability theory for fractional systems. Stability and high resistance against the external disturbances and uncertainties of the system and to minimize chattering are the main features of this method.

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Some results on Ricci-Harmonic Bourguignon solitons and applications

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Abstract. In this paper, we give some results for the almost Ricci-harmonic Bourguignon solitons which is a generalization of Ricci-harmonic solitons.
We also find some integral equations rely on analytic techniques for the compact gradient Ricci-harmonic Bourguignon almost solitons and by this we get rigidity result for a compact gradient Ricci-harmonic Bourguignon almost soliton.

Keywords: Ricci soliton, Gradient Ricci soliton.

AMS Mathematics Subject Classification [2010]: 53C25, 53C44

1. Introduction

Let (M, g) and (N, h) be complete Riemannian manifolds and $\varphi : M \longrightarrow N$ be a critical point the energy integral $E(\varphi) = \int_M |\nabla\varphi|_g^2 dv$, where N is isometrically embeded in $\mathbb{R}^d$, $d \geq n$. By a one parameter family of Riemannian metrics $(g(x,t), \varphi(x,t))$, $t \in [0, T)$ and a family of smooth maps $\varphi(x,t)$, we recall that the Ricci-harmonic Bourguignon flow defined as

$$
\begin{aligned}
\frac{\partial}{\partial t} g(x,t) &= -2\mathrm{Ric}(x,t) + 2\rho R(x,t) + 2\alpha\nabla\varphi(x,t) \otimes \nabla\varphi(x,t), \\
\frac{\partial}{\partial t} g(x,t) &= \tau_g \varphi(x,t).
\end{aligned} \tag{1}
$$

Here α is positive constant and $\tau_g\varphi$ is the intrinsic Laplacian of φ which denotes the tension field of map φ [6]. If in (1) we have $\alpha = 0$ then it defines Ricci-Bourguignon flow.

The Ricci-harmonic Bourguignon soliton. Let M denote a smooth n-dimensional manifold, g a Riemannian metric, and X a smooth vector field on M. Then the system $(M, g, X, \lambda, \rho, \varphi)$ is said to define a Ricci-harmonic Bourguignon soliton (RHBS for short) when it satisfies in the following coupled equation

$$
\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g + \rho R g + \alpha\nabla\varphi \otimes \nabla\varphi,
$$

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$$\tau_g\varphi - \mathcal{L}_X\nabla\varphi = 0, \tag{2}$$

where λ, α and ρ are constants, R is scalar curvature and φ is a smooth function $\varphi : (M, g) \to (N, h)$ where M and N are static Riemannian manifolds. Here are some special examples of solitons:
(1) If in first equation of (2) $\alpha = \rho = 0$ or $\rho = 0$ and φ is a constant map then M is Ricci soliton and equation becomes

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g.$$

(2) If $\alpha = 0$ or φ is a constant map then M is Ricci-Bourguignon soliton and equation becomes

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g + \rho R g.$$

(3) If $\rho = 0$ then M is Ricci-harmonic soliton and equation changes as

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g + \alpha\nabla\varphi \otimes \nabla\varphi.$$

Especially if λ be a smooth function then we call manifold in (2) as almost Ricci-harmonic Bourguignon soliton. We refer the reader to [2–4] for background on Ricci solitons and their connection to the Ricci flow.
In definition of RHBS if $X = \nabla f$, which f is a smooth function on M, then we say M is a gradient Ricci-harmonic Bourguignon soliton (GRHBS for short). In this case we have

$$\begin{aligned} \mathrm{Ric} + \mathrm{Hess} f - \rho R g - \alpha\nabla\varphi \otimes \nabla\varphi &= \lambda g \\ \tau_g\varphi - < \nabla\varphi, \nabla f > &= 0. \end{aligned} \tag{3}$$

The function f is called the potential. The GRHBS soliton is steady, expanding or shrinking if $\lambda = 0$, $\lambda < 0$ or $\lambda > 0$ respectively. Note that for all examples of soliton which we defined above we could define the gradient Ricci soliton. In this paper we denote $\mathrm{Ric} - \alpha\nabla\varphi \otimes \nabla\varphi$ by Sc, its components in local coordinates by $S_{ij} := R_{ij} - \alpha\nabla_i\varphi\nabla_j\varphi$ and the metric trace of S_{ij} by $S := R - \alpha|\nabla\varphi|^2$.
Some basic structural equations for compact Ricci and Ricci almost solitons were proved in [1] and [5]. Actually in [5], S. Dwivedi proved some integral formulas and showed that a compact gradient Ricci-Bourguignon soliton is isometric to an Euclidean sphere if it has constant scalar curvature or it associated vectore field be conformal. The aim of this paper is to generalize the results obtained in those papers for RHBS, GRHBS and almost RHBS and GRHBS. Motivated by this we prove some identities and integral formula for almost RHBS and GRHBS.

2. Main results

First we express important propositions and lemmas which is necessary for main theorems.

PROPOSITION 2.1. *Let $(M^n, g, f, \lambda, \rho, \varphi)$ be an almost GRHBS. Then the following equations hold*

$$(1 - n\rho)R + \Delta f = n\lambda + \alpha|\nabla\varphi|^2, \tag{4}$$

$$(1 - \rho(n-1))\nabla_i R = R_{il}\nabla_l f + (n-1)\nabla_i\lambda + \alpha\nabla_i|\nabla\varphi|^2 - \alpha\nabla_j\nabla_i\varphi\nabla_j\varphi, \tag{5}$$

$$\begin{aligned} \nabla_j R_{ik} - \nabla_k R_{ij} &= R_{jkil}\nabla_l f + \rho(\nabla_j R g_{ik} - \nabla_k R g_{ij}) + (\nabla_j\lambda g_{ik} - \nabla_k\lambda g_{ij}) \\ &\quad + \alpha(\nabla_j\nabla_i\varphi\nabla_k\varphi - \nabla_k\nabla_i\varphi\nabla_j\varphi), \end{aligned} \tag{6}$$

$$\nabla_i[(1-2\rho(n-1))R+|\nabla f|^2-2(n-1)\lambda-2\alpha|\nabla\varphi|^2+2\alpha\nabla_i\varphi\nabla_j\varphi] \\ =(2\lambda+2\rho R+2\alpha\nabla_i\varphi\nabla_j\varphi)\nabla_i f. \tag{7}$$

PROPOSITION 2.2. *For an almost GRHBS $(M^n,g,f,\lambda,\rho,\alpha)$ the following identities hold*

$$divSc=\frac{1}{2}\nabla S-\alpha\tau_g(\varphi)\nabla\varphi, \tag{8}$$

$$g(\nabla S,\nabla f)=2(n-1)g(\nabla\lambda,\nabla f)+2(n-1)\rho g(\nabla R,\nabla f)+2Sc(\nabla f,\nabla f), \tag{9}$$

$$-\frac{1}{2}\nabla S+(n-1)\nabla\lambda+Sc(\nabla f,.)-n\rho\nabla R=0, \tag{10}$$

$$\nabla(S+|\nabla f|^2)=2(n-1)[\nabla\lambda+\rho\nabla R]+2\lambda\nabla f+2\rho R\nabla f, \tag{11}$$

(12)
$$\frac{1}{2}\Delta|\nabla f|^2=|\mathrm{Hess}f|^2-(n-2)g(\nabla\lambda,\nabla f)-2(n-1)\rho g(\nabla R,\nabla f)-Sc(\nabla f,\nabla f)+\alpha|g(\nabla\varphi,\nabla f)|^2.$$

Depend on Lemma 2.3 in [5] we prove next Lemma.

LEMMA 2.3. *Let $(M^n,g,X,\lambda,\rho,\varphi)$ be an almost RHBS. Then*

$$\begin{aligned}\frac{(1-n\rho)}{2}\Delta|X|^2 &= (1-n\rho)|\nabla X|^2+(n\rho-1)\mathrm{Ric}(X,X)+n\rho\nabla_X divX\\ &\quad+2\rho(1-n\rho)g(\nabla R,X)-(n(2\rho+1)-2)g(\nabla\lambda,X)\\ &\quad+2\alpha(1-n\rho)g(\tau_g(\varphi)\nabla\varphi,X)-\alpha n\rho g(\nabla|\nabla\varphi|^2,X),\end{aligned} \tag{13}$$

and

$$\begin{aligned}\frac{(1-n\rho)}{2}(\Delta-\nabla_X)|X|^2 &= (1-n\rho)|\nabla X|^2+\lambda(n\rho-1)|X|^2\\ &\quad+\rho(n\rho-1)R|X|^2+\alpha(n\rho-1)|\nabla_X\varphi|^2\\ &\quad+n\rho\nabla_X divX+2\rho(1-n\rho)g(\nabla R,X)\\ &\quad-(n(2\rho+1)-2)g(\nabla\lambda,X)\\ &\quad+2\alpha(1-n\rho)g(\tau_g(\varphi)\nabla\varphi,X)-\alpha n\rho g(\nabla|\nabla\varphi|^2,X).\end{aligned} \tag{14}$$

For a compact almost GRHBS due to the equations (9), (11) and (12) and using identity $|\mathrm{Hess}f-\frac{1}{n}g\Delta f|^2=|\mathrm{Hess}f|^2-\frac{1}{n}(\Delta f)^2$ and integrating over compact M we get

THEOREM 2.4. *Let $(M^n,g,f,\lambda,\rho,\alpha)$ be a compact almost GRHBS. Then we have*

$$\begin{aligned}2\int_M|\mathrm{Hess}f-\frac{1}{n}g\Delta f|^2dv &= (\frac{n-2}{n}+2n\rho)\int_M<\nabla R,\nabla f>dv\\ &\quad-\frac{n-2}{n}\int_M\alpha<\nabla|\nabla\varphi|^2,\nabla f>dv\\ &\quad-2\alpha\int_M|<\nabla\varphi,\nabla f>|^2dv+\rho\int_M\Delta Rdv.\end{aligned} \tag{15}$$

By integrating of (13) on compact M we have

Theorem 2.5. *Let* $(M^n, g, X, \lambda, \rho, \varphi)$ *be a compact almost RHBS with* $n \geq 3$. *If* $\rho \neq \frac{1}{n}$ *and*

$$\int_M [\mathrm{Ric}(X,X) + \frac{n\rho}{n\rho - 1}\nabla_X divX - 2\rho g(\nabla R, X) - \frac{(n-(2\rho+1)-2)}{n\rho-1} g(\nabla\lambda, X) \\ -2\alpha g(\tau_g(\varphi)\nabla\varphi, X) - \frac{\alpha n\rho}{n\rho - 1} g(\nabla|\nabla\varphi|^2, X)]dv \leq 0$$

then X *is a Killing vector field and* M *is a trivial RHBS.*

With identities in proposition 2.2 we estabilish next two theorems.

Theorem 2.6. *Let* $(M^n, g, X, \lambda, \rho, \varphi)$ *be a compact almost GRHBS shuch that* $\Delta\lambda \leq 0$ *on* M. *We put* $S_{min} := \min_M S$, *then*
(1) if M *be steady with* $R = 0$, *then* $S_{min} = 0$,
(2) if M *be steady and* $R > 0$, *then* $0 \leq S_{min} \leq n\rho R$,
(3) if M *be steady and* $R < 0$, *then* $n\rho R \leq S_{min} < 0$.

The next result is the generalization of theorem 2.3 in [1] for Ricci-harmonic soliton to almost GRHBS.

Theorem 2.7. *Let* $(M^n, g, f, \lambda, \rho, \varphi)$ *be an almost GRHBS with* $n \geq 3$, *if*

$$\int_M (Sc(\nabla f, \nabla f) + (n-2)g(\nabla\lambda, \nabla f) + 2(n-1)\rho g(\nabla R, \nabla f)dv \leq 0,$$

then M *is travialiy rigid.*

3. Conclusion

We show that some identities and especially integral formulas for Ricci-harmonic ($(HR)_\alpha$ for short) and Ricci-Bourguignon (RB for short) solitons could be generalized for almost RHBS and GRHBS and in especiall case when $\nabla\varphi$ be constant we will conclude that if X be a conformal vector feild then for both compact and non-compact M, X could be a Killing vector feild.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Minimum-maximum programming with bipolar max-product fuzzy relation equation constraints

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Abstract. This paper studies the minimum-maximum programming subject to Bipolar Fuzzy Relation Equation (BFRE) constraints with the max-product composition. The characteristics of its feasible domain is expressed. It is shown that there exists an optimal solution for the problem such that each its component is either the corresponding component of the lower or upper bound vector on its nonempty feasible domain. We create a value matrix based on the useful property and apply a modified branch-and-bound method to find the optimal solution of the problem.

Keywords: Bipolar fuzzy relation equation, Min-max programming, Branch-and-bound method.

AMS Mathematics Subject Classification [2010]: 90Cxx, 90C70, 90C10

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1. Introduction

Fuzzy relation equations were first studied by Sanchez [5]. He determined the structure of its feasible solution set. Its complete solution set can be determined by a maximum solution and a finite number of minimal solutions. The fuzzy relation programming problem with a nonlinear objective function has been developing very slowly although many of the real world problems cannot be formulated in terms of the linear programming problems. We have to apply the genetic algorithm to obtain an approximate solution. To overcome the difficulties and finding exact optimal solution, some researchers focused on designing algorithms for resolution of special classes of the nonlinear optimization problems. The min-max programming problems provided to FRE constraints have been investigated by Zhou et al. [6]. We often need to variables with a bipolar characterization in some applications such as the covering and investing problem and the public awareness in revenue management. Freson et al. [2] formulated the system of bipolar max-min FREs, for the first time. The solution set of these equations can be characterized by a finite set of maximal and minimal solution pairs. Li and Jin [4] showed that checking the consistency of the system of bipolar max-min FREs is NP-complete. Therefore, the resolution of the linear

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optimization problem with constraints of bipolar FREs will be NP-hard. Some optimization problems subject to the BFRE were investigated using the max-Lukasiewicz [3] and max-product [1] with the negation of Lukasiewicz n_l.
As an extension of models [1,6] and its important in wireless communication and BitTorrent-like Peer-to-Peer file sharing system, we want to study the max-min programming problem with BFRE constraints:

$$\begin{aligned} \min\ Z(x) &= \bigvee_{j=1}^{n} (f_j(x_j)), \\ \text{s.t.}\ \ & A^+ \circ x \vee A^- \circ \neg x = b, \\ & x \in [0,1]^n, \end{aligned} \tag{1}$$

where $A^+ = (a_{ij}^+)$ and $A^- = (a_{ij}^-)$ are two $m \times n$ fuzzy relation matrices with $0 \le a_{ij}^+, a_{ij}^- \le 1$ for $i \in I = \{1, 2, \dots, m\}$ and $j \in J = \{1, 2, \dots, n\}$. Let $b = (b_1, \dots, b_m)^T \in [0,1]^m$ and $x = (x_1, \dots, x_n)^T \in [0,1]^n$ be the vector of decision variables to be determined and $\neg x$ denotes the negation of x, i.e., $\neg x = (1 - x_1, \dots, 1 - x_n)^T$. Also $f_j(x_j)$, for each $j \in J$, are one-variable increasing functions with respect to x_j. The operator of "$\circ$" represents the max-product composition operator. The notation of $S(A^+, A^-, b)$ is defined as $\{x \in [0,1]^n \mid A^+ \circ x \vee A^- \circ \neg x = b\}$ which consists of finding a set of solution vectors $x \in [0,1]^n$ such that

$$\max_{j \in J}\ \max\ \left\{a_{ij}^+ . x_j, a_{ij}^- . (1 - x_j)\right\} = b_i,\ \forall i \in I. \tag{2}$$

In the rest of paper, we investigate the structure of the feasible domain of the problem in Section 2. Section 3 presents an interesting property from its optimal solution. In Section 4, an algorithm is proposed to solve the problem based on a value matrix and the modified branch-and-bound method based on the property. Conclusions are finally given in Section 5.

2. Characteristics of feasible domain of problem (1)

System (2) is called consistent if its solution set is nonempty. Otherwise, it is inconsistent. The system (2) is equivalent to a set of vectors $x \in [0,1]^n$ satisfying the following conditions:
(i) $\forall i \in I, \forall j \in J:\ max\{a_{ij}^+ . x_j, a_{ij}^- . (1 - x_j)\} \le b_i$ and
(ii) $\forall i \in I, \exists j_i \in J:\ max\{a_{ij_i}^+ . x_{j_i}, a_{ij_i}^- . (1 - x_{j_i})\} = b_i$.
The lower and upper bound on the solution set of system (2) are obtained by the following lemma.

Lemma 2.1. *[1] Suppose that $S(A^+, A^-, b) \neq \emptyset$. The vector of $\check{x} = (\check{x}_1, \dots, \check{x}_n)^T$ is the lower bound on the solution set of equations (2) where $\check{x}_j = \max_{i \in I} \left\{1 - \frac{b_i}{a_{ij}^-} \mid a_{ij}^- > b_i\right\}$, for each $j \in J$. Also, the vector of $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)^T$ is the upper bound on the solution set of equations (2) where $\hat{x}_j = \min_{i \in I} \left\{\frac{b_i}{a_{ij}^+} \mid a_{ij}^+ > b_i\right\}$, for each $j \in J$. It is assumed that* $\max\ \emptyset = 0$ *and* $\min\ \emptyset = 1$ *are defined.*

If system (2) is consistent, then we have $\check{x} \le \hat{x}$. Moreover, If $x \in S(A^+, A^-, b)$, then $\check{x} \le x \le \hat{x}$, but its converse is not true. Without loss of generality, we can assume that

$\check{x}_j < \hat{x}_j$, for each $j \in J$, and $b_i > 0$, for each $i \in I$.

Definition 2.2. [1] Define two characteristic matrices $Q^+ = (q_{ij}^+)_{m\times n}$ and $Q^- = (q_{ij}^-)_{m\times n}$ such that for each $i \in I$ and $j \in J$,

$$q_{ij}^+ = \begin{cases} 1, & \text{if } a_{ij}^+.\hat{x}_j = b_i, \\ 0, & \text{otherwise,} \end{cases} \quad \textit{and } q_{ij}^- = \begin{cases} 1, & \text{if } a_{ij}^-.(1-\check{x}_j) = b_i, \\ 0, & \text{otherwise.} \end{cases}$$

Also, a series of index sets is defined as follows:

$$I_j^+(x) = \{i \in I \mid x_j = \hat{x}_j \textit{ and } q_{ij}^+ = 1\} \textit{ and } J_i^+(x) = \{j \in J \mid x_j = \hat{x}_j \textit{ and } q_{ij}^+ = 1\}.$$

$$I_j^-(x) = \{i \in I \mid x_j = \check{x}_j \textit{ and } q_{ij}^- = 1\} \textit{ and } J_i^-(x) = \{j \in J \mid x_j = \check{x}_j \textit{ and } q_{ij}^- = 1\},$$

for each $i \in I$ *and* $j \in J$. Furthermore, let $I_j(x) = I_j^+(x) \cup I_j^-(x)$, for each $j \in J$.
Let $I_j^+ = I_j^+(\hat{x})$, $J_i^+ = J_i^+(\hat{x})$, $I_j^- = I_j^-(\check{x})$, and $J_i^- = J_i^-(\check{x})$, for each $i \in I$ and $j \in J$.

The necessary and sufficient conditions for checking the consistency of system (2) is presented in the following theorem.

Theorem 2.3. *[3] A system of bipolar max-T_p FREs $A^+ \circ x \vee A^- \circ \neg x = b$ is consistent if and only if its characteristic boolean formula $C = \bigwedge_{i\in I} C_i$ is well-defined and satisfiable, where $C_i = \bigvee_{j\in J_i^+} y_j \vee \bigvee_{j\in J_i^-} \neg y_j$. for each $j \in J$, the value $\hat{x}_j$ is labeled with the positive literal y_j and the value $\check{x}_j$ is labeled with the negative literal $\neg y_j$, respectively.*

We are now ready to focus on the resolution procedure of problem (1) in the next section.

3. A procedure for solving problem (1)

First of all, we present an interesting property about the optimal solution of problem (1) which can help us to find its optimal solution.

Lemma 3.1. *Suppose that $S(A^+, A^-, b) \neq \emptyset$. Then there exists an optimal solution $x^* = (x_1^*, \ldots, x_n^*)^T$ for the problem (1) such that for each $j \in J$ either $x_j^* = \hat{x}_j$ or $x_j^* = \check{x}_j$.*

Proof. Suppose that x^{**} is an optimal solution for the problem (1) and there exists an index $k \in J$ such that $\check{x}_k < x_k^{**} < \hat{x}_k$ and assume that $x_j^{**} = \hat{x}_j$ or $x_j^{**} = \check{x}_j$, for each $j \in J \setminus \{k\}$. We create a feasible solution x^* with regard to x^{**}. Put $x_k^* = \check{x}_k$ and $x_j^* = x_j^{**}$, for each $j \in J \setminus \{k\}$. The vector x^* is a feasible solution for the bipolar system of (2). On the other hand, we have:

$$Z(x^*) = \left(\bigvee_{j\in J\setminus\{k\}} (f_j(x_j^{**}))\right) \vee (f_k(\check{x}_k)) \leq \left(\bigvee_{j\in J\setminus\{k\}} (f_j(x_j^{**})\right) \vee (f_k(x_k^{**})) = Z(x^{**}).$$

The inequality holds true since f_k is an increasing function and $x_k^{**} > \check{x}_k$. Therefore, there exists an optimal solution x^* such that for each $j \in J$ either $x_j^* = \hat{x}_j$ or $x_j^* = \check{x}_j$.

The value matrix of $M = (m_{ij})_{m\times 2n}$ can be defined based on the objective function of problem (1) and the characteristic matrices as follows:

$$(3)\qquad m_{i,2j-1} = \begin{cases} f_j(\hat{x}_j), & \text{if } q_{ij}^{+} = 1, \\ \infty, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad m_{i,2j} = \begin{cases} f_j(\check{x}_j), & \text{if } q_{ij}^{-} = 1, \\ \infty, & \text{otherwise,} \end{cases}$$

for each $i \in I$ and $j \in J$.

We will employ the branch-and-bound method on the value matrix M to solve problem (1). In this method, along branches from node 0 to node k, we cannot apply both $\hat{x}_j$ and $\check{x}_j$. The method is terminated if one of the following cases occurs: 1. reaching to the last row of matrix M, 2. all the selected variables along node 0 to node k together with other $\check{x}_j$'s create a feasible solution, and 3. lack of candidates to satisfy an equation. If the method terminates, then the remaining variables x_j are assigned the corresponding value of $\check{x}_j$. An algorithm is now proposed based on the above points.

Algorithm 1. An algorithm for resolution of the problem of (1).

Step 1. Compute the lower and upper bound of $\check{x}$ and $\hat{x}$ using Lemma 2.1.
Step 2. Create two characteristic matrices Q^{+} and Q^{-} using Definition 2.2.
Step 3. Compute the index sets of I_j^{+}, I_j^{-}, J_i^{+}, and J_i^{-}.
Step 4. Check the consistency of the bipolar system (2) applying Theorem 2.3. If it is inconsistent, then stop! Otherwise, go to Step 5.
Step 5. Create the value matrix of M using the relation (2.2).
Step 6. Employ the modified branch-and-bound method with the jump-tracking technique on the matrix of M to solve the optimization problem.
Step 7. Produce the optimal solution and the optimal objective value of the problem of (1).

4. Conclusion

In this paper, the min-max programming problem subject to bipolar max-product fuzzy relation equation constraints was studied. An important property was proposed about one of its optimal solutions. A value matrix was created based the property. The modified branch-and-bound was applied on the matrix to solve the problem.

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Faculty of Mathematics and Computer
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Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Asymptotic expansion of the number of derangements in terms of Bell numbers

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Abstract. In this paper we review our study of the difference $D_n - \frac{n!}{e}$, where D_n denotes the number of derangements on n objects. First, we consider some explicit formulas for D_n. Then, using an integral representation for this reference, we compute the moments of this difference, and we also get an asymptotic expansion for D_n with coefficients in terms of the Bell numbers B_n.

Keywords: derangement, permutation, Bell number, integration by parts

AMS Mathematics Subject Classification [2010]: 05A05, 05A16, 11B73

1. Introduction

A permutation of $S_n = \{1, 2, 3, \cdots, n\}$ that has no fixed points is a "derangement" of S_n. Let D_n denote the number of derangements of S_n. The problem of counting derangements was first considered by Pierre Raymond de Montmort in 1708, who solved the problem in 1713, as did Nicholas Bernoulli at about the same time by using the inclusion-exclusion principle. It is known that

$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

For the above information and some more details see [6, p. 702, entry "Derangement"]. Since $\sum_{i=0}^{\infty}(-1)^i/i! = 1/e$, a good approximation for D_n is $n!/e$. In this paper we review our study of the sharpness of this approximation, which eventually lead us to an asymptotic expansion for the number of derangements in terms of Bell numbers. We recall that [6, p. 178, entry "Bell Number"] the number of ways a set of n elements can be partitioned into nonempty subsets is called a Bell number and is denoted B_n.

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2. Explicit formulas for the number of derangements

The first observation on the difference of D_n and $n!/e$ asserts that their distance never exceeds 1/2. Indeed, it is known [6, p. 702] that

$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} = \left\| \frac{n!}{e} \right\|,$$

where $\|x\|$ denotes the nearest integer to the real number x, that is $\|x\| = \lfloor x + 1/2 \rfloor$. Thus,

$$D_n = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor.$$

The author [2, 4] proved that this floor-function relation also holds for each $n \geqslant 1$ if we replace 1/2 by any $\eta \in [1/3, 1/2]$. By taking $\eta = 1/e$ we obtain

$$D_n = \left\lfloor \frac{n! + 1}{e} \right\rfloor.$$

More precisely, we observe that for each $n \geqslant 2$,

$$\begin{aligned} \left| D_n - \frac{n!}{e} \right| &\leqslant \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots \\ &< \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} \cdots = \frac{1}{n}. \end{aligned}$$

Therefore, the difference of D_n and $n!/e$ tends to 0 as $n \to \infty$. Finally, we recall that the author [4, Corollary 2.2] proved the following explicit formula for each $n \geqslant 2$,

$$D_n = \lfloor (e + e^{-1}) n! \rfloor - \lfloor e n! \rfloor.$$

3. Moments of the difference of D_n and $n!/e$

Although the above observation shows that $|D_n - n!/e| < 1/n$ for each $n \geqslant 2$, we can obtain see even more about the difference $D_n - n!/e$ by considering its moments. Recently, the author [3, Theorem 2] proved that

$$\sum_{n=1}^{\infty} \left(D_n - \frac{n!}{e} \right) = -1 + \frac{1}{e} + \frac{Ei(2) - Ei(1)}{e^2} \approxeq -0.218114,$$

where Ei denotes the exponential integral function defined by the Cauchy principal value of the integral

$$Ei(x) = -\int_{-x}^{\infty} \frac{e^{-z}}{z}\, dz,$$

and

$$\sum_{n=1}^{\infty} \left(D_n - \frac{n!}{e} \right)^2 = -\frac{(e-1)^2}{e^2} + \frac{4}{e^2} \int_0^{\frac{1}{2}} h(z)\, dz \approxeq 0.433113,$$

where

$$h(z) = \frac{e^{2z}}{\sqrt{1-z^2}} \arctan \frac{z}{\sqrt{1-z^2}} + \frac{e^{2-2z}}{\sqrt{2z-z^2}} \arctan \frac{z}{\sqrt{2z-z^2}}.$$

Moreover, for each integer $k \geqslant 1$ the following multiple integral representation holds:

$$\sum_{n=1}^{\infty} \left(D_n - \frac{n!}{e} \right)^k = -\frac{(e-1)^k}{e^k} + \frac{1}{e^k} \int_0^1 \cdots \int_0^1 \frac{e^{x_1 + \cdots + x_k}}{1 - (-1)^k x_1 \cdots x_k}\, d\mathbf{X},$$

where $\mathbf{X}$ represents the k-tuple $(x_1, \ldots, x_k)$.

The key idea to prove the above moment results is the following representation of the difference $D_n - n!/e$ by an integration,

$$D_n = \frac{n!}{e} + (-1)^n \frac{L_n}{e},$$

for each integer $n \geqslant 1$, where

$$L_n = \int_1^e \log^n t \, dt.$$

4. Asymptotic expansion of the number of derangements in terms of Bell numbers

In the last integral, we apply the change of variable $z = \log t$, satisfying $t = e^z$ and $dt = e^z dz$. Accordingly,

$$L_n = \int_0^1 z^n e^z \, dz = \int_0^1 z^n \sum_{j=0}^{\infty} \frac{z^j}{j!} \, dz = \int_0^1 \sum_{j=0}^{\infty} \frac{z^{n+j}}{j!} \, dz.$$

Since the last sum converges uniformly for $0 \leqslant z \leqslant 1$, we may change the order of sum and integral. Therefore,

$$L_n = \sum_{j=0}^{\infty} \int_0^1 \frac{z^{n+j}}{j!} \, dz = \sum_{j=0}^{\infty} \frac{1}{j!(n+j+1)}.$$

Given any positive integer r, an easy computation shows that

$$\frac{1}{n+b} = \sum_{k=1}^{r} (-1)^{k-1} \frac{b^{k-1}}{n^k} + \frac{(-1)^r}{n+b} \left(\frac{b}{n}\right)^r,$$

for $n + b \neq 0$. If we take $b = j + 1$, then

$$\begin{aligned} L_n &= \sum_{j=0}^{\infty} \sum_{k=1}^{r} \frac{(-1)^{k-1}}{n^k} \frac{(j+1)^{k-1}}{j!} + (-1)^r \sum_{j=0}^{\infty} \frac{1}{j!(n+j+1)} \left(\frac{j+1}{n}\right)^r \\ &= \sum_{k=1}^{r} \frac{(-1)^{k-1}}{n^k} \sum_{j=0}^{\infty} \frac{(j+1)^{k-1}}{j!} + \frac{(-1)^r}{n^r} \sum_{j=0}^{\infty} \frac{(j+1)^r}{j!(n+j+1)}. \end{aligned}$$

Dobiński's formula [6, p. 178] states that the k-th Bell number B_k equals

$$B_k = \frac{1}{e} \sum_{j=0}^{\infty} \frac{j^k}{j!}.$$

On account of this formula, we have

$$\sum_{j=0}^{\infty} \frac{(j+1)^{k-1}}{j!} = \sum_{j=0}^{\infty} \frac{(j+1)^k}{(j+1)!} = \sum_{j=1}^{\infty} \frac{j^k}{j!} = \sum_{j=0}^{\infty} \frac{j^k}{j!} = e\, B_k,$$

and

$$\sum_{j=0}^{\infty} \frac{(j+1)^r}{j!(n+j+1)} \leqslant \frac{1}{n} \sum_{j=0}^{\infty} \frac{(j+1)^r}{j!} = \frac{1}{n} \sum_{j=0}^{\infty} \frac{(j+1)^{r+1}}{(j+1)!} = \frac{e\, B_{r+1}}{n}.$$

Therefore, given any positive integer r, for any integer $n \geqslant 1$ we obtain the asymptotic expansion

$$D_n = \frac{n!}{e} + \sum_{k=1}^{r} (-1)^{n+k-1} \frac{B_k}{n^k} + O\left(\frac{1}{n^{r+1}}\right),$$

where B_k denotes the k-th Bell number and the constant of O-term does not exceed B_{r+1} in both expansions.

5. Conclusion

In this paper we give an example of the power of analytic methods in studying combinatorial and counting problems. Analytic methods usually ends in some very surprising results, which are far from elementary counting methods. The content of the present paper gives an example of "how we can count by integration!"

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On Connes-amenability of a class of dual quotient Banach algebras

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Abstract. Let $\mathfrak{A}$ be a Banach algebra with the multiplier algebra $M(\mathfrak{A})$. It is known that, for a closed submodule Z of $\mathfrak{A}^*$, the quotient space $\mathfrak{A}^{**}/Z^{\perp}$ with the product induced by the first Arens product is a dual Banach algebra if and only if $Z \subseteq WAP(\mathfrak{A}^*)$. When $M(\mathfrak{A})$ is a dual Banach algebra, under some conditioins, we show that amenability of $\mathfrak{A}$ is equivalent to Connes-amenability of $\mathfrak{A}^{**}/Z^{\perp}$, where Z is isometrically isomorphic to some predual of $M(\mathfrak{A})$.

Keywords: Amenability, Connes-amenability, weakly almost periodic functions.

AMS Mathematics Subject Classification [2010]: 46H20, 46H25, 47L10.

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1. Introduction

In [4], the Johnson's original defintion for general Banach algebras modified in the sense that it took the dual space structure of a von Neumann algebra into account. This leads to a notion of amenability which is called Connes-amenability. V. Runde extended the notion of Connes-amenability to the larger class of dual Banach algebras [5]. As a result, he proved that for a locally compact group G, the measure algebra $M(G)$ is Connes-amenable if and only if G is amenable [6].

For a Banach algebra $\mathfrak{A}$ the set of weakly almost periodic functions on $\mathfrak{A}$ is denoted by $WAP(\mathfrak{A}^*)$. It is known that, for a closed submodule Z of $\mathfrak{A}^*$, the quotient space $\mathfrak{A}^{**}/Z^{\perp}$ is a dual Banach algebra if and only $Z \subseteq WAP(\mathfrak{A}^*)$ [2] . So, in case that $\mathfrak{A}^{**}/Z^{\perp}$ is a dual Banach algebra, we are intrested in knowing the answer of this question: Is amenability of $\mathfrak{A}$ equivalent to Connes-amenabilty of such dual quotient Banach algebras for some closed submodules?

In case that the multiplier algebra is a dual Banach algebra and the closed summodule $Z \subseteq WAP(\mathfrak{A}^*)$ is isometrically isomorphic to some predual of multiplier algebra, we give a positive answer to this question.

In general, we note that the answer is not positive. Take $E := l^2 \hat{\otimes} l^2$; the dual of E is the space $B(l^2)$ of all bounded linear operators on l^2. $K(E)$ does not have bounded approximate identity [5, Corollary 3.1.5] and, so is not amenable. Take $Z = WAP(K(E)^*)$, since E is not reflexive, then $K(E)^{**}/Z^{\perp} = WAP(K(E)^*)^* = \{0\}$; which is Connes-amenable.

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2. Preliminaries

For a Banach algebra $\mathfrak{A}$ and a Banach $\mathfrak{A}$-bimodule E, a continuous linear map $D : \mathfrak{A} \to E$ such that

$$D(ab) = Da \cdot b + a \cdot Db \qquad (a, b \in \mathfrak{A})$$

is called a *derivation* from $\mathfrak{A}$ into E. The space of all derivations of $\mathfrak{A}$ into E is denoted by $\mathfrak{Z}^1(\mathfrak{A}, E)$. For each $x \in E$, the map $a \mapsto a \cdot x - x \cdot a$ is a derivation, and these maps form the space $\mathfrak{N}^1(\mathfrak{A}, E)$ of *inner* derivations.

The quotient space $\mathfrak{H}^1(\mathfrak{A}, E) = \mathfrak{Z}^1(\mathfrak{A}, E)/\mathfrak{N}^1(\mathfrak{A}, E)$ is the first cohomology group of $\mathfrak{A}$ with coefficients in E and $\mathfrak{A}$ is called *amenable* if $\mathfrak{H}^1(\mathfrak{A}, E^*) = \{0\}$, for every Banach $\mathfrak{A}$-bimodule E, E^* is a dual Banach $\mathfrak{A}$-bimodule via the actions:

$$\langle x, a \cdot x^* \rangle = \langle x \cdot a, x^* \rangle, \quad \langle x, x^* \cdot a \rangle = \langle a \cdot x, x^* \rangle \qquad (a \in \mathfrak{A},\ x \in E,\ x^* \in E^*).$$

A Banach algebra $\mathfrak{A}$ is called a *dual Banach algebra* if it is dual as a Banach $\mathfrak{A}$-bimodule. One can see that a Banach algebra which is also a dual space is a dual Banach algebra if and only if the multiplication map is separately w^*-continuous [5].

The algebra $B(E) = (E\hat{\otimes}E^*)^*$ of all bounded operators on a reflexive Banach space E, von Neumann algebras, the measure algebra $M(G) = C_0(G)^*$ are all examples of dual Banach algebras.

For a Banach algebra $\mathfrak{A}$, let $\mathcal{L}^2(\mathfrak{A}; \mathbb{C})$ stand for the space of all bounded bilinear maps on $\mathfrak{A} \times \mathfrak{A}$. A dual Banach $\mathfrak{A}$-bimodule E is called *normal* if for each $x \in E$, the maps $a \mapsto a \cdot x$ and $b \mapsto x \cdot b$ from $\mathfrak{A}$ into E are w^*-continuous, and *Connes-amenable* if for every normal dual Banach $\mathfrak{A}$-bimodule E, every w^*-continuous derivation $D : \mathfrak{A} \to E$ is inner [5].

For a dual Banach algebra $\mathfrak{A}$ with preual $\mathfrak{A}_*$, let $\Delta_{\mathfrak{A}} : \mathfrak{A}\hat{\otimes}\mathfrak{A} \to \mathfrak{A}$ be the diagonal operator induced by $a \otimes b \mapsto ab$, $a, b \in \mathfrak{A}$. The multiplication in $\mathfrak{A}$ is separately w^*-continuous, so $\Delta^*_{\mathfrak{A}}(\mathfrak{A}_*) \subset \mathcal{L}^2_{w^*}(\mathfrak{A}; \mathbb{C})$. We denote $\mathcal{L}^2_{w^*}(\mathfrak{A}; \mathbb{C})$ for the set of all w^*-continuous bilinear maps from $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ into $\mathbb{C}$. Taking the adjoint of $\Delta^*_{\mathfrak{A}}|_{\mathfrak{A}_*}$, we may extend $\Delta_{\mathfrak{A}}$ to a $\mathfrak{A}$-bimodule homomorphism Δ_{w^*} on $\mathcal{L}^2_{w^*}(\mathfrak{A}; \mathbb{C})^*$. An element $M \in \mathcal{L}^2_{w^*}(\mathfrak{A}; \mathbb{C})^*$ is called a *normal, virtual diagonal* for $\mathfrak{A}$ if

$$a \cdot M = M \cdot a, \qquad a\Delta_{w^*}M = a \qquad (a \in \mathfrak{A}).$$

Let $\mathfrak{A}$ be an algebra, the *multiplier algebra* of $\mathfrak{A}$ is denoted by $M(\mathfrak{A})$ and consists of all pairs (L, R) where L and R are mappings from $\mathfrak{A}$ into $\mathfrak{A}$ with the following property:

$$L(a)b = aR(b) \qquad (a, b \in \mathfrak{A}).$$

Let $\mathfrak{A}$ be a Banach algebra, for each $\mu \in \mathfrak{A}^*$, the bounded linear maps L_μ and R_μ from $\mathfrak{A}$ into $\mathfrak{A}^*$ are defined by

$$L_\mu(a) = \mu \cdot a, \quad R_\mu(a) = a \cdot \mu \quad (a \in \mathfrak{A}).$$

An element $\mu \in \mathfrak{A}^*$ belongs to $WAP(\mathfrak{A}^*)$ if and only if L_μ or equivalently R_μ is weakly compact operator from $\mathfrak{A}$ into $\mathfrak{A}^*$. Let $\Box$ and $\Diamond$ stand for the first and second Arens product respectively; by Daws [1, Lemma 2.3],

$$WAP(\mathfrak{A}^*) = \{\mu \in \mathfrak{A}^* : \ \langle \Phi\Box\Psi, \mu \rangle = \langle \Phi\Diamond\Psi, \mu \rangle \qquad \Phi, \Psi \in \mathfrak{A}^{**}\}.$$

Let X be a Banach space; for a subspace Y of X, and a subspace Z of X^*, we define $Y^\perp = \{\mu \in X^* : \langle \mu, y \rangle = 0 \quad y \in Y\}$ and ${}^\perp Z = \{x \in X : \langle \mu, x \rangle = 0 \quad \mu \in Z\}$. It is well known that if Y is closed, Y^* is isometrically isomorphic to $X^*/Y^\perp$, while $(X/Y)^*$ is isometrically isomorphic to $Y^\perp$.

3. Main results

The following theorem is due to Daws [1, Proposition 2.4].

THEOREM 3.1. *Let $\mathfrak{A}$ be a Banach algebra, and let $Z \subseteq \mathfrak{A}^*$ be a closed submodule. Then the followings are equivalent:*

i) *the first Arens products drops to a well-defined product on $Z^* = \mathfrak{A}^{**}/X^{\perp}$ turning (Z^*, Z) into a dual Banach algebra;*

ii) *$Z \subseteq WAP(\mathfrak{A}^*)$.*

Let $\mathfrak{A}$ be a Banach algebra with a contractive approximate identity such that its multiplier algebra $M(\mathfrak{A})$ is a dual Banach algebra. We denote its predual with $M(\mathfrak{A})_*$ and recall that this space is a closed submodule of $M(\mathfrak{A})^*$. Let $\iota : \mathfrak{A} \to M(\mathfrak{A})$ be the canonical embeding map, then ι induces a map $\iota_* : M(\mathfrak{A})_* \to \mathfrak{A}^*$ with

$$\langle \iota_*(\mu), a\rangle = \langle \mu, \iota(a)\rangle \qquad (a \in \mathfrak{A}, \mu \in M(\mathfrak{A})_*)$$

A net (τ_α) in $M(\mathfrak{A})$ converges to τ in wst-topology if $\langle \varphi, (\tau_\alpha - \tau)b\rangle \to 0$ for each $\varphi \in \mathfrak{A}^*,\ b \in \mathfrak{A}$.

THEOREM 3.2. *Let $\mathfrak{A}$ be a Banach algebra with a contractive approximate identity and let $M(\mathfrak{A})$ be a dual Banach algebra with predual $M(\mathfrak{A})_*$. Also let the identity map on $M(\mathfrak{A})$ be $w^* - wst$-continuous. Then, for $Z = \overline{\iota_*(M(\mathfrak{A})_*)}$, the followings are equivalent:*

i) *$\mathfrak{A}$ is amenable;*

ii) *$\mathfrak{A}^{**}/Z^{\perp}$ is Connes-amenable;*

iii) *$\mathfrak{A}^{**}/Z^{\perp}$ has a normal, virtual diagonal.*

PROOF. One can see that $\iota_*(M(\mathfrak{A})_*)) \subseteq \iota_*(WAP(M(\mathfrak{A})^*)) \subseteq WAP(\mathfrak{A}^*)$. Then $\iota_*^* : WAP(\mathfrak{A}^*)^* \to M(\mathfrak{A})$ is a homomorphism which extends ι.

Define $\eta : (\mathfrak{A}\hat{\otimes}M(\mathfrak{A})_*) \oplus_1 (\mathfrak{A}\hat{\otimes}M(\mathfrak{A})_*) \to WAP(\mathfrak{A}^*)$ by

$$\eta((a \otimes \mu) \oplus (b \otimes \lambda)) = a \cdot \iota_*(\mu) + \iota_*(\lambda) \cdot b \qquad (a, b \in \mathfrak{A},\ \mu, \lambda \in M(\mathfrak{A})_*)$$

and linearity and continuity. It is easy to see that η is a contraction.

Let

$$X = span\{(b \otimes \mu \cdot \iota(a)) \oplus (-a \otimes \iota(b) \cdot \mu) :\ a, b \in \mathfrak{A},\ \mu \in M(\mathfrak{A})_*\}.$$

Then $X \subseteq (\mathfrak{A}\hat{\otimes}M(\mathfrak{A})_*) \oplus_1 (\mathfrak{A}\hat{\otimes}M(\mathfrak{A})_*)$ and $\eta(X) = \{0\}$; so η induces a map

$$\tilde{\eta} : Y = (\mathfrak{A}\hat{\otimes}M(\mathfrak{A})_*) \oplus_1 (\mathfrak{A}\hat{\otimes}M(\mathfrak{A})_*)/\bar{X} \to WAP(\mathfrak{A}^*).$$

One can see that $\overline{Im(\tilde{\eta})}$ is a closed submodule of $WAP(\mathfrak{A}^*)$, so as above $(\overline{Im(\tilde{\eta})})^*$ is a dual Banach algebra. Indeed, $Z = \overline{\iota_*(M(\mathfrak{A})_*)} = \overline{Im(\tilde{\eta})}$ and we are going to show that $\overline{Im(\tilde{\eta})}$ is isometric isomorph with $M(\mathfrak{A})_*$. To see, it suffices to prove that $\tilde{\eta}$ is an isometic isomorphism onto Z.

Following the same approach presented in [2], let $(e_\alpha)_\alpha$ be a contractive approximate identity of $\mathfrak{A}$ and let $\Phi_0 \in \mathfrak{A}^{**}$ be a w^*-accumulation point of it. Consider the map $\sigma : M(\mathfrak{A}) \to \mathfrak{A}^{**}$ with $(L, R) \mapsto L^{**}(\Phi_0)$. The map σ is a homomorphism for the second Arens product. Let $q : \mathfrak{A}^{**} \to \mathfrak{A}^{**}/WAP(\mathfrak{A}^*)^{\perp} = WAP(\mathfrak{A}^*)^*$ be the quotient map; q is a $\mathfrak{A}$-bimodule homomorphism. Take $\sigma_w = q \circ \sigma$, since q is a homomorphism for either Arens product, it follows that $\sigma_w : M(\mathfrak{A}) \to WAP(\mathfrak{A}^*)^*$ is a homomorphism.

Let $\sigma_Z : M(\mathfrak{A}) \to Z^*$ be the composition of the map σ_w and the quotient map $q_Z : WAP(\mathfrak{A}^*)^* \to Z^*$. As $\mathfrak{A}$ has a contractive approximate identity, σ_Z is a contraction.

By [2, Theorem 7.1] and its proof, for $a, b \in \mathfrak{A}$, $\mu, \lambda \in M(\mathfrak{A})_*$ and $(L, R) \in M(\mathfrak{A})$, we have

$$\begin{aligned}\langle \eta^* \circ \sigma_Z(L,R), (a \otimes \mu) \oplus (b \otimes \lambda)\rangle &= \langle L^{**}(\Phi_0), a \cdot \iota_*(\mu) + \iota_*(\lambda) \cdot b\rangle \\ &= \langle \iota_*(\mu), L(a)\rangle + \langle \imath_*(\lambda), R(a)\rangle \\ &= \langle (L,R), (a \otimes \mu) \oplus (b \otimes \lambda)\rangle.\end{aligned}$$

Hence $\eta^* \circ \sigma_Z : M(\mathfrak{A}) \to X^\perp$ is the canonical map, which is an isometric isomorphism. Therefore $\sigma_Z : M(\mathfrak{A}) \to Z^*$ must be an isometry, and we see that $\tilde{\eta}^*$ is an isometric isomorphism between the image of σ_Z and $X^\perp$.

It follows that $\tilde{\eta}$ is an isometry and hence an isometric isomorphism onto Z. Indeed, for $\tau \in (\mathfrak{A}\hat{\otimes} M(\mathfrak{A})_*) \oplus_1 (\mathfrak{A}\hat{\otimes} M(\mathfrak{A})_*)$, we can find $T \in X^\perp$ with $\| T \|= 1$ and $\langle T, \tau\rangle =\| \tau \|$, the norm in the quotient $(\mathfrak{A}\hat{\otimes} M(\mathfrak{A})_*) \oplus_1 (\mathfrak{A}\hat{\otimes} M(\mathfrak{A})_*)/\bar{X}$. Then, there exist $\Phi \in Z^*$ in the image of σ_Z with $\tilde{\eta}^*(\Phi) = T$ and $\| \Phi \|= 1$. Thus

$$\| \tau \|= \langle T, \tau\rangle = \langle \Phi, \tilde{\eta}(\tau)\rangle \leq\| \tilde{\eta}(\tau) \|\leq\| \tau \|,$$

that is, $\| \tilde{\eta}(\tau) \|=\| \tau \|$ for each τ and so $\tilde{\eta}$ is isometric. Hence $\tilde{\eta}^* : Z^* \to X^\perp$ is also an isometric isomorphism. We conclude that $\sigma_Z : M(\mathfrak{A}) \to Z^*$ is surjective, and is hence an isometric isomorphism. By [3, Theorem 3.5], the proof is established. □

Example 3.3. Let G be a compact group. Take $\mathfrak{A} = L^1(G)$, then $M(\mathfrak{A}) = M(G) = C(G)^*$ and hence $M(\mathfrak{A})_* = C(G)$. The map $\iota : \mathfrak{A} \to M(G)$ is the canonical embedding which takes $f \in L^1(G)$ to $f\lambda$, where λ is the left Haar measure on G. Also the identity map on $M(G)$ is w^*-wst-continuous [3, Example 3.6]. Since $\mathfrak{A} = L^1(G)$ is amenable, then $L^1(G)^{**}/C(G)^\perp$ is Connes-amenable and has a normal, virtual diagonal.

Question. *Does there exist any other closed submodule $Z \subseteq WAP(\mathfrak{A}^*)$ satisfying Theorem 3.2?*

4. Conclusion

This research has made an attempt to prove that, for a Banach algebra $\mathfrak{A}$, there exists a dual quotient Banach algebra $\mathfrak{A}^{**}/Z^\perp$ such that amenability of $\mathfrak{A}$, Connes-amenability of $\mathfrak{A}^{**}/Z^\perp$ and the existence of a normal, virtual diagonal for $\mathfrak{A}^{**}/Z^\perp$ are all equivalent.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

The Muntz-Galerkin method for numerical solution of the generalized Abel-integral equation

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Abstract. In this work, we propose an efficient numerical method based on the Muntz-Legendre polynomial for solving the generalized Abel-integral equations. This equation is very important in several branches of Physics, engineering and mathematics. A lot of works have been done to study this equation. We show that the presented method is very capable and accurate for solving these kinds of problems. Also, the superiority and effectiveness of the Muntz-Galerkin method will be shown through an example.

Keywords: Generalized Abel-integral equations, Muntz-Legendre polynomial, Muntz-Galerkin method

AMS Mathematics Subject Classification [2010]: 41E10, 49K20, 65R20

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1. Introduction

The generalized Abel-integral equation in the following form

$$f(x) = \int_0^x \frac{K(x,t)}{(x^2-t^2)^\alpha} G\big(t,u(t)\big)dt, \tag{1}$$

where $f(x)$ is an arbitrary continues function, $K(x,t)$ is the Kernel of integral and $G(t,u(t))$ is a linear or nonlinear function will be studied in this work. Also, $u(x)$ is an unknown function. This equation will be study in the domain $I=[0,1]$. The existence and uniqueness of the solution for this equation on the interval I was studied in [6]. The author showed that under the following assumptions, these important properties are guarantied:

- $f \in C(I)$,
- $K \in C(D)$ where $D = \{(x,t) \mid 0 \le x \le t < 1\}$ and $K(x,t) \neq 0$,
- for all $u_1, u_2 \in C(I)$, there exists nonnegative real number c such that

$$\mid G(t,u_1(t)) - G(t,u_2(t)) \mid \le c \mid u_1(t) - u_2(t) \mid \text{ on } I.$$

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The Abel-integral equation was introduced by Niels Henrik Abel, norwegian mathematician, in 1823. These types of equations are appeared and used in many branches of sciences and engineering, such as radio astronomy, radar ranging, plasma diagnostic, X-ray radiography and so on. In this work, the Muntz-Legendre polynomials will be used to investigate the numerical solution of this equation. There are many analytical and numerical methods for solving equation (1) such as orthogonal polynomials [3], Laplace transform [1], Fourier transform [2], Hilbert transform [2], Tau method [4] and ....

1.1. **Definition** For all $m = 0, 1, 2, ...,$ function

$$L_m(x) = \frac{1}{2\pi i}\int_I \prod_{k=0}^{m-1} \frac{t+\xi_k+1}{t-\xi_k}\frac{x^t}{t-\xi_m}dt, \tag{2}$$

is the m-th Muntz-Legendre polynomials, where I have all the zeros of denominator of the integrand, $0 = \xi_0 < \xi_1 < \xi_2 < ... < \xi_m < ...$ with $\lim \xi_m = +\infty$ and $\Sigma_{m=1}^{+\infty} \gamma_m u^m(t) = +\infty$.

These polynomials are orthogonal with the weight function $w(x) = 1$, i.e.,

$$\int_I L_\imath(x)L_\jmath(x)dx = \frac{\delta_{\imath,\jmath}}{2\xi_\imath+1}, \qquad \imath \geq \jmath, \tag{3}$$

where $\delta_{\imath,\jmath}$ is the delta Kronecker function. One can see that the Muntz-Legendre polynomials can be written as follows [5]

$$L_m(x) = \sum_{k=0}^{m} \rho_{k,m}x^{\xi_k}, \tag{4}$$

where

$$\rho_{k,m} = \frac{\prod_{\jmath=0}^{m-1}(\xi_k+\xi_\jmath+1)}{\prod_{\jmath=0,\jmath\neq k}^{m}(\xi_k-\xi_\jmath)}, \qquad m = 0, 1, 2, ... \ . \tag{5}$$

Therefore, these Muntz-Legendre polynomials are in the Muntz space generated by $Span\{1, x^{\xi_1}, x^{\xi_2}, ...x^{\xi_m}\}$. Moreover, the Muntz-Legendre polynomials are constructed by the following recurrence relation [5].

$$L_m(x) = L_{m-1}(x)-(\xi_k+\xi_j+1)x^{\xi_m}\int_x^1 t^{-\xi_m-1}L_{m-1}(t)dt, \quad x \in (0,1], \ m = 0, 1, 2, ... \ . \tag{6}$$

2. Method of solution

Let $\xi_\imath = 2\imath$, $\imath = 0, 1, 2, ...$ and assume

$$u_N(x) = \sum_{\imath=0}^{N} u_\imath L_\imath(x) = \underline{u}L\underline{X}, \tag{7}$$

where $\underline{u} = \begin{bmatrix} u_0 & u_1 & u_2 & \dots & u_N \end{bmatrix}$ is a vector of unknowns, L is the Muntz-Legendre polynomial and $\underline{X} = \begin{bmatrix} 1 & x^{\xi_1} & x^{\xi_2} & \dots & x^{\xi_N} \end{bmatrix}^T$ is the base elements. The functions $f(x)$ and $G\big(t, u(t)\big)$ will be approximated as follows $f(x) \approx \Sigma_{\imath=0}^{N} f_\imath x^{\xi_\imath} = \underline{f}\underline{X}$, where $\underline{f} = \begin{bmatrix} f_0 & f_1 & f_2 & \dots & f_N \end{bmatrix}^T$, and, $G\big(t, u(t)\big) \approx \Sigma_{\ell=0}^{N} z_\ell(t)u^\ell(t)$. Also, we set $K(x,t)z_\ell(t) =$

$\sum_{\imath=0}^{+\infty}\sum_{\imath=0}^{+\infty} k_{\imath,\jmath}^{\ell}x^{\imath}t^{\jmath}, \quad \forall \ell \epsilon \mathbb{N}.$
Substituting the above relations in (1), results in

$$\underline{f}\underline{X} = \int_0^x \frac{K(x,t)z_0(t)}{(x^2-t^2)^{\alpha}}dt + \sum_{\imath=0}^{+\infty}\sum_{\jmath=0}^{+\infty}\sum_{\ell=1}^{N}\int_0^x \frac{k_{\imath,\jmath}^{\ell}x^{\imath}t^{\jmath}}{(x^2-t^2)^{\alpha}}u_N^{\ell}dt. \tag{8}$$

Using the following formulae

$$u_N^{\ell}(x) = \underline{u}LQ^{\ell-1}\underline{X}, \quad \forall \ell \epsilon \mathbb{N}, \tag{9}$$

$$\int_0^x \frac{t^{\jmath+2n}}{(x^2-t^2)^{\alpha}}dt = \frac{\Gamma(1-\alpha)\Gamma\big(n+\frac{1+\jmath}{2}\big)}{2\Gamma\big(n+\frac{\jmath+3}{2}-\alpha\big)}x^{2n+\jmath+1-2\alpha}, \tag{10}$$

equation (8) can be rewritten as

$$\underline{u}L\underline{Q}\underline{X} = \big(\underline{f} - \underline{z_0}A_0\big)\underline{X}, \tag{11}$$

where $\underline{z_0} = \begin{bmatrix} z_0^0 & z_0^1 & z_0^2 & \dots & z_0^N \end{bmatrix}$, $\underline{Q} = \sum_{\ell=1}^{N} Q^{\ell-1}A_{\ell}$,

$$[A_{\ell}]_{\alpha\alpha} = \sum_{\imath+\jmath=\alpha} k_{\imath\jmath}^{\ell}A_{\jmath}^{\alpha}, \quad \alpha = 0,1,2,..., \qquad A_{\jmath}^{n} = \frac{\Gamma(1-\alpha)\Gamma\big(n+\frac{1+\jmath}{2}\big)}{2\Gamma\big(n+\frac{\jmath+3}{2}-\alpha\big)}, \tag{12}$$

$$Q = \begin{bmatrix} \underline{\underline{u_0}} & \underline{\underline{u_1}} & \underline{\underline{u_2}} & \cdots & \underline{\underline{u_N}} \\ & \underline{\underline{u_0}} & \underline{\underline{u_1}} & \cdots & \underline{\underline{u_{N-1}}} \\ & & \underline{\underline{u_0}} & \cdots & \underline{\underline{u_{N-2}}} \\ & O & & \ddots & \vdots \\ & & & & \underline{\underline{u_0}} \end{bmatrix}, \qquad \underline{\underline{u_{\imath}}} = \underline{u}L_{\imath}, \quad \imath = 0,1,2,.... \tag{13}$$

In particular case $G\big(t,u(t)\big) = u^{\ell}(t)$, system (11) will be as follows

$$\underline{u}LQ^{\ell-1}A_{\ell} = \underline{f}. \tag{14}$$

In the case of $K(x,t) = \ell = 1$, this system is reduced to

$$\underline{u}LA = \underline{f}, \tag{15}$$

in which

$$A = diag\left(\pi, \frac{\pi}{2}, \frac{3\pi}{8}, ..., \frac{\sqrt{\pi}\Gamma\big(N+\frac{1}{2}\big)}{\Gamma(N+1)}\right).$$

The matrices L and A are nonsingular, therefore, equation (14) has a unique solution $\underline{u} = \underline{f}(LA)^{-1}$. Substituting the obtained vector $\underline{u}$ in (15), an approximate solution $u_N(x)$ for the equation (1) will be determined.

3. Numerical experiments

In this section we implement our method on an example. All the experiments are done by using **Mathematica 12.2** on a personal computer with operational Windows 10.

3.1. **Example** We consider the following linear first kind generalized Abel-integral equation [1]

$$x^2 = \int_0^x \frac{u(t)}{\sqrt{x^2-t^2}}dt, \tag{16}$$

with the exact solution $u(x) = \frac{4}{\pi}x^2$.

The numerical results by using the Muntz Galerkin method are shown in Table 1 and Figure 1.

TABLE 1. the results of the Muntz-Galerkin method for example 3.1.

N	Numerical errors
1	0.56941
≥ 2	0.0

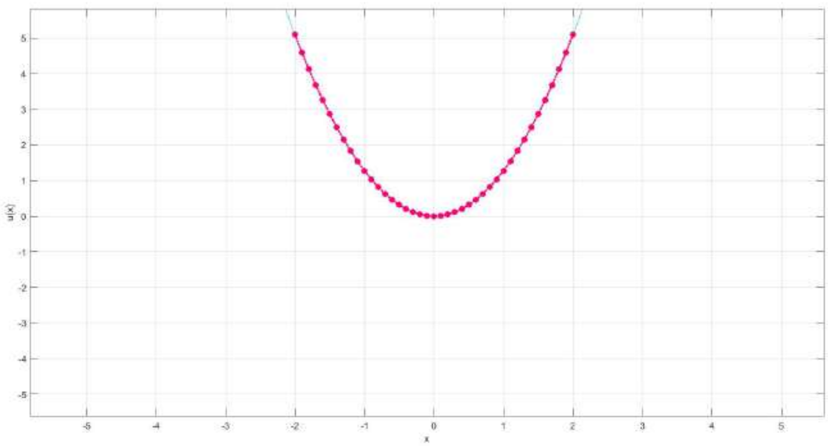

FIGURE 1. The solid line indicates the exact solution and dashes line indicates the numerical solution for Example 3.1 by the Muntz-Galerkin method.

4. Conclusion

In this presented work, an efficient numerical algorithm based on the Muntz-Legendre polynomials for solving the generalized Abel-integral equation was introduced. In this scheme the Muntz-Galerkin method was used to solve this equation. Also, we investigated the capability and accuracy of the method by an example. The obtained results showed that our presented algorithm was efficient with a good accuracy to handle this kind of equation.

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پنجاه و دومین کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

When $q(X)$ is a PF-ring

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Abstract. We examine the situations in which every maximal ideal of the classical ring of quotients $q(X)$ of the ring of real-valued continuous functions on a Tychonoff space X contains a unique minimal prime ideal. In other words, we present some topological and algebraic conditions for which $q(X)$ is a PF-ring.

Keywords: classical rings of quotients, PF-rings, rings of continuous functions.

AMS Mathematics Subject Classification [2010]: 54C40

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1. Introduction

In this article, rings are considered to be commutative unitary rings and topological spaces are assumed to be *completely regular Hausdorff* (Tychonoff) spaces. We denote by $C(X)$ the ring of all real-valued continuous functions on a space X. For each $f \in C(X)$ the *zero-set* $Z(f)$ is the set of zeros of f and its complement, denoted by $\operatorname{coz} f$, is called *cozero-set* of f. An element r of a ring R is called *regular* (non-zerodivisor) if $ra = 0$, $a \in R$ implies that $a = 0$. An ideal of a ring is said to be *regular* if it contains a regular element. We denote by $r(X)$ the set of all regular elements of $C(X)$ and it is easy to see that $f \in C(X)$ is regular if and only if $\operatorname{int}_X Z(f) = \emptyset$ or equivalently, $\operatorname{coz} f$ is dense in X. The classical ring of quotients $q(R)$ of a ring R is the ring of all equivalence classes of formal fractions $\frac{a}{r}$, for $a, r \in R$, where r is regular and the equivalence relation is the obvious one. $q(C(X))$ is denoted by $q(X)$ for simplicity.

A ring R is called a PF-ring (or PIF-ring by Matlis [7]) if every principal ideal of R is a flat R-module. It is known that a commutative ring R is a PF-ring if and only if R is reduced and every maximal ideal of R contains only one minimal prime ideal; see [7, Proposition 2.1]. Thus, a reduced ring R is a PF-ring if and only if for every prime ideal $P \subseteq R$ the ideal $\mathcal{O}(P)$ is prime, where $\mathcal{O}(P)$ is the intersection of all minimal prime ideals of R contained in P; see [4, Theorem 8]. We also see in the same paper that $\mathcal{O}(P) = \{a \in R : ab = 0 \text{ for some } b \notin P\}$. In case $R = C(X)$, Theorem 7.3 in [3] shows that every maximal ideal is precisely of the form $M^p = \{f \in C(X) : p \in \operatorname{cl}_{\beta X} Z(f)\}$, for some $p \in \beta X$, where βX is the *Stone-Čech compactification* of X. Using Part 7.12 of the same reference, for every $p \in \beta X$, $\mathcal{O}(M^p)$, or briefly O^p is of the form $\{f \in C(X) : p \in \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z(f)\}$. Therefore, $C(X)$ is a PF-ring if and only if for every $p \in \beta X$ the ideal O^p is prime which

*Speaker. Email address: a.r.salehi@put.ac.ir implies that X is an F-space, i.e., every finitely generated ideal in $C(X)$ is principal or equivalently any two disjoint cozero-sets of X are completely separated; see [3, Theorem 14.25] for more equivalent conditions.

In literature, there are several equivalent conditions for a ring to be a PF-ring; for example, see [1, 4, 6, 7]. In this article, using the structure of $q(X)$, we characterize topological spaces for which $q(X)$ is a PF-ring. Also, we give some algebraic equivalent conditions on $q(X)$ to be a PF-ring. To do this, We need the following result.

Lemma 1.1. *([6, Proposition 2.11]) Let R be a ring. Then $q(R)$ is a PF-ring if and only if for any two elements $a, b \in R$ with $ab = 0$, the ideal $Ann_R(a) + Ann_R(b)$ is regular.*

2. Main results

To prove the main result of this section we need the following lemmas.

Lemma 2.1. *The following statements are equivalent for f, g in $C(X)$.*

1. *The ideal $(\frac{f}{1}, \frac{g}{1})$ in $q(X)$ is principal.*
2. *For every $r, s \in r(X)$ the ideal $(\frac{f}{r}, \frac{g}{s})$ in $q(X)$ is principal.*
3. *There exists $p \in r(X)$ such that the ideal $(f|_{coz p}, g|_{coz p})$ is principal in $C(coz p)$.*

Proof. The equivalence of (1) and (2) is evident.

(1)⇒(3). There exsist $k_1, ..., k_4, h \in C(X)$ and $t_1, ..., t_4 \in r(X)$ such that $\frac{f}{1} = \frac{k_1}{t_1}\frac{h}{1}$, $\frac{g}{1} = \frac{k_2}{t_2}\frac{h}{1}$ and $\frac{h}{1} = \frac{k_3}{t_3}\frac{f}{1} + \frac{k_4}{t_4}\frac{g}{1}$. If we take $p = t_1t_2t_3t_4$, then using these equivalities on $\operatorname{coz} p$, we conclude that the ideal generated by the contraction of f and g on $\operatorname{coz} p$ is principal in $C(\operatorname{coz} p)$.

(3)⇒(1). Using the assumption, there exist $k_1, ..., k_4$ and h in $C(\operatorname{coz} p)$ such that $f = k_1h, g = k_2h$ and $h = k_3f + k_4g$ on $\operatorname{coz} p$. If for every $i = 1, ..., 4$, we take

$$\hat{k}_i(x) = \begin{cases} \frac{k_i}{1+|k_i|}(x)p(x) & x \in \operatorname{coz} p \\ 0 & x \in Z(p) \end{cases}$$

and

$$\bar{k}_i(x) = \begin{cases} \frac{1}{1+|k_i|}(x)p(x) & x \in \operatorname{coz} p \\ 0 & x \in Z(p), \end{cases}$$

then for every $i = 1, ..., 4$ we have $\bar{k}_i, \hat{k}_i \in C(X)$ and $\bar{k}_i \in r(X)$ as $\operatorname{coz} \bar{k}_i = \operatorname{coz} p$. Similarly, we define $\hat{h} \in C(X)$ and $\bar{h} \in r(X)$. Now, on $\operatorname{coz} p$ we have $\bar{h}\bar{k}_1 f = \hat{k}_1\hat{h}, \bar{h}\bar{k}_2 g = \hat{k}_2\hat{h}$ and $\bar{h}\bar{k}_3\bar{k}_4 h = \hat{k}_3\bar{k}_4\hat{h}f + \bar{k}_3\hat{k}_4\hat{h}g$. Since $\operatorname{coz} p$ is dense, these equalites hold on X. But, $\bar{h}$ and $\bar{k}_i$'s are regular elements, and thus we are done. □

Lemma 2.2. *The following statements are equivalent for every $f \in C(X)$.*

1. *There exists $s \in r(X)$ such that $pos f|_{coz s}$ and $neg f|_{coz s}$ are completely separated in $coz s$.*
2. *There exist $s \in r(X)$ and $k \in C(coz s)$ such that $f = k|f|$ on $coz s$.*
3. *There exist $s \in r(X)$ such that the ideal generated by the contraction of f and $|f|$ on $coz s$ is principal in $C(coz s)$.*
4. *For any two elements $r, s \in r(X)$ the ideal $(\frac{f}{r}, \frac{|f|}{s})$ is principal in $q(X)$.*
5. *There exists $\frac{k}{t} \in q(X)$ such that $\frac{f}{1} = \frac{k}{t}\frac{|f|}{1}$.*
6. *For any two elements $r, s \in r(X)$ there exists $\frac{k}{t} \in q(X)$ such that $\frac{f}{r} = \frac{k}{t}\frac{|f|}{s}$.*

Proof. The equivalence of (1), (2) and (3) is due to Corollary 14.22 in [3]. Using the above lemma, (3) and (4) are equivalent, and clearly (5) $\Rightarrow$ (6) $\Rightarrow$ (2). To complete the proof, it is enough to show that (4) $\Rightarrow$ (5). Let (4) holds. There exists $\frac{g}{p} \in q(X)$ such that $(\frac{f}{1}, \frac{|f|}{1}) = (\frac{g}{p})$. Thus, there are $\frac{h_1}{q_1}, ..., \frac{h_4}{q_4} \in q(X)$ such that $\frac{f}{1} = \frac{h_1}{q_1}\frac{g}{p}$, $\frac{|f|}{1} = \frac{h_2}{q_2}\frac{g}{p}$ and $\frac{g}{p} = \frac{h_3}{q_3}\frac{f}{1} + \frac{h_4}{q_4}\frac{|f|}{1}$. Taking $t = pq_1q_2q_3q_4$ and $Y = \operatorname{coz} t$, the ideal generated by the contraction of f and $|f|$ on Y is principal in $C(Y)$. Thus by Corollary 14.22 in [3], there is $k \in C^*(Y)$ such that $\frac{f}{1}(x) = k(x)\frac{|f|}{1}(x)$, for every $x \in Y = \operatorname{coz} t$. Now, if we define

$$\hat{k}(x) = \begin{cases} k(x)s(x) & x \in \operatorname{coz} t \\ 0 & x \in Z(t), \end{cases}$$

then $\hat{k} \in C(X)$, and $\frac{f}{1}(x) = \frac{\hat{k}}{t}(x)\frac{|f|}{1}(x)$ for every $x \in \operatorname{coz} t$, which implies that $\frac{f}{1} = \frac{\hat{k}}{t}\frac{|f|}{1}$. $\square$

Now, we are ready to examine the situation in which every maximal ideal of $q(X)$ contains only one minimal prime ideal. Notice that for every space X the ring $q(X)$ is reduced.

Theorem 2.3. *The following statements are equivalent for any space X.*

1. *$q(X)$ is a PF-ring.*
2. *For any two disjoint cozero-sets C_1 and C_2 in X, there exists zero-sets Z_1 and Z_2 such that $C_1 \subseteq Z_1, C_2 \subseteq Z_2$ and $int_X(Z_1 \cap Z_2) = \emptyset$.*
3. *If $f_1, f_2 \in C(X)$ and $coz\, f_1 \subseteq Z(f_2)$, then there exist $g_1, g_2 \in C(X)$ such that*
$$coz\, f_1 \subseteq int_X Z(g_1) \subseteq cl_X coz\, g_2 \subseteq Z(f_2).$$
4. *For every $f \in C(X)$ there exists $r \in r(X)$ such that $pos f|_{coz\, r}$ and $neg f|_{coz\, r}$ are completely separated in $coz\, r$.*
5. *For any two disjoint cozero-sets C_1 and C_2 in X, there exists $r \in r(X)$ such that $C_1 \cap coz\, r$ and $C_2 \cap coz\, r$ are completely separated in $coz\, r$.*
6. *For every $f \in C(X)$ there exist $s \in r(X)$ and $k \in C(coz\, s)$ such that $f = k|f|$ on $coz\, s$.*
7. *For every $f \in C(X)$ there exist $s \in r(X)$ such that the ideal generated by the contraction of f and $|f|$ on $coz\, s$ is principal in $C(coz\, s)$.*
8. *For every $f \in C(X)$ and any two elements $r, s \in r(X)$ the ideal $(\frac{f}{r}, \frac{|f|}{s})$ is principal in $q(X)$.*
9. *The ideal $(\frac{f}{1}, \frac{|f|}{1})$ is principal in $q(X)$.*
10. *For every $f \in C(X)$ and any two elements $r, s \in r(X)$ there exists $\frac{k}{t} \in q(X)$ such that $\frac{f}{r} = \frac{k}{t}\frac{|f|}{s}$.*

Proof. (1)$\Rightarrow$(2). Let $C_1 = \operatorname{coz} f_1$ and $C_2 = \operatorname{coz} f_2$. Then $C_1 \cap C_2 = \emptyset$ implies that $f_1 f_2 = 0$. Using Lemma 1.1 there exist $g_1 \in \operatorname{Ann}(f_1)$ and $g_2 \in \operatorname{Ann}(f_2)$ such that $g_1 + g_2$ is regular. Thus, $\operatorname{coz} f_1 \subseteq Z(g_1)$, $\operatorname{coz} f_2 \subseteq Z(g_2)$ and $\operatorname{int}_X(Z(g_1) \cap Z(g_2)) \subseteq \operatorname{int}_X Z(g_1 + g_2) = \emptyset$.

(2)$\Leftrightarrow$(3). The equivalence of (2) and (3) is clear since the first, the second and the third inclusions in part (3) are equivalent to $f_1 g_1 = 0$, $\operatorname{int}_X(Z(g_1) \cap Z(g_2)) = \emptyset$ and $f_2 g_2 = 0$, respectively.

(2)$\Rightarrow$(4). Let $f \in C(X)$. Since $\operatorname{pos} f = \operatorname{coz} f \wedge 0 := C_1$ and $\operatorname{neg} f = \operatorname{coz} f \vee 0 := C_2$ so it follows by (2) that there exists zero-sets Z_1 and Z_2 in X such that $C_1 \subseteq Z_1, C_2 \subseteq Z_2$ and $int_X(Z_1 \cap Z_2) = \emptyset$. Let $r \in r(X)$ be such that $Z(r) = Z_1 \cap Z_2$, then $C_1 \cap \operatorname{coz} r = Z_1 \cap \operatorname{coz} r$,

$C_2 \cap \operatorname{coz} r = Z_2 \cap \operatorname{coz} r$ and $Z_1 \cap Z_2 \cap \operatorname{coz} r = \emptyset$, i.e., $\operatorname{pos} f|_{\operatorname{coz} r}$ and $\operatorname{neg} f|_{\operatorname{coz} r}$ are completely separated in $\operatorname{coz} r$ by Theorem 1.15 in [3].

(4)⇒(5). Let C_1 and C_2 be two disjoint cozero-sets. Suppose that $C_1 = \operatorname{coz} u$ and $C_2 = \operatorname{coz} v$, where $0 \leq u, v \in C(X)$. If we take $f = u - v$, then clearly $C_1 = \operatorname{pos} f$ and $C_2 = \operatorname{neg} f$ as $C_1 \cap C_2 = \emptyset$. Now, using the hypothesis we are done.

(5)⇒(1). Let $u, v \in C(X)$ and $uv = 0$, or equivalently $\operatorname{coz} u \cap \operatorname{coz} v = \emptyset$. We want to show that $\operatorname{Ann}(u) + \operatorname{Ann}(v)$ is a regular ideal, and thus by Lemma 1.1 we conclude that $q(X)$ is a PF-ring. Using the hypothesis there exists $w \in C(\operatorname{coz} r)$, $0 \leq w \leq 1$, such that $w(\operatorname{coz} u \cap \operatorname{coz} r) = 0$ and $w(\operatorname{coz} v \cap \operatorname{coz} r) = 1$. Now, we define

$$w_r(x) = \begin{cases} r(x)w(x) & x \in \operatorname{coz} r \\ 0 & x \in Z(r). \end{cases}$$

It is not hard to see that $w_r \in C(X)$, $\operatorname{coz} u = (\operatorname{coz} u \cap \operatorname{coz} r) \cup (\operatorname{coz} u \cap Z(r)) \subseteq Z(w_r)$ and $\operatorname{coz} v \subseteq Z(r - w_r)$. Thus, $w_r \in \operatorname{Ann}(u)$ and $r - w_r \in \operatorname{Ann}(v)$, which implies that $\operatorname{Ann}(u) + \operatorname{Ann}(v)$ is regular.

To complete the proof, notice that by the above lemmas parts (4), (6), (7), (8), (9) and (10) are equivalent. □

Using part (2) of the above result and Theorem 6.2 in [5], the following corollary is now evident.

Corollary 2.4. *$q(X)$ is a PF-ring if and only if the space of maximal ideals of $q(X)$ endowed with the hull-kernel topology is an F-space.*

Maximal z°-ideals of $C(X)$ which are in fact maximal in the realm of ideals consisting entirely of zerodivisors are in a one-to-one correspondence with the maximal ideals of $q(X)$. Thus, using the above result we have the following corollary.

Corollary 2.5. *$q(X)$ is a PF-ring if and only if the space of maximal z°-ideals of $C(X)$ endowed with the hull-kernel topology is an F-space.*

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A partially described inverse eigenvalue problem for pseudo-symmetric periodic Jacobi matrices

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Abstract. In this paper, pseudo-symmetric periodic Jacobi matrices are studied. A partially described inverse eigenvalue problem is solved and some properties of the such matrices are proved. The necessary conditions under which the problem is solvable are given and then a numerical example is given demonstrate efficiency of the method.

Keywords: Inverse eigenvalue problem, leading principal minors, Pseudo-symmetric periodic Jacobi matrix

AMS Mathematics Subject Classification [2010]: 65F18,05C50

1. Introduction

An inverse eigenvalue problem (IEP) is the problem of reconstructing a matrix with a special structure from prescribed spectral data. By structure we mean the pattern of entries that are either zero or nonzero. There are many different types of IEPs and the level of their difficulty depends on the structure of the matrices which are to be reconstructed and the available eigen information. In [2] Chu gave a perfect characterization of IEPs and at present many scholars studied different types of IEPs [4, 7]. Periodic Jacobi matrices are important class of matrices and have been studied in different papers. Heydari et al. in [5] investigated spectral properties of such matrices and other studies can be found in [1, 3, 8]. Pseudo-symmetric matrices are mainly appear in perturbation analysis and investigated by Qifang in [6]

A pseudo-symmetric periodic Jacobi matrix is a real matrix of the following form

$$A_n = \begin{bmatrix} a_1 & \epsilon_1 b_1 & 0 & 0 & \cdots & \epsilon_n b_n \\ b_1 & a_2 & \epsilon_2 b_2 & 0 & \cdots & 0 \\ 0 & b_2 & a_3 & \epsilon_3 b_3 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & b_{n-2} & a_{n-1} & \epsilon_{n-1} b_{n-1} \\ b_n & 0 & 0 & 0 & b_{n-1} & a_n \end{bmatrix}, \tag{1}$$

where $b_i > 0$ and $\epsilon_i = \pm 1$.

In this paper we shall use the following notations:

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- A_n denotes the matrix defined as in (1) and by A_j we denote the $j \times j$ leading principle submatrix of A_n,
- B_j denotes a $(j-1) \times (j-1)$ matrix obtained by removing the first row and column of matrix A_j,
- $\phi_j(\lambda) = \det(\lambda I_j - A_j), j = 1, 2, \ldots, n$, i.e, the characteristic polynomial of A_j, and I_j denotes the identity matrix of order j, for convenience let $\phi_0(\lambda) = 1$,
- $\chi_j(\lambda) = \det(\lambda I_{j-1} - B_j), j = 2, 3, \ldots, n-1$, for convenience let $\chi_1(\lambda) = 1$

The following problem will be investigated:

Problem 1 For a given list of real numbers $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, a vector $X_n = (x_1, x_2, \ldots, x_n)^T$ and a real nonzero number c, find an $n \times n$ matrix A_n of form (1) such that λ_i is an eigenvalue of A_i, (λ_n, X_n) is an eigenpair of A_n and $\epsilon_n b_n = c$.

2. Preliminaries

Lemma 2.1. *For any monic polynomial $P(\lambda)$ of order n such that λ_1 and λ_n are its minimal and maximal roots, the followings relations are true:*

- $\forall x > \lambda_n : P(x) > 0$,
- $\forall x < \lambda_1 : (-1)^n P(x) > 0$.

Lemma 2.2. *For a pseudo-symmetric Jacobi matrix J_n, the sequence $\{\phi_j(\lambda)\}$ of characteristic polynomials satisfies the following relations:*

- $\phi_1(\lambda) = \lambda - a_1$,
- $\phi_j(\lambda) = (\lambda - a_j)\phi_{j-1}(\lambda) - \epsilon_{j-1} b_{j-1}^2 \phi_{j-2}(\lambda), j = 2, \ldots, n$,

to write the recurrences let $\phi_0(\lambda) = 1$.

Proof. The recurrence relations can be verified by expanding the determinant. □

Lemma 2.3. *For a pseudo-symmetric Jacobi matrix J_n, There is not two successive leading principle minors of J_n having common eigenvalues.*

Proof. We prove this lemma by contradiction. For any j, $3 \le j \le n$, if for any scalar $\lambda, \phi_j(\lambda) = \phi_{j-1}(\lambda) = 0$, it implies by Lemma 2.2 that

$$(\lambda - a_j)\phi_{j-1}(\lambda) - \epsilon_{j-1} b_{j-1}^2 \phi_{j-2} = 0,$$

since $\epsilon_{j-1} \neq 0, b_{j-1} \neq 0$, hence $\phi_{j-2}(\lambda) = 0$. By continuing this way, it will end up with $\phi_2(\lambda) = \phi_1(\lambda) = 0$, and it implies

$$(\lambda - a_2)\phi_1(\lambda) - \epsilon_1 b_1^2 = 0.$$

Because $\phi_1(\lambda) = 0$, therefore $\epsilon_1 b_1^2 = 0$, which is a contradiction to the conditions of the pseudo-symmetric Jacobi matrix. □

3. Main results

In this section the main results and the solution to the problem are given.

Lemma 3.1. *For a pseudo-symmetric periodic Jacobi matrix A_n, the sequence $\{\phi_j(\lambda)\}$ of characteristic polynomials satisfies the following relations:*

- $\phi_1(\lambda) = \lambda - a_1$,
- $\phi_j(\lambda) = (\lambda - a_j)\phi_{j-1}(\lambda) - \epsilon_{j-1} b_{j-1}^2 \phi_{j-2}(\lambda), j = 2, \ldots, n-1$,
- $\phi_n(\lambda) = (\lambda - a_n)\phi_{n-1}(\lambda) - \epsilon_{n-1} b_{n-1}^2 \phi_{n-2}(\lambda) - b_n \prod_{i=1}^{n-1} \epsilon_i b_i - \epsilon_n \prod_{i=1}^{n} b_i - \epsilon_n b_n^2 \det(\lambda I_{n-2} - B_{n-1})$.

PROOF. The recurrence relations can be verified by expanding the determinant. □

LEMMA 3.2. *Let* $X = (x_1, x_2, \ldots, x_n)^T$ *and* (λ, X) *be an eigenpair of* A_n*, then* $|x_1| + |x_n| > 0$ *and every component* $x_j, j = 2, 3, \ldots, n-1$ *of* X *is given by*

$$x_j = \frac{\phi_{j-1}(\lambda)x_1 - \chi_{j-1}(\lambda)\epsilon_n b_n x_n}{\prod_{i=1}^{j-1} \epsilon_i b_i}. \tag{2}$$

PROOF. From the eigenidentity $A_n X = \lambda X$, we obtain the following relations:

$$a_1 x_1 + \epsilon_1 b_1 x_2 + \epsilon_n b_n x_n = \lambda x_1, \tag{3}$$

$$b_{j-1}x_{j-1} + a_j x_j + \epsilon_j b_j x_{j+1} = \lambda x_j, \text{ for } j = 2, \ldots, n-1, \tag{4}$$

$$b_n x_1 + b_{n-1}x_{n-1} + a_n x_n = \lambda x_n. \tag{5}$$

We prove the lemma by induction on x_js. By (3) one has

$$x_2 = \frac{(\lambda - a_1)x_1 - \epsilon_n b_n x_n}{\epsilon_1 b_1} = \frac{\phi_1(\lambda)x_1 - \chi_1(\lambda)\epsilon_n b_n x_n}{\epsilon_1 b_1}. \tag{6}$$

Now let the lemma holds for $x_2, x_3, \ldots, x_j$, we prove it for x_{j+1} such that $j+1 \le n-1$. By (4) we get

$$\begin{aligned}
x_{j+1} &= \frac{(\lambda - a_j)x_j - b_{j-1}x_{j-1}}{\epsilon_j b_j} \\
&= \frac{(\lambda - a_j)}{\epsilon_j b_j}x_j - \frac{b_{j-1}}{\epsilon_j b_j}x_{j-1} \\
&= \frac{(\lambda - a_j)}{\epsilon_j b_j}\left(\frac{\phi_{j-1}(\lambda)x_1 - \chi_{j-1}(\lambda)\epsilon_n b_n x_n}{\prod_{i=1}^{j-1} \epsilon_i b_i}\right) - \frac{b_{j-1}}{\epsilon_j b_j}\left(\frac{\phi_{j-2}(\lambda)x_1 - \chi_{j-2}(\lambda)\epsilon_n b_n x_n}{\prod_{i=1}^{j-2} \epsilon_i b_i}\right) \\
&= \frac{(\lambda - a_j)\phi_{j-1}(\lambda)x_1 - (\lambda - a_j)\chi_{j-1}(\lambda)\epsilon_n b_n x_n}{\prod_{i=1}^{j} \epsilon_i b_i} - \frac{\epsilon_{j-1}b_{j-1}^2\phi_{j-2}(\lambda)x_1 - \epsilon_{j-1}b_{j-1}^2\chi_{j-2}(\lambda)\epsilon_n b_n x_n}{\prod_{i=1}^{j} \epsilon_i b_i} \\
&= \frac{(\lambda - a_j)\phi_{j-1}(\lambda) - \epsilon_{j-1}b_{j-1}^2\phi_{j-2}(\lambda)}{\prod_{i=1}^{j} \epsilon_i b_i}x_1 - \frac{(\lambda - a_j)\chi_{j-1}(\lambda) - \epsilon_{j-1}b_{j-1}^2\chi_{j-2}(\lambda)}{\prod_{i=1}^{j} \epsilon_i b_i}\epsilon_n b_n x_n \\
&= \frac{\phi_j(\lambda)x_1 - \chi_j(\lambda)\epsilon_n b_n x_n}{\prod_{i=1}^{j} \epsilon_i b_i}.
\end{aligned}$$

□

THEOREM 3.3. *There is a unique solution to Problem 1 if for* $j = 2, 3, \ldots, n$, $x_j \neq 0, \lambda_j \neq \lambda_{j-1}$ *and* $\lambda_n \notin \sigma(A_{n-1})$*, and the unique solution is given by*

$$\begin{aligned}
&a_1 = \lambda_1, \epsilon_n b_n = c, b_n = |c|, \\
&a_j = \lambda_j - \epsilon_{j-1}b_{j-1}^2\frac{\phi_{j-2}(\lambda_j)}{\phi_{j-1}(\lambda_j)}, j = 2, 3, \ldots, n-1, \\
&\epsilon_{j-1}b_{j-1} = \frac{\phi_{j-1}(\lambda_n)x_1 - \chi_{j-1}(\lambda_n)\epsilon_n b_n x_n}{x_j \prod_{i=1}^{j-2} \epsilon_i b_i}, b_{j-1} = |\epsilon_{j-1}b_{j-1}|, j = 2, 3, \ldots, n, \\
&a_n = \lambda_n - \frac{\epsilon_{n-1}b_{n-1}^2\phi_{n-2}(\lambda) + b_n \prod_{i=1}^{n-1} \epsilon_i b_i + \epsilon_n \prod_{i=1}^{n} b_i + \epsilon_n b_n^2 \det(\lambda I_{n-2} - B_{n-1})}{\phi_{n-1}(\lambda_n)}.
\end{aligned}$$

4. Numerical Results

Let $\Lambda = \{1, 2, -3, 4, -7\}, X_5 = \{0, -2, -13, 2, -5\}, c = -2$, By applying the results of Theorem 3.3 we get the following as solution:

$$A_5 = \begin{bmatrix} 1 & 5 & 0 & 0 & -2 \\ 5 & -23 & 2.4615 & 0 & 0 \\ 0 & 2.4615 & -3.2308 & -26.9612 & 0 \\ 0 & 0 & 26.9612 & 109.2535 & -116.6005 \\ 2 & 0 & 0 & 116.6005 & -53.6402 \end{bmatrix}.$$

We compute the spectra of all of the leading principle submatrices of A_5.

$$\sigma(A_5) = \{29.8375 + 88.4911i, 29.8375 - 88.4911i, -24.3057, 2.0133, \mathbf{-7.0000}\},$$
$$\sigma(A_4) = \{-24.2205, 1.8765, \mathbf{4.0000}, 102.3666\},$$
$$\sigma(A_3) = \{-24.2769, \mathbf{-3.0000}, 2.0461\},$$
$$\sigma(A_2) = \{-24, \mathbf{2}\},$$
$$\sigma(A_1) = \{\mathbf{1}\}.$$

It is easy to see $A_5X_5 = \lambda_5X_5 = (0, 14, 91, 14, 35)^T$, which verifies that the conditions of the Problem 1 are satisfied.

5. Conclusion

In this work a PDIEP for pseudo-symmetric periodic Jacobi matrices is studied. The problem involves construction of the matrix by one eigenvector and one eigenvalue of each of leading principal minor of the required matrix and one extra piece of data. An algorithm was derived and a numerical example was given to test the efficiency of the algorithm.

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۸ تا ۱۱ شهریور ۱۴۰۰

Vital Prime Injective S-acts

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Abstract. In this paper, we consider a kind of monomorphism, namely vital monomorphism and study injectivity with respect to this class of monomorphism, which is denoted by vital prime injective. We investigate some properties of vital injective and study the behaviour of vital prime injective with respect to products, coproducts and direct sum.

Keywords: vital rime injectives, vital prime subact, vital monomorphism

AMS Mathematics Subject Classification [2010]: 20M30, 20M50, 18A20

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1. Introduction

Let S be a monoid. A (*right*) *S-act* is a non-empty set A together with a map $A \times S \to A, (a, s) \mapsto as$, such that for all $a \in A, s, t \in S$, $(as)t = a(st)$ and $a1 = a$. A non-empty subset $B \subseteq A$ is called a *subact* of A if $bs \in B$ for all $b \in B$ and $s \in S$. An element θ in an S-act A is said to be a *zero* or *fixed element* if $\theta s = \theta$ for all $s \in S$. Let A and B be two S-acts. A mapping $f : A \to B$ is called an *S-homomorphism* if $f(as) = f(a)s$ for all $a \in A, s \in S$. We denote the category of all S-acts and S-homomorphisms between them by **Act-S**. We recall from [3], a subact B of right *S-act* A is vital, if for every $a \in A$ there exists $s \in S$ such that $as \in B$. An S-homomorphism $f : A \to B$ is vital S-homomorphism if $f(A)$ is a vital subact of B. A right *S-act* M is vital injective if it is injective relative to vital monomorphisms and it is called right *S-act* M is (fg ,principally) weakly vital injective if injective relative to monomorphism of (fg, principally) vital right ideal to S-act S. Let B be a subact of *S-act* A, the set $(B : A) = \{s \in S : As \subseteq B\}$ is an ideal of S, which is called the associated ideal of B. We recall from [7], an subact B of an *S-act* A is a prime subact if for any $a \in A$ and $s \in S$, the inclusion $aSs \subseteq B$ implies $a \in B$ or $s \in (B : A)$. A right ideal I of S is prime if and only if I is a prime subact of S.

2. Main results

Definition 2.1. A prime subact B of *S-act* A is vital prime subact, abbreviation v-prime subact, if B is vital subact of A. An S-homomorphism $f : A \longrightarrow B$ is said to be v-prime homomorphism, if $f(A)$ is a v-prime subact of B.

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Definition 2.2. An *S-act* A is said to be vital prime injective, shortly v-prime injective if it is injective relative to v-prime monomorphism.

Example 2.3. Let S be a group. Clearly, any *S-act* is v-prime injective.

By the definition of v-prime injective the following lemma is clear.

Lemma 2.4. *An S-act A is v-prime injective if and only if for any S-act C, for any v-prime subact B and S-homomorphism $f : B \longrightarrow A$, there exists a S-homomorphism $\overline{f} : C \longrightarrow A$ which extends f.*

Similary to [4, Proposition 3.4.2], the following theorem is clear.

Theorem 2.5. *An S-act A is weakly v-prime injective if and only if for any S-homomorphism $f : I \longrightarrow S$, where $I \subseteq S$ is v-prime right ideal of monoid S, there exists an element $a \in A$ such that $f(s) = as$ for every $s \in I$.*

Definition 2.6. An S-act A is called v-prime retract of an *S-act* B, if for every v-prime monomorphism $f : A \longrightarrow B$ there exists an S-homomorphism $g : B \longrightarrow A$ such that $gf = id_A$.

Theorem 2.7. *Every v-prime retract of a v-prime injective S-act is a v-prime injective.*

Proof. Let B be a v-prime retract of v-prime injective A. We show that B is a v-prime injective. Consider the following diagram.

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "f"'] & D \arrow[d, dotted, "\overline{g}"] \\
B \arrow[r, bend left, "g"] & A \arrow[l, "g^{-1}"]
\end{tikzcd}$$

Since A is a v-prime injective there exists an S-homomorphism $\overline{g} : D \longrightarrow A$ such that $\overline{g}|_C = gf$. Let $\overline{f} = g^{-1}\overline{g}$. So, we have $\overline{f}|_C = g^{-1}\overline{g}|_C = g^{-1}gf = f$.

□

Lemma 2.8. *Let A be a v-prime injective and be a v-prime subact of an S-act B, then A is a v-retract of B.*

Proof. It is clear. □

By the [8, Theorem 3.24], we have the following result.

Theorem 2.9. *Let $\{A_i | i \in I\}$ be a family of v-prime injective then $A = \prod_{i\in I} A_i$ is a v-prime injective S-act.*

Theorem 2.10. *Let S be a monoid with zero and $\{A_i | i \in I\}$ be a family of right S-acts. If $\prod\limits_{i\in I} A_i$ is a v-prime injective then any A_i is v-prime injective.*

Proof. Let B be a v-prime subact of C and $f : B \longrightarrow A_k$ be an S-homomorphism. Define $\overline{f} : B \longrightarrow \prod\limits_{i\in I} A_i$ such that for every $x \in B$, $\overline{f}(x)(i) = \begin{cases} f(x) & i = k \\ \theta_i & i \neq k \end{cases}$. Since $\prod\limits_{i\in I} A_i$ is a v-prime injective, $\overline{f}$ can be extended to $\overline{g} : C \longrightarrow \prod\limits_{i\in I} A_i$. Now, the S-homomorphism $\rho_k\overline{g} : C \longrightarrow A_i$ extends f, where k is k th projection S-homomorphism and we have

$\rho_k\overline{g}(b) = \rho_k\overline{f}(b) = \rho_k(\cdots, \lambda_{a_{k-1}}, f(b), \lambda_{a_{k+1}}, \cdots) = f(b)$ for any $b \in B$, so A_k is v-prime injective. $\square$

Theorem 2.11. *Let $\{A_i | i \in I\}$ be a family of right S-acts. If $A = \coprod_{i\in I} A_i$ is a v-prime injective, then $A_i, i \in I$ is a v-prime injective.*

Proof. Suppose that $\coprod\limits_{i\in\ I} A_i$ is a v-prime injective, we show that A_i, $i \in I$ is a v-prime injective. Consider the following diagram

$$\begin{tikzcd}
B \arrow[r, hook] \arrow[d, "f"'] & C \arrow[dl, dotted, "\overline{f}"] \arrow[ddl, dotted, "f^*"] \\
A_i \arrow[d, "\iota_i"'] & \\
\coprod\limits_{i\in\ I} A_i &
\end{tikzcd}$$

Since $\coprod\limits_{i\in\ I} A_i$ is a v-prime injective, then there exists an S-homomorphism $f^* : C \longrightarrow \coprod\limits_{i\in\ I} A_i$ such that $f^*|_B = \iota_i f$. We have $f^*(x) \in A_i$ for any $x \in C$. Otherwise if there exists $j \in I$ such that $f^*(x) \in A_j$, then for any $s \in S$ we have $f^*(xs) = f^*(x)s \in A_j$. Now since $x \in C - B$, there exists $s \in S$ such that $xs \in B$, so $f^*(xs) = \iota_i f(xs) \in A_i$ and we have $f^*(xs) \in A_i \cap A_j$ that contradiction since $A_i \cap A_j = \emptyset$, so $f^*(x) \in A_i$.

$\square$

Conversly of Theorrem 2.10, is obtained by similar proof of [4, Proposition 3.1.13]

Theorem 2.12. *Let S be a left reversible monoid. All coproducts of family of v-prime injective S-acts is a v-prime injective S-act.*

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Solving optimization problems with nonconvex feasible sets by using neurodynamic optimization

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Abstract. In this paper, a nonconvex optimization problem which the feasible region is nonconvex set, is considered. A novel neural network model for solving nonconvex optimization problem is proposed. It is proved that the equilibrium points of the neural network model coincides with the alternative optimal solutions of the constrained nonconvex optimization problem. Furthermore, it is shown that under suitable assumptions this model is globally convergent and stable in the sense of Lyapunov at each equilibrium points. Numerical simulation for a nonconvex otimization problem is discussed.

Keywords: Nonconvex optimization, Recurrent neural network, Global optimality conditions, Invex functions

AMS Mathematics Subject Classification [2010]: 90C26, 90C30

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1. Introduction and Preliminaries

Consider the following nonconvex constraiend optimization problem:

$$\begin{aligned} &\min\ f(\boldsymbol{x}) \\ &\text{s.t. } G(\boldsymbol{x}) \le \mathbf{0}, \\ &\qquad H(\boldsymbol{x}) = \mathbf{0}, \quad \boldsymbol{x} \in C, \end{aligned} \tag{1}$$

where $G(\boldsymbol{x}) = [g_1(\boldsymbol{x}), g_2(\boldsymbol{x}), ..., g_m(\boldsymbol{x})]$, $H(\boldsymbol{x}) = [h_1(\boldsymbol{x}), h_2(\boldsymbol{x}), ..., h_l(\boldsymbol{x})]$, f, g_i and h_j are continuously differentiable functions and C is a closed and convex subset of $\mathbb{R}^n$. Note that problem (1) is nonconvex if at least one function from $f,\ g_i,\ i = 1, ..., m$, is not convex or at least one function from $h_j,\ j = 1, ..., l$, is not affine. Let $S = \{\boldsymbol{x} \in C | G(\boldsymbol{x}) \le \mathbf{0}, H(\boldsymbol{x}) = \mathbf{0}\}$ is nonconvex set. The collection of optimal solutions of problem (1) are called alternative optimal solutions and denoted by S^*. In the optimization context, the conventional way of tackling this problem is the Lagrangian approach which leads to well-known optimality conditions like the Kuhn-Tucker necessary conditions or saddle point theorem. Thus a global optimal solution of the problem must be a KKT point and the KKT points are easier to characterize. In terms of developing neural network models for solving general

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nonconvex optimization, it is very hard to find global optima at the beginning; and a more attainable goal at present is to design neural networks for seeking local optima first with the aid of KKT conditions. In last decade, neurodynamic optimization approaches have been extended to nonconvex and Min-Max optimization problems [2], [3] and [4]. In this paper a modified neural network model is proposed for searching KKT points of nonconvex constraint optimization problems. It is shown that any equilibrium point $\boldsymbol{y}^*$ of the proposed neural network model corresponds to a KKT point $(\boldsymbol{x}^*, \lambda^*, \mu^*)$ of the nonconvex problem and contrariwise. Also under some standard assumptions in the optimization context, the state of the proposed network converges to the global optimal solution.

Definition 1.1. [1] Let $f : \mathcal{D} \to \mathbb{R}$, where $\mathcal{D}$ is a nonempty convex set in $\mathbb{R}^n$. The function f is said to be quasiconvex at $\bar{\boldsymbol{x}} \in \mathcal{D}$ if

$$f(\lambda \bar{\boldsymbol{x}} + (1-\lambda)\boldsymbol{x}) \le \max\{f(\bar{\boldsymbol{x}}), f(\boldsymbol{x})\},$$

for each $\lambda \in (0,1)$ and each $\boldsymbol{x} \in \mathcal{D}$.

Definition 1.2. [1] Let $\mathcal{D}$ be a nonempty convex set in $\mathbb{R}^n$, and let $f : \mathcal{D} \to \mathbb{R}$ be a differentiable on $\mathcal{D}$. The function f is said to be pseudoconvex at $\bar{\boldsymbol{x}} \in \mathcal{D}$ if $\nabla f(\bar{\boldsymbol{x}})^{\mathrm{T}}(\boldsymbol{x} - \bar{\boldsymbol{x}}) \ge 0$ for $\boldsymbol{x} \in \mathcal{D}$ implies that $f(\boldsymbol{x}) \ge f(\bar{\boldsymbol{x}})$.

Definition 1.3. [1] Assume $X \subseteq \mathbb{R}^n$ is an open set.The differentiable function $f : X \to \mathbb{R}$ is said to be an η-invex function if there exists some function $\eta : X \times X \to \mathbb{R}^n$ such that for each $\boldsymbol{x}_1,\ \boldsymbol{x}_2 \in X$,

$$f(\boldsymbol{x}_2) \ge f(\boldsymbol{x}_1) + \nabla f(\boldsymbol{x}_1)^{\mathrm{T}} \eta(\boldsymbol{x}_1, \boldsymbol{x}_2).$$

Moreover, f is said to be an η-pseudoinvex function if $\nabla f(\boldsymbol{x}_1)^{\mathrm{T}} \eta(\boldsymbol{x}_1, \boldsymbol{x}_2) \ge 0$ implies that $f(\boldsymbol{x}_2) \ge f(\boldsymbol{x}_1)$. Similiarly f is said to be an η-quasi-invex if $f(\boldsymbol{x}_2) \le f(\boldsymbol{x}_1)$ implies that $\nabla f(\boldsymbol{x}_1)^{\mathrm{T}} \eta(\boldsymbol{x}_1, \boldsymbol{x}_2) \le 0$.

2. Global optimality conditions

From know on let $\boldsymbol{x}^*$ be a feasible solution for problem (1) and there exist scalars $\lambda^* \in \mathbb{R}^m_+$ and $\mu^* \in \mathbb{R}^l$ such that $(\boldsymbol{x}^{\mathrm{T}*}, \lambda^{\mathrm{T}*}, \mu^{\mathrm{T}*})$ is a KKT point. Assume that $\mathcal{I} = \{i | g_i(\boldsymbol{x}^*) = 0\}$, $\mathcal{I}^+ = \{i \in \mathcal{I} | \lambda_i^* > 0\}$, $\mathcal{J}^+ = \{j | \mu_j > 0\}$ and $\mathcal{J}^- = \{j | \mu_j < 0\}$.

Definition 2.1. [1] A feasible solution $\boldsymbol{x}^*$ is said to be a regular point if $\nabla g_i(\boldsymbol{x}^*)$, for $i \in \mathcal{I}$, and $\nabla H(\mathbf{x}^*)$ are linearly independent.

Theorem 2.2. *[1] In problem (1), let that the KKT conditions hold at $\boldsymbol{x}^*$. Then $\boldsymbol{x}^*$ is a global optimal solution, if one of the following conditions hold:*

(I) *f is a pseudoconvex at $\boldsymbol{x}^*$, g_i is quasiconvex at $\boldsymbol{x}^*$ for $i \in \mathcal{I}$, h_i is quasiconvex at $\boldsymbol{x}^*$ for $i \in \mathcal{J}^+$ and h_i is quasiconcave at $\boldsymbol{x}^*$ for $i \in \mathcal{J}^-$.*

(II) *f is pseudoconvex at $\boldsymbol{x}^*$ and ϕ is quasiconvex at $\boldsymbol{x}^*$, where $\phi(\boldsymbol{x}) = \sum_{i \in \mathcal{I}} \lambda_i^* g_i(\boldsymbol{x}) + \sum_{i=1}^{l} \mu_i^* h(\boldsymbol{x})$.*

(III) *$f + \sum_{i \in \mathcal{I}} \lambda_i^* g_i + \sum_{i=1}^{l} \mu_i^* h_i$ is pseudoconvex function.*

Theorem 2.3. *[1] Suppose that $\boldsymbol{x}^* \in int\ C$ is a feasible point and $\boldsymbol{x}^*$, λ^* and μ^* satisfy the KKT conditions. Assume further that*

$$\boldsymbol{d}^{\mathrm{T}} \nabla_{\boldsymbol{x}}^2 L(\boldsymbol{x}^*, \lambda^*, \mu^*) \boldsymbol{d} > 0,\ \forall \boldsymbol{d} \in \mathcal{P}(\boldsymbol{x}^*),\ \ \boldsymbol{d} \ne 0,$$

where:

$$\mathcal{P}(\boldsymbol{x}^*) \ = \ \{\boldsymbol{d} \in \mathbb{R}^n | \nabla g_i(\boldsymbol{x}^*)^{\mathrm{T}} \boldsymbol{d} \le 0,\ \forall i \in \mathcal{I};\ \nabla H(\boldsymbol{x}^*)^{\mathrm{T}} \boldsymbol{d} = 0;\ \nabla g_i(\boldsymbol{x}^*)^{\mathrm{T}} \boldsymbol{d} = 0,\ \forall i \in \mathcal{I}^+\};$$

then $\boldsymbol{x}^*$ *is a strict minimum point of problem* (1).

Note 1. Let $\mathcal{N}(\boldsymbol{x}^*) = \{\boldsymbol{d} \in \mathbb{R}^n | \nabla g_i(\boldsymbol{x}^*)^{\mathrm{T}}\mathbf{d} = 0, \ \forall i \in \mathcal{I}, \ \nabla H(\mathbf{x}^*)^{\mathrm{T}}\mathbf{d} = 0\}$ and $\mathcal{N}^+(\boldsymbol{x}^*) = \{\boldsymbol{d} \in \mathbb{R}^n | \nabla H(\mathbf{x}^*)^{\mathrm{T}}\mathbf{d} = 0; \ \nabla g_i(\mathbf{x}^*)^{\mathrm{T}}\mathbf{d} = 0, \ \forall i \in \mathcal{I}^+\}$. We have $\mathcal{N}(\boldsymbol{x}^*) \subseteq \mathcal{P}(\boldsymbol{x}^*) \subseteq \mathcal{N}^+(\boldsymbol{x}^*)$. Note that if in Theorem (2.3) substitute the set $\mathcal{P}(\boldsymbol{x}^*)$ with the set $\mathcal{N}^+(\boldsymbol{x}^*)$, we still have a sufficient condition. Furthermore, if $\mathcal{I} = \mathcal{I}^+$ then $\mathcal{N}(\boldsymbol{x}^*) = \mathcal{P}(\boldsymbol{x}^*) = \mathcal{N}^+(\boldsymbol{x}^*)$.

Theorem 2.4. *[1] Consider problem (1) without* $H(\boldsymbol{x}) = 0$. *Let* $(\boldsymbol{x}^{\mathrm{T}*}, \lambda^{\mathrm{T}*})$ *be a KKT point. Then* $\boldsymbol{x}^*$ *is a global optimal solution if one of the following conditions hold:*
(i) f *and* g_i *for* $i \in \mathcal{I}$ *are all* η*-invex.*
(ii) f *is* η*-pseudoinvex and* g_i, $i \in \mathcal{I}$, *are* η*-quasi-invex.*

3. Neural network model

Let $\boldsymbol{x}(.)$, $\lambda(.)$, $\mu(.)$ and $\boldsymbol{y}(.)$ be some time dependent variables. We propose a modified recurrent neural network model for solving (1), whose dynamical system for initial point $(\boldsymbol{x}_0^{\mathrm{T}}, \lambda_0^{\mathrm{T}}, \mu_0^{\mathrm{T}})^T$ is defined as follows:

$$D(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{x} - P_C(\nabla f(x) - (\alpha\lambda + \beta(\lambda + G(\boldsymbol{x}))^+)\nabla G(\boldsymbol{x}) - \nabla H(\boldsymbol{x})^{\mathrm{T}}\mu)) \\ (\lambda + G(\boldsymbol{x}))^+ - \lambda \\ H(\boldsymbol{x}) \end{pmatrix}, \tag{2}$$

where $\alpha > 0$, $\beta > 0$ and $\alpha + \beta = 1$. We propose the following neural network model:

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \mathbf{M}D(\boldsymbol{y}), & \text{Dynamical system;} \\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0, & \text{Initial point;} \\ E(\boldsymbol{y}) = \frac{1}{2}(\boldsymbol{y} - \tilde{\boldsymbol{y}})^{\mathrm{T}}\mathbf{M}^{-1}(\boldsymbol{y} - \tilde{\boldsymbol{y}}), & \text{Energy function;} \end{cases} \tag{3}$$

where $\mathbf{M} \in S^{n+m+l}$ is nonsingular matrix, $\boldsymbol{y}(t) = (\boldsymbol{x}(t)^{\mathrm{T}}, \lambda(t)^{\mathrm{T}}, \mu(t)^{\mathrm{T}})^{\mathrm{T}}$ is the state vector, $\boldsymbol{x}(t) = (\mathbf{I}_n, \mathbf{0}, \mathbf{0})\boldsymbol{y}(t)$ is the output vector and $\tilde{\boldsymbol{y}}$ is an equilibrium point.

Corollary 3.1. *[4] Let* Ω^* *be a set of equilibrium points of the recurrent neural model* (3) *in* $\mathbb{R}^{n+m+l}$, *then* $\boldsymbol{y}^* \in \Omega^*$ *if and only if* $\boldsymbol{y}^* = (\boldsymbol{x}^{*\mathrm{T}}, \lambda^{*\mathrm{T}}, \mu^{*\mathrm{T}})^{\mathrm{T}}$ *satisfies the KKT conditions.*

3.1. Stability and convergence analysis.

Lemma 3.2. *In model* (3), *for any initial point* $\boldsymbol{y}_0 = (\boldsymbol{x}^{\mathrm{T}}(t_0), \lambda^{\mathrm{T}}(t_0), \mu^{\mathrm{T}}(t_0))^{\mathrm{T}}$ *there exists a unique continuous solution .*

Theorem 3.3. *[5] Let there exsists a convex set* $Y \subseteq \mathbb{R}^{n+m+l}$ *which* $\nabla D(\boldsymbol{y}) \preceq 0$ *on* Y. *Suppose that* $\Omega^* \subseteq Y$ *and* $\boldsymbol{y}^* \in \Omega^*$. *Then*
(i) $\boldsymbol{y}^*$ *is stable in the sense of Lyapunov.*
(ii) $\boldsymbol{y}(t)$ *has an upper bound.*
(iii) $\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t}$ *converges to* 0 *and any positive limit point of* $\boldsymbol{y}(t)$ *is an equlibrium point of* (3).
(iv) $\lim_{t\to+\infty} dist(\boldsymbol{y}(t), \Omega^*) = 0$.

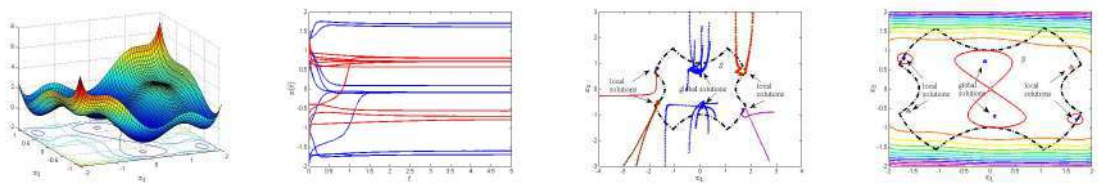
Corollary 3.4. *Let* $\boldsymbol{y}^* \in \mathbb{R}^{n+m+l}$ *be an equilibrium point for the RNN model* (3). *Then* $\boldsymbol{x}^* = (\mathbf{I}_n, \mathbf{0}_{n\times m}, \mathbf{0}_{n\times l})\boldsymbol{y}^*$ *is an optimal solution for* (1) *if one of the conditions in Theorems 2.2 and 2.4 hold.*

4. Numerical examples

Example 4.1. Consider the following nonconvex optimization problem:

$$\begin{aligned}
\min\ & f(x) = 4x_1^2 - 2.1x_1^4 + \frac{x_1^6}{3} + x_1x_2 + 4x_2^4 - 4x_2^2 \\
\text{s.t.}\ & (\frac{x_1}{1.9})^2 + (\frac{x_2}{1.9})^2 \leq 1, \quad -\frac{x_1^2}{2} - x_2 \leq 1, \\
& -(x_1+0.4) - x_2^2 \leq 1, \quad (x_1 - 0.4) - x_2^2 \leq 1, \quad -\frac{x_1^2}{2} + x_2 \leq 1.
\end{aligned}$$

The objective function $f(\boldsymbol{x})$ is a Six-Hump Camel Back function. As shown in Fig. 1 (A), there are two global minima and four additional local minima in this problem. Alternative optimal solutions $S^* = \{(-0.0898, 0.7127), (0.0898, -0.7127), (-1.7036, 0.7961),$ $(1.7036, -0.7961), (-1.6071, -0.5687), (1.6071, 0.5687)\}$. Let $\mathbf{M} = \mathbf{I}_7$ and $\alpha = \beta = 1/2$. All simulation results show that the output trajectory of the proposed model converges to the collection of optimal solutions of this problem. Since $f(\boldsymbol{x})$ is a pseudoconvex at $\pm(-0.0898, 0.7127)$ therefore $(-0.0898, 0.7127)$ and $(0.0898, -0.7127)$ are two global optimal solutions by Theorem 2.2. Fig. 1 (B) shows that the trajectories of the neural network model with ten random initial points converge to alternative optimal solutions of this problem. Figs. 1 (C) and 1 (D) depicte the feasible region and the transient behavior of $\boldsymbol{x}(t)$ with twenty initial points and the contours of the objective function, respectively.



(A) (B) (C) (D)

FIGURE 1. (A) Visualization of objective function (B) The output transient behavior of the neural network mode (3) with ten various initial points (C) The feasible region and the trajectories of the $(x_1(t), x_2(t))$ started from different points (D) The objective function contours and the feasible region.

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Kerman, Iran
30 August - 02 September 2021

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Numerical solution of generalized Gross-Pitaevskii equation by a meshless local method

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Abstract. In this paper, a meshless local radial point interpolation technique is applied for solving 2D generalized Gross-Pitaevskii equation. An efficient fourth-order time differencing Runge-Kutta method is utilized for the time discretization. The main aim of this paper is to show that the meshless local radial point interpolation method is an appropriate technique for solving the non-linear partial differential equations especially generalized Gross-Pitaevskii equation. To show the efficiency of the proposed method, a comparison between this method and Lattice Boltzman and RBF-DQ methods is done.

Keywords: Generalized Gross-Pitaevskii equation, Local radial point interpolation technique, Fourth-order time differencing Runge-Kutta method.

AMS Mathematics Subject Classification [2010]: 65M99; 65N99

1. Introduction

The nonlinear Schrödinger equation (NLSE) is a very important equation in physics, and many problems can be described by this equation. The Gross-Pitaevskii equation (GPE) is one of the NLSE with trapping potential. The GPE describes the dynamics of Bose-Einstein Condensate at temperature much smaller than the critical condensation temperature. The generalized Gross-Pitaevskii equation is as follows [5]:

$$i\psi_t(\mathbf{x},t)+\beta\nabla^2\psi(\mathbf{x},t)+\gamma|\psi(\mathbf{x},t)|^2\psi(\mathbf{x},t)+V(\mathbf{x},t)\psi(\mathbf{x},t)+W(\mathbf{x},t)=0,\quad \mathbf{x}\in\Omega,\ t>0, \tag{1}$$

with Dirichlet boundary condition

$$\psi(\mathbf{x},t)=g(\mathbf{x},t),\quad \mathbf{x}\in\partial\Omega,\quad t>0, \tag{2}$$

and initial condition

$$\psi(\mathbf{x},0)=\psi_0(\mathbf{x}),\quad \mathbf{x}\in\Omega. \tag{3}$$

The Gross-Pitaevskii equation is presented for the first time by Gross [2] and Pitaevskii [4] that describes the ground state of a quantum system of identical bosons using the Hartree-Fock approximation and the pseudo-potential interaction model. The Gross-Pitaevskii equation has attracted the attention of many researchers and has been solved by different

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methods. In this paper, we consider the meshless local radial point interpolation technique for finding the numerical solution of generalized Gross-Pitaevskii equation (GGPE).
In recent years, meshless methods for the solution of partial differential equations (PDEs) has received more and more attention and have become increasingly popular. This is due to the fact that, no meshing is generally required in meshless methods. In the meshless methods, the unknown field is approximated by a linear combination of shape functions built without having recourse to a mesh of the domain. Instead, nodes are scattered in the domain and a certain weight function with a local support is associated with each of these nodes.
One of the most important techniques for approximating unknown field in meshless methods is moving least-squares (MLS) approximation. Although the MLS based methods are efficient meshless methods, there exists an inconvenience to enforce the essential boundary conditions because the shape function constructed by the moving least-squares (MLS) approximation lacks the delta function property [3]. In order to eliminate this shortcoming of the MLS shape functions, the radial point interpolation technique, which has the Kronecker delta function and consistency properties, can be employed instead of the traditional MLS approximation to construct the meshless shape functions. In this paper, we apply the meshless local radial point interpolation method for finding numerical solution of two-dimensional generalized Gross-Pitaevskii equation. The main aim of this paper is to show that the meshless local radial point interpolation technique is suitable for solving non-linear PDEs especially generalized Gross-Pitaevskii equation.

2. Discretization process

In meshless methods, the computational domain Ω is discretized via $n = n_b + n_I$ scattered points $X = \{\mathbf{x}_k\}_{k=1}^{n_b} \cup \{\mathbf{x}_k\}_{k=n_b+1}^{n_b+n_I}$, where n_b is number of boundary points and n_I is number of interior points. In the meshless radial point interpolation method, trial functions $\psi(\mathbf{x}, t)$ are written as a linear combination of RPIM shape functions $\phi_1, \phi_2, \ldots, \phi_n$ in the following form

$$\psi(\mathbf{x}, t) = \sum_{j=1}^{n} \phi_j(\mathbf{x})\, \psi_j(t), \tag{4}$$

where $\psi_j(t) = \psi(\mathbf{x}_j, t)$ and RPIM shape functions are defined as follows

$$\Phi(\mathbf{x}) := [\phi_1(\mathbf{x}), ..., \phi_n(\mathbf{x}), \underbrace{0, ..., 0}_{m}] = \begin{bmatrix} \mathbf{r}(\mathbf{x}) & \mathbf{p}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} R_0 & P \\ P^T & 0 \end{bmatrix}^{-1}, \tag{5}$$

where

$$\mathbf{r}(\mathbf{x}) = [r_1(\mathbf{x}), r_2(\mathbf{x}), ..., r_n(\mathbf{x})], \tag{6}$$

$$\mathbf{p}(\mathbf{x}) = [p_1(\mathbf{x}), p_2(\mathbf{x}), ..., p_m(\mathbf{x})], \tag{7}$$

$$R_0 := \begin{bmatrix} \mathbf{r}(\mathbf{x}_1) \\ \mathbf{r}(\mathbf{x}_2) \\ \vdots \\ \mathbf{r}(\mathbf{x}_n) \end{bmatrix}, \quad P := \begin{bmatrix} \mathbf{p}(\mathbf{x}_1) \\ \mathbf{p}(\mathbf{x}_2) \\ \vdots \\ \mathbf{p}(\mathbf{x}_n) \end{bmatrix}, \tag{8}$$

In the above relations, $r_i(\mathbf{x}) = \left(\|\mathbf{x} - \mathbf{x}_i\|^2 + (\alpha_c h)^2 \right)^q$ is a radial basis function (RBF), $p_j(\mathbf{x})$ is the monomial in the space Cartesian coordinates $\mathbf{x} = [x, y]^T$, and m is the number

of monomial basis functions.
By using the approximation (4) in Eq. (1) and collocating interior points $\mathbf{x}_k$, $k = n_b + 1, \ldots, n$, we obtain the following

$$
\begin{aligned}
&i\,\delta_k\left[\psi_t(\mathbf{x},t)\right] + \beta\,\xi_k\left[\psi(\mathbf{x},t)\right] + \gamma|\delta_k\left[\psi(\mathbf{x},t)\right]|^2\delta_k\left[\psi(\mathbf{x},t)\right] + \\
&\delta_k\left[V(\mathbf{x},t)\right]\delta_k\left[\psi(\mathbf{x},t)\right] + \delta_k\left[W(\mathbf{x},t)\right] = 0.
\end{aligned}
\tag{9}
$$

where δ_k are point evaluation functionals at $\mathbf{x}_k$ and functionals ξ_k are defined as follows

$$
\xi_k\left[\psi\left(\mathbf{x},t\right)\right] = \sum_{j=1}^{n}\Delta\phi_j\left(\mathbf{x}_k\right)\psi_j(t). \tag{10}
$$

Dirichlet boundary conditions (2) are applied as follows

$$
\delta_k\left[\psi(\mathbf{x},t)\right] = g\left(\mathbf{x}_k,t\right), \qquad k = 1,2,\ldots,n_b. \tag{11}
$$

According to Eqs. (9), for obtaining the unknown coefficients $\psi_j(t)$, we have a linear system of ordinary differential equations as

$$
\begin{cases}
\frac{d}{dt}\Psi(t) = f(\Psi,t), \\
\Psi(t^0) = \Psi^0.
\end{cases}
$$

We apply the fourth-order Runge-Kutta method for solving this linear first-order ODE.

3. Numerical results

In this example, we consider two-dimensional GGPE (1) with parameters $\beta = \frac{1}{2}$, $\gamma = -1$. To simulate the interaction of two-dimensional solitons, we set [5]

$$
V(x,y,t) = \exp\left(V_1 + V_2\right), \tag{12}
$$

$$
W(x,y,t) = \exp\left[i\left(x+y-t\right)\right]\left\{
\begin{array}{l}
\frac{1}{2}\exp\left(2V_1+V_2\right) + \frac{1}{4}\exp\left(V_1+2V_2\right) + \\
\exp\left(3V_1\right) + \frac{1}{8}\exp\left(3V_2\right) \\
+\exp\left(V_1\right)\left[2iw_1\left(a_1+1\right) + 2iw_2\left(a_2+1\right) + 2V_1 + 2\right] \\
+\frac{1}{2}\exp\left(V_2\right)\left[2iw_3\left(a_3+1\right) + 2iw_4\left(a_4+1\right) + 2V_2 + 2\right]
\end{array}
\right\}, \tag{13}
$$

where

$$
V_1 = -w_1^2 - w_2^2, \quad V_2 = -w_3^2 - w_4^2, \tag{14}
$$

$$
w_1 = x + a_1t + b_1, \quad w_2 = y + a_2t + b_2, \quad w_3 = x + a_3t + b_3, \quad w_4 = y + a_4t + b_4. \tag{15}
$$

Here the exact solution of GGPE (1) is [5]

$$
u(x,y,t) = \exp\left[V_1 + i\left(x+y-t\right)\right] + \frac{1}{2}\exp\left[V_2 + i\left(x+y-t\right)\right]. \tag{16}
$$

The initial and Dirichlet boundary conditions are obtained from the above exact solution. We solve this problem with parameters $a_1 = a_2 = -20$, $a_3 = a_4 = 20$, $b_1 = b_2 = -8$ and $b_3 = b_4 = -12$. In this problem the computational domain is $[0,20]^2$. The numerical computations are done with $q = 1.03$, $\alpha_c = 5$, $\tau = 10^{-4}$ and L_∞ error norms at $T = 0.05$

TABLE 1. L_∞ errors at $T = 0.05$.

Mesh size	Present method	CPU(s)	RBF-DQ methd [1]	Lattice Boltzman method [5]	
				Latic size	$\|\|e_{\|u\|}\|\|_\infty$
20×20	4.0326×10^{-3}	0.42	2.3190×10^{-4}	150×150	3.0443×10^{-2}
40×40	2.2434×10^{-4}	1.35	1.0054×10^{-4}	200×200	2.1927×10^{-2}
60×60	1.0634×10^{-5}	3.64	5.4082×10^{-5}	250×250	1.7868×10^{-2}
80×80	1.1623×10^{-6}	7.13	2.0417×10^{-5}	300×300	1.5705×10^{-2}

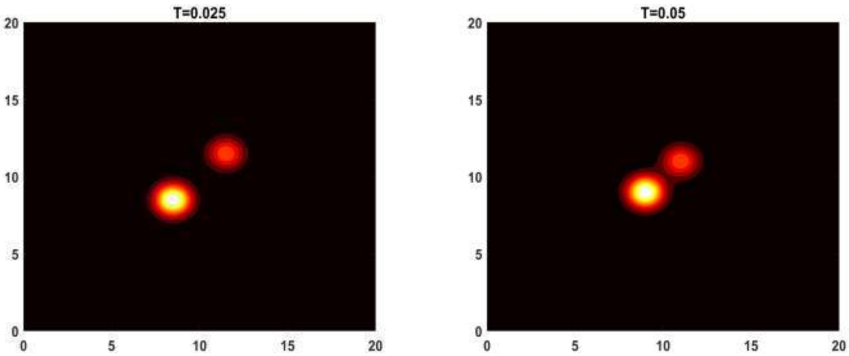

FIGURE 1. Interaction of two solitons for 2D GGPE.

for different mesh size are shown in Table 1. In addition, Table 1 presents a comparison between the obtained errors using local radial point interpolation technique and local RBF-DQ procedure [1] and the Lattice Boltzman method [5] with $\tau = 10^{-4}$ at $T = 0.05$. As comes from Table 1, the meshless local radial point interpolation technique gives better results in comparison with local RBF-DQ procedure [1] and the Lattice Boltzman method [5]. Figure 1 shows the interaction of two solitons at final times $T = 0.025$ and $T = 0.05$.

4. Conclusion

In this paper, the meshless local radial point interpolation method was applied for solvind 2D generalized Gross-Pitaevskii equation. This method was compared with Lattice Boltzman and RBF-DQ techniques. The results show that this method is suitable for solving non-linear PDEs especially generalized Gross-Pitaevskii equation.

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On Hankel matrices

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Abstract. A Hankel matrix is a square matrix in which each ascending skew-diagonal from left to right is constant. A matrix R is called integral row stochastic, if each row has exactly a nonzero entry, $+1$, and other entries are zero. In the present paper, we describe L-ray of a matrix and characterize L-rays of integral row stochastic Hankel matrices. We provide an algorithm for constructing integral row stochastic Hankel matrices.

Keywords: Majorization; Integral row stochastic; Hankel matrices.

AMS Mathematics Subject Classification [2010]: 15A36.

1. Introduction

In linear algebra, a Hankel matrix, named after Hermann Hankel, is a square matrix in which each ascending skew-diagonal from left to right is constant. If $A = [a_{ij}]$ is a Hankel matrix, then we have $a_{ij} = a_{ji} = a_{i+j-2}$. If each row of a matrix R has exactly a nonzero entry, $+1$, and its other entries zero, R is called integral row stochastic. The collection of all n-by-n integral row stochastic Hankel matrices is denoted by $\mathcal{HR}(n)$. Let $\mathbf{M}_n$ be the set of all n-by-n real matrices, and $\mathbb{R}^n$ be the set of all n-by-1 real column vectors. Let $e = (1, \ldots, 1)^t \in \mathbb{R}^n$. For each $1 \le k \le n$ let

$$L^{(k)} = \{(k,1), (k,2), \ldots, (k,k), (k-1,k), \ldots, (1,k)\}.$$

We observe that $L^{(k)}$ consists of the first k positions in row k and column k.
We define the linear function

$$\sigma : \mathbf{M}_n \to \mathbb{R}^n$$
$$\sigma(A) = (\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A))^t,$$

by

$$\sigma_k(A) = \sum_{(i,j)\in L^{(k)}} a_{ij}.$$

$\sigma(A)$ is called the L-ray of A.

In this paper, we investigate the image set $\sigma(\mathcal{A}) = \{\sigma(A) : A \in \mathcal{A}\}$, for each of the classes of integral row stochastic Hankel matrices. A main reference concerning majorization is [2].

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2. Main results

In this section, we provide an algorithm for constructing integral row stochastic Hankel matrices.
Let $P_n = [p_{ij}]$ be the n-by-n permutation matrix where

$$p_{1n} = p_{2n-1} = \cdots = p_{n-12} = p_{n1} = 1,$$

and all other entries equal to zero. Let n_i be the number of i in the vector x.
Note that $x = (x_1, x_2, \ldots, x_n)^t \in \{0, 1, 2\}^n$ means that $x_i \in \{0, 1, 2\}$ for each $1 \leq i \leq n$.
Following [1] we use the following variation of majorization.

Definition 2.1. Let $x = (x_1, x_2, \ldots, x_n)^t$, $y = (y_1, y_2, \ldots, y_n)^t \in \mathbb{R}^n$. Then $x \prec^* y$ if $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$ for $k < n$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$.

We can obtain the following proposition directly from the definition of an integral row stochastic Hankel matrices.

Proposition 2.2. *Let $A \in \mathcal{HR}(n)$. Then*

(i) $0 \leq \sigma_1(A) \leq 1$ *and for each* $2 \leq k \leq n$ *we have* $0 \leq \sigma_k(A) \leq 2$.

(ii) *There exists* $1 \leq k \leq n$ *such that* $A = P_k \oplus P_{n-k}$, *and*

$$\sigma(A) = (\sigma_1(P_k), \ldots, \sigma_k(P_k), \sigma_1(P_{n-k}), \ldots, \sigma_{n-k}(P_{n-k}))^t.$$

(iii) $\mathcal{HR}(n) = \{P_k \oplus P_{n-k} : k = 1, 2, \ldots, n\}$.

(iv) $\sigma(P_{2k}) = \{(0, \ldots, 0, 2, \ldots, 2)^t\}$ *and* $\sigma(P_{2k+1}) = \{(0, \ldots, 0, 1, 2, \ldots, 2)^t\}$, *where* $n_0 = n_2$.

Suppose t is the number of zeros from the left, and $k = n_0 = n_2$. Let $x \in \mathbb{R}^n$.

(1) $k = \dfrac{n-2}{2}$, $n_1 = 2$, and

$$\begin{aligned} x &= (0, \ldots, 0, 1, 2, \ldots, 2, 0, \ldots, 0, 1, 2, \ldots, 2)^t \\ &= e_{t+1} + e_{t+k+2} + 2(\textstyle\sum_{i=t+2}^{2t+1} e_i + \sum_{i=t+k+3}^{n} e_i). \end{aligned}$$

(2) $k = \dfrac{n}{2}$, $n_1 = 0$, and

$$\begin{aligned} x &= (0, \ldots, 0, 2, \ldots, 2, 0, \ldots, 0, 2, \ldots, 2)^t \\ &= 2(\textstyle\sum_{i=t+1}^{2t} e_i + \sum_{i=t+k+1}^{n} e_i). \end{aligned}$$

(1)$'$ $k = \dfrac{n-1}{2}$, $n_1 = 1$, and

$$\begin{aligned} x &= (0, \ldots, 0, 2, \ldots, 2, 0, \ldots, 0, 1, 2, \ldots, 2)^t \\ &= e_{t+k+1} + 2(\textstyle\sum_{i=t+1}^{2t} e_i + \sum_{i=t+k+2}^{n} e_i). \end{aligned}$$

$(2)'$ $\quad k=\dfrac{n-1}{2}$, $n_1=1$, and

$$\begin{aligned} x &= (0,\ldots,0,1,2,\ldots,2,0,\ldots,0,2,\ldots,2)^t \\ &= e_{t+1}+2(\textstyle\sum_{i=t+2}^{2t+1} e_i+\sum_{i=t+k+2}^{n} e_i). \end{aligned}$$

We observe that in (1) and (2) n is even, and in $(1)'$ and $(2)'$ n is odd.

In the following, we will show that the matrix constructed by this algorithm is in $\mathcal{HR}(n)$.

Consider the following Algorithm. Theorem 2.3 ensures that Algorithm offers an integral row stochastic Hankel matrix A with $\sigma(A)=x$. e denotes an all ones vector.

Algorithm

Input: A vector $x=(x_1,x_2,\ldots,x_n)^t\in\{0,1,2\}^n$, $x\prec^* e$, and with one of the cases (1), (2), $(1)'$, or $(2)'$.

1. Initialize: Let $A=(a_{ij})=0_n$ (the zero matrix).
2. for $k=1,2,\ldots,n$ do
   (a) If $x_k=1$, let $a_{kk}=1$.
   Do not use the used rows.
   (b) If $x_k=2$, let l be maximal with $l<k$ and $\sigma_l(A)=0$.
   Let $a_{kl}=a_{lk}=1$.

Output: A.

In the following theorem we describe $L-$rays of integral row stochastic Hankel matrices.

Theorem 2.3. $\sigma(\mathcal{HR}(n))=\{x\in\{0,1,2\}^n\mid x\prec^* e,$ and one of *(1),(2),(1)′*, or $(2)'$ holds$\}$.

Furthermore, if $x\in\{0,1,2\}^n$, $x\prec^* e$, and one of the cases (1), (2), $(1)'$, or $(2)'$ occurs, then Algorithm offers an integral row stochastic Hankel matrix A with $\sigma(A)=x$.

Proof. First, assume that $x\in\sigma(\mathcal{HR}(n))$. There exists some $A\in\mathcal{HR}(n)$ such that $x=\sigma(A)$. Then

$$\sum_{r=1}^{k} x_r=\sum_{r=1}^{k}\sigma_r(A)=\sum_{i,j\le k} a_{ij}\le k$$

for each $k\le n$, and equality holds for $k=n$. This shows that $x\prec^* e$. Proposition 2.2 ensures that $x\in\{0,1,2\}^n$ and there exists some $1\le k\le n$ such that $A=P_k\oplus P_{n-k}$. We consider two steps.

Step 1. If n is even; this implies that both k and $n-k$ are even or odd.

Case 1. k and $n-k$ are even. Proposition 2.2 ensures that

$$\sigma(P_{2k})=\{(0,\ldots,0,2,\ldots,2)^t\}$$

and

$$\sigma(P_{n-k})=\{(0,\ldots,0,2,\ldots,2)^t\},$$

where $n_0=n_2$. We observe that (2) holds.

Case 2. k and $n-k$ are odd. Similar to case 1, we prove it.

Step 2. If n is odd; In a similar fashion, we can prove it.

For the converse, let $x\in\{0,1,2\}^n$, $x\prec^* e$, and one of the cases (1), (2), $(1)'$, or $(2)'$ occurs. We claim that Algorithm constructs some $A\in\mathcal{HR}(n)$ such that $x=\sigma(A)$.

Claim: After each iteration k (of step 2) the present matrix A has the property $\sigma_i(A)=x_i$ for each $i=1,2,\ldots,k$.

Proof of Claim: Use induction on k. For $k=1$ there is nothing to prove. Suppose that $k \leq n$ and the statement holds for $k' < k$.
We consider three cases.

Case 1. If $x_k = 0$; then A is not modified, and so $\sigma_k(A) = 0$.

Case 2. If $x_k = 1$, then $a_{kk} = 1$, and so $\sigma_k(A) = 1$.

Case 2. If $x_k = 2$, as $x \prec^* e$, we see that $\sum_{i=1}^{k} x_i \leq k$, and hence $\sum_{i=1}^{k-1} x_i \leq k-2$. If $x_1, \ldots, x_{k-1} \neq 0$, we have $x_1, \ldots, x_{k-1} \geq 1$, and hence

$$k-1 = \sum_{i=1}^{k-1} 1 \leq \sum_{i=1}^{k-1} x_i.$$

This follows that $k-1 \leq k-2$, which is a contradiction. Thus, there exists $1 \leq i \leq k-1$ such that $x_i = 0$. Let l be maximal $l < k$ with $\sigma_l(A) = 0$. Algorithm ensures that $a_{kl} = a_{lk} = 1$, and so $\sigma_k(A) = 2$.

We see in each case $\sigma_k(A) = x_k$. So the statement holds by induction, and we have $\sigma(A) = x$. One can prove $A \in \mathcal{HR}(n)$, similarly. $\square$

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Efficient explicit methods for nonstiff ordinary differential equations

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Abstract. In this paper, we use Albrecht technique to construct a subclass of variable stepsize general linear methods which have large region of absolute stability. Such methods are considered as an alternative to the Nordsieck technique.

Keywords: Initial value problem, General linear methods, Zero-stability, Variable stepsize

AMS Mathematics Subject Classification [2010]: 65L05

1. Introduction

Consider the initial-value problem (IVP) for a system of autonomous ordinary differential equations (ODEs)

$$\begin{cases} y'(x) = & f(y(x)), \quad x \in [x_0, \overline{x}], \\ y(x_0) = & y_0, \end{cases} \tag{1}$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$, $y : \mathbb{R} \to \mathbb{R}^m$, and m is the dimensionality of the system and f is sufficiently smooth function. In 1966, Butcher [2] provided general linear methods (GLMs) as a middle ground between two traditional methods; Runge–Kutta (RK) methods and linear multistep methods (LMMs). To achieve the optimized calculations and effective implementation, using the variable stepsize technique is necessary. In this direction, Jackiewicz [3] studied a class of variable stepsize diagonally implicit multistage integration methods with Runge–Kutta stability (RKS) property for the numerical solution of ODEs by using Albrecht's technique [1]. The goal of this paper is to construct and implement a class of variable stepsize GLMs (VS-GLMs) without RKS property which have large region of stability. Similarly as in [3, 4], consider a nonuniform mesh

$$x_{-\rho} < \cdots < x_{-1} < x_0 < x_1 < \cdots < x_N, \quad x_N > \bar{x}, \tag{2}$$

and assume $h_n = x_{n+1} - x_n$, $n = -\rho, \ldots, 0, \ldots, N+1$, $\sigma_{n,i} = \frac{h_{n-i}}{h_n}$, $i = 1, 2, \ldots, \rho$. Here, the points $x_{-\rho}, \ldots, x_{-1}$ are introduced in order to simplify the formulas of order conditions.

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It should be noted that we only consider the grids $x_0, x_1, \ldots, x_N$ and start the integration process at x_ρ for some integer ρ. In this paper, we are going to find methods of the form

$$
\text{(3)} \qquad \begin{cases} Y^{[n+1]} = h_n(A(\sigma_n) \otimes I_m) f(Y^{[n+1]}) + (U(\sigma_n) \otimes I_m) y^{[n]}, \\ y^{[n+1]} = h_n(B(\sigma_n) \otimes I_m) f(Y^{[n+1]}) + (V(\sigma_n) \otimes I_m) y^{[n]}, \end{cases}
$$

$n = 0, 1, \ldots, N-1$ where $Y_i^{[n+1]} \simeq y(x_n + c_i(\sigma_n)h_n)$, $i = 1, 2, \ldots, s$, $c(\sigma_n) = [c_1(\sigma_n), \ldots, c_s(\sigma_n)]^T$ and the starting values $y_i^{[0]}$, $i = 1, 2, \ldots, r$ are approximations to linear combinations of $y(x_{-\rho})$, $y(x_{-\rho+1})$, $\ldots$, $y(x_0)$. The coefficients matrices $A(\sigma_n) \in \mathbb{R}^{s\times s}$, $U(\sigma_n) \in \mathbb{R}^{s\times r}$, $B(\sigma_n) \in \mathbb{R}^{r\times s}$, $V(\sigma_n) \in \mathbb{R}^{r\times r}$ and the vector $c(\sigma_n) \in \mathbb{R}^s$ depend on the ratios of the current stepsize and the past stepsizes.
Through the paper, we assume that $p = q = r = s = \rho$ and $V(\sigma_n)$ is a rank-one matrix, i.e $V(\sigma_n) = ev(\sigma_n)^T$ where $v(\sigma_n) = [v_1(\sigma_n)\ v_2(\sigma_n)\ \ldots\ v_r(\sigma_n)]^T$, and $v(\sigma_n)^T e = 1$. The product of matrices $V(\sigma_n)$ determines stability properties of the method (3).

Definition 1.1. The method (3) is zero-stable if the product $\prod_{j=0}^{n} V(\sigma_j)$ is bounded uniformly with respect to n, i.e. $\| \prod_{j=0}^{n} V(\sigma_j)\| \leq L$, where L is a scaler.

2. Order conditions

In this section, to derive the order conditions of VS-GLMs (3), assume that the stage vector $Y^{[n]}$ is an approximation of order $q = p$ and at least one to the vector $z_1(x_n) := y(x_n + bh_{n-1})$, i.e.

$$
Y^{[n+1]} = y(x_n + bh_{n-1}) + \mathcal{O}(h^{p+1}),
$$

where y is the solution to system (1) and $b = c(\sigma_{n-1}) - e$, with $c(\sigma_{n-1}) = [c_1(\sigma_{n-1}), \ldots, c_s(\sigma_{n-1})]^T$, and $e = [1, \ldots, 1]^T$. To obtain the order conditions for VS-GLMs (3), we assume that

$$
y^{[n]} = \sum_{l=0}^{\rho} \beta_l y(x_{n-l}) + \mathcal{O}(h^{p+1}), \quad \textit{where} \quad h = \max_{0\leq n\leq N-1} |h_n|,
$$

for the some vectors $\beta_l = [\beta_{i,l}]_{i=1}^r$, and require that

$$
y^{[n+1]} = \sum_{l=0}^{\rho} \beta_l y(x_{n-l+1}) + \mathcal{O}(h^{p+1}),
$$

for the same vectors β_l. It means that the correct function is defined by $z_2(x_n) := \sum_{l=0}^{\rho} \beta_l y(x_{n-l})$. This leads to a method of order p and provide a starting procedure to compute the initial vector $y^{[0]}$ such that

$$
y^{[0]} = \sum_{l=0}^{\rho} \beta_l y(x_{-l}) + \mathcal{O}(h^{p+1}).
$$

Define $h_n d^{[n+1]}$ and $h_n \widehat{d}^{[n+1]}$ as the local discretization errors by

$$
\text{(4)} \qquad \begin{cases} z_1(x_{n+1}) = h_n A(\sigma_n) f(z_1(x_{n+1})) + U(\sigma_n) z_2(x_n) + h_n d^{[n+1]}, \\ z_2(x_{n+1}) = h_n B(\sigma_n) f(z_1(x_{n+1})) + V(\sigma_n) z_2(x_n) + h_n \widehat{d}^{[n+1]}. \end{cases}
$$

By substituting the equivalent values $z_1(x_{n+1})$, $z_2(x_n)$ and $z_2(x_{n+1})$ into equations (4) and expanding $y(x_n + c(\sigma_n)h_n)$ and $y'(x_n + c(\sigma_n)h_n)$ around the point x_n, after some computations, we get

$$
(5) \qquad \begin{cases} h_n d^{[n+1]} = C_0(\sigma_n) y(x_n) + \displaystyle\sum_{\mu=1}^{p} C_\mu(\sigma_n) h_n^\mu y^{(\mu)}(x_n) + \mathcal{O}(h^{p+1}), \\ h_n \widehat{d}^{[n+1]} = \widehat{C}_0(\sigma_n) y(x_n) + \displaystyle\sum_{\mu=1}^{p} \widehat{C}_\mu(\sigma_n) h_n^\mu y^{(\mu)}(x_n) + \mathcal{O}(h^{p+1}), \end{cases}
$$

where coefficients $C_\mu(\sigma_n)$ and $\widehat{C}_\mu(\sigma_n)$, $\mu = 0, 1, \ldots, p$ are given by

$$
\begin{cases} C_0(\sigma_n) = \displaystyle\sum_{l=0}^{\rho} \beta_l - U(\sigma_n) \sum_{l=0}^{\rho} \beta_l, \\ C_\mu(\sigma_n) = \dfrac{C(\sigma_n)}{\mu!} - A(\sigma_n) \dfrac{C(\sigma_n)^{\mu-1}}{(\mu-1)!} - \dfrac{(-1)^\mu}{\mu!} U(\sigma_n) \displaystyle\sum_{l=0}^{\rho} (\sum_{\nu=1}^{l} \sigma_{n,\nu})^\mu \beta_l, \end{cases}
$$

and

$$
\begin{cases} \widehat{C}_0(\sigma_n) = (I_r - V(\sigma_n)) \displaystyle\sum_{l=0}^{\rho} \beta_l, \\ \widehat{C}_\mu(\sigma_n) = \dfrac{\beta_0}{\mu!} - \dfrac{(-1)^\mu}{\mu!} \displaystyle\sum_{l=2}^{\rho} \beta_l (\sum_{\nu=1}^{l-1} \sigma_{n,\nu})^\mu - B(\sigma_n) \dfrac{C(\sigma_n)^{\mu-1}}{(\mu-1)!} - \dfrac{(-1)^\mu}{\mu!} V(\sigma_n) \sum_{l=1}^{\rho} \beta_l (\sum_{\nu=1}^{l} \sigma_{n,\nu})^\mu, \end{cases}
$$

for $\mu = 1, 2, \ldots, p$ where I_r is identity matrix of dimension equal to r.

Definition 2.1. The method (3) is said to be preconsistent if $C_0(\sigma_n) = \widehat{C}_0(\sigma_n) = 0$. Also, it is said to be consistent if $C_1(\sigma_n) = \widehat{C}_1(\sigma_n) = 0$.

We constract VS-SGLMs up to order four by using the order conditions (5), and assume $c = [0 \ \frac{1}{s-1} \ \cdots \ \frac{s-2}{s-1} \ 1]$. We will have some free components in the coefficients matrices, after applying the order and stage order conditions (5). Thus, using MATLAB subroutine `fminsearch`, we obtain methods in such a way that the underlying fixed stepsize methods have large regions of stability. The coefficients of the methods for $p = q = r = s = \rho \le 4$ take the form

$$
\left[\begin{array}{c|c} A(\sigma_n) & U(\sigma_n) \\ \hline B(\sigma_n) & V(\sigma_n) \end{array} \right] = \left[\begin{array}{cccc|ccccc} 0 & 0 & \cdots & 0 & u_{1,1} & u_{1,2} & \cdots & u_{1,r-1} & 1 - \sum_{j=1}^{r-1} u_{1,j} \\ a_{2,1} & 0 & \cdots & 0 & u_{2,1} & u_{2,2} & \cdots & u_{2,r-1} & 1 - \sum_{j=1}^{r-1} u_{2,j} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{s,1} & a_{s,2} & \cdots & 0 & u_{s,1} & u_{s,2} & \cdots & u_{s,r-1} & 1 - \sum_{j=1}^{r-1} u_{s,j} \\ \hline b_{1,1} & b_{1,2} & \cdots & b_{1,s} & v_1 & v_2 & \cdots & v_{r-1} & 1 - \sum_{j=1}^{r-1} v_j \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{r,1} & b_{r,2} & \cdots & b_{r,s} & v_1 & v_2 & \cdots & v_{r-1} & 1 - \sum_{j=1}^{r-1} v_j \end{array} \right],
$$

where all of the components $a_{i,j}$, $u_{i,j}$, $b_{i,j}$ and v_j are depend on the σ_n.

3. Numerical results

In order to show the efficiency of the constructed method, we solve the non-stiff problem

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}, \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \tag{6}$$

with the exact solution $y_1(x) = 3\sin(x) + 2\cos(x)$, $y_2(x) = \cos(x) - 5\sin(x)$, on the generated meshes according to the pattern $h_{n+1} = \rho^{\theta_n} \cdot h_n$, $n = 0, 1, \ldots, N$, where $\theta_n = (-1)^n \sin(5\pi n/(\overline{x} - x_0))$ and $\rho = 2$, with $h_0 = (\overline{x} - x_0)/N$. Moreover, we provide a numerical estimation to the order of convergence, p, which is computed by the formula

$$O_N = \log\left(\frac{\mathtt{ge}_1}{\mathtt{ge}_2}\right)/\log\left(\frac{N_2}{N_1}\right),$$

where ge_1 and ge_2 respectively stand for the global error of the methods corresponding to N_1 and N_2 grid points.

TABLE 1. Numerical results of explicit VS-GLMs (5) up to order four

N	1000	2000	4000	8000	16000
method with $p = 1$	7.57e−01	3.73e−01	1.80e−01	8.57e−02	4.22e−02
O_N		1.02	1.06	1.06	1.02
method with $p = 2$	5.02e−03	1.26e−03	3.15e−04	7.87e−05	1.97e−05
O_N		2.00	2.00	2.00	2.00
method with $p = 3$	1.07e−06	1.35e−07	1.81e−08	2.31e−09	2.66e−10
O_N		3.00	2.89	2.97	3.11
method with $p = 4$	5.20e−7	3.49e−08	2.19e−09	1.40e−10	1.03e−11
O_N		3.90	3.99	3.98	3.77

4. Conclusion

We introduced the GLMs in a variable stepsize environment in which the coefficients matrices of the methods depend on the rations of the current stepsize and the past stepsizes. By formulating such methods, we derived their order conditions of order p and high stage order $q = p$. Finally, some numerical experiments were provided demonstrating the efficiency and high accuracy of the proposed methods.

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Generalized minimum residual method based on the knowledge of a frame in a Hilbert space

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Abstract. The goal of this paper is to study the application of frames in generalized minimum residual method for solving the operator equation $Lu = f$ where $L : H \to H$ is a bounded, invertible and self-adjoint linear operator on a separable Hilbert space H. Convergence rate in this approach is formed by upper and lower bounds of a frame, so we can control the convergence rate by choosing an appropriate frame with desired values of bounds.

Keywords: separable Hilbert space, frame, GMRES method, operator equation.

AMS Mathematics Subject Classification [2010]: 65F10, 65F08.

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1. Introduction and Preliminaries

Projection methods are the most recently practical iterative techniques for solving large linear system of equations

$$Lu = f, \tag{1}$$

where $L : H \to H$ is a bounded, invertible and self-adjoint linear operator on a separable Hilbert space H. By using this approach, we can extract canonically an approximation u_n to the exact solution u of the linear system from a subspace $\mathcal{K} \subseteq H$, called search subspace, provided that

$$f - Lu_n \perp \mathcal{L},$$

where $\mathcal{L} \subseteq H$ is another (maybe the same) subspace, called the subspace of constraints, of the equal dimension. we refer the interested reader to the book by Saad [5]. In the meantime, GMRES (Generalized Minimum Residual Method) is of the great importance in projection methods which utilizes Krylov subspaces $\mathcal{K} = \mathcal{K}_m(L, r_0)$ with Arnoldi orthonormal basis.

In this paper we study the application of frames in GMRES method for solving operator equation (1). In [3, 4] some numerical algorithms for solving this system have been developed by using wavelets and frames. The method is designed on the basic of preconditioning the operator equation Lu = f, using frames and then applying GMRES iteration method but with an orthonormal basis other than Arnoldi type.

We will now give a brief review about the definitions and basic properties of frames. For

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more information we refer the reader to the book by Christensen [2]. Throughout this paper H will be a separable Hilbert space and Λ denotes a countable index set. We begin here defining the concept of the frame.

Definition 1.1. Let $(\psi_\lambda)_{\lambda\in\Lambda} \subset H$. Then $(\psi_\lambda)_{\lambda\in\Lambda}$ is a frame for H, if there exist constants $0 < A \leq B < \infty$ such that

$$A \parallel f \parallel_H^2 \leq \sum_{\lambda\in\Lambda} |< f, \psi_\lambda >|^2 \leq B \parallel f \parallel_H^2, \quad \forall f \in H.$$

The constants A and B are called the lower and upper frame bounds, respectively. If $A_\Psi = B_\Psi$, we call $(\psi_\lambda)_{\lambda\in\Lambda}$ an A-tigh frame. For a frame $\Psi = (\psi_\lambda)_{\lambda\in\Lambda}$, the operator

$$S =: H \to H, \ S(f) = \sum_{\lambda\in\Lambda} < f, \psi_\lambda > \psi_\lambda,$$

is called *frame operator* which is positive, self-adjoint and invertible, which satisfies

$$AI_H \leq S \leq BI_H. \tag{2}$$

Also it has been shown that if $(\psi_\lambda)_{\lambda\in\Lambda}$ is a frame for H and if L is a bounded onto operator on H, then the sequence $(L(\psi_\lambda))_{\lambda\in\Lambda}$ would be a frame for H too. Moreover, if L is also a self-adjoint operator and S is the frame operator of $(\psi_\lambda)_{\lambda\in\Lambda}$, then LSL is the frame operator of $(L(\psi_\lambda))_{\lambda\in\Lambda}$. For more details we refer the reader to [1, 2].
In the reminder of the discussion we consider alternatively the following preconditioned operator equation

$$\frac{2}{A+B} LSLu = \frac{2}{A+B} LSf. \tag{3}$$

2. GMRES method by using frames

First of all for any given frame $\Psi = (\psi_\lambda)_{\lambda\in\Lambda}$ with frame bounds A and B and frame operator S, we note that since LSL is a positive definite operator, we can thus define the following LSL-norm

$$\|f\|_{LSL} = \langle LSLf, f\rangle^{\frac{1}{2}}, \ \forall f \in H,$$

with corresponding inner product

$$\langle f, g\rangle_{LSL} = \langle LSLf, g\rangle, \ \forall f, g \in H.$$

To continue, we define the recurrence sequence

$$v_{n+1} := LSLv_n - \frac{\langle LSLv_n, LSLv_n\rangle}{\langle v_n, LSLv_n\rangle} v_n - \frac{\langle LSLv_n, LSLv_{n-1}\rangle}{\langle v_{n-1}, LSLv_{n-1}\rangle} v_{n-1} \ n \geq 0, \tag{4}$$

with $v_{-1} = 0$, $v_0 = \frac{2}{A+B} LSLu$. For this sequence, we have some pleasant properties exhibited in the two following lemmas.

Lemma 2.1. *Let u be the exact solution of (3) and let us define the space*

$$\mathcal{K}_n := span\left\{ \left(\frac{2}{A+B} LSL\right)^i u : 1 \leq i \leq n \right\} = span\left\{ (LSL)^i u : 1 \leq i \leq n \right\},$$

then for vectors v_i defined by (4), we have

$$\{v_0, v_1, \ldots, v_{n-1}\} \subset \mathcal{K}_n. \tag{5}$$

Lemma 2.2. *The system $\{v_0, v_1, \ldots, v_{n-1}\}$, forms an orthogonal basis for $\mathcal{K}_n$ with respect to the inner product $\langle\cdot,\cdot\rangle_{LSL}$.*

To continue, for each m, we define the tridiagonal matrix $H_m = [h_{ij}]_{m+1\times m}$ such that $h_{ij} = 0$ for each $i \neq j-1, j, j+1$, and $h_{j-1,j} = \frac{\langle LSLv_j, LSLv_{j-1}\rangle}{\langle v_{j-1}, LSLv_{j-1}\rangle}$, $h_{jj} = \frac{\langle LSLv_j, LSLv_j\rangle}{\langle v_j, LSLv_j\rangle}$, and $h_{j+1,j} = 1$. Let $\mathbb{V}_m$ denotes the $n \times m$ matrix with column vectors $v_1, \ldots, v_m$.
Concerning to above discussion, if $\Psi = (\psi_\lambda)_{\lambda\in\Lambda}$ is a frame for H with frame operator S, and L be as in (1) and if A, B are the frame bounds of the frame $L\Psi = (L(\psi_\lambda))_{\lambda\in\Lambda}$, FGMRES can be defined algorithmically as follows:

FGMRES $[L, S, \epsilon, A, B] \rightarrow u_\epsilon$

***(1)*:** $i := 0,\ u_0^i = 0,\ v_0 = \frac{2}{A+B} LSf$
***(2)*:** Compute $r_0^i = (LS)f - (LSL)u_0^i$
***(3)*:** $i := i+1,\ i \leq m$
***(4)*:** $h_{i-1,i} = \frac{\langle LSLv_i, LSLv_{i-1}\rangle}{\langle v_{i-1}, LSLv_{i-1}\rangle}$, $h_{ii} = \frac{\langle LSLv_i, LSLv_i\rangle}{\langle v_i, LSLv_i\rangle}$, $h_{i+1,i} = 1$
***(5)*:** For $j = i-1, i, i+1$ Do $v_i := (LSL)v_{i-1} + h_{ji}v_{i-1}$
***(6)*:** Put tridiagonal matrix $H_i = \{h_{ji}\}_{i-1\leq j\leq i+1, 1\leq i\leq m}$
***(7)*:** Compute x_i the minimizer of $\left\|H_i x - \frac{A+B}{2} e_1\right\|_{LSL}$ and $u_i = \mathbb{V}_i x_i$.
***(8)*:** If $\left(\frac{B^4-A^4}{B^4}\right)^{i/2} \left\|r_0^i\right\|_{LSL} < \epsilon$ stop and set $u_\epsilon := u_i$, if else set $u_0^i = u_i$ and Goto *(2)*.

3. Convergence analysis

Here, we study the convergence of FGMRES under the already known assumption that LSL is a positive definite operator, where L is as in (1) and S is the frame operator of a frame $(\psi_\lambda)_{\lambda\in\Lambda}$. As one may expect, the convergence rate obtained via FGMRES is directly computed by using frame bounds of $(L(\psi_\lambda))_{\lambda\in\Lambda}$.

First of all, we present here an auxiliary lemma.

LEMMA 3.1. *[5] Let A be a positive definite operator and assume that $\mathcal{L} = A\mathcal{K}$. Then a vector u_m is the result of an (oblique) projection method onto $\mathcal{K}$ LSL-orthogonally to $\mathcal{L}$ with the starting vector u_0^k if and only if it minimizes the LSL-norm of the residual vector $b - Au$ over $u \in u_0^k + \mathcal{K}$, i.e., if and only if*

$$\left\|b - Au_m^{k+1}\right\|_{LSL} = \min_{u\in u_0^k+\mathcal{K}} \|b - Au\|_{LSL}.$$

THEOREM 3.2. *Let LSL be as mentioned, then for each m the residual vector*

$$r_m^{k+1} = (LS)f - (LSL)u_m^{k+1},$$

of FGMRES method satisfies

$$\left\|r_m^{k+1}\right\|_{LSL} \leq \left(\frac{B^4 - A^4}{B^4}\right)^{1/2} \left\|r_0^k\right\|_{LSL}. \tag{6}$$

Since $\left(\frac{B^4-A^4}{B^4}\right)^{1/2} < 1$, FGMRES converges to the exact solution of (1) for any initial guess. This convergence rate suggests also that the more closely to be to a tight frame, the faster convergence of $\{u_n\}$ to the exact solution of (1) is expected.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Non-isolated resolving number of corona product of some families of graphs

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Abstract. Let G be a connected graph and $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered subset of vertices of G. For any vertex v of G, the ordered k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

is called the metric representation of v with respect to W, where $d(x, y)$ is the distance between the vertices x and y. A set W is called a resolving set for G if distinct vertices of G have distinct metric representations with respect to W. A resolving set W is called a non-isolated resolving set for G if the induced subgraph $\langle W\rangle$ of G has no isolated vertices. The minimum cardinality of a non-isolated resolving set for G is called the non-isolated resolving number of G and denoted by $nr(G)$. The aim of this paper is to investigate resolving number of corona product graphs of some families of graphs.

Keywords: non-isolated resolving sets, adjacency dimension, corona product.

AMS Mathematics Subject Classification [2010]: 05C12

1. Introduction

In this section, we present some definitions and notations which are necessary to prove main results. Throughout this paper, G is a simple nontrivial graph with vertex set $V(G)$ and edge set $E(G)$. $\overline{G}$ denotes the complement of the graph G. The distance between two vertices u and v, denoted by $d(u, v)$, is the length of a shortest path between u and v. The number of all neighbors of a vertex v is $\deg(v)$. We use symbols $(v_1, v_2, \ldots, v_n)$ and $(v_1, v_2, \ldots, v_n, v_1)$ for a path of order n, P_n, and a cycle of order n, C_n, respectively.

For an ordered subset $W = \{w_1, \ldots, w_k\}$ of $V(G)$ and a vertex v of G, the *metric representation* of v with respect to W is

$$r(v|W) = (d(v, w_1), \ldots, d(v, w_k)).$$

The set W is a *resolving set* for G if the distinct vertices of G have different metric representations, with respect to W. A resolving set W for G with minimum cardinality is a *metric basis* of G, and its cardinality is the *metric dimension* of G, denoted by $\dim(G)$. A resolving set W is called a *non-isolated resolving set* for G if the induced subgraph $\langle W\rangle$ of G has no isolated vertices. The minimum cardinality of a non-isolated resolving set for G is called the *non-isolated resolving number* of G and denoted by $nr(G)$.

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The concepts of resolving sets and metric dimension of a graph were introduced independently by Slater [6] and by Harary and Melter [3]. Resolving sets have applications in diverse areas such as coin weighing problem, robot navigation in networks, network discovery and verification, strategies for mastermind game, and chemical structures in pharmacy. Several variations of resolving sets were introduced by imposing conditions on the subgraph induced by a resolving set. One of these variations is the concept of non-isolated resolving sets. The concepts of non-isolated resolving sets and non-isolated resolving number were introduced in [5]. The non-isolated resolving number of some families of graphs such as paths, complete graphs, bipartite graphs and some friendship graphs are obtained in [5]. For more results about non-isolated resolving sets see [1, 2, 5].

The *corona product* of graphs G and H, denoted by $G \odot H$, is obtained by taking one copy of G and $n(G)$ copies of H, and by joining each vertex of the i^{th} copy of H to the i^{th} vertex of G, $1 \leq i \leq n(G)$.

In this paper we consider non-isolated resolving number of $(G \odot H)$. Our tool is the *adjacency dimension* of graphs.

Definition 1.1. [4] Let G be a graph, and let $W = \{w_1, \ldots, w_k\} \subseteq V(G)$. For each vertex $v \in V(G)$, the *adjacency representation* of v with respect to W is the k-vector

$$r_2(v|w) = (a_G(v, w_1), \ldots, a_G(v, w_k)),$$

where $a_G(v, w_i) = \min\{2, d(v, w_i)\}$. The set W is an *adjacency resolving set* for G if the vectors $r_2(v|W)$ for $v \in V(G)$ are distinct. The minimum cardinality of an adjacency resolving set is the adjacency dimension of G, denoted by $\dim_2(G)$. An adjacency resolving set of cardinality $\dim_2(G)$ is an *adjacency basis* of G.

We will show the relation between $nr(G \odot H)$ and adjacency dimension of H. Using this relation $nr(G \odot H)$ can be computed for many families of graphs.

2. Main results

This section is aimed to investigate non-isolated resolving number of corona product nontrivial graphs G and H based on adjacency dimension of graphs. The next Lemma states the connection between non-isolated resolving number of a graph and its adjacency dimension.

Lemma 2.1. *Let G be a connected graph of order $n \geq 2$ and H be an arbitrary graph of order at least 2. Then*

$$n \dim_2(H) \leq nr(G \odot H) \leq n(1 + \dim_2(H)).$$

To find the exact value of $nr(G \odot H)$ we need to recognize adjacency bases of H.

Theorem 2.2. *Let G be a connected graph of order $n \geq 2$ and H be an arbitrary graph of order at least 2. Then $nr(G \odot H) = n \dim_2(H)$ if and and only if H has an adjacency basis without any non-isolated vertex.*

Using Lemma 2.1 and Theorem 2.2 the exact value of $nr(G \odot H)$ can be computed for many families of graphs. Let us start with some useful results on the adjacency dimension of graphs.

Lemma 2.3. [4] Let H be a graph of order m.

- If $\operatorname{diam}(H) = 2$, then $\dim_2(H) = \dim(H)$.
- If H is connected, then $\dim_2(H) \geq \dim(H)$.

- $1 \leq \dim_2(H) \leq m-1$.
- $\dim_2(H) = m-1$ if and only if $H = K_m$ or $H = \overline{K}_m$.
- $\dim_2(H) = 1$ if and only if $H \in \{P_1, P_2, P_3, \overline{P}_2, \overline{P}_3\}$.
- If $m \geq 4$, then $\dim_2(C_m) = \dim_2(P_m) = \lfloor \frac{2m+2}{5} \rfloor$.
- If $K_{m_1,\ldots,m_t}$ is the complete t-partite graph with r parts of size at least 2 and the other parts of size 1 and $\sum_{i=1}^{t} m_i = m$, then

$$\dim_2(K_{m_1,\ldots,m_t}) = \dim(K_{m_1,\ldots,m_t}) = \begin{cases} m-r-1 & \text{if } r \neq t, \\ m-r & \text{if } r = t. \end{cases}$$

Lemmas 2.3, 2.1 and Theorem 2.2 conclude the following corollary.

Corollary 2.4. *Let G be a connected graph of order $n \geq 2$ and H be a graph of order $m \geq 2$.*

- *If H is a graph with $\operatorname{diam}(H) = 2$, then $n\dim(H) \leq nr(G \odot H) \leq n(1+\dim(H))$.*
- *If H is connected, then $nr(G \odot H) \geq n\dim(H)$.*
- *$2n \leq nr(G \odot H) \leq nm$.*
- *$nr(G \odot H) = nm$ if and only $H = \overline{K}_m$.*
- *$nr(G \odot K_m) = n(m-1)$, for $m \geq 3$.*

If $H \in \{P_m, \overline{P}_m\}$, for some $m \geq 2$ then $nr(G \odot H)$ is obtained by the following theorem.

Theorem 2.5. *Let G be a connected graph of order $n \geq 2$. Then*

$$nr(G \odot \overline{P}_m) = \begin{cases} n\lfloor \frac{2m+2}{5} \rfloor & \text{if } m \geq 4, \\ 2n & \text{if } m \in \{2,3\}. \end{cases}$$

And

$$nr(G \odot P_m) = \begin{cases} n\lfloor \frac{2m+2}{5} \rfloor & \text{if } m \in \{4,5,9\}, \\ n(\lfloor \frac{2m+2}{5} \rfloor + 1) & \text{otherwise.} \end{cases}$$

Now we consider $nr(G \odot H)$, in the case H is a cycle or its complement.

Theorem 2.6. *Let G be a connected graph of order $n \geq 2$. Then*

$$nr(G \odot \overline{C}_m) = \begin{cases} n\lfloor \frac{2m+2}{5} \rfloor & \text{if } m \geq 4, \\ 3n & \text{if } m \in \{3,4\}. \end{cases}$$

And

$$nr(G \odot C_m) = \begin{cases} n\lfloor \frac{2m+2}{5} \rfloor & \text{if } m \in \{3,4,5,9\}, \\ n(\lfloor \frac{2m+2}{5} \rfloor + 1) & \text{otherwise.} \end{cases}$$

The following theorem computes the non-isolated resolving number of the complete t-partite graphs.

Theorem 2.7. *If $K_{m_1,\ldots,m_t}$ is the complete t-partite graph with r parts of size at least 2 and the other parts of size 1 and $\sum_{i=1}^{t} m_i = m$, then*

$$nr(G \odot \overline{K}_{m_1,\ldots,m_t}) = n(m-r).$$

And

$$nr(G \odot K_{m_1,\ldots,m_t}) = \begin{cases} n(m-r) & \text{if } r = t, \\ n(m-1) & \text{if } t = 2, r = 1, \\ n(m-r-1) & \text{otherwise.} \end{cases}$$

The next lemma gives a lower bound for $nr(G \odot H)$, in terms of order of G and $nr(H)$.

Lemma 2.8. *Let G be connected graph of order $n(G) \geq 2$ and H be a graph of order at least 2. Then*

$$n(G)nr(H) \leq nr(G \odot H).$$

A set $S \subseteq V(H)$ is a *dominating set* for H if for each $x \in V(H) \setminus S$, x has a neighbour in S. The minimum cardinality of a dominating set, $\gamma(H)$, is *domination number* of H. If B is an adjacency resolving set for H such that the adjacency representation of no vertex of H is $(2, 2, \ldots, 2)$, then B is a dominating set for H. This means $\dim_2(H) + 1 \geq \gamma(H)$. Therefore we have the following lower bound for $nr(G \odot H)$.

Theorem 2.9. *Let G be a connected graph of order at least 2, and H be a graph of order at least 2. Then*

$$n(G)(\gamma(H) - 1) \leq nr(G \odot H).$$

3. Conclusion

The non-isolated resolving number of corona product of graphs G and H of order at least two is considered in this paper. The structure of $G \odot K_1$ is special and there is no any results about $nr(G \odot K_1)$ in the context. Therefore the study of $nr(G \odot K_1)$ is an interesting work for future.

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Optimality conditions for nonsmooth optimization problems using semi-quasidifferentials

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Abstract. In this paper, semi-quasidifferentiability, as a generalization of well-known quasidifferentiability, is considered to obtain some optimality conditions for a nonsmooth optimization problem. We show that, under some constraint qualifications and a non-degeneracy condition, the KKT-type optimality conditions are achievable. This is done without imposing any locally Lipschitz or continuity or convexity assumption on the objective and constraint functions.

Keywords: Nonsmooth optimization, Semi-quasidifferentiable function, Nonconvex programming

AMS Mathematics Subject Classification [2010]: 49J52, 90C46, 90C30

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1. Introduction

The quasidifferentiability notion has experienced significant development in nonsmooth analysis to deal with nonsmooth optimization problems with directionally differentiable objective(s) and/or constraint functions. This notion was firstly introduced by Pshenichnyi [5] and after that was developed by Demyanov and Rubinov [1]. Nonsmooth functions even under locally Lipschitz condition, which is a necessity condition in many nonsmooth optimization problems, are not necessarily directionally differentiable. Recently, Kabgani and Soleimani-damaneh [4] have introduced the notion of semi-quasidifferentiability based on the quasidifferentiability and the convexificator notion developed by Jeyakumar and Luc [2]. They have shown that, if $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz, then f is semi-quasidifferentiable on a dense subset of $\mathbb{R}^n$. Moreover, Kabgani [3] has shown that semi-quasidifferentiability is useful for characterization of generalized convex functions. In this paper, we consider a nonsmooth constrained optimization problem and obtain some KKT optimality conditions for it. In the rest of this section, we recall some definitions and preliminaries. The main results are presented in Section 2.
For a set $S \subseteq \mathbb{R}^n$, we use the notations $\operatorname{co} S$ and $\operatorname{cl} S$ to denote the convex hull and the closure of S, respectively. Throughout the paper, we use the conventions $\infty - \infty = \infty$,

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$0 \cdot \infty = 0$, and $\infty \cdot \infty = \infty$. The notation $\langle \cdot, \cdot \rangle$ is utilized to denote the standard inner product.

For a nonempty set $S \subseteq \mathbb{R}^n$, the tangent cone to S at $\bar{x} \in \operatorname{cl} S$, denoted by $T_S(\bar{x})$ is defined as

$$T_S(\bar{x}) := \{d \in \mathbb{R}^n : \exists t_n \downarrow 0,\ \exists \{d_n\} \subseteq \mathbb{R}^n\ s.t.\ d_n \to d,\ \bar{x} + t_n d_n \in S\}.$$

The polar cone of a set S is defined by

$$S^\circ := \{d \in \mathbb{R}^n : \langle d, x \rangle \leq 0,\ \forall x \in S\}.$$

The convex cone generated by $S \subseteq \mathbb{R}^n$, is defined as follows:

$$\operatorname{pos}(S) := \left\{ y \in \mathbb{R}^n\ : \exists l \in \mathbb{N}\ s.t.\ y = \sum_{i=1}^{l} \lambda_i y_i,\ \lambda_i \geq 0,\ y_i \in S,\ i = 1, 2, \ldots, l \right\}.$$

The upper Dini directional derivative and the upper Dini-Hadamard directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at $x \in \mathbb{R}^n$ in direction $d \in \mathbb{R}^n$ are respectively defined by

$$f^+(x; d) := \limsup_{t \downarrow 0} \frac{f(x + td) - f(x)}{t},$$

$$f^{DH}(\bar{x}; d) = \limsup_{\substack{d' \to d \\ t \downarrow 0}} \frac{f(\bar{x} + td') - f(\bar{x})}{t}.$$

Definition 1.1. [4] The function $f : \mathbb{R}^n \to \mathbb{R}$ is called DH-regular at $\bar{x} \in \mathbb{R}^n$ if $f^{DH}(\bar{x}; d) = f^+(\bar{x}; d)$ for any $d \in \mathbb{R}^n$.

The class of DH-regular functions contains convex and locally Lipschitz functions.

Definition 1.2. [4] The function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be semi-quasidifferentiable at $\bar{x} \in \mathbb{R}^n$ if there are two nonempty closed sets $\partial_* f(\bar{x}), \partial^* f(\bar{x}) \subseteq \mathbb{R}^n$ such that

$$f^+(\bar{x}; d) = \sup_{\eta \in \partial_* f(\bar{x})} \langle \eta, d \rangle + \inf_{\zeta \in \partial^* f(\bar{x})} \langle \zeta, d \rangle,\ \forall d \in \mathbb{R}^n.$$

The pair of closed sets $(\partial_* f(\bar{x}), \partial^* f(\bar{x}))$ is said to be a semi-quasidifferential (SQD) of f at $\bar{x}$. Moreover, $\partial_* f(\bar{x})$ and $\partial^* f(\bar{x})$ are called the SQD parts. An SQD is called compact if both its parts are compact.

2. Main results

We consider the following optimization problem:

$$\min f(x) \qquad (1)$$
$$s.t.\ g_i(x) \leq 0,\ i \in I := \{1, \ldots, m\},$$

where $f, g_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in I$ are real-valued functions. The set of feasible solutions of Problem (1) is

$$K := \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i \in I\}. \qquad (2)$$

We assume $K \neq \emptyset$. For a given $\bar{x} \in K$, set $I(\bar{x}) := \{i \in I : g_i(\bar{x}) = 0\}$. We do not impose any locally Lipschitz or continuity or convexity assumption on the objective and constraint functions. We assume that g_i for $i \in I(\bar{x})$ admits an SQD $(\partial_* g_i(\bar{x}), \partial^* g_i(\bar{x}))$ at $\bar{x}$, and define

$$\mathcal{S}(\bar{x}) := \bigcup_{i \in I(\bar{x})} \left[\partial_* g_i(\bar{x}) + \partial^* g_i(\bar{x})\right].$$

Theorem 2.1 provides a necessary optimality condition under DH-regularity.

THEOREM 2.1. *Assume that f is DH-regular at $\bar{x}$ and has a compact SQD at $\bar{x}$ as $(\partial_* f(\bar{x}), \partial^* f(\bar{x}))$. If $\bar{x}$ is an optimal solution of Problem (1), $[\mathcal{S}(\bar{x})]^\circ \subseteq T_K(\bar{x})$, and $\operatorname{pos}(\mathcal{S}(\bar{x}))$ is closed, then there exist $\mu_i \geq 0$, $(i \in I)$ such that*

$$0 \in \operatorname{co}(\partial_* f(\bar{x})) + \operatorname{co}(\partial^* f(\bar{x})) + \sum_{i \in I} \mu_i \left(\partial_* g_i(\bar{x}) + \partial^* g_i(\bar{x})\right).$$

In Theorem 2.5, another necessary optimality condition for optimal solutions in terms of SQDs is provided. In this theorem, the SQD of the objective function is not necessarily compact. Assume that $\bar{\zeta} = (\bar{\zeta}_1, \ldots, \bar{\zeta}_m) \in \prod_{i \in I} \partial^* g_i(\bar{x})$ is given. Define

$$\mathcal{Z}(\bar{\zeta}) := \left[\bigcup_{i \in I(\bar{x})} \left(\operatorname{co}(\partial_* g_i(\bar{x})) + \bar{\zeta}_i\right)\right].$$

and $\mathcal{Q}(\bar{\zeta}) := \left(\mathcal{Z}(\bar{\zeta})\right)^\circ$.

DEFINITION 2.2. We say that the constraint qualification 1 (CQ1) at $\bar{x}$ holds if there exists $d \in \mathbb{R}^n$ such that

$$\max_{\eta \in \partial_* g_i(\bar{x})} \langle \eta, d \rangle + \max_{\zeta \in \partial^* g_i(\bar{x})} \langle \zeta, d \rangle < 0, \ \forall i \in I(\bar{x}). \tag{3}$$

DEFINITION 2.3. [4] We say that the constraint qualification 2 (CQ2) at $\bar{x}$ holds if for each $\bar{\zeta} = (\bar{\zeta}_1, \ldots, \bar{\zeta}_m) \in \prod_{i \in I} \partial^* g_i(\bar{x})$, $\operatorname{pos}\left(\mathcal{Z}(\bar{\zeta})\right)$ is closed.

REMARK 2.4. If there exists $d \in \mathbb{R}^n$ such that (3) holds, then

$$0 \notin \operatorname{co}\left(\bigcup_{i \in I(\bar{x})} \left(\partial_* g_i(\bar{x}) + \partial^* g_i(\bar{x})\right)\right).$$

Thus,

$$\operatorname{pos}\left(\operatorname{co}\left(\bigcup_{i \in I(\bar{x})} \left(\partial_* g_i(\bar{x}) + \partial^* g_i(\bar{x})\right)\right)\right) = \operatorname{pos}\left(\bigcup_{i \in I(\bar{x})} \left(\partial_* g_i(\bar{x}) + \partial^* g_i(\bar{x})\right)\right),$$

is a closed set. Similarly, for each $\bar{\zeta} = (\bar{\zeta}_1, \ldots, \bar{\zeta}_m) \in \prod_{i \in I} \partial^* g_i(\bar{x})$, $\operatorname{pos}\left(\bigcup_{i \in I(\bar{x})} (\partial_* g_i(\bar{x}) + \bar{\zeta}_i)\right)$ is closed. Thus, CQ1 implies CQ2.

Consider the Assumptions A as follows.
Assumptions A:
There exists a convex neighbourhood U of $\bar{x} \in K$ such that:
i) For each $i \in I(\bar{x})$, the function g_i is semi-quasidifferentiable on U.
ii) For each $i \in I(\bar{x})$, the set-valued mapping $\partial_* g_i(\cdot) \cup \partial^* g_i(\cdot)$ is deleted locally bounded at $\bar{x}$ with neighborhood U.
iii) For each $i \in I(\bar{x})$, the function g_i is lower semicontinuous on U. For each $i \notin I(\bar{x})$, g_i is upper semicontinuous at $\bar{x}$.

Set

$$\Upsilon(\bar{x}) := \{d \in \mathbb{R}^n : g_i^+(\bar{x}; d) < 0, \ \forall i \in I(\bar{x})\},$$

$$\Xi(\bar{x}) := \{d \in \mathbb{R}^n : g_i^+(\bar{x}; d) \leq 0, \ \forall i \in I(\bar{x})\}.$$

We say the non-degeneracy condition holds at $\bar{x}$ if

$$\Upsilon(\bar{x}) \neq \emptyset \;\&\; \Xi(\bar{x}) = \operatorname{cl}\Upsilon(\bar{x}). \tag{4}$$

THEOREM 2.5. *Assume that f has an SQD at $\bar{x}$ as $(\partial_* f(\bar{x}), \partial^* f(\bar{x}))$, assumptions **A** are fulfilled and the non-degeneracy condition (4) holds at $\bar{x}$. If $\bar{x}$ is an optimal solution of Problem (1), $\operatorname{co}(\partial_* f(\bar{x})) + \operatorname{co}(\partial^* f(\bar{x})) + \mathcal{Q}(\bar{\zeta})$ is a closed set, and CQ1 or CQ2 hold, then for each $(\zeta_1, \ldots, \zeta_m) \in \prod_{i\in I} \partial^* g_i(\bar{x})$, there exist some vector $(\mu_1, \ldots, \mu_m) \in \mathbb{R}^m_{\geqq}$ such that*

$$0 \in \operatorname{co}(\partial_* f(\bar{x})) + \operatorname{co}(\partial^* f(\bar{x})) + \sum_{i\in I} \mu_i(\operatorname{co}(\partial_* g_i(\bar{x})) + \zeta_i).$$

EXAMPLE 2.6. Consider the following optimization problem:

$$\min f(x) \;\; s.t. \;\; g(x) \leq 0, \tag{5}$$

where

$$f(x) = \begin{cases} |x|, & x \in \mathbb{Q}, \\ 0, & otherwise. \end{cases} \qquad g(x) = \begin{cases} 1-x, & x > 0, \\ x, & x \leq 0. \end{cases}$$

The feasible set is $K = (-\infty, 0] \cup [1, +\infty)$. Let $\bar{x} = 0$. Here, f is DH-regular at $\bar{x}$. Furthermore, $(\{-1, 1\}, \{0\})$ is an SQD for f. Also, $([0, \infty), \{1\})$ is an SQD of g at $\bar{x}$. We have $T_K(\bar{x}) = (-\infty, 0]$, $[\mathcal{S}(\bar{x})]^\circ \subseteq T_K(\bar{x})$, and $\operatorname{pos}(\mathcal{S}(\bar{x}))$ is closed. Here, $\bar{x}$ is an optimal solution and

$$0 \in \operatorname{co}(\partial_* f(\bar{x})) + \operatorname{co}(\partial^* f(\bar{x})) + \mu\left(\operatorname{co}(\partial_* g(\bar{x})) + \operatorname{co}(\partial^* g(\bar{x}))\right),$$

for $\mu = 1$.

3. Conclusion

The semi-quasidifferentiability notion, recently introduced by the authors, is a useful generalization of the well-known quasidifferentiability notion. This paper shows how this new notion could be applicable to obtain optimality conditions for nonsmooth optimization problems whose the objective and/or constraint functions are not necessarily locally Lipschitz or continuous or convex.

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۸ تا ۱۱ شهریور ۱۴۰۰

Binary corona of asymptotic resemblance spaces

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Abstract. In this paper, we introduce the notion of the binary corona of asymptotic resemblance spaces as a new large scale property of asymptotic resemblance spaces (coarse spaces). In special cases, binary corona can be considered as a generalization of the notion of space of ends of locally compact Hausdorff topological groups.

Keywords: Asymptotic resemblance spaces, Binary corona, ends of groups, large scale properties

AMS Mathematics Subject Classification [2010]: 51F99, 53C23, 54C20, 18B30

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1. Introduction and Preliminaries

The notion of *coarse structures* has been introduced by Roe and it is widely known as an appropriate way for defining large scale structures on sets ([5]). A coarse structure on a set X is a family of subsets of $X \times X$ with some additional properties and it can be considered as a large scale counterpart of the notion of *uniformity*. Honari and Kalantari defined the concept of *asymptotic resemblance* as a large scale structure such that it can be considered as a large scale analogous of the small scale notion *proximity* ([3]). From now on we denote the family of all subsets of a set X by $\mathcal{P}(X)$.

Definition 1.1. Let X be a set and let λ be an equivalence relation on $\mathcal{P}(X)$. The relation λ is called an asymptotic resemblance relation (an AS.R) on X if,
i) $A_i \lambda B_i$ for each $i \in \{1,2\}$ then $(A_1 \cup A_2)\lambda(B_1 \cup B_2)$.
ii) $A\lambda(B_1 \cup B_2)$ and $B_1, B_2 \neq \emptyset$ then there are $A_1, A_2 \subseteq X$ such that $A = A_1 \cup A_2$ and $A_i \lambda B_i$, for $i \in \{1,2\}$.
If λ is an AS.R on the set X then the pair (X, λ) is called an asymptotic resemblance space (an AS.R space).

Each coarse structure on a set X can induce an AS.R on X and relatively all concepts of large scale geometry can be carried out to AS.R spaces (see for example [3]).

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Definition 1.2. Let (X, λ) be an AS.R space. A subset D of X is called *bounded* if $D = \emptyset$ or $D\lambda\{x\}$, for some $x \in X$. Assume that (X, λ) and (Y, λ') are two AS.R spaces. A map $f : X \to Y$ is called an *AS.R mapping* if the inverse image of each bounded subset of Y is a bounded subset of X and $A\lambda B$ implies $f(A)\lambda' f(B)$, for all $A, B \subseteq X$. An AS.R mapping $f : X \to Y$ is called an *asymptotic equivalence* between AS.R spaces (X, λ) and (Y, λ') if there exists some AS.R mapping $g : Y \to X$ such that $f \circ g(B)\lambda' B$ and $g \circ f(A)\lambda A$, for all $A \subseteq X$ and $B \subseteq Y$. If there exists an asymptotic equivalence between AS.R spaces (X, λ) and (Y, λ') then they are called to be asymptotic equivalent.

Large scale properties are those properties of AS.R spaces that are invariant under asymptotic equivalences.
Classifying finitely generated groups by using the large scale properties of their *Cayley graphs* is one of the main questions in *geometric group theory*. One of the well-known such properties is the notion of *ends of finitely generated groups* defined by Freudenthal ([2]). The space of ends of a finitely generated group G is a totally disconnected compact topological space and the cardinality of this space is called the number of ends of G. It is known that a finitely generated group has $0, 1, 2$ or infinitely many ends, and the space of ends of finitely generated groups do not depend on the choice of generating sets. For more details about the space of ends of finitely generated groups, see §8 of [1]. It worth mentioning that the structure of groups with $0, 2$ or infinitely many ends is completely known. By using the *Stone Representation Theorem* Specker generalized the concept of ends of finitely generated groups to all locally compact Hausdorff topological groups ([6]).
Before going further, let us recall that if X is a nonempty set, then $\mathcal{P}(X)$ can be considered as a Boolean ring with two operations mentioned below:

$$A + B = A\Delta B = (A \setminus B) \cup (B \setminus A)$$

$$AB = A \cap B$$

where $A, B \subseteq X$. Now we can mention Specker's definition of space of ends.

Definition 1.3. Suppose that G is a locally compact Hausdorff topological group and let $\mathcal{K}$ denote the family of all relatively compact subsets of G. Clearly, $\mathcal{K}$ is an ideal in the Boolean ring $\mathcal{P}(G)$. Assume that $\mathcal{P}_G(G)$ denotes the set of all $[A] \in \mathcal{P}(G)/\mathcal{K}$ such that $AK\Delta A \in \mathcal{K}$, for all $K \in \mathcal{K}$. It can be easily seen that $\mathcal{P}_G(G)$ is a Boolean ring, and the Stone representation theorem shows that there exists a totally disconnected compact topological space $\mathfrak{E}(G)$ such that the ring $\mathcal{P}_G(G)$ is isomorphic to the ring of all clopen subsets of $\mathfrak{E}(G)$. The topological space $\mathfrak{E}(G)$ is called the space of ends of the topological group G, and its cardinality is called the number of ends of G.

Contrary to the geometric way of defining the notion of space of ends of finitely generated groups, the Definition 1.3 is somehow algebraic. It is also known that each locally compact Hausdorff topological group has $0, 1, 2$ or infinitely many ends.
As we mentioned before, one can try to classify finitely generated groups by large scale properties of their Cayley graphs, and thus, finitely generated groups are one of the main objects that large scale geometry tries to investigate. For investigating more general groups in large scale geometry, we can use the following definition (see [4]).

Definition 1.4. Suppose that G is a group. The subsets $\mathcal{F}$ of $\mathcal{P}(G)$ is called a *generating family* on the group G if $\mathcal{F}$ contains a nonempty subset of G and each subset of a member of $\mathcal{F}$ is in $\mathcal{F}$ and in addition

$$A^{-1}, AB, A \cup B \in \mathcal{F}$$

for all $A, B \in \mathcal{F}$.

Now let G be a group and assume that $\mathcal{F}$ is a generating family on G. If $A, B \subseteq G$, define $A\lambda_{\mathcal{F}}B$ if there exists some $F \in \mathcal{F}$ such that $A \subseteq BF$ and $B \subseteq AF$. It can be shown that $\lambda_{\mathcal{F}}$ defines an AS.R on G and if $\bigcup_{F\in\mathcal{F}} F = G$ then the family of all bounded subsets of $(G, \lambda_{\mathcal{F}})$ is equal to $\mathcal{F}$.

Example 1.5. Suppose that G is a group, then the family of all finite subsets of G is a generating family on G. If G is a locally compact group, then the family of all relatively compact subsets of G is a generating family on G.

2. Definition of binary corona and some results

We begin by defining the binary corona of asymptotic resemblance spaces.

Definition 2.1. Suppose that (X, λ) is an AS.R space. We say two subsets A and B of X are *asymptotically disjoint* if $L_1 \subseteq A$ and $L_2 \subseteq B$ and $L_1\lambda L_2$ then L_1 and L_2 are both bounded. By this definition each bounded subset of X is asymptotically disjoint from all subsets of X. We denote the family of all $A \subseteq X$ such that A and $A^c = X \setminus A$ are asymptotically disjoint by $\mathcal{D}_\lambda(X)$.

Definition 2.2. Suppose that (X, λ) is an AS.R space and let $\mathcal{B}$ denote the family of all bounded subsets of X. It is easy to see that $\mathcal{B}$ is an ideal in the Boolean ring $\mathcal{P}(X)$. Denote the quotient Boolean ring $\mathcal{P}(X)/\mathcal{B}$ by $\mathcal{P}^*(X)$. Assume that $\mathcal{P}_\lambda(X)$ denotes the family of all $[A] \in \mathcal{P}^*(X)$ such that $A \in \mathcal{D}_\lambda(X)$. It can be shown that $\mathcal{P}_\lambda(X)$ is well defined, and it is a Boolean ring. By using the Stone representation theorem, there exists a totally disconnected compact topological space $\mathfrak{E}_\lambda(X)$ such that $\mathcal{P}_\lambda(X)$ is isomorphic to the Boolean ring of the family of all clopen subsets of $\mathfrak{E}_\lambda(X)$. We call $\mathfrak{E}_\lambda(X)$ the *binary corona* of the AS.R space (X, λ).

The following theorem shows that Definition 2.2 offers a large scale property.

Theorem 2.3. *Suppose that (X, λ) and (Y, λ') are two asymptotically equivalent AS.R spaces. Then $\mathcal{P}_\lambda(X)$ and $\mathcal{P}_{\lambda'}(Y)$ are two isomorphic Boolean rings.*

Corollary 2.4. *Two asymptotic equivalent AS.R spaces have homeomorphic binary coronas.*

The following proposition shows that Definition 2.2 is a generalization of the definition of ends of locally compact Hausdorff topological groups (Definition 1.3).

Proposition 2.5. *Suppose that G is a locally compact Hausdorff topological group and let $\mathcal{K}$ denote the family of all relatively compact subsets of G. Assume that $\lambda = \lambda_{\mathcal{K}}$. Then $\mathfrak{E}(G) = \mathfrak{E}_\lambda(G)$.*

A subset D of an AS.R space (X, λ) is called a *large scale continuum* if $D = D_1 \cup D_2$, for two asymptotically disjoint subsets D_1 and D_2 of X, then D_1 is bounded or D_2 is bounded.

Proposition 2.6. *Suppose that (X, λ) is an AS.R space. Then $\mathfrak{E}_\lambda(X)$ is singleton if and only if X is a large scale continuum.*

Example 2.7. Suppose that $\mathcal{D}$ denotes the family of all bounded subsets of $\mathbb{Q}$ with respect to the standard metric of $\mathcal{Q}$. It can be easily seen that $\mathcal{D}$ is a generating family on G. Assume that $\mathcal{F}$ denotes the family of all finite subsets of $\mathbb{Q}$. Let $\lambda = \lambda_{\mathcal{D}}$ and $\lambda' = \lambda_{\mathcal{F}}$. It can be shown that $\mathfrak{E}_\lambda(\mathbb{Q})$ has two elements and $\mathfrak{E}_{\lambda'}(\mathbb{Q})$ has infinite elements.

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Mean Ergodic Multiplication Operators on Hardy Spaces

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Abstract. In this paper, the power boundedness and mean ergodicity of multiplication operators are investigated on the Hardy spaces $H^p(\mathbb{D})$, $1 \leq p < \infty$. Let $\mathbb{D}$ be the open unit disk on the complex plane $\mathbb{C}$ and ψ be a function in the space of holomorphic functions $H(\mathbb{D})$. We provide the necessary and sufficient conditions under which a multiplication operator M_ψ is power bounded, mean ergodic and uniformly mean ergodic on the Hardy spaces.

Keywords: mean ergodicity, multiplication operator, Hardy space

AMS Mathematics Subject Classification [2010]: 47B38, 46E15, 47A35

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{D}$. For $0 < p < \infty$, the Hardy space $H^p(\mathbb{D})$ is defined by

$$H^p(\mathbb{D}) = \{f \in H(\mathbb{D}) : ||f||_p^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p < \infty\}.$$

When $p \geq 1$, $H^p(\mathbb{D})$ is a Banach space with norm $||.||_p$.
If ψ is a holomorphic function on $\mathbb{D}$, the *multiplication operator* M_ψ on $H(\mathbb{D})$ is defined by

$$M_\psi(f) = \psi f.$$

Following Proposition states the necessary and sufficient conditions for boundedness of multiplication operators on the Hardy spaces. For the proof one can refer to [5].

Proposition 1.1. *Let $1 \leq p < \infty$ and $\psi \in H(\mathbb{D})$. Then $M_\psi : H^p(\mathbb{D}) \to H^p(\mathbb{D})$ is bounded if and only if $\psi \in H^\infty(\mathbb{D})$. In this case $||M_\psi|| = ||\psi||_\infty$, where $||\psi||_\infty = \sup_{z\in\mathbb{D}} |\psi(z)|$.*

Proof. See [5]. □

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Let $L(X)$ be the space of all linear bounded operators from locally convex Hausdorff space X into itself and $T \in L(X)$. The Cesáro means of T is defined by

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m, \ n \in \mathbb{N}.$$

An operator T is mean ergodic if $\{T_{[n]}\}_{n=0}^{\infty}$ is a convergent sequence in the strong operator topology and is said to be uniformly mean ergodic if the convergence is in the norm operator topology. Also T is power bounded if the sequence $\{T^n\}_{n=0}^{\infty}$ is bounded in $L(X)$, i.e. $\sup_{n\in\mathbb{N}} ||T^n|| < \infty$.
The study of mean ergodicity of linear operators on Banach spaces goes back to 1931, when Von Numann proved that for a unitary operator T on a Hilbert space H, there is a projection P on H, such that $T_{[n]}$ converges to P in the strong operator topology. In 1939 Lorch demonstrated that for reflexive Banach spaces, power bounded operators are mean ergodic. Dunford in 1943 stated the connection between the spectral properties of an operator and its uniform mean ergodicity. Recall that by $\sigma(T)$ (spectrum of T) we mean the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible. The following Theorem represent Lorch and Dunford Theorems together.

THEOREM 1.2. *If an operator T on a Banach space X is uniformly mean ergodic, then both $(||T^n||/n)_n$ converges to 0 and either $1 \in \mathbb{C} \setminus \sigma(T)$ or 1 is a pole of order 1 of the resolvent $R_T : \mathbb{C} \setminus \sigma(T) \to L(X), R_T(\lambda) = (T - \lambda I)^{-1}$. Consequently if 1 is an accumulation of $\sigma(T)$, then T is not uniformly mean ergodic.*

PROOF. See [1]. □

Bonet and Ricker [3], characterized the mean ergodicity of multiplication operators in weighted spaces of holomorphic functions and recently Bonet, Jordá and Rodríuez [2] extended the results to the weighted space of continuous functions.
A good reference for information on ergodic theory is the monograph [4].

2. Main results

We first investigate the power boundedness of a multiplication operator acting on $H^p(\mathbb{D})$, $p \geq 1$.

THEOREM 2.1. *Let $\psi \in H^\infty(\mathbb{D})$ and $1 \leq p < \infty$. Then M_ψ is power bounded on $H^p(\mathbb{D})$ if and only if $||\psi||_\infty \leq 1$.*

PROOF. Since for all $n \in \mathbb{N}$, $||M_\psi^n|| = ||\psi^n||_\infty = ||\psi||_\infty^n$, the conclusion follows immediately. □

THEOREM 2.2. *Let $\psi \in H^\infty(\mathbb{D})$ and $1 \leq p < \infty$. Then M_ψ is mean ergodic on $H^p(\mathbb{D})$ if and only if it is power bounded if and only if $||\psi||_\infty \leq 1$.*

PROOF. Suppose M_ψ is mean ergodic. Then for all $f \in H^p(\mathbb{D})$ we have $\frac{||M_\psi^n f||_p}{n} \to 0$ as $n \to \infty$. Let $f \equiv 1$, so $\frac{||\psi^n||_p}{n} \to 0$, on the other hand, for all $n \in \mathbb{N}$ and all $z \in \mathbb{D}$ by [5, Lemma 1] we have:

$$\frac{(1 - |z|^2)^{\frac{1}{p}} |\psi^n(z)|}{n} \leq \frac{||\psi^n||_p}{n}.$$

It forces $|\psi(z)| \leq 1$ for all $z \in \mathbb{D}$ or equivalently $||\psi||_\infty \leq 1$. In the case $1 < p < \infty$, $H^p(\mathbb{D})$ is a reflexive Banach space and by the Lorch theorem power boundedness implies mean ergodicity. So in this case there is nothing left to prove. Now, suppose $p = 1$ and $||\psi||_\infty \leq 1$. If $|\psi(z_0)| = 1$ for some $z_0 \in \mathbb{D}$, by Maximum Modulus Principle $\psi \equiv \eta$ for some $\eta \in \partial\mathbb{D}$. If $\eta = 1$, clearly $(M_\psi)_{[n]} = I$ and $||(M_\psi)_{[n]} - I|| \to 0$ as $n \to \infty$. If $\eta \neq 1$, $(M_\psi)_{[n]} = \frac{\eta(1-\eta^n)}{n(1-\eta)}$ and $||(M_\psi)_{[n]}||_p \leq \frac{2\eta}{n|1-\eta|} \to 0$. So in both cases M_ψ is uniformly mean ergodic. Let $|\psi(z)| < 1$ for all $z \in \mathbb{D}$. For $f \in H^p(\mathbb{D})$ and $z \in \mathbb{D}$, we have:

$$|(M_\psi)_{[n]} f(z)| = |\frac{\psi(z)f(z)}{n} \frac{1-\psi^n(z)}{1-\psi(z)}| \leq \frac{2|f(z)|}{n|1-\psi(z)|} \to 0, \quad as \;\; n \to \infty.$$

Thus the limit of $\{(M_\psi)_{[n]}\}_n$, if exists, has to be zero. For $k \in \mathbb{N}$,

$$||(M_\psi)_{[n]}(z^k)||_1 \leq ||(M_\psi)_{[n]}(z^k)||_2,$$

and we have the mean ergodicity of M_ψ on $H^2(\mathbb{D})$. So $||(M_\psi)_{[n]}(z^k)||_1 \to 0$, as $n \to \infty$. By linearity for all polynomial P, $||(M_\psi)_{[n]}(P)||_1 \to 0$, as $n \to \infty$. M_ψ is power bounded and polynomials are dense in $H^1(\mathbb{D})$ so the result follows. □

Theorem 2.3. *Let $\psi \in H^\infty(\mathbb{D})$ and $1 \leq p < \infty$. Then M_ψ is uniformly mean ergodic on $H^p(\mathbb{D})$, if and only if $||\psi||_\infty \leq 1$ and either $\psi \equiv \eta$ for some $\eta \in \partial\mathbb{D}$ or $\frac{1}{1-\psi} \in H^\infty(\mathbb{D})$.*

Proof. Let $||\psi||_\infty \leq 1$. If $\psi \equiv \eta$ for some $\eta \in \partial\mathbb{D}$, it was shown in the proof of Theorem 2.2, that M_ψ is uniformly mean ergodic. If $\frac{1}{1-\psi} \in H^\infty(\mathbb{D})$, then $(1-\psi)^{-1}\{0\} = \emptyset$. So for all $f \in H^p(\mathbb{D})$ and all $z \in \mathbb{D}$, $(M_\psi)_{[n]} f(z) = \frac{\psi(z)f(z)}{n} \frac{1-\psi^n(z)}{1-\psi(z)}$, from this we get

$$||(M_\psi)_{[n]}|| \leq \frac{2||\psi||_\infty}{n||1-\psi||_\infty},$$

and hence $||(M_\psi)_{[n]}|| \to 0$. Conversely, suppose M_ψ is uniformly mean ergodic. By Theorem 2.2, $||\psi||_\infty \leq 1$. Suppose ψ is not a uni-modular constant function. By Theorem 1.2 we have $1 \notin \sigma(M_\psi)$, so $I - M_\psi = M_{1-\psi}$ is an invertible operator from $H^p(\mathbb{D})$ onto itself. But $(M_{1-\psi})^{-1} = M_{\frac{1}{1-\psi}}$, so by Proposition 1.1, $\frac{1}{1-\psi} \in H^\infty(\mathbb{D})$. □

3. Conclusion

We completely characterized power bounded, uniformly mean ergodic and mean ergodic bounded multiplication operators on $H^p(\mathbb{D})$ where $1 \leq p < \infty$. M_ψ is mean ergodic, if and only if, it is power bounded, if and only if, $||\psi||_\infty \leq 1$. Also M_ψ is uniformly mean ergodic, if and only if $||\psi||_\infty \leq 1$ and either $\psi \equiv \eta$ for some $\eta \in \partial\mathbb{D}$ or $\frac{1}{1-\psi} \in H^\infty(\mathbb{D})$.

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The action of automorphisms of groups on fuzzy subgroups of dihedral groups

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Abstract. In this paper, we determine fuzzy subgroups of dihedral groups in some particular cases by the new equivalence relation which has a consistent group theoretical foundation. In this case, the corresponding equivalence classes of fuzzy subgroups of a group G are closely connected to the automorphism group and the chains of subgroups of G.

Keywords: equivalence relation, fuzzy subgroup, chain of subgroups, automorphism group, dihedral group

AMS Mathematics Subject Classification [2010]: 20N25, 20F28

1. Introduction and Preliminaries

The concept of fuzzy sets was introduced by Zadeh and in 1971, Rosenfeld used this concept to develop the theory of fuzzy groups. A fuzzy subset μ of a group G is called a fuzzy subgroup if $\min\{\mu(x), \mu(y)\} \leq \mu(xy)$ and $\mu(x) \leq \mu\left(x^{-1}\right)$, for all $x, y \in G$. Since the notion of fuzzy group is a generalization of the notion of group, many basic properties in group theory extended to fuzzy groups. The level subsets defined as $U(\mu, t) = \{x \in G \mid \mu(x) \geq t\}$, where $t \in [0,1]$, are useful in characterization of fuzzy subgroups.

The number of fuzzy subgroups of a finite group is infinite even for the trivial group $\{e\}$. Therefore, the fuzzy subgroups of G must be classified up to some equivalence relations on the set $FL(G)$ consisting of all fuzzy subgroups of G. Starting point for our discussion is the natural equivalence $\sim$ which is introduced in [6] as follows: for two fuzzy subgroups μ and η of G, $\mu \sim \eta$ if and only if for all $x, y \in G$, $\mu(x) > \mu(y)$ if and only if $\eta(x) > \eta(y)$. According to this equivalence relation, for counting all distinct fuzzy subgroups of G with respect to $\sim$ it is sufficient to find the number of all chains of subgroups of G that terminate in G and this number is denoted by $F(G)$. Utilizing the above statements, some remarkable papers has treated to the number of fuzzy subgroups of groups with respect to $\sim$, for example see [1, 3, 4]. One of the largest class of groups for which it was completely solved is constituted by finite cyclic groups.

Theorem 1.1. *[3, Corollary 4] If G is a finite cyclic group of order n (that is $G \cong \mathbb{Z}_n$) and $n = p_1^{m_1} p_2^{m_2} \ldots p_s^{m_s}$ is the decomposition of n as a product of prime factors, then the*

*Speaker. Email address: l.kamali@ardakan.ac.ir *number of all distinct fuzzy subgroups of G is given by the equality*

$$F(G) = 2^{\sum_{\alpha=1}^{s} m_\alpha} \sum_{i_2=0}^{m_2} \sum_{i_3=0}^{m_3} \cdots \sum_{i_s=0}^{m_s} (-1/2)^{\sum_{\alpha=2}^{s} i_\alpha} \prod_{\alpha=2}^{s} \binom{m_\alpha}{i_\alpha} \binom{m_1 + \sum_{\beta=2}^{\alpha} (m_\beta - i_\beta)}{m_\alpha},$$

where the above iterated sums are equal to 1 *for* $s = 1$.

Recently, Tărnăuceanu has treated the problem of classifying the fuzzy subgroups of a finite group by a new equivalence relation $\approx$ introduced in [5] as follows: suppose that G is a finite group and ρ is the following action of $Aut(G)$ on $FL(G)$:

$\rho : FL(G) \times Aut(G) \longrightarrow FL(G)$, such that $\rho(\mu, f) = \mu \circ f$, for all $(\mu, f) \in FL(G) \times Aut(G)$.

We denote by $\approx_\rho$ the equivalence relation on $FL(G)$ induced by ρ, namely

$$\mu \approx_\rho \eta \text{ if and only if there exists } f \in Aut(G) \text{ such that } \eta = \mu \circ f.$$

Now, consider the equivalence relation $\approx$ on $FL(G)$ which is described with chains of subgroups of G as follows [5]: let $\mu, \eta \in FL(G)$ and put $\mu(G) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that $\alpha_1 > \alpha_2 > \ldots > \alpha_n$, $\eta(G) = \{\beta_1, \beta_2, \ldots, \beta_m\}$ such that $\beta_1 > \beta_2 > \ldots > \beta_m$. Then, μ and η determine the following chains of subgroups of G which ends in G:
$\mathcal{C}_\mu : U(\mu, \alpha_1) \subset U(\mu, \alpha_2) \subset \ldots \subset U(\mu, \alpha_n) = G$ and $\mathcal{C}_\eta : U(\eta, \beta_1) \subset U(\eta, \beta_2) \subset \ldots \subset U(\eta, \beta_m) = G$. The equivalence relation $\approx$ on $FL(G)$ is defined as follows:

$$\mu \approx \eta \text{ iff } \exists f \in Aut(G) \text{ such that } f(\mathcal{C}_\eta) = \mathcal{C}_\mu.$$

Obviously, this is a little more general than $\approx_\rho$. In fact, if $\mu \approx \eta$, then their images are not necessarily equal, but certainly there is a bijection between $Im(\mu)$ and $Im(\eta)$. Moreover, we also remark that $\approx$ generalizes the equivalence relation $\sim$ defined in [3] excepting the case when G is cyclic which we have $\approx = \sim$.

Next, we will focus on computing the number $\mathcal{N}$ of distinct fuzzy subgroups of G with respect to $\approx$, that is the number of distinct equivalence classes of $FL(G)$ modulo $\approx$. Denote by $\bar{\mathcal{C}}$ the set consisting of all chains of subgroups of G terminated in G. Then, the previous action ρ of $Aut(G)$ on $FL(G)$ can be seen as an action of $Aut(G)$ on $\bar{\mathcal{C}}$ and $\approx_\rho$ as the equivalence relation induced by this action. An equivalence relation on $\bar{\mathcal{C}}$ which is similar with $\approx$ can also be constructed in the following manner: for two chains

$$\mathcal{C}_1 : H_1 \subset H_2 \subset \cdots H_m = G \text{ and } \mathcal{C}_2 : K_1 \subset K_2 \subset \cdots \subseteq K_n = G$$

of $\bar{\mathcal{C}}$, we put

$$\text{(1)} \qquad \mathcal{C}_1 \approx \mathcal{C}_2 \text{ iff } m = n \text{ and } \exists f \in Aut(G) \text{ such that } f(H_i) = K_i,\ 1 \le i \le n.$$

In this case the orbit of a chain $\mathcal{C} \in \bar{\mathcal{C}}$ is $\{f(\mathcal{C}) \mid f \in Aut(G)\}$, while the set of all chains in $\bar{\mathcal{C}}$ that are fixed by an automorphism f of G is $Fix_{\bar{\mathcal{C}}}(f) = \{\mathcal{C} \in \bar{\mathcal{C}} \mid f(\mathcal{C}) = \mathcal{C}\}$. Now, the Burnside's lemma leads to the following Theorem:

Theorem 1.2. *The number $\mathcal{N}$ of all distinct fuzzy subgroups with respect to $\approx$ of a finite group G is given by the equality*

$$\mathcal{N} = \frac{1}{|Aut(G)|} \sum_{f \in Aut(G)} |Fix_{\bar{\mathcal{C}}}(f)|.$$

The above formulas can successfully be used to calculate $\mathcal{N}$ for any finite group G whose subgroup lattice $L(G)$ and automorphism group $Aut(G)$ are known. In [2, 5], the number $\mathcal{N}$ is explicitly determined for some particular classes of dihedral groups, by Theorem 1.2. But using Theorem 1.2 for counting $\mathcal{N}(D_{2n})$ in the general case, needs extended calculations. Therefore in the next section, we use the relation (1) and compute explicitly the number $\mathcal{N}$ of all distinct fuzzy subgroups of dihedral groups with respect to $\approx$.

2. Main results

First, we recall that the dihedral group D_{2n} $(n \geq 2)$ is the symmetry group of a regular polygon with n sides and has the order $2n$. More abstractly, one can define the dihedral group D_{2n} as any group having the presentation

$$D_{2n} =< x, y \mid x^n = y^2 = e,\ y^{-1}xy = x^{-1} > .$$

Note that D_4 is considered as a group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

By the maximal subgroups described in [4], the structure of the subgroup lattice of D_{2n}, i.e. $L(D_{2n})$, is as follows: for every divisor r of n, D_{2n} possesses a subgroup isomorphic to $\mathbb{Z}_r$, namely ${H_0}^r =< x^{\frac{n}{r}} >$ and $\frac{n}{r}$ subgroups isomorphic to D_{2r}, namely ${H_i}^r =< x^{\frac{n}{r}}, x^{i-1}y >$, where $1 \leq i \leq \frac{n}{r}$.

By the properties of automorphism group and the order of elements of D_{2n}, we find that $Aut(D_{2n}) = \{f_{\alpha,\beta} \mid 0 \leq \alpha \leq n-1 \text{ s.t. } (\alpha, \mathrm{n}) = 1,\ 0 \leq \beta \leq \mathrm{n}-1\}$, where

$$f_{\alpha,\beta} = \begin{cases} x \longrightarrow x^{\alpha} \\ y \longrightarrow x^{\beta}y \end{cases} .$$

This implies that $|Aut(D_{2n})| = n\varphi(n)$, which φ is Euler's Totient function [5].

Suppose that $\mathcal{V}_1$ is the set of all chains of cyclic subgroups of D_{2n} terminating in D_{2n}. Also, assume that $\mathcal{V}$ is the set of all chains of subgroups of ${H_0}^n =< x >\cong \mathbb{Z}_n$ terminating in ${H_0}^n$, then $|\mathcal{V}| = F(\mathbb{Z}_n)$ which is determined in Theorem 1.1. It is clear that every chain of $\mathcal{V}_1$ is obtained by adding D_{2n} to the end of a chain of $\mathcal{V}$ or putting D_{2n} instead of ${H_0}^n$ in a chain of $\mathcal{V}$. By the structure of $\mathcal{V}_1$ and $Aut(D_{2n})$, all chains of $\mathcal{V}_1$ are distinct under $\approx$. Consequently, $\mathcal{N}(\mathcal{V}_1) = 2\,|\mathcal{V}| = 2F(\mathbb{Z}_n)$. Now, assume that $\mathcal{V}_2$ is the set of all chains of subgroups of D_{2n} which last two elements of them are ${H_i}^1 =< x^{i-1}y >$ and D_{2n}, where $1 \leq i \leq n$. More precisely, the chains terminating in ${H_i}^1 \subset D_{2n}$ and containing the elements of $L({H_i}^1)$. Therefore, we get $\mathcal{V}_2 = \{\mathcal{C}_{i1}, \mathcal{C}_{i2} \mid 1 \leq i \leq n\}$, where $\mathcal{C}_{i1}:\ {H_i}^1 \subset D_{2n}$ and $\mathcal{C}_{i2}:\ \{e\} \subset {H_i}^1 \subset D_{2n}$. Obviously, for all $1 \leq i_1, i_2 \leq n$, there is $f_{\alpha,\beta} \in Aut(D_{2n})$ such that $f_{\alpha,\beta}(\mathcal{C}_{i_1j}) = \mathcal{C}_{i_2j}$ and this implies that $\mathcal{C}_{i_1j} \approx \mathcal{C}_{i_2j}$, where $j = 1, 2$. Then, there are two distinct chains with respect to $\approx$ in $\mathcal{V}_2$ which imply that $\mathcal{N}(\mathcal{V}_2) = 2$. Similar to the above discussions, we will determine $\mathcal{N}(D_{2n})$, where $n = p_1^{m_1}p_2^{m_2}\cdots p_s^{m_s}$ is the decomposition of n as a product of distinct prime factors.

If $m_1 = m_2 = \cdots = m_s = 1$, then by the structure of $L(D_{2n})$, we get $\mathcal{N}(D_{2p_1p_2\cdots p_s}) = \mathcal{N}(\mathcal{V}_1)+\mathcal{N}(\mathcal{V}_2)+\sum_{k=1}^{s-1}\mathcal{N}(\mathcal{W}_k)$, where $\mathcal{W}_k$ is defined as follows: suppose that $1 \leq j_1, \cdots, j_k \leq s$ and for all $1 \leq i \leq \frac{n}{p_{j_1}\cdots p_{j_k}}$, consider $\mathcal{W}_{ki}^{j_1,j_2,\cdots,j_k}$ as the set of all chains which the last two elements of them are ${H_i}^{p_{j_1}\cdots p_{j_k}}$ and D_{2n}, exactly the chains terminating in ${H_i}^{p_{j_1}\cdots p_{j_k}} \subset D_{2n}$ and containing the elements of $L({H_i}^{p_{j_1}\cdots p_{j_k}})$. Also, assume that $\mathcal{W}_k$ is a union of the sets $\mathcal{W}_k^{j_1,j_2,\cdots,j_k}$, where $1 \leq j_1, j_2, \cdots, j_k \leq s$ and

$$\mathcal{W}_k^{j_1,j_2,\cdots,j_k} = \bigcup_{i=1}^{n/(p_{j_1}\cdots p_{j_k})} \mathcal{W}_{ki}^{j_1,j_2,\cdots,j_k}.$$

One can easily show that for all $\mathcal{C}_{\mathcal{W}_{ki_1}^{j_1,j_2,\cdots,j_k}} \in \mathcal{W}_{ki_1}^{j_1,j_2,\cdots,j_k}$ there exists $\mathcal{C}_{\mathcal{W}_{ki_2}^{j_1,j_2,\cdots,j_k}} \in \mathcal{W}_{ki_2}^{j_1,j_2,\cdots,j_k}$ such that $\mathcal{C}_{\mathcal{W}_{ki_1}^{j_1,j_2,\cdots,j_k}} \approx \mathcal{C}_{\mathcal{W}_{ki_2}^{j_1,j_2,\cdots,j_k}}$, where $1 \leq i_1, i_2 \leq \frac{n}{p_{j_1}\cdots p_{j_k}}$. Then, for counting the distinct chains with respect to $\approx$ in $\mathcal{W}_k^{j_1,j_2,\cdots,j_k}$, it is sufficient to determine $\mathcal{N}(\mathcal{W}_{k1}^{j_1,j_2,\cdots,j_k})$ which is equal to $\mathcal{N}(D_{2p_1p_2\cdots p_k})$. Hence, $\mathcal{N}(\mathcal{W}_k) = \binom{s}{k}\mathcal{N}(\mathcal{W}_k^{j_1,j_2,\cdots,j_k}) = \binom{s}{k}\mathcal{N}(D_{2p_1p_2\cdots p_k})$. Since $\mathcal{N}(\mathcal{V}_1) = 2F(\mathbb{Z}_{p_1p_2\cdots p_s})$ and $\mathcal{N}(\mathcal{V}_2) = 2$, by the above results we get the following theorem.

THEOREM 2.1. *Let $p_1, p_2, \cdots, p_s$ be prime numbers. The number $\mathcal{N}$ of all distinct fuzzy subgroups of the group $D_{2p_1p_2\cdots p_s}$ is given by the equality*

$$\mathcal{N}(D_{2p_1p_2\cdots p_s}) = 2F(\mathbb{Z}_{p_1p_2\cdots p_s}) + 2 + \sum_{k=1}^{s-1}\binom{s}{k}\mathcal{N}(D_{2p_1p_2\cdots p_k}),$$

where $s \geq 1$ and the above iterated sum is equal to 0 for $s = 1$.

Similar to the above discussions, we get the following Theorem.

THEOREM 2.2. *The number $\mathcal{N}$ of all distinct fuzzy subgroups of the group D_{2n} with respect to $\approx$ is given by the equality*

$$\mathcal{N}(D_{2n}) = 2F(\mathbb{Z}_n) + 2 + \sum_{\substack{r|n \\ r\neq 1,n}} \mathcal{N}(D_{2r}).$$

In particular,

$$\mathcal{N}(D_{2p^m}) = 2^{m+1} + 2 + \sum_{j=1}^{m-1}\mathcal{N}(D_{2p^j}),$$

where the above iterated sum is equal to 0 for $m = 1$.

3. Conclusion

In this paper, we obtained a recurrence relation that permit us to determine the distinct fuzzy subgroups of dihedral groups relative to a certain equivalence relation these groups.

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52nd Annual Iranian Mathematics Conference
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Some results on Ramsey numbers of K_n-good graphs

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Abstract. Let G and G_1, G_2 be given graphs. By $G \to (G_1, G_2)$ we mean if the edges of G are arbitrarily colored by red and blue, then there is either a red copy of G_1 or a blue copy of G_2 in G. The Ramsey number $R(G_1, G_2)$ is defined as the smallest positive integer n such that $K_n \to (G_1, G_2)$. Also, the star-critical Ramsey number $R_*(G_1, G_2)$ is defined as $\min\{\delta(G) : \ G \subseteq K_r, \ G \to (G_1, G_2)\}$, where, $r = R(G_1, G_2)$. If G is a connected vertex transitive graph on n vertices and G is a K_m-good graph i.e. $R(G, K_m) = (n-1)(m-1)+1$, then the Ramsey number and the star-critical Ramsey number of K_m versus G^{+e} is determined exactly, where G^{+e} is obtained from G by adding a leaf to G.

Keywords: star-critical Ramsey number, complete graph, transitive graph

AMS Mathematics Subject Classification [2010]: 05C55, 05D10

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1. Introduction

In this note, we are only concerned with simple finite graphs and we follow [1] for terminology and notations not defined here. For a given graph G, we denote its *vertex set*, *edge set*, *maximum degree*, *minimum degree* and *chromatic number* of G by $V(G)$, $E(G)$, $\Delta(G)$, $\delta(G)$ and $\chi(G)$, respectively. For a vertex $v \in V(G)$, we use $\deg(v)$ and $N(v)$ to denote the degree and the set of neighborhoods of v in G, respectively. Also, for a given subset A of the vertex set of G, we use $G[A]$ to denote the induced subgraph of G spanned by A and for given disjoint subsets A and B of $V(G)$, we use $E[A, B]$ to denote the set of all edges in the bipartite graph whose partite sets are A and B. A *transitive* graph, is a graph such that every pair of vertices is equivalent under some element of its automorphism group. The *complete graph* on n vertices is denoted by K_n and a *tree* is an undirected, connected and acyclic graph. A *leaf* or *end-vertex* of a tree is a vertex of degree 1. In addition, for a given red/blue coloring of the edges of a graph G, we use G^r and G^b to denote the spanning subgraphs of G induced by the edges of colors red and blue, respectively.

Let G and G_1, G_2 be given graphs. By $G \to (G_1, G_2)$ we mean if the edges of G are arbitrarily colored red and blue, then there is either a red copy of G_1 or a blue copy of G_2 in G. A red/blue coloring of the edges of G is called a (G_1, G_2)-*free coloring* if $G_1 \nsubseteq G^r$ and $G_2 \nsubseteq G^b$. The *Ramsey number* $R(G_1, G_2)$ is defined as the smallest positive integer

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n such that $K_n \to (G_1, G_2)$. The existence of such a positive integer is guaranteed by the Ramsey's classical result [7]. For a survey on Ramsey theory and results in this area, we refer the reader to the regularly updated survey by Radziszowski [6]. Also, the *star-critical Ramsey number* $R_*(G_1, G_2)$ is defined as $\min\{\delta(H) : H \subseteq K_r, H \to (G_1, G_2)\}$, where $r = R(G_1, G_2)$. The concept of the star-critical Ramsey number was first defined by Hook and Isaak in [3]. For more details check [2, 4, 8]. Let $K_n \sqcup K_{1,k}$ be the graph obtained from K_n by adding a new vertex v adjacent to k vertices of K_n. It is easy to see that the star-critical Ramsey number $R_*(G_1, G_2)$ is equivalent to finding the smallest integer k such that $K_{r-1} \sqcup K_{1,k} \to (G_1, G_2)$, where $r = R(G_1, G_2)$.

For given connected graphs G_1 and G_2, we say G_1 is a G_2-*good graph* if

$$R(G_1, G_2) = (\chi(G_2) - 1)(|V(G_1)| - 1) + s(G_2),$$

where $s(G_2)$ is the *chromatic surplus* of G_2, i.e., the minimum cardinality of color classes over all chromatic colorings of $V(G_2)$.

2. Main Results

Let G be a connected vertex transitive graph on n vertices which is K_m-good graph, i.e. $R(G, K_m) = (n-1)(m-1)+1$. By G^{+e} we mean the graph obtained from G by adding a leaf neighbor to a vertex of G. In the following, we determine the Ramsey number of $R(G^{+e}, K_m)$ and also the star-critical Ramsey number $R_*(G^{+e}, K_m)$. For this purpose, first we show that any (G^{+e}, K_m)-free coloring of the complete graph $K_{n(m-1)}$ is unique and we use this uniqueness to determine the Ramsey number $R(G^{+e}, K_m)$.

Definition 2.1. Let $m \geq 2$ and $n \geq 3$ be given integers. Let $\mathrm{F} = K_{n(m-1)}$ be the complete graph with the following red/blue coloring.

$$\mathrm{F}: \quad \begin{aligned} \mathrm{F}^r &= (m-1)K_n \\ \mathrm{F}^b &= K_{m-1}(n, n, \ldots, n). \end{aligned}$$

where $K_{m-1}(n, n, \ldots, n)$ is the complete $(m-1)$-partite graph with n vertices in each part.

Lemma 2.2. *Let $m \geq 2$ and $n \geq 3$ be given integers and let G be a connected vertex transitive graph on n vertices which is K_m-good. If c be a (G^{+e}, K_m)-free coloring of $K_{n(m-1)}$, then c is the coloring described in Definition 2.1.*

Proof. Let G be a graph with $R(G, K_m) = (n-1)(m-1)+1$ and c be a (G^{+e}, K_m)-free coloring of $F = K_{n(m-1)}$. We use induction on m. Let $m = 2$. Since c is a (G^{+e}, K_2)-free coloring, then c does not contain blue copy of K_2 and so, F is a monochromatic red copy of K_n. Thus, the c coloring is as described in Definition 2.1.

$$\mathrm{F}: \quad \begin{aligned} \mathrm{F}^r &= K_n \\ \mathrm{F}^b &= K_1(n). \end{aligned}$$

Now, let $m \geq 3$ and c be a red/blue coloring on F. Since $n, m \geq 3$, then $n(m-1) \geq R(G, K_m)$ and so, c contains either a monochromatic blue copy of G or a monochromatic red copy of K_m. Since c is (G^{+e}, K_m)-free coloring, thus, c must contains a red copy of

G, name this copy as G_0. Delete the vertices of red G_0 from F and let H be the resulting graph. We have

$$V(H) = n(m-1) - n = n(m-2).$$

Let c' be the induced red/blue coloring of c on the edges of H. Clearly, c' is a (G^{+e}, K_{m-1})-free coloring because otherwise, if there is a blue copy of K_{m-1} in H and $E[V(G_0), V(H)] \subseteq F^b$, then $K_m \subseteq F^b$ or if there is red edge $e \in E[V(G_0), V(H)]$, then $e \cup G_0$ form a red copy of G^{+e}, a contradiction. So, c' is a (G^{+e}, K_{m-1})-free coloring on H with $|V(H)| = n(m-2)$, thus, by the induction hypothesis, c' is red/blue coloring of H as described in Definition 2.1, such that

$$H: \quad \begin{aligned} H^r &= (m-2)K_n \\ H^b &= K_{m-2}(n, \ldots, n). \end{aligned}$$

Since G_0 is a vertex transitive graph, then an edge $xy \in E[V(G_0), V(H)]$ of color red yields a monochromatic red copy of G^{+e}. Thus, $E[V(G_0), V(H)] \in F^b$ and so, c is the red/blue coloring of F, described in the Definition 2.1. □

In the following, we show that for a given connected vertex transitive K_m-good graph G, we have $R(G^{+e}, K_m) = n(m-1) + 1$.

Theorem 2.3. *Let $m \geq 2$ and $n \geq 3$ be given integers and let G be an arbitrary connected vertex transitive K_m-good graph on n vertices. Then,*

$$R(G^{+e}, K_m) = n(m-1) + 1.$$

Proof. Let r be the claimed number for $R(G^{+e}, K_m)$. To see r is a lower bound for $R(G^{+e}, K_m)$, we represent a (G^{+e}, K_m)-free coloring of $H = K_{r-1}$. Since $r-1 = n(m-1)$, consider the red/blue coloring of $K_{n(m-1)}$ described in Definition 2.1. Since $\Delta(H^r) \leq n-1$, then H^r does not contain G^{+e}. On the other hand, the $\chi(H^b) \leq m-1$ and so, H^b does not contain blue copy of K_m. Therefore, $K_{r-1} \nrightarrow (G^{+e}, K_m)$.

For the upper bound, let c be an arbitrary red/blue coloring of $F = K_r$. Delete an arbitrary vertex v from F and let H be the resulting graph with $(r-1)$ vertices. Let c' be the coloring induced by c on H. If c' contains a red copy of G^{+e} or a blue copy of K_m, we are done. So, we may assume that c' is a (G^{+e}, K_m)-free coloring of H and so, by Lemma 2.2, this coloring is unique as described in Lemma 2.2.

$$H: \quad \begin{aligned} H^r &= (m-1)K_n \\ H^b &= K_{m-1}(n, \ldots, n). \end{aligned}$$

Now, if the edge $vu \in F^r$, for some $u \in V(H)$, then we have a red copy of G^{+e}, so, $E[v, V(F)] \in F^b$, which form a blue copy of K_m. Therefore, for any red/blue arbitrary coloring of K_r, we have a red copy of G^{+e} or a blue copy of K_m and thus, $F \rightarrow (G^{+e}, K_m)$. □

In squall, we determine that star-critical Ramsey number $R_*(G^{+e}, K_m)$.

Theorem 2.4. *If $m \geq 2$ and $n \geq 3$ be given integers and let G be a connected vertex transitive K_m-good graph on n vertices, then $R_*(G^{+e}, K_m) = n(m-2) + 1$.*

Proof. Let $r = r(G^{+e}, K_m) = n(m-1)+1$ and r_* be the claimed number for $r_*(G^{+e}, K_m)$. For the lower bound, we represent a red/blue coloring of graph $H = K_{r-1} \sqcup K_{1,r_*-1}$, which is (G^{+e}, K_m)-free coloring. Partition the vertices of K_{r-1} into $(m-1)$ parts $V_1, V_2, \ldots, V_{m-1}$ such that for every i, $1 \leq i \leq m-1$, $|V_i| = n$. Color all edges contained in V_i, $1 \leq i \leq m-1$, by color red and the rest of edges by color blue. Now, add a vertex v adjacent to every vertex in V_i, $1 \leq i \leq m-2$, by color blue. Since $\chi(H^b) \leq m-1$, then H^b does not contain K_m as a subgraph. Also, $\Delta(H^r) \leq n-1$ and thus, the subgraph H^r does not contain a red copy of G^{+e}. Therefore, we have a (G^{+e}, K_m)-free coloring of H and so, $H \nrightarrow (G^{+e}, K_m)$.

For the upper bound, let $F = K_{r-1} \sqcup K_{1,r^*}$ and let v_0 be the vertex of degree r^* in F. Consider an arbitrary red/blue coloring of F which induced a red/blue coloring of $F \setminus \{v_0\} \cong K_{r-1}$, called c'. If c' contains a red copy of G^{+e} or a blue copy of K_m, we are done. So, let c' be a (G^{+e}, K_m)-free coloring of $F \setminus \{v_0\}$ and by Lemma 2.2, this coloring is unique as described in Lemma 2.2.

Since $G \subseteq K_n$ and $(F \setminus \{v_0\})^r = (m-1)K_n$, if $uv_0 \in F^r$, for some $u \in V(F \setminus \{v_0\})$, then $G^{+e} \in F^r$. Therefore, we may assume that all $r^* = n(m-2)+1$ edges incident with v_0 are color blue. As $\deg(v_0) \geq r^*$, it is easy to see that v_0 has at least one neighbour in each V_i, $1 \leq i \leq m-1$, and hence, $K_m \in F^b$. Thus, $F \rightarrow (G^{+e}, K_m)$. □

In [5] it is proved that the cycle C_n is K_m-good graph, i.e. for $m \geq 4$ and $n \geq 4m+2$, $R(C_n, K_m) = (n-1)(m-1)+1$. Since C_n is a connected vertex transitive graph, then by Theorems 2.3 and 2.4, we have $R(C_n^{+e}, K_m) = n(m-1)+1$ and $R_*(C_n^{+e}, K_m) = n(m-2)+1$, for $m \geq 4$ and $n \geq 4m+2$.

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A Hermite interpolation method for Duffing equations involving both integral and non-integral forcing terms

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Abstract. In this work, an efficient method is presented for the numerical solution of the nonlinear Duffing equation as an important equation for the mathematical modeling of real-life phenomena. The presented method is based upon the two-point Taylor formula. It is tried to utilize the problem structure in order to extract the needed data for finding the approximate solution. The efficiency and accuracy of the method is demonstrated through the numerical results.

Keywords: Two-point Taylor formula, Duffing equation, Integral and non-integral forcing terms

AMS Mathematics Subject Classification [2010]: 34B15, 34K28, 65D05

1. Introduction

Integro-differential equations frequently appear in the mathematical modeling of real-life phenomena. A wide range of problems in the fluid mechanics, electromagnetics, neural networks, nuclear reactors, biological populations, and many other areas of science and engineering can be modeled with the help of integro-differential equations, (e.g., see [2]). Hence, many efforts have been devoted to study this diverse class of mathematical equations. The aim of this paper is to present an approach for the numerical solution of the Duffing equation involving both integral and non-integral forcing terms which can be formulated in the following general form:

$$(1)\qquad \begin{cases} u''(x) + \sigma u'(x) + f\left(x, u(x), u'(x)\right) + \displaystyle\int_0^x k\left(x, t, u(t)\right) dt = 0, & x \in [0,1], \\ p_0 u(0) - q_0 u'(0) = a, \quad p_1 u(1) + q_1 u'(1) = b, \end{cases}$$

where $\sigma \in \mathbb{R} - \{0\}$, $a, b \in \mathbb{R}$, and $p_0, q_0, p_1, q_1 \in \mathbb{R}^+$. Furthermore, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$, and $k : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are assumed to be known continuous functions, and u is the unknown solution of the problem.

Duffing equation is a famous nonlinear model in the interpretation of the periodic orbit extraction, signal processing, mechanical oscillators, fuzzy modelling, brain modelling, etc [5]. Over the last few years, the application of the Duffing equation in the simulation of chaotic phenomena has been significantly considered in the study of the human body

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and the prediction of diseases. The use of the Duffing equation as a mathematical model to predict the arrhythmia, and to measure the bloodstream speed can be mentioned as two attractive examples in this field [1]. Moreover, the Duffing equation is an interesting tool for investigating the efficiency of numerical methods in solving nonlinear integro-differential equations.

In this paper, the two-point Taylor formula as a certain case of the Hermite interpolant is utilized to construct a numerical method for solving Eq. (1). In the proposed method, an approximation of u, the unknown solution of Duffing equation (1), is provided by estimating the values of u and its derivatives up to an adequate order at $x = 0$ and $x = 1$. In order to estimate these unknown values, the properties of the equation and the existed boundary conditions are applied. The method of extracting the needed data and the construction of the approximation are described in the next section.

2. The computational method

In this section, a polynomial interpolant based on some estimated data is presented to approximate the solution of Duffing equation (1).

Two-point Taylor formula. Two-point Taylor formula is a special case of the Hermite interpolant in which the values of a function f, and its derivatives up to $(n-1)$th order at the endpoints of the interval $[0,1]$ are utilized to obtain a $(2n-1)$th degree polynomial P as an approximation of f over this interval. Indeed, in this interpolation method, f is approximated by polynomial P which satisfies the following interpolation conditions:

$$\mathrm{P}^{(j)}(x) = f^{(j)}(x), \quad j = 0, 1, \dots, n-1, \quad x \in \{0, 1\}.$$

In the following theorem, the unique solution of the mentioned interpolation problem is presented.

Theorem 2.1. [3] *Let the values of $f \in C^{2n}[0,1]$ and its first $(n-1)$ derivatives be available at $x = 0$ and $x = 1$. Then, f can be approximated using the polynomial*

$$\mathrm{P}_{2n-1}(f;x) = \sum_{j=0}^{n-1} \left[C_{n,j}(x) f^{(j)}(0) + (-1)^j C_{n,j}(1-x) f^{(j)}(1) \right],$$

where

$$C_{n,j}(x) = \frac{x^j}{j!}(1-x)^n \sum_{k=0}^{n-j-1} \binom{n+k-1}{k} x^k, \quad j = 0, 1, \dots, n-1.$$

Furthermore, the error term of the approximation can be computed as

$$\mathrm{R}_n(f;x) = \frac{f^{(2n)}(\xi_x)}{(2n)!} x^n (1-x)^n, \quad \xi_x \in (0,1).$$

Now, assume that $u \in C^{2n}[0,1]$ is the exact solution of problem (1), then, according to Theorem 2.1, an approximation of u can be determined as

$$\mathrm{P}_{2n-1}(u;x) = \sum_{j=0}^{n-1} \left[C_{n,j}(x) u^{(j)}(0) + (-1)^j C_{n,j}(1-x) u^{(j)}(1) \right]. \tag{2}$$

It is obvious that since u is unknown, the values of $u^{(j)}(x), j = 0, 1, \dots, n-1, x \in \{0, 1\}$ are not generally available. Therefore, the main challenge in computing (2) is to obtain suitable estimations of these unknown values.

Computation of derivatives. Let us denote the unknown values $u'(0)$ and $u'(1)$ by β_0 and β_1, respectively, then, $u(0)$ and $u(1)$ can be computed from the separated boundary conditions of (1) as

$$u(0) = \frac{a + q_0\beta_0}{p_0}, \quad u(1) = \frac{b - q_1\beta_1}{p_1}.$$

Moreover, from Eq. (1), $u''(x)$ can be written in the form

$$u''(x) = -\left(\sigma u'(x) + f\left(x, u(x), u'(x)\right) + \int_0^x k\left(x, t, u(t)\right) dt\right). \tag{3}$$

Therefore, $u''(0)$ and $u''(1)$ can be obtained as

$$u''(0) = -\sigma\beta_0 - f\left(0, \frac{a + q_0\beta_0}{p_0}, \beta_0\right), \quad u''(1) = -\left(\sigma\beta_1 + f\left(1, \frac{b - q_1\beta_1}{p_1}, \beta_1\right) + \alpha_0\right),$$

where

$$\alpha_0 = \int_0^1 k\left(x, t, u(t)\right) dt\bigg|_{x=1}.$$

Indeed, $u''(0)$ is obtained in one unknown β_0, while $u''(1)$ is determined in two unknowns β_1 and α_0. Note that j times differentiating (3) leads to

$$u^{(j+2)}(x) = -\left(\sigma u^{(j+1)}(x) + \frac{d^j}{dx^j} f\left(x, u(x), u'(x)\right) + \left(D^j k\right)(x)\right), \quad j = 0, 1, \ldots n-3, \tag{4}$$

where

$$\left(D^j k\right)(x) = \sum_{i=0}^{j-1} \left(\frac{\partial^i k(x, t, u(t))}{\partial x^i}\bigg|_{t=x}\right)^{(j-i-1)} + \int_0^x \frac{\partial^j k(x, t, u(t))}{\partial x^j} dt. \tag{5}$$

Subsequently, the values of $u^{(j)}(0)$ and $u^{(j)}(1)$ for $j = 2, 3, \ldots, n-1$ can be calculated recursively from Eq. (4) in some unknowns. It should be noted that when the derivatives are computed at $x = 0$, the integral term of (5) is equal to zero. Thus, all the derivatives at $x = 0$ can be obtained in only one unknown β_0. However, the values of $u^{(j)}(1)$ for $j = 2, 3, \ldots, n-1$ are computed in the unknown values β_1 and α_{j-2}, where

$$\alpha_j = \int_0^1 \frac{\partial^j k\left(x, t, u(t)\right)}{\partial x^j} dt\bigg|_{x=1}, \quad j = 0, 1, \ldots, n-3.$$

Now, we should compute n unknowns β_0, β_1, and $\alpha_j, j = 0, 1, \ldots, n-3$, to obtain the needed data for determining (2). Hence, we should construct a system of n equations to evaluate these unknowns. To this aim, we employ (4) to get $n-2$ equations

$$u^{(j+2)}(b) = -\left(\sigma u^{(j+1)}(b) + \frac{d^j}{dx^j} f\left(x, u(x), u'(x)\right)\bigg|_{x=b} + \left(\tilde{D}^j k\right)(b)\right), \quad j = 0, 1, \ldots n-3,$$

where $\left(\tilde{D}^j k\right)(x)$ is obtained by substituting P_{2n-1} for u in the integral term of (5). Also the remaining two equations can be constructed as

$$\begin{cases} \operatorname{res}(x_1) = 0, \\ \operatorname{res}(x_2) = 0, \end{cases} \quad x_1, x_2 \in (0, 1),$$

where $\operatorname{res}(x)$ is the residual function defined by substituting P_{2n-1} for u in Eq. (1). Finally, by solving the provided system, polynomial (2) can be determined as an estimation of u.

3. Illustrative examples

In order to study the utility of the method, we apply it to solve two problems and report the results in this section. The computations are performed by using the Maple software. It should be emphasized that in order to increase the speed of the algorithm, the integrals which appear in the approximations are estimated by using the Gauss-Legendre quadrature formulas.

Example 3.1. [4] As the first example consider

$$\begin{cases} u''(x) - 1.72u'(x) + e^{-u(x)} - \displaystyle\int_0^x (1-2t)\, u(t)dt = f(x), \\ u(0) - 3u'(0) = -1, \quad u(1) + 3u'(1) = -1, \end{cases}$$

where f is chosen so that the exact solution of the problem is $u(x) = \frac{1}{3}\sin(x - x^2)$.

Example 3.2. The second example is the nonlinear Duffing equation

$$\begin{cases} u''(x) + u'(x) + u(x)\left(u(x) + u'(x)\right) + \displaystyle\int_0^x t^2 u^3(t)dt = f(x), \\ u(0) - u'(0) = -1, \quad u(1) + u'(1) = 7e, \end{cases}$$

where f has been chosen such that the exact solution of the problem is $u(x) = e^x(x+2)$.

The absolute errors of the proposed method for solving both the examples and the used CPU times for different choices of n are reported in Table 1.

Table 1. The results of the presented method.

	Example 3.1		Example 3.2	
n	Absolute Error	CPU time (s)	Absolute Error	CPU time (s)
4	2.12e-05	0.47	1.00e-06	0.58
8	1.54e-11	1.05	9.00e-16	1.44
12	4.00e-19	3.82	4.00e-30	4.01

4. Conclusion

Two-point Taylor formula was utilized as the basis of a numerical method to approximate the solution of the Duffing equation. The proposed method is easy to implement and provides acceptable results without the need to choose a large n. The illustrative examples demonstrated the utility of the method for solving such integro-differential equations.

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Commutative rings which every module has a prime submodule

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Abstract. In this paper, we study rings in which every nonzero module has a prime submodule. At first, we show that if every nonzero submodule of an R-module M has a prime submodule, then $N(R)$ is T-nilpotent on M. Finally, we prove every nonzero R-module has at least one prime submodule if and only if R is a Max ring.

Keywords: prime submodule, maximal submodule, Max ring

AMS Mathematics Subject Classification [2010]: 13A15, 13C60

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1. Introduction

Let R be a commutative ring with non-zero identity, and let M be a unitary R-module. $N(R)$ will denote nil radical of R, and $J(R)$ will denote Jacobson radical of R. A proper submodule N of an R-module M is called to be prime whenever $rm \in N$, then $m \in N$ or $rM \subseteq N$ [4]. The concept of prime submodule is important in commutative algebra since it is used to classify modules. We study modules that have at least one prime submodule. In Proposition 2.8, it is proved that if every nonzero submodule of M has at least one prime submodule, then $N(R)$ is T-nilpotent on M. According to Theorem 2.14, every nonzero R-module has a prime submodule if and only if $N(R)$ is T-nilpotent and $R/N(R)$ is a von Neumann regular ring.

The ring R is a Max ring if every nonzero R-module M has at least one maximal submodule.

Hamsher in [2] proved R is a Max ring if and only if $J(R)$ is T-nilpotent and $R/J(R)$ is a von Neumann regular ring. In Theorem 2.15 shows that every nonzero R-module has a prime submodule if and only if R is a Max ring.

2. Main results

The proof of the following theorem is routine

Theorem 2.1. *Let R be a ring and let $\{M_\alpha\}_{\alpha\in\Lambda}$ be a family of R-modules. Then the following are equivalent:*
(i) Every nonzero submodule of M_α has a prime submodule for each $\alpha \in \Lambda$.

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(ii) Every nonzero submodule of $\oplus_{\alpha\in\Lambda}M_\alpha$ *has a prime submodule.*
(iii) Every nonzero submodule of $\prod_{\alpha\in\Lambda}M_\alpha$ *has a prime submodule.*

Let M be an R-module. $Rad(0_M)=\cap\{K|K$ is a prime submodule of $M\}$. For each ordinal α, we shall define $Rad_\alpha(0_M)$ in the following manner:
$Rad_0(0_M)=M$,
$Rad_{\alpha+1}(0_M)=Rad(Rad_\alpha(0_M))$,
$Rad_\alpha(0_M)=\cap_{\beta<\alpha}Rad_\beta(0_M)$ when α is limit ordinal.

Example 2.2. Consider $\mathbb{Q}$ to be a $\mathbb{Z}$-module. Every nonzero submodule of $\mathbb{Q}$ has a prime submodule. However, $\mathbb{Q}$ does not have a maximal submodule.

Theorem 2.3. *Let M be an R-module. Then the following statements are equivalent:*
(i) Every nonzero submodule of M has a prime submodule.
(ii) $Rad_\beta(0_M)$ has a prime submodule, or is 0, for every ordinal β.
(iii) $Rad_\alpha(0_M)=0$ for some α.

Definition 2.4. Let M be an R-module and I be an ideal of R. I is T-nilpotent on M if for every $x\in M$ and every sequence $a_1,a_2,\dots\in I$ there exists an integer n such that $a_1a_2\dots a_{n-1}a_nx=0$.

Note that an ideal I of R is T-nilpotent if for every sequence $a_1,a_2,\dots\in I$ there exists an integer n such that $a_1a_2\dots a_{n-1}a_n=0$.

Example 2.5. Let $\mathbb{Z}$ be a set of integers and p be a prime integer. Put $S=\mathbb{Z}\setminus p\mathbb{Z}$. $N(S^{-1}\mathbb{Z})$ is T-nilpotent, but $J(S^{-1}\mathbb{Z})$ is not T-nilpotent.

Proposition 2.6. *Let M be an R-module. If every nonzero submodule of M has a prime submodule, then $N(R)$ is T-nilpotent on M.*

Proof. Let $0\neq x\in M$ and let $o(x)$ be the smallest ordinal for which $x\notin Rad_\beta(0_M)$. It is clear that $0(x)$ is not a limit ordinal. Hence for some α, $o(x)=\alpha+1$. We get $x\in Rad_\alpha(0_M)$. Assume P is a prime submodule of $Rad_\alpha(0_M)$. We have $N(R)Rad_\alpha(0_M)\subset P$. As a result, $N(R)Rad_\alpha(0_M)$ is a subset of $Rad_{\alpha+1}(0_M)$. It implies that $o(ax)<o(x)$ for all nonzero $x\in M$ and $a\in N(R)$. Assume that $a_1,a_2,\dots\in N(R)$ and $a_1a_2\dots a_nx\neq 0$ for all $n\in\mathbb{N}$. So $o(x)>o(a_1x)>\dots$ is a strictly descending chain of ordinals, which is a contradiction. Therefore there is an integer n such that $a_1\dots a_nx=0$, and so $N(R)$ is T-nilpotent on M. □

Proposition 2.7. *Assume M is a nonzero R-module. If every nonzero submodule of M has a prime submodule, then $N(R)M\neq M$.*

Proof. Assume that $N(R)M=M$ on the contrary. Since $M\neq 0$, there are $m\in M$ and $a\in N(R)$ such that $am\neq 0$. Hence $m=\Sigma_{i=1}^k a_im_i$ where $a_i\in N(R)$ and $m_i\in M$. Since $am\neq 0$, $aa_lm_l\neq 0$ for some $1\leq l\leq k$. We can produce element $a_2\in N(R)$ and $m_2\in M$ with $a_1a_2m_2\neq 0$, since $N(R)M\neq 0$. By induction, there exists a sequence $\{a_i\}_{i=1}^\infty$ in $N(R)$ and sequence $\{m_i\}_{i=1}^\infty$ in M with $a_1a_2\dots a_km_k\neq 0$ for $k=1,2,\dots$. This contradicts the fact that $N(R)$ is T-nilpotent on M. □

Corollary 2.8. *If every nonzero R-module has a prime submodule, then $N(R)$ is T-nilpotent.*

By [2, Theorem] we have the next proposition:

Proposition 2.9. *Let M be an R-module.*
(i) If $R/Ann(M)$ is von Neumman regular, then every nonzero submodule of M has a prime submodule.
(ii) If R is a von Neumann regular ring, then every nonzero submodule of M has a prime submodule.

Lemma 2.10. *Let R be a ring. If every nonzero R-module has a prime submodule, then every nonzero R/I-module has a prime submodule for any ideal I of R.*

Theorem 2.11. *Assume every nonzero R-module has a prime submodule. If x is not a zero divisor element of R, then x is a unit.*

Proof. Assume x is an element of R that is not a zero divisor.
Let $A = \oplus_{i=1}^{\infty} Ry_i$, with $Ry_i \cong R/Rx^i$. So $(0 :_R y_i) = Rx^i$. Put $B = \Sigma_{i=1}^{\infty} R(y_i - xy_{i+1})$. Then $A/B = \Sigma_{i=1}^{\infty} R\overline{y_i}$ where $\overline{y_i} = y_i + B$. We'll show that $A = B$. Assume that $A \neq B$. Then A/B has a prime submodule P/B. So there exists an integer i such that $\overline{y_i} \notin P/B$. Since $x^i y_i = 0$, $x^i \overline{y_i} = 0$. We obtain $x^i(P/B : A/B)$ because P/B is a prime submodule and $\overline{y_i} \notin P/B$. Therefor $\overline{y_i} = x^i \overline{y_i} \in P/B$, which is a contradiction.. As a result, A equals B. Hence there are $r_1, r_2, \ldots, r_n \in R$ with $y_1 = \Sigma_{i=1}^{n} r_i(y_i - xy_{i+1})$. Since y_i's are independent, $y_1 = -r_1 y_1$ and $r_{i-1}x - r_i) \in (0 : y_i) = Rx^i$ for $i = 2, \ldots, n$ and $r_n x \in (0 : y_{n+1}) = Rx^{n+1}$. So there is $a \in R$ such that $r_n x = ax^{n+1}$. Hence $r_n \in Rx^n$. We can conclude $r_k \in Rx^k$ where $2 \leq k \leq n$. So $r_1 x = r_2 = a_2 x^2$ because $r_2 \in Rx^2$. As a result, $r_1 \in Rx$. Then $y_1 = -r_1 y_1 = 0$. It suggests that $R/Rx \cong Ry_1 = 0$. Hence x is a unit. □

Lemma 2.12. *Let R be a ring. If every R-module has a prime submodule, then every prime ideal in R is maximal.*

Proof. Let P be a prime ideal of R. Then R/P is an integral domain and every R/P-module has a prime submodule. According to Theorem 2.11, every nonzero element of R/P is a unit. Hence R/P is a field. So P is maximal. □

Theorem 2.13. *Let R be a ring. The following statements are equivalent:*
(i) Every nonzero R-module has a prime submodule;
(ii) $N(R)$ is T-nilpotent and every nonzero $R/N(R)$-module has a prime submodule;
(iii) There is a cogenerator R-module U such that every nonzero submodule of U has a prime submodule;
(iv) For every simple R-module S, every submodule of $E(S)$ has a prime submodule.

Proof. $(i) \Longrightarrow (ii)$ By Corollary 2.8, $N(R)$ is T-nilpotent. Every nonzero $R/N(R)$-module has a prime submodule, according to Lemma 2.10.
$(ii) \Longrightarrow (i)$ Now assume that $N(R)$ is T-nilpotent and every nonzero $R/N(R)$-module has a prime submodule. Let M be an R-module. By Proposition 2.7, $N(R)M \neq M$. Thus $M/N(R)M$ is a nonzero $R/N(R)$-module. As a result, $M/N(R)M$ has a prime submodule as $R/N(R)$-module, and M also has a prime submodule as R-module.
$(iii) \Longrightarrow (i)$ Let M be an R-module. So, there is a nonzero homomorphism $f : M \longrightarrow U$. $f(M)$ has a prime submodule Q because it is a submodule of U. So $f^{-1}(Q)$ is a prime submodule of M.
$(iv) \Longrightarrow (iii)$ Let $E = \oplus E(S)$ as S range over all simple R-modules. Then E is a cogenerator R-module. By Theorem 2.1, every nonzero submodule of E has a prime submodule. □

Theorem 2.14. *Every nonzero R-module has a prime submodule if and only if $N(R)$ is T-nilpotent and $R/N(R)$ is a von Neumann regular ring.*

Proof. $\Longrightarrow$ According to Theorem 2.13, $N(R)$ is T-nilpotent and every nonzero $R/N(R)$-module has a prime submodule. Put $S := R/N(R)$. Let $0 \neq a \in S$. Because there are no nilpotent ideals in S, $S \cap (0 :_S a) = 0$. Assume $T := S/(0 :_S a)$ and $0 \neq a \in S$. Since S does not have nilpotent element, $Sa \cap (0 :_S a) = 0$. We shall show that $a + (0 :_S a) \in S/(0 :_S a)$ is no zero divisor. Assume that $(s + (0 :_S a))(a + (0 :_S a)) = 0$ for some $s + (0 :_S a) \in S/(0 :_S a)$. So $sa \in Sa \cap (0 :_S a) = 0$. Hence $s \in (0 :_S a)$. Thus $s + (0 :_S a) = 0$. So $a + (0 :_S a)$ is not zero divisor. Since by Lemma 2.10 every nonzero S-module has a prime submodule and $a + (0 :_S a)$ is not a zero divisor, by Theorem 2.11 $a + (0 :_S a)$ is a unit. So $S(a + (0 :_S a)) = S$ and we have that $Sa \oplus (0 :_S a) = S$. This shows that $R/N(R)$ is von Neumann regular.
$\Longleftarrow$ By Proposition 2.9, every nonzero $R/N(R)$-module has a prime submodule. As a result, according to Theorem 2.13, every nonzero R-module has a prime submodule. $\square$

Theorem 2.15. *Every nonzero R-module has a prime submodule if and only if R is a Max ring.*

Proof. $\Longrightarrow$ By Lemma 2.12, every prime ideal of R is maximal. So $N(R) = J(R)$ and $R/N(R) = R/J(R)$. Hence by Theorem 2.14, $N(R) = J(R)$ is T-nilpotent and $R/N(R) = R/J(R)$ is a von Neumann regular ring. So, according to [**2**, Theorem], R is a Max ring.
$\Longleftarrow$ Because the maximal submodule is also the prime submodule, every nonzero R-module has a prime submodule.

$\square$

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Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
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۸ تا ۱۱ شهریور ۱۴۰۰

Smoothing transformation and a Nystrom method for two-dimensional weakly singular Volterra integral equations

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Abstract. Nonlinear weakly singular Volterra integral equations often have non-smooth solutions, particularly at t = 0: to overcome this difficulty we propose a smoothing change of variable and then employ Navot's quadrature formula for solving the transformed equation. By using smoothing an equation is obtained which, while still weakly singular, can have a solution as smooth as required. Numerical example shows the efficient of the method.

Keywords: Weakly singular Volterra integral equations, Smoothing transformation, Navot's quadrature.

AMS Mathematics Subject Classification [2010]: 65R20, 45G10

1. Introduction

Volterra integral equations with weakly singular kernels typically have solutions which are nonsmooth near the initial point of the interval of the integration [2], therefore there is difficulty for the chosen numerical approach in order to obtain an optimal rate of convergence.

Various regularity and smoothing strategies have been proposed by number of authors for overcome the difficulty caused by the nonsmooth behavior of the solutions in these equations. To mention a few most relevant among many, in [3, 5] authors considered product integration and fractional linear multistep methods. Also Tao and Yong [7] applied a smoothing change of variable so that the solution of the transformed equation is smooth. During the last decades there have been some numerical methods for solving two-dimensional weakly singular Volterra integral equations (for example see [1,6,8]), but in most of them, authors deal with the linear weakly singular equations or have assumed that these equations have smooth solutions. In this paper we use the change of variable technique for smoothing nonlinear two-dimensional weakly singular Volterra integral equations with nonsmooth solution and then employ a Nystrom method for solving them. For this purpose consider

$$u(t,s) = y(t,s) + \int_a^t \int_b^s (t-\xi)^\alpha (s-\eta)^\beta k(t,s,\xi,\eta,u(\xi,\eta))d\eta d\xi,$$

$$a \le \xi \le t \le T, \quad b \le \eta \le s \le S. \tag{1}$$

where $-1 < \alpha, \beta < 0$, $u(t,s)$ is an unknown function, $y(t,s)$ and $k(t,s,\xi,\eta,u)$ are given continuous functions on $[0,T]\times[0,S]$ and $\Lambda \times \mathbb{R}$ ($\Lambda := \{(t,s,\xi,\eta) : a \le \xi \le t \le T,\ b \le$

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$\eta \leq s \leq S\}$), respectively. Moreover, $k(t,s,\xi,\eta,u)$ satisfy Lipschitz condition with respect to u.

The rest of the paper is organized as follows. In the next section we employ smoothing to eliminate the singularity of the solution. In Section 3, a Nystrom method is constructed and the numerical results is given in Section 4.

2. Smoothing

Consider the change of variables $\gamma(t) = (t-a)^q + a$ and $\theta(s) = (s-b)^p + b$ in equation (1), with suitable positive constants p, q. Also let $\xi \to \gamma(\xi)$, $\eta \to \theta(\eta)$, then we have

$$u(\gamma(t),\theta(s)) = y(\gamma(t),\theta(s)) + \int_a^t \int_b^s (\gamma(t)-\gamma(\xi))^\alpha (\theta(s)-\theta(\eta))^\beta k(\gamma(t),\theta(s),\gamma(\xi),\theta(\eta),u(\gamma(\xi),\theta(\eta)))d\theta(\eta)d\gamma(\xi), \tag{2}$$

$$a \leq \xi \leq t \leq \gamma^{-1}(T), \quad b \leq \eta \leq s \leq \theta^{-1}(S),$$

Equation (2) can be simplified as

$$U(t,s) = Y(t,s) + \int_a^t \int_b^s (t-\xi)^\alpha (s-\eta)^\beta K(t,s,\xi,\eta,U(\xi,\eta))d\eta d\xi, \tag{3}$$

where

$$Y(t,s) := y(\gamma(t),\theta(s)), \qquad U(t,s) := u(\gamma(t),\theta(s)),$$
$$K(t,s,\xi,\eta,U) := \delta_\alpha(t,\xi)\delta_\beta(s,\eta)k(\gamma(t),\theta(s),\gamma(\xi),\theta(\eta),u(\gamma(\xi),\theta(\eta)))\gamma'(\xi)\theta'(\eta),$$

with

$$\delta_\alpha(t,\xi) := \begin{cases} (\frac{\gamma(t)-\gamma(\xi)}{t-\xi})^\alpha, & t \neq \xi, \\ (\gamma'(t))^\alpha, & t = \xi, \end{cases} \qquad \delta_\beta(s,\eta) := \begin{cases} (\frac{\theta(s)-\theta(\eta)}{s-\eta})^\beta, & s \neq \eta, \\ (\theta'(s))^\beta, & s = \eta, \end{cases}$$

equation (3) has a kernel which is still weakly singular and has a unique continuous solution.

3. Nystrom method

Let uniform mesh $s_i = b + ih, i = 0,1,...,N$, $Nh = \theta^{-1}(S) - b$ and $t_j = a + j\tau$, $j = 0,1,...,M$, $\tau M = \gamma^{-1}(T) - a$. By collocating the equation (3) on the first grid points, we obtain

$$U(t,s_i) = Y(t,s_i) + \int_a^t \int_b^{s_i} (t-\xi)^\alpha (s_i-\eta)^\beta K(t,s_i,\xi,\eta,U(\xi,\eta))d\eta d\xi, \qquad i = 0,1,...,N, \tag{4}$$

using the Navot's quadrature rule ([4]) reduce to

$$U(t,s_i) \simeq Y(t,s_i) + \int_a^t (t-\xi)^\alpha \left[h\sum_{k=0}^{i-1} w_{ik}(s_i-s_k)^\beta K(t,s_i,\xi,s_k,U(\xi,s_k)) + w_{ii}hK(t,s_i,\xi,s_i,U(\xi,s_i)) \right] d\xi,$$

where

$$w_{ik} = \begin{cases} \frac{1}{2} & k = 0, \\ 1 & 0 < k < i, \\ -h^\beta \zeta(-\beta) & k = i, \end{cases}$$

and $\zeta(x)$ is the Rimann-zete function. In the following by collocating on the second grid points we can write

$$U(t_j,s_i) \simeq Y(t_j,s_i) + h\sum_{k=0}^{i-1} w_{ik}(s_i-s_k)^{\beta}\int_a^{t_j}(t_j-\xi)^{\alpha}K(t_j,s_i,\xi,s_k,U(\xi,s_k))d\xi$$
$$+ w_{ii}h\int_a^{t_j}(t_j-\xi)^{\alpha}K(t_j,s_i,\xi,s_i,U(\xi,s_i))d\xi,$$

Again the Navot's quadrature rule leads to

$$U_{ji} = Y(t_j,s_i) + h\sum_{k=0}^{i-1} w_{ik}(s_i-s_k)^{\beta}\left[\tau\sum_{l=0}^{j-1}\omega_{jl}(t_j-t_l)^{\alpha}K_{lk} + \omega_{jj}\tau K_{jk}\right]$$
$$(5) \qquad + w_{ii}h\left[\tau\sum_{l=0}^{j-1}\omega_{jl}(t_j-t_l)^{\alpha}K_{li} + \omega_{jj}\tau K_{ji}\right], \quad i=1,2,...,N, j=1,2,...,M.$$

where $U_{ji} \simeq U(t_j,s_i)$, $K_{lk} = K(t_j,s_i,t_l,s_k,U_{lk})$ and

$$\omega_{jl} = \begin{cases} \frac{1}{2}, & l=0, \\ 1, & 0<l<j, \\ -\tau^{\alpha}\zeta(-\alpha), & l=j, \end{cases}$$

By knowing the values $U_{00} = Y(0,0)$, $U_{j0} = Y(t_j,0)$, $j=1,2,...,M$ and $U_{0i} = Y(0,s_i)$, $i=1,2,...,N$, we can obtain other values of the unknown function in the mesh points by solving equation (5) for $i=1,...,N$, $j=1,...,M$. Obviously for nonlinear integral equations, these equations are nonlinear which can be solved by using Newton's iterative method. It is well known that the initial guesses for Newton's method are very important: thus we choose $U_{j-1,i}$ as initial guesses for compute $U_{j,i}$.

4. Numerical results

In this section, in order to test experimentally the convergence of the proposed method and measure the error accuracy, we consider a test problem.

Example 4.1. Consider two-dimensional nonlinear WSVIE

$$u(t,s) = \sqrt{st} - \frac{256}{441}(ts)^{\frac{7}{4}} + \int_0^t\int_0^s \frac{u^2(\xi,\eta)}{(t-\xi)^{0.25}(s-\eta)^{0.25}}d\eta d\xi, \quad 0\le\xi\le t\le 1,\ 0\le\eta\le s\le 1,$$

with exact solution $u(t,s) = \sqrt{ts}$.

The Table 1 displays the absolute error and ratios of the errors for $h=\tau=0.02$ and $h=\tau=0.01$ in some mesh points. Note that for $p=q=1$ smoothing was not done. As it shown in table, the error is reduced at the origin with smoothing $(p,q=2,3)$. In these cases, the ratios can be seen to be close to 5.

Table 1 : *Numerical results of Example 4.1*

$h=\tau$	(t_i, s_i)	$p=q=1$	*Ratio*	$p=q=2$	*Ratio*	$p=q=3$	*Ratio*
0.02	$(0.02, 0.02)$	$9.156e{-8}$	2.76	$1.951e{-12}$	4.80	$4.936e{-11}$	4.85
	$(0.06, 0.06)$	$8.366e{-7}$	2.95	$4.978e{-10}$	4.12	$5.986e{-14}$	4.83
	$(0.1, 0.1)$	$2.260e{-6}$	3.02	$5.954e{-9}$	3.40	$4.859e{-12}$	3.91
	$(0.3, 0.3)$	$1.885e{-5}$	3.13	$1.885e{-6}$	3.36	$6.228e{-8}$	3.36
	$(0.5, 0.5)$	$5.483e{-5}$	3.16	$2.809e{-5}$	3.36	$5.454e{-6}$	3.36
	$(0.8, 0.8)$	$1.889e{-4}$	3.17	$4.086e{-4}$	3.36	$3.750e{-4}$	3.37
	$(1, 1)$	$4.238e{-4}$	3.18	$2.148e{-3}$	3.37	$4.563e{-3}$	3.38
0.01	$(0.02, 0.02)$	$3.312e{-8}$		$4.063e{-13}$		$1.017e{-18}$	
	$(0.06, 0.06)$	$2.827e{-7}$		$1.206e{-10}$		$1.238e{-14}$	
	$(0.1, 0.1)$	$7.466e{-7}$		$1.250e{-9}$		$1.241e{-12}$	
	$(0.3, 0.3)$	$6.018e{-6}$		$5.602e{-7}$		$1.450e{-8}$	
	$(0.5, 0.5)$	$1.733e{-5}$		$8.354e{-6}$		$1.622e{-6}$	
	$(0.8, 0.8)$	$5.941e{-5}$		$1.213e{-4}$		$1.114e{-4}$	
	$(1, 1)$	$1.330e{-4}$		$6.369e{-4}$		$1.348e{-3}$	

5. conclusion

In this article we considered numerical methods for singular two-dimensional Volterra integral equations of the second kind where typically non-smooth solutions are the norm. To overcome this, we employed a smoothing change of variables followed by the idea of Navot's quadrature rule. Numerical examples were given to illustrate the theoretical results.

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Faculty of Mathematics and Computer
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Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
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دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Liouville-type theorems for p-harmonic maps with potential

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Abstract. In the present paper, $p-$harmonic maps with potential from a complete Riemannian manifold of non-negative Ricci curvature to a complete Riemannian manifold are studied. First, we compute the first and second variational formulas for this kind of harmonic maps. Then, a Liouville-type theorem for p-harmonic maps with potential is given .

Keywords: $p-$ harmonic maps, Liouville theorems, Calculus of variations.

AMS Mathematics Subject Classification [2010]: Primary 53C43; Secondary 58E20

1. Introduction

Harmonic maps with potential between Riemannian complete manifolds are first introduced by Ratto in 1997. This new type of harmonic maps more than usual harmonic maps plays a key role in many branch of mathematical physics such as Neumann motion and equilibrium system of ferromagnetic spin chain, [8].
Let $f : (P, \rho) \longrightarrow (K, \ell)$ be a smooth map between complete Riemannian manifolds. The map f is called harmonic map if f is a critical point of the energy functional

$$E(f) = \int_P e(f) dV_\rho, \tag{1}$$

where $e(f) = \frac{1}{2}\sum_{\alpha=1}^{p}\langle df(e_\alpha), df(e_\alpha)\rangle$. Here p is the dimension of P and $\{e_\alpha\}_{\alpha=1,\cdots,p}$ is a local orthonormal frame in P.
Now, regard an extension of energy functional for a smooth function $G\ inC^\infty(K)$. Setting

$$E_G(F) = \int_P [e(f) - G\circ f] dV_\rho. \tag{2}$$

The Euler-Lagrange equation associated to $E_G(F)$ is obtained as follows

$$\tau_G(f) = \tau(f) + \nabla G(f) = 0 \tag{3}$$

where ∇G is the gradient of G on K and $\tau(f) = trace_\rho \nabla df$ is the tension field of f. Any smooth map f satisfying in (2.5) is said to be a harmonic map with potential G . Recently, many scholars have done research on this topic. For instance, Qun obtained Liouville type results and gradient estimates for these maps, [2]. In [4], the authors studied the heat

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equation associated to $E_G(f)$ and investigated phenomena of blowing up solution.
A natural extension of the concept of harmonic map is that of a p-harmonic map for a number $p > 2$. $p-$harmonic maps are critical points of $p-$ energy functional, which is defined as follows

$$E_p(f) = \int_P \mid df \mid^p dV_\rho$$

This type of harmonic map is applied in many fields of physics and mechanics such as glaciology, non-linear elasticity, non-Newtonian fluids, [6]. In the last decade many developments have been witnessed in the theory of $p-$harmonic maps. In [7], the relative Dirichlet problem for any smooth $p-$harmonic maps from a compact manifold into a negative sectional curvature is solved. Moreover, in [3], many various geometric applications of $p-$harmonic maps from a positive Ricci curvature complete manifolds to manifolds with bounded sectional curvature are given.
In this paper, motivated by [1] and [5], $p-$energy functional with potential G is introduced. Then, the first and second variational formulas are calculated. Finally, a Liouville-type theorem for p-harmonic maps with potential is given.

2. Main results

In this section, the variational formulas of $p-$energy functional with potential G is calculated . Then, a Liouville-type theorem for $p-$harmonic maps with potential is given.

Let $f : (P, \rho) \longrightarrow (K, \ell)$ be a smooth map between complete Riemannian manifolds. Throughout this paper, we will denote the Levi-Civita connection of P, K and $f^{-1}TK$ by ${}^P\nabla, {}^K\nabla$ and $\dot{\nabla}$. considering that the induced connection $\dot{\nabla}$ on $f^{-1}TK$ defined by $\dot{\nabla}_X Z = {}^K\nabla_{df(X)} Z$, where $X \in \chi(P)$ and $Z \in \Gamma(f^{-1}TK)$.

Definition 2.1. Let $G \in C^\infty(K)$. The $p-$energy functional of f with potential G is denoted by $E_{p,G}(f)$ and defined by

$$E_{p,G}(f) = \int_P [\mid df \mid^p - G(f)] dV_\rho. \tag{4}$$

The smooth map f is said to be $p-$harmonic with potential G if it is a critical point of $E_{p,G}$.

By considering a local orthonormal frame field $\{e_\alpha\}$ on P, the $p-$tension field of f with potential G, $\tau_{p,G}(f)$, is defined by

$$\tau_{p,G}(f) = \mid df \mid^{p-2} \tau(f) + df(grad \mid df \mid^{p-2}) + {}^K\nabla G \circ f, \tag{5}$$

where $\tau(f)$ is the tension field of f. Based on the above notations, it is obtained that

Lemma 2.2. *(The first variation formula) Let $f : (P, \rho) \longrightarrow (K, \ell)$ be a smooth map. Then*

$$\frac{d}{dt} E_{p,G}(f_t) \mid_{t=0} = - \int_P \ell(\tau_{p,G}(f), W) dV_\rho, \tag{6}$$

where $W = \frac{df_t}{dt} \mid_{t=0}$.

Definition 2.3. A map f is called $p-$ harmonic with potential G if $\tau_{p,G}(f) = 0$.

Definition 2.4. Let $f : (P, \rho) \longrightarrow (K, \ell)$ be a $p-$harmonic map with potential G, and let $f_t : P \longrightarrow K$ $(-\varepsilon < t < \varepsilon)$ be a compctly supported variation such that $f_0 = f$ and $W = \frac{\partial f_t}{\partial t} \mid_{t=0}$. Setting

$$I(W) = \frac{d^2}{dt^2} E_{p,G}(f_t) \mid_{t=0},$$

The map f is said to be stable if $I(W) \geq 0$ for any vector field W along f.

By calculating the second variational formula of $E_{p,G}$, and make use of Green's Theorem and divergence theorem, $I(W)$ is obtained as follows

$$\begin{aligned} I(W) &= \int_P (p-2) \mid df \mid^{\frac{p-4}{2}} \langle \dot{\nabla} W, df \rangle^2 dV_\rho \\ &+ \int_P \mid df \mid^{\frac{p-4}{2}} \Big\{ \langle \mid \dot{\nabla} W \mid^2 - \ell(trace_\rho {}^K R(W, df) df \\ &- (\nabla^K_W grad^K G) \circ f, W) \Big\} dV_\rho \end{aligned} \tag{7}$$

where $\mid \dot{\nabla} V \mid$ denotes the Hilbert-Schmidt norm of the $\dot{\nabla} W \in \Gamma(T^* P \times f^{-1} TK)$. According to the above equations, we have

Theorem 2.5. *Let $f : (P, \rho) \longrightarrow (K, \ell)$ be a $p-$ harmonic map with potential G from a complete manifold with a positive Ricci curvature to a complete Riemannian manifold. Assume that*

$$\Delta\, grad(\mid df \mid^{\frac{p-4}{2}}) + \frac{1}{2} grad \mid grad \mid df \mid^{\frac{p-4}{2}} \mid^2 = 0. \tag{8}$$

Then, f is a harmonic map.

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On matching energy of graphs

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Abstract. The matching energy of a graph G, denoted by $ME(G)$, is defined as the sum of absolute values of the zeros of the matching polynomial of G. In this paper, we would like to present some lower bounds for $ME(G)$. For any connected graph G, it is proved that $ME(G) \geq 2\mu(G)$, where $\mu(G)$ is the matching number of G. Also it is shown that if G has no perfect matching, then $ME(G) \geq 2\mu(G) + 1$, except for $K_{1,2}$. Moreover, we characterize some class of graphs whose matching energy is at least equal to the number of vertices.

Keywords: matching energy, matching polynomial, matching number.

AMS Mathematics Subject Classification [2010]: 05C31, 05C70

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1. Introduction

All graphs we consider are finite, simple and undirected. Let G be a graph. By *order* and *size* of G, we mean the number of vertices and the number of edges of G, respectively. We denote the complete graph and the cycle of order n, by K_n and C_n, respectively. A complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. A $\{1, 2\}$-*factor* is a spanning subgraph of G all of whose components are 1-regular or 2-regular. A *traceable graph*, is a graph with a Hamilton path. A graph is called *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. An r-*matching* in a graph G is a set of r pairwise non-incident edges. The number of r-matchings in G is denoted by $m(G, r)$. The *matching number* of G, $\mu(G)$, is the number of edges in a maximum matching of G.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a graph G, i.e the eigenvalues of its adjacency matrix. The energy of the graph G denoted by $\mathcal{E}(G)$, is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The theory of graph energy is well developed nowadays, for details see [5]. The Coulson integral formula [2] plays an important role in the study on graph energy, its version for an acyclic graph T is as follows:

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$$\mathcal{E}(T) = \frac{2}{\pi}\int_0^{+\infty} \frac{1}{x^2} \ln\left(\sum_{r\geq 0} m(T,r)x^{2r}\right) dx. \tag{1}$$

Motivated by formula (1), Gutman and Wagner in 2012 defined the matching energy of a graph G as

$$ME(G) = \frac{2}{\pi}\int_0^{+\infty} \frac{1}{x^2} \ln\left(\sum_{r\geq 0} m(G,r)x^{2r}\right) dx, \tag{2}$$

see [3]. Energy and matching energy of graphs are closely related, and they are two quantities of relevance for chemical applications, [3]. Recall that the *matching polynomial* of G is defined by

$$\alpha(G,x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r m(G,r)x^{n-2r},$$

where n is the order of G and $m(G,0)$ is considered to be 1. For any graph G, all zeros of $\alpha(G,x)$ are real [4]. Furthermore, if μ is a matching zero of G, then so is $-\mu$. The following result gives an equivalent definition of matching energy:

Theorem 1.1. *[3] Let G be a graph and let $\mu_1,\dots,\mu_n$ be the zeros of its matching polynomial. Then*

$$ME(G) = \sum_{i=1}^{n} |\mu_i|.$$

Since 2012 matching energy of graphs has been studied by several authors and a series of results concerning the extremal matching energy of graphs have been obtained. For details, we refer to [3,7]. In this paper, we present a lower bound for the matching energy of a graph in terms of the matching number of the graph. We prove that for a connected graph G, $ME(G) \geq 2\mu(G)$. Also it is shown that if G has no perfect matching, then $ME(G) \geq 2\mu(G)+1$, except for $K_{1,2}$. Among other results, we characterize some class of graphs whose matching energy is at least equal to the number of vertices. We prove that if G is a graph of order n such that G has a $\{1,2\}$-factor, then $ME(G) \geq n$. Also we show that if a connected graph G is traceable or claw-free, then its matching energy exceeds the number of vertices, except for K_2 and $K_{1,2}$. The following theorems and lemmas are needed in the sequel.

Lemma 1.2. *[1] Let G be a connected graph. If the zeros of $\alpha(G,x)$ are ≥ -1, then G is either K_1 or K_2.*

Lemma 1.3. *[7] If H is a subgraph of G, then $ME(H) \leq ME(G)$, with equality if H and G are the same except possibly for isolated vertices.*

Lemma 1.4. *[3] Let G_1 and G_2 be two vertex disjoint graphs. Then $ME(G_1 \cup G_2) = ME(G_1) + ME(G_2)$.*

THEOREM 1.5. *[3] The matching energy of a graph G with m edges and matching number L is bounded as:*

$$2\sqrt{m + L(L-1)m(G,L)^{1/L}} \leq ME(G) \leq 2\sqrt{(L-1)m + Lm(G,L)^{1/L}}.$$

The following lemma can be easily proved.

LEMMA 1.6. *If $T \neq K_1$ is a tree with no perfect matching, then T has at least two maximum matchings.*

2. Main results

In this section, we present some lower bounds for matching energy of graphs. Moreover, we characterize some graphs whose matching energy exceeds the number of vertices.

THEOREM 2.1. *Let G be a connected graph. Then $ME(G) \geq 2\mu(G)$ and equality holds if and only if $G \in \{K_1, K_2\}$.*

PROOF. Let G be of order n. Assume that $\mu_1 > \mu_2 \geq \ldots \geq \mu_L$ are all positive zeros of $\alpha(G,x)$. Hence

$$\alpha(G,x) = x^{n-2L}(x^{2L} + a_1 x^{2L-2} + \cdots + a_L),$$

and $ME(G) = 2\sum_{i=1}^{L} \mu_i$. Note that L is the size of the maximum matching of G and $|a_L| = m(G,L) \geq 1$. Then the arithmetic-geometric inequality implies that

$$\frac{\mu_1 + \cdots + \mu_L}{L} \geq \sqrt[L]{\mu_1\mu_2\cdots\mu_L} = \sqrt[2L]{\mu_1^2\mu_2^2\cdots\mu_L^2} = \sqrt[2L]{|a_L|} \geq 1. \tag{3}$$

From (3), it follows that $ME(G) \geq 2L$. If $G \in \{K_1, K_2\}$, then obviously $ME(G) = 2L$. Conversely, suppose that $ME(G) = 2L$. So the equality holds in (3), that is $\mu_1 = \ldots = \mu_L = 1$. Now, by Lemma 1.2, $G \in \{K_1, K_2\}$. □

COROLLARY 2.2. *For any connected graph G apart form K_1 and $K_{1,i}$, $1 \leq i \leq 3$, $ME(G) \geq 4$.*

COROLLARY 2.3. *Let G be a connected graph of order n witch has a perfect matching. Then $ME(G) \geq n$ and the equality holds only if $G = K_2$.*

THEOREM 2.4. *Let G be a connected graph with at least two vertices. If G has no perfect matching, then $ME(G) \geq 2\mu(G) + 1$, except for $K_{1,2}$.*

PROOF. Let G be of order n and size m. Let L be the size of the maximum matching of G. Since G has no perfect matching, $2L \leq n-1$. If G is a tree, then by Lemma 1.6, $m(G,L) \geq 2$. Now, using Theorem 1.5, we obtain:

$$ME(G) \geq 2\sqrt{m + L(L-1)m(G,L)^{1/L}} \geq \sqrt{4(n-1) + 4L(L-1)2^{1/L}}.$$

Note that if $L = 1$, then by Corollary 2.2 and the facts that $ME(K_{1,2}) = 2.82$ and $ME(K_{1,3}) = 3.46$, we are done. If $L \geq 2$, then since $2^{1/L} > \exp(1/2\mathrm{L}) > 1 + \frac{1}{2\mathrm{L}} \geq 1 + \frac{1}{4\mathrm{L}(\mathrm{L}-1)}$, we have

$$ME(G) \geq \sqrt{8L + 4L(L-1)[1 + \frac{1}{4L(L-1)}]} \geq \sqrt{4L^2 + 4L + 1} \geq 2L + 1.$$

If G is not a tree, then Theorem 1.5 implies that

$$\begin{aligned} ME(G) &\geq \sqrt{4m + 4L(L-1)} \geq \sqrt{4n + 4L(L-1)} \\ &\geq \sqrt{4L^2 + 4L + 4} \geq 2L + 1. \end{aligned}$$

and the proof is complete. □

LEMMA 2.5. *Let $n \geq 3$. Then $ME(C_n) > n$. In particular, if n is even, then $ME(C_n) > n + 1$.*

THEOREM 2.6. *Let G be a graph of order n. If G has a $\{1, 2\}$-factor, then $ME(G) \geq n$. Equality occurs if and only if $G = \frac{n}{2}K_2$.*

THEOREM 2.7. *[6] Let r be a positive integer. Then every r-regular graph has a $\{1, 2\}$-factor.*

The following corollary is an immediate consequence of Theorems 2.6 and 2.7.

COROLLARY 2.8. *Let r be a positive integer. If G is an r-regular graph of order n, then $ME(G) \geq n$ and equality holds if and only if G is 1-regular.*

THEOREM 2.9. *Let G be a connected traceable graph of order n. Then $ME(G) > n$, except for K_2 and $K_{1,2}$.*

THEOREM 2.10. *Let G be a connected claw-free graph of order n. Then $ME(G) > n$, except for K_2 and $K_{1,2}$.*

3. Conclusion

In this paper, we studied further properties of the matching energy of a graph. For a graph G, we obtained a lower bound for $ME(G)$ in terms of the matching number of G. Moreover, we characterized some class of graphs whose matching energy is at least equal to the number of vertices.

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۸ تا ۱۱ شهریور ۱۴۰۰

The discrete variant of a kind of continuous problem

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Abstract. In this paper, we present the following fractional discrete boundary-value problem

$$\begin{cases} -\Delta\left(\frac{1}{2}{}_0\Delta_k^{-\beta}(\Delta u(k)) + \frac{1}{2}{}_k\Delta_T^{-\beta}(\Delta u(k))\right) = \lambda f(k, u(k)), & k \in [1, T]_{\mathbb{N}_0}, \\ u(0) = u(T+1) = 0, \end{cases}$$

where $T \geq 2$ is a fixed positive integer, $0 \leq \beta < 1$ and ${}_0\Delta_k^{-\beta}$ and ${}_k\Delta_T^{-\beta}$ are the β-th left and right discrete fractional sum, respectively, and $\Delta u(k) = u(k+1) - u(k)$ is the forward and $[1, T]_{\mathbb{N}_0}$ is the discrete set $\{1, 2, \cdots, T-1, T\}$ and $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$, difference operator $f : [1, T]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\lambda > 0$ is a parameter.

Keywords: Discrete fractional calculus; Discrete nonlinear boundary value problem; Continuous nonlinear boundary value problem.

AMS Mathematics Subject Classification [2010]: 26A33, 39A10, 34B15

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1. Introduction

There has been surge in the interest for boundary value problems with fractional differential equations in many fields because of their applications. This kind of problems play a fundamental role in different fields of research, such as mechanical, economics, computer science, physics, chemistry, aerodynamics, ecology and many others. The importance and role of studies in discrete space requires that we present these types of continuous problems in a discrete way. The aim of this paper is to apply the continuous model

$$\text{(1)} \qquad \begin{cases} -\frac{d}{dt}\left(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))\right) = \lambda \nabla F(t, u(t)), & \text{a.e.} t \in [0, T], \\ u(0) = u(T) = 0, \end{cases}$$

to the discrete model

$$\text{(2)} \qquad \begin{cases} -\Delta\left(\frac{1}{2}{}_0\Delta_k^{-\beta}(\Delta u(k)) + \frac{1}{2}{}_k\Delta_T^{-\beta}(\Delta u(k))\right) = \lambda f(k, u(k)), & k \in [1, T]_{\mathbb{N}_0}, \\ u(0) = u(T+1) = 0, \end{cases}$$

where $T \geq 2$ is a fixed positive integer, $0 \leq \beta < 1$ and ${}_0\Delta_k^{-\beta}$ is the β-th left discrete fractional sum and ${}_k\Delta_T^{-\beta}$ is the β-th right discrete fractional sum and $\Delta u(k) = u(k+$

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$1) - u(k)$ is the forward and T is a fixed positive integer, $[1,T]_{\mathbb{N}_0}$ is the discrete set $\{1,2,\cdots,T-1,T\}$ and $\mathbb{N}_0 = \{0,1,2,\cdots\}$, difference operator $f : [1,T]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\lambda > 0$ is a parameter.
${}_0D_t^\beta$ and ${}_tD_T^\beta$ are the left and right Riemann-Liouville fractional integrals of order $0 \leq \beta < 1$ respectively, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function and $\nabla F(t,x)$ is the gradient of F at x and $\lambda > 0$ is real number. Jiao and Zhou [1] studied the problem (1) by using the mountain pass theorem and in [2] the problem (1) with additional nonlinear term studied by using Nehari manifold. When $\beta = 0$, this boundary value problem reduces to the standard second-order boundary value problem of the following form

$$\begin{cases} -\Delta^2(u(k)) = \lambda f(k,u(k)), & k \in [1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{3}$$

2. Preliminaries

we recall that the falling factorial is defined as $(k-s-1)^{\underline{-\nu-1}} = \frac{\Gamma(k-s)}{\Gamma(k-s+\nu+1)}$, hence for $s = k+\nu$, one has: $(-\nu-1)^{\underline{-\nu-1}} = \frac{\Gamma(-\nu)}{\Gamma(1)} = \Gamma(-\nu)$ and for $s = k+\nu-1$, one has:

$$(-\nu)^{\underline{-\nu-1}} = \frac{\Gamma(-\nu+1)}{\Gamma(2)} = \Gamma(-\nu+1) = -\nu\Gamma(-\nu).$$

We define left and right discrete fractional sum operators as follows.

Definition 2.1. ([4, Definition 3.1]) Let $0 < \nu \leq 1$ and $u : \mathbb{N}_0 \to \mathbb{R}$ be any real-valued function the ν-th left discrete fractional sum of u is defined

$${}_k\Delta_0^{-\nu}(u(k)) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{k} (k-s-1)^{\underline{\nu-1}} u(s),$$

$k \equiv \nu \pmod 1$, and the ν-th right discrete fractional sum of u is defined

$${}_T\Delta_k^{-\nu}(u(k)) = \frac{1}{\Gamma(\nu)} \sum_{s=k+\nu}^{T} (k-s-1)^{\underline{\nu-1}} u(s),$$

$k \in_{T-\nu} \mathbb{N} = \{T-\nu, T-\nu-1, T-\nu-2, ...\}$.

3. Main results

Lemma 3.1. *([3, Theorem 2.1]) For any $\nu > 0$, the following equality holds:*

$$\Delta_a^{-\nu}\Delta f(t) = \Delta\Delta_a^{-\nu} f(t) - \frac{(t-a)^{\underline{\nu-1}}}{\Gamma(\nu)} f(a)$$

where f is defined on $\mathbb{N}_a = \{a, a+1, a+2, ...\}$.

Now, corresponding to Lemma 3.1, for right fractional sum of order $\nu > 0$, we provide next theorem.

Theorem 3.2. *For any $\nu > 0$, the following equality holds:*

$${}_b\Delta^{-\nu}(\Delta f(t)) = \Delta_b\Delta^{-\nu}(f(t)) + \frac{1}{\Gamma(\nu)}(t-b-1)^{\underline{\nu-1}} f(b+1)$$

where f is defined on ${}_b\mathbb{N} = \{b, b-1, b-2, ...\}$.

PROOF. By similar argument in [3], First recall the summation by parts formula:

$$\Delta_s((t-s)^{\underline{\nu-1}}f(s)) = (t-s-1)^{\underline{\nu-1}}\Delta_s f(s) - (\nu-1)(t-s-1)^{\underline{\nu-2}}f(s)$$

Indeed,

$$\Delta_s((t-s)^{\underline{\nu-1}}f(s)) = (t-s-1)^{\underline{\nu-1}}\Delta_s f(s) + \Delta_s((t-s)^{\underline{\nu-1}})f(s)$$

and by $\Delta_s(t-s)^{\underline{\nu-1}} = -(\nu-1)(t-s-1)^{\underline{\nu-2}}$, one has

$$\Delta_s((t-s)^{\underline{\nu-1}}f(s)) = (t-s-1)^{\underline{\nu-1}}\Delta_s f(s) - (\nu-1)(t-s-1)^{\underline{\nu-2}}f(s)$$

Sum by parts to obtain

$$\begin{aligned}
{}_b\Delta^{-\nu}(\Delta f(t)) &= \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b}(t-s-1)^{\underline{\nu-1}}\Delta_s f(s) \\
&= \frac{1}{\Gamma(\nu)}(t-s)^{\underline{\nu-1}}f(s)|_{t+\nu}^{b+1} \\
&+ \frac{\nu-1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b}(t-s-1)^{\underline{\nu-2}}f(s) \\
&= \frac{1}{\Gamma(\nu)}(t-b-1)^{\underline{\nu-1}}f(b+1) - \frac{1}{\Gamma(\nu)}(-\nu)^{\underline{\nu-1}}f(t+\nu) \\
&+ \frac{\nu-1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b}(t-s-1)^{\underline{\nu-2}}f(s) \\
&= \frac{1}{\Gamma(\nu-1)}\sum_{s=t+\nu}^{b}(t-s-1)^{\underline{\nu-2}}f(s) - \frac{1}{\Gamma(\nu)}(-\nu)^{\underline{\nu-1}}f(t+\nu) \\
&+ \frac{1}{\Gamma(\nu)}(t-b-1)^{\underline{\nu-1}}f(b+1)
\end{aligned}$$

On the other hand

$$\begin{aligned}
\Delta_b\Delta^{-\nu}(f(t)) &= {}_b\Delta^{-\nu}(f(t+1)) -_b \Delta^{-\nu}(f(t)) \\
&= \frac{1}{\Gamma(\nu)}\sum_{s=t+1+\nu}^{b}(t+1-s-1)^{\underline{\nu-1}}f(s) - \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b}(t-s-1)^{\underline{\nu-1}}f(s) \\
&= \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b}(t-s)^{\underline{\nu-1}}f(s) - \frac{1}{\Gamma(\nu)}(t-(t+\nu))^{\underline{\nu-1}}f(t+\nu) \\
&- \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b}(t-s-1)^{\underline{\nu-1}}f(s) \\
&= \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b}\left[(t-s)^{\underline{\nu-1}} - (t-s-1)^{\underline{\nu-1}}\right]f(s) - \frac{1}{\Gamma(\nu)}(-\nu))^{\underline{\nu-1}}f(t+\nu) \\
&= \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b}\left[-\Delta_s(t-s)^{\underline{\nu-1}}\right]f(s) - \frac{1}{\Gamma(\nu)}(-\nu))^{\underline{\nu-1}}f(t+\nu)
\end{aligned}$$

$$= \frac{1}{\Gamma(\nu)} \sum_{s=t+\nu}^{b} \left[(\nu-1)(t-s-1)^{\underline{\nu-2}}\right] f(s) - \frac{1}{\Gamma(\nu)}(-\nu))^{\underline{\nu-1}} f(t+\nu)$$

$$= \frac{1}{\Gamma(\nu-1)} \sum_{s=t+\nu}^{b} (t-s-1)^{\underline{\nu-2}} f(s) - \frac{1}{\Gamma(\nu)}(-\nu))^{\underline{\nu-1}} f(t+\nu)$$

So the desired equality follows. $\square$

Since $u(0) = 0 = u(T+1)$, hence by Lemma 3.1 and Lemma 3.2, we have

$${}_0\Delta_k^{-\beta}(\Delta u(k)) = \Delta_0 \Delta_k^{-\beta} u(k), \quad {}_k\Delta_T^{-\beta}(\Delta u(k)) = \Delta_k \Delta_T^{-\beta} u(k)$$

LEMMA 3.3. *([6, Lemma 6]) Let $f : \mathbb{N}_a \to \mathbb{R}$ be given. For any $k \in \mathbb{N}_0$ and $\nu > 0$ we have*

$$\Delta^k \Delta_a^{-\nu} f(t) = \Delta_a^{k-\nu} f(t), \qquad \textit{for } t \in \mathbb{N}_{a+\nu}.$$

Therefore by Lemma 3.3, one has $\Delta_0 \Delta_k^{-\beta}(u(k)) =_0 \Delta_k^{1-\beta} u(k)$. Also by Definition 2.1, one has $\Delta_k \Delta_T^{-\beta}(u(k)) = -_k\Delta_T^{1-\beta} u(k)$. By above properties and putting $\alpha = 1-\beta$, BVP (2) transforms to

$$\begin{cases} -\Delta\left(\frac{1}{2}{}_0\Delta_k^{\alpha}(u(k)) - \frac{1}{2}{}_k\Delta_T^{\alpha}(u(k))\right) = \lambda f(k, u(k)), & k \in [1,T]_{\mathbb{N}_0}, \\ u(0) = u(T+1) = 0. \end{cases}$$

where $0 < \alpha \leq 1$.When $\alpha = 1$, since ${}_k\Delta_T^1 u(k) = (-\Delta)u(k)$ and ${}_0\Delta_k^1 u(k) = \Delta u(k)$, this boundary value problem reduces to

$$\begin{cases} -\Delta\left(\frac{1}{2}\Delta u(k) - \frac{1}{2}(-\Delta)u(k)\right) = \lambda f(k, u(k)), & k \in [1,T]_{\mathbb{N}_0}, \\ u(0) = u(T+1) = 0, \end{cases}$$

hence for $\alpha = 1$, the problem (2) reduces to the standard second-order boundary value problem (3), this standard second-order boundary value problem has been studied by many researchers; see, for instance, [5].

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Some new results for residual Fisher information distance

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Abstract. Fisher information plays a pivotal role throughout statistical inference especially in optimal and large sample studies in estimation theory. It also plays a key role in physics, thermodynamic, information theory and other applications. In this paper, we establish some new results on residual Fisher information distance (RFID) between residual density functions of two systems. Further, some results on RFID and their relations to other reliability measures are investigated along with some comparison of systems based on stochastic ordering. A lower bound for RFID measure is provided based on quadratic form of hazards functions. In addition, RFID measure for equilibrium distributions are studied.

Keywords: Equilibrium distribution; escort distribution; Fisher information distance; residual density function; stochastic ordering.

AMS Mathematics Subject Classification [2010]: 18A32, 18F20, 05C65

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1. Introduction

The Fisher information (FI) is not only integral to statistical inference but also is considered fundamental in statistics, information theory, physics, and allied disciplines (see, for example, [4] and [3]). Let us consider a random variable X (continuous or discrete) with a distribution function F_θ having a probability density function f_θ, where $\theta \in \Theta \subseteq R$. We assume throughout the paper that $f_\theta(x)$ is differentiable with respect to both θ and x. The Fisher information of random variable X (or distribution F_θ) about the parameter θ , based on an observation x of X, is defined as

$$I(\theta) \quad = \quad E\left[\frac{\partial \log f_\theta(X)}{\partial \theta}\right]^2. \tag{1}$$

If a random variable X has density $f(x)$, under the condition that the derivative of f exists for all values on its support, the Fisher information of the density is defined as

$$I(f) = E\left[\rho^2(X)\right], \tag{2}$$

where $\rho(x) = \frac{f'(x)}{f(x)}$ is called the score function corresponding to f.

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Given two random variables X and Y with absolutely continuous density functions f and g, respectively, the Fisher information distance (FID) between X and Y (or f and g) is defined by

$$D(f,g) = E_f\left[\left(\rho_f(X) - \rho_g(X)\right)^2\right], \tag{3}$$

where $\rho_f(x)$ and $\rho_g(x)$ are the score functions corresponding to f and g, respectively, for pertinent details, see [1]. Recently, time-dependent version of (2) and (3) have been proposed by [6]. These information measures so-called RFI and RFID. Recently, the cumulative versions of information measures in (1), (2) and (3) have proposed by [3]. The authors have provided some interesting connections between proposed measures with other well-know information quantities such as chi-square, cumulative Kullback-Leibler and Jefferry's divergence measures.

The purpose of this work is to establish some new properties of RFID measure between two residual lifetime distributions. Let X be a nonnegative random variable denoting a duration such as a lifetime where we assume that it has the distribution function F, and the probability density function f. The random variable of interest is the *residual random variable*, $X_t = X|X > t$, on the set

$$S_t = \{x : x > t\} \quad t \in (0,b) \;\; b \leq \infty.$$

Hence, the distribution of interest for computing information is the residual distribution with survival function

$$\bar{F}_t(x) = \begin{cases} \frac{\bar{F}(x)}{\bar{F}(t)} & x \in S_t \\ 1 & \text{otherwise,} \end{cases} \tag{4}$$

provided that $\bar{F}(t) < \infty$, where $\bar{F} = 1 - F$ denotes the survival function of X; see [7] for pertinent details.

In this work, our main objective is to establish some new results for RFID measure. We examine the relationship between RFID and some of reliability quantities. Moreover, we study RFID measure for equilibrium and escort distributions.

2. New results for Residual Fisher information distance (RFID)

Assuming that $f_t(x)$ and $g_t(x)$ denote the density functions corresponding to residual lifetimes variables X and Y, respectively. We now introduce RFI and RFID measures that will be used in the sequel.

Definition 2.1. Let X_t be a residual random variable with an absolutely continuous density function $f_t(x)$. The residual Fisher information of $f_t(x)$ is defined as

$$I(f;t) \;\; = \;\; E\big(\rho^2(X)|X > t\big), \tag{5}$$

where $t > 0$ and $b \leq \infty$ the right extremity of the support of X, i.e., $F(b) = 1$.

Definition 2.2. The RFID between f_t and g_t is defined as

$$D(f,g;t) \;\; = \;\; E\left[\left(\rho_f(X) - \rho_g(X)\right)^2 |X > t\right]. \tag{6}$$

Clearly for a non-negative random variable X, RFID reduce to FID, when $t \to 0$.

For more details, see [6]. Next, we establish some new results associated with RFID measure.

Theorem 2.3. *Given two random variables X and Y with RFID measure $D(f,g;t)$. Then under the condition of Lemma 2.1 from [20], we have*

$$D(f,g;t) \geq \Big(r_f(t) + E\big(\rho_g(X)|X>t\big)\Big)^2, \tag{7}$$

where $r_f(t) = \frac{f(t)}{\bar{F}(t)}$ is hazard function of variable X.

Given two random variables X and Y with absolutely continuous density functions f and g, respectively. The variable X is said to be less than Y in hazard rate order, $X \leq_{hr} Y$, if $r_f(x) \geq r_g(x)$, for all x in the union of supports of X and Y, where $r_f(x)(r_g(x))$ is the hazard rate of $X(Y)$. The following theorem provides an interesting lower bound for RFID measure based on the hazard functions.

Theorem 2.4. *Given two random variables X and Y with hazard functions $r_f(x)$ and $r_g(x)$, respectively. If $\rho_g(x)$ is decreasing and $X \leq_{hr} Y$, then*

$$D(f,g;t) \geq \big(r_f(t) - r_g(t)\big)^2. \tag{8}$$

In a similar way, we can show that if $\rho_f(x)$ is decreasing and $Y \leq_{hr} X$, then

$$D(g,f;t) \geq \left(r_f(t) - r_g(t)\right)^2.$$

From Theorem 2.4, we have

$$\Big(r_f(t) - r_g(t)\Big)^2 \leq \frac{D(f,g;t) + D(g,f;t)}{2}.$$

Example 2.5. Let X_i, $i = 1,2$, be distributed as gamma distribution with density

$$f_i(x) = \frac{\lambda^{\alpha_i}}{\Gamma(\alpha_i)} x^{\alpha_i - 1} e^{-\lambda x}, \quad x > 0,\ \alpha_i, \lambda > 0, \quad i = 1,2.$$

For $\alpha_1 > 2$, we have

$$D(f_1, f_2; t) \quad = \quad \lambda^2 \frac{(\alpha_2 - \alpha_1)^2 \Gamma(\alpha_1 - 2, \lambda t)}{\Gamma(\alpha_1, \lambda t)},$$

where $\Gamma(\alpha_1, t) = \int_t^\infty x^{\alpha_1 - 1} e^{-x} dx$, is incomplete gamma function.

We now give some theorems corresponding to RFID measure between densities of two transformed variables.

Theorem 2.6. *Given two random variables X and Y with absolutely continuous density functions f and g, respectively, and ϕ be a nonnegative increasing, twice differentiable and invertible function. Then*

$$D(f_\phi, g_\phi; t) = E\bigg[\frac{1}{\phi'(X)^2}\left[\frac{f'(X)}{f(X)} - \frac{g'(X)}{g(X)}\right]^2 \big| X > \phi^{-1}(t)\bigg]. \tag{9}$$

Let X and Y be continuous variables with density functions f and g, respectively. Variable X is said to be less than variable Y in likelihood ratio order, $X \leq_{lr} Y$, if $\frac{g(x)}{f(x)}$ is increasing in x for all x in the union of supports of X and Y.

Definition 2.7. Let X and Y be two random variables with residual Fisher information distance $D(f,g;t)$ and $D(g,f;t)$, respectively. X is said to be less than Y in residual Fisher information distance, denoted by $X \leq_{RFID} Y$, if $D(g,f;t) \leq D(g,f;t)$, for all t.

Theorem 2.8. *Let X and Y have densities f and g, respectively.*

(i) *Assume that* $\frac{f}{g}$ *is log-concave. If* $X \leq_{lr} Y$, *then* $X \leq_{RFID} Y$,
(ii) *Assume that* $\frac{g}{f}$ *is log-convex. If* $Y \leq_{lr} X$, *then* $X \leq_{RFID} Y$.

3. RFID measure for equilibrium distributions

Assume that $\bar{F}(x)$ is the survival function of a nonnegative continuous random variable X with finite mean μ. The random variable X_e is said to be the equilibrium random variable corresponding to the random variable X, if the density function of X_e is given by

$$f_e(x) = \frac{\bar{F}(x)}{\mu}, \quad x > 0. \tag{10}$$

The equilibrium distributions arise in renewal theory as the asymptotic distributions of the waiting time till the next event and the time since the last event at time t. Before presenting the theorem, we recall that the mean residual lifetime (MRL) of continuous random variable X with survival function $\bar{F}$ is defined at time t as

$$m(t) = E(X - t|X > t) = \frac{\int_t^b \bar{F}(x)dx}{\bar{F}(t)},$$

provided that $\bar{F}(t) > 0$. Note that $m(0) = \mu$ is the mean of X.

Let X and Y be two continuous random variables with density functions f and g and corresponding equilibrium densities f_e and g_e, respectively.

THEOREM 3.1. *The RFID between two equilibrium distributions* f_e *and* g_e *can be represented as*

$$D(f_e, g_e; t) = \frac{E\left[\frac{\left(r_f(X)-r_g(X)\right)^2}{r_f(X)}|X > t\right]}{m_f(t)},$$

where $m_f(t)$ *denote the MRL of variable* X.

4. Conclusion

In this paper, we have considered RFID measure and established some new properties associated with stochastic ordering in order to compare the lifetimes of two systems. We have shown that a lower bound for RFID measure can be expressed based on quadratic form of the corresponding hazard functions. In addition, we have provided some results of RFID measure in context of equilibrium distributions. Besides, we have shown that RFID measure between two equilibrium distributions is connected with hazard and mean residual functions of underlying variable.

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A mixed type functional equation in p-Banach spaces

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Abstract. In this paper, we consider and solve a mixed type functional equation in connection with a characterization of inner product spaces. This is applied to give a solution to the stability problem for the quadratic functional equation in the class of mappings from a quasi-normed space into a p-Banach space.

Keywords: p-Banach space, quasi-norm, mixed type functional equation

AMS Mathematics Subject Classification [2010]: 46B20, 39B52, 46A16

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1. Introduction and preliminaries

Lately many researchers have been interested about diverse issues related to quasi-Banach spaces. These spaces arise in a natural way as a generalization of Banach spaces, where the triangular inequality of the norm is changed by a weaker condition. Quasi-Banach spaces are an important class of metrizable topological vector spaces (often, not locally convex), see [5] for fundamental facts in quasi-Banach spaces.

Let X be a vector space over a real or complex field $\mathbb{F}$. A quasi-norm on X is a function $\|\cdot\|$ from X to $[0,\infty)$ satisfying

(i) $\|x\| > 0$ if $x \neq 0$ in X;

(ii) $\|\alpha x\| = |\alpha|\|x\|$ for $x \in X$, $\alpha \in \mathbb{F}$;

(iii) $\|x+y\| \leq \kappa(\|x\|+\|y\|)$ for $x, y \in X$, where $\kappa = \kappa(X) \geq 1$ is a constant, the modulus of concavity of X.

A quasi-norm is p-subadditive $(0 < p \leq 1)$, and then is called p-norm, if besides,

(iv) $\|x+y\|^p \leq \|x\|^p + \|y\|^p$ for $x, y \in X$.

A quasi-norm clearly defines a metrizable vector topology on X. If such topology is complete then we say that $(X, \|\cdot\|)$ is a quasi-Banach space. If the quasi-norm is also p-subadditive, then X is a p-Banach space.

The main difference of a quasi-normed space compared with a normed space is the modulus of concavity $\kappa \geq 1$. This causes a quasi-norm to not be necessarily continuous [5, Example 3]. Also, a quasi-normed space is not necessarily normable [5, Examples 1 and 2]. However, every quasi-normed space is p-normable in the sense that there exists a p-norm equivalent to the given quasi-norm by Aoki-Rolewicz theorem [5, Theorem 1]. Since it is much easier to work with p-norms, authors often restrict their attention to p-norms.

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The most important class of quasi-Banach spaces which are not already Banach spaces is the class of $L^p(\mu)$ spaces for $0 < p < 1$ with the usual quasi-norm $\|f\|_p = \left(\int |f|^p d\mu\right)^{\frac{1}{p}}$. In this case, $\|f+g\|_p \leq 2^{\frac{1}{p}-1}\left(\|f\|_p + \|g\|_p\right)$, i.e., the modulus of concavity of $L^p(\mu)$ is $2^{\frac{1}{p}-1}$.

The question of how much a function satisfying an equation approximately (for example, a difference, differential, functional or integral equation) may differ from a solution to the equation arises naturally in applications of mathematics. The theory of Ulam stability provides some effcient tools to evaluate such errors (see [1] for further details and references).

A functional equation of the form

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1}$$

was introduced by Jun and Kim [3] which is said to be a cubic functional equation and every solution of (1) is called a cubic mapping. One of the solutions of (1) is the function f defined by $f(x)=cx^3$ for all $x \in \mathbb{R}$, where c is an arbitrary real constant. They proved that a mapping f between two real vector spaces X and Y is a solution of the functional equation (1) if and only if there exists a mapping $C : X \times X \times X \longrightarrow Y$ such that $f(x) = C(x,x,x)$ for all $x \in X$, and C is symmetric for each fixed one variable and additive for fixed two variables. The mapping C is given by

$$C(x,y,z)=\frac{1}{24}\left(f(x+y+z)+f(x-y-z)-f(x+y-z)-f(x-y+z)\right)$$

for all $x, y, z \in X$. They also investigated its stability problem for the functional equation (1) on Banach spaces.

It is well known [4] that a mapping $f : X \to Y$ between two real vector spaces X and Y satisfies the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{2}$$

if and only if there exists a unique symmetric biadditive mapping $B : X \times X \to Y$ such that $f(x) = B(x,x)$ for all $x \in X$. The biadditive mapping B is given by

$$B(x,y)=\frac{1}{4}\left(f(x+y)-f(x-y)\right).$$

It is natural that the functional equation (2) is called a quadratic functional equation. In particular, every solution of (2) is said to be a quadratic mapping.

The following functional equation

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(z+x) \tag{3}$$

was solved by Kannappan [4] in connection with a characterization of inner product spaces. Let X be a vector space over a field K of characteristic zero (or characteristic different from 2). He proved that a function $f : X \to K$ is a solution of (3) if and only if there exist a symmetric biadditive mapping B and an additive mapping A such that $f(x) = B(x,x) + A(x)$ for all $x \in X$.

Chahbi et al. [2] have obtained some results concerning the stability of the k-quadratic functional equation

$$f(x+ky)+f(x-ky)=2f(x)+2k^2f(y), \ (k \in \mathbb{N}),$$

in the class of mappings from an abelian group into a Lipschitz space.

As a generalization of all the above functional equations, we treat the mixed type of cubic, quadratic and additive functional equation

$$\begin{aligned} f(x+y+kz)&+f(x-y+kz)+f(x+y-kz)+f(x-y-kz) \\ &= 2\left(f(x+y)+f(x-y)\right)+2k^2\left(f(x+z)+f(x-z)\right)-4k^2 f(x) \end{aligned} \tag{4}$$

for any fixed positive integer $k>1$. We establish the general solution of (4) in connection with a characterization of inner product spaces. Furthermore, we give a solution to the stability problem for the mixed type functional equation (4) in the framework of p-Banach spaces.

2. Main results

First, we present the general solution of (4) in the class of all mappings between real vector spaces.

Theorem 2.1. *Let X and Y be real vector spaces. A mapping $f : X \to Y$ with $f(0)=0$ is a solution of (4) if and only if there exist mappings $C : X \times X \times X \longrightarrow Y$, $B : X \times X \longrightarrow Y$ and $A : X \to Y$ such that*

$$f(x) = C(x,x,x) + B(x,x) + A(x) \tag{5}$$

for all $x \in X$, where the mapping C is symmetric for each fixed one variable and is additive for fixed two variables, B is symmetric biadditive and A is additive.

Proof. Let f with $f(0)=0$ satisfies (4). We decompose f into the even part and odd part by putting

$$f_e(x) = \frac{f(x)+f(-x)}{2}, \qquad f_o(x) = \frac{f(x)-f(-x)}{2}$$

for all $x \in X$. It is clear that $f(x) = f_e(x)+f_o(x)$ for all $x \in X$. Since f satisfies (4), the mappings f_e and f_o satisfy (4). In (4), replace f by f_e and set $x=y=0$ to obtain $f_e(kz) = k^2 f_e(z)$ for all $z \in X$. Hence, by [6, Theorems 2.1 and 2.2], we achieve that the mappings f_e and f_o are quadratic and cubic-additive, respectively. Therefore there exist a symmetric bi- additive mapping $B : X \times X \longrightarrow Y$ such that $f_e(x) = B(x,x)$ for all $x \in X$, and the mapping $C : X \times X \times X \longrightarrow Y$ and additive mapping $A : X \to Y$ such that $f_o(x) = C(x,x,x)+A(x)$, for all $x \in X$, where the mapping C is symmetric for each fixed one variable and is additive for fixed two variables. Hence we obtain (5).

Conversely, let $f(x) = C(x,x,x)+B(x,x)+A(x)$ for all $x \in X$, where C is symmetric for each fixed one variable and is additive for fixed two variables, B is biadditive and A is additive. By a simple computation one can show that the mappings $x \mapsto C(x,x,x)$ and $x \mapsto B(x,x)$ and A satisfy the functional equation (4). So f satisfies (4). $\square$

Now, let us assume that $(X, \|\cdot\|)$ is a real normed space and

$$\begin{aligned} &\|x+y+kz\|^2 + \|x-y+kz\|^2 + \|x+y-kz\|^2 + \|x-y-kz\|^2 \\ &\quad = 2\left(\|x+y\|^2+\|x-y\|^2\right) + 2k^2\left(\|x+z\|^2+\|x-z\|^2\right) - 4k^2\|x\|^2 \end{aligned}$$

holds for any fixed positive integer $k>1$ and all $x,y,z \in X$. Define $g : X \to \mathbb{R}$ by $g(x) = \|x\|^2$. Then $g(0)=0$, g is even, and g satisfies (4). Then, by Theorem 2.1,

$g(x) = B(x, x)$, where B is a symmetric, biadditive mapping. Also, it is easy to see that B is bilinear and so X is an inner product space.

Now before taking up the stability of the mixed type functional equation (4), we define the following difference operator for a given mapping $f : X \to Y$,

$$\begin{aligned} Df\,(x, y, z) :=& f(x + y + kz) + f(x - y + kz) + f(x + y - kz) + f(x - y - kz) \\ & - 2\,(f(x + y) + f(x - y)) - 2k^2\,(f(x + z) + f(x - z)) + 4k^2 f(x) \end{aligned}$$

for any fixed positive integer $k > 1$ and all $x, y, z \in X$, where X is a quasi-normed space with quasi-norm $\|\cdot\|_X$ and Y is a p-Banach space with p-norm $\|\cdot\|_Y$.

THEOREM 2.2. *Assume that $\alpha : X \times X \times X \longrightarrow [0, \infty)$ is a function such that $\lim_{m\to\infty} k^{2m}\alpha\left(\frac{x}{k^m}, \frac{y}{k^m}, \frac{z}{k^m}\right) = 0$ and $\beta(x) := \sum_{i=1}^{\infty} k^{2ip}\alpha^p\left(0, 0, \frac{x}{k^i}\right) < \infty$ for all $x, y, z \in X$, where $\alpha^p(x, y, z) = (\alpha(x, y, z))^p$. Suppose that an even mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality $\|Df(x, y, z)\|_Y \leq \alpha(x, y, z)$ for all $x, y, z \in X$. Then there exists a unique quadratic mapping $G : X \to Y$ such that $\|f(x) - G(x)\|_Y$ is bounded by $\frac{\sqrt[p]{\beta(x)}}{4k^2}$ for all $x \in X$.*

3. Conclusion

This research has made an attempt to analyze the general solution and the stability problem of a mixed type functional equation in p-Banach spaces.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

On non-linear maps preserving Drazin invertible operator matrices

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Abstract. Let $\mathcal{H}$ be a complex Hilbert space and $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a bijection, not necessarily linear and unital. We give the general form of Ψ under some preserving conditions on operator matrices in $\mathcal{B}(\mathcal{H}^2)$ related to the square zero or Drazin invertible (of index at most 1) generalized Schur complements.

Keywords: preserver problem, Drazin invertible operator, Schur complement, Square zero operator

AMS Mathematics Subject Classification [2010]: Primary 47A08, 15A99, Secondary 47B49

1. Introduction

Motivated by the recent results concerning Drazin invertibility preserving maps (see for example [4, 5, 7, 8]) on operator matrices, we studied, in [3], certain non-linear preserver problems concerning Drazin invertibility of 2×2 operator matrices. For an infinite-dimensional Hilbert space $\mathcal{H}$, the main result of [3] characterizes unital bijections $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, not necessarily linear, preserving Drazin invertible 2×2 operator matrices of index one that their generalized Schur complements are either invertible or Drazin invertible of index one. It was shown that for such a map Ψ there exists a bounded invertible linear or conjugate linear operator $U : \mathcal{H} \to \mathcal{H}$ such that either $\Psi(S) = USU^{-1}$ or $\Psi(S) = US^*U^{-1}$ for every $S \in \mathcal{B}(\mathcal{H})$.

In this note, by considering an additional preserving condition, we give the same result without assuming that Ψ is unital.

For a complex Banach space $\mathcal{X}$, let $\mathcal{X}^*$ be its dual space, and $\mathcal{B}(\mathcal{X})$ be the Banach algebra of all bounded linear operators on $\mathcal{X}$. We denote the identity operator on $\mathcal{X}$ by I. We use the notation $\mathcal{N}_1(\mathcal{X})$ for the set of all rank one nilpotent operators in $\mathcal{B}(\mathcal{X})$. Hence each element of $\mathcal{N}_1(\mathcal{X})$ can be written as $x \otimes f$ for some non-zero $x \in \mathcal{X}$ and non-zero $f \in X^*$ with $f(x) = 0$. We recall that $x \otimes f$ is the rank one operator on $\mathcal{X}$ defined by $(x \otimes f)(y) = f(y)x$ for all $y \in \mathcal{X}$. For an operator $T \in \mathcal{B}(\mathcal{X})$, the notations $\operatorname{ran}(T)$ and

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$\ker(T)$ stand for the range and the kernel of T, respectively, and $\operatorname{rank}(T)$ denotes the rank of T.

An operator $T \in \mathcal{B}(\mathcal{X})$ is called Drazin invertible, if there exists (a unique) operator $T^D \in \mathcal{B}(\mathcal{X})$ and a non-negative integer k such that

$$TT^D = T^D T,\ T^D T T^D = T^D, \text{ and } T^{k+1}T^D = T^k. \tag{1}$$

If $T \in \mathcal{B}(\mathcal{X})$ is Drazin invertible, then the smallest non-negative integer k satisfying (1) is called the index of T and it is denoted by $\operatorname{Ind}(T)$.

The notations $\mathcal{D}(\mathcal{X})$ and $\mathcal{D}_n(\mathcal{X})$ will be used for the set of all Drazin invertible operators and the set of all Drazin invertible operators of index n in $\mathcal{B}(\mathcal{X})$, respectively. If $\operatorname{Ind}(T) = 0$, then T is invertible and T^D is, indeed, the inverse of T. In the case where $\operatorname{Ind}(T) \leq 1$, T^D is called the group inverse of T and may be denoted by $T^{\#}$.

We note that if $T \in \mathcal{B}(\mathcal{X})$ is a non-zero nilpotent operator, then $T \notin \mathcal{D}(\mathcal{X})$. In particular, if $x \otimes f \in \mathcal{N}_1(\mathcal{X})$, then $x \otimes f \notin \mathcal{D}(\mathcal{X})$. However if $f(x) = 1$, then $x \otimes f \in \mathcal{D}_1(\mathcal{X})$ and its Drazin inverse is $x \otimes f$ itself.

For an operator $T \in \mathcal{B}(\mathcal{X})$, by $\operatorname{asc}(T)$ and $\operatorname{dsc}(T)$ we mean the ascent and the descent of T, respectively which are defined as follows

$$\operatorname{asc}(T) = \inf\{n \geq 0 : \ker(T^n) = \ker(T^{n+1})\},$$
$$\operatorname{dsc}(T) = \inf\{m \geq 0 : \operatorname{ran}(T^m) = \operatorname{ran}(T^{m+1})\}.$$

If no such n or m exists, we set $\operatorname{asc}(T) = \infty$ or $\operatorname{dsc}(T) = \infty$.

In the study of (not necessarily linear) maps on operator algebras that preserve Drazin invertible operator matrices, it is essential to have a description of Drazin invertible operator matrices. For invertible operator matrices, such a description is provided by applying Schur complement.

Let $\mathcal{X}$ be a complex Banach space and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an operator matrix with entries in $\mathcal{B}(\mathcal{X})$ such that A is invertible. Then the Schur complement of M is the operator $S_M \in \mathcal{B}(\mathcal{X})$ defined by $S_M = D - CA^{-1}B$.

Lemma 1.1. *[6, Problem 1.6.7] Let $\mathcal{X}$ be a complex Banach space and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{B}(\mathcal{X}^2)$ such that A is invertible. Then M is invertible if and only if S_M is invertible. Furthermore, in this case*

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS_M^{-1}CA^{-1} & -A^{-1}BS_M^{-1} \\ -S_M^{-1}CA^{-1} & S_M^{-1} \end{pmatrix}.$$

It would be convenient if we had a similar result as the above lemma for Drazin invertible operator matrices. In the sequel, we apply the following results from [2], for determining the Drazin invertibility of an operator matrix and computing its Drazin inverse. For any Drazin invertible $T \in \mathcal{B}(\mathcal{X})$, the spectral idempotent T^π of T corresponding to $\{0\}$ is defined by $T^\pi = I - TT^D$.

If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an operator matrix with entries in $\mathcal{B}(\mathcal{X})$ such that A is Drazin invertible, then the generalized Schur complement of M, denoted again by S_M is defined by $S_M = D - CA^DB$. When S_M is Drazin invertible, as discussed in [2], the operator matrix

$$\widetilde{\Gamma}_M = \begin{pmatrix} AA^\pi - A^\pi BS_M^DCA^\pi & A^\pi BS_M^\pi \\ S_M^\pi CA^\pi & S_M S_M^\pi \end{pmatrix}$$

plays a crucial role in determining the representation of $M^{\#}$, especially when $\widetilde{\Gamma}_M = 0$.

THEOREM 1.2. *[2, Theorem 9]Let $\mathcal{X}$ be a complex Banach space and* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *with entries in $\mathcal{B}(\mathcal{X})$ such that $A, S_M \in \mathcal{D}(\mathcal{X})$, $\widetilde{\Gamma}_M = 0$ and $CA^\pi B = 0$. Then $M \in \mathcal{D}_0(\mathcal{X}^2) \cup \mathcal{D}_1(\mathcal{X}^2)$ if and only if the operator $W_M = I + A^D B S_M^\pi C A^D$ is invertible in $\mathcal{B}(\mathcal{X})$. Moreover, in this case*

$$M^{\#} = \left[\begin{pmatrix} 0 & A^\pi B \\ S_M^\pi C & S_M^\pi D \end{pmatrix} R + I\right] \cdot W \cdot R \cdot W \cdot \left[I + R\begin{pmatrix} 0 & BS_M^\pi \\ CA^\pi & DS_M^\pi \end{pmatrix}\right],$$

where $W = W_M^{-1} \oplus I$ and

$$R = \begin{pmatrix} A^D + A^D B S_M^D C A^D & -A^D B S_M^D \\ -S_M^D C A^D & S_M^D \end{pmatrix}.$$

2. Main results

Throughout this section $\mathcal{H}$ is an infinite-dimensional complex Hilbert space. We first introduce certain subsets $\mathcal{D}_1^*(\mathcal{H}^2)$ and $\mathcal{S}_0(\mathcal{H}^2)$ of $\mathcal{B}(\mathcal{H}^2)$ with $\mathcal{D}_1^*(\mathcal{H}^2) \cap \mathcal{S}_0(\mathcal{H}^2) = \{0\}$. Then we characterize bijections $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ preserving $\mathcal{D}_1^*(\mathcal{H}^2)$ and $\mathcal{S}_0(\mathcal{H}^2)$ in both directions, in the sense that for the operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with entries in $\mathcal{B}(\mathcal{H})$ we have $M \in \mathcal{D}_1^*(\mathcal{H}^2)$ (resp. $M \in \mathcal{S}_0(\mathcal{H}^2)$) if and only if

$$\Psi_2(M) = \begin{pmatrix} \Psi(A) & \Psi(B) \\ \Psi(C) & \Psi(D) \end{pmatrix} \in \mathcal{D}_1^*(\mathcal{H}^2)$$

(resp. $\Psi_2(M) \in \mathcal{S}_0(\mathcal{H}^2)$). We should note that such a map Ψ is not assumed to be neither linear nor unital.

Let $\mathcal{S}_q(\mathcal{H})$ stands for the set of all square zero operators in $\mathcal{B}(\mathcal{H})$. Now we fix the following notations:

$$\mathcal{S}_0(\mathcal{H}^2) = \left\{N = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \mathcal{B}(\mathcal{H}^2) : A_1 \in \mathcal{D}(\mathcal{H}),\ N \neq 0,\ S_M \in \mathcal{S}_q(\mathcal{H})\right\},$$

$$\mathcal{D}_1^*(\mathcal{H}^2) = \left\{M = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \mathcal{D}_1(\mathcal{H}^2) : A_1 \in \mathcal{D}(\mathcal{H}),\ S_M \in \mathcal{D}_1(\mathcal{H}) \cup \mathcal{D}_0(\mathcal{H})\right\},$$

where S_M is the generalized Schur complement of M. Note that $\mathcal{S}_0(\mathcal{H}^2) \bigcap \mathcal{D}_1^*(\mathcal{H}^2) = \{\emptyset\}$, since $S_q(\mathcal{H}) \cap (D_1(\mathcal{H}) \cup \mathcal{D}_0(\mathcal{H})) = \{0\}$.

Now we state and prove our main result, which is a modification of [3, Theorem 3.7], under some additional assumption.

THEOREM 2.1. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and Ψ be a (not necessarily linear) bijection on $\mathcal{B}(\mathcal{H})$, preserving $\mathcal{D}_1^*(\mathcal{H}^2)$ and $\mathcal{S}_0(\mathcal{H}^2)$ in both directions. Then there exists a non-zero scalar $\beta \in \mathbb{C}$, a bounded invertible linear or conjugate linear operator $U : \mathcal{H} \to \mathcal{H}$ such that either*

$$\Psi(S) = \beta U S U^{-1} \qquad (S \in \mathcal{B}(\mathcal{H}))$$

or

$$\Psi(S) = \beta U S^* U^{-1} \qquad (S \in \mathcal{B}(\mathcal{H})).$$

PROOF. Since by assumption, Ψ preserves $\mathcal{D}_1^*(\mathcal{H}^2)$ in both directions, it follows from step 1 and step 2 in the proof of [3, Theorem 3.7], that $\Psi(0) = 0$ and Ψ preserves invertibility and $\mathcal{D}_1(\mathcal{H})$ in both directions. In particular, $\Psi(I)$ is an invertible operator. We claim that $\Psi(I)$ is a scalar operator. Assume on the contrary that $\Psi(I)$ is not a scalar operator. Then, by Theorem 1.4 in [1], there exists $x \in \mathcal{H}$ such that x and $\Psi(I)x$ are linearly independent. Choose $y \in \mathcal{H}$ such that $\langle x, y\rangle = 1$ and $\langle \Psi(I)x, y\rangle = 0$. Then $x \otimes y$ belongs to $D_1(\mathcal{H})$ and $(x \otimes y)^D = x \otimes y$, since $x \otimes y$ is a rank one idempotent. Similarly, $\Psi(I)x \otimes y \notin D_1(\mathcal{H})$ since $Sx \otimes y$ is a square zero rank one operator.

Consider the operator matrix

$$M = \begin{pmatrix} x \otimes y & x \otimes y \\ \Psi(I) & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H}^2).$$

Then $S_M = -\Psi(I)x \otimes y$ which is a square zero operator. Consequently $M \in \mathcal{S}_0(\mathcal{H}^2)$. Since by assumption, Ψ preserves $\mathcal{S}_0(\mathcal{H}^2)$ in both directions, it follows that

$$N = \begin{pmatrix} \Psi^{-1}(x \otimes y) & \Psi^{-1}(x \otimes y) \\ I & 0 \end{pmatrix} \in \mathcal{S}_0(\mathcal{H}^2).$$

But this is a contradiction, since $S_N = -\Psi^{-1}(x \otimes y)^D \Psi^{-1}(x \otimes y) \in \mathcal{D}_1(\mathcal{H})$ with $(S_N)^D = S_N$. Hence $\Psi(I) = \beta I$ for some non-zero complex scalar β. Now the rest of the proof follows from the same steps in the proof of [3, Theorem 3.7]. $\square$

3. Conclusion

In this paper we introduced a special subset $\mathcal{S}_0(\mathcal{H}^2)$ of $\mathcal{B}(\mathcal{H}^2)$ including operator matrices that their generalized Schur complement are square zero operators. Then we proved that the assumption of preserving $\mathcal{S}_0(\mathcal{H}^2)$ in both directions is a replacement for the property that Ψ is a unital map.

Acknowledgement

The authors were partially supported by Iran National Science Foundation: INSF (Grant No. 97013493).

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Precover of S-posets over pomonoids

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Abstract. In this paper we introduce the concept of an X-precover for a class of S-posets X. Then we prove that for those classes that are closed under isomorphisms and directed colimits, directed colimits of precovers are precovers. We also provide the necessary and coefficient conditions for S-posets to have X-precovers. Finally, we show that every S-poset has a projective precover.

Keywords: S-posets, Pomonoids, Precover, Cover, Projective

AMS Mathematics Subject Classification [2010]: 20M30, 06F05

1. Introduction

A *pomonoid* S is a monoid equipped with a partial order relation which is compatible with the binary semigroup operation. Let A be a partially ordered set. We say that A is a *right S-poset* if A_S is a right S-act A and, in addition, for all $s, t \in S$ and $a, b \in A$, if $s \leq t$ then $as \leq at$, and if $a \leq b$ then $as \leq bs$. An *S-subposet* of a right S-poset A is a subset of A that is closed under the S-action. Moreover, *S-morphisms* are the functions that preserve both the action and the order. Let $f : A \to B$ be an S-morphism, the subkernel of an S-morphism f is defined by

$$\overrightarrow{\ker} f := \{(a, a') \in A \times A : f(a) \leq f(a')\}.$$

Then $\nu(\overrightarrow{\ker} f) = \ker f := \{(a, a') \in A \times A : f(a) = f(a')\}$. As in the the category of S-acts, the coproduct of a family $\{A_i\}_{i \in I}$ of S-posets will be denoted by $\coprod_{i \in I} A_i$ that is the disjoint union, with S-action and order given componentwise.

Directed colimits of families of right S-posets are introduced in [1]. Then, in [5], it is proved that every class of S-posets having a flatness property is closed under directed colimits. Let I be a quasi-ordered (that is, a reflexive and transitive relation) set. *A direct system* in Pos-S is a collection of right S-posets $(A_i)_{i \in I}$ and a collection of right S-poset morphisms $\phi_{i,j} : A_i \to A_j$ $(i \leq j)$ with the following properties:

(1) $\phi_{i,i} = 1_{A_i}$ for all $i \in I$;
(2) $\phi_{j,k} \circ \phi_{i,j} = \phi_{i,k}$, whenever $i \leq j \leq k$.

The colimit of the system $(A_i, \phi_{i,j})$ is a right S-poset A_S together with right S-poset morphisms $\alpha_i : A_i \to A$ such that

(1) $\alpha_j \circ \phi_{i,j} = \alpha_i$, whenever $i \leq j$;

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(2) If B_S is a right S-poset and $f_i : A_i \to B$ are right S-poset morphisms such that $f_j \circ \phi_{i,j} = f_i$ whenever $i \leq j$, then there exists a unique S-poset morphism $f : A \to B$ such that the diagram

$$\begin{tikzcd} A_i \arrow[r, "\alpha_i"] \arrow[d, "f_i"'] & A \arrow[dl, "f"] \\ B & \end{tikzcd}$$

commutes for all $i \in I$.

Further, if the indexing set I satisfies the Condition that for all $i, j \in I$ there exists $k \in I$ such that $k \geq i, j$ then we say that I is *directed*. In this case, we call that the colimit (A, ϕ_i) is a *directed colimit*. In Pos-S, directed colimits of directed systems of S-posets exist.

A right S-poset A_S satisfies *condition (P)* if, for all $a, b \in A$ and $s, t \in S$, $as \leq bt$ implies $a = a'u, b = a'v$ for some $a' \in A$, $u, v \in S$ with $us \leq vt$, and it satisfies *condition (E)* if, for all $a \in A$ and $s, t \in S$, $as \leq at$ implies $a = a'u$ for some $a' \in A$, $u \in S$ with $us \leq ut$. A right S-poset is called *strongly flat* if it satisfies both conditions (P) and (E). Projectivity is defined in the standard categorical manner. The reader is referred to the monograph [4] for a complete discussion of acts over monoids.

In [3], Gould and Shaheen characterized poperfect pomonoids. Then, in [2], pomonoids over which all (cyclic) S-posets have strongly flat or condition (P) covers are characterized. For further information of S-posets, we refer the reader to [6] and [1]. In Section 2, we investigate precover of S-posets over pomonoids. Indeed pomonoids over which all right S-posets have X-precovers are characterized where X is an S-poset property which is preserved under isomorphism and coproduct.

2. Main results

Let S be a pomonoid. A right S-poset B_S is called a *cover* of an S-poset A_S if there exists an epimorphism $f : B_S \to A_S$ such that for any proper subposet C_S of B_S the restriction $f\mid_{C_S}$ is not an epimorphism. An epimorphism with this property is called a *coessential epimorphism*. We say that B_S is an X-cover if it also satisfies property X. In this section we focus our attention on X-precover of S-posets, where X be an S-poset property which is preserved under coproduct and decompositon.

Definition 2.1. Let X be a class of S-acts closed under isomorphisms. An *X-precover* of A is an S-morphism $g : P \to A$ for some $P \in X$ such that for every S-morphism $h : Q \to A$, for $Q \in X$, there exists an S-morphism $f : Q \to P$ with $h = gf$, i.e., the following diagrams commutes.

$$\begin{tikzcd} P \arrow[r, "g"] & A \\ Q \arrow[u, "f"] \arrow[ur, "h"'] & \end{tikzcd}$$

An object G in the category Pos-S is called a generator if the functor $\mathrm{Hom}_S(G_S, -)$ is faithful, It is shown that G is generator if and only if there exists an epimorphism $\pi : G \to S$. For every S-poset A_S and its element a, $\lambda_a : S_S \to A_S$ will denote the S-morphism defined by $\lambda_a(s) = as$ for every $s \in S$.

Proposition 2.2. *Let S be a pomonoid and let X be a class of S-posets which contains a generator G. If $g : C \to A$ is an X-precover of A then g is an epimorphism.*

From now on we use variable X to indicate the properties that can be transferred from S-posets to their coproducts and vice versa.

Proposition 2.3. *Let S be a pomonoid. Then each S-poset A_i has an X-precover then $\coprod_{i\in I} A_i$ has an X-precover. The converse is valid when $\coprod_{i\in I} A_i$ has an epimorphic X-precover*

We now show that directd colimits of X-precovers are X-precovers.

Theorem 2.4. *Let S be a pomonoid and X be a class of S-posets closed under directed colimits. Suppose that $(A_i, \phi_{i,j})$ is a direct system of S-posets with $A_i \in X$ for each $i \in I$ and with colimit (A, α_i). If B is an S-poset, and for each $i \in I$, $f_i : A_i \to B$ is an X-precover of B such that $f_j\phi_{i,j} = f_i$ whenever $i \leq j$, then there exists an X-precover $f : A \to B$ such that $f\alpha_i = f_i$ for all $i \in I$.*

Proof. Suppose that B is an S-poset and (A, α_i) is directed colimit of direct system $(A_i, \phi_{i,j})$ of S-posets with $A_i \in X$ for each $i \in I$, in addition there exists an X-precover $f_i : A_i \to B$ of B with $f_j \circ \phi_{i,j} = f_i$ for $i \leq j$. The property of directed colimit implies that there exists a unique S-morphism $f : A \to B$ such that the diagram

$$\begin{tikzcd} A_i \arrow[r, "\alpha_i"] \arrow[d, "f_i"'] & A \arrow[dl, "f"] \\ B & \end{tikzcd}$$

commutes for all $i \in I$. So $f \circ \alpha_i = f_i$ for all $i \in I$. To show that f is an X-precover of B, assume that $C \in X$ and $g : C \to B$ is an S-morphism. Since $f_i : A_i \to B$ is an X-precover of B, there exists $h_i : C \to A_i$ such that $f_i h_i = g$ for each $i \in I$. Fix $i \in I$ and let $h : C \to A$ be given by $h = \alpha_i h_i$. Therefore, $fh = f\alpha_i h_i = f_i h_i = g$ as required.

□

The monomorphisms are exactly the injective morphisms and regular monomorphisms are exactly the order embeddings.

Lemma 2.5. *Let S be a pomonoid and X a class of S-posets closed under directed colimits. Let A be an S-poset and suppose that $g : P \to A$ is an X-precover of A. Then there exists an X-precover $h : Q \to A$ and an S-morphism $f : P \to Q$ with $hf = g$ such that for any X-precover $h' : Q' \to A$ and any S-morphism $k : Q \to Q'$ with $h'k = h$ then $k|_{f(P)}$ is a regular monomorphism.*

Lemma 2.6. *Let S be a monoid and X a class of S-posets closed under directed colimits. Let A be an S-poset and suppose that $g : P \to A$ is an X-precover of A. Then there exists an X-precover $h : Q \to A$ such that for any X-precover $h' : Q' \to A$ and any S-morphism $k : Q \to Q'$ with $h'k = h$ then k is a regular monomorphism.*

Let S be a pomonoid. A class X of S-posets satisfies the (weak) solution set condition if for every S-poset A there exists a set $S_A \subseteq X$ such that for each (indecomposable) $X \in X$ and each S-morphism $h : X \to A$ there exists $Y \in S_A$, $f : X \to Y$ and $g : Y \to A$ such that $h = gf$.

Theorem 2.7. *Let S be a pomonoid. Then every S-poset has an X-precover if and only if*

(1) *for every S-poset A there exists $B \in X$ such that $Hom_S(B, A) \neq \emptyset$.*
(2) *X satisfies the weak solution set condition.*

Note from the proof of Theorem 2.7 that we can also deduce the following result.

THEOREM 2.8. *Let S be a pomonoid.Then every S-poset has an X-precover if and only if*
(1) *for every S-poset A there exists $B \in X$ such that $Hom_S(B, A) \neq \emptyset$;*
(2) *X satisfies the solution set condition.*

Finally, we conclude the following result.

PROPOSITION 2.9. *Let S be a pomonoid. Every S-poset has a projective precover.*

3. Conclusion

Let S be a pomonoid and X a class of S-posets which is closed under coproducts. This paper is devoted to study X-precovers of S-posets.We have shown that all S-posets have projective precovers.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

On Set-Valued F-contraction Mappings and Volterra-type Integral Equations

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ABSTRACT. In this paper, we present some new fixed point theorems involving set-valued contractions in the setting of quasi-ordered metric spaces. We generalize Banach contraction principle in a different way than in the known results from the literature. Some examples and an application to the existence of solution of Volterra-type integral equations are given to support the obtained results. In particular, we refer to the results of Wardowski [Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory and Appl. 2012, 2012:94].

Keywords: Fixed point, Quasi-ordered complete metric spaces, F-contraction Maps, order-closed, approximative values.

AMS Mathematics Subject Classification [2010]: 47H10; 47S50.

1. Introduction

In 1969, Nadler [4] extended the Banach contraction principle from single-valued mapping to set-valued mapping. Then Ćirić [3] generalized Nadler's result. In 2011, Amini-Harandi [1] gave a fixed point theorem for set-valued quasi-contraction maps in metric spaces.

In 2012, Wardowski [6] gave a new fixed point theorem concerning F-contraction for single valued mapping. In this paper, we give a fixed point theorem for set-valued contraction maps in quasi-ordered metric spaces. In the present article, using a mapping $F : R^+ \longrightarrow R$ we define a new type of contraction called F-contraction and prove some new fixed point theorems concerning F-contraction. Throughout this paper, by $CB(X)$ we denote the family of all nonempty closed and bounded subsets of X and we denote the family of all nonempty subsets of X by $N(X)$.

DEFINITION 1.1. Let (X, d) be a metric space with a quasi-order "$\preceq$". We say that X is sequentially complete if every Cauchy sequence whose consecutive terms are comparable in X converges.

DEFINITION 1.2. [5] Let $(X, \leq)$ be a partially ordered set, and A and B be two nonempty subsets of $N(X)$. The relation between A and B are defined as follows:

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(r_1) If for every $a \in A$, there exists $b \in B$ such that $a \leq b$, then $A \sqsubseteq_1 B$.
(r_2) If for every $b \in B$, there exists $a \in A$ such that $a \leq b$, then $A \sqsubseteq_2 B$.
(r_3) If $A \sqsubseteq_1 B$ and $A \sqsubseteq_2 B$, then $A \sqsubseteq B$.

Definition 1.3. Let (X, d) be a metric space with a quasi-order " $\preceq$ ". A subset $D \subset X$ is said to be approximative if the set-valued mapping:

$$P_D(x) = \{p \in D : d(x, D) = d(p, x)\},$$

for all $x \in X$ has nonempty value.

The set-valued mapping $G : X \longrightarrow N(X)$ is said to have approximative values, AV for short, if Gx is approximative for each $x \in X$.

The set-valued mapping $G : X \longrightarrow N(X)$ is said to have comparable approximative values, CAV for short, if for each $x \in X$, Gx has approximative values and for each $z \in X$ there exists $y \in P_{Gz}(x)$ such that y is comparable to z.

The set-valued mapping $G : X \longrightarrow N(X)$ is said to have upper comparable approximative values, UCAV for short (resp. lower comparable approximative values, LCAV for short) if Gx has approximative values and for each $z \in X$ there exists $y \in P_{Gz}(x)$ such that $y \succeq z$ (resp. $y \preceq z$).

Definition 1.4. [5] The set-valued mapping G is said to has a fixed point if there exists $x \in X$ such that $x \in Gx$.

Definition 1.5. [5] A set-valued operator $G : X \to N(X)$ is called order-closed if for monotone sequences $\{x_n\}$, $\{y_n\} \subset X$, $x_0, y_0 \in X$, $x_n \to x_0$, $y_n \to y_0$ and $y_n \in G(x_n)$ imply $y_0 \in G(x_0)$.

In this paper, we give some fixed point theorems for set-valued F-contraction maps in quasi-ordered metric spaces.

2. Main result

Definition 2.1. [6] Let $F : R^+ \longrightarrow R$ be a mapping satisfying :

(F1) F is strictly increasing, i.e., for all $a, b \in R^+$ such that $a < b$, $F(a) < F(b)$,
(F2) for each sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\longrightarrow\infty} a_n = 0$ if and only if

$$\lim_{n\longrightarrow\infty} F(a_n) = -\infty,$$

(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to0^+} \alpha^k F(\alpha) = 0$.

A mapping $G : X \longrightarrow N(X)$ is said to be an F-contraction if there exists $\tau > 0$ such that for all $x, y \in X$,

$$H(Gx, Gy) > 0 \Longrightarrow \tau + F(H(Gx, Gy)) \leq F(d(x, y)). \tag{1}$$

Theorem 2.2. *Let $(X, d, \preceq)$ be a sequentially complete metric space. Suppose that the map $G : X \longrightarrow N(X)$ be a ordered-close set-valued F-contraction and has UCAV. Then G has a fixed point $x^* \in X$.*

Theorem 2.3. *Let $(X, d, \preceq)$ be a sequentially complete metric space. Suppose that the map $G : X \to N(X)$ be a ordered-close set-valued F-contraction and has LCAV. Then G has a fixed point $x^* \in X$.*

Theorem 2.4. *Let* $(X, d, \preceq)$ *be a sequentially complete metric space with the property any non-decreasing sequence* $\{x_n\}$ *with* $x_n \to x^*$ *implies* $x_n \preceq x^*$ *for each* $n \in \mathbb{N}$. *Suppose that the non-decreasing mapping* $G : X \to N(X)$ *be a ordered-close set-valued* F*-contraction and has* AV. *If there exists* $x_0 \in X$ *such that* $\{x_0\} \sqsubseteq Gx_0$, *then* G *has a fixed point* $x^* \in X$.

Theorem 2.5. *Let* $(X, d, \preceq)$ *be a sequentially complete metric space with the property any non-increasing sequence* $\{x_n\}$ *with* $x_n \to x^*$ *implies* $x^* \preceq x_n$ *for each* $n \in \mathbb{N}$. *Suppose that the non-increasing mapping* $G : X \to N(X)$ *be a ordered-close set-valued* F*-contraction and has AV. If there exists* $x_0 \in X$ *such that* $Gx_0 \sqsubseteq \{x_0\}$, *then* G *has a fixed point* $x^* \in X$.

Theorem 2.6. *Let* $(X, d, \preceq)$ *be a sequentially complete metric space. Suppose that the map* $G : X \longrightarrow N(X)$ *be a ordered-close set-valued and has UCAV. If we have*

$$F(H(Gx, Gy)) \le F(M(x, y)) - \tau. \tag{2}$$

Where

$$M(x, y) = max\left\{d(x, y), D(x, Gx), D(y, Gy), \frac{1}{2}[D(x, Gy) + D(y, Gx)]\right\}.$$

Then G *has a fixed point* $x^* \in X$.

Theorem 2.7. *Let* $(X, d, \preceq)$ *be a sequentially complete metric space. Suppose that the map* $G : X \longrightarrow N(X)$ *is an ordered-close set-valued and has LCAV and*

$$F(H(Gx, Gy)) \le F(M(x, y)) - \tau$$

where

$$M(x, y) = max\left\{d(x, y), D(x, Gx), D(y, Gy), \frac{1}{2}[D(x, Gy) + D(y, Gx)]\right\}.$$

Then G *has a fixed point* $x^* \in X$.

Theorem 2.8. *Let* $(X, d, \preceq)$ *be a sequentially complete metric space with the property any non-decreasing sequence* $\{x_n\}$ *with* $x_n \to x^*$ *implies* $x_n \preceq x^*$ *for each* $n \in \mathbb{N}$. *Suppose that the non-decreasing mapping* $G : X \to N(X)$ *is an ordered-close set-valued and has* AV *and*

$$F(H(Gx, Gy)) \le F(M(x, y)) - \tau,$$

where

$$M(x, y) = max\left\{d(x, y), D(x, Gx), D(y, Gy), \frac{1}{2}[D(x, Gy) + D(y, Gx)]\right\}.$$

If there exists $x_0 \in X$ *such that* $\{x_0\} \sqsubseteq Gx_0$, *then* G *has a fixed point* $x^* \in X$.

Theorem 2.9. *Let* $(X, d, \preceq)$ *be a sequentially complete metric space with the property any non-increasing sequence* $\{x_n\}$ *with* $x_n \to x^*$ *implies* $x^* \preceq x_n$ *for each* $n \in \mathbb{N}$. *Suppose that the non-increasing mapping* $G : X \to N(X)$ *be a ordered-close set-valued and has* AV *and*

$$F(H(Gx, Gy)) \le F(M(x, y)) - \tau,$$

where

$$M(x, y) = max\left\{d(x, y), D(x, Gx), D(y, Gy), \frac{1}{2}[D(x, Gy) + D(y, Gx)]\right\}.$$

If there exists $x_0 \in X$ *such that* $Gx_0 \sqsubseteq \{x_0\}$, *then* G *has a fixed point* $x^* \in X$.

3. Application

As an application of our results, we will consider the following Volterra type integral equation:

$$fx(t) = \int_0^t K(t,s,x(s))ds + g(t), \quad t \in I \tag{3}$$

for all $t \in I = [0,1]$.

Let $I = [0,1]$ be a given real interval, $C(I,\mathbb{R})$ the Banach space of all real continuous functions defined on I with the sup norm

$$||x||_\infty = \max_{t\in I} |x(t)|, \quad x \in C(I,\mathbb{R})$$

and $C(I \times I \times C(I,\mathbb{R}),\mathbb{R})$ the space of all continuous functions defined on $I \times I \times C(I,\mathbb{R})$. Alternatively, the Banach space $C(I,\mathbb{R})$ can be endowed with Bielecki norm

$$||x||_B = \sup_{t\in I}\{|x(t)|e^{-\tau t}\}, \quad x \in C(I,\mathbb{R}), \quad \tau > 0$$

and the induced metric $d_B(x,y) = ||x-y||_B$ for all $x,y \in C(I,\mathbb{R})$, see [2]. Define $f : C(I,\mathbb{R}) \to C(I,\mathbb{R})$, by the formula

$$fx(t) = \int_0^t K(t,s,x(s))ds + g(t), \quad g \in C(I,\mathbb{R}).$$

THEOREM 3.1. *Let $(C(I,\mathbb{R}), d_B, \preceq)$ be a sequentially complete metric space. Suppose $G : C(I,\mathbb{R}) \to N(C(I,\mathbb{R}))$ is a set valued operator such that $G(x) = \{fx(t)\}$ and has UCAV. $K \in C(I \times I \times \mathbb{R},\mathbb{R})$ be an operator satisfying the following conditions*

(i) *K is continuous;*
(ii) *$\int_0^t K(t,s,.)$, is increasing, for all $t,s \in I$;*
(iii) *there exists $\tau > 0$ such that for all $x,y \in C(I,\mathbb{R})$ and all $t,s \in I$ we have*

$$|K(t,s,fx(s)) - K(t,s,fy(s))| \le e^{-\tau}|x(s) - y(s)|.$$

Then, the Volterra-type integral equation 3 has a solution in $C(I,\mathbb{R})$.

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Depth of a pair of ideals on ZD-modules

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Abstract. Let R be a Noetherian ring, I and J be two ideals of R, and S be a Serre subcategory of the category of R-modules satisfying the condition C_I. We extend the notion of S-depth of I on a finitely generated R-module M, denoted by $S-\operatorname{depth}(I, M)$, to the class of ZD-modules. Next, as a generalization of $S - \operatorname{depth}(I, M)$ and $\operatorname{depth}(I, J, M)$, the $S -$ depth of (I, J) on a ZD-module M is defined as $S - \operatorname{depth}(I, J, M) = \inf\{S - \operatorname{depth}(\mathfrak{a}, M) : \mathfrak{a} \in \tilde{W}(I, J)\}$, and some properties of this concept are investigated. Also, the relations between $S - \operatorname{depth}(I, J, M)$ and $H^i_{I,J}(M)$ are studied.

Keywords: Depth, Local cohomology, Serre subcategory, ZD-Module

AMS Mathematics Subject Classification [2010]: 13C15, 13C60, 13D45

1. Introduction

Throughout this lecture, R is a commutative Noetherian ring with non-zero identity, I and J are to ideals of R, M is an R-module, and t is an integer.

An R-module M is called a ZD-module (zero-divisor module) if for any submodule N of M, the set of zero-divisors of M/N is a finite union of the associated prime ideals of M/N. According to [3, Example 2.2], the class of ZD-modules contains finitely generated, Laskerian, weakly Laskerian, linearly compact, Matlis reflexive and minimax modules. Also, it contains modules whose quotients have finite Goldie dimension, and modules with finite support, in particular, Artinian modules.

Let S be a Serre subcategory of the category of R-modules. As a generalization of the regular sequences, Aghapournahr and Melkersson [1] introduced the S-sequences as follows. An element a of R is called S-regular on M, if $0 :_M a \in S$. A sequence $a_1, \ldots, a_t$ is an S-sequence on M, if a_i is S-regular on $M/(a_1, \ldots, a_{i-1})M$ for $i = 1, \ldots, t$.

Let S satisfy the condition C_I, and M be finitely generated such that $M/IM \notin S$. They showed that all maximal S-sequences on M in I, have the same length equal to $\inf\{i : \operatorname{Ext}^i_R(R/I, M) \notin S\}$. This common length, denoted by $S - \operatorname{depth}(I, M)$, is called the $S -$ depth of I on M. We generalize this concept to the ZD-modules. Let S satisfy the condition C_I, M be a ZD-module, and let I contain a maximal S-sequence on M. It is shown, in Theorem 2.2, that all maximal S-sequences on M in I, have the same length, equal to $\inf\{i : \operatorname{Ext}^i_R(R/I, M) \notin S\}$. Also, it is proved that if $M/IM \notin S$, then I contains maximal S-sequences on M; see Proposition 2.4.

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The local cohomology theory has been a useful and significant tool in Commutative Algebra and Algebraic Geometry. As a generalization of the ordinary local cohomology modules, Takahashi, Yoshino, and Yoshizawa [6] defined the local cohomology modules with respect to a pair of ideals. To be more precise, let $\Gamma_{I,J}(M) = \{x \in M : \exists\, t \in \mathbb{N}, I^t x \subseteq Jx\}$. It is easy to see that $\Gamma_{I,J}(M)$ is a submodule of M, and $\Gamma_{I,J}(-)$ is a covariant, R-linear functor from the category of R-modules to itself. For integer i, the local cohomology functor $H^i_{I,J}(-)$ with respect to (I,J), is defined to be the i-th right derived functor of $\Gamma_{I,J}(-)$. Also $H^i_{I,J}(M)$ is called the i-th local cohomology module of M with respect to (I,J). If $J = 0$, then $H^i_{I,J}(-)$ coincides with the ordinary local cohomology functor $H^i_I(-)$.

Let $\tilde{\mathrm{W}}(I,J) = \{\mathfrak{a} \leq R : I^t \subseteq J + \mathfrak{a}$ for some positive integer $t\}$. One can see that $x \in \Gamma_{I,J}(M)$ if and only if $\mathrm{Ann}_R(x) \in \tilde{\mathrm{W}}(I,J)$. Let $\mathrm{W}(I,J) = \{\mathfrak{p} \in \mathrm{Spec}(R) : I^t \subseteq J + \mathfrak{p}$ for some positive integer $t\}$. It is shown in [6, Corollary 1.8] that $x \in \Gamma_{I,J}(M)$ if and only if $\mathrm{Supp}_R\, Rx \subseteq \mathrm{W}(I,J)$.

The concept of depth of a pair of ideals (I,J) on a finitely generated R-module M was introduced, in [2], as $\mathrm{depth}(I,J,M) = \inf\{\mathrm{depth}(\mathfrak{a},M) : \mathfrak{a} \in \tilde{\mathrm{W}}(I,J)\}$. Let S be a Serre subcategory of the category of R-modules satisfying the condition C_I and M be a ZD-module. We define the $S-\mathrm{depth}$ of a pair of ideals (I,J) on M as $S-\mathrm{depth}(I,J,M) = \inf\{S-\mathrm{depth}(\mathfrak{a},M) : \mathfrak{a} \in \tilde{\mathrm{W}}(I,J)\}$. It is easy to see that $S-\mathrm{depth}(I,J,M)$ is a generalization of $S-\mathrm{depth}(I,M)$ and $\mathrm{depth}(I,J,M)$. We also investigate some properties of $S-\mathrm{depth}(I,J,M)$.

Also, the relations between the local cohomology modules of M with respect to (I,J) and $S-\mathrm{depth}(I,J,M)$ are studied. Let S be a Serre subcategory closed under taking injective hulls, and M be a ZD-module. As one of the main results of this lecture, it is shown that $S-\mathrm{depth}(I,J,M) = \inf\{i : H^i_{I,J}(M) \notin S\}$; see Theorem 2.15.

2. Main results

Throughout this section, S denotes a Serre subcategory of the category of R-modules.

Lemma 2.1. *Let S satisfy the condition C_I and M be a ZD-module. Then the following conditions are equivalent:*

(i) *There is an S-sequence on M in I of length t.*
(ii) *$H^i_I(M) \in S$ for all $i < t$.*

Note that an S-sequence $a_1, \ldots, a_t$ (contained in ideal I) is maximal (in I), if $a_1, \ldots, a_t, b$ is not an S-sequence for any $b \in R$ ($b \in I$).

Theorem 2.2. *Let S satisfy the condition C_I and M be a ZD-module. Let I contain a maximal S-sequence on M. Then all maximal S-sequences on M in I have the same length, and this length is equal to $\inf\{i : \mathrm{Ext}^i_R(R/I,M) \notin S\}$.*

Definition 2.3. Let S satisfy the condition C_I, M be a ZD-module, and let I contain a maximal S-sequence on M. The common length of all maximal S-sequences on M in I is called the $S-\mathrm{depth}$ of I on M, denoted by $S-\mathrm{depth}(I,M)$.

Let S satisfy the condition C_I and M be a ZD-module. We complement the above definition by setting $S-\mathrm{depth}(I,M) = \infty$, whenever there is no maximal S-sequence on M in I. This is consistent with Theorem 2.2:

$$S-\mathrm{depth}(I,M) = \infty \quad \Leftrightarrow \quad \mathrm{Ext}^i_R(R/I,M) \in S \ \text{ for all } \ i.$$

The next result provides an important case that I contains a maximal S-sequence on M.

PROPOSITION 2.4. *If $M/IM \notin S$, then every S-sequence on M in I can be extended to a maximal one.*

EXAMPLE 2.5. [1, Example 2.16] Let M be a ZD-module. The following are some examples of $S-\operatorname{depth}(I,M)$.

(a) If S is the class of zero modules, then it is the same as ordinary $\operatorname{depth}(I,M)$.
(b) If S is the class of Artinian R-modules, then it is the same as $f-\operatorname{depth}(I,M)$ (filter-depth).
(c) If S is the class of R-modules with finite support, then it is the same as $g-\operatorname{depth}(I,M)$ (generalized depth).

COROLLARY 2.6. *Let S satisfy the condition C_I, and M be a ZD-module. Then*

$$\begin{aligned} S-\operatorname{depth}(I,M) &= \inf\{i : \operatorname{Ext}^i_R(R/I,M) \notin S\} \\ &= \inf\{i : H^i_I(M) \notin S\}. \end{aligned}$$

COROLLARY 2.7. *Let $\mathfrak{p}$ be a prime ideal of R, and let M be a ZD-module. Then* $\operatorname{depth} M_{\mathfrak{p}} = \inf\{i : \mu^i(\mathfrak{p},M) \neq 0\}$, *where $\mu^i(\mathfrak{p},M)$ denotes the i-th Bass number of M with respect to $\mathfrak{p}$.*

The notion of depth of a pair of ideals (I,J) on a finitely generated module M was introduced, in [2], as $\operatorname{depth}(I,J,M) = \inf\{\operatorname{depth}(\mathfrak{a},M) : \mathfrak{a} \in \tilde{\mathrm{W}}(I,J)\}$. Let S satisfy the condition C_I. We define the $S-$ depth of a pair of ideals (I,J) on a ZD-module M as $S-\operatorname{depth}(I,J,M) = \inf\{S-\operatorname{depth}(\mathfrak{a},M) : \mathfrak{a} \in \tilde{\mathrm{W}}(I,J)\}$. Let S satisfy the condition C_I, and M be a ZD-module. If S is the class of zero modules, then $S-\operatorname{depth}(I,J,M)$ coincides with $\operatorname{depth}(I,J,M)$. Also if $J=0$, then $S-\operatorname{depth}(I,J,M) = S-\operatorname{depth}(I,M)$.

PROPOSITION 2.8. *Let S satisfy the condition C_I. Let M be a ZD-module, and I' and J' be two ideals of R. Then*

(i) *If $I \subseteq I'$, then $S-\operatorname{depth}(I,J,M) \leq S-\operatorname{depth}(I',J,M)$.*
(ii) *If $J \subseteq J'$, then $S-\operatorname{depth}(I,J,M) \geq S-\operatorname{depth}(I,J',M)$.*
(iii) $S-\operatorname{depth}(I,J,M) = S-\operatorname{depth}(\sqrt{I},J,M) = S-\operatorname{depth}(I,\sqrt{J},M)$.
(iv) $S-\operatorname{depth}(II',J,M) = S-\operatorname{depth}(I\cap I',J,M)$.
(v) $S-\operatorname{depth}(I,JJ',M) = S-\operatorname{depth}(I,J\cap J',M)$.
(vi) *If $\mathfrak{a} \in \tilde{\mathrm{W}}(I,J)$, then*
$$S-\operatorname{depth}(I,J,M) \leq S-\operatorname{depth}(\mathfrak{a},J,M) \leq S-\operatorname{depth}(\mathfrak{a},M).$$
(vi) *If $\underline{a} = a_1,\ldots,a_t$ is an S-sequence on M in I, then*
$$S-\operatorname{depth}(\frac{I}{(\underline{a})},\frac{M}{(\underline{a})M}) = S-\operatorname{depth}(I,\frac{M}{(\underline{a})M}) = S-\operatorname{depth}(I,M)-t.$$

PROPOSITION 2.9. *Let S satisfy the condition C_I, and let $0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of ZD-modules. Then*

(i) $S-\operatorname{depth}(I,J,M) \geq \min\{S-\operatorname{depth}(I,J,U), S-\operatorname{depth}(I,J,N)\}$.
(ii) $S-\operatorname{depth}(I,J,U) \geq \min\{S-\operatorname{depth}(I,J,M), S-\operatorname{depth}(I,J,N)+1\}$.
(iii) $S-\operatorname{depth}(I,J,N) \geq \min\{S-\operatorname{depth}(I,J,U)-1, S-\operatorname{depth}(I,J,M)\}$.

Now, we study the relations between local cohomology modules of M with respect to (I,J) and $S-\operatorname{depth}(I,J,M)$. It is well-known that if a Serre subcategory is closed under taking injective hulls, then it satisfies the condition C_I; see [1, Lemma 2.2].

Proposition 2.10. *Let S be a Serre subcategory closed under taking injective hulls. Let $0 \to M \to E^0 \to E^1 \to \cdots$ be a minimal injective resolution of M, where $E^i \cong \bigoplus_{\mathfrak{p}\in \mathrm{Spec(R)}}(E_R(R/\mathfrak{p}))^{\mu^i(\mathfrak{p},M)}$ is a decomposition of E^i as the direct sum of indecomposable injective R-modules and $E_R(R/\mathfrak{p})$ denotes the injective hull of $R/\mathfrak{p}$. The following conditions are equivalent:*

(i) $H^i_{I,J}(M) \in S$ *for all* $i < t$.

(ii) $\Gamma_{I,J}(E^i) \in S$ *for all* $i < t$.

Proposition 2.11. *Let S be a Serre subcategory closed under taking injective hulls, and M be a ZD-module. Then*

$$\inf\{i : H^i_{I,J}(M) \notin S\} = \inf\{\operatorname{depth} M_{\mathfrak{p}} : \mathfrak{p} \in \mathrm{W}(I,J) \text{ and } R/\mathfrak{p} \notin S\}.$$

Corollary 2.12. *Let M be a ZD-module. Then*

$$\inf\{i : H^i_{I,J}(M) \text{ is not Artinian}\} = \inf\{i : \operatorname{Supp}_R H^i_{I,J}(M) \nsubseteq \operatorname{Max} R\}.$$

Corollary 2.13. *Let M be a ZD-module. If $\operatorname{Supp}_R \Gamma_{I,J}(E^i) \subseteq \operatorname{Max}(R)$ for all $i < t$, then $H^i_{I,J}(M)$ is Artinian for all $i < t$.*

Now, we get a formula on the relation between $S-\operatorname{depth}(I,J,M)$ and local cohomology modules of M with respect to (I,J).

Lemma 2.14. *Let S satisfy the condition C_I, and let M be a ZD-module. Then S-$\operatorname{depth}(I,J,M) \geq \inf\{i : H^i_{I,J}(M) \notin S\}$.*

Theorem 2.15. *Let S be a Serre subcategory closed under taking injective hulls, and M be a ZD-module. Then $S - \operatorname{depth}(I,J,M) = \inf\{i : H^i_{I,J}(M) \notin S\}$.*

Corollary 2.16. *Let S be a Serre subcategory closed under taking injective hull. Let M be a ZD-module, and J' be an ideal of R such that $J' \subseteq J$. Then $S - \operatorname{depth}(I + J', J, M) = S - \operatorname{depth}(I,J,M)$. In particular, $S - \operatorname{depth}(I+J,J,M) = S - \operatorname{depth}(I,J,M)$.*

Proposition 2.17. *Let S be a Serre subcategory closed under taking injective hulls, and M be a ZD-module. Then*

$$\begin{aligned} S - \operatorname{depth}(I,J,M) &= \inf\{\operatorname{depth} M_{\mathfrak{p}} : \mathfrak{p} \in \mathrm{W}(I,J) \text{ and } R/\mathfrak{p} \notin S\} \\ &= \inf\{S - \operatorname{depth}(\mathfrak{p},M) : \mathfrak{p} \in \mathrm{W}(I,J) \text{ and } R/\mathfrak{p} \notin S\}. \end{aligned}$$

Corollary 2.18. *Let S be a Serre subcategory closed under taking injective hulls. Let M be a ZD-module, and I' be an ideal of R. Then*

$$S - \operatorname{depth}(I \cap I', M) = \min\{S - \operatorname{depth}(I,M), S - \operatorname{depth}(I',M)\}.$$

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Cartan-Brauer-Hua type conditions over the division rings with involution

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Abstract. Cartan-Brauer-Hua Theorem is a well-known theorem which states that if R is a subdivision ring of a division ring D which is invariant under all elements of D or $DRD^{-1} \subseteq R$ for all $d \in D\backslash\{0\}$, then either $R = D$ or R is contained in the center of D. The invariance idea of this basic theorem is the main notion of this paper. We prove that if D is a division ring with involution * and M is a subspace of D which is invariant under all symmetric elements of D, then either M is contained in the center of D or is a Lie ideal of D.

Keywords: Division ring, Lie ideal, involution.

AMS Mathematics Subject Classification [2010]: 16K40, 16W10

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1. Introduction

Cartan-Brauer-Hua Theorem (briefly CBH Theorem) as follows, is a well known theorem due independently to Cartan, Brauer and Hua [5, p. 211].

Theorem 1.1. (CBH Theorem) *Let D be a division ring and R a subdivision ring of D. If $dRd^{-1} \subseteq R$ for all $d \in D \setminus \{0\}$, then either $R = D$ or R is contained in the center of D.*

Let D be a division ring with an *involution* *, that is a map of D into itself satisfying $a^{**} = a$, $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$, for every $a, b \in D$. We consider the field F to be the center of D or $Z(D) = F$. The characteristic of D is denoted by $char(D)$ and the group automorphisms of D is denoted by $Aut(D)$ and the group of F-automorphisms is denoted by $Aut_F(D)$. Let $S = S(D) = \{a \in D \mid a^* = a\}$ and $K = K(D) = \{a \in D \mid a^* = -a\}$ be the set of *symmetric* elements and *skew-symmetric* elements of D, respectively. The involution is called of the *first kind* if $F \subseteq S$ and otherwise is called of the *second kind* [3]. Clearly, D is a vector space over F and all subspaces are considered in this vector space. For a pair $a, b \in D$ we denote by $[a, b] = ab - ba$ the *Lie product* of a and b. An additive subgroup I of D is said to be a *Lie ideal* if $[d, i] \in I$ for every $d \in D$ and $i \in I$. Also, for subsets A, B of D we denote by $[A, B]$ the additive subgroup of D generated by all $[a, b]$, where $a \in A$ and $b \in B$. When T is a subset of a group G, then we denote by $\langle T \rangle$ the subgroup generated by T in G. We denote the cardinality of a set A by $|A|$.

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An element $a \in D$ is said to be *algebraic* over F if a satisfies a non-zero polynomial in $F[x]$. A subset $A \subseteq D$ is called algebraic over F if each of its elements is algebraic over F. By an algebraic division algebra we mean a division algebra algebraic over its center. Let A, B be two subsets of D, we say that A is *B-invariant* if $bAb^{-1} \subseteq A$, for all $b \in B \setminus \{0\}$. For two subsets $A, B \subseteq D$ we define the *normalizer* of A in B by $N_B(A) = \{b \in B \setminus \{0\} \mid bAb^{-1} \subseteq A\}$ and the *centralizer* of A in D by $C(A) = C_D(A) = \{d \in D \mid da = ad \ for \ all \ a \in A\}$. For a subset $A \subseteq D$ let $U(A) = A \setminus \{0\}$. We say a subset $A \subseteq D$ is *self-invariant* if $N_D(A) = A$. If $a \in D$, then $F(a)$ denotes the subfield of D generated by $F \cup \{a\}$.

There are many different generalizations of CBH Theorem, in the literature. For instance Faith [5, p. 211] proved that the theorem is valid, as well, if the multiplicative group index $[U(D) : N_D(R)]$ is finite. Herstein [4] proved that if D has an involution and $dim_F D > 4$, then one can reduce the D-invariance of R to the S-invariance or K-invariance. Schenkman and Scott [6] proved that if R is a subdivision ring of D such that $U(R)$ is a subnormal subgroup of $U(D)$, then either $R = D$ or R is in the center of D. Also we need to recall the following theorem of Asano [2], which has a similar idea.

Theorem 1.2. (AS Theorem) *Let D be a division algebra with center F which is algebraic over an (infinite) field K and let M be a non-central D-invariant K-subspace of D. Then $[D, D] \subseteq M$. In particular M is a Lie ideal of D.*

In this paper we consider some substructures of division rings with a more limited invariance conditions. For example we show that if T is a self-invariant subfield of D with a nontrivial automorphism, then T contains at least one non-central proper subfield of D. Also we are interested in knowing the other substructures of a division ring that may have a similar property as M has in the AS Theorem. When D has an involution we apply Herstein's method [4] to restrict the D-invariant of M to some special subsets. We prove that if the subspace M is S-invariant, then M is a Lie ideal. We also present a variation of this theorem when the subspace M is K-invariant.

2. Main results

In this section we consider subfields of a division ring with least D-invariance, in other words we study self-invariant subfields of a division ring. The Skolem-Noether Theorem shows that if T is a subfield in finite-dimensional central division algebra D, then $Aut_F(T) \neq 1$, if and only if $N_D(T) \neq T$. Naturally a question arises: what would happen if we remove the finite dimensionality of division ring in this theorem? Clearly, when $N_D(T) \neq T$, then always (finite or infinite-dimensional case) $Aut_F(T) \neq 1$. In the following theorem we study the other direction.

Theorem 2.1. [1] Let D be a non-commutative division ring with center F and T be a subfield of D with a nontrivial F-automorphism or $Aut_F(T) \neq 1$. If $N_D(T) = T$, then T contains at least one non-central proper subfield.

Self-invariant subfields with nontrivial automorphisms in division rings, also have the following properties.

Theorem 2.2. [1] Let D be a division ring with center F and let T be a self-invariant subfield of D with a nontrivial F-automorphism or $Aut_F(T) \neq 1$. Then we have the following:

(i) Any subalgebra K of D either is contained in T or $dim_T T(K) > \infty$, where by $T(K)$ we mean the division subring generated by $T \cup K$.

(ii) If D is algebraic, then all T-invariant finite-dimensional subalgebras of D over F are contained in T.

We continue with the following lemma that is an extension of a lemma due to Asano [2].

Lemma 2.3. [1] Let D be an algebraic division algebra with center F. If $N \subseteq D$ is an additive subgroup with $|F \cap N| > \infty$ and M is an N-invariant subspace of D, then M contains $[N, M]$.

We need the following technical lemma, as well.

Lemma 2.4. [1] Let D be an algebraic non-commutative division algebra with involution and $char(D) \neq 2$. Then

(i) $[[S, S], S] \subseteq S$.
(ii) $[[S, S], K] \subseteq [S, S]$.
(iii) $[[S, S], M] \subseteq M$ for every S-invariant subspace M of D.

Using above lemmas, in the following two theorems we give a generalization of AS Theorem, based on Herstein method [4] to restrict the D-invariance of a subspace M to S-invariance and K-invariance.

Theorem 2.5. [1] Let D be an algebraic division algebra with involution and $char(D) \neq 2$. If M is an S-invariant subspace of D and neither S nor M is contained in F, then M contains $[D, D]$. In particular, M is a Lie ideal of D.

Theorem 2.6. [1] Let D be an algebraic division algebra with involution such that $char(D) \neq 2$. If M is a subspace of D which is K-invariant and neither S nor M is contained in F and one of the following conditions holds, then M contains $[D, D]$. In particular, M is a Lie ideal of D.

(i) The involution is of the second kind.
(ii) $S \subseteq \langle \bigcup_{u \in K} C_S(u), K \rangle$.
(iii) For each $y \in S$, there exists $x \in K$, such that $xy = cyx$ for some $c \in F \setminus \{-1\}$.

3. Conclusion

In this paper we consider some substructures of division rings with a more limited invariance conditions. Also we give a generalization of AS Theorem 1.2 , based on Herstein method [4] to restrict the D-invariance of a subspace M to S-invariance and K-invariance.

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Find the extreme points of the Nonlinear Algebraic Equation based on the improved bisection method and the Monte Carlo method

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Abstract. Obtaining Extreme points of Nonlinear Algebraic Equation is widely used in various sciences in optimization. Derivatives are often used to find extreme points. When the function is complex or not derivable, meta-heuristic algorithms can be used to find the extreme points of Nonlinear Algebraic Equation. In this paper, we first modify the bisection method in line with the intended purpose, and then present a new method using the integration of the new bisection method and the Monte Carlo method to find the extreme points of the functions. The advantage of this method is its use in complex functions. It is indivisible.

Keywords: Extreme points, Nonlinear Algebraic Equation, Monte Carlo, Bisection method, R Software

AMS Mathematics Subject Classification [2010]: 11K45, 11H55, 13P25

1. Introduction

Many physical problems in basic sciences and engineering are modeled in the form of nonlinear equations. In this regard, solving them is a challenge in various sciences, so it is easier to use numerical methods. The study of solving nonlinear equations is an important part of applied mathematics. Because the most of real-world phenomena can be modeled according to a nonlinear equation or systems of nonlinear equations [1].

An efficient method for obtaining extreme points depends on the use of the derivative, but its use in complex equations is very costly. The lack of analytical answers for such complex and nonlinear equations has led to the development of numerical solution methods. Today, due to the development of scientific and computer programs, the use of numerical methods and simulations in solving complex problems is very practical and efficient [2, 5, 6].

In this paper, the bisection method is used to find the extreme points of the function. This method can be efficient in complex equations. The organization of this research is as follows: In Section 2, some preliminaries are described. In Section 3, the modified bisection method is first defined, and then the algorithm for using the new bisection method and

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the Monte Carlo method for finding the extreme points of a function are described. Then the research findings were presented and finally concluded.

2. Some Preliminaries

2.1. Bisection Method. The bisection method is one of the most important methods in numerical calculations to find the root of a continuous function, which we know has a different sign at two points. Repeating this method on the functions with the mentioned feature will take us to the root if they are not equal in the range of the interval. If f(x) is continuous on [a, b], and f(a), f(b) have opposite signs, then by the intermediate value theorem it follows that f(c) = 0 for some a < c < b. The bisection method simply checks the sign of f(x) at the midpoint x = (a+b)/2 of the interval at each iteration. If f(a), f(x) have opposite signs, then the interval is replaced by [a, x] and otherwise it is replaced by [x, b]. At each iteration, the length of the interval containing the root decreases by half. The method cannot fail, and the number of iterations needed to achieve a specified tolerance is known in advance. If the initial interval [a, b] contains more than one root, then bisection will find one of the roots. The rate of convergence of the bisection algorithm is linear [4].

2.2. Monte Carlo methods. Monte Carlo methods refer to a diverse collection of methods in statistical inference and numerical analysis where simulation is used. Many statistical problems can be approached through some form of Monte Carlo integration. For an interesting discussion of the history of the Monte Carlo method and scientific computing, see Eckhart [3]

3. Method

First we modify the bisection method to find the extreme points, then using the Monte Carlo method we divide the desired distance into smaller parts. In each section where the slope function changes, we use the modified Bisection method to find the extreme point.

3.1. Modified bisection method. To find the extreme points, we follow the bisection method, except that we find the places where the function changes the slope. For any extreme point in the distance [a, b]:

$$\exists x_i \ s.t: (f(x_i+\varepsilon)-f(x_i)*(f(x_{i+1}+\varepsilon)-f(x_{i+1})\leq 0$$

The bisection method simply checks the sign of f(x) at the midpoint x = (a+b)/2 of the interval at each iteration. If f(a), f(x) have opposite signs, then the interval is replaced by [a, x] and otherwise it is replaced by [x, b]. At each iteration, the length of the interval containing the extreme point decreases by half.

3.2. Proposed algorithm to find the extreme points of the Nonlinear Algebraic Equation. Step 1:
Generating random sample numbers from a uniform distribution in the interval.
Step 2:
We use the Modified Bisection method to find the extreme point at distance.
Step 3:
To find the next extreme point, we repeat the first to Second steps with . Until all the extreme points are found in the distance.

4. Result and Discussion

The advantage of the proposed method is finding extreme points in the all nonlinear functions. In other methods, including derivatives and metacognitive algorithms, there are limitations such as the non-extractability of the function or getting stuck in local optimal points, but the proposed method has the ability. The advantages of this method can almost cover the disadvantages of other methods. The proposed algorithm is implemented in R software and some examples are provided.

5. Conclusion

In this paper, we tried to find the extreme points of the Nonlinear Algebraic Equation without any restrictions. For this purpose, we first modified the bisection method for the purpose of the research and then by merging the Monte Carlo method and the modified Bisection method, we were able to obtain the extreme points of the functions.

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Characterization Of Some Quasisimple And Almost Simple Groups By Their Character Degrees

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ABSTRACT. Let G be a finite group and $cd(G)$ denote the set of complex irreducible character degrees of G. Let G be a sporadic quasisimle group or an almost simple group with socle $PSp_4(q)$ or $PSL_4(q)$. The main result of this paper is to characterize G by $cd(G)$.

Keywords: Character degrees, sporadic quasisimple groups, projective conformal symplectic groups, projective general linear groups, Huppert's conjecture.

AMS Mathematics Subject Classification [2010]: primary 20C15; secondary 20C33, 20G40.

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1. Introduction

Let G be a finite group and $Irr(G)$ denote the set of complex irreducible characters of G. The set of irreducible character degrees is indicated by:

$$cd(G) = \{\chi(1) | \chi \in Irr(G)\}.$$

It is known that G cannot completely characterize the structure of G. For example, two nonisomorphic famous groups, Q_8 and D_8, not only have the same character degrees, but also the same character tables. So the set of character degrees does not have enough power to characterize the structure of groups completely.

However $cd(G)$ has a strong influence on the properties of G and we can extract useful information from it. In this regard, some researches have been done on $cd(G)$ for characterizing G or at least finding some properties of it. In this way, group theorists found that although $cd(G)$ contains some information, but cannot recognize some properties of groups. For instance $cd(G)$ cannot recognize the solvability or nilpotency of G. For example Q_8 is nilpotent and S_3 is solvable, but the set of character degrees of both groups is $\{1, 2\}$.

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Although $cd(G)$ does not generally characterize the group structure up to isomorphism, in the late 1990, Huppert discussed the possibility of characterizing finite non-Abelian simple groups:

Huppert's Conjecture: Let S be a finite non-Abelian simple group and G be a finite group. Then $cd(G) = cd(S)$ if and only if $G \cong S \times A$ for an Abelian group A.

Huppert proved his conjecture for some non-Abelian simple groups in [1], [2] and [3]. After him, some other group theorists, tried to prove this conjecture for various non-Abelian simple groups. For example:

- Huppert verified the conjecture for $PSL_4(2) \cong \mathbb{A}_8$.
- Nguyen and his colleagues proved the conjecture for $PSL_4(q)$ when $q \geqslant 13$.
- Huppert verified the conjecture for $PSp_4(q)$ when $q = 3, 4, 5$ or 7.
- Wakefield proved the conjecture for $PSp_4(q)$ when $q > 7$.
- Also this conjecture has been proved for the all sporadic simle groups in papers by various athors.

However this conjecture is still open. Inpired by the Huppert's conjecture, there is growing interest in investigating the similar results for the class of quasisimple and almost simple groups.

A quasisimple group is a group that is a perfect central extension H of a non-Abelian simple group H_0.

A group H is said to be an almost simple group with socle H_0 if $H_0 \leqslant H \leqslant Aut(H_0)$, where H_0 is a non-Abelian simple group.

Nguyen and his colleagues presented the following conjecture in order to generalize the Huppert's conjecture to quasisimple groups:

Conjecture 1.2 of [7]: Let G be a finite group and H a finite quasisimple group with $Mult(H/Z(H))$ cyclic. Then $cd(G) = cd(H)$ if and only if $G \cong HoA$, a central product of H and an Abelian group A.

The authors proved their conjecture for $SL_2(q)$ and $SL_3(q)$ when $q \geqslant 5$. Also They pointed out that the condition related to $Mult(H/Z(H))$ is essential. For example, let $G \cong 2^2{\cdot}\Omega_8^+(2)$ and $H = 2{\cdot}\Omega_8^+(2)$, Then $cd(G) = cd(H)$, but G is not a central product of H with an Abelian subgroup.

Authors in [9] expressed the following conjecture for characterizing almost simple groups of Lie type by the set of irreducible character degrees. Also they confirmed it for projective general linear and unitary groups of dimension 3. Before that, they showed by giving an example that in contrast to Huppert's conjecture, the converse of their conjecture does not necessarily hold. In other words, there exists some finite groups G whose quotient with an Abelian subgroup is isomorphic to an almost simple group H, but $cd(G) \neq cd(H)$.

Conjecture 1.1 of [9]: Let G be a finite group, and let H be an almost simple group of Lie type whith $cd(G) = cd(H)$. Then $G/A \cong H$ for an Abelian normal subgroup of G.

Clearly any simple group is also an quasisimple and almost simple group. Also central products and group extensions are generalizations of direct products, so the last two conjectures are actually generalizations for Huppert's conjecture.

2. Main results

The main result of this paper is to prove the last two conjectures for sporadic quasisimple groups, projective conformal symplectic groups $PCSp_4(q)$ and projective general linear groups $PGL_4(q)$:

THEOREM 2.1. *(Theorem 1.3 of [6]) Let G be a finite group and H be a sporadic quasisimple group except $2 \cdot M_{12}$. Then $cd(G) = cd(H)$ if and only if $G \cong H \circ A$, a central product of H and an Abelian group A.*

THEOREM 2.2. *(Theorem 1.3 of [5] and Theorem 1.3 of [4]) Let G be a finite group and H be a projective conformal symplectic group which is extended from $PSp_4(q)$ by it's diagonal outomorphism, or a projective general linear group, which is extended from $PSL_4(q)$ by it's diagonal outomorphism. If $cd(G) = cd(H)$, then G/A is isomorphic to H, where A is an Abelian normal subgroup of G.*

REMARK 2.3. Recall that Theorem 2.1, for the case where $H = H_0$ is a sporadic simple group, has already been settled. Therefore, we only need to consider the remaining cases where H is a sporadic quasisimple group with $Z(H) \neq 1$.

REMARK 2.4. If q is even, then $PSp_4(q)$ does not have any diagonal outomorphism, so $PCSp_4(q) \cong PSp_4(q)$. But Hupper't conjecture already has been proved for $PSp_4(q)$ by Huppert for $3 \leqslant q \leqslant 7$ in [3] and by Wakefield for $q > 7$ in [10]. Therefore, we only need to consider the cases which q is odd and d is the diagonal outomorphism of $PSp_4(q)$ of order 2.

REMARK 2.5. If q is even, then $PSL_4(q)$ does not have any diagonal outomorphism, so $PGL_4(q) \cong PSL_4(q)$. But Hupper't conjecture already has been proved for $PSL_4(2)$ by himself in [3] and for $PSL_4(q)$ when $q \geqslant 13$ by Nguyen and his colleagues in [8]. Therefore, we only need to consider the cases which q is odd and d is the diagonal outomorphism of $PSL_4(q)$ of order $(q-1, 4)$. Furthermore, we have fixed all the bugs in [8] for $3 \leqslant q \leqslant 11$, so the proof of Huppert's conjecture for $PSL_4(q)$ over these fields, can be another result of this paper.

3. Conclusion

Sporadic quasisimple groups and almost simple groups with socle $PSp_4(q)$ or $PSL_4(q)$, can be characterized by their set of irreducible character degrees.

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A local Multiquadric quasi-interpolation operator based on the piecewise linear hat functions

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Abstract. In this paper, we introduce a local quasi-interpolation operator by generalization of the hat functions to Multiquadrics. It possesses linear reproducing property and preserves positivity and monotonicity. It is an improvement of the piecewise linear interpolation in the sense that it is in C^∞. The local property of the proposed operator offers an advantage in terms of computational complexity rather than the global Multiquadric (MQ) quasi-interpolation formula. We also prove that the proposed scheme converges with a rate of $O(h^2)$. Numerical results give an accurate reconstruction of the original function in the well-known Runge phenomenon.

Keywords: Quasi-interpolation, Multiquardic RBF, Piecewise linear hat functions

AMS Mathematics Subject Classification [2010]: 65D05, 65D15, 65D20

1. Introduction

Given a set of n distinct (scattered) points $\{x_j\}_{j=0}^n \in \Omega \subseteq \mathbb{R}^d$ and corresponding data values $\{f_j\}_{j=0}^n \in \mathbb{R}$, a standard way to interpolate a function $f : \Omega \to \mathbb{R}$ is by using

$$\mathcal{L}f(x) = \sum_{j=0}^{n} \lambda_j \mathcal{X}(x - x_j), \tag{1}$$

with the coefficients λ_j determined by the interpolation conditions $\mathcal{L}f(x_j) = f_j$, $j = 0, \dots, n$, where $\mathcal{X}(\cdot)$ is an interpolation kernel [7]. Many authors use MQ radial basis function (RBF) $\phi(r) = \sqrt{r^2 + c^2}$ to solve the interpolating problem (1), that is $\mathcal{X}(x-x_j) = \phi(\|x - x_j\|)$. Then, the coefficients λ_j are determined by solving a symmetric linear system $A\lambda = f$, where $A = [\phi(\|x_i - x_j\|)]_{0\le i,j\le n}$. Although the MQ interpolation is always solvable, the resulting matrix quickly becomes ill-conditioned as the number of points increases. So researchers concentrated on a weaker form of (1), known as *quasi-interpolation*, that holds only for polynomials of some low degree m, i.e.,

$$\mathcal{L}p_m(x_j) = p_m(x_j), \quad \forall p_m \in \Pi_m^d,$$

for all $0 \le j \le n$, where Π_m^d denotes the space of polynomials of degree less and equal to m in $\mathbb{R}^d$. Global MQ quasi-interpolation is constructed by a linear combination of the

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MQ RBF and low degree polynomials. Since $\sqrt{x^2+c^2}$ tends to $|x|$ as c tends to zero, and RBF interpolation based on $|x|$ is piecewise linear interpolation, the shape-preserving properties of piecewise linear interpolation can be expected to hold for quasi-interpolation with Multiquadrics, too. Beatson and Powell [1,5] proposed three univariate global MQ quasi-interpolations $\mathcal{L}_A$, $\mathcal{L}_B$, and $\mathcal{L}_C$. Later, Wu and Schaback [8] proposed the univariate global MQ quasi-interpolation $\mathcal{L}_D$ and proved that the scheme is shape preserving and the approximation order is two at most. Moreover, MQ quasi-interpolation operator has been successfully applied in a wide range of fields. For example, Wang and Wu [6] applied the operator to tackle approximate implicitization of parametric curves. Hon and Wu [3], Chen and Wu [2], Jiang and Wang [4], and other researches provided some successful examples of using it to solve different types of partial differential equations. In this paper, we devise a local MQ quasi-interpolation operator based on the piecewise linear hat functions. The main advantage of the method is that it does not require the solution of any linear system. Instead, the formula uses the function values f_j at x_j as its coefficients. The local property of the proposed operator offers an advantage in terms of computational complexity rather than the global MQ quasi-interpolation formula.

2. Construction

Definition 2.1. Local quasi-interpolation of a function $f:[a,b]\longrightarrow\mathbb{R}$ with Multiquadrics on the scattered points

$$a=x_0<x_1<\cdots<x_n=b,\quad h:=\max_{1\le j\le n}(x_j-x_{j-1}) \tag{2}$$

has the form

$$(\mathcal{S}f)(x)=\sum_{j=0}^{n} f(x_j)\psi_j(x), \tag{3}$$

where

$$\psi_0(x)=\begin{cases}\dfrac{\phi_1(x)}{|x_0-x_1|}, & x_0\le x\le x_1,\\ 0, & \text{o.w}\end{cases}$$

$$\psi_j(x)=\begin{cases}\dfrac{\phi_{j-1}(x)}{|x_j-x_{j-1}|}, & x_{j-1}\le x\le x_j,\\ \dfrac{\phi_{j+1}(x)}{|x_j-x_{j+1}|}, & x_j\le x\le x_{j+1},\\ 0, & \text{o.w}\end{cases} \qquad j=1,\cdots,n-1$$

$$\psi_n(x)=\begin{cases}\dfrac{\phi_{n-1}(x)}{|x_n-x_{n-1}|} & x_{n-1}\le x\le x_n,\\ 0 & \text{otherwise}\end{cases}$$

$$\phi_j(x)=\sqrt{c^2+(x-x_j)^2},\quad j=0,\ldots,n,\quad c\in\mathbb{R}^+.$$

Note that for $c=0$ the expression (3) is a piecewise linear function and the basis functions are cardinal. For a general quasi-interpolation operator $\mathcal{S}$ we can state the following definitions.

Definition 2.2. The quasi-interpolation operator $\mathcal{S}$ constructed at the data points (x_j,f_j), is called to be positivity-preserving if $f_j>0$ implies that $(\mathcal{S}f)>0$.

Definition 2.3. The quasi-interpolation operator $\mathcal{S}$ constructed at the data points $\{(x_j, f_j)\}$, is called to be monotonicity-preserving, if the first order divided difference $f[x_j, x_{j+1}]$ is nonnegative (non-positive) implies that $(\mathcal{S}f)'$ is also nonnegative (non-positive).

Definition 2.4. We say that the quasi-interpolation $(\mathcal{S}f)(x)$ possesses linear reproducing property on $[x_0, x_n]$, if $(\mathcal{S}f)(x) = px + q$ as $f_j = px_j + q$, $j = 0, \dots, n$, for all $p, q \in \mathbb{R}$.

Theorem 2.5. *The local quasi-interpolation operator $\mathcal{S}$ constructed by data points $\{(x_j, f_j)\}$ is positivity preserving.*

Proof. Let $x \in [x_j, x_{j+1}]$ for some j, and $f_j,\ f_{j+1} > 0$. Then $(\mathcal{S}f)(x) = f_j\psi_j(x) + f_{j+1}\psi_{j+1}(x) > 0$, due to positivity of the MQ RBF and absolute value function. □

Theorem 2.6. *The local quasi-interpolation operator $\mathcal{S}$ constructed by data points $\{(x_j, f_j)\}$ is monotonicity preserving for c small enough.*

Proof. Let $x \in [x_j, x_{j+1}]$ for some j, and $f[x_j, x_{j+1}] = \frac{f_{j+1}-f_j}{x_{j+1}-x_j} \geq 0$. Then

$$\begin{aligned}
(\mathcal{S}f)'(x) \quad &= f_j\psi_j'(x) + f_{j+1}\psi_{j+1}'(x) \\
&= f_j\frac{\phi_{j+1}'(x)}{|x_j - x_{j+1}|} + f_{j+1}\frac{\phi_j'(x)}{|x_{j+1} - x_j|} \\
&= \frac{f_j(x - x_{j+1})}{|x_j - x_{j+1}|(c^2 + (x - x_{j+1})^2)^{\frac{1}{2}}} + \frac{f_{j+1}(x - x_j)}{|x_{j+1} - x_j|(c^2 + (x - x_j)^2)^{\frac{1}{2}}} \\
&= \frac{f_{j+1} - f_j}{x_{j+1} - x_j} \geq 0,
\end{aligned}$$

for c small enough. □

Theorem 2.7. *The local quasi-interpolation operator $\mathcal{S}$ has the linear reproducing property for c small enough.*

Proof. Let $x \in [x_j, x_{j+1}]$ for some j, and $f_j = px_j + q$, $j = 0, \dots, n$, for some $p, q \in \mathbb{R}$. Then

$$(\mathcal{S}f)(x) = (px_j + q)\frac{\sqrt{(x - x_{j+1})^2 + c^2}}{|x_j - x_{j+1}|} + (px_{j+1} + q)\frac{\sqrt{(x - x_j)^2 + c^2}}{|x_{j+1} - x_j|} = px + q,$$

for c small enough. □

3. Accuracy of the local quasi-interpolation $\mathcal{S}$

Theorem 3.1. *For $f \in C[a, b]$, the quasi-interpolation operator $\mathcal{S}f$, at the point set (2) as $h \to 0$, converges as follows*

$$\|f - \mathcal{S}f\|_\infty \leq kh^2,$$

where k is independent of h and c, provided that $c = O(h^3)$.

Proof. Let $\mathcal{L}f$ be the piecewise linear interpolant of f. Then

$$\|\mathcal{S}f - f\|_\infty \leq \|\mathcal{S}f - \mathcal{L}f\|_\infty + \|\mathcal{L}f - f\|_\infty.$$

Let $x \in [x_j, x_{j+1}]$ for some j. Then we have

$$
\begin{aligned}
(\mathcal{S}f - \mathcal{L}f)(x) &= f_j \frac{\phi_{j+1}(x)}{x_{j+1}-x_j} + f_{j+1}\frac{\phi_j(x)}{x_{j+1}-x_j} - f_j \frac{x_{j+1}-x}{x_{j+1}-x_j} - f_{j+1}\frac{x-x_j}{x_{j+1}-x_j} \\
&= \frac{1}{(x_{j+1}-x_j)} \left[f_j \left(\phi_{j+1}(x) - |x - x_{j+1}| \right) + f_{j+1} \left(\phi_j(x) - |x - x_j| \right) \right] \\
&\leq k\frac{c}{h},
\end{aligned}
$$

due to the boundedness of f and the inequality

$$
\sqrt{c^2 + y^2} - |y| \leq c, \quad c \geq 0, \ y \geq 0.
$$

So we get

$$
\|\mathcal{S}f - f\|_\infty \leq kh^2,
$$

because of the $O(h^2)$ convergence of $\mathcal{L}$ and the fact that $c = O(h^3)$. □

4. Numerical results

In this section, we use the new local quasi-interpolation operator for interpolating the Runge function. We take equidistant center points and choose different shape parameters c and also different step sizes h. We choose $m = 200$ equidistant evaluation points.

Test problem. (Runge function). Let us consider the Runge function on $[-1, 1]$, that is $f(x) = \dfrac{1}{1+25x^2}$. Figure 1 shows the exact and approximate values of f for $c = 0.01$, $h = 0.1,\ 0.02$. In Figure 1, we see that the Runge phenomenon has disappeared by decreasing h. Relative errors of the proposed method are shown in Figure 2 for $h = 0.02$, $c = 0.01,\ 0.001$. Figure 2 shows a steady decrease in the error by decreasing shape parameter c. This supports the theoretical claims. In Figure 3 the relative errors are plotted by the classical global MQ RBF interpolation method [7] for $h = 0.02$, $c = 0.01,\ 0.001$. It can be noted from Figures 2-3 that the proposed method leads to more accurate results by decreasing c.

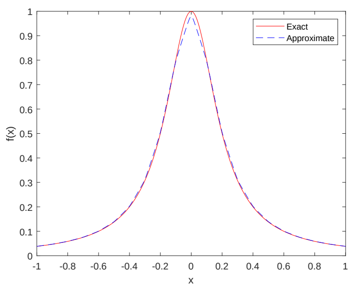
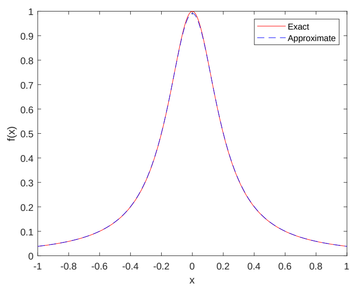
(a)  (b) 

FIGURE 1. Local quasi-interpolation of $f(x) = \frac{1}{1+25x^2}$; $h = 0.1$ (a), $h = 0.02$ (b), and $c = 0.01$.

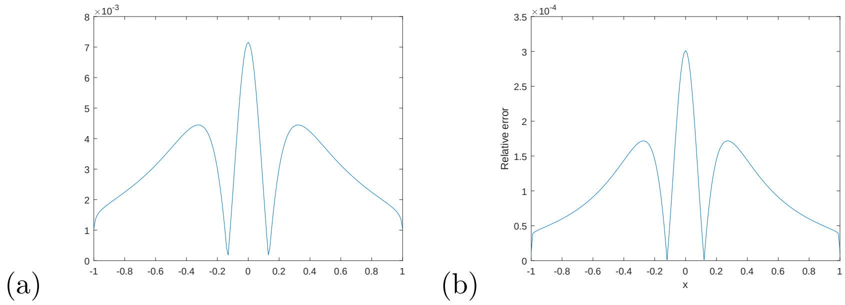


FIGURE 2. Relative error of the proposed method: $c = 0.01$ (a), $c = 0.001$ (b), and $h = 0.02$.

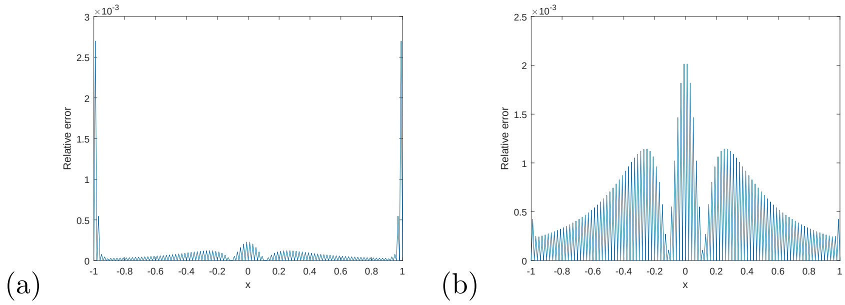


FIGURE 3. Relative error of the global MQ RBF interpolation method: $c = 0.01$ (a), $c = 0.001$ (b), and $h = 0.02$.

5. Conclusion

In this paper, we develop a local MQ quasi-interpolation which has the properties of linear reproducing and positivity and monotonicity preserving. It is based on the reformulation of the linear spline basis by Multiquadrics. Numerical experiments reveal that the proposed operator gives accurate results and it does not suffer from the Runge phenomenon. As a future work we are working on a local Multiquadric quasi-interpolation operator based on the quadratic spline basis which preserves convexity, too.

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Collocation method for linear and nonlinear Volterra integro-differential equations

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Abstract. A numerical method based on quintic B-spline is developed to solve the linear and nonlinear Volterra integro-differential equations up to order 4. The solution and its derivatives are collocated by quintic B-spline and then the integral equation is approximated by Gauss-Kronrod-Legendre quadrature formula of degree 2. The error analysis of proposed numerical method is studied theoretically. Numerical results are given to illustrate the efficiency of the proposed method which shows that our method can be applied for large values of N. The results are compared with those obtained by other methods which show that our method is accurate.

Keywords: Linear and nonlinear Volterra integro-differential equations, Quintic B-spline, Gauss-Kronrod-Legendre quadrature formula

AMS Mathematics Subject Classification [2010]: 34K05, 34K30, 41A15

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1. Introduction

In this paper, we will develop an approximation method based on B-spline to obtain numerical solution of the following integro-differential equation

$$\sum_{j=0}^{m} P_j(t)y^{(j)}(t) = f(t) + \int_a^t k(t,x,y(x))dx, \ t \in [a,T], \ T \in (a,b], \ 1 \le m \le 4, \tag{1}$$

with the boundary conditions,

$$\sum_{j=0}^{m-1} [\alpha_{i,j}y^{(j)}(a) + \beta_{i,j}y^{(j)}(b)] = \gamma_i, \quad 0 \le i \le m-1, \tag{2}$$

where $\alpha_{i,j}$, $\beta_{i,j}$ and γ_i are given real constants. The given kernel k is continuous on $[a,b]$ and satisfies a uniform Lipschitz, $f(t)$ and $P_j(t)$ are the known functions and y is unknown function. To solve the integro-differential equations, several numerical approaches have been proposed such as [1, 2]. In this paper we will use quintic B-spline collocation to approximate the unknown functions of up to 4th order and Gauss-Kronrod-Legendre quadrature formula to approximate the integral equation in the boundary value problems of linear and nonlinear Volterra integro-differential equations of second kind.

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2. Quintic B-spline collocation method

The construction of the quintic B-spline interpolate s to the analytical solution y for (1)-(2) can be performed with the help of the ten additional knots such that

$$t_{-5} < t_{-4} < t_{-3} < t_{-2} < t_{-1} \text{ and } t_{N+1} < t_{N+2} < t_{N+3} < t_{N+4} < t_{N+5}.$$

Following [3] we consider a quintic B-spline $s(t)$ of the form

$$s(t) = \sum_{i=-2}^{N+2} c_i B_i^5(t), \tag{3}$$

where

$$\begin{cases} B_i^0(t) = \begin{cases} 1, & if\ t_i \le t \le t_{i+1}, \\ 0, & \text{otherewise} \end{cases} \\ B_i^k(t) = (\frac{t-t_i}{t_{i+k}-t_i})B_i^{k-1}(t) + (\frac{t_{i+k+1}-t}{t_{i+k+1}-t_{i+1}})B_{i+1}^{k-1}(t),\ k \ge 1, \end{cases}$$

satisfying the following interpolatory conditions

$$s(t_i) = y(t_i),\ 0 \le i \le N,$$

and the end conditions

$$D^j s(t_0) = D^j s(t_N),\ j = 1, 2, 3, 4. \tag{4}$$

3. On quadrature formulas of the Gauss-Kronrod-Legendre

The Gauss-Kronrod quadrature formula is

$$\int_{\mathbb{R}} f(x)d\mu(x) = \sum_{k=1}^{2n+1} \overline{\delta}_k f(\overline{\tau}_k) = \sum_{\nu=1}^{n} \delta_\nu f(\tau_\nu) + \sum_{\rho=1}^{n+1} \delta_k^* f(\tau_\rho^*) + R_{2n+1}(f),$$

where τ_ν are the Gaussian nodes, i.e., the zeros of the orthogonal polynomial P_n with respect to μ and the nodes τ_ρ^* and weights $\delta_\nu, \delta_\rho^*$ are chosen so that they maximize the polynomial degree of exactness of the quadrature formula. It turns out that

$$E_{n+1}R_n(f) = 0 \text{ for all } f \in P_{3n+1},$$

(see [4]). Here we use the Legendre polynomial of degree 2 and $w(x) = 1$, i.e. $d\mu(x) = dx, [-1, 1]$ and we can obtain the weights δ_ν and δ_ρ^* by solving the system, $\int_{-1}^{1} x^j dx = \sum_{\nu=1}^{2} \delta_\nu x^j(\tau_\nu) + \sum_{\rho=1}^{3} \delta_k^* x^j(\tau_\rho^*)$ $j = 0, \dots, 7$. Finally we obtain

$$\begin{cases} \tau_1 = \frac{\sqrt{3}}{3},\ \tau_2 = \frac{\sqrt{3}}{3},\ \tau_1^* = -\sqrt{\frac{6}{7}},\ \tau_2^* = 0,\ \tau_3^* = \sqrt{\frac{6}{7}}, \\ \delta_1 = \frac{27}{55},\ \delta_2 = \frac{27}{55},\ \delta_1^* = \frac{98}{495},\ \delta_2^* = \frac{28}{45},\ \delta_3^* = \frac{98}{495}. \end{cases} \tag{5}$$

4. Nonlinear Volterra integro-differential equations

In the given nonlinear Volterra integro-differential equations (1) and (2) we can replaced the unknown function and its derivatives by quintic B-spline (4), then we collocate (1) at collocation points $t_i = a + ih, h = \frac{T-a}{N}, i = 0, 1, \dots, N$, and by partitioning the interval [a, T] to N equal subintervals we obtain:

$$\begin{cases} \sum_{j=0}^{m} P_j(t_i)s^{(j)}(t_i) = f(t_i) + \sum_{p=0}^{i-1} \int_{t_p}^{t_{p+1}} k(t_i, x, s(x))dx, \\ i = 1, \dots, N,\ 1 \le m \le 4, \\ \sum_{j=0}^{m-1} [\alpha_{i,j}s^{(j)}(a) + \beta_{i,j}s^{(j)}(b)] = \gamma_i,\ 0 \le i \le m-1. \end{cases} \tag{6}$$

For using the Gauss-Kronrod-Legendre formula we need to change each subinterval $[t_p, t_{p+1}]$ to the interval $[-1, 1]$. Then by the following change of variable, we have $x = \frac{1}{2}[(t_{p+1} - t_p)y + (t_{p+1} - t_p)]$, $dx = \frac{t_{p+1}-t_p}{2}dy = \frac{h}{2}dy$. To approximate the integral (6) , we can use the Gauss-Kronrod-Legendre quadrature formula in the case $n = 2$, then we get the following $(N + m) \times (N + 5)$, nonlinear system

$$(7) \qquad \begin{cases} \sum_{j=0}^{m} P_j(t_i)s^{(j)}(t_i) = f(t_i) + \frac{h}{2}\sum_{p=0}^{i-1}\sum_{\nu=1}^{5} \bar{\delta}_\nu k(t_i, \zeta_{p\nu}, s(\zeta_{p\nu})), \\ 1 \le m \le 4,\ i = 1, \dots, N \\ \sum_{j=0}^{m-1}[a_{i,j}s^{(j)}(a) + b_{i,j}s^{(j)}(b)] = d_i,\ 0 \le i \le m-1, \end{cases}$$

where $\zeta_{p\nu} = \frac{(t_{p+1}-t_p)\bar{\tau}_\nu + (t_{p+1}+t_p)}{2}$, and we have the nodes $\bar{\tau}_\nu$ and coefficients $\bar{\delta}_\nu$ of previous section. We need $5 - m$ more equations to obtained the unique solution for equation (7). We impose the end conditions (4). Hence by associating equation (7) with (4) we have the following nonlinear system $(N + 5) \times (N + 5)$,

$$(8) \qquad \begin{cases} \sum_{j=0}^{m} P_j(t_i)s^{(j)}(t_i) = f(t_i) + \frac{h}{2}\sum_{p=0}^{i-1}\sum_{\nu=1}^{5} \bar{\delta}_\nu k(t_i, \zeta_{p\nu}, s(\zeta_{p\nu})), \\ 1 \le m \le 4,\ i = 1, \dots, N \\ \sum_{j=0}^{m-1}[a_{i,j}s^{(j)}(a) + b_{i,j}s^{(j)}(b)] = d_i,\ 0 \le i \le m-1, \\ s^{(j+1)}(t_0) = s^{(j+1)}(t_N),\ 0 \le j \le 4-m, \end{cases}$$

by solving the above nonlinear system via iterative method we determine the coefficients $c_i, i = -2, \dots, N+2$. By substituting c_i in (3), we obtain the approximate solution for (1).

5. Numerical examples

In order to test the viability of the presented method, we consider two linear and nonlinear Volterra integro-differential equations with the end conditions.

Example 5.1. Consider the following linear Volterra integro-differential equation with the exact solution $y(t) = e^t$.

$$y''(t) + ty'(t) - ty(t) = e^t + \frac{1}{2}t\cos t - \frac{1}{2}\int_0^t \cos te^{-x}y(x)dx,\ y(0) = 1,\ y'(0) = 1,\ 0 \le x, t \le 1.$$

Example 5.2. Consider the following nonlinear Volterra integro-differential equation with exact solution $y(t) = e^t$,

$$y^{(4)}(t) = 1 + \int_0^t e^{-x}y^2(x)dx,\ y(0) = 1,\ y(1) = e,\ y''(0) = 1,\ y''(1) = e,\ t \in [0, 1].$$

We apply the system of (8) to solve examples with different values of N, and also we compare our obtained results with the results in [5, 6] that the maximum absolute errors are tabulated in tables 1 and 2.

6. Conclusions

This paper presents method to compute the solution of linear and nonlinear Volterra integro differential equations by using Gauss-Kronrod-Legendre quadrature formula and collocating by quintic B-spline. These equations are converted to a system of linear or nonlinear algebraic equations in terms of the linear combination coefficients appearing in the representation of the solution in spline basic functions.

TABLE 1. The maximum absolute errors $\|E\|_\infty$ in solution of Example 5.1.

x	Our Method	Method in [5]
0	1.33447(-14)	0
0.2	2.74185(-09)	1.7613(-04)
0.4	9.67209(-09)	8.0643(-04)
0.6	9.51870(-09)	5.1126(-04)
0.8	5.07328(-07)	4.4919(-03)
1	8.13245(-06)	2.0920(-02)

TABLE 2. The errors $\|E\|$ in solution of Example 5.2 at particular points for $N = 10$.

x	Our Method	Method in [6]
0	1.34337(-14)	0
0.2	1.07163(-05)	1.11470(-03)
0.4	1.74573(-05)	1.88476(-03)
0.6	1.77865(-05)	2.00882(-03)
0.8	1.12839(-05)	1.32855(-03)
1	0	0

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Adjoint of certain weighted composition operators on certain weighted Hardy Spaces

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Abstract. Let υ be an analytic function on the unit disk $\mathbb{D}$, and ϕ be a holomorphic self-map of $\mathbb{D}$. The weighted composition operator with symbols ϕ and υ is defined by $C_{\upsilon,\phi} = \upsilon f o \phi$. In this paper, we characterize the adjoint of certain weighted composition operators on certain weighted Hardy spaces.

Keywords: Dirichlet space, weighed composition operator, adjoint, Bergman space, weighted Hardy space

AMS Mathematics Subject Classification [2010]: 47B33, 47B38

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane. A weighted Hardy space $H^2(\beta)$ is a Hilbert space whose elements are functions analytic on $\mathbb{D}$ such that the set $\{z^n | n \geq 0\}$ constitute a complete orthonormal set in $H^2(\beta)$. In this notation the weight $\beta = \{\beta(j)\}$, where $\beta(j) = \|z^j\|$ for $j = 0, 1, 2, \cdots$ and $\beta(0) = 1$. If $f(z) = \Sigma_0^\infty a_n z^n$ belongs to $H^2(\beta)$, then $\|f\|^2 = \Sigma_0^\infty |a_n|^2 |\beta(n)|^2$ and the inner product is given by $\langle \Sigma_0^\infty a_n z^n, \Sigma_0^\infty b_n z^n \rangle = \Sigma_0^\infty a_n \overline{b_n} |\beta(n)|^2$. It is well known that the evaluation of functions at $w \in \mathbb{D}$ is a bounded linear functional and if k_w denotes the unique vector in $H^2(\beta)$ such that $f(w) = \langle f, k_w \rangle$, then $k_w(z) = \Sigma_0^\infty \frac{\overline{w}^n z^n}{\beta(n)^2} = k(\overline{w} z)$, where $k(z) = \sum_0^\infty \frac{z^n}{\beta_n^2}$ and k is analytic on $\mathbb{D}$. Let ϕ and υ be functions analytic on the unit disk and $\phi(\mathbb{D}) \subset \mathbb{D}$. The composition operator with symbol ϕ defined on $H^2(\beta)$ by $C_\phi f = f \circ \phi$ and the weighted composition operator by symbols ϕ, υ is defined by $C_{\upsilon,\varphi} f = \upsilon f o \varphi$ for each $f \in H^2(\beta)$. For more information about the weighted Hardy spaces see for instance [4] and the references therein. Some classical examples of such spaces run as follows:

1-The *Dirichlet space* $\mathcal{D}$ is the space of all analytic functions $f : \mathbb{D} \to C$ such that

$$\|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty,$$

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*Speaker. Email address: math.mehrangiz@gmail.com.com where A denotes the area measure on $\mathbb{D}$, normalized to have the total mass 1. If $f(z) = \sum_{n=0}^{\infty} c_n z^n$, then

$$\|f\|_{\mathcal{D}}^2 = |c_0|^2 + \sum_{n=1}^{\infty} n|c_n|^2.$$

It is easy to see that $\mathcal{D}$ is a weighted Hardy space with $\beta(j) = j^{\frac{1}{2}}$ for each positive integer j.

2-The Bergman space $A^2(\mathbb{D})$ is the space of all analytic functions f on $\mathbb{D}$ for which the norm

$$\|f\|_{A^2} = \{\int_{\mathbb{D}} |f|^2 dA\}^{\frac{1}{2}},$$

is finite. The space $A^2(\mathbb{D})$ is a Hilbert space with inner product

$$< f, g >= \int_{\mathbb{D}} f(z)\overline{g(z)} \frac{dA(z)}{\pi}.$$

The space $A^2(\mathbb{D})$ is a weighted Hardy space with $\beta(j) = (j+1)^{\frac{-1}{2}}$ for each positive integer j.

In 1988, Cowen determined the formula for C_φ^* on the Hardy space H^2, when φ is a linear fractional self-map of $\mathbb{D}$. He showed that if $\varphi(z) = \frac{az+b}{cz+d}$ is a linear fractional self map of $\mathbb{D}$, then

$$C_\varphi^* = M_\phi C_\sigma M_\psi^*,$$

where $\sigma(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}$ is the Kreĭn adjoint of φ, $\phi(z) = (-\overline{b}z + \overline{d})^{-1}$, $\psi(z) = cz + d$ and M_ϕ and M_ψ are multiplication operators. Later, Hurts [6] obtained the Cowen's formula on weighted Bergman spaces. Such formulas initiated more studies of the adjoint of linear fractional composition operators on different spaces of analytic functions and on H^2 for general rational symbols.

In [5], Gallardo-Gutiérrez and Montes-Rodríguez gave an explicit formula in the Dirichlet space $\mathcal{D}$ for C_ϕ^*, when ϕ is a linear fractional symbol. They have shown that C_ϕ^* acting on the Dirichlet space is given by the formula

$$C_\phi^* f = f(0)K_{\phi(0)} - (C_{\phi^*} f)(0) + C_{\phi^*} f, \quad f \in \mathcal{D}. \tag{1.1}$$

In 2008, A. Abdollahi consider automorphic composition operators C_ϕ acting on the Dirichlet space. By using the E. Gallardo and A. Montes adjoint formula on the Dirichlet space. He has completely determined the spectrum, essential spectrum and point spectrum for self-commutators of such operators. In [1] and [2] the authors do the same work for monomial symbols on some Hilbert spaces of analytic functions.

For more information about the adjoint of composition operators, we refer the reader to [3] and the references therein.

2. Main results

In this section we state main theorems and results of the article.

THEOREM 2.1. *Let $H^2(\beta)$ be a weighted Hardy space, $\phi(z) = z^m$ and $\upsilon(z) = a_1 z^{k_1} + a_2 z^{k_2} + \ldots + a_l z^{k_l}$ where m, $k_1, \ldots k_l$ are positive integers and $a_1, \ldots, a_l$ are complex numbers. For an arbitrary point $w \in \mathbb{D}$, the adjoint of $C_{\upsilon,\phi}$ on $H^2(\beta)$ is given by*

$$C_{\upsilon,\phi}^* f(w) = \sum_{n=0}^{\infty} (\overline{a_1} \frac{f^{(nm+k_1)}(0)}{(nm+k_1)!} \frac{\beta(nm+k_1)^2}{\beta(n)^2} + \ldots + \overline{a_l} \frac{f^{(nm+k_l)}(0)}{(nm+k_l)!} \frac{\beta(nm+k_l)^2}{\beta(n)^2}) w^n.$$

Corollary 2.2. *Let $\upsilon(z) = z^m$ and $\phi(z) = z^n$, where m and n are positive integers. For an arbitrary point $w \in \mathbb{D}$, the adjoint of $C_{\upsilon,\phi}$(viewed as an operator on the Dirichlet space) is given by the formula*

$$C^*_{\upsilon,\phi} f(w) = m\frac{f^{(m)}(0)}{m!} + \sum_{k=1}^{\infty}(\frac{m}{k} + n)\frac{f^{(m+nk)}(0)}{(m+nk)!}w^k.$$

Corollary 2.3. *Let $\upsilon(z) = z^{mk+1} + z^{mk+3}$ and $\phi(z) = z^l$, where k, m and l are positive integers. For an arbitrary point $w = re^{i\theta}$ in $\mathbb{D}$, the adjoint of $C_{\upsilon,\phi}$ (viewed as an operator on the Bergman space) is given by the formula*

$$C^*_{\upsilon,\phi} f(w) = \sum_{n=0}^{\infty}(\frac{n+1}{ln+mk+2}\frac{f^{(ln+mk+1)}(0)}{(ln+mk+1)!} + \frac{n+1}{ln+mk+4}\frac{f^{(ln+mk+3)}(0)}{(ln+mk+3)!})w^n.$$

Theorem 2.4. *Let υ be any analytic rational function with poles off $\overline{\mathbb{D}}$ and ϕ be an analytic rational self-map of the unit disk such that $C_{\upsilon,\phi}$ is bounded on the Dirichlet space. Assume that $\hat{\upsilon}(z) = \overline{\upsilon(\frac{1}{\bar{z}})}$, $\hat{\upsilon'}(z) = \overline{\upsilon'(\frac{1}{\bar{z}})}$, $\hat{\phi}'(z) = \overline{\phi'(\frac{1}{\bar{z}})}$ and $\hat{\phi}(z) = \overline{\phi(\frac{1}{\bar{z}})}$. Then the adjoint formula for $C_{\upsilon,\phi}$ on the Dirichlet space is given by*

$$\begin{aligned} C^*_{\upsilon,\phi} f(w) &= f(0)\overline{\upsilon(0)} + \sum Res(\frac{f(z)\hat{\upsilon'}(z)}{z^2}, u_m) + \sum_{n=1}^{\infty}\frac{w^n}{n}(f(0)\overline{\upsilon(0)\phi(0)^n} \\ &+ \sum Res(\frac{f(z)\hat{\upsilon'}(z)(\hat{\phi}(z))^n}{z^2}, v_k) + n\sum Res(\frac{f(z)\hat{\upsilon}(z)\hat{\phi}'(z)(\hat{\phi}(z))^{n-1}}{z^2}, w_l)), \end{aligned}$$

where u_m, v_k and w_l are respectively poles of the functions $\frac{f(z)\hat{\upsilon'}(z)}{z^2}$, $\frac{f(z)\hat{\upsilon'}(z)(\hat{\phi(z)})^n}{z^2}$ and $\frac{f(z)\hat{\upsilon}(z)\hat{\phi}'(z)(\hat{\phi}(z))^{n-1}}{z^2}$ in $\mathbb{D}$.

Theorem 2.5. *Let υ be any analytic rational function with poles off $\overline{\mathbb{D}}$ and ϕ be an analytic rational self-map of the unit disk such that $C_{\upsilon,\phi}$ is bounded on the Bergman space. Assume that $\hat{\upsilon}(z) = \overline{\upsilon(\frac{1}{\bar{z}})}$ and $\hat{\phi}(z) = \overline{\phi(\frac{1}{\bar{z}})}$. Then the adjoint formula for $C_{\upsilon,\phi}$ on the Bergman space is given by the formula*

$$C^*_{\upsilon,\phi} f(w) = \sum_{n=0}^{\infty}(n+1)\sum Res(\frac{F(z)\hat{\upsilon}(z)(\hat{\phi}(z))^n}{z^2}, z_k)w^n,$$

where $F(z)$ is holomorphic on the unit disk $\mathbb{D}$, such that for each $z \in \mathbb{D}$, $F'(z) = f(z)$, and z_k are poles of the functions $\frac{F(z)\hat{\upsilon}(z)(\hat{\phi}(z))^n}{z^2}$ in $\mathbb{D}$.

Theorem 2.6. *If $\rho : \hat{C} \longrightarrow \hat{C}$ denotes inversion in the unit circle, $\rho(z) = \frac{1}{\bar{z}}$, and $\upsilon(z) = 1$ and $\phi(z) = \frac{z+z^2+...+z^n}{n}$, then $C_{\upsilon,\phi} = C_\phi$. Assume that $w_0 \in \mathbb{D}$ is a regular value of $\phi_e = \rho o \phi o \rho$ and $V \subset \mathbb{D}$ is any connected neighborhood of w_0 on which are defined n distinct branches $\{\sigma_j\}_{j=1}^n$ of ϕ_e^{-1}. Then for all non zero $w \in V$ the adjoint formula for C_ϕ on the Dirichlet space is given by*

$$C^*_\phi f(w) = \sum_{j=1}^{n}\frac{f(\sigma_j(w))}{\sigma_j(w)} - (n-1)f(0),$$

and for $w = 0$,

$$C^*_\phi f(0) = f(0).$$

3. Conclusion

In this manuscript under certain conditions on the symbols of a weighted composition operator, we determined the formula for it's adjoint, when the spaces is a general weighted Hardy space, or in particular when the space is the classical Bergman space or the classical Dirichlet space.

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پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
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دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Some classes of continuous maps in terms of closure and interior operators

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Abstract. In this paper, we give some generalized and modification categories of topological spaces by monotone operators and investigate some categorical properties of them. In particular, we study the properties of some classes of morphisms, such as final, initial, closed and open morphisms in these categories.

Keywords: Generalized topology, monotone operator, closure and interior operator

AMS Mathematics Subject Classification [2010]: 65F05, 46L05

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1. Introduction

We first give some notions and notations. A mapping $\gamma : P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is said to be *monotone* provided that $A \subseteq B \subseteq X$ implies $\gamma A \subseteq \gamma B$, where we write γA for $\gamma(A)$. The pair (X, γ) is called a Γ-*space*. A set $A \subseteq X$ is said to be γ-*open* provided that $A \subseteq \gamma A$; γ-*closed* provided that $\gamma A \subseteq A$ and the collection μ_γ of all γ-open sets is a generalized topology in the sense of [2], where a subset μ of $P(X)$ is called a *generalized topology* (briefly GT) on X and the pair (X, μ) is called a *generalized topological space* (briefly GTS) if $\emptyset \in \mu$ and any union of elements of μ belongs to μ. A GTS (X, μ) is called *strong* if $X \in \mu$. Also the collection $\mu^*_\gamma = \{A \mid \gamma(X - A) \subseteq X - A\}$ is a GT on X.

A monotone map $\gamma : P(X) \to P(X)$ is said to be:

(1) *idempotent* if $\gamma^2 A = \gamma\gamma A = \gamma A$ for $A \subseteq X$;
(2) *restricting* if $\gamma A \subseteq A$ for $A \subseteq X$;
(3) *enlarging* if $A \subseteq \gamma A$ for $A \subseteq X$;
(4) $\vee$-*additive* if $\gamma(A \cup B) = \gamma A \cup \gamma B$ for $A, B \subseteq X$;
(5) $\wedge$-*additive* if $\gamma(A \cap B) = \gamma A \cap \gamma B$ for $A, B \subseteq X$.

The conjugate of a monotone map γ is defined by $\gamma^* A = X - \gamma(X - A)$ for $A \subseteq X$. Clearly (X, γ^*) is a Γ-space. If μ is a GT on X, then the interior operator $i_\mu : P(X) \to P(X)$ defined by $i_\mu A = \bigcup\{M \in \mu \mid M \subseteq A\}$ is monotone, idempotent and restricting; and the closure operator $c_\mu : P(X) \to P(X)$ defined by $c_\mu A = \bigcap\{N \mid A \subseteq N, X - N \in \mu\}$ is monotone, idempotent and enlarging.

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A mapping $f : (X, \mu_X) \to (Y, \mu_Y)$ between GTS's is said to be *g-continuous* if $f^{-1}(B) \in \mu_X$ whenever $B \in \mu_Y$ [2–4]. We denote by **Top** and **GenTop** the category of all topological spaces with continuous maps; and the category of all generalized topological spaces with g-continuous maps, respectively. In the following, readers are suggested to refer to [1] for some categorical notions.

2. Main results

We first present some generalized and modification categories of topological spaces in terms of closure and interior operators. Recall that every monotone and restricting operator is called an interior operator, and every monotone and enlarging operator is called a closure operator.

Definition 2.1. Let $f : (X, \gamma) \to (Y, \delta)$ be a mapping between Γ-spaces. We say that f is *i-continuous* if $f^{-1}(\delta B) \subseteq \gamma f^{-1}(B)$ for all subset B of Y; and *c-continuous* if $\gamma f^{-1}(B) \subseteq f^{-1}(\delta B)$ for all subset B of Y, or equivalently, $f(\gamma A) \subseteq \delta f(A)$ for all subset A of X.

We denote by:

(1) $\mathbf{Mon_i}$ and $\mathbf{Mon_c}$ the category of all Γ-spaces and i-continuous maps, and the category of all Γ-spaces and c-continuous maps, respectively;
(2) **Int** and **Clo** the full subcategories of $\mathbf{Mon_i}$ and $\mathbf{Mon_c}$ of all restricting maps, and of all enlarging maps, respectively;
(3) $\mathbf{Int_2}$ and $\mathbf{Clo_2}$ the full subcategories of **Int** and **Clo** of all idempotent maps, respectively;
(4) $\mathbf{Int_2}_\wedge$ and $\mathbf{Clo_2}_\vee$ the full subcategories of $\mathbf{Int_2}$ and $\mathbf{Clo_2}$ of all $\wedge$-additive maps; and of all $\vee$-additive maps, respectively.

The following diagram summarizes the relation between the above categories, where we use the notations $\cong$, $\uparrow$ and $\hookrightarrow$ for isomorphic, full subcategory and reflective full subcategory, respectively [6].

$$\begin{array}{ccccc}
\mathbf{Mon_i} & \hookleftarrow & \mathbf{GenTop} & \hookrightarrow & \mathbf{Mon_c} \\
\uparrow & & \shortparallel & & \uparrow \\
\mathbf{Int} & \hookleftarrow & \mathbf{GenTop} & \hookrightarrow & \mathbf{Clo} \\
\uparrow & & \shortparallel & & \uparrow \\
\mathbf{Int_2} & \cong & \mathbf{GenTop} & \cong & \mathbf{Clo_2} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{Int}_{\wedge\mathbf{2}} & \cong & \mathbf{Top} & \cong & \mathbf{Clo_2}_\vee
\end{array}$$

In the following, we study the notions of initial, final, open and closed morphisms with respect to closure and interior operators. In an arbitrary category with a subject structure and a closure operator, the notions of such morphisms were introduced in [5]. Similarly, we have the following definitions.

Definition 2.2. A mapping $f : (X, \gamma) \to (Y, \delta)$ between Γ-spaces is called:

(1) *c-final* if $\delta B = f(\gamma f^{-1}(B))$ for all subset B of Y;
(2) *c-initial* if $\gamma A = f^{-1}(\delta f(A))$ for all subset A of X;
(3) *i-final* if $\delta^* B = f(\gamma^* f^{-1}(B))$ for all subset B of Y;
(4) *i-initial* if $\gamma^* A = f^{-1}(\delta^* f(A))$ for all subset A of X.

Definition 2.3. A mapping $f : (X, \gamma) \to (Y, \delta)$ between Γ-spaces is called:

(1) *c-closed* or Γ*-preserving* if $f(\gamma A) = \delta f(A)$ for all subset A of X;
(2) *c-open* or Γ*-reflecting* if $f^{-1}(\delta B) = \gamma f^{-1}(B)$ for all subset B of Y;
(3) *i-closed* or Γ^**-preserving* if $f(\gamma^* A) = \delta^* f(A)$ for all subset A of X;
(4) *i-open* or Γ^**-reflecting* if $f^{-1}(\delta^* B) = \gamma^* f^{-1}(B)$ for all subset B of Y.

It is easy to show that c-final, c-initial, c-closed and c-open maps are c-continuous and i-final, i-initial, i-closed and i-open maps are i-continuous. Thus we study the properties of c-final, c-initial, c-closed and c-open maps in $\mathbf{Mon_c}$ and its full subcategories (i.e., **Clo**, $\mathbf{Clo_2}$, $\mathbf{Clo_{2\vee}}$); and the properties of i-final, i-initial, i-closed and i-open maps in $\mathbf{Mon_i}$ and its full subcategories (i.e., **Int**, $\mathbf{Int_2}$, $\mathbf{Int_{2\wedge}}$).

THEOREM 2.4. (1) *In the construct* $\mathbf{Mon_c}$ *or any of its full subcategories, a mapping f is c-initial if and only if it is an initial morphism.*
(2) *In the construct* $\mathbf{Mon_i}$ *or any of its full subcategories, a mapping f is i-initial if and only if it is an initial morphism.*

By Theorem 2.4, we have the following result.

COROLLARY 2.5. *In the construct* **Top** (**GenTop**) *a continuous (g-continuous) map* $f : (X, \tau) \to (Y, \sigma)$ *is initial if and only if* $c_\tau A = f^{-1}(c_\sigma f(A))$ *for every subset A of X.*

REMARK 2.6. We point out that in the constructs **Clo**, $\mathbf{Clo_2}$ and $\mathbf{Clo_{2\vee}}$ finality does not characterize c-final maps. For example, let $X = Y = \{1, 2\}$, $\tau = \{\emptyset, \{2\}, X\}$, $\sigma = P(Y)$ and $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(1) = f(2) = 1$. Then, f is a final morphism in **Top** and hence in **GenTop**. Since every isomorphism functor between concrete categories preserves final morphisms, it follows that $f : (X, c_\tau) \to (Y, c_\sigma)$ is a final morphism in $\mathbf{Clo_2}$ and $\mathbf{Clo_{2\vee}}$. But $c_\sigma\{2\} = \{2\}$ and $f(c_\tau f^{-1}(\{2\}) = \emptyset$. Thus f is not c-final.

THEOREM 2.7. (1) *In the construct* $\mathbf{Mon_c}$ *a mapping f is c-final if and only if it is a final morphism.*
(2) *In the construct* **Clo** *a mapping f is c-final if and only if it is a surjective final morphism.*
(3) *In any of the constructs* $\mathbf{Clo_2}$ *and* $\mathbf{Clo_{2\vee}}$ *a mapping f is c-final if it is a bijective final morphism.*

THEOREM 2.8. (1) *In the construct* $\mathbf{Mon_i}$ *a mapping f is i-final if and only if it is a final morphism.*
(2) *In the construct* **Int** *a mapping f is i-final if and only if it is a surjective final morphism.*
(3) *In any of the constructs* $\mathbf{Int_2}$ *and* $\mathbf{Int_{2\wedge}}$ *a mapping f is i-final if it is a bijective final morphism.*

By Theorems 2.7 and 2.8, the following result holds.

COROLLARY 2.9. *In the construct* **Int**, *i-final maps and in the construct* **Clo**, *c-final maps are precisely quotient morphisms.*

THEOREM 2.10. (1) *If* $f : (X, \gamma) \to (Y, \delta)$ *is c-closed in* $\mathbf{Mon_c}$ *or any of its full subcategories, then f maps γ-closed subsets to δ-closed subsets.*
(2) *Let* $f : (X, \gamma) \to (Y, \delta)$ *be a c-continuous mapping in any of the categories* $\mathbf{Clo_2}$ *and* $\mathbf{Clo_{2\vee}}$. *Then f is c-closed if and only if f maps γ-closed subsets to δ-closed subsets.*
(3) *If* $f : (X, \gamma) \to (Y, \delta)$ *is i-closed in* $\mathbf{Mon_i}$ *or any of its full subcategories, then f maps γ^*-closed subsets to δ^*-closed subsets.*

(4) *Let $f : (X, \gamma) \to (Y, \delta)$ be an i-continuous mapping in any of the categories* $\mathbf{Int_2}$ *and* $\mathbf{Int_{2\wedge}}$. *Then f is i-closed if and only if f maps γ^*-closed subsets to δ^*-closed subsets.*

Theorem 2.11. (1) *If $f : (X, \gamma) \to (Y, \delta)$ is c-open in* $\mathbf{Mon_c}$ *or any of its full subcategories, then f maps γ^*-open subsets to δ^*-open subsets.*

(2) *Let $f : (X, \gamma) \to (Y, \delta)$ be a c-continuous mapping in any of the categories* $\mathbf{Clo_2}$ *and* $\mathbf{Clo_{2\vee}}$. *Then f is c-open if and only if f maps γ^*-open subsets to δ^*-open subsets.*

(3) *If $f : (X, \gamma) \to (Y, \delta)$ is i-open in* $\mathbf{Mon_i}$ *or any of its full subcategories, then f maps γ-open subsets to δ-open subsets.*

(4) *Let $f : (X, \gamma) \to (Y, \delta)$ be an i-continuous mapping in any of the categories* $\mathbf{Int_2}$ *and* $\mathbf{Int_{2\wedge}}$. *Then f is i-open if and only if f maps γ-open subsets to δ-open subsets.*

Now, by Theorems 2.10 and 2.11, the following results holds.

Corollary 2.12. *Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous mapping in any of the categories* **Top** *and* **GenTop**. *Then $f : (X, c_\tau) \to (Y, c_\sigma)$ is c-closed if and only if f maps c_τ-closed subsets to c_σ-closed subsets; and $f : (X, i_\tau) \to (Y, i_\sigma)$ is i-closed if and only if f maps $(i^*_\tau = c_\tau)$-closed subsets to $(i^*_\sigma = c_\sigma)$-closed subsets. Thus f is c-closed or i-closed if and only if f is a closed map.*

Corollary 2.13. *Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous mapping in any of the categories* **Top** *and* **GenTop**. *Then $f : (X, c_\tau) \to (Y, c_\sigma)$ is c-open if and only if f maps $(c^*_\tau = i_\tau)$-open subsets to $(c^*_\sigma = i_\sigma)$-open subsets; and $f : (X, i_\tau) \to (Y, i_\sigma)$ is i-open if and only if f maps i_τ-open subsets to i_σ-open subsets. Thus f is c-open or i-open if and only if f is an open map.*

3. Conclusion

We have given some isomorphic and generalized categories of **Top** and **GenTop**; and studied the properties of final, initial, closed and open maps with respect to closure operators by defining c-final, c-initial, c-closed, c-open maps; and with respect to interior operators by defining i-final, i-initial, i-closed and i-open maps, respectively.

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پنجاه و دومین کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Primary decomposition of submodules of a free module of finite rank over a domain

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Abstract. Let R be a commutative ring with identity.In this paper, we study the existence of primary decomposition of a submodule of F over a domain and characterize the minimal primary decomposition of this submodule.

Keywords: Primary decompositions, Free modules, GCD domains, Bézout domains, .

AMS Mathematics Subject Classification [2010]: 13C10, 13C99, 13G99.

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1. Introduction

Throughout this article, all rings are assumed to be commutative with identity and F denotes a free R-module of finite rank n ($n \geqslant 2$). Let R be a commutative domain and K be the quotient field of R. An integral domain R is a GCD domain if any two elements of R have a greatest common divisor. A Bézout domain is an integral domain in which the sum of two principal ideals is again principal. Note that a Bézout domain is a GCD domain, see [3, 4]. Furthermore, by [1, Theorem 4.8], a one dimensional GCD domain is Bézout. Any PID is a Bézout domain but a Bézout domain need not be a PID or a UFD. Let R be the ring of entire complex valued functions. By [5, Fact 2.3], R is a Bézout domain and so is a GCD domain. Since the irreducible elements of R are linear polynomials and there are functions with infinitely many roots, R is not a UFD.

Note that a one dimensional GCD domain is not necessarily a UFD. For example, suppose that R is the ring of algebraic integers. By [3, Theorem 102], R is a Bézout domain and hence is a GCD domain. For every non-zero and non-unit element $a \in R$, we have $a = \sqrt{a}\sqrt{a}$. But $\sqrt{a}$ is a non-unit in R and hence R is not a UFD. Moreover, $dim(R) = dim(\mathbb{Z}) = 1$.
In this paper, we study the existence of primary decomposition of a submodule of F of finite rank, where R is a domain and characterize its minimal primary decomposition.

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2. Primary Decomposition of submodules of a free module of finite rank over a domain.

The following notations and results obtained from [7], will be frequently used in this article. Let $F = R^{(n)}$ be a free R-module and $X = (x_{i1}, \ldots, x_{in}) \in F$, for some $x_{ij} \in R$ $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n,\ 1 \leqslant m \leqslant n)$. We put

$$B_{m\times n} = [X_1 \ldots X_m] = \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ & \ddots & & \\ x_{m1} & x_{m2} & \ldots & x_{mn} \end{pmatrix} \in M_{m\times n}(R).$$

Thus the jth row of the matrix $[X_1 \ldots X_m]$ consists of the components of element X_j in F. We use $B(j_1, \ldots, j_k) \in M_{m\times k}(R)$ to denote the submatrix of B consisting of the columns $j_1, \ldots, j_k \in \{1, \ldots, n\}$. Setting $\psi = \{X_i = (x_{i1}, \ldots, x_{in}) \in F \mid i \in \Omega\}$, where $\Omega(\subseteq \mathbb{N})$ is an index set.We use $\langle B\rangle$ for the submodule of F generated by the rows of B.

Lemma 2.1. *[6, Lemma 1.1] Let R be a domain. Suppose that $B \in M_{n\times n}(R)$, $detB \neq 0$ and $B' = (b'_{ij})$ be the adjoint matrix of B. Then $(x_1, \ldots, x_n) \in \langle B\rangle$, for some $x_i \in R$ $(1 \leqslant i \leqslant n)$ if and only if $\sum_{i=1}^{n} x_i b'_{ij} \in \langle detB\rangle$ for every j $(1 \leqslant j \leqslant n)$.*

Lemma 2.2. *Suppose R is a domain with a principal maximal ideal $m = \langle p\rangle$, n, s and α are positive integers such that $s < n$. Also suppose that $A \in M_{n\times s}(R)$, $Y \in M_{n\times 1}(R)$ and $X = [x_1 \cdots x_s]^T \in M_{s\times 1}(R)$. Let $C \in M_{n\times(s+1)}(R)$ be the augmented matrix $[A : Y]$. If p does not divide the determinant of at least one $s \times s$ submatrix of A, then the system of equations $AX \equiv Y (mod\, p^\alpha)$ has a solution if and only if p^α divides the determinants of all $(s + 1) \times (s + 1)$ submatrices of C.*

Proposition 2.3. *Let R be a domain, n be a positive integer and $F = R^n$ $(n \geqslant 2)$. Suppose that $B \in M_{n\times n}(R)$ and $0 \neq detB$ has a prime decomposition. Then $\langle B\rangle$ is a primary submodule of F if and only if $detB = up^\alpha$ for some unit $u \in R$, a prime element $p \in R$ and a positive integer α.*

Let $m \leqslant n$ be positive integers and $B \in M_{m\times n}(R)$. Suppose that t $(1 \leqslant t \leqslant m)$, $1 \leqslant i_1 < \cdots < i_t \leqslant m$ and $1 \leqslant j_1 < \cdots < j_t \leqslant n$ be some integers. Then $B\begin{bmatrix} i_1 & \cdots & i_t \\ j_1 & \cdots & j_t \end{bmatrix}$ denotes the determinant of the $t \times t$ submatrix of B consisting of rows $i_1, \cdots, i_t$ and columns $j_1, \cdots, j_t$.

Theorem 2.4. *Let $m \leqslant n$ be positive integers and let $B \in M_{m\times n}(R)$. Suppose that $p \in R$ is a prime element and let α be the greatest integer such that $p^\alpha \mid B\begin{bmatrix} 1 & \cdots & m \\ 1 & \cdots & m \end{bmatrix}$. Then there exists an upper triangular matrix $A \in M_{n\times n}(R)$ with $detA = p^\alpha$ such that $\langle B\rangle \subseteq \langle A\rangle$.*

Proof. Let B_j $(1 \leqslant j \leqslant m)$ be the submatrix consisting of the first j columns of B and $B_m = (b_{ij})$. Suppose that $\alpha_1 \geqslant 0$ is the greatest integer such that $p_1^\alpha \mid b_{i1}$, for all i $(1 \leqslant i \leqslant m)$. Assume that $\alpha_1, \cdots, \alpha_{j-1}$ are defined for some $j \leqslant m$ and define $\alpha_j \geqslant 0$ to be the greatest integer such that $p^{\alpha_1+\cdots+\alpha_j}$ divides the determinants of all $j \times j$ submatrices of B_j. Hence $\alpha_1 + \cdots + \alpha_m = \alpha$. By induction on j, we justify the following assertion:

for every j $(1 \leqslant j \leqslant m)$, there exist $C_j \in M_{m\times j}(R)$ and $A_j \in M_{j\times j}(R)$, such that $C_jA_j = B_j$, A_j is upper triangular and $detA_j = p^{\alpha_1+\cdots+\alpha_j}$.

For $j = 1$, we put $C_1 = (\frac{b_{11}}{p^{\alpha_1}}, \cdots, \frac{b_{m1}}{p^{\alpha_1}})^t$ and $A_1 = (p^{\alpha_1})$. Clearly $C_1A_1 = B_1$. Now assume the assertion to be true for some j $(1 \leqslant j < m)$. Let $1 \leqslant i_1 < \cdots < i_j \leqslant m$ be arbitrary integers. We have

$$\begin{aligned} B_j \begin{bmatrix} i_1 & \cdots & i_j \\ 1 & \cdots & j \end{bmatrix} &= C_j \begin{bmatrix} i_1 & \cdots & i_j \\ 1 & \cdots & j \end{bmatrix} (\det A_j) \\ &= C_j \begin{bmatrix} i_1 & \cdots & i_j \\ 1 & \cdots & j \end{bmatrix} p^{\alpha_1+\cdots+\alpha_j}. \end{aligned}$$

Hence p does not divide the determinant of at least one $j \times j$ submatrix of C_j. Also, it implies that

$$\begin{aligned} &p^{\alpha_1+\cdots+\alpha_{j+1}} \mid B_{j+1} \begin{bmatrix} i_1 & \cdots & i_{j+1} \\ 1 & \cdots & j+1 \end{bmatrix} \\ &= (\sum_{k=1}^{j+1} (-1)^{k+j+1} b_{i_k(j+1)}) B_j \begin{bmatrix} i_1 & \cdots & i_{k-1} & i_{k+1} \cdots & i_{j+1} \\ 1 & \cdots & \cdots & \cdots & j \end{bmatrix} \\ &= (p^{\alpha_1+\cdots+\alpha_j} \sum_{k=1}^{j+1} (-1)^{k+j+1} b_{i_k(j+1)}) C_j \begin{bmatrix} i_1 & \cdots & i_{k-1} & i_{k+1} \cdots & i_{j+1} \\ 1 & \cdots & \cdots & \cdots & j \end{bmatrix} \end{aligned}$$

for all positive integers $i_1 < \cdots < i_{j+1} \leqslant m$. It follows that

$$p^{\alpha_{j+1}} \mid (\sum_{k=1}^{j+1} (-1)^{k+j+1} b_{i_k(j+1)}) C_j \begin{bmatrix} i_1 & \cdots & i_{k-1} & i_{k+1} \cdots & i_{j+1} \\ 1 & \cdots & \cdots & \cdots & j \end{bmatrix}.$$

Let Y_{j+1} be the $(j+1)$th column of B. By Lemma 2.2, there exist $a_{1(j+1)}, \cdots, a_{j(j+1)} \in R$ such that $C_j(a_{1(j+1)}, \cdots, a_{j(j+1)})^t \equiv Y_{j+1}(mod\, p^{\alpha_{j+1}})$. Put

$$C_{j+1} = \left(C_j : \frac{1}{p^{\alpha_{j+1}}}(Y_{j+1} - C_j \begin{pmatrix} a_{1(j+1)} \\ \vdots \\ a_{j(j+1)} \end{pmatrix}) \right), \quad A_{j+1} = \begin{pmatrix} & a_{1(j+1)} \\ A_j & \vdots \\ & a_{j(j+1)} \\ 0 \cdots 0 & p^{\alpha_{j+1}} \end{pmatrix}$$

It is easy to see that $C_{j+1}A_{j+1} = B_{j+1}$, $detA_{j+1} = p^{\alpha_1+\cdots+\alpha_{j+1}}$ and A_{j+1} is upper triangular. Hence the assertion is true for $j+1$ and so by induction for all j $(1 \leqslant j \leqslant m)$. In particular for $j = m$, there exist $A_m, C_m \in M_{m\times m}(R)$ such that $C_mA_m = B_m$. If $m < n$, let B_0 be the submatrix consisting of columns $m+1, \cdots, n$ of B. Let $C \in M_{m\times n}(R)$ be the augmented matrix $(C_m : B_0)$ and let

$$A = \begin{pmatrix} A_m & 0 \\ 0 & I_{n-m} \end{pmatrix} \in M_{n\times n}(R).$$

Clearly A is upper triangular and $detA = detA_m = p^{\alpha}$. Also we have $CA = B$ and so $\langle B \rangle \subseteq \langle A \rangle$. □

THEOREM 2.5. *Let R be a domain and $F = R^{(n)}$. Let $B \in M_{n\times n}(R)$ such that $detB$ is non-unit and non-zero. Suppose that $detB = p_1^{\beta_1} \cdots p_t^{\beta_t}$ is a prime decomposition, $p_i \neq p_j \in R$ and $\beta_i \in \mathbb{N}$ $(1 \leqslant i, j \leqslant t)$. Let A_k with $detA_k = p_k^{\beta_k} (1 \leqslant k \leqslant t)$ be the* *triangular matrix in Theorem 2.4. Then* $\bigcap_{k=1}^{t}\langle A_k\rangle$ *is a minimal primary decomposition of* $\langle B\rangle$.

Example 2.6. Let $B = \begin{pmatrix} x+2 & x+2 & 0 \\ x+2 & x+3 & x \\ x^2(x+2) & x & x^2 \end{pmatrix} \in M_{3\times 3}(R)$. We shall find a minimal primary decomposition of $\langle B\rangle$. Since $detB = x^3(x+2)^2$, by Theorem 2.4, there exist upper triangular matrices

$$\begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & x^3 \end{pmatrix} \quad , \quad \begin{pmatrix} x+2 & a'_{12} & a'_{13} \\ 0 & 1 & a'_{23} \\ 0 & 0 & x+2 \end{pmatrix}$$

such that $C_1(a_{12}) \equiv Y_2(mod\,1)$ and $C'_1(a'_{12}) \equiv Y_2(mod\,1)$. Where $C_1 = Y_2 = (x+2, x+2, x^2(x+2))$ and $C'_1 = (1, 1, x^2)$.So

$$\begin{cases} (x+2)a_{12} \equiv (x+1)(mod\,1) \\ x^2(x+2)a_{12} \equiv x^2(x+2)a_{12}(mod\,1) \end{cases} \quad , \quad \begin{cases} a'_{12} \equiv (x+2)(mod\,1) \\ x^2a'_{12} \equiv x^2(x+2)(mod\,1) \end{cases}$$

A solution for the above systems is $a_{12} = a'_{12} = 1$.
Also $C_2(a_{13}, a_{23})^t \equiv Y_3(mod\,x^3)$ and $C'_2(a'_{13}, a'_{23})^t \equiv Y_3(mod\,x{+}2)$, where $\mathrm{Y}_3 = (0, x, x^2)$, $C_2 = \begin{pmatrix} x+2 & 0 \\ x+2 & 1 \\ x^2(x+2) & -x^3-2x^2+x \end{pmatrix}$ and $C'_2 = \begin{pmatrix} 1 & x+1 \\ 1 & x+2 \\ x^2 & x-x^2 \end{pmatrix}$. Then

$$\begin{cases} (x+2)a_{13} \equiv 0(mod\,x^3) \\ (x+2)a_{13} + a_{23} \equiv x(mod\,x^3) \\ x^2(x+2)a_{13} + (-x^3-2x^2+x)a_{23} \equiv x^3(mod\,x^3) \end{cases} \quad , \quad \begin{cases} a'_{13} + (x+1)a'_{23} \equiv 0(mod\,x+2) \\ a'_{13} + (x+2)a'_{23} \equiv x(mod\,x+2) \\ x^2a'_{13} + (x-x^2)a'_{23} \equiv x^2(mod\,x+2). \end{cases}$$

A solution for the above systems is

$$a_{13} = x^3, \quad a_{23} = -x^4 - x^3 + x, \quad a'_{13} = 3x+4 \quad and \quad a'_{23} = -2.$$

Hence

$$A_1 = \begin{pmatrix} 1 & 1 & x^3 \\ 0 & 1 & -x^4-x^3+x \\ 0 & 0 & x^3 \end{pmatrix} \quad , \quad A_2 = \begin{pmatrix} x+2 & 1 & 3x+4 \\ 0 & 1 & -2 \\ 0 & 0 & x+2 \end{pmatrix}.$$

By Theorem 2.5, $\langle B\rangle = \langle A_1\rangle \bigcap \langle A_2\rangle$ is a minimal primary decomposition.

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پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Stability theory and representations

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Abstract. Many model theoretic aspects of the dynamical representations of the action of the groups (in particular $\mathbb{Z}$) on the spaces of types or models were investigated in a few works such as the one from the present paper's author. In this paper, we deal with higher order and more complex actions raised from more general groups (such as both automorphism and definable groups) acting on the structures or spaces of types. We will consider the class of NIP theories (which is is amongst the most important classes of first order theories studied in model theory) from the point of view of the defined representations.

Keywords: Model theory, stability theory, representations of dynamics of group actions, NIP

AMS Mathematics Subject Classification [2010]: 03C45, 03C95, 03C98

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1. Introduction

Investigation of the dynamical representations of the actions of the groups (in particular for the case of the action of $\mathbb{Z}$) on the spaces of types in the context of model theory has been carried out in a few works such as [4]. Also, some characterizations for NIP theories as well as some aspects of the symbolic representations were given by the author in the paper [4]. In this work, we elaborate those studies and deal with higher order and more complex actions raised from more general groups G, including both actions of automorphism groups and definable groups on the structures and spaces of types.

We first recall some essential definitions from stability theory and then introduce some new notions.

There are certain fundamental classes of theories studied in model theory. The class of stable theories is an important one of them which had received a lot of attentions in the classical model theory. One can see [6] as one of the main sources about stable class. Then, some other classes started to appear as central areas of research. The class of NIP theories is one of the most important classes of first order theories studied in the nowadays model theory. In recent years, the machinery of modern stability theory has been used to analyze several aspects of this class. The interested reader can refer to for example [1]

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and [2] for more details about the NIP theories. Also in [3], the notion of measures in the context of models and definable sets are introduced.

From another perspective, the theory of dynamical systems is recently involved with model theory, in particular stable theories and NIP theories, in several directions.

We will consider the class of NIP theories from the perspective of dynamics of actions of model theoretic objects and prove some results. These result also helps one to have more connections between model theory and other fields of mathematics.

We work in the setting of first order logic. Assume that T is a first order theory. Let $\phi(x, y)$ be a formula in the theory T. By $|x|$ and $|y|$ we mean the arity of tuples x and y. We use the notation $\mathbb{N}$ for the set of natural numbers starting from 1.

Definition 1.1. We say that ϕ has the *independence property or IP* if there exists some model M of the theory T such that for every natural number $n \in \mathbb{N}$, there exist $a_1, \ldots, a_n$, which are all $|x|$-tuples, such that witness independence property of length n for ϕ, which means that for every $J \subseteq \{1, \ldots, n\}$, there exists some $|y|$-tuple b_J such that $M \models \phi(a_i, b_J)$ if and only if $i \in J$. A theory is called *NIP* if no formula in it has IP.

Definition 1.2. A first order theory is called a *NIP* theory if no formula in that theory has the independence property.

2. Some definitions and main results

Definition 2.1. By a *Keisler measure* μ on M^n over parameter set A we mean a finitely additive probability measure on the set of definable sets with parameters from A namely, $Def_A(M^n)$. When $n = 1$, we use M instead of M^1 in all of the above notations.

Note that one can extract a countable additive Borel probably measure from each Keisler measure.

Now we want to consider a definable context and also a representation introduced by Newelski.

Let G be a definable group in model N and C a monster model. $S_G(N)$ is a compact topological space by logic topology. Newelski considers the dynamic of the G-flow obtained from the (left) action of G on $S_G(N)$.

Definition 2.2. Let G be definable group in a model M and $S_{ext,G}(M)$ be the space of types over external definables of G. Also let $p, q \in S_{ext,G}(M)$ and $U \in Def_{ext,G}(M)$. Then we let $U \in p.q$ if and only if $d_q(U) \in p$ where $d_q(U) = \{g \in G, g.q \in U\}$.

The above product makes $S_{ext,G}(M)$ a semigroup.

Now we introduce the following notions which are the main notions of the present work. Note that these notions are in the same sprits of the classical convolutions and try to bring ideas and adapt them to the context of model theory.

Definition 2.3. Let $(X, \mathcal{B})$ be a set equipped with a sigma algebra (in particular the space of types in a monster model), G be a group acting on X (in particular, a definable group or subgroup of automorphism group) and $\mathcal{A}$ be an G-closed space of measurable real valued functions on M. Let $\mathcal{M}(X)$ be the space of all probably measures on X (usually Borel when X is a topological space). For every $f \in \mathcal{A}$ define

$$\pi_{G,f} : \mathcal{M}(X) \to \mathbb{R}^G$$
$$\mu \to f * \mu$$

where

$$f * \mu : G \to \mathbb{R}$$

$$g \to \int_M g^{-1}.f \ d\mu.$$

We call π the *representation* of $\mathcal{A}$ with respect to G.

DEFINITION 2.4. Define

$$\pi_{G,U}(S(M)) := \{\pi_{G,U}(p) : p \in S(M)\}$$

and

$$\pi_{G,U}(\mathcal{M}(M)) := \{\pi_{G,U}(\mu) : \mu \in \mathcal{M}(X)\}.$$

Note that $\pi_{G,U}(S(M))$ and $\pi_{G,f}(\mathcal{M}(M))$ are associated dynamical invariant (G-flows).

REMARK 2.5. The representation defined in in Definitions 2.3 and 2.4 above generalizes Newelski's representations of types and product of external types. More precisely

$$d_q(U) = \chi_U * q.$$

In the following we give characterization of NIP via above descibed representations.

THEOREM 2.6. *Let M be a monster model of a theory T and $\phi(\bar{x}, \bar{y})$ be a formula. Then the followings are equivalent;*

(1) *ϕ is NIP.*
(2) *For every instance of ϕ, say $U = \phi^M(\bar{x}, \bar{a})$, and every subgroup G of $Aut(M)$ containing at least one non-periodic element, $\pi_{G,U}(M)$ is not dense in 2^G.*
(3) *For every instance of ϕ, say $U = \phi^M(\bar{x}, \bar{a})$, and every infinite $H \subseteq Aut(M)$, the set $\pi_{H,U}(\mathcal{M}(M))$ is not dense in $[0,1]^H$.*

PROOF. We give a sketch of the proof and roughly point out the main ideas. The absence of IP configuration for a formula causes the absence of various configurations in the orbits of the action of H as a subgroup of $Aut(M)$. Such absence will be induced on the space $\mathcal{M}(M)$ in a suitable and canonical way. Therefor, the representation $\pi_{H,U}(\mathcal{M}(M))$ will get the similar property of lacking certain configurations. It implies that $\pi_{H,U}(\mathcal{M}(M))$ can not be dense in $[0,1]^H$.

On the other hand, non-density in the space of representations implies that certain codes are never used in the members since otherwise every possible element would be produced as a limiting object. Such mentioned codes can be translated model theoretically in terms of configurations in the indiscernible sequences and causes restriction on indiscernibles. In turn, using model theoretic techniques, those restrictions imply the absence of IP which is indeed the property of NIP. As it is clear, two techniques of Ramsey and compactness have been used frequently. □

Acknowledgement

The author is indebted to Institute for Research in Fundamental Sciences, IPM, for support. This research was in part supported by a grant from IPM (No.1400030117)

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
دانشگاه شهید باهنر کرمان
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z_c°-ideals in $C_c(X)$ VS z°-ideals in $C(X)$

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Abstract. In this article, we compare z_c°-ideals in $C_c(X)$ and z°-ideals in $C(X)$. Every minimal prime ideal in $C_c(X)$ is a z_c°-ideal. Also, If X is a CP-space, then every ideal in $C_c(X)$ is a z_c°-ideal. Every z_c°-ideal is a z_c-ideal but the converse is not necessarily true. We prove that every z_c°-ideal is a contraction of a z°-ideal. Furthermore,if X is a strongly zero-dimensional space, then every z_c°-ideal is a contraction of a unique z°-ideal. Moreover, in the class of almost CP-spaces, X is strongly zero-dimensional space if and only if every z_c°-ideal is a contraction of a unique z°-ideal.

Keywords: zero-dimensional space, strongly zero-dimensional space, $z_c^\circ-$ideals
AMS Mathematics Subject Classification [2010]: 54C40

1. Introduction

Let $C_c(X)$ be the ring of all continuous real-valued functions with countable image on the topological space X, also $C_c^*(X)$ as a subring of $C_c(X)$ is the ring of bounded elements of $C_c(X)$, i.e., $C_c^*(X) = C^*(X) \cap C_c(X)$, so we have $C_c^*(X) \subseteq C_c(X) \subseteq C(X)$.We recall that a zero-dimensional space is a Hausdorff space with a base consisting of clopen sets. It is shown that for any topological space X, there exists a zero-dimensional space Y which is a continuous image of X and $C_c(X) \cong C_c(Y)$, see [4], so we can assume that X is a zero-dimensional space.
For each $f \in C_c(X)$, the zero-set of f is denoted by $Z(f)$. For each $f \in C_c(X)$,the set of all zero-sets in X is denoted by $Z_c(X)$. Also, $Z_c(X)$ is closed under countable intersection property. Furthermore, $Z_c(X) = Z(X)$ if and only if X is strongly zero-dimensional, see [6, Proposition 2.4]. Banaschewski has shown that for every zero-dimensional space X, there is a unique zero-dimensional compactification, denoted by $\beta_0 X$ in which each continuous function from X into a compact and zero-dimensional space T, has a continuous extension from $\beta_0 X$ into T. For more results, see [3]. As in $C(X)$, similar to the concept of the ideal O^p, $p \in \beta X$, for the zero-dimensional space X, we have the ideal O_c^p in $C_c(X)$:

$$O_c^p = \{f \in C_c(X) : p \in int_{\beta_0 X}(cl_{\beta_0 X} Z(f))\} \quad (p \in \beta_0 X)$$

Furthermore, $O_c^p = O_{cp} = \{f \in C_c(X) : p \in int(Z(f))\}$ if $p \in X$. A space X is a $CP-$space when $C_c(X)$ is a regular ring if and only if each prime ideal in $C_c(X)$ is a maximal ideal. Furthermore, a Tychonoff space X is called strongly zero-dimensional

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if each pair of disjoint zero-sets are contained in disjoint clopen sets. A space X is an almost $CP-$space if for each nonempty $Z(f) \in Z_c(X)$, we have $int(Z(f)) \neq \phi$. An ideal I in $C_c(X)$ is a z_c-ideal if for each $f \in I$, $g \in C_c(X)$ and $Z(f) = Z(g)$ we have $g \in I$. If I is a z-ideal in $C(X)$, then $I^c = I \cap C_c(X)$ is a z_c-ideal. For more results about $CP-$space,almost $CP-$space and z_c-ideal,see [3], [4]. The space of minimal prime ideals of $C_c(X)$ with Zariski topology is denoted by $Min(C_c(X))$. We recall that a proper ideal I in a ring R is a z°-ideal if for each $a \in I$, we have $P_a \subseteq I$ in which $P_a = \bigcap\{P : P \in V(a)\}$. Furthermore, if a is a zero divisor, then P_a is a z°-ideal that is called a basic z°-ideal. We denote P^c_f as a basic z°_c-ideal in $C_c(X)$ for each $f \in C_c(X)$.
Also, the properties of z°-ideals in $C(X)$ is considered, see [1], [2].
Similar to the concept of z°-ideals in $C(X)$, see [1], we introduce z°_c-ideals in $C_c(X)$ and compare z°-ideals and z°_c-ideals, also the conditions when the minimal prime ideals in $C_c(X)$ and prime z°_c-ideals coincide.

2. Main results

NOTATION 2.1. For each $f \in C_c(X)$,
$Ann_c(f) = \{g \in C_c(X) : fg = 0\}$ is used for an annihilator of f.
,
$V_c(f) = \bigcap\{P \in Min(C_c(X)) : f \in P\}$ is used for a basic element for open sets in $Min(C_c(X))$.
,
$D_c(f) = \bigcap\{P \in Min(C_c(X)) : f \notin P\}$ is used for a basic element for closed sets in $Min(C_c(X))$.
and

$P^c_f = \bigcap\{P : P \in V_c(f)\}$ is used for a basic z°_c-ideal in $C_c(X)$.

LEMMA 2.2. *Let $f, g \in C_c(X)$, then:*

$$Ann_c(f) \subseteq Ann_c(g) \text{ if and only if } int(Z(f)) \subseteq int(Z(g))$$

PROPOSITION 2.3. *Suppose I is a proper ideal of $C_c(X)$. The following statements are equivalent:*
(1) I is a z°_c-ideal.
(2) If $P^c_f = P^c_g$, $g \in C_c(X)$, $f \in I$ then $g \in I$.
(3) If $V_c(f) = V_c(g)$, $g \in C_c(X)$, $f \in I$ then $g \in I$.
(4) If $Ann_c(f) = Ann_c(g)$, $g \in C_c(X)$, $f \in I$ then $g \in I$.
(5) If $int(Z(f)) = int(Z(g))$, $g \in C_c(X)$, $f \in I$ then $g \in I$.

PROPOSITION 2.4. *For each $f \in C_c(X)$, we have:*
$P^c_f = \{g \in C_c(X) : Ann_c(f) \subseteq Ann_c(g)\}$.

EXAMPLE 2.5. (1) If I is a nonzero ideal in $C_c(X)$, then $Ann_c(I)$ is a z°_c-ideal.
(2) If A is a regular closed set $(cl(intA)) = A)$ in X, then $\mathcal{M}_{cA} = \{f \in C_c(X) : A \subseteq Z(f)\}$ is a z°_c-ideal.
(3) The ideal O_{cp} for $p \in X$, and more generally the ideal O^p_c for $p \in \beta_0 X$ are z°_c-ideal.

COROLLARY 2.6. *The following statements are valid:*
(1) Every minimal prime ideal in $C_c(X)$ is a z°_c-ideal.
(2) If I is a z°_c-ideal in $C_c(X)$ and $P \in Min(I)$, then P is a z°_c-ideal.
(3) If X is a CP-space, then every ideal in $C_c(X)$ is a z°_c-ideal.

Corollary 2.7. *Let X be a CP-space and $f, g \in C_c(X)$ then we have:*

$$P_f^c = P_g^c \text{ if and only if } D_c(f) = D_c(g).$$

Remark 2.8. (1) Every z_c°-ideal is a z_c-ideal. The converse is not necessarily true. For that, let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and $\sum = \mathbb{N} \cup \{\sigma\}$ where $\sigma \notin \mathbb{N}$, and define a topology on $\sum$ as follows: all points of $\mathbb{N}$ are isolated, and neighbourhoods of σ are the sets $U \cup \{\sigma\}$ for $u \in \mathcal{U}$. The space $\sum$ is extremally disconnected and every closed set in $\sum$ is a zero-set, see [5, 4M]. The ideal $\mathbb{M}_\sigma$ is not a z°-ideal. Since $\{\sigma\}$ is closed, then it is a zero-set, i.e., $Z(f) = \{\sigma\}$ in which $f \in C(\sum)$, so $f \in \mathcal{M}_\sigma$ but $int(Z(f)) = \phi$. Furthermore, $\sum$ is countable, so $C_c(\sum) = C(\sum)$ and $\mathcal{M}_{c\sigma} = \mathcal{M}_\sigma$. Thus, $\mathcal{M}_\sigma$ is not a z_c°-ideal. Obviously, $\mathcal{M}_\sigma$ is both z-ideal and z_c-ideal. Moreover, if X is a CP-space, every z_c-ideal is a z_c°-ideal.
(2) Each contraction of a z°-ideal in $C(X)$ is a z_c°-ideal in $C_c(X)$.
(3) Every z_c° -ideal of $C_c(X)$ contracts to a z°-ideal of $C_c^*(X)$.

Proposition 2.9. *Every z_c°-ideal in $C_c(X)$ is a contraction of a z°-ideal in $C(X)$.*

Corollary 2.10. *An ideal J in $C_c(X)$ is a z_c°-ideal if and only if it is a contraction of a z°-ideal in $C(X)$.*

Corollary 2.11. *Let X be a strongly zero-dimensional space, then every z_c°-ideal in $C_c(X)$ is a contraction of a unique z°-ideal in $C(X)$.*

Proposition 2.12. *The following statements are equivalent:*
(1) The minimal prime ideals in $C_c(X)$ are the only prime ideals-containing zero-divisors
(2) Every prime z_c°-ideal in $C_c(X)$ is a minimal prime ideal in $C_c(X)$
(3) $q_c(X)$, the classical ring of quotients of $C_c(X)$, is a regular ring.
(4) For each $f \in C_c(X)$, $Ann_c(f)$ is a basic z_c°-ideal.

Proposition 2.13. *The following statments are equivalent.*
(1) X is an almost CP-space .
(2) Every z_c-ideal in $C_c(X)$ is a z_c°-ideal
(3) Every maximal ideal (prime z_c-ideal) in $C_c(X)$ is a z_c°-ideal
(4) Every maximal ideal in $C_c(X)$ consists entirely of zero divisors.

Proposition 2.14. *Let X be an almost CP-space. The following statements are equivalent.*
(1) X is strongly zero-dimensional.
(2) Every z_c°-ideal in $C_c(X)$ is a contraction of a unique z°-ideal in $C(X)$.

Acknowledgement

We would like to thank the referee for his/her suggestions, for improving the quality of our exposition.

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Applying the Attention Mechanism on Deep Galerkin Method for Option Pricing Problems

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Abstract. Partial differential equations have many limitations as well as high performance in high dimensions. In this paper, we first use machine learning algorithms to transfer the high-dimensional challenge in partial differential equations to the machine learning challenge. Here, we employ the neural network architecture similar to the architecture for long-short term memory (LSTM) networks and highway networks. Finally, since the attention mechanism is one of the techniques that can be used to improve accuracy and specially speed, we apply it to improve the method.

Keywords: Deep Galerkin Method, Neural Networks, Attention Mechanism, LSTM Networks, Vanilla Options

AMS Mathematics Subject Classification [2010]: 65M99, 91G20, 91G80

1. Introduction

Partial differential equations (PDEs) have a significant role in modeling real-world problems. Some common and well-known examples are the Hamilton-Jacobi-Bellman equation in engineering, and the Black-Scholes equation in financial science for pricing financial derivatives. The finite difference method is one of the traditional and widely used methods to solve this type of equations. However, it will not meet the researchers needs in high dimensions due to the creation of the network and as the number of the network dimensions increases, the computational cost will grow exponentially. Many researchers were inspired by [1] and today Attention Mechanism is known as an effective method in deep learning and has been widely used in many fields such as Natural Language Processing (NLP) and sequence-to-sequence (Seq2Seq) models. Many financial institutions are interested in pricing options on portfolios with large number of stocks. Therefore, we study the Attention Mechanism Deep Galerkin method (AttDGM) on a class of partial differential equations with free boundaries in high dimensions.

2. Black-Scholes Model and AttDGM Algorithm

Suppose that u is an unknown function of time and space on $[0, T] \times \Omega$ where $x \in \Omega \subset \mathbb{R}^d$ and $\partial\Omega$ is the domain boundary of Ω and also suppose that u satisfies the following

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PDE:

$$
\begin{aligned}
&(1) \qquad \partial_t u(t,x) + \mathcal{L}u(t,x) = 0, && (t,x) \in [0,T] \times \Omega,\\
&(2) \qquad u(0,x) = u_0(x), && x \in \Omega,\\
&(3) \qquad u(t,x) = g(t,x), && x \in \partial\Omega.
\end{aligned}
$$

The goal is to approximate the solution of $u(t,x)$ by using the deep neural network $f(t,x;\theta)$ where $\theta \in \mathbb{R}^k$ is the neural network parameter. We know that the solution of $u(t,x)$ is unknown, but by calculating the L^2 error and minimizing it, we can find an approximation solution. This error consists of three main parts: 1) the differential operator error, 2) the boundary condition error, and 3) the initial / final condition error of the problem. The objective function is:

$$
\begin{aligned}
J(f) &= \|\frac{\partial f}{\partial t}(t,x;\theta) + \mathcal{L}f(t,x;\theta)\|^2_{[0,T]\times\Omega,\upsilon_1}\\
&+ \|f(t,x;\theta) - g(t,x)\|^2_{[0,T]\times\partial\Omega,\upsilon_2} + \|f(0,x,\theta) - u_0(x)\|^2_{\Omega,\upsilon_3}.
\end{aligned}
$$

Here $\|f(y)\|^2_{\mathcal{Y},\upsilon} = \int |f(y)|^2 \upsilon(y)\,dy$ where $\upsilon(y)$ is a positive probability density on $y \in \mathcal{Y}$. So, if this value tends more to zero, the solution of the equation will be more accurate. The algorithm is as follows:

(1) Initialize θ_0 and learning rate α_0.
(2) Generate random points (t_n, x_n) on $[0,T]\times\Omega$ with probability density υ_1, random points (τ_n, z_n) on $[0,T] \times \partial\Omega$ with probability density υ_2, and points w_n on Ω with probability density υ_3.
(3) Calculate the square error of $G(\theta_n, s_n)$ at random sampled points $s_n = \{(t_n, x_n), (\tau_n, z_n), w_n\}$ such that:

$$
\begin{aligned}
G(\theta_n, s_n) &= \left(\frac{\partial f}{\partial t}(t_n, x_n;\theta_n) + \mathcal{L}f(t_n, x_n;\theta_n)\right)^2\\
&+ \left(f(\tau_n, z_n;\theta_n) - g(\tau_n, z_n)\right)^2 + \left(f(0, w_n;\theta_n) - u_0(w_n)\right)^2
\end{aligned}
$$

(4) Take a descending step at the random point s_n:

$$\theta_{n+1} = \theta_n - \alpha_n \nabla_\theta G(\theta_n, s_n).$$

(5) Repeat steps 3 and 4 to achieve the desired result.

European call option. European option is a financial derivative on a stock portfolio. This type of option gives the holder the right, to buy or sell the asset only on the expiration date itself. Therefore, it can only be exercised by its holder at the expiration date. The PDE of a one-dimensional European call option is as follows:

$$
\begin{gathered}
\partial_t g(t,x) + rx.\partial_x g(t,x) + \tfrac{1}{2}\sigma^2 x^2.\partial_{xx} g(t,x) = r.g(t,x),\\
g(T,x) = G(x).
\end{gathered}
$$

Note that the problem has a exact solution in the following form:

$$
\begin{gathered}
g(t,x) = x\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-),\\
d_\pm = \frac{\ln(\frac{x}{K}) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.
\end{gathered}
$$

where x is the stock price, r is the risk-free interest rate, σ is the volatility, K is the strike price, T is the expiration date or, in other words, the maturity of the option and $g(x) : \mathbb{R}^d \to \mathbb{R}$ is the payoff function. Suppose that $X_t \in \mathbb{R}^d$ is the price of d stocks. If the stock price is $X_t = x$ at T, then the option price will be $u(T, x)$. The $u(T, x)$ price function satisfies the PDE with free boundary on $[0, T] \times \mathbb{R}^d$.

American option: An American option gives the holder the right, to buy or sell the option anytime, be it on or before the expiration date. Therefore, the holder is able to exercise the it more freely over time. The PDE of a one-dimensional American put option is as follows:

$$\partial_t g(t,x) + rx.\partial_x g(t,x) + \frac{1}{2}\sigma^2 x^2.\partial_{xx} g(t,x) = r.g(t,x) \qquad \{(t,x) : g(t,x) > G(x)\} \tag{4}$$

$$g(t,x) \geq G(x) \qquad (t,x) \in [0,T] \times \mathbb{R} \tag{5}$$

$$g(T,x) = G(X) \qquad x \in \mathbb{R} \tag{6}$$

where $G(x) = (K - x)_+$.

Solution: No analytical solution.

Theorem 2.1. ***(Neural Network Approximation Theorem for PDEs. [5])*** *Let the L^2 error $J(f)$ measure how well the neural network f satisfies the differential operator, boundary condition, and initial condition. Define ξ^n as the class of neural networks with n hidden units and let f^n be a neural network with n hidden units which minimizes $J(f)$. Also in [5] have proved that there exists $f^n \in \xi^n$ such that $J(f^n) \to 0$, as $n \to \infty$, and $f^n \to u$ as $n \to \infty$, in the appropriate sense, for a class of quasilinear parabolic PDEs with the principle term in divergence form under certain growth and smoothness assumptions on the nonlinear terms.*

3. Numerical results

We trained the network to learn the value of European call option and American put option. We used $S_0 = 0.05$, $K = 50$, $r = 0.05$, $\sigma = 0.25$ and $T = 1$. The value of the European call option and American put option is approximated by the analytical method based on the neural network in Figure 1.

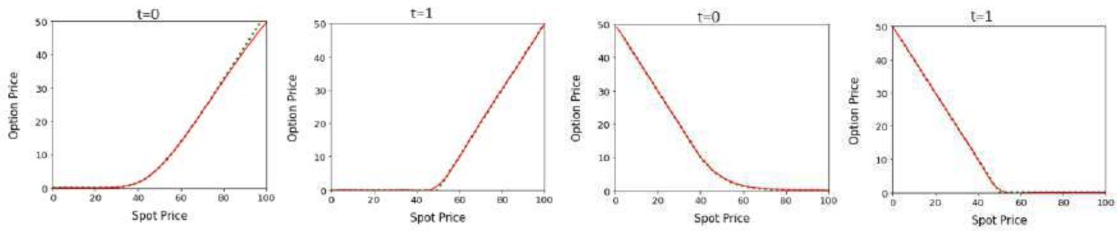

Figure 1. Left: European Call Option. Right: American Put Option. The deep learning solution is in red. The exact solution found via the Black-Scholes model for European Call Option and the semi-analytic solution found via Binomial model for American Put Option , are in green. Solutions are reported at times $t = 0$ and $t = 1$ and dimension $d = 1$.

3.1. System Environment. We used the TensorFlow library in Python to implement the algorithm. The system environment information is given in the left panel of Figure 3. In all the examples, we used the network architecture which is introduced in the right panel of Figure 3. The weights are initialized by using Xavier initialization method. Adam algorithm is also used to update the parameters. The network is trained for a number of iterations (Epochs) and this number can be different in each example. In this

Number of dimensions	Error
1	%1.4
3	%1.4
7	%1.5

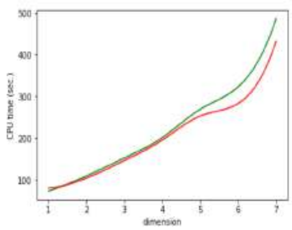

FIGURE 2. Left: The AttDGM algorithm solution is compared with the DGM solution for the Black-Scholes model. Right: The reported CPU time for the AttDGM algorithm and the DGM algorithm are in red and green, respectively.

paper, every 10 iterations are found appropriate by randomly resampling the points in the initial and boundary conditions and after many tests, we found $\alpha = 0.001$ as the best learning rate for the network training.

System	SDD 1TB NVMe PCIe 4.0
CPU	Ryzen 3900x
RAM	64GB
GPU	RTX 3090
Python Version	Python 3.6.9
Tensorflow Version	Tensorflow 2.5.0

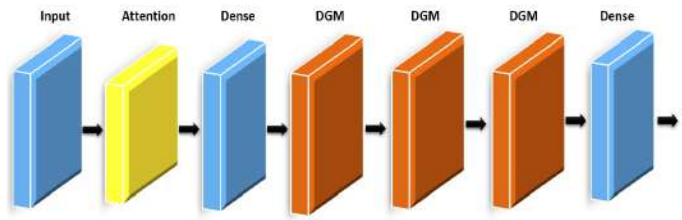

FIGURE 3. Left: System Environment. Right: Attention DGM network architecture.

4. Conclusion

So far, the deep learning approach has had a significant impact on solving high-dimensional PDE problems. In this paper, this approach was implemented for European and American options and compared with Deep Galerkin method. The results showed that algorithm was improved.

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Some combinatorial properties of the derivative operator

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Abstract. Studying expressions of the form $(f(x)D)^p$, where $D = \dfrac{d}{dx}$ is the derivation operator, goes back to Scherk's Ph.D. thesis in 1823. Some new problems in quantum physics motivated physicists to publish many papers in this area based on combinatorial methods. This has led some mathematicians and computer scientists to continue such studies. In this manuscript, we discuss about expansion of $(f(x)D)^p$ and related coefficients. Particularly we discuss about the values and the combinatorial meaning and the values of these coefficients.

Keywords: derivative operator, expansion, increasing trees

AMS Mathematics Subject Classification [2010]: 05B20, 05E30

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1. Introduction

The derivative operator $\frac{d}{dx}$ (or briefly D) plays an important role in the theory of formal power series. There are some results about iterating this operator and related ones on expressions. The most famous results among these identities are Liebniz and Faà di Bruno formulas: While the first identity is a generalization of the "product rule", the second one is an extension of the "chain rule". The coefficients appearing in both these results have combinatorial interpretations.

There are less-famous but much-extensively studied results about expressions of the form $(f(x)D)^p$. Studying such expressions goes back to Scherk's Ph.D. thesis in 1823 [9]. Motivated by the normal ordering problem, these studies have been extremely emerged by some quantum physicists in the three last decades (See [2–7] and the references there in). They have extensively used combinatorial objects in their studies. These works have been reviewed and continued by some mathematicians and computer scientists such as Philip Flajolet [4], Toufik Mansour [6] and others. Several combinatorial objects are considered and used in these studies, among which, the "increasing trees" are used in this work.

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2. Notation and Definitions

Notation 1. The set of integers (resp. nonnegative integers) is denoted by $\mathbb{Z}$ (resp. $\mathbb{N}$). For integers m and n we denote the set $\{x \in \mathbb{Z} : m \leq x \leq n\}$ by $[m, n]$. We denote the set of infinite row vectors of nonnegative integers by $\mathbb{N}^\infty$, so each element $\mathbf{a} \in \mathbb{N}^\infty$ is represented as $\mathbf{a} = (\mathbf{a}(0), \mathbf{a}(1), \cdots, \mathbf{a}(p-1), \cdots)$. The vectors $\mathbf{j}, \mathbf{e_m}, \mathbf{n} \in \mathbb{N}^\infty$ are defined respectively by $\mathbf{j}(i) = 1, \mathbf{e_m}(i) = \delta_{m,i}$ and $\mathbf{n}(i) = i$ for any integer $i \geq 0$. The value of $f^{\mathbf{a}}$, for a vector $\mathbf{a} \in \mathbb{N}^\infty$ with finitely many nonzero components, is defined as

$$f^{\mathbf{a}} = \prod_i (f^{(i)})^{\mathbf{a}(i)},$$

where $f^{(0)} = f$ and for $j \geq 1$, we have $f^{(j)} = D^j f$. Also we define the set Λ_p by

$$\Lambda_p = \{\mathbf{a} \in \mathbb{N}_p^\infty : \mathbf{a}.\mathbf{j}^\top = p - 1, \ \mathbf{a}.\mathbf{n}^\top < p\}.$$

It is obvious that any vector $\mathbf{a} \in \Lambda_p$ has only finitely many nonzero components; In fact, from $\mathbf{a} \in \Lambda_p$, one concludes that $\mathbf{a}(i) = 0$ for each $i \geq p$.

Definition 1. Let V be a finite ordered set with $v_0 = \min V$ (for instance, V can be considered as a finite set of integers). An increasing tree on V, is a tree T rooted at v_0 with $V(T) = V$, such that for any $v \in V$, the vertices in the unique $v_0 - v$ path P in T, appear increasingly. A *starlike increasing tree* is an increasing tree,in which, any vertex (except possibly the root) has at most one child. The increasing trees are widely studied in the literature (See Section 1.3 of [10]; for more information see [1]). For a vertex v of an increasing tree, we denote the number of its children by $d'(v)$.

3. An expansion of $(f(x)D)^p$

After testing some small cases, one can guess that $(fD)^p$ is expressed in the following form

$$(fD)^p = \sum_{\sum_i a_i = p-1, \sum_i ia_i < p} \gamma_{p;a_0,a_1,\cdots,a_{p-1}} (f^{(0)})^{a_0+1} (f^{(1)})^{a_1} \cdots (f^{(p-1)})^{a_{p-1}} D^{p-\sum_i ia_i}, \tag{1}$$

where the constants $\gamma_{p;a_0,a_1,\cdots,a_{p-1}}$ are nonnegative integers. We rewrite this formula in terms of the notation of the previous section in Theorem 1. Furthermore, we give a combinatorial description of the coefficients.

Theorem 1.. *Let p be a positive integer.*

(i) *We have*

$$(fD)^p = \sum_{\mathbf{a}} \gamma_{p;\mathbf{a}} f^{\mathbf{a}+\mathbf{e_0}} D^{p-\mathbf{a}.\mathbf{n}^\top} \tag{2}$$

where the summation runs over the elements $\mathbf{a} \in \Lambda_p$. Equivalently, one can say that $\mathbf{a}$ runs over $\mathbb{N}^\infty$ but $\gamma_{p;\mathbf{a}} \neq 0$ only if $\mathbf{a} \in \Lambda_p$.

(ii) *the value of $\gamma_{p;\mathbf{a}}$ equals the number of increasing trees on $\{0, 1 \cdots, p\}$ in which*
   (a) The number of the leaves is $\mathbf{a}(0) + 1$.
   (b) The number of the nodes which have exactly i children is $\mathbf{a}(i)$ for $i = 1, \cdots, p$.

Proof. See Theorem 1 and Proposition 11 of [8]. □

Now we are interested to find a values $\gamma_{p;\mathbf{a}}$ based on Part (ii) of the above theorem. It is notable that the number of all increasing trees on $[0, p]$ is $p!$ [10]. But here we should count a special subset of these trees which are described by conditions (a) and (b)

mentioned above. Recall that the number of children of a vertex v in an increasing tree is denoted as $d'(v)$. The next step is finding the answer of this question: Given a sequence $\ell_1, \ldots, \ell_{p-1}$ of nonnegative integers, do these exist increasing trees on $V = [0, p]$ satisfying $d'(i) = \ell_i$ for $i = 1, \cdots, p-1$? (Note that since $d'(p) = 0$ and $d'(0) = p - \sum_{i=1}^{p-1} d'(i)$, these values are excluded from the sequence ℓ_i.) Both existence and enumeration questions are answered in Theorem 2 (resp. in Part

(i) and Part (ii)). Before stating that theorem, we introduce some notation.

Notation 2. Let p be a positive integer and let $\ell_1, \ldots, \ell_{p-1}$ be a given sequence of nonnegative integers. Then the value $g(\ell_1, \cdots, \ell_{p-1})$ is defined as

$$g(\ell_1, \cdots, \ell_{p-1}) = (2 - \ell_{p-1})_*(3 - \ell_{p-1} - \ell_{p-2})_* \cdots (p - 1 - \sum_{i=2}^{p-1} \ell_i)_* \tag{3}$$

where for a real number x, the value of $(x)_*$ is defined to be x if $x > 0$ and 0 otherwise.

Theorem 2.. *[[8], Proposition 10] Let $\ell_1, \ell_2, \cdots, \ell_{p-1}$ be a sequence of nonnegative integers and let $V = [0, p]$. Then*

(i) *There exists an increasing tree T on $V = [0, p]$ with $d'_T(v) = \ell_v$ for $v = 1, \cdots, p-1$ if and only if $\sum_{i=j}^{p-1} \ell_i \leq p - j$ for $j = 1, \cdots, p-1$.*

(ii) *The number of increasing trees mentioned in part (i) is obtained as $\frac{g(\ell_1, \cdots, \ell_{p-1})}{\ell_1! \cdots \ell_{p-1}!}$ where $g(\ell_1, \cdots, \ell_{p-1})$ is as given in (3).*

The following theorem gives a nonrecursive formula to compute the coefficient $\gamma_{p;\mathbf{a}}$.

Theorem 3.. *[[8], Theorem 14] The coefficient $\gamma_{p;\mathbf{a}}$ can be computed as*

$$\gamma_{p;\mathbf{a}} = \frac{1}{(0!)^{\mathbf{a}(0)}(1!)^{\mathbf{a}(1)} \ldots ((p-1)!)^{\mathbf{a}(p-1)}} \sum g(\ell_1, \ell_2, \ldots, \ell_{p-1}),$$

where the summation runs over all $(p-1)$-tuple $(\ell_1, \ell_2, \ldots, \ell_{p-1})$ of integers satisfying

$$\{\ell_1, \ell_2, \ldots, \ell_{p-1}\} = \{\mathbf{a}(0).0, \mathbf{a}(1).1, \ldots, \mathbf{a}(p-1).(p-1)\}$$

which means that the number of i's appearing in the sequence $\{\ell_i\}_{1 \leq i \leq p-1}$ is $\mathbf{a}(i)$ for $i = 0, \cdots, p-1$.

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b-Birkhoff Orthogonal Elements In 2-Normed Linear Spaces

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Abstract. In this paper, we define the notion of a bilinear 2-operator on the cartesian product of two subspaces of a 2-normed spaces and obtain some Corollary about it. Also, we discuss the relationships between 2-functionals and the existence of b-Birkhoff orthogonal elements in 2-normed linear spaces. Moreover, we obtain some characterizations of 2-inner product spaces by b-Birkhoff orthogonality.

Keywords: b-Birkhoff orthogonal, 2-functionals, 2-hyperplane, 2-inner product, 2-normed linear spaces.

AMS Mathematics Subject Classification [2010]: 46C05.

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1. Introduction

The concept of 2-normed linear spaces has been investigated by S. Gähler in 1960's [2] and has been developed extensively in different subjects by many authors. Let X be a linear space of dimension greater than 1. Suppose $\|.,.\|$ is a real-valued function on $X \times X$ satisfying the following conditions:

(1) $\|x, y\| = 0$, if and only if x and y are linearly dependent vectors,
(2) $\|x, y\| = \|y, x\|$, for all $x, y \in X$,
(3) $\|\lambda x, y\| = |\lambda| \|x, y\|$, for all $\lambda \in \mathbb{R}$ and all $x, y \in X$,
(4) $\|x + y, z\| \leqslant \|x, z\| + \|y, z\|$, for all $x, y, z \in X$.

Then $\|.,.\|$ is called a 2-norm on X and $(X, \|.,.\|)$ is called a 2-normed linear space. A 2-norm is non-negative and the basic property of a 2-norm is $\|x, y + \alpha x\| = \|x, y\|$, for all $x, y \in X$ and all $\alpha \in \mathbb{R}$. Note that $(X, \|.,.\|)$ with the formula $\|x, y\| = \|x\|\|y\|$, for each $x, y \in X$, is not a 2-normed space. So the relationship $\|x, y + \alpha x\| = \|x, y\|$, is not valid. for example, let $x \neq 0$ and $\alpha \neq 0$. Then

$$0 = \|x, 0\| = \|x, 0 + \alpha x\| = \|x, \alpha x\| = \|x\|\|\alpha x\| = ||\alpha|\|x\|^2 > 0.$$

Example 1.1. Let $X = E^3$ be an Euclidean 3-dimensional linear space. The formula $\|x, y\| = |x \times y|$ defines a 2-norm on X, where x, y are two vector in E^3 and $x \times y$ means the vector product of x and y.

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As an example of a 2-normed space, take $X = \Re^2$, equipped with $\|x, y\|$ which is defined as the erea of the parallelogram spanned by the vectors x, y (i.e. the parallelogram whose adjacent sides are the vectors a and b) which may be given explicitly by the formula $\|x, y\| =| x_1y_2 - x_2y_1 |$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$.
Along with the 2-norm, we have the standard 2-inner product space. Let X be a real vector space of dimension≥ 2. The real-valued function $\langle ., .|.\rangle : X \times X \times X \to \mathbb{R}$, which satisfies the following properties on X^3 is called 2-inner product on X:

(1) $\langle x, x|z\rangle \geqslant 0$ for every $x, z \in X$ and $\langle x, x|z\rangle = 0$ if and only if x and z are linearly dependent,
(2) $\langle x, y|z\rangle = \langle y, x|z\rangle$ for every $x, y, z \in X$,
(3) $\langle x, x|z\rangle = \langle z, z|x\rangle$ for every $x, z \in X$,
(4) $\langle \alpha x, y|z\rangle = \alpha\langle x, y|z\rangle$ for every $x, y, z \in X$ and $\alpha \in \mathbb{R}$,
(5) $\langle x_1 + x_2, y|z\rangle = \langle x_1, y|z\rangle + \langle x_2, y|z\rangle$ for every $x_1, x_2, y, z \in X$.

Under these conditions, the pair $(X, \langle ., .|.\rangle)$ is called an inner product space [1]. Also, we observe that $\|x, y\| = \langle x, x|y\rangle^{1/2}$ and the Cauchy-Schwarz inequality $\langle x, y|z\rangle^2 \leqslant \|x, z\|^2\|y, z\|^2$ for every $x, y, z \in X$ is valid.
In [4], Khan and Siddiqui defined the notion of P, I, and BJ-orthogonality in 2-normed spaces $(X, \|., .\|)$ as follows:

P-orthogonality: $x \perp_P y$ if only if $\|x + y, z\|^2 = \|x, z\|^2 + \|y, z\|^2$ for every z.

I-orthogonality: $x \perp_I y$ if only if $\|x + y, z\| = \|x - y, z\|$ for every $z \neq 0$.

BJ-orthogonality: $x \perp_{BJ} y$ if only if $\|x+\alpha y, z\| \geqslant \|x, z\|$ for every $z \neq 0$ and $\alpha \in \mathbb{R}$.

Also we have the followin definition in [6].

Definition 1.2. Let $(X, \|., .\|)$ be a 2-normed space and $x, y \in X$. If there exists $b \in X$ such that $\|x, b\| = 0$ and $\|x, b\| \geqslant \|x + \alpha y, b\|$ for each scalar $\alpha \in \Re$, then x is b-orthogonal to y (denoted by $x \perp_b y$).

Now, let $(X, \|., .\|)$ be a 2-normed space and W_1 and W_2 be two subspaces of X. A map $f : W_1 \times W_2 \to \mathbb{R}$ is called a bilinear 2-functional ([6]) on $W_1 \times W_2$ whenever for all $x_1, x_2 \in W_1, y_1, y_2 \in W_2$ and all $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

(1) $f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_2, y_2) + f(x_2, y_1) + f(x_2, y_2)$,
(2) $f(\lambda_1x_1, \lambda_2y_1) = \lambda_1\lambda_2 f(x_1, y_1)$.

A bilinear 2-functional $f : W_1 \times W_2 \to \mathbb{R}$ is called bounded if there exists a non-negative real number M (M is called a Lipschitz constant for f) such that $|f(x, y)| \leqslant M\|x, y\|$ for all $x \in W_1$ and all $y \in W_2$. Also, the norm of a bilinear 2-functional is defined by

$$\|f\| = \inf\{M \geqslant 0 : M \textit{is a Lipschitz constant for} f\}.$$

For example, Let $(E^3, \|, \|)$ be the 2-normed space with $\|x, y\| = |x \times y|$. Define $f(x, y) = x \cdot y$, where $x \cdot y$ is the dot product of vector analysis. Then f is an unbounded linear 2-functional. Now, define $f(x, y) = (|x|^2|y|^2 - |(x.y)|^2)^{\frac{1}{2}}$, where $|a|$ denotes the length of a. Since $|x|^2|y|^2 - |(x.y)|^2 = |x \times y|^2$ so, f is a bounded 2-functional.
For a 2-normed space $(X, \|., .\|)$ and $0 \neq b \in X$, we denote by X_b^* the Banach space of all bounded bilinear 2-functionals on $X \times \langle b\rangle$, where $\langle b\rangle$ is the subspace of X generated by b ([5]).

2. Existence of b-Birkhoff orthogonal elements

Let X be a 2-normed linear space. Also, let $0 \neq b \in X$ and $0 \neq f$ be a nonzero bilinear 2-functional on $X \times \langle b \rangle$. Then we define the 2-hyperplane H through the origin by $H = \{x \in X; f(x,b) = 0\}$.

Theorem 2.1. *Under the above conditions,* $|f(x,b)| = \|f\|\|x,b\|$ *if and only if* $x \perp_b H$, *where* H *is a 2-hyperplane of all* h *for which* $f(h,b) = 0$.

Example 2.2. Let $X = (E^3, \|,\|)$ be the 2-normed space with $\|x,y\| = |x \times y|$. Suppose $b = (1,0,0)$ and define $f : X \times < b > \to \mathbb{R}$ with $f(x,y) = |x \times y|$, where $x \in X$ and $y \in < b >$. So $\|f\| = 1$ and for each $x \in X$, we have $|f(x,b)| = \|f\|\|x,b\|$. On the other hand, the 2-hyperplane H through the origin is as follows:

$$H = \{x \in X; f(x,b) = 0\} = \{x \in X; |x \times b| = 0\} = \{x \in X; x = (a,0,0), \forall a \in \mathbb{R}\}.$$

Now, for each $\alpha \in \mathbb{R}$, $(x,y,z) \in X$ and $h = (a,0,0) \in H$, we have

$$\|x + \alpha h, b\| = \|(x + \alpha a, y, z), (1,0,0)\| = \sqrt{z^2 + y^2} = \|x,b\|.$$

That means $x \perp_b H$.

Now, let X be a 2-normed linear space. For $X_0 \subseteq X$, put

$$M^b_{X_0} = \{f \in X^*_b; \quad \|f\| = 1, f(x,b) = \|x,b\|, \forall x \in X_0\}.$$

One can find the proof of the following theorem in [6].

Theorem 2.3. *Let* X *be a 2-normed linear space,* $b \in X$, $y \in X$ *and* $x \in X \setminus \langle b \rangle$. *Then* $x \perp_b y$ *if and only if there exists* $f \in M^b_x$ *such that* $f(y,b) = 0$.

Example 2.4. Let$X = \mathbb{R}^3$, $W = \{(0,x,x), x \in \mathbb{R}\}$ and

$$\|(x_1,x_2,x_3),(y_1,y_2,y_3)\| = max\{|x_1y_2 - x_2y_1| + |x_1y_3 - x_3y_1|, |x_1y_2 - x_2y_1| + |x_2y_3 - x_3y_2|\}$$

for all $(x_1,x_2,x_3),(y_1,y_2,y_2) \in X$. Then $\|\cdot,\cdot\|$ is a 2-norm on X. If $x = (1,0,1)$ and $b = (2,2,0)$, it is clear that $x \perp_b W$.

Theorem 2.5. *Let* X *be a 2-normed linear space and* $(0 \neq)b \in X$. *Then there exist an element b-orthogonal to each closed 2-linear subset of* X *if and only if for each bilinear 2-functional* f *defined on* $X \times \langle b \rangle$, *there is an element* x *with* $f(x,b) = \|f\|\|x,b\|$.

3. Characterization of 2-Inner Product Spaces by b-Birkhoff Orthogonality

Definition 3.1. let $(X, \|.,.\|)$, $(Y, \|.,.\|)$ be two 2-normed spaces, and W_1 and W_2 be two subspaces of X. A map $T : W_1 \times W_2 \longrightarrow Y$ is called a bilinear 2-operator on $W_1 \times W_2$ whenever for all $x_1, x_2 \in W_1$ and $y_1, y_2 \in W_2$ and all $\lambda_1, \lambda_2 \in \mathbb{R}$,

i) $T(x_1 + x_2, y_1 + y_2) = T(x_1,y_1) + T(x_1,y_2) + T(x_2,y_1) + T(x_2,y_2)$,
ii) $T(\lambda_1 x_1, \lambda_2 y_2) = \lambda_1 \lambda_2 T(x_1,y_1)$.

Note thar if $Y = \mathbb{R}$, then T is called a bilinear 2-functional. Also, a bilinear 2-operator T is called a 2-projection if $T^2 = T$.

Theorem 3.2. *let* X *be a 2-normed linear space and* $0 \neq b \in X$. *For any* $x,y \in X$, *there exists a number* a *such that* $ax + y \perp_b x$. *This number* a *is a value of* k *for which* $\|kx + y, b\|$ *takes on its absolute minimum.*

Proof. By Definition 1.2, $ax + y \perp_b x$ if and only if $\|(ax + y) + kx, b\| \geqslant \|ax + y, b\|$, for each k, or if and only if $\|ax+y, b\|$ is the smallest value of $\|kx+y, b\|$. Since $\|kx+y, b\|$ is continuous in k, it must take on its minimum. □

Theorem 3.3. *Let X be a 2-normed space and $0 \neq b \in X$. If $\dim X \geqslant 3$, then b-orthogonality is symmetric if and only if a 2-inner product can be defined in X.*

Proof. Suppose that $\dim X_0 = 3$ where X_0 is a subspace of X. Also, let x_1 and x_2 be any two elements of $X_0 \setminus (\langle b \rangle)$ and H_0 be the linear hull of x_1 and x_2. By Theorem 2.5 and Theorem 2.1, there is an element $y \in X_0$ that is b-orthogonal to H_0. Conversely, suppose that b-orthogonality is symmetric. Then $H_0 \perp_b y$ and by Theorem 3.2, there is a number a_z such that we can define $P : X_0 \times \langle b \rangle \longrightarrow H_0 \times \langle b \rangle$ by $P(z, b) = (z - a_z y, b)$ for each $z \in X_0$. So P is a bilinear 2-operator. Also, since H_0 is the linear hull of x_1 and x_2 and $H_0 \perp_b y$, we have

$$\|P(z, b)\| = \|z - a_z y, b\| \leqslant \|z, b\| \qquad \forall z \in X_0.$$

Thus, $\|P\| = 1$. In addition, since $P(a_z y, b) = 0$ for each $z \in X_0$, we have

$$P^2(z, b) = P(P(z, b)) = P(z - a_z y, b) = P(z, b) - P(a_z y, b) = P(z, b).$$

Therefore, P is a 2-projection of $X_0 \times \langle b \rangle$ on $H_0 \times \langle b \rangle$ with $\|P\| = 1$. Now, according to the points stated before this theorem, a 2-inner product can be defined in a 2-normed linear space of three or more dimensions if there is a 2-projection of norm 1 on any given closed linear subspace. Thus a 2-inner product can be defined in any three-dimensional subspace of X and hence in X itself. □

Corollary 3.4. *Let x and y in a 2-normed space X with $\dim X \geqslant 3$, and $0 \neq b \in X$. If there exists a nonzero bilinear 2-functional f with $f(x, b) = \|f\|\|x, y\|$ and $f(y, b) = 0$, then there exists a nonzero bilinear 2-functional g such that $g(y, b) = \|g\|\|y, b\|$ and $g(x, b) = 0$.*

Corollary 3.5. *Let X be a 2-normed space and $0 \neq b \in X$, and $x, y \in X$. If f is a bilinear 2-functional such that $f(x, b) = \|f\|\|x, b\|$, then $\|ax + y, b\|$ is minimum when $a = -\dfrac{f(y, b)}{f(x, b)}$.*

Acknowledgement

The author would like to thank the Ahvaz Branch, Islamic Azad University, Research Council for their financial support.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

An alternative perspective on FC-pure injectivity of modules

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Abstract. Given modules M and A, M is called A-FC-pure-subinjective if for every FC-pure extension B of A, each homomorphism from A to M can be extended to a homomorphism from B to M. The FC-pure-subinjectivity domain of M is defined to be the collection of all modules A such that M is A-FC-pure-subinjective. Basic properties of FC-pure-subinjectivity domains are investigated. In particular, we obtain characterizations for various types of rings and modules, including f-injective modules, von Neumann regular rings, Köthe rings, semisimple rings, and right Noetherian rings in terms of FC-pure-subinjectivity domains.

Keywords: FC-pure injective module; FC-pure-subinjectivity domain; Köthe ring

AMS Mathematics Subject Classification [2010]: 16D80; 16D10; 16D50

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1. Introduction

In [1], Aydoğdu and López-Permouth studied the notion of subinjectivity. Namely, a module M is called *A-subinjective* if for every extension B of A, every homomorphism from A to M can be extended to a homomorphism from B to M. For a module M, the *subinjectivity domain* of M, $\underline{\mathfrak{I}}^{-1}(M)$, is defined to be the collection of all modules A such that M is A-subinjective. In contrast to the notion of pure-injectivity, López-Permouth et al. studied in [5] the notion of pure-subinjectivity. Namely, a module M is called *A-pure-subinjective* if for every pure extension B of A, every homomorphism from A to M can be extended to a homomorphism from B to M. For a module M, the *pure-subinjectivity domain* of M, $\underline{\mathfrak{PI}}^{-1}(M)$, is defined to be the collection of all modules A such that M is A-pure-subinjective. Clearly, the subinjectivity domain $\underline{\mathfrak{I}}^{-1}(M)$ of a module M is contained in $\underline{\mathfrak{PI}}^{-1}(M)$.

The concept of FC-pure submodule, with the related notions of FC-pure projective and FC-pure injective module, was introduced by Puninski, use the phrase W-purity (Warfield purity) to mean FC-purity. The goal of this paper is to initiate the study of an alternative perspective on the analysis of the FC-pure injectivity of a module. In contrast to the well-known notion of FC-pure injectivity, we introduce the notion of FC-pure-subinjectivity. Namely, a module M is said to be *A-FC-pure-subinjective* if for every FC-pure extension B of A, every homomorphism from A to M can be extended to a homomorphism from

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B to M. For every module M, the *FC-pure-subinjectivity domain* of M consists those modules A such that M is A-FC-pure-subinjective.

Throughout this paper, R denotes an associative ring with identity and all modules will be assumed to be unitary. In what follows $\mathrm{E}(M)$, $\mathrm{PE}(M)$ and $\mathrm{FCE}(M)$ denote the injective hull, the pure injective hull and the FC-pure injective hull of a module M, respectively. A cyclic right R-module $M_R \cong R/I$ is called *finitely presented cyclic* if I is a finitely generated right ideal of R.

2. Main Results

Recall that an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of right R-modules is said to be *pure exact* (resp., *FC-pure exact*) if the induced homomorphism $\mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, C)$ is surjective for any finitely presented (resp., finitely presented cyclic) right R-module M. A submodule A of a right R-module B is called a *pure submodule* (resp., *FC-pure submodule*) if the exact sequence $0 \longrightarrow A \hookrightarrow B \longrightarrow B/A \longrightarrow 0$ is pure (resp., FC-pure). An R-module M is said to be *pure-injective* (resp., *FC-pure injective*) if it is injective with respect to pure exact (resp., FC-pure exact) sequences. Also, an R-module M is said to be *pure-projective* (resp., *FC-pure projective*) if it is projective with respect to pure exact (resp., FC-pure exact) sequences (see [4], [2] and [6]).

Definition 2.1. Given modules M and A, we say that M is *A-FC-pure-subinjective* if for every FC-pure extension B of A, every homomorphism from $\varphi : A \to M$ there exists a homomorphism $\phi : B \to M$ such that $\phi|_A = \varphi$. The *FC-pure-subinjectivity domain* of a module M, $\underline{\mathfrak{FCI}}^{-1}(M)$, is defined to be the collection of all modules A such that M is A-FC-pure-subinjective.

Proposition 2.2. *For any right R-modules M and A, the following are equivalent:*

(1) $A \in \underline{\mathfrak{FCI}}^{-1}(M)$.
(2) *Every homomorphism from A to M can be extended to a homomorphism from $FCE(A)$ to M.*
(3) *There exists an FC-pure injective extension B of A such that every homomorphism from A to M can be extended to a homomorphism from B to M.*

Applying Proposition 2.2 to the identity $M \to M$, one sees that a module M is M-FC-pure-subinjective if and only if it is FC-pure injective. Thus, we have:

Corollary 2.3. *For any right R-module M, M is FC-pure injective if and only if $\underline{\mathfrak{FCI}}^{-1}(M)$= Mod-$R$, if and only if, $M \in \underline{\mathfrak{FCI}}^{-1}(M)$.*

Since every pure exact is FC-pure exact, for a module M, we have the following relationships: $\underline{\mathfrak{I}}^{-1}(M) \subset \underline{\mathfrak{FCI}}^{-1}(M) \subset \underline{\mathfrak{PI}}^{-1}(M)$.

Recall that a ring R is called *right pure-semisimple* (resp., *right Köthe*) if every right R-module is a direct sum of finitely generated (resp., cyclic) right R-modules. The following example shows that the above relationships need not be equal (see [3]).

Example 2.4. (a) Assume that R is a pure-semisimple ring that is not Köthe. Thus, every R-module is pure-injective and so by [4, Theorem 3.10], there exists a right (left) R-module M that it is pure-injective and is not FC-pure injective. So, by Corollary 2.3, $M \in \underline{\mathfrak{PI}}^{-1}(M) \setminus \underline{\mathfrak{FCI}}^{-1}(M)$.
(b) Assume that $M := \mathbb{Z}_2$. Thus M is a FC-pure injective $\mathbb{Z}$-module and so $M \in \underline{\mathfrak{FCI}}^{-1}(M)$ by Corollary 2.3. But M is not an injective $\mathbb{Z}$-module and so $M \notin \underline{\mathfrak{I}}^{-1}(M)$.

Recall that for a module M, its *injectivity domain*, denoted by $\mathfrak{I}^{-1}(M)$, consists of all modules N such that M is injective relative to N (or N-injective). Also, the *FC-pure injectivity domain* (resp., *pure-injectivity domain*) of a module M, denoted by $\mathfrak{FCI}^{-1}(M)$ (resp., $\mathfrak{PI}^{-1}(M)$), consists of those modules N such that M is N-FC-pure-injective (resp., N-pure-injective). So, we have $\mathfrak{I}^{-1}(M) \subset \mathfrak{FCI}^{-1}(M) \subset \mathfrak{PI}^{-1}(M)$.

Proposition 2.5. *A ring R is von Neumann regular if and only if for every R-module M, $\underline{\mathfrak{FCI}}^{-1}(M) \subseteq \underline{\mathfrak{I}}^{-1}(M)$, if and only if, for every R-module M, $\mathfrak{FCI}^{-1}(M) \subseteq \mathfrak{I}^{-1}(M)$.*

Recall that a right R-module A is *f-injective* if $\mathrm{Ext}^1_R(R/I, A) = 0$ for all finitely generated right ideal I of R. In the next theorem, we see that A-FC-pure-subinjectivity and A-subinjectivity coincide for an f-injective module A. Moreover, this condition is a characterization of A being a f-injective module.

Theorem 2.6. *The following statements are equivalent for any right R-module A:*

(1) *A is an f-injective module.*
(2) *Every module M_R is A-FC-pure-subinjective if and only if it is A-subinjective.*
(3) *FCE(A) is A-subinjective.*

Proof. (1) $\Rightarrow$ (2) Assume that M is an A-FC-pure-subinjective right R-module and $f : A \to M$ homomorphism. Since A is f-injective, [4, Proposition 2.15], A is an FC-pure submodule of $\mathrm{E}(A)$. Thus, by hypothesis, the homomorphism f can be extended to a homomorphism from $\mathrm{E}(A)$ to M. Therefore, by [1, Lemma 2], M is A-subinjective. The converse is clear.

(2) $\Rightarrow$ (3) By Corollary 2.3, $\underline{\mathfrak{FCI}}^{-1}(\mathrm{FCE}(A))=$ Mod-R. Hence, FCE(A) is A-FC-pure-subinjective and so by hypothesis, FCE(A) is also A-subinjective.

(3) $\Rightarrow$ (1) Assume that FCE(A) is A-subinjective and $i_1 : A \hookrightarrow \mathrm{E}(A)$ and $i_2 : A \hookrightarrow \mathrm{FCE}(A)$ are the inclusion maps. By hypothesis, there exists a homomorphism $\varphi : \mathrm{E}(A) \to \mathrm{FCE}(A)$ such that $\varphi i_1 = i_2$. Since i_1 is an essential monomorphism and φi_1 is a monomorphism, φ is also a monomorphism. Also, $A = \varphi(A) \subseteq \varphi(\mathrm{E}(A)) \subseteq \mathrm{FCE}(A)$. It follows that $\varphi(A)$ is an FC-pure submodule of $\varphi(\mathrm{E}(A))$. So, since φ is a monomorphism, A is also an FC-pure submodule of $\mathrm{E}(A)$. Therefore, by similar to the proof of [2, Proposition 2.1], we can obtain that A is an f-injective module, as required. ☐

Proposition 2.7. *A right R-module M is FC-pure injective if and only if $\underline{\mathfrak{FCI}}^{-1}(M)$ is closed under FC-pure submodules, if and only if, $\underline{\mathfrak{FCI}}^{-1}(M) \subseteq \mathfrak{FCI}^{-1}(M)$.*

Theorem 2.8. *For a ring R, the following statements are equivalent:*

(1) *R is a Köthe ring.*
(2) *For every right and left R-module M, $\underline{\mathfrak{FCI}}^{-1}(M) \subseteq \mathfrak{FCI}^{-1}(M)$.*
(3) *For every right and left R-module M, $\underline{\mathfrak{FCI}}^{-1}(M) = \mathfrak{PI}^{-1}(M)$.*
(4) *For every right and left R-module M, $\underline{\mathfrak{PI}}^{-1}(M) \subseteq \mathfrak{FCI}^{-1}(M)$.*

Proof. (1) $\Leftrightarrow$ (2) follows from Proposition 2.7 and [4, Theorem 3.10].

(1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) Assume that R is a Köthe ring. Thus, by [4, Theorem 3.10], every right and left R-module is FC-pure injective. It follows that $\underline{\mathfrak{FCI}}^{-1}(M) = \mathrm{Mod} - R = \mathfrak{PI}^{-1}(M)$ and $\underline{\mathfrak{FCI}}^{-1}(M) = \mathrm{Mod} - R = \underline{\mathfrak{PI}}^{-1}(M)$ since always every pure exact is FC-pure exact.

(3) $\Rightarrow$ (1) Assume that K is a (right or left) R-module. Since PE(K) is pure-injective, $\mathrm{Mod}{-}R = \mathfrak{PI}^{-1}(\mathrm{PE}(K))$ by [5, Theorem 2.3]. Thus, by hypothesis, $\mathrm{Mod}{-}R = \underline{\mathfrak{FCI}}^{-1}(\mathrm{PE}(K))$ and so PE(K) is FC-pure injective. So, $\mathrm{PE}(K) \in \underline{\mathfrak{FCI}}^{-1}(K) \subseteq \mathfrak{PI}^{-1}(K)$.

Thus, K is PE(K)-pure-injective and so [5, Theorem 2.2] implies that $K \in \underline{\mathfrak{PI}}^{-1}(K)$. Hence, by [5, Theorem 2.3], K is pure injective, i.e., $\mathfrak{PI}^{-1}(K) = \mathrm{Mod}-R$. So, by hypothesis, $\underline{\mathfrak{FCI}}^{-1}(K) = \mathrm{Mod}-R$ and so by Corollary 2.3, K is FC-pure injective. Since every right and left R-module is FC-pure injective, R is a Köthe ring by [4, Theorem 3.10].

(4) $\Rightarrow$ (1) Assume that K is a (right or left) R-module. Since every FC-pure injective module is pure-injective, $\mathrm{FCE}(K) \in \underline{\mathfrak{PI}}^{-1}(K)$. Thus, by hypothesis, $\mathrm{FCE}(K) \in \mathfrak{FCI}^{-1}(K)$, i.e., K is FCE(K)-FC-pure-injective and so Proposition 2.2 implies that $K \in \underline{\mathfrak{FCI}}^{-1}(K)$. Hence, by Corollary 2.3, K is FC-pure injective. Since every right and left R-module is FC-pure injective, R is a Köthe ring by [4, Theorem 3.10]. □

Recall that a right R-module M is called *absolutely pure* if $\mathrm{Ext}^1_R(N, M) = 0$ for every finitely presented right R-module N.

PROPOSITION 2.9. *A right R-module M is injective if and only if $\underline{\mathfrak{FCI}}^{-1}(M) \subseteq \mathfrak{I}^{-1}(M)$, if and only if, M is absolutely pure and $\underline{\mathfrak{FCI}}^{-1}(M) \subseteq \mathfrak{PI}^{-1}(M)$, if and only if, M is f-injective and $\underline{\mathfrak{PI}}^{-1}(M) \subseteq \underline{\mathfrak{FCI}}^{-1}(M)$.*

COROLLARY 2.10. *A ring R is semisimple if and only if for every R-module M, $\underline{\mathfrak{FCI}}^{-1}(M) \subseteq \mathfrak{I}^{-1}(M)$, if and only if, for every R-module M, $\mathfrak{FCI}^{-1}(M) = \underline{\mathfrak{I}}^{-1}(M)$.*

COROLLARY 2.11. *A ring R is right Noetherian if and only if for every absolutely pure right R-module M, $\underline{\mathfrak{FCI}}^{-1}(M) \subseteq \mathfrak{PI}^{-1}(M)$.*

Note that FC-pure-subinjectivity domain of a module M is not closed under factor modules in general. Now, we have the following two results:

PROPOSITION 2.12. *For right R-modules M and A, if every factor of* FCE(A) *is FC-pure injective and M is A-FC-pure-subinjective, then M is A/K-FC-pure-subinjective for any $K \leq A$.*

THEOREM 2.13. *For a ring R, the following statements are equivalent:*

(1) *Every factor of FC-pure injective right R-modules is FC-pure injective.*
(2) *The FC-pure-subinjectivity domain of each right R-module is closed under factors.*

PROPOSITION 2.14. *If A is an FC-pure submodule of right R-module M such that $A \in \underline{\mathfrak{FCI}}^{-1}(M)$, then FCE($A$) is a direct summand of M.*

COROLLARY 2.15. *If A is an f-injective submodule of right R-module M and $A \in \underline{\mathfrak{FCI}}^{-1}(M)$, then $M = \mathrm{E}(A) \oplus L$ for some submodule L of M*

COROLLARY 2.16. *A is an FC-pure submodule of M and $A \in \underline{\mathfrak{FCI}}^{-1}(M)$ if and only if $M = \mathrm{FCE}(A) \oplus L$ for some submodule L of M and $A \in \underline{\mathfrak{FCI}}^{-1}(L)$.*

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Generalized projective dimension and the Generalized Intersection Theorem

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Abstract. Let R be a commutative Noetherian ring, M a finitely generated R-module and $\mathfrak{a}$ be an arbitrary ideal of R. For an arbitrary integer $k \geq -1$, we introduce a generalization of projective dimension named the k-projective dimension denoted by k-$\mathrm{pd}_R M$. The finite k-projective dimension of M is at least k-$\mathrm{depth}(\mathfrak{a}, R) - k$-$\mathrm{depth}(\mathfrak{a}, M)$. If N is a finitely generated R-module, in certain conditions, it is shown that $\dim N \leq k$-$\mathrm{pd}_R M$ which is a generalization of the Intersection Theorem.

Keywords: local cohomology modules, generalized projective dimension, the Auslander-Buchsbaum formula, the Generalized Intersection Theorem

AMS Mathematics Subject Classification [2010]: 13D22

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1. Introduction

Throughout this paper, R denotes a commutative Noetherian ring with non-zero identity, M denotes a finitely generated R-module, and $k \geq -1$ is an arbitrary integer. For a subset T of $\mathrm{Spec}R$, we set

$$(T)_{>k} := \{\mathfrak{p} \in T \mid \dim R/\mathfrak{p} > k\},$$
$$(T)_{\leq k} := \{\mathfrak{p} \in T \mid \dim R/\mathfrak{p} \leq k\}.$$

This paper is essentially devoted to generalize an interesting conjecture in commutative algebra, which deal with the concept of depth of a module. An effective instrument for the computation of the depth of a module is the Auslander-Buchsbaum Formula which is related to its projective dimension. There are various generalizations of the depth of a module. The notion of k-regular sequence was introduced by Chinh and Nhan [2] which is an extension of the well-known notion of filter regular sequence introduced by Schenzel, Trung, and Cuong [4] and the notion of regular sequence as well.
To generalize the Intersection Theorem, we need to generalize the Auslander-Buchsbaum

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Formula. For this purpose, firstly, we introduce the concept of k-projective dimension which is an extension of the well-known notion of projective dimension (for $k \geq -1$).

2. Main results

In this section, we shall deal with a particular generalization of the concept of projective dimension called k-projective dimension.

DEFINITION 2.1. Let M be an R-module. The k-projective dimension of M denoted by $k\text{-pd}_R M$, is defined
$k\text{-pd}_R M = \sup\{i \in \mathbb{N}_0 | \exists N \neq 0, (\mathrm{Supp}N)_{>k} \neq \emptyset \ s.t \ \dim \mathrm{Ext}^i_R(M,N) > k\}$,
if sup exists; otherwise, we define $k\text{-pd}_R M = \infty$.

In the case $k = -1$, the notion of $k\text{-pd}_R M$ is the same as $\mathrm{pd}_R M$, the projective dimension of M.

By the above definition, we have the following results.

LEMMA 2.2. *Let M be an R-module and t be a non-negative integer. Then $k\text{-pd}_R M \leq t$ if and only if $\dim \mathrm{Ext}^i_R(M,N) \leq k$, for all $i > t$ and all R-modules $N \neq 0$ with $(\mathrm{Supp}N)_{>k} \neq \emptyset$.*

REMARK 2.3. For every R-module M, if $j \geq k$ is an integer, then $j\text{-pd}_R M \leq k\text{-pd}_R M$. Specially, $j\text{-pd}_R M \leq \mathrm{pd}_R M$ for all $j \geq -1$.

EXAMPLE 2.4. It is notable that, the k-projective dimension of an R-module is not necessarily equal to its projective dimension. It is clear that $\mathbb{Z}_2$ is not projective module over $\mathbb{Z}$, in fact $\mathrm{pd}_{\mathbb{Z}}\mathbb{Z}_2 = 1$; but $0\text{-pd}_{\mathbb{Z}}\mathbb{Z}_2 \neq 1$.

Now, we give a relation between the concept of projective dimension and k-projective dimension.

PROPOSITION 2.5. *Let M be a finitely generated R-module. Then*

$$k\text{-pd}_R M = \sup\{\mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in (\mathrm{Supp}M)_{>k}\}.$$

The following theorem is a generalization of a part of the Auslander-Buchsbaum Formula. This formula shows the relation between k-depth and k-pd.

THEOREM 2.6. *Let M be a finitely generated R-module with finite k-projective dimension, and $\mathfrak{a}$ be an ideal of R such that $(\mathrm{Supp}R/\mathfrak{a})_{>k} \neq \emptyset$. Then*

$$k\text{-pd}_R M \geq k\text{-depth}(\mathfrak{a}, R) - k\text{-depth}(\mathfrak{a}, M).$$

PROOF. We prove by induction on $k\text{-pd}_R M = n$. □

In case of $k = -1$, we get a relation between the projective dimension of a finitely generated R-module, with the depth of an ideal on an arbitrary (not necessarily local) ring.

COROLLARY 2.7. *Let M be a finitely generated R-module with $\mathrm{pd}_R M < \infty$ and $\mathfrak{a}$ be an proper ideal of R. Then*

$$\mathrm{pd}_R M \geq \mathrm{depth}(\mathfrak{a}, R) - \mathrm{depth}(\mathfrak{a}, M).$$

In 1965, Serre [5] proved the following theorem.

THEOREM 2.8. (Dimension Inequality). Let R be a regular local ring, M and N be finitely generated R-modules with $\ell(M \otimes N) < \infty$. Then

$$\dim M + \dim N \leq \dim R.$$

Applying the above inequality, Serre concluded that $\dim N \leq \mathrm{pd}_R M$.

Definition 2.9. Let $\mathfrak{a}$ be an ideal of R with $(\mathrm{Supp} M/\mathfrak{a}M)_{>k} \neq \phi$. The k-height of $\mathfrak{a}$ with respect to M is defined by

$$k\text{-ht}_M \mathfrak{a} = \min\{\mathrm{ht} M_{\mathfrak{p}} | \mathfrak{p} \in (\mathrm{Supp} M/\mathfrak{a}M)_{>k}\}.$$

For an ideal $\mathfrak{a}$ of R with $(\mathrm{Supp} M/\mathfrak{a}M)_{>k} = \phi$, we set $k\text{-ht}_M \mathfrak{a} = \infty$. In the case $k = -1$, the notion of $k\text{-ht}_M \mathfrak{a}$ is the same as $\mathrm{ht}_M \mathfrak{a}$, the height of ideal $\mathfrak{a}$ with respect to M.

Definition 2.10. Let M be a finitely generated R-module. M is called a k-Cohen-Macaulay module, and denoted by k-C.M., if either $k\text{-depth}(\mathfrak{a}, M) = k\text{-ht}_M \mathfrak{a}$ for all ideal $\mathfrak{a}$ of R with $(\mathrm{Supp} M/\mathfrak{a}M)_{>k} \neq \phi$ or $(\mathrm{Supp} M)_{>k} = \phi$.
In the case $k = -1$, (-1)-modules are exactly Cohen-Macaulay modules.

Theorem 2.11. (Generalized Intersection Theorem). Let R be a k-Cohen-Macaulay ring. Let M and N be finitely generated R-modules and $\mathfrak{a}$ be an ideal of R with $(\mathrm{Supp} R/\mathfrak{a})_{>k} \neq \emptyset$, such that $\dim M + \dim N \leq k\text{-ht}_R \mathfrak{a}$. Assume that $k\text{-pd}_R M < \infty$. Then

$$\dim N \leq k\text{-pd}_R M.$$

Proof. As R is a k-Cohen-Macaulay ring, by using Theorem 2.6, we deduce the result. □

3. Conclusion

The Intersection Theorem shows that k-projective dimension of M might be more near upper bound for $\dim N$ than projective dimension of M.

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Faculty of Mathematics and Computer
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Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

On Matrix Range

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Abstract. Let $\mathcal{A}$ be a C^*-algebra, $a \in \mathcal{A}$ and $M_n = M_n(\mathbb{C})$. The C^*-algebra $n-$dimensional matrix range of a, is defined as all matrices of the form $\varphi(a)$ where φ range over all completely positive linear maps of $C^*(a)$ into M_n which preserve the identity. In this paper we discuss some properties of $V_n(a)$,

1. Introduction

Let T be a Hilbert space operator, and let $C^*(T)$ denote the C^*-algebra generated by T and the identity. It is well known that, as φ runs over the state space of $C^*(T)$, the complex numbers $\varphi(T)$ fill out the closure of the numerical range of T. The following definition generalizes this notion.

Definition 1.1. Let T be a Hilbert space operator, and let n be a positive integer. $W_n(T)$ is defined as all $n \times n$ matrices of the form $\varphi(T)$ where φ ranges over all completely positive linear maps of $C^*(T)$ which preserve the identity. Completely positive maps are discussed in [1] and [5]

If $n = 1$, $W_1(T) = \overline{W(T)}$ and thus $W_n(T)$ is a generalization of the numerical range. In [4] Pollack uncover information about $W_n(T)$. These facts are recorded here.

Let $T \in B(\mathcal{H})$ and $\|T\| = 1$ and denoted by B_n the solid unit ball of $B(\mathcal{H}_n)$.

Proposition 1.2. *If $T \in B(\mathcal{H})$ and $\|T\| = 1$, then the following conditions are equivalent:*
(i) $\|\alpha T + \beta I\| = |\alpha| + |\beta|$ for all $\alpha, \beta \in \mathbb{C}$,
(ii) $W_n(T) = B_n$ for all n,
(iii) $W_{n_0}(T) = B_{n_0}$ for some n_0.

Recall that the numerical radius $w(T) := \sup\{|\lambda| : \lambda \in W(T)\}$ and the relation $\frac{1}{2}\|T\| \leq w(T) \leq \|T\|$ [3]. If we analogously define $|W_n(T)| := \sup\{\|S\| : S \in W_n(T)\}$ for $n \geq 2$, we have

Proposition 1.3. $|W_n(T)| = \|T\|$ *for* $n \geq 2$.

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Proposition 1.4. *Let T be a normal operator and let n be a positive integer. Then $W_n(T)$ is the closure of the set*

$$\{\Sigma_{i=1}^{r}\lambda_i K_i : r \geq 1, \lambda_i \in \sigma(T),\ K_i \in B(\mathcal{H}_n),\ K_i \geq 0, \Sigma_{i=1}^{r} K_i = I\}$$

If T is self-adjoint, then since completely positive maps preserves adjoint, $W_n(T)$ consists entirely of self-adjoint operators. There are several natural questions. Is the converse true? When does $W_n(T)$ consists entirely of normal operators? If $W_n(T)$ consists of normal operators must T be normal? The next propositions resolves these questions.

Proposition 1.5. *If $T \in B(\mathcal{H})$, the following conditions are equivalent:*
(i) $W_n(T)$ consists entirely of normal operators for all n,
(ii) $W_{n_0}(T)$ consists entirely of normal operators for some $n_0 \geq 2$,
(iii) T is normal and $\sigma(T)$ is contained in a line.

Proposition 1.6. *If $W_n(T)$ consists entirely of self-adjoint operators. Then T is self-adjoint.*

Definition 1.7. For an arbitrary $T \in B(\mathcal{H})$, $n \geq 1$ $S_n(T)$ is the closure of the set

$$\{\Sigma_{i=1}^{r}\lambda_i K_i : r \geq 1, \lambda_i \in \sigma(T),\ K_i \in B(\mathcal{H}_n),\ K_i \geq 0, \Sigma_{i=1}^{r} K_i = I\}$$

Proposition 1.8. *If $T \in B(\mathcal{H})$, then $S_n(T) \subset W_n(T)$ for all n.*

One more important property of the numerical range map, $A \to \overline{W(A)}$ is the continuity of it. For the convergence of compact subsets of the plane, we use the topology induced by the Hausdorff metric. Recently in [2] the authors extended this result to matrix range, indeed, they have shown that the matrix range of a tuple generating a continuous field of C^*-algebras is continuous in the sense that every level is continuous in the Hausdorff metric.

2. Main results

Let $\mathcal{A}$ be a C*-algebra and $a \in \mathcal{A}$. We define C*-algebra $n-$dimensional matrix range for an element $a \in \mathcal{A}$ (generalization of $n-$dimensional matrix range of operators) by

$$V_n(a) := \{\varphi(a) : \varphi \in CP(C^*(a),\ \mathcal{H}_n\ ;\ 1)\},$$

where $C^*(a)$ is the C*-algebra generated by $\{a, 1\}$, $\mathcal{H}_n$, for positive integer n, the $n-$dimensional Hilbert space and $CP(C^*(a),\ \mathcal{H}_n;\ 1)$ is the set of all completely positive maps of $C^*(a)$ into $B(\mathcal{H}_n)$ which preserve the identity.

We analogously define $|V_n(a)| := \sup\{\|S\| : S \in V_n(a)\}$ for $n \geq 2$, and $S_n(a)$ is the closure of the set

$$\{\Sigma_{i=1}^{r}\lambda_i K_i : r \geq 1, \lambda_i \in \sigma(a),\ K_i \in B(\mathcal{H}_n),\ K_i \geq 0, \Sigma_{i=1}^{r} K_i = I\}$$

Theorem 2.1. *Let $\mathcal{A}$ be a C*-algebra and $a \in \mathcal{A}$. Then*

(1). If $\|a\| = 1$, the following conditions are equivalent:
(i) $\|\alpha a + \beta 1\| = |\alpha| + |\beta|$ for all $\alpha, \beta \in \mathbb{C}$,
(ii) $V_n(a) = B_n$ for all n,
(iii) $V_{n_0}(a) = B_{n_0}$ for some n_0.

(2). $|V_n(a)| = \|a\|$ for $n \geq 2$.

(3). If a is normal, $V(a) = convh(\sigma(a))$(i.e. a is convexoid), in general, if a is hyponormal(i.e. $a^*a \geq aa^*$) then a is convexoid.

(4). The following conditions are equivalent:
(i) $V_n(a)$ consists entirely of normal operators for all n,
(ii) $V_{n_0}(a)$ consists entirely of normal operators for some $n_0 \geq 2$,
(iii) a is normal and $\sigma(a)$ is contained in a line.

(5). If $V_n(a)$ consists entirely of self-adjoint operators. Then a is self-adjoint.

(6). $S_n(a) \subset V_n(a)$ for all n.

(7). Let $a \in \mathcal{A}$ be normal and let n be a positive integer. Then $V_n(a)$ is the closure of the set

$$\{\Sigma_{i=1}^{r} \lambda_i K_i : r \geq 1, \lambda_i \in \sigma(a),\ K_i \in B(\mathcal{H}_n),\ K_i \geq 0, \Sigma_{i=1}^{r} K_i = I\}$$

Proof. It is trivial that $V_n(a) = W_n(T_a)$, where $T_a = \varphi(a)$ and φ is the faithful representation of $\mathcal{A}$ on some Hilbert space constructed by GNS. Then the proof follows from above propositions and isometry of faithful representations(i.e. $\|a\| = \|T_a\|$). Details have been omitted for brevity □

Remark 2.2. If J_n denoted the $n \times n$ shift matrix and k is an integer with $1 \leq k \leq n-1$, then

$$V(J_n^k) = \bar{\mathbb{D}}(0; cos(\frac{\pi}{[(n-1)/k] + 2})).$$

If k fix and n large enough, then $V_1(J_n^k) = B_1$. Since $\|J_n^k\|_1 = \|J_n^k\|_\infty = 1$, we have $V_m(J_n^k) = B_m$ for all positive integer m.

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Composition operators on weighted variable exponent Bergman spaces

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Abstract. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and let φ be a holomorphic function from disk $\mathbb{D}^m$ into $\mathbb{D}^n$. We study the composition operator C_φ on the weighted variable exponent Bergman space with classical radial weight and give a sufficient condition for the boundedness of this operator on $A^{p(\cdot)}_\alpha(\mathbb{D}^n)$.

Keywords: composition operator, Bergman space, weighted variable exponent Lebesgue space, polydisk

AMS Mathematics Subject Classification [2010]: 47B33, 30H20, 46B50

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. We mean by a *variable exponent*, a measurable function $p : \Omega \to [1, \infty)$. We shall write

$$p_+ = p^+_\Omega := ess\ sup_{x\in\Omega}\ p(x),$$
$$p_- = p^-_\Omega := ess\ inf_{x\in\Omega}\ p(x).$$

Let $\mathcal{P}(\Omega)$ denote the set of all variable exponents $p(\cdot)$ for which $p_+ < \infty$. For a complex-valued measurable function $f : \Omega \to \mathbb{C}$ and the weight $\omega : \Omega \to [0, \infty)$ we define the modular $\rho_{p(\cdot),\omega}$ by

$$\rho_{p(\cdot),\omega}(f) := \int_\Omega \omega(x)|f(x)|^{p(x)} d\mu(x)$$

where μ is the Lebesgue measure on Ω. The norm induced by this modular is given by

$$\|f\|_{L^{p(\cdot)}_\omega} := \inf \left\{ \lambda > 0 : \rho_{p(.),\omega}\left(\frac{f}{\lambda}\right) \le 1 \right\}.$$

Definition 1.1. The weighted variable exponent Lebesgue space $L^{p(\cdot)}_\omega(\Omega)$ consists of all complex-valued functions $f : \Omega \to \mathbb{C}$ for which $\rho_{p(\cdot),\omega}(f) < \infty$.

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It is well-known that if $p(\cdot) \in \mathcal{P}$, then $L^{p(\cdot)}_{\omega}$ equipped with the above norm is a Banach space [1]. Moreover, the dual of $L^{p(\cdot)}$ is $L^{p'(\cdot)}$ where the conjugate exponent $p'(\cdot)$ satisfies

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

It can also be verified that the conjugate exponent satisfies the following equalities:

$$p'(\cdot)_+ := \left(p'(\cdot)\right)_+ = (p_-)',$$
$$p'(\cdot)_- := \left(p'(\cdot)\right)_- = (p_+)'.$$

Definition 1.2. A function $p : \Omega \to \mathbb{R}$ is said to be locally log-Holder continuous on Ω if there exists a positive constant C such that for all $x, y \in \Omega$ that $|x - y| < \frac{1}{2}$, we have

$$LH_0 : |p(x) - p(y)| \leq \frac{C}{\log(\frac{1}{|x-y|})}.$$

and is log-holder continuous at infinity if there exists p_∞, $1 < p_\infty < \infty$, such that

$$LH_\infty : |p(x) - p_\infty| \leq \frac{C}{\log\left(e + |x|\right)}.$$

We denote by $\mathcal{P}^{log}(\Omega)$ the set of all locally log-Holder continuous functions in Ω for which $1 < p_- \leq p_+ < \infty$.

Definition 1.3. Given $f \in L^1_{loc}(\Omega)$, the weighted Hardy-Littlewood maximal function of f is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| d\mu(y), \quad x \in \Omega.$$

It is known that for each $f : \Omega \to \mathbb{C}$ we have $|f(x)| \leq Mf(x)$ (see [1]).

Definition 1.4. we say the weight ω is of class A_1 and denote $\omega \in A_1$ if

$$[\omega]_{A_1} := ess\ sup_{z\in\mathbb{D}^n} \frac{M\omega(z)}{\omega(z)} < \infty.$$

Theorem 1.5. *([1]) Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy $1 < p_- \leq p_+ < \infty$. Then Hardy-Littlewood maximal function M is bounded on $L^p(\cdot)(\Omega)$ if and only if M is bounded on $L^{p'(\cdot)}(\Omega)$.*

Theorem 1.6. *(The Rubio de Francia extrapolation)([1]): Suppose that for some $p_0 \geq 1$ the family $\mathcal{F}$ (of non-negative measurable pairs of functions) is such that for all ω in the Muckenhoupt weight class A_1 there exists $C_0 > 0$ such that*

$$\int_\Omega F(x)^{p_0} \omega(x) dx \leq C_0 \int_\Omega G(x)^{p_0} \omega(x) dx; \qquad F, G \in \mathcal{F}.$$

Let $p(\cdot) \in \mathcal{P}(\Omega)$, $p_0 \leq p_- \leq p_+ < \infty$ and the maximal operator M is bounded on the space $L^{\left(\frac{p(.)}{p_0}\right)'}(\Omega)$. Then there exists $C_{p(\cdot)}$ such that

$$\|F\|_{L^{p(\cdot)}} \leq C_{p(\cdot)} \|G\|_{L^{p(\cdot)}}.$$

We now turn to the n-dimensional complex plane, and consider the unit polydisk

$$\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D} = \{z = (z_1, ..., z_n) \in \mathbb{C}^n : |z_k| < 1, 1 \leq k \leq n\}$$

in $\mathbb{C}^n$. From now on, we assume that $\Omega = \mathbb{D}^n$, so that we can use all the above arguments with $x \in \Omega$ replaced by $z \in \mathbb{D}^n$.

DEFINITION 1.7. For $p \in \mathcal{P}(\mathbb{D}^n)$ and $-1 < \alpha < \infty$, the weighted variable exponent Bergman space $A_\alpha^{p(\cdot)}$ consists of all holomorphic functions $f : \mathbb{D}^n \to \mathbb{C}$ for which

$$\int_{\mathbb{D}^n} |f(z)|^{p(z)} d\upsilon_\alpha(z) < \infty$$

where

$$d\upsilon_\alpha(z) = d\upsilon_\alpha(z_1, ..., z_n) = dA_\alpha(z_1) \cdots dA_\alpha(z_n),$$

and

$$dA_\alpha(z_k) = \frac{\alpha+1}{\pi}(1 - |z_k|^2)^\alpha dx_k\, dy_k, \quad z_k = x_k + iy_k.$$

Note that with $\Omega = \mathbb{D}^n$, the maximal operator M takes the form

$$Mf(z) := \sup_{z \in Q} \frac{1}{|Q|} \int_Q |f(z)| d\upsilon_\alpha(z), z \in \mathbb{D}^n,$$

where the supremum is taken over all cubes Q with sides parallel to the coordinate axes and containing z.

2. Composition operators

We fix two positive integers m and n. We consider a holomorphic mapping $\varphi : \mathbb{D}^m \to \mathbb{D}^n$ given by

$$\varphi(z_1, z_2, ..., z_m) = (\varphi_1(z_1, z_2, ..., z_m), ..., \varphi_n(z_1, z_2, ..., z_m))$$

where for each $1 \le k \le n$, $\varphi_k : \mathbb{D}^m \to \mathbb{D}$ is a holomorphic function. We study the composition operator

$$C_\varphi : A_\alpha^{p(\cdot)}(\mathbb{D}^n) \to A_\beta^{p(\cdot)}(\mathbb{D}^m)$$

defind by

$$C_\varphi(f) = f \circ \varphi, \quad f \in A_\alpha^{p(\cdot)}(\mathbb{D}^n).$$

DEFINITION 2.1. Let μ be a positive Borel measure on $\mathbb{D}^n$. We say that μ is a Carleson measure for$A^{p(\cdot)}(\mathbb{D}^n)$ if there exist a constant $C > 0$ such that

$$\int_{\mathbb{D}^n} |f(z)|^{p(z)} d\mu(z) \le C \int_{\mathbb{D}^n} |f(z)|^{p(z)} d\upsilon_\alpha(z).$$

It is well-known that the Carleson measure does not depend on the exponent.

THEOREM 2.2. *([2, 5]): Suppose $\alpha > -1$ and μ is a positive Borel measure on $\mathbb{D}^n$. Then, the following two conditions are equivalent.*
(a) μ is a Carleson measure for $A_\alpha^p(\mathbb{D}^n)$ for some $p > 0$,
(b) μ is a Carleson measure for $A_\alpha^p(\mathbb{D}^n)$ for every $p > 0$.

This was used by Stessin and Zhu to prove the following theorem on the boundedness of composition operators on usual (constant exponent) Bergman spaces of the polydisk. Indeed, Stessin and Zhu proved that the operator C_φ maps $A_\alpha^p(\mathbb{D}^n)$ boundedly into $A_\alpha^p(\mathbb{D}^n)$ if and only if $\mu_{\varphi,\alpha}$ is Carleson measure for $A_\alpha^p(\mathbb{D}^n)$; here $\mu_{\varphi,\alpha}(E) = \int_{\varphi^{-1}(E)} d\upsilon_\alpha$. Moreover,

THEOREM 2.3. *([3]) Suppose $p > 0$, $\alpha > -1$ and $\varphi : \mathbb{D}^n \to \mathbb{D}^n$ is an analytic self-map of the unit polydisk. Then the operator $C_\varphi : A_\alpha^p(\mathbb{D}^n) \to A_\alpha^p(\mathbb{D}^n)$ is bounded.*

For different weights, they even found the best possible constants determining the weights; that is, α and *beta*.

Theorem 2.4. *([3]) For any $p > 0$ and $\alpha > -1$ the composition operator C_φ maps $A_\alpha^{p(\cdot)}(\mathbb{D}^n)$ boundedly into $A_\beta^{p(\cdot)}(\mathbb{D}^n)$ where $\beta = n(2+\alpha) - 2$.*

The main result of this paper is to generalize some of the above results to variable exponent Bergman space $A_\alpha^{p(\cdot)}(\mathbb{D}^n)$. To do this we need to study the Muckenhoupt weight classes, and its relevance to Carleson measures in this new setting. Indeed, we prove that

Theorem 2.5. *Let $\varphi : \mathbb{D}^m \longrightarrow \mathbb{D}^n$ be a holomorphic function and $p(\cdot) \in \mathcal{P}^{log}(\mathbb{D}^n)$. Then the composition operator $C_\varphi : A_\alpha^{p(\cdot)}(\mathbb{D}^n) \longrightarrow A_\beta^{p(\cdot)}(\mathbb{D}^m)$ is bounded on $A_\alpha^{p(\cdot)}$ where $\beta = n(\alpha + 2) - 2$.*

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Faculty of Mathematics and Computer
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Coefficient estimates for a new subclass of meromorphic bi-univalent functions by using Faber polynomial

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Abstract. In this paper, we find the coefficient bounds for meromorphic bi-univalent functions of subclass $\mathcal{N}_{\Sigma_{\mathfrak{B}}}(\lambda, \beta, \alpha)$ by using the Faber polynomial expansions which will be defined on the domain $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. The results presented in this paper would generalize and improve some works of earlier authors.

Keywords: Analytic functions, Univalent and bi-univalent functions, Meromorphic bi-univalent functions, Coefficient estimates, Faber polynomial.

AMS Mathematics Subject Classification [2010]: 30C45, 30C50

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1. Introduction

Let Σ denote the family of meromorphic univalent functions f of the form

$$f(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}, \tag{1}$$

which defined on the domain $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. Since $f \in \Sigma$ is univalent, it has an inverse f^{-1}, that satisfy

$$f^{-1}(f(z)) = z \ (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \ \ (M < |w| < \infty, \ M > 0).$$

Furthermore, the coefficients of g, the inverse map of f, are given by the Faber polynomial ([1, 11]):

$$g(w) = f^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} = w - b_0 - \sum_{n=1}^{\infty} \frac{1}{n} K_{n+1}^n \frac{1}{w^n}, \tag{2}$$

where $M < |w| < \infty$,

$$K_{n+1}^n = K_{n+1}^n(b_0, b_1, ..., b_n) = nb_0^{n-1}b_1 + n(n-1)b_0^{n-2}b_2 + \frac{1}{2}n(n-1)(n-2)b_0^{n-3}(b_3 + b_1^2)$$

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$$+\frac{n(n-1)(n-2)(n-3)}{3!}b_0^{n-4}(b_4+3b_1b_2)+\sum_{j\geq 5}b_0^{n-j}V_j$$

and V_j with $5\leq j\leq n$ is a homogeneous polynomial of degree j in the variables $b_1, b_2, ..., b_n$. A function $f\in\Sigma$ is said to be meromorphic bi-univalent if $f^{-1}\in\Sigma$. The family of all meromorphic bi-univalent functions is denoted by $\Sigma_{\mathfrak{B}}$.

Estimates on the coefficient of meromorphic univalent functions were widely studied in the literature; for instance, the estimate $|b_2|\leq 2/3$ for meromorphic univalent functions $f\in\Sigma$ with $b_0=0$ was obtained by Schiffer [9] and the inequality $|b_n|\leq 2/(n+1)$ for $f\in\Sigma$ with $b_k=0$, $1\leq k\leq n/2$ was proven by Duren [3].

For the coefficients of the inverse of meromorphic univalent functions, Springer [10] proved that

$$|B_3|\leq 1 \textit{ and } |B_3+\frac{1}{2}B_1^2|\leq\frac{1}{2}$$

and conjectured that

$$|B_{2n-1}|\leq\frac{(2n-2)!}{n!(n-1)!}\ (n=1,2,3,...).$$

In 1977, Kubota [6] has proved that the Springer's conjecture is correct for $n=3,4,5$ and afterwards sharp bounds for the coefficients B_{2n-1}, $1\leq n\leq 7$ were obtained by Schober [7].

The goal of the present paper is to define a general subclass of meromorphic bi-univalent functions which includes the two subclasses of meromorphic bi-univalent functions studied in [2, 4, 5, 8, 12]. For this subclass, we find bounds for $|b_0|$ and $|b_1|$ to show the unpredictability of the coefficients of meromorphic bi-univalent functions.

2. Main results

In this section, we define and investigate the general subclass $M_{\Sigma_{\mathfrak{B}}}(\lambda,\mu,\gamma,\alpha)$.

Definition 2.1. For $\lambda\geq 1$, $0\leq\mu<1$, $\gamma\in\mathbb{C}-\{0\}$ and $0\leq\alpha<1$, a function $f(z)\in\Sigma_{\mathfrak{B}}$ given by (1) be in the class $M_{\Sigma_{\mathfrak{B}}}(\lambda,\mu,\gamma,\alpha)$, if the following conditions are satisfied:

$$f\in\Sigma_{\mathfrak{B}},\ Re\left\{1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}-1\right]\right\}>\alpha \tag{3}$$

and

$$Re\left\{1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}-1\right]\right\}>\alpha, \tag{4}$$

where $z,w\in\Delta$ and the function g is the inverse of f given by (2).

Remark 2.2. There are many selections of the parameters λ, μ and γ which would provide interesting subclasses of the meromorphic bi-univalent function class $\Sigma_{\mathfrak{B}}$. For example:

If we put $\gamma = \lambda = 1$ and $\mu = 0$, the class $M_{\Sigma_{\mathfrak{B}}}(\lambda, \mu, \gamma, \alpha)$ changes to the class $\Sigma^*_{\mathfrak{B}}(\alpha)$ that was defined by Hamidi et al. [5].

If we put $\gamma = \lambda = 1$, the class $M_{\Sigma_{\mathfrak{B}}}(\lambda, \mu, \gamma, \alpha)$ changes to the class $B(\alpha, \mu)$ that was defined by Halim et al. [4].

If we put $\gamma = 1$, the class $M_{\Sigma_{\mathfrak{B}}}(\lambda, \mu, \gamma, \alpha)$ changes to the class $M_{\Sigma}(\lambda, \mu, \alpha)$ that was studied by Bulut et al. [2].

Theorem 2.3. *Let $f(z) \in \Sigma_{\mathfrak{B}}$ given by (1) be in the class $M_{\Sigma_{\mathfrak{B}}}(\lambda, \mu, \gamma, \alpha)$ $(\lambda \geq 1,\ 0 \leq \mu < 1,\ \gamma \in \mathbb{C} - \{0\},\ 0 \leq \alpha < 1)$. Then*

$$|b_0| \leq \begin{cases} 2\sqrt{\frac{(1-\alpha)|\gamma|}{|(\mu-2\lambda)(\mu-1)|}}; & 0 \leq \alpha < 1 - \frac{(\mu-\lambda)^2}{|\gamma||(\mu-2\lambda)(\mu-1)|} \\ \frac{2(1-\alpha)|\gamma|}{|\mu-\lambda|}; & 1 - \frac{(\mu-\lambda)^2}{|\gamma||(\mu-2\lambda)(\mu-1)|} \leq \alpha < 1 \end{cases}$$

and

$$|b_1| \leq \frac{2(1-\alpha)|\gamma|}{|\mu - 2\lambda|}.$$

If we put $\gamma = \lambda = 1$ in Theorem 2.3, we get the subsequent corollary.

Corollary 2.4. *Let $f(z) \in \Sigma_{\mathfrak{B}}$ given by (1) be in the class $B(\alpha, \mu)$ $(0 \leq \alpha < 1,\ 0 \leq \mu < 1)$. Then*

$$|b_0| \leq \begin{cases} 2\sqrt{\frac{(1-\alpha)}{(2-\mu)(1-\mu)}}; & 0 \leq \alpha < \frac{1}{2-\mu} \\ \frac{2(1-\alpha)}{1-\mu}; & \frac{1}{2-\mu} \leq \alpha < 1 \end{cases}$$

and

$$|b_1| \leq \frac{2(1-\alpha)}{2-\mu}.$$

If we put $\mu = 0$ in Corollary 2.4, we get the subsequent corollary.

Corollary 2.5. *Let $f(z) \in \Sigma_{\mathfrak{B}}$ given by (1) be in the class $\Sigma^*_{\mathfrak{B}}(\alpha)$ $(0 \leq \alpha < 1)$. Then*

$$|b_0| \leq \begin{cases} \sqrt{2(1-\alpha)}; & 0 \leq \alpha < \frac{1}{2} \\ 2(1-\alpha); & \frac{1}{2} \leq \alpha < 1 \end{cases}$$

and

$$|b_1| \leq 1 - \alpha.$$

Remark 2.6. The above bounds for $|b_0|$ and $|b_1|$ demonstrate that Corollary 2.5 is an improvment of the bounds given by Hamidi et al. [5, Theorem 2].

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30 August - 02 September 2021

پنجاه و دومین
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۸ تا ۱۱ شهریور ۱۴۰۰

Automatic continuity of bijective Jointly linear Separating maps between Banach Modules

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Abstract. Let A and B be Banach algebras. Linear maps $T, S : A \to B$ are called *jointly separating* whenever $a \cdot b = 0$ implies $Ta \cdot Sb = 0$, for all $a, b \in A$. In this paper, first we generalize the definition of jointly linear separating maps to Banach module cases. Then we give the characterization of such maps. Finally, we prove that under certain conditions on Banach modules, both jointly separating maps are continuous if at least one of them is bijective.

Keywords: Banach module, separating, hyper maximal, jointly separating

AMS Mathematics Subject Classification [2010]: 46H25, 46H40, 47B37

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1. Introduction

The well-known notion, separating maps between two spaces of the function A and B, is defined such that $f.g = 0$ implies $Tf.Tg = 0$ for all f, g in A. It is equivalent to this fact that $\mathrm{coz}(Tf) \cap \mathrm{coz}(Tg) = \emptyset$ if $\mathrm{coz}(f) \cap \mathrm{coz}(g) = \emptyset$, for all f, g in A. In recent years the definition of separating maps has been generalized to different kinds of spaces, see [1, 5–7]. In [8, 9], we generalized this concept for Banach module cases. In [10], authors studied a pair of linear maps S, T from a subspace $A(X, E)$ of $C(X, E)$ to $C(X, F)$, called jointly separating operators in the sense that $\mathrm{coz}(Tf) \cap \mathrm{coz}(Sg) = \emptyset$, if $\mathrm{coz}(f) \cap \mathrm{coz}(g) = \emptyset$, for all f, g in $A(X, E)$, when E is a Banach space and F is a locally convex topological vector space. In this paper, we generalize the notion of jointly linear separating maps to Banach module cases, and then we prove some results and automatic continuity of these maps

2. Preliminaries

Let A be a commutative Banach algebra with or without unit. The set of all non-zero multiplicative linear functional on A, is denoted by $\sigma(A)$, and is called maximal ideal space of A. We identify the maximal ideal space of A_1 (the standard unitization of A) by $\sigma_0(A) = \sigma(A) \cup \{0\}$. For a commutative Banach algebra A, recall that a left Banach A-module X is *essential* if it is the closure of linear span of the set $\{a.x : a \in A, x \in X\}$. Obviously, each unital left Banach module is essential. Let A be a commutative Banach algebra, and X be a left Banach A-module, following [2, 4], for a point $\varphi \in \sigma_0(A)$ a functional $\xi \in X^*$ is called a *point multiplier* at φ, if $\langle \xi, a.x \rangle = \varphi(a)\langle \xi, x \rangle$, for all $a \in A$

and $x \in X$. By [3] the kernel of each non-trivial point multiplier is a closed left submodule of X with codimension one which is called a *hyper maximal* left submodule. Conversely, each closed hyper maximal left submodule P of X is the kernel of some point multiplier $\xi \in X^*$ at some $\varphi \in \sigma_0(A)$, see([3], Prop. 4.3). The set of all closed hyper maximal left submodules of X is denoted by $\Delta_A(X)$ and is called the *hyper maximal left submodule space*, see [4]. Supposing X an essential left Banach module, it is seen there is no non-trivial point multiplier at 0. Hence each closed hyper maximal left submodule of X is the kernel of a point multiplier at some point of $\sigma(A)$. In this case, the *natural map* $\nu_X : \Delta_A(X) \to \sigma(A)$ is defined so that it associates to each $P \in \Delta_A(X)$ a unique point $\nu_X(P)$ in $\sigma(A)$ [4]. Supposing, $\Delta^1_A(X) = \Delta_A(X) \cup X$ we define the extended natural map by $\widetilde{\nu}_X : \Delta^1_A(X) \to \sigma_0(A)$, which $\widetilde{\nu}_X(X) = 0$. It is possible that $\Delta_A(X) = \emptyset$, (see examples 4.4, 4.8 in [4]). In this paper we regard essential left Banach modules with non-empty hyper maximal left submodule spaces. Let A be a commutative Banach algebra and X be a left Banach A-module. For each $x \in X$, we set $\mathrm{Coz}(x) = \nu_X(\{P \in \Delta_A(X) : x \notin P\})$. Suppose A and B are commutative Banach algebras and X and Y are two left Banach modules over A and B, respectively with non empty hyper maximal left submodule spaces. A linear map $T : X \longrightarrow Y$ is called separating if $\mathrm{Coz}(x_1) \cap \mathrm{Coz}(x_2) = \emptyset$ implies $\mathrm{Coz}(Tx_1) \cap \mathcal{C}oz(Tx_2) = \emptyset$ for all $x_1, x_2 \in X$, [8]. A bijective linear map $T : X \longrightarrow Y$ is a biseparating map when both T and T^{-1} are separating [9].

Definition 2.1. Let A and B be commutative Banach algebras, and X and Y be essential left Banach modules over A and B, respectively. Linear maps $T, S : X \to Y$ are called *jointly separating* whenever, $\mathrm{Coz}(x_1) \cap \mathrm{Coz}(x_2) = \emptyset$ implies $\mathrm{Coz}(Tx_1) \cap \mathrm{Coz}(Sx_2) = \emptyset$, for all $x_1, x_2 \in X$. In addition, if T, S are bijective jointly separating maps such that $T^{-1}, S^{-1} : Y \to X$ are jointly separating, then T, S are called *jointly biseparating* maps.

3. Jointly linear Separating maps between Banach Modules

In this section, we suppose that A and B are commutative Banach algebras, X and Y are essential left Banach modules over A and B with non-empty hyper maximal left submodule spaces and $T, S : X \to Y$ are jointly separating maps.

Definition 3.1. The subset $\Delta_1(Y)$ of $\Delta_B(Y)$ is defined as follows:

$$\Delta_1(Y) = \Big(\bigcup_{x \in X} \{P \in \Delta_B(Y) : Tx \notin P\} \Big) \bigcap \Big(\bigcup_{x \in X} \{P \in \Delta_B(Y) : Sx \notin P\} \Big).$$

Definition 3.2. Let $\Delta_1(Y) \neq \emptyset$. For any $P \in \Delta_1(Y)$ we define the jointly support set of T, S, $J_{\{T,S\}}(P)$, as the set of all $Q \in \Delta^1_A(X)$ such that for each open neighborhood U of $\widetilde{\nu}_X(Q)$ in $\sigma_0(A)$, there exists an element $x \in X$ with $\mathrm{Coz}(x) \subseteq U$ and $Tx \notin P$ or $Sx \notin P$.

Lemma 3.3. *For each $P \in \Delta_1(Y)$, the jointly support set $J_{\{T,S\}}(P) \neq \emptyset$ is nonempty.*

Definition 3.4. Let A be a commutative Banach algebra and X be a left Banach A-module. It is said that X is a PHS left Banach A- module whenever for each $x \in X$ and $Q \in \Delta^1_A(X)$ with $x \in Q$, there is a sequence $\{x_n\}$ in X such that for a neighborhood U_n of $\widetilde{\nu}_X(Q)$ in $\sigma_0(A)$, $\mathrm{Coz}(x_n) \subseteq \sigma_0(A) \backslash U_n$ and $\|x_n - x\| \to 0$.

Example 3.5. i) If A is a commutative Banach algebra which satisfies Ditkin's condition, we have $\Delta_A(A) = \{\ker \varphi : \varphi \in \sigma(A)\}$, and A is a PHS left Banach A- module.
ii) Let A be a non-unital commutative Banach algebra and X be a dense ideal of A. If

$(X, \|.\|_X)$ is a Banach algebra satisfying Ditkin's condition, such that $\|ax\|_X \leq \|a\|\|x\|_X$, then X is a PHS left Banach A-module.
iii) Each Segal algebra on a locally compact abelian group G can be regarded as a PHS left Banach $L^1(G)$-module.
iv) For a compact metric space (X, d) and a Banach space E and $\alpha \in (0, 1]$, we denote the space of E-valued Lipschitz functions of order α on X by $Lip^\alpha(X, E)$. The Banach space $Lip^\alpha(X, E)$ with respect to the lipschitz norm $\|.\|_\alpha$ is a PHS left Banach $Lip^\alpha(X)$-module.

Lemma 3.6. *Let X be a PHS left Banach A- module, then for each $P \in \Delta_1(Y)$ the jointly support set $J_{\{T,S\}}(P)$ is a singleton.*

Regarding the above Lemma, in the case that X is a PHS module we can define a map $H : \Delta_1(Y) \to \Delta^1_A(X)$ by $\{H(P)\} = J_{\{T,S\}}(P)$ for all $P \in \Delta_1(Y)$.

For each $P \in \Delta_1(Y)$ and y in Y the map $\chi_P : \Delta_1(Y) \longrightarrow Y/P$ is defined by $\chi_P(y) = y + P$. Now, we define two subsets of $\Delta_1(Y)$,

$$\Delta_{CT}(Y) = \Big\{P \in \Delta_1(Y) : \chi_P \circ T \text{ is continuous on } X\Big\}$$

$$\Delta_{CS}(Y) = \Big\{P \in \Delta_1(Y) : \chi_P \circ S \text{ is continuous on } X\Big\}$$

Lemma 3.7. *Let X be a hyper semisimple PHS left Banach A- module. Then $P \in \Delta_{CT}(Y)$ (resp. $P \in \Delta_{CS}(Y)$) if and only if $H(P) = T^{-1}(P)$ (resp. $H(P) = S^{-1}(P)$).*

Consider the following subset of $\prod_{P\in\Delta_A(X)} X/P$,

$$\underline{X} = \Big\{\underline{x} = \big(x_P + P\big)_{P\in\Delta_A(X)} : \|\underline{x}\| = \sup_{P\in\Delta_A(X)} \|x_P + P\| < \infty\Big\}.$$

It is seen that $\underline{X}$ is a Banach space under the defined norm $\|\underline{x}\|$ and also, is a left Banach A-module by the following action,

$$a.\underline{x} = a.\big(x_P + P\big)_{P\in\Delta_A(X)} = \big(a.x_P + P\big)_{P\in\Delta_A(X)}$$

It is important to note that for each $x \in X$ and the family $\mathcal{F} \subseteq \Delta_A(X)$, $\big(x + Q\big)_{Q\in\mathcal{F}}$ may be considered as an element $\big(x_Q + Q\big)_{Q\in\Delta_A(X)}$ of $\underline{X}$, where $x_Q = x$, for all $Q \in \mathcal{F}$ and $x_Q = 0$, for the other elements $Q \in \Delta_A(X)$. Clearly, the set $\big\{\big(x + Q\big)_{Q\in\mathcal{F}} : x \in X\big\}$ is a submodule of $\underline{X}$.

Now, we give a representation of jointly separating maps $T, S : X \to Y$ similar to the composition maps.

Corollary 3.8. *Let X be a PHS left Banach A-module and $T, S : X \to Y$ be surjective jointly separating maps. Then, there exist two submodules $\underline{X}^T$ and $\underline{Y}^T$ (resp, $\underline{X}^S$ and $\underline{Y}^S$) of $\underline{X}$ and $\underline{Y}$, respectively and a bijective linear map $J_T : \underline{X}^T \to \underline{Y}^T$ (resp, $J_S : \underline{X}^S \to \underline{Y}^S$) such that*

$$\big(Tx + P\big)_{P\in\Delta_{CT}(Y)} = J_T\Big(\big(x + H(P)\big)_{P\in\Delta_{CT}(Y)}\Big),$$

$$\big(Sx + P\big)_{P\in\Delta_{CS}(Y)} = J_S\Big(\big(x + H(P)\big)_{P\in\Delta_{CS}(Y)}\Big).$$

Proof. Define

$$\underline{X}^T = \Big\{\big(x + H(P)\big)_{P\in\Delta_{CT}(Y)} : x \in X\Big\},$$

and

$$\underline{Y}^T = \Big\{\big(Tx + P\big)_{P\in\Delta_{CT}(Y)} : y \in Y\Big\}.$$

As it was indicated, $\underline{X}^T$ and $\underline{Y}^T$ are submodules of $\underline{X}$ and $\underline{Y}$, respectively. Now, we define $J_T : \underline{X}^T \to \underline{Y}^T$ by $J_T\Big(\big(x + H(P)\big)_{P\in\Delta_{CT}(Y)}\Big) = \big(Tx + P\big)_{P\in\Delta_{CT}(Y)}$. By Lemma 3.7, J_T is a well defined linear bijective map. The results are similarly hold for S. $\square$

Now we divide $\Delta_B(Y)$ into three disjoint parts as follows:

$$\Delta_0(Y) = \Delta_B(Y)\backslash\Delta_1(Y), \Delta_c(Y) = \Delta_{CT}(Y) \cap \Delta_{CS}(Y), \Delta_d(Y) = \Delta_1(Y)\backslash\Delta_c(Y).$$

Definition 3.9. A Banach left A-module X satisfies the $\mathcal{H}$ condition whenever, for each closed submodules M and N of X, $(M :_A X) \subseteq (N :_A X)$ implies $M \subseteq N$, where $(M :_A X) = \{a \in A : aX \subseteq M\}$

Example 3.10. Let $S(G)$ be a Segal algebra on a locally compact abelian group G. Consider $S(G)$ as a left Banach $L^1(G)$-module. Then $S(G)$ is an essential PHS left Banach A-module satisfying $\mathcal{H}$ condition.

Lemma 3.11. *Let X be a PHS left Banach A-module satisfying $\mathcal{H}$ condition then, $\widetilde{\nu}_X \circ H$ is continuous on $\Delta_c(Y) \cup \Delta_d(Y)$ and H is continuous on $\Delta_c(Y)$.*

In the following, we suppose that A and B are commutative Banach algebras, X and Y are essential hyper semisimple left Banach module over A and B, respectively.

Theorem 3.12. *Let A be regular, and X be a PHS left Banach A-module satisfying $\mathcal{H}$ condition, and for each $y \in Y$, the set $\big\{P \in \Delta_B(Y) : y \notin P\big\}$ is an open set in Zariski topology of $\Delta_B(Y)$. If one of jointly separating maps $T, S : X \to Y$ is bijective then,*

(i) *Both T and S are continuous.*
(ii) *If T and S are jointly biseparating maps and B, Y have the same conditions of A, X then, H is a homeomorphism.*

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Some results on the open locating-total domination number in graphs

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ABSTRACT. In this paper, we introduce a concept as open locating-total dominating set (OLTD-set) in graphs. $S \subseteq V(G)$ is an OLTD-set if and only if S is a total dominating set of G and for any pair of distinct vertices x and y in $V(G)$, we have $N(x) \cap S \neq N(y) \cap S$. So, the open locating-total domination number $\gamma_t^{OL}(G)$ is the minimum cardinality of an OLTD-set for G. In this paper, we determine the open locating-total dominating set of some families of graphs. Also, the open locating-total domination number is calculated for two families of trees.

Keywords: Open neighborhood locating domination, Total dominating set

AMS Mathematics Subject Classification [2010]: 05C69

1. Introduction

Graph theory is used as a theoretical tool to consider actual networks. One of the studies based on graphs, finding the location of monitoring devices to safeguard a system serves. Locating sets for such studies were introduced by salter in [4] and the locating-total dominating set in the graph was introduced by Haynes and Hening [2]. The problem of open locating dominating sets was introduced by Honkala et al. [3] in the context of coding theory for binary hypercubes. Some more results on the locating-total domination number are obtained in [6].

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The open neighborhood of vertex $v \in V$ is denoted by $N(u) = \{u \in V | uv \in E\}$, while its closed neighborhood is given by $N[v] = N(u) \cup \{u\}$. A set S of vertices of a graph G is a dominating set (DS) of G if every vertex in $V \setminus S$ is adjacent to a vertex of S, and S is a total dominating set (TDS) if every vertex in V has a neighbor in S. In [5], a subset S of V is an open neighborhood locating dominating set (OLDS) of G if and only if for each vertex $w \in V(G)$ there is at least one vertex v in $S \cap N(w)$ (that is, S is an open-dominating set) and for any pair of distinct vertices x and y in V we have $N(x) \cap S \neq N(y) \cap S$. The open neighborhood locating domination number $\gamma^{OL}(G)$ is the minimum cardinality of an OLD-set for G. Motivated by the definition of locating dominating set and open dominating set, we introduce the open locating-total dominating set of a graph G. This is

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equivalent to that $S \subseteq V(G)$ is open locating-total dominating set (OLTDS) if and only if S is a total dominating set of G and for any pair of distinct vertices x and y in V, we have $N(x) \cap S \neq N(y) \cap S$. So, the open locating-total domination number $\gamma_t^{OL}(G)$ is the minimum cardinality of an OLTD-set for G. An OLTD-set for G of order $\gamma_t^{OL}(G)$ will be called an $\gamma_t^{OL}(G)$-set. It is clear that $\gamma_t^{OL}(G)$=$\gamma^{OL}(G)$.
In this paper, we investigate the open locating-total dominating set for some of the families of the graph. All graphs considered in this paper are simple without isolated vertices.

2. Main results

In the following theorem, we obtain the bounds on the open locating-total dominating set in complementary prisms. For a graph G, its complementary prism, denoted $G\overline{G}$, is formed from a copy of G and a copy of $\overline{G}$ by adding a perfect matching between corresponding vertices.

THEOREM 2.1. *For any graph G,*

$$max\{\gamma_t^{OL}(G), \gamma_t^{OL}(\overline{G})\} \leq \gamma_t^{OL}(G\overline{G}) \leq \gamma_t^{OL}(G) + \gamma_t^{OL}(\overline{G}).$$

PROOF. If $G = K_n$, then the result holds. Thus, we may assume G is not complete and D is an OLTD-set in $G\overline{G}$. Let $D_1 = D \cap V(G)$ and $D_2 = D \cap V(\overline{G})$. If D_1 is an open locating-total dominating in G, then we are finished. So, assume there exists a set $T \subseteq V(G)$ such that T is not open locating-total dominated by D_1. Thus, T will get these features by D_2. Since each vertex in D_2 is adjacent to at most one vertex in T, then $|T| \leq |D_2|$. But, set $T \cup D_1$ is an open locating-total dominating set in G. Assume that without loss of generality, that $\gamma_t^{OL}(G) \geq \gamma_t^{OL}(\overline{G})$. Thus,

$$\gamma_t^{OL}(G) \leq |T \cup D_1| \leq |T| + |D_1| \leq |D_2| + |D_1| = |D| = \gamma_t^{OL}(G\overline{G}).$$

Therefore,

$$max\{\gamma_t^{OL}(G), \gamma_t^{OL}(\overline{G})\} \leq \gamma_t^{OL}(G\overline{G}).$$

For the upper bound, let S_1 be an OLTD-set in G and S_2 be an OLTD-set for $\overline{G}$. Let $S = S_1 \cup S_2$. Since every vertex of $G(\overline{G})$ is dominated by $S_1(S_2)$ then, S is a total dominating set. It is sufficient to show that S is an open locating set in $G\overline{G}$ for every two vertices $u \in V(G)$ and $\overline{v} \in V(\overline{G})$. Because sets S_1 and S_2 are open locating set in G and $\overline{G}$, respectively. We have, $N_{G\overline{G}}(u) = N_G(u) \cup \{\overline{u}\}$, and $N_{G\overline{G}}(\overline{v}) = N_{\overline{G}}(\overline{v}) \cup \{v\}$. Since $N_{G\overline{G}}(u) \cap S_1 \neq N_{G\overline{G}}(\overline{v}) \cap S_2$ we have, $N_{G\overline{G}}(u) \cap S \neq N_{G\overline{G}}(\overline{v}) \cap S$. Thus, S is an open locating set. Therefore,

$$\gamma_t^{OL}(G\overline{G}) \leq |S_1 \cup S_2| = |S_1| + |S_2| = \gamma_t^{OL}(G) + \gamma_t^{OL}(\overline{G}).$$

□

Now, we investigate the open locating-total dominating set for the join and the corona of graphs. Let G and H be graphs of order m and n, respectively. The corona of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and m copies of H and then joining the i'th vertex of G to every vertex of the i'th copy of H.

THEOREM 2.2. *Let G and H be nontrivial connected graphs and $m = |V(G)| \geq 4$ and $n = |V(H)| \geq 4$. Then $S \subseteq V(G \circ H)$ is an OLTD-set in $G \circ H$ if and only if $S = \bigcup_{x \in V(G)} S_x$ where S_x is OLTD-set in H^x and H^x is the copy of H whose vertices are attached one by one to the vertex x.*

Proof. Let $S \subseteq V(G \circ H)$ be an OLTD-set in $G \circ H$ and $S_x = V(H^x) \cap S$. In the graph $G \circ H$, every vertex of H^x is adjacent to x in G. Since S is an OLTD-set in $G \circ H$, for every $u, v \in H^x \subseteq V(G \circ H)$, we can obtain $N_H(u) \cap S_x \neq N_H(v) \cap S_x$. So, S_x is an open locating set in H^x.
If $x \notin S$, then S_x is OLTD-set in H^x. Because S is an OLTD-set in $G \circ H$ and S_x is open locating set.
If $x \in S$, since S_x is an open locating set we get, $N_G(u) \cap S_x \neq \emptyset$ for each vertex $u \in V(H^x)$. Therefore, S_x is the dominating set. If $u \in S_x$ is not adjacent to any vertex in S_x then it is contrary to S_x is an open locating set. Thus, $S = \bigcup_{x \in V(G)} S_x$ is an OLTD-set in $G \circ H$.
For the converse, suppose that $S = \bigcup_{x \in V(G)} S_x$ in which S_x is an OLTD-set in H^x. Since S_x are the OLTD-set in H^x for every vertex $x \in V(G)$, S is a total dominating set. Thus, it is sufficient to show that S is an open locating set. For every two distinct vertices $u, v \in (G \circ H)$ we have $N_{G \circ H}(u) \cap S \neq N_{G \circ H}(v) \cap S$. Hence, S is an $\gamma_t^{OL}(G \circ H) - set$. □

Theorem 2.3. *Let G and H be nontrivial connected graphs. If $S \subseteq V(G + H)$ is an OLTD-set in $G + H$ then $S_1 = S \cap V(G)$ and $S_2 = S \cap V(H)$ are open locating sets of G and H, respectively.*

Proof. Let $S \subseteq V(G + H)$ and $S_1 = \emptyset$ and $S = S_2 = S \cap V(H)$. Since every vertex of G is adjacent to $V(H)$, for any distinct vertices u and v in $V(G)$ we have $N_{G+H}(u) \cap S = N_{G+H}(v) \cap S$. Thus, it is contrary to the assumption of S. Thus, $S_1 \neq \emptyset$. Similarly, $S_2 \neq \emptyset$. Suppose that one of S_1 and S_2 is not an open locating set, say S_1 is not an open locating set in G. Then, there exist distinct vertices u and v of G such that $N_G(u) \cap S_1 = N_G(v) \cap S_1$. Since $S_2 \subseteq N_{G+H}(u)$ and $S_2 \subseteq N_{G+H}(v)$ it follows that $N_{G+H}(u) \cap S = N_{G+H}(v) \cap S$. Thus, S is not a locating set in $G + H$, contrary to our assumption. Therefore, S_1 and S_2 are locating sets in G and H, respectively. □

Theorem 2.4. *For any positive integers m, n such that $m = 3t$ or $m = 3t + 1$ where $t \geq 1$ and $n \geq 2$,*

$$\gamma_t^{OL}(C_m \Box P_n) \leq \frac{2}{3}mn.$$

Proof. Let G be the Cartesian product $C_m \Box P_n$ where $m = 3t$ for a positive integer t. For every $1 \leq j \leq n$, we define $D_j = B_j - \cup_{l=0}^{t-1}(v_{(3l+1)j})$ such that $B_j = \{v_{1j}, v_{2j}, \ldots, v_{mj}\}$ and $1 \leq j \leq n$. We show that $S = \cup_{j=1}^{n} D_j$ is an open locating-total dominating set in $C_m \Box P_n$. We complete the proof by induction on n. If $n = 2$, then $\gamma_t^{OL}(C_m \Box P_2) \subseteq D_1 \cup D_2$. Thus, we can obtain $\gamma_t^{OL}(C_m \Box P_2) \leq 4t = \frac{2}{3}mn$. If $n = 3$, then $S = \cup_{j=1}^{3} D_j$ is an OLTD-set for $C_m \Box P_3$. Thus, $|S| = 6t = \frac{2}{3}mn = \gamma_t^{OL}(C_m \Box P_3)$. Assume that for $n - 1$, $\gamma_t^{OL}(C_m \Box P_{n-1}) \leq \frac{2}{3}m(n-1)$. We add a leaf to the last vertex from P_{n-1} to obtain path P_n. So, one cycle C_m is added to graph $C_m \Box P_{n-1}$ where every vertex in new cycle C_m is adjacent to m the last vertex from P_{n-1}'s. So, the new graph is a $C_m \Box P_n$. Therefore, we have

$$\gamma_t^{OL}(C_m \Box P_n) \leq 2t + \gamma_t^{OL}(C_m \Box P_{n-1}) \leq 2t + \frac{2}{3}m(n-1)$$

Therefore, the result holds. For $m = 3t + 1$, by a similar proof as above the result is true. □

Theorem 2.5. *For any positive integers m, n such that $m = 3t + 2$ where $t \geq 1$ and $n \geq 2$,*

$$\gamma_t^{OL}(C_m \Box P_n) \leq \frac{2}{3}(m+1)n.$$

Proof. We proceed by induction on n. Assume that $D_j = B_j - \cup_{l=1}^{t}(v_{(3l)j})$ where $B_j = \{v_{1j}, v_{2j}, \ldots, v_{mj}\}$ and $1 \leq j \leq n$. We show that $S = \cup_{j=1}^{n} D_j$ is an open locating-total dominating set in $C_m \Box P_n$. If $n = 2$, then we have $OLTD - set = S = \cup_{j=1}^{2} D_j = D_1 \cup D_2$. Therefore, $|S| = 4t + 4 = \frac{2}{3}(m+1)n$.
Let $\gamma_t^{OL}(C_m \Box P_n) \leq \frac{2}{3}(m+1)(n-1)$, for $n-1$. According to the method of proof Theorem 2.4, by adding the vertices C_m to each of the last vertices of P_n.

$$\begin{aligned}\gamma_t^{OL}(C_m \Box P_n) &\leq 2(t+1) + \gamma_t^{OL}(C_m \Box P_{n-1})\\ &\leq 2(t+1) + \frac{2}{3}(m+1)(n-1)\\ &= \frac{2}{3}(m+1)n.\end{aligned}$$

□

Finally, we obtain the OLTD for a family of trees. For any tree T, let $L(T)$ denotes the set of leaves of a tree. Also, let n and l denote the order of the tree and the number of leaves, respectively. We consider the family Γ of labeled trees that is introduced in [1].

Theorem 2.6. *If $T \in \Gamma$, then $\gamma_t^{OL}(T) = \frac{3n-l}{4}$ and $S = B(T) \cup \{L(T) \cap A(T)\} \cup \{N(B(T)) \cap C(T)\}$ is an open locating-total dominating set for T where $L(T)$ is set of leaves of T.*

Proof. Let $T = T_k$ for $k \geq 0$. We proceed by induction on the order k. Let D be an OLTD-set in T_k. For $k = 0$ that $T_k = P_6$, it is clearly. For $k = 1$, $T = T_1$ is obtained from T_0 by two operations τ_1 and τ_2. It is clear that $S = B(T_1) \cup \{L(T_1) \cap A(T_1)\} \cup \{N(B(T_1)) \cap C(T_1)\}$ is an OLTD-set in T_1.Thus, $|S| = |B(T_1)| + |C(T_1)|$. There is a support vertex for each of the leaves in the tree. So, in $T \in \Gamma$, $|B(T)| = |C(T)|$. For tree T in the family of Γ, we have $|B(T)| = 2|A(T)| - l$. We can obtain $\gamma_t^{OL}(T_1) = 3|A(T)| - l = 3(\frac{n+l}{4}) - l = \frac{3n-l}{4}$. Assume that every tree $T_{k'}$ where $0 \leq k' \leq k-1$ with l' leaves satisfies in this Theorem. Let T_{k-1} be a tree of order n' having l' leaves. By doing some operations in tree T_{k-1} (see [1]), we obtain tree T_k of order $n = \frac{5}{2}(n' + l')$ with $l - \frac{n'+l'}{2}$ leaves. It is clear that $v \in D$ and $y \notin \gamma_t^{OL}(T_{k-1})$. Since D is total dominating set, $\{w, v\} \subseteq D$. We have $N(x) \cap D = \{w\} = N(v) \cap D$, which is a contradiction. So, $y \in D$. Since y is a leaf in T_{k-1}, its support is in $\gamma_t^{OL}(T_{k-1})$. Thus, we can have

$$|D| = \frac{3n' - l'}{4} + 3l' + 2(\frac{n' + l'}{2} - l') = \frac{3n - l}{4}.$$

□

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پنجاه و دومین کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran
30 August - 02 September 2021
دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Lattices of radical subacts of acts over semigroups

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Abstract. Our objective in this paper is to give some properties of radical subacts and prime subacts of any act over a semigroup S. We prove that the radical subacts of an $S-$act form an $S-$lattice. We also determine the relationship between $S-$lattice of radical subacts and the set of prime subacts of an $S-$act.

Keywords: S-act, prime subact, radical subact

AMS Mathematics Subject Classification [2010]: 20M30, 20M50

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1. Introduction and Preliminaries

Throughout this article, unless otherwise stated, S will denote a semigroup and an S-act A_S (or act A for short) is a right S-act which is unitary when S is a monoid. For any subact B of an $S-$act A, the set $\{s \in S | As \subseteq B\}$, denoted by $(B : A)$, is said *the colon of B in A*. It is easy to see that $(B : A)$ is a right ideal of S, whenever it is not empty. Any proper subact B of A is called *prime*, if for any $a \in A$ and $s \in S$, B contains the translation aSs then $a \in B$ or $s \in (B : A)$.

The notions of prime ideals of rings and prime submodules of modules over rings are the remarkable subjects in the study of rings and modules. Recently a great deal of work has been done on the concept of prime subacts in the category of $S-$acts (**Act**-S) (for instance see [1] and [2]). The notions of free acts and different kinds of flat acts are defined by using the translations of semigroups on acts. So in some sense, the study of prime subacts in **Act**-S is more important than the category of modules over rings. Also because of the differences between structures of underlying sets of objects in these categories, there are many differences in final results and the proving techniques about prime subacts and submodules. For basic results and definition relating to acts over monoids in this paper, we refer the reader to [3].

2. Main results

Definition 2.1. Let S be a semigroup, P an ideal of S and A be an S-act. P is called a *prime ideal* if for $a, b \in S$, $aSb \subseteq P$ implies $a \in P$ or $b \in P$. A proper subact B of A is called *prime*, if for any $a \in A$ and $s \in S$, $aSs \subseteq B$ implies $a \in B$ or $As \subseteq B$ (that is, $a \in B$ or $s \in (B : A)$). Also a centered $S-$act A is called a prime act if the one element

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subact $\{\theta\}$ is a prime subact of A. The set of all prime subacts of A with colon P is called the $P-$prime spectrum of A and denoted by $Spec_P(A)$.

For an arbitrary subact B of an $S-$act A the intersection of all prime subacts of A which are containing B, is called the *prime radical* of B (radical of B for short) and is denoted by $rad(B)$. Also B is called a *radical subact* if $rad(B) = B$. Clearly every prime subact is a radical subact and $rad(rad(B)) = rad(B)$, for any subact B of A. The set of all radical subacts of A is denoted by $Rad(A)$.

In the continuation of this section we assume that all semigroups are commutative. However many of the results are true for arbitrary semigroups.

Lemma 2.2. *Let S be a commutative semigroup and A be an S act. Then every non-empty intersection of prime subacts of A is a radical subact of A.*

Let P be a prime subact of an $S-$act A and B be any subact of A. Then P is called a *minimal prime of* B if $B \subseteq P$ and there is no prime subact strictly between B and P. If A is a centered act, any minimal prime subact of $\{\theta\}$ is called minimal prime in A.

Proposition 2.3. *Let A be a centered $S-$act. Every prime subact of A contains a minimal prime subact of A.*

Corollary 2.4. *Suppose that B is a subact of a centered $S-$act A, for which $rad(B) \neq A$. Then $rad(B)$ is the intersection of all minimal prime subacts of B.*

Lemma 2.5. *Let A be an $S-$act and B a subact of A, F be a free $S-$act and P and Q be ideals of S. The following hold.*

(i) *$FP \subseteq FQ$ if and only if $P \subseteq Q$;*
(ii) *$(FP : F) = P$;*
(iii) *P is a prime ideal of S, if and only if, FP is a prime subact of F.*
(iv) *If B is a prime subact of A, then $(B : A)$ is a prime ideal of S.*

Lemma 2.6. *Suppose that B is a subact and P is a prime subact of an $S-$act A, and I is any ideal of S with $BI \neq A$. Then $BI \subseteq P$ if and only if $I \subseteq (P : A)$ or $B \subseteq P$.*

Using the notation of Lemma 2.6, it is easily checked that,

$$rad(BI) = rad(AI) \cap rad(B). \quad \textbf{(*)}$$

Therefore,

$$rad(rad(B)I) = rad(AI) \cap rad(rad(B)) = rad(AI) \cap rad(B) = rad(BI). \quad \textbf{(**)}$$

suppose that B and C are two subacts of an $S-$act A. Clearly $rad(B\cap C) \subseteq rad(B) \cap rad(C)$, because every prime subact containing B and C also contains $B \cap C$. But the equality is not true in general. However in the following proposition some of the particular cases for equality are considered.

Proposition 2.7. *Suppose that B and C are two subacts of an $S-$act A. For each of the following cases we have $rad(B \cap C) = rad(B) \cap rad(C)$.*

(i) *Both of subacts B and C are radical (in particular when B and C are prime);*
(ii) *$B \subseteq A(B : A)$ and $C \subseteq A(C : A)$;*
(iii) *$(B : A) \cup (C : A) = S$.*

Proposition 2.8. *Suppose that B and C are radical subacts of an $S-$act A and I is any ideal of S. Then $rad((B \cap C)I) = rad(BI) \cap rad(CI)$.*

PROOF. By Proposition 2.7 (i) we have, $rad(B \cap C) = rad(B) \cap rad(C)$. Now we have,

$$rad((B \cap C)I) = rad(AI) \cap rad((B \cap C) = rad(AI) \cap (B \cap C) =$$
$$(rad(AI) \cap rad(B)) \cap (rad(AI) \cap rad(C)) = rad(BI) \cap rad(CI),$$

using Lemma 2.6. □

PROPOSITION 2.9. *Suppose that B and C are radical subacts of an $S-$act A and I is any ideal of S. Then $rad(rad(B \cup C)I) = rad(rad(BI) \cup rad(CI))$.*

PROOF. First it is easily seen that, $rad(B \cup C) = rad(rad(B) \cup rad(C))$. Now by using the equations (**) and (*) in the paragraph after Lemma 2.6, we have,

$$rad(rad(B \cup C)I) = rad((B \cup C)I) = rad(BI \cup CI) = rad(rad(BI) \cup rad(CI)).$$

□

Suppose that S is a semigroup and $(L, \vee, \wedge)$ is a lattice of $S-$acts. L is called an $S-$lattice if the set of ideals of S acts on L by an operation $\cdot$ with the following properties. For any $S-$acts A and B in L and any ideals I and J of S we have,

$$A \cdot I \in L;$$
$$A \cdot (IJ) = (A \cdot I) \cdot J;$$
$$(A \vee B) \cdot I = (A \cdot I) \vee (B \cdot I);$$
$$(A \wedge B) \cdot I = (A \cdot I) \wedge (B \cdot I).$$

Let A be an $S-$act. Then the set of all radical subacts of A with the inclusion is a partially ordered set, denoted by $(Rad(A), \subseteq)$ form a lattice. The following theorem gives more details.

THEOREM 2.10. *Suppose that S is a semigroup and A is an $S-$act.Then the poset $(Rad(A), \subseteq)$, of all radical subacts of A is an $S-$lattice with the following operetions:*

$$B \cdot I = rad(BI) \text{ and } B \wedge C = B \cap C \text{ and } B \vee C = rad(B \cup C),$$

for every radical subacts $B, C \in Rad(A)$ and every ideal I of S.

PROOF. Suppose that B and C are arbitrary radical subacts of A and I and J are any ideals of S. First note that by definition of radical subacts and Lemma 2.7(i), $B \cdot I$ and $B \vee C$ and $B \wedge C$ are in $Rad(A)$. Also it is easily checked that $Rad(A), \vee, \wedge)$ is a lattice. Moreover, we have the following properties for $Rad(A)$ to be an $S-$lattice by using equation (**) and Proposition 2.8 and Proposition 2.9.

$$B \cdot (IJ) = rad(B(IJ)) = rad((BI)J) = rad(rad(BI)J) = rad((B \cdot I)J) = (B \cdot I) \cdot J,$$

and

$$(B \wedge C) \cdot I = rad((B \cap C)I) = rad(BI) \cap rad(CI) = (B \cdot I) \wedge (C \cdot I),$$

and

$$(B \vee C) \cdot I = rad(rad(B \cup C)I) = rad(rad(BI) \cup rad(CI)) = rad(BI) \vee rad(CI)$$
$$= (B \cdot I) \vee (C \cdot I).$$

□

By an $S-$homomorphism between two $S-$lattices we mean a function which preserves operations $\cdot$ and $\vee$ and $\wedge$. Also every bijective $S-$homomorphism is called an $S-$isomorphism. The following theorem and its converse show that relation between lattices of radical subacts of two $S-$acts can be reduced to their prime spectrum.

Theorem 2.11. *Suppose that A and B are $S-$acts and $f : Rad(A) \longrightarrow Rad(B)$ is an isomorphism of $S-$lattices. Then for any arbitrary prime ideal P of S, the restriction of f to $Spec_P(A)$, denoted by $f_P : Spec_P(A) \longrightarrow Spec_P(B)$ is a bijection.*

The converse of the above theorem completes our expectation of relationship between lattices of radical subacts and prime spectrum.

Theorem 2.12. *Let A and B be $S-$acts and $f : Rad(A) \longrightarrow Rad(B)$ be a homomorphism of $S-$lattices. If for each prime ideal P of S the restriction of f to $Spec_P(A)$, gives a bijection $f_P : Spec_P(A) \longrightarrow Spec_P(B)$, then f is an $S-$isomorphism.*

Proof. Let $D \in Rad(B)$. Then $D = rad(D) = \bigcap_{i\in I} B_i$, where the intersection is taken over all prime subacts $B_i \in Spec_{P_i}(B)$, such that $B_i \supseteq D$ for each $i \in I$. Since $f_{P_i} : Spec_{P_i}(A) \longrightarrow Spec_{P_i}(B)$, is a bijection, so there exists $A_i \in Spec_{P_i}(A)$ such that $f_{P_i}(A_i) = B_i$. Now let $C = \bigcap_{i\in I} A_i$. By lemma 2.2, $C \in Rad(A)$. We claim that $f(C) = D$. Since f is $S-$homomorphism, it preserves $\wedge$. So we have,

$$f(C) = f(\bigcap_{i\in I} A_i) = f(\bigwedge_{i\in I} A_i) = \bigwedge_{i\in I} f(A_i) = \bigcap_{i\in I} f(A_i) = \bigcap_{i\in I} B_i = D.$$

So f is onto. Now let

$$C = rad(C) = \bigcap_{C\leq X_i\leq A} X_i \text{ and } C' = rad(C') = \bigcap_{C'\leq Y_i\leq A} Y_i,$$

are radical subacts of A, where X_i's and Y_i's are prime subacts of A, with $(X_i : A) = P_i$ and $(Y_i : A) = Q_i$. Let $f(C) = f(C')$. Then $f(\bigcap_{C\leq X_i\leq A} X_i) = f(\bigcap_{C'\leq Y_i\leq A} Y_i)$. Hence

$$f^{-1}(f(\bigcap_{C\leq X_i\leq A} X_i)) = f^{-1}(f(\bigcap_{C'\leq Y_i\leq A} Y_i)).$$

Since f preserves intersections,

$$\bigcap_{C\leq X_i\leq A} (f_{P_i}^{-1}(f_{P_i}(X_i))) = \bigcap_{C\leq X_i\leq A} (f^{-1}(f(X_i)))$$
$$= \bigcap_{C'\leq Y_i\leq A} (f^{-1}(f(Y_i))) = \bigcap_{C'\leq Y_i\leq A} (f_{Q_i}^{-1}(f_{Q_i}(Y_i))),$$

because for each prime ideal P of S the restriction f_p of f to prime subacts of A is a bijection. So $\bigcap_{C\leq X_i\leq A} X_i = \bigcap_{C'\leq Y_i\leq A} Y_i$, that is, $C = C'$. Thus f is one-to-one. □

3. Conclusion

By studying the relation between $S-$lattices of radical subacts of non-isomorphic acts we can find many of common properties between them, as we can see an example of this for prime subacts in Theorem 2.12.

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پنجاه و دومین
کنفرانس ریاضی ایران
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دانشگاه شهید باهنر کرمان
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۸ تا ۱۱ شهریور ۱۴۰۰

Two stage shrinkage pretest procedure under a reflected gamma loss function

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Abstract. In this paper, we consider the problem of two stage shrinkage pretest (TSP) estimation for the scale parameter σ of a Rayleigh distribution under the reflected gamma loss (RGL) function. We define a TSP estimator using a prior point information σ_0 and compare its risk with the pooled estimator of σ under the RGL function numerically and graphically. The usefulness of this estimator with respect to the pooled estimator in the presence of σ_0 under different cases is discussed.

Keywords: Rayleigh distribution, Reflected gamma loss function, Two stage estimation.

AMS Mathematics Subject Classification [2010]: 62F15

1. Introduction

Let $X_1, \cdots, X_n$ be a random sample of size n taken from a Rayleigh distribution with probability density function (p.d.f)

$$f(x|\sigma) = \frac{x}{\sigma} e^{-\frac{x^2}{2\sigma}}, \quad x > 0. \tag{1}$$

The maximum likelihood estimator of σ is $\hat{\sigma} = \frac{1}{2n}\sum_{i=1}^{n} X_i^2$. Suppose that we have a priori about the parameter σ in form of a point guess σ_0, i.e., the sample data come from a distribution that is close to the Rayleigh distribution with parameter σ_0. This information may be regarded as a nuisance parameter in the statistical estimation of the model. Such information about the parameter is called nonsample information or uncertain prior information. Following [6] we can take the estimator $\hat{\sigma}$ and shrink it toward σ_0 as $\hat{\sigma}^s = k\hat{\sigma} + (1-k)\sigma_0$, $0 \leq k \leq 1$, to construct the shrinkage estimator. The shrinkage pretest estimator is proposed as

$$\hat{\sigma}^{sp} = [k\hat{\sigma} + (1-k)\sigma_0]I(A) + \hat{\sigma}I(A^c) \tag{2}$$

where A is the acceptance region of a test for $H_0 : \sigma = \sigma_0$ against $H_1 : \sigma \neq \sigma_0$, A^c is the complement of A and $I(.)$ is the indicator function. He show that this estimator dominates $\hat{\sigma}$ in a large portion of the parameter space. This strategy of estimation is widely used in literature, see [1].

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In some situations, the researcher can employ a TSP estimator using prior information for achieving a minimum cost of experimentation: he/she consider a small first stage sample and an additional second stage sample for estimation. The earliest work on two stage estimation procedure in the exponential distribution is [3] who considered estimating the mean of an exponential distribution under the SEL. They used a shrinkage estimator on the first sample if the null hypothesis $H_0 : \sigma = \sigma_0$ is accepted; otherwise used the pooled mean of the two samples if H_0 is rejected. [4] used the general entropy loss and compared a TSP estimator with the conventional estimator $\bar{X}_1$.

In this paper, we consider the reflected gamma loss (RGL) function of the form

$$L(\Delta) = l\big\{1 - \Delta^{\gamma} e^{-\gamma(\Delta-1)}\big\},\ \Delta = \frac{\delta}{\sigma}, \tag{3}$$

where $l > 0$ is the maximum loss and $\gamma > 0$ is a shape parameter. This loss is bounded function of Δ by 0 and l. Also, it is asymmetric function of Δ but not convex in Δ and has a unique minimum 0 for $\Delta = 1$, is strictly decreasing on (0,1) and increasing on $(1, \infty)$. $L(0) = l$ and $\lim_{\Delta\to\infty} L(\Delta) = l$. This loss is scale invariant, which is appropriate for estimating scale parameter, and it penalizes heavily under estimation, see [2].

We compute the risk of the pooled estimator and TSP estimator under the RGL in section 2 and compare them numerically and graphically in Section 3. An illustrative example is provided in section 4. We end the paper with some concluding remarks in Section 5.

2. The proposed TSP estimator

Let $X_{11}, X_{12}, \cdots, X_{1n_1}$ be the first sample of size n_1 taken from the Rayleigh distribution with p.d.f given in (1). The MLE of σ is then given by $\hat{\sigma}_1 = \frac{1}{2n_1}\sum_{i=1}^{n_1} X_{1i}^2$. Now, suppose that it is suspected a priori that $\sigma = \sigma_0$ may hold. This information can be tested based on $H_0 : \sigma = \sigma_0$ against $H_1 : \sigma \neq \sigma_0$ at the level of significance α. A likelihood ratio test (LRT) statistic is $\frac{2n_1\hat{\sigma}_1}{\sigma_0} \sim \chi^2_{2n_1}$ under H_0 which has an acceptance region $A = \{\hat{\sigma}_1 : \frac{q_1\sigma_0}{2n_1} \leq \hat{\sigma}_1 \leq \frac{q_2\sigma_0}{2n_1}\}$ where q_1 and q_2 are the values of the lower and upper $100\alpha/2\%$ points of the chi-square distribution with $2n_1$ degrees of freedom, i.e., $q_1 = \chi^2_{2n_1}(\frac{\alpha}{2}),\ q_2 = \chi^2_{2n_1}(1 - \frac{\alpha}{2})$.

If H_0 is accepted, we stop sampling and take the estimator $k\hat{\sigma}_1 + (1-k)\sigma_0$, $0 \leq k \leq 1$. If not so, we take additional observations $X_{21}, X_{22}, \cdots, X_{2n_2}$ of size n_2 and compute the pooled estimator of σ as $\hat{\sigma}_p = \frac{n_1\hat{\sigma}_1 + n_2\hat{\sigma}_2}{n_1+n_2}$, where $\hat{\sigma}_2 = \frac{1}{2n_2}\sum_{i=1}^{n_2} X_{2i}^2$ is the MLE of σ based on data in stage two. Therefore, a version of TSP estimator of σ is

$$\hat{\sigma}^{tsp} = \big[k\hat{\sigma}_1 + (1-k)\sigma_0\big] I(A) + \hat{\sigma}_p I(A^c). \tag{4}$$

Before computing the risk of $\hat{\sigma}^{tsp}$, we calculate the risk function of $\hat{\sigma}_p$ under the RGL function as

$$\begin{aligned} R(\sigma, \hat{\sigma}_p) &= 1 - E\big[(\frac{\hat{\sigma}_p}{\sigma})^{\gamma} e^{-\gamma(\frac{\hat{\sigma}_p}{\sigma}-1)}\big] \\ &= 1 - E\big[U^{\gamma} e^{-\gamma(U-1)}\big] = 1 - \frac{e^{\gamma}(n_1+n_2)^{n_1+n_2}\Gamma(\gamma+n_1+n_2)}{(n_1+n_2+\gamma)^{n_1+n_2+\gamma}\Gamma(n_1+n_2)}, \end{aligned} \tag{5}$$

where $U = \frac{\hat{\sigma}_p}{\sigma} \sim Gamma(n_1+n_2, n_1+n_2)$. The risk function under the RGL for the estimator $\hat{\sigma}^{tsp}$ is given by

$$R(\sigma, \hat{\sigma}^{tsp}) = 1 - E\big[(\frac{\hat{\sigma}^{tsp}}{\sigma})^{\gamma} e^{-\gamma(\frac{\hat{\sigma}^{tsp}}{\sigma}-1)}\big]$$

$$
\begin{aligned}
&= 1 - E\big\{\big(\frac{kU_1}{n_1} + (1-k)\sigma^\star\big)^\gamma e^{-\gamma(\frac{kU_1}{n_1}+(1-k)\sigma^\star-1)}\mathtt{I}(B)\big\} \\
&+ E\big\{\big(\frac{U_1+U_2}{n_1+n_2}\big)^\gamma e^{-\gamma(\frac{U_1+U_2}{n_1+n_2}-1)} I(B)\big\} - E\big\{U^\gamma e^{-\gamma(U-1)}\big\} \\
&= 1 - \frac{e^{\gamma(1-\sigma^\star(1-k))}}{\Gamma(n_1)}\int_{w_1}^{w_2}\big(\frac{ku_1}{n_1} + (1-k)\sigma^\star\big)^\gamma u_1^{n_1-1} e^{-u_1(1+\frac{k\gamma}{n_1})} du_1 \\
&+ \frac{e^\gamma}{(n_1+n_2)^\gamma\Gamma(n_1)\Gamma(n_2)}\int_{w_1}^{w_2}\int_0^\infty (u_1+u_2)^\gamma u_1^{n_1-1}u_2^{n_2-1} e^{-(u_1+u_2)(\frac{\gamma}{n_1+n_2}+1} du_2 du_1 \\
&- \frac{e^\gamma\Gamma(n_1+n_2+\gamma)(n_1+n_2)^{n_1+n_2}}{\Gamma(n_1+n_2)(n_1+n_2+\gamma)^{n_1+n_2+\gamma}} \qquad (6)
\end{aligned}
$$

where $U_i = \frac{n_i\hat{\sigma}_i}{\sigma} \sim Gamma(n_i, 1)$, $i = 1, 2$, $\sigma^\star = \frac{\sigma_0}{\sigma}$, $w_1 = \frac{q_1\sigma^\star}{2}$, $w_2 = \frac{q_2\sigma^\star}{2}$, $B = \{U_1 : w_1 \leq U_1 \leq w_2\}$.

3. Performance analysis

For comparison purposes, the relative efficiency of $\hat{\sigma}^{tsp}$ with respect to $\hat{\sigma}_p$ is calculated as

$$
RE(\hat{\sigma}_{tsp}, \hat{\sigma}_p) = \frac{R(\sigma, \hat{\sigma}_p)}{R(\sigma, \hat{\sigma}^{tsp})}. \qquad (7)
$$

The relative efficiency (7) is drown in Figure 1 for the values $n_1 = 25$, $n_2 = 15$, $\alpha = 0.01, 0.05$, $k = 0.2(0.2)0.8$ and $\gamma = 1$ with respect to $\sigma^\star = \sigma_0/\sigma$ (more figures are provided, but not presented here). It is observed from Figure 1 that the estimator $\hat{\sigma}^{tsp}$ dominates the pooled estimator in the neighborhood of null hypothesis ($\sigma^\star = 1$).

We held fix $\sigma^\star = 1$ and plotted the relative efficiency with respect to k in a Figure but it is not presented. This figure show that relative efficiency is a decreasing function of k for fixed n_1, n_2, γ and α. Moreover, a TSP estimator with smaller level of significance performs better than other estimators with fixed values of another parameters. It can be conclude that for fixed n_1, the relative efficiency increases when n_2 increases.

4. A numerical example

In this section, a numerical example from [5] is provided to illustrate the proposed estimators. Data is related to the recovery time of leukemia patients treated with medication.The MLE of σ is $\hat{\sigma} = 2.882$. The proposed estimator $\hat{\sigma}_{tsp}$ and $RE(\hat{\sigma}_{tsp}, \hat{\sigma}_p)$ are computed for selected values of $\sigma_0 = 2, 3$, $n_1 = 5$ and $n_2 = 5, 7, 10$ when $k = 0.4$, $\gamma = 2$ and $\alpha = 0.05$ and summarized in Table 1.

5. Conclusion

In the present paper, we deal with point shrinkage estimation on two stage in Rayleigh distribution under the RGL function. A TSP estimator of σ is proposed and its risk is computed numerically. For comparison purposes, we compute the R.E. between a TSP and the pooled estimator and study the performance of them graphically. Our findings show that the proposed TSP estimator outperforms the pooled estimator in neighborhood σ_0.

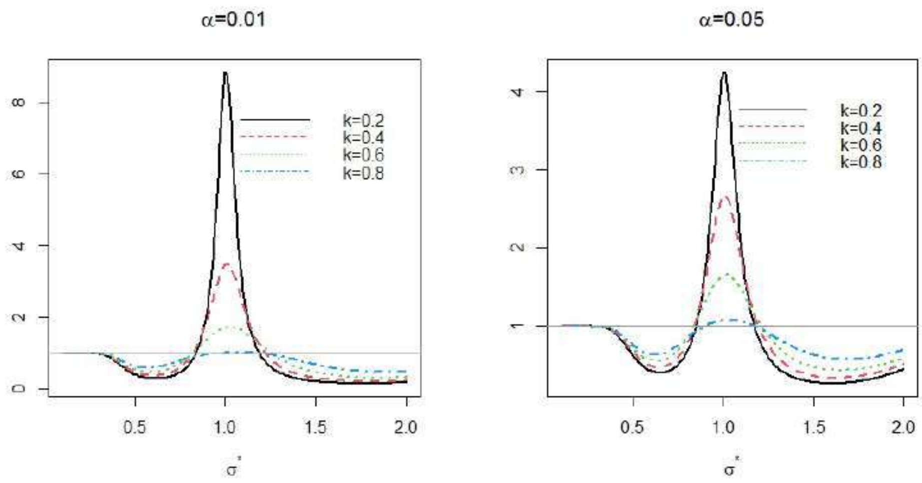

FIGURE 1. Plot of the relative efficiency for the values $n_1 = 25$, $n_2 = 15$, $\gamma = 1$ and selected values of $\alpha = 0.01, 0.05$ and $k = 0.2, 0.4, 0.6, 0.8$ with respect to $\sigma^\star$.

TABLE 1. $RE(\hat{\sigma}_{tsp}, \hat{\sigma}_p)$ and $\hat{\sigma}_{tsp}$ for selected values σ_0, n_1 and n_2, $k = 0.4$, $\gamma = 2$ and $\alpha = 0.05$.

			n_2		
n_1	σ_0		5	7	10
5	3	RE	2.63	2.44	2.17
		$\hat{\sigma}_{tsp}$	3.52	3.52	3.52
	2	RE	0.939	0.839	0.719
		$\hat{\sigma}_{tsp}$	3.17	3.95	4.18

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Engel orthomodular lattices

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Abstract. In this paper, the concept of engel ortomodular lattices are defined and their properties are investigated. The notion of n-Engel ortomodular lattices as a natural generalization of distributive ortomodular lattices is introduced, and we discuss Engel ortomodular lattices, which is defined by left and right normed commutators.

Keywords: Engel ortomodular lattices, commutator, solvable, Engel element.

AMS Mathematics Subject Classification [2010]: 06D35, 03G12, 03B50

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1. Introduction and Preliminary

In 1936 G. Birkhoff suggested taking the lattice of closed subspaces of a Hilbert space as a suitable model for the logic of quantum mechanics. This lattice equipped with the relation of orthogonal complement can be described as an ortholattice. Since then the theory of orthomodular lattices has been developed; the monographs Kalmbach and Beran are highly recommended for the following and other facts about orthomodular lattices.
By an ortholattice we shall mean an algebra $L = (L, \vee, \wedge, ', 0, 1)$ satisfying the following postulates [3]:
(i) the algebra $(L, \vee, \wedge)$ is a lattice, (ii) the unary operation $' : L \longrightarrow L$ is such that the relations $a \vee a^{'} = 1, a \wedge a^{'} = 0$ hold for every $a \in L$, (iii) if $a \leq b$, then $b^{'} \leq a^{'}$, and $(iv)(a')' = a$ for every $a \in L$.
A lattice L is said to be distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for every $a, b, c \in L$.
A lattice L is modular if it satisfies the implication $a \leq c \Longrightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$.
A lattice L which is distributive end complemented is said to be a Boolean lattice. Every element $a \in L$ has in this case exactly one complement which is denoted by a'. A non-empty subset I of the base set L of an ortholattice $(L, \vee, \wedge, ', 0, 1)$ is called an ideal of L if and only if the following conditions hold in L:
$i)$ $a \vee b \in I$, for any $a, b \in I$, (ii) if $a \in I$ and $b \in L$ is such that $b \leq a$, then $b \in I$.
A p-ideal of L is an ideal such that for any $i \in I$ and any $a \in L$, $(i \vee a^{'}) \wedge a \in I$.
As usual, for a, b of ortholattice L, we write aCb if and only if $a = (a \wedge b) \vee (a \wedge b^{'})$ and in this case we say that a and b commute. (For more details we refere redears to [1-4])

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Definition 1.1. [4] For elements a and b of the orthomodular lattice L, we define the commutator of a and b by $(a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$ denoted by $com(a, b)$. Nemly,

$$com(a, b) = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b').$$

The n-th commutator sublattice L_n of a generalized orthomodular lattice L is defined by induction in the following way: $L_0 = L$ and for $n \geq 1$, L_n is by definition the p-ideal generated in L_{n-1} by all the commutators of the generalized orthomodular lattice L_{n-1}. The lattice L is said to be solvable (in the sense of Marsden) if there exists $n \in N$ such that $L_n = \{0\}$. Recall that [3] L is solvable if and only if it is distributive.

Proposition 1.2. [1] *For elements a, b of an orthomodular lattice aCb holds if and only if $com(a, b) = 0$.*

Theorem 1.3. [4] *An orthomodular lattice L is a Boolean algebra iff L is distributive iff aCb for every $a, b \in L$.*

2. Engel elements and Engel sets in orthomodular lattices

Suppose that L be any orthomodular lattice and n be a non-negative integer. For any two elements a and b of L, we define inductively $com(a, {}_nb)$, the *n-Engel left normed commutator* of the pair (a, b), as follows:

$$com(a, {}_0b) = a, ..., com(a, {}_nb) = com(com(a, {}_{n-1}b), b)$$

n-Engel right normed commutator $com({}_na, b)$ of the pair (a, b) is defined by induction as

$$com({}_0a, b) = b, ..., com({}_na, b) = com(a, com({}_{n-1}a, b)).$$

Especially, $com(a, {}_1b) = com({}_1a, b) = com(a, b) = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$.

Definition 2.1. For a positive integer k, an element a of L is called a *right k-Engel element* of L whenever $com(a, {}_kb) = 0$ for all $b \in L$. An element a of L is called a *right Engel element* if it is right k-Engel for some non-negative integer k. We denote by $R(L)$ and $R_k(L)$ the set of right Engel elements and right k-Engel elements, respectively. So

$$R_k(L) = \{a \in L : com(a, {}_kb) = 0, \forall b \in L\} \text{ and } R(L) = \bigcup_{k \in N} R_k(L).$$

Notice that the variable element b appears on the right of bracket and if n can be chosen independently of b, then a is a right n-Engel element of L. Left Engel elements are defined in a similar way. For a positive integer k, an element a of L is called a *left k-Engel element* of L whenever $com(b, {}_ka) = 0$ for all $b \in L$. Also a is said to be a *left Engel element* of L if it is left k-Engel for some non-negative integer k. We denote by $L(L)$ and $L_k(L)$ the set of left Engel elements and left k-Engel elements, respectively.

$$L_k(L) = \{a \in L : com(b, {}_ka) = 0, \forall y \in L\} \text{ and } L(L) = \bigcup_{k \in N} L_k(L).$$

So, an element $a \in L$ is a left Engel element if for all $b \in L$ there exists a positive integer n such that $com(a,_n b) = 0$, where the variable b is on the left of bracket. Also, since $com(a, 0) = com(a, 1) = 0$, for every $a \in L$, $0, 1 \in R(L) \cap L(L)$.

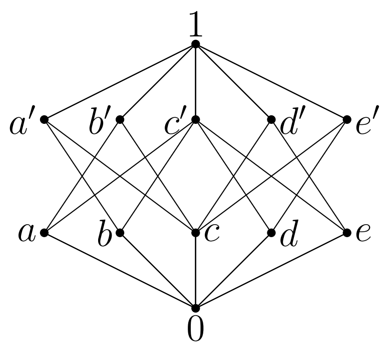
Definition 2.2. An element a of L that is both the left and right Engel element is said to be an *Engel element.* The set of all Engel elements of L is denoted by $En(L)$. Obviously, $0, 1 \in En(L)$.

Theorem 2.3. *$L(L) = L$ if and only if $R(L) = L$ if and only if $En(L) = L$.*

Proof. Suppose that $L = L(L)$. Then $L = \{a \in L : \forall b \in L\, \exists n \in N\ s.t\ com(b, {}_n a) = 0\}$. i.e., for all $a \in L$ and for every $b \in L$ exists $n \in N$ such that $com(b, {}_n a) = 0$. With substituting a to b and b instead a for any $b \in L$, there is positive integer n such that $com(a, {}_n b) = 0$ for all $a \in L$. Hence $L = R(L)$. Now, let $R(L) = L$ and $a \in L = R(L)$. Then, by definition of $R(L)$, for all $b \in L$, there exists $n \in N$ such that $com(a, {}_n b) = 0$. So, for each $b \in L$ there exists $n \in N$ such that $com(b, {}_n a) = 0$. Thus $a \in R(L) \cap L(L)$ and hence $L \subseteq En(L)$. Since $En(L) \subseteq L$ we have $L = En(L)$. If $En(L) = L$, then $R(L) \cap L(L) = L$. So $R(L) = L(L) = L$. □

Corollary 2.4. *$L_n(L) = L$ if and only if $R_n(L) = L$.*

Example 2.5. We consider the orthomodular lattice L given as follows:



By routine calculations, we obtain $com(a, {}_0 d) = a, com(a, {}_1 d) = com(a, d) = c'$ and $com(a, {}_2 d) = com(com(a, d), d) = com(c', d) = 0$. Also $com(b, {}_0 d) = a, com(b, {}_1 d) = com(b, d) = 0$ and $com(b, {}_2 d) = com(com(b, d), d) = com(0, d) = 0$. So for every positive integer $n \geq 3$, we have $com(a, {}_n d) = 0 = com(b, {}_n d) = 0$. Also we obtain $com(a, {}_2 d) = com(com(a, d), d) = com(c', d) = e, com(a, {}_3 d) = com(com(a, {}_2 d), d) = com(e, d) = 0$, $com(b, {}_2 d) = com(com(b, d), d) = com(0, d) = 0$. For every positive integer $n \geq 3$, we have $com(a, {}_n d) = 0, com(b, {}_n d) = 0$ and hence $a, b \in R(L)$. We saw that $R(L) = L(L) = L$. Thus $En(L) = L$.

By Theorem 2.3 and Corollary 2.4 the definition of an Engel orthomodular lattice is rewritten as follows:

Definition 2.6. L is said to be an Engel orthomodular lattice, if for all $a, b \in L$, there is a non-negative integer n such that $com(a, {}_n b) = 0$. Also L is called n-Engel if $L = L_n(L)$ or equivalently $L = R_n(L)$.

Theorem 2.7. *Any solvable orthomodular lattice is an Engel orthomodular lattice.*

Proof. Let L be a solvable orthomodular lattice of class n. Then $L_n = \{0\}$. Hence $com(com(a_1, a_2, ..., a_{n-1}), a_n) = 0$ for all $a_i \in L$. Hence $com(a, {}_n b) = 0$ for all $a, b \in L$. Therefore L is Engel. □

Remark 2.8. The converse of Theorem 2.7, is not true in general. The orthomodular lattice in Example 2.5 is a 2-Engel orthomodular lattice which is not solvable.

Proposition 2.9. *Let I be an ideal of L. Then I and L/I are Engle iff L is an Engel orthomodular lattice.*

Proof. Let L be an Engel orthomodular lattice. Since $I \subseteq L$ and any subset of an Engel orthomodular lattice is an Engel by the definition, I is an Engel. Let a, b be elements of L, then $a/I, b/I \in L/I$. Since L is Engel, then there exists $n \in N$ such that $com(a,{}_n b) = 0$. We claim that $com(a/I,{}_n b/I) = com(a,{}_n b)/I$. By induction on n, if $n = 1$, then

$$\begin{aligned} com(a,b)/I =& ((a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b'))/I \\ =&(a \vee b)/I \wedge (a \vee b')/I \wedge (a' \vee b)/I \wedge (a' \vee b')/I \\ =&(a/I \vee b/I) \wedge (a/I \vee b'/I) \wedge (a'/I \vee b/I) \wedge (a'/I \vee b'/I) \\ =&com(a/I, b/I). \end{aligned}$$

Now, let $com(a/I,{}_{n-1} b/I) = com(a,{}_{n-1} b)/I$. Therefore $com(a/I,{}_n b/I) = com(a/I,{}_{n-1} b/I), b/I) = com(com(a,{}_{n-1} b)/I, b/I) = com(com(a,{}_{n-1} b), b)/I = com(a,{}_n b)/I$. Hence the above claim holds for every positive integer n. Then $com(a/I,{}_n b/I) = com(a,{}_n b)/I = 0/I$. So L/I is Engel. Conversely, let I and L/I be Engle. If a is an arbitrary element of L, then $a/I \in L/I$. Therefore, for every $b \in L$ there exists a positive integer n such that $com(a/I,{}_n b/I) = 0/I$. But $com(a/I,{}_n b/I) = com(a,{}_n b)/I$, so $com(a,{}_n b)/I = 0/I$. Hence $com(a,{}_n b) \in I$. Since I is Engel, then there exists $m \in N$ such that $com(com(a,{}_n b),{}_m b) = 0$. Whence $com(a,{}_{n+m} b) = 0$. Then L is Engel. □

Obvious, if L is Engel, then any subalgebra and sublattice of L is Engel, too. Also the intersection of any two Engel subalgebras of L, is Engel.

Proposition 2.10. *The product of two Engel orthomodular lattices is again an Engel orthomodular lattice.*

Proposition 2.11. *L is 1-Engel iff L is distributive .*

Proof. Let L be an 1-Engel orthomodular lattice and $a, b \in L$. Then $com(a, b) = com({}_1 a, b) = com(a,{}_1 b) = 0$ for all $a, b \in L$. So aCb for all $a, b \in L$ and hence L is distributive. Conversely, if L is a distributive orthomodular lattice, then for all $a, b \in L$ we obtain aCb, which implies $com(a, b) = 0$. Therefore L is 1-Engel orthomodular lattice. □

Theorem 2.12. *The following conditions on orthomodular lattice are equivalent:*
(i) L is a Boolean algebra, (ii) L is distributive, (iii) aCb for every $a, b \in L$,
(iv) C is an equivalence relation, (vi) L is 1-Engel, (vii) L is solvable of class 1.

Proof. The proof is obvious by Propositions 1.2, 2.11 and Theorem 1.3. □

3. Conclusion

In the present paper, we have introduced the concepts of Engel elements and Engel sets in orthomodular lattices and investigated some of their properties. To develop the theory of orthomodular lattices, one of the most encouraging ideas could be investigating the Engel degree and finding a relation diagram between subclasses of orthomodular lattices. For instance, 1-Engel orthomodular lattices are strictly distributive orthomodular lattices.

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پنجاه و دومین کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
Faculty of Mathematics and Computer
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۸ تا ۱۱ شهریور ۱۴۰۰

Application of confluent hypergeometric distribution series to univalent functions defined by the means of convolution structure

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Abstract. In the present paper, we introduce a new subclass of univalent functions associated with confluent hypergeometric distribution. We obtain main results of this subclass related with sharp coefficient estimate and convolution preserving properties. Also we study neighborhood structure and convexity of our subclass.

Keywords: Univalent function, Convolution, confluent hypergeometric function, Convex set, neighborhood, Coefficient bound.

AMS Mathematics Subject Classification [2010]: 30C45, 30C50.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk:

$$\mathbb{U} = \{z \in \mathbb{C} : |z|\},$$

and normalized by $f(0) = f'(0) - 1 = 0$. Also $\mathcal{N}$ be the main subclass of $\mathcal{A}$ consisting the functions of the type:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad (a_k \geqslant 0, \quad z \in \mathbb{U}), \tag{2}$$

see [?] and [?].

A variable X is said to be Poisson distribution if it takes the values 0,1,2,... with probabilities e^{-m}, $\frac{me^{-m}}{1!}$, $m^2\frac{e^{-m}}{2!}$, ... respectively, where m is called the parameter. Thus

$$\mathcal{P}(X = k) = \frac{m^k e^{-m}}{k!}, \qquad (k = 0, 1, 2, \ldots). \tag{3}$$

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Recently, Porwal [?] introduced a power series whose coefficients are probabilities of the Poisson distribution as follows:

$$P(m,z) = z + \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} e^{-m} z^k. \tag{4}$$

For more details, see [?].

Porwal and Kumar [?], introduced the confluent hypergeometric distribution whose probabilities mass function is:

$$\frac{(a)_k}{(c)_k k! F(a;c;m)}, \tag{5}$$

where a and c are complex numbers such that $c \neq 0, -1, -2, \ldots$, $(x)_k$ is the Pochhammer symbol defined by:

$$(x)_k = \begin{cases} 1 & , \quad k = 0 \\ x(x+1)\cdots(x+k-1) & , \quad k \in \mathbb{N} = \{1,2,3,\ldots\} \end{cases}$$

and $F(a;c;m) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k (1)_k} z^k$ is the well-known confluent hypergeometric function which is convergent for all finite values of z, see also [?].

If we put $a = c$, then it reduce to the Poisson distribution. Now we consider a series $\mathcal{CH}(a;c;m;z)$ whose coefficients are probabilities of confluent hypergeometric distribution:

$$\mathcal{CH}(a;c;m;z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1} m^{k-1}}{(c)_{k-1}(k-1)! F(a;c;m)} z^k, \tag{6}$$

where $a, c, m \geqslant 0$.

The Hadamard product (Convolution) for functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad \text{and} \qquad g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

belongs to $\mathcal{A}$ denoted by $f * g$ is defined as follows:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{7}$$

Furthermore $f(z)$ is said to be subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$ if there exists a function w analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = 0$ and $f(\mathbb{U}) \subset g(\mathbb{U})$, see [?] and [?].

We consider $\mathcal{C}_f(\beta, \gamma)$ as a subclass of $\mathcal{N}$ consisting of all functions for which:

$$\operatorname{Re}\left\{ \frac{z\big(Q_f(z)\big)' + \beta z^2 \big(Q_f(z)\big)''}{\beta z \big(Q_f(z)\big)' + (1-\beta) Q_f(z)} \right\} > \gamma, \tag{8}$$

where $0 \leqslant \beta \leqslant 1$, $0 \leqslant \gamma < 1$ and for $f(z) \in \mathcal{N}$:

$$Q_f(z) = \left[\Big(2z - \mathcal{CH}(a;c;m;z) \Big) * f \right](z). \tag{9}$$

2. Main results

First, we state a sharp coefficient bound on the class $\mathcal{C}_f(\beta,\gamma)$. Furthermore, the convolution preserving property on β and γ is investigated.

THEOREM 2.1. *Let $f(z)=z-\sum_{k=2}^{\infty}a_k z^k$ be analytic in $\mathbb{U}$. Then $f\in\mathcal{C}_f(\beta,\gamma)$ if and only if*

$$\sum_{k=2}^{\infty}\frac{(k-\gamma)(1-\beta+\beta k)(a)_{k-1}m^{k-1}}{(1-\gamma)(c)_{k-1}(k-1)!F(a;c;m)}a_k\leqslant 1. \tag{10}$$

PROOF. By making use of (9), we obtain:

$$Q_f(z)=z-\sum_{k=2}^{\infty}\frac{(a)_{k-1}m^{k-1}}{(c)_{k-1}(k-1)!F(a;c;m)}a_k z^k, \tag{11}$$

so if $f\in\mathcal{C}_f(\beta,\gamma)$, then

$$\operatorname{Re}\left\{\frac{z-\sum\limits_{k=2}^{\infty}(k+\beta k(k-1))\frac{(a)_{k-1}m^{k-1}}{(c)_{k-1}(k-1)!F(a;c;m)}a_k z^k}{z-\sum\limits_{k=2}^{\infty}(\beta k+1-\beta)\frac{(a)_{k-1}m^{k-1}}{(c)_k(k-1)!F(a;c;m)}a_k z^k}\right\}>\gamma.$$

By choosing the values of z on the real axis and then letting $z\to 1^-$ through real values, we have:

$$(1-\gamma)-\sum_{k=2}^{\infty}\Big[k+\beta k(k-1)-\gamma(\beta k+1-\beta)\Big]\frac{(a)_{k-1}m^{k-1}}{(c)_k(k-1)!F(a;c;m)}a_k\geqslant 0,$$

or

$$\sum_{k=2}^{\infty}\frac{(k-\gamma)(1-\beta+\beta k)(a)_{k-1}m^{k-1}}{(1-\gamma)(c)_{k-1}(k-1)!F(a;c;m)}a_k\leqslant 1.$$

Conversely, we suppose that (10) holds true. We will show that (8) is satisfied. Using the fact that $\operatorname{Re}W>\gamma$ if and only if $|W-(1+\gamma)|<|W+1-\gamma|$, it is enough to show that:

$$\begin{aligned}\mathcal{L}&=\left|\frac{z\big(Q_f(z)\big)'+\beta z^2\big(Q_f(z)\big)''}{\beta z\big(Q_f(z)\big)'+(1-\beta)Q_f(z)}-1-\gamma\right|\\&<\left|\frac{z\big(Q_f(z)\big)'+\beta z\big(Q_f(z)\big)''}{\beta z\big(Q_f(z)\big)'+(1-\beta)Q_f(z)}+1-\gamma\right|=\mathcal{R}.\end{aligned}$$

But if $\beta z\big(Q_f(z)\big)'+(1-\beta)Q_f(z)=X$, we have:

$$\mathcal{L}=\frac{1}{|X|}\left|z\big(Q_f(z)\big)'+\beta z^2\big(Q_f(z)\big)''-(1+\gamma)X\right|.$$

Thus

$$\begin{aligned}\mathcal{L}=\frac{1}{|X|}\Bigg|&-\gamma z-\sum_{k=2}^{\infty}\Big[\beta k(k-1)+\big(1-(1+\gamma)\beta\big)k-(1+\gamma)(1-\beta)\Big]\times\\&\times\frac{m^{k-1}(a)_{k-1}a_k z^k}{(c)_{k-1}(k-1)!F(a;c;m)}\Bigg|\end{aligned}$$

$$< \frac{|z|}{|X|}\left[\gamma + \sum_{k=2}^{\infty}(1+\beta k-\beta)(k-\gamma-1)\frac{(a)_{k-1}m^{k-1}}{(c)_{k-1}(k-1)!F(a;c;m)}a_k|z|^{k-1}\right],$$

and

$$\begin{aligned}\mathcal{R} &= \frac{1}{|X|}\left|\left[z\big(Q_f(z)\big)' + \beta z^2\big(Q_f(z)\big)'' + (1-\gamma)X\right]\right| \\ &= \frac{1}{|X|}\Bigg|\Bigg[(2-\gamma)z - \sum_{k=2}^{\infty}\big[\beta k(k-1) + \big(1+(1-\gamma)\beta\big)k(1-\gamma)(1-\beta)\big]\times \\ &\qquad \times \frac{m^{k-1}(a)_{k-1}a_k z^k}{(c)_{k-1}(k-1)!F(a;c;m)}\Bigg]\Bigg| \\ &\geqslant \frac{|z|}{|X|}\left[(2-\gamma) - \sum_{k=2}^{\infty}(1+\beta k-\beta)(k-\gamma+1)\frac{m^{k-1}(a)_{k-1}a_k|z|^{k-1}}{(c)_{k-1}(k-1)!F(a;c;m)}\right].\end{aligned}$$

When $z \in \partial\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, it is easy to verify that $\mathcal{R} - \mathcal{L} > 0$ if (10) holds and so the proof is complete. □

Remark 2.2. The result (10) is sharp for the function $H(z)$ given by:

$$H(z) = z - \frac{(1-\gamma)(c)_1F(a;c;m)}{m(2-\gamma)(1+\beta)(a)_1}z^2. \tag{12}$$

Theorem 2.3. *Let the functions $f(z) = z - \sum_{k=2}^{\infty}a_kz^k$ and $g(z) = z - \sum_{k=2}^{\infty}b_kz^k$ be in the class $\mathcal{C}_f(\beta,\gamma)$, then $(f*g)(z)$ belnogs to $\mathcal{C}_f(\beta,\gamma_0)$, where:*

$$\gamma_0 \leqslant 1 - \frac{k-1}{\left(\frac{k-\gamma}{1-\gamma}\right)^2\frac{(1-\beta+\beta k)(a)_{k-1}m^{k-1}}{(c)_{k-1}(k-1)!F(a;c;m)} - 1}. \tag{13}$$

Proof. It is sufficient to prove that

$$\sum_{k=2}^{\infty}\frac{(1-\beta+\beta k)(a)_{k-1}m^{k-1}}{(c)_{k-1}(k-1)!F(a;c;m)}\left(\frac{k-\gamma_0}{1-\gamma_0}\right)a_kb_k \leqslant 1.$$

By using the Cauchy–Schwarz inequality from (10), we obtain:

$$\sum_{k=2}^{\infty}\frac{(1-\beta+\beta k)(a)_{k-1}m^{k-1}}{(c)_{k-1}(k-1)!F(a;c;m)}\left(\frac{k-\gamma}{1-\gamma}\right)\sqrt{a_kb_k} \leqslant 1.$$

Hence, we find the largest γ_0 such that:

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{(1-\beta+\beta k)(a)_{k-1}m^{k-1}}{(c)_{k-1}(k-1)!F(a;c;m)}\left(\frac{k-\gamma_0}{1-\gamma_0}\right)a_kb_k \\ &\leqslant \sum_{k=2}^{\infty}\frac{(1-\beta+\beta k)(a)_{k-1}m^{k-1}}{(c)_{k-1}(k-1)!F(a;c;m)}\left(\frac{k-\gamma}{1-\gamma}\right)\sqrt{a_kb_k} \\ &\leqslant 1,\end{aligned}$$

or equivalently

$$\sqrt{a_kb_k} \leqslant \frac{(1-\gamma_0)(k-\gamma)}{(k-\gamma_0)(1-\gamma)}, \qquad (k \geqslant 2).$$

This inequality holds if

$$\frac{(c)_{k-1}(k-1)!F(a;c;m)}{(1-\beta+\beta k)(a)_{k-1}m^{k-1}}\left(\frac{1-\gamma}{k-\gamma}\right) \leqslant \frac{(1-\gamma_0)(k-\gamma)}{(k-\gamma_0)(1-\gamma)}.$$

After a simple calculation, we get the required result, so the proof is complete. □

THEOREM 2.4. *With the same conditions as in Theorem 2.3,* $(f*g)(z) \in \mathcal{C}_f(\beta_0,\gamma)$, *where:*

$$\beta_0 \leqslant \frac{(1-\beta+\beta k)^2(a)_{k-1}(k-\gamma)m^{k-1}}{(c)_{k-1}(k-1)!F(a;c;m)(1-\gamma)(k-1)} - \frac{1}{k-1}. \tag{14}$$

PROOF. By using the same techniques as in the Theorem 2.3, we can easily prove, so the proof is omitted. □

3. Neighborhood and convexity

In this section, we derive some geometric properties of $\mathcal{C}_f(\beta,\gamma)$ such as convexity and neighborhood structure.

THEOREM 3.1. $\mathcal{C}_f(\beta,\gamma)$ *is a convex set.*

PROOF. It is enough to show that if $f_j(z)$ $(j=1,2,\dots,t)$ be in the class $\mathcal{C}_f(\beta,\gamma)$, then $F(z)=\sum_{j=1}^{t}\lambda_j f_j(z)$ is also in $\mathcal{C}_f(\beta,\gamma)$ where $\lambda_j \geqslant 0$ and $\sum_{j=1}^{t}\lambda_j = 1$. But we have:

$$\begin{aligned} F(z) &= \sum_{j=1}^{t}\lambda_j f_j(z) \\ &= \sum_{j=1}^{t}\lambda_j\Big(z-\sum_{k=2}^{\infty}a_{k,j}z^k\Big) \\ &= z-\sum_{k=2}^{\infty}\Big(\sum_{j=1}^{t}\lambda_j a_{k,j}\Big)z^k. \end{aligned}$$

Since

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{(k-\gamma)(1-\beta+\beta k)(a)_{k-1}m^{k-1}}{(1-\gamma)(c)_{k-1}(k-1)!F(a;c;m)F(a;c;m)}\Big(\sum_{j=1}^{t}\lambda_j a_{k,j}\Big) \\ &= \sum_{j=1}^{t}\lambda_j\left\{\sum_{k=2}^{\infty}\frac{(k-\gamma)(1-\beta+\beta k)(a)_{k-1}m^{k-1}}{(1-\gamma)(c)_{k-1}(k-1)!F(a;c;m)F(a;c;m)}a_{k,j}\right\} \\ &\leqslant \sum_{j=1}^{t}\lambda_j = 1. \end{aligned}$$

So by Theorem 2.1, we conclude that $F(z)\in\mathcal{C}_f(\beta,\gamma)$. Hence the proof is complete. □

Now, we define the (k,δ)–neighborhood of a function $f\in\mathcal{N}$ by:

$$\mathcal{N}_{k,\delta}(f) = \left\{g\in\mathcal{N} : g(z)=z-\sum_{k=2}^{\infty}b_k z^k \quad\text{and}\quad \sum_{k=2}^{\infty}k\,|a_k-b_k|\leqslant\delta\right\}. \tag{15}$$

For the identity function $I(z) = z$, we have:

(16) $$\mathcal{N}_{k,\delta}(I) = \left\{ g \in \mathcal{N} : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty} k\,|b_k| \leqslant \delta \right\}.$$

THEOREM 3.2. *Let* $\delta = \frac{2(1-\gamma)(c)_1 F(a;c;m)}{(2-\gamma)(1+\beta)(a)_1 m}$, *then* $\mathcal{C}_f(\beta,\gamma) \subset \mathcal{N}_{k,\delta}(I)$.

PROOF. By Theorem 2.1, we have:

$$\left|g'(z)\right| \leqslant 1 + \frac{2(1-\gamma)(c)_1 F(a;c;m)}{(2-\gamma)(1+\beta)(a)_1 m} \,|z|^{k-1}.$$

Indeed

$$\left|g'(z)\right| = \left|1 - \sum_{k=2}^{\infty} k b_k z^{k-1}\right| \leqslant 1 + \sum_{k=2}^{\infty} k b_k \,|z|^{k-1}.$$

By choosing the values of z on the real axis and then letting $z \to 1^-$ through real values, we obtain:

$$\sum_{k=2}^{\infty} k b_k \leqslant \frac{2(1-\gamma)(c)_1 F(a;c;m)}{(2-\gamma)(1+\beta)(a)_1 m} = \delta,$$

so $g(z) \in \mathcal{N}_{k,\delta}(I)$. □

The function $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ is said to be a member of $\mathcal{C}_f^{\theta}(\beta,\gamma)$ if there exists a function $f \in \mathcal{C}_f(\beta,\gamma)$ such that:

(17) $$\left|\frac{g(z)}{f(z)} - 1\right| \leqslant 1 - \theta, \qquad (z \in \mathbb{U}, \quad 0 \leqslant \theta < 1).$$

THEOREM 3.3. *If* $f \in \mathcal{C}_f(\beta,\gamma)$ *and*

(18) $$\theta = 1 - \frac{\delta(2-\gamma)(1+\beta)(a)_1 m}{2\big[(2-\gamma)(1+\beta)(a)_1 m - (1-\gamma)(c)_1 F(a;c;m)\big]},$$

then $\mathcal{N}_{k,\delta}(f) \subset \mathcal{C}_f^{\theta}(\beta,\gamma)$.

PROOF. Let $f \in \mathcal{N}_{k,\delta}(f)$, then $\sum_{k=2}^{\infty} k\,|a_k - b_k| < \delta$, which implies the coefficient inequality $\sum_{k=2}^{\infty} |a_k - b_k| \leqslant \frac{\delta}{2}$. Also since $f \in \mathcal{C}_f(\beta,\gamma)$ we have from (10):

$$\sum_{k=2}^{\infty} a_k \leqslant \frac{(1-\gamma)(c)_1 F(a;c;m)}{(2-\gamma)(1+\beta)(a)_1 m},$$

so that

$$\begin{aligned}
\left|\frac{g(z)}{f(z)} - 1\right| &< \left|\frac{\sum\limits_{k=2}^{\infty}(b_k - a_k) z^k}{z - \sum\limits_{k=2}^{\infty} a_k z^k}\right| \\
&< \frac{\sum\limits_{k=2}^{\infty} |b_k - a_k|}{1 - \sum\limits_{k=2}^{\infty} a_k} \\
&\leqslant \frac{\delta}{2}\,\frac{(2-\gamma)(1+\beta)(a)_1 m}{(2-\gamma)(1+\beta)(a)_1 m - (1-\gamma)(c)_1 F(a;c;m)}
\end{aligned}$$

$$= 1 - \theta.$$

Thus by definition $g \in \mathcal{C}_f^{\theta}(\beta, \gamma)$ for θ given by (18). □

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Factorization Properties of Quotients of Polynomial Rings by Monomial Ideals

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Abstract. Suppose that K is a field, $S = K[x_1, \ldots, x_n]$ and I is a monomial ideal of S. We study factorization properties of the ring $R = S/I$. In particular, we present conditions equivalent to R being présimplifiable or a bounded factorization ring or a finite factorization ring or a unique factorization ring. We also present necessary conditions and sufficient conditions for R to be a half-factorial ring.

Keywords: Unique factorization ring, Bounded factorization ring, Finite Factorization ring, présimplifiable ring, Monomial ideal

AMS Mathematics Subject Classification [2010]: 13F15, 13A05, 13F55

1. Introduction

Throughout this paper, all rings are commutative with identity and K is always a field. Also $\mathrm{U}(R)$ denotes the set of units of the ring R.

Although factorization theory in commutative rings has a long history (see, for example [6]), it still gets much attention. For more recent papers on this topic, we refer the reader to [1, 4, 7, 8] and the references therein. At first, the focus of research on this subject was factorization properties of integral domains. In the late nineties, Anderson and Valdes-Leon, generalized this theory to commutative rings with zero-divisors and to modules (see [2, 3]). More recently, in [8], factorization properties of an element of a module with respect to a multiplicatively closed subset of the ring has been investigated. In [7], it is shown that using these generalizations, one can get new results and insights on the classic case of factorization theory in integral domains.

Assume that $S = K[x_1, \ldots, x_n]$ is a polynomial ring in n indeterminates, I is an ideal of S and $R = S/I$. Here we are going to study factorization properties of R, when I is generated by a set of monomials of S. Note that if we let I to be general then R can be any finitely generated K-algebra and hence the question in this case is quite hard. Thus we assume that I is a monomial ideal. Monomial ideals, although are much simpler than general ideals, but they have many interesting properties and have been studied a lot (see for example [5]).

At the end of this introduction, we recall some definitions from factorization theory in rings with zero-divsors. Two elements r, s of R are *associates*, when $Rr = Rs$. We denote

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being associates by $\sim$. Also a nonunit element $r \in R$ is called irreducible, if for $s, t \in R$, $r = st$ implies $r \sim s$ or $r \sim t$. We should mention that there are several other equivalent conditions for being associates or irreducible when R is a domain. But these conditions are not equivalent when we let R to have zero-divisors and hence these lead to different types of associativity and irreducibility in rings with zero-divisors. We refer the reader to [**2**], for a detailed study of these conditions. There is a condition under which all types of associativity (and hence all types of irreducibility) coincide. This is when the ring R is *présimplifiable*, that is, when from $x = xy$ for some $x, y \in R$, we can deduce that either $x = 0$ or $y \in \mathrm{U}(R)$.

By an *atomic factorization* of an element $r \in R$, we mean an equation $r = r_1 r_2 \cdots r_t$ in R, such that every r_i is irreducible. The *length of this factorization* is t. Also the ring R is called *atomic*, when every nonzero nonunit of R has an atomic factorization. Moreover, R is called a *bounded factorization ring* (*BFR*, for short) when for every nonzero nonunit $r \in R$, there is a positive integer N_r, such that the length of any atomic factorization of r is at most N_r. If R is atomic and for each nonzero nonunit element of $r \in R$, the length of all atomic factorizations of r are the same, then R is called a *half-factorial ring* (*HFR*, for short).

Two factorizations $r = r_1 \cdots r_t = s_1 \cdots s_l$ of $r \in R$ are called *isomorphic*, if $l = t$ and after a possible reordering of the r_i's, we have $r_i \sim s_i$ for each $1 \le i \le t$. If every nonzero nonunit element of R has exactly one atomic factorization up to isomorphism, then R is called a *unique factorization ring* (*UFR*, for short). Furthermore, we say that R is a *finite factorization ring* (*FFR*, for short), when R is atomic and for every nonzero nonunit $r \in R$, the number of atomic factorizations of r is finite up to isomorphism.

2. Main results

We recall that here $R = K[x_1, \ldots, x_n]/I$ where I is a monomial ideal. We also assume that $I \subseteq \mathfrak{M}^2$, where $\mathfrak{M} = \langle x_1, \ldots, x_n \rangle$, since if an indeterminate, say x_n, is in I, then $R \cong K[x_1, \ldots, x_{n-1}]/I'$ for a monomial ideal of $K[x_1, \ldots, x_{n-1}]$. Note that R is a Noetherian ring and hence it satisfies the ascending chain condition on its principal ideals. Therefore by [**2**, Theorem 3.2], R is atomic. Also by [**2**, Theorem 3.9], for Noetherian rings being présimplifiable is equivalent to being a BFR. Our first result states when R is a BFR or equivalently is présimplifiable. Note that a monomial ideal has a unique minimal monomial generating set which we denote by $G(I)$, see [**5**, Proposition 1.1.6]. Here we assume that the variables $x_1, \ldots, x_r$ appear in an element of $G(I)$ and $x_{r+1}, \ldots, x_n$ do not appear in any such element.

THEOREM 2.1. *The following are equivalent:*

(1) *R is a BFR;*
(2) *R is présimplifiable;*
(3) *$x_i^{k_i} \in I$ for each $1 \le i \le r$ and for some positive integers k_i.*

Next we investigate when R is a UFR.

THEOREM 2.2. *The ring R is a UFR if and only if one of the following holds:*

(1) *$r = n = 1$;*
(2) *$I = \mathfrak{M}^2$.*

Recall that R is called *irreducible-divisor-finite* when every nonzero element of R has at most a finite number of nonassociate irreducible divisors. Also R is called a weak FFR, when every nonzero nonunit of R has only a finite number of nonassociate divisors. Clearly a FFR is a weak FFR and a weak FFR is irreducible-divisor-finite.

THEOREM 2.3. *The following are equivalent:*
(1) *R is irreducible-divisor-finite;*
(2) *R is a weak FFR;*
(3) *R is a FFR;*
(4) *one of the following cases hold:*
  (a) $n = r = 1$*;*
  (b) $n = r$ *and* $I = \mathfrak{M}^2$*;*
  (c) $n = r$ *and K is a finite field;*
  (d) $n = r + 1$ *and K is a finite field and* $I = \langle x_1, \ldots, x_r\rangle^2$*;*
  (e) $n = r + 1 = 2$ *and* $I = \langle x_1^2, x_2^2\rangle$*;*
  (f) $n = r + 1 = 3$ *and* $I = \langle x_1^2, x_2^2\rangle$*.*

Finally, we investigate when R is a HFR. Yet, we have not fully characterized when R is a HFR, but we can state the following necessary conditions.

THEOREM 2.4. *If R is a HFR, then $r = n$ and $\mathfrak{M}^5 \subseteq I$.*

In the case that I is a power of the maximal ideal $\mathfrak{M}$, we have the following.

THEOREM 2.5. *If $r = n$ and $I = \mathfrak{M}^i$ for some positive integer i, then R is a HFR if and only if either $i = 2$ or $i = 3$.*

At the end it should be noted that we could not find an example of I such that R is a HFR and $\mathfrak{M}^3 \not\subseteq I$. Therefore we guess that if R is a HFR then $r = n$ and $\mathfrak{M}^3 \subseteq I \subseteq \mathfrak{M}^2$.

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The persistence property for submodules of a Dedekind module

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Abstract. Let R be an integral domain and M be a faithful multiplication Dedekind R-module. We proved that every proper submodule of M has the persistence property and for non-zero proper ideals $I_1, \cdots, I_n$ of R, $Ass^{\infty}(I_1^{k_1} \cdots I_n^{k_n} M) = \bigcup_{i=1}^{n} Ass^{\infty}(I_i M)$, where $k_1, \cdots, k_n \geq 1$. We also show that every non-zero submodule of M is Ratlif-Rush closed.

Keywords: Dedekind modules, faithful modules, multiplication modules, persistence property, Ratlif-Rush closed

AMS Mathematics Subject Classification [2010]: 13F05, 13C13, 13C99

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1. Introduction

Throughout this paper all rings are commutative with a non-zero identity and all modules are unitary. A prime ideal P of a ring R is called an *associated prime* of an R-module M if $P = \mathrm{Ann}_R(m)$, where m is a non-zero element of M. The set of all associated primes of an R-module M is usually denoted by $\mathrm{Ass}(M)$.

A proper submodule N of an R-module M is called a *prime submodule* of M, if for every $r \in R, x \in M; rx \in N$ we have $x \in N$ or $r \in (N : M)$, where $(N : M) = \{r \in R : rM \subseteq N\}$. In such a case $P := (N : M)$ is a prime ideal of R and N is said to be P*-prime*. The set of all prime submodules of M is denoted by $\mathrm{Spec}(M)$ (see [8]). Recall that, an integral domain R is called a *Dedekind domain* if every non-zero proper ideal of R is a product of prime ideals. It is well known that Dedekind domains are Noetherian.

Let S be the set of all non-zero divisors of R and R_S be the total quotient ring of R. For any ideal I of R, let $I' := \{x \in R_S | xI \subseteq R\}$. If $II' = R$, then I is said to be an *invertible ideal*. Let R be an integral domain, then R is a Dedekind domain if and only if every non-zero ideal of R is invertible, see [3, Theorem 6.10]. Let S be a multiplicative closed subset of a ring R and M be an R-module. Now let $T := \{s \in S | sm = 0 \; for \; m \in M, \; implies \; that \; m = 0\}$. Then T is a multiplicative closed subset of R. Let R_T be the localization of R at T. Let N be a submodule of M. Put $N' := \{x \in R_T | xN \subseteq M\}$. We say that N is *invertible* in M if $N'N = M$. Note that Bourbaki, in Commutative Algebra,

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$II.5.6.$ has a different definition of invertible submodule. A non-zero module M is said to be a *Dedekind module* if each non-zero submodule of M is invertible (see [4]).

Assume that R is a Noetherian ring and I is an ideal of R. It is known by Brodmann in [1] that $Ass(\frac{R}{I^k})$, stabilize, that is, there exists a positive integer k_0 such that $Ass(\frac{R}{I^k}) = Ass(\frac{R}{I^{k_0}})$ for all $k \geq k_0$. The smallest number k_0 for which this equalities hold is called *the index of stability of* I and $Ass(\frac{R}{I^{k_0}})$ is called *the stable set of associated prime ideals of* I, which is denoted by $Ass^\infty(I)$. An ideal I is said to satisfy the persistence property if $Ass(\frac{R}{I}) \subseteq Ass(\frac{R}{I^2}) \subseteq \cdots \subseteq Ass(\frac{R}{I^k}) \subseteq \cdots$. See [2] for example of ideals that does not satisfy the persistence property. An R-module M is called a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$, it can be shown that $N = (N : M)M$ in this case. Let $N = IM$ is a submodule of a multiplication R-module M. The submodule N is said to *satisfy the persistence property* if $Ass(\frac{M}{IM}) \subseteq Ass(\frac{M}{I^2M}) \subseteq \cdots \subseteq Ass(\frac{M}{I^kM}) \subseteq \cdots$. The smallest number k_0 for which $Ass(\frac{M}{I^kM}) = Ass(\frac{M}{I^{k_0}M})$ $(k \geq k_0)$ is called *the index of stability of* N and $Ass(\frac{M}{I^{k_0}M})$ is called *the stable set of associated prime ideals of* N, which is denoted by $Ass^\infty(N)$. Let I be an ideal of R. The ideal $I^* := \bigcup_{n\in\mathbb{N}}(I^{n+1} : I^n)$ is an interesting ideal first studied by Ratlif and Rush in [7]. The ideal I^* is called *the Ratlif-Rush ideal associated with* I or *the Ratlif-Rush closure of* I. An ideal I for which $I^* = I$ is called *Ratlif-Rush closed.* In [6] it is proved every non-zero ideal in a Dedekind ring is Ratlif-Rush closed. Let I be an ideal of a ring R and M be an R-module. We set $(IM)^* := \bigcup_{n\in\mathbb{N}}(I^{n+1}M : I^nM)$. The ideal $(IM)^*$ is called *the Ratlif-Rush closure of* IM. A submodule IM for which $(IM)^* = I$ is called *Ratlif-Rush closed.*

2. Main results

Recall that an R-module M is called a *faithful* R*-module* if $Ann_R(M) = 0$, where $Ann_R(M) := \{r \in R | rM = 0\}$.

PROPOSITION 2.1. *Let R be a ring and M be a faithful multiplication R-module. Then M is flat.*

PROOF. Since $Ann_R(M) = 0$, by [5, Theorem 4.1], M is flat. □

PROPOSITION 2.2. *Let R be an integral domain and M be a multiplication R-module. Then M is faihtful if and only if M is flat.*

PROOF. $\Rightarrow$) It follows from Proposition 2.1.
$\Leftarrow$) By [9, Proposition 2.5.4], every flat module is torsion-free. Since R is an integral domain, it is easy to see that $Ann_R(M) = 0$. □

PROPOSITION 2.3. *Let R be an integral domain and M be a faithful multiplication R-module. Then M is a finitely generated projective R-module.*

PROOF. By [8, Corollary 2.6], M is a finitely generated R-module. So by Proposition 2.1, M is flat and hence by [9, Theorem 2.6.18], M is projective. □

THEOREM 2.4. *Let R be a domain and M be a faithful multiplication R-module. Then the following conditons are equivalent.*
(1) *R is a Dedekind domain.*
(2) *M is a Dedekind R-module.*
(3) *M is a Noetherian R-module and every submodule of M is projective.*

Proof. (1) $\Leftrightarrow$ (2) It follows from [4, Theorem 3.4] and [4, Theorem 3.5].
(2) $\Rightarrow$ (3) Assume that N is a sunmodule of M and hence there exists an ideal I of R such that $N = IM$. By (2), N is invertible and by [8, Corollary 2.6], M is finitely generated. Hence by [4, Lemma 3.3], I is invertible. Thus by [9, Theorem 5.2.6], I is finitely generated. Since M and I are finitely generated, then $N = IM$ is finitely generated. So M is Noetherian. Since R is Dedekind, I is projective. By Proposition 2.3, M is projective and hence M is flat. So we have $IM \cong I \otimes M$. By [9, Theorem 2.3.8], N is projective.
(3) $\Rightarrow$ (2) Let P be a prime ideal of R and $N = IM$ be a submodule of M, where I is an ideal of R. By Proposition 2.3, M is projective and so $N = IM \cong I \otimes M$. By (3), N is projective. Now since every projective module over a local ring is free, we have $N_P \cong \bigoplus_i A_i$ and $M_P \cong \bigoplus_j B_j$, where $A_i \cong B_j \cong R_P$. Then we have $\bigoplus_i A_i \cong N_P \cong I_P \otimes M_P \cong I_P \otimes (\bigoplus_j B_j) \cong \bigoplus_j (I_P \otimes B_j)$ and hence $\bigoplus_i A_i \cong \bigoplus_j I_P$. Thus I_P is a flat R_P-module for every prime ideal P and so I is a flat R-module. Since N is finitely generated, by [4, Note 3.7], I is finitely generated and by [9, Theorem 5.2.6], I is invertible. It is clear that $N = IM$ is invertible. Therefore M is a Dedekind module.

$\square$

Corollary 2.5. *Let R be an integral domain and M be a faithful multiplication Dedekind R-module. Then every non-zero prime submodule of M is maximal.*

Proof. Assme that $N = IM$ is a non-zero prime submodule of M, where I is a non-zero ideal of R and $K = JM$ is a proper submodule of M such that $N \subsetneq K$, where J is a non-zero proper ideal of R. Since M is Dedekind, N, K, I and J are invertible. By Proposition 2.1, M is flat and by [9, Proposition 2.5.4], M is torsion-free. Hence by [4, Lemma 3.1], $N' = I'$ and $K' = J'$. Since $N \subseteq K$, we have $K'N \subseteq K'K = M$. So $K' \subseteq N'$ and hence $IJJ' \subseteq IJI'$. Thus $I \subseteq J$. By [8, Corollary 2.3], I is prime and by Theorem 2.4, I is maximal. So we have $I = J$ and hence $N = K$. $\square$

Theorem 2.6. *Let R be an integral domain and M be a faithful multiplication Dedekind R-module. Suppose that $N = IM$ is a non-zero proper submodel of M, where I is a non-zero proper ideal of R. Then N has the persistence property. Furthermore, $Ass^\infty(N) = Ass(\frac{M}{N}) = Ass^\infty(I) = Ass(\frac{R}{I})$*

Proof. By Theorem 2.4, R is a Dedekind domain and so $I = P_1^{\alpha_1} \cdots P_n^{\alpha_n}$, where $P_1, \cdots, P_n$ are distinct non-zero prime ideals of R and $\alpha_1, \cdots, \alpha_n$ are positive integers. By Theorem 2.4, P_i is maximal for all $i = 1, \cdots, n$ and so by [9, Theorem 1.2.11], we have $\frac{R}{I^k} \cong \frac{R}{P_1^{k\alpha_1}} \oplus \cdots \oplus \frac{R}{P_n^{k\alpha_n}}$. We know that $M \otimes \frac{R}{I^k} \cong \frac{M}{I^k M}$. So $\frac{M}{I^k M} \cong M \otimes (\frac{R}{P_1^{k\alpha_1}} \oplus \cdots \oplus \frac{R}{P_n^{k\alpha_n}}) \cong (M \otimes \frac{R}{P_1^{k\alpha_1}}) \oplus \cdots \oplus (M \otimes \frac{R}{P_n^{k\alpha_n}}) \cong \frac{M}{P_1^{k\alpha_1} M} \oplus \cdots \oplus \frac{M}{P_n^{k\alpha_n} M}$. Hence $Ass(\frac{M}{I^k M}) = Ass(\frac{M}{P_1^{k\alpha_1} M}) \cup \cdots \cup Ass(\frac{M}{P_n^{k\alpha_n} M})$. Since P_i is maximal for all $i = 1 \cdots n$, we have $Ass(\frac{M}{I^k M}) = \{P_1, \cdots, P_n\}$. Thus $Ass(\frac{M}{I^k M}) = Ass(\frac{M}{IM})$. Therefore $Ass^\infty(N) = Ass(\frac{M}{IM}) = Ass^\infty(I)$. $\square$

Corollary 2.7. *Let R be an integral domain, $I_1, \cdots, I_n$ be non-zero proper ideals of R and M be a faithful multiplication Dedekind R-module. Then $Ass^\infty(I_1^{k_1} \cdots I_n^{k_n} M) = \bigcup_{i=1}^n Ass(\frac{M}{I_i M})$ for all $k_1, \cdots, k_n \in \mathbb{N}$.*

Proof. By Theorem 2.4, R is a Dedekind domain and so by [6, Theorem 2.6], we have $Ass^\infty(I_1^{k_1} \cdots I_n^{k_n}) = \bigcup_{i=1}^n Ass(\frac{R}{I_i})$. By Theorem 2.6, $Ass^\infty(I_1^{k_1} \cdots I_n^{k_n}) = Ass^\infty(I_1^{k_1} \cdots I_n^{k_n} M)$ and $Ass(\frac{R}{I_i}) = Ass(\frac{M}{I_i M})$. Thus the proof is completed. $\square$

Corollary 2.8. *Let R be an integral domain, $I_1, \cdots, I_n$ be non-zero proper ideals of R and M be a faithful multiplication Dedekind R-module. Then $Ass^\infty(I_1^{k_1} \cdots I_n^{k_n} M) = \bigcup_{i=1}^n Ass^\infty(I_i M)$ for all $k_1, \cdots, k_n \in \mathbb{N}$.*

Proof. It follows from Theorem 2.6 and Corollary 2.7. □

Theorem 2.9. *Let R be an integral domain, I be a non-zero proper ideal of R and M be a faithful multiplication Dedekind R-module. Then $(I^{k+i}M : I^i M) = I^k$ for all $i, k \in \mathbb{N}$.*

Proof. It is clear that $I^k \subseteq (I^{k+i}M : I^i M)$. Now we show that $(I^{k+i}M : I^i M) \subseteq I^k$. Let $r \in (I^{k+i}M : I^i M)$. So we have $rI^i M \subseteq I^{k+i}M$. By Theorem 2.4, $I^i M$ is inverible and hence $rM \subseteq I^k M$. Then $r \in (I^k M : M)$. Since M is faithful multiplication, by [8, Corollary 2.7], $(I^k M : M) = I^k$. Thus $r \in I^k$. □

Proposition 2.10. *Let I be a non-zero ideal of a ring R and M be an R-module. If $(IM)^*$ is Ratlif-Rush closed, then I^* is Ratlif-Rush closed.*

Proof. We know that $I \subseteq I^*$. Let $r \in I^* = \bigcup_{n \in \mathbb{N}} (I^{n+1} : I^n)$. So $r \in (I^{n+1} : I^n)$ for some $n \in \mathbb{N}$ and hence $rI^n \subseteq I^{n+1}$. We have $rI^n M \subseteq I^{n+1}M$. Thus $r \in (IM)^* = I$ and so $I^* = I$. □

Corollary 2.11. *Let R be an integral domain and M be a faithful multiplication Dedekind R-module. Then every non-zero submodule of M is Ratlif-rush closed.*

Proof. Let $N = IM$ be a non-zero submodule of M, where I is a non-zero ideal of R. By Theorem 2.9, we have $(IM)^* = \bigcup_{n \in \mathbb{N}} (I^{n+1}M : I^n M) = I$. □

3. conclusion

Let R be an integral domain and M be a faithful multiplication Dedekind R-module. we proved that every proper submodule of M has the persistence property and for non-zero proper ideals $I_1, \cdots, I_n$ of R and $Ass^\infty(I_1^{k_1} \cdots I_n^{k_n} M) = \bigcup_{i=1}^n Ass^\infty(I_i M)$, where $k_1, \cdots, k_n \geq 1$. We also proved that every non-zero submodule of M is Ratlif-Rush closed.

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Faculty of Mathematics and Computer
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پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Numerical solutions of a mathematical model for the spread of computer virus using an artificial neural networks

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Abstract. Computer virus is a harmful computer program that enters the victim computer without authorization. In this work, we intend to consider an epidemiological model of computer virus. The model consists of three nonlinear first order ordinary differential equations. We solve the model with the aid of the theory of universal approximation capability of artificial neural networks. To do this, we propose a three layer feedforward neural networks to approximate the system of nonlinear ordinary differential equations. The numerical solutions are presented in order to show the efficiency and accuracy of the proposed method.

Keywords: Computer virus, Modified epidemiological model, System of ordinary differential equations, Numerical solutions, Artificial neural networks.

AMS Mathematics Subject Classification [2010]: 34Fxx, 68Uxx, 68Wxx

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1. Introduction

Computer virus is a harmful computer program that can be written with different aims. This program enters the victim computer without authorization. The modified Susceptible-Infectious-Recovered (SIR) model as a classical epidemiological model were proposed to model the spread of computer virus [1]. Some methods are presented to solve this model such as collocation method [1], homotopy analysis method [2], adomian decomposition method [3], and variational method [4].
In this work, we are motivated to investigate the modified SIR epidemiological model for the spread of computer virus by using an artificial neural networks.
The rest of this work is organized as follows. In Section 2, we introduce the modified SIR model. In Section 3, we deal with the mathematical framework of the proposed artificial neural networks. In Section 4, we present numerical simulations to substantiate our theoretical results with the aid of the proposed method. Finally, Section 5, outlines the given work.

2. The modified SIR model for the spread of computer virus

The modified SIR epidemiological model for computer virusesus is defined by [1]. Let S(t), I(t), and R(t) denote the number of suceptible, infected, and recovered computers.

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TABLE 1. Parameters and functions values of the modified SIR model

Parameter and functions	Values
λ	0.001
ϵ	0.1
d	0.1
f_1, f_2, f_3	0

This model consists of three first order ordinary differential equations as:

$$\begin{cases} \frac{dS}{dt} = f_1(t) - \lambda S(t)I(t) - dS(t), \\ \\ \frac{dI}{dt} = f_2(t) + \lambda S(t)I(t) - \epsilon I(t) - dR(t), \\ \\ \frac{dR}{dt} = f_3(t) + \epsilon I(t) - dR(t), \end{cases} \tag{1}$$

with initial values

S(0)=20, I(0)=15 , and R(0)=10.

We consider the values for the parameters and functions from Table 1 [1].

3. The proposed method for solving the modified SIR model

The universal approximation capability of feedforward artificial neural network is an important fact. It states that any smooth function can be approximated by feedforward artificial neural networks arbitrarily well. The universal approximation capability of feedforward neural networks in the weighted space of continuous function can be found in [5]. Elfwing et al. [6] proposed sigmoid-weighted neural networks. They showed that sigmoid-weighted neural networks are more accurate than feedforward sigmoid neural networks. We use these networks for solving systems of first order ordinary differential equations. The activation function of these neural networks is introduced as follows:

$$a(x) = \sigma(x)(1 + x(1 - \sigma(x))). \tag{2}$$

Here, we decribe the formuiation of differential equations using our neural networks. We use feedforward three layer sigmoid-weighted neural networks to approximate solutions of systems of three first order ordinary differential equations with initial value conditions. We have:

$$\begin{cases} \frac{dy_r}{dx} = f_r(x, y_1, ..., y_n) \qquad r = 1, 2, ..., n \qquad \text{and} \qquad x \in [a, b], \\ y_r(a) = A_r, \qquad r = 1, 2, ..., n. \end{cases} \tag{3}$$

The trial solution $y_{t_r}(x, p_r)$ satisfies the initial conditions is provied as:

$$y_{t_r}(x, p_r) = A_r + (x - a)N_r(x, p_r) \qquad r = 1, 2, ..., n. \tag{4}$$

For each $r, N_r(x, p_r)$ is the output of the feedforward sigmoid-weighted neural network with input x and the network adjustable parameters (weights) p_r. The initial network

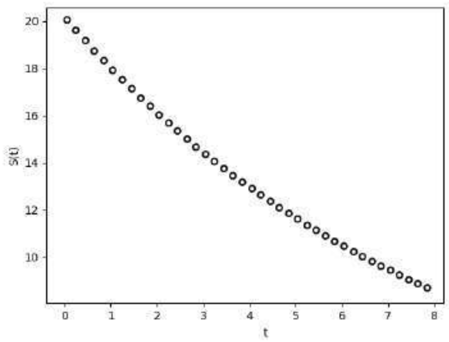

FIGURE 1. The approximate solutions of the number of suceptible computers

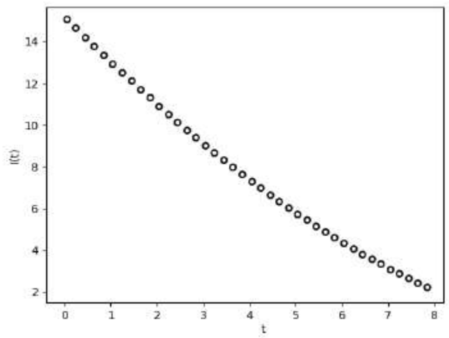

FIGURE 2. The approximate solutions of the number of infected computers

parameters are taken as random. From Eq. (4), we conclude that the derivative of $y_{t_r}(x, p_r)$ with respect to x is as follows:

$$\frac{dy_{t_r}(x, p_r)}{dx} = N_r(x, p_r) + (x - a)\frac{dN_r(x, p_r)}{dx} \qquad r = 1, 2, ..., n. \tag{5}$$

The error function is given as: [7, 8]

$$E(x, p) = \sum_{i=1}^{h}\sum_{r=1}^{n}\frac{1}{2}\bigg(\frac{dy_{t_r}(x_i, p_r)}{dx} - f_r\big(x_i, y_{t_1}(x_i, p_1), ..., y_{t_n}(x_i, p_n)\big)\bigg)^2. \tag{6}$$

For minimizing the error function $E(x, p)$ i.e. to update the feedforward sigmoid-weighted neural network parameters (weights), we differentiate $E(x, p)$ with respect to the parameters.

4. Numerical results

We use the data set tabulated in Table 1. The numerical results of the proposed atrificial neural networks for solving the system of Eqs. (1) are presented in Figs. 1-3.

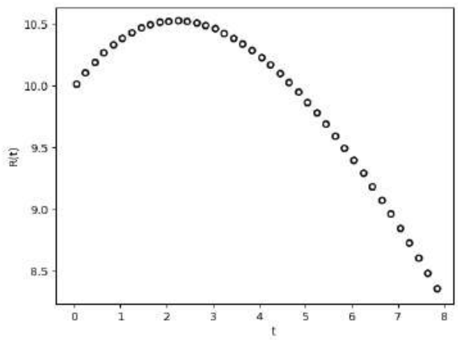

FIGURE 3. The approximate solutions of the number of recoverd computers

5. Conclusions

We have investigated the modified SIR epidemiological model for the spread of computer virus. The artificial neual networks are employed to solve the given model. The obtained results has been provided a good starting step for describing the spread of computer virus. We have shown several figures using these results. Thus, we have established the approximation capability of the proposed method. For future work, we are interested to fractional calculus incorporating into this modified epidemiological model.

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Global existence and uniform stability for a system of wave equations of Kirchhoff type with degenerate damping effects and nonlinear sources

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Abstract. In this article we are concerned with asymptotic stability and lower bounds of blow up solutions for a class of coupled wave equations of Kirchhoff type with degenerate damping effects and nonlinear sources. Under appropriate assumptions on initial datum we show existence of global solutions and obtain an energy decay estimate by employing a lemma of Komornik [1].

Keywords: Kirchhoff equation, stability, instability

AMS Mathematics Subject Classification [2010]: 35B40, 35L20, 35B35

1. Introduction

In this work we investigate the following system of Kirchhoff wave equations

$$\partial_t^2 u_i - M\left(\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2\right)\Delta u_i + \theta_i \partial_t u_i + \Big(|u_1|^{\kappa_i} + |u_2|^{\varrho_i}\Big)|\partial_t u_i|^{q_i-1}\partial_t u_i = f_i(u_1,u_2), \tag{1}$$

for $i = 1,2$, $(x,t) \in \Omega \times (0,T)$ with the initial-boundary conditions

$$\begin{cases} u_i(x,0) = u_0^i(x), \quad \partial_t u_i(x,0) = u_1^i(x), \quad x \in \Omega, \quad i = 1,2, \\ u_i(x,t) = 0, \qquad (x,t) \in \Gamma \times (0,T), \quad i = 1,2, \end{cases} \tag{2}$$

where Ω is an open bounded domain in $\mathbb{R}^N$ $(N = 1,2,3)$, $M(s) = 1+s^\gamma$ where γ, $\theta_i, \kappa_i, \varrho_i, q_i$ $(i = 1,2)$ are some positive constants and

$$\begin{cases} f_1(u_1,u_2) = a|u_1+u_2|^{2(r+1)}(u_1+u_2) + b|u_1|^r u_1 |u_2|^{r+2}, \\ f_2(u_1,u_2) = a|u_1+u_2|^{2(r+1)}(u_1+u_2) + b|u_2|^r u_2 |u_1|^{r+2}. \end{cases} \tag{3}$$

For different kinds of the parameters γ, $\theta_i, \kappa_i, \varrho_i, q_i$ $(i = 1,2)$, the above problem investigated by many authors. To have an overview of results related to nonlinear wave equations during the last half century we refer the interested reader to the recent review work by Messaoudi and Talahmeh [3]. Wu in [5] considered (1)-(2) for $M \equiv 1$ and $\theta_i = 0$, $(i = 1,2)$ in presence the terms $\int_0^t g_i(t-s)\Delta u_i(s)ds$, $(i = 1,2)$. By employing the perturbed energy

*Speaker. Email address: peyravi@shirazu.ac.ir.com technique the author proved that the decay rate of energy is similar to those of the relaxation functions when $0 < q_i < 1, (i = 1, 2)$. Under suitable assumptions on the initial data, the relaxation functions and degenerate damping terms, the author obtained global existence and general decay of weak solutions. In this article, instead of using the Lyapunov perturbed energy method and defining Nehari functional types, we investigate the uniform stability of solutions by using a technical lemma (Lemma 2.4 below). First, we define weak solutions associated to the problem (1)-(2). For simplicity we assume that $a = b = \theta_i = 1, (i = 1, 2)$.

Definition 1.1. A pair (u_1, u_2) is said to be a weak solution of (1)-(2) on $[0, T]$ if $u_i \in C_w([0, T], H_0^1(\Omega))$, $\partial_t u_i \in C_w([0, T], L^2(\Omega)) \cap L^2(\Omega \times [0, T])$, $u_0^i(x) \in H_0^1(\Omega)$, $u_1^i(x) \in L^2(\Omega)$, and

$$\begin{aligned}\langle \partial_t u_i, \phi_i\rangle_{L^2(\Omega)} &+ \int_0^t M\big(\|\nabla u_1(s)\|_2^2 + \|\nabla u_2(s)\|_2^2\big)\langle \nabla u_i(s), \nabla\phi_i\rangle_{L^2(\Omega)} ds\\ &+ \langle u_i(s), \phi_i\rangle_{L^2(\Omega)} ds + \int_0^t \langle \Big(|u_1(s)|^{\kappa_i} + |u_2(s)|^{\varrho_i}\Big)|\partial_t u_i(s)|^{q_i-1}\partial_t u_i(s), \phi_i\rangle_{L^2(\Omega)} ds\\ &= \langle u_1^i, \phi_i\rangle_{L^2(\Omega)} + \langle u_0^i, \phi_i\rangle_{L^2(\Omega)} + \int_0^t \langle f_i(u_1(s), u_2(s)), \phi_i\rangle_{L^2(\Omega)} ds,\end{aligned}$$

holds for all $\phi_i \in H_0^1(\Omega)$, $i = 1, 2$.

Next, we provide the following local existence result which can be obtained by adopting the arguments in [4]:

Theorem 1.2. *Suppose that, for $i = 1, 2$, $u_0^i \in H_0^1(\Omega)$, $u_1^i \in L^2(\Omega)$, $\gamma \geq 1$ and (4) holds. If $0 < q_i < 1$, $(i = 1, 2)$ and*

$$\begin{cases} for\ N = 1, 2: & \kappa_i, \varrho_i \geq 1, \quad i = 1, 2,\\ for\ N = 3: & \max\{\kappa_i, \varrho_i\} \leq 3(1 - q_i), \quad i = 1, 2.\end{cases}$$

Then there exists a unique local weak solution (u_1, u_2) defined on $[0, T]$ for the problem (1)-(2) in the sense of definition 1.1.

By (3) we have

$$u_1 f_1(u_1, u_2) + u_2 f_2(u_1, u_2) = 2(r + 2)F(u_1, u_2), \qquad \forall (u_1, u_2) \in \mathbb{R}^2,$$

where

$$F(u_1, u_2) = \frac{1}{2(r + 2)}\Big(a|u_1 + u_2|^{2(r+2)} + 2b|u_1 u_2|^{r+2}\Big),$$

where r satisfies

$$\begin{cases} r = 0, & \text{if} \quad N = 3,\\ r \geq 0, & \text{if} \quad N = 1, 2.\end{cases} \tag{4}$$

Associated to problem (1)-(2) we define

$$J(u)(t) = J(t) = \frac{1}{2}\big(\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2\big) + \frac{1}{2(\gamma + 1)}\big(\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2\big)^{(\gamma+1)} - \int_\Omega F(u_1, u_2)dx,$$

on $H_0^1(\Omega)$ and the energy identity

$$E(t) = \frac{1}{2}\big(\|\partial_t u_1\|_2^2 + \|\partial_t u_2\|_2^2\big) + J(t). \tag{5}$$

Multiplying the first equation in (1) by $\partial_t u_1$ and the second one by $\partial_t u_2$ and using initial and boundary conditions, it is straightforward to see

$$(6)\quad E(t)-E(0)=-\int_0^t\int_\Omega\sum_{i=1}^2\Big[\big(|u_1(s)|^{\kappa_i}+|u_2(s)|^{\varrho_i}\big)|\partial_t u_i(s)|^{q_i+1}+\|\partial_t u_i(s)\|_2^2\Big]dxds,$$

for all $t\geq 0$. By using (4) and following [2], there exists a positive constant η such that

$$(7)\qquad 2(r+2)\int_\Omega F(u_1,u_2)dx\leq \eta\Big(\|\nabla u_1\|_2^2+\|\nabla u_2\|_2^2\Big)^{r+2}.$$

2. Global existence and asymptotic stability

In this section we investigate asymptotic stability of solutions to the problem (1)-(2). Setting $\zeta(t)=\big(\|\nabla u_1(t)\|_2^2+\|\nabla u_2(t)\|_2^2\big)^{1/2}$, then by (5) and (7) we have

$$(8)\qquad E(t)\geq J(t)\geq\frac{1}{2}\big(\zeta(t)\big)^2-\frac{\eta}{2(r+2)}\big(\zeta(t)\big)^{2(r+2)}\triangleq Z(\zeta(t)),\qquad \forall t\geq 0.$$

It is not difficult to check that Z is increasing over $(0,\zeta_1)$, decreasing over $(\zeta_1,+\infty)$ where $\zeta_1=\Big(\eta^{-1}\Big)^{\frac{1}{2(r+1)}}$, and $Z(\zeta)\to-\infty$ as $\zeta\to+\infty$ and $Z(\zeta)\to 0$ as $\zeta\to 0^+$. So, it is not difficult to check that Z takes its maximum at ζ_1 and this maximum is $E_1:=\frac{r+1}{2(r+2)}\Big(\frac{1}{\eta}\Big)^{\frac{1}{r+1}}$.

Lemma 2.1. *Let $u_0^i\in H_0^1(\Omega)$ and $u_1^i\in L^2(\Omega)$, $(i=1,2)$. If $\zeta(0)<\zeta_1$ and $E(0)<E_1$, then $\zeta(t)<\zeta_1$ for all $t\geq 0$.*

Proof. Using the assumptions and taking (8) into account, a contradiction argument gives the result. □

Remark 2.2. From the Lemma 2.1 we have

$$(9)\qquad Z(\zeta(t))\geq\zeta^2(t)\left(\frac{1}{2}-\frac{\eta}{2(r+2)}\zeta_1^{2(r+1)}\right)=\zeta^2(t)\left(\frac{r+1}{2(r+2)}\right),\qquad\forall t\geq 0.$$

Therefore, by (6) and $E(0)<E_1$ we find

$$(10)\qquad \|\nabla u_1(t)\|_2^2+\|\nabla u_2(t)\|_2^2\leq\left(\frac{2(r+2)}{r+1}\right)E(t)\leq\left(\frac{2(r+2)}{r+1}\right)E(0),\qquad\forall t\geq 0.$$

This shows that solutions are global and bounded in time.

Remark 2.3. By (9) it is clear that $Z(\zeta(t))\geq 0$, for all $t\geq 0$. Hence, by (8) we deduce that, for any $t\geq 0$, $J(t)\geq 0$ and thus

$$\|\partial_t u_1(t)\|_2^2+\|\partial_t u_2(t)\|_2^2\leq E(t),\qquad\forall t\geq 0.$$

Next, we show that the energy solution to the problem (1)-(2) uniformly goes to zero as an exponential function. To this end we use the following lemma by Komornik [1]:

Lemma 2.4. *Let $E:\mathbb{R}_+\to\mathbb{R}_+$ be a non-increasing function and there exists $\omega>0$ such that*

$$\int_t^{+\infty}E(s)ds\leq\frac{1}{\omega}E(t),\qquad\forall t\geq 0,$$

then $E(t)\leq E(0)e^{1-\omega t}$ for all $t\geq 0$.

Our main result reads in the following theorem:

Theorem 2.5. *Let $u_0^i \in H_0^1(\Omega)$ and $u_1^i \in L^2(\Omega)$, $(i = 1, 2)$. Assume that $\zeta(0) < \zeta_1$ and $E(0) < E_1$. Then for the solution energy to the problem* (1)-(2) *the following decay estimate holds, for some positive constants c_0 and γ,*

$$E(t) \le E_1 \exp\left(\frac{\gamma - c_0 t}{\gamma}\right), \qquad \forall t \ge 0.$$

Proof. Multiplying both sides of the equations in (1) by u_i $(i = 1, 2)$, integrating over $(t_1, t_2) \times \Omega$, $0 < t_1 < t_2 < T$, using Young's and Hölder's inequalities, by the Remarks 2.2 and 2.3 and taking the imbeddings $H_0^1(\Omega) \hookrightarrow L^{\kappa_i + q_i + 1}(\Omega), L^{2\rho_i}(\Omega), L^{2(q_i+1)}(\Omega)$, into account we find, for all $\varepsilon, \delta_1, \delta_2 > 0$,

$$\begin{aligned} &2\int_{t_1}^{t_2} E(t)dt - 2(r+1)\int_{t_1}^{t_2}\int_\Omega F(u_1, u_2)dxdt \\ &\qquad \le \Lambda_1(\varepsilon, \delta_1, \delta_2)E(t_1) + \Lambda_2(\varepsilon, \delta_1, \delta_2)\int_{t_1}^{t_2} E(t)dt, \end{aligned} \tag{11}$$

where

$$\begin{cases} \Lambda_1(\varepsilon, \delta_1, \delta_2) = 4\left(1 + \frac{B^2(r+2)}{r+1}\right) + 2 + \frac{1}{\varepsilon} + \frac{2q_1}{q_1+1}\delta_1^{-\frac{q_1+1}{q_1}} + \frac{2q_2}{q_2+1}\delta_2^{-\frac{q_2+1}{q_2}}, \\ \Lambda_2(\varepsilon, \delta_1, \delta_2) = \varepsilon\left(\frac{2(r+2)}{r+1}B\right) + \frac{\delta_1^{q_1+1}}{q_1+1}K_1 + \frac{\delta_2^{q_2+1}}{q_2+1}K_2, \end{cases}$$

in which K_1 and K_2 are some positive constants depending on E_1 and the imbedding constants and B denotes the best constant in Poincaré inequality. On the other hand, by (7) and (10), we have

$$2\int_{t_1}^{t_2} E(t)dt - 2(r+1)\int_{t_1}^{t_2}\int_\Omega F(u_1, u_2)dxdt \ge 2D_0\int_{t_1}^{t_2} E(t)dt,$$

where

$$D_0 = 1 - \eta\left(\frac{2(r+2)}{r+1}E(0)\right)^{r+1}.$$

By the assumption $E(0) < E_1$ we see that $D_0 > 0$. Thus, we can choose ε, δ_1 and δ_2 sufficiently small such that $c_0 := 2D_0 - \Lambda_2(\varepsilon, \delta_1, \delta_2) > 0$. Then, from (11), we get

$$\int_{t_1}^{t_2} E(t)dt \le \frac{\Lambda_1(\varepsilon, \delta_1, \delta_2)}{c_0}E(t_1).$$

Therefore, letting t_2 goes to infinity and taking Lemma 2.4 into account we obtain the decay estimate in Theorem 2.5. ☐

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Kerman, Iran
30 August - 02 September 2021

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On invariant submanifolds of generalized quasi Sasakian manifolds

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Abstract. In this paper, we prove that every invariant submanifold of a generalized quasi Sasakian manifold is again a generalized quasi Sasakian manifold and give some characterization results for the second fundamental form h and the shape operator A. Finally, we show that invariant submanifolds of a G.Q.S manifold are minimal.

Keywords: Generalized quasi Sasakian manifolds, Invariant submanifolds, Minimal submanifolds

AMS Mathematics Subject Classification [2010]: 53C25, 53C40

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1. Introduction

The geometry of Riemannian submanifolds is an active and important area of research in differential geometry. J. Simons studied the immersed minimal hypersurfaces in S^n or E^n. He obtained a second order elliptic partial differential equation for the second fundamental form h and showed that all immersed minimal hypersurfaces in S^n or E^n must satisfy Simons' inequality. In [5], M. Okumura considered the invariant submanifolds of a contact Riemannian manifold. Let $M \subseteq \overline{M}$ be a submanifold of the almost contact metric manifold $(\overline{M}, \varphi, \xi, \eta, g)$. The submanifold M is named invariant if $\varphi(X) \in \Gamma(TM)$ for any $X \in \Gamma(TM)$. Further, invariant submanifolds of a Sasakian manifold were studied in [3]. K. Kenmotsu [3] proved that the only η-Einstein connected complete invariant submanifolds in S^{2n+1} are S^{2n-1} and (S^n, Q^{n-1}). Also, V. Mangione [4] studied the invariant submanifolds of a Kenmotsu manifold and proved that every invariant submanifold of a Kenmotsu manifold is again a Kenmotsu manifold. Moreover, C. Calin [1] proved that the second fundamental form and the shape operator of an invariant submanifold of the Kenmotsu manifold $(\overline{M}, \varphi, \xi, \eta, g)$ satisfy the following equations:

(1)
$$h(\varphi X, Y) = \varphi h(X, Y), \qquad A_{\varphi(N)}(X) = \varphi A_N(X),$$
$$A_N(\varphi X) = -\varphi A_N(X),$$

where $N \in \Gamma(T^{\perp}M)$ and $X, Y \in \Gamma(TM)$. Indeed, in the case where M is tangent to ξ we also have

(2)
$$h(\xi, X) = A_N(\xi) = 0,$$

*Speaker. Email address: v.pirhadi@kashanu.ac.ir for any $X \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$. In [6], S. S. Eum defined the sense of generalized quasi Sasakinan manifolds (in short G.Q.S) and studied the Kaehlerian hypersurfaces which are isometrically immersed in a G.Q.S manifold. The Riemannian manifold $(\overline{M}, g)$ with a 1-form η, a vector field ξ and a $(1,1)$ tensor field φ such that

$$\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi, \tag{3}$$
$$g(\varphi X, Y) = -g(X, \varphi Y),$$

is called an almost contact metric manifold. The almost contact metric manifold $(\overline{M}, \varphi, \xi, \eta, g)$ is called a G.Q.S if it satisfies:

$$(\overline{\nabla}_X \varphi)Y = g(\overline{\nabla}_{\varphi X}\xi, Y)\xi - \eta(Y)\overline{\nabla}_{\varphi X}\xi, \tag{4}$$

for any $X, Y \in \Gamma(T\overline{M})$. Thereafter, C. Calin [2] showed that G.Q.S manifolds are normal and satisfy the following equations:

$$\overline{F}(\xi) = 0, \quad \overline{\nabla}_{\xi}\varphi = 0, \quad \overline{F} \circ \varphi = \varphi \circ \overline{F}, \quad \eta \circ \overline{F} = 0, \tag{5}$$

in which $\overline{F}(X) := \overline{\nabla}_X \xi$. Motivation by these works, in this paper we prove that every invariant submanifold of a G.Q.S manifold is again a G.Q.S manifold. Next, we show that invariant submanifolds of a G.Q.S manifold are minimal.

2. Preliminaries

Let M be a submanifold of the Riemannian manifold $(\overline{M}, g)$. Consider $T\overline{M}$ as a vector bundle on the base manifold M. Hence, every vector field $X \in \Gamma(T\overline{M})$ decomposes into two parts, the horizontal part and the normal part. Suppose $T^{\perp}M$ is the vector bundle of all normal vectors on the base manifold M. Thus, the tangent bundle $T\overline{M}$ can be written as follows:

$$T\overline{M} = TM \oplus T^{\perp}M. \tag{6}$$

According to the above decomposition, the Gauss and Weingarten formulas are given as follows:

$$\overline{\nabla}_X Y = h(X, Y) + \nabla_X Y, \tag{7}$$

$$\overline{\nabla}_X N = -A_N(X) + \nabla^{\perp}_X Y, \tag{8}$$

in which $\overline{\nabla}$ and ∇ are the Levi-Civita connections of $\overline{M}$ and M, respectively and $\nabla^{\perp}$ is the induced connection on the normal bundle $T^{\perp}M$. In the equations above, h and A are named the second fundamental form and the shape operator of M, respectively and are related by

$$g(h(X, Y), N) = g(A_N(X), Y), \tag{9}$$

where $N \in \Gamma(T^{\perp}M)$ and $X, Y \in \Gamma(TM)$. A submanifold of the Riemannian manifold $(\overline{M}, g)$ with $h = 0$ is called totally geodesic. Suppose $\{e_1, ..., e_n\}$ is an orthonormal basis for T_pM $(p \in M)$. Then, the normal vector

$$H := \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{10}$$

is called the mean curvature vector. A submanifold of the Riemannian manifold $(\overline{M}, g)$ which satisfies $H = 0$ for any $p \in M$ is called a minimal submanifold. Henceforth, we assume that all submanifolds are tangent to the vector field ξ $(\xi \in \Gamma(TM))$.

3. Invariant submanifolds of G.Q.S manifolds

Theorem 3.1. *Let $(\overline{M}, \varphi, \xi, \eta, g)$ be a G.Q.S manifold. Then, every invariant submanifold of $\overline{M}$ is again a G.Q.S manifold.*

Proof. It follows immediately from (7) that

$$
\begin{aligned}
(\overline{\nabla}_X \varphi)Y &= \overline{\nabla}_X \varphi Y - \varphi \overline{\nabla}_X Y \\
&= \nabla_X \varphi Y + h(X, \varphi Y) - \varphi(\nabla_X Y) - \varphi h(X, Y).
\end{aligned} \tag{11}
$$

This together with (4) implies that

$$
\begin{aligned}
g(\overline{\nabla}_{\varphi X}\xi, Y)\xi - \eta(Y)\overline{\nabla}_{\varphi X}\xi &= g(\nabla_{\varphi X}\xi, Y)\xi - \eta(Y)\nabla_{\varphi X}\xi - \eta(Y)h(\varphi X, \xi) \\
&= \nabla_X \varphi Y + h(X, \varphi Y) - \varphi(\nabla_X Y) - \varphi h(X, Y).
\end{aligned} \tag{12}
$$

Separate the horizontal and vertical parts of the above equation, we get

$$
(\nabla_X \varphi)Y = g(\nabla_{\varphi X}\xi, Y)\xi - \eta(Y)\nabla_{\varphi X}\xi, \tag{13}
$$

$$
h(X, \varphi Y) - \varphi h(X, Y) = -\eta(Y)h(\varphi X, \xi), \tag{14}
$$

which complete the proof. □

Further, since h is symmetric, we have

$$
\begin{aligned}
h(X, \varphi Y) + \eta(Y)h(\varphi X, \xi) &= \varphi h(X, Y) \\
&= h(Y, \varphi X) + \eta(X)h(\varphi Y, \xi).
\end{aligned} \tag{15}
$$

Lemma 3.2. *Let M be an invariant submanifold of the G.Q.S manifold $(\overline{M}, \varphi, \xi, \eta, g)$. Then*

$$
h(\varphi X, \xi) = \varphi h(X, \xi), \qquad h(\xi, \xi) = 0, \qquad \eta(A_N(\xi)) = 0, \tag{16}
$$

for any $N \in \Gamma(T^{\perp}M)$ and $X \in \Gamma(TM)$.

Proof. The result follows from (14) and the Equation (9). □

Theorem 3.3. *Suppose M is an invariant submanifold of the G.Q.S manifold $(\overline{M}, \varphi, \xi, \eta, g)$. Then, the shape operator A satisfies*

$$
\begin{aligned}
&a)\ A_N(\varphi Y) = -A_{\varphi N}Y + \eta(Y)\varphi(A_N(\xi)), \\
&b)\ \varphi A_N(Y) = A_{\varphi N}Y - g(h(Y, \xi), \varphi N)\xi, \\
&c)\ A_N(\varphi Y) = -\varphi A_N(Y) - g(h(Y, \xi), \varphi N)\xi + \eta(Y)\varphi(A_N(\xi)).
\end{aligned} \tag{17}
$$

Proof. To prove the first assertion, we compute

$$
\begin{aligned}
g(A_N(\varphi Y), X) = g(h(X, \varphi Y), N) &= g(\varphi h(X, Y) - \eta(Y)h(\varphi X, \xi), N) \\
&= -g(h(X, Y), \varphi N) - \eta(Y)g(h(\varphi X, \xi), N) \\
&= -g(A_{\varphi N}Y, X) - \eta(Y)g(A_N(\xi), \varphi X) \\
&= -g(A_{\varphi N}Y - \eta(Y)\varphi(A_N(\xi)), X).
\end{aligned} \tag{18}
$$

Similarly, for the second assertion we get

$$
\begin{aligned}
-g(\varphi(A_N(X)), Y) = g(A_N(X), \varphi Y) &= g(h(X, \varphi Y), N) \\
&= g(\varphi h(X, Y) - \eta(Y)h(\varphi X, \xi), N) \\
&= -g(h(X, Y), \varphi N) + \eta(Y)g(h(X, \xi), \varphi N) \\
&= -g(A_{\varphi N}X, Y) + g(h(X, \xi), \varphi N)g(\xi, Y)
\end{aligned}
$$

$$= -g(A_{\varphi N}X - g(h(X,\xi),\varphi N)\xi, Y). \tag{19}$$

Finally, the last assertion follows from (17.a) and (17.b). □

Theorem 3.4. *Every invariant submanifold of a G.Q.S manifold is minimal.*

Proof. Let M be an invariant submanifold of the G.Q.S manifold $(\overline{M},\varphi,\xi,\eta,g)$. Suppose $\{e_0 := \xi, e_1, ..., e_m, e_{m+1} := \varphi e_1, ..., e_{2m} := \varphi e_m\}$ is a φ-basis for T_pM. Using (15) and (16), we infer that

$$\begin{aligned} h(\varphi e_i,\varphi e_i) &= \varphi^2 h(e_i,e_i) \\ &= -h(e_i,e_i), \end{aligned} \tag{20}$$

for any $i = 1, ..., m$. This asserts that

$$\begin{aligned} H &= \frac{1}{n}\sum_{i=0}^{2m} h(e_i,e_i) = \frac{1}{n}h(\xi,\xi) + \frac{1}{n}\sum_{i=1}^{m} h(e_i,e_i) + \frac{1}{n}\sum_{i=1}^{m} h(\varphi e_i,\varphi e_i) \\ &= \frac{1}{n}h(\xi,\xi) + \frac{1}{n}\sum_{i=1}^{m} h(e_i,e_i) - \frac{1}{n}\sum_{i=1}^{m} h(e_i,e_i) = 0, \end{aligned} \tag{21}$$

and proves the theorem. □

Theorem 3.5. *Let M be an invariant submanifold of the G.Q.S manifold $(\overline{M},\varphi,\xi,\eta,g)$ which satisfies $\overline{R}(X,Y)\xi = R(X,Y)\xi$. Then*

$$a)\ \ h(X,\xi) = 0, \qquad b)\ \ h(X,FY) = h(FX,Y). \tag{22}$$

Proof. Using the relation between $\overline{R}$ and R, we conclude that

$$A_{h(X,\xi)}Y = A_{h(Y,\xi)}X, \qquad (\overline{\nabla}_X h)(Y,\xi) = (\overline{\nabla}_Y h)(X,\xi). \tag{23}$$

The first equality implies

$$g(h(Y,Z),h(X,\xi)) = g(h(X,Z),h(Y,\xi)). \tag{24}$$

Setting $Y = \xi$ and $Z = X$, we find that $h(X,\xi) = 0$. This together with $(\overline{\nabla}_X h)(Y,\xi) = (\overline{\nabla}_Y h)(X,\xi)$, results

$$-h(Y,\nabla_X\xi) = -h(X,\nabla_Y\xi), \tag{25}$$

which completes the proof. □

4. Conclusion

Every invariant submanifold of a generalized quasi Sasakian manifold is minimal and is again a generalized quasi Sasakian manifold.

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On the number of solutions of commutator equation $[x, y] = g$ in a finite group of nilpotency class two

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Abstract. In this paper, we consider the finitely presented groups H_m as follows;

$$H_m = \langle a, b | a^{m^2} = b^m = 1, b^{-1}ab = a^{1+m} \rangle, m \geq 2.$$

For $g \in G$, we consider $\rho_g(G) = \{(x, y) | x, y \in G, [x, y] = g\}$. Then the probability that the commutator equation $[x, y] = g$ has solution in a finite group G, written $P_g(G)$, is equal to $\frac{|\rho_g(G)|}{|G|^2}$. By using the numerical solutions of the equation $xy - zu \equiv t (\text{mod } \ n)$, we derive formulas for calculating the probability of $\rho_g(G)$ where $G' \leq Z(G)$.

Keywords: Finite groups, nilpotent groups, commutativity degree, GAP

AMS Mathematics Subject Classification [2010]: 20F12, 20D15

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1. Introduction

In the last years there has been a growing interest in the use of probability in finite group theory. One of the most important aspects that have been studied is the probability that two elements of a finite group G commute. This is denoted by $P(G)$ and is called the commutativity degree of G. In obtaining the properties of $P(G)$, Gustafson [3] proved that for a non-abelian finite group G. M. Hashemi [4] gave some explicit formulas of $P(G)$ for some particular finite groups G. Also Hashemi and et al. [5] derived formulas for calculating the probability of $P_g(G)$ where G is a two generated group of nilpotency class 2.

Definition 1.1. Let G be a finite group. The commutativity degree of G, written $P(G)$, is defined as the ratio

$$P(G) = \frac{|\{(x, y) \in G \times G : xy = yx\}|}{|G|^2}.$$

In [6], Pournaki and R. Sobhani have studied and generalized this concept for the group G and $g \in G$ as follows:

$$P_g(G) = \frac{|\{(x, y) \in G \times G : [x, y] = g\}|}{|G|^2}.$$

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Note that for every $g \in G$, we have $0 \leq P_g(G) \leq 1$. In particular for $g \in G - G'$, we get $P_g(G) = 0$ and $P_g(G) = 1$ if and only if G is abelian and $g = e$.

In this paper, we obtain explicit formula for the commutativity degree of generalized relative g the element of G of the finite group H_m. In Section 2, we state some lemmas and theorems are needed in the proofs of main results. Section 3 is devoted to compute the formula for $P_g(G)$, where $G = H_m$. These results can be checked for some small values by GAP [2].

2. Preliminary

In this section, we state some lemmas and theorems which will be used in the next section. First, we state lemmas that establishes some properties of groups of nilpotency class two, where $[x, y] = x^{-1}y^{-1}xy$.

Lemma 2.1. *If G is a group and $G' \subseteq Z(G)$, then the following hold for every integer k and $u, v, w \in G$:*

1. $[uv, w] = [u, w][v, w]$ *and* $[u, vw] = [u, v][u, w]$.
2. $[u^k, v] = [u, v^k] = [u, v]^k$.
3. $(uv)^k = u^k v^k [v, u]^{k(k-1)/2}$.
4. *If* $G = \langle a, b\rangle$ *then* $G' = \langle [a, b]\rangle$.

The following lemma can be seen in [1].

Lemma 2.2. *Let $G = H_m$. Then*

1. *Every element of H_m may be uniquely represented by $b^j a^i$, where $0 \leq i \leq m^2 - 1$ and $0 \leq j \leq m - 1$.*
2. $Z(G) = G' = \langle a^m\rangle$ *and* $|Z(G)| = m$.
3. $|H_m| = m^3$.
4. $x^s y^r = y^r x^{s+mrs}$.

The following results are of interest to consider and one may see the proof in [4].

Corollary 2.3. *For the integer $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\ldots{p_k}^{\alpha_k}$ and variables x, y, z and u, the number of solutions of the equation $xy \equiv zu \pmod{n}$ is*

$$\prod_{i=1}^{k} p_i^{2\alpha_i-1}(p_i^{\alpha_i+1} + p_i^{\alpha_i} - 1).$$

Corollary 2.4. *Let m, n be integers and x, y, z and u be variables where $1 \leq x, z \leq n$ and $1 \leq y, u \leq m$. Then the number of solutions of the equation $xy \equiv zu \pmod{d}$ is*

$$(\frac{m}{d})^2(\frac{n}{d})^2\prod_{i=1}^{k} p_i^{2\alpha_i-1}(p_i^{\alpha_i+1} + p_i^{\alpha_i} - 1).$$

where $d = gcd(m, n) = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\ldots{p_k}^{\alpha_k}$.

3. Computations on 2-generated groups of nilpotency class two

In the present part, we get explicit formulas for the commutativity degree of generalized relative g the element of G of the finite group H_m. First, we need the following Theorem.

THEOREM 3.1. *[5] For the integers t, n and variables x, y, u and z, the number of solutions of the equation $xy - uz \equiv t \pmod{n}$ is*

$$\sum_{d|n}[\sum_{d_2|d_1}(\frac{n^2}{d}\phi(\frac{n}{d})\phi(\frac{d}{d_2}) \times d_2)].$$

By elementary concepts of number theory, we have the following corollary:

COROLLARY 3.2. *Let t, n be integers and i, j, r and s be variables, when $0 \leq i, s < n$ and $0 \leq r, j < n^2$. Then the number of solutions of the equation $ri - sj \equiv t \pmod{n}$ is*

$$n^3 \sum_{d|n}[\sum_{d_2|d_1}(\frac{n}{d}\phi(\frac{n}{d})\phi(\frac{d}{d_2}) \times d_2)].$$

Now, these facts leads us to prove the main results.

THEOREM 3.3. *For the group $G = H_m$ and $g \in G^{'}$, $P_g(G) = \alpha/m^6$, where $\alpha = m^3[\sum_{d|m}(\sum_{d_2|(d,t_g)} \frac{m}{d}\phi(\frac{m}{d})\phi(\frac{d}{d_2}) \times d_2)]$.*

PROOF. Let x, $y \in H_m$. Then by the first part of Lemma 2.2, we have $x = b^{r_1}a^{s_1}$, $y = b^{r_2}a^{s_2} \in H_m$, where $0 \leq r_1$, $r_2 \leq m-1$ and $0 \leq s_1$, $s_2 \leq m^2 - 1$. Now, using Lemma 2.1 and the relations of H_m, we get

$$\begin{aligned} xy =& b^{r_1}a^{s_1}b^{r_2}a^{s_2} = b^{r_1+r_2}a^{s_1+s_2}[a^{s_1}, b^{r_2}] = b^{r_1+r_2}a^{s_1+s_2}[a, b]^{s_1r_2} \\ =& b^{r_1+r_2}a^{s_1+s_2+ms_1r_2}, \end{aligned}$$

and

$$\begin{aligned} [x, y] =& a^{-s_1}b^{-r_1}a^{-s_2}b^{-r_2}b^{r_1}a^{s_1}b^{r_2}a^{s_2} \\ =& a^{-s_1-s_2}b^{-r_1-r_2}[b^{-r_1}, a^{-s_2}]b^{r_1+r_2}a^{s_1+s_2}[a^{s_1}, b^{r_2}] \\ =& [a, b]^{r_2s_1-r_1s_2}. \end{aligned}$$

On the other hand, for $x, y, g \in G$ where $g = [x, y] \in G^{'} = \langle [a, b] \rangle$ there is $1 \leq t_g \leq m$ such that $g = [x, y] = [a, b]^{t_g}$. Now, for $g \in G^{'}$, we obtain

$$\begin{aligned} |\rho_g(G)| =& |\{(x, y) \in G \times G;\ [x, y] = g\}| \\ =& |\{(x, y) \in G \times G;\ a^{m(r_2s_1-r_1s_2)} = a^{mt_g}\}| \\ =& |\{(r_1, s_1, r_2, s_2);\ r_2s_1 - r_1s_2 \equiv t_g \pmod{m}\}|. \end{aligned}$$

So that, by Corollary 3.2, we have
$|\rho_g(G)| = m^3 \sum_{d|m}[\frac{m}{d}\phi(\frac{m}{d})(\sum_{d_2|d_1} \phi(\frac{d}{d_2}) \times d_2)]$, where $d|m$, $d_1 = (d, t_g)$ and the result follows. □

Let s_t be the number of solutions of the equation of $r_2s_1 - r_1s_2 \equiv t \pmod{m}; 0 \leq s_1, s_2 < m^2, 0 \leq r_1, r_2 < m$. The table 1 is a verified result of GAP [2] , where $2 \leq m \leq 10$ and $1 \leq t \leq 5$.

TABLE 1. The number of solutions of $r_2 s_1 - r_1 s_2 \equiv t (\mathrm{mod}\ \ m)$.

$m \setminus t$	s_1	s_2	s_3	s_4	s_5
2	24	40	24	40	24
3	216	216	297	216	216
4	768	1152	768	1408	768
5	3000	3000	3000	3000	3625
6	5184	8640	7128	8640	5184
7	16464	16464	16464	16464	16464
8	24576	36864	24576	43008	24576
9	52488	52488	69984	52488	52488
10	72000	120000	72000	12000	87000

Acknowledgement

The authors would like to thank reviewers for the reading and their useful comments in this paper.

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Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
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۸ تا ۱۱ شهریور ۱۴۰۰

New methods to accelerate the frame algorithm

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Abstract. The frame algorithm is one way for approximating the function f in a Hilbert space based on the knowledge of some frame bounds. In this paper we design two algorithms in order to improve the acceleration of the frame algorithm. These algorithms have a faster convergence rate than the classical frame algorithm.

Keywords: frame, frame algorithm, Chebyshev polynomials

AMS Mathematics Subject Classification [2010]: 65F10, 65F05

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1. Introduction and preliminaries

One of the attraction of frames is due to the representation of functions. In fact, the set up of frames provides great flexibility in approximating and representing functions. These representations convey quantitative information about the components of such functions. This has led to important applications in signal analysis and image compression. Some surveys of frames and their applications were provided in [1, 5].

Let us first, following Cassazza [2], provide a brief overview of the basics of frame theory.

Definition 1.1. Let $\mathcal{H}$ be a separable Hilbert space. A family of $\{\mathrm{f}_i\}_{i=1}^{\infty}$ of elements in $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2.$$

A and B are called the lower and respectively the upper bounds for the frame. If $A = B$, it is called **A-tight frame**.

Given a frame $\{\mathrm{f}_i\}_{i=1}^{\infty}$, the frame operator is defined as,

$$S : \mathcal{H} \longrightarrow \mathcal{H},\ Sf = \sum_{i=1}^{\infty}\langle f, f_i\rangle f_i.$$

It is proved that the frame operator is positive definite, self adjoint and invertible. In fact, $AI \leq S \leq BI$, where I denotes the identity operator on $\mathcal{H}$. Moreover, the family $\{\mathrm{S}^{-1}f_i\}_{i=1}^{\infty}$ is also a frame for $\mathcal{H}$, that is called the canonical dual frame. More precisely every $f \in \mathcal{H}$ can be represented as

$$f = \sum_{i=1}^{\infty}\langle f, S^{-1}f_i\rangle f_i = \sum_{i=1}^{\infty}\langle f, f_i\rangle S^{-1}f_i \tag{1}$$

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The formula (1) shows that we can find an element $f \in \mathcal{H}$ based on the knowledge of the coefficients $\{\langle f, f_i\rangle\}_{i=1}^{\infty}$. However, in order to this formula to be useful we need to compute the inverse of the frame operator which can be complicated.

Another option is to approximate f. Frame algorithm [4] is one way to obtain an approximation of f. the convergence rate of this algorithm is $\left(\frac{B-A}{B+A}\right)$. Also in [6] we can see an algorithm, based on the conjugate gradient method , that has a faster convergence. The convergence rate of this algorithm depends on $\left(\frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}\right)$.

In this paper, in order to approximate every $f \in \mathcal{H}$ based on the knowledge of the coefficients $\{\langle f, f_i\rangle\}_{i=1}^{\infty}$, we design two algorithms with the better convergence rate, where $\{\mathrm{f}_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$.

2. Proposed algorithm

In this section we propose a new recursive formula for approximating any element f based on the knowledge of a frame in a Hilbert space $\mathcal{H}$.
For given $f \in \mathcal{H}$ consider the following recursive formula,

$$g_0 = 0, \qquad g_i = g_{i-1} + \frac{4}{A+B}\left(I - \frac{1}{A+B}S^2\right)S^2\left(f - g_{i-1}\right). \tag{2}$$

. In this case the following lemma holds.

Lemma 2.1. *Let $\{f_i\}_{i=1}^{\infty}$ be a frame for Hilbert space $\mathcal{H}$ with frame bounds A and B, respectively, and also the frame operator S. Then*

$$\|I - \frac{4}{A+B}\left(I - \frac{1}{A+B}S^2\right)S^2\| \leq \left(\frac{B-A}{B+A}\right)^2.$$

Now we can conclude the following theorem.

Theorem 2.2. *Let $\{f_i\}_{i=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame bounds A and B respectively. Given $f \in \mathcal{H}$, the sequence $\{g_i\}_{i=1}^{\infty}$ in (2) converges to f and,*

$$\|f - g_i\| \leq \left(\frac{B-A}{B+A}\right)^{2i}\|f\|.$$

Note that this theorem shows that the squence $\{\mathrm{g}_i\}_{i=1}^{\infty}$ converges to f with the convergence rate $\left(\frac{B-A}{B+A}\right)^2$. However the convergence rate of the frame algorithm is $\frac{B-A}{B+A}$.
We summarize these results in the following algorithm which generates an approximation for $f \in \mathcal{H}$ with an arbitrary accuracy $\epsilon > 0$.

Algorithm 1
Let $\epsilon > 0$ be given.

(1) Let $\rho = \left(\frac{B-A}{B+A}\right)^2$.
(2) $i = 0, g_i = 0$.
(3) a) $i = i+1, r_{i-1} := f - g_{i-1}, v_{i-1} := S^2 r_{i-1}$.
    b) $g_i = g_{i-1} + \frac{4}{A+B}\left(v_{i-1} - \frac{1}{A+B}S^2 v_{i-1}\right)$.
(4) If $\frac{\rho^i}{A}\|f\| < \epsilon$ stop and set g_i as an approximation for f. Else go to step 3.

3. Acceleration by the Chebyshev polynomials

Consider the sequence $\{\mathrm{h}_n\}_{n=1}^{\infty}$ as $h_n = \sum_{i=1}^{n} a_{n_i} g_i$ where $\{\mathrm{g}_i\}_{i=1}^{\infty}$ is the sequence of approximations obtaining in (2) and $\{\mathrm{a}_{n_i}\}_{i=1}^{n}$ is a finite sequence of real or complex numbers corresponding to any h_n such that $\sum_{i=1}^{n} a_{n_i} = 1$.
By using the properties of the Chebyshev polynomials, we design an algorithm with the better convergence rate. These polynomials have the important minimization property that makes them useful for convergence acceleration. These polynomials are defined by,

$$c_n(x) = \begin{cases} \cos(n \arccos(x)) & |x| \leq 1 \\ \cosh(ncosh^{-1}(x)) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^{n} + \left(x+\sqrt{x^2-1}\right)^{-n}\right] & |x| > 1 \end{cases}$$

which satisfy the following recurrence relations,

$$c_0(x) = 1, c_1(x) = x, c_n(x) = 2xc_{n-1}(x) - c_{n-2}(x), \forall n \geq 2.$$

For more details see [3].
If we define $\rho_n = \frac{\frac{2}{\rho} c_{n-2}(\frac{1}{\rho})}{c_n(\frac{1}{\rho})}$, then a straightforward computing gives the following recursive relation,

$$\rho_n = \left(1 - \frac{\rho^2}{4}\rho_{n-1}\right)^{-1}.$$

Now we consider the following algorithm.

Algorithm 2

(1) Let $\rho = \left(\frac{B-A}{B+A}\right)^2$, $\sigma = \frac{\sqrt{A^2+B^2}-\sqrt{2AB}}{\sqrt{A^2+B^2}+\sqrt{2AB}}$.

(2) Set $h_0 = 0$, $h_1 = \dfrac{4}{A+B}\left(I - \dfrac{1}{A+B}S^2\right)S^2 f$, $\rho_1 = 2$, $n = 1$.

(3) While $\dfrac{2\sigma^n}{1+\sigma^{2n}}\dfrac{\|f\|}{A} > \epsilon$ Do,

i) $n = n + 1$;

ii) $\rho_n = \left(1 - \dfrac{\rho^2}{4}\rho_{n-1}\right)^{-1}$;

iii) $h_n = \rho_n\left[h_{n-1} - h_{n-2} + \dfrac{4}{A+B}\left(I - \dfrac{1}{A+B}S^2\right)S^2\left(f - h_{n-1}\right)\right] + h_{n-2}$.

(4) $u_\epsilon = h_n$.

The following theorem investigate the convergence of the Algorithm 2.

THEOREM 3.1. *If $\{f_i\}_{i=1}^{\infty}$ is a frame for Hilbert space $\mathcal{H}$ with frame bounds A and B respectively and also the frame operator S, then the sequence $\{h_n\}_{n=1}^{\infty}$, obtaining from the Algorithm 2, converges to f with,*

$$\|f - h_n\| \leq \frac{2\sigma^n}{1+\sigma^{2n}}\frac{\|f\|}{A}.$$

Also the output u_ϵ of the Algorithm 2 satisfies $\|f - u_\epsilon\| < \epsilon$.

This Algorithm guarantees a faster convergence than the Algorithm 1, when B is much more larger than A. The following example shows a frame with frame bounds A and B, so that under these frame conditions, Algorithm 1 can converge at least in 57 steps but under the same conditions, Algorithm 2 can converge in 3 steps.

EXAMPLE 3.2. Let $\mathcal{H}$ be a Hilbert space of dimension 100. Assume that $\{\mathrm{e}_i\}_{i=1}^{100}$ is an orthonormal basis for $\mathcal{H}$ and let $\{\mathrm{f}_i\}_{i=1}^{5050} = \{e_1, e_2, e_2, e_3, e_3, e_3, ..., e_{100}, ..., e_{100}\}$, that is the sequence where each vector e_i is repeated i times. Thus $\{\mathrm{f}_i\}_{i=1}^{5050}$ is a frame for $\mathcal{H}$ with frame bounds $A = 1$ and $B = 100$ respectively. Assume that $f \in \mathcal{H}$ and suppose for the convenience of calculations, $\|f\| = 1$. Then by using Algorithm 1 with $\epsilon = 0.1$ we have $\rho = \left(\frac{B-A}{B+A}\right)^2 \simeq 0.961$. By applying the stop condition of the Algorithm 1, we can conclude $n \geq 57$. It means at least 57 steps are required for the Algorithm 1 to converge. But this time, using Algorithm 2 and applying its stop condition $\dfrac{2\sigma^n}{1+\sigma^{2n}}\dfrac{\|f\|}{A} \leq \epsilon$, with $\sigma \simeq 0.382$ and the same ϵ, we can conclude that $n \geq 3$. Therefore the approximations obtained from the Algorithm 1 won't converge in less than 57 steps , while the Algorithm 2 can converge in only three steps.

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A Relationship Between Algebraic Graph Theory And Associated Prime Ideals

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Abstract. In this paper, we define the torsion graph determined by equivalence classes of torsion elements, denoted by $A_E(M)$. We shall prove that for every torsion finitely generated module M over a Dedekind domain R, a vertex of $A_E(M)$ has degree two if and only if it is an associated prime of M.

Keywords: Associated prime ideals, Dedekind domain, Zero-divisor graph, Chromatic number, Clique number.

AMS Mathematics Subject Classification [2010]: 13F05, 16D10, 05C15, 05C69.

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unitary. The notion of zero-divisor graph $G(R)$ of a ring R, was introduced by I. Beck in [1]. We follow the ideas from Mulay, Spiroff and Wickham in [3], who studied the graph of equivalence classes of zero-divisors of a ring R. Let M be an R-module and $T(M)$ be the set of all torsion elements of M and $T(M)^* = T(M) - \{0\}$. For every $x, y \in M$, we say that $x \sim y$ if $ann(x) = ann(y)$. The relation " $\sim$ " is an equivalence relation. The equivalence class of x is denoted by $[x]$. Here we define a graph whose set of vertices is the set of equivalence classes $\{[x] \mid x \in T(M)^*\}$, and two distinct torsion elements $x, y \in T(M)^*$ are equivalent if $ann(x) = ann(y)$. Also, two distinct classes $[x]$ and $[y]$ are adjacent provided that $ann(x)ann(y)M = 0$. This graph will be denoted by $A_E(M)$. For an $R-$module M(a ring R), we denote the set of all $ann(x)$ such that $0 \neq x \in M(R)$, by $\Omega_R(M)(\Omega(R))$. There is a natural bijective map from $\Omega_R(M)$(or $\Omega(R)$) to the set of vertices of $A_E(M)$(or $A_E(R)$) given by $I \to [x]$, where $I = ann(x)$. We will slightly abuse terminology and refer to $[x]$ as an element of Ω. A colour-partition of a graph G is a partition of $V(G)$ into colour-classes $V_1, \ldots, V_l$ such that each $V_i (1 \leq i \leq l)$, contains no pair of adjacent vertices. The chromatic number of G, is the least natural number l for which such a partition is possible and denoted by $\nu(G)$.

In this paper we show that a vertex of $A_E(M)$ has degree two if and only if it is an associated prime of M. Finally, we determine $\nu(A_E(M))$, where $|V(A_E(M))| > 1$ and prove that the chromatic number of $A_E(M)$ equals its clique number.

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2. Main results

Definition 2.1. The graph of equivalence classes of torsion elements of an R-module M, denoted by $A_E(M)$, is the graph whose vertices are the classes of elements in $T(M)^*$. Also, each pair of distinct classes $[x]$ and $[y]$ are joined by an edge, if $ann(x)ann(y)M = 0$.

Lemma 2.2. *Let M be a torsion finitely generated module over a Dedekind domain R. Suppose that $ann(M) = P_1^{\alpha_1} \ldots P_k^{\alpha_k}$ is the decomposition of $ann(M)$ into the prime ideals of R. Then $V(A_E(M)) = \{P_1^{\beta_1} \ldots P_k^{\beta_k} \neq R \mid 0 \leq \beta_i \leq \alpha_i\}$ and $|Ass(M)| = k$ and $|V(A_E(M))| = (\prod_{i=1}^{k}(\alpha_i + 1)) - 1$ and*

$$deg(P_1^{\beta_1} \ldots P_k^{\beta_k}) = \begin{cases} (\prod_{i=1}^{k}(\beta_i + 1)) - 2 & ; \textit{if } \beta_i = \alpha_i \ ; \ \forall i \\ (\prod_{i=1}^{k}(\beta_i + 1)) - 1 & ; \textit{if } \beta_i \geq \dfrac{\alpha_i}{2} \ ; \ \forall i \\ \prod_{i=1}^{k}(\beta_i + 1) & ; \textit{if } \exists i \ ; \ \beta_i < \dfrac{\alpha_i}{2} \end{cases}$$

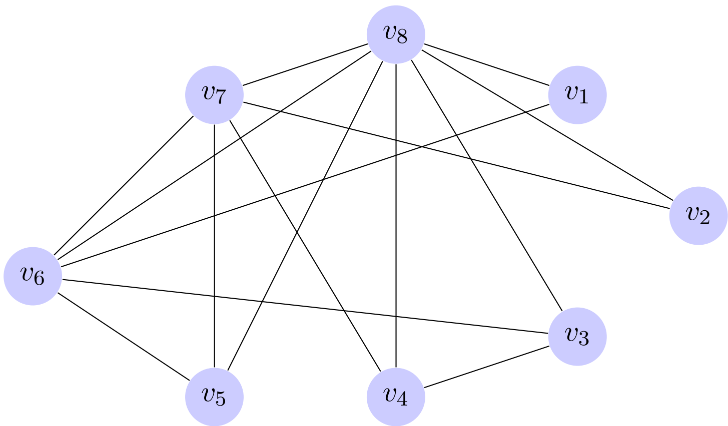
Proof: Since R is a Dedekind domain and M is a torsion finitely generated R-module , by [2], there exist torsion submodules of M, $< x_i >$, $1 \leqslant i \leqslant n$, such that $M \cong \bigoplus_{i=1}^{n} < x_i >$. If $x = x_1 + \ldots + x_n$, then $ann(x) = ann(M)$. For every $i (1 \leqslant i \leqslant k)$, let $T_i = P_i^2 \cup P_1 \cup \ldots \cup P_{i-1} \cup P_{i+1} \cup \ldots \cup P_k$. Then there exists $r_i \in P_i - T_i$ such that $P_i R_{P_i} = (\dfrac{r_i}{1})$. Let $r = r_1^{\beta_1} \ldots r_k^{\beta_k}$, where $0 \leq \beta_i \leq \alpha_i$. We have $ann(rx) = P_1^{\gamma_1} \ldots P_k^{\gamma_k}$, where $\gamma_i = \alpha_i - \beta_i, (1 \leqslant i \leqslant k)$. Hence $T = \{P_1^{\gamma_1} \ldots P_k^{\gamma_k} \neq R | 0 \leqslant \gamma_i \leqslant \alpha_i\} \subseteq \Omega_R(M)$. Clearly $\Omega_R(M) \subseteq T$ and so $V(A_E(M)) = T$. Hence $|\Omega_R(M)| = |V(A_E(M))| = (\prod_{i=1}^{k}(\alpha_i + 1)) - 1$ and $|Ass(M)| = k$. Also by Definition 1.1, two distinct vertices $P_1^{\beta_1} \ldots P_k^{\beta_k}$ and $P_1^{\gamma_1} \ldots P_k^{\gamma_k}$ are adjacent if and only if $\beta_i + \gamma_i \geq \alpha_i (1 \leq i \leq k)$. Then for every $P_1^{\beta_1} \ldots P_k^{\beta_k} \in V(A_E(M))$, its neighbourhood is the set $A = \{P_1^{\gamma_1} \ldots P_k^{\gamma_k} \neq R \mid \alpha_i - \beta_i \leq \gamma_i \leq \alpha_i, \forall i, 1 \leq i \leq k\}$ and the proof is complete.

Theorem 2.3. *Let M be a torsion finitely generated module over a Dedekind domain R. If $|V(A_E(M))| \geqslant 5$, then a vertex of $A_E(M)$ has degree two if and only if it is an associated prime of M.*

Proof: By Lemma 2.2, we have $Ass(M) = \{P_1, \ldots, P_k\}$ and $\deg(P_i) = 1 + 1 = 2 (1 \leq i \leq k)$. Conversely, let $deg(P_1^{\beta_1} \ldots P_k^{\beta_k}) = 2$. If $(\prod_{i=1}^{k}(\beta_i + 1)) - 2 = 2$ or $(\prod_{i=1}^{k}(\beta_i + 1)) - 1 = 2$, we have $|V(A_E(M))| \leq 4$, which is a contradiction. Now let $\prod_{i=1}^{k}(\beta_i + 1) = 2$. So there exists $i (1 \leq i \leq k)$ such that $\beta_i + 1 = 2$ and for every $j \neq i$ $(1 \leq j \leq k)$, $\beta_j + 1 = 1$. Then $\beta_i = 1$ and for every $j \neq i$ $(1 \leq j \leq k)$, $\beta_j = 0$. Therefore, $P_1^{\beta_1} \ldots P_k^{\beta_k} = P_i \in Ass(M)$.

Example 2.4. Let $R = \mathbb{Z}[\sqrt{10}]$, $I =< 10, 10\sqrt{10} >$ and $M = \dfrac{R}{I}$. We know that R is a Dedekind domain, but it is not a PID. We have $ann(5\sqrt{10} + I) =< 2, \sqrt{10} >$, $ann(2\sqrt{10} + I) =< 5, \sqrt{10} >$, $ann(5 + I) =< 2, 2\sqrt{10} >$, $ann(2 + I) =< 5, 5\sqrt{10} >$, $ann(\sqrt{10} + I) =< 10, \sqrt{10} >$, $ann(2 + 5\sqrt{10} + I) =< 10, 5\sqrt{10} >$, $ann(5 + 2\sqrt{10} + I) =< 10, 2\sqrt{10} >$ and $ann(1 + I) =< 10, 10\sqrt{10} >$.

Put $v_1 = [5\sqrt{10} + I]$, $v_2 = [2\sqrt{10} + I]$, $v_3 = [5 + I]$, $v_4 = [2 + I]$, $v_5 = [\sqrt{10} + I]$, $v_6 = [2 + 5\sqrt{10} + I]$, $v_7 = [5 + 2\sqrt{10} + I]$ and $v_8 = [1 + I]$.



$A_E(M)$

Then $Ass(M) = \{P_1 =< 2, \sqrt{10} >, P_2 =< 5, \sqrt{10} >\}$.

THEOREM 2.5. *Let M_1 and M_2 be torsion finitely generated modules over a Dedekind domain R such that $A_E(M_1) \cong A_E(M_2)$ and $|V(A_E(M_1))| = |V(A_E(M_2))| \geq 5$. If $ann(M_1) = P_1^{\alpha_1}...P_k^{\alpha_k}$ and $ann(M_2) = Q_1^{\beta_1}...Q_s^{\beta_s}$ are the decompositions of $ann(M_i), i = 1, 2,$ into prime ideals of R such that $\alpha_1 \geq ... \geq \alpha_k$ and $\beta_1 \geq ... \geq \beta_s$, then $k = s$ and $|Ass(M_1)| = |Ass(M_2)| = k$. Furthermore, for every i, $1 \leq i \leq k$, $\alpha_i = \beta_i$.*

Proof: By Lemma 2.2, $k = |Ass(M_1)| = |Ass(M_2)| = s$. Let $\alpha_1 > 1$. By Lemma 2.2, $deg(P_1^{\alpha_1-1}P_2^{\alpha_2}...P_k^{\alpha_k}) = \alpha_1(\alpha_2 + 1)...(\alpha_k + 1) - 1$ is the second maximum degree of $A_E(M_1)$. Then, $\alpha_1(\alpha_2+1)...(\alpha_k+1)-1 = \beta_1(\beta_2+1)...(\beta_k+1)-1$ and we have $\alpha_1 = \beta_1$. Now for every $0 \leq s \leq \alpha_1$, we have $deg(P_1^{\alpha_1-s}P_2^{\alpha_2}...P_k^{\alpha_k}) = deg(Q_1^{\alpha_1-s}Q_2^{\beta_2}...Q_k^{\beta_k})$ and there exists s such that

$deg(P_1^{\alpha_1-s-1}P_2^{\alpha_2}...P_k^{\alpha_k}) < deg(P_1^{\alpha_1}P_2^{\alpha_2-1}P_3^{\alpha_3}...P_k^{\alpha_k}) \leq deg(P_1^{\alpha_1-s}P_2^{\alpha_2}...P_k^{\alpha_k})$.

Therefore, $deg(P_1^{\alpha_1}P_2^{\alpha_2-1}P_3^{\alpha_3}...p_k^{\alpha_k}) = deg(Q_1^{\alpha_1}Q_2^{\beta_2-1}Q_3^{\beta_3}...Q_k^{\beta_k})$. So $\alpha_2 = \beta_2$. Let $\alpha_i = \beta_i$, for every i, $1 \leq i \leq t-1$. Then there exist s_i, $0 \leq s_i \leq \alpha_i(1 \leq i \leq t-1)$, such that $deg(P_1^{\alpha_1}...P_{t-1}^{\alpha_{t-1}}P_t^{\alpha_t-1}P_{t+1}^{\alpha_{t+1}}...P_k^{\alpha_k}) \leq deg(P_1^{\alpha_1-s_1}...P_{t-1}^{\alpha_{t-1}-s_{t-1}}P_t^{\alpha_t}...P_k^{\alpha_k})$. Also for every $i(1 \leq i \leq t-1)$, we have

$$deg(P_1^{\alpha_1-s_1}...P_{i-1}^{\alpha_{i-1}-s_{i-1}}P_i^{\alpha_i-s_i-1}P_{i+1}^{\alpha_{i+1}-s_{i+1}}...P_{t-1}^{\alpha_{t-1}-s_{t-1}}P_t^{\alpha_t}...P_k^{\alpha_k}) <$$

$$deg(P_1^{\alpha_1}...P_{t-1}^{\alpha_{t-1}}P_t^{\alpha_t-1}P_{t+1}^{\alpha_{t+1}}...P_k^{\alpha_k}).$$

Hence $deg(P_1^{\alpha_1}...P_{t-1}^{\alpha_{t-1}}P_t^{\alpha_t-1}P_{t+1}^{\alpha_{t+1}}...P_k^{\alpha_k}) = deg(Q_1^{\alpha_1}...Q_{t-1}^{\alpha_{t-1}}Q_t^{\beta_t-1}Q_{t+1}^{\beta_{t+1}}...Q_k^{\beta_k})$ and it follows that $\alpha_t = \beta_t$. Therefore for every $i(1 \leq i \leq k)$, $\alpha_i = \beta_i$.

By observations above it is easy to see that the number of graphs Γ such that there exist a Dedekind domain R and a torsion finitely generated $R-$module M with $A_E(M) \cong \Gamma$ and $|V(A_E(M))| = n$, is equal the number of ways that $n+1$ can be written as a product of the form $n + 1 = \prod_{i=1}^{k} a_i$, where $k \in \mathbb{N}$ and $a_i \geq 2(1 \leq i \leq k)$. The number 3 is an exception, because $A_E(\mathbb{Z}_{p^3}) \cong A_E(\mathbb{Z}_{pq}) \cong K_3$, where p, q are prime numbers.

THEOREM 2.6. *Let M be a finitely generated module over a Dedekind domain R with $ann(M) \notin Spec(R)$. Suppose that $ann(M) = P_1^{\alpha_1}...P_k^{\alpha_k}Q_1^{\beta_1}...Q_t^{\beta_t}$ is the decomposition of $ann(M)$ into prime ideals of R such that for every $i(1 \leq i \leq k)$, α_i is even and for every*

$j(1 \leq j \leq t)$, β_j is odd. Then $\nu(A_E(M)) = (\prod_{i=1}^{k}(\frac{\alpha_i}{2}+1)\prod_{j=1}^{t}(\frac{\beta_j+1}{2})) + t$. Also the clique number and the chromatic number of $A_E(M)$ are equal.

Proof: Let $ann(M) = T_1^{\gamma_1}...T_s^{\gamma_s}$, where $\{T_1, ..., T_s\} = \{P_1, ..., P_k, Q_1, ..., Q_t\}$ and $\{\gamma_1, ..., \gamma_s\} = \{\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_t\}$. We define the function $f : \mathbb{N} \longrightarrow \mathbb{N}$ by

$$f(\gamma) = \begin{cases} \frac{\gamma}{2}+1 & ;\text{if } \gamma \text{ is even} \\ \frac{\gamma+1}{2} & ;\text{if } \gamma \text{ is odd} \end{cases}$$

Now we consider $\{(i_1, ..., i_s) \mid 0 \leq i_j \leq \gamma_j \text{ and } 1 \leq j \leq s\}$ with the following ordering

$(0, ..., 0) < (1, 0, ..., 0) < ... < (\gamma_1, 0, ..., 0) < (0, 1, 0, ...,) < (1, 1, 0, ..., 0) < (\gamma_1, 1, 0, ..., 0) < (0, 2, ..., 0) < (1, 2, ..., 0) < ... < (0, \gamma_2, 0..., 0) < (1, \gamma_2, 0, ..., 0) < ... < (0, ..., 0, \gamma_s) < (1, 0, ..., 0, \gamma_s) < ... < (\gamma_1, ..., \gamma_s)$.

For every $(i_1, ..., i_s)$, we consider the subsets $V_{(i_1,...,i_s)}$ of $V(A_E(M))$ that satisfy the following three conditions:

i) $T_1^{\gamma_1-i_1}...T_s^{\gamma_s-i_s} \in V_{(i_1,...,i_s)}$;

ii) for every $(l_1, ..., l_s) < (i_1, ..., i_s), V_{(i_1,...,i_s)} \bigcap V_{(l_1,...,l_s)} = \emptyset$;

iii) for every $v \in V(A_E(M))$ such that $v \notin \bigcup_{(l_1,...,l_s)<(i_1,...,i_s)} V_{(l_1,...,l_s)}$ and v and $T_1^{\gamma_1-i_1}...T_s^{\gamma_s-i_s}$ are not adjacent, then $v \in V_{(i_1,...,i_s)}$.

We have $V_{(i_1,...,i_s)} \neq \emptyset$, when $0 \leq i_j \leq f(\gamma_j) - 1$ and if γ_j is odd,

we have $V_{(0,...,0,\frac{\gamma_j+1}{2},0,...,0)} \neq \emptyset$. Let $A = \{V_{(i_1,...,i_s)} \mid V_{(i_1,...,i_s)} \neq \emptyset\}$ and $|A| = a$. We have the set A is a colour partition of $A_E(M)$, hence $\nu(A_E(M)) \leq a$. On the other hand, the induced subgraph generated by $\{T_1^{\gamma_1-i_1}...T_s^{\gamma_s-i_s} \mid T_1^{\gamma_1-i_1}...T_s^{\gamma_s-i_s} \in V_{(i_1,...,i_s)}\}$ is the complete graph K_a. So $\nu(A_E(M)) \geq a$ and hence

$\nu(A_E(M)) = a = (\prod_{i=1}^{s} f(\gamma_i)) + t = (\prod_{i=1}^{k}(\frac{\alpha_i}{2}+1)\prod_{j=1}^{t}(\frac{\beta_j+1}{2})) + t$.

Also it is easy to see that the clique number and the chromatic number of $A_E(M)$ are equal.

3. Conclusion

If the graph $A_E(M)$ of a torsion finitely generated module M over a Dedekind domain R is known, we can find $Ass(M)$ and the decomposition of $ann(M)$ into prime ideals of R.

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On the Random Bony attractors

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Abstract. An open set of skew products over the Bernoulli shift with fiber [0, 1] is constructed such that maximal attractors of these skew products are either a continuous invariant graph or a bony attractor. Moreover, maximal attractors carry an invariant ergodic measure that projects to the Bernoulli measure in the base. These skew products have negative fiber Lyapunov exponents and their fiber maps are non-uniformly contracting, hence the non-uniform contraction rate are measured by Lyapnnov exponents.

Keywords: skew product, invariant graph, maximal attractor, bony attractor

AMS Mathematics Subject Classification [2010]: 37C70; 37C40; 37H15; 37A25

1. Introduction

In [1], Kudryashov introduced a new type of attractors which is called bony attractors, then he presented an open set in the space of step skew products over the Bernoulli shift such that any of them had a bony attractor. Following [1], an attractor A of a skew product is *bony* if A is the union of the graph of a continuous function on some subset of the base and an uncountable set of vertical closed intervals (bones) contained in the closure of the graph. The objective of this article is to extend aforementioned result from [1] to the random case, where the skew products are general (not necessarily step). One novelty here is that, in our context, in contrast the Kudryashov' case, fiber maps are non-uniformly contracting, therefore the contraction rates are non-uniform and hence measured by Lyapunov exponents.

Dynamical systems under the external forcing are modeled, in discrete time, as skew products,

$$F : \Omega \times M \to \Omega \times M, \quad F(\omega, x) = (\theta\omega, f_\omega(x)), \tag{1}$$

where the dynamics of the forcing process are described by the base transformation θ which is assumed to be a measure-preserving transformation of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (random forcing).

An *invariant graph* of F is the graph of a measurable function $\gamma : \Omega \to M$ which satisfies $f_\omega(\gamma(\omega)) = \gamma(\theta(\omega))$, for $\mathbb{P}$-almost all $\omega \in \Omega$.

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Assume that X is a metric measure space. Denote by $\mathrm{int}(D)$ and $\mathrm{Cl}(D)$, respectively, the interior and the closure of any set D.

Let $(X; \mathcal{B}; \mu; f)$ be a measure preserving dynamical system. If f is invertible then, based on [2, 3], the system is *Bernoulli* if it is isomorphic to a Bernoulli shift. Clearly invertible systems cannot be isomorphic to non-invertible systems. But there is a construction to make a non-invertible system invertible, namely by passing to the natural extension. For non-invertible case, being Bernoulli means that the natural extension is isomorphic to a Bernoulli shift.

The map f is *mixing* (or *strong mixing*) if

$$\mu(f^{-n}(A) \cap B) \to \mu(A)\mu(B), \ as\ n \to +\infty,$$

for every $A, B \in \mathcal{B}$. Every mixing system [3] is necessarily ergodic.

For a metric space X, putting

$$\mathrm{Lip}_1(X) = \{f : X \to \mathbb{R} : |f(x) - f(y)| \leq d(x, y) \text{ for all } x, y \in X\},$$

define the *Hutchinson metric* on the set $\mathcal{M}(X)$, the space of all Borel probability measures, by

$$d_H(\nu, \mu) = \sup\{|\int_X f d\nu - \int_X f d\mu : f \in \mathrm{Lip}_1(X)|\}. \tag{2}$$

For every metric space X, the topology $\mathcal{T}$ on $\mathcal{M}(X)$ generated by $d_H(\nu, \mu)$ coincides with the topology $\mathcal{W}$ of weak convergence if and only if $\mathrm{diam}(X) < \infty$. Moreover, the space $\mathcal{M}(X)$ is complete in the metric d_H if and only if X is complete.

A *random map* with base $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$, in the sense of Arnold [4], is a skew product of the form (1) where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\theta : \Omega \to \Omega$ is a bi-measurable and ergodic measure-preserving bijection and M is a measurable space. If M is a smooth manifold and all fibre maps f_ω are C^r, we call F a random C^r-map.

Take $\Sigma_k^+ = \{0, \ldots, k-1\}^{\mathbb{N}}$ and $\Sigma_k = \{0, \ldots, k-1\}^{\mathbb{Z}}$ endowed with the product topology and equip them with the Bernoulli measures ν^+ and ν, respectively, corresponding to some distribution of probabilities $p_0, \ldots, p_{k-1}$, which gives us the probability with which we apply f_i. Here, assume that the probabilities p_i, $i = 0, \ldots, k-1$, are the same and equal to $1/k$. Let $\sigma : \Sigma_k \to \Sigma_k$ and $\sigma^+ : \Sigma_k^+ \to \Sigma_k^+$ denote the one-sided and two-sided left shift. It is well known that σ^+ and σ are ergodic transformations preserving the probabilities ν^+ and ν, respectively.

Let M be a compact smooth manifold. Here, we consider skew products of the form

$$F : \Sigma_k \times M \to \Sigma_k \times M; \quad (\omega, x) \to (\sigma\omega, f_w(x)) \tag{3}$$

which is called a *skew product over the Bernoulli shift*, where $\omega \in \Sigma_k$, $x \in M$ and the maps f_ω are C^r diffeomorphisms on M. The space Σ_k is called the *base*, the space M is called the *fiber*, and the maps f_ω are called the *fiber maps*. Thus each skew product of the form (3) is a random C^r-map.

A skew product over the Bernoulli shift is a *step skew product* if the fiber maps f_ω depend only on the digit ω_0 and not on the whole sequence ω. We emphasise, in contrast to step skew products, the fiber maps of (general) skew products of the form (3) depend on the whole sequence ω. When treating a step skew product for one sided time $\mathbb{N}$, this results in the skew product system F^+ on $\Sigma_k^+ \times M$:

$$F^+ : \Sigma_k^+ \times M \to \Sigma_k^+ \times M; \quad (\omega, x) \to (\sigma^+\omega, f_{w_0}(x)). \tag{4}$$

We denote iterates of a skew product system F of the form (3) as $F^n(\omega, x) = (\sigma^n(\omega), f^n_\omega(x))$. Here, for $n \geq 1$

$$f^n_\omega(x) := f_{\sigma^{n-1}\omega} \circ \ldots \circ f_\omega(x).$$

For a step skew product system this becomes

$$f^n_\omega(x) := f_{\omega_{n-1}} \circ \ldots \circ f_{\omega_0}(x),$$

where $\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots, \omega_n, \ldots) \in \Sigma_k$.

In the rest of this article we assume that the fiber M is always the unit interval I.

Take $\mathcal{C}(I)$ the space of all random C^2-maps (general skew products) acting on $\Sigma_k \times I$ defined by C^2 interval diffeomorphisms. We equip $\mathcal{C}(I)$ with the following metric:

$$\mathrm{dist}_{C^2}(F, G) := \sup_{\omega \in \Sigma_k} (\mathrm{dist}_{C^2}(f^{\pm 1}_\omega, g^{\pm 1}_\omega)), \text{ for each } F, G \in \mathcal{C}(I), \tag{5}$$

where f_ω and g_ω are the fiber maps of F and G, respectively.

Let $F : \Sigma_k \times I \to \Sigma_k \times I$ be a homeomorphism onto its image, but suppose its image is contained strictly in $\Sigma_k \times I$. The (global) *maximal attractor* of F is defined as:

$$A_{max}(F) := \bigcap_{n=0}^{\infty} F^n(\Sigma_k \times I). \tag{6}$$

2. Main results

To state the main result precisely, the concept of a bony attractor may need to be introduced.

DEFINITION 2.1. Following [1], an attractor Λ of a skew product F is a bony graph attractor if Λ is the union of the graph of a continuous function γ defined on some set of full measure of the base and a set of vertical closed intervals ("bones") contained in the closure of the graph.

This feature is similar to porcupine horseshoes discovered by Diaz and Gelfert in [5]. In this article, we will show that maximal attractors of a certain class of general skew products (random maps) are either a continuous invariant graph or a bony attractor. Our novelty here is that the fiber maps of such systems depend on the whole sequence ω and hence they are not necessarily step skew products. Moreover, the fiber contraction rates are non-uniform and hence measured by Lyapunov exponents, in addition, the attractors carry an ergodic measure. Our result thus extends work by Kudryashov in [1] who treated step skew products over the Bernoulli shift having bony attractors.

THEOREM 2.2. *There exists an open nonempty set $\mathcal{U}$ in the space C^2 random maps $\mathcal{C}(I)$ given by (5) such that any system G belonging to this set has a maximal attractor $A_{max}(G)$ satisfies the following properties:*

1. *the maximal attractor $A_{max}(G)$ is either a continuous invariant graph or a bony graph attractor;*
2. *there exists an invariant ergodic measure μ_G whose support is the closure of the graph Γ_G, in particular, (G, Γ_G, μ_G) is Bernoulli and therefore it is mixing, additionally, the invariant measure for the perturbed system is continuous in the Hutchinson metric;*
3. *the fiber Lyapunov exponent of G is negative;*

Moreover, the set of random maps of $\mathcal{U}$ which admit a bony graph attractor is nonempty.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Frame-Type Expansion of Functions and Modulation Spaces

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Abstract. In this paper we introduce a great class of wavelet systems and their duals (based on MRA) which generally are not frames in $L^2(R^d)$. Then we propose Frame-type expansions that could happen in different senses of series convergence. As a new point of view, these expansions are considered in modulation spaces which is our novelty

Keywords: Frames, dual wavelets, modulation spaces, Frame-type wavelets.

AMS Mathematics Subject Classification [2010]: 46C20, 42C15, 30E20

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1. Introduction

During the early 1980's Feichtinger introduced the class of *modulation spaces*. As an straight definition we can say that a modulation space M^p, contains of those functions or distributions whose windowed Fourier transform is in L^P. A width rang of explanation and application of modulation spaces could be find in [5] and several references therein and refereed to.
Now modulation spaces are known as the exact framework of time-frequency analysis. In this paper we consider modulation spaces M^p, as our function space and will state some convergence results according to them. From now on S denotes the Schwartz class, S' is its dual, i.e., the space of tempered distributions and by H, we mean a Hilbert space with inner product $\langle , \rangle$.

On the other hand, we know that expanding and approximating the functions, are historically and scientifically attractive subjects. Although expanding of signals is an interesting and important subject for engineers, but also it has a grate benefit for mathematicians to study and investigate this, since it helps peoples to increase the big deal with science and technology. The most famous tools for this aim were bases and corresponding coefficients. But scientists realized very soon that there are many more ways and tools to do that in faster and better ways. Nowadays it is very popular and relevant way to employing frames and wavelets in function expansion. By definition, that was first given by Duffin and Schaefer [4], a frame is a family of vectors $\{f_j\}_{j\in J}$ in a Hilbert space H where J is a countable set in Hilbert space H, if there exist constants $A, B > 0$, such that

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$$A\|f\|^2 \le \sum_{j\in J} |\langle f, f_j\rangle|^2 \le B\|f\|^2,$$

for all $f \in H$ and constants $0 < A \le B < \infty$ independent of f. A frame has an important property: for any $f \in H$ we have the decomposition $f = \sum_j \langle f, \tilde{f}_j\rangle f_j$ where $\{\tilde{f}_j\}$ is a dual frame in H.
One desirable family of frames arises from wavelet frames. Wavelet frames are considered actively during last twenty years in engineering and mathematics, [1], [2] . Constructing a dual wavelet systems $\psi(v), \tilde{\psi}(v)$, based on an MRA, is a famous and known subject, but it should be noticed that it generally does not lead to wavelet frames, since it is necessary to provide vanishing moment property for a wavelet systems to be a frame in H and this is a complicate manner. However engineers employ such frames (which are not really frames!)for approximation goals without any care either they satisfy the frame property or not.
Recently many researchers start to employ frame-type wavelet systems results. As a notable work in this area the authors in [2] have studied MRA based frame-type wavelet systems, their corresponding expansion and convergence in different sense, such as in Schwartz class or in $L^2(R)$.

2. Main results

At first consider unitary operators $T_x f(t) = f(t-x)$ and $M_\omega f(t) = e^{2\pi i\omega t} f(t)$, in $L^2(R)$ and define

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle,$$

with L^2 inner product. We can state the following definition.

Definition 2.1. For a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, a v-moderate weight function m on $\mathbb{R}^{2d}$ and $1 \le p < \infty$, the modulation space $M_m^p(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L_m^p(\mathbb{R}^{2d})$ with $\|f\|_{M_m^p} = \|V_g f\|_{L_m^p}$. $M_m^p(\mathbb{R}^d)$ is Banach space which its norm is independent of the function g, [5].

Remark: An important result is that modulation space M_m^2 is identical with L_m^2, through Wilson bases.
Now we are going to mention some notes on wavelet system construction, [5].

Definition 2.2. Let $\phi \in L^2(R)$ and m_0 be a 1-periodic function in $L^2([0,1])$. Then ϕ is called refinement function if it satisfies in the refinement equation

$$\hat{\phi}(\xi) = \hat{\phi}(\xi/2) m_0(\xi/2),$$

where the $\hat{\phi}$ is the dual function.

Theorem 2.3. *Let m_0 be a 1-periodic function in $L^2([0,1])$. Then there exist 1-periodic functions $m_1, m_2, ...m_r$ such that $M^*M = I_2$ for matrix*

$$M = \begin{pmatrix} m_0(\xi) & ... & m_r(\xi) \\ m_0(\xi+1/2) & ... & m_r(\xi+1/2) \end{pmatrix}$$

Definition 2.4. Let $\hat{\psi}^v(\xi) = m_v(\xi/2)\hat{\phi}(\xi/2)$. Then functions $\psi^1, \psi^2, ...\psi^r$ are called the wavelet functions corresponding with ϕ.
Also we have $\psi_{jk}^v x = 2^{j/2}\psi(2^j x - k)$ for $j, k \in \mathbb{Z}$

Definition 2.5. Let $\{\psi^v_{jk}\}, \{\tilde{\psi}^v_{jk}\}$ be dual wavelet systems and A be a class of functions f for which $\langle f, \tilde{\psi}^v_{jk}\rangle$ have meaning. We say that the family $\{\psi^v_{jk}\}_{j,k\in\mathbb{Z}}, v = 1, 2, ..., r$ is frame-type if:

$$f = \sum_{j,k\in\mathbb{Z}} \sum_{v=1}^{r} \langle f, \tilde{\psi}^v_{jk}\rangle \psi^v_{jk}, \forall f \in A,$$

where the series converges in some natural sense.

Theorem 2.6. *Let $f \in S, \phi \in L^2(\mathbb{R}), \tilde{\phi} \in S'$, also suppose that $\phi, \tilde{\phi}$ are compactly supported and refinable, $\hat{\phi}(0) = \hat{\tilde{\phi}}(0) = 1$ and $\psi^v, \tilde{\psi}^v, v = 1, ..., r$ be associated wavelet functions. Then the above equation holds with the series convergence in L^2-norm, [2].*

Another version of this theorem can be stated in $L^p(\mathbb{R})$, with just some justifications.

Now we are ready to explain our main theorems. Here we consider refinable function in modulation spaces $M^2(\mathbb{R})$ and $M^p(\mathbb{R})$ respectively and investigate the convergence properties.

Theorem 2.7. *Let $f \in M^1, \phi \in L^2(\mathbb{R}), \tilde{\phi} \in M^\infty$, also suppose that $\phi, \tilde{\phi}$ are compactly supported and refinable, $\hat{\phi}(0) = \hat{\tilde{\phi}}(0) = 1$ and $\psi^v, \tilde{\psi}^v, v = 1, ..., r$ be associated wavelet functions. Then*

$$f = \sum_{j,k\in\mathbb{Z}} \sum_{v=1}^{r} \langle f, \tilde{\psi}^v_{jk}\rangle \psi^v_{jk}, \forall f \in M^1,$$

with the series convergence in M^2-norm.

proof. This is an straight result from equivalence of L^2 norm and M^2 norm.

Theorem 2.8. *Let $f \in S, \phi \in M^p_m(\mathbb{R}), \tilde{\phi} \in S'$, also suppose that $\phi, \tilde{\phi}$ are compactly supported and refinable, $\hat{\phi}(0) = \hat{\tilde{\phi}}(0) = 1$ and $\psi^v, \tilde{\psi}^v, v = 1, ..., r$ be associated wavelet functions. Then*

$$f = \sum_{j,k\in\mathbb{Z}} \sum_{v=1}^{r} \langle f, \tilde{\psi}^v_{jk}\rangle \psi^v_{jk}, \forall f \in S,$$

with the series convergence in M^p_m-norm.

proof: [6].

Note that there are different ways to extend MRA-based wavelet system from $\mathbb{R}$ to $\mathbb{R}$. For these standard ways, all results which are given here, are valid. Finally we finish our work with mention to another problem would be considered: question about higher dimensions! which we hope to work after [6].

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Application of Jacobi wavelets for solving nonlinear stochastic Itô-Volterra integral equations

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Abstract. We propose an optimization method, based on the Jacobi wavelets along with the Gauss-Legendre quadrature and Itô approximation, for solving nonlinear stochastic Itô-Volterra integral equations (SIVIEs). By applying these basis and approximations the nonlinear SIVIEs convert to a system of algebraic equations which can be solved by an appropriate numerical method.

Keywords: Jacobi wavelets, Stochastic Itô integral equations, Numerical method.

AMS Mathematics Subject Classification [2010]: 60H20, 45D05, 33C47.

1. Introduction

Nonlinear stochastic equations (NSEs) or random functional have been extensively studied by many scholars in the last years due to the fact that these equations are successfully used for modeling different physical and engineering phenomena like reactor dynamics, the growth of biological populations, automatic systems, HIV internal virus dynamics and mathematical ecology science. These systems often depend on a noise source, like a Gaussian white noise, governed by certain probability laws. Because of the difficulty and complexity, solving such problems analytically are usually difficult, so several numerical schemes are emerged to solve nonlinear stochastic Itô-Volterra integral equations.

2. Preliminaries and fundamentals

2.1. Stochastic calculus.

Definition 2.1. (Brownian motion process) [1]
Brownian motion ($B(t)$) is a stochastic process which satisfies in the following properties
(i): $B(t)-B(s)$ for $t>s$ is independent of the past such that $B(t)-B(s)$ and $B(v)-B(u)$ for $0<u<v<s<t$ are independent.
(ii) $B(t)-B(s)$ for $t>s$ has normal distribution with mean 0 and variance $t-s$. That means

$$B(t)-B(s)\sim\sqrt{t-s}N(0,1),$$

where $N(0,1)$ demonstrates normal distribution with mean 0 and variance 1.
(iii) $B(t), t\geq 0$ is continuous function of t. Here we consider $B(0)=0$.

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2.2. Jacobi wavelets. The Jacobi wavelets $\Psi_{n,m}(t) = \Psi(k, n, m, t)$ are defined over $[0, 1)$ as [2]

$$\Psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}} \sqrt{\frac{1}{\gamma_m^{(\alpha,\beta)}}} P_m^{(\alpha,\beta)}(2^k t - (2n-1)), & \frac{\hat{n}}{2^{k-1}} \le t < \frac{\hat{n}+1}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $n = 1, 2, \ldots, 2^{k-1}$ and $m = 0, 1, 2, \ldots, M-1, \hat{m} = 2^{k-1}M, \hat{n} = n-1$ and $P_m^{(\alpha,\beta)}(t)$ are the Jacobi polynomials of order m given on the interval $[0, 1]$ as [3]

$$P_m^{(\alpha,\beta)}(t) = \frac{\Gamma(m+\alpha+1)}{m!\Gamma(m+\alpha+\beta+1)} \sum_{i=0}^{m} \binom{m}{i} \frac{\Gamma(m+\alpha+\beta+i+1)}{\Gamma(i+\alpha+1)} \left(\frac{t-1}{2}\right)^i. \tag{2}$$

These polynomials have the following orthogonality property with respect to the wight function $\omega^{(\alpha,\beta)}(t) = (1-t)^\alpha (1+t)^\beta$ as

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(t) P_m^{(\alpha,\beta)}(t) \omega^{(\alpha,\beta)}(t) dt = \gamma_m^{(\alpha,\beta)} \delta_{nm},$$

where

$$\gamma_m^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{m!(2m+\alpha+\beta+1)\Gamma(m+\alpha+\beta+1)}.$$

3. Description of the proposed computational method

In this section, we obtain a new technique for solving the following nonlinear stochastic Itô-Volterra integral equation as ([4], [5])

$$X(t) = f(t) + \int_0^t \Lambda(\tau, X(\tau)) d\tau + \int_0^t \Upsilon(\tau, X(\tau)) dB(\tau), \quad t, \tau \in [0, 1), \tag{3}$$

where $X(t)$ is an unknown stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$, $f(t)$ is a known stochastic process over $(\Omega, \mathcal{F}, P)$, $B(t)$ is a Brownian motion process and Λ and Υ are analytic functions. For solving problem (3), we approximate $X(t)$ by the Jacobi wavelets as

$$X(t) \simeq \tilde{X}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \Psi_{n,m}(t), \tag{4}$$

where the coefficients $c_{n,m}$ are unknown. By using Eqs. (3)-(4) and zeros of the shifted Legendre polynomials (t_i) of order ι, we yield

$$Res(t_i, C) = \tilde{X}(t_i) - f(t_i) - \int_0^{t_i} \Lambda(\tau, \tilde{X}(\tau)) d\tau - \int_0^{t_i} \Upsilon(\tau, \tilde{X}(\tau)) dB(\tau), \tag{5}$$

where C is a unknown vector as

$$C = [c_{1,0}, c_{1,1}, \ldots, c_{1,M-1}, c_{2,0}, \ldots, c_{2^{k-1},M-1}]^T.$$

The first integral terms in Eq. (5) can be approximated by applying the Gauss-Legendre quadrature formula and the second integral terms (Itô integral) can be approximated similar to Ref. [5] as

$$\begin{aligned} Res(t_i, C) \simeq\ & \tilde{R}(t_i, C) = \tilde{X}(t_i) - f(t_i) - \frac{t_i}{2} \sum_{j=1}^{\tilde{n}} \omega_j \Lambda\left(\frac{t_i}{2}\eta_j + \frac{t_i}{2}, \tilde{X}\left(\frac{t_i}{2}\eta_j + \frac{t_i}{2}\right)\right) \\ & - \sum_{j=1}^{n^*} \Upsilon(\vartheta_{j-1}, \tilde{X}(\vartheta_{j-1}))(B(\vartheta_j) - B(\vartheta_{j-1})), \end{aligned} \tag{6}$$

here, we consider $\tilde{n} = 2M$ and $n^* = 100$.
So, the following optimization should be obtained

$$C^* = min_C \frac{1}{2} \sum_{i=1}^{\iota} \tilde{R}^2(t_i, C). \tag{7}$$

The above relation is an unconstrained parametric optimization problem which can be stated as follows. Find vector C so that C^* is minimized. The necessary conditions for minimum of C^* are obtained as

$$\frac{\partial}{\partial c_{n,m}} C^* = 0, \qquad n = 1, 2, \ldots, 2^{k-1}; m = 0, 1, \ldots, M-1. \tag{8}$$

We can solve the above system by any numerical method. Finally, we get numerical solution $\tilde{X}(t)$.

4. Illustrative test problems

EXAMPLE 4.1. Consider the following nonlinear SIVIE as [4]

$$X(t) = X_0 - \frac{\xi^2}{2} \int_0^t \tanh(X(\tau)) sech^2(X(\tau)) d\tau + \xi \int_0^t sech(X(\tau)) dB(\tau), \tag{9}$$

where the exact solution is given by

$$X(t) = arcsinh(\xi B(t) + sinh(X_0)).$$

We report the absolute errors of Legendre wavelets method [4] with $\hat{m} = 48, 96$ and Jacobi wavelets method with $\hat{m} = 4$ for $X_0 = 0, \xi = \frac{1}{30}, T = 1$ in Table 1. Also, Table 2 displays the absolute errors of our method for $\hat{m} = 4$ and different values of α and β.

TABLE 1. The absolute errors with $X_0 = 0, \xi = \frac{1}{30}, k = 2, M = 4$ and $T = 1$ (Example 1).

	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	CPU
Legendre wavelets					
Absolute error ($\hat{m} = 48$)	$6.8570E-4$	$1.0130E-3$	$1.3281E-3$	$1.5258E-3$	–
Absolute error ($\hat{m} = 96$)	$8.4116E-7$	$1.7767E-6$	$4.7233E-6$	$6.6628E-6$	–
Jacobi wavelets					
Absolute error ($\hat{m} = 4$)	$1.7717E-7$	$4.1526E-7$	$9.5786E-6$	$1.8178E-6$	0.340
Numerical solution ($\hat{m} = 4$)	0.0019	0.0039	0.0059	0.0078	–

EXAMPLE 4.2. Consider the following nonlinear SIVIE as ([4], [5])

$$X(t) = X_0 - \xi^2 \int_0^t X(\tau)(1 - X^2(\tau)) d\tau + \xi \int_0^t (1 - X^2(\tau)) dB(\tau), \tag{10}$$

where the exact solution is given by

$$X(t) = \tanh(\xi B(t) + arctanh(X_0)).$$

The comparison of the absolute error is obtained by Jacobi wavelets scheme with Legendre wavelets [4] method and piecewise collocation method [5] for $X_0 = \frac{1}{100}, \alpha = \frac{1}{30}$ and $T = 1$ are shown in Table 3. Also, Table 4 displays the absolute errors of our method for $\hat{m} = 8$ and different values of α and β.

Table 2. The absolute errors with $X_0 = 0, \xi = \frac{1}{30}, k = 2, M = 2$ and $T = 1$ (Example 1).

	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	CPU
Jacobi wavelets					
$\alpha = \beta = 0$	$1.7717E-7$	$4.1526E-7$	$9.5786E-7$	$1.8178E-6$	0.340
$\alpha = \beta = 0.2$	$3.1466E-7$	$7.1536E-7$	$1.6408E-6$	$3.1576E-6$	0.484
$\alpha = \beta = 0.4$	$1.6568E-6$	$2.3207E-6$	$4.7015E-6$	$1.1998E-5$	0.594
$\alpha = \beta = 0.6$	$1.1249E-6$	$1.9052E-6$	$4.0893E-6$	$9.2017E-6$	0.484
$\alpha = \beta = 0.8$	$1.3336E-6$	$2.0913E-6$	$4.3921E-6$	$1.0373E-5$	0.641
$\alpha = \beta = 1$	$1.8165E-6$	$2.4132E-6$	$4.7978E-6$	$1.2734E-5$	0.516

Table 3. The absolute errors with $X_0 = \frac{1}{100}, \xi = \frac{1}{30}, k = 2, M = 4$ and $T = 1$ (Example 2).

	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	CPU
Legendre wavelets ($\hat{m} = 96$)	$1.2937E-7$	$1.2398E-6$	$1.1533E-6$	$5.6874E-6$	$-$
Piecewise collocation ($\hat{m} = 13$)	$1.7000E-5$	$2.4000E-5$	$2.9000E-5$	$4.5000E-5$	$-$
Jacobi wavelets					
Absolute error ($\hat{m} = 8$)	$2.2381E-6$	$4.5086E-6$	$6.8115E-6$	$9.1466E-6$	2.718
Numerical solution ($\hat{m} = 8$)	0.0101	0.0102	0.0104	0.0105	$-$

Table 4. The absolute errors with $X_0 = \frac{1}{100}, \xi = \frac{1}{30}, k = 2, M = 4$ and $T = 1$ (Example 2).

	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	CPU
Jacobi wavelets					
$\alpha = \beta = 0$	$2.2381E-6$	$4.5086E-6$	$6.8115E-6$	$9.1466E-6$	2.718
$\alpha = \beta = 0.2$	$3.3655E-6$	$8.9290E-6$	$1.6555E-5$	$2.6107E-5$	2.234
$\alpha = \beta = 0.4$	$2.3025E-6$	$4.7660E-6$	$7.3903E-6$	$1.0175E-5$	2.235
$\alpha = \beta = 0.6$	$2.8698E-6$	$7.0198E-6$	$1.2426E-5$	$1.9066E-5$	2.328
$\alpha = \beta = 0.8$	$3.1504E-6$	$8.1117E-6$	$1.4810E-5$	$2.3184E-5$	2.328
$\alpha = \beta = 1$	$3.7549E-6$	$1.0343E-5$	$1.9412E-5$	$3.0606E-5$	2.328

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K-frames and Bessel sequences

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Abstract. The K-frames were introduced by L. Găvruţa for Hilbert spaces to study atomic systems with respect to a bounded linear operator. In this article, we study the extensions of Bessel sequences in a Hilbert space H to K-frames. Indeed, we provide some conditions that under them, we can extend a Bessel sequence to a K-frame by adding finitely many elements of H.

Keywords: Bessel sequence, frame, K-frame, compact operator

AMS Mathematics Subject Classification [2010]: 42C15, 47A05

1. Introduction

Duffin and Schaeffer in 1952 introduced the concept of frames to study some problems in non-harmonic Fourier series. The notion of frames in a Hilbert spaces is a kind of generalization of the bases and orthonormal bases in Hilbert spaces. The fundamental paper written by Daubechies, Grossmann and Meyer, caused frame theory to be used more and more, especially in the field of wavelet frames and Gabor frames, signal processing, image and data compression and sampling theory. More details about frames are discussed in [2].

Găvruţa introduced K-frames in Hilbert spaces while studying the atomic decomposition systems [4]. This type of frame are obtained by restricting the lower frame bound to the range of a bounded linear operator on a Hilbert space. Building new frames from existing Bessel sequences or frames has already been reviewed in some articles such as [1], [5] and [6].

Throughout this article, H is a separable Hilbert space, $B(H)$ is the set of all bounded linear operators on H and $B(H,K)$ shows the set of all bounded linear operators from Hilbert space H into Hilbert space K. Also, for $V \in B(H,K)$, $N(V)$ denotes the kernel of V and $R(V)$ denotes the range of V.

Definition 1.1. Let $K \in B(H)$ and $\{f_i\}_{i=1}^{\infty} \subseteq H$. Then $\{f_i\}_{n=1}^{\infty}$ is called a K-frame for H, if there exist positive constants A, B such that

$$A\|K^* f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad \forall f \in H. \tag{1}$$

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A, B are called the lower and the upper frame bounds of the K-frame $\{f_i\}_{i=1}^{\infty}$, respectively. If $A = B$, then $\{f_i\}_{i=1}^{\infty}$ is called a tight K-frame and if $A = B = 1$, then $\{f_i\}_{i=1}^{\infty}$ is called a Parseval K-frame. In the case that only the right inequality (1) holds, $\{f_i\}_{i=1}^{\infty}$ is called a Bessel sequence in H. If K is the identity operator on H, then $\{f_i\}_{i=1}^{\infty}$ is a frame for H.

Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence in H. The synthesis operator of $\{f_i\}_{i=1}^{\infty}$ is defined by

$$T : \ell^2 \longrightarrow H$$

$$T(\{c_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} c_i f_i.$$

The synthesis operator is bounded and its adjoint will be expressed as bellow:

$$U : H \longrightarrow \ell^2$$

$$U(f) = \{\langle f, f_i \rangle\}_{i=1}^{\infty}.$$

The operator U is bounded and it is called the analysis operator of $\{f_i\}_{i=1}^{\infty}$.

Definition 1.2. If V is an operator on the Hilbert space H and M is a closed subspace of H, then M is an invariant subspace for V if $V(M) \subseteq M$ and a reducing subspace for V if, in addition, $V(M^{\perp}) \subseteq M^{\perp}$.

The following result from [3], characterizes the invariant and reducing subspaces for an operator.

Proposition 1.3. *If V is an operator on H, M is a closed subspace of H and P_M is the orthogonal projection onto H, then M is an invariant subspace for V if and only if $P_M V P_M = V P_M$ if and only if $M^{\perp}$ is an invariant subspace for V^*; further, M is a reducing subspace for V if and only if $P_M V = V P_M$ if and only if M is an invariant subspace for both V and V^*.*

Definition 1.4. An operator $V \in B(H, K)$ is called Fredholm if $N(V)$ is of finite dimension and $R(V)$ of finite codimension.

Theorem 1.5. *(Atkinson) The operator $V \in B(H, K)$ is Fredholm if and only if there exist operators $T_1, T_2 \in B(K, H)$ such that the operators $I - T_1 V$ and $I - V T_2$ are compact.*

Now we state a generalization of the Atkinson's Theorem.

Proposition 1.6. *If the operators $V, L \in B(H, K)$ and L is invertible, then V is a Fredholm operator if and only if there exists an operator $R \in B(K, H)$ such that $L - RV$ and $L - VR$ are both compact.*

Theorem 1.7. ([4]) *Let H be a (separable) Hilbert space and $K \in B(H)$. Then $\{f_i\}_{i=1}^{\infty}$ is a K -frame for H if and only if there exists a linear bounded operator $L : \ell^2 \longrightarrow H$ such that $f_n = Le_n$ and $R(K) \subseteq R(L)$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for ℓ^2.*

2. Main Results

In this section, we state some conditions on Bessel sequences in Hilbert spaces which lead us to construct new K-frames.

At first we show that for each bounded operator K on any finite dimensional Hilbert space H, we can find a K-frame.

Proposition 2.1. *Let H be a finite dimensional Hilbert space and $K : H \longrightarrow H$ be a bounded operator. If A, B are positive real numbers so that $A \leq B\|K\|^{-2}$, then there exist some finite number of elements in H that form a K-frame for H with bounds A, B. Moreover, It is possible to find a finite K-Parseval frame for H.*

As a result Proposition 2.1, we have the following corollary.

Corollary 2.2. *If H is a infinite dimensional Hilbert space, $K \in B(H)$ and M is a finite dimensional subspace of H. Then there exist finitely many elements $f_1, f_2, ..., f_N$ in M such that $\{f_n\}_{n=1}^N$ is a PKP-frame for M, where P is the orthogonal projection of H onto M.*

Suppose that H is a finite dimensional Hilbert space, $K \in B(H)$ and $\{f_i\}_{i=1}^\infty$ is a Bessel sequence in H. Proposition 2.1 states that we can find a finite K-frame $\{h_i\}_{i=1}^N$ for H. So the sequence $\{h_n\}_{n=1}^N \bigcup \{f_n\}_{n=1}^\infty$ is a K-frame for H. In other words, for each Bessel sequence in H, we can find a finite extension to a K-frame.

Now we verify this topic for infinite dimensional Hilbert spaces. Next proposition is one side of our goal to find finite number of elements to expand a Bessel sequence to a K-frame.

In the following of this article, to avoid repetition, we suppose that Hilbert space H is infinite dimensional.

Proposition 2.3. *Suppose that $K \in B(H)$ and $\{f_i\}_{i=1}^\infty$ is a Bessel sequence in H. If there exists finite number of elements $h_1, h_2, ...h_N$ in H such that the sequence $\{h_i\}_{i=1}^N \bigcup \{f_i\}_{i=1}^\infty$ is a K-frame for H, then there exists a Bessel sequence $\{g_n\}_{n=1}^\infty$ in H such that the operator $K^* - U_2^* U_1$ is a finite-rank operator, where U_1 and U_2 are the analysis operators of $\{f_i\}_{i=1}^\infty$ and $\{g_i\}_{i=1}^\infty$, respectively.*

Similar to Atkinson's theorem, we have the next Lemma.

Lemma 2.4. *Let $U \in B(H, K)$ and there exists $V \in B(K, H)$ such that the operator $I - VU$ is compact. Then $R(U)$ is a closed subspace of K and $\dim(N(U)) < \infty$.*

As a corollary of above lemma, we have the next lemma.

Lemma 2.5. *If $L, U \in B(H, K)$, L is invertible and there exists a $V \in B(K, H)$ such that the operator $L - VU$ is compact, then $R(U)$ is a closed subspace of K and $\dim(N(U)) < \infty$.*

By Lemma 2.4, we have the next theorem.

Theorem 2.6. *Suppose that $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ are Bessel sequences for H with bounds B_1 and B_2 and analysis operators U_1 and U_2, respectively. If $K \in B(H)$, $I - U_2^* U_1$ is a compact operator and P is the orthogonal projection of H onto $N(U_1)$, then there exist finite number of elements $x_1, x_2, ..., x_r$ and $y_1, y_2, ..., y_t$ such that the sequences $\{x_i\}_{i=1}^r \bigcup \{f_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^t \bigcup \{g_i\}_{i=1}^\infty$ are KP-frames for H with upper bounds B_1 and B_2, respectively.*

From Lemma 2.5, we get the next result.

Corollary 2.7. *Suppose that $\{f_i\}_{i=1}^\infty$ and $\{g_i\}_{i=1}^\infty$ are Bessel sequences for H with bounds B_1 and B_2 and analysis operators U_1 and U_2, respectively. Let $K \in B(H)$ be invertible, $K^* - U_2^* U_1$ be a compact operator and P be the orthogonal projection of H onto $N(U_1)$. Then there exist finite number of elements $x_1, x_2, ..., x_r$ and $y_1, y_2, ..., y_t$ such that $\{x_i\}_{i=1}^r \bigcup \{f_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^t \bigcup \{g_i\}_{i=1}^\infty$ are KP-frames for H with upper bounds B_1 and B_2, respectively.*

Now, under some conditions, we extend a Bessel sequence to a K-frame by adding finitely many elements to it.

Theorem 2.8. *Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence for H with bound B and analysis operator U and it is not a frame. Suppose that there exists $R \in B(H, \ell^2)$ such that $I - R^*U$ is a compact operator. If $K \in B(H)$ is such that $N(U)$ is a reducing subspace for K and P is the orthogonal projection of H onto $N(U)$, then there exist finite number of elements $x_1, x_2, ..., x_r$ such that the sequence $\{x_i\}_{i=1}^{r} \bigcup \{f_i\}_{i=1}^{\infty}$ is K-frame for H with upper bound B.*

Remark 2.9. In the Theorem 2.8, if $\{f_i\}_{i=1}^{\infty}$ is a frame for H, then by method of its proof, we can't find such an extension. In the case that $\{f_i\}_{i=1}^{\infty}$ is a frame for H, Theorem 1.7 implies that for every $K \in B(H)$, the sequence $\{f_i\}_{i=1}^{\infty}$ is also a K-frame for H.

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An extension of Wang's protein design model using Blosum62 substitution matrix

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Abstract. One of the problems that help us understand the relation between protein structures is the well-known protein design problem which attempts to find an amino acid sequence that can fold into a desired tertiary structure. However, despite having an acceptable accuracy in protein design, this accuracy is an identical percentage of amino acid retrieving. At the same time, it is well-known that amino acids can replace each other in evolution while the function and structure of protein stay the same. Thus the designed sequence does not have the opportunity to be close to the target in the evolutionary aspect. This paper presents an extension to Wang's deep learning model and uses evolutionary information in the Blosum62 substitution matrix to take amino acid replacement probability into account while designing a sequence.

Keywords: Protein Design, Blosum Matrix, Deep Learning

AMS Mathematics Subject Classification [2010]: 92-08

1. Introduction

Protein's vast majority of functionalities, e.g., helping the olfactory system and catalyzing metabolism reactions, made this macromolecule an essential topic of study in biology. The protein's functionality comes from its most significant structure, the tertiary structure, which has an almost unique shape for the protein. With all this importance in mind and PDB saturation in tertiary structures, understanding the relationship between primary and tertiary structures helps us in protein tertiary structure prediction and genome sequence functionality prediction. One of the approaches for understanding the relationship between primary and tertiary structures is discovering a sequence of amino acids that can get a desired tertiary structure, referred to as the protein design problem (PDP). Some applications of PDP are designing proteins that can interact with specific targets and designing biosensors. Besides, this is also notable that PDP is an NP-hard problem.

There are three broad classes of PDP algorithms, approximation algorithms, heuristic algorithms, and machine learning. The first class contains algorithms such as dead-end elimination with a good quality solution but no guarantee for runtime. Algorithms in the

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second class use a heuristic algorithm to optimize a designed energy function and find a sequence that minimizes this energy function. For example, PDP tools like EvoDesign [2] use this approach, but they have the downside of designing an energy function which is hard to do. The third class of algorithms uses machine learning methods, especially Deep Learning methods, and tends to find a mapping from input to the desired output.

There is no need to design an energy function in the machine learning approaches, and the only needed material is data. In 2014 Li et al. used a simple neural network and manually extracted structural features as input [1]. After that, in 2018, O'Connell et al. developed a deep neural network and more manually extracted features as the network input [3]. In the same year, Wang et al. provided a more complex deep learning model and used feature extraction as input [4].These models, despite having acceptable accuracies lack quality in produced sequences. This inadequateness in the quality of generated sequences has two main reasons. The first is a supervised training method where the targets for output are one-hot encodings of amino acid classes like in all models. The second reason is the lack of information in the input data like first three papers.

One-hot encoded targets for training cause the network to restore the wild-type sequence of structure. However, some non-identical protein sequences have similar foldings due to their mutations in evolution, and such information about sequences is available in substitution matrices like Blosum.

In this paper, we review Wang's model in detail and then extend this model to use available information in Blosum matrices and obtain generated sequences with higher quality.

2. Material and Methods

We can explain PDP as a problem with inputs, outputs, and goals. The input is a backbone-only model of tertiary structure for a protein, and the output is a sequence of amino acids. The aim is to design the output sequence so that the resulting structure is similar to the input structure when it folds into a tertiary structure.

Among the provided machine learning methods for PDP so far, Wang's model has the highest accuracy among reviewed models, therefore we decided to extend Wang's model to obtain better higher accuracy which has the quality of a wild sequence.

2.1. Dataset. Data collection happens by retrieving protein structure and corresponding sequences from PDB with the following criteria as Wang's model does. (1) The structure determination method is x-ray crystallography, (2) The resolution of the tertiary structure is better than 2, (3) The length of the protein sequence is more than 50 amino acids, (4) The entry does not contain any DNA or RNA molecules, and (5)Amino acid sequences of all found entries have less than 30% pairwise identity [4].

Furthermore, each entry containing D-amino acids vanishes from the dataset, and the non-amino acid residues of each protein also exclude from the structure. In the next step, each protein with a sequence length of L_s split into L_s clusters, where each corresponds to one of the residues. Each cluster contains a target residue and its 15 nearest neighbors regarding the $C_\alpha - C_\alpha$ distance. For each cluster, all the neighbor and target residues rotate and translate such that the C_α atom of the target residue locates on the $(0, 0, 0)$ point, the N atom of the target residue lies on the $-x$ axis, and the C atom of the target residue takes place in the $z = 0$ plane.

2.2. Wang's model.

Input and Output. For input, feature extraction happens on each cluster, and each one of the clusters would have one set of features for the target residue, and another set of features correspond to each of the neighbors in the cluster. Target residue feature set contains sine and cosine of three backbone dihedral angles ϕ, ψ, and ω, the total solvent accessible surface area(SASA) of backbone atoms, and the three stated secondary structures (helix, sheet, or coil) represented with a one-hot vector. As for the neighboring residues, the feature set for each one contains the exact features like target and other features. Other features include $C_\alpha - C_\alpha$ Euclidean distance to the target, unit $C_\alpha - C_\alpha$ vector from the target to the neighbor, unit $C - N$ and $C - C$ vector in the under process neighbor residue, and the number of hydrogen bonds between the target and neighbor. Feature extraction results in 10 features for the target residue and 24 features for each neighboring residue in a cluster. As for outputs, we perform one-hot encoding of the amino acid type for each cluster's target residue as output targets. Thus, the model outputs a vector of size 20 that we interpret as probabilities of different amino acid types for the input cluster.

Model. The presented model by wang et al., as presented in Figure1, has two sub-networks and three final layers before the output layer. The sub-networks are called residue probability network and weight network. The residue probability network tends to find primal probabilities for the class of target residue by seeing this residue and one of its neighbor residues features, and the Weight network produces a weight by considering the same input as the residue probability network as well. The output of two subnetworks is multiplied by each other and kept as part of the input for later layers. This procedure executes concurrently for 15 nearest neighbors of the target amino acid, and then the multiplied result of all is concatenated to each other. The concatenated result feeds into three layers of densely connected layers, and at the end, a softmax layer containing 20 nodes outputs a probability vector.

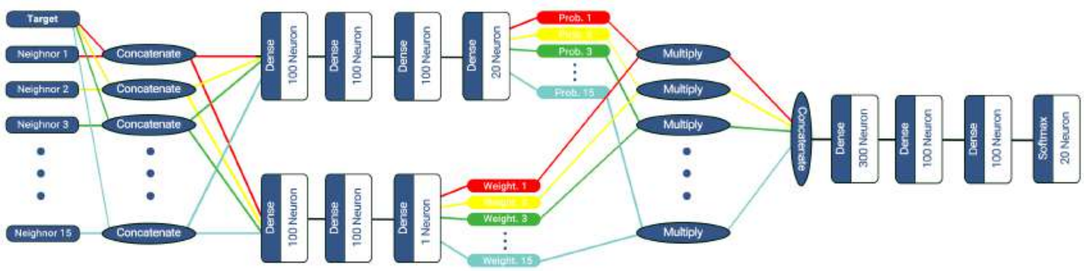

FIGURE 1. Wang's model architecture

2.3. Extended Model.

Input and Architecture. The input and architecture for our extended model are the same as Wang's model.

output. Instead of training the network with one-hot encoded targets, it is rational to use a vector that considers other probabilities. We use the characteristic of the Blosum matrices and present a new target of training that contains probabilities of multiple amino acid classes. We chose Blosum62, which contains substitution information from proteins with less than 62% identity. To present these scores as targets of training the network, we applied the softmax function to each row of the Blosum62 matrix. Eventually, these converted rows are considered as targets and replaced with one-hot encoded vectors for loss calculation.

3. Results

To implement both networks, we used the Keras tool and all the same hyperparameters as the Wang et al. paper suggested. Because of numerous clusters, we selected 100,000 clusters randomly and split them into three non-redundant train, test, and validation datasets, respectively, with 0.7, 0.15, and 0.15 of selected data. The same results as the original model were first regenerated. Then, after modifying and training the network on the same data, we compared the results of both the original and extended models on the test set. Comparision shows a remarkable improvement in sequence identity and accuracy compared to Wang's model, as shown in table1. For a better perception of quality change, we compared 15 positions of natural sequence and the sequences generated by Wang's and our extended model as shown in table2.

Table 1. Accuracy of different models on test set

Model	Accuracy
Wang's Model	33.39%
Extended Wang's Model	44.89%

Table 2. Comparison of 15 positions on a sample sequence

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Real sequence	Y	A	V	K	L	K	T	D	F	D	N	P	R	W	I
Wang's Model	A	A	P	A	A	A	P	A	A	D	D	P	R	A	A
Extended Wang's Model	K	A	P	R	L	K	P	D	E	D	D	P	R	Y	E

4. Conclusion

Despite having acceptable identity between output sequence of machine learning methods and the natural target sequences, the generated sequences have a low quality. In this paper, we provided an extension to Wang's model and achieved much better results to such a minimal extent.

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On the construction of second derivative methods with inherent quadratic stability

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Abstract. The purpose of this paper is to construct explicit Nordsieck second derivative general linear methods with inherent Quadratic Stability property which have large region of absolute stability. Examples of such methods of order $p = q + 1 = r = s$ are given and the stability regions are plotted together with those for general linear methods of the same order.

Keywords: General linear methods. second derivative general linear methods. Nordsieck representation. Order conditions. Inherent Rung-Kutta stability, Inherent Quadratic Stability

1. Introduction

In this paper, we focus on constructing a class of explicit second derivative general linear methods (SGLMs) for the numerical solution of a system of ordinary differential equations (ODEs):

$$y'(x) = f(y(x)), \quad y(x_0) = y_0, \tag{1}$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$. We consider the class of SGLMs in the following form [4]

$$\begin{cases} Y_i^{[n]} = h\sum_{j=1}^{s} a_{ij} f(Y_j^{[n]}) + h^2 \sum_{j=1}^{s} \overline{a}_{ij} g(Y_j^{[n]}) + \sum_{j=1}^{r} u_{ij} z_j^{[n-1]}, \\ z_i^{[n]} = h\sum_{j=1}^{s} b_{ij} f(Y_j^{[n]}) + h^2 \sum_{j=1}^{s} \overline{b}_{ij} g(Y_j^{[n]}) + \sum_{j=1}^{r} v_{ij} z_j^{[n-1]}, \end{cases} \tag{2}$$

$n = 1, 2, \ldots, N$. Here, $Y^{[n]}$ is an approximation of the stage order q to $y(x_n + c_i h)$, $g(.) = f^{'}(.)f(.)$ and $z_i^{[n]}$ is an approximation of order p to the element $h^{i-1}y^{(i-1)}(t_n)$ of the Nordsieck vector defined by $z(t, h) = [y(t) \ hy'(y) \ \cdots \ h^{r-1}y^{(r-1)}(t)]^T$. Such schemes are characterized by six coefficients matrices; $A = [a_{ij}] \in \mathbb{R}^{s\times s}$, $\overline{A} = [\overline{a}_{ij}] \in \mathbb{R}^{s\times s}$, $U = [u_{ij}] \in \mathbb{R}^{s\times r}$, $B = [b_{ij}] \in \mathbb{R}^{r\times s}$, $\overline{B} = [\bar{b}_{ij}] \in \mathbb{R}^{r\times s}$ and $V = [v_{ij}] \in \mathbb{R}^{r\times r}$, the abscissa vector $c = [c_1 \ c_2 \ \ldots \ c_s]^T$, and the four integers: the order p, the stage order q, the number of

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internal stages s, and the number of external stages r. In what follows, we will assume that $p = q+1 = s = r$, the coefficients matrices A and $\overline{A}$ are strictly lower triangular and the matrix V is supposed to take the form

$$V = \begin{bmatrix} 1 & v_{12} & v_{13} & \cdots & v_{1r} \\ 0 & 0 & v_{23} & \cdots & v_{2r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v_{sr} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

which ensures that the SGLM (2) is zero-stable.

Assuming

$$z_i^{[n-1]} = \sum_{k=0}^{p} q_{ik} h^k y^{(k)}(t_{n-1}) + O(h^{p+1}), \quad i = 1, 2, \ldots, r,$$

the SGLM (2) has order p and stage order $q = p$ if

$$Y_i^{[n]} = y(t_{n-1} + c_i h) + O(h^{p+1}), \quad i = 1, 2, \ldots, s,$$
$$z_i^{[n]} = \sum_{k=0}^{p} q_{ik} h^k y^{(k)}(t_n) + O(h^{p+1}), \quad i = 1, 2, \ldots, r,$$

for some parameters q_{ik}. Introduce the matrices $C \in \mathbb{R}^{s\times(p+1)}$, $K \in \mathbb{R}^{(p+1)\times(p+1)}$ and $E \in \mathbb{R}^{(p+1)\times(p+1)}$ as follow

$$C := \left[e \ \ \frac{c}{1!} \ \ \frac{c^2}{2!} \ \ \cdots \ \ \frac{c^p}{p!} \right], \quad K := [0 \ \ e_1 \ \ e_2 \ \ \cdots \ \ e_p],$$

$$E := \exp(K) = \begin{bmatrix} 1 & \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{p!} \\ 0 & 1 & \frac{1}{1!} & \cdots & \frac{1}{(p-1)!} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \frac{1}{1!} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

with c^j as the component-wise powers of abscissa vector c, $e = [1 \ 1 \ \cdots \ 1]^T \in \mathbb{R}^s$, and e_j as the jth unit vector in $\mathbb{R}^{p+1}$ and put $W = [q_0 \ q_1 \ \ldots \ q_p]$ with $q_k = [q_{1k} \ q_{2k} \ \ldots \ q_{rk}]^T$, $k = 0, 1, 2, \ldots, p$. It was demonstrated in [2] that the SGLM (2) has order and stage order p iff

$$UW = C - ACK - \overline{A}CK^2,$$
$$VW = WE - BCK - \overline{B}CK^2.$$

In the case of methods with $r = p+1$, if $W = I_r$ the methods are in the Nordsieck form. For methods with $r = p$, the order and stage order conditions are determined in [1] by choosing the last column of W as zero vector of relevant dimension, $W = [I_p \ \ \mathbf{0}]$.

Theorem 1.1. *The SGLM (2) in Nordsieck form has order and stage order $p = q+1 = r$ iff*

$$\begin{aligned} U &= \widetilde{C} - A\widetilde{C}\widetilde{K} - \overline{A}\widetilde{C}\widetilde{K}, \\ V &= \widetilde{E} - B\widetilde{C}\widetilde{K} - \overline{B}\widetilde{C}\widetilde{K}, \end{aligned} \tag{3}$$

and

$$B\frac{c^{p-1}}{(p-1)!}+\overline{B}\frac{c^{p-2}}{(p-2)!}=\widehat{E}. \tag{4}$$

Here, the matrices $\widetilde{K}$ and $\widetilde{E}$ stand for the pth order leading principal submatrix of K and E, respectively, and the matrix $\widetilde{C}$ shows the first p columns of the matrix C and the vector $\widehat{E}$ are defined by

$$\widehat{E}=\begin{bmatrix}\frac{1}{p!} & \frac{1}{(p-1)!} & \cdots & \frac{1}{1!}\end{bmatrix}^T.$$

Applying such methods to the test equation $y'=\xi y$, $t\geq 0$, $\xi\in\mathbb{C}$, leads to a recurrence equation $z^{[n]}=M(z)z^{[n-1]}$, $n\geq 1$, where $z=h\xi$ and the stability matrix $M(z)$ is given by

$$M(z)=V+(zB+z^2\overline{B})(I-zA-z^2\overline{A})^{-1}U.$$

Moreover, the stability function is defined by $p(w,z)=\det(wI-M(z))$. The method is said to possess Runge–Kutta stability (RKS) if the stability function has special form $p(w,z)=w^{r-1}(w-R(z))$, where $R(z)$ has the same role as the stability function of a Runge–Kutta (RK) method. Imposing RKS conditions directly results in complicated equations in terms of coefficients matrices of the methods which is difficult to solve. To overcome this drawback, Movahednejad et al. in [5] determined some interrelations between the coefficients matrices of the SGLMs to guarantee the methods have RKS property. Considering these conditions, some A- and L-stable SGLMs with inherent RKS (SIRKS) were constructed up to order five. The aim of this paper is to relax the concept of SIRKS to the concept of inherent quadratic stability (IQS) and search for methods with a large area stability. Property of IQS is a weaker property than SIRKS but compared to the methods with SIRKS, we need to solve fewer equations which makes construction to be easier and provides some additional free parameters. We are going to use these free parameters in order to maximize the area of absolute stability region . The concept of IQS for SGLMs which was first introduced in [6] means that there exists a matrix $X\in\mathbb{R}^{s\times s}$ in such a way that

$$B\overline{A}\equiv X\overline{B},\quad BA+\overline{B}\equiv XB,\quad BU\equiv XV-VX. \tag{5}$$

Here, the relation $P\equiv Q$ means that the matrices P and Q are identical except possibly in their first row, and the matrix X appearing in these conditions is given by

$$X=\left[\begin{array}{ccccc|c} x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,r-1} & x_{1,r}\\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots & x_{2,r-1} & x_{2,r}\\ \hline 0 & 1 & 0 & \cdots & 0 & x_{3,r}\\ 0 & 0 & 1 & \cdots & 0 & x_{4,r}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & x_{r,r}\end{array}\right].$$

2. Example of methods with a large area stability

In this section, we search for methods of order $p=q+1=s=r=2$ with IQS by solving the minimization problem min `-area` by using the Matlab program `fminsearch`. It should be noted that in this case all the methods possess the quadratic stability property. Solving the order and stage order conditions (3) and (4) lead to an 7-parameter family of methods. We use these free parameters to obtain methods with largest area stability. The

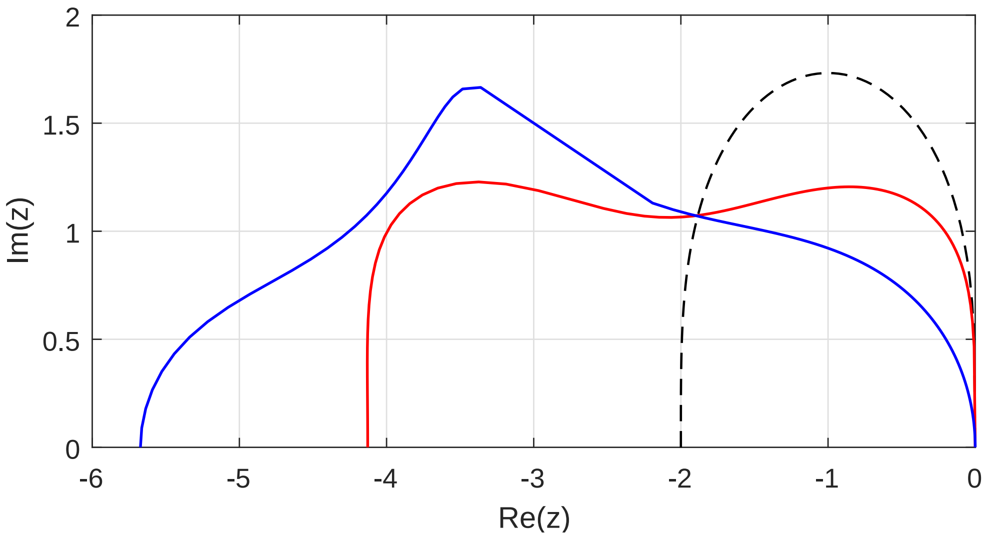

FIGURE 1. Stability regions of SGLM with IQS (blue line) of order $p = q+1 = s = r = 2$ together with that for GLM with IQS of order $p = q+1 = s = r = 2$ (red line) and RK method (dashed-line) of order $p = s = 2$.

coefficients matrices of the constructed method with $c = [\frac{1}{2} \;\; 1]^T$ which has stability area equal to 17.92 are given by

$$A = \begin{bmatrix} 0 & 0 \\ \frac{416}{833} & 0 \end{bmatrix}, \quad \overline{A} = \begin{bmatrix} 0 & 0 \\ \frac{109}{10000} & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & \frac{417}{833} \end{bmatrix},$$

$$B = \begin{bmatrix} -\frac{257}{500} & \frac{718}{877} \\ \frac{692}{159} & -\frac{533}{159} \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} -\frac{699}{1817} & -\frac{13}{250} \\ \frac{2685}{1613} & \frac{23}{2000} \end{bmatrix}, \quad V = \begin{bmatrix} 1 & \frac{429}{617} \\ 0 & 0 \end{bmatrix}.$$

The stability region of the derived method together with that for GLMs with IQS of the same order, investigated in [3], and RK method of order $p = s = 2$ are plotted in Figure 1.

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30 August - 02 September 2021
52nd Annual Iranian Mathematics Conference

Approximately C-controlled g-dual frames

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Abstract. In this paper, approximately for controlled g-dual frames is defined and some of their properties are investigated. Finally, we characterize the relationship between approximately C-controlled dual and C-controlled g-dual.

Keywords: frames, controlled frames, g-dual frame, approximate g-dual
AMS Mathematics Subject Classification [2010]: 42C15, 42C99

1. Introduction

Frames for Hilbert space were formally defined by Duffin and Schaeffer [4] in 1952 for studying some problems in non harmonic Fourier series. Recall that for a Hilbert space $\mathcal{H}$ and a countable index set J, a collection $\{f_j\}_{j\in J} \subset \mathcal{H}$ is called a frame for the Hilbert space $\mathcal{H}$, if there exist two positive constants c, d, such that for all $f \in \mathcal{H}$

$$c\|f\|^2 \leq \sum_{j\in J} |\langle f, f_j\rangle|^2 \leq d\|f\|^2; \tag{1}$$

c and d are called the lower and upper frame bounds, respectively.

Dehghan and Hasankhani Fard [3] introduced and characterized g-duals of a frame in a separable Hilbert space and Ramezani and Nazari [6] extended this concept for generalized frame. A frame $\{g_j\}_{j\in J}$ is called a g-dual frame of the frame $\{f_j\}_{j\in J}$ for $\mathcal{H}$ if there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that, for all $f \in \mathcal{H}$

$$f = \sum_{j\in J} \langle Af, g_j\rangle f_j,$$

where $\mathcal{B}(\mathcal{H})$ denotes the set of all bounded operators on $\mathcal{H}$. Let $GL(\mathcal{H})$ be the set of all bounded operators with a bounded inverse. A frame controlled by the operator C or C-controlled frame is a family of vectors $\{f_j\}_{j\in J} \subseteq \mathcal{H}$, such that there exist two constants $A_c > 0$ and $B_c < \infty$ satisfying

$$A_c\|f\|^2 \leq \sum_{j\in J} \langle f, f_j\rangle \langle Cf_j, f\rangle \leq B_c\|f\|^2; \tag{2}$$

*Speaker. Email address: m.ramezani@yu.ac.ir for every $f \in \mathcal{H}$, where $C \in GL(\mathcal{H})$. Every frame is an I-controlled frame. Hence the controlled frames are generalizations of frames. The controlled frame operator S_c is defined by

$$S_c f = \sum_{j\in J} \langle f, f_j\rangle C f_j = CS, \qquad (f \in \mathcal{H}), \tag{3}$$

where S is the frame operator of $\{f_j\}_{j\in J}$. The synthesis operator for a C-controlled frame $\{f_j\}_{j\in J}$ is defined as follows

$$T_c(\{\alpha_j\}_{j\in J}) = \sum_{j\in J} \alpha_j C f_j = CT,$$

where T is the synthesis operator of $\{f_j\}_{j\in J}$ and $S_c = T_c T^*$. C-Controlled frame $\{f_j\}_{j\in J}$ and Bessel sequences$\{g_j\}_{j\in J}$ are said to be C-controlled duals for $\mathcal{H}$ if the following equality holds

$$f = \sum_{j\in J} \langle f, g_j\rangle C f_j, \text{ for all f} \in \mathcal{H}.$$

Ramezani [5] introduced the notion of controlled g-dual frames in Hilbert spaces and characterized all controlled g-dual frames for a given controlled frame. in this paper we define approximate controlled g-duals for the controlled frames and using this concept, we establish relationship between approximately controlled g-dual frames and controlled dual frames and controlled g-dual frames.

2. Main Results

C-controlled g-dual frames are stable under some perturbations.

Definition 2.1. [1] Bessel sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are said to be approximately dual frames if $\|I - TU^*\| < 1$ or $\|I - UT^*\| < 1$

Definition 2.2. [2] The Bessel sequence $\{g_j\}_{j\in J}$ is called an approximate C-controlled dual of a C-controlled frame $\{f_j\}_{j\in J}$ whenever

$$\|f - \sum_{j\in J} \langle f, g_j\rangle C f_j\| < 1, \qquad (f \in \mathcal{H}).$$

On the other hands $\|I - T_c U^*\| < 1$.

The above definitions led us to define the following definitions.

Definition 2.3. Two Bessel sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ with synthesis operator T and U, respectively, are approximately g-dual frames for $\mathcal{H}$ if there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that $\|I - TU^*A\| < 1$ or $\|I - UT^*A\| < 1$.

Definition 2.4. The Bessel sequence $\{g_j\}_{j\in J}$ is called an approximate C-controlled g-dual of a C-controlled frame $\{f_j\}_{j\in J}$ if there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that

$$\|f - \sum_{j\in J} \langle Af, g_j\rangle C f_j\| < 1, \qquad (f \in \mathcal{H}).$$

On the other hands $\|I - T_c U^* A\| < 1$.

Theorem 2.5. *If two Bessel sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are approximately C-controlled dual frames for $\mathcal{H}$, then $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are C-controlled g-dual frames.*

Proof. Since $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are approximately C-controlled dual frames, $\|I - T_cU^*\| < 1$, where T and U are synthesis operators of $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$, respectively. Hence T_cU^* is an invertible operator. Then for all $f \in \mathcal{H}$ we have

$$f = T_cU^*\big(T_cU^*\big)^{-1}(f) = C\left(\sum_{j\in J}\langle (T_cU^*)^{-1}f, g_j\rangle f_j\right) = \sum_{j\in J}\langle Af, g_j\rangle Cf_j,$$

where $A = (T_cU^*)^{-1}$ is an invertible operator. So for all $f \in \mathcal{H}$ we have

$$f = \sum_{j\in J}\langle Af, g_j\rangle Cf_j,$$

□

as claimed.

The following example illustrates that the set of approximately C-controlled duals of a frame is a proper subset of the set of its C-controlled g-duals.

Example 2.6. Let $\{e_j\}_{j\in J}$ be an orthonormal basis for $\mathcal{H}$. Set

(1) $\{f_j\}_{j\in J} = \{e_1, e_1, e_1, e_2, e_3, \cdots\}$

(2) $\{g_j\}_{j\in J} = \{\frac{1}{3}e_1, \frac{1}{3}e_1, \frac{1}{3}e_1, e_2, , e_3, \cdots\}$

and consider the operator $C : \mathcal{H} \longrightarrow \mathcal{H}$ given by $C(f) = \frac{1}{2}f$. Now we have

$$\sum_{j\in J}\langle f, f_j\rangle\langle Cf_j, f\rangle = \langle f, e_1\rangle\langle e_1, f\rangle + \frac{1}{2}\sum_{j\in J}\langle f, e_j\rangle\langle e_j, f\rangle$$

So

$$\frac{1}{2}\|f\|^2 \le \sum_{j\in J}\langle f, f_j\rangle\langle Cf_j, f\rangle \le \frac{3}{2}\|f\|^2$$

Therefore $\{f_j\}_{j\in J}$ is a C-controlled frame and $\{g_j\}_{j\in J}$ is a Bessel sequence and

$$\|f - \sum_{j\in J}\langle f, g_j\rangle Cf_j\| = \frac{1}{2}\|f\|, \text{ for all } f \in \mathcal{H}$$

Hence $\{g_j\}_{j\in J}$ is not a approximately C-controlled dual frame of $\{f_j\}_{j\in J}$ but a C-controlled g-dual frame for $\{f_j\}_{j\in J}$ with the invertible operator $A(f) = 2f$; because $\sum_{j\in J}\langle Af, g_j\rangle Cf_j = f$, for any $f \in \mathcal{H}$.

The following theorem shows under what conditions the opposite of Theorem 2.5 is established.

Theorem 2.7. *If $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are C-controlled g-dual frames with invertible operator A such that $\|I - A^{-1}\| < 1$, then $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are approximately C-controlled dual frames.*

Proof. Since $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are C-controlled g-dual frames, so for all $f \in \mathcal{H}$, $f = \sum_{j\in J}\langle Af, g_j\rangle Cf_j$. Hence

$$\begin{aligned}\|f - \sum_{j\in J}\langle f, g_j\rangle Cf_j\| &= \|f - \sum_{j\in J}\langle AA^{-1}f, g_j\rangle Cf_j\| \\ &= \|f - A^{-1}f\| < 1.\end{aligned}$$

□

Corollary 2.8. *Let $\mathcal{H}$ be a Hilbert space and $C \in GL(\mathcal{H})$. Also let $\{f_j\}_{j\in J}$ be a C-controlled frame and $\{g_j\}_{j\in J}$ be an approximate C-controlled g-dual of $\{f_j\}_{j\in J}$ with invertible operator A. Then*

(i) *$\{(UT_c^*)^{-1}g_j\}_{j\in J}$ is a C-controlled dual of $\{f_j\}_{j\in J}$ and*

$$(UT_c^*)^{-1}g_j = g_j + \sum_{n=1}^{+\infty}(I - UT_c^*)^n g_j.$$

(ii) *$\{g_j\}_{j\in J}$ is a C-controlled g-dual of $\{f_j\}_{j\in J}$ with invertible operator $(T_cU^*)^{-1}$.*

(iii) *$\{g_j\}_{j\in J}$ is an approximately C-controlled dual of $\{f_j\}_{j\in J}$.*

Proof. To prove (i), by the definition of an approximate C-controlled g-dual, we have

$$\|I - T_cU^*A\| < 1,$$

which implies that T_cU^*A is an invertible operator. By assumption, A is an invertible operator, so T_cU^* is an invertible operator. Therefore similar argument in proof Theorem 3.2 from [2], $\{(UT_c^*)^{-1}g_j\}_{j\in J}$ is a C-controlled dual of $\{f_j\}_{j\in J}$ and

$$(UT_c^*)^{-1}g_j = g_j + \sum_{n=1}^{+\infty}(I - UT_c^*)^n g_j.$$

Now we prove (ii), we have already seen in parts (i) that T_cU^* is an invertible operator, the remainder of proof (ii) follows immediately from proof of Theorem 2.5.

Finally, to prove (iii), by part (ii), $\{g_j\}_{j\in J}$ is a C-controlled g-dual of $\{f_j\}_{j\in J}$ with invertible operator $(T_cU^*)^{-1}$. Also,

$$\|I - ((T_cU^*A)^{-1})^{-1}\| = \|I - T_cU^*A\| < 1.$$

Now directly using Theorem 2.7, $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are approximately C-controlled dual frames and this completes the proof. □

3. Conclusion

In the 51^{st} Annual Iranian Mathematics Conference, we introduced and characterized controlled g-duals of a frame in a separable Hilbert space $\mathcal{H}$. This article is a continuation of our previous work and in it approximately for controlled g-dual frames is defined and the relationship between approximately C-controlled dual and C-controlled g-dual is characterized.

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Second derivative multistage methods

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Abstract. In this paper, we describe construction of a class of explicit second derivative Runge–Kutta methods which have extensive region of absolute stability. Examples of such methods with $p = q = s = 1$ and 2 are given in which p and q stand for order and stage order, and s is the number of stages. These methods, because of their extensive stability region, can compete with the traditional explicit Runge–Kutta methods of the same order in solving initial value problems.

Keywords: Ordinary differential equations, Two-derivative Runge–Kutta methods, Order conditions, Stability

AMS Mathematics Subject Classification [2010]: 65L05

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1. Introduction

Two-derivative Runge–Kutta (TDRK) methods for solving initial value problems (IVPs) of ordinary differential equations (ODEs)

$$(1) \qquad \begin{cases} y'(x) = f\big(y(x)\big), \quad x \in [x_0, \overline{X}], \\ y(x_0) = y_0, \end{cases}$$

are defined by

$$(2) \qquad \begin{cases} Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j) + \sum_{j=1}^{s} \widehat{a}_{ij} g(Y_j), \quad i = 1, 2, \dots, s, \\ y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f\big(Y_i\big) + \sum_{i=1}^{s} \widehat{b}_i g(Y_i), \end{cases}$$

where $y'' = g(y) := f'(y)f(y)$ with $f, g : \mathbb{R}^m \to \mathbb{R}^m$ and the internal stage value Y_i approximate $y(x_n + c_i h)$ and y_{n+1} is the update value which approximates $y(x_n + h)$. The coefficients of these methods can be represented by Butcher tableau

$$\begin{array}{c|c||c} c & A & \widehat{A} \\ \hline & b^T & \widehat{b}^T \end{array}$$

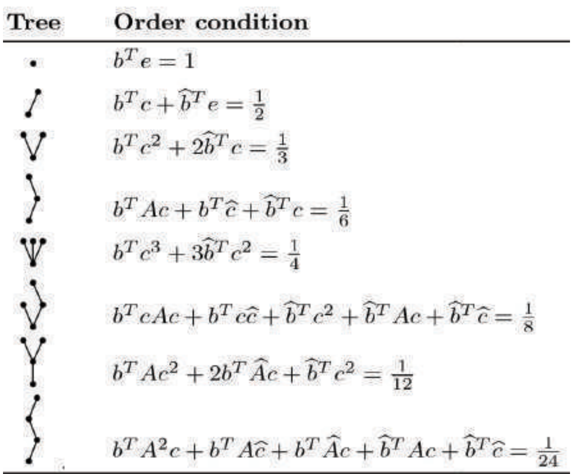
*Speaker. Email address: s.rasi@tabrizu.ac.ir where $A = (a_{ij})_{s\times s}$, $\widehat{A} = (\widehat{a}_{ij})_{s\times s}$, the vector forms $b = (b_i)_{s\times 1}$ and $\widehat{b} = (\widehat{b}_i)_{s\times 1}$ are the vectors of weights and $c = (c_i)_{s\times 1}$ is the abscissa vector. The internal stage vector is $Y = [Y_1, \ldots, Y_s]^T$, the vectors of the first and second derivatives evaluated at the internal stage points are $F(Y) = [f(Y_1), \ldots, f(Y_s)]^T$ and $G(Y) = [g(Y_1), \ldots, g(Y_s)]^T$ respectively. The TDRK method (2) can be written in a more compact vector form

$$\begin{aligned} Y &= e \otimes y_n + h(A \otimes I_m)F(Y) + h^2(\widehat{A} \otimes I_m)G(Y), \\ y_{n+1} &= y_n + h(b^T \otimes I_m)F(Y) + h^2(\widehat{b}^T \otimes I_m)G(Y). \end{aligned} \tag{3}$$

TDRK methods as well as second derivative multistep methods have been studied in a unifying framework by introducing second derivative general linear methods collectively named by SGLMs [1, 3]. Some efficient methods have been constructed in large class of SGLMs. However, modifications of second derivative methods have been directly done on TDRK methods [6]. In this paper, we are going to construct explicit TDRK methods with a large stability regions. To do this, we use the remaining free parameters, after imposing the order conditions, and maximize the area of the regions. The novelty of the constructed methods lies in this fact that they can be applied to non-stiff and mildly stiff initial value problems with smaller stepsizes in comparison with similar methods. This advantage allows us to solve a wider range of problems at a lower computational cost.

2. Order conditions and stability properties of TDRK methods

Chan and Tsai [4] derived the order conditions of TDRK methods based on the tree theory, including mappings and composition of trees, developed in [2, 5]. The order conditions of TDRK methods up to order four are given in Figure 1.



Tree	Order condition
	$b^T e = 1$
	$b^T c + \widehat{b}^T e = \frac{1}{2}$
	$b^T c^2 + 2\widehat{b}^T c = \frac{1}{3}$
	$b^T Ac + b^T \widehat{c} + \widehat{b}^T c = \frac{1}{6}$
	$b^T c^3 + 3\widehat{b}^T c^2 = \frac{1}{4}$
	$b^T cAc + b^T c\widehat{c} + \widehat{b}^T c^2 + \widehat{b}^T Ac + \widehat{b}^T \widehat{c} = \frac{1}{8}$
	$b^T Ac^2 + 2b^T \widehat{A}c + \widehat{b}^T c^2 = \frac{1}{12}$
	$b^T A^2 c + b^T A\widehat{c} + b^T \widehat{A}c + \widehat{b}^T Ac + \widehat{b}^T \widehat{c} = \frac{1}{24}$

FIGURE 1. The order conditions of TDRK methods up to order four.

Also, the stage order conditions are given by

$$Ac^{k-1} + (k-1)\widehat{A}c^{k-2} = \frac{c^k}{k}, \quad k = 1, \ldots, q.$$

The stability properties of (2) are studied by applying (1) to the linear test problem

$$y' = \xi y,$$

where $\xi \in \mathbb{C}$. This leads to the matrix recurrence relation

$$y_{n+1} = R(z)y_n,$$

in which

$$R(z) = 1 + (zb^T + z^2\widehat{b}^T)(I - zA - z^2\widehat{A})^{-1}e,$$

with $z = h\xi \in \mathbb{C}$, is the stability function. In our proposed methods, this function contains some free parameters which are used to construct methods with a large region of absolute stability.

3. Construction of the methods

After satisfying the appropriate order and stage order conditions, we find the free parameters such that the resulting method has a large area of absolute stability. Here, we illustrate the construction of methods with $s = p = q = 2$. Such methods with the abscissa vector $c = [0 \;\; c_2]^T$ are given by the Butcher tableau

$$\begin{array}{c|cc||cc} 0 & & & & \\ c_2 & a_{21} & & \widehat{a}_{21} & \\ \hline & b_1 & b_2 & \widehat{b}_1 & \widehat{b}_2 \end{array}$$

with seven parameters which must satisfy the order and stage order conditions

$$\begin{aligned} &b_1 + b_2 = 1, && \frac{1}{2}b_2 + \widehat{b}_1 + \widehat{b}_2 = \frac{1}{2}, \\ &a_{21} = \frac{1}{2}, && \widehat{a}_{21} = \frac{1}{8}. \end{aligned}$$

The explicit two-stage method of order 2 with tableau

$$\begin{array}{c|cc||cc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{8} & 0 \\ \hline & 2(\widehat{b}_1 + \widehat{b}_2) & 1 - 2(\widehat{b}_1 + \widehat{b}_2) & \widehat{b}_1 & \widehat{b}_2 \end{array}$$

is derived by the equations. The stability function of the resulting two-parameter methods is

$$R(z) = 1 + z + \frac{1}{2}z^2 + (\frac{1}{8} - \frac{1}{4}\widehat{b}_1 + \frac{1}{4}\widehat{b}_2)z^3 + \frac{1}{8}\widehat{b}_2 z^4.$$

The method with a large region of absolute stability can be found with the values $\widehat{b}_1 = 0.1844$ and $\widehat{b}_2 = 0.0412$. The area of the stability region of resulted method is approximately 37.8559. This region is plotted in Figure 2 and compared with that for the explicit Runge–Kutta method with $p = s = 2$.

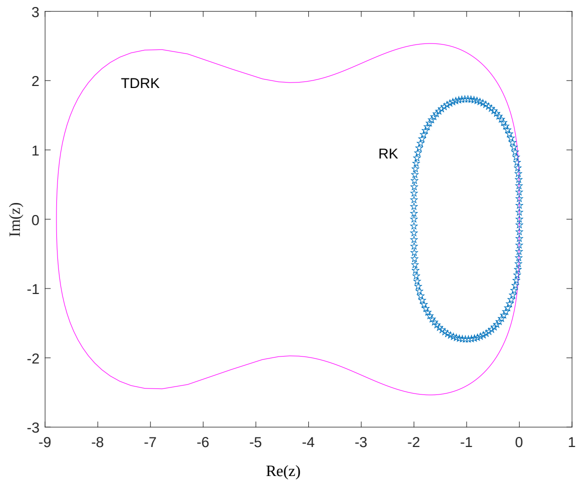


FIGURE 2. Stability regions TDRK and RK methods of the order two.

By some numerical experiments, we show capability of the constructed methods and the effect of their wider the stability region comparison with the similar methods.

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Cycle structure of elements of permutation groups with movement m or $m-3$

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Abstract. Let G be a permutation group on a set Ω which has no fixed points in Ω and let m be a positive integer. Suppose that G has bounded movement m and every non-identity element of it has movement m or $m-3$. In this paper, we determine the cycle structure of elements of G.

Keywords: Permutation group, transitive, movement, cycle

AMS Mathematics Subject Classification [2010]: 20B05

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1. Introduction

Let G be a permutation group on a set Ω which has no fixed points in Ω and let m be a positive integer. If for a subset Γ of Ω the size $|\Gamma^g - \Gamma|$ is bounded, for $g \in G$, the *movement* of Γ is defined as

$$\text{move}(\Gamma) := \max_g |\Gamma^g - \Gamma|.$$

If $\text{move}(\Gamma) \leq m$ for all $\Gamma \subseteq \Omega$, then G is said to have *bounded movement* m and the movement of G is defined as the

$$\text{move}(G) := \max_{\Gamma, g} |\Gamma^g - \Gamma|.$$

This notion was introduced in [4]. Similarly, for each $g \in G$, we define the movement of g as the

$$\text{move}(g) := \max_{\Gamma} |\Gamma^g - \Gamma|.$$

If all non-identity elements of G have the same movement, then we say that G has constant movement (see [1]). It is obvious that every permutation group with constant movement m has bounded movement m.

The purpose of this paper is to find the cycle structure of elements of permutation groups G with movement m or $m-3$. Clearly every permutation group in which every non-identity element has movement m or $m-3$, is a permutation group with bounded movement m. Moreover, by Theorem 1 of [4], if G has movement equal to m, then Ω

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is finite, and its size is bounded by a function of m. We note that for $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the integer part of x.

2. Preliminaries

In this section, we give some preliminary results that will be used in the proof of our main results. First, we present a technique to calculate the movement of elements of a permutation group.
Let $1 \neq g \in G$ and suppose that g in its disjoint cycle representation has s (s is a positive integer) nontrivial cycles of lengths $l_1, ..., l_s$, say. We might represent g as

$$g = (a_1 a_2 ... a_{l_1})(b_1 b_2 ... b_{l_2}) ... (z_1 z_2 ... z_{l_s}).$$

Let $\Gamma(g)$ denote a subset of Ω consisting of $\lfloor l_i/2 \rfloor$ points from the i^{th} cycle, for each i, chosen in such a way that $\Gamma(g)^g \cap \Gamma(g) = \emptyset$. For example, we could choose

$$\Gamma(g) = \{a_2, a_4, \ldots, a_{k_1}, b_2, b_4, \ldots, b_{k_2}, ..., z_2, z_4, \ldots, z_{k_s}\},$$

where $k_i = l_i - 1$ if l_i is odd and $k_i = l_i$ if l_i is even. Note that $\Gamma(g)$ is not uniquely determined as it depends on the way each cycle is written. For any set $\Gamma(g)$ of this kind, we say that $\Gamma(g)$ *consists of every second point of every cycle of* g. From the definition of $\Gamma(g)$ we see that

$$|\Gamma(g)^g - \Gamma(g)| = |\Gamma(g)| = \sum_{i=1}^{s} \lfloor l_i/2 \rfloor.$$

The next lemma shows that this quantity is an upper bound for $|\Gamma^g - \Gamma|$ for an arbitrary subset Γ of Ω.

Lemma 2.1. *(Lemma 2.1 of [3]) Let G be a permutation group on a set Ω and suppose that $\Gamma \subseteq \Omega$. Then for each $g \in G$, $|\Gamma^g - \Gamma| \leq \sum_{i=1}^{s} \lfloor \frac{l_i}{2} \rfloor$, where l_i is the length of the i^{th} cycle of g and s is the number of non-trivial cycles of g in its disjoint cycle representation. This upper bound is attained for $\Gamma = \Gamma(g)$ defined above.*

Remark 2.2. ([2]) Let g be an element of a permutation group G on a set Ω. Assume that the set Ω is the disjoint union of G-invariant sets Ω_1 and Ω_2. Then every subset Γ of Ω is a disjoint union of subsets $\Gamma_i = \Gamma \cap \Omega_i$ for $i = 1, 2$. Let g_i be the permutation on Ω_i induced by g for $i = 1, 2$. Since $|\Gamma^g - \Gamma| = |\Gamma_1^{g_1} - \Gamma_1| + |\Gamma_2^{g_2} - \Gamma_2|$, we have:

$$\text{move}_\Omega(g) = \sum_{i=1}^{2} \max\{|\Gamma_i^{g_i} \setminus \Gamma_i| | \Gamma_i \subseteq \Omega_i\} = \text{move}_{\Omega_1}(g_1) + \text{move}_{\Omega_2}(g_2).$$

Definition 2.3. A group G is called a CP-group if every non-identity element of G has prime power order.

The classification of finite CP-groups is given in Lemma 0.4 of [5].

3. Main results

In this section we present our main results. let G be a permutation group on a set Ω with bounded movement m, for some positive integer m. Also, suppose that every non-identity element has movement m or $m - 3$. In the following theorem, we give the cycle structure of elements of G.

THEOREM 3.1. *Let m be a positive integer and G be a permutation group on a set Ω with bounded movement m, in which every non-identity element has movement m or $m-2$. Further, suppose that $1 \neq g \in G$ and $g = c_1 ... c_s$ is the decomposition of g into its disjoint non-trivial cycles such that $|c_i| = l_i$ for $1 \leq i \leq s$. Then one of the following holds:*
1) *$l := l_1 = l_2 = ... = l_s$, where l is an odd prime or a power of 2;*
2) *$s = 1$, such that g is a cycle of length 49;*
3) *$s = 3$, such that g has three cycles of length 9;*
4) *$s = 3$, such that g has three cycles of length $2p$, for some odd prime p;*
5) *$s = 3$, such that g has two cycles of length 2 and one cycle of length $2p$, for some odd prime p;*
6) *$s = 3$, such that g has one cycle of length 2 and two cycles of length $2p$, for some odd prime p;*
7) *$s = 3$, such that g has two cycles of length 3 and one cycle of length 21;*
8) *$s = 4$, such that g has three cycles of length 2 and one cycle of length 7;*
9) *$s = 4$, such that g has three cycles of length 3 and one cycle of length 7;*
10) *$s = 4$, such that g has three cycles of length 14 and one cycle of length 7;*
11) *$s = 6$, such that g has three cycles of length 2 and three cycle of length 3;*
12) *$s = 6$, such that g has three cycles of length 3 and three cycle of length 6;*
13) *g has three cycles of length 2 and $(s-3)$-cycles of length a power of 2 for $s \geq 4$;*
Moreover, the order of g is either 6, 9, 14, 21, 49, p, $2p$, or a power of 2.

Now, we give an example of a transitive permutation group with movement m or $m-3$.

EXAMPLE 3.2. The cyclic group $\mathbb{Z}_{49}$ in its transitive action on 49 points is a group which satisfies in the above theorem. In fact, if $G = \mathbb{Z}_{49}$, then every element of G has movement 24 or 21.

If the cases (1), (2), (3) or (13) happen, then G is a CP-group. By Lemma 0.4 of [5], one can see that there is no transitive simple group with movement m or $m-3$. The other cases are still remain to verify.

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Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Irreducible filters of eRM-algebras

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Abstract. In this paper, we introduce and study the concept of irreducible filters of eRM-algebras and investigate some of them properties. In particular, we show that the set of all filters of a eRM-algebra $\mathcal{X}$ is a chain if and only if every proper filter of $\mathcal{X}$ is prime.

Keywords: RM-algebra, eRM-algebra, (irreducible, prime, maximal) filter.

AMS Mathematics Subject Classification [2010]: 06D20, 06F35, 03G25

1. Introduction

Researchers proposed several kinds of algebraic structures related to some axioms in many-valued logic for investigation in many-valued logics. Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras (see [1], [2]). All of the algebras mentioned above are contained in the class of RM-algebras (a RM-algebra is an algebra $(X; \to, 1)$ of type (2,0) satisfying the axioms: (R) $x \to x = 1$ and (M) $1 \to x = x$) ([3]). A. Walendziak introduced the notion of strongly p-semisimple RM-algebras and proved that these algebras are equivalent with involutive moons ([6]). Recently, A. Rezaei and A. Borumand Saeid defined a new extension of RM-algebras, wRM/eRM-algebras, by considering the non-empty subset instead of one constant and got the algebraic structure with a set of constants. Also, they defined the concept of a positive implicative eRM-algebra and study its properties ([5]). In this paper, the concept of irreducible and prime filters are introduced and investigated some of them properties.

2. Preliminaries

We recall the basic notions and results regarding wRM/eRM-algebras used in the paper.

Definition 2.1. ([5]) Let X be a non-empty set. By a *weak* RM-*algebra* or wRM-*algebra*, for short, we shall mean an algebra $(X; \to, A)$ such that $\to$ is a binary operation on X and A is a non-empty subset of X satisfies the following axioms:

(wRM$_1$) $x \to x \in A$,

*Speaker. Email address: rezaei@pnu.ac.ir (wRM$_2$) $x \in A \to x$, for all $x \in X$.

A wRM-algebra $(X; \to, A)$ is called an *extended* RM-*algebra* or eRM-*algebra*, for short, if (wRM$_2$) replaced with:

(eRM) $A \to x = \{x\}$, for all $x \in X$,

where, $A \to x = \{a \to x : a \in A\}$.

It is obvious that every eRM-algebra is a wRM-algebra. A wRM-algebra which is not an eRM-algebra will be called *proper*.

From now on, $\mathcal{X}$ denotes the eRM-algebra $(X; \to, A)$, unless otherwise stated.

Definition 2.2. ([5]) *A subset F of X is called a* filter *of $\mathcal{X}$ if it satisfies:*

(F1) $A \subseteq F$;

(F2) $x \in F$ *and* $x \to y \in F$ *imply* $y \in F$.

We denote by $\mathsf{Fil}(\mathcal{X})$ the set of all filters of $\mathcal{X}$. Since X, $A \in \mathsf{Fil}(\mathcal{X})$, we get $\mathsf{Fil}(\mathcal{X}) \neq \emptyset$. We say that a filter F of $\mathcal{X}$ is *proper* if $F \neq X$. For any $x, y \in X$ and every $F \in \mathsf{Fil}(\mathcal{X})$, if $x \leqslant y$ and $x \in F$, then $y \in F$.

Let $\mathcal{X}$ be an eRM-algebra and $Y \subseteq X$. The set $\mathfrak{F}(Y) := \bigcap\{F \in \mathsf{Fil}(\mathcal{X}) : Y \subseteq F\}$ is a filter of $\mathcal{X}$, called the *filter generated by* Y. For $x \in X$, we write $\mathfrak{F}(x)$ instead of $\mathfrak{F}(\{x\})$. Define $\wedge$ and $\vee$ on $\mathsf{Fil}(\mathcal{X})$ by $E \wedge F = E \cap F$ and $E \vee F = \mathfrak{F}(E \cup F)$. It is easy to see that under these operations $\mathsf{Fil}(\mathcal{X})$ is a lattice. Moreover, this lattice is complete, since it is closed under arbitrary intersections.

Let $(L; \wedge, \vee)$ be a lattice. An element $a \in L$ is said to be (see [4]) :

- *meet irreducible* if $a = b \wedge c$ always implies $a = b$ or $a = c$,
- *meet prime* if $b \wedge c \leq a$ always implies $b \leq a$ or $c \leq a$.

If L has a greatest element 1, then the lower covers of 1 are called *coatoms* of L.

Lemma 2.3. ([4], Lemma 2.54) *In any distributive lattice, an element is meet irreducible if and only if it is meet prime.*

Denote by $\mathsf{NF}(\mathcal{X})$ the set of all normal filters of $\mathcal{X}$.

3. Some basic properties of irreducible filters of eRM-algebras

In this section, we define irreducible, prime and maximal filters of eRM-algebras.

Definition 3.1. *A proper filter F of an eRM-algebra $\mathcal{X}$ is called:*

- *irreducible* if $E_1 \cap E_2 = F$ implies $F = E_1$ or $F = E_2$ for any $E_1, E_2 \in \mathsf{Fil}(\mathcal{X})$,
- *prime* if $E_1 \cap E_2 \subseteq F$ implies $E_1 \subseteq F$ or $E_2 \subseteq F$ for any $E_1, E_2 \in \mathsf{Fil}(\mathcal{X})$,
- *maximal* if $F \subseteq E \subseteq X$ implies $E = F$ or $E = X$ for every $E \in \mathsf{Fil}(\mathcal{X})$.

Proposition 3.2. *A filter F of an eRM-algebra $\mathcal{X}$ is irreducible (prime or maximal) if and only if it is a meet irreducible element (meet prime element or coatom, respectively) of the lattice* $\mathsf{Fil}(\mathcal{X})$.

Example 3.3. *Let $X = \{a, b, c, d, e, f\}$ and $A = \{a, b\}$. We define the binary operation $\to_1$ on X by the following table: Then $(X; \to_1, A)$ is an eRM-algebra. It is easy to check that $\mathsf{Fil}(\mathcal{X}) = \{A, F_1, F_2, F_3, X\}$, where $F_1 = \{a, b, d\}$, $F_2 = \{a, b, e\}$, $F_3 = \{a, b, d, f\}$. The lattice $\mathsf{Fil}(\mathcal{X})$ is diagramed in Figure 1. It is not modular (hence also not distributive), since it contains the pentagon lattice.*

TABLE 1. eRM-algebra $(X; \to_1, A)$

$\to_1$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	a	b	c	d	e	f
c	a	b	b	b	b	b
d	a	b	e	b	e	f
e	a	b	c	c	b	f
f	a	b	c	f	e	b

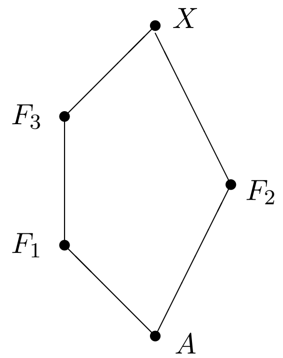

Figure 1

It is easy to check that filters F_1, F_2 and F_3 are irreducible; F_2 and F_3 are also maximal and prime.

REMARK 3.4. *Since every meet prime element is meet irreducible, we conclude that every prime filter is irreducible. The converse does not hold in general. In Example 3.3, the filter F_1 is irreducible but not prime.*

REMARK 3.5. *It is known that if* L *is a lattice with element 1 and x is a coatom of* L*, then x is a meet irreducible element. Therefore, every maximal filter is irreducible.*

PROPOSITION 3.6. *Let $\mathcal{X}$ be an eRM-algebra such that $\mathsf{Fil}(\mathcal{X})$ is a distributive lattice. Then a filter of $\mathcal{X}$ is irreducible if and only if it is prime.*

THEOREM 3.7. *Let $\mathcal{X}$ be an eRM-algebra, let F be a filter of $\mathcal{X}$, and let $a \in X - F$. Then there is an irreducible filter E such that $F \subseteq E$ and $a \notin E$.*

PROOF. Set $\mathfrak{G} = \{G \in \mathsf{Fil}(\mathcal{X}) : F \subseteq G$ and $a \notin G\}$. Then $F \in \mathfrak{G}$, that is $\mathfrak{G} \neq \emptyset$. Applying Kuratowski-Zorn's Lemma, we see that there exists a maximal member of $\mathfrak{G}$. Denote it by E. We shall prove that E is irreducible. On the contrary suppose that there are two filters E_1 and E_2 such that $E_1 \cap E_2 = E$ and $E \subset E_i$ for $i = 1, 2$. Since E is a maximal member of $\mathfrak{G}$, we conclude that $a \in E_1$ and $a \in E_2$. Therefore, $a \in E_1 \cap E_2 = E$, a contradiction. Consequently, E is irreducible. □

COROLLARY 3.8. *Every filter of an eRM-algebra is the intersection of all irreducible filters containing it.*

THEOREM 3.9. *Let F be a proper filter of an eRM-algebra $\mathcal{X}$. Then the following conditions are equivalent:*

(i) *F is prime,*
(ii) *$\mathfrak{F}(x) \cap \mathfrak{F}(y) \subseteq F$ implies $x \in F$ or $y \in F$, for any $x, y \in X$.*

Theorem 3.10. *Let $\mathcal{X}$ be an eRM-algebra. Then* $\mathsf{Fil}(\mathcal{X})$ *is a chain if and only if every proper filter of $\mathcal{X}$ is prime.*

Proof. Let $\mathsf{Fil}(\mathcal{X})$ be a chain and F be a proper filter of $\mathcal{X}$. Let $x, y \in X$ and suppose that $\mathfrak{F}(x) \cap \mathfrak{F}(y) \subseteq F$. Since $\mathsf{Fil}(\mathcal{X})$ is a chain, it follows that $\mathfrak{F}(x) \subseteq F$ or $\mathfrak{F}(y) \subseteq F$. Consequently, $x \in F$ or $y \in F$. By Theorem 3.9, F is prime.

Conversely, let any proper filter of $\mathcal{X}$ be prime. Let F and G be two proper filters of $\mathcal{X}$. Then $F \cap G$ is proper and, by assumption, prime. Thus $F \subseteq F \cap G$ or $G \subseteq F \cap G$. Hence $F \subseteq G$ or $G \subseteq F$. Therefore, $\mathsf{Fil}(\mathcal{X})$ is a chain. $\square$

Example 3.11. *Let $X = \{a, b, c, d, e\}$ and $A = \{a, b\}$. We define the binary operation $\rightarrow_2$ on X by the following table: Then $(X; \rightarrow_2, A)$ is an eRM-algebra. We check at once*

Table 2. eRM-algebra $(X; \rightarrow_2, A)$

$\rightarrow_2$	a	b	c	d	e
a	a	b	c	d	e
b	a	b	c	d	e
c	a	b	b	d	e
d	a	b	b	b	b
e	a	b	c	e	b

that $\mathsf{Fil}(\mathcal{X}) = \{F_1, F_2, X\}$, where $F_1 = A$, $F_2 = \{a, b, c\}$ and $F_3 = X$ *is a chain. Hence, every proper filter of $\mathcal{X}$ is prime.*

Definition 3.12. *A filter F of $\mathcal{X}$ is said to be normal if it satisfies the following condition: for all $x, y, z \in X$,*
(NF) $\quad x \rightarrow y \in F \Longrightarrow [(z \rightarrow x) \rightarrow (z \rightarrow y) \in F$ *and* $(y \rightarrow z) \rightarrow (x \rightarrow z) \in F]$.

Theorem 3.13. *Let $\mathcal{X}$ be an eRM-algebra and let $F, G \in \mathsf{NF}(\mathcal{X})$ such that $F \subseteq G$. We have:*

(i) *if G is an irreducible filter of $\mathcal{X}$, then G/F is an irreducible filter $\mathcal{X}/F$,*
(ii) *if G is a prime filter of $\mathcal{X}$, then G/F is a prime filter of $\mathcal{X}/F$.*

4. Conclusion

In this paper, we studt the concept of irreducible filters of eRM-algebras and investigate some of them properties. Also, we show that the set of all filters of a eRM-algebra $\mathcal{X}$ is a chain if and only if every proper filter of $\mathcal{X}$ is prime.

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A characterization of p-normed spaces based on some quasi norm inequalities

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Abstract. In this talk, we state a characterization of p-normed spaces which is based on the generalized triangle inequality of the second type and its reverse in quasi normed spaces. More exactly, for a quasi normed space $(X, \|.\|)$ and $0 < p \leq 1$ we obtain some regions of $\mathbb{R}^n$ which contain the set of all n-tuples $(\mu_1, \ldots, \mu_n)$ satisfying $\sum_{i=1}^n \frac{\|x_i\|^p}{\mu_i} \leq \|\sum_{i=1}^n x_i\|^p$, for all $x_1, \ldots, x_n \in X$.

Keywords: p-normed space, quasi normed space, generalized triangle inequality of the second type.

AMS Mathematics Subject Classification [2010]: 46A16, 47A30, 46B20

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1. Introduction

The triangle inequality is considered to be one of the most fundamental inequalities in mathematics. There are many interesting generalizations, refinements and reverses of the triangle inequality in normed spaces, quasi normed spaces, inner product spaces, pre-Hilbert C^*-modules. The generalized triangle inequality are useful for studying the geometrical structure of normed spaces. In this direction some results have been based on the triangle inequality of the second type

$$\|x + y\|^2 \leq 2(\|x\|^2 + \|y\|^2)$$

in normed space; see [1, 2, 6]. In [1, 2], the authors characterized some regions of $\mathbb{R}^n$ for which the generalized triangle inequality of the second type

$$\|x_1 + \cdots + x_n\|^p \leq \frac{\|x_1\|^p}{\mu_1} + \cdots + \frac{\|x_n\|^p}{\mu_n} \tag{1}$$

and its reverse holds in normed spaces. Very recently, in [5], the authors investigated inequality (1) in the framework of quasi normed spaces.

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In this talk, we state a characterization of p-normed spaces which is based on the generalized triangle inequality of the second type and its reverse in quasi normed spaces. More exactly, for a quasi normed space $(X, \|.\|)$ and $0 < p \leq 1$ we obtain some regions of $\mathbb{R}^n$ which contain the set of all n-tuples $(\mu_1, \dots, \mu_n)$ satisfying $\sum_{i=1}^{n} \frac{\|x_i\|^p}{\mu_i} \leq \| \sum_{i=1}^{n} x_i\|^p$, for all $x_1, \dots, x_n \in X$.

The remainder of this section contains some theoretical basis and symbols of related notions. The concept of quasi norm is a generalization of a norm in which the triangle inequality is replaced by a weaker inequality under the name quasi triangle inequality. The formal definition is as follows.
A quasi norm on a vector space X is a function $\|\cdot\| : X \to [0, \infty)$ with the following properties:

(i) If $\|x\| = 0$, then $x = 0$,
(ii) $\|\lambda x\| = |\lambda| \|x\|$, for all $\lambda \in \mathbb{R}$ and all $x \in X$,
(iii) $\|x + y\| \leq C(\|x\| + \|y\|)$, for all $x, y \in X$,
where $C \geq 1$ is a constant independent of $x, y \in X$.

The smallest possible C in (iii) is called the modulus of concavity of $\|\cdot\|$ and the pair $(X, \|\cdot\|)$ is called a quasi normed space.
A quasi norm $\|\cdot\|$ is called a p-norm $(0 < p \leq 1)$ if it satisfies the inequality $\|x + y\|^p \leq \|x\|^p + \|y\|^p$, for all $x, y \in X$. In this case, a quasi normed space is called a p-normed space. The case where $p = 1$ corresponds to the well known class of normed spaces. If is noted that every p_1-norm is also a p_2-norm for all $p_2 \leq p_1$.
A quasi norm gives rise to some equivalent metric topologies on the underlying space one of which is given by the following Aoki-Rolewicz theorem (see [3]).

Theorem 1.1. *[3] Let $(X, \|\cdot\|)$ is a quasi normed space with the modulus of concavity C, then there is a number $p \in (0, 1]$ such that the functional*

$$|||x||| := \inf \left\{ \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}} : n > 0,\, x_1, \dots, x_n \in X,\, x = \sum_{i=1}^{n} x_i \right\},$$

defines a p-norm equivalent to the quasi norm $\|.\|$. Moreover $|||x||| \leq \|x\| \leq 2C|||x|||$ and $2^{\frac{1}{p}-1} \leq C$.

So every quasi norm is equivalent to some p-norms $(0 < p \leq 1)$ and $d(x, y) := |||x - y|||^p$ defines a metric topology on X. A complete quasi normed space (p-normed space) is called a quasi Banach space (p-Banach space). We refer the reader to [3, 4] for more information on quasi Banach spaces.
Suppose that $(X, \|\cdot\|)$ be a quasi normed space with the modulus of concavity C and let $0 < p \leq 1$ be given. We put

$$F_{\|\cdot\|}(p) = \{(\mu_1, \dots, \mu_n) : \mu_i > 0 \quad \textit{and} \quad \| \sum_{i=1}^{n} x_i \|^p \leq \sum_{i=1}^{n} \frac{\|x_i\|^p}{\mu_i} \quad \text{for all}\, x_1, \dots, x_n \in X\}$$

and

$$G_{\|\cdot\|}(p) = \{(\mu_1, \dots, \mu_n) : \exists j = 1, \dots, n; \mu_j > 0,\, \mu_i < 0\, (i \neq j) \text{ and}$$
$$\sum_{i=1}^{n} \frac{\|x_i\|^p}{\mu_i} \leq \| \sum_{i=1}^{n} x_i \|^p \quad \text{for all}\, x_1, \dots, x_n \in X\}.$$

2. Main results

Dealing with p-norms and related p-norm inequalities are simpler than quasi norms. In the case where $0 < p \leq 1$, we can take advantage of this feature by Theorem 1.1. So we regard the Aoki-Rolewicz theorem which offers a remarkable surrogate, an equivalent quasi norm $|||\cdot|||$ satisfying $|||x+y|||^p \leq |||x|||^p + |||y|||^p$, for some $0 < p \leq 1$. By doing this, in Theorem 2.2, we provide some sufficient conditions for n-tuples $(\mu_1, \dots, \mu_n)$ that belong to $G_{\|\cdot\|}(p)$ and some necessary conditions are also presented.
Let us first introduce the following lemma which is given in [1] and it generalizes some results due to Takagi et al.

LEMMA 2.1. *[1] Let $0 < r \leq 1$, $\Omega \subseteq \{(s_1, \dots, s_n) : s_1, \dots, s_n \geq 0, \sum_{i=1}^n s_i \geq 1\}$ and let $D_r(\Omega) := \{(a_1, \dots, a_n) : a_1 s_1^r + \cdots + a_n s_n^r \geq 1 \quad \text{for all } (s_1, \dots, s_n) \in \Omega\}$. Then the following hold:*

(i): $\{(a_1, \dots, a_n) : a_1 \geq 1, \dots, a_n \geq 1\} \subseteq D_r(\Omega)$;

(ii): *If $\{(e_1, \dots, e_n)\} \subseteq \overline{\Omega}$ where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$, then*

$$D_r(\Omega) = \{(a_1, \dots, a_n) : a_1 \geq 1, \dots, a_n \geq 1\}.$$

THEOREM 2.2. *Let $\|.\|$ and $|||.|||$ be a quasi norm and a p-norm $(0 < p \leq 1)$, respectively, on a nonzero vector space X which are equivalent. Also let $\hat{\alpha} := \inf\{\alpha > 0 : \|x\| \leq \alpha|||x||| \text{ for all } x \in X\}$, $\hat{\beta} := \inf\{\beta > 0 : |||x||| \leq \beta\|x\| \text{ for all } x \in X\}$. Then the following assertions are valid:*

(i) *If an n-tuple $(\mu_1, \dots, \mu_n)$ satisfies $\mu_j \geq \max\{\{\hat{\alpha}^p \hat{\beta}^p |\mu_i| : i = 1, \dots, n, \ i \neq j\} \cup \{\hat{\alpha}^p \hat{\beta}^p\}\}$ for some $1 \leq j \leq n$ and $\mu_i < 0$ for all $i \neq j$, then $(\mu_1, \dots, \mu_n) \in G_{\|.\|}(p)$.*

(ii) *If $(\mu_1, \dots, \mu_n) \in G_{\|.\|}(p)$, then $\mu_j \geq \max\{\{\frac{|\mu_i|}{\hat{\alpha}^p \hat{\beta}^p} : i = 1, \dots, n, \ i \neq j\} \cup \{\frac{1}{\hat{\alpha}^p \hat{\beta}^p}\}\}$ for some $1 \leq j \leq n$.*

In Corollary 2.3, $F(p)$ is completely characterized in the setting of p-normed spaces.

COROLLARY 2.3. *[5] Let $(X, \|.\|)$ be a p-normed space $(0 < p \leq 1)$. Then*

$$F(p) = (0, 1] \times \cdots \times (0, 1].$$

Now we are ready to state the following theorem as a consequence of Theorem 2.2 and Corollary 2.3. It is a characterization of p-normed spaces which is based on the generalized triangle inequality of the second type and its reverse in quasi Banach spaces. Our results generalize some already known results due to [1, Theorem 2.5 and Theorem 2.7].

THEOREM 2.4. *Let $(X, \|.\|)$ be a quasi Banach space and $0 < p \leq 1$. Then the following assertions are mutually equivalent:*

(i) *$G(p) = \{(\mu_1, \dots, \mu_n) : \mu_j \geq \max\{\{|\mu_i| : i = 1, \dots, n, \ i \neq j\} \cup \{1\}\} \quad \text{for some } 1 \leq j \leq n \text{ and } \mu_i < 0 \quad \text{for all } i \neq j\}$;*

(ii) *$F(p) = (0, 1] \times \cdots \times (0, 1]$;*

(iii) *$(X, \|.\|)$ is a p-Banach space.*

3. Conclusion

In the framework of quasi normed spaces, by using the generalized triangle inequality of the second type and its reverse, we present a characterization of p-normed spaces. The results provide a better understanding of the behaviors of some inequalities with the source of the triangle inequality in some spaces such as $\mathbb{R}^n, l_p, \ldots$.

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30 August - 02 September 2021

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Poisson Burning of Graphs

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Abstract. Graph burning is a deterministic discrete-time graph process that is defined on the vertex set of a simple finite graph and can be considered as a model for the spread of social contagion. Its corresponding graph parameter is called the burning number and can be interpreted as a measure for the speed of contagion. In this paper, we define the **Poisson burning** process on the vertex set of a finite simple graph as a random variation of the graph burning process. We also define its corresponding graph parameter, called **burning time**. We then obtain a general asymptotic upper bound on the burning time of connected graphs, and we find the asymptotic order of the burning time for the paths.
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Keywords: Poisson burning, burning time
AMS Mathematics Subject Classification [2010]: 05C57, 05C85

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1. Introduction

In this section we define the Poisson burning process and the burning time of graphs. We start by providing some background and terminology. *Graph burning* is a discrete time graph process that is defined on the vertex set of simple finite graphs and was introduced by Anthony Bonato et al in 2014 in [2]. By starting such a process on a graph G, we say that we are *burning* G and it can be interpreted as a model for spread of influence (a fire) in social networks. The *burning number* of G, denoted by $b(G)$, is then the minimum number of rounds that is needed for burning G over the set of all possible burning processes (that can be seen as a measure for the speed of social contagion). Here is the definition of this process:

We have a simple finite graph $G = (V, E)$ and a fire that we aim to spread among the vertices of G. We assume that throughout this process every vertex has two states: it is either unburned or burning. Initially all the vertices of G are unburned, and once a vertex is burned, it remains burning until the end of the process. At each time step $i \geq 1$, the process progresses as follows:

At time step $i = 1$, we choose a vertex x_1 and we burn it. At each time step $i \geq 2$, we do two things: We choose a vertex x_i that is not burned in the previous steps and we burn it; at the same time, the fire spreads from the burning vertices of the previous stage to their unburned neighbours. The process ends once all the vertices are burned.

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Each vertex x_i in the above definition is called a *fire source*, and if the process ends in k steps, then the sequence $(x_1, x_2, \dots, x_k)$ is called a *burning sequence* for G. Hence, the burning number of G equals the length of the smallest burning sequence for G. For more information on the graph burning problem see [1–4].

As an example, the following result is known on the burning number of paths.

THEOREM 1.1 ([3]). *For every path P_n of order $n \geq 1$, $b(P_n) = \lceil \sqrt{n} \rceil$.*

By the results in [1–4], we have the following corollary.

COROLLARY 1.2. *For every connected graph G of order n, $b(G) = O(\sqrt{n})$.*

A *Homogeneous Poisson process* with parameter λ is a counting (or point) stochastic process that counts the number of events that occur in the given intervals of time. We denote a Poisson process by $\{N(t)\}_{t\geq 0}$, where the parameter t represents the time, and $N(t)$ denotes the number of events that happen in the time interval $[0, t]$. The sequence $\{N(t)\}_{t\geq 0}$ satisfies the following properties:
(i) $N(0) = 0$.
(ii) The number of events in non-overlapping time intervals are independent.
(iii) For a positive real number h, we have that

$$\mathbb{P}\left(N(t+h) - N(t) = 1\right) = \lambda h + o(h),$$

and

$$\mathbb{P}\left(N(t+h) - N(t) \geq 2\right) = o(h).$$

The properties (i), (ii), and (iii) imply that

$$\mathbb{P}\left(N(t+h) - N(t) = k\right) = \mathbb{P}\left(N(t) - N(0) = k\right) = \mathbb{P}\left(N(t) = k\right).$$

Let T_1 denote the time of the occurrence of the first event in a Poisson process with parameter λ, and for $i \geq 2$, suppose that T_i denotes the time that we wait after the occurrence of the $(i-1)$-th event until the i-th event happens. Since the occurrence of each event happens randomly, each T_i is a random variable that is called the *i-th inter-arrival time*. The time that the i-th event occurs is called the *i-th waiting time* and is denoted by S_i. Thus, by definition, $S_1 = T_1$, and for $j \geq 2$, $S_j = \sum_{i=1}^{j} T_j$. Moreover, for $j \geq 1$, $T_j = S_j - S_{j-1}$.

It is known in probability theory that for every natural number n, the random variables $T_1, T_2, \dots, T_n$ are independent and identically distributed exponential random variables with rate λ; that is, the probability density function of each T_i is defined by the function $f(x) = \lambda \exp(-\lambda x)$, where $x \geq 0$. Hence, the expectation or mean of each T_i equals $\frac{1}{\lambda}$, and its variance equals $\frac{1}{\lambda^2}$. Moreover, each S_n is a gamma random variable $\Gamma(n, \lambda)$; that is, the probability density function of S_n is defined by $f(x) = \frac{\lambda^n x^{n-1} \exp(-\lambda x)}{\Gamma(\lambda)}$, for $x \geq 0$. In this formula, $\Gamma(\lambda)$ is the well-known *gamma function*.

Also, $N(t)$ is a Poisson random variable with rate λt; that is, the probability distribution function of S_n is defined by $\mathbb{P}(N(t) = i) = \frac{\exp(-\lambda t)(\lambda t)^i}{i!}$, where $i \geq 0$ is an integer.

Hence, each Poisson process is uniquely determined by one of the sequences $\{N(t)\}_{t\geq 0}$, or $\{T_i\}_{i=1}$, or $\{S_i\}_{i=1}$. We can also see that the event $N(t) < i$ happens if and only if the event $S_i > t$. Thus,

$$\mathbb{P}(N(t) < i) = \mathbb{P}(S_i > t). \tag{1}$$

For more details on Poisson process see [5]. The following concentration inequality is called the *Chebyshev's inequality*; see [6].

Theorem 1.3 ([6]). *Suppose that X is a random variable with finite mean μ, and finite non-zero variance σ^2. Then for any $t > 0$,*

$$\mathbb{P}\left(|X-\mu| \geq t\right) \leq \frac{\sigma^2}{t^2}.$$

Now, we are ready to define the *Poisson graph burning process* as follows. Suppose that we consider the burning process for a graph G of order n, with $b(G) = k$. However, the time for choosing the i-th source of fire is the waiting time for the i-th event in a Poisson process with parameter λ, and we choose the i-th source of fire uniformly at random to be any vertex in G. We continue this until at some time t the whole graph G is burning. We may or may not consider to continue the Poisson process after time t as every vertex in G will be burning after t. Such a burning process on G is called a Poisson burning process for G. We call the time t at which the whole graph G is burning in a Poisson burning process the *burning time* of G, and we denote it by $b_{p,\lambda}(G)$, or simply by $b_p(G)$ when we know the parameter λ.

Note that in a regular random burning process for G, we choose the i-th source of fire at time $t = i$. However, in a Poisson burning process the time for choosing the i-th source of fire can be any time t either before i or after i. Moreover, here $b_p(G)$ is a random variable as it depends on the Poisson process, and also on the way that we choose the fire sources. Our goal is here to consider the asymptotic behavior of $b_p(G)$.

2. Main results

In this section, we consider the asymptotic behavior of the Poisson burning process and the burning time of connected graphs in general, and for the paths in particular.

Theorem 2.1. *If G is a connected graph of order n with $b(G) = k$, then in a Poisson burning process with $\lambda = 1$ on G,*

$$b_p(G) = O(k\sqrt{\log n}).$$

Proof. Assume that $(x_1, x_2, \ldots, x_k)$ is a fixed burning sequence for G (in the regular burning process), and $t = k + k\sqrt{\log n}$. In a Poisson burning process for G let A be the event that "all the vertices in G are burning at time t". Also, let B be the event "$\bigcap_{i=1}^{k} A_i$", where A_i is the event that "$S_i \leq t-(k-i)$ and the vertex x_i is burned in the i-th step" (or equivalently, $t - S_i \geq (k-i)$ and the vertex x_i is burned in the i-th step), for $1 \leq i \leq k$. We can see that B is a special case of the event A (or equivalently, $B \subseteq A$), and hence, $\mathbb{P}(B) \leq \mathbb{P}(A)$. Namely, if B occurs, then it implies that $S_k \leq t$, and for $1 \leq i \leq k$, the vertex x_i is burning at time $t-(k-i)$. Therefore, $\{N_{t-S_i}[x_i]\}_{i=1}^{k}$ forms a covering for the vertex set of G. Hence, G must be burning at time t in such a case.

Thus, if we show that $\lim_{n\to\infty} \mathbb{P}(B) = 1$, then it implies that $\lim_{n\to\infty} \mathbb{P}(A) = 1$. Consequently, it shows that a.a.s., $b_p(G) \leq t$. For this, it suffices to prove that $\lim_{n\to\infty} \mathbb{P}(\overline{B}) = 0$. In the rest of the argument, we will try to prove this by Chebyshev's inequality.

Assume that $\lambda = 1$. Therefore, $\mathbb{E}\left[T_i\right] = \mathrm{Var}(T_i) = 1$, for each $1 \leq i \leq k$. Note that $S_i = \sum_{j=1}^{i} T_j$. Since T_j's are independent exponentially distributed random variables, then for $1 \leq i \leq k$,

$$\mathrm{Var}(S_i) = \sum_{j=1}^{i} \mathrm{Var}(T_j) = i = \left(\sqrt{i}\right)^2.$$

Moreover, $\mathbb{E}[S_i] = \sum_{j=1}^{i} \mathbb{E}[T_j] = i$. Thus, by Chebyshev's inequality, for $1 \leq i \leq k$,

$$\begin{aligned}\mathbb{P}(\overline{A_i}) &= \mathbb{P}(S_i > t - (k - i)) = \mathbb{P}\left(S_i - i > k\sqrt{\log n}\right) \\ &\leq \mathbb{P}\left(|S_i - \mathbb{E}[S_i]| \geq k\sqrt{\log n}\right) \leq \frac{i}{k^2 \log n}.\end{aligned}$$

Since $B = \bigcap_{i=1}^{k} A_i$, then

$$\mathbb{P}(\overline{B}) = \mathbb{P}(\bigcup_{i=1}^{k} \overline{A_i}) \leq \sum_{i=1}^{k} \frac{i}{k^2 \log n} = \frac{k(k+1)}{2k^2 \log n}.$$

Therefore, $\lim_{n\to\infty} \mathbb{P}(\overline{B}) = \lim_n \frac{k(k+1)}{2k^2 \log n} = 0$, and consequently, $\mathbb{P}(A) = \mathbb{P}(B) = 1$, a.a.s.; that is, a.a.s., all the vertices of G are burning at time $t = k + k\sqrt{\log n}$. Since $k \leq k\sqrt{\log n}$, then the proof follows. □

Note that the whole probability argument above is only dependent on the Poisson process and a fixed optimum burning sequence for G. Since by Corollary 1.2, the burning number of every connected graph G of order n, is of order $O(\sqrt{n})$, then we conclude the following result on the Poisson burning for $b_p(G)$.

THEOREM 2.2. *If G is a connected graph of order n, then in a Poisson burning process with $\lambda = 1$ on G,*

$$b_p(G) = O(\sqrt{n \log n}).$$

We now consider the burning time of paths as the most simple-structured connected graphs.

THEOREM 2.3. *In a Poisson burning process with $\lambda = 1$ on the path P_n,*

$$b_p(P_n) = \Theta(\sqrt{n \log n})$$

PROOF. By Theorem 2.2, we have the upper bound. To prove the lower bound, we use a technical argument on the decomposing of a Poisson process on paths into smaller processes. For the shortness of the paper we avoid bringing the complete proof here. □

3. Conclusion

We introduced a random variation of the graph burning process and its corresponding parameter, using a stochastic Poisson process, and we obtained an upper bound on the asymptotic value of the burning time of graphs, and also an asymptotically tight bound on the burning time of paths. There are left tones of open problems on the burning time of graphs such as the asymptotic bounds on the burning time of other known families of graphs.

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۸ تا ۱۱ شهریور ۱۴۰۰

Some results of effectiveness in metric model theory

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Abstract. This article presents an approach to verify the effective versions of theorems and concepts in metric model theory. Metric model theory (continuous logic) is a framework to study mathematical analysis and the corresponding spaces. So, a suitable way to study computability is Type-two-theory of effectivity(TTE). By TTE, effective version of some theorems in metric model theory can be obtained.

Keywords: Metric model theory, TTE, omitting types theorem, definability

AMS Mathematics Subject Classification [2010]: 03C99, 03D78

1. Introduction

2. Preliminaries

2.1. Metric model theory (Continuous logic). In the following, a logic which is suitable to study metric structures is explained, [1].

Assume (M, d) is a complete metric space. A metric structure $\mathcal{M}$ based on (M, d) denoted by

$$\mathcal{M} = (M, P_i^{\mathcal{M}}, f_j^{\mathcal{M}}, c_k^{\mathcal{M}} \mid \; i \in I, j \in J, k \in K)$$

is defined as follows; $P^{\mathcal{M}}$, $f^{\mathcal{M}}$ and $c^{\mathcal{M}}$ are the interpretations of the predicate symbol P, the function symbol f and the constant symbol c, respectively. $P^{\mathcal{M}} : M^n \to I$ and $f^{\mathcal{M}} : M^n \to M$ are uniformly continuous, for some arity n and a bounded interval I in $\mathbb{R}$. Moreover, $P^{\mathcal{M}}$ and $f^{\mathcal{M}}$ are uniformly continuous with modulus Δ_P and Δ_f, respectively. Also, L consists of a real number D_L which is the diameter of (M, d). Note that the metric d can be assumed as a binary predicate symbol and interpreted as the metric of M.

Terms are defined as in first-order logic. An atomic formula is of the form $P(t_1, \ldots, t_n)$, for terms t_i and a predicate symbol P. Also, $d(t_1, t_2)$ is an atomic formula for every two terms t_1 and t_2. Every atomic formula is a formula. Moreover, for every formula $\varphi_1, \ldots, \varphi_n$ and every continuous function $u : [0, 1]^n \to [0, 1]$, $u(\varphi_1, \ldots, \varphi_n)$ is a formula. And, for every formula φ and every variable x, $\sup_x \varphi$ and $\inf_x \varphi$ are formulas. Note that continuous functions u are connectives. The interpretation of each formula without free variables, a sentence, is as usual and defined by induction. A structure $\mathcal{M}$ is a model of a sentence φ

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if $\varphi^{\mathcal{M}} = 0$.

Definition 2.1. *Assume $\mathcal{M}$ is a metric structure and $A \subseteq M$.*

(1) *A predicate $P : M^n \to [0,1]$ is* definable *in $\mathcal{M}$ over A, if there is a sequence $(\varphi_k(x) \mid\ k \geq 1)$ of $L(A)$-formulas such that*

$$\forall\varepsilon > 0\ \exists N\ \forall k \geq N\ \forall x \in M^n\ (|\ \varphi_k^{\mathcal{M}}(x) - P(x) \ |\leq \varepsilon).$$

(2) *A function $f : M^n \to M$ is definable in $\mathcal{M}$ over A if and only if the function $d(f(\bar{x}), y)$ on M^{n+1} is a definable predicate in $\mathcal{M}$ over A.*

(3) *A set $D \subseteq M^n$ is definable in $\mathcal{M}$ over A if the distance predicate $d(\bar{x}, D)$ is definable in $\mathcal{M}$ over A.*

2.2. Type-two theory of the effectivity (TTE). In this section, the approach used to study the effectivity is introduced briefly, [6].

Definition 2.2. *Let Σ be a finite alphabet. A naming system on a set M is a surjective function $\nu :\subseteq X \to M$ where $X \in \{\Sigma^*, \Sigma^\omega\}$. If $X = \Sigma^*$, ν is called a notation and if $X = \Sigma^\omega$, ν is called a representation.*

The definitions of computable function on Σ^ω and every other arbitrary set can be found in [5].

2.3. Effective metric model theory. In the following, the concepts of computable and decidable metric structures are explained. This approach to study the effectiveness of the metric structures is firstly introduced in [4]. the definitions of en effective and computable metric space sre in [6]. The Cauchy representation for them can be defined. There exists a representation η for $F^{\omega\omega}$, the set of all partial continuous functions $f :\subseteq \Sigma^\omega \to \Sigma^\omega$ with G_δ-domain. It means $p \in \Sigma^\omega$ is a name for a continuous function $\eta_p :\subseteq \Sigma^\omega \to \Sigma^\omega$ with a G_δ-domain which on input q returns the value $\eta_p(q)$. For more details of this representation, see [3] and [4]. By the above representation, a continuous function $f \in F^{\omega\omega}$ is computable if there is a computable $p \in \Sigma^\omega$ such that $f = \eta_p$.

Definition 2.3. *[3] Let $\gamma_1 :\subseteq \Sigma^\omega \to M_1$ and $\gamma_2 :\subseteq \Sigma^\omega \to M_2$ be two representations. For the set $C(M_1, M_2)$ of continuous total functions $f : M_1 \to M_2$, define a representation $[\gamma_1 \to \gamma_2] :\subseteq \Sigma^\omega \to C(M_1, M_2)$ as follows:*

$$[\gamma_1 \to \gamma_2](p) = f :\Longleftrightarrow (f \circ \gamma_1)(q) = (\gamma_2 \circ \eta_p)(q),$$

for every $q \in \Sigma^\omega$ such that $(f \circ \gamma_1)(q)$ exists.

Definition 2.4. *[4] A countable signature L is* effectively presented *if*

(1) *The sets of variable, predicate, function and constant symbols are computable. It means if $c_V :\subseteq \Sigma^* \to Var$, $c_P :\subseteq \Sigma^* \to \mathcal{P}$, $c_F :\subseteq \Sigma^* \to \mathcal{F}$ and $c_C :\subseteq \Sigma^* \to \mathcal{C}$ are the naming systems for the sets of variables, predicate, function and constant symbols, respectively, then $dom(c_V)$, $dom(c_P)$, $dom(c_F)$ and $dom(c_C)$ are computable subsets of Σ^*.*

(2) *Moduli of uniform continuity of predicate and function symbols are (ρ_C, ρ_C)-computable functions.*

Similar to computability theory, a notation c for $Form$, the set of L-formulas exists such that $dom(c)$ is a c.e. set. So, let $\{\varphi_n \mid n \in \mathbb{N}\}$ be an effective list of the set of all L-formulas.

Now, let (M, d, A, α) be an effective metric space. Put the Cauchy representations δ_M on M and ρ_C on $[0,1]$. Let $\mathcal{M}$ be a metric L-structure based on (M, d, A, α). Assume

$$Form(\mathcal{M}, L) = \{\varphi^{\mathcal{M}} :\ M^{n_\varphi} \to [0,1] \mid\ \varphi \text{ is an } L\text{-formula with } n_\varphi \text{ free variables}\}.$$

To define a representation on $Form(\mathcal{M}, L)$, take the representation $\beta_n = [[\delta_M]^n \to \rho_C] :\subseteq \Sigma^\omega \to Form(\mathcal{M}, L)_n$, where

$$Form(\mathcal{M}, L)_n = \{\varphi^{\mathcal{M}} : M^{n_\varphi} \to [0,1] \mid\ \varphi \text{ is an } L\text{-formula with } n \text{ free variables}\},$$

for any $n \in \mathbb{N}$. Since $Form(\mathcal{M}, L) = \bigcup_{n\in\mathbb{N}} Form(\mathcal{M}, L)_n$ it follows that the function $\beta :\subseteq \Sigma^\omega \to Form(\mathcal{M}, L)$ defined by $\beta(0^n 1p) = \beta_n(p)$ for each $p \in dom(\beta)$, is a representation for $Form(\mathcal{M}, L)$. A similar representation β_{at} can be defined for the set of all interpretations of atomic L-formulas in $\mathcal{M}$, $Form_{at}(\mathcal{M}, L)$, instead of the set $Form(\mathcal{M}, L)$.

Therefore, a computable and a decidable metric structure can be defined.

Definition 2.5. (1) *With the preceding assumption, a metric structure $\mathcal{M}$ is computable iff the sequence*

$$(\varphi_n^{\mathcal{M}} : M^{n_\varphi} \to [0,1] \mid \varphi \textit{ is an atomic } L\textit{-formula with } n_\varphi \textit{ free variables})_{n\in\mathbb{N}}$$

has a computable $[\beta_{at}]^\omega$-name.

(2) *Respectively, a metric structure $\mathcal{M}$ is decidable iff the sequence*

$$(\varphi_n^{\mathcal{M}} : M^{n_\varphi} \to [0,1] \mid \varphi \textit{ is an } L\textit{-formula with } n_\varphi \textit{ free variables})_{n\in\mathbb{N}}$$

has a computable $[\beta]^\omega$-name.

Actually, $[\beta]^\omega$ is a naming system for $Form(\mathcal{M}, L)^\omega$ which is the set of all sequences on $Form(\mathcal{M}, L)$. Hence, for a decidable metric structure $\mathcal{M}$, there is an algorithm such that for a given L-formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$, it returns a good approximation of $\varphi^{\mathcal{M}}(a_1, \ldots, a_n)$ in rational numbers. This means that, for each $\varepsilon > 0$, $r, s \in \mathbb{Q}$ is computably found such that $r < \varphi^{\mathcal{M}}(a_1, \ldots, a_n) < s$ and $s - r < \varepsilon$.

3. Computably definable predicates

Proposition 3.1. *Let $(x_i)_{i\in\mathbb{N}}$ be a $(\nu_\mathbb{N}, \rho)$-computable sequence of real numbers with computable modulus of convergence $e : \mathbb{N} \to \mathbb{N}$. Then, its limit $x = \lim_{i\to\infty} x_i$ is computable.*

Definition 3.2. *An L-formula φ with n free variables is computable in M when $\varphi^{\mathcal{M}} : M^n \to [0,1]$ is a (δ_M, ρ)-computable function.*

Definition 3.3. *A predicate $P : M^n \to [0,1]$ (with n-arity) is computably definable in $\mathcal{M}$ (over $\emptyset$) iff there is a sequence $(\varphi_k(x) \mid k \geq 1)$ of computable L-formulas such that the sequence of predicates $(\varphi_k^{\mathcal{M}}(x) : M^n \to [0,1] \mid k \geq 1)$ is a $(\nu_\mathbb{N}, \rho)$-computable sequence with a computable modulus of convergence and $P(a) = \lim_{k\to\infty} \varphi_k^{\mathcal{M}}(a)$, for every $a \in M^n$.*

Obviously, if an n-arity predicate P is computably definable in $\mathcal{M}$ then by Proposition 3.1, $P(a)$ is computable for every $a \in M^n$.

Proposition 3.4. *Every (δ, ρ)-computable function $f :\subseteq M \to \mathbb{R}$ with co-r.e domain has a (δ, ρ)-computable total (δ, ρ)-computable extension $f : M \to \mathbb{R}$ with the same* sup *and* inf.

Assume

$$\mathcal{C} = \{(a_k)_{k\in\mathbb{N}} \in [0,1]^{\mathbb{N}} \mid \forall N \in \mathbb{N}\ \forall i,j > N \ \ | a_i - a_j | \leq 2^{-N}\}.$$

Also, let $([0,1]^{\mathbb{N}}, d)$ be a metric space such that the metric d is defined by

$$d((a_k),(b_k)) = \Sigma_{k=0}^{\infty} 2^{-k} \mid a_k - b_k \mid,$$

for every $(a_k),(b_k) \in [0,1]^{\mathbb{N}}$. Since $([0,1]^{\mathbb{N}}, d)$ is compact, it is separable. Therefore, let A be a countable and dense subset of $[0,1]^{\mathbb{N}}$ and α be a notation for A. So, $\mathcal{N} = ([0,1]^{\mathbb{N}}, d, A, \alpha)$ is an effective metric space.

Thus, the Cauchy representation δ can be defined for $[0,1]^{\mathbb{N}}$ as follows

$$\begin{aligned} \delta(p) = (a_k)_{k\in\mathbb{N}} :\longleftrightarrow\ & \exists p_0, p_1, \dots \in dom(\alpha), \\ & p := \iota(p_0)\iota(p_1)\dots, \\ & d(\alpha(p_i), \alpha(p_j)) \leq 2^{-j} (i < j), \\ & (a_k)_{k\in\mathbb{N}} = \lim_{n\to\infty} \alpha(p_n). \end{aligned} \tag{1}$$

Every sequence in $\mathcal{C}$ is Cauchy and so its limit exists in $[0,1]$. We can define a function $f : N \to [0,1]$ by $f((a_k)_{k\in\mathbb{N}}) = \lim_{k\to\infty} a_k$ and $dom(f) = \mathcal{C}$.

LEMMA 3.5. *The above function has a closed and co-r.e domain and is (δ,ρ)-computable.*

The next theorem says that in which situation a predicate is computably definable in metric structures.

THEOREM 3.6. *Let M be an effective metric space. Assume $P : M^k \to [0,1]$ is a predicate. Then, P is computably definable iff there are a (δ,ρ)-computable function $u : [0,1]^{\mathbb{N}} \to [0,1]$ and computable L-formulas $(\psi_l(x) \mid l \in \mathbb{N})$ such that for all $a \in M^k$, $P(a) = u(\psi_l^{\mathcal{M}}(a) \mid l \in \mathbb{N})$.*

COROLLARY 3.7. *An operator $T : M \to M$ on an effective metric space M is computably definable if and only if there are a (δ,ρ)-computable function $u : [0,1]^{\mathbb{N}} \to [0,1]$ and computable L-formulas $(\psi_k(x,y) \mid k \in \mathbb{N})$ such that for all $a,b \in M$, $d(T(a),b) = u(\psi_k^{\mathcal{M}}(a,b) \mid k \in \mathbb{N})$.*

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On Coregualr S-Acts

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Abstract. This paper shall be concerned with the notion of coregular S-acts (acts which all their cyclic subacts are injective) over a monoid S as a dual concept of regular acts. We present various properties and some homological classifications of coregular S-acts. Also the relations between these kinds of acts and some properties around injectivity are investigated.

Keywords: right S-act, coregular, smiretractable

AMS Mathematics Subject Classification [2010]: 20M30

1. Introduction

Throughout this paper S will denote a monoid and an S-act A is a right S-act. Following [1], an S-act A is called *regular* if for every element $a \in A$, there exists a homomorphism $f : aS \longrightarrow S$ such that $af(a) = a$. Also it is known that a right S-act A is regular if and only if for every $a \in A$, the cyclic subact aS is projective (see [1]). In this paper we define the notion of *coregular* acts as the dual notion of regular acts which are the acts all their cyclic subacts are injective. Herein we investigate the relations between coregular acts and some properties around injectivity. So let us recall them. We refer the reader to [1] for all concepts and basic properties of S-acts not defined here. A right S-act A is called *injective* if for any S-act B, any subact C of B and any homomorphism $f : C \longrightarrow A$, there exists a homomorphism $\bar{f} : B \longrightarrow A$ such that $\bar{f}\mid_C = f$ (see [1]). Also a right S-act A is called *quasi-injective (cyclic quasi-injective)* if it is injective relative to all inclusions from its subacts (cyclic subacts). For the sake of simplicity, we denote "quasi-injective" and "cyclic quasi-injective", by "Q-injective" and "CQ-injective", respectively. Recall from [5], the right S-act A is called *C-injective* if it is injective relative to every inclusion from cyclic acts. Moreover for right S-acts A and B a homomorphism $f : A \longrightarrow B$ is called a *semiretraction* if for every element $x \in f(A)$, there exists a homomorphism $g : B \longrightarrow A$ such that $f(g(x)) = x$. If B be a subact of an S-act A, then B is said to be a *semiretractable* subact of A if the inclusion map $i : B \hookrightarrow A$ is a semiretraction. Also a right S-act A is called *fully semiretractable* (or FSR) if every subact of A is semiretractable(see [3]).

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For an S-act A_S, by $E(A)$, we mean the injective envelope of A. We provide some useful information concerning C-injective and coregular acts. We prove that a right S-act is coregular if and only if it is C-injective and FSR. Also it is shown that over a commutative monoid S every torsion free S-act is coregular if and only if every cyclic S-act is injective.

2. Main Results

This section is devoted the study of coregular acts. This leads to study the relationship between these kinds of acts and the other classes of acts, such as FSR, C-injective and Q-injective.

DEFINITION 2.1. Let A be a right S-act. We say that A is *coregular* if every cyclic subact of A is injective.

Recall that over a monoid S a right S-act A is cyclic if for some $a \in A$,
$A = aS = \{as : s \in S\}$.

LEMMA 2.2. *Suppose $i : aS \hookrightarrow B$ is a monomorphism. If aS and B contain zero elements and $aS \times B$ is CQ-injective, then aS is a retract of B.*

PROOF. Suppose $i : aS \hookrightarrow B$ is a monomorphism. By CQ-injectivity of $aS \times B$, there exists a homomorphism $f : aS \times B \longrightarrow aS \times B$ such that $fi_2 i = i_1$ where i_1 and i_2 are the canonical injections. Take $g = fi_2$, then $gi = i_1$ and so $p_1 gi = p_1 i_1 = 1_{aS}$ where $p_1 : aS \times B \longrightarrow aS \times B$ is the canonical projection. Let $h = p_1 g$, we get $hi = 1_{aS}$ which means that aS is a retract of B. □

COROLLARY 2.3. *Suppose aS is a cyclic right S-act with a zero element. Then aS is injective if and only if $aS \times E(aS)$ is CQ-injective.*

From [3] recall that a homomorphism $f : A \longrightarrow B$ is called a *semiretraction* if for every element $x \in f(A)$, there exists a homomorphism $g : B \longrightarrow A$ such that $f(g(x)) = x$. Also if B be a subact of an S-act A, then B is said to be a *semiretractable* subact of A if the inclusion map $i : B \hookrightarrow A$ is a semiretraction, i.e., for every $x \in B$, there exists a homomorphism $f : A \longrightarrow B$ such that $f(x) = x$. Moreover, the right S-act A is called *fully semiretractable* (or *FSR*) if every subact of A is semiretractable.

THEOREM 2.4. *For a right S-act A, the following conditions are equivalent:*
(i) *A is coregular.*
(ii) *A is C-injective and FSR.*
(iii) *Every subact of A is C-injective.*
(iv) *Every finitely generated subact of A is C-injective.*
(v) *Every cyclic subact of A is C-injective.*
(vi) *For every element $a \in A$, aS contains a zero element and $aS \times E(aS)$ is CQ-injective.*

By Corollary 1 and Proposition 2 of [3] and also by the previous theorem, we have the following corollary.

COROLLARY 2.5. *The right S-act S_S is coregular if and only if S is a regular self injective monoid.*

The following example illustrates that the implications "C-injectivie$\longrightarrow$ FSR" and "FSR $\longrightarrow$ C-injective" are not true in general.

EXAMPLE 2.6. As we know, $S = (\mathbb{N}, \max)$ is a regular monoid which is not injective. Then by Corollary 1 of [3], S_S is FSR but it is not C-injective. Conversely, in [4] it is shown that a finite monogenic semigroup with identity and a zero adjoined is a self-injective monoid, but it can not be regular. Then by Corollary 1 of [3], S_S is not FSR.

PROPOSITION 2.7. *The following hold over a monoid S:*
(i) *Every subact (retract) of any coregular act is coregular.*
(ii) *For any family of right S-acts $\{A_i\}_{i\in I}$, $\coprod_{i\in I} A_i$ is coregular if and only if A_i is coregular for each $i \in I$.*
(iii) *For any family of right S-acts $\{A_i\}_{i\in I}$, $\prod_{i\in I} A_i$ is coregular if and only if for any $a_i \in A_i (i \in I), \prod_{i\in I} a_i S$ is coregular.*

Regarding parts (i) and (ii) of the previous proposition and Corollary 2.5, the next result can be proved.

PROPOSITION 2.8. *The following are equivalent over a monoid S:*
(i) *Every (finitely generated) projective S-act is coregular.*
(ii) *Every (finitely generated) free S-act is coregular.*
(iii) *The right S-act S_S is coregular .*
(iv) *S is a regular self injective monoid.*

The next proposition is easily checked using Theorem 2.4.

PROPOSITION 2.9. *The following are equivalent over a monoid S:*
(i) *Every S-act is coregular.*
(ii) *Every cyclic S-act is injective (C-injective).*
(iii) *Every cyclic S-act contains a zero element and all S-acts are CQ-injective.*

Note that in the above proposition (part (ii)), we can replace "injectivity" with every property which is weaker than injectivity.

It is well-known that over a commutative regular monoid S, every S-act is flat. Now using Corollary 2.5 and considering the fact that S_S is flat, we have the following result.

PROPOSITION 2.10. *Over a commutative monoid S, the following conditions are equivalent:*
(i) *Every torsion free S-act is coregular.*
(ii) *Every principally weakly flat S-act is coregular.*
(iii) *Every weakly flat S-act is coregular.*
(iv) *Every flat S-act is coregular.*
(v) *Every cyclic S-act is injective.*

In general "coregularity" does not imply "injectivity" for example if S is a monoid which is not left reversible, then clearly $\Theta \sqcup \Theta$ is coregular which is not injective. In the next propositions, we investigate conditions under which a coregular act is injective. Evidently if B is a subact of a cyclic S-act aS, then $I = \{s \in S : as \in B\} \subseteq S$ is a right ideal. Thus over a principal right ideal monoid S, any subact of every cyclic S-act is cyclic. Now the next result.

PROPOSITION 2.11. *Suppose S is a principal right ideal monoid. Then every C-injective (coregular) S-act is injective. .*

Recall from [1] that a monoid S is called *right hereditary*, if every right ideal of S is projective.

Proposition 2.12. *Over a hereditary monoid S, the following are equivalent.*
(i) *Every coregular S-act is injective.*
(ii) *Every coregular S-act is weakly injective.*
(iii) *Every coregular S-act is finitely generated weakly injective.*
(iv) *S is left reversible.*

Proof. (i)$\longrightarrow$(ii) and (ii)$\longrightarrow$(iii) are clear.
(iii)$\longrightarrow$(iv) Since $\Theta \sqcup \Theta$ is coregular, Proposition 3.4.3 of [1] implies the result.
(iv)$\longrightarrow$(i) By assumption, we conclude that every right ideal of S is indecomposable. Also since S is hereditary, every right ideal of S is principal and by the previous proposition, we obtain the result. □

Proposition 2.13. *Suppose A is a coregular S-act which contains a nonzero cyclic large subact. Then A is a cyclic injective S-act.*

3. Conclusion

The main body of the paper is related to the study of coregular acts as the dual notion of regular acts and a useful tool for the study of properties of injective acts and some related notions.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

The oriented coloring of generalized Theta graphs

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Abstract. We study oriented coloring of Theta graphs. An oriented coloring of an oriented graph $\overrightarrow{G}$ is a vertex coloring of $\overrightarrow{G}$ such that (i) no two adjacent vertices have the same color and (ii) all the arcs between any two color classes have the same direction. The oriented chromatic number of $\overrightarrow{G}$ is the smallest integer k such that $\overrightarrow{G}$ admits an oriented coloring with k colors. In this paper we prove that oriented chromatic number of any oriented generalized Theta graph lies between 2 and 6 and that these bounds are tight.

Keywords: Oriented coloring, oriented chromatic number and generalized Theta graph.

AMS Mathematics Subject Classification [2010]: 05C15, 05C20

1. Introduction

In this paper, let $G = (V(G), E(G))$ be a simple graph. An orientation of G is a directed graph $\overrightarrow{G} = (V(\overrightarrow{G}), A(\overrightarrow{G}))$ obtained from G by ordering every edge $uv \in E(G)$ either from u to v (resulting in an arc $\overrightarrow{uv} \in A(\overrightarrow{G})$), or conversely (resulting in an arc $\overrightarrow{vu} \in A(\overrightarrow{G})$). An orientation of a graph is called an oriented graph.

An oriented coloring of an oriented graph $\overrightarrow{G}$ is a mapping c from $V(\overrightarrow{G})$ to a set of k colors such that:

(i) $c(u) \neq c(v)$ whenever $\overrightarrow{uv} \in A(\overrightarrow{G})$ and
(ii) $c(v) \neq c(x)$ whenever $\overrightarrow{uv}, \overrightarrow{xy} \in A(\overrightarrow{G})$ and $c(u) = c(y)$.

Note that these two conditions ensure that any two vertices linked by an oriented path of length one ($\rightarrow$) or two ($\rightarrow\rightarrow$) are assigned distinct colors in any oriented coloring. The oriented chromatic number of $\overrightarrow{G}$, denoted by $\chi_o(\overrightarrow{G})$, is the smallest integer k such that $\overrightarrow{G}$ admits an oriented coloring with k colors. An oriented coloring of G using k colors is denoted by k-oriented coloring. If G is an undirected graph, the oriented chromatic number $\chi_o(G)$ of G is defined as the maximum oriented chromatic number of its orientations:

$$\chi_o(G) = \max\{\chi_o(\overrightarrow{G}) | \overrightarrow{G} \text{ is an orientation of } G\}.$$

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Let $\overrightarrow{G}$ and $\overrightarrow{H}$ be two oriented graphs. A homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$ is a mapping c from $V(\overrightarrow{G})$ to $V(\overrightarrow{H})$ that preserves the arcs (that is $\overrightarrow{c(u)c(v)} \in A(\overrightarrow{H})$ whenever $\overrightarrow{uv} \in A(\overrightarrow{G})$). A k-oriented coloring of an oriented graph $\overrightarrow{G}$ can thus be viewed as a homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$, where $\overrightarrow{H}$ is an oriented graph of order k. The existence of such a homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$ is denoted by $\overrightarrow{G} \to \overrightarrow{H}$. The vertices of $\overrightarrow{H}$ are called colors, and we say that $\overrightarrow{G}$ is a $\overrightarrow{H}$-colorable or has a $\overrightarrow{H}$-oriented coloring.

The oriented chromatic number of $\overrightarrow{G}$ can then be equivalently defined as the smallest order of an oriented graph $\overrightarrow{H}$ such that $\overrightarrow{G} \to \overrightarrow{H}$.

The notion of oriented coloring introduced by Courcelle in [1]. Oriented coloring has been studied by several authors. A survey on the study of oriented colorings has been done by Sopena in 2001 and recently updated [2].

Let $\overrightarrow{P_n}$ be an oriented path of length n and the vertices of $\overrightarrow{P_n}$ be $v_0, v_1, \ldots, v_n$, each vertex v_i, $0 \le i \le n$, is connected with the next vertex by an arc, either by $\overrightarrow{v_iv_{i+1}} \in A(\overrightarrow{P_n})$ (forward arc) or by $\overrightarrow{v_{i+1}v_i} \in A(\overrightarrow{P_n})$ (backward arc). For each pair of consecutive vertices v_i and v_{i+1}, we define $\lambda(v_i, v_{i+1}) = 1$ if $\overrightarrow{v_iv_{i+1}} \in A(\overrightarrow{P_n})$, and $\lambda(v_i, v_{i+1}) = -1$ if $\overrightarrow{v_{i+1}v_i} \in A(\overrightarrow{P_n})$. Now, we define $\lambda(\overrightarrow{P_n}) = \Sigma_{i=0}^{n-1}\lambda(v_i, v_{i+1})$. In other words $\lambda(\overrightarrow{P_n})$ is the number of forward arcs minus the number of backward arcs in $\overrightarrow{P_n}$. Similarly $\lambda(\overrightarrow{C_n})$ is defined.

The generalized Theta graph $\Theta_{l_1,\ldots,l_p}$ consists two end-vertices joined by $p \ge 2$ internally vertex-disjoint paths with respective lengths $1 \le l_1 \le \cdots \le l_p$. We denote by u and v the end-vertices of the generalized Theta graph $\Theta_{l_1,\ldots,l_p}$ and by $P^i = ux_1^i \ldots x_{l_i-1}^i v$ the corresponding paths of length l_i for every i, $1 \le i \le p$.

In this paper, we determine the oriented chromatic number of every generalized Theta graph.

2. Main results

We determine the oriented chromatic number of generalized Theta graphs $\Theta_{l_1,\ldots,l_p}, p \ge 2$. In the following, let tournament T_5 in Figure 1, be the oriented graph with the vertex set $V(T_5) = \{0, 1, \ldots, 4\}$ and the arc set $A(T_5) = \{(x, (x+k)(\text{mod}5)) | x \in V(T_5), k = 1, 2\}$. Let tournament T_6 in Figure 1, be the oriented graph with the vertex set $V(T_6) = \{0, 1, \ldots, 5\}$ and the arc set

$$\begin{aligned} A(T_6) = &\{(x, (x+1)(\text{mod}6)) | x \in V(T_6)\} \bigcup \\ &\{(x, (x+k)(\text{mod}6)) | x \in V(T_6), 2|x, k = 2, 3\} \bigcup \\ &\{(x, (x+4)(\text{mod}6)) | x \in V(T_6), 2 \nmid x\}. \end{aligned}$$

Theorem 2.1. *Let $\overrightarrow{\Theta} = \overrightarrow{\Theta_{l_1,\ldots,l_p}}$, $p \ge 2$, be a generalized Theta graph. Then $\chi_o(\overrightarrow{\Theta}) = 2$ if and only if all P^i's, $1 \le i \le p$, are oriented alternating paths ($\rightarrow\leftarrow$ or $\leftarrow\rightarrow$) from u to v and every vertex of $\overrightarrow{\Theta}$ is a source or sink.*

Theorem 2.2. *Let $\overrightarrow{\Theta} = \overrightarrow{\Theta_{l_1,\ldots,l_p}}$, $p \ge 2$, be a generalized Theta graph and $\overrightarrow{P^i}$, $1 \le i \le p$, be the oriented paths from u to v of $\overrightarrow{\Theta}$. Then $\chi_o(\overrightarrow{\Theta}) = 3$ if and only if one of the following conditions holds:*

(i) *for $p = 2$, $\lambda(\overrightarrow{P^1}) + \lambda(\overrightarrow{P^2}) = 0(\text{mod}3)$ or*

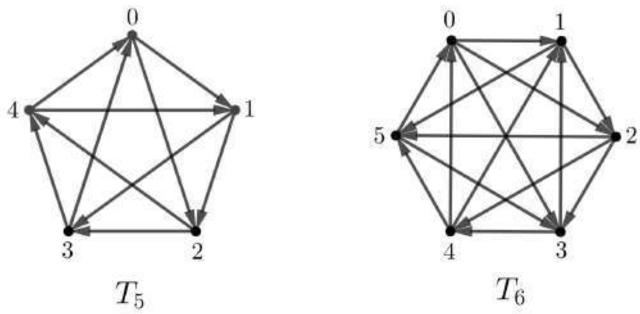

FIGURE 1. The tournaments T_5 and T_6.

(ii) *for every path* $\overrightarrow{P^i}$ *and* $\overrightarrow{P^j}$, $1 \leq i, j \leq p$, $\lambda(\overrightarrow{P^i}) = \lambda(\overrightarrow{P^j})(\mathrm{mod}3)$ *or*

(iii) $\overrightarrow{\Theta}$ *does not contain three consecutive arcs going in the same direction* ($\rightarrow\rightarrow\rightarrow$ or $\leftarrow\leftarrow\leftarrow$).

THEOREM 2.3. *For every generalized Theta graph* $\Theta = \Theta_{l_1,\ldots,l_p}, p \geq 2$ *and* $l_i \geq 3$, *whenever* $1 \leq i \leq p$, *we have* $2 \leq \chi_o(\Theta) \leq 5$. *Moreover, this upper bound is tight.*

To prove Theorem 2.3, we obtain a homomorphism from $\overrightarrow{\Theta}$ to T_5.

THEOREM 2.4. *For every generalized Theta graph* $\Theta = \Theta_{l_1,\ldots,l_p}, p \geq 2$, *we have* $2 \leq \chi_o(\Theta) \leq 6$. *Moreover, this upper bound is tight.*

To prove Theorem 2.4, we obtain a homomorphism from $\overrightarrow{\Theta}$ to T_6.

3. Conclusion

In this paper, we prove that oriented chromatic number of any oriented generalized Theta graph lies between 2 and 6 and that these bounds are tight.

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30 August - 02 September 2021

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approximating the efficient border for nondifferential multi-objective problems

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Abstract. In multi-objective convex optimization, we need to compute an infinite set of nondominated points. The proposed method for approximating an nondominated set of multi-objective nonlinear programming problem, is the extension of Benson's external approximation algorithm for multi-objective linear programming problems. In the case that the objective functions and constraints are differentiable, for implement the main step, we describe the effective method of "constructing a hyperplane separating an external point from the feasible set in the target space". In the case of non-differentiablility of objective functions and constraints, this method is not applicable. Therefore, we will use the generalized directinal derivative and subdifferentials, then we re-examine and explaine the algorithm for this case.

Keywords: Multi-objective optimization, differentiability, sub-differential of convex functions, ε -nondominated point

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1. Introduction

In multi-objective optimization, several objective functions must be minimized simultaniously. Usually these goals are in conflict with each other. So in most cases, there is no one solution that optimizes all the objectives at the same time. Therfor the purpose of multi-objective optimization is to obtain nondominated points (instead of optimal solutions).

The problem of multi-objective convex programming: Suppose $y^1, y^2 \in R^p$, here we use symbol $y^1 \leqq y^2$ to show $y_i^1 \leqq y_i^2$, for all $i = 1, ..., p$. In addition $y^1 \leq y^2$ shows that $y^1 \leqq y^2$, $y^1 \neq y^2$. while $y^1 < y^2$ means that $y_i^1 < y_i^2$, for each $i = 1, ..., p$. Suppose $A \subseteq R^p$, point of $y \in A$ is called weak nondominated if $(\{y\} - intR^p_{\geqq}) \cap A = \phi$.
Consider the following multi-objective program MOP :

$$min\ f(x) = (f_1(x), ..., f_p(x))$$

$$x \in \mathcal{X} = \{x \in R^n : g(x) = (g_1(x), ..., g_m(x))^T \leqq 0\} \quad (1)$$

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$\mathcal{X}$ is feasible set in the decision space R^n ,that we assume is nonempty.
Theorem 1: The following statements are hold:
1) The set $\mathcal{P} \subset R^p$ is a nonempty polygon with dimension P and the feasible set $\mathcal{P}$ is $R^p_{\geqq}$ bounded from below
2. $\mathcal{Y}_N = \mathcal{P}_N$
3. Every vertex of $\mathcal{P}$ belongs to $\mathcal{Y}_N$.
4. $\mathcal{P}_{WN} = bd\mathcal{P}$

Clearly $\mathcal{P}$ is an nonempty, convex, closed set and the point $s \notin \mathcal{P}$, so there is a unique point $y \in bd\mathcal{P}$ such that it has the shortest distance to the point s.
To make a cut that separate $s \notin \mathcal{P}$ from $\mathcal{P}$, Benson proposed the following method for two objective case: the Points s and $\hat{p}$ are connected by a line segment, this line segment intersects the boundary of $\mathcal{P}$ at a unique point y and this is the closest unique point.
The idea of making this hyperplane is that support(Rely) on $\mathcal{P}$ at the border point y and therefore separate s from $\mathcal{P}$. To find this hyperplane, initial and doul programs $P(y)$, $D(y)$ from the linear program dependent to $y \in R^p$ are required:

$$P(y) \qquad min\{z : Ax \geqq b,\ \ Cx - ez \leqq y\}$$

$$D(y) \qquad max\{b^T u - y^T \lambda : Au - C^T \lambda = 0,\ \ e^T \lambda = 1,\ u, \lambda \geqq 0\}$$

Note that if $y \notin \mathcal{P}$ target value $P(y)$ be strictly greater than zero and $y \in \mathcal{P}_{WN}$ then this target value will be equal to zero and if $y \in \mathcal{P} \setminus \mathcal{P}_{WN}$ then the value of this target is strictly less than zero.
So Benson has shown that an optimal point of $D(y)$ can be used to obtain a supporting hyperplane if y be a weak nondominated point of $\mathcal{P}$.
Algorithm 1: (External approximation algorithm for MOP Convex)
i1) put: $S^0 = y^I + R^p_{\geqq}$ ($y^I_i = \{y_i : y \in \mathcal{Y}\}$) Then the corner is $S^0 = \{y^I\}$.

i2) An inner point $\hat{p}$ From $\mathcal{P}$ is finded.

i3) Put $\mathcal{O} := \emptyset$ and $\mathcal{I} := \emptyset$ and $k = 0$.

Repeat steps.
k1) If for any $s \in VertS^k$, $s \in \mathcal{P} \cup \mathcal{O}$ be established then go to step(k6). Otherwise select $s^k \in VertS^k \setminus (\mathcal{P} \cup \mathcal{O})$ and continue.
k2) Calculate unique border point $y^k := \rho_k s^k + (1 - \rho_k)\hat{p} \in \mathcal{P}$ where $\rho_k \in (0, 1)$.
k3) If the distance $d(s^k, y^k)$ From s^k up y^k be at most ε , then add s^k to $\mathcal{O}$ and add y^k to set $\mathcal{I}$ and go (k1).
k4) If the distance $d(s^k, y^k) > \varepsilon$ then determine the hyperplane

$$H := \{y \in R^p : y^T \lambda^k \geqq (g(x^k) - \nabla g(x^k)^T x^k - \nabla^2 g(x^k)\frac{(x^k)^2}{2})u^k\}$$

that separates s^k from $\mathcal{P}$, and put:

$$S^{k+1} = S^k \cap \{y \in R^p : y^T \lambda^k \geqq (g(x^k) - \nabla g(x^k)^T x^k - \nabla^2 g(x^k)\frac{(x^k)^2}{2})u^k\}$$

k5) Determine the vertex s^{k+1} , put $k = k + 1$ and come back to step(k1) .

k6) Define the set of external approximation points $V_0(S^k) = Vert\ S^k$ and define the set

of internal approximation points $V_i(S^k) = (Vert\ S^k \setminus \mathcal{O}) \cup \mathcal{I}$.
Results :

r1) Suppose $\mathcal{P}^i = convV_i(S^k) + R^p_{\geqq}$
that $\mathcal{P}^i$ shows the internal approximation of $\mathcal{P}$, ($\mathcal{P}^i \subseteq \mathcal{P}$).
r2) Suppose $P^0 = convV_0(S^k) + R^p_{\geqq}$

that $\mathcal{P}^0$ shows external approximation of $\mathcal{P}$, ($\mathcal{P} \subseteq \mathcal{P}^0$).
Proposition: Suppose $\hat{p} \in int\mathcal{P}$ and $s^k \notin \mathcal{P}$, Suppose (x^k, ρ^k) be the optimal solution of nonlinear program (6) , then x^k is weak efficient point of MOP (1) .
Quality guarantee of this approximation is possible because this algorithm making an external approximation $\mathcal{P}$ and also internal approximation $\mathcal{P}$ ie, $\mathcal{P}^i \subseteq \mathcal{P} \subseteq \mathcal{P}^0$.

Step (k4) for the general case of differentiability and non-differentiability functions:
(k4)
a) Determin the subgradients $\partial f(x^k)$ and $\partial g(x^k)$ and $\partial^2 g(x^k)$.
b) Solve $LD(x^k, s^k)$ and suppose (u^k, λ^k) be the optimal solution.
c) If the target value $LD(x^k, sk)$ is less than zero then

$$S^{k+1} = S^k \cap \{y \in R^p : y^T\lambda \geqq b^T u^T\}$$

d) Otherwise choose new subgradients $\partial f(x^k)$ and $\partial g(x^k)$ and go step (k4) part (b) or choose the other corner s^k and go to step (k2) .
remark 1: The gradient is always a member of the subdifferential $\nabla f(x) \in \partial f(x)$.
Also when the function f is differentiable, the only member of the subdifferential is the gradient

$$\partial f(x) = \{\nabla f(x)\}$$

remark 2: Given that the functions are convex and as a result the epigraph of these functions is a convex set, using the existence of a superplane that supports on this convex set, is proved that the subdifferential of these functions is never empty;

$$\partial f(x) = \{\xi \in R^n | f^0(x; d) \geqslant \xi^T d, \ \ for\ all \quad d \in R^n\}$$

When the function f be convex we have:

$$\partial_c f(x) = \{\xi \in R^n | f^{'}(x; d) \geqslant \xi^T d, \ , for\ all \quad d \in R^n\}$$

necessary and sufficient conditions $K.K.T$ for nonsmooth functions:
Suppose the problem (1) satisfies in the Slater conditions and assume $f : R^n \to R^p$ and $g : R^n \to R^m$ be convex. Then x^k is the minimal solution for problem (1) if and only if there exist coefficients $\lambda_i \geqslant 0$, $i = 1, ..., p$ and $\mu_j \geqslant 0$, $j = 1, ..., m$ such that

$$\mu_j g_j(x^k) = 0$$

and

$$0 \in \sum_{i=1}^{p} \lambda_i \ \ \partial_c f_i(x^k) + \sum_{j=1}^{m} \mu_j \ \ \partial_c g_j(x^k)$$

Theorem 2: If the function f be convex in this case or adopts its global minimum in x or there exist a decended direction such as d for f at the point x.
Theorem 3: If the function f be locally Lipschitz continuous at the point x Lipschitz then $0 \in \partial f(x)$ or there exists a decreasing direction $d \in R^n$ for f at the point x.

Suppose f be locally lipschitz continuous.
we say d is a decreasing direction of f at the point x if and only if

$$\xi^T d < 0 \ \ ,\forall \xi \in \partial f(x) \qquad or \qquad f^0(x;d) < 0$$

because a derivative of direction, ie $f^{'}(x;d)$, does not exist for all functions, so we use the generalized derivative of $f^0(x;d)$.

$$f^{\,'}(x^k;d) := \lim_{t\downarrow 0} \frac{f(x^k+td)-f(x^k)}{t}$$
$$f^{\,0}(x^k;d) := \limsup_{\substack{y^k\to x^k \\ t\downarrow 0}} \frac{f(x^k+td)-f(x^k)}{t}$$

(the second one, is for the general case of derivatives or non-derivatives of functions; the same as the supremacy of directional derivatives)
For building a hyperplane for the general case of differentiability and non-differentiability of functions we put :

$$f_{x^k} := f(x^k) - \partial f(x^k)^T x^k \qquad , \qquad b_{x^k} := g(x^k) - \partial g(x^k)^T x^k$$
$$C_{x^k} := \partial f(x^k)^T \qquad , \qquad A_{x^k} := -\partial g(x^k)^T$$

that is, for different scenarios of each member of ∂f We will run the algorithm to obtain the hyperplane:

$$H := \{y \in R^p | \ \ (y - f_{x^k})^T \lambda^* = b_{x^k}^T u^*\}$$

2. Conclusion

In this paper, we revise the developed Benson's external approximation method to solve multi-objective convex nonlinear programming problems. We explained the algorithm for both case of differentiablility and nondifferentiablility of objective functions and the constraint functions and proved that this algorithm guarantees finding weak ε - nondominated points with an approximation error ϵ .
Our suggestion for next works is that after finding the hyperplane, approximate the efficient border of $\mathcal{P}$ with a quadratic surface, because approximating the efficient border with a quadratic surface instead of a line, will increase the quality of the approximation.

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پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Group actions on irreducible characters: applications to local-global conjectures for groups of type A

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Abstract. The aim of this paper is to study the Brauer program on classifying finite groups with isomorphic complex group algebras. More precisely, we first show that a large family of almost simple groups of type A are uniquely determined up to isomorphism by the structure of their complex group algebras. For this purpose, we need to understand the action of automorphisms on irreducible characters. The second part of the paper is devoted to study the action of automorphisms of quasi-simple groups of type A on the set of their irreducible characters. Consequently, we give a short proof of the global side of the inductive McKay condition for irreducible characters of groups of type A.

Keywords: Automorphisms, Irreducible characters, Finite groups of Lie type

AMS Mathematics Subject Classification [2010]: 20C15; 20C33

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1. Introduction

A fundamental question in representation theory of finite groups is the extent to which complex group algebra of a finite group determines the group or some of its properties. In 1963, R. Brauer asked "when do non-isomorphic groups have isomorphic complex group algebras?". Although the question seems to be too general to be solved completely, it has initiated a program aimed at determining all finite groups (up to isomorphism) with isomorphic complex group algebras to that of a given group G. Note that it is fairly possible for two non-isomorphic solvable groups to have isomorphic complex group algebras. However, it seems that non-abelian simple groups or more generally quasi-simple and almost simple groups have a very close connection with their complex group algebras. Indeed, Tong-Viet proved in [6] that non-abelian simple groups are uniquely determined by the structure of their complex group algebras. He also posed the following question:

Question. *Which finite groups are determined uniquely by the structure of their complex group algebras?*

In this paper, we propose a conjecture to extend these results to almost simple groups.

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Conjecture. [2] Every finite almost simple group is uniquely determined up to isomorphism by the structure of its complex group algebra.

We recall that a finite group G is said to be almost-simple if $S \leq G \leq Aut(S)$ for a non-abelian simple group S. By the classification of finite simple groups, most of the almost simple groups are of Lie type and they can be considered as (disconnected) reductive groups over finite fields. The current state of knowledge of the ordinary representation theory of finite reductive groups is still incomplete, even after the ground-breaking work of Deligne and Lusztig. Questions on the behaviour of irreducible characters under automorphisms, and hence question of determining character degrees of almost simple groups still remain unsolved. So, investigating the above conjecture for almost simple groups seems to be a more difficult task.

In this paper, we will study the above conjecture for almost simple groups of type A. The verification of the conjecture consists of two main steps: Assume H is a finite group and G is an almost simple group of type A with $\mathbb{C}H \cong \mathbb{C}G$. In the first step, we reduce the question to almost simple groups by showing that $H/C_H(H')$ is an almost simple group with the same socle as G. The second step is devoted to comparing the characters degrees of all two almost simple groups of type A with those of G, for which we need to understand the action of $Aut(PSL_n(q))$ on $\mathrm{Irr}(PSL_n(q))$. As the second goal of this paper, we will study this question in a more general form.

Question. [1, Problem 2.33] For G a quasi-simple group of Lie type, determine the action of $Aut(G)$ on $\mathrm{Irr}(G)$.

For $\epsilon \in \{\pm\}$, we set $SL_n^-(q) = SU_n(q)$ and $SL_n^+(q) = SL_n(q)$. In this paper, we answer the above question for $G = SL_n^\epsilon(q)$. This will be extremely useful in the so-called local global conjecture such as the McKay conjecture. As an application, we give a short proof of the global side of the inductive McKay condition for irreducible characters of groups of type A.

2. Main Results

In this section we state our results concerning the above conjecture and question. In 1963, Richard Brauer posed the following fundamental question:

Question. *When do two finite groups have isomorphic complex group algebras?*

Although by the present knowledge of representation theory it is not possible to settle Brauer's question, it is possible to obtain some significant progress in this regard.

Definition 2.1. A group G is said to be uniquely determined by the structure of its complex group algebra if for any group H, the $\mathbb{C}$-algebra isomorphism $\mathbb{C}H \cong \mathbb{C}G$ yields the group isomorphism $H \cong G$.

In the sequel we will partially answer Brauer's question in affirmative for some almost simple groups of type A (both linear and unitary) of arbitrary large ranks.

Theorem 2.2. *[3, 5, Main Theorem] Let q be a prime power and $n \geq 2$. Let G be a finite group such that* $\mathrm{PSL_n(q)} \leq \mathrm{G} \leq \mathrm{PGL_n(q)}$. *Then for any finite group H with $\mathbb{C}H = \mathbb{C}G$, we have $H \cong G$.*

We then extend this theorem to an analogous result for simple groups of type ${}^2A$.

THEOREM 2.3. *[2] Let q be a prime power and $n \geq 3$. Let G be a finite group such that $\mathrm{PSU_n(q)} \leq \mathrm{G} \leq \mathrm{PGU_n(q)}$, where $q+1$ divides neither of n and $n-1$. Then for any finite group H with $\mathbb{C}H = \mathbb{C}G$, we have $H \cong G$.*

We now focus on studying the action of automorphisms on irreducible characters. A finite reductive group is the fixed-point subgroup $G := \mathbf{G}^F$ of a connected reductive group $\mathbf{G}$ defined over the finite field $\mathbb{F}_q$ of characteristic $p > 0$, where $F : \mathbf{G} \to \mathbf{G}$ is the Frobenius map corresponding to this $\mathbb{F}_q$-structure. It has been shown by Lusztig that the irreducible characters of $\mathbf{G}^F$ can be partitioned into the so-called geometric Lusztig series, labelled by the semisimple $\mathbf{G}^*$-classes of ${\mathbf{G}^*}^{F^*}$, where $(\mathbf{G}^*, F^*)$ denotes a pair dual to $(\mathbf{G}, F)$. If such a series is labelled by a semisimple class with representative s, then it contains $|A_{\mathbf{G}^*}(s)^{F^*}|$ semisimple characters, where $A_{\mathbf{G}^*}(s) = C_{\mathbf{G}^*}(s)/C^{\circ}_{\mathbf{G}^*}(s)$ is the component group of s. The set of semisimple (and regular) characters of $G = \mathbf{G}^F$ can be naturally parametrized by pairs (s, ξ) where s runs over a set of representatives of the semisimple classes of ${\mathbf{G}^*}^{F^*}$ and $\xi \in \mathrm{Irr}(A_{G^*}(s))$, where $A_{G^*}(s) := A_{\mathbf{G}^*}(s)^{F^*}$.

To state our result, we need to introduce some more notation. For any $\lambda \in \mathrm{Irr}(C^{\circ}_{G^*}(s))$, we denote by $A_{G^*}(s)_\lambda$ the stabilizer of λ under $A_{G^*}(s)$. Moreover, the outer automorphism group $Out(G)$ is well known to be generated by diagonal, field, and graph automorphisms. In the case of diagonal automorphisms, the action on $\mathrm{Irr}(G)$ is well understood by work of Lusztig. In the sequel, we write F_p for a generator of the group of field automorphisms, and write γ for a generator of the group of graph automorphisms of G.

THEOREM 2.4. *[4, Main Theorem] Assume that $G = SL^{\epsilon}_n(q)$. For any semisimple element $s \in G^*$ and any unipotent character $\lambda \in \mathrm{Irr}(C^{\circ}_{G^*}(s))$, there exists a morphism $\omega^0_{s,\lambda} : H^1(F, Z(\mathbf{G})) \to \mathrm{Irr}(A_{G^*}(s)_\lambda)$ such that for the irreducible character $\chi_{s,\lambda,\omega^0_{s,\lambda}(z)} \in \mathrm{Irr}(G)$ parametrized by triple $(s, \lambda, \omega^0_{s,\lambda}(z))$ for some $z \in H^1(F, Z(\mathbf{G}))$, one has*

$$^{\sigma}\chi_{s,\lambda,\omega^0_{s,\lambda}(z)} = \chi_{\sigma^{*-1}(s),\sigma^*(\lambda),\omega^0_{s,\lambda}(\sigma(z))},$$

where $\sigma \in \langle F_p, \gamma\rangle$ and $\sigma^* \in Aut(G^*)$ is its dual automorphism.

Consequently, using the above theorem, we obtain a short and explicit proof of the global side of the so-called inductive McKay condition for the irreducible characters of $G = SL^{\epsilon}_n(q)$. In what follows, let $\tilde{G} = GL^{\epsilon}_n(q)$ where $GL^{-}_n(q) = GU_n(q)$.

COROLLARY 2.5. *[4, Corollary] If $\tilde{\chi} = \chi_{s,\lambda} \in \mathrm{Irr}(\tilde{G})$, then for the irreducible character $\chi_0 = \chi_{s,\lambda,1} \in \mathrm{Irr}(G|\chi)$ we have*

$$(\tilde{G} \times \langle F_p, \gamma\rangle)_{\chi_0} = \tilde{G}_{\chi_0} \times (\langle F_p, \gamma\rangle)_{\chi_0}.$$

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On the eigenvalues inclusion sets of stochastic tensors

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Abstract. The purpose of this paper is to locate and estimate the eigenvalues of stochastic tensors. We present several estimation theorems about the eigenvalues of stochastic tensors. Meanwhile, we obtain the distribution theorem for the eigenvalues of the tensor product of two stochastic tensors. We will conclude the paper with the distribution for the eigenvalues of generalized stochastic tensors.

Keywords: Inclusion sets, Nonnegative tensors, Stochastic tensors.

AMS Mathematics Subject Classification [2010]: 15A18, 15A69

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1. Introduction

Tensors have numerous applications in many branches of mathematics and physics. In late studies of numerical multilinear algebra eigenvalue problems for tensors have been brought to special attention. The concept of eigenvalues for tensors was first introduced and studied by Qi [3] and Lim [2] independently in 2005 and initiated the rapid developments of the spectral theory of tensors. Eigenvalue localization has been a hot topic in tensor theory and its applications. This article discusses location, distribution and estimate of the eigenvalues for stochastic tensor. In continue we introduce the concepts of generalized stochastic tensor and discuss the eigenvalue distribution for generalized stochastic tensors. A tensor can be regarded as a higher order generalization of a matrix, which takes the form

$$\mathbb{A} = (a_{i_1,\dots,i_m}), \quad a_{i_1,\dots,i_m} \in \Re, \quad 1 \le \ i_1,\dots,i_m \ \le n,$$

where $\Re$ is the real field. Such a multi-array $\mathbb{A}$ is said to be an mth order n-dimensional square real tensor with n^m entries $a_{i_1,\dots,i_m}$. We denote the set of all nonnegative mth order n-dimensional tensors by $\Re_+^{[m,n]}$. For a vector $x = (x_1,\dots,x_n)^T$, let $\mathbb{A}x^{m-1}$ be a vector in $\Re^n$ whose ith component is defined as the following:

$$(1) \qquad \left(\mathbb{A}x^{m-1}\right)_i = \sum_{i_2,\dots,i_m=1}^{n} a_{ii_2\dots i_m} x_{i_2}\dots x_{i_m},$$

and let $x^{[m]} = (x_1^m,\dots,x_n^m)^T$.

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Definition 1.1. [3] A pair $(\lambda, x) \in \mathcal{C} \times (\mathcal{C}^n \setminus \{0\})$ is called an eigenvalue and an eigenvector of $\mathbb{A} \in \Re^{[m,n]}$, if they satisfy

(2) $$\mathbb{A}x^{m-1} = \lambda x^{[m-1]}.$$

Definition 1.2. [4] Let $\mathbb{A}$ (and $\mathbb{B}$) be an order $m \geq 2$ (and order $k \geq 1$), dimension n tensor, respectively. The product $\mathbb{AB}$ is defined to be the following tensor $\mathbb{C}$ of order $(m-1)(k-1)+1$ and dimension n:

$$c_{i\alpha_1\ldots\alpha_{m-1}} = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} b_{i_2\alpha_1} \ldots b_{i_m\alpha_{m-1}},$$

where $(i \in [n], \alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1})$.

It is easy to check from the definition that $I_n\mathbb{A} = \mathbb{A} = \mathbb{A}I_n$, where I_n is the identity matrix of order n. When $k = 1$ and $\mathbb{B} = x \in \mathcal{C}^n$ is a vector of dimension n, then $(m-1)(k-1)+1 = 1$. Thus $\mathbb{AB} = \mathbb{A}x$ is still a vector of dimension n, and we have

$$(\mathbb{A}x)_i = (\mathbb{AB})_i = c_i = \sum_{i_2\ldots i_m=1}^{n} a_{ii_2\ldots i_m} x_{i_2} \ldots x_{i_m} = \left(\mathbb{A}x^{m-1}\right)_i,$$

thus we have $\mathbb{A}x^{m-1} = \mathbb{A}x$. So the first application of the tensor product defined above is that now $\mathbb{A}x^{m-1}$ can be simply written as $\mathbb{A}x$.

Definition 1.3. [2] A tensor $\mathbb{A} \in \Re^{[m,n]}$ is called reducible, if there exists a nonempty proper index subset $I \subset \{1, \ldots, n\}$ such that

$$a_{i_1,\ldots,i_m} = 0, \;\; \forall i_1 \in I, \;\; \forall i_2, \ldots, i_m \notin I,$$

if $\mathbb{A}$ is not reducible, then we call $\mathbb{A}$ irreducible.

2. Main results

Definition 2.1. [5] A nonnegative tensor $\mathbb{A}$ of order m, and dimension n is called stochastic provided that

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} = 1, i = 1, 2, \ldots, n.$$

Obviously, when $\mathbb{A}$ is stochastic, 1 is the spectral radius of $\mathbb{A}$ and e is an eigenvector corresponding to 1, where e is an all ones vector; if, further, $\mathbb{A}$ is irreducible, then e is the unique positive eigenvector corresponding to 1.

Theorem 2.2. *Suppose $\mathbb{A} \in \Re_+^{[m,n]}$ is a stochastic tensor and $M = \min\{a_{ii\ldots i} : i = 1, 2, \ldots, n\}$, then*

$$\sigma(\mathbb{A}) \subset G(\mathbb{A}) = \{z : |z - M| \leq 1 - M\},$$

where $\sigma(\mathbb{A})$ is denoted the whole eigenvalues of tensor $\mathbb{A}$, $G(A)$ is the Gerschgorin disc of tensor $\mathbb{A}$.

Definition 2.3. [4] Let $\mathbb{A}$ and $\mathbb{B}$ be two order k tensors with dimension n and m, respectively. Define the direct product $\mathbb{A} \otimes \mathbb{B}$ to be the following tensor of order k and dimension nm (the set of subscripts is taken as $[n] \times [m]$ in the lexicographic order):

$$(\mathbb{A} \otimes \mathbb{B})_{(i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)} = a_{i_1i_2\ldots i_k} b_{j_1j_2\ldots j_k}.$$

THEOREM 2.4. *[4] Let $\mathbb{A}$ and $\mathbb{B}$ be two order k-tensors with dimension n and m, respectively. Suppose that, we have $\mathbb{A}u^{k-1} = \lambda u^{[k-1]}$, and $\mathbb{B}v^{k-1} = \mu v^{[k-1]}$, and we also write $w = u \otimes v$. Then we have:*

$$(\mathbb{A} \otimes \mathbb{B})\, w^{[k-1]} = (\lambda\mu)\, w^{[k-1]}.$$

THEOREM 2.5. *Let $\mathbb{A}, \mathbb{B} \in \Re_+^{[m,n]}$ are stochastic tensors, $M_1 = \min\{a_{ii\ldots i} : i = 1, 2, \ldots, n\}$ and $M_2 = \min\{b_{ii\ldots i} : i = 1, 2, \ldots, n\}$, then*

$$\sigma(\mathbb{A} \otimes \mathbb{B}) \subset G(\mathbb{A} \otimes \mathbb{B}) = \{z : |z - M_1| \le 1 - M_1\} . \{z : |z - M_2| \le 1 - M_2\},$$

where $\sigma(\mathbb{A} \otimes \mathbb{B})$ is denoted the whole eigenvalues of tensor product for tensor $\mathbb{A}$ and tensor $\mathbb{B}$, $G(\mathbb{A} \otimes \mathbb{B})$ is the oval region of the product for elements of Gerschgorin disc whose center is M_1 and radius is $1 - M_1$ and Gerschgorin disc whose center is M_2 and radius is $1 - M_2$.

THEOREM 2.6. *Suppose $\mathbb{A} \in \Re_+^{[m,n]}$ is a stochastic tensor and $M_i = \max\{a_{ii_2\ldots i_m} : 1 \le i_2, \ldots, i_m \le n\}$, then*

$$\sigma(\mathbb{A}) \subset G(\mathbb{A}) = \bigcup_{i=1}^{n} \left\{ z : |z - a_{ii\ldots i}| \le \sqrt{(m-1)(n-1) M_i (1 - a_{ii\ldots i})} \right\}.$$

LEMMA 2.7. *[1] Assume that $a_1 \le a_2 \le \cdots \le a_n < k$. Each of the ovals*

$$|z - a_i|\, |z - a_j| \le (k - a_i)(k - a_j), \quad (i, j = 1, 2, \ldots, n;\ i < j) \tag{3}$$

is either identical with the oval

$$|z - a_1|\, |z - a_2| \le (k - a_1)(k - a_2) \tag{4}$$

or lies in the interior of (4). The point $z = k$ is the only common point of the boundaries of two different ovals (3).

THEOREM 2.8. *Let $a_{ii\ldots i}$ and $a_{jj\ldots j}$ be the smallest elements of the main diagonal of a stochastic tensor $\mathbb{A} \in \Re_+^{[m,n]}$. Then all the eigenvalues lie in the interior or on the boundary of the following oval*

$$\{z : |z - a_{ii\ldots i}|\, |z - a_{jj\ldots j}| \le (1 - a_{ii\ldots i})(1 - a_{jj\ldots j})\}. \tag{5}$$

DEFINITION 2.9. **(i)** A nonnegative tensor $\mathbb{A}$ of order m and dimension n is called first generalized stochastic, if

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} = k, \quad i = 1, 2, \ldots, n.$$

(ii) A tensor $\mathbb{A}$ of order m and dimension n is called second generalized stochastic if

$$\sum_{i_2,\ldots,i_m=1}^{n} |a_{ii_2\ldots i_m}| = 1, \quad i = 1, 2, \ldots, n.$$

(iii) A tensor $\mathbb{A}$ of order m and dimension n is called third generalized stochastic if

$$\sum_{i_2,\ldots,i_m=1}^{n} |a_{ii_2\ldots i_m}| = k, \quad i = 1, 2, \ldots, n.$$

THEOREM 2.10. *Suppose $\mathbb{A}, \mathbb{B} \in \Re_+^{[m,n]}$ are first generalized stochastic tensors, then*

$$\begin{aligned}\sigma(\mathbb{A} \otimes \mathbb{B}) \subset G(\mathbb{A} \otimes \mathbb{B}) = &\{z : |z - a_{ii\ldots i}|\, |z - a_{jj\ldots j}| \le (k - a_{ii\ldots i})(k - a_{jj\ldots j})\} \\ &. \{z : |z - b_{ii\ldots i}|\, |z - b_{jj\ldots j}| \le (k - b_{ii\ldots i})(k - b_{jj\ldots j})\}.\end{aligned}$$

THEOREM 2.11. *Suppose* $\mathbb{A} \in \Re^{[m,n]}$ *is third generalized stochastic tensor, and* $M = \min\{|a_{ii\ldots i}| : i = 1, 2, \ldots, n\}$*, then*

$$\sigma(\mathbb{A}) \subset G(\mathbb{A}) = \{z : |z - M| \leq k + M\}.$$

THEOREM 2.12. *Let* $\mathbb{A}, \mathbb{B} \in \Re^{[m,n]}$ *be third generalized stochastic tensors,* $M_1 = \min\{|a_{ii\ldots i}| : i = 1, 2, \ldots, n\}$ *and* $M_2 = \min\{|b_{ii\ldots i}| : i = 1, 2, \ldots, n\}$*, then*

$$\sigma(\mathbb{A} \otimes \mathbb{B}) \subset G(\mathbb{A} \otimes \mathbb{B}) = \{z : |z - M_1| \leq k + M_1\}.\{z : |z - M_2| \leq k + M_2\}.$$

THEOREM 2.13. *Let* $a_{ii\ldots i}$ *and* $a_{jj\ldots j}$ *be the smallest elements of the main diagonal of a third generalized stochastic tensor* $\mathbb{A} \in \Re^{[m,n]}$*. Then*

$$\sigma(\mathbb{A}) \subset G(\mathbb{A}) = \{z : |z - a_{ii\ldots i}|\,|z - a_{jj\ldots j}| \leq (k + |a_{ii\ldots i}|)(k + |a_{jj\ldots j}|)\}.$$

THEOREM 2.14. *Let* $\mathbb{A}, \mathbb{B} \in \Re_+^{[m,n]}$ *are first generalized stochastic tensors, then*

$$\begin{aligned}\sigma(\mathbb{A} \otimes \mathbb{B}) \subset G(\mathbb{A} \otimes \mathbb{B}) = &\{z : |z - a_{ii\ldots i}|\,|z - a_{jj\ldots j}| \leq (k + a_{ii\ldots i})(k + a_{jj\ldots j})\} \\ &.\{z : |z - b_{ii\ldots i}|\,|z - b_{jj\ldots j}| \leq (k + b_{ii\ldots i})(k + b_{jj\ldots j})\}.\end{aligned}$$

EXAMPLE 2.15. Let $\mathbb{A} = (a_{ijk}) \in \Re^{[3,2]}$ for $1 \leq i, j, k \leq 2$ such that:

$$a_{ijk} = \begin{cases} 1 & \text{if} \quad i = j = k \\ 0 & \text{o.w} \end{cases}$$

Thus $\mathbb{A}$ is a first generalized stochastic tensor, and suppose $\mathbb{B} = (b_{ijk})$ for $1 \leq i, j, k \leq 2$ such that:

$$b_{ijk} = \begin{cases} 1 & \text{if} \quad i = 1, j = k = 2 \\ 0 & \text{o.w} \end{cases}$$

Then by theorem 2.14, we have

$$\sigma(\mathbb{A} \otimes \mathbb{B}) \subset \{z : |z - 1|^2 \leq 4\}.\{z : |z|^2 \leq 1\}$$

where (A $\otimes\mathbb{B}$) $\in\Re^{[3,4]}$ has 81 entries.

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Some results on the essential graph of $\mathbb{Z}_n$

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ABSTRACT. Let R be a commutative ring. The essential graph of R, denoted by $EG(R)$ is a simple graph associated to R with vertex set $Z(R) \setminus \{0\} = Z(R)^*$, and a pair of distinct vertices x and y are adjacent if and only if $\mathrm{Ann}(xy)$ is an essential ideal of R. In this paper, we investigate the concept of the dominating set for the essential graph of $\mathbb{Z}_n$.

Keywords: Essential graph, zero divisor graph, domination number.

AMS Mathematics Subject Classification [2010]: 13Cxx , 05C25

1. Introduction

The study of algebraic structures by using the properties of a simple graph is a topic which becomes more attention in last decades and leads many authors to study and explore its properties. The zero divisor graph of R, denoted by $\Gamma(R)$ is a simple graph with vertex set $Z^*(R)$ and two distinct vertices x and y are adjacent if and only if $xy = 0$, see [1, 2]. Recently, the essential graph of a commutative ring was introduced and studied in [3]. Also, the concept of the essential graph for modules has been defined and studied in [5]. In this paper, we study the domination number of the essential graph of $\mathbb{Z}_n$.

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. A graph with no vertex is called an empty graph. For every $u, v \in V(G)$, the distance between u and v is defined as the length of a shortest path from u to v and is denoted by $\mathrm{d}(u, v)$. We write $u \sim v$ if $\mathrm{d}(u, v) = 1$. The vertex u is said to be a universal vertex if it is adjacent to every other vertices of G. The graph is connected if there is a path between any two distinct vertices. A complete graph is a graph in which each pair of vertices is connected by an edge and a complete graph with n vertices, denoted by K_n. A bipartite graph is one whose vertex set can be partitioned into two subsets so that an edge has both ends in no subset. A complete bipartite graph is a bipartite graph in which each vertex is adjacent to every vertex that is not in the same subset. The complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. The join $G_1 \vee G_2$ of two vertex-disjoint graphs G_1 and G_2 has $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv \,|\, u \in V(G_1), v \in V(G_2)\}$. Let D is a subset of $V(G)$. The subset D is called dominating set of G whenever every vertex of $V(G) \setminus D$ is adjacent to some vertex of D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set.

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Throughout this paper, all rings are assumed to be commutative with nozero identity. By $Z(R)$ and $\mathrm{Ass}(R)$ we denoted the set of zero-divisors and associated prime ideals of R, respectively.

2. Main results

In this section, we calculate the domination number of the essential graph of the ring $\mathbb{Z}_n$.

DEFINITION 2.1. [3] Let R be a commutative ring. The essential graph of R, denoted by $EG(R)$ is a simple graph with vertices set $Z(R)\setminus\{0\} = Z(R)^*$ and two distinct vertices $x, y \in Z(R)^*$ are adjacent if and only if $\mathrm{Ann}(xy)$ is an essential ideal of R.

Let $n = p$ be a prime number. Then $EG(\mathbb{Z}_n)$ is an empty graph .

LEMMA 2.2. *Let $n = p^\alpha$, where p is a prime number and α is a positive integer. Then $EG(\mathbb{Z}_n)$ is a connected graph and $\gamma(EG(\mathbb{Z}_n)) = 1$ if and only if $\alpha \geq 2$.*

PROOF. Suppose that $n = p^\alpha$, where p is a prime number and $\alpha \geq 2$ is an integer. Then $Z(\mathbb{Z}_n)^* = \mathrm{Nil}(\mathbb{Z}_n)^* = p\mathbb{Z}_n$. Thus every vertex of $EG(\mathbb{Z}_n)$ is a universal vertex, see [5, Lemma 2.2]. Hence, $EG(\mathbb{Z}_n)$ is a complete graph so is connected and $\gamma(EG(\mathbb{Z}_n)) = 1$.
The converse is obvious since $EG(\mathbb{Z}_n)$ is an empty graph whenever $\alpha = 1$. □

Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be an integer, where $p_1, \cdots, p_k$ are prime numbers, $\alpha_1, \cdots, \alpha_k$ are positive integers and $k \geq 2$. If $\alpha_i \geq 2$, for some i with $1 \leq i \leq k$, then in view of Lemma 2.2, $\gamma(EG(\mathbb{Z}_n)) = 1$. In the following we consider the case $k \geq 2$ with $\alpha_1 = \cdots = \alpha_k = 1$.

THEOREM 2.3. *Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be an integer, where $p_1, \cdots, p_k$ are prime numbers, $\alpha_1, \cdots, \alpha_k$ are positive integers, $k \geq 2$ and let $EG(\mathbb{Z}_n)$ be a connected graph. Then $\alpha_1 = \cdots = \alpha_k = 1$ if and only if $\gamma(\mathbb{Z}_n) = k$.*

PROOF. Suppose that $\alpha_1 = \cdots = \alpha_k = 1$. Then $\mathrm{Ass}(\mathbb{Z}_n) = \{p_1\mathbb{Z}_n = \mathrm{Ann}(x_1), \cdots, p_k\mathbb{Z}_n = \mathrm{Ann}(\mathrm{x_k})\}$, where $x_i \in Z(R)^*$, for all $1 \leq i \leq k$. Set $D = \{x_1, \cdots, x_k\}$. We show that D is a dominating set for $EG(\mathbb{Z}_n)$. Assume that $y \in Z(R)^*$. Since $EG(\mathbb{Z}_n)$ is a connected graph by [5, Corollary 3.4], it follows that $\mathrm{Ann}(y) \nsubseteq \bigcap_{i=1}^k p_i\mathbb{Z}_n$. Hence, there is i with $1 \leq i \leq k$ such that $\mathrm{Ann}(y) \nsubseteq p_i\mathbb{Z}_n$. Therefore, $yx_i = 0$ and so y, x_i are adjacent, it follows from [4, Lemma 3.1(ii)]. Now, assume that $D' = \{x'_1, \cdots, x'_{k-1}\} \subseteq Z(R)^*$. To prove the assertion, it is enough to show that D' is not a dominating set for $EG(\mathbb{Z}_n)$. Assume in contrary that D' is a dominating set for $EG(\mathbb{Z}_n)$ and we achieve a contradiction. By the hypothesis $\bigcap_{i=1}^k p_i\mathbb{Z}_n = 0$ so for all $1 \leq j \leq k-1$ there exists $1 \leq i \leq k$ such that $x'_j \notin p_i\mathbb{Z}_n$. Without loss of generality, we may assume that $x'_j \notin p_j\mathbb{Z}_n$, for every $1 \leq j \leq k-1$. Suppose that $x \in \bigcap_{i=1}^{k-1} p_i\mathbb{Z}_n \setminus p_k\mathbb{Z}_n$ so $x \neq x'_j$, for all $1 \leq j \leq k-1$ and x is adjacent to x'_t for some $1 \leq t \leq k-1$. Then $xx'_t = 0$ so $xx'_t \in \bigcap_{i=1}^{k-1} p_i\mathbb{Z}_n$. Hence, $xx'_t \in p_t\mathbb{Z}_n$, which is a contradiction. Therefore, D' is not a dominating set for $EG(\mathbb{Z}_n)$ and the proof is completed.
The converse is obvious by Lemma 2.2. □

EXAMPLE 2.4. (i) Consider the ring $\mathbb{Z}_{16}$. It is clear that $\mathrm{Ass}(\mathbb{Z}_{16}) = \{2\mathbb{Z}_{16}\}$ and $Z(\mathbb{Z}_{16})^* = \mathrm{Nil}(\mathbb{Z}_{16})^* = 2\mathbb{Z}_{16}$. Thus $EG(\mathbb{Z}_{16})$ is a complete graph with 7 vertices and $D = \{2\}$ is a dominating set for it, see Figure 1.
(ii) Consider the ring $\mathbb{Z}_{12}$. It is clear that $\mathrm{Ass}(\mathbb{Z}_{12}) = \{2\mathbb{Z}_{12}, 3\mathbb{Z}_{12}\}$, $\mathrm{Nil}(\mathbb{Z}_{12})^* = 2\mathbb{Z}_{12} \cap 3\mathbb{Z}_{12} = 6\mathbb{Z}_{12}$. Then by the paragraph previous that Theorem 2.3, $D = \{6\}$ is a dominating set for connected graph $EG(\mathbb{Z}_{12})$, see Figure 2.

In the following we show that more results on complete bipartite and join graphs over the essential graph $EG(\mathbb{Z}_n)$.

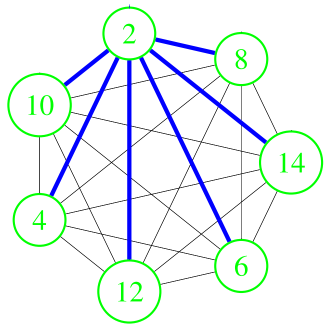

FIGURE 1. Essential graph of $EG(\mathbb{Z}_{16})$ with domination number 1.

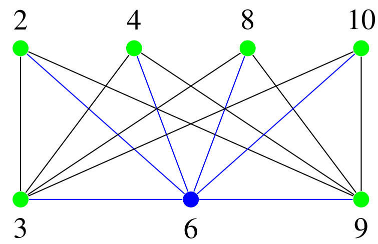

FIGURE 2. Essential graph of $EG(\mathbb{Z}_{12})$ with domination number 1.

THEOREM 2.5. *Let R be a semi-local ring with maximal ideals M_1 and M_2. Then the following statements are true:*

(i) *If $M_1 \cap M_2 = 0$, then $EG(R) = K_{|M_1^*|,|M_2^*|}$ is a complete bipartite graph.*
(ii) *If $M_1 \cap M_2 \neq 0$, then $EG(R) = K_{|M_1 \setminus M_2|,|M_2 \setminus M_1|} \vee K_{|(M_1 \cap M_2)^*|}$.*

PROOF. (i) By the assumption $Z(R) = M_1 \cup M_2$ moreover in this case $EG(R) = \Gamma(R)$, see [5, Theorem 4.6]. Let $x, y \in Z(R)^*$, then x and y are adjacent if and only if $xy = 0$. Thus either $x \in M_1$ or $x \in M_2$ and a same assertion is true for y. Hence, it follows that distinct elements of M_i are not adjacent to each other, for $i = 1,2$ and any element of M_1 is adjacent to each element of M_2. So $EG(R) = K_{|M_1^*|,|M_2^*|}$.

(ii) It is an immediate consequence of (i), [5, Theorem 2.5] and [5, Lemma 2.2]. □

COROLLARY 2.6. *Let p and q be distinct prime numbers. Then the following statements are true:*

(i) *$EG(\mathbb{Z}_{pq}) = K_{|p\mathbb{Z}_{pq}^*|,|q\mathbb{Z}_{pq}^*|}$ is a complete bipartite graph.*
(ii) *$EG(\mathbb{Z}_{p^\alpha q^\beta}) = K_{|p\mathbb{Z}_{pq} \setminus q\mathbb{Z}_{pq}|,|q\mathbb{Z}_{pq} \setminus p\mathbb{Z}_{pq}|} \vee K_{|(p\mathbb{Z}_{pq} \cap q\mathbb{Z}_{pq})^*|}$, where either $\alpha \geq 2$ or $\beta \geq 2$.*

PROOF. It is an immediate consequence of Theorem 2.5. □

COROLLARY 2.7. *Let p and q be distinct prime numbers. Then the following statements are true:*

(i) *$\gamma(EG(\mathbb{Z}_{pq})) = 2$.*
(ii) *$\gamma(EG(\mathbb{Z}_{p^\alpha q^\beta})) = 1$, where either $\alpha \geq 2$ or $\beta \geq 2$.*

PROOF. It is an immediate consequence of Theorem 2.5. □

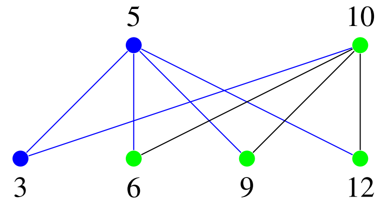

FIGURE 3. Essential graph of $EG(\mathbb{Z}_{15})$ with domination number 2.

EXAMPLE 2.8. Consider the ring $\mathbb{Z}_{15}$. It is clear that $\mathrm{Ass}(\mathbb{Z}_{15}) = \{3\mathbb{Z}_{15}, 5\mathbb{Z}_{15}\}$ and $\mathrm{Nil}(\mathbb{Z}_{15}) = 3\mathbb{Z}_{15} \cap 5\mathbb{Z}_{15} = 0$. By Corollary 2.6, $EG(\mathbb{Z}_{15}) = K_{2,4}$ the complete bipartite graph with 6 vertices and $D = \{3,5\}$ is a dominating set for $EG(\mathbb{Z}_{15})$, see Figure 3.

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Left contractibility FC-algebras in terms of multi-norms

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Abstract. The notion of a multi-norm space was introduced by Dales and Polyakov. It generalizes that of a normed linear space E, which has one norm, by a taking a sequence of norms, one on each of the n-fold product spaces of E with itself. In this paper we investigate equivalent condition left contractibility on FC-algebras to the language of multi-norms. We then apply this result for the group algebra $L^1(G)$ for a locally compact group G.

Keywords: Multi-norms, F-algebras, locally compact group

AMS Mathematics Subject Classification [2010]: 43A07, 43A20, 22D15, 46H25

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1. Introduction

Let $\mathcal{E}$ be a Banach space, and $n \in \mathbb{N}$, we denote by $\mathcal{E}^n$ the vector space Cartesian product of n copies of $\mathcal{E}$, and by $\mathfrak{G}_n$ the group of all permutations of the set $\{1, \ldots, n\}$. A multi-norm based on $\{\mathcal{E}^n : n \in \mathbb{N}\}$ is a sequence $(\|.\|_n) = (\|.\|_n : n \in \mathbb{N})$ such that $\|.\|_n$ is a norm on $\mathcal{E}^n$, such that $\|x\|_1 = \|x\|$ for each $x \in \mathcal{E}$, and such that the following Axioms (A1) $-$ (A4) are satisfied for each $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathcal{E}$:

$$\text{(A1)}\|(\mathrm{x}_{\sigma(1)}, \ldots, \mathrm{x}_{\sigma(\mathrm{n})})\|_{\mathrm{n}} = \|(\mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{n}})\|_{\mathrm{n}} \qquad (\sigma \in \mathfrak{G}_n);$$

$$\text{(A2)}\|(\alpha_1\mathrm{x}_1, \ldots, \alpha_{\mathrm{n}}\mathrm{x}_{\mathrm{n}})\|_{\mathrm{n}} \leq (\max_{1\leq \mathrm{i}\leq \mathrm{n}} \mid \alpha_{\mathrm{i}} \mid)\|(\mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{n}})\|_{\mathrm{n}} \qquad (\alpha_1, \ldots, \alpha_n \in \mathbb{C});$$

$$\text{(A3)}\|(\mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{n}-1}, 0)\|_{\mathrm{n}} = \|(\mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{n}-1})\|_{\mathrm{n}-1};$$

$$\text{(A4)}\|(\mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{n}-1}, \mathrm{x}_{\mathrm{n}-1})\|_{\mathrm{n}} = \|(\mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{n}-1})\|_{\mathrm{n}-1}.$$

In this case, we say that $((\mathcal{E}^n, \|.\|_n) : n \in \mathbb{N})$ is a multi-normed space.

The theory of multi-norms spaces was first introduced and studied by Dales and Polyakov in [2].Also, Dales, Daws, Pham and Ramsden characterized amenability of a locally compact group G by the theory of multi-norms in [1]. The theory of the equivalences of multi-norms has recently been described by several authors. Pham [6] has introduced a new combinatorial condition that characterized the amenability for locally

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*Speaker. Email address: simasoltani@iut.ac.ir compact groups. Indeed he has used a new notion of almost (p,q)-multiboundedness for a subset of a Banach space.

A Banach algebra $\mathcal{A}$ is called F-algebra if the dual space $\mathcal{A}'$ of $\mathcal{A}$ is a W^*-algebra and the identity element of $\mathcal{A}'$ is a multiplicative linear functional on $\mathcal{A}$. The wide range of F-algebras includes the Fourier algebra $A(G)$, the Fourier- Stieltjes algebra $B(G)$, the group algebra $L^1(G)$ of a locally compact group G, and the measure algebra of a locally compact semigroup or hypergroup. In particular, it includes the semigroup algebra $\ell^1(S)$ of a discrete semigroup S.

Let $\mathcal{A}$ be a Banach algebra, $\Delta(\mathcal{A})$ be the set of all nonzero characters on $\mathcal{A}$ and $\phi \in \Delta(\mathcal{A})$. The Banach algebra $\mathcal{A}$ is called left ϕ-contractible if the first cohomology group $H^1(\mathcal{A}, \mathcal{X})$ vanishes for any Banach $\mathcal{A}$-bimodule $\mathcal{X}$ such that its right module product is defined by $x \cdot a = \phi(x)\, a$ $(a \in \mathcal{A}, x \in \mathcal{X})$. The notion of left ϕ-contractibility was introduced and studied in [4] as right ϕ-contractibility. We say F-algebra $\mathcal{A}$ is left contractible, if $\mathcal{A}$ is left u-contractible for the identity element u of $\mathcal{A}'$. The cocepet of left contractibilty of certian Banach algebras characterized by Hamidi and Soltani [3] by the notions of a (p,q)-multi-norms and a multi bounded set.

In this paper, we investigate equivalent condition left contractibility on FC-algebras to the language of multi-norms by the notion of almost (p, q)-multi bounded sets.

2. Main Results

In this part, we recall the definition of the weak $p-$summing norm on a normed space; following the notation of [1], [2] we denote the weak $p-$summing norm (for $1 \le p < \infty$) on $\mathcal{E}^n$ by

$$\mu_{p,n}(x) = \sup \left\{ \left(\sum_{i=1}^{n} \mid \langle x_i, \lambda \rangle \mid^p \right)^{\frac{1}{p}} : \lambda \in \mathcal{E}'_{[1]} \right\}$$

Where $x = (x_1, \ldots, x_n) \in \mathcal{E}^n$ and the closed unit ball of $\mathcal{E}$ is denoted by $\mathcal{E}_{[1]}$ and the action of $\lambda \in \mathcal{E}'$ on $x \in \mathcal{E}$ is written as $\langle x, \lambda \rangle$. In the sequel we state that an important class of multi-norms. The following definition was first given in [[2],§4.1]. Let $\mathcal{E}$ be a normed space, and take p, q with $1 \le p, q < \infty$. For each $n \in \mathbb{N}$ and each $x = (x_1, \ldots, x_n) \in \mathcal{E}^n$, we denote by

$$\|x\|_n^{(p,q)} := \sup \left\{ \left(\sum_{i=1}^{n} \mid \langle x_i, \lambda_i \rangle \mid^q \right)^{\frac{1}{q}} : \lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathcal{E}')^n, \mu_{p,n}(\lambda) \le 1 \right\}.$$

Where the supremum is take over all $\lambda_1, \ldots, \lambda_n \in (\mathcal{E}')^n$.
It is clear that $\|.\|_n^{(p,q)}$ is a norm on $\mathcal{E}^n$. As noted in [[2], Theorem 4.1], in the case where $1 \le p \le q < \infty$, the sequence $(\|.\|_n^{(p,q)} : n \in \mathbb{N})$ is a multi-norm based on $\mathcal{E}$; it is called the (p,q)-multi-norm.

Now, we characterize left contractibility of FC-algebras in terms of multi-norms.

Let $\mathcal{A}$ be a F-algebra and let u be the identity element of $\mathcal{A}'$. Let $P(\mathcal{A})$ be the set of all elements a in $\mathcal{A}$ that induce positive functionals on the W^*-algebra $\mathcal{A}'$, and let $P_1(\mathcal{A})$ be the set of all elements a in $P(\mathcal{A})$ such that $\langle u, a \rangle = 1$; note that

$$P(\mathcal{A}) = \{a \in \mathcal{A} : \|a\| = \langle u, a \rangle\};$$

And hence span $\mathcal{A}$. Note that $\mathcal{A}$ is called an FC-algebra if the W^*-algebra $\mathcal{A}'$ is commutative. In this case, $\mathcal{A} \cong L^1(\Omega, \mu)$ (isometrically isomorphic), and $\mathcal{A}'$ may be regarded as the W^*-algebra $L^\infty(\Omega, \mu)$ for some measure space (Ω, μ).

Proposition 2.1. *Let $\mathcal{A}$ be an* FC-*algebra and* $1 \leq p, q < \infty$. *Then the following statements are equivalent:*

(a) *$\mathcal{A}$ is left contractible;*

(b) *There is an element $m \in P_1(\mathcal{A})$ such that* $\lim_{n\to\infty} \frac{\|(a_1m,\dots,a_nm)\|_n^{(p,q)}}{n^{\frac{1}{q}}} = 0$, *for all sequence (a_n) in $P_1(\mathcal{A})$.*

Proof. Let $\mathcal{A}$ be a left contractible FC-algebra. Then by [5] there exists a topological left invariant mean in $\mathcal{A}$; that is, an element $m \in P_1(\mathcal{A})$ such that

$$am = m \ (a \in P_1(\mathcal{A})).$$

Thus it is clear that

$$\lim_{n\to\infty} \frac{\|(a_1m,\dots,a_nm)\|_n^{(p,q)}}{n^{\frac{1}{q}}} = \lim_{n\to\infty} \frac{||m||}{n^{\frac{1}{q}}} = 0,$$

for all sequence (a_n) in $P_1(\mathcal{A})$.

Now, suppose that (b) hold , then by Theorem 3.10 of [6]

$$\mathcal{A}m = \{am \ : \ a \in P_1(\mathcal{A})\}$$

is a almost (p, q)-multi bounded set. Therefore Theorem 4.5 of [6] shows that $\mathcal{A}m$ is relatively weakly compact, and also, by the $Krein - Smulian$ Theorem, the following set is weakly compact

$$K := \overline{co(\mathcal{A}m)} = \overline{\left\{\sum_{i=1}^{n} t_i a_i m : 0 \leq t_i \leq 1, \sum_{i=1}^{n} t_i = 1, a_i \in P_1(\mathcal{A})\right\}}.$$

Now, let $\Sigma = \{L_b : b \in P_1(\mathcal{A})\}$ be a semigroup of affine maps from the weakly compact convex set K into itself defined by

$$L_b(\Lambda) = b\Lambda \quad (b \in P_1(\mathcal{A}), \Lambda \in K).$$

L_b is isometric map since, for every $b \in P_1(\mathcal{A})$

$$\begin{aligned} \|L_b(\Lambda)\| &= \|b\Lambda\| \\ &= \|\Lambda\|. \end{aligned}$$

Hence, by the $Ryll - Nardzewski$ fixed point Theorem, there exists $m_0 \in K$ which is a common fixed point for the set Σ such that $L_b(m_0) = m_0$. This implies that m_0 is a topological left invariant mean in $\mathcal{A}$. □

In the following, we apply the result of Proposition 2.1 for the group algebras.
Let G be a locally compact group with left Haar measure λ_G and let $L^1(G) = L^1(G, \lambda_G)$ be the group algebra of G endowed with the norm $\|.\|_1$ and the convolution product $*$ given by

$$(\varphi * \psi)(s) = \int_G \varphi(t)\psi(t^{-1}s)d\lambda_G(t) \quad (s \in G),$$

where $\varphi, \psi \in L^1(G)$ and the integral is defined for almost all $s \in G$.

equivalent the concept of left contractible of the group algebra $L^1(G)$ with the compression of G that has been shown in [[5], Theorem 6.1].

Corollary 2.2. *Let G be a locally compact group and $1 \leq p, q < \infty$. Then the following statements are equivalent:*

(a) *G is compact;*

(b) *There is a mean* $m \in P_1(L^1(G))$ *such that* $\lim_{n\to\infty} \frac{\|(f_1 m,\ldots,f_n m)\|_n^{(p,q)}}{n^{\frac{1}{q}}} = 0$, *for all sequence* (f_n) *in* $L^1(G)$.

PROOF. The result follows immediately from Proposition 2.1 and the fact that $L^1(G)$ is left contractible if and only if G is compact, see Theorem 6.1 of [5]. □

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Amenability and Super-amenability of generalized Feichtinger algebras

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Abstract. Let G be a locally compact group (not necessarily abelian) and B be a homogeneous Banach space on G, which is in a good situation with respect to a homogeneous function algebra on G. Feichtinger showed that there exists a minimal Banach space B_{min} in the family of all homogenous Banach spaces C on G, containing all elements of B with compact support. In this paper, we study the amenability and super amenability of B_{min}.

Keywords: Amenability, Super-Amenability ,Homogenous Banach space, Homogeneous function algebra, IN-group)

AMS Mathematics Subject Classification [2010]: [2020]Primary 47J30; Secondary 30H05, 46A18.

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1. Introduction

In 1980, Feichtinger [1] introduced an especial Banach space $W(B, C)$, for any Banach space B on G that is in a good situation with respect to a homogeneous function algebra $\mathcal{A}$ and any solid, translation invariant Banach space C on G.
Afterwards, Feichtinger [2] showed that if B is a homogeneous Banach space on G, which is in a good situation, then there exists a minimal space B_{min} in the family of all homogeneous Banach spaces C on G that contains all elements of B with compact support. Moreover $B_{min} = W(B, L^1(G))$, whenever B is in a good situation with respect to a homogeneous function algebra; see [2].

We recall some known concepts and frameworks, which will be used throughout the paper. Let G be a locally compact Hausdorff group with the fixed Haar measure λ and $L^\infty(G)$ be the Banach algebra of all essentially bounded Borel-measurable functions on G. Moreover, for $0 < p < \infty$, $L^p(G)$ indicates the usual Lebesgue spaces, as defined in [5]. Also $L_{loc}(G)$ is the space, consisting of all (equivalent classes of) measurable functions f on G such that $f\chi_K \in L^1(G)$, for any compact subset $K \subseteq G$. Let $C_0(G)$ (resp. $C_b(G), C_c(G)$) be the space of all continuous, complex valued functions on G vanishing at infinity (resp. bounded, with compact support). Moreover, $A(G)$ and $B(G)$ are the

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Fourier and Fourier-Stieltjes algebra of G, as defined in [1]. Also, $A_p(G)$ are The Figà-Talamanca-Herz algebras for $1 < p < \infty$; see [1]. Let $\mathcal{L}A(G) := A(G) \cap L^1(G)$ be the Lebesgue-Fourier algebra of G; see [3].

We say that a Banach space $(B, \|.\|_B)$ is a BF-space if it is continuously embedded into $L_{loc}(G)$, that is the space consisting of all locally integrable functions on G. Indeed, for each compact subset K of G there exists some constant $C_K > 0$ such that

$$\|f\chi_K\|_1 \leq C_K\|f\|_B,$$

for each $f \in B$ with $\mathrm{supp}(f) \subseteq K$. Further, let

$$B_\Omega = \{f \in B : \text{ supp(f) is compact}\},$$

and

$$B_K := \{f \in B : supp(f) \subseteq K\},$$

for any $K \subseteq G$. We say that $f \in B_K$ almost every where (for abbreviation a.e) if $supp(f) \subseteq K$ a.e on G. For the remainder, we emphasize that two measurable functions are identifying if they are equal locally almost everywhere (for abbreviation l.a.e) on G, i.e. they are equal a.e on any compact subset $K \subseteq G$. A BF-space on G is called left translation invariant if $L_yf \in B$ for any $y \in G$ and $f \in B$, where $L_yf(x) = f(y^{-1}x)$ $(x \in G)$. A left invariant BF-space will be called a homogenous Banach space on G if the following conditions are satisfied:

(i) $\|L_yf\|_B = \|f\|_B$ for any $y \in G$;
(ii) the map $y \to L_yf$ from G into $(B, \|.\|_B)$ is continuous for all $f \in B$.

Moreover, we say that G acts by right translation isometrically on B and right translation is continuous in B, if $R_xf \in B$ and $\|R_xf\|_B = \|f\|_B$ for all $x \in G$ and $f \in B$, and also $\lim_{y\to e}\|R_yf - f\|_B = 0$ for all $f \in B$. A Banach algebra $(A, \|.\|_A) \subseteq C_b(G)$ with the pointwise multiplication is a homogenous function algebra on G, if

(i) $\mathcal{A}$ is a homogeneous Banach space on G;
(ii) $\mathcal{A}$ separates the points of G from closed sets;
(iii) $\mathcal{A}$ is a self-adjoint Banach algebra with respect to complex conjugate.

Let G be a locally compact group and B be a homogenous Banach space on G that is in a good situation with respect to a homogenous function algebra $\mathcal{A}$. Then, there exists a minimal space B_{min} in the family of all homogeneous Banach spaces C satisfying $B_\Omega \subseteq C$. Also, B_{min} can be characterized as follows. Let an open, relatively compact set $Q \subseteq G$ be given. $f \in B_{min}$ if and only if $f = \sum_n L_{y_n}f_n$ for some $y_n \in G$ and $f_n \in B$ such that $\sum_n \|f_n\|_B < \infty$ and $supp(f_n) \subseteq y_nQ$ for each $n \in N$ and

$$\|f\|_{min} = inf\{\sum_n \|f_n\|_B : f = \sum_n L_{y_n}f_n\}, \tag{1}$$

where the infimum is taken over all representations of f as in (1.1). The space $B \cap C_c(G)$ is a dense subspace of B_{min}. If B contains positive elements with arbitrary small support, i.e. for any open subset U of G, there exists a (nonzero) positive element $f \in A$ such that $\mathrm{supp}(f) \subseteq U$, then B_{min} is a Segal algebra; see [2]. The space B_{min} is independent of the choice of Q; see [2, Theorem 3]).

Several authors have introduced various notions of amenability in the recent years; see [5]. The first notion of amenability has been presented for locally compact groups. Let G be a locally compact group, then G is called amenable if there exists a functional $m \in L^\infty(G)^*$ such that

(i) $\langle 1, m\rangle = \|m\| = 1$;

(ii) $\langle L_x\varphi, m\rangle = \langle \varphi, m\rangle$, for all $\varphi \in L^\infty(G)$ and $x \in G$

This definition has been generalized for the Banach algebras, as follows. Let $\mathcal{A}$ be a Banach algebra and E be a Banach $\mathcal{A}$-bimodule, then E^*, the dual space of E, becomes a Banach $\mathcal{A}$-bimodule via

$$< x, \varphi.a > := < a.x, \varphi >, \quad < x, a.\varphi > := < x.a, \varphi >$$

for any $a \in \mathcal{A}$ and $\varphi \in E^*$. A derivation from $\mathcal{A}$ into E is a bounded linear map $D : \mathcal{A} \to E$ satisfying

$$D(a.b) = a.D(b) + D(a).b \quad (a, b \in \mathcal{A}).$$

The derivation D is called inner, if there is $x \in E$ such that $D(a) = a.x - x.a$, for any $a \in \mathcal{A}$. We now say that $\mathcal{A}$ is amenable if each continuous derivation from $\mathcal{A}$ into E^* is inner for all Banach $\mathcal{A}$-bimodules E. Also, $\mathcal{A}$ is called super-amenable if every continuous derivation from $\mathcal{A}$ into E is inner for every Banach A-bimodule E. Note that every amenable Banach algebra has a bounded approximate identity and every super-amenable Banach algebra is unital. Hence, $L^1(G)$ is super-amenable if and only if G is finite; see [5].

2. Main results

In this section, we proceed to the main results of the paper. For the Lebesgue Fourier algebras, the following results were studied by Ghahramani and Lau [5]. We now extend those to some Feichtinger algeras.

THEOREM 2.1. *Let G be a locally compact group, $1 < p < \infty$ and B be a homogenous Banach space on G which is in a good situation with respect to a homogeneous function algebra on G. Moreover, suppose that B contains positive elements with arbitrary small support. Moreover, let $B_\Omega \subseteq L^p(G)$. Then, B_{min} is amenable with respect to the convolution product if and only if G is discrete and amenable.*

PROOF. Let B_{min} be amenable, with the convolution product. Then, B_{min} has a bounded approximate identity. Also, B_{min} is a Segal algebra. It follows that $B_{min} = L^1(G)$. On the other hand, there exists $C_Q > 0$ such that

$$\|f\|_1 \le C_Q \|f\|_B,$$

for any $f \in B$ with supp$f \subseteq Q$. Thus,

$$\|f\|_1 \le C_Q \|f\|_{min},$$

for any $f \in B_{min}$. Therefore, the inclusion map

$$j : B_{min} \to L^1(G)$$

is a continuous homomorphism. Thus, G is amenable by [5, Theorem 2.1.8] and [5, Proposition 2.3.1]. Also, $B_{min} \subseteq L^p(G)$. Consequently, $L^1(G) \subseteq L^p(G)$, which implies the discreteness of G.

Conversely, let G be discrete and amenable. Hence, $L^1(G)$ with the convolution product is amenable by [5, Theorem 2.1.8] and there exists a nonzero positive element $f \in B$ such that $supp(f) \subseteq \{e\}$. Thus, $\delta_e = \frac{f}{f(e)} \in B$ and consequently, $L^1(G)_\Omega \subseteq B$. Thus, there exists $C_Q > 0$ such that

$$\|\delta_e\|_B \le C_Q \|\delta_x\|_1 = C_Q \lambda(\{e\}),$$

for any $x \in G$ by considering $Q := \{e\}$. Take $f \in L^1(G)$ to be arbitrary. Then,

$$f = \sum a_n \delta_{x_n} = \sum L_{x_n}(a_n \delta_e)$$

such that $\sum |a_n| < \infty$. Thus,

$$\sum \|L_{x_n}(a_n\delta_e)\|_B = \sum \|a_n\delta_e\|_B \leq C_Q\lambda(\{e\}) \sum |a_n| < \infty.$$

It follows that $f \in B_{min}$. Again by considering $Q := \{e\}$, we have $L^1(G) \subseteq B_{min}$. Thus, $B_{min} = L^1(G)$ and so the inclusion map $j : L^1(G) \to B_{min}$ is a continuous homomorphism, by applying the open mapping theorem. Therefore, B_{min} is amenable; see [**5**, Proposition 2.3.1]. □

Corollary 2.2. *Let G be a locally compact group, $1 < p < \infty$ and B be a homogenous Banach space on G which is in a good situation with respect to a homogeneous function algebra on G. Moreover, suppose that B contains positive elements with arbitrary small support. Furthermore, let $B_\Omega \subseteq L^p(G)$. Then, B_{min} is super-amenable with respect to the convolution product if and only if G is finite.*

Corollary 2.3. *Let G be a locally compact group and $\mathcal{A}$ is a homogeneous function algebra on G. Then, $\mathcal{A}_{min}$ is amenable with respect to the convolution product if and only if G is discrete and amenable.*

Proof. This is due to $\mathcal{A} \subseteq L^2(G)$ and Theorem 2.6. □

Corollary 2.4. *Let G be a locally compact group and A is a homogeneous function algebra on G. Then, A_{min} is super-amenable with respect to the convolution product if and only if G is finite.*

Proof. This is due to $\mathcal{A} \subseteq L^2(G)$ and Corollary 2.7. □

By using of the above assertions, the following result is immediate.

Corollary 2.5. *Let G be a locally compact group and $1 < p < \infty$. Then,*
(i) $L^p(G)_{min}$, $C_0(G)_{min}$ and $A(G)_{min}$ are amenable with the convolution product if and only if G is discrete and amenable;
(ii) $L^p(G)_{min}$, $C_0(G)_{min}$, and $A(G)_{min}$ are super-amenable with the convolution product if and only if G is finite.

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۸ تا ۱۱ شهریور ۱۴۰۰

Fixed points of expansive mappings in modular G-metric spaces

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Abstract. In this paper, we define the concept of modular G-metric spaces introduced by Azadifar and Maramaei [1]. Further, we prove some fixed point theorems for expansive mappings in modular G-metric spaces.

Keywords: Modular G-metric space; expansive mapping; G-metric space; fixed point.

AMS Mathematics Subject Classification [2010]: 47H10,46A80

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1. Introduction

In 1992, Dhage [4] in his PhD thesis introduced a new class of generalized metric space, which called D-metric spaces (see [5, 6] for more details). In 2003, Mustafa and Sims [8] showed that most of the claims concerning the fundamental topology of Dhage's D-metric spaces are incorrect. They [9] also introduced a valid generalization of metric space (X, d), which they called G-metric spaces.

Definition 1.1 ([9]). Let X be a nonempty set, and $G : X \times X \times X \to [0, +\infty)$ be a function satisfying:

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$; for any $x, y \in X$, with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$; for any $x, y, z \in X$ with $z \neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, (symmetry in any three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for any $x, y, z, a \in X$, (rectangle inequality),

then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is a G-metric space.

Clearly these properties are satisfied when $G(x, y, z)$ is the perimeter of the triangle with vertices at x, y and z in $\mathbb{R}^2$, further taking a in the interior of the triangle shows that $(G5)$ is best possible.

The notion of a modular space on a linear space, as a generalization of metric space, was introduced and studied by Nakano [10].

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In 2010, Chistyakov [2, 3] introduced the notion of a modular metric space, as a generalization of a metric space.

Definition 1.2 ([3]). Let X be a nonempty set, and let $w : (0, \infty) \times X \times X \to [0, \infty]$ be a function satisfying:

(i) given $x, y \in X, w_\lambda(x, y) = 0$ for any $\lambda > 0$ if and only if $x = y$,
(ii) $w_\lambda(x, y) = w_\lambda(y, x)$ for any $x, y \in X$ and $\lambda > 0$,
(iii) $w_{\lambda+\mu}(x, y) \leq w_\lambda(x, z) + w_\mu(z, y)$ for any $\lambda, \mu > 0$ and $x, y, z \in X$,

then the function w_λ is called a modular metric on X.

The main idea behind this new concept is the physical interpretation of the modular. Modular w allow different interpretations. A natural modular on a metric space (X, d) is given by $w_\lambda(x, y) = d(x, y)/\lambda$, which $w_\lambda(x, y)$ is the real average velocity between points x and y in time $\lambda > 0$.

In 2013, Azadifar and Maramaei [1] introduced the notion of modular G-metric spaces and proved some known fixed point theorems on the modular G-metric spaces.

Definition 1.3 ([1]). Let X be a nonempty set, and let $w : (0, \infty) \times X \times X \times X \to [0, \infty]$ be a function satisfying:

(W1) $w_\lambda(x, y, z) = 0$ for any $\lambda > 0$ if $x = y = z$,
(W2) $w_\lambda(x, x, y) > 0$ for any $x, y \in X$ and $\lambda > 0$ with $x \neq y$,
(W3) $w_\lambda(x, x, y) \leq w_\lambda(x, y, z)$ for any $x, y, z \in X$ and $\lambda > 0$ with $z \neq y$,
(W4) $w_\lambda(x, y, z) = w_\lambda(x, z, y) = w_\lambda(y, z, x) = ...$, for any $\lambda > 0$ (symmetry in any three variables),
(W5) $w_{\lambda+\mu}(x, y, z) \leq w_\lambda(x, a, a) + w_\mu(a, y, z)$ for any $x, y, z, a \in X$ and $\lambda, \mu > 0$,

then the function w_λ is called a modular G-metric on X.

Remark 1.4 ([1]). Note that for $x, y, z \in X$ the function $0 < \lambda \to w_\lambda(x, y, z) \in [0, \infty]$ is nonincreasing on $(0, \infty)$. Suppose $0 < \mu < \lambda$, then $(W1)$ and $(W5)$ imply

$$w_\lambda(x, y, z) \leq w_{\lambda-\mu}(x, x, x) + w_\mu(x, y, z) = w_\mu(x, y, z).$$

It follows that each point $\lambda > 0$ the right limit $w_{\lambda+0}(x, y, z) = lim_{\mu\to\lambda+0} w_\mu(x, y, z)$ and left limit $w_{\lambda-0}(x, y, z) = lim_{\varepsilon\to 0} w_{\lambda-\varepsilon}(x, y, z)$ exist in $[0, \infty)$ and following two inequalities hold:

$$w_{\lambda+0}(x, y, z) \leq w_\lambda(x, y, z) \leq w_{\lambda-0}(x, y, z).$$

Definition 1.5 ([1]). Let w be a modular G-metric on a set X. Then the binary relation $\sim$ on X defined for $x, y, z \in X$ by

$$x \sim y \text{ if and only if } \lim_{\lambda\to\infty} w_\lambda(x, y, z) = 0 \quad \text{for some } z \in X$$

is an equivalence relation. Denote by $\frac{X}{\sim}$ the quotient-set of X with respect to $\sim$ and for a fixed element $x_0 \in X$ set

$$X_w = X_w(x_0) = \{y \in X : y \sim x_0\}$$

Definition 1.6 ([1]). Let (X, w) be a modular G-metric space then for $x_0 \in X_w$ and $r > 0$, the w-bany with center x_0 and radius $r > 0$ is

$$B_w(x_0, r) = \{y \in X_w : w_\lambda(x_0, y, y) < r \text{ for any } \lambda > 0\}.$$

Definition 1.7 ([1]). Let (X, w) be a modular G-metric space.

(i) The sequence $\{x_n\}$ in X_w is said to be w-convergent if for any $\varepsilon > 0$, there exist $x \in X_w$ and $n_\varepsilon \in \mathbb{N}$ such that $w_\lambda(x, x_n, x_m) < \varepsilon$, for any $n, m \geq n_\varepsilon$ and $\lambda > 0$.
(ii) The sequence $\{x_n\}$ in X_w is said to be w-Cauchy if for any $\varepsilon > 0$, there exist $n_\varepsilon \in \mathbb{N}$ such that $w_\lambda(x_n, x_m, x_l) < \varepsilon$, for any $n, m, l \geq n_\varepsilon$ and $\lambda > 0$.
(iii) X_w is said to be w-complete if every w-Cauchy in X_w is a w-convergent sequence in X_w.

PROPOSITION 1.8 ([1]). *Let (X, w) be a modular G-metric space, then for any $x_0 \in X_w$ and $r > 0$, we have:*
(1) *if $w_\lambda(x_0, x, y) < r$, for any $\lambda > 0$, then $x, y \in B_w(x_0, r)$,*
(2) *if $y \in B_w(x_0, r)$, then there exists a $\delta > 0$ such that $B_w(y, \delta) \subseteq B_w(x_0, r)$.*

PROPOSITION 1.9 ([1]). *Let (X, w) be a modular G-metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X_w . Then the following are equivalent:*
(1) *$\{x_n\}$ is w-convergent to x,*
(2) *$w_\lambda(x_n, x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$ for any $\lambda > 0$,*
(3) *$w_\lambda(x_n, x, x) \longrightarrow 0$ as $n \longrightarrow \infty$ for any $\lambda > 0$,*
(4) *$w_\lambda(x_m, x_n, x) \longrightarrow 0$ as $m, n \longrightarrow \infty$ for any $\lambda > 0$.*

The concept of expansive mappings in G-metric spaces is introduced by Mustafa and Awawdeh [11]. They defined the concept of expansive mappings for single valued mappings in G-metric spaces and proved the existence of fixed points.

DEFINITION 1.10 ([11]). Let (X, G) be a G-metric space and T be a self mapping on X. Then T is called expansive mapping if there exists a constant $a > 1$ such that for all $x, y, z \in X$, we have

$$G(Tx, Ty, Tz) \geq aG(x, y, z).$$

The following example shows that expansive mapping on G-metric space need not be G-continuous.

EXAMPLE 1.11 ([11]). Let $T : (\mathbb{R}, G) \to (\mathbb{R}, G)$ be defined by

$$Tx = \left\{ \begin{array}{ll} 5x \quad ; & x \leq 3 \\ 5x + 2; & x > 3 \end{array} \right\} \tag{1}$$

where $G(x, y, z) = \max\{|x - y|, |y - z|, |x - z|\}$. Then (R, G) is a complete G-metric space and T is expansive mapping where T is not G-continuous.

THEOREM 1.12 ([11]). *Let (X, G) be a complete G-metric space. If there exists a constant $a > 1$ and a surjective self mapping T on X, such that for all x, y, zX*

$$G(Tx, Ty, Tz) \geq aG(x, y, z),$$

then T has a unique fixed point.

In this article, we will prove some fixed point theorems for expansive mappings in modular G-metric spaces.

2. Main Results

THEOREM 2.1. *Let X_w be a complete modular G-metric space. If there exists a constant $a > 1$ and a surjective self mapping T on X_w, such that*

$$w_\lambda(Tx, Ty, Tz) \geq aw_\lambda(x, y, z)$$

for all $x, y, z \in X_w$ and $\lambda > 0$, then T has a unique fixed point.

Theorem 2.2. *Let X_w be a complete modular G-metric space. If there exists a constant $c > 1$ and a surjective self mapping T on X_w, such that*

$$w_\lambda(Tx, Ty, Ty) \geq cw_\lambda(x, y, y)$$

for all $x, y \in X_w$ and $\lambda > 0$, then T has a unique fixed point.

Corollary 2.3. *Let X_w be a complete modular G-metric space. If there exists a constant $k > 1$ and a surjective self mapping on X_w, such that*

$$w_\lambda(Tx, Ty, Tz) \geq k\{w_\lambda(x, z, z) + w_\lambda(y, z, z)\} \tag{2}$$

for all $x, y, z \in X_w$ and $\lambda > 0$, then T has a unique fixed point.

Proof. Follows from Theorem (2.2), by taking $z = y$ in Condition (2).

Theorem 2.4. *Let X_w be a complete modular G-metric space and let $T : X_w \to X_w$ be an onto mapping satisfying the following condition for all $x, y, z \in X_w$ and $\lambda > 0$,*

$$w_\lambda(Tx, Ty, Tz) \geq aw_\lambda(x, y, z) + bw_\lambda(x, x, Tx) + cw_\lambda(y, y, Ty) + dw_\lambda(z, z, Tz) \tag{3}$$

where $a + b + c + d > 1$ and $b + c < 1$. Then T has a fixed point.

Corollary 2.5. *Let X_w be a complete modular G-metric space and let $T : X_w \to X_w$ be an onto mapping satisfying the following condition for all $x, y, z \in X_w$ and $\lambda > 0$,*

$$w_\lambda(Tx, Ty, Tz) \geq \alpha w_\lambda(x, y, z) + \beta\{w_\lambda(x, x, Tx) + w_\lambda(y, y, Ty) + w_\lambda(z, z, Tz)\} \tag{4}$$

where $\alpha + 3\beta > 1$ and $\beta < 1/2$. Then T has a fixed point.

Proof. In Theorem (2.4), If $a = \alpha$, and $b = c = d = \beta$, then the condition (3) reduced to Condition (4), so the proof follows from Theorem (2.4).

Theorem 2.6. *Let X_w be a complete modular G-metric space and let $T : X_w \to X_w$ be an onto mapping satisfying the following condition for all $x, y, z \in X_w$ and $\lambda > 0$,*

$$w_\lambda(Tx, Ty, Tz) \geq k \max\{w_\lambda(x, Tx, Tx), w_\lambda(y, Ty, Ty), w_\lambda(z, Tz, Tz)\}$$

where $k > 1$. Then T has a fixed point.

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On the essential graph of a finite commutative ring

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Abstract. Let R be a commutative ring. The essential graph of R, denoted by $EG(R)$ is a graph associated to R with vertex set $Z(R) \setminus \{0\}$ and a pair of distinct vertices x and y are adjacent if and only if $\text{Ann}(xy)$ is an essential ideal of R. In this paper, we calculate the domination, the clique and the chromatic numbers of the essential graph of a finite commutative ring.

Keywords: essential graph, domination number, clique number, chromatic number.

AMS Mathematics Subject Classification [2010]: 13Cxx , 05C25

1. Introduction

The idea of a zero-divisor graph of a commutative ring was introduced by I. Beck in [2], where he was mainly interested in colorings. Let R be a commutative ring and let $Z(R)$ denote the set of zero-divisors of R. The zero-divisor graph of R, denoted by $\Gamma(R)$ is a simple graph with vertex set $Z(R)^*$, the set of nonzero zero-divisors of R, and distinct vertices x and y are adjacent if and only if $xy = 0$, see [1]. The main object for introducing the zero divisor graph is to study the interplay between of ring theoretic properties of R and graph theoretic properties of $\Gamma(R)$. The essential graph of R, denoted by $EG(R)$ is a simple graph associated to R whit vertex set $Z(R)^*$ and two distinct vertices x and y are adjacent if and only if $\text{Ann}(xy)$ is an essential ideal of R, see [4]. In this paper, we calculate the domination, the clique and the chromatic numbers of the essential graph of a finite commutative ring.

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. We write $u - v$, to denote an edge with ends u, v. Also G is called an empty graph if it has no any vertices. Recall that a graph is connected if there exists a path connecting between any two distinct vertices. The distance between two distinct vertices x and y, denoted by $\text{d}(x, y)$, is the length of the shortest path connecting them (if such a path does not exist, then we set $\text{d}(x, y) = \infty$). The diameter of a connected graph G, denoted by $\text{diam}(G)$, is the maximum distance between any pair of vertices of G. A graph in which each pair of vertices is joined by an edge is called a complete graph.

Throughout this paper, all rings are assumed to be commutative with nonzero identity. We denote by $\text{Max}(R)$ and $\text{Nil}(R)$, the set of all maximal ideals of R and the set of all

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nilpotent elements of R, respectively. For every ideal I of R, we denote the annihilator of I by $\mathrm{Ann}(I) = \{r \in R : rI = 0\}$. A nonzero ideal I of R is called essential, if has a nonzero intersection with any nonzero ideal of R.

2. Domination, clique and chromatic numbers of essential graph

In this section, we will study the domination, the clique and the chromatic numbers of the essential graph of a finite commutative ring. Let R be a finite commutative ring. Then R is Noetherian with finite maximal ideals. Suppose that $\mathrm{Spec}(R) = \mathrm{Max}(R)\{M_1, \cdots, M_k\}$, where k is a positive integer.

Lemma 2.1. *Let (R, M_1) be a finite local ring. Then $EG(R)$ is a complete graph and $\Gamma(R)$ has a universal vertex so* $\mathrm{diam}(\Gamma(R)) \leq 2$.

Proof. Suppose that R is a finite local ring with maximal ideal M_1. If $M_1 = 0$, then $EG(R) = \Gamma(R)$ is an empty graph. Now, suppose that $M_1 \neq 0$. In this case $Z(R) = \mathrm{Nil}(R) = M_1$ hence $EG(R)$ is a complete graph, by [6, Lemma 2.2]. Moreover, $M_1 \in \mathrm{Ass}(R)$ so there is a nonzero element $x \in R$ such that $M_1 = \mathrm{Ann}(x)$. Thus for each $y \in Z(R)^*$ we have $xy = 0$ thus x is a universal vertex for $\Gamma(R)$ and $\mathrm{diam}(\Gamma(R)) \leq 2$. □

Theorem 2.2. *Let (R, M_1) be a finite local ring. Then $\Gamma(R)$ is a complete graph if and only if $M_1^2 = 0$. Hence,* $\mathrm{diam}(\Gamma(R)) = 2$ *if and only if $M_1^2 \neq 0$.*

Proof. $\Rightarrow$) Suppose that $\Gamma(R)$ is a complete graph and x, y are two distinct vertices of $\Gamma(R)$. By the hypotheses $xy = 0$, $x - y \in Z(R)^*$ and $x \neq x - y$. Thus $x(x - y) = 0$ and so $x^2 = 0$. Hence, for each $x \in Z(R)$, $x^2 = 0$ and therefore $M_1^2 = 0$.
$\Leftarrow$) It is obvious since $Z(R) = M_1$.
The second assertion is the contrapositive of the first one. □

A dominating set of a graph G is a subset D of $V(G)$ such that every vertex in $V(G) \setminus D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. So by the above results for a finite local ring R we have $\gamma(EG(R)) = \gamma(\Gamma(R)) = 1$.

Theorem 2.3. *Let R be a commutative Noetherian ring such that $|\mathrm{Ass}(R)| = k \geq 2$. Then*

$$\gamma(EG(R)) = \begin{cases} 1 & \mathrm{Nil}(R) \neq 0 \\ |\mathrm{Ass}(R)| & \mathrm{Nil}(R) = 0. \end{cases}$$

Proof. Suppose that $\mathrm{Nil}(R) \neq 0$. Then every nonzero element of $\mathrm{Nil}(R)$ is a universal vertex of $EG(R)$, by [6, Lemma 2.2], so $\gamma(EG(M)) = 1$.

Now, suppose that $\mathrm{Nil}(R) = 0$ and $\mathrm{Ass}(R) = \{P_1, \cdots, P_k\}$. In this case, by [6, Theorem 2.5] we have $EG(R) = \Gamma(R)$ and every elements of $\mathrm{Ass}(R)$ is a minimal prime ideal of R. Let $x_i \in \bigcap_{j=1, j\neq i}^{k} P_j \setminus P_i$, for all i with $1 \leq i \leq k$ and $D = \{x_1, \cdots, x_k\}$. We show that D is a dominating set for $EG(R)$. Assume that $x \in Z(R) = \bigcup_{i=1}^{k} P_i$. Then $x \in P_t$ for some t with $1 \leq t \leq k$. Then $xx_t \in \bigcap_{i=1}^{k} P_i = \mathrm{Nil}(R) = 0$. So x, x_t are adjacent in $\Gamma(R)$ and are adjacent in $EG(R)$. Hence, D is a dominating set for $EG(R)$. Now, suppose that $D' = \{x'_1, \cdots, x'_{k-1}\}$ is a dominating set for $EG(R)$. By the assumption for every $1 \leq j \leq k-1$ there is $1 \leq i \leq k$ such that $x'_j \notin P_i$ otherwise $x'_j \in \bigcap_{i=1}^{k} P_i = 0$ which is a contradiction. Without loss of generality we may assume that $x'_j \notin P_j$, for all j with $1 \leq j \leq k-1$. Assume that $x' \in P_k \setminus \bigcup_{i=1}^{k-1} P_i$. Thus x' is not adjacent with x'_j, for all $1 \leq j \leq k-1$. Since $x' - x'_j$ implies that $x'x'_j = 0 \in P_j$ which is a contradiction. Therefore, a set with leas than k elements can not be a dominating set for $EG(R)$. Hence, $\gamma(EG(M)) = k = |\mathrm{Ass}(R)|$. $\square$

Corollary 2.4. *Let R be a finite commutative ring such that $|\mathrm{Max}(R)| = k \geq 2$. Then*

$$\gamma(EG(R)) = \begin{cases} 1 & \mathrm{Jac}(R) \neq 0 \\ |\mathrm{Max}(R)| & \mathrm{Jac}(R) = 0. \end{cases}$$

Proof. It is an immediate consequence of Theorem 2.3. $\square$

A clique of G is a complete subgraph of G and the number of vertices in a largest clique of G, denoted by $\omega(G)$, is called the clique number of G. For a finite local ring R with maximal ideal M_1 we have $\omega(EG(R)) = |M_1^*|$ and $\omega(\Gamma(R)) = |M_1^*|$ whenever $M_1^2 \neq 0$.

Theorem 2.5. *Let R be a finite commutative ring such that $|\mathrm{Max}(R)| = k \geq 2$ and let $EG(R)$ be a connected graph. Then*

$$\omega(EG(R)) = \begin{cases} k & \mathrm{Jac}(R) = 0 \\ |\mathrm{Jac}(R)| - 1 & Z(R) = \mathrm{Jac}(R) \neq 0 \\ |\mathrm{Jac}(R)| + k - 1 & Z(R) \neq \mathrm{Jac}(R) \neq 0. \end{cases}$$

Proof. Suppose that $\mathrm{Max}(R) = \{M_1 = \mathrm{Ann}(x_1), \cdots, M_k = \mathrm{Ann}(x_k)\}$, where $x_i \in Z(R)^*$, for all $1 \leq i \leq k$ and $\mathrm{Jac}(R) = 0$. Then $\Gamma(R) = EG(R)$ and $X = \{x_1, \cdots, x_k\}$ is a clique for $EG(R)$, see [6, Theorem 2.5] and [5, Lemma 3.1 (i)]. Assume $X' = \{x'_1, \cdots, x'_n\}$ ($n > k$) is a maximal clique. Thus there are $1 \leq i \neq j \leq n$ such that $\mathrm{Ann}(x'_i), \mathrm{Ann}(x'_j) \subseteq \mathrm{Ann}(x_t)$, for some $1 \leq t \leq k$, see [3, Theorem 3.6]. Since x'_i, x'_j are adjacent so $x'_i x'_j = 0$. Hence, $x'_i R \subseteq \mathrm{Ann}(x'_j) \subseteq \mathrm{Ann}(x_t)$. Therefore, $x_t x'_i = 0$ so $x_t R \subseteq \mathrm{Ann}(x'_i)$. Thus $x_t^2 = 0$ which is a contradiction. Therefore, $\omega(EG(R)) = |\mathrm{Max}(R)|$.

Let $\mathrm{Jac}(R) \neq 0$ and $Z(M) = \mathrm{Jac}(R)$. Then by [6, Theorem 2.5], $EG(R)$ is a complete graph. Hence, $\mathrm{Jac}(R)^*$ is a clique and the result follows.

Now, assume that $Z(R) \neq \mathrm{Jac}(R)$ and $\mathrm{Max}(R) = \{M_1, \cdots, M_k\}$. It is easy to see that $\mathrm{Jac}(R)^* \cup \{x_1, \cdots, x_k\}$ is a clique for $EG(R)$, where $x_i \in \bigcap_{j=1, j\neq i}^{k} M_j \setminus M_i$, for all $i = 1, \cdots, k$. Moreover, if $y \in Z(R) \setminus \mathrm{Jac}(R)$, then the set $\mathrm{Jac}(R)^* \cup \{x_1, \cdots, x_k, y\}$ is not a clique since for $y \in \bigcap_{i=1}^{t} M_i \setminus \cup_{i=t+1}^{k} M_i$ with $1 \leq t < k$ it is clear that $yx_{t+1} \notin \mathrm{Jac}(R)$ so y and x_{t+1} are not adjacent. Suppose that X is a clique for $EG(R)$. Thus in view of [6, Theorem 2.5], $\mathrm{Jac}(R)^* \subseteq X$. Let $X = \mathrm{Jac}(R)^* \cup \{y_1, \cdots, y_{k+1}\}$. Then there are $1 \leq i \neq j \leq k+1$ such that $y_i, y_j \notin M_t$, for some $1 \leq t \leq k$, but $y_i y_j \in \mathrm{Jac}(R) \subseteq M_t$ which is a contradiction. Therefore, X is not a clique. $\square$

The chromatic number of G, denoted by $\chi(X)$, is the minimal number of colours, which can assigned to the vertices of G in such a way that two adjacent vertices have different colours.

Theorem 2.6. *Let R be a finite commutative ring such that $|\mathrm{Max}(R)| = k \geq 2$ and let $EG(R)$ be a connected graph. Then*

$$\chi(EG(R)) = \begin{cases} |\mathrm{Jac}(R)| - 1 & Z(R) = \mathrm{Jac}(R) \neq 0 \\ |\mathrm{Jac}(R)| + k - 1 & Z(R) \neq \mathrm{Jac}(R) \neq 0 \\ k & \mathrm{Jac}(R) = 0. \end{cases}$$

Proof. (i) In this case $EG(R)$ is a complete graph thus $\chi(EG(R)) = \omega(EG(R) = \mid \mathrm{Jac}(R) \mid -1$.

(ii) Suppose that $X = \mathrm{Jac}(R)^* \cup \{x_1, \cdots, x_k\}$ is a maximal clique for $EG(R)$. So $\mid \mathrm{Jac}(R)^* \mid +k \leq \chi(EG(M))$. Assume that $Y = \mathrm{Jac}(R)^* \cup \{y_1, \cdots, y_n\}$ is a clique for $EG(R)$. Thus $n \leq k$. For all j with $1 \leq j \leq n$ there is i with $1 \leq i \leq k$ such that y_j and x_i are not adjacent, otherwise $X' = \mathrm{Jac}(R)^* \cup \{y_1, x_1, x_2, \cdots, x_k\}$ is a clique which is a contradiction. Without loss of generality, we may assume y_1 and x_1 are not adjacent. Hence, y_1 and x_1 have same colour. Now, assume that $j = 2$ there is i with $1 \leq i \leq k$ such that y_2 and x_i are not adjacent, we have $i \neq 1$. If $i = 1$, then $X'' = \mathrm{Jac}(R)^* \cup \{y_1, y_2, x_2, \cdots, x_k\}$ is a clique which is a contradiction. Hence, we may assume y_2 and x_2 are not adjacent. Therefore, y_2 and x_2 have same colour. If we continue this procedure, then we obtain that the elements of $\{y_1, \cdots, y_n\}$ have same colours with $\{x_1, \cdots, x_n\}$. Hence, $\chi(EG(R)) \leq \mid \mathrm{Jac}(R) \mid +k$.

(iii) The proof is similar to that of (ii). □

The graph G is called weakly perfect whenever $\chi(G) = \omega(G)$.

Corollary 2.7. *Let R be a finite commutative ring and let $EG(R)$ be a connected graph. Then $EG(R)$ is a weakly perfect graph.*

Proof. It is an immediate consequence of Theorems 2.2, 2.5 and 2.6. □

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

ADI Spectral Element Method for the Solution of Nonlinear Time-Fractional Schrödinger Equation

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Abstract. In this work, we propose a numerical scheme for the solution of two dimensional time fractional nonlinear Schrödinger equation. To this end, for the time stepping, an alternating direction implicit (ADI) method based on a scheme of order $\mathcal{O}(\tau)$ is given and for space discretization, spectral element method is used. We present the error estimate of proposed method. To demonstrate the accuracy and efficiency of method, a test problem is presented.

Keywords: Time-fractional Schrödinger equations, ADI spectral element method, Error estimate.

AMS Mathematics Subject Classification [2010]: 65M12, 65M06, 65M60.

1. Introduction

In the current work, we propose an ADI spectral element method for the solution of two dimensional time fractional nonlinear Schrödinger equation [3]

$$
\begin{cases} i{}_0^C D_t^\alpha u(\mathbf{x},t) + \Delta u(\mathbf{x},t) = f(|u(\mathbf{x},t)|^2)u(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times (0,T], \\ u(\mathbf{x},0) = g_1(\mathbf{x}), & \mathbf{x} \in \Omega, \\ u(\mathbf{x},t) = h(\mathbf{x},t), & (\mathbf{x},t) \in \partial\Omega \times (0,T], \end{cases} \tag{1}
$$

where $\Omega \in \mathbf{R}^2$ and ${}_0^C D_t^\alpha u(\mathbf{x},t)$ is the Caputo fractional derivative with $0 < \alpha < 1$. Time fractional Schrödinger equations are important PDEs to describe many phenomena, such as soliton propagation through optical fibers in nonlinear optics plasma physics, superconductivity and quantum mechanics [1], the fractional dynamics in quantum mechanics, the fractional Planck quantum energy relation [3], non-Markovian evolution of a free particle in quantum physics [4] and so on. In this paper, we use L_1 scheme to approximate the Caputo fractional derivative and obtain a scheme of order $\mathcal{O}(\tau)$ to discretize Eq. (1) in time component. To obtain a high order method, the spectral element is used to discretize this equation in spatial direction. Thereafter, the ADI version of the given scheme is constructed. Combining the ADI scheme with spectral element method reduces the complexity in high dimensions. Therefore, the proposed method is a fast and high order scheme for the solution of Eq. (1).

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The layout of the current paper is as follows: In Section 2, time discrete scheme and Galerkin formulation of Eq. (1) are given. ADI scheme is constructed in this section. Fully discrete scheme is obtained using Legendre spectral element method in Section 3. Also, we peresent an error estimate in this section. Accuracy and efficiency of the ADI-SEM is illustrated with an example in Section 4. Finally, a breif conclusion is expressed in Section 5.

2. Time discretization, Galerkin formulation and constructing ADI Scheme

2.1. Preliminareis and time discrete scheme. Consider the following functioal space with the inner product and norm

$$L^2(\Omega) = \left\{ u : \int_\Omega |u|^2 d\Omega < \infty \right\}, (u,v) = \int_\Omega uv d\Omega, \quad \|v\| = (v,v)^{\frac{1}{2}}.$$

Also, define the following Sobolev spaces

$$H^1(\Omega) = \{v \in L^2(\Omega),\, \nabla v \in L^2(\Omega)\}, \quad H_0^1(\Omega) = \{v \in H^1(\Omega),\, v|_{\partial\Omega} = 0\},$$

whith

$$(u,v)_1 = (u,v) + (\nabla u, \nabla v), \quad \|v\|_1 = (v,v)_1^{\frac{1}{2}}, \quad |v|_1 = (\nabla v, \nabla v)^{\frac{1}{2}}.$$

In this work, we use the following notations

$$t_n = n\tau,\ n = 0,1,...,N,\ T = N\tau,\ u(\mathbf{x}, t_n) = u^n,\ \delta_t u^n = (u^n - u^{n-1})/\tau.$$

The L_1 scheme to aproximate the Caputo derivative is given by [5]

$$ {}_0^C D_t^\alpha u(\mathbf{x}, t_n) = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} a_{n-k} \delta_t u^k + r, \tag{2}$$

where $a_k = (k+1)^{1-\alpha} - k^{1-\alpha}$ and $|r| \le C\tau^{2-\alpha}$.
Using the L1 scheme, we can write the following sheme to discretize Eq. (1) at $t = t_n$

$$iu^n + \chi \Delta u^n = iu^n - i \sum_{k=1}^{n} a_k (u^{n-k+1} - u^{n-k}) + \chi f(|u^{n-1}|^2 u^{n-1}) + R \tag{3}$$

where $|R| \le C\tau$ and $\chi = \tau^\alpha \Gamma(2-\alpha)$.

2.2. Galerkin formulation and ADI scheme. The Galerkin formulation of Eq. (3) is given by

$$\begin{aligned} i(u^n, v) - \chi(\nabla u^n, \nabla v) &= i(u^n, v) - i \sum_{k=1}^{n} a_k (u^{n-k+1} - u^{n-k}, v) + \chi(f(|u^{n-1}|^2 u^{n-1}), v) \\ &+ (R, v),\ v \in H_0^1, \end{aligned} \tag{4}$$

Adding the small term $\frac{\tau\chi^2}{i} \left(\frac{\partial^2 \delta_t u^n}{\partial x \partial y}, \frac{\partial^2 v}{\partial x \partial y} \right) = \mathcal{O}(\tau^{1+\alpha})$ on the left hand side of Eq. (4), gives

$$i(u^n, v) - \chi(\nabla u^n, \nabla v) + \frac{\chi^2}{i} \left(\frac{\partial^2 u^n}{\partial x \partial y}, \frac{\partial^2 v}{\partial x \partial y} \right) = (F^n, v) + (R, v),\ v \in H_0^1, \tag{5}$$

where

$$(G^n, v) = i(u^n, v) - i\sum_{k=1}^{n} a_k(u^{n-k+1} - u^{n-k}, v) + \chi f(|u^{n-1}|^2 u^{n-1}, v) + \frac{\chi^2}{i}\left(\frac{\partial^2 u^{n-1}}{\partial x \partial y}, \frac{\partial^2 v}{\partial x \partial y}\right).$$

Omitting the small term R, we have

$$i(U^n, v) - \chi(\nabla U^n, \nabla v) + \frac{\tau\chi^2}{i}\left(\frac{\partial^2 U^n}{\partial x \partial y}, \frac{\partial^2 v}{\partial x \partial y}\right) = (G_*^n, v)\ v \in H_0^1, \tag{6}$$

in which

$$(G_*^n, v) = i(U^n, v) - i\sum_{k=1}^{n} a_k(U^{n-k+1} - U^{n-k}, v) + \chi f(|U^{n-1}|^2 U^{n-1}, v) + \frac{\chi^2}{i}\left(\frac{\partial^2 U^{n-1}}{\partial x \partial y}, \frac{\partial^2 v}{\partial x \partial y}\right).$$

Consider L_s^x and L_s^y as finite dimensional subspaces of H_0^1. Let $\{\lambda_i\}_{i=0}^{M_x}$ and $\{\psi_j\}_{j=0}^{M_y}$ be the basis of these spaces, respectively. Therefore, we can consider $L = L_s^x \otimes L_s^y$ with the base $\{\lambda_i \psi_j\}_{i,j=0}^{M_x, M_y}$. Suppose

$$U^n(x, y) = \sum_{i=0}^{M_x} \sum_{j=0}^{M_y} \eta_{ij}^n \lambda_i(x) \psi_j(y). \tag{7}$$

We put $v = \lambda_r \psi_k (r = 0, 1, ..., M_x,\ k = 0, 1, ..., M_y)$ and consider the following matrices and vectors

$$\begin{aligned}
&B_x = ((\lambda_i, \lambda_k)_x)_{i,k=0}^{M_x},\ B_y = \left((\psi_j, \psi_l)_y\right)_{j,l=0}^{M_y},\\
&D_x = \left(\left(\frac{\partial \lambda_i}{\partial x}, \frac{\partial \lambda_k}{\partial x}\right)_x\right)_{i,k=0}^{M_x},\ D_y = \left(\left(\frac{\partial \psi_j}{\partial y}, \frac{\partial \psi_l}{\partial y}\right)_y\right)_{j,l=0}^{M_y},\\
&\mathfrak{g}^{(k)} = \left[\left(G_*^k, \lambda_0 \psi_0\right), \left(G_*^k, \lambda_0 \psi_1\right), \ldots, \left(G_*^k, \lambda_{M_x M_y} \psi_{M_x M_y}\right)\right]^T,\ k = 0, ..., n,
\end{aligned} \tag{8}$$

We can write Eq. (6) as

$$\left[i\left(B_x \otimes B_y\right) - \chi^2\left(D_x \otimes B_y + B_x \otimes D_y\right) + \frac{\chi^2}{i}\left(D_x \otimes D_y\right)\right]\eta^{(l)} = \mathfrak{g}^{(l)},\ l = 0, ..., n, \tag{9}$$

or

$$\left[\left(\sqrt{i}B_x - \frac{\chi}{\sqrt{i}}D_x\right) \otimes I_{M_y}\right]\left[I_{M_x} \otimes \left(\sqrt{i}B_y - \frac{\chi}{\sqrt{i}}D_y\right)\right]\eta^{(l)} = \mathfrak{g}^{(l)},\ l = 0, ..., n. \tag{10}$$

Now the solution can be obtained by solving two sets of independent one-dimensional problems as

$$\left(\sqrt{i}B_x - \frac{\chi}{\sqrt{i}}D_x\right)\hat{\eta}_m^{(l)} = \mathfrak{g}_m^{(l)}, \quad m = 0, ..., M_y, \tag{11}$$

in x direction and

$$\left(\sqrt{i}B_y - \frac{\chi}{\sqrt{i}}D_y\right)\eta_s^{(l)} = \hat{\eta}_s^{(l)}, \quad s = 0, ..., M_x, \tag{12}$$

in y direction.

3. Legendre spectral element method and error estimate

To implement LSEM, at first, we divide the domain into N_e non-overlapping subdomains. Then, the unknown function U is approximated on each element as

$$U^e(x,t_n) = \sum_{i=0}^{M} U(x_i,t_n)\omega_i(x), \quad 1 \le e \le N_e, \quad 1 \le n \le N, \tag{13}$$

in which $\omega_i(x)$ is the i^{th} Lagrange polynomial based on $M+1$ Gauss-Lobatto-Legendre points $\{\kappa_i\}_{i=0}^{M}$, i.e. [6]

$$\omega_k(\lambda) = \frac{1}{M(M+1)L_M(\kappa_k)} \frac{(\kappa^2-1)L_M'(\kappa)}{\kappa-\kappa_k}, \quad 0 \le k \le M, \ -1 \le \kappa \le 1. \tag{14}$$

Using the map function into $[-1,1]$ and it's inverse, we can get the entries mass and stiffness matrices using the Gauss-Lobatto-Legendre quadrature as

$$B_{ij}^e = \frac{h_e}{2}\int_{-1}^{1} \omega_i(\kappa)\omega_j(\kappa)d\omega = \frac{h_e}{2}\delta_{ij}w_i, D_{ij}^e = \frac{2}{h_e}\int_{-1}^{1} \frac{d\omega_i(\kappa)}{d\kappa}\frac{d\omega_j(\kappa)}{d\omega}d\kappa = \frac{2}{h_e}\sum_{l=0}^{M} d_{il}d_{jl}w_l, \tag{15}$$

where h_e is the length of e^{th} element, $\{w_i\}_{i=0}^{M}$ are the Gauss-Lobatto-Legendre quadrature weights and the matrix d is the transpose of the differentiation matrix in spectral method [2].

3.1. Error estimate. Consider $\mathbb{P}_M(\Omega)$ as the space of polynomials defined on Ω with the degree no greater than $M \in N$. Define

$$\mathfrak{O}_h^0 = \{u \in H_0^1 : u|_{\Omega_e} \in \mathbb{P}_M\}, \tag{16}$$

and the following Ritz projection

$$\begin{aligned} &\aleph_h : H_0^1 \to \mathfrak{O}_h^0 \\ &(\nabla u, \nabla v) = (\nabla \aleph_h u, \nabla v), \quad u \in H_0^1, \quad \forall\, v \in \mathfrak{O}_h^0. \end{aligned} \tag{17}$$

Lemma 3.1. *[2] Suppose H^ϱ with the norm $||.||_\varrho$ be the Sobolev space, then we have*

$$||(u-\aleph_h u)|| \le C\left[\sum_{i=1}^{N_e} h_i^{2(\min(M_i+1,\varrho)-1)} M_i^{2(1-\varrho)} ||u||_\varrho^2\right]^{\frac{1}{2}}, \qquad \forall u \in H^\varrho, \tag{18}$$

and if h_i satisfies $h \le h_i \le c'h$ for all i and $M_i = M$, then

$$||(u-\aleph_h u)|| \le C h_i^{(\min(M+1),\varrho)-1} M^{1-\varrho}||u||_\varrho. \tag{19}$$

Theorem 3.2. *Let $u(x,y,t_n) = u^n$ and U^n are the exact solutions of Eq. (1) and the ADI spectral element method (6), respectively. Let $e^n = u^n - U^n$, then we have*

$$||e^n|| \le C(\tau + M^{1-\varrho}). \tag{20}$$

where C is a positive constant, independent of n, τ, and M.

4. Numerical results

In curent section, we report the results of proposed method. If e_1 and e_2 are error corespond to steps τ_1 and τ_2, the computational order of given method is calculated as

$$\text{C-order}= \frac{log\frac{e_1}{e_2}}{log\frac{\tau_1}{\tau_2}}.$$

EXAMPLE 4.1. We perform the given method for the solution of two dimensional fractional Schrödinger equation as [3]

$$i{}_0^C D_t^\alpha u(\mathbf{x},t) + \Delta u(\mathbf{x},t) = |u|^2 u - |u|^4 u + g(x,y,t),\ (x,y) \in (0,1)\times(0,1), t \in [0,1], \tag{21}$$

with the exact solution $u(x,y,t) = t^2 x(1-x)y(1-y)$. In Table 1, the L_∞ error and comutational order of ADI-SEM with $N_e = 5$ and $M = 6$ is presented. We depict the graph of error as a function of M with $N_e = 2$, and the error as a function of N_e with $M = 3$.

TABLE 1. Errors and computational orders for Test problem 1.

	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.8$		CPU(s)
τ	L_∞	C-order	L_∞	C-order	L_∞	C-order	
1/20	1.4617×10^{-2}		8.1270×10^{-3}		6.4442×10^{-3}		0.9852
1/40	7.0323×10^{-3}	1.0323	3.6023×10^{-3}	1.1738	3.1310×10^{-3}	1.0414	1.1270
1/80	3.1625×10^{-3}	1.1529	1.6773×10^{-3}	1.1028	1.5556×10^{-3}	1.0092	1.9117
1/160	1.4287×10^{-3}	1.1487	8.0638×10^{-4}	1.0566	6.9149×10^{-4}	1.1669	2.9814

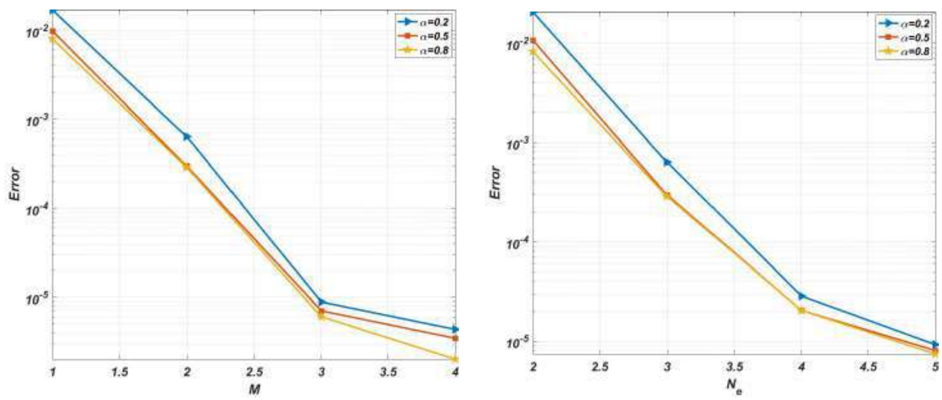

FIGURE 1. Error as function of M (left panel) and Ne (right panel) with $\tau = 0.001$.

5. Conclusion

At the present paper, we investigated a numerical method using spectral element method in spatial direction and an ADI scheme of order $\mathcal{O}(\tau)$ for the solution of two dimensional time fractional time-fractional Schrödinger equation. Using an example, accuracy of this method is shown.

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On amenability-like properties of certain matrix algebras with applications

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Abstract. In this paper, we show that the class of $I \times I$ matrices with finite ℓp-norm is always pseudo-amenable, where $1 \leq p \leq 2$. As an application, for the case $p = 1$, we give some applications for semigroup algebras.

Also we study approximate homological notions for the class of upper triangular matrix algebras with respect to the Esslamzadeh-Munn algebras.

Keywords: matrix algebras, pseudo-amenability, approximate biprojectivity

AMS Mathematics Subject Classification [2010]: 46M10, 43A20, 46H05

1. Introduction

A Banach algebra A is biflat if there exists a bounded A-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ such that $\pi_A^{**} \circ \rho(a) = a$, where $\pi_A : A \otimes_p A \to A$ is the product morphism given by $\pi_A(a \otimes b) = ab$ for every $a, b \in A$. It is worth mentioning that the biflatness has direct relation with the amenability of a Banach algebra. Note that a Banach algebra A is amenable if and only if A is biflat and posses a bounded approximate identity. For the history of amenability and biflatness, see [5]

Ghahramani and Zhang [2] introduced pseudo-amenability and pseudo-contractibility for Banach algebras. In fact a Banach algebra $\mathcal{A}$ is pseudo-amenable (pseudo-contractible) if there exists a net (m_α) in $\mathcal{A} \otimes_p \mathcal{A}$ such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ ($a \cdot m_\alpha = m_\alpha \cdot a$) and $\pi_{\mathcal{A}}(m_\alpha)a \to a$ for all $a \in \mathcal{A}$, respectively.

G. H. Esslamzadeh in [1] introduced and studied a class of matrix algebras, named ℓ^1-Munn algebras. Then Esslamzadeh in [1] studied their general analytic form and applied them to solve some harmonic analysis problems. Also, he investigated some cohomological properties of these algebras like amenability and the existence of bounded approximate identity. M. Lashkarizadeh Bami and S. Naseri [3] extended the notion of ℓ^1-Munn algebras to ℓ^p-Munn algebras, see [3] and [4].

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Let I be any non-empty set. Then we denote $\mathcal{LM}_I^p(\mathbb{C})$ for the vector space of all $I \times I$ matrices $A = [a_{ij}]$ over $\mathbb{C}$ such that $||A||_p = (\sum_{i,j\in I}(|a_{ij}|^p)^{\frac{1}{p}} < \infty$. With the matrix operations and $||\cdot||_p$ as a norm, $\mathcal{LM}_I^p(\mathbb{C})$ becomes a Banach algebra, provided that $1 \leq p \leq 2$. In the case I is a totally ordered set, we denote

$$\mathcal{UP}_I^p(\mathbb{C}) = \{[a_{ij}] \in \mathcal{LM}_I^p(\mathbb{C}) | a_{ij} = 0 \quad \text{whenever} \quad i > j\}.$$

2. Main results

THEOREM 2.1. *Suppose that I is a non-empty index set and $1 \leq p \leq 2$. Then $\mathcal{LM}_I^p(\mathbb{C})$ has a central approximate identity if and only if I is finite.*

THEOREM 2.2. *Suppose that I is a non-empty index set and $1 \leq p \leq 2$. Then $\mathcal{LM}_I^p(\mathbb{C})$ is pseudo-contractible if and only if I is finite.*

THEOREM 2.3. *Let I be a non-empty index set and $1 \leq p \leq 2$. Then $\mathcal{LM}_I^p(\mathbb{C})$ is biflat.*

In the case $p = 1$ we denote $\mathcal{LM}_I^p(\mathbb{C})$ by $M_I(\mathbb{C})$. That is the Banach algebra of $I \times I$ with finite ℓ^1-norm.

For a locally compact group G and a non-empty set I, set

$$M^0(G, I) = \{(g)_{i,j} : g \in G, i, j \in I\} \cup \{0\},$$

where $(g)_{i,j}$ denotes the $I \times I$ matrix with g in (i, j)-position and zero elsewhere. With the following multiplication $M^0(G, I)$ becomes a semigroup

$$(g)_{i,j} * (h)_{k,l} = \begin{cases} (gh)_{il} & j = k \\ 0 & j \neq k, \end{cases}$$

It is known that $\ell^1(M^0(G, I)) \cong (M_I(\mathbb{C}) \otimes_p \ell^1(G)) \oplus_1 \mathbb{C}$. So there is a direct relation between the structure of the semigroup algebra and the matrix algebra, where the semigroup is Brandt.

COROLLARY 2.4. *Let $S = M^0(G, I)$ be a Brandt semigroup. Then the following are equivalent:*

(i) *$\ell^1(S)$ is pseudo-amenable;*
(ii) *G is amenable.*

LEMMA 2.5. *Let I be a non-empty index set and $1 \leq p \leq 2$. Then $\mathcal{LM}_I^p(\mathbb{C})$ has an approximate identity.*

PROOF. Let $F(I)$ be the set of all finite subsets of I and $\lambda \in F(I)$. Then put $e_\lambda := [e_{ij}]$, where $e_{ii} = 1$ for $i \in \lambda$ and $e_{ij} = 0$ elsewhere. Also, let $a = [a_{ij}] \in \mathcal{LM}_I^p(\mathbb{C})$ and $\epsilon > 0$. Then there exists a finite subset $\lambda_0 \in F(I)$ such that $(\sum_{i,j\in I-\lambda_0} |a_{ij}|^p)^{\frac{1}{p}} < \epsilon$. It Follows that $||e_{\lambda_0} a - a|| = (\sum_{i,j\in I-\lambda_0} |a_{ij}|^p)^{\frac{1}{p}} < \epsilon$. So, $(e_\lambda)_{\lambda\in F(I)}$ is an approximate identity for $\mathcal{LM}_I^p(\mathbb{C})$. □

LEMMA 2.6. *Suppose that $\mathcal{A}$ is a biflat Banach algebra with an approximate identity. Then $\mathcal{A}$ is pseudo-amenable.*

Using previous two lemmas we have the following corollary.

COROLLARY 2.7. *Let I be a non-empty index set and $1 \leq p \leq 2$. Then $\mathcal{LM}_I^p(\mathbb{C})$ is pseudo-amenable.*

A Banach algebra $\mathcal{A}$ is called approximately biprojective if there exists a net of $\mathcal{A}$-bimodule morphisms from $\mathcal{A}$ into $\mathcal{A}\otimes_p\mathcal{A}$ such that $\pi_{\mathcal{A}}\circ\rho_\alpha(a)\to a$ for all $a\in\mathcal{A}$, see [7].

THEOREM 2.8. *Suppose that I is a totally ordered set with a smallest element and $1\leq p\leq 2$. Then $\mathcal{UP}^p_I(\mathbb{C})$ is approximately biprojective if and only if I is singleton.*

A Banach algebra $\mathcal{A}$ is called approximately biflat if there exists the net (ρ_α) of $\mathcal{A}$-bimodule morphisms from $(\mathcal{A}\otimes_p\mathcal{A})^*$ into $\mathcal{A}^*$ such that $\rho_\alpha\circ\pi^*_{\mathcal{A}}\xrightarrow{W^*OT} id_{\mathcal{A}^*}$. Here we denote W^*OT for the weak* operator topology, see [6].

THEOREM 2.9. *Suppose that I is a totally ordered set with a smallest element and $1\leq p\leq 2$. Then $\mathcal{UP}^p_I(\mathbb{C})$ is approximately biflat if and only if I is a singleton.*

THEOREM 2.10. *Suppose that I is a totally ordered set and $1\leq p\leq 2$. Then $\mathcal{UP}^p_I(\mathbb{C})$ is amenable if and only if I is a singleton.*

THEOREM 2.11. *Let I be a totally ordered set and $1\leq p\leq 2$. Then $\mathcal{UP}^p_I(\mathbb{C})$ is not pseudo-contractible unless it is trivial.*

PROOF. Suppose that $\mathcal{UP}^p_I(\mathbb{C})$ is pseudo-contractible. Then $\mathcal{UP}^p_I(\mathbb{C})$ posses a central approximate identity. By Theorem 2.1 we have that I must be finite. Similar to the arguments as in Theorem 2.10, deduces that I is singleton. $\square$

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Faculty of Mathematics and Computer
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Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
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52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
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۸ تا ۱۱ شهریور ۱۴۰۰

Existence and k-Mittag-Leffler-Ulam-Hyers stability results of k-generalized ψ-Hilfer Boundary value problem

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Abstract. In this paper, we present a generalized ψ-Hilfer fractional derivative and set some of the generalized operator's properties. We give a generalized Gronwall inequality and present the definitions of the k-Mittag-Leffler-Ulam-Hyers stability and some related remarks. We prove some existence, uniqueness and k-Mittag-Leffler-Ulam-Hyers stability results for a class of boundary value problem for implicit nonlinear fractional differential equations and k-Generalized ψ-Hilfer fractional derivative. In addition, various examples are given in order to justify our results.

Keywords: ψ-Hilfer fractional derivative, Generalized Gronwall Inequality, Mittag-Leffler function, Ulam-Hyers stability.

AMS Mathematics Subject Classification [2010]: 26A33, 34A12.

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1. Introduction

Fractional calculus is a classical mathematical branch that concerns the generalization of the integer order differentiation and integration of a function to non-integer order, its theory and application is a solid and growing work. Existence, uniqueness and stability to fractional differential equations was investigated in a variety of papers [1,4,5]. Recently, in [2] Diaz *et al.* have presented k-gamma and k-beta functions and proved a variety of their properties. In [6], Sousa *et al.* introduce another so-called ψ-Hilfer fractional derivative with respect to another function, and gave some important properties concerning this type of fractional operator.

Motivated by the papers mentioned above and by using the functions k-Gamma, k-Beta and k-Mittag-Leffler, we give the definition to the k-generalized ψ-Hilfer fractional derivative and prove some of its properties. Then, we propose a generalized Gronwall inequality to be used in the k-Mittag-Leffler-Ulam-Hyers stability. We consider the boundary valued problem with nonlinear implicit k-generalize ψ-Hilfer type fractional differential equation :

$$\left({}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}x\right)(t)=f\left(t,x(t),\left({}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}x\right)(t)\right),\quad t\in(a,b], \tag{1}$$

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$$(2) \qquad c_1 \left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi} x\right)(a^+) + c_2 \left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi} x\right)(b) = c_3,$$

where ${}^H_k\mathcal{D}_{a+}^{\vartheta,r;\psi}, \mathcal{J}_{a+}^{k(1-\xi),k;\psi}$ are the k-generalize ψ-Hilfer fractional derivative of order $\vartheta \in (0,k)$ and type $r \in [0,1]$ defined later, and k-generalize ψ-fractional integral of order $k(1-\xi)$ defined in [3] respectively, where $\xi = \frac{1}{k}(r(k-\vartheta)+\vartheta)$, $k > 0$, $f \in C([a,b]\times\mathbb{R}^2,\mathbb{R})$ and $c_1, c_2, c_3 \in \mathbb{R}$ such that $c_1 + c_2 \neq 0$.

1.1. Preliminaries. First, we present the weighted spaces, notations, definitions, and preliminary facts which are used in this article. Let $0 < a < b < \infty$, $J = [a,b]$, $\vartheta \in (0,k)$, $r \in [0,1]$, $k > 0$ and $\xi = \frac{1}{k}(r(k-\vartheta)+\vartheta)$. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|x\|_\infty = \sup\{|x(t)| : t \in J\}.$$

Consider the weighted Banach space

$$C_{\xi,k;\psi}(J) = \left\{x : (a,b] \to \mathbb{R} : t \to (\psi(t)-\psi(a))^{1-\xi} x(t) \in C(J,\mathbb{R})\right\},$$

with the norm

$$\|x\|_{C_{\xi,k;\psi}} = \sup_{t\in J}\left|(\psi(t)-\psi(a))^{1-\xi} x(t)\right|,$$

Consider the space $X_\psi^p(a,b)$, $(c \in \mathbb{R},\ 1 \leq p \leq \infty)$ of those real-valued Lebesgue measurable functions g on $[a,b]$ for which $\|g\|_{X_\psi^p} < \infty$, where the norm is defined by

$$\|g\|_{X_\psi^p} = \left(\int_a^b \psi'(t)|g(t)|^p dt\right)^{\frac{1}{p}},$$

where ψ is an increasing and positive function on $[a,b]$ such that ψ' is continuous on $[a,b]$ with $\psi(0) = 0$. In particular, when $\psi(x) = x$, the space $X_\psi^p(a,b)$ coincides with the $L_p(a,b)$ space.

Definition 1.1. ([2]) The k-gamma function is defined by

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \alpha > 0.$$

When $k \to 1$ then $\Gamma(\alpha) = \Gamma_k(\alpha)$. Furthermore k-beta function is defined as follows

$$B_k(\alpha,\beta) = \frac{1}{k}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1} dt.$$

The Mittag-Leffler function can also be refined into the k-Mittag-Leffler function defined as follows

$$\mathbb{E}_k^{\alpha,\beta}(x) = \sum_{i=0}^\infty \frac{x^i}{\Gamma_k(\alpha i+\beta)}, \alpha, \beta > 0.$$

Definition 1.2. ([3]) (k-Generalized ψ-fractional Integral) Let $g \in X_\psi^p(a,b)$ and $[a,b]$ be a finite or infinite interval on the real axis $\mathbb{R} = (-\infty,\infty)$, $\psi(t) > 0$ be an increasing function on $(a,b]$ and $\psi'(t) > 0$ be continuous on (a,b), $k > 0$ and $\vartheta > 0$. The generalized k-fractional integral operator of a function f of order ϑ is defined by

$$\mathcal{J}_{a+}^{\vartheta,k;\psi} g(t) = \frac{1}{k\Gamma_k(\vartheta)}\int_a^t \frac{\psi'(s)g(s)ds}{(\psi(t)-\psi(s))^{1-\frac{\vartheta}{k}}}.$$

2. Main results

2.1. Definition of the new generalized derivative. We are now able to define the k-generalized ψ-Hilfer derivative as follows.

Definition 2.1. (k-Generalized ψ-Hilfer Derivative) Let $n-1<\dfrac{\vartheta}{k}\leq n$ with $n\in\mathbb{N}$, $J=[a,b]$ an interval such that $-\infty\leq a<b\leq\infty$ and $g,\psi\in C^n([a,b],\mathbb{R})$ two functions such that ψ is increasing and $\psi'(t)\neq 0$, for all $t\in J$. The k-generalized ψ-Hilfer fractional derivative ${}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}(\cdot)$ of a function g of order ϑ and type $0\leq r\leq 1$, with $k>0$ is defined by

$$\begin{aligned}{}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}g\left(t\right)&=\left(\mathcal{J}^{r(kn-\vartheta),k;\psi}_{a+}\left(\frac{1}{\psi'\left(t\right)}\frac{d}{dt}\right)^n\left(k^n\mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+}g\right)\right)(t)\\&=\left(\mathcal{J}^{r(kn-\vartheta),k;\psi}_{a+}\delta^n_\psi\left(k^n\mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+}g\right)\right)(t)\end{aligned}$$

where $\delta^n_\psi=\left(\dfrac{1}{\psi'\left(t\right)}\dfrac{d}{dt}\right)^n$.

2.2. The generalized Gronwall inequality. Now, we give a generalized Gronwall inequality. We prove this result with the properties of the functions k-gamma, k-beta and k-Mittag-Leffler taken into account.

Theorem 2.2. *Let x, y be two integrable functions and g continuous, with domain $[a,b]$. Let $\psi\in C^1\left[a,b\right]$ an increasing function such that $\psi'\left(t\right)\neq 0$, $t\in[a,b]$ and $\vartheta>0$ with $k>0$. Assume that*

(1) *x and y are nonnegative;*
(2) *w is nonnegative and nondecreasing.*

If

$$x\left(t\right)\leq y\left(t\right)+\frac{w\left(t\right)}{k}\int_a^t\psi'\left(s\right)\left[\psi(t)-\psi(s)\right]^{\frac{\vartheta}{k}-1}x\left(s\right)ds,$$

then

$$x\left(t\right)\leq y\left(t\right)+\int_a^t\sum_{i=1}^{\infty}\frac{\left[w\left(t\right)\Gamma_k\left(\vartheta\right)\right]^i}{k\Gamma_k\left(\vartheta i\right)}\psi'\left(s\right)\left[\psi(t)-\psi(s)\right]^{\frac{i\vartheta}{k}-1}y\left(s\right)ds,\tag{3}$$

for all $t\in[a,b]$. And if y is a nondecreasing function on $[a,b]$. Then, we have

$$x\left(t\right)\leq y\left(t\right)\mathbb{E}^{\vartheta}_k\left(w\left(t\right)\Gamma_k\left(\vartheta\right)\left(\psi\left(t\right)-\psi\left(a\right)\right)^{\frac{\vartheta}{k}}\right).$$

2.3. Existence of solutions. The following hypotheses will be used in the sequel :

(*Ax1*) The function $f:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and
$$f(\cdot,x(\cdot),y(\cdot))\in C^1_{\xi,k;\psi}(J),\text{ for any }x,y\in C_{\xi,k;\psi}(J).$$

(*Ax2*) There exist constants $\eta_1>0$ and $0<\eta_2<1$ such that
$$|f(t,x,y)-f(t,\bar{x},\bar{y})|\leq\eta_1|x-\bar{x}|+\eta_2|y-\bar{y}|$$
for any $x,y,\bar{x},\bar{y}\in\mathbb{R}$ and $t\in J$.

We are now in a position to state and prove our existence result for the problem (1)-(2) based on based on Banach's fixed point theorem.

Theorem 2.3. *Assume* (Ax1) *and* (Ax2) *hold. If*

$$\mathcal{L} = \frac{\eta_1 \left(\psi(b) - \psi(a)\right)^{\frac{\vartheta}{k}}}{1-\eta_2}\left[\frac{|c_2|}{|c_1+c_2|\Gamma_k(k+\vartheta)} + \frac{\Gamma_k(k\xi)}{\Gamma_k(\vartheta+k\xi)}\right] < 1, \tag{4}$$

then the problem (1)-(2) *has a unique solution in* $C_{\xi,k;\psi}(J)$.

2.4. k-Mittag-Leffler-Ulam-Hyers stability. Let $x \in C^1_{\xi,k;\psi}(J)$, $\epsilon > 0$. We consider the following inequality :

$$\left|\left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}x\right)(t) - f\left(t, x(t), \left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}x\right)(t)\right)\right| \leq \epsilon\mathbb{E}^{\vartheta}_k\left(\left(\psi(t)-\psi(a)\right)^{\frac{\vartheta}{k}}\right),\ t \in (a,b]. \tag{5}$$

Definition 2.4. Problem (1)-(2) is k-Mittag-Leffler-Ulam-Hyers stable with respect to $\mathbb{E}^{\vartheta}_k\left(\left(\psi(t)-\psi(a)\right)^{\frac{\vartheta}{k}}\right)$ if there exists a real number $a_{\mathbb{E}^{\vartheta}_k} > 0$ such that for each $\epsilon > 0$ and for each solution $x \in C^1_{\xi,k;\psi}(J)$ of inequality (5) there exists a solution $y \in C^1_{\xi,k;\psi}(J)$ of (1)-(2) with

$$|x(t)-y(t)| \leq a_{\mathbb{E}^{\vartheta}_k}\epsilon\mathbb{E}^{\vartheta}_k\left(\left(\psi(t)-\psi(a)\right)^{\frac{\vartheta}{k}}\right), \qquad t \in J.$$

3. Conclusion

The present work, we have achieved the following: We defined the k-generalize ψ-Hilfer type fractional derivative and gave some necessary theorems and lemmta. We presented a generalized Gronwall inequality. Then, we established an existence, uniqueness and k-Mittag-Leffler-Ulam-Hyers stability result for the problem (1)-(2). Finally, we gave examples to illustrate the feasibility of our results and present the different particular cases of our problem.

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$\mathbb{F}_q\mathbb{F}_q[u]$-Additive skew cyclic codes

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Abstract. In this paper, we characterize the algebraic structure of $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes, where $u^2 = 0$. Also, we provide new methods to determine the structure of skew cyclic codes of length s over $\mathbb{F}_q[u]$. We classify that there are eight different types of explicit generators of $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes.

Keywords: Skew cyclic code, Additive code.

AMS Mathematics Subject Classification [2010]: 94B15, 16S36.

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1. Introduction

Abualrub et al. defined cyclic codes for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes in [1]. The algebraic structure and some properties of these codes over finite chain rings and their Euclidean and Hermitian dual codes have been established in [3]. Skew cyclic codes of length p^s over R_2 have been studied in [2].

This paper has been organized as follows. Section 2, contains some basic definitions, some notations and previous results related to our work. In Section 3, we specify the $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes, where $u^2 = 0$. We classify that there are eight different types of explicit generators of $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes.

2. Preliminaries

A ring R is a *principal left ideal ring* if it has unity and every left ideal is principally generated. R is called a *local ring* if R has a unique maximal right (left) ideal. Furthermore, a ring R is called a (*left*) *chain ring* if the set of all (left) ideals of R is linearly ordered under set-theoretic inclusion.

A code C of length n over a commutative ring R is a non-empty subset of R^n and the ring R is referred to as the alphabet of C. If C is also an R-submodule of R^n, then C is called a linear code.

For a given automorphism σ of a finite commutative ring R, the set $R[x, \sigma] = \{a_0 + a_1x + a_2x^2 + ... + a_nx^n : \quad a_i \in R \text{ and } n \in \mathbb{N}_0\}$ of formal polynomials forms a ring under the usual addition of polynomials and where the multiplication is defined using the rule $xa = \sigma(a)x$. The multiplication is extended to all element in $R[x;\sigma]$ holding associativity and distributivity. The ring $R[x;\sigma]$ is called a *skew polynomial ring* over R

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and an element in $R[x;\sigma]$ is called a *skew polynomial.* It is easily seen that the ring $R[x;\sigma]$ is non-commutative unless σ is the identity automorphism on R.

For a given automorphism σ of R a code C over R is called skew σ-cyclic, if C is closed under σ-cyclic shift $\rho_\sigma : R^n \longrightarrow R^n$ which is defined by

$$\rho_\sigma\big((a_0, a_1, ..., a_{n-1})\big) = \big(\sigma(a_{n-1}), \sigma(a_0), ..., \sigma(a_{n-2})\big).$$

Each codeword $c = (c_0, c_1, ..., c_{n-1})$ is customarily identified with its polynomial representation $c(x) = c_0 + c_1x + ... + c_{n-1}x^{n-1}$.

Let $\mathbb{F}_q$ denote the finite field with q elements and δ is a primitive $(q-1)$-th root of unity in $\mathbb{F}_q$, i.e,

$$\mathbb{F}_q = \{0, \delta, \cdots, \delta^{q-2}, \delta^{q-1} = 1\}.$$

Suppose that $R_2 = \mathbb{F}_q + u\mathbb{F}_q$, with $u^2 = 0$. It is known that R_2 is a chain ring with the unique maximal ideal $u\mathbb{F}_q$.

Moreover, let θ be an automorphism of $\mathbb{F}_q$ and Θ be an automorphism of R_2, such that $\Theta = \theta^k$, for some $0 \le k \le o(\theta) - 1$.

Lemma 2.1. *[4] Let $f, g \in \mathbb{F}_q[x;\theta]$ with $f \neq 0$. Then there exist $q, r \in \mathbb{F}_q[x,\theta]$ with $g = qf + r$, where $r = 0$ or $\deg(r) < \deg(f)$. In particular, $\mathbb{F}_q[x,\theta]$ is a principal ideal domain.*

Definition 2.2. Suppose $f(x), g(x)$ are skew polynomials in $\mathbb{F}_q[x;\theta]$. The *greatest common right divisor* of $f(x)$ and $g(x)$ is the monic polynomial $d_r(x) \in \mathbb{F}_q[x;\theta]$, where $d_r(x) \mid_r f(x), d_r(x) \mid_r g(x)$ and for any $d'_r(x) \in \mathbb{F}_q[x;\theta]$ such that $d'_r(x) \mid_r f(x)$ and $d'_r(x) \mid_r g(x)$, hence $d'_r(x) \mid_r d_r(x)$. We denote $d_r(x)$ by $\text{gcrd}(f(x), g(x))$.

Throughout this paper, we use the following symbols for simplicity:

- $\mathbb{F}_q[x;\Theta] = \mathbb{F}_q[x;\theta]$.
- $R_2 = \mathbb{F}_q[u] = \mathbb{F}_q + u\mathbb{F}_q$.
- $\mathcal{R}_{1,k} = \frac{\mathbb{F}_q[x;\Theta]}{\langle x^k - 1\rangle}$, for $k = r, s$.
- $\mathcal{R}_s = \frac{R_2[x;\Theta]}{\langle x^s - 1\rangle}$.
- $\mathcal{R}_n = \frac{R_2[x;\Theta]}{\langle x^n - 1\rangle}$.
- $\mathcal{R} = \mathcal{R}_{1,r} \times \mathcal{R}_s$.

Let r and s be positive integers. By [3, Proposition 2.2], we assume that $o(\Theta) = o(\theta) = l \mid \gcd(r, s)$. (since $x^r - 1$ and $x^s - 1$ are monic central skew polynomials, hence by P [3, Proposition 2.3], right divisors of $x^r - 1$ and $x^s - 1$ are two-sided divisor).

Let $\mu : R_2 \longrightarrow \mathbb{F}_q$, be the natural ring morphism, where $\mu(a_0 + ua_1) = a_0$. We construct the set

$$\mathbb{F}_qR_2 = \{(a|b) : \quad a \in \mathbb{F}_q, b \in R_2\},$$

by the following scalar multiplication, $\mathbb{F}_qR_2$ is an R_2-module,

$$\begin{aligned} . : R_2 \times \mathbb{F}_qR_2 &\longrightarrow \mathbb{F}_qR_2, \\ \nu.(a|b) &= (\mu(\nu)a|b). \end{aligned}$$

This multiplication be generalized over the set $\mathbb{F}_q^rR_2^s$ in the following way. For any $\nu \in R_2$ and $(a_0, a_1, ..., a_{r-1}|b_0, ..., b_{s-1}) \in \mathbb{F}_q^rR_2^s$ define

$$\nu.(a_0, a_1, ..., a_{r-1}|b_0, ..., b_{s-1}) = (\mu(\nu)a_0, \mu(\nu)a_1, ..., \mu(\nu)a_{r-1}|\nu b_0, ..., \nu b_{s-1}).$$

Definition 2.3. A non-empty subset C of $\mathbb{F}_q^rR_2^s$ is called a $\mathbb{F}_q\mathbb{F}_q[u]$-*additive skew cyclic code* of length (r, s) if C is an R_2-submodule of $\mathbb{F}_q^rR_2^s$.

Corollary 2.4. *There is a bijective map between* $\mathbb{F}_q^r R_2^s$ *and* $\mathcal{R}$ *given by*

$$(a_0, ..., a_{r-1}|b_0, ..., b_{s-1}) \longmapsto (a_0 + \cdots + a_{r-1}x^{r-1}|b_0 + \cdots + b_{s-1}x^{s-1}) = (a(x)|b(x)).$$

Suppose $(f(x)|g(x)) \in \mathcal{R}$ and $\nu(x) \in R_2[x;\Theta]$, we have

$$. : R_2[x;\Theta] \times \mathcal{R} \longrightarrow \mathcal{R},$$
$$\nu(x).(f(x)|g(x)) = (\mu(\nu(x))f(x)|\nu(x)g(x)),$$

where

$$\mu(\nu(x)) = \mu(\sum_{j=0}^{r-1} \nu_j x^j) = \sum_{j=0}^{r-1} \mu(\nu_j)x^j,$$

and $\nu_j \in R_2$.

3. $\mathbb{F}_q\mathbb{F}_q[u]$-Additive skew cyclic codes

In this section we characterize the algebraic structure of $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes of length (r, s).

Note that the skew polynomial ring $\mathbb{F}_q[x;\Theta]$ is not a unique factorization domain. In fact, many different factorizations may be possible. For example in the ring $\mathbb{F}_{27}[x;\Theta]$, $x-1, x-\delta^2, x-\delta^6$ and $x-\delta^{18}$ are factors of x^3-1 (for any $a \in \mathbb{F}_{27}$, $\theta(a) = a^3$).

Set $\mathcal{F}_k := \{a(x) \in \mathbb{F}_q[x;\Theta] :\ \ a(x)$ is a monic factor of $x^k - 1\}$, for $k = r, s$.

It is well known that R_2 is a finite chain ring of nilpotency index 2 and the unique maximal ideal $u\mathbb{F}_q$. In this section we determine of $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes of length (r, s) which r and s are multiple of the order of Θ.

Remark 3.1. We extend the natural ring morphism $\mu : R_2 \longrightarrow \mathbb{F}_q$, where $\mu(a_0 + ua_1) = a_0$, as follows:

$$\mu : R_2[x;\Theta] \longrightarrow \mathbb{F}_q[x;\Theta],$$
$$\sum_{i=0}^{n-1}(a_{0i} + ua_{1i})x^i \longmapsto \sum_{i=0}^{n-1} a_{0i}x^i.$$

It can be even extended to $\mu : \mathcal{R}_s \longrightarrow \mathcal{R}_{1,k}$.

Proposition 3.2. *A linear code* C *of length* s *is skew cyclic over* R_2 *if and only if* C *is left ideal of* $\mathcal{R}_s$.

As corollaries 4.3, 4.4 in [2] and 4.2 in [3], we have the following theorems.

Theorem 3.3. *Every left ideal of* $\mathcal{R}_s$ *is of the form*

$$\mathcal{I} = \mathcal{R}_s\big(a_1(x) + ug_1(x)\big) + \mathcal{R}_s\big(ua_2(x)\big),$$

where $a_i(x) \in \mathcal{F}_s$, $a_2(x) \mid_r a_1(x)$ *and* $g_1(x)$ *is a polynomial in* $\mathcal{R}_{1,s}$ *such that* $\deg(g_1(x)) < \deg(a_2(x))$. *Moreover,* $g_1(x)$ *with the above conditions is unique.*

Theorem 3.4. *Skew cyclic codes of lenrth* s *over* R_2*, i.e., left ideals of the ring* $\mathcal{R}_s$ *can be separated into the following types:*

- *Type* 1 (*trivial ideals*)*:* 0*,* $\mathcal{R}_s$.
- *Type* 2 : $\mathcal{R}_s\big(ua(x)\big)$*, where* $a(x) \in \mathcal{F}_s, 0 \leqslant \deg(a(x)) \leqslant s-1$.
- *Type* 3 : $\mathcal{R}_s\big(a(x) + ug(x)\big)$*, where* $a(x) \in \mathcal{F}_s, 1 \leqslant \deg(a(x)) \leqslant s-1$ *and* $\deg(g(x)) <$

$\deg(a(x))$. *Moreover, $g(x)$ with the above conditions is unique.*
• *Type* 4 : $\mathcal{R}_s\big(a_1(x)+ug_1(x)\big)+\mathcal{R}_s\big(ua_2(x)\big)$, *where* $a_1(x), a_2(x) \in \mathcal{F}_s$, $1 \leqslant \deg(a_1(x)) \leqslant s-1$, $a_2(x) \mid_r a_1(x)$ *and* $\deg(g_1(x)) < \deg(a_2(x))$. *Moreover* $a_2(x) \mid_r \frac{x^s-1}{a_1(x)}g_1(x)$ *and* $g_1(x)$ *with the above conditions is unique.*

Lemma 3.5. *A code C is an $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic code of length (r,s) if and only if C is a left $R_2[x;\Theta]$-submodule of $\mathcal{R}$.*

We can list all $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes of length (r,s) as follows.

Theorem 3.6. *$\mathbb{F}_q\mathbb{F}_q[u]$-Additive skew cyclic codes of length (r,s) are*

• *Type* 1 : 0, $\mathcal{R}$.

• *Type* 2 : $\mathcal{R}_n\big((\hat{a}(x)|0)\big)$, *where* $\hat{a}(x) \in \mathcal{F}_r$ *and* $0 \le \deg(\hat{a}(x)) \le r-1$.

• *Type* 3 : $\mathcal{R}_n\big((k(x)|a(x)+ug(x))\big)$, *where* $k(x) \in \mathcal{R}_{1,r}$, $a(x) \in \mathcal{F}_s$, $0 \le \deg(a(x)) \le s-1$, $g(x) \in \mathcal{R}_{1,s}$ *and* $\deg(g(x)) < \deg(a(x))$. *Moreover, $g(x)$ with the above conditions is unique.*

• *Type* 4 : $\mathcal{R}_n\big((k(x)|ua(x))\big)$, *where* $k(x) \in \mathcal{R}_{1,r}$, $a(x) \in \mathcal{F}_s$ *and* $0 \le \deg(a(x)) \le s-1$.

• *Type* 5 : $\mathcal{R}_n\big((\hat{a}(x)|0)\big)+\mathcal{R}_n\big((k(x)|a(x)+ug(x))\big)$, *where* $\hat{a}(x) \in \mathcal{F}_r$, $0 \le \deg(\hat{a}(x)) \le r-1$, $k(x) \in \mathcal{R}_{1,r}$, $a(x) \in \mathcal{F}_s$, $0 \le \deg(a(x)) \le s-1$, $g(x) \in \mathcal{R}_{1,s}$, $\deg(k(x)) < \deg(\hat{a}(x))$ *and* $\deg(g(x)) < \deg(a(x))$. *Moreover, $g(x)$ with the above conditions is unique.*

• *Type* 6 : $\mathcal{R}_n\big((\hat{a}(x)|0)\big)+\mathcal{R}_n\big((k(x)|ua(x))\big)$, *where* $\hat{a}(x) \in \mathcal{F}_r$, $0 \le \deg(\hat{a}(x)) \le r-1$, $k(x) \in \mathcal{R}_{1,r}$, $a(x) \in \mathcal{F}_s$, $0 \le \deg(a(x)) \le s-1$ *and* $\deg(k(x)) < \deg(\hat{a}(x))$.

• *Type* 7 : $\mathcal{R}_n\big((k_1(x)|a_1(x)+ug_1(x))\big)+\mathcal{R}_n\big((k_2(x)|ua_2(x))\big)$, *where* $k_i(x) \in \mathcal{R}_{1,r}$, $a_i(x) \in \mathcal{F}_s$, $a_2(x) \mid_r a_1(x)$, $0 \le \deg(a_1(x)) \le s-1$, $g_1(x) \in \mathcal{R}_{1,s}$ *and* $\deg(g_1(x)) < \deg(a_2(x))$. *Moreover, $g_1(x)$ with the above conditions is unique.*

• *Type* 8 : $\mathcal{R}_n\big((\hat{a}(x)|0)\big)+\mathcal{R}_n\big((k_1(x)|a_1(x)+ug_1(x))\big)+\mathcal{R}_n\big((k_2(x)|ua_2(x))\big)$, *where* $\hat{a}(x) \in \mathcal{F}_r$, $0 \le \deg(\hat{a}(x)) \le r-1$, $a_i(x) \in \mathcal{F}_s$, $0 \le \deg(a_i(x)) \le s-1$, $a_2(x) \mid_r a_1(x)$, $k_i(x) \in \mathcal{R}_{1,r}$, $\deg(k_i(x)) < \deg(\hat{a}(x))$ *and* $\deg(g_1(x)) < \deg(a_2(x))$. *Moreover, $g_1(x)$ with the above conditions is unique.*

4. Conclusion

In this paper, we have determined additive skew cyclic codes over $\mathbb{F}_q\mathbb{F}_q[u]$.

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To investigate a multi-singular pointwise defined fractional q-integro-differential equation with application

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Abstract. In this study, by using the Caputo type q–derivative and the Riemann-Liouville type fractional q–derivative, we investigate a multi-singular pointwise defined fractional q–integro-differential equation under some boundary conditions on a time scale. New existence results rely on α-admissible map and the fixed point theorem for α-ψ-contraction map. Lastly, we present an example with application and some algorithms illustrate the primary effects.

Keywords: Singularity, pointwise defined equations, integral boundary conditions, Caputo q–derivation

AMS Mathematics Subject Classification [2010]: 34A08, 34B16, 39A13

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1. Introduction

The subject of fractional differential equations ranges from the theoretical views of existence and uniqueness of solutions to the analytical and mathematical methods for finding solutions [?, ?]. The quantum calculus is an old subject that was first developed by Jackson [?]. Recently, some researchers studied q–difference equations [?, ?, ?].

We investigate the existence of solutions for the following nonlinear pointwise defined fractional q–integro-differential equation

$$\mathbb{D}_q^{\alpha} u(t) + w\left(t, u(t), \mathbb{D}_q^{\beta} u(t), \int_0^t f(r) u(r)\, \mathrm{d}r, \varphi(u(t))\right) = 0, \tag{1}$$

on a time scale $\mathbb{T}_{t_0} = \{t : t = t_0 q^n\} \cup \{0\}$ here $t_0 \in \mathbb{R}$ and $q \in J = (0,1)$, under boundary conditions $\int_0^b u(r)\, \mathrm{d}r = 0$, $u'(1) = u(a)$ and $u^{(j)}(0) = 0$ for $j \geq 2$, here $\alpha \geq 2$, $a, b, \beta \in J$, $\varphi : \overline{\mathcal{B}} \to \overline{\mathcal{B}}$ is a map such that $\|\varphi(u_1) - \varphi(u_2)\| \leq c_1 \|u_1 - u_2\| + c_2 \|u_1' - u_2'\|$, for some non-negative real numbers c_1 and c_2 belong to $[0, \infty)$ and all $u_1, u_2 \in \overline{\mathcal{B}} = C^1(\overline{J})$, where $\mathbb{D}_q$ is the Caputo fractional q–derivative and $w \in \overline{\mathcal{L}} = L^1(\overline{J})$ is singular at some points $t \in \overline{J} = [0,1]$. We say that, $\mathbb{D}_q^{\alpha} u(t) + g(t) = 0$ is a pointwise defined equation on $\overline{J}$ if there exists set $S \subset \overline{J}$ such that the measure of S^c is zero and the equation holds on S.

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2. Preliminaries

We consider the fractional q-calculus on the specific time scale $\mathbb{T} = \mathbb{R}$ where $\mathbb{T}_{s_0} = \{0\} \cup \{t : t = s_0 q^n\}$, for nonnegative integer n, $s_0 \in \mathbb{R}$ and $q \in J$. Let $a \in \mathbb{R}$. Define [?] $[a]_q = (1-q^a)/(1-q)$. The power function $(r-s)_q^n$ with $n \in \mathbb{N}_0$ is defined by $(r-s)_q^{(n)} = \prod_{k=0}^{n-1}(r - sq^k)$, for $n \geq 1$ and $(r-s)_q^{(0)} = 1$, where r and s are real numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ [?]. Also, for $\sigma \in \mathbb{R}$ and $a \neq 0$, we have $(r-s)_q^{(\sigma)} = r^\sigma \prod_{k=0}^{\infty}(r - sq^k)/(r - sq^{\sigma+k})$. If $s = 0$, then it is clear that $r^{(\sigma)} = r^\sigma$ [?]. The q–Gamma function is given by $\Gamma_q(\nu) = ((1-q)^{(\nu-1)})/((1-q)^{\nu-1})$, where $\nu \in \mathbb{R} \setminus \{0, -1, -2, \cdots\}$ [?]. Note that, $\Gamma_q(\nu+1) = [\nu]_q \Gamma_q(\nu)$. The q–derivative of function h, is defined by $\mathbb{D}_q y(\nu) = (y(\nu) - y(q\nu))/((1-q)\nu)$, and $\mathbb{D}_q y(0) = \lim_{\nu \to 0} \mathbb{D}_q y(\nu)$ [?]. Furthermore, the higher order q–derivative of a function y is defined by $\mathbb{D}_q^n y(\nu) = \mathbb{D}_q(\mathbb{D}_q^{n-1} y)(\nu)$, for $n \geq 1$, where $\mathbb{D}_q^0 y(\nu) = y(\nu)$ [?]. The q–integral of a function y is defined on $[0, \delta]$ by $\mathbb{I}_q y(\nu) = \int_0^\nu y(\xi)\, \mathrm{d}_q \xi = \nu(1-q)\sum_{k=0}^{\infty} q^k y(\nu q^k)$, for $0 \leq \nu \leq \delta$, provided the series is absolutely converges [?]. If ν in $[0, \tau]$, then $\int_\nu^\tau y(\nu)\, \mathrm{d}_q r = (1-q)\sum_{k=0}^{\infty} q^k[\tau y(\tau q^k) - \nu y(\nu q^k)]$, whenever the series exists. The operator $\mathbb{I}_q^n$ is given by $\mathbb{I}_q^0 y(\nu) = y(\nu)$ and $\mathbb{I}_q^n y(\nu) = \mathbb{I}_q(\mathbb{I}_q^{n-1} y)(\nu)$ for $n \geq 1$ and $y \in C([0, \tau])$ [?]. It has been proved that $\mathbb{D}_q(\mathbb{I}_q y)(\nu) = y(\nu)$ and $\mathbb{I}_q(\mathbb{D}_q y)(\nu) = y(\nu) - y(0)$ whenever y is continuous at $\nu = 0$ [?]. The fractional Riemann–Liouville type q-integral of the function y on $J = (0, 1)$ for $\sigma \geq 0$ is defined by $\mathbb{I}_q^0 y(t) = y(t)$ and

$$\mathbb{I}_q^\sigma y(t) = \frac{1}{\Gamma_q(\sigma)} \int_0^t (t - q\xi)^{(\sigma-1)} y(\xi)\, \mathrm{d}_q \xi = t^\sigma (1-q)^\sigma \sum_{k=0}^{\infty} q^k \frac{\prod_{i=1}^{k-1}(1 - q^{\sigma+i})}{\prod_{i=1}^{k-1}(1 - q^{i+1})} y(tq^k), \tag{2}$$

for $t \in J$ [?]. Also, the Caputo fractional q-derivative of a function y is defined by

$${}^c\mathbb{D}_q^\sigma y(t) = \mathbb{I}_q^{[\sigma]-\sigma}(\mathbb{D}_q^{[\sigma]} y)(t) = \frac{1}{\Gamma_q([\sigma] - \sigma)} \int_0^t (t - q\xi)^{([\sigma]-\sigma-1)} \mathbb{D}_q^{[\sigma]} y(\xi)\, \mathrm{d}_q \xi, \tag{3}$$

where $t \in J$ and $\sigma > 0$ [?]. It has been proved that $\mathbb{I}_q^\beta [\mathbb{I}_q^\sigma y](\nu) = \mathbb{I}_q^{\sigma+\beta} y(\nu)$ and ${}^c\mathbb{D}_q^\sigma (\mathbb{I}_q^\sigma y)(\nu) = y(\nu)$, where $\sigma, \beta \geq 0$ [?]. So from (3), we can write ${}^c\mathbb{D}_q^\sigma y(t) = \mathbb{I}_q^{2-\sigma} y''(t)$, for $t \in \overline{I} = [s_1, s_2] \subset \mathbb{R}$ and $y \in C^2(\overline{I})$. Now, we present some necessary notions. Throughout this article, we use the norms $\|.\|$, $\|(u, u')\|_* = \max\{\|u\|, \|u'\|1\}$, and $\|.\|_1$ for $\overline{\mathcal{A}} = C(\overline{J})$, $\overline{\mathcal{B}}$ and $\overline{\mathcal{L}}$, respectively. Let Ψ be the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, for all $t > 0$. One can check that $\psi(t) < t$ for all $t > 0$ [?]. The map $T : \mathcal{X} \to \mathcal{X}$ is called an α-admissible whenever $\alpha(x_1, x_2) \geq 1$ implies $\alpha(T(x_1), T(x_2) \geq 1$ where α maps $\mathcal{X}^2$ to $[0, \infty)$. Let $(\mathcal{X}, \rho)$ be a metric space, where $\psi \in \Psi$ and $\alpha : \mathcal{X}^2 \to [0, \infty)$ is a map. A self-map T define on $\mathcal{X}$ is called an α-ψ-contraction whenever $\alpha(x_1, x_2)\rho(T(x_1), T(x_2)) \leq \psi(\rho(x_1, x_2))$, for each $x_1, x_2 \in \mathcal{X}$ [?].

Lemma 2.1. *[?] Let $(\mathcal{X}, \rho)$ be a complete metric space. If $T : \mathcal{X} \to \mathcal{X}$ is continuous then T has a fixed point whenever there exists $x_0 \in \mathcal{X}$ such that $\alpha(x_0, T(x_0)) \geq 1$, $\psi \in \Phi$, $\alpha : \mathcal{X}^2 \to [0, \infty)$ a map and $T : \mathcal{X} \to \mathcal{X}$ an α-admissible α-ψ-contraction.*

Lemma 2.2. *[?] If $x \in \overline{\mathcal{A}} \cap \overline{\mathcal{L}}$ with $\mathbb{D}_q^\alpha x \in \mathcal{A} \cap \mathcal{L}$, then $\mathbb{I}_q^\alpha \mathbb{D}_q^\alpha x(t) = x(t) + \sum_{i=1}^{n} c_i t^{\alpha-i}$, where $[\alpha] \leq n < [\alpha] + 1$ and c_i is some real number.*

3. Main results

Lemma 3.1. *Let $\alpha \geq 2$, $[\alpha] = n - 1$, $a, b, q \in J$ and $g \in \overline{\mathcal{L}}$. The solution of the problem $\mathbb{D}_q^\alpha u(t) + g(t) = 0$ with the boundary conditions $u^{(j)}(0) = 0$ for $j \geq 2$, $u'(1) = u(a)$ and $\int_0^b u(r)\, \mathrm{d}r = 0$ is $u(t) = \int_0^1 G_q(t, s) g(s)\, \mathrm{d}_q s$, on a time scale $\mathbb{T}_{t_0}$ where $G_q(t, s)$ is given by*

$$\begin{cases} -A_0(t - qs)^{(\alpha-1)} + A_1(t)(1 - qs)^{(\alpha-2)} + A_2(t)(a - qs)^{(\alpha-1)} + A_3(t)(b - qs)^{(\alpha)}, & s \leq a, s \leq b, \\ -A_0(t - qs)^{(\alpha-1)} + A_1(t)(1 - qs)^{(\alpha-2)} + A_2(t)(a - qs)^{(\alpha-1)}, & b \leq s \leq a, \\ -A_0(t - qs)^{(\alpha-1)} + A_1(t)(1 - qs)^{(\alpha-2)} + A_3(t)(b - qs)^{(\alpha)}, & a \leq s \leq b, \\ -A_0(t - qs)^{(\alpha-1)} + A_1(t)(1 - qs)^{(\alpha-2)}, & a \leq s, b \leq s, \end{cases} \tag{4}$$

whenever $0 \le s \le t \le 1$,

$$(5)\qquad \begin{cases} A_1(t)(1-qs)^{(\alpha-2)} + A_2(t)(a-qs)^{(\alpha-1)} + A_3(b-qs)^{(\alpha)}, & s \le a, s \le b,\\ A_1(t)(1-qs)^{(\alpha-2)} + A_2(t)(a-qs)^{(\alpha-1)}, & b \le s \le a,\\ A_1(t)(1-qs)^{(\alpha-2)} + A_3(t)(b-qs)^{(\alpha)}, & a \le s \le b,\\ A_1(t)(1-qs)^{(\alpha-2)}, & a \le s, b \le s, \end{cases}$$

whenever $0 \le t \le s \le 1$, *here* $A_0 = \frac{1}{\Gamma_q(\alpha)}$, $A_1(t) = \frac{b(1-a+t)-\mu(a,b)}{\mu(a,b)\Gamma_q(\alpha-1)}$, $A_2(t) = \frac{\mu(a,b)+b(a+t-1)}{\mu(a,b)\Gamma_q(\alpha)}$, $A_3(t) = \frac{\mu(a,b)(1-a)+t}{\mu(a,b)\Gamma_q(\alpha+1)}$ *and* $\mu(a,b) = b(1-a) + \frac{b^2}{2}$.

Note that, the mappings G and $\frac{\partial G}{\partial t}$ are continuous respect to t. Let w be a map on $\overline{J} \times \overline{\mathcal{B}}^2$ such that w is singular at some points of $\overline{J}$. Define the function $\Theta : \overline{\mathcal{B}} \to \overline{\mathcal{B}}$ by

$$\begin{aligned}\Theta_u(t) = &-\mathbb{I}_q^{\alpha} w\left(t, u(t), \mathbb{D}_q^{\beta}u(t), \int_0^t f(r)u(r)\,\mathrm{d}r, \varphi(u(t))\right) + A_1(t)\mathbb{I}_q^{\alpha-1} w\left(1, u(1), \mathbb{D}_q^{\beta}u(1), \int_0^1 f(r)u(r)\,\mathrm{d}r, \varphi(u(1))\right)\\ &+ A_2(t)\mathbb{I}_q^{\alpha} w\left(a, u(a), \mathbb{D}_q^{\beta}u(a), \int_0^a f(r)u(r)\,\mathrm{d}r, \varphi(u(a))\right) + A_3(t)\mathbb{I}_q^{\alpha+1} w\left(b, u(b), \mathbb{D}_q^{\beta}u(b), \int_0^b f(r)u(r)\,\mathrm{d}r, \varphi(u(b))\right)\end{aligned}$$

for all $t \in \overline{J}$. Then, we obtain

$$\begin{aligned}\Theta_u'(t) &= \int_0^1 \frac{\partial G}{\partial t}(t, qs) w\left(s, u(s), \mathbb{D}_q^{\beta}u(s), \int_0^s f(r)u(r)\,\mathrm{d}r, \varphi(u(s))\right)\,\mathrm{d}_q s\\ &= -\mathbb{I}_q^{\alpha-1} w\left(t, u(t), \mathbb{D}_q^{\beta}u(t), \int_0^t f(r)u(r)\,\mathrm{d}r, \varphi(u(t))\right) + \frac{b}{\mu(a,b)}\mathbb{I}_q^{\alpha-1} w\left(1, u(1), \mathbb{D}_q^{\beta}u(1), \int_0^1 f(r)u(r)\,\mathrm{d}r, \varphi(u(1))\right)\\ &\quad + \frac{b}{\mu(a,b)}\mathbb{I}_q^{\alpha} w\left(a, u(a), \mathbb{D}_q^{\beta}u(a), \int_0^a f(r)u(r)\,\mathrm{d}r, \varphi(u(a))\right)\\ &\quad + \frac{1}{\mu(a,b)}\mathbb{I}_q^{\alpha+1} w\left(b, u(b), \mathbb{D}_q^{\beta}u(b), \int_0^b f(r)u(r)\,\mathrm{d}r, \varphi(u(b))\right).\end{aligned}$$

Our key note is that the singular pointwise defined equation (1) has a solution if and only if the map Θ has a fixed point. Now, we give our main result.

THEOREM 3.2. *Let* $\alpha \ge 2$, $[\alpha] = n-1$, $a, b, q \in J$, $f \in \overline{\mathcal{L}}$ *with* $\|f\|_1 = m$, $\varphi : \overline{\mathcal{B}} \to \mathbb{R}$ *be such that* $|\varphi(u(t)) - \varphi(v(t))| \le c_1|u(t) - v(t)| + c_2|u'(t) - v'(t)|$, *for some* $c_1, c_2 \in [0, \infty)$. *Let* $\Omega : \overline{J} \times \overline{\mathcal{B}}^5 \to \mathbb{R}$ *is a mapping which is singular on some points* $\overline{J}$ *and* $|w(t, u_1, \dots, u_5) - w(t, v_1, \dots, v_5)| \le \sum_{i=1}^{k_0} \mu_i(t)\Omega_i(u_1 - v_1, \dots, u_5 - v_5)$, *for all* $u_1, u_2, v_1, v_2 \in \overline{\mathcal{B}}$ *and each* $t \in \overline{J}$, *where* k_0 *is a natural number,* $\mu_i : \overline{J} \to \mathbb{R}^+$, $\hat{\mu}_i \in \overline{\mathcal{L}}$, $\hat{\mu}_i(s) = (1-qs)^{\alpha-2}\mu_i(s)$, $\Omega_i : \overline{\mathcal{B}}^5 \to \mathbb{R}^+$ *is a nondecreasing mapping respect to all components with* $\frac{\Omega_i(\nu,\nu,\nu,\nu,\nu)}{\nu^{\gamma_i}} \to p_i$, *as* $\nu \to 0^+$ *for some* $\gamma_i > 0$, $p_i \in \mathbb{R}^+$ *with* $1 \le i \le k_0$. *Suppose that* $|w(t, u_1, \dots, u_5)| \le h(t)T(u_1, \dots, u_5)$, *for all* $(u_1, \dots, u_5) \in \overline{\mathcal{B}}^5$ *and almost all* $t \in \overline{J}$, *where* $h : \overline{J} \to \mathbb{R}^+$, $\hat{h} \in \overline{\mathcal{L}}$, $T : \overline{\mathcal{B}}^5 \to \mathbb{R}^+$ *is a nondecreasing mapping respect all their components such that* $\lim_{\nu \to 0^+} \frac{T(\nu,\nu,\nu,\nu,\nu)}{\nu} \in [0, \tau)$, *where* $\tau = (\ell\|\hat{h}\|_1 M_{\alpha,a,b})^{-1}$, $\ell = \max\{1, \frac{1}{\Gamma_q(2-\beta)}, m, c_1 + c_2\}$, $\mu(a,b)$ *is defined in Lemma 3.1 and*

$$\begin{aligned}M_{\alpha,a,b} = \max\Bigg\{&\frac{1}{\Gamma_q(\alpha)} + \frac{b(2-a)-\mu(a,b)}{\mu(a,,b)\Gamma_q(\alpha-1)} + \frac{\mu(a,b)+ab}{\mu(a,b)\Gamma_q(\alpha)} + \frac{\mu(a,b)(1-a)+1}{\mu(a,b)\Gamma_q(\alpha+1)},\\ &\frac{1}{\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha)} + \frac{1}{\mu(a,b)\Gamma_q(\alpha+1)}\Bigg\}.\end{aligned}$$

If $M_{\alpha,a,b}\sum_{i=1}^{k_0} p_i \ell^{\gamma_i}\|\hat{\mu}_i\|_{\overline{J}} < 1$, *then the pointwise defined equation* (1) *under boundary conditions* $u^{(j)}(0) = 0$ *for* $j \ge 2$, $\int_0^b u(r)\,\mathrm{d}r = 0$ *and* $u'(1) = u(a)$ *has a solution.*

4. An illustrative example with application

We consider a constrained motion of a particle along a straight line restrained by two linear springs with equal spring constant (stiffness coefficient) under external force and fractional damping along the t-axis (Fig. 1).

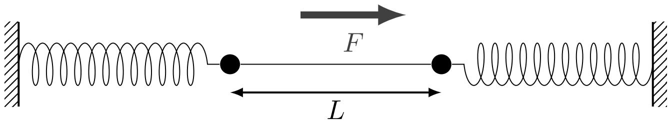

FIGURE 1. A particle along a straight line restrained by two linear springs with equal spring constant.

The vibration of the system is represented by a system of equations with the first equation having similar form of simple harmonic oscillator which cannot produce instability. Hence, we consider the pointwise defined equation

$$100\theta(t)^c\mathbb{D}_q^{2.5}u(t)+p(t)u(t)=-p(t)\left(+|u'(t)|+\left|\mathbb{D}_q^{\frac{1}{2}}u(t)\right|+\left|\int_0^t\frac{u(r)}{\sqrt{r}}\,\mathrm{d}r\right|+|\sin(u(t))|\right), \tag{6}$$

here $p(t)=\frac{1}{8}[2-2L-\eta^2L-\eta^2L\cos t]$, and η is constant and L is the unstretched length of the spring. We change Eq. (6) into form of the problem (1) as follow;

$$\mathbb{D}_q^{\frac{5}{2}}u(t)=\frac{1}{100\,\theta(t)}\left(|u(t)|+|u'(t)|+\left|\mathbb{D}_q^{\frac{1}{2}}u(t)\right|+\left|\int_0^t\frac{u(r)}{\sqrt{r}}\,\mathrm{d}r\right|+|\sin(u(t))|\right) \tag{7}$$

with boundary conditions $\int_0^{\frac{1}{3}}u(r)\,\mathrm{d}r=0$, $u'(1)=u(\frac{1}{4})$ and $u''(0)=0$, where $\theta(t)=0$ whenever $t\in\overline{J}\cap\mathbb{Q}$ and $\theta(t)=1-t$ whenever $t\in\overline{J}\cap\mathbb{Q}^c$. Take $f(\xi)=\frac{u(\xi)}{\sqrt{\xi}}$ and $\varphi(x)=\sin(x)$. Then, we get $p_1=0.01$, $\mu_1,h\in L^1$, $m=\|h\|_1=2$, $\|\hat{h}\|_{\overline{J}}=\|\hat{\mu}_1\|_{\overline{J}}=2$, T,Ω_1 are nonnegative and nondecreasing respect to $u_1,\dots,u_5$, $\mu(a,b)=\frac{11}{36}$, $\ell=2$, $M_{\alpha,a,b}=\max\{\frac{25}{11\Gamma_q(\frac{5}{2})}+\frac{10}{11\Gamma_q(\frac{3}{2})}+\frac{177}{44\Gamma_q(\frac{7}{2})},\frac{23}{11\Gamma_q(\frac{3}{2})}+\frac{12}{11\Gamma_q(\frac{5}{2})}+\frac{36}{11\Gamma_q(\frac{7}{2})}\}$. We put $\Lambda_1=\frac{25}{11\Gamma_q(\frac{5}{2})}+\frac{10}{11\Gamma_q(\frac{3}{2})}+\frac{177}{44\Gamma_q(\frac{7}{2})}$, $\Lambda_2=\frac{23}{11\Gamma_q(\frac{3}{2})}+\frac{12}{11\Gamma_q(\frac{5}{2})}+\frac{36}{11\Gamma_q(\frac{7}{2})}$. We can see that $M_{\alpha,a,b}=33.170478$, 21.551855, 16.363257, 15.234356, for $q=\frac{1}{8}$, $\frac{1}{2}$, $\frac{4}{5}$ and $\frac{8}{9}$, respectively. One can check that $\lim_{\nu\to0^+}\frac{T(\nu,\nu,\nu,\nu,\nu)}{\nu}=0.01\in[0,\tau)$, and $M_{\alpha,a,b}\sum_{i=1}^{k_0}\|\hat{\mu}_i\|_{\overline{J}}p_i\ell^{\gamma_i}<1$, for all $q\in J$. Now, with the results obtained, Theorem 3.2 implies that problem (7) has a solution.

5. Conclusion

One of such equations are pointwise defined multi-singular fractional q–differential equations.In this work, we study the existence of solutions for a pointwise defined multi-singular fractional q–integro-differential equations pointwise defined equations (1) on a time scale under some boundary conditions. The paper is appended with an application that describe the motion of a particle in the plane.

References

BSE properties of little Bloch and Zygmund type spaces

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Abstract. In this paper, we investigate the Bochner-Schoenberg-Eberlein or briefly *BSE* properties of the little Bloch type spaces. We also give the corresponding results for the little Zygmund type spaces. In order to get our results, we use the relations between these spaces, the analytic Lipschitz algebras and the differentiable Lipschitz algebras on the closed unit ball of the complex plane.

Keywords: little Bloch type spaces, little Zygmund type spaces, analytic Lipschitz algebras, differentiable Lipschitz algebras, *BSE*-algebras

AMS Mathematics Subject Classification [2010]: 46J15, 46E25, 46J10

1. Introduction

Let $\mathbb{D}$ denote the open unit ball of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denote the space of all complex-valued analytic functions on $\mathbb{D}$. For each $0 < \alpha < \infty$, the *Bloch type space* $\mathcal{B}^\alpha$ consists of all functions $f \in H(\mathbb{D})$ for which

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty.$$

The Bloch type space $\mathcal{B}^\alpha$ is a complex Banach space equipped with the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|, \tag{1}$$

for each $f \in \mathcal{B}^\alpha$. In the special case of $\alpha = 1$, we get the *classic Bloch space* $\mathcal{B}^1$ which is simply denoted by $\mathcal{B}$.

For each $0 < \alpha < \infty$, the *little Bloch type space* $\mathcal{B}_0^\alpha$ is the closed subspace of $\mathcal{B}^\alpha$ consisting of those functions $f \in \mathcal{B}^\alpha$ satisfying

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha|f'(z)| = 0.$$

For general background on the classic Bloch space, Bloch type spaces and little Bloch type spaces, see [2, 5] and references therein.

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The *classic Zygmund space* $\mathcal{Z}$ consists of all functions $f \in H(\mathbb{D})$ which are continuous on the closed unit ball $\overline{\mathbb{D}}$ and

$$\sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty,$$

where the supremum is taken over all $\theta \in \mathbb{R}$ and $h > 0$. By [2, Theorem 5.3], an analytic function f belongs to $\mathcal{Z}$ if and only if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)| < \infty.$$

Motivated by this, for each $0 < \alpha < \infty$, the *Zygmund type space* $\mathcal{Z}^\alpha$ is defined to be the space of all functions $f \in H(\mathbb{D})$ for which

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f''(z)| < \infty.$$

The Zygmund type space $\mathcal{Z}^\alpha$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{Z}^\alpha} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f''(z)|, \tag{2}$$

for each $f \in \mathcal{Z}^\alpha$. The *little Zygmund type space* $\mathcal{Z}_0^\alpha$ is the closed subspace of $\mathcal{Z}^\alpha$ consisting of those functions $f \in \mathcal{Z}^\alpha$ satisfying

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha|f''(z)| = 0.$$

More generally, for each $n \in \mathbb{N}$ and $0 < \alpha < \infty$, the space $\mathcal{Z}_n^\alpha$ consists of all functions $f \in H(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f^{(n+1)}(z)| < \infty.$$

The space $\mathcal{Z}_n^\alpha$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{Z}_n^\alpha} = |f(0)| + |f'(0)| + \cdots + |f^{(n)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f^{(n+1)}(z)|, \tag{3}$$

for each $f \in \mathcal{Z}_n^\alpha$. The little version of $\mathcal{Z}_n^\alpha$, denoted by $\mathcal{Z}_{n,0}^\alpha$, is the closed subspace of $\mathcal{Z}_n^\alpha$ consisting of those functions $f \in \mathcal{Z}_n^\alpha$ for which

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha|f^{(n+1)}(z)| = 0.$$

For more information about classic Zygmund space, Zygmund type spaces and little Zygmund type spaces, see [2, 5] and references therein.

For the spaces $\mathcal{B}^\alpha$ and $\mathcal{Z}^\alpha$, Hosseini, Feizi and Sanatpour [3] investigated the *BSE* property, defined in the next chapter. In this paper we investigate *BSE* properties of the corresponding little cases $\mathcal{B}_0^\alpha$ and $\mathcal{Z}_0^\alpha$.

2. Main results

Let $\mathcal{A}$ be a commutative Banach algebra with maximal ideal space $\Phi_\mathcal{A}$ and $C_0(\Phi_\mathcal{A})$ denote the space of all continuous functions on $\Phi_\mathcal{A}$ vanishing at infinity. The algebra $\mathcal{A}$ is embedded in $C_0(\Phi_\mathcal{A})$ by considering the Gelfand transform $a \mapsto \widehat{a}$, where $\widehat{a}(\varphi) = \varphi(a)$ for each $\varphi \in \Phi_\mathcal{A}$. A commutative Banach algebra $\mathcal{A}$ is called *without order* if $a \in \mathcal{A}$ and $a\mathcal{A} = \{0\}$ implies that $a = 0$. Given a without order commutative Banach algebra $\mathcal{A}$, a bounded linear operator $T : \mathcal{A} \to \mathcal{A}$ is called a *multiplier* if $a(Tb) = T(ab)$ for all $a, b \in \mathcal{A}$. The set of all multipliers on $\mathcal{A}$ is denoted by $M(\mathcal{A})$ which is a commutative unital Banach subalgebra of $\mathcal{B}(\mathcal{A})$, the space of all bounded linear operators on $\mathcal{A}$. Larsen in 1971 proved that for every $T \in M(\mathcal{A})$ there exists a unique bounded continuous function $\widehat{T}$ on $\Phi_\mathcal{A}$ such that $\widehat{(Tx)} = \widehat{T}\widehat{x}$ for all $x \in \mathcal{A}$. As an another definition of the multiplier algebra of $\mathcal{A}$, a complex-valued continuous function $T : \Phi_{\mathcal{A}} \to \mathbb{C}$ is a multiplier if $T \cdot \widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}$, that is

$$\mathcal{M}(\mathcal{A}) = \{T : \Phi_{\mathcal{A}} \to \mathbb{C} \mid T \text{ is continuous and } T \cdot \widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}\}.$$

A bounded continuous function σ on $\Phi_{\mathcal{A}}$ is called a *BSE-function* if there exists a positive constant $\beta > 0$ such that for any finite numbers of $\varphi_1, \varphi_2, \ldots, \varphi_n$ in $\Phi_{\mathcal{A}}$ and any complex numbers $c_1, c_2, \ldots, c_n$, the following inequality holds:

$$|\sum_{i=1}^{n} c_i \sigma(\varphi_i)| \leq \beta \| \sum_{i=1}^{n} c_i \varphi_i \|_{\mathcal{A}^*}.$$

The *BSE-norm* of σ is defined to be the infimum of all such β in the above inequality and $C_{BSE}(\Phi_{\mathcal{A}})$ denotes the set of all BSE-functions. Takahasi and Hatori [6, Lemma 1] proved that $C_{BSE}(\Phi_{\mathcal{A}})$ with the BSE-norm is a commutative semisimple Banach subalgebra of $C^b(\Phi_{\mathcal{A}})$, the space of all bounded continuous functions on $\Phi_{\mathcal{A}}$. The next definition is given by Takahasi and Hatori in [6].

Definition 2.1. A without order commutative Banach algebra $\mathcal{A}$ is called a *BSE-algebra* if $\widehat{M(\mathcal{A})} = C_{BSE}(\Phi_{\mathcal{A}})$, where $\widehat{M(\mathcal{A})} = \{\widehat{T} : T \in M(\mathcal{A})\}$.

Bochner and Schoenberg in 1934 studied these algebras on the real line and then Eberlein in 1955 gave the extension for locally compact abelian groups G. Takahasi, Hatori, Kaniuth, Ülger and some other mathematicians studied this topic for the commutative Banach algebras, Banach function algebras and some other well-known algebras [1, 4, 6]. Hosseini, Feizi and Sanatpour in [3] investigated BSE properties of Bloch type spaces $\mathcal{B}^\alpha$ and Zygmund type spaces $\mathcal{Z}^\alpha$. We next focus on the little cases $\mathcal{B}_0^\alpha$ and $\mathcal{Z}_0^\alpha$.

Our first result is the next lemma which plays an important roll in the rest of this paper.

Lemma 2.2. *Let $0 < \alpha < \infty$. Then, the closed unit ball of $\mathcal{B}_0^\alpha$ is closed with the pointwise convergence topology τ_{pw}.*

Using Lemma 2.2, we next prove that $\mathcal{B}_0^\alpha$ is a BSE-algebra if equipped with a suitable algebra norm. It is worth mentioning that in order to prove the next theorem, we use the relation between the little Bloch type space $\mathcal{B}_0^\alpha$ and the analytic little Lipschitz algebra $\ell ip_A(\overline{\mathbb{D}}, 1-\alpha)$ (see, for example, [5]). Recall that for each $0 < \alpha \leq 1$ the *Lipschitz algebra* of order α on $\overline{\mathbb{D}}$, denoted by $Lip(\overline{\mathbb{D}}, \alpha)$, is the algebra of all complex-valued functions f on $\overline{\mathbb{D}}$ for which

$$p_\alpha(f) = \sup \left\{ \frac{|f(z) - f(w)|}{|z - w|^\alpha} : z, w \in \overline{\mathbb{D}} \text{ and } z \neq w \right\} < \infty.$$

For each $0 < \alpha < 1$, the subalgebra of those functions $f \in Lip(\overline{\mathbb{D}}, \alpha)$ for which

$$\lim_{|z-w| \to 0} \frac{|f(z) - f(w)|}{|z - w|^\alpha} = 0,$$

is called the *little Lipschitz algebra* of order α, denoted by $\ell ip(\overline{\mathbb{D}}, \alpha)$. The algebras $Lip(\overline{\mathbb{D}}, \alpha)$ for $0 < \alpha \leq 1$ and $\ell ip(\overline{\mathbb{D}}, \alpha)$ for $0 < \alpha < 1$ are Banach function algebras if equipped with the norm

$$\|f\|_\alpha = \|f\|_\infty + p_\alpha(f),$$

where

$$\|f\|_\infty = \sup_{z \in \overline{\mathbb{D}}} |f(z)|.$$

Let $A(\overline{\mathbb{D}})$ denote the disk algebra. Then, for each $0 < \alpha < 1$, the *analytic Lipschitz algebra* of order α is defined by

$$Lip_A(\overline{\mathbb{D}}, \alpha) = A(\overline{\mathbb{D}}) \cap Lip(\overline{\mathbb{D}}, \alpha),$$

and similarly, for each $0 < \alpha < 1$, the *little analytic Lipschitz algebra* of order α is defined by

$$\ell ip_A(\overline{\mathbb{D}}, \alpha) = A(\overline{\mathbb{D}}) \cap \ell ip(\overline{\mathbb{D}}, \alpha).$$

Theorem 2.3. *Let $0 < \alpha < 1$. Then, there exists an algebra norm on the little Bloch type space $\mathcal{B}_0^\alpha$ which is equivalent to the norm (1) and makes $\mathcal{B}_0^\alpha$ a BSE-algebra.*

In order to get the result of Theorem 2.3 for the little Zygmund type spaces $\mathcal{Z}_{n,0}^\alpha$ and $\mathcal{Z}_0^\alpha$, we use the desired relations between these spaces and differentiable Lipschitz algebras $\ell ip^n(\overline{\mathbb{D}}, 1-\alpha)$ (see, for example, [5]). Recall that for each $n \in \mathbb{N}$ and $0 < \alpha \leq 1$, the algebra of all complex-valued functions f on $\overline{\mathbb{D}}$ whose derivatives up to order n exist and $f^{(k)} \in Lip(\overline{\mathbb{D}}, \alpha)$ for each k $(0 \leq k \leq n)$, is denoted by $Lip^n(\overline{\mathbb{D}}, \alpha)$. Also, for each $0 < \alpha < 1$, the algebra $\ell ip^n(\overline{\mathbb{D}}, \alpha)$ is defined. These *differentiable Lipschitz algebras* equipped with the norm

$$\|f\|_{n,\alpha} = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\alpha}{k!} = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\infty + p_\alpha(f^{(k)})}{k!},$$

are Banach function algebras.

Corresponding to Lemma 2.2, we next give the desired result for the little Zygmund type spaces $\mathcal{Z}_{n,0}^\alpha$.

Lemma 2.4. *Let $n \in \mathbb{N}$ and $0 < \alpha < \infty$. Then, the closed unit ball of $\mathcal{Z}_{n,0}^\alpha$ is closed with the pointwise convergence topology τ_{pw}.*

By applying Lemma 2.4, we get the following result for the general little Zygmund type spaces $\mathcal{Z}_{n,0}^\alpha$.

Theorem 2.5. *Let $n \in \mathbb{N}$ and $0 < \alpha < 1$. Then, there exists an algebra norm on the little Zygmund type space $\mathcal{Z}_{n,0}^\alpha$ which is equivalent to the norm (3) and makes $\mathcal{Z}_{n,0}^\alpha$ a BSE-algebra.*

Finally, as a consequence of Theorem 2.5 in the special case of $n = 1$, we get the following result for the little Zygmund type space $\mathcal{Z}_0^\alpha$.

Theorem 2.6. *Let $n \in \mathbb{N}$ and $0 < \alpha < 1$. Then, there exists an algebra norm on the little Zygmund type space $\mathcal{Z}_0^\alpha$ which is equivalent to the norm (2) and makes $\mathcal{Z}_0^\alpha$ a BSE-algebra.*

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Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

On the normally torsion-freeness of König ideals

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Abstract. Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K, $\mathfrak{m} = (x_1, \ldots, x_n)$ be the graded maximal ideal of R, and $\{u_1, \ldots, u_{\beta_1(I)}\}$ be a maximal independent set of minimal generators of I such that $\mathfrak{m} \setminus x_i \notin \mathrm{Ass}(R/(I \setminus x_i)^t)$ for all $x_i \mid \prod_{i=1}^{\beta_1(I)} u_i$ and some positive integer t, where $I \setminus x_i$ denotes the deletion of I at x_i and $\beta_1(I)$ denotes the maximum cardinality of an independent set in I. In this paper, we prove that if $\mathfrak{m} \in \mathrm{Ass}(R/I^t)$, then $t \geq \beta_1(I) + 1$. As an application, we verify that under certain conditions, every unmixed König ideal is normally torsion-free, and so has the strong persistence property.

Keywords: Associated primes, Normally torsion-free ideals, Strong persistence property, König ideals, Corner-elements

AMS Mathematics Subject Classification [2010]: 13B25, 13F20, 05E40

1. Introduction

Suppose that R is a commutative Noetherian ring, I is an ideal of R, and $\mathrm{Ass}_R(R/I)$ denotes the set of all prime ideals associated to I. Brodmann [1] proved that the sequence $\{\mathrm{Ass}_R(R/I^k)\}_{k\geq 1}$ of associated prime ideals is stationary for large k, i.e., there exists a positive integer k_0 such that $\mathrm{Ass}_R(R/I^k) = \mathrm{Ass}_R(R/I^{k_0})$ for all $k \geq k_0$. The minimum such k_0 is called the *index of stability of* I and $\mathrm{Ass}_R(R/I^{k_0})$ is called the *stable set of associated prime ideals of* I, which is denoted by $\mathrm{Ass}^\infty(I)$.

Many questions arise in the context of Brodmann's result. Recall that if I is an ideal in a commutative Noetherian ring R, then I is said to have the *persistence property* if $\mathrm{Ass}(R/I^k) \subseteq \mathrm{Ass}(R/I^{k+1})$ for all positive integers k. Moreover, an ideal I satisfies the *strong persistence property* if $(I^{k+1} : I) = I^k$ for all positive integers k. Furthermore, we say that I has the *symbolic strong persistence property* if $(I^{(k+1)} : I^{(1)}) = I^{(k)}$ for all k, where $I^{(k)}$ denotes the k-th symbolic power of I. An ideal I is called *normally torsion-free* if $\mathrm{Ass}(R/I^k) \subseteq \mathrm{Ass}(R/I)$ for all k, see [3, Definition 1.4.5]. In particular, Kaiser, Stehlík, and Škrekovski have shown that not all square-free monomial ideals have the persistence property. However, by applying combinatorial methods, it has been shown that many large families of square-free monomial ideals satisfy the persistence property and the strong persistence property. It has been shown that the persistence property and

*Speaker. Email address: msayedsadeghi@gmail.com also the strong persistence property holds for edge ideals of finite simple graphs, edge ideals of finite graphs with loops, and polymatroidal ideals. Furthermore, cover ideals of perfect graphs have the persistence property. In addition, a few examples of normally torsion-free monomial ideals appear from graph theory. It has been already proved that a finite simple graph G is bipartite if and only if its edge ideal is normally torsion-free. Moreover, it is well-known that the cover ideals of bipartite graphs are normally torsion-free. In addition, it has been verified that every transversal polymatroidal ideal is normally torsion-free. One of our motivations in this paper is to give a large class of square-free monomial ideals which satisfy normality, normally torsion-freeness, and the (symbolic) (strong) persistence property.

Now, let I be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$, and $\mathfrak{m} = (x_1, \ldots, x_n)$ be the graded maximal ideal of R. One motivating question in this field is the existence of the graded maximal ideal in the set of associated primes. It should be noted that little is known on this subject in literature. As another motivation of this paper, we give some results in this theme.

In what follows, we introduce notation and definitions which will be used in the rest of this paper.

Let I be a square-free monomial ideal and $\Gamma \subseteq \mathcal{G}(I)$, where $\mathcal{G}(I)$ denotes the unique minimal set of monomial generators of the monomial ideal I. We say that Γ is an *independent* set in I if $\gcd(f, g) = 1$ for each $f, g \in \Gamma$ with $f \neq g$. We denote the maximum cardinality of an independent set in I by $\beta_1(I)$. Furthermore, if I is a monomial ideal, then the *deletion* of I at x_i with $1 \leq i \leq n$, denoted by $I \setminus x_i$, is obtained by setting $x_i = 0$ in every minimal generator of I, that is, we delete every minimal generator such as $u \in \mathcal{G}(I)$ with $x_i \mid u$, see [2, page 303] for definitions of deletion and independent set.

Recall from [5, Definition 6.1.5] that if $u = x_1^{a_1} \cdots x_n^{a_n}$ is a monomial in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K, then the *support* of u is given by $\mathrm{supp}(u) := \{x_i | \ a_i > 0\}$.

Notice that a *König* ideal is a square-free monomial ideal I with $\mathcal{G}(I) = \{u_1, \ldots, u_r\}$ such that the maximum number of pairwise disjoint monomials $u_1, \ldots, u_r$ is equal to the height of I. Note that edge ideals of bipartite graphs form a large class of König ideals.

Let R be a unitary commutative ring and I an ideal in R. An element $f \in R$ is *integral* over I, if there exists an equation

$$f^k + c_1 f^{k-1} + \cdots + c_{k-1} f + c_k = 0 \text{ with } c_i \in I^i.$$

The set of elements $\overline{I}$ in R which are integral over I is the *integral closure* of I. The ideal I is *integrally closed*, if $I = \overline{I}$, and I is *normal* if all powers of I are integrally closed.

2. Main results

Theorem 2.1. *[4, Theorem 3.3] Let $I_1 \subset R_1 = K[x_1, \ldots, x_n]$ and $I_2 \subset R_2 = K[y_1, \ldots, y_m]$ be two monomial ideals in disjoint sets of variables. Let*

$$I = I_1 R + I_2 R \subset R = K[x_1, \ldots, x_n, y_1, \ldots, y_m].$$

Then $\mathfrak{p} \in \mathrm{Ass}(R/I)$ if and only if $\mathfrak{p} = \mathfrak{p}_1 R + \mathfrak{p}_2 R$, where $\mathfrak{p}_1 \in \mathrm{Ass}(R_1/I_1)$ and $\mathfrak{p}_2 \in \mathrm{Ass}(R_2/I_2)$.

Theorem 2.2. *[4, Theorem 3.4] Let $I \subset R = K[x_1, \ldots, x_n]$ be a monomial ideal, $\mathfrak{m} = (x_1, \ldots, x_n)$, t a positive integer, and $y_1, \ldots, y_s$ be distinct variables in R such that, for each $i = 1, \ldots, s$, $\mathfrak{m} \setminus y_i \notin \mathrm{Ass}(R/(I \setminus y_i)^t)$, where $I \setminus y_i$ denotes the deletion of I at y_i. Then $\mathfrak{m} \in \mathrm{Ass}(R/I^t)$ if and only if $\mathfrak{m} \in \mathrm{Ass}(R/(I^t : \prod_{i=1}^{s} y_i))$.*

Corollary 2.3. *[4, Corollary 3.5] Let $I \subset R = K[x_1, \ldots, x_n]$ be a square-free monomial ideal, $\mathfrak{m} = (x_1, \ldots, x_n)$, and $\{u_1, \ldots, u_{\beta_1(I)}\}$ be a maximal independent set of minimal generators of I such that $\mathfrak{m} \setminus x_i \notin \mathrm{Ass}(R/(I \setminus x_i)^t)$ for all $x_i \mid \prod_{i=1}^{\beta_1(I)} u_i$ and some positive integer t, where $I \setminus x_i$ denotes the deletion of I at x_i. If $\mathfrak{m} \in \mathrm{Ass}(R/I^t)$, then $t \geq \beta_1(I) + 1$.*

Theorem 2.4. *[4, Theorem 3.6] Let I be an unmixed König ideal in the polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K, $\mathfrak{m} = (x_1, \ldots, x_n)$, and $\{u_1, \ldots, u_{\beta_1(I)}\}$ be a maximal independent set of minimal generators of I such that $\mathfrak{m} \setminus x_i \notin \mathrm{Ass}(R/(I \setminus x_i)^t)$ for all t and $x_i \mid \prod_{i=1}^{\beta_1(I)} u_i$. Then the following statements hold:*

(i) *I is normally torsion-free.*
(ii) *I is normal.*
(iii) *I has the strong persistence proeprty.*
(iv) *I has the persistence property.*
(v) *I has the symbolic strong persistence property.*

Theorem 2.5. *[4, Theorem 3.7] Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K and $\mathfrak{m} = (x_1, \ldots, x_n)$. If there exists a square-free monomial $v \in I$ such that $v \in \mathfrak{p} \setminus \mathfrak{p}^2$ for any $\mathfrak{p} \in \mathrm{Min}(I)$, and $\mathfrak{m} \setminus x_i \notin \mathrm{Ass}(R/(I \setminus x_i)^s)$ for all s and $x_i \in \mathrm{supp}(v)$, then the following statements hold:*

(i) *I is normally torsion-free.*
(ii) *I is normal.*
(iii) *I has the strong persistence proeprty.*
(iv) *I has the persistence property.*
(v) *I has the symbolic strong persistence property.*

Definition 2.6. [4, Definition 3.10] Let F be a non-empty subset of $[n] = \{1, \ldots, n\}$. We denote by $\mathfrak{p}_F$ the monomial prime ideal $(x_j : j \in F)$. A *transversal polymatroidal ideal* is an ideal I of the form $I = \mathfrak{p}_{F_1} \cdots \mathfrak{p}_{F_r}$, where $F_1, \ldots, F_r$ is a collection of non-empty subsets of $[n]$ with $r \geq 1$.

Theorem 2.7. *[4, Theorem 3.11] Every square-free transversal polymatroidal ideal is normally torsion-free.*

Lemma 2.8. *[4, Lemma 4.4] Let I be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K. Let $\mathfrak{p} = (I^t : h)$ for some positive integer t and some monomial h in R such that $x_i \nmid h$ for some $1 \leq i \leq n$. Then $\mathfrak{p} \setminus x_i = ((I \setminus x_i)^t : h)$, and so $\mathfrak{p} \setminus x_i \in \mathrm{Ass}(I \setminus x_i)^t$.*

Corollary 2.9. *[4, Corollary 4.5] Let I be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K. If $\mathfrak{p} = (I^t : h)$ for some positive integer t and some monomial h in R such that $\mathfrak{p} \setminus x_i \notin \mathrm{Ass}(I \setminus x_i)^t$ for some $1 \leq i \leq n$, then $x_i \mid h$.*

Proposition 2.10. *[4, Proposition 4.8] Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K. Let $\mathfrak{p} = (I^s : h) \in \mathrm{Ass}(R/I^s)$ for some positive integer s and some monomial h in R. Then $\deg_{x_i} h \leq s - 1$ for each $i = 1, \ldots, n$.*

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پنجاه و دومین کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

A Sannon's entropy bound

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Abstract. Entropy of a dynamical system is a scientific concept, as well as a measurable physical property that is most commonly associated with a state of disorder. Estimation of topological entropies from naive numerical simulation of dynamical systems is difficult. In this paper we give upper and lower bounds for Shannon's entropy of a probability distribution (with the use of uniformly convex functions).

Keywords: Shannon's entropy, Jensen's inequality, uniformly convex function, bounds, refinements

AMS Mathematics Subject Classification [2010]: 37B40, 26A51, 37A35

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1. Introduction and Basic notions

The concept of entropy is an active research area in thermodynamics, dynamical systems, statistics, code theory, physics, and also in some other fields of mathematics [1,3,13]. Numerical calculations of entropy are still difficult. In [11, 12], the authors presented a strong upper bound for the classical Shannon entropy. In [7], the author obtain upper bound and lower bound for Shannon's entropy of information sources. In [11], the author obtained new and more precise bounds for Shannon's entropy. An upper global bound for a differentiable convex function was given by Dragomir in [4]. In this paper, we establish some new and strong bounds for Shannon's entropy. Extensions of this result are discussed in [4, 8, 11, 12].

Definition 1.1. Let $\{x_i\}_{i=1}^n \subseteq I := [a, b]$, and let $\{p_i\} \subseteq [0, 1]$ be coefficients such that $\sum_{i=1}^n p_i = 1$. The sum $\sum_{i=1}^n p_i x_i$ is called the convex combination of points x_i.

Definition 1.2. [2] Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Then f is uniformly convex with modulus $\psi : R_{\geq 0} \longrightarrow [0, +\infty)$ if is increasing, vanishes only at 0, and

$$f(\lambda x + (1-\lambda)y) + \lambda(1-\lambda)\psi(|x-y|) \leq \lambda f(x) + (1-\lambda) f(y)$$

for every $\lambda \in [0, 1]$ and $x, y \in [a, b]$.

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2. Main Results

We start this section with an extension of Jensen's inequality to a more general class of uniformly convex functions

LEMMA 2.1. *If* $a > 0$ *and* $f : [a, b] \longrightarrow \mathbb{R}$ *defined by* $f(x) = -\log x$, *then* f *is uniformly convex with modulus* $\psi(r) := \frac{r^2}{2b^2}$.

LEMMA 2.2. *If* $a > 0$ *and* $f : [a, b] \longrightarrow \mathbb{R}$ *defined by* $x \log(x)$, *then* f *is uniformly convex with modulus* $\psi(r) := \frac{r^2}{2b}$.

THEOREM 2.3. *Let* $f : I \longrightarrow \mathbb{R}$ *be an uniformly convex function with modulus* $\psi : \mathbb{R}_+ \longrightarrow [0, +\infty]$ *on* I, $\{x_k\}_{k=1}^n \subseteq [a, b]$ *be a non-decreasing sequence. Then the inequality*

$$f(\sum_{k=1}^{n} p_k x_k) + \sum_{k=1}^{n-1} p_k p_{k+1} \psi(x_{k+1} - x_k) \leq \sum_{k=1}^{n} p_k f(x_k) \tag{1}$$

holds for every convex combination $\sum_{k=0}^{n} p_k x_k$ *of points* $x_k \in I$.

THEOREM 2.4. *If* f *is uniformly convex with modulus* $\psi : \mathbb{R}_+ \longrightarrow [0, +\infty]$ *on* I *and* $x_1 \leq x_2 \leq ... \leq x_n$. *Then the inequality*

$$\begin{aligned}\sum_{i=1}^{n} p_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i) \geq \max_{1 \leq \mu < \nu \leq n} \{ & p_\mu f(x_\mu) + p_\nu f(x_\nu) \\ & - (p_\mu + p_\nu) f(\frac{p_\mu x_\mu + p_\nu x_\nu}{p_\mu + p_\nu}) + J_1^{\psi,\overline{x}}(x_\mu, x_\nu)\} \geq 0\end{aligned}$$

holds for every convex combination $\sum_{i=1}^{n} p_i x_i$ *of points* $x_i \in I$, *where*

$$\begin{aligned} J_1^{\psi,\overline{x}}(x_\mu, x_\nu) := & \frac{1}{\sum_{i \neq \mu,\nu} p_i} \sum_{i \notin A_{\mu,\nu}} p_i p_{i+1} \psi(x_{i+1} - x_i) + \frac{p_{\mu-1} p_{\mu+1} \psi(x_{\mu+1} - x_{\mu-1})}{\sum_{i \neq \mu,\nu} p_i} \\ & + \frac{p_{\nu-1} p_{\nu+1} \psi(x_{\nu+1} - x_{\nu-1})}{\sum_{i \neq \mu,\nu} p_i} + (p_\mu + p_\nu)(\sum_{i \neq \mu,\nu} p_i) \psi(|\frac{\sum_{i \neq \mu,\nu} p_i x_i}{\sum_{i \neq \mu,\nu} p_i} - \frac{p_\mu x_\mu + p_\nu x_\nu}{p_\mu + p_\nu}|), \end{aligned}$$

$A_{\mu,\nu} := \{\mu.\nu, \mu - 1, \nu - 1, n\}$ *for every* $\mu, \nu \in \{1, ..., n\}$, $p_0 p_2 \psi(|x_0 - x_2|) := 0$ *and* $p_{n-1} p_{n+1} \psi(|x_{n-1} - x_{n+1}|) := 0$.

LEMMA 2.5. *If* f *is uniformly convex with modulus* ψ *on* $[a, b]$ *and* $0 \leq p, q \leq 1$; $p + q = 1$, *then*

(1) $f(pa + qb) + f(qa + pb) \geq 2f(\frac{a+b}{2}) + \frac{1}{2}\psi(|(b - a)(p - q)|)$.
(2) $pf(a) + qf(b) - f(qa + pb) \leq f(a) + f(b) - 2f(\frac{a+b}{2}) - pq\psi(b - a) - \frac{1}{2}\psi(|(b - a)(p - q)|)$.

THEOREM 2.6. *If* f *is uniformly convex with modulus* ψ *on* I *and* $a \leq x_1 \leq x_2 \leq ... \leq x_n \leq b$, *then*

$$\sum_{i=1}^{n} p_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i) \leq f(a) + f(b) - 2f(\frac{a+b}{2}) - J_2^{\psi,\overline{x}}(a, b),$$

where

$$\begin{aligned} J_2^{\psi,\overline{x}}(a, b) := & \frac{(b - x_n)(x_1 - a)}{(b - a)^2} \psi(b - a) + \frac{1}{2} \psi(|a + b - 2 \sum_{i=1}^{n} p_i x_i|) \\ & + \frac{(b - \sum_{i=1}^{n} p_i x_i)(\sum_{i=1}^{n} p_i x_i - a)}{(b - a)^2} \psi(b - a). \end{aligned}$$

THEOREM 2.7. *If f is uniformly convex with modulus ψ on I and $a \le x_1 \le x_2 \le ... \le x_n \le b$, then*

$$\begin{aligned}
&\frac{1}{n}(f(x_1)+f(x_n)-2f(\frac{x_1+x_n}{2}))+\frac{1}{n(n-2)}\sum_{i=2}^{n-2}\psi(x_{i+1}-x_i)\\
&+\frac{2(n-2)}{n^2}\psi(|\frac{\sum_{i=2}^{n-1}x_i}{n-2}-\frac{x_1+x_n}{2}|)\\
&\le \frac{1}{n}\sum_{i=1}^{n}f(x_i)-f(\frac{\sum_{i=1}^{n}x_i}{n})\\
&\le f(x_1)+f(x_n)-2f(\frac{x_1+x_n}{2})-\frac{1}{2}\psi(2|\frac{\sum_{i=1}^{n}x_i}{n}-\frac{x_1+x_n}{2}|)\\
&-\frac{(x_n-\frac{1}{n}\sum_{i=1}^{n}x_i)(\frac{1}{n}\sum_{i=1}^{n}x_i-x_1)}{(x_n-x_1)^2}\psi(x_n-x_1),
\end{aligned} \tag{2}$$

DEFINITION 2.8. The Shannon's entropy of a positive probability distribution $X = (p_1, ..., p_n)$ is defined by $H(X) := -\sum_{i=1}^{n} p_i \log p_i$.

PROPOSITION 2.9. *[11] If $\mu := \min_{1\le i\le n}\{p_i\}$ and $\nu := \max_{1\le i\le n}\{p_i\}$, then*

$$\begin{aligned}
m(\mu,\nu) := \mu\log(\frac{2\mu}{\mu+\nu})+\nu\log(\frac{2\nu}{\mu+\nu}) &\le \log n - H(X)\\
&\le \log(\frac{(\mu+\nu)^2}{4\mu\nu}) := M(\mu,\nu).
\end{aligned} \tag{3}$$

PROPOSITION 2.10. *Define $\mu := \min_{1\le i\le n}\{p_i\}$ and $\nu := \max_{1\le i\le n}\{p_i\}$. Then*

$$\log n - \tilde{M}(\mu,\nu) \le H(X) \le \log n - \tilde{m}(\mu,\nu), \tag{4}$$

where

$$\tilde{m}(\mu,\nu) : \mu\log(\frac{2\mu}{\mu+\nu})+\nu\log(\frac{2\nu}{\mu+\nu})+\frac{\mu^2(2-n\mu-n\nu)^2}{2(\mu+\nu)(1-\mu-\nu)},$$

and

$$\tilde{M}(\mu,\nu) := \log(\frac{(\mu+\nu)^2}{4\mu\nu})-\frac{(\mu+\nu-2n\mu\nu)^2+2\mu\nu(1-\mu n)(\nu n-1)}{4\nu^2}.$$

EXAMPLE 2.11. Let $n = 10^k, \mu = 10^{-k-1}$ and $\nu = 10^{-k+1}(k > 2)$. Then $M(\mu,\nu) \simeq 1.406$, but $\tilde{M}(\mu,\nu) \simeq 1.202$.

EXAMPLE 2.12. Let $n = m^k, \mu = m^{-k-1}$ and $\nu = m^{-k+1}$. Then for every $k \in \mathbb{N}(k > 2)$.

$$M(\mu,\nu)-\tilde{M}(\mu,\nu)=\frac{(m^2+1)(m-1)^2}{4m^4}\longrightarrow 0.25,$$

as $m \longrightarrow +\infty$

PROPOSITION 2.13. *Define $\mu := \min_{1\le i\le n}\{p_i\}$ and $\nu := \max_{1\le i\le n}\{p_i\}$. Then*

$$\log n - \overline{M}(\mu,\nu) \le H(X) \le \log n - \overline{m}(\mu,\nu), \tag{5}$$

where

$$\overline{m}(\mu,\nu) := \mu\log(\frac{2\mu}{\mu+\nu})+\nu\log(\frac{2\nu}{\mu+\nu})+\frac{(2-n\mu-n\nu)^2}{4\nu n(n-2)}$$

and

$$\overline{M}(\mu,\nu) := n\mu\log(\frac{2\mu}{\mu+\nu}) + n\nu\log(\frac{2\nu}{\mu+\nu}) - \frac{(2-n\mu-n\nu)^2 + 2(n\nu-1)(1-n\mu)}{4\nu n}.$$

COROLLARY 2.14. *If* $\mu := \min_{1\le i\le n}\{p_i\}$ *and* $\nu := \max_{1\le i\le n}\{p_i\}$*, then*

$$\max\{\tilde{m}(\mu,\nu), \overline{m}(\mu,\nu)\} \le \log n - H(X) \le \min\{\tilde{M}(\mu,\nu), \overline{M}(\mu,\nu)\}.$$

EXAMPLE 2.15. Let $n = 10^k, \mu = 10^{-k-1}$ and $\nu = 10^{-k+1}(k > 2)$. Then

$$nm(\mu,\nu) - \overline{M}(\mu,\nu) > 2.$$

EXAMPLE 2.16. Let $n = 100^k, \mu = 100^{-k-1}$ and $\nu = 100^{-k+1}(k > 2)$. Then

$$nm(\mu,\nu) - \overline{M}(\mu,\nu) > 24.$$

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

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An efficient technique for solving state dependent delay differential equations

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Abstract. This work is devoted to introducing a new numerical method based on hybrid functions approximation for solving the neutral delay differential equations. First, we present the main properties of hybrid functions consisting of the block–pulse functions and Bernoulli polynomials. Then we utilize these properties to reduce the solution of neutral delay differential equations to a set of algebraic equations by first expanding the candidate function as hybrid functions with unknown coefficients. We use the collocation method to obtain the coefficients of the hybrid functions and discuss the error analysis. Finally, we solve some examples to demonstrate the high precision of the new technique.

Keywords: Neutral delay differential equation, Bernoulli polynomials, Error analysis

AMS Mathematics Subject Classification [2010]: 41A10, 65G99, 34K28

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1. Introduction

It is well known that it is difficult to analytically solve a delay system. Many researchers have devoted considerable effort to find numerical methods for solving neutral delay differential equations. These methods have included discontinuous Galerkin method [1], multistep methods [3], homotopy perturbation method [5], one-leg θ-method [6], Runge-Kutta method [7], etc. In this investigation, a numerical method is proposed to obtain an approximate solution of the state-dependent delay differential equations (DDEs).

2. Hybrid of block-pulse functions and Bernoulli polynomials

Hybrid functions $b_{nm}(t),\ \ n = 1, 2, \ldots, N,\ \ m = 0, 1, \ldots, M$ are defined on the interval $[0, t_f)$ as

$$b_{nm}(t) = \begin{cases} \beta_m(\frac{N}{t_f}t - n + 1), & t \in [\frac{n-1}{N}t_f, \frac{n}{N}t_f), \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where n and m are the order of block-pulse functions and Bernoulli polynomials. In Eq. (1), $\beta_m(t),\ \ m = 0, 1, 2, \ldots$ are the Bernoulli polynomials of order m, which can be defined by $\beta_m(t) = \sum\limits_{k=0}^{m} \binom{m}{k} \wp_{m-k} t^k$, where $\wp_k,\ \ k = 0, 1, \ldots, m$, are Bernoulli numbers.

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Suppose that $H = L^2[0,1]$ and $\{b_{10}(t), b_{11}(t), ..., b_{NM}(t)\} \subset H$ be the set of hybrid of block-pulse functions and Bernoulli polynomials then for $f \in H$ the best approximation of f using the set of hybrid of block-pulse functions and Bernoulli polynomials has the form

$$f \simeq P_M^N f = \sum_{m=0}^{M}\sum_{n=1}^{N} c_{nm} b_{nm}(t) = C^T B(t). \tag{2}$$

3. Problem statement

Consider the neutral delay differential equation with state dependent delays

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-\alpha_1(t,x(t))), x(t-\alpha_2(t,x(t))), \ldots, x(t-\alpha_j(t,x(t))), \\ \quad \dot{x}(t-\theta_1(t,x(t))), \ldots, \dot{x}(t-\theta_j(t,x(t)))), \qquad 0 \le t \le t_f, \\ x(0) = x_0, \end{cases} \tag{3}$$

where $x \in \mathbb{R}^l$ is a real valued l-vector function, f is assumed to be a sufficiently smooth real valued l-vector function, α_κ, θ_κ, $\kappa = 1, 2, ..., j$, τ_ν are assumed to be continuous functions.

4. Numerical method

In this section, we develop a numerical method for solving the problems given in Eqs. (3). For these problems we expand each of $\dot{x}_i(t)$, $i = 1, 2, \cdots, l$ by the hybrid of block-pulse functions and Bernoulli polynomials. Let

$$x(t) = [x_1(t), x_2(t), \cdots, x_l(t)]^T, \;\; f(t) = [f_1(t), f_2(t), \cdots, f_l(t)]^T. \tag{4}$$

Using Eqs. (2) and (4) we obtain

$$\dot{x}(t) = \hat{B}(t)X, \tag{5}$$

where X is a $l(M+1)N \times 1$ vector given by $X = [X_1, X_2, \cdots, X_l]^T$, and $\hat{B}(t) = I_l \otimes B^T(t)$, in which I_l is the l dimensional identity matrix, $\hat{B}(t)$ is $l \times l(M+1)N$ matrix as well, and $\otimes$ denotes Kronecker product. Using Eq (5), we get $x(t) = \hat{B}(t,1)X + x(0)$.

4.1. Solution of state dependent DDEs. Consider the state-dependent delay differential Eq. (3). By replacing Eqs. (5) in (3), we have

$$\begin{aligned} &\hat{B}(t)X = f(t, \hat{B}(t,1)X + x_0, \hat{B}(t-\alpha_1(t,\hat{B}(t,1)X + x_0),1)X + x_0, \hat{B}(t-\alpha_2(t,\hat{B}(t,1)X \\ &+x_0),1)X + x_0, \ldots, \hat{B}(t-\alpha_j(t,\hat{B}(t,1)X + x_0),1)X + x_0, \hat{B}(t-\theta_1(t,\hat{B}(t,1)X + x_0))X, \\ &\quad \ldots, \hat{B}(t-\theta_j(t,\hat{B}(t,1)X + x_0))X). \end{aligned} \tag{6}$$

For suitable collocation points, we choose N(M + 1) points as

$$t_{nm} = (\frac{t_f}{2N})(t_m + 2n - 1), \;\; n = 1, ..., N, \;\; m = 0, 1, 2, ..., M, \tag{7}$$

where t_m are the $M+1$ Legendre nodes for $[-1,1]$ and the collocation points t_{nm} are the shifted of t_m into $I_n = [\frac{n-1}{N}t_f, \frac{n}{N}t_f)$, $n = 1, ..., N$. We now collocate Eq. (6) at $(M+1)N$ points t_{nm} as

$$\begin{aligned} &\hat{B}(t_{nm})X - f(t_{nm}, \hat{B}(t_{nm},1)X + x_0, \hat{B}(t_{nm}-\alpha_1(t_{nm},\hat{B}(t_{nm},1)X + x_0),1)X + x_0, \\ &\hat{B}(t_{nm}-\alpha_2(t_{nm},\hat{B}(t_{nm},1)X + x_0),1)X + x_0, \ldots, \hat{B}(t_{nm}-\alpha_j(t_{nm},\hat{B}(t_{nm},1)X + x_0),1)X \\ &+x_0, \hat{B}(t_{nm}-\theta_1(t_{nm},\hat{B}(t_{nm},1)X + x_0))X, \ldots, \hat{B}(t_{nm}-\theta_j(t,\hat{B}(t_{nm},1)X + x_0))X) = 0, \end{aligned}$$

which corresponds to a system of $l(M+1)N$ nonlinear equations which can be solved for the elements of X using the well Newton's iterative method. Finally, we calculate $x(t)$.

5. Error bounds

In this section, we give the convergence results for NFDEs with state-dependent delays. For the sake of simplicity, we describe convergence analysis for $l = 1,\ x_1 = x$ and $f_1 = f$. At first, we provide the following notations and theorem which we use in this section. We also assume that the problem is sufficiently smooth and there are Lipschitz constants, $\lambda,\ \lambda_\kappa,\ \tilde{\lambda}_\kappa,\ \lambda_{w_k},\ \lambda_{\dot{w}_k},\ \lambda_{\alpha_\kappa}$ and $\lambda_{\theta_\kappa},\ \kappa = 1, 2, ..., j$ for which the following inequalities hold.

$$\begin{aligned} &\|f(t, w, w_1, w_2, ..., w_j, \dot{w}_1, \dot{w}_2, ..., \dot{w}_j) - f(t, u, u_1, u_2, ..., u_j, \dot{u}_1, \dot{u}_2, ..., \dot{u}_j)\|_{L^2(0,t_f)} \leq \\ &\quad +\lambda\|w - u\|_{L^2(0,t_f)} + \lambda_1\|w_1 - u_1\|_{L^2(0,t_f)} + ... + \lambda_j\|w_j - u_j\|_{L^2(0,t_f)} \\ &\quad +\tilde{\lambda}_1\|\dot{w}_1 - \dot{u}_1\|_{L^2(0,t_f)} + ... + \tilde{\lambda}_j\|\dot{w}_j - \dot{u}_j\|_{L^2(0,t_f)} \end{aligned} \tag{8}$$

$$\|w_\kappa(t_2) - w_\kappa(t_1)\|_{L^2(0,t_f)} \leq \lambda_{w_k}\|t_2 - t_1\|_{L^2(0,t_f)}, \tag{9}$$

$$\|\dot{w}_\kappa(t_2) - \dot{w}_\kappa(t_1)\|_{L^2(0,t_f)} \leq \lambda_{\dot{w}_k}\|t_2 - t_1\|_{L^2(0,t_f)}, \tag{10}$$

$$\|\alpha_\kappa(t, w) - \alpha_\kappa(t, u)\|_{L^2(0,t_f)} \leq \lambda_{\alpha_\kappa}\|w - u\|_{L^2(0,t_f)}, \tag{11}$$

$$\|\theta_\kappa(t, w) - \theta_\kappa(t, u)\|_{L^2(0,t_f)} \leq \lambda_{\theta_\kappa}\|w - u\|_{L^2(0,t_f)}. \tag{12}$$

Theorem 1 Suppose $x \in H^\mu(0, t_f)$ with $\mu \geq 0$, then [4]

$$\|x - P_M^N x\|_{L^2(0,t_f)} \leq cM^{-\mu}N^{-\mu}\left\|x^{(\mu)}\right\|_{L^2(0,t_f)}, \tag{13}$$

and for $1 \leq r \leq \mu$,

$$\|x - P_M^N x\|_{H^r(0,t_f)} \leq cM^{2r-\frac{1}{2}-\mu}N^{r-\mu}\left\|x^{(\mu)}\right\|_{L^2(0,t_f)}. \tag{14}$$

Now, we establish sufficient conditions for convergence of the proposed methods.

5.1. Convergence results for NFDEs with state-dependent delays. Throughout this section, we shall use these inequalities

$$\begin{aligned} &\|y(t - \alpha_\kappa(t, y(t))) - x(t - \alpha_\kappa(t, x(t)))\|_{L^2(0,t_f)} \leq \\ &\|y(t - \alpha_\kappa(t, y(t))) - x(t - \alpha_\kappa(t, y(t)))\|_{L^2(0,t_f)} + \lambda_x\lambda_{\alpha_\kappa}\|y(t) - x(t)\|_{L^2(0,t_f)}, \end{aligned} \tag{15}$$

for the delay term associated with state-dependent delay problems where the functions x and $\alpha_\kappa,\ \kappa = 1, 2, ..., j$ are assumed to satisfy the Lipschitz conditions (9) and (11). These inequalities follow by applying the triangle inequality and the Lipschitz conditions, and

$$\begin{aligned} &\|\dot{y}(t - \theta_\kappa(t, y(t))) - \dot{x}(t - \theta_\kappa(t, x(t)))\|_{L^2(0,t_f)} \leq \\ &\|\dot{y}(t - \theta_\kappa(t, y(t))) - \dot{x}(t - \theta_\kappa(t, y(t)))\|_{L^2(0,t_f)} + \lambda_{\dot{x}}\lambda_{\theta_\kappa}\|y(t) - x(t)\|_{L^2(0,t_f)}, \end{aligned} \tag{16}$$

for the derivative delay term associated with state-dependent delay problems where the functions $\dot{x}$ and θ_κ are assumed to satisfy the Lipschitz conditions (10) and (12).

Theorem 2: Let $x(t)$ be the exact solution of the state-dependent NFDE (3) and $u(t)$ be the approximate solution obtained by the proposed method in section 4. If $x \in H^\mu(I),\ I = (0, t_f)$ then for $\mu \geq 0$, we have

$$\|x(t) - u(t)\|_{L^2(I)} \leq \varepsilon\frac{\Upsilon}{1 - \sum_{\kappa=1}^{j}\tilde{\lambda}_\kappa}M^{-\mu}N^{-\mu}\|x^{(\mu)}\|_{L^2(I)}, \tag{17}$$

for $M \geq \mu - 1$ and for $r \geq 1$ we have,

$$\|x(t) - u(t)\|_{L^2(I)} \leq \varepsilon \frac{\Upsilon}{1 - \sum_{\kappa=1}^{j} \tilde{\lambda}_\kappa} M^{2r-\frac{1}{2}-\mu} N^{r-\mu} \left\| x^{(\mu)} \right\|_{L^2(I)}, \tag{18}$$

provided that N is sufficiently large, where λ, λ_κ, $\tilde{\lambda}_\kappa$, λ_x, $\lambda_{\dot{x}}$, λ_{α_κ} and λ_{θ_κ}, are Lipschitz constants, $\Upsilon = \lambda + \sum_{\kappa=1}^{j} (\lambda_\kappa(1 + \lambda_x \lambda_{\alpha_\kappa}) + \tilde{\lambda}_\kappa(1 + \lambda_{\dot{x}} \lambda_{\theta_\kappa}))$.

5.2. Example 1. Consider the following artificial problem with vanishing Delay

$$\begin{aligned} \dot{x}(t) &= cos(t)(1 + x(tx^2(t))) + \lambda x(t)\dot{x}(tx^2(t)) \\ &+(1 - \lambda)sin(t)cos(tsin^2(t)) - sin(t + sin^2 t), \; x(0) = 0. \end{aligned}$$

The exact solution is $x(t) = sin(t)$. In Tables 1 we compare the errors of the the present method with the Radau IIA method [2], by code Radar 5 for $\lambda = 0.3$.

TABLE 1. Error analysis of Example 1 for $\lambda = 0.3$ at the point $t = \pi$.

Methods	*Errors*
The Radau IIA method with ($Tol\ 10^{-8}$)	
$Number\ of\ steps = 120$	0.10×10^{-8}
Present method	
$N = 1, M = 11$	4.6×10^{-13}
$N = 2, M = 11$	1.5×10^{-15}

6. Conclusion

We have introduced a numerical algorithm for solving a large class of delay problems that includes neutral delay differential equations and differential equations with state delays. The error analysis carried out in this paper indicates the validity and applicability of the proposed method for smooth functions.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Injectivity and regular injectivity in PosAct-S

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Abstract. In this paper, we first study monomorphisms and regular monomorphisms and show that monomorphisms in **PosAct**-S are exactly one-one morphisms and regular monomorphisms in **PosAct**-S are exactly order embeddings. Then recalling the fact that the category **Pos** does not have any non-trivial (non-singleton) injective object with respect to monomorphisms, we see that **PosAct**-S has no non-trivial injective object, too. Then we study regular injectivity, that is, injectivity with respect to regular monomorphisms.

Keywords: poset act, injective poset act, regular injective poset act

AMS Mathematics Subject Classification [2010]: 06F05, 18G05, 20M30

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1. Introduction

The action of a monoid S on a set, namely S-act, is an important algebraic structure in mathematics and other mathematical areas such as graph theory and algebraic automata theory as well as in computer science. For example, computer scientists use the notion of a projection algebra (sets with an action of the monoid $(\mathbb{N}^\infty, min)$) as a convenient means of algebraic specification of process algebras (see [2,3]). Combining the notions of a poset and an act, many algebraic and categorical properties of the category of actions of a pomonoid on a poset, namely S-poset, have been studied. In fact, S-posets appear naturally in the study of mappings between posets (see [1]). More precisely, as S-acts correspond to representations of monoids by transformations of sets, S-posets correspond to order preserving representations of pomonoids by order preserving transformations of posets. Preliminary work on properties of S-posets was done by Fakhruddin in the 1980s and was continued in recent years. In the present paper, actions of a pomonoid S on a set, S-acts as unary algebraic structures, are investigated as algebras in the category **Pos**. Even when S is a pogroup the notions of S-poset and poset act are not the same and this motivates the author to study poset acts as a generalization of S-posets. The category of poset acts with action preserving monotone maps between them, first has been introduced and studied by Skornyakov in [4] where it is shown that the category of posets is fundamental and fundumentality is defined via the category **PosAct**-S. In [5], it is shown that the category of poset acts has enough regular injectives. Probably this is the first paper where regular injectivity of poset acts has been considered. Finally in [6], it is proved that every

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regular injective poset act is complete as a poset. Also, it is proved that all complete poset acts over a monoid S are injective if and only if S is a group. In this paper, we first study monomorphisms and regular monomorphisms and show that monomorphisms in **PosAct**-S are exactly one-one morphisms and regular monomorphisms in **PosAct**-S are exactly order embeddings. Then recalling the fact that the category **Pos** does not have any non-trivial (non-singleton) injective object with respect to monomorphisms, we see that **PosAct**-S has no non-trivial injective object, too. Then we study regular injectivity, that is, injectivity with respect to regular monomorphisms.

In this section, we briefly recall the preliminary notions about the actions of a monoid on a set and a pomonoid on a poset.

The category of all partially ordered sets (posets) with order preserving (monotone) maps between them is denoted by **Pos**. A poset is said to be *complete* if each of its subsets has an infimum and a supremum.

Let S be a monoid with 1 as its identity. A *right* S*-act* is a set A equipped with an action $\lambda : A \times S \to A, (\lambda(a, s)$ is denoted by $as)$ such that $a1 = a$ and $a(st) = (as)t$, for all $a \in A$ and $s, t \in S$. An S*-map* $f : A \to B$ between S-acts is an action preserving map, that is $f(as) = f(a)s$ for each $a \in A, s \in S$. The category of all S-acts and S-maps between them is denoted by **Act**-S.

Recall that a monoid (semigroup) S is said to be a *pomonoid* (*posemigroup*) if it is also a poset whose partial order $\leq$ is compatible with its binary operation (that is, $s \leq t$, $s' \leq t'$ imply $ss' \leq tt'$).

A *right* S*-poset* over a pomonoid S is a poset A which is also an S-act whose action $\lambda : A\times S \to A$ is order-preserving, where $A\times S$ is considered as a poset with componentwise order. An S*-poset map* (or *morphism*) is an action preserving monotone map between S-posets. Moreover, regular monomorphisms (equalizers) are exactly *order-embeddings*; that is, morphisms $f : A \to B$ for which $f(a) \leq f(a')$ if and only if $a \leq a'$, for all $a, a' \in A$. The category of all S-posets and S-poset maps between them is denoted by **Pos**-S.

2. The category PosAct-S

In the following, we introduce the category of acts in **Pos** and then the congruences in this category are introduced.

Definition 2.1. A *poset act* over a pomonoid S is a poset A together with a mapping $A \times S \to A, (a, s) \mapsto as$ such that
1. $a(st) = (as)t$,
2. $a1 = a$,
3. $a \leq a'$ implies $as \leq a's$ for every $a, a' \in A$ and $s, t \in S$.

This makes a poset act an ordered algebra, where all operations R_s are unary.

By a *poset act map* between poset acts, we mean an order preserving map which is also an S-map.

In Skornyakov's definition of poset acts in [4], S is a monoid, but since if S is a monoid, then the notions of S-poset and poset act coincide, we suppose that S to be a pomonoid and we study and compare the categories **Pos**-S of S-posets and **PosAct**-S of poset acts.

The category of all poset acts with action-preserving monotone maps between them is denoted by **PosAct**-S. It is easily seen that the category **Pos**-S is a full subcategory of **PosAct**-S.

As we mentioned above, each S-poset is a poset act but the converse is not true generally. For example, let $G = \{0, 1\}, 00 = 11 = 1, 01 = 10 = 0, 0 < 1$ be the two element pogroup and $A = \{a, b, c\}$ with the order $b < c$ be a poset. Then with the action $0a = a, 0b = b, 0c = c, 1a = 1b = 1c = b$, A becomes a poset act which is not an S-poset. This example shows that even when S is a pogroup the notions of S-poset and poset act are not the same.

Definition 2.2. If A is a poset act, a *congruence* θ on A is an equivalence relation on A that is compatible with the S-action, and has the further property that A/θ can be equipped with a partial order so that A/θ is a poset act and the natural map $A \to A/\theta$ is a poset act morphism.

Recall that if θ is any binary relation on A, we write $a \leq_\theta a'$ if a so-called θ-chain

$$a \leq a_1 \theta a_1' \leq a_2 \theta a_2' \leq \cdots \theta a_m' \leq a'$$

from a to a' exists in A.

3. Monomorphisms and regular monomorphisms

First, we show that the monomorphisms in **PosAct**-S are just the injective poset act maps. Note that a *monomorphism* in **PosAct**-S is a morphism that is left cancellable under composition.

Theorem 3.1. *Monomorphisms in* **PosAct**-S *are exactly one-one monotone* S*-maps.*

Notice that poset act order embeddings are injective, but the converse is not true. For example, the identity map from the discrete two element set $\mathbf{1} \sqcup \mathbf{1} = \{0, 1\}$ onto the two element chain $\mathbf{2} = \{0, 1\}$ with $0 < 1$, both considered as poset acts over a one-element pomonoid, is a monomorphism but it is not an order embedding.

Recall that a monomorphism f is called *regular* if it is the equalizer of a pair of morphisms, and f is *extremal* if, whenever $f = h \circ g$ and g is an epimorphism, then g is an isomorphism. Also, a poset act map $f : A \to B$ is called an order embedding if $f(a) \leq f(a')$ implies $a \leq a'$, for all $a, a' \in A$.

Similar to the case for S-posets, one can show that the classes of regular and extremal monomorphisms coincide with each other, and in fact are exactly poset act order embeddings.

Theorem 3.2. *For a monomorphism* $f : A \to B$ *in* **PosAct**-S*, the following are equivalent:*
(i) f *is regular,*
(ii) f *is extremal,*
(iii) f *is an order embedding.*

Definition 3.3. A poset act monomorphism is called *subregular* if it is the subequalizer of a pair of poset act maps.

As in the case of equalizers, one can prove that a subequalizer is always a monomorphism. Also, it is shown that in **PosAct**-S, not all monomorphisms are subregular. In fact, by showing that subregular monomorphisms are exactly order embedding morphisms and then applying Theorem 3.2, it is shown that, in **PosAct**-S, the regular monomorphisms coincide with the subregular monomorphisms.

Theorem 3.4. *A poset act monomorphism* $f : A \to B$ *is subregular if and only if it is an order embedding.*

4. Injective and regular injective poset acts

Theorem 4.1. **PosAct**-S *has no non-trivial injective object.*

We now study regular injectivity of poset acts. Recall that a regular injective object in **Pos**-S has zero bottom and top elements. As for poset acts, we have

Proposition 4.2. *Every non-trivial* (*non-singleton*) *regular injective poset act A is bounded by two zero elements.*

Theorem 4.3. *If A_S is a regular injective poset act then the S-poset $A^{(S)}$ is regular injective .*

In the following we give answer to the question that does **PosAct**-S have enough regular injectives? That is for any $A \in \mathbf{PosAct} - S$, does there exist a regular injective $E \in \mathbf{PosAct} - S$ with a regular monomorphisms $A \to E$? First recall the following:

Proposition 4.4. *Regular injective posets are exactly complete posets.*

Theorem 4.5. *Each poset act can be regularly embedded into a regular injective poset act.*

Theorem 4.6. *A poset act is regular injective if and only if every regular embedding $A \to B$ has a left inverse.*

Acknowledgement

Special thanks goes to Professor Mojgan Mahmoudi which the author is indebted to her for useful conversations during the preparation of this paper.

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On the some properties of generalized groups

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Abstract. In this article we introduce generalized groups as an extension of the structures of groups. It is an algebraic structure which has a background in the unified gauge theory and it has been studied first in 1999 by M.R. Molaei. We will review their structures and properties and we will give some examples and obtain some results in this subject.

Keywords: generalized group, generalized subgroup, group.

AMS Mathematics Subject Classification [2010]: 20Nxx.

1. Introduction

Generalized groups was introduced by Molaei in [3]. It is as an extension of groups. A generalized group is a non-empty set G admitting an operation called multiplication subject to the set of rules given below:
i) $x(yz) = (xy)z$; for all $x, y \in G$;
ii) For each $x \in G$, there exists a unique $e(x) \in G$ such that $xe(x) = e(x)x = x$;
iii) For each $x \in G$, there exists $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = e(x)$.

According to Araujo and Konieczny [2], generalized groups are equivalent to the notion of completely simple semigroups. In fact a semigroup G is called a completely simple semigroup if for all $g \in G$, $GgG = G$, and if a and b are idempotents in G such that $ab = ba$ then $a = b$. Here we call them as generalized groups. Some of the structures and properties of generalized groups have been studied by Vagner [6], Molaei [4], and Agboola [1]. Also, various applications of these algebraic structures are studied in the some recent papers.

In [5], Shajareh Poursalavati, introduced the concept of Molaei's generalized hypergroups by using construction of the Rees matrix semigroup over a polygroup P and a matrix with entries in P.

2. Properties of generalized groups

Definition 2.1. Let G be a non-empty set and "$\cdot$"be a binary operation on G. $(G, \cdot)$ is called a groupoid. If the equations $g \cdot x = h$ and $y \cdot g = h$ have unique solutions relative to x and y respectively and for all $g, h \in G$, then $(G, \cdot)$ is called a quasigroup. If $(G, \cdot)$

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be a groupoid, and for all g, h and $k \in G$, $(g \cdot h) \cdot k = g \cdot (h \cdot k)$, then $(G, \cdot)$ is called a semigroup.

Definition 2.2. A generalized group $(G, \cdot)$ is a semigroup, which is satisfy the following conditions:
(i) For each $g \in G$ there exists a unique $e(g) \in G$ such that $g \cdot e(g) = e(g) \cdot g = g$;
(iii) For each $g \in G$, there exists $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e(g)$.

Example 2.3. Assume that G be a group and e be the identity element of G, and Γ, I be nonempty sets. Let $P = (g_{\gamma i})$ be a $\Gamma \times I$ matrix with entries in G. Define a binary operation "" $\cdot$ "" on the set $I \times G \times \Gamma$ by $(i, k, \gamma) \cdot (j, h, \mu) = (i, kg_{\gamma j}h, \mu)$, for all $i, j \in I$, and $\gamma, \mu \in \Gamma$, and $k, h \in G$. It is easy to see that: $e((i, k, \gamma)) = (i, g_{\gamma i}^{-1}, \gamma)$ and $(i, k, \gamma)^{-1} = (i, g_{\gamma i}^{-1} k^{-1} g_{\gamma i}^{-1}, \gamma)$, so $(I \times G \times \Gamma, \cdot)$ is a generalized group. More ever, $I \times G \times \Gamma = \bigcup_{i \in I,\ \gamma \in \Gamma} \{i\} \times G \times \{\gamma\}$, which $\{i\} \times G \times \{\gamma\}$ isomorphic to G.

Definition 2.4. Let $(G, \cdot)$ be a generalized group. If $e(g \cdot h) = e(g) \cdot e(h), \forall g, h \in G$, then $(G, \cdot)$ is called normal generalized group.

Example 2.5. In general, by the notion of Example 2.3, $(I \times G \times \Gamma, \cdot)$ is not normal generalized group. In fact $e((i, k, \gamma) \cdot (j, h, \mu)) = e((i, kg_{\gamma j}h, \mu)) = (i, g_{\mu i}^{-1}, \mu)$ and $e((i, k, \gamma)) \cdot e((j, h, \mu)) = (i, g_{\gamma i}^{-1}, \gamma) \cdot (j, g_{\mu j}^{-1}, \mu) = (i, g_{\gamma i}^{-1} g_{\gamma j} g_{\mu j}^{-1}, \mu)$.

Example 2.6. Let F be a field and assume that $H = \left\{ \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \mid 0 \neq y, x \in F \right\}$, then H is a normal generalized group with ordinary matrices product. It is easy to see that: $e(\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ xy^{-1} & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 \\ x^2y^{-1} & y^{-1} \end{pmatrix}$, we have:

$$e(\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}\begin{pmatrix} 0 & 0 \\ z & t \end{pmatrix}) = e(\begin{pmatrix} 0 & 0 \\ yz & yt \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ zt^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ xy^{-1} & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ zt^{-1} & 1 \end{pmatrix}$$
$$= e(\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix})e(\begin{pmatrix} 0 & 0 \\ z & t \end{pmatrix}).$$

Definition 2.7. Let $(G, \cdot)$ be a generalized group. If $g \cdot h = g \cdot h, \forall g, h \in G$, then $(G, \cdot)$ is called Abelian generalized group. If G be an Abelian generalized group, then the cardinal of set $\{e(g) \mid g \in G\}$ is one, so G is an Abelian group.

In the next Theorem, we reviewed some properties of generalized group.

Theorem 2.8. *Let $(G, \cdot)$ be a generalized group and $g, h \in G$. Then, we have:*
1. *$e(e(g)) = e(g)$, i.e., $e(g)$ is unique;*
2. *$e(g) \cdot e(g) = e(g)$*
3. *g^{-1} is unique and $(g^{-1})^{-1} = g$;*
4. *If $(G, \cdot)$ be a normal generalized group in which $e(g) \cdot h^{-1} = h^{-1} \cdot e(g)$. Then, $(g \cdot h)^{-1} = h^{-1} \cdot g^{-1}$;*
5. *For all integer number n, $e(g^n) = e(g)$.*
6. *$G_g := \{\ x \in G\ :\ e(x) = e(g)\ \}$ is a group; and, $G = \bigcup_{g \in G} G_g$, therefore G is the union of disjoint some groups;*
7. *If G be a finite generalized group, then, there is a positive integer number n, such that, $g^n = e(g)$.*

Theorem 2.9. *Let $(G,\cdot)$ be a finite generalized group and $g \in G$. Then, the cardinal of G_g divided cardinal of G.*

Theorem 2.10. *Let $(G,\cdot)$ be a finite generalized group and $G = \bigcup_{g\in A} G_g$, where $A \subset G$ such that for all $a, b \in A$ and $a \neq b$ implies $G_a \neq G_b$. Then, the cardinal of A divided cardinal of G.*

Definition 2.11. Let $(G,\cdot)$ be a generalized group and S be a non-empty subset of G such that $(S,\cdot)$ is a generalized group. We recalled that S is a generalized subgroup of G and denoted by $S \leq G$.

Theorem 2.12. *Let $(G,\cdot)$ be a generalized group and S be a non-empty subset of G. Then $S \leq G$ if and only if for all $x,\ y \in S$, $x \cdot y^{-1} \in S$.*

Theorem 2.13. *Let G be a generalized group and $S \leq G$. Assume that $x \in S$, then S_x is a subgroup of G_x. In special case, if G be finite, then $card(S_x)$ divided $card(G_x)$, therefore $card(S_x)$ divided $card(G)$.*

Theorem 2.14. *Let G be a generalized group and $S \leq G$. Then there are $B \subseteq A \subseteq G$ such that*

$$G = \bigcup_{y\in A} G_y \quad \textit{and} \quad S = \bigcup_{x\in B} S_x$$

such that for all $y, z \in A$; $y \neq z$ implies $G_y \cap G_z = \emptyset$. In special case, if G be finite, then $card(S)$ divided $card(G)$ if and only if $card(B)$ divided $card(A)$.

Theorem 2.15. *Let G be a generalized group and $g \in G$. Then the sets $L_g = \{\ x \in G \mid xg = g\ \}$ and $R_g = \{\ x \in G \mid gx = g\ \}$ are generalized subgroups of G. More over, $\{e(g)\} = G_g \cap L_g = G_g \cap R_g = L_g \cap R_g$.*

Theorem 2.16. *Let G be a generalized group. Then H is a generalized subgroup of G, if and only if $H = \bigcup_{x\in G} H^x$, where that H^x is a subgroup of G_x, for all $x \in G$.*

Corollary 2.17. *Let G be a finite generalized group and H be a generalized subgroup of G. Then the generalized Lagrang Theorem may be not true for H and G, i.e., it may be $card(H)$ not divided $card(G)$.*

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Category of Graph Automaton

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Abstract. In this note, at first, by using the notion of zero-forcing set, we present the notion of graph automata, which is called GA. We show that for a given graph and for some zero forcing sets, various GA will be obtained. Also, we show that there exists a functor from the category of graphs to the category of nondeterministic automata. Thereafter, we prove that there is a functor from the category of graphs to the category of general fuzzy automata.

Keywords: Automata, Category, Graph

AMS Mathematics Subject Classification [2010]: 18B20, 68Q70

1. Introduction

An important branch of automata theory, itself one of the classical subdisciplines of computer science, concerns the study of finite automata as devices for classifying infinite, or possibly infinite, objects. Finite automata have many applications in plenty of areas of computer science such as databases, functional languages, bisimulation, and biology, for more information, see [5]. This current study aims to establish a connection between graphs and automata theory, which apparently show different mathematical structures.

2. Preliminaries

Definition 2.1. [1] Recall that a system (Q, δ) is called a non-deterministic automaton (NDA) over a monoid $(X, *)$, if Q is a set of states and $\delta : Q \times X \to 2^Q$ is a non-deterministic transition, where 2^Q is the power set of Q, such that:

(1) for every $q \in Q, \delta(q, 1_X) = \{q\}$,
(2) for every $x, y \in X, q \in Q, \delta(q, x * y) = \bigcup_{p \in \delta(q,x)} \delta(p, y)$.

An NDA over a monoid $(X, *)$ with initial and terminal states is a system (Q, δ, I, T) such that (Q, δ) is an NDA over X and $I, T \subseteq Q$. By NA_m, we denote the category of NDAs. A morphism in NA_m is defined as $\alpha : (Q_1, \delta_1) \to (Q_2, \delta_2)$, i.e., $\alpha : Q_1 \to Q_2$ is a map such that:

$$\alpha(\cup_{q' \in Q_1, \alpha(q') = \alpha(q)} \delta_1(q', x)) = \delta_2(\alpha(q), x) \cap \alpha(Q_1), \text{ for every } q \in Q_1, x \in X.$$

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By NA_m, we denote the category of NDAs with initial and terminal states.

A morphism $\alpha : (Q_1, \delta_1, I_1, T_1) \to (Q_2, \delta_2, I_2, T_2)$ is defined by: (i) $\alpha \in Mor((Q_1, \delta_1), (Q_2, \delta_2))$, (ii) $\alpha(I_1) = I_2 \cap \alpha(Q_1)$, (iii) $\alpha(T_1) = T_2 \cap \alpha(Q_1)$.

THEOREM 2.2. *[1] There exists a functor $F : NA_m \to K_G$, where K_G is the category of general fuzzy automata.*

DEFINITION 2.3. [3] A general fuzzy automaton (GFA) $\tilde{F}$ is an eight-tuple machine denoted by $\tilde{F} = (Q, X, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where (i) Q is a finite set of states, (ii) X is a finite set of input symbols, (iii) $\tilde{R}$ is a set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$, where $\tilde{P}(Q)$ is the fuzzy power set of Q, (iv) Z is a finite set of output symbols, (v) $\tilde{\delta} : (Q \times [0,1]) \times X \times Q \to [0,1]$ is the augmented transition function, (vi) $\omega : Q \to Z$ is the output function, (vii) $F_1 : [0,1] \times [0,1] \to [0,1]$ is called the membership assignment function. $(viii)$ $F_2 : [0,1]^* \to [0,1]$ is called the multi-membership resolution function.

Let the set of all transitions of $\tilde{F}$ be denoted by Δ. Now, suppose that $Q_{act}(t_i)$ is the set of every active states at time t_i, for every $i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$ and $Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) \big| \exists q' \in Q_{act}(t_{i-1}), \exists a \in X, \delta(q', a, q) \in \Delta\}$, for every $i \geq 1$, where $\mu^{t_i}(q)$ is the membership value of state q at time t_i.

DEFINITION 2.4. [2]

- Color-change rule: If G is a graph with each vertex colored either white or black, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black.
- Given a coloring of G, the derived coloring is resulted by the color-change rule until no more changes are possible.
- A zero forcing set for a graph G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived coloring of G is all black.
- $Z(G)$ refers to the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$.

We say that vertex u forces vertex v if v got black with u. If black vertex u of G changes the color of vertex v to black, then we say that vertex u forces vertex v.

3. Graph automata

DEFINITION 3.1. Let $G = (V, E)$ be a graph. A graph automaton (GA) is a five-tuple machine denoted by $\mathcal{A} = (Q, A, \varphi, I, T)$, where (i) $Q = V$ is the finite set of states, (ii) $A = \{f, n\}$ is the set of alphabet, (iii) $\varphi : Q \times A \to P(Q)$ is the transition function, where if vertex u forces vertex v in G, then define $\varphi(u, f) = v$ in $\mathcal{A}$ and if $uv \in E$ and u and v do not force each other, then $\varphi(u, n) = v$ and $\varphi(v, n) = u$, (iv) $I = Z(G)$ is the set of initial states, (v) T is the set of final states, which $u \in T$ if and only if u does not force any vertex. Naturally, φ can be extended to $\varphi^* : Q \times A^* \to P(Q)$.

Note that if $\mathcal{A}(Z(G))$ is a GA, then $\mathcal{A}(Z(G))$ recognizes a word w in A^* if $\varphi^*(i, w) \cap T \neq \emptyset$, for some $i \in I$.

EXAMPLE 3.2. Consider graph G as Figure 1, where $V(G) = \{p_1, p_2, p_3, p_4, p_5, p_6\}$. If $Z(G) = \{p_1, p_6\}$, then we have the GA $A(Z(G))$ as in Figure 2, where $Q = \{p_1, p_2, p_3, p_4, p_5, p_6\}, I = \{p_1, p_6\}, X = \{n, f\}, T = \{p_4\}$ and

$$\begin{array}{llll} \varphi(p_1, n) = p_6 & \varphi(p_6, n) = p_1 & \varphi(p_1, f) = p_2 & \varphi(p_6, f) = p_5 \\ \varphi(p_1, n) = p_5 & \varphi(p_5, n) = p_1 & \varphi(p_2, n) = p_5 & \varphi(p_5, n) = p_2 \end{array}$$

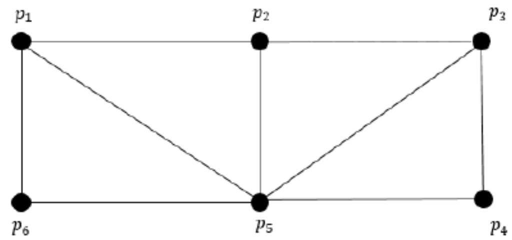


FIGURE 1. The graph G of Example ??

$\varphi(p_2, f) = p_3$ $\quad$ $\varphi(p_5, f) = p_4$ $\quad$ $\varphi(p_3, n) = p_5$ $\quad$ $\varphi(p_5, n) = p_3$
$\varphi(p_3, f) = p_4$.

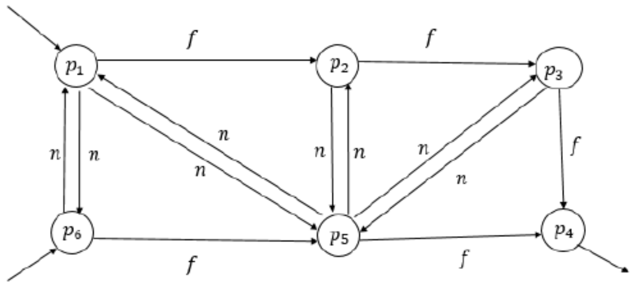


FIGURE 2. The GA $A(Z(G))$ of Example 3.2

EXAMPLE 3.3. Let graph G be as Figure 3 and choosing $Z_1(G) = \{p_2, p_3, p_4, p_5\}$ and $Z_2(G) = \{p_2, p_3, p_5, p_6\}$. Then the GA of them have the same language.

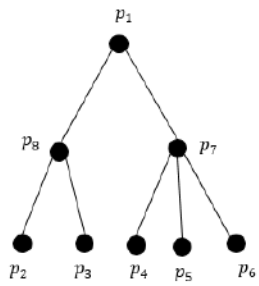
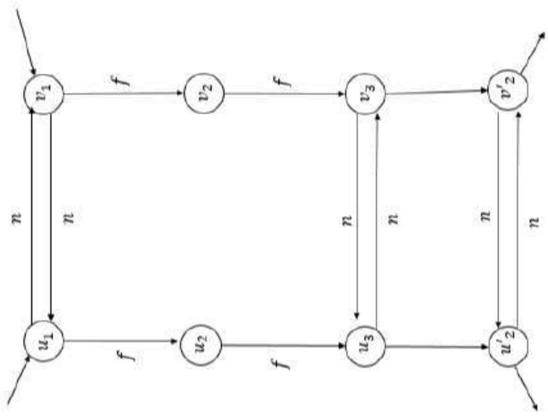


FIGURE 3. The graph G of Example 3.3

As we see in Example 3.2, it is clear that in a given symmetric graph, for some different symmetric zero forcing sets like $Z_1(G)$ and $Z_3(G)$, the Z-F-finite automata of them are equivalent. The graph of Example 3.3 is a nonsymmetric graph. In this example, we show that for some nonsymmetric graphs we also can find zero forcing sets in which the Z-F-finite automata of them have the same language.

LEMMA 3.4. *Let G be a symmetric graph. For every symmetric zero forcing sets $Z_1(G)$ and $Z_2(G)$, $\mathcal{A}(Z_1(G))$ and $\mathcal{A}(Z_2(G))$ are isomorphic. Also, $\mathcal{L}(\mathcal{A}(Z_1(G))) = \mathcal{L}(\mathcal{A}(Z_2(G)))$.*



4. Category of Graph Automata

Definition 4.1. Let $G = (V_G, E_G, Z(G))$ and $H = (V_H, E_H, Z(H))$. An isomorphism between G and H is a surjective and one-one function $\alpha : V_G \to V_H$ such that $\alpha(x)\alpha(y)$ belongs E_H if and only if xy belongs E_H, and $\alpha(Z(G)) = Z(H) \cap \alpha(G)$.

By ξ_G, we denote the category of graphs.

Example 4.2. Let $G = (V_G, E_G, Z(G))$ and $H = (V_H, E_H, Z(H))$ such that $V(G) = (p_1, p_2, p_3, p_4), E_G = (p_1p_2, p_2p_3, p_3p_4, p_4p_1)$ and $Z(G) = \{p_1, p_2\}$ and $V(H) = (q_1, q_2, q_3, q_4)$, $E_H = (q_1q_2, q_2q_3, q_3q_4, q_4q_1)$ and $Z(H) = \{q_3, q_4\}$. Consider $\alpha : V_G \to V_H$ by $\alpha(p_1) = q_3, \alpha(p_2) = q_4, \alpha(p_3) = q_1, \alpha(p_4) = q_2$. Clearly, α is a surjective and one-one function. Also, α is an isomorphism between G and H.

Theorem 4.3. *There exists a functor* $F : \xi_G \to NA_M$.

Corollary 4.4. *There exists a functor* $F : \xi_G \to K_G$.

5. Conclusion

In this note, by considering the notion of zero-forcing set, we give the notion of G-Automata. We show that there exists a functor from the category of graphs to the category of nondeterministic automata.

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Construction of Implicit-Explicit multivalue methods of high order and stage order for ODEs

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ABSTRACT. In this paper, we construct a class of numerical methods for solving initial value problems of differential equations which have both non-stiff and stiff parts. Such systems can be solved by a class of implicit-explicit (IMEX) diagonally implicit multistage integration methods (DIMSIMs), where the non-stiff part and stiff part are treated by explicit and implicit formulas, respectively. Assuming that the implicit part of methods is $L-$stable, we construct methods of order $p = 5$ and $p = 6$ and stage order $q = p$, with large absolute stability regions and show their efficiency by applying to some well-known problems.

Keywords: IMEX methods, General linear methods, Order conditions, Stability analysis

AMS Mathematics Subject Classification [2010]: 65L05

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1. Introduction

For many systems of ordinary differential equations (ODEs) there are natural splitting of the right-hand side of differential systems into two parts. Such systems can be written in the form

$$\begin{cases} y'(x) = f_1\big(y(x)\big) + f_2\big(y(x)\big), & x \in [x_0, \bar{x}], \\ y(x_0) = y_0, \end{cases} \tag{1}$$

where $f_1(y)$ represents the non-stiff part and $f_2(y)$ represents the stiff part of (1). The non-stiff part is solved by the explicit GLMs introduced by Butcher [3] which takes the form

$$\begin{aligned} Y_i^{[n]} &= h\sum_{j=1}^{s} a_{ij}F\big(Y_j^{[n]}\big) + \sum_{j=1}^{r} u_{ij}y_j^{[n-1]},\ i = 1, 2, \dots, s, \\ y_i^{[n]} &= h\sum_{j=1}^{s} b_{ij}F\big(Y_j^{[n]}\big) + \sum_{j=1}^{r} v_{ij}y_j^{[n-1]},\ i = 1, 2, \dots, r, \end{aligned} \tag{2}$$

*Speaker. Email address: moh.sharifi@tabrizu.ac.ir for $n = 1, 2, \dots, N$ and $a_{ii} = 0$ for $j \geq i$. Here $\big[Y_i^{[n]}\big]_{i=1}^{s}$ are approximations of stage order q to $y(x_{n-1} + c_i h)$, i.e., $Y_i^{[n]} = y(x_{n-1} + c_i h) + \mathcal{O}(h^{q+1})$, and $\big[y_i^{[n]}\big]_{i=1}^{r}$ are approximations of order p to the linear combinations of the derivative of the solution y at the point x_n,

$$y_i^{[n]} = \sum_{k=0}^{p} q_{ik} h^k y^{(k)}(x_n),$$

for some real vectors $q_k = [q_{ik}]_{i=1}^{r}$, $k = 0, 1, \dots, p$. The stiff part is solved by the implicit GLMs which takes the form

$$\begin{aligned} Y_i^{[n]} &= h \sum_{j=1}^{s} \hat{a}_{ij} F\big(Y_j^{[n]}\big) + \sum_{j=1}^{r} u_{ij} y_j^{[n-1]}, \ i = 1, 2, \dots, s, \\ y_i^{[n]} &= h \sum_{j=1}^{s} \hat{b}_{ij} F\big(Y_j^{[n]}\big) + \sum_{j=1}^{r} v_{ij} y_j^{[n-1]}, \ i = 1, 2, \dots, r, \end{aligned} \tag{3}$$

where $\hat{a}_{ii} = \lambda > 0$ and also where both methods have the same abscissa vector c and the coefficient matrices U and V.

2. A class of IMEX-DIMSIMs

First, we consider the transformation $y = x + z$, where

$$x' = f_1(x + z), \qquad z' = f_2(x + z). \tag{4}$$

For the system (4), the non-stiff and stiff parts will be solved by the explicit DIMSIMs and implicit DIMSIMs respectively, i.e.,

$$\begin{aligned} X^{[n]} &= h(A \otimes I_m) f_1\big(X^{[n]} + Z^{[n]}\big) + (U \otimes I_m) x^{[n-1]}, \\ x^{[n]} &= h(B \otimes I_m) f_1\big(X^{[n]} + Z^{[n]}\big) + (V \otimes I_m) x^{[n-1]}, \end{aligned} \tag{5}$$

and

$$\begin{aligned} Z^{[n]} &= h(\hat{A} \otimes I_m) f_2\big(X^{[n]} + Z^{[n]}\big) + (U \otimes I_m) z^{[n-1]}, \\ z^{[n]} &= h(\hat{B} \otimes I_m) f_2\big(X^{[n]} + Z^{[n]}\big) + (V \otimes I_m) z^{[n-1]}. \end{aligned} \tag{6}$$

Combining (5) and (6) leads to the class of so-called IMEX-DIMSIMs defined by

$$\begin{aligned} Y^{[n]} &= h(A \otimes I_m) f_1\big(Y^{[n]}\big) + h(\hat{A} \otimes I_m) f_2\big(Y^{[n]}\big) + (U \otimes I_m) y^{[n-1]}, \\ y^{[n]} &= h(B \otimes I_m) f_1\big(Y^{[n]}\big) + h(\hat{B} \otimes I_m) f_2\big(Y^{[n]}\big) + (V \otimes I_m) y^{[n-1]}, \end{aligned} \tag{7}$$

for $n = 1, 2, \dots, N$, where

$$Y^{[n]} = \big(X^{[n]} + Z^{[n]}\big), \qquad y^{[n]} = \big(x^{[n]} + z^{[n]}\big).$$

It was proved in [2] that if explicit and implicit method has order p and stage order $q = p$, then the overall method (7) has also order p and stage order $q = p$. See also [1].

3. Stability analysis of IMEX-DIMSIMs

To analyze the stability properties of IMEX-DIMSIMs, we will imply (7) to the test equation

$$y'(x) = \lambda_0 y(x) + \lambda_1 y(x), \quad x \geq 0,$$

where λ_0 and λ_1 are complex parameters corresponding to the non-stiff and stiff parts of (1), we obtain $y^{[n]} = M(z_0, z_1)y^{[n]}$, $n = 0, 1, \ldots,$ where $z_0 = \lambda_0 h$, $z_1 = \lambda_1 h$. Here $M(z_0, z_1)$ is the stability matrix defined by

$$M(z_0, z_1) = V + (z_0 B + z_1 \hat{B})(I - z_0 A - z_1 \hat{A})^{-1} U, \tag{8}$$

and also we define the stability function as the stability polynomial of $M(z_0, z_1)$, by

$$p(w, z_0, z_1) = \det\big(wI - M(z_0, z_1)\big),$$

where $w \in \mathbb{C}$. We say that the IMEX-DIMSIMs (7) is stable for given $z_0, z_1 \in \mathbb{C}$ if all roots $w_i(z_0, z_1)$ of stability polynomial $p(w, z_0, z_1)$ are inside of the unit circle. We will be mainly interested in IMEX-DIMSIMs which are $A-$, $A(\alpha)-$, or $L-$stable with respect to the implicit part $z_1 \in \mathbb{C}$.

4. Numerical experiment

Our test problem is the famous van der Pol system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= \left((1 - y_1^2)y_2 - y_1)\right)/\epsilon, \end{aligned} \tag{9}$$

$x \in [0, 055139]$, where the first component is non-stiff and the second component is stiff for small values of the parameter ϵ. The initial values are

$$y_1(0) = 2, \qquad y_2 = -\frac{2}{3} + \frac{10}{81}\epsilon - \frac{292}{2187}\epsilon^2 - \frac{1814}{19683}\epsilon^3 + O(\epsilon^4).$$

Table 1. Numerical results of the IMEX-DIMSIMs of orders $p = 5, 6$ with $\alpha = \frac{\pi}{2}$, for the problem (9) with $\epsilon = 10^{-6}$.

h	IMEX-DIMSIM5($S_{\frac{\pi}{2}}$)		IMEX-DIMSIM6($S_{\frac{\pi}{2}}$)	
	$\Vert error \Vert_1$	p	$\Vert error \Vert_1$	p
1.72e-3	4.75e-7		1.39e-2	
8.62e-4	1.48e-8	5.00	2.11e-4	6.04
4.31e-4	4.26e-10	5.12	3.27e-6	6.01
2.15e-4	2.21e-11	4.27	5.04e-8	6.02
1.08e-4			7.36e-10	6.10
5.38e-5			8.38e-12	6.46

In Table 1 we have presented the results of numerical experiments for $\epsilon = 10^{-6}$ with the methods IMEX-DIMSIMs of orders $p = 5,\ 6$ with $\alpha = \frac{\pi}{2}$.

5. Conclusion

In this paper we constructed IMEX DIMSIMs of order $p = 5$ and $p = 6$ and satge order $q = p$. It was demonstrated by some numerical experiments that these methods do not suffer from order reduction which is the case for some IMEX RK (Runge-Kutta) [4] methods.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

On modules with only finitely many small submodules

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ABSTRACT. In this article we introduce and study concept of modules with only finitely many small submodules (briefly, fs-module). Using this concept, we show that M is a fs-module if and only if $J(M)$ has only finitely submodules. Also, we show that if M is a fs-module, with non-zero small submodule, then $Soc(M) \neq 0$ and M is not semisimple. In particular, we prove that M is a fs-module if and only if $M = M_1 \oplus M_2$, where M_1 is semisimple and M_2 is a fs-module that $soc(M_2) \ll M$. Further, we prove if R is a right us-ring, then the Jacobson radical $J = J(R)$ is a minimal right ideal of R and $J^2 = 0$ and each right ideal A of R is either simple or non-small. Also, we show that if R is a commutative ring, then R has finite hollow dimension if and only if $\frac{R}{J(R)} \cong \oplus_1^n F_i$, where F_i is a field and conclude that if R be a semiprime local right fs-ring with finite hollow dimension, then R is a division ring.

Keywords: us-modules, fs-modules

AMS Mathematics Subject Classification [2010]: 16P60, 16P20, 16P40

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1. Introduction

It is known that Jacobson radical of R is a small ideal of R but in general, it is not true for an R-module. Motivated by this, one is led to consider one of the modules whose Jacobson radicas are small. In this way, an R-module M is defined a finite small module (briefly, fs-module) if it has only finitely many non-zero small submodules and one small (briefly, us-module) if it has only a non-zero small submodule. A ring R is called a fs-ring (us-ring), if as an R-module is a fs-ring (us-ring). In this article, we study some properties of fs-modules. For instance, we prove that An R-module M is a fs-module if and only if $J(M)$ has only finitely many submodules. Also, we show that if M is a fs-module with finite hollow dimension, then M is Noetherian and Artinian but with an example, we will see that there exist an R-module with finite hollow dimension such that it is not fs-module. We show that R is an us-ring if and only if the Jacobson radical of R is a minimal right ideal of R and $J^2 = 0$ ($J = J(R)$) if and only if each right ideal of R is either minimal or non-small. We also prove that every commutative ring R has finite hollow dimension if and only if $\frac{R}{J(R)} \cong \oplus_1^n F_i$, where F_i is a field. We prove that if R is a commutative ring with finite hollow dimension, then R has only finitely many maximal ideals.

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Throughout this paper rings are associative with identity and all modules are unital right R-modules. As usual, $J(M)$, $Soc(M)$ denote the Jacobson radical, the socle, of a module M.

2. fs-modules and their properties

We begin with the following definition.

Definition 2.1. Let M be an R-module. A proper submodule S of M is small in M if $S + N \neq M$ for every proper submodule N of M. We will indicate that S is a small submodule of M by notation $S \ll M$. Also, M is called hollow if $M \neq 0$ and every proper submodule S of M is small in M.

In the following, we recall some basic properties of small submodules, see [1].

Lemma 2.2. *Let M be a module and suppose $K \subseteq N \subseteq M$ and $P \subseteq M$. Then*

(1) *$N \ll M$ if and only if $K \ll M$ and $\frac{N}{K} \ll \frac{M}{K}$.*
(2) *$P + K \ll M$ if and only if $P \ll M$ and $K \ll M$.*
(3) *If $K \ll N$, then $K \ll M$.*
(4) *$J(M)$ is the sum of all the small submodules of M.*
(5) *If $S \ll M$, then M is finitely generated if and only if $\frac{M}{S}$ is finitely generated.*
(6) *$Soc(J(M))$ is small submodule of M. More generally, if N is small in M and $Soc(\frac{J(M)}{N}) = \frac{L}{N}$, then L is small in M.*
(7) *We have always $hdim(\frac{M}{N}) \leq hdim(M)$.*
(8) *If $N \ll M$, then $hdim(M) = hdim(\frac{M}{N})$.*
(9) *If M has finite hollow dimension and $hdim(M) = hdim(\frac{M}{N})$, then $N \ll M$.*
(10) *If $M = M_1 \oplus ... \oplus M_k$, then $hdim(M) = \sum_{i=1}^{n} hdim(M_i)$.*
(11) *If $M = M_1 \oplus M_2$ and $S_1 \subseteq M_1,\ S_2 \subseteq M_2$, then $S_1 \oplus S_2 \ll M_1 \oplus M_2$ if and only if $S_1 \ll M_1$ and $S_2 \ll M_2$.*

We give our definition of fs-modules and prove that interest results.

Definition 2.3. A right R-module M with only finitely many non-zero small submodules is said to be a fs-module. In particular, M is an us-module, if it has an unique non-zero small submodule. A ring R is called an fs-ring (us-ring), if as an R-module, it is an fs-module (us-module).

We note that every module which has not non-zero small submodule, is fs-module.

Definition 2.4. Let M be an R-module and $S \ll M$, then $\frac{M}{S}$ is called small quotient module of M.

The following result shows that the class of all fs-modules are closed under sumodules and small quotient modules.

Proposition 2.5. *The following are equivalent for any R-module M.*

(1) *M is a fs-module.*
(2) *Every submodule of M is a fs-module.*
(3) *Every small quotient module of M is a fs-modules.*

We note that if M is a fs-module, then $S \ll M$ if and only if $S \subseteq J(M)$.

Proposition 2.6. *Let R be a ring and M be an R-module. Then M is a fs-module if and only if $J(M)$ has only finitely submodules.*

PROOF. Since $J(M)$ is the sum of all small submodules of M, it follows that $J(M)$ contains all small submodules of M. Thus, if $J(M)$ has only finitely many submodules, then M is a fs-module. Now assume that M is a fs-module. Then $J(M) \ll M$, so every submodue S of M contained in $J(M)$ is a small submodule of M. This shows that $J(M)$ has only finitely many small submodules. $\square$

The following result is now immediate.

COROLLARY 2.7. *If M is a fs-module, then:*
(1) *$J(M)$ has finite length, so it is both Artinian and Noetherian.*
(2) *M is Noetherian (Artinian) if and only if $\frac{M}{J(M)}$ is Noetherian (Artinian).*

DEFINITION 2.8. A fs-module M, is said to be a homogeneuse fs-module if every submodule of M has non-zero small submodule.

PROPOSITION 2.9. *Let M be a fs-module, with non-zero small submodule. We have the following.*
(1) *$Soc(M) \neq 0$.*
(2) *$Soc(M) \neq M$, i.e., M is not semisimple.*
(3) *$Soc(M) \cap J(M) \neq 0$.*

PROOF. Since set of all small submodules of M has minimal elements which are minimal submodules of M and so $Soc(M) \neq 0$ and $Soc(M) \cap J(M) \neq 0$. Suppose, by way of contradiction, that $Soc(M) = M$. According to what was said, there exists minimal and small submodule S of M such that $S \subseteq Soc(M)$, thus $S \oplus K = Soc(M) = M$, for some submodule K of M, the contradiction required. $\square$

The following result is now immediate.

COROLLARY 2.10. *Let M is an Artinian fs-module. Then $J(M) \neq 0$, that is M has non-zero small submodules.*

The previous proposition, immediately yield the next result.

COROLLARY 2.11. *Let M is a homogeneous fs-module. Then $Soc(M) \subseteq_e M$.*

PROOF. By above proposition, for every submodule N of M, $0 \neq Soc(N) \subseteq N \cap Soc(M)$, and hence we are done. $\square$

THEOREM 2.12. *([5, 7.14], [6, 1.11]) If M has finite hollow dimension, then $\frac{M}{J(M)}$ is finitely generated semisimple.*

If M is a fs-module with finite hollow dimension then $\frac{M}{J(M)}$ is is both Noetherian and Artinian, by previous theorem. We may invoke the part (2) of Corollary 2.7 to infer that any fs-module M with finite hollow dimension is both Noetherian and Artinian. This implies that $J(M)$ is finitely generated. Moreover, such a module M, has a finite composition series. Therefore it has finite length and finite uniform dimension, too. The next theorem is devoted to these facts.

THEOREM 2.13. *Let M be a fs-module with finite hollow dimension over a ring R. The following holds.*
(1) *M is an Artinian module.*
(2) *M is a Noetherian module.*
(3) *$J(M)$ is finitely generated.*

(4) *M has a finite composition series.*
(5) *M has finite length.*
(6) *M has finite uniform dimension.*

LEMMA 2.14. *[3, Lemma 6] or [2] Let M be $AB5^*$. Then M is a qfd-module if and only if every submodule of M has finite hollow dimension.*

The following result is now immediate.

COROLLARY 2.15. *Let M be an $AB5^*$, qfd and fs-module. Then M is Artinian and Noetherian.*

REMARK 2.16. In general, every module with finite hollow dimension is not fs-module. For example, let $R = F[x_1, x_2, ..., x_n]$, where F is a field. So R is a commutative Noetherian ring and every maximal ideal M of R, has rank exactly n (i.e., there exists a chain of prime ideals of length n descending from M, but no longer chain). Let $A = E(S)$ be the injective envelope of a simple R-module S. By [7, Theorem 2], A is an Artinian module and by [4, Poroposion 5], $n - dimA = Rank(M)$, where M is a maximal ideal of R such that $S \cong \frac{R}{M}$. Thus there exists Artinian modules with Noetherian dimension of any natural number. Therefore, by taking A to be an Artinian module over R with $n - dimA > 1$. We infer that A is not fs-module. By Theorem 2.13, we know that every fs-module with finite hollow dimension is Noetherian. Thus Artinian modules that is not fs-module always exist.

DEFINITION 2.17. A right R-module with only finitely many minimal and small (unique minimal and small) submodules is called a fsm-module (usm-module). A ring with only finitely many small and minimal right ideals is called an fsm-ring (usm-ring).

Clearly every fs-module is fsm-module.

PROPOSITION 2.18. *Let M be a non-simple R-module. Then M is fsm if and only if $J(M)$ has only finitely many minimal submodules.*

PROOF. At first, we show that a minimal submodule N of M is small if and only if it contained in $J(M)$. For this, let N be a minimal submodule of M. If N is small, it is clear that $N \subseteq J(M)$. Conversely, let $N \subseteq J(M)$ and suppose that $N + K = M$ for submodule K of M. Since N is a minimal submodule and $N \cap K \subseteq N$, either $N \cap K = 0$ or $N \cap K = N$. If $N \cap K = 0$, then $N \oplus K = M$, i.e., K is a maximal submodule of M, so $N \subseteq J(M) \subseteq K$. This gives $K = M$. In case, $N \cap K = N$, it follows that $K = N + K = M$. Thus N is small in M, as we claimed. In particular, M has only finitely many small and minimal submodules if and only if $J(M)$ has only finitely many minimal submodules. □

REMARK 2.19. We note that for any semisimple submodule N of M that is contained in $J(M)$, we have $N = Soc(N) \subseteq Soc(J(M))$, so N is small, by part (6) of Lemma 2.2. This show that not only every minimal submodule of M contained in $J(M)$ is small, but also every semisimple submodule of M contained in $J(M)$ is small.

Next we give a structure theorem for fs-modules.

THEOREM 2.20. *Let M be an R-module . Then M is a fs-module if and only if $M = M_1 \oplus M_2$, where M_1 is semisimple and M_2 is a fs-module that $soc(M_2) \ll M$.*

Proof. We suppose that M is a fs-module and $T = \{S_1, S_2, \cdots, S_n\}$ is the set of all its non-zero small submodules. Let $T' = \{S_1, S_2, \cdots, S_m\}, m \leq n$ be set of all minimal elements of T and so $T' \subseteq Soc(M)$, by Proposition 2.9. If $T' = Soc(M)$, then $Soc(M) \ll M$, then $M = 0 \oplus M$ and we are done. But if $T' \neq Soc(M)$, then there exists a minimal submodule N_1 of M which is non-small. So there exist a proper submodule K_1 of M such that $N_1 + K_1 = M$. Since $N_1 \cap K_1 \subseteq N_1$ and N_1 is a minimal submodule of M. Thus either $N_1 \cap K_1 \neq 0$ or $N_1 \cap K_1 = N_1$. If $N_1 \cap K_1 = N_1$, then $N_1 \subseteq K_1$ and so $K_1 = N_1 + K_1 = M$, this is a contradiction. Thus $M = N_1 \oplus K_1$, where N_1 is a minimal and non-small submodule of M and K_1 is a maximal submodule of M. Similarly, if K_1 has a minimal submodule, N_2 say, which is non-small in K_1, there exist a submodule K_2 of K_1 such that $K_1 = K_2 \oplus N_2$. This implies that $M = N_1 \oplus N_2 \oplus K_2$. In the same way $M = M_1 \oplus M_2$, where $M_1 = \sum \oplus N_i$ is semisimple and M_2 is a fs-module which every its minimal submodules is small and so $soc(M_2) \ll M$. Conversely, since M_1 is semisimple, by Lemma 2.2(11) every small submodule of M is to form $0 \oplus S_2$ such that $S_2 \ll M_2$ and so M is a fs-module. □

the following result is now immediate.

Corollary 2.21. *Let M be an R-module with finite Goldie dimension. Then M is a fs-module if and only if $M = M_1 \oplus M_2$, where M_1 is a finitely generated semisimple module and M_2 is a fs-module that $soc(M_2) \ll M$.*

Theorem 2.22. *Let M be a fs-module. Then M is either $Soc(M) \ll M$ or there exists $N \subseteq Soc(M)$ such that $\frac{Soc(M)}{N} \ll \frac{M}{N}$.*

3. Some properties of fs-rings

We recall that R is an us-ring, if it has an unique non-zero small submodule.

Theorem 3.1. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is a right us-ring.*
(2) *The Jacobson radical $J = J(R)$ is a minimal right ideal of R and $J^2 = 0$.*
(3) *Each right ideal A of R is either simple or non-small.*

Proof. (1) ⇒ (2). It is easy to see that J, the right Jacobson of R is the unique small right ideal which is minimal. So $J \subseteq Soc(R)$ and $Soc(R).J = 0$ implies that $J^2 \subseteq Soc(R).J = 0$. Therefore $J^2 = 0$.
(2) ⇒ (3). Let A be a right ideal of R. If A is a right small ideal of R, then $A \subseteq J(R)$ and by(2), $A = J(R)$ is small. Otherwise, A non-small.
(3) ⇒ (1). We infer that always $J(R)$ is a small ideal of R and by (3), $J(R)$ is simple right ideal of R. Therefore R is an us-ring. □

Corollary 3.2. *If R is local and right us-ring, then $J(R)$ is minimal and maximal right ideal, so it is the unique non-trivial ideal of R.*

Corollary 3.3. *For a ring R the following statements are equivalent:*

(1) *R is a right fs-ring.*
(2) *Each right ideal A of R is either uniserial of finite length or it is non-small.*

The previous theorem is also true for us-modules.

Theorem 3.4. *Let R be a commutative ring. Then R has finite hollow dimension iff $\frac{R}{J(R)} \cong \oplus_1^n F_i$ where F_i is a field.*

Proof. By [8, Theorem 17.2.8], if R is a commutative ring, $\frac{R}{J(R)} \cong \oplus_{i\in I} F_i$, where F_i is a field and since R has finite hollow dimension, $\frac{R}{J(R)}$ is a finitely generated semisimple R-module, by Theorem 2.12. and we are done. Conversely, we suppose that $\frac{R}{J(R)} \cong \oplus_1^n F_i$, where F_i is a field. For F_i is Artinian hence $hdim(F_i) < \infty$ for any $i, 1 \leq i \leq n$. By Theorem 2.2(10), $hdim(R) = hdim(\frac{R}{J(R)}) = \sum_{i=1}^n hdim(F_i)$ and so R has finite hollow dimension. □

Corollary 3.5. *Let R be a commutative ring with finite hollow dimension. Then R is zero-dimensional.*

Corollary 3.6. *Let R be a semiprime local right fs-ring with finite hollow dimension. Then R is a division ring.*

Proof. By Theorem 2.13, R is an Artinian ring. Therefore $J(R) = M$ is nilpotent such that M is a maximal ideal. But R be a semiprime and so $M = 0$. □

Corollary 3.7. *Let R be a reduced right fs-ring with finite hollow dimension. Then only small ideal of R is zero, i.e., $J(R) = 0$.*

Proposition 3.8. *Let R be a ring. Then R is a right us-ring if R is either a local ring with unique maximal right ideal M such that $M^2 = (0)$, or R is a right usm-ring (i.e., $J(R)$ is only minimal small right ideal of R) and if R is a right usm-ring, then R is a right us-ring.*

Proposition 3.9. *Let R be a domain with finite hollow dimension. Then R is a right fs-ring if and only if R is a division ring. In particular a domain R with finite hollow dimension is a right fs-ring if and only if it is a left fs-ring.*

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Dominating sets in the perpendicular graphs of modules

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Abstract. Let R be a ring with identity and M be an R-module. Two modules A and B are called orthogonal, written $A \perp B$, if they do not have non-zero isomorphic submodules. Perpendicular graph of M defined as the graph $\Gamma_\perp(M)$, with the vertex set $\mathcal{M}_\perp = \{(0) \neq A \lneq M \mid \exists (0) \neq B \lneq M \text{ such that } A \perp B\}$ and two distinct vertices A and B are adjacent if and only if $A \perp B$. In this paper we study dominating set of $\Gamma_\perp(M)$.

Keywords: Dominating set, Orthogonal Submodules, Perpendicular Graph

AMS Mathematics Subject Classification [2010]: 05C25,16D10.

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1. Introduction

The investigation of graphs associated to algebraic structures is very important. Bosak [3] in 1964 defined the graph of semigroups. Inspired by his work, Csakany and Pollak [4] in 1969, studied the graph of subgroups of a finite group. Fundamental papers devoted to graphs assigned to a ring have appeared, see for example [1, 2]. In this paper, R be a ring with identity and M be an R-module, $\mathcal{M}_\perp = \{(0) \neq A \lneq M \mid \exists (0) \neq B \lneq M \text{ such that } A \perp B\}$ is the set of all vertices of perpendicular graph. As [5], we say that two modules A and B are orthogonal, written $A \perp B$, if they do not have non-zero isomorphic submodules. The perpendicular graph of M, denoted by $\Gamma_\perp(M)$, is an undirected simple graph with the vertex set $\mathcal{M}_\perp$ in which every two distinct vertices A and B are adjacent if and only if $A \perp B$ (see [5] for more details). If M is a simple R-module, then $\Gamma_\perp(M)$ has no vertices so that $\Gamma_\perp(M) = \emptyset$. We can see that every two non-isomorphic simple submodules of M are mutually orthogonal. A module M is called *atomic* if $M \neq 0$ and for any $x, y \in M \setminus \{0\}$, xR and yR have non-zero isomorphic submodules. A module M is *semi-artinian* if for every submodule $N \neq M$, we have $\mathrm{Soc}(\frac{M}{N}) \neq 0$. A ring R is left semi-artinian , if R is a semi-artinian left R-module. A module M is semi-artinian if and only if M is a loewy module. We say an R-module N is *subisomorphic submodule* of an R-module M, and denoted by $N \lesssim M$, when N is isomorphic to a submodule of M.

Inasmuch as Abelian groups are precisely $\mathbb{Z}$-modules, it is natural to try to relate perpendicular graph to an Abelian group M. Let M be a finitely generated Abelian group, so there exist prime number's $p_1, p_2, ..., p_m$ and positive integer's $\alpha_1, \alpha_2, ..., \alpha_m$ such that

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$$M \cong \underbrace{\mathbb{Z}_{p_1{}^{\alpha_1}} \oplus \mathbb{Z}_{p_2{}^{\alpha_2}} \oplus ... \oplus \mathbb{Z}_{p_m{}^{\alpha_m}}}_{tr(M)} \underbrace{\oplus \mathbb{Z} \oplus ... \oplus \mathbb{Z}}_{\beta(M)}$$

beti number of M denoted by $\beta(M)$, which the number of $\mathbb{Z}$'s and torsion part of M denoted by $tr(M)$.

For every graph G, we say that G is connected if there is a path between any two distinct vertices. In [5], we have shown that $\Gamma_{\perp}(M)$ is connected graph and also, we showed that graph $\Gamma_{\perp}(M)$ is empty if and only if M is atomic module. A *complete graph* is a graph in which every pair of distinct vertices are adjacent. A complete graph with n vertices is denoted by K_n. By a complete subgraph we mean a subgraph which is complete as a graph. A *bipartite graph* (or bigraph) is a graph whose vertices can be divided into two disjoint sets V_1 and V_2 (that is, V_1 and V_2 are each independent sets) such that every edge connects a vertex in V_1 to one in V_2. Assume that $K_{m,n}$ denoted the complete bipartite graph on two non-empty disjoint sets V_1 and V_2 with $|V_1| = m$ and $|V_2| = n$ (here m and n may be infinite cardinal number). A $K_{1,n}$ graph is often called a *star graph*. A *clique* of a graph is a maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the *clique number* of G. Let $\chi(G)$ denote the *chromatic number* of the graph G, that is, the minimal number of colors need to color the vertices of G so that no two adjacent vertices have the same color. Obviously $\omega(G) \leq \chi(G)$.

Let G be a graph with the vertex set V(G). A subset D of $V(G)$ is called a *dominating set* if every vertex of G is either in D or adjacent to at least a vertex in D, denoted by $\gamma - set$. The *domination number* of G, denoted by $\gamma(G)$, is the number of vertices in a smallest dominating set of G. A *total dominating set* of a graph G is a set S of vertices of G such that every vertex is adjacent to a vertex in S, denoted by $\gamma_t - set$. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. If G has no isolated vertices, then $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$. A dominating set C is said to be a *clique dominating set*, if the induced subgraph $< C >$ is a clique. The *clique domination number* $\gamma_{cl}(G)$ is the minimum cardinality of a clique dominating set of G. Clearly, if $\gamma(G) = 1$, then $\gamma_{cl}(G) = 1$. The porpose of this article is to study some properties of dominating sets of the perpendicular graphs of modules.

2. Main results

We investigate the dominating sets of perpendicular graph of R-module M. Before we state and prove our first main result, we express an auxiliary lemma.

Lemma 2.1. *Let A, B and C are submodules of M as R-module. Then the following facts hold.*

(1) *If $A \perp B$, then $A \cap B = 0$.*
(2) *If $B \cong C$ and $A \perp B$, then $A \perp C$.*

Let M be semi-artinian R-module such that $\Gamma_{\perp}(M) \neq \emptyset$ and D be a dominating set in $\Gamma_{\perp}(M)$. Now, assume that $A \in D$, since M is semi-artinian so there exists simple submodule S_A such that $S_A \subset A$ also it is clear that there exists simple submodule L_A such that $L_A \not\subset A$ (because, if A contains every simple submodule, then A is not adjacent to any non-trivial submodule of M and so $\Gamma_{\perp}(M) = \emptyset$ which is a contradiction). Let

$$D' = \{S_A | A \in D\} \cup \{L_A | A \in D\}$$

then D' is also a dominating set, because for every vertex B of $\Gamma_\perp(M)$, with $B \notin D'$, we have two cases:
(Case 1) If $B \notin D$, then there exists $B' \in D$ such that $B' \perp B$. But M is semi-artinian so there exists simple submodule $S_{B'}$ such that $S_{B'} \subset B'$. Thus $B \perp S_{B'}$ and since $S_{B'} \in D'$, i.e., B is adjacent to a vertex of D'.
(Case 2) If $B \in D$, so there exist simple submodules S_B and L_B such that $S_B \subset B$ and $L_B \not\subset B$. Thus $B \perp L_B$ and since $L_B \in D'$, i.e., B is adjacent to a vertex of D'.

Now we state the main result of this section, which is the dominating set of $\Gamma_\perp(M)$ of semi-artinian modules.

Theorem 2.2. *If M is semi-artinian R-module and $\mathcal{S}$ be a collection of non-isomorphic simple submodules of M such that $|\mathcal{S}| > 2$. Then the following hold.*

(1) *$\mathcal{S}$ is dominating set of $\Gamma_\perp(M)$.*
(2) *$\mathcal{S} \setminus \{T\}$ is not dominating set of $\Gamma_\perp(M)$.*

Proof. (1) Let $A \in \mathcal{M}_\perp \setminus \{\mathcal{S}\}$ then for any $S \in \mathcal{S}$, $A \neq S$ and since M is semi-artinian module so A contains a simple submodule. That is either $S \subset A$ or $S \cong T \subset A$. But there exists simple submodule S' of M such that $S' \in \mathcal{S}$ and $S' \not\subset A$. Hence $S' \perp A$, i.e., A is adjacent to a vertex of $\mathcal{S}$.
(2) On the contrary, assume that $\mathcal{B} = \mathcal{S} \setminus \{T\}$ is γ-set in $\Gamma_\perp(M)$. It is clear that $T \perp \Sigma_{S\in\mathcal{B}}S$, i.e., $\Sigma_{S\in\mathcal{B}}S$ is a vertex in $\Gamma_\perp(M)$ such that for any $S \in \mathcal{B}$, $S \not\perp \Sigma_{S\in\mathcal{B}}S$. That is $\Sigma_{S\in\mathcal{B}}S$ is not adjacent to any vertex in $\mathcal{B}$, i.e., $\mathcal{B}$ is not γ-set in $\Gamma_\perp(M)$, which is a contradiction. □

Remark 2.3. Let M be a semi-artinian R-module and $\mathcal{S}$ be a collection of non-isomorphic simple submodules of M such that $|\mathcal{S}| > 2$. Since all of elements of $\mathcal{S}$ are mutually adjacent, then subgraph $< \mathcal{S} >$ is a clique in $\Gamma_\perp(M)$. Thus $\mathcal{S}$ is γ_{cl}-set in $\Gamma_\perp(M)$ and we give $\gamma_{cl} = |\mathcal{S}|$.

The next results are immediate by previous fact.

Corollary 2.4. *If M is semi-artinian R-module and $\mathcal{S}$ be a collection of non-isomorphic simple submodules of M such that $|\mathcal{S}| > 2$. Then $\mathcal{S}$ is γ-set (γ_t-set and γ_{cl}-set) in $\Gamma_\perp(M)$.*

Corollary 2.5. *If M is semi-artinian R-module and $\mathcal{S}$ be a collection of non-isomorphic simple submodules of M such that $|\mathcal{S}| > 2$. Then $\gamma(\Gamma_\perp(M)) = \gamma_t(\Gamma_\perp(M)) = \gamma_{cl}(\Gamma_\perp(M)) = |\mathcal{S}|$.*

The condition of M to be a semi-artinian module in Theorem 2.2 is necessary: see the next example.

Example 2.6. Let $R = \mathbb{Z}$ and consider the R-module $M = \mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_q$ such that p, q are distinct prime numbers. Note, abelian group M is not semi-artinian and has two non-isomorphic simple submodules. We show that $D = \{\mathbb{Z}_p, \mathbb{Z}_q, \mathbb{Z}_{pq}\}$ is minimal dominating set in $\Gamma_\perp(M)$. To see this, for any $X \in \mathcal{M}_\perp \setminus \{D\}$, there exists $Y \in \mathcal{M}_\perp$ such that $X \perp Y$. Two cases may happen:
(Case 1) If $\beta(X) \neq 0$, then $\mathbb{Z} \lesssim X$ and we give $\beta(Y) = 0$. In this case, $tr(Y) \neq 0$ and so $tr(Y)$ is $\mathbb{Z}_p$ or $\mathbb{Z}_q$ or $\mathbb{Z}_{pq}$. That is three cases may happen:
(1) If $Y \cong \mathbb{Z}_p$, then $X \perp \mathbb{Z}_p$ such that $\mathbb{Z}_p \in D$.
(2) If $Y \cong \mathbb{Z}$, then $X \perp \mathbb{Z}_q$ such that $\mathbb{Z}_q \in D$.
(3) If $Y \cong \mathbb{Z}_{pq}$, then $X \perp \mathbb{Z}_{pq}$ such that $\mathbb{Z}_{pq} \in D$.
(Case 2) If $\beta(X) = 0$, then $tr(X) \neq 0$ and so $X \cong \mathbb{Z}_p$ or $X \cong \mathbb{Z}_q$ or $X \cong \mathbb{Z}_{pq}$. Since $X \perp Y$

by Lemma 2.1, $\mathbb{Z}_p \perp Y$ or $\mathbb{Z}_q \perp Y$ or $\mathbb{Z}_{pq} \perp Y$. Hence D is minimal dominating set in $\Gamma_\perp(M)$ and so we give $\gamma(\Gamma_\perp(M)) = 3$. But subgraph $< D >$ is not clique in $\Gamma_\perp(M)$. We can see that $C = \{\mathbb{Z}_p, \mathbb{Z}_q\}$ is γ_{cl}-set in $\Gamma_\perp(M)$ and we give $\gamma_{cl}(\Gamma_\perp(M)) = 2$.

Also the condition $|\mathcal{S}| > 2$, where $\mathcal{S}$ be a collection of non-isomorphic simple submodules of M in Theorem 2.2 is necessary: see the next example.

Example 2.7. Let $R = \mathbb{Z}$ and consider the R-module $M = \mathbb{Z}_6$. We can see that M is semi-artinian R-module and $\gamma(\Gamma_\perp(M)) = 1$ such that $|\mathcal{S}| = 2$.

Theorem 2.8. *Let M be an R-module. If $\gamma(\Gamma_\perp(M)) = 1$ then $\gamma_t(\Gamma_\perp(M)) = 2$.*

Proof. Suppose that $\gamma(\Gamma_\perp(M)) = 1$, hence $D = \{A\}$ is a minimum dominating set of $\Gamma_\perp(M)$. That is for any $K \in \mathcal{M}_\perp \setminus \{D\}$, $K \perp A$. On the other hand, by $\gamma(\Gamma_\perp(M)) \leq \gamma_t(\Gamma_\perp(M)) \leq 2\gamma(\Gamma_\perp(M))$ we give $1 \leq \gamma_t(\Gamma_\perp(M)) \leq 2$. Now we show that $\gamma_t(\Gamma_\perp(M)) \neq \gamma(\Gamma_\perp(M))$. On the contrary, $\gamma_t(\Gamma_\perp(M)) = \gamma(\Gamma_\perp(M))$ then $D' = \{A'\}$ is a minimum total dominating set of $\Gamma_\perp(M)$. That is for any $K \in \mathcal{M}_\perp$, $A' \perp K$. But $A' \in \mathcal{M}_\perp$, and so $A' \perp A'$ which is a contradiction. Thus $\gamma_t(\Gamma_\perp(M)) = 2$. $\square$

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

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52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

On p-subgroups of central automorphism group of a finite p-group

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ABSTRACT. Let G be a group and $L_c(G)$ be the central kernel of G, that is the set of all elements of G fixed by all central automorphisms of G. Also let $\mathrm{Aut}_{L_c}(G)$ denote the group of all central automorphisms of G fix $G/L_c(G)$ elementwise. In this paper, we give some results on the group $\mathrm{Aut}_{L_c}(G)$, where G is a finite p-group.

Keywords: Automorphism group, central kernel, central autocommutator subgroup, finite p-group.

AMS Mathematics Subject Classification [2010]: 20D45, 20D25, 20D15

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1. Introduction

Throughout the paper all groups are assumed to be finite and p denotes a prime number. By G', $Z(G)$, $\Phi(G)$, $\mathrm{Inn}(G)$ and $\mathrm{Aut}(G)$, respectively we denote the commutator subgroup, the center, the Frattini subgroup, the group of all inner automorphisms and the group of all automorphisms of G. For each $x \in G$ and $\alpha \in \mathrm{Aut}(G)$, the element $[x, \alpha] = x^{-1}x^{\alpha}$ is called the autocommutator of x and α. An automorphism α of G is called a central automorphism if $[x, \alpha] \in Z(G)$ for each $x \in G$. An automorphism α of G is called an IA-automorphism if $[x, \alpha] \in G'$ for all $x \in G$. An automorphism α of G is called a class preserving automorphism if $x^{\alpha} \in x^G$ for all $x \in G$, where x^G is the conjugacy class of x in G. Let $\mathrm{Aut}^Z(G)$, $\mathrm{Aut}^{G'}(G)$ and $\mathrm{Aut}_c(G)$ respectively, denote the group of all central automorphisms, IA-automorphisms and class preserving automorphisms of G. In 1994, Hegarty [4] introduced the concept of absolute center subgroup of a group G, as follows:

$$L(G) = \{x \in G \mid [x, \alpha] = 1, \forall \alpha \in \mathrm{Aut}(G)\}.$$

It is easy to check that the absolute center of G is a characteristic subgroup contained in the center of G.

Haimo in [3] introduced the following subgroup of a given group G, which is similarly [1], denoted by $L_c(G)$ as follows:

$$L_c(G) = \{x \in G \mid [x, \alpha] = 1, \forall \alpha \in \mathrm{Aut}^Z(G)\},$$

*Speaker. Email address: r_soleimani@pnu.ac.ir & rsoleimanii@yahoo.com and it is called the central kernel of G. Also

$$K_c(G) = \langle [x, \alpha] : x \in G, \alpha \in \mathrm{Aut}^Z(G) \rangle,$$

is said the central autocommutator subgroup of G ([1]). One can easily check that $L_c(G)$ is a characteristic subgroup of G contains $L(G)$ and $K_c(G)$ is a central characteristic subgroup of G. According [1] let

$$\mathrm{Aut}_{L_c}(G) = \{\alpha \in \mathrm{Aut}^Z(G) \mid [x, \alpha] \in L_c(G), \forall x \in G\},$$

which is a normal subgroup of $\mathrm{Aut}^Z(G)$. Clearly, by the definition of $L_c(G)$, $\mathrm{Aut}_{L_c}(G)$ acts trivially on the central kernel of G. Since the central automorphisms of G fix G' elementwise, it follows that $G' \leq L_c(G)$ and $G/L_c(G)$ is abelian. There are some results on the central kernel subgroup of a finite group G, see for example [1], [2].

Recall an abelian finite p-group A has invariants or is of type $(n_1, n_2, ..., n_k)$ if it is the direct product of cyclic subgroups of orders $p^{n_1}, p^{n_2}, ..., p^{n_k}$, where $n_1 \geq n_2 \geq ... \geq n_k > 0$.

In this paper, first we give a necessary and sufficient condition on a finite non-abelian p-group G for the groups $\mathrm{Aut}_{L_c}(G)$ and $\mathrm{Inn}(G)$ coincide. Also we characterize finite non-abelian p-group G such that $\mathrm{Aut}_{L_c}(G) = \mathrm{Aut}^{G'}(G)$. Finally, we give a necessary and sufficient condition for a finite p-group G such that $\mathrm{Aut}_{L_c}(G) = \mathrm{Aut}_c(G)$.

2. Main results

In this section, we provide some results concerning the group $\mathrm{Aut}_{L_c}(G)$, where G is a finite p-group.
First, we define two subgroups of $\mathrm{Aut}^Z(G)$ and G as follows:

$$C_{L_c}(G) = C_{\mathrm{Aut}^Z(G)}(\mathrm{Aut}_{L_c}(G)) = \{\alpha \in \mathrm{Aut}^Z(G) : \alpha\beta = \beta\alpha, \ \forall \beta \in \mathrm{Aut}_{L_c}(G)\},$$

and

$$E_{L_c}(G) = [G, C_{L_c}(G)].$$

Obviously, $E_{L_c}(G)$ is characteristic in G, which is contained in $K_c(G)$. Also, if $G/Z(G)$ is abelian, then $G' = [G, \mathrm{Inn}(G)] \leq [G, C_{L_c}(G)] \leq E_{L_c}(G)$.

The following lemma gives the important property of $E_{L_c}(G)$ which will be needed in our investigation.

Lemma 2.1. *If G be an arbitrary group, then $\mathrm{Aut}_{L_c}(G)$ acts trivially on the subgroup $E_{L_c}(G)$ of G.*

Proof. Let $\alpha \in \mathrm{Aut}_{L_c}(G)$, then $g^{-1}g^\alpha \in L_c(G)$, for all $g \in G$ and so $g^\alpha = gt_g$, for some $t_g \in L_c(G)$. Next, by take an automorphism $\beta \in C_{L_c}(G)$, we have

$$\begin{aligned}[g, \beta]^\alpha &= (g^{-1}g^\beta)^\alpha = (g^{-1})^\alpha (g^\beta)^\alpha = (g^{-1})^\alpha (g^\alpha)^\beta \\ &= t_g^{-1} g^{-1} g^\beta t_g^\beta = g^{-1} g^\beta t_g^{-1} t_g = [g, \beta],\end{aligned}$$

which completes the proof. □

Lemma 2.2. *Let G be a finite p-group. Then*

$$\mathrm{Aut}_{L_c}(G) \cong \mathrm{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G)).$$

In the following result we characterize finite non-abelian p-groups G of arbitrary class such that $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$.
Let G be a finite non-abelian p-group and $G/E_{L_c}(G)L_c(G)$ is of type $(a_1, a_2, ..., a_k)$. Also if $G/Z(G)$ is abelian, then it has invariants $(b_1, b_2, ..., b_n)$.

By fixed the above notation, we have the following result:

THEOREM 2.3. *Let G be a finite non-abelian p-group. Then* $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$ *if and only if $L_c(G)$ is cyclic, $k = n$ and one of the following conditions holds:*

(1) $E_{L_c}(G)L_c(G) = Z(G)$

(2) $b_t = f$ *and* $a_s = b_s$ *for* $s = t+1, ..., k$ *where,* $\exp(L_c(G)) = p^f$ *and t is the largest integer between 1 and k such that* $a_t > f$.

Let G be a finite non-abelian p-group and G/G' is of type $(a_1, a_2, ..., a_k)$. Also if $G/Z(G)$ is abelian, then G' has invariants $(e_1, e_2, ..., e_n)$.
Keeping fixed the above notation, we prove the following theorem:

THEOREM 2.4. *Let G be a finite non-abelian p-group. Then the following two conditions holds: Then*

(i) $\mathrm{Aut}_{L_c}(G) = \mathrm{Aut}^{G'}(G)$;

(ii) $\mathrm{Aut}_{L_c}(G) \cong \mathrm{Hom}(G, L_c(G))$, $L_c(G) \leq Z(G)$ *and one of the following conditions holds:*

  (1) $G' = L_c(G)$ *or*

  (2) $G' < L_c(G)$, $n = m$, *where* $(b_1, b_2, ..., b_m)$ *be invariants of* $L_c(G)$ *and* $a_1 = e_t$, *where t is the largest integer between 1 and n such that* $b_t > e_t$.

LEMMA 2.5. *Let G be a finite non-abelian p-group such that* $\mathrm{Aut}_{L_c}(G) = \mathrm{Aut}_c(G)$. *Then*

(i) $L_c(G) \leq Z(G) \cap \Phi(G)$;

(ii) $\mathrm{Aut}_{L_c}(G) \cong \mathrm{Hom}(G/L_c(G), G')$.

Let G be a finite non-abelian p-group such that $G/L_c(G)$ is of type $(b_1, b_2, ..., b_m)$. By fixed this notation, we have the following theorem:

THEOREM 2.6. *Let G be a finite non-abelian p-group. Then the following statements are equivalent:*

(i) $\mathrm{Aut}_{L_c}(G) = \mathrm{Aut}_c(G)$;

(ii) $G' = L_c(G) \leq Z(G)$, $\mathrm{Aut}_c(G) \cong \mathrm{Hom}(G/L_c(G), G')$ *and one of the following conditions holds:*

  (1) $L_c(G) = E_{L_c}(G)$ *or*

  (2) $L_c(G) < E_{L_c}(G)$, $m = k$ *and* $e_1 \leq c_t$, *where* $G/E_{L_c}(G)$ *and* G' *are of types* $(c_1, c_2, ..., c_k)$ *and* $(e_1, e_2, ..., e_n)$ *and t is the largest integer between 1 and m such that* $b_t > c_t$.

Let G be a finite group and N be non-trivial proper normal subgroup of G. Then (G, N) is called a Camina pair if $xN \subseteq x^G$ for all $x \in G\backslash N$. It follows that (G, N) is a Camina pair if and only if $N \subseteq [x, G]$ for all $x \in G\backslash N$. A group G is called a Camina group if (G, G') is a Camina pair.

COROLLARY 2.7. *Let G be a finite non-abelian p-group such that $L_c(G)$ is elementary abelian. Then* $\mathrm{Aut}_{L_c}(G) = \mathrm{Aut}_c(G)$ *if and only if* $G' = L_c(G)$ *and G is a Camina p-group.*

COROLLARY 2.8. *Let G be a finite p-group such that $L_c(G)$ is cyclic. Then* $\mathrm{Aut}_{L_c}(G) = \mathrm{Aut}_c(G)$ *if and only if* $G' = L_c(G)$ *and* $Z(G) = E_{L_c}(G)G^{p^n}$ *where* $\exp(L_c(G)) = p^n$.

In the following result we give a sufficient condition such that the group $\mathrm{Aut}_{L_c}(G)$ acts trivially on $K_c(G)$.

Theorem 2.9. *Let G be a group such that $K_c(G)$ is a torsion-free subgroup of G and $K_c(G)/E_{L_c}(G)$ is a torsion group. Then $\mathrm{Aut}_{L_c}(G)$ acts trivially on $K_c(G)$ and moreover is a torsion-free abelian group.*

3. Conclusion

In this paper we study closely the group $\mathrm{Aut}_{L_c}(G)$ for a finite non-abelian p-group G. We give a necessary and sufficient condition on G for the groups $\mathrm{Aut}_{L_c}(G)$ and $\mathrm{Inn}(G)$ coincide. Also we characterize all groups G such that $\mathrm{Aut}_{L_c}(G) = \mathrm{Aut}^{G'}(G)$. Finally, we give a necessary and sufficient condition on G such that $\mathrm{Aut}_{L_c}(G) = \mathrm{Aut}_c(G)$.

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30 August - 02 September 2021

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Right Tr-Contractibility of the Convolution Algebra of Trace Class Operators

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Abstract. Let G be a locally compact group. The convolution algebra of trace class operators $(\mathcal{T}(L^p(G)), *)$ with $p \in (1, \infty)$, is a new Banach algebra, that was introduced by M. Neufang [5]. In this paper, we study $(\mathcal{T}(L^p(G)), *)$ in the view of one of the important properties of Banach algebras, that is a notion of contractibility.

Keywords: locally compact group, nuclear operators, right ϕ-contractibility.

AMS Mathematics Subject Classification [2010]: 22D05, 47B10, 43A07.

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1. Introduction

Let G be a locally compact group with the left Haar measure λ. A classical Banach space is the Banach space of all trace class or nuclear operators. We denote $\mathcal{T}(L^p(G))$ for the Banach space of all trace class operators on $L^p(G)$; the space of all functions on G whose p-th powers are integrable, where $p \in (1, \infty)$. An operator ξ is called a trace class operator on $L^p(G)$ if it belongs to the image of the map J as follows

$$\begin{cases} J : L^q(G)\hat{\otimes}L^p(G) \longrightarrow \mathcal{T}(L^p(G)) \\ \displaystyle\sum_{n=1}^{\infty} f_n \otimes g_n \mapsto \begin{cases} J\left(\displaystyle\sum_{n=1}^{\infty} f_n \otimes g_n\right) : L^p(G) \longrightarrow L^p(G) \\ J\left(\displaystyle\sum_{n=1}^{\infty} f_n \otimes g_n\right)(h) = \displaystyle\sum_{n=1}^{\infty} \langle f_n, h\rangle\, g_n = \sum_{n=1}^{\infty}\left(\int f_n h \, d\lambda\right) g_n, \end{cases} \end{cases}$$

where $n \in \mathbb{N}$, $f_n \in L^q(G)$, $g_n, h \in L^p(G)$ and q is the convex conjugate of p and $\hat{\otimes}$ is the Banach space projective tensor product. The map J is an isometric isomorphism between $\mathcal{T}(L^p(G))$ and $L^q(G)\hat{\otimes}L^p(G)$. In other words,

$$\mathcal{T}(L^p(G)) \cong L^q(G)\hat{\otimes}L^p(G) = \left\{ \sum_{n=1}^{\infty} f_n \otimes g_n \in \mathcal{B}(L^p(G)) \quad ; \quad \sum_{n=1}^{\infty} \|f_n\|_q \|g_n\|_p < \infty \right\}.$$

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For detailed information see also [6, §2.6].

The Banach space of trace class operators was introduced for the first time by A. Grothendieck [2]. Indeed, Grothendieck had worked on tensor products of locally convex linear spaces and had proved a version of «Schwartz's Kernel Theorem». The terminology «kernel» conduced the concept «nuclear». On the other hand, A. F. Ruston and Grothendieck had extended the «Fredholms Determinant Theory» to operators in Banach Spaces. Here, the word «Determinant» conduced the concept «trace».

This Banach space was paid attention by M. Neufang, who introduced a new convolution product on the Banach space $\mathcal{T}(L^p(G))$ that is different from the usual composition of operators [5]. This convolution is denoted by $*$ and makes $\mathcal{T}(L^p(G))$ to a Banach algebra. Its formula is as follows

$$\xi * \eta = \int_G^w L_{t^{-1}}\, \xi\, L_t\, \sigma(\eta)(t)\, d\lambda(t) \qquad (\, \xi, \eta \in \mathcal{T}(L^p(G))\,)\,, \tag{1}$$

such that for every $t \in G$, $L_t : L^p(G) \longrightarrow L^p(G)$ is the left translation operator on $L^p(G)$ and has the formula $L_t(h)(x) = h(t^{-1}x)$ $(t, x \in G, h \in L^p(G))$ and

$$\sigma : L^q(G)\hat{\otimes}L^p(G) \cong \mathcal{T}(L^p(G)) \twoheadrightarrow L^1(G)$$

is a Banach algebra epimorphism which is a connection between the Banach algebra $\mathcal{T}(L^p(G))$ and the group algebra $L^1(G)$ with the formula

$$\sigma(f \otimes g) = \widetilde{fg} \qquad (\, f \in L^q(G)\,,\; g \in L^p(G)\,),$$

such that for every $t \in G$, $f \in L^q(G)$ and $g \in L^p(G)$, $\widetilde{fg}(t) = \Delta(t^{-1})\, f(t^{-1})\, g(t^{-1})$ and Δ is the modular function on G. The integral in (1) is the weak integral with respect to λ. $(\mathcal{T}(L^p(G)), *)$ is an associative Banach algebra and it is considered as a non-commutative version of the group algebra $(L^1(G), *)$. The reader can refer to [5] for more information and more details about $(\mathcal{T}(L^p(G)), *)$.

Let $\mathcal{A}$ be a Banach algebra, $\Delta(\mathcal{A})$ be the set of all characters on $\mathcal{A}$ and $\phi \in \Delta(\mathcal{A})$. The Banach algebra $\mathcal{A}$ is called right ϕ-contractible if the first cohomology group $H^1(\mathcal{A}, \mathcal{X})$ vanishes for any Banach $\mathcal{A}$-bimodule $\mathcal{X}$ such that its left module product is defined by $a \cdot x = a\,\phi(x)$ $(a \in \mathcal{A},\ x \in \mathcal{X})$. Left ϕ-contractibility of $\mathcal{A}$ is defined analogously. The notion of ϕ-contractibility of a Banach algebra was defined and studied in [3] and [7]. This concept was introduced with a new approach by the first author and Nasr-Isfahani in [4]. Indeed, by this new approach, the Banach algebra $\mathcal{A}$ is called right ϕ-contractible (resp. left ϕ-contractible), if there exists an element $a_0 \in \mathcal{A}$ with $\phi(a_0) = 1$, such that for every $a \in \mathcal{A}$

$$a_0\, a = a_0\, \phi(a) \qquad (\text{ resp.} \quad a\, a_0 = \phi(a)\, a_0\ ).$$

In this paper, we study the convolution algebra of trace class operators $(\mathcal{T}(L^p(G)), *)$ in the view of right ϕ-contractibility, where $\phi \in \Delta(\mathcal{T}(L^p(G)))$.

2. Main Results

Let G be a locally compact group. In this section, we investigate right ϕ-contractibility of $(\mathcal{T}(L^p(G)), *)$, where $\phi \in \Delta(\mathcal{T}(L^p(G)))$ and equal it to a property of the group G. At first, we use [4, Lemma 3.11]. This important result relates contractibility of two Banach algebras as follows.

PROPOSITION 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras, $\phi \in \Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{B})$. Suppose that there exists a continuous epimorphism $\kappa : \mathcal{A} \twoheadrightarrow \mathcal{B}$ such that $\phi = \psi \circ \kappa$. If $\mathcal{A}$ is ϕ-contractible, then $\mathcal{B}$ is ψ-contractible.*

It is appropriate to identify a certain character for the convolution algebra of trace class operators $(\mathcal{T}(L^p(G)), *)$. Indeed, we can deduce that there always exists a nonzero character on $(\mathcal{T}(L^p(G)), *)$, that is denoted by $Tr : \mathcal{T}(L^p(G)) \longrightarrow \mathbb{C}$. Further, there is an important relation between Tr and the augmentation character

$$\begin{cases} \varepsilon : L^1(G) \longrightarrow \mathbb{C} \\ f \mapsto \varepsilon(f) = \displaystyle\int_G f \, d\lambda \end{cases}$$

mentioned in [1], as follows

$$\begin{cases} Tr : \mathcal{T}(L^p(G)) \twoheadrightarrow L^1(G) \longrightarrow \mathbb{C} \\ Tr = \varepsilon \circ \sigma. \end{cases}$$

In other words, for every $\xi \in \mathcal{T}(L^p(G))$

$$Tr(\xi) = \varepsilon(\sigma(\xi)) = \int_G \sigma(\xi)(t) \, d\lambda(t).$$

By setting the Banach algebras $\mathcal{A} = \mathcal{T}(L^p(G))$, $\mathcal{B} = L^1(G)$ and the epimorphism $\sigma : \mathcal{T}(L^p(G)) \twoheadrightarrow L^1(G)$ instead of $\kappa : \mathcal{A} \twoheadrightarrow \mathcal{B}$ in Proposition 2.1, we can relate contractibility of two Banach algebras $\mathcal{T}(L^p(G))$ and $L^1(G)$. So we have the following theorem.

THEOREM 2.2. *Let G be a locally compact group. If the Banach algebra $(\mathcal{T}(L^p(G)), *)$ is right Tr-contractible, then G is compact.*

PROOF. Suppose that $(\mathcal{T}(L^p(G)), *)$ is a right Tr-contractible Banach algebra. By setting the epimorphism $\sigma : \mathcal{T}(L^p(G)) \twoheadrightarrow L^1(G)$ in Proposition 2.1, we deduce that the group algebra $(L^1(G), *)$ is right ε-contractible. Finally, by applying [4, Theorem 6.1], we get compactness of G. □

It is natural to ask weather the converse of Theorem 2.2 holds. In the following theorem, we shall prove the converse of previous theorem is also true.

THEOREM 2.3. *Let G be a compact group with a normalized Haar measure λ. Then the algebra of trace class operators $(\mathcal{T}(L^p(G)), *)$ is right Tr-contractible.*

PROOF. Suppose that G is a compact group. So $\mathbf{1} \in L^q(G)$ and $\mathbf{1} \in L^p(G)$. Therefore

$$\mathbf{1} \otimes \mathbf{1} \in L^q(G) \hat{\otimes} L^p(G) \cong \mathcal{T}(L^p(G)).$$

We show that $\xi_0 := \mathbf{1} \otimes \mathbf{1}$ is the desired right Tr-mean for $(\mathcal{T}(L^p(G)), *)$. First, we calculate $Tr(\xi_0)$ as follows

$$\begin{aligned} Tr(\xi_0) &= Tr(\mathbf{1} \otimes \mathbf{1}) = \varepsilon(\sigma(\mathbf{1} \otimes \mathbf{1})) = \int_G \sigma(\mathbf{1} \otimes \mathbf{1})(t) \, d\lambda(t) \\ &= \int_G \tilde{\mathbf{1}}(t) \, d\lambda(t) = \int_G \frac{1}{\Delta(t)} \mathbf{1}(t^{-1}) \, d\lambda(t) = \lambda(G) = 1. \end{aligned} \tag{2}$$

Further, for every $\xi \in \mathcal{T}(L^p(G))$,

$$\xi_0 * \xi = (\mathbf{1} \otimes \mathbf{1}) * \xi = \int_G^w L_{t^{-1}} (\mathbf{1} \otimes \mathbf{1}) L_t \, \sigma(\xi)(t) \, d\lambda(t) = \int_G^w (\mathbf{1} \otimes \mathbf{1}) \, \sigma(\xi)(t) \, d\lambda(t)$$

$$= (\mathbf{1} \otimes \mathbf{1}) \int_G \sigma(\xi)(t) \, d\lambda(t) = (\mathbf{1} \otimes \mathbf{1}) \, \varepsilon(\sigma(\xi)) = (\mathbf{1} \otimes \mathbf{1}) \, Tr(\xi) = \xi_0 \, Tr(\xi). \tag{3}$$

Finally, the equations (2) and (3) show that $(\mathcal{T}(L^p(G)), *)$ is right Tr-contractible. $\square$

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

On some properties of intuitionistic fuzzy modules

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Abstract. In this talk at first we recall the class of intuitionistic fuzzy modules and then present some properties of them. Also we introduce a special class of intuitionistic fuzzy modules and study on them. Exact sequences in category of intuitionistic fuzzy modules is introduced and investigated.

Keywords: Intuitionistic fuzzy set, Intuitionistic fuzzy modules, Exact Sequences

AMS Mathematics Subject Classification [2010]: 03E72, 06F25

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1. Introduction

The concept of fuzzy intuitionistic sets was introduced by K. T. Atanassov in [1]. By a *fuzzy set (or fuzzy subset)* of a module M, we mean the μ from M to $[0,1]$. By $[0,1]^M$ we will denote the set of all fuzzy subsets of M.

An *intuitionistic fuzzy set* (briefly an IFS) A of a non-void set X is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$, where the maps $\mu_A : X \longrightarrow [0,1]$ and $\nu_A : X \longrightarrow [0,1]$, are fuzzy subsets of X, denote respectively the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$, and $0 \le \mu_A(x) + \nu_A(x) \le 1$ for every $x \in X$.

For the sake of simplicity, we denote an IFS, $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$ of the set X by $A = (\mu_A, \nu_A)$ or briefly A, and the set of all IFS of X by $IFS(X)$.

Definition 1.1. Let M be an $R-$module and $A = (\mu_A, \nu_A)$ an IFS of M. Then A is called an *intuitionistic fuzzy submodule of M* if A satisfies the following conditions:

(1) $\mu_A(0) = 1$, $\nu_A(0) = 0$
(2) $\mu_A(x+y) \ge \mu_A(x) \wedge \mu_A(y)$, for every $x, y \in M$
$\nu_A(x+y) \le \nu_A(x) \vee \nu_A(y)$, for every $x, y \in M$
(3) $\mu_A(rx) \ge \mu_A(x)$, for every $x \in M$ and $r \in R$
$\nu_A(rx) \le \nu_A(x)$, for every $x \in M$ and $r \in R$

If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy submodule of an R-module M, we write A is an IFM of M and denote by $A \le_{IF} M$. In this case we say A is an intuitionistic fuzzy

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module too.
We use by $IFS(M)$, the set of all IFM of M and $IFR - Mod$, the category of all IF R-modules.

Definition 1.2. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFM's of M. Then the IFM, $A + B$ of M is $A + B = \{(x, \mu_{A+B}(x), \nu_{A+B}(x); x \in M\}$ defined as

$$\mu_{A+B}(x) = \bigvee\{\mu_A(y) \wedge \mu_B(z) \mid x = y + z;\ y, z \in M\}$$

$$\nu_{A+B}(x) = \bigwedge\{\nu_A(y) \vee \nu_B(z) \mid x = y + z;\ y, z \in M\}$$

Proposition 1.3. *Let M be an R-module and $N \subseteq M$. Then $N \leq M$ if and only if $\chi_N^{IF} \leq_{IF} M$.*

Example 1.4.
(1) Since $n\mathbb{Z} \leq \mathbb{Z}$ so $\chi_{n\mathbb{Z}}^{IF} = (\chi_{n\mathbb{Z}},\ \chi_{n\mathbb{Z}}^c)$ is IFM of $\mathbb{Z}$ for every $n \in \mathbb{Z}$.
(2) $\mathbb{Z} \leq \mathbb{Q}$ and hence $\chi_{\mathbb{Z}}^{IF}$ is an IF submodule of $\mathbb{Q}$.
(3) $\mathbb{Z}_p^\infty \leq \frac{\mathbb{Q}}{\mathbb{Z}}$ and hence $\chi_{\mathbb{Z}_p^\infty}^{IF}$ is an IF submodule of $\frac{\mathbb{Q}}{\mathbb{Z}}$.

2. Main Results

Definition 2.1. Let R be a ring and $M,\ N$ be R-modules such that $A = (\mu_A, \nu_A) \leq_{IF} M$ and $B = (\mu_B, \nu_B) \leq_{IF} N$. The function $\tilde{f} : A \to B$ is called an *intuitionistic fuzzy R-homomorphism*, if $f : M \longrightarrow N$ is an R-homomorphism and $\mu_B(f(x)) \geq \mu_A(x)$ and $\nu_B(f(x)) \leq \nu_A(x)$ for every $x \in M$.

Definition 2.2. An intuitionistic fuzzy R-homomorphism $\tilde{f} \in Hom_{IF-R}(A, B)$ is called *fuzzy split*, if there is an intuitionistic fuzzy R-homomorphism $\tilde{t} \in Hom_{IF-R}(B, A)$ such that the composition $\tilde{t}\tilde{f} = id$.

Definition 2.3. An intuitionistic fuzzy R-homomorphism $\tilde{f} \in Hom_{IF-R}(A, B)$ is called *intuitionistic fuzzy quasi-isomorphism* if f is an isomorphism.

If $\tilde{f} : A \to B$ is an IF R-homomorphism, define $Ker\tilde{f} = \left\{ a \in A \mid \begin{array}{c} \mu_B(\tilde{f}(a)) = 1; \\ \nu_B(\tilde{f}(a)) = 0 \end{array} \right\}$

and $Im\tilde{f} = \{\tilde{f}(a) | a \in A\}$.

Definition 2.4. An intuitionistic fuzzy R-homomorphism $\tilde{f} \in Hom_{IF-R}(A, B)$ is called *intuitionistic fuzzy isomorphism*, if f is an isomorphism and $\mu_B(\bar{f}(a)) = \mu_A(a)$, $\nu_B(\bar{f}(a)) = \nu_A(a)$ for every $a \in M$.

Note that $ker\tilde{f} = kerf$ is not true in general, but $kerf \subseteq ker\tilde{f}$.
If $ker\tilde{f} = \{0\}$ then $\tilde{f}$ is monomorphism because if

$$f(x) = f(y) \Rightarrow f(x-y) = 0 \Rightarrow \left\{ \begin{array}{l} \mu_N(\tilde{f}(x-y)) = 1; \\ \nu_N(\tilde{f}(x-y)) = 0. \end{array} \right. \Rightarrow x - y \in ker\tilde{f} = \{0\} \Rightarrow x = y.$$

But the reverse is not true, it means if $\tilde{f}$ is a monomorphism then it need not that $ker\tilde{f} = \{0\}$.

EXAMPLE 2.5. If $B = 1_M^{IF}$, then $Ker\tilde{f} = M$, for every $A \in IFR - Mod$ and $\tilde{f} \in Hom_{IF-R}(A, B)$. Especially let $M = N = \mathbb{Z}$, $A = B = 1_M^{IF}$ and $\tilde{f} : A \to B$ be the identity map. Then $kerf = \{0\}$ but $ker\tilde{f} = \mathbb{Z}$.

PROPOSITION 2.6. *Let R be a ring. If $\tilde{f} \in Hom_{IF-R}(A, B)$, where A and B are two IF R-modules, such that $A = (\mu_A, \nu_A) \leq_{IF} M$ and $B = (\mu_B, \nu_B) \leq_{IF} N$, then*
(1) $Ker\tilde{f}$ is a submodule of M,
(2) Define $\mu' \mid_{ker\tilde{f}}: Ker\tilde{f} \to [0,1]$, $\nu' \mid_{ker\tilde{f}}: Ker\tilde{f} \to [0,1]$ by $\left\{ \begin{array}{l} \mu'(k) = \mu(k); \\ \nu'(k) = \nu(k). \end{array} \right.$ for every $k \in ker\tilde{f}$.

Then $A' = (\mu'|_{ker\tilde{f}} \; , \; \nu'|_{ker\tilde{f}})$ is an IF submodule of A.

DEFINITION 2.7. Let A, B and C be IF R-modules of M, N and K respectively. A short exact sequence is a sequence of the form

$$\bar{0} \longrightarrow A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$$

where $\tilde{f}$ is a monomorphism, $\tilde{g}$ is an epimorphism and $Im\tilde{f} = ker\tilde{g}$.
Note that $Ker\tilde{f}$ is usually larger than $\{0\}$ by Definition 2.4. Hence, the crisp case of the definition is different from the well-known notion of short exact sequence in the category IFR-mod.

If $C = 1_K$, we get that $Im\tilde{f} = Ker\tilde{g} = N$. As $\tilde{f}$ is monic, we can get that $\tilde{f}$ is quasi-isomorphism.

THEOREM 2.8. *Let R be a commutative ring and $A = (\mu_A, \nu_A) \leq_{IF} M$,*
$B = (\mu_B, \nu_B) \leq_{IF} N$ be two intuitionistic fuzzy R- modules. Then
$Hom_{IF-R}(A, B) = (\alpha, \beta)$ is an IF R-module with membership function
$\alpha : Hom_{IF-R}(A, B) \longrightarrow [0,1]$ and non-membership function
$\beta : Hom_{IF-R}(A, B) \longrightarrow [0,1]$ defined by

$$\alpha(\tilde{f}) = \bigwedge\{\mu_B(\tilde{f}(x)) \mid x \in M\} \text{ and } \beta(\tilde{f}) = \bigvee\{\nu_B(\tilde{f}(x)) \mid x \in M\}$$

THEOREM 2.9. *Let R be a commutative ring and let*

$$\bar{0} \longrightarrow A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C$$

be an exact sequence in IFR-Mod, where $\tilde{f}$ is IF split homomorphism. Then $Hom_{IF-R}(D,$-
) preserves the sequence, for every $D \in \mathrm{IFR-Mod}$.

LEMMA 2.10. *Let R be a commutative ring and $A \in IFR - Mod$. Then*
$\Gamma_A : Hom(0_{Re}^{IF}, A) \longrightarrow eA$ defined by $\tilde{f} \longmapsto \tilde{f}(e)$, is an IF R-module
isomorphism.

PROPOSITION 2.11. *Let R be a ring and the following diagram of IF R-modules is commutative:*

$$\begin{array}{ccccccccc} \bar{0} & \longrightarrow & A & \xrightarrow{\tilde{f}} & B & \xrightarrow{\tilde{g}} & C & \longrightarrow & \bar{0} \\ & & \downarrow\tilde{\alpha} & & \big\downarrow\tilde{\beta} & & \downarrow\tilde{\gamma} & & \\ \bar{0} & \longrightarrow & D & \xrightarrow{\tilde{h}} & E & \xrightarrow{\tilde{p}} & F & \longrightarrow & \bar{0} \end{array}$$

where $\tilde{\alpha}$, $\tilde{\gamma}$ are IF isomorphisms and $\tilde{\beta}$ is an IF quasi-isomorphism. Then the bottom row is a short exact sequence if and only if so is the top row.

Definition 2.12. Let M, N be two $R-$modules and $A \leq_{IF} M$, $B \leq_{IF} N$. If $\tilde{f}: A \longrightarrow B$ is an IF homomorphism and $e \in E(R)$, we define $\tilde{ef}: eA \longrightarrow eB$ by $\tilde{ef}(em) = \tilde{f}(em) = e\tilde{f}(m)$, for every $m \in M$.

Proposition 2.13. *Let R be a commutative ring and $\bar{0} \longrightarrow A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$ be a short exact sequence of IF R-module. Let $e \in E(R)$, $\tilde{ef} = \tilde{f}|_{eA}$ and $\tilde{eg} = \tilde{g}|_{eB}$. Then the sequence $\bar{0} \longrightarrow eA \xrightarrow{\tilde{ef}} eB \xrightarrow{\tilde{eg}} eC \longrightarrow \bar{0}$ is exact.*

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Symmetric methods based on triple-jump composition for solving periodic differential equations

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Abstract. In this paper, symmetric methods by composition technique are constructed which are applicable to general time-reversible ordinary differential equations (ODEs). Here, the aim is to increase the order while preserving some desirable properties of the basic method. To show the advantages of the proposed methods, some periodic problems are tested.

Keywords: Ordinary differential equations, Symmetric methods, Composition methods

AMS Mathematics Subject Classification [2010]: 65L05

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1. Introduction

This paper, is concerned with symmetric methods for solving first order autonomous initial value problem (IVP) of the form

$$\begin{aligned} y'(t) &= f(y(t)), \quad t \in [t_0, T], \\ y(t_0) &= y_0, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$ and $T > 0$. The solution $y(t)$ of (1) is the flow map $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ such that $y(t) = \varphi_t(y_0)$.

Symmetric methods play an essential role in the geometric solution of differential equations. The construction of symmetric methods goes back to many years. When Lambert [2], for the first time introduced symmetric linear multistep methods. Later, symmetric Runge–Kutta methods were investigated [1].

The main aim of this paper is to construct and test higher-order symmetric Runge–Kutta methods by triple-jump composition. Here, we consider Hamiltonian systems as our IVPs. The equation of motion are called Hamiltonian systems with Hamiltonian H which is a function of $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ and defines the differential system

$$\dot{p} = -H_q(p, q), \qquad \dot{q} = H_p(p, q), \quad p, q \in X, \tag{2}$$

having n degrees of freedom.

For autonomous Hamiltonian systems, the total energy remains conserved. This means that the value of Hamiltonian H remains constant along the solution of the system.

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1.1. Runge–Kutta methods. Runge–Kutta methods are one-step methods for the numerical solutions of IVPs (1). The general form of a Runge–Kutta method is in the form

$$Y_i = y_{n-1} + h\sum_{j=1}^{s} a_{ij} f(Y_j),$$

$$y_n = y_{n-1} + h\sum_{i=1}^{s} b_i f(Y_i),$$

where $Y_i, i = 1, 2, \ldots, s$, are stages calculated during the integration.

2. Symmetric Runge–Kutta methods

Definition 2.1. The adjoint method Φ_h^* is the inverse of Φ_h with reversed step size $-h$.

A numerical one-step method Φ_h is said to be symmetric, if it is equal to its adjoint method. In the other words, the method is symmetric if it satisfies $\Phi_h = \Phi_h^*$.

A one-step method Φ_h is said to be of order p if

$$\Phi_h(y) = \varphi_h(y) + \mathbf{C}(y)h^{p+1} + O(h^{p+2}),$$

and also, the adjoint method Φ_h^* is of order p if

$$\Phi_h^*(y) = \varphi_h(y) - \mathbf{C}(y)(-h)^{p+1} + O(h^{p+2}),$$

Thus the symmetry condition for one-step method implies that a symmetric method is of even order.

Theorem 2.2. *A Runge–Kutta method is symmetric if there exists a permutation matrix P such that*

$$b^T = b^T P,$$

$$A + PAP = eb^T,$$

where $e = [1, 1, \ldots, 1]^T \in \mathbb{R}^s$.

2.1. Symmetric methods achieved by composition. An important class of symmetric methods is included of symmetric compositions of low-order methods.
Let $\Phi_h : X \to X$ denote a Runge–Kutta method. Then, a composition method with step sizes $\alpha_1 h$, $\alpha_2 h$,...,$\alpha_k h$ is given by

$$\Psi_h(y_0) = \Phi_{\alpha_k h} \circ \ldots \circ \Phi_{\alpha_1 h}, \tag{3}$$

where it is assumed that $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$.
If Φ_h is of even order p, then the composed method is at least of order $p+1$ if

$$\begin{aligned} \alpha_1 + \alpha_2 + \cdots + \alpha_k &= 1, \\ \alpha_1^{p+1} + \alpha_2^{p+1} + \cdots + \alpha_k^{p+1} &= 0. \end{aligned}$$

This observation is the key to triple-jump composition. The most important issue in the composition theory is that for certain choices of α_i, for $i = 1, \ldots, k$, the composed method is of higher order than the base method.

Notation 2.3. (Triple-jump [1]): For a symmetric Runge–Kutta method of even order p, the triple-jump method is in the form

$$\Psi_h(y_0) = \Phi_{\alpha_1 h} \circ \Phi_{\alpha_2 h} \circ \Phi_{\alpha_1 h}(y_0).$$

where α_1 and α_2 can be calculated as

$$\alpha_1 = \frac{1}{2 - 2^{1/(p+1)}}, \quad \alpha_2 = \frac{-2^{1/(p+1)}}{2 - 2^{1/(p+1)}}. \tag{4}$$

Here, we have considered the Trapezoidal method as the base method, which is symmetric Runge–Kutta method of order 2.
We start with a symmetric Runge–Kutta method (Trapezoidal) of order 2 and then apply equations (4) with $p = 2$ to obtain order 3; but due to the symmetry of α's this new method is of order 4. This procedure can be repeated. In the second iteration, we use the new method of order 4 and apply (4) with $p = 4$ and obtain a symmetric method of order 6.

3. Numerical results

We restrict our numerical experiments to the composition methods of order 4 and 6 based on the Trapezoidal method, to demonstrate the effectiveness of the proposed methods and confirm the theoretical order by applying methods on the following problems.

P1. The first problem is the Kepler's problem also known as the one-body problem which describes the motion of a single planet moving around a heavy sun. The equations of motion define a separable Hamiltonian system

$$H(p_1, p_2, q_1, q_2) = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{\sqrt{q_1^2 + q_2^2}},$$

and the initial values are prescribed to be

$$p_1(0) = 0, \quad p_2(0) = \sqrt{\frac{1+e}{1-e}}, \quad q_1(0) = 1 - e, \quad q_2(0) = 0,$$

where e is the eccentricity of an ellipse on which the orbit lies and is fixed to be 0.3. With these initial values, all points in the orbit lie on the ellipse

$$(p_1 + e)^2 + \frac{p_2^2}{1 - e^2} = 1.$$

P2. This problem describes the motion of a simple pendulum with unit mass and length. The corresponding Hamiltonian is written as

$$H(p, q) = \frac{1}{2}p^2 - \cos(q),$$

and the initial values are prescribed to be

$$p(0) = 0, \quad q(0) = 2.3.$$

The results of numerical experiments for the kepler problem is presented in Table 1. The convergence rates of the methods are obtained by

$$\mathcal{P} = \log_2 \frac{\|\mathcal{E}_h\|}{\|\mathcal{E}_{\frac{h}{2}}\|},$$

where $\mathcal{E}_h$ is the global error of the methods with step size h. Also the results of numerical experiments for the problem P2 are presented in Figures 1 and 2. It is known that simple

Table 1. Numerical results for problem P1 on the interval $[0, 10\pi]$

h	$\|\mathcal{E}_h\|$ for method of order 4	$\mathcal{P}$	$\|\mathcal{E}_h\|$ for method of order 6	$\mathcal{P}$
$\frac{1}{10}$	4.05×10^{-5}		6.64×10^{-7}	
$\frac{1}{20}$	2.57×10^{-6}	3.9751	1.06×10^{-8}	5.9601
$\frac{1}{40}$	1.62×10^{-7}	3.9912	1.68×10^{-10}	5.9873
$\frac{1}{80}$	1.01×10^{-8}	3.9994	2.63×10^{-12}	5.9968

pendulum is a Hamiltonian problem and by Figures 1 and 2 we can see that the proposed methods approximately preserve the structure of the Hamiltonian problem P2.

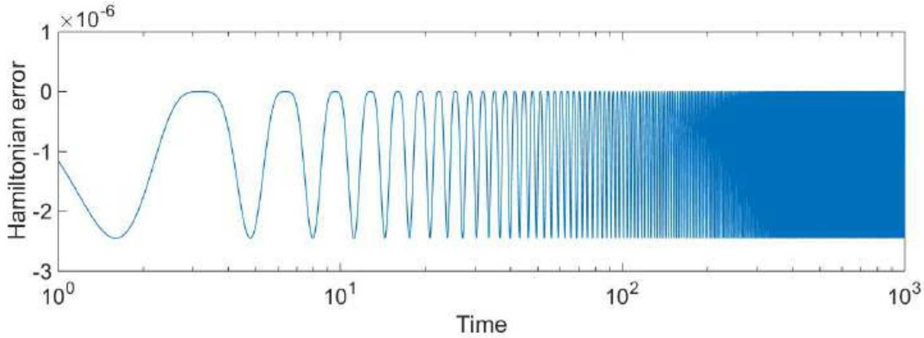

Figure 1. Hamiltonian preservation for the method of order 4 for simple pendulum with $h = 0.05$.

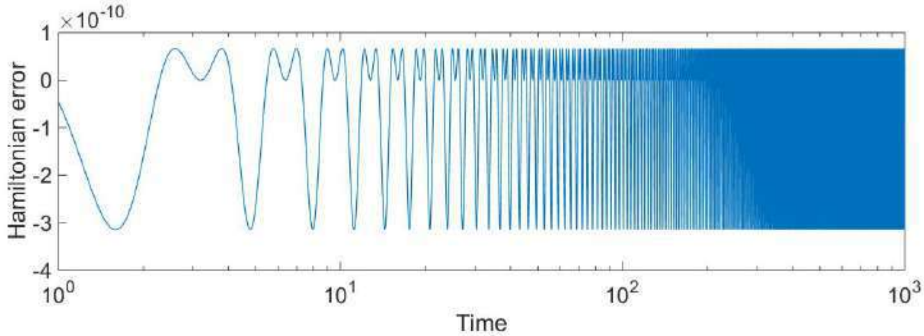

Figure 2. Hamiltonian preservation for the method of order 6 for simple pendulum with $h = 0.05$.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Analytic Torsion on manifolds with fibred boundary metrics

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Abstract. In this paper, we construct the renormalized analytic torsion in the setup of manifold endowed with fibred boundary metrics. The method of construction is to determine the asymptotic of heat kernel, both in short time regime and long time regime and apply these asymptotics together with renormalization to determine the renormalized zeta function and the determinant of Hodge Laplacian.

Keywords: Spectral invariants, Manifold with fibred boundary metrics, Analytic torsion, Heat kernel, Zeta function

AMS Mathematics Subject Classification [2010]: Spectral theory, Differential geometry, Analysis of PDEs

1. Introduction

Analytic torsion was introduced by Ray and Singer [4] as analytic counterpart of Reidemeister torsion in topology. Ray and Singer conjectured that these two torsions are equivalent on closed manifolds. Cheeger and Müller, later Bismut and Zhang, proved this conjecture independently. Assume that (M, g) is a closed Riemannian manifold and $e^{-t\Delta_g}(x, y) := H(t, x, y)$ is the heat kernel with respect to Hodge Laplacian $\Delta_g^q : \Omega^q(M) \longrightarrow \Omega^q(M)$, acting on the space of q forms,

$$\begin{aligned} \partial_t H(t, x, y) + \Delta_{g,x}^q H(t, x, y) &= 0, \\ H(t = 0, x, y) &= \delta(x - y). \end{aligned} \tag{1}$$

One define the heat trace to be, $Tr(e^{-t\Delta_g^q}) = \int_M e^{-t\Delta_g}(x, x) dvol_g$. Assume Δ_g^q is Hodge Laplacian acting on the space of q-forms. The corresponding zeta function is defined as,

$$\zeta_q^M(s) := \frac{1}{\Gamma(s)} \int_M Tr(e^{-t\Delta_g^q} - \dim\ker(\Delta)) t^{s-1} dt. \tag{2}$$

So defined (2) is defined on $Re(s) > \frac{n}{2}$ where $n = dim(M)$ but can be extended holomorphically to complex plane $\mathbb{C}$ with regular point at $s = 0$. The determinant of Laplacian is defined to be,

$$det(\Delta_g^q) := e^{-\zeta_q'(0)},$$

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One defines the analytic torsion as,

$$\log T(M) := \frac{1}{2}\sum_{q=0}^{n}(-1)^q q\zeta_q'(0),$$

In this work we consider manifolds with fibred boundary metrics 2.1. In this set up, we are going to define the analytic torsion. The main difficulty arises when we consider the heat trace,

$$Tr(e^{-t\Delta^q_{g_\phi}}) = \int_{\mathbb{R}^+} e^{-t\Delta_{g_\phi}}(x,x)dvol_\phi,$$

i.e in ϕ set up, the boundary is located at infinity and therefore the integration over diagonal diverges. To address this problem, the renormalized heat trace is applied in the Hadamard manner [6], which essentially takes into account the integration of the heat kernel along the diagonal on the finite component. The heat kernel structure theorem may be employed to take the finite part of this integral at zero to be heat trace renormalized . In order to describe analytic torsion in the set up of ϕ manifolds, we can explicitly describe renormalized zeta function and Laplacian determinant by means of renormalized heat trace.

The paper is organized as follows. In section 2, we provide our main result. We introduce the set up manifolds with fibred boundary metrics in 2.1. In section 3, we give main results which we need for the definition of Analytic Torsions from [5], [1] and [6] and define renormalized Zeta function.

2. Main results

Theorem 2.1. *For $(\overline{M}, g_\phi)$ fibred boundary manifold, denote Δ^k_ϕ to be Hodge Laplacian acting on the space of k forms. One may define the renormalized analytic torsion by,*

$$Log\,{}^R T_{M,g_\phi} := \frac{1}{2}\sum_{q=0}^{n}(-1)^q q\,\frac{d}{ds}\,{}^R\zeta^q_{M,\phi}(s)|_{s=0}. \tag{3}$$

2.1. Manifolds with fibred boundary metrics.

Definition 2.2. Assume $\overline{M}$ is a compact manifold with boundary ∂M and ∂M has fibration structure i.e, $\partial M \overset{\phi}{-} B - F$, where ϕ is trivialization of fibration. B is base manifold and F is closed manifold as fibre. Near the boundary one may take the product $[0,\epsilon)\times\partial M$ by collar neighborhood theorem, and fix local coordinates on $\overline{M}$ to be, $(x, y = (y_1 \cdots y_b), z = (z_1, \cdots z_f))$. Here $x = \rho_{\partial M}$ is the boundary defining function of ∂M, i.e, $\partial M = \{x = 0\}$, $dx \neq 0$, $x \geq 0$. Consider the metric,

$$g_\phi = \frac{dx^2}{x^4} + \frac{\phi^* g_B}{dx^2} + g_F,$$

on $\overline{M}$ where g_B is Riemannian metric on base B and g_F is symmetric bilinear form which restricts to Riemannian metric on fibre F. We assume further that $\varphi : (\partial M, g_F + \varphi^* g_B) \longrightarrow (B, g_B)$ is Riemannian submersion. Such a geometric set up is called fibred boundary ϕ metric manifolds. Intuitively the boundary is fibre bundle with base B and fibre F where the boundary is viewed to be located at infinity.

EXAMPLE 2.3. Examples of these manifolds include, scattering manifolds where fibre F is trivial. Gravitational instantons as hyperkähler 4 manifolds [2]. Locally symmetric spaces of rank 1 and in other cases, finite processing of fibre bundles over cones.

3. Heat kernel for manifolds with fibred boundary metrics

In this section, we states two main Theorems for Asymptotics of Heat kernel in finite and long time regimes on manifolds with fibred boundary metrics. We apply the methods of Geometric microlocal analysis [1] (Section 2).

THEOREM 3.1 (Heat kernel for finite time). *[5](Theorem 1.1) With the same assumptions as in [5], the fundamental solution of the heat equation, for finite time $t < \infty$,*

$$\partial_t H(t, x, x') + \Delta^q_{g_\phi, x} H(t, x, x') = 0,$$
$$H(t = 0, x, x') = \delta(x - x'),$$

lifts to polyhomogeneous conormal distribution on HM_ϕ with leading asymptotics 0 at fd and $-n$ at td and vanishing to infinite order on other hypersurfaces of HM_ϕ. Here $n = dim\,\overline{M}$.

THEOREM 3.2 (Resolvent at low energy for phi Manifolds). *[1](Theorem 1.6) Under the same assumptions as in [1], The resolvent $(\Delta_\phi + k^2)^{-1}$ as $k \longrightarrow 0^+$ is an element of the split calculus (defined as in [1]) where,*

$$\mathcal{E}_{sc} \geq 0,\ \mathcal{E}_{\phi f_0} \geq 0,\ \mathcal{E}_{bf_0} \geq -2,\ \mathcal{E}_{lb_0}, \mathcal{E}_{rb_0} > 0, \mathcal{E}_{zf} \geq -2.$$

The leading terms at sc, ϕf_0, bf_0 and zf are of order $0, 0, -2, -2$.

THEOREM 3.3 (Heat kernel for infinite time). *[6](Theorem 3.7) The heat kernel which is given by*

$$H^M(t, x, x') = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}(\Delta_\phi + \lambda)^{-1} d\lambda,$$

is polyhomogeneous conormal at $t = \omega^{-\frac{1}{2}}$ at $\omega \longrightarrow 0$ on $M^2_{\omega,\phi}$ with index sets given in terms of index sets of resolvent $(\Delta_\phi + \lambda)^{-1}$ at low energy level. More explicitly the asymptotics of heat kernel in long time regime are of leading order 0 at sc face and of order 0 at zf and bf_0 faces. More over the leading order at the face ϕf_0 is 2. In long time regime the heat kernel vanishes to infinite order at lb, rb, and bf faces of $M^2_{\omega,\phi}$. The explicit index sets are as follows,

$$\mathcal{E}_{sc} \geq 0,\ \mathcal{E}_{\phi f_0} \geq 2,\ \mathcal{E}_{bf_0} \geq 0,\ \mathcal{E}_{lb_0}, \mathcal{E}_{rb_0} > 0, \mathcal{E}_{zf} \geq 0.$$

Consider formally,

$$\frac{1}{\Gamma(s)}\int_0^\infty {}^R\mathrm{Tr}(H^k_{M,\phi})(t) t^{s-1} dt. \tag{4}$$

Apriori (4) is not defined for any $s \in \mathbb{C}$. By breaking (4) at some constant c we may express (4) as sum of two integrals. The integral,

$${}^R_0\zeta^k_{M,\phi}(s) := \frac{1}{\Gamma(s)}\int_0^c {}^R\mathrm{Tr}(H^k_{M,\phi})(t) t^{s-1} dt, \tag{5}$$

is defined for $Re(s) > \frac{n}{2}$. Each summand can directly be evaluated to show that (5) admits meromorphic extension to complex plane $\mathbb{C}$. The second integral is denoted as ${}^R_\infty\zeta^k_{M,\phi}(s)$,

$${}^R_\infty\zeta^k_{M,\phi}(s) := \frac{1}{\Gamma(s)}\int_c^\infty {}^R\mathrm{Tr}(H^k_{M,\phi})(t) t^{s-1} dt. \tag{6}$$

One can show that the integral converges for $Re(s) < 0$ and by evaluating directly (6) the meromorphic extension to complex plane $\mathbb{C}$ follows.

One may now define the renormalized Zeta function and determinant of Laplacian in order to obtain the main result. For more detail on the proofs we refer to [6].

4. Conclusion

On manifolds with fibred boundary metrics, using the methods of geometric microlocal analysis and functional calculation together with resolvent kernel we determined the asymptotics of the Heat kernel in finite time and long-term regimes on suitable spaces of manifolds with corners. Using these asymptotics, we determined the heat trace asymptotics and applied renormalization to define the renormalized zeta function and the determinant of Laplace. The definition of analytic torsion is then straightforward.

Remark 4.1. One still has to show that this definition is canonical and then the open question will be the Cheeger-Müller-Type statement for manifolds with fibred boundary metrics.

Acknowledgement

The author thanks the AICM for the opportunity to present his work. He also acknowledges and thanks very useful discussions and comments from Werner Müller, Collin Guillarmou, Daniel Grieser and Dominic Joyce.

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On χ-Connes module amenability and χ-module normal virtual diagonals

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Abstract. In this paper, we define χ-Connes module amenability of a semigroup algebra $l^1(S)$, where χ is a bounded module homomorphism from $l^1(S)$ to itself and S is an inverse weakly cancellative semigroup with subsemigroup E_S of idempotents. We investigate and study of χ-module normal, virtual diagonals. Realy, we obtain some inherited properties for semigroup algebra $l^1(S)$ over $l^1(E_S)$ via mentioned diagonals.

Keywords: χ-Connes module amenable, χ-module normal virtual diagonal, inverse semigroup algebra, module ψ-derivation, weakly cancellative semigroup

AMS Mathematics Subject Classification [2010]: 43A20, 43A10, 22D15

1. Introduction

The concept of module amenability for a Banach algebra is introduced by Amini [1]. Amini show that if S is an inverse semigroup with subsemigroup E_S of idempotents, then semigroup S is amenable if and only if semigroup algebra $l^1(S)$ over $l^1(E_S)$ is module amenable. In [4], Johnson has previously explored these concepts. In this paper, we introduce the concept of χ-Connes module amenability for semigroup algebra $l^1(S)$ and give a characterization of χ-Connes module amenability in terms of χ-modul normal virtual diagonals.

2. Main results

Let S be a semigroup. We say that S is cancellative semigroup, if for every $r, p \neq q \in S$ we have $rs \neq rq$ and $pr \neq qr$.

Let S be a semigroup. We say that S is an inverse semigroup if for each $x \in S$ there exists a unique element $x^* \in S$ such that $xx^*x = x$ and $x^*xx^* = x^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotent elements of S is denoted by E. For $p \in S$, we define $L_p, R_p : S \to S$ by $L_s(q) = pq, R_p(q) = qp; (q \in S)$. If for each $p \in S$, L_p and R_p are finite-to-one maps, in this case S is named weakly cancellative.

Remark 2.1. Let S be a weakly cancellative semigroup, then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$ [3]. In other word $l^1(S) = c_0(S)^*$.

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Let $\mathcal{A}$ be a dual Banach algebra and $\mathcal{A}_*$ be predual of $\mathcal{A}$, namely $\mathcal{A} = (\mathcal{A}_*)^*$. Let $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-bimodule via,

$$\alpha.(ab) = (\alpha.a).b, \quad (\alpha\beta).a = \alpha.(\beta.a) \qquad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

Let I be the closed ideal of $\mathcal{A}\widehat{\otimes}\mathcal{A}$ that $I =< \alpha.(a\otimes b)-(a\otimes b).\alpha >$, for $a, b \in \mathcal{A}$ and $\alpha \in \mathcal{U}$. Suppose that $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{A} \equiv \frac{\mathcal{A}\widehat{\otimes}\mathcal{A}}{I}$. Let J be the closed ideal of $\mathcal{A}$ that $J =< (\alpha.a).b-a.(b.\alpha) >$. Since J is ω^*-closed, then the quotient algebra $\frac{\mathcal{A}}{J}$ is again dual with predual ${}^{\perp}J = \{\phi \in \mathcal{A}_* : \langle \phi, a\rangle = 0 \ for\ all\ \ a \in J\}$. Also we have $J^{\perp} = \{\phi^* \in \mathcal{A}^* : \langle \phi, a\rangle = 0 \ for\ all\ \ \phi \in J\}$.

In this paper we let that $\mathcal{L}^2_{\omega^*}(\frac{\mathcal{A}}{J}, \mathbb{C})$ denote the separately ω^*-continuous two-linear maps from $\frac{\mathcal{A}}{J} \times \frac{\mathcal{A}}{J}$ to $\mathbb{C}$, $\tilde{\omega}^* : \mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{A} \to \frac{\mathcal{A}}{J}$ be the multiplication operator with $\tilde{\omega}(a\otimes b+I) = ab + J$ and $\tilde{\psi} : \frac{\mathcal{A}}{J} \to \frac{\mathcal{A}}{J}$ be the map that is defined by $\tilde{\psi}(a + J) = \psi(a) + J, \ \ a \in \mathcal{A}$.

Definition 2.2. Let $\mathcal{A}$ be a dual Banach algebra. A module homomorphism from $\mathcal{A}$ to $\mathcal{A}$ is a map $\psi : \mathcal{A} \to \mathcal{A}$ with

$$\psi(\alpha.a + b.\beta) = \alpha.\psi(a) + \psi(b).\beta, \quad \psi(ab) = \psi(a)\psi(b) \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

Definition 2.3. Let $\mathcal{A}$ be a dual Banach algebra and $\psi : \mathcal{A} \to \mathcal{A}$ be a bounded module homomorphism. An element $\mathcal{M} \in \mathcal{L}^2_{\omega^*}(\frac{\mathcal{A}}{J}, \mathbb{C})^*$ is called a ψ-module normal virtual diagonal for $\mathcal{A}$ if $\tilde{\omega}^{**}(\mathcal{M}).(\psi(a) + J) = \psi(a) + J$ and $\mathcal{M}.\tilde{\psi}(a+J) = \tilde{\psi}(a+J).\mathcal{M} \ \ (a \in \mathcal{A})$.

Let $\mathcal{X}$ be a dual Banach $\mathcal{A}$-bimodule. we say that $\mathcal{X}$ is normal if for every $x \in \mathcal{X}$, the maps

$$\mathcal{A} \to \mathcal{X}; \qquad a \to a.x, \quad a \to x.a$$

are ω^*-continuous. If moreover $\mathcal{X}$ is a $\mathcal{U}$-bimodule such that for $a \in \mathcal{A}, \alpha \in \mathcal{U}$ and $x \in \mathcal{X}$

$$\alpha.(a.x) = (\alpha.a).x, \quad (a.\alpha).x = a.(\alpha.x), \quad (\alpha.x).a = \alpha.(x.a),$$

then $\mathcal{X}$ is called a normal Banach left $\mathcal{A}$-$\mathcal{U}$-module. Similarly for the right and two sided actions. Also, we say that $\mathcal{X}$ is symmetric, if $\alpha.x = x.\alpha \qquad (\alpha \in \mathcal{U}, x \in \mathcal{X})$.

Throughout this paper $\mathcal{H}_{\omega^*}(\mathcal{A})$ will denotes the space of all ω^*-continuous bounded module homomorphisms from $\mathcal{A}$ to itself.

Definition 2.4. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, $\psi \in \mathcal{H}_{\omega^*}(\mathcal{A})$ and let that $\mathcal{X}$ be a dual Banach $\mathcal{A}$-bimodule. A bounded map $D_{\mathcal{U}} : \mathcal{A} \to \mathcal{X}$ is called a module ψ-derivation if for every $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathcal{U}$, we have

$$D_{\mathcal{U}}(\alpha.a \pm b.\beta) = \alpha.D_{\mathcal{U}}(a) \pm D_{\mathcal{U}}(b).\beta, \quad D_{\mathcal{U}}(ab) = D_{\mathcal{U}}(a).\psi(b) + \psi(a).D_{\mathcal{U}}(b).$$

When $\mathcal{X}$ is symmetric, each $x \in \mathcal{X}$ defines a module ψ-derivation

$$(D_{\mathcal{U}})_x(a) = \psi(a).x - x.\psi(a) \qquad (a \in \mathcal{A}).$$

In this case we say $D_{\mathcal{U}}$ is an inner module ψ-derivation.

Definition 2.5. Let $\mathcal{A}$ be a dual Banach algebra, $\psi \in \mathcal{H}_{\omega^*}(\mathcal{A})$ and $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-module. we say that $\mathcal{A}$ is ψ-Connes module amenable if for any symmetric normal Banach $\mathcal{A}$-$\mathcal{U}$-module $\mathcal{X}$, each ω^*-continuous module ψ-derivation $D_{\mathcal{U}} : \mathcal{A} \to \mathcal{X}$ is inner.

Theorem 2.6. *Let $\mathcal{A}$ and $\mathcal{U}$ be dual Banach algebras, let $\mathcal{A}$ be an unital dual Banach $\mathcal{U}$-module and let $\mathcal{A}$ has an id-module normal virtual diagonal. Then $\mathcal{A}$ is id-Connes module amenable.*

Proof. Let $\mathcal{X}$ be a symmetric normal Banach $\mathcal{A}$-$\mathcal{U}$-module. By assumption we know that $\mathcal{A}$ has an identity. It is therefore sufficient for $\mathcal{A}$ to be id-Connes module amenable that we suppose that $\mathcal{X}$ is unital. Let $D_{\mathcal{U}}:\mathcal{A}\to\mathcal{X}$ be a module derivation that is ω^*-continuous. It is routine to see that E is a normal Banach $\frac{\mathcal{A}}{J}$-$\mathcal{U}$-module. Let $\mathcal{X}=(\mathcal{X}_*)^*$. Since $\mathcal{X}$ is symmetric, then $D_{\mathcal{U}}|_J=0$. We define $\tilde{D_{\mathcal{U}}}:\frac{\mathcal{A}}{J}\to\mathcal{X}$; $\tilde{D_{\mathcal{U}}}(a+J):=D_{\mathcal{U}}(a)$ $(a\in\mathcal{A})$. To each $x\in\mathcal{X}_*$, there corresponds $V_x:\frac{\mathcal{A}}{J}\times\frac{\mathcal{A}}{J}\to\mathbb{C}$ via $V_x(a+J,b+J)=\langle x,(a+J)\tilde{D_{\mathcal{U}}}(b+J)\rangle(a,b\in\mathcal{A})$. It is clearly that $V_x\in\mathcal{L}^2_{\omega^*}(\frac{\mathcal{A}}{J},\mathbb{C})$. For each $a,b\in\mathcal{A}$ and $a_*\in\mathcal{A}_*$ we have

$$\langle\int ab+Jd\mathcal{M},a_*+J^{\perp}\rangle=\langle\mathcal{M},\tilde{\omega}^*(a_*+J^{\perp})\rangle=\langle\tilde{\omega}^{**}(\mathcal{M}),a_*+J^{\perp}\rangle.$$

Now, put $f(x)=\langle\mathcal{M},\nu_x\rangle(x\in\mathcal{X}_*)$. Let $c\in\mathcal{A}$. After a little calculation, we obtain

$$\langle(c+J).f-f.(c+J)\rangle=\int\langle(ab+J)\tilde{D_{\mathcal{U}}}(c+J),x\rangle d\mathcal{M}=\langle\tilde{\omega}^{**}(\mathcal{M}).\tilde{D_{\mathcal{U}}}(c+J),x\rangle.$$

All in all, $D_{\mathcal{U}}(c)=c.f-f.c$ holds. □

In Theorem 2.6 it is shown that if an unital Banach algebra $\mathcal{A}$ has an id-module normal virtual diagonal, then $\mathcal{A}$ is id-Connes module amenable. Let S be a semigroup, it would be interesting to know that the converse holds for inverse semigroup algebra $l^1(S)$. Thus for an inverse semigroup S, we define an equivalence relation on S where $p\approx q$ if and only if there is $e\in E_S$ with $pe=qe$. The quotient semigroup $S_G=\frac{S}{\approx}$ is a group [5]. Also, E_S is a symmetric subsemigroup of S. Thus, $l^1(S)$ is a Banach $l^1(E_S)$-module. Let $l^1(E_S)$ acts on $l^1(S)$ via

$$\delta_e.\delta_p=\delta_p,\quad \delta_p.\delta_e=\delta_{pe}=\delta_p*\delta_e\qquad(p\in S,e\in E_S).$$

With recent notation, $l^1(S_G)$ is a quotient of $l^1(S)$ and so the above action of $l^1(E_S)$ on $l^1(S)$ lifts to an action of $l^1(E_S)$ on $l^1(S_G)$, making it a Banach $l^1(E_S)$-module [1].

The following theorem is the main result of the present paper.

Theorem 2.7. *Let S be an inverse weakly cancellative semigroup with idempotents E_S, let $l^1(S)$ be a Banach $l^1(E_S)$- module and let $\chi\in\mathcal{H}_{\omega^*}(l^1(S))$. If $l^1(S)$ is χ-Connes module amenable, then $l^1(S)$ has a χ-module normal virtual diagonal.*

Proof. Let $\pi:S\to S_G$ be the quotient map. By [1, Lemma 3.2], we define a bimodule action of $l^1(S)$ on $l^\infty(S_G)$ by

$$\delta_p.x=\delta_{\pi(p)}*x,\quad x.\delta_p=x*\delta_{\pi(p)}\qquad(p\in S,x\in l^\infty(S_G)).$$

It is clearly that $l^1(S_G)^*$ is a normal Banach $l^1(S)$-$l^1(E)$-module. Choose $\eta\in l^1(S_G)^*$ such that $\langle\eta,1\rangle=1$, and we considering $D:l^1(S)\to l^1(S_G)^*$ by $D(\delta_p)=\chi(\delta_p).\eta-\eta.\chi(\delta_p)$. Since $l^1(S)$ is χ-Connes module amenable, then D is inner. Therefore, there exists $\tilde{\eta}\in(\frac{l^\infty(S_G)}{\mathbb{C}})^*$ with $D(\delta_s)=ad_{\tilde{\eta}}$, so

$$\tilde{\chi}(\delta_{\pi(p)}).\eta-\eta.\tilde{\chi}(\delta_{\pi(p)})=\tilde{\chi}(\delta_{\pi(p)}).\tilde{\eta}-\tilde{\eta}.\tilde{\chi}(\delta_{\pi(p)})$$

Then we may define

$$\langle\mathcal{M},f\rangle=\lim_{\alpha}\int f\big(\tilde{\chi}(\delta_{\pi(x^*)}),\tilde{\chi}(\delta_{\pi(x)})\big)f_\alpha(x)dx.$$

Also for each p we obtain

$$\tilde{\omega}^{**}(\mathcal{M}).\tilde{\chi}(\delta_{\pi(p)})=\langle\mathcal{M},\tilde{\omega}^*(\tilde{\chi}(\delta_{\pi(p)}))\rangle=\lim_{\alpha}\int(\omega^*(\tilde{\chi}(\delta_{\pi(p)})))(\tilde{\chi}(\delta_{\pi(x^*)})),\tilde{\chi}(\delta_{\pi(x)}))f_\alpha(x)dx$$

$$= \lim_{\alpha} \tilde{\chi}(\delta_{\pi(p)}) \int f_\alpha(x)dx = \tilde{\chi}(\delta_{\pi(x)}).$$

Consequently, $\mathcal{M}$ is a χ-normal module virtual diagonal for $l^1(S)$. □

THEOREM 2.8. *Let S be a weakly cancellative semigroup with idempotents E_S, let $l^1(S)$ be an unital dual Banach $l^1(E_S)$-module and let $l^1(S)\widehat{\otimes}_{l^1(E_S)}l^1(S)$ be a dual Banach $l^1(E_S)$-module and $\chi \in \mathcal{H}_{\omega^*}(l^1(S))$. If $l^1(S)$ is χ-Connes module amenable, then $l^1(S)\widehat{\otimes}_{l^1(E_S)}l^1(S)$ is $\chi\otimes_{l^1(E_S)}\chi$-Connes module amenable.*

COROLLARY 2.9. *Let S be an inverse weakly cancellative semigroup, with idempotents E_S and let $l^1(S)$ be a Banach $l^1(E_S)$-module. Then $l^1(S)$ is Connes module amenable if and only if $l^1(S)$ has a module normal virtual diagonal.*

PROOF. This follows immediately from Theorem 2.6 and Theorem 2.7. □

EXAMPLE 2.10. Let $(\mathbb{N}, \vee)$ be a semigroup with maximum operation. It is clearly that $\mathbb{N}$ is weakly cancellative, thus $l^1(\mathbb{N})$ is a dual Banach algebra that $l^1(\mathbb{N}) = c_0(\mathbb{N})^*$. By [3, Theorem 5.13], $l^1(\mathbb{N})$ is not Connes amenable. Moreover $l^1(\mathbb{N})$ is module amenable on $l^1(E_\mathbb{N})$, therefore $l^1(\mathbb{N})$ is Connes module amenable.

3. Conclusion

In this paper we show that if S is an inverse weakly cancellative semigroup with idempotents E, χ is a bounded module homomorphism from semigroup algebra $l^1(S)$ to itself that is ω^*-continuous and $l^1(S)$ as a Banach module over $l^1(E)$ is χ-Connes module amenable, then it has χ-module normal virtual diagonal.

Acknowledgement

The author would like to thank the referee for his/her careful reading of the paper and for many valuable suggestions.

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۸ تا ۱۱ شهریور ۱۴۰۰

A polynomial-time algorithm for finding most vital edges in min-max spanning tree problems

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Abstract. A min-max spanning tree problem is to find a spanning tree in a weighted graph so that the maximum amount of its weights is minimized. This paper considers the problem in the presence of a proactive adversary. His goal is to remove some edges of the graph under a budget constraint so that the optimal value of the problem is increased as much as possible. Such the edges is called most vital edges. In this paper, a polynomial-time algorithm is proposed to find most vital edges in min-max spanning tree problems.

Keywords: Most vital edges, Spanning tree, Polynomial-time algorithm, Cut

AMS Mathematics Subject Classification [2010]: 90C27, 05C85

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1. Introduction

Two main classes of optimization problems are defined on spanning tree structures:

- minimum spanning tree (MST) problems,
- min-max spanning tree (MMST) problems.

The first is to find a spanning tree in a given weighted network $G(V, E, \mathbf{w})$ so that the summation of its weights is minimized. The second is to look for a spanning tree so as to minimize the maximum value of its weights. It is well known that MST problems can be solved in $O(|E| + |V| \log |V|)$ time by Kruskal's algorithm and Prime's algorithm, and min-max spanning tree problems can be solved in $O(|E|)$ time by a recursive algorithm [2].

A natural extension of the MST problem is to find k edges whose removal increases the optimal value of the problem as much as possible. This situation occurs whenever there is an adversary which would like to prevent us for optimizing our objective function. Such the edges are referred to as k most vital edges. It is proved that the problem can be solved in polynomial time for $k = 1$ [4], whereas it is strongly NP-hard for $k > 1$ [5]. In spite of the fact that there are several papers for finding most vital edges in MST problems [1], to the best our knowledge, there is not any work to consider the problem of finding most vital edges in MMST problems. This paper focuses on this issue in the case that there is a

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fixed cost for removing any edge, and the adversary has to satisfy a budget constraint with respect to these costs. A polynomial-time algorithm is developed to solve the problem.

2. Preliminaries and problem statement

Suppose that a network $G(V, E)$ is given, where $V = \{1, 2, \dots, n\}$ is the node set and $E = \{e_1, e_2, \dots, e_m\}$ is the edge set. We say that an edge e_k is incident to a node i if one of its endpoints is i. We use the notation (i, j) to denote edge e_k whose two endpoints are i and j. A path from i to j is a sequence of edges $e_{k_1} - e_{k_2} - \cdots - e_{k_l}$ so that any two consecutive edges intersect in one of their endpoints and additionally, e_{k_1} and e_{k_l} are incident to i and j, respectively. We admit the convention that any path contains no two repetitive nodes.

A graph $G'(V', E')$ is said to be a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$. A subgraph is called spanning if it contains all the nodes of G, i.e., $V' = V$. A (sub)graph is said to be connected if there is at least a path between any two nodes. Any maximal connected subgraph of G is referred to its connected component. So, G contains only one connected component if it is connected. Throughout this paper, we assume that G is connected. A cut is a set of edges whose removal exactly converts the graph into two connected components. A path from a node to itself is called a cycle. A set of edges which does not contain any cycle is referred to as a forest. A forest of G with n nodes and $n - 1$ edges is called a spanning tree. On the other word, a spanning tree of G is a spanning and connected subgraph which does not include any cycle.

Suppose that any edge (i, j) is associated with a nonnegative weight w_{ij}. The min-max spanning tree problem is to find a spanning tree in a way that the maximum value of its weights is minimized. The following lemma states an optimality condition for MMST problems.

Lemma 2.1. *A spanning tree T is a min-max spanning tree with optimal value p if and only if there is a cut C so that*

$$\begin{cases} w_{ij} \leq p & (i, j) \in T, \\ w_{ij} \geq p & (i, j) \in C, \end{cases} \qquad \forall (i, j) \in E. \tag{1}$$

Proof. For a proof, see [6]. □

Now let us introduce the main problem. Consider the situation in which an adversary wants to remove some edges to increases the objective function of a given MMST problem as much as possible. Any edge is associated with a removal cost c_{ij}, and the adversary has to remove edges such that their total cost does not exceed a given budget B. So, the problem is as follows:

$$\max_{E' \in X} \min_{T \in \mathbb{T}(E')} \max_{(i,j) \in T} \{w_{ij}\}, \tag{2}$$

in which $X = \{E' \subseteq E : \sum_{(i,j) \in E \setminus E'} c_{ij} \leq B\}$ and $\mathbb{T}(E')$ is the set of all spanning trees in $G(V, E')$. In the case that $\mathbb{T}(E')$ does not contain any spanning tree, then it is assumed that the optimal value of problem (2) is $+\infty$.

3. Algorithm

To develop an efficient algorithm, we introduce a new edge set corresponding to a cut C and a value p. This is denoted by $E^{(C,p)}$ and is defined as

$$E^{(C,p)} = \{(i, j) \in C : w_{ij} \leq p\}. \tag{3}$$

It is easy to see that the objective value of the MMST problem is at least equal to p in $G(V, E\backslash E^{(C,p)})$. The following lemma clarifies the reason of defining $E^{(C,p)}$.

LEMMA 3.1. *If there is a most-vital edge set S which decreases the optimal value of the MMST problem to p, then there is a cut C so that $E^{(C,p)}$ also decreases the optimal value to p*

PROOF. The proof follows from the cut notion. □

Based on Lemma 3.1, we can restrict ourselves to the sets defined in (3) to find an optimal set. The proposed approach is to find the greatest value of p so that $E^{(C,p)}$ satisfies the budget constraints. For this purpose, we introduce a new cost vector $\bar{\mathbf{c}}^p$ as follows:

$$\bar{c}^p_{ij} = \begin{cases} 0 & p \geq w_{ij}, \\ c_{ij} & p < w_{ij}, \end{cases} \qquad (i,i) \in E. \tag{4}$$

The following lemma states the relationship between $\bar{\mathbf{c}}^p$ and $E^{(C,p)}$.

LEMMA 3.2. *For a given cut C, the capacity of C with respect to $\bar{\mathbf{c}}^p$, that is $\sum_{(i,j)\in C} \bar{c}^p_{ij}$, is less than or equal to B if and only if $E^{(C,p)}$ satisfies the budget constraint.*

PROOF. The proof is straightforward. □

We are now ready to state our proposed algorithm in complete details. This algorithm uses the divide-and-conquer technique on a set of possible objective values to look for the greatest value p so that $E^{(C,p)}$ satisfies the budget constraint for some cut C. From Lemma 3.2, it can finds a minimum cut with respect to $\bar{\mathbf{c}}^p$. If the capacity of the minimum cut is less than or equal to B, then $E^{(C,p)}$ is feasible and otherwise, there is no feasible solution with the objective value greater than or equal to p. The following lemma determines the search space for finding the optimal value.

LEMMA 3.3. *The optimal value of problem (2) belongs to $\{+\infty\} \cup \bigcup_{(i,j)\in E}\{w_{ij}\}$.*

PROOF. By definition, the proof is immediate. □

Our proposed algorithm is stated formally in Algorithm 1. The correctness of Algorithm 1 follows from the above argument. Let us discuss about its complexity. The number of iterations is $O(\log(k)) = O(\log(|V|))$ in while-loop. Moreover, it is obvious that the bottleneck operation is finding a minimum cut in while-loop. Since a minimum cut can be iteratively constructed by $|V| - 1$ maximum flow computations in $(|V| - 1)O(|V||E|\log(|V|)) = O(|E||V|^2\log(|V|))$ time, the following result is immediate.

THEOREM 3.4. *Algorithm 1 solves problem (2) in $O(|E||V|^2\log^2(|V|))$ time.*

4. Conclusion

This paper considered the problem of finding most vital edges in min-max spanning tree problems. An algorithm is presented to solve the problem in polynomial time. This result is a significant observation since finding most vital edges in minimum spanning tree problems is NP-hard in general [5].

As a future work, it will be meaningful to investigate finding most vital edges and most vital nodes in other bottleneck optimization problems, such as the min-max shortest path problem, the min-max matching problem.

Algorithm 1

Input: An instance of problem (2).
Output: A set E^* of most vital edges.
Set $S = \{+\infty\} \cup \bigcup_{(i,j)\in E}\{w_{ij}\}$.
Sort elements of S in an increasing order. Let $w_1 < w_2 < \ldots < w_k$ be the sorted list.
Set $l = 1$ and $u = k$.
while $u \neq l$ **do**
  Set $mid = [\frac{l+u}{2}]$
  Set $p = w_{mid}$ and obtain $\bar{\mathbf{c}}^p$.
  Find a minimum cut C in $G(V, E)$ with respect to $\bar{\mathbf{c}}^p$.
  if the capacity of C is less than or equal to B **then**
    Set $u = mid$.
    Obtain $E^* = \{(i,j) \in C : w_{ij} \leq p\}$.
  else
    Set $l = mid$
  end if
end while

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Three-point methods with memory based on Kung-Traub's method for solving nonlinear equations

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Abstract. In this work, we have proposed a family of with-memory Kung-Traub-like three-step methods. Also, by using an accelerator parameter, we have increased the convergence order of the with-memory methods to twelve. We have increased the degree of convergence from 8 to 12, which shows a 50% improvement. Several examples are considered to illustrate the proposed method is accurate and efficient.

Keywords: With-memory method , Accelerator parameter, Nonlinear equation

AMS Mathematics Subject Classification [2010]: 65B99, 65BXX

1. Introduction

One of the most studied problems in Numerical Analysis is the conjecture of nonlinear equations. A robust tool is the use of iterative methods. It is famous that Newton's method, is one of the most used iterative methods to approximate the solution x^* of $f(x) = 0$. Because this method has no memory and has a lower efficiency index, the researchers used with-memory methods to solve nonlinear equations. In this work, we will convert the following three-step without memory method, into a single-parameter with memory method. Soleymani and Shateyi in [3] proposed the following one-parameter three-point methods for solving nonlinear equations:

(1)

$$\begin{cases} w_k = x_k + \beta f(x_k),\ y_k = x_k - \frac{f(x_k)}{f[x_k,w_k]},\ z_k = y_k - \frac{f(y_k)f(w_k)}{f[x_k,y_k](f(w_k)-f(y_k))}, \\ \phi_k = \frac{f(z_k)}{f(y_k)},\ \tau_k = \frac{f(z_k)}{f(w_k)},\ \sigma_k = \frac{f(z_k)}{f(x_k)},\ p_k = \frac{f(y_k)}{f(w_k)} k = 0, 1, 2, \cdots, \\ G(\phi_k) = 1 + \phi_k,\ H(\tau_k) = 1 + \tau_k,\ Q(\sigma_k) = 1 + \sigma_k,\ L(p_k) = 1 + (1 + \beta f[x_k, w_k])p_k^2, \\ x_{k+1} = z_k - \frac{f(z_k)f(w_k)}{f[x_k,y_k](f(w_k)-f(y_k))}(G(\phi_k)H(\tau_k)Q(\sigma_k)L(p_k)). \end{cases}$$

Theorem 1.1. *Assume that f is a sufficiently differentiable real function. Let one suppose that $\xi \in D$ is a simple zero of f. If the initial estimation x_0 is close enough to ξ, then the sequence x_n generated by any method of the family (1) converges to x^* with eighth-order of convergence if G, H, Q, and L are real sufficiently differentiable functions*

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satisfying $G(0) = G'(0) = 1, H(0) = H'(0) = 1, Q(0) = Q'(0) = 1, L(0) = 1, L'(0) = 0,$ *and* $L''(0) = 2 + 2\beta f[x_k, w_k]$.

$$e_{n+1} = (1+\beta f'(\xi))^4 c_2(2c_2^2 - c_3)(7c_2^4 - 8c_2^2c_3 + c_3^2 + c_2c_4)e_n^8 + O(e_n^9). \tag{2}$$

This work is organized as follows. In Section 2, we are going to construct new iterative methods with memory. Section 3 recounts the numerical results for solving some test systems of nonlinear equations, including matching with other existing iterative methods. Finally, Section 4 gets the main conclusions of this paper.

2. Main results

The authors(Soleymani,Shateyi) showed that the convergence order of the without methods (1) is 8. This paper aims to find more efficient methods using the same number of evaluations. For this goal, we have approximated the self-accelerating parameter in the above methods. We observe from (2) that the order of convergence of the family (1) is eight when $\beta \neq \frac{-1}{f'(\alpha)}$. With the choice $\beta = \frac{-1}{f'(\alpha)}$, it can be proved that the order of the family (1) would be 12. However, the exact value of $f'(\alpha)$ is not available in practice and such acceleration of convergence can not be realized. We have approximated the parameter β by β_k and $\beta_k = \frac{-1}{N_4'(x_k)} \approx \frac{-1}{f'(\alpha)}$, where $N_4(t) = N_4(t; x_k, x_{k-1}, w_{k-1}, y_{k-1}, z_{k-1})$. Finally, we propose one-parameter family with memory method:

(3)
$$\begin{cases} \beta_k = -\frac{1}{N_4'(x_k)},\ k = 1,2,3,\dots, \\ w_k = x_k + \beta_k f(x_k),\ y_k = x_k - \frac{f(x_k)}{f[x_k,w_k]},\ z_k = y_k - \frac{f(y_k)f(w_k)}{f[x_k,y_k](f(w_k)-f(y_k))}, \\ \phi_k = \frac{f(z_k)}{f(y_k)},\ \tau_k = \frac{f(z_k)}{f(w_k)},\ \sigma_k = \frac{f(z_k)}{f(x_k)},\ p_k = \frac{f(y_k)}{f(w_k)} k = 0,1,2,\cdots, \\ G(\phi_k) = 1+\phi_k,\ H(\tau_k) = 1+\tau_k,\ Q(\sigma_k) = 1+\sigma_k,\ L(p_k) = 1 + (1+\beta_k f[x_k,w_k])p_k^2, \\ x_{k+1} = z_k - \frac{f(z_k)f(w_k)}{f[x_k,y_k](f(w_k)-f(y_k))}(G(\phi_k)H(\tau_k)Q(\sigma_k)L(p_k)). \end{cases}$$

Lemma 2.1. *If* $\gamma_k = \frac{-1}{N_4'(x_k)}$, *then:*

$$(1+\gamma_k f'(\alpha)) \sim c_5 e_{k-1} e_{k-1,w} e_{k-1,y} e_{k-1,z}. \tag{4}$$

Theorem 2.2. *If a primary approximation x_0 is sufficiently close to the zero α of $f(x) = 0$ and the parameter γ_k in the iterative scheme (3) is recursively calculated by lemma in (2.1), then the R-order of convergence is at least 12.*

Proof. Suppose the sequence x_k is a sequence of approximations created by an iterative (IM) method. If this sequence converges to the root α, we have the equation $f(x) = 0$ with R-order, $O_r((IM), \alpha) \geq r$:

$$e_{k+1} \sim D_{k,r} e_k^r,\ e_k = x_k - \alpha. \tag{5}$$

Where $D_{k,r}$ tends to the constant asymptotic error D_k of the iterative method (IM) when $k \to \infty$. Therefore

$$e_{k+1} \sim D_{k,r} e_k^r = D_{k,r}(D_{k-1,r} e_{k-1}^r)^r = D_{k,r} D_{k-1,r} e_{k-1}^{r^2}. \tag{6}$$

If we assume that the minimum R-order duplicate sequences w_k and y_k are equal to p, and q, respectively, then we have:

$$e_{k,w} \sim D_{k,r} e_k^p = D_{k,r}(D_{k-1,r} e_{k-1}^r)^p = D_{k,r} D_{k-1,r} e_{k-1}^{rp}. \tag{7}$$

And

$$e_{k,y} \sim D_{k,r}e_k^q = D_{k,r}(D_{k-1,r}e_{k-1}^r)^q = D_{k,r}D_{k-1,r}e_{k-1}^{rq}. \tag{8}$$

$$e_{k,z} \sim D_{k,r}e_k^s = D_{k,r}(D_{k-1,r}e_{k-1}^r)^s = D_{k,r}D_{k-1,r}e_{k-1}^{rs}. \tag{9}$$

Now, using Lemma (2.1) and R-orders we will have:

$$\begin{aligned}(1+\beta_k f'(\alpha)) &\sim e_{k-1}e_{k-1,w}e_{k-1,y}e_{k-1,z}\\ &\sim D_{k-1,p}e_{k-1}^p D_{k-1,q}e_{k-1}^q D_{k-1,s}e_{k-1}^s e_{k-1}\\ &= D_{k-1,p}D_{k-1,q}D_{k-1,s}e_{k-1}^{1+p+q+s}.\end{aligned} \tag{10}$$

Now, we have :

$$\begin{cases} e_{k,w} \sim (1+\beta_k f'(\alpha))e_k,\\ e_{k,y} \sim (1+\beta_k f'(\alpha))e_k^2,\\ e_{k,z} \sim (1+\beta_k f'(\alpha))^2 e_k^4,\\ e_{k+1} \sim (1+\beta_k f'(\alpha))^4 e_k^8. \end{cases} \tag{11}$$

Therefore, considering the relations (10), and (11) we obtain:

$$\begin{cases} e_{k,w} \sim (1+\beta_k f'(\alpha))e_k \sim D_{k-1,p}D_{k-1,q}D_{k-1,s}D_{k-1,r}e_{k-1}^{(1+p+q+s)+r},\\ e_{k,y} \sim (1+\beta_k f'(\alpha))e_k^2 \sim D_{k-1,p}D_{k-1,q}D_{k-1,s}D_{k-1,r}e_{k-1}^{(1+p+q+s)+2r},\\ e_{k,z} \sim (1+\beta_k f'(\alpha))^2 e_k^4 \sim D_{k-1,p}D_{k-1,q}D_{k-1,s}D_{k-1,r}e_{k-1}^{2(1+p+q+s)+4r},\\ e_{k+1} \sim (1+\beta_k f'(\alpha))^4 e_k^8 D_{k-1,p}D_{k-1,q}D_{k-1,r}D_{k-1,s}e_{k-1}^{4(1+p+q+s)+8r}. \end{cases} \tag{12}$$

Combining (6)-(12), (7)-(12), (8)-(12), and (9)-(12), we conclude

$$\begin{cases} e_{k,w} \sim e_{k-1}^{(1+p+q+s)+r},\\ e_{k,y} \sim e_{k-1}^{(1+p+q+s)+2r},\\ e_{k,z} \sim e_{k-1}^{2(1+p+q+s)+4r},\\ e_{k+1} \sim e_{k-1}^{4(1+p+q+s)+8r}. \end{cases} \tag{13}$$

Therefore, by comparing exponents of e_{k-1} appearing in three pairs of relations ((6),(13)), ((7),(13)), ((8),(13)) and ((9),(13)), considering the power equations, we will finally reach the system of the following three unknown equations:

$$\begin{cases} rp - r - (p+q+s+1) = 0,\\ rq - 2r - (p+q+s+1) = 0,\\ rs - 4r - 2(p+q+s+1) = 0,\\ r^2 - 8r - 4(p+q+s+1) = 0. \end{cases} \tag{14}$$

The only positive answer of this system equations nonlinear is: $r = 12$, $s = 6$, $q = 3$, $p = 2$. So, the proof of Theorem 2.2 finish. We show this method with TM12. □

3. Numerical results

This section demonstrates the convergence behavior of the with-memory methods (3). All computations are performed using the programming package Mathematica with multiple-precision arithmetic. Table 1 also include, for each test function, the initial estimation values and the last value of the computational order of convergence r_c [4] computed by the expression

$$r_c \approx \frac{\log |f(x_n)/f(x_{n-1})|}{\log |f(x_{n-1})/f(x_{n-2})|}. \tag{15}$$

3.1. Table. We use the following functions, most of which are the same as in [5]

$$\begin{cases} f_1(x) = x\log(1 + x\sin(x)) + e^{-1+x^2+x\cos(x)} \sin(\pi x),\ \alpha = 0,\ x_0 = 0.6, \\ f_2(x) = \dfrac{1}{x^4} - x^2 - \dfrac{1}{x} + 1,\ \alpha = 1,\ x_0 = 1.4 \end{cases} \tag{16}$$

TABLE 1. Comparison of the absolute error of the proposed method with other methods

$f_1(x) = x\log(1 + x\sin(x)) + e^{-1+x^2+x\cos(x)} \sin(\pi x),\ \alpha = 0,\ x_0 = 0.6$					
Methods	$\lvert x_1 - \alpha\rvert$	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	r_c	EI
KTM [2]	0.23230(−1)	0.33703(−13)	0.13863(−107)	8.00000	1.68179
CLMTM [1]	0.74137(−18)	0.43686(−144)	0.63508(−1154)	8.00000	1.68179
SSM [3]	0.19880(−3)	0.14391(−32)	0.10773(−265)	8.00000	1.68179
TM12, $\beta_0 = 0.1$	0.10773(0)	0.69686(−14)	0.49537(−163)	12.00000	1.86121
$f_2(x) = \frac{1}{x^4} - x^2 - \frac{1}{x} + 1,\ \alpha = 1,\ x_0 = 1.4$					
Methods	$\lvert x_1 - \alpha\rvert$	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	r_c	EI
KTM [2]	0.10721(−1)	0.45584(−12)	0.50318(−95)	8.00000	1.68179
SSM [3]	0.59024(−2)	0.29737(−13)	0.11094(−103)	8.00000	1.68179
CLMTM [1]h_1, g_1	0.37731(−21)	0.41016(−169)	0.79993(−1353)	8.00000	1.68179
TM12, $\beta_0 = 0.1$	0.63131(−4)	0.22066(−46)	0.61197(−556)	12.00000	1.86121

4. Conclusion

In this paper, we have obtained a new class of with memory methods. The order of convergence of the new methods with memory is 12. Also, the Newtons interpolatory polynomials of 4 degree is applied for constructing considerably faster methods employing information from the current and previous iteration without any additional evaluations of the function. The results show that this new methods is useful to nd an acceptable approximation of the exact solution. The efciency index of the proposed family with memory is $12^{\frac{1}{4}} = 1.86121$.

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Gradient Einstein-type manifold

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Abstract. We derive a lower bound for potential function of extended gradient Einstein-type metrics on a complete non-compact Riemannian manifold. Moreover, it's proved that any gradient Einstein-type manifold is homeomorphic to interior of a compact manifold with boundary, that is, it's of finite topological type, provided that either $Ric \geqslant \delta^{-1} g$ and the injectivity radius $inj(M, g) \geqslant \delta > 0$ for some $\delta > 0$ or the Ricci tensor Ric is bounded above.

Keywords: Ricci soliton; Einstein-type manifold; quasi-Einstein

AMS Mathematics Subject Classification [2010]: 53C25, 53C20

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1. Introduction

Riemannian manifolds endowed with metrics satisfying some structural equations, possibly involving curvature and some globally defined vector fields are subject of great interest in geometry and physics. Recently, Einstein-type manifolds were introduced by G. Catino et al. that generalized Einstein manifolds, quasi-Einstein metrics, Ricci solitons, Ricci almost soliton, m-quasi Einstein metric, Yamabe solitons, Yamabe quasi-solitons and conformal gradient solitons, see [2]. Take (M, g) as a Riemannian manifold; then (M, g) is an Einstein-type manifold if there is a vector field V and smooth real function λ on M such that g satisfies the equation

$$\alpha Ric + \beta \mathcal{L}_V g - \gamma V^\flat \otimes V^\flat = \lambda(x) g, \tag{1}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. If the vector field V is the gradient of a potential function f , then (M, g) is said to be gradient Einstein-type manifold and (1) takes the familiar form

$$\alpha Ric + \beta \nabla\nabla f - \gamma df \otimes df = \lambda(x) g.$$

It is conjectured, by J. Milnor, that the fundamental group of a complete Riemannian manifold with positive mean curvature Ricci tensor must be finitely generated, see [5]. Such a manifold may not be homeomorphic to interior of a compact manifold with boundary, that is, may not have finite topological type, see [4]. F. Fang, J. Man, and Z. Zhang show that a complete shrinking gradient Ricci soliton with suitable assumptions on the Ricci tensor is of finite topological type, see [3]. Moreover J.Y. Wu has studied extended Yamabe solitons for inequalities and elaborated that a complete extended gradient Yamabe soliton, that is a complete Riemannian manifold satisfying $Hess(f) \geqslant (\lambda - R)g$, in

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certain cases has finite topological type, see [6]. Next, the present author inspiring the Bonnet-Myers Theorem has extended the equation Yamabe soliton for inequalities and among the others they have shown that a Riemannian complete non-compact shrinking Yamabe soliton (M, g, V, λ) has finite fundamental group and its first cohomology group vanishes, provided that the scalar curvature is strictly bounded above by λ, see [1].
In the peresent paper, we have showed that a complete extended gradient Einstein-type manifold with suitable assumptions on the Ricci tensor has finite topological type.

2. Finiteness of topological type

Here, we consider extended gradient Einstein-type manifolds as follows:

$$\alpha Ric + \beta Hess(f) - \gamma df \otimes df \geqslant \lambda(x) g, \tag{2}$$

and considering some conditions on this we prove it has finite topological type. That is, M is homeomorphic to interior of a compact manifold with boundary.

THEOREM 2.1. *Let (M, g) be a complete Riemannian manifold satistfying (2), where $\alpha, \gamma \geqslant 0$ and $\lambda(x) \geqslant \Lambda > 0$ for some constant $\Lambda > 0$. Then for any $p, q \in M$,*

$$\|\nabla f\|_q \geqslant \frac{\Lambda}{2|\beta|} T - \|\nabla f\|_p - C,$$

provided that either $Ric \geqslant \delta^{-1} g$ and $inj(M, g) \geqslant \delta > 0$ for some $\delta > 0$ or the Ricci tensor Ric is bounded above.

PROOF. Consider any geodesic $\theta : [0, +\infty) \longrightarrow M$ parametrized by arc length s emanating from a point p in M. Then along θ we have

$$\alpha Ric(\theta', \theta') \geq \lambda(x) - \beta Hess(f)(\theta', \theta') + \gamma g(\theta', \nabla f)^2.$$

On the other side

$$Hess(f)(\theta', \theta') = \mathcal{L}_{\nabla f} g(\theta', \theta') = 2g(\nabla_{\theta'} \nabla f, \theta') = 2\frac{d}{ds} g(\nabla f, \theta').$$

Therefore

$$\alpha \int_0^T Ric(\theta', \theta') ds \geq \int_0^T \lambda(x) ds - 2\beta g(\nabla f, \theta'(s))\Big|_0^T + \gamma \int_0^T g(\theta'(s), \nabla f)^2 ds.$$

Since $\gamma \geqslant 0$, we have $\gamma \int_0^T g(\theta'(s), \nabla f)^2 ds \geqslant 0$ and by assumption $\lambda(x) \geqslant \Lambda > 0$, we conclude that

$$\alpha \int_0^T Ric(\theta', \theta') ds \geq \Lambda T - 2\beta\big(g(\nabla f, \theta'(T)) - g(\nabla f, \theta'(0))\big) \tag{3}$$

By using the Cauchy-Schwarz inequality $|g(\theta'(s), \nabla f)| \leq \|\nabla f\|_{\theta(s)}$, we have $-\|\nabla f\|_p \leqslant g(\theta'(0), \nabla f) \leqslant \|\nabla f\|_p$ and $-\|\nabla f\|_q \leqslant g(\theta'(T), \nabla f) \leqslant \|\nabla f\|_q$. Hence, we get

$$-(\|\nabla f\|_q + \|\nabla f\|_p) \leqslant g(\theta'(T), \nabla f) - g(\theta'(0), \nabla f) \leqslant \|\nabla f\|_q + \|\nabla f\|_p.$$

If $\beta \geqslant 0$, we have $-2\beta(g(\theta'(T), \nabla f) - g(\theta'(0), \nabla f)) \geqslant -2\beta(\|\nabla f\|_q + \|\nabla f\|_p)$ and consequently (3) reduces to

$$\alpha \int_0^T Ric(\theta', \theta') ds \geqslant \Lambda T - 2\beta\big(g(\nabla f, \theta'(T)) - g(\nabla f, \theta'(0))\big) \geqslant \Lambda T - 2\beta(\|\nabla f\|_q + \|\nabla f\|_p).$$

If $\beta \leqslant 0$, we have $-2\beta(g(\theta'(T),\nabla f)-g(\theta'(0),\nabla f)) \geqslant 2\beta(\|\nabla f\|_q+\|\nabla f\|_p)$ and consequently (3) reduces to

$$\alpha \int_0^T Ric(\theta',\theta')ds \geqslant \Lambda T - 2\beta\big(g(\nabla f,\theta'(T)) - g(\nabla f,\theta'(0))\big) \geqslant \Lambda T + 2\beta(\|\nabla f\|_q + \|\nabla f\|_p).$$

Therefore for all $\beta \in \mathbb{R}$ we have

$$\alpha \int_0^T Ric(\theta',\theta')ds \geqslant \Lambda T - 2|\beta|(\|\nabla f\|_q + \|\nabla f\|_p).$$

Hence

$$2|\beta|\|\nabla f\|_q \geqslant \Lambda T - 2|\beta|\|\nabla f\|_p - \alpha \int_0^T Ric(\theta',\theta')ds.$$

If the integral $\alpha \int_0^T Ric(\theta',\theta')ds < C$ for some real positive constant C, then

$$2|\beta|\|\nabla f\|_q \geqslant \Lambda T - 2|\beta|\|\nabla f\|_p - C,$$

as we have claimed. □

THEOREM 2.2. *Under the assumptions of Theorem (2.1), M is of finite topological type.*

PROOF. By using Theorem (2.1), $\|\nabla f\|_q$ has a linear growth in $T = d(p,q)$. Since f is a smooth function so it is continuous and consequently $f^{-1}\big((-\infty,a]\big)$ is closed. On the other hand, for any $p,q \in f^{-1}\big((-\infty,a]\big)$ we have

Obviously, $f^{-1}\big((-\infty,a]\big)$ is compact for any $a < \infty$ and so f is a proper function. Also, one can easily check that f has no critical points outside of a compact set. In fact, it's enough to consider a compact set $\bar{B}(p,\frac{2|\beta|\|\nabla f\|_p+C}{\Lambda})$. The deformation lemma(Isotopy Lemma) of Morse theory leads to M has finite topological type. □

COROLLARY 2.3. *[3] If $Ric \geqslant \delta^{-1}g$ and $inj(M,g) \geqslant \delta > 0$ for some $\delta > 0$, then the integral $\int_0^T Ric(\theta',\theta')ds$ is bounded above.*

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An unprecedented monadic filter in monadic BL-algebras

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Abstract. In this paper, we introduce and consider a new type of monadic filters in monadic BL-algebras, Gödel monadic filters. We define the notion of Gödel monadic filters on monadic BL-algebras and some basic properties of them are determined. Therefore, using this concept and their vital properties, representable monadic BL-algebras are characterized.

Keywords: (Monadic) BL-algebras, monadic filters, Gödel monadic filters.

AMS Mathematics Subject Classification [2010]: 08B50, 03G25, 03B52.

1. Introduction

Certain information processing approaches, especially inferences based on certain information, is based on the classical logic. Fuzzy logics are generalizations of classical logic that allow us to sake inchmeal.Several new algebras playing the role of the structures of truth values have been introduced and axiomatize. BL-algebras were introduced in the 1990s by Hájek as the equivalent algebraic semantics for his basic fuzzy logic.

Monadic algebras have been investigated since Halmos introduced monadic Boolean algebras [4]. Monadic MV-algebras were introduced by Rutledge and studied by Di Nola, Grigolia, Cimadamore, and Díaz Varela [2, 3]. D. Castaño et al. introduced the variety of monadic BL-algebras avowed with monadic operators $\forall$ and $\exists$, providing the complete characterization of the range of the monadic operators [1].

In the present paper, we define and study Gödel monadic filters in monadic BL-algebras. Definition of Gödel monadic filters in monadic BL-algebras is distinct from that of filter in another algebraic structures such as BL-algebras. In Section 3, we determine the concept of Gödel monadic filters and give their basic properties that will be used in the extant of the article.

2. Preliminaries

Definition 2.1. [1] A residuated lattice is an algebra $\mathcal{L} = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ with four binary operations and two constants 0,1 such that:

- $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
- operation $*$ is commutative and associative, with 1 as neutral element, and

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• $x * y \leq z$ iff $x \leq y \to z$, for all x, y and z in L.

A residuated lattice $\mathcal{L} = (L, \vee, \wedge, *, \to, 0, 1)$ is called a BL-algebra if it satisfies the following identities, for all $x, y \in L$:

$(x \to y) \vee (y \to x) = 1$ (prelinearity),

$x \wedge y = x * (x \to y)$(divisibility).

Definition 2.2. [1] A monadic BL-algebra is a structure $\mathcal{A} = (A, \vee, \wedge, *, \to, \forall, \exists, 0, 1)$ such that $(A, \vee, \wedge, *, \to, 0, 1)$ is a BL-algebra, and $\forall$, $\exists$ are unary operations on A satisfying the following conditions, for all $x, y \in A$:

(M_1) $\forall x \to x = 1$.

(M_2) $\forall(x \to \forall y) = \exists x \to \forall y$.

(M_3) $\forall(\forall x \to y) = \forall x \to \forall y$.

(M_4) $\forall(\exists x \vee y) = \exists x \vee \forall y$.

(M_5) $\exists(x * x) = \exists x * \exists x$.

From now on $\mathcal{A} = (A, \vee, \wedge, \to, *, \forall, \exists, 0, 1)$ or simply A is a monadic BL-algebra unless otherwise specified.

Definition 2.3. [1] A monadic filter of a monadic BL-algebra A is a non-empty subset F of A satisfying the following conditions, for all $x, y \in A$:

(F_1) if $x \in F, y \in A$ and $x \leq y$, then $y \in F$,

(F_2) if $x, y \in F$, then $x * y \in F$,

(F_3) if $x \in F$, then $\forall x \in F$.

A monadic filter F of A is called proper if $F \neq A$. If F satisfies conditions (F_1), (F_2), then F is a filter of A. For all $x, y \in A$, we write $x \sim_F y$ iff $x \to y$ and $y \to x$ are both in F. $\sim_F$ is always a congruence relation. Note that $(F.3)$ is a necessary condition for this statement. Indeed, if $\sim_F$ is a congruence relation on a monadic BL-algebra A and $x \in F$, than $x \sim_F 1$ and therefor $\forall x \sim_F \forall 1 = 1$, which is equivalent with $\forall x \in F$.

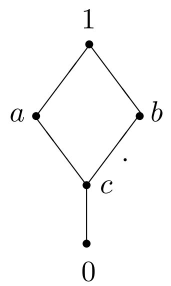
3. Gödel filters on monadic BL-algebras

Definition 3.1. A monadic BL-algebra A in which $x^2 = x$, for every $x \in A$ is called a monadic Gödel algebra.

Example 3.2. Let $A = \{0, a, b, c, 1\}$ and the operations $*$, $\to$ defined as follows:

$*$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1



Then $(A, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra.

Consider the maps $\forall_j, \exists_k : A \longrightarrow A$, $j = 1, 2, 3$, $k = 1, 2, 3$, given in the tables below:

x	0	c	a	b	1
$\forall_1 x$	0	0	0	0	1
$\forall_2 x$	0	c	c	c	1
$\forall_3 x$	0	c	a	b	1

x	0	c	a	b	1
$\exists_1 x$	0	1	1	1	1
$\exists_2 x$	0	c	1	1	1
$\exists_3 x$	0	c	a	b	1

Clearly, $(A, \wedge, \vee, *, \to, \forall_j, \exists_k, 0, 1)$ are monadic BL-algebras. Since $x^2 = x$, for every $x \in A$, $(A, \wedge, \vee, *, \to, \forall_j, \exists_k, 0, 1)$ are monadic Gödel algebras.

Definition 3.3. A filter F of a monadic BL-algebra A is called Gödel monadic filter if A/F is a monadic Gödel algebras.

In the following, we give an example of Gödel monadic filter in monadic BL-algebras.

Example 3.4. Let $A = \{0, a, 1\}$. We define operators $\forall, \exists, *, \to$ as follow:

x	$\forall x$
0	0
a	0
1	1

x	$\exists x$
0	0
a	1
1	1

$*$	0	a	1
0	0	0	0
a	0	0	a
1	0	a	1

$\to$	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

$\mathcal{A} = (A, \vee, \wedge, *, \to, \forall, \exists, 0, 1)$ is a monadic BL-algebra. It is clear that, $F = \{1\}$ is a Gödel monadic filter of $\mathcal{A}$.

Definition 3.5. A monadic filter F of monadic BL-algebra A is called an implicative monadic filter if $\forall x \to (\forall y \to \forall z) \in F$ and $\forall x \to \forall y \in F$ imply $\forall x \to \forall z \in F$, for all $x, y, z, \in A$.

Proposition 3.6. *F is an implicative monadic filter of monadic BL-algebra A if and only if $\forall x \to (\forall x)^2 \in F$ for every $x \in A$.*

Proposition 3.7. *For a monadic filter F of a monadic BL-algebra A the following conditions are equivalent:*
(i) F is a Gödel monadic filter of A,
(ii) $\forall x \to (\forall x)^2 \in F$ for every $x \in A$,
(iii) If $\forall y \to (\forall y \to \forall x) \in F$, then $\forall y \to \forall x \in F$,
(iv) If $\forall x \to (\forall y \to \forall z) \in F$, then $(\forall x \to \forall y) \to (\forall x \to \forall z) \in F$, for every $x, y, z \in A$.

By Proposition 3.6 and Proposition 3.7, we have:

Theorem 3.8. *For any monadic filter F, F is an implicative monadic filter if and only if F is Gödel monadic filter.*

Using Theorem 3.8, we have:

Example 3.9. In Example 3, $F = \{1\}$ is an implicative monadic filter of A.

Theorem 3.10. *For a monadic filter F of a monadic BL-algebra A the following statements are equivalent:*
(i) F is a Gödel monadic filter of A,
(ii) $(\forall x \wedge \forall y) \to (\forall x * \forall y) \in F$, for every $x, y \in A$,
(iii) if $\forall x \to (\forall y \to \forall z) \in F$, then $(\forall x \to \forall z) \vee (\forall y \to \forall z) \in F$.

Theorem 3.11. *For a monadic filter F of a monadic BL-algebra A the following conditions are equivalent:*
(i) If $(\forall x \to \forall y) \to (\forall x \to \forall z) \in F$, then $(\forall x \to \forall z) \vee (\forall y \to \forall z) \in F$,
(ii) $((\forall x \to \forall y) \to (\forall x \to \forall z)) \to ((\forall x \to \forall z) \vee (\forall y \to \forall z)) \in F$, for every $x, y, z \in A$.

Proposition 3.12. *Let F be a monadic filter of a monadic BL-algebra A. If F is a Gödel monadic filter of A, then $[\forall x \to (\forall x * \forall y)] \vee [\forall y \to (\forall x * \forall y)] \in F$, for every $x, y \in A$.*

Proof. Let F be a monadic filter of a monadic BL-algebra A. Using following Theorem 3.10, (ii), $(\forall x \wedge \forall y) \to (\forall x * \forall y) \in F$, for every $x, y \in A$, Since $\forall x * (\forall x \to \forall y) \leq \forall x \wedge \forall y$, then

$$\begin{aligned}(\forall x \wedge \forall y) \to (\forall x * \forall y) &\leq [\forall x * (\forall x \to \forall y)] \to (\forall x * \forall y)\\ &= (\forall x \to \forall y) \to [\forall x \to (\forall x * \forall y)]\end{aligned}$$

By hypothesis, F is a monadic filter, so $(\forall x \to \forall y) \to [\forall x \to (\forall x * \forall y)] \in F$. Using Theorem 3.11 (i), we get $[\forall x \to (\forall x * \forall y)] \vee [\forall y \to (\forall x * \forall y)] \in F$, for every $x, y \in A$. $\square$

Corollary 3.13. *F be a monadic filter of a monadic BL-algebra A if and only if $[\forall x \to (\forall x * \forall y)] \vee [\forall y \to (\forall x * \forall y)] \in F$, for every $x, y \in A$.*

4. Conclusion

In this paper, motivated by the previous research of monadic BL-algebras, we extended the concept of Gödel monadic filters in monadic BL-algebras. We introduce and study these types of monadic filters and given some characterizations and several examples of them. Therefore, we used to these results to find some classification for monadic BL-algebras.

In our future work, we will continue our study of algebraic structures, especially monadic BL-algebras, with the view to identify a classification for these structures.

Acknowledgements. The authors are very grateful to the referees for the valuable suggestions in obtaining the final form of this paper.

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On Skew power series McCoy rings

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ABSTRACT. Let R be a ring with an endomorphism α. A ring R is a skew power series McCoy ring if whenever any non-zero power series $f(x)=\sum_{i=0}^{\infty} a_i x^i, g(x)=\sum_{j=0}^{\infty} b_j x^j \in R[[x;\alpha]]$ satisfy $f(x)g(x)=0$, then there exists a non-zero element $c \in R$ such that $a_i c=0$, for all $i=0,1,\ldots$. We investigate relations between the skew power series ring and the standard ring-theoretic properties. Moreover, we obtain some characterizations for skew power series ring $R[[x;\alpha]]$, to be McCoy, zip, strongly AB and has Property (A).

Keywords: Noetherian ring, α-compatible ring, Skew Power series McCoy ring, Zip ring, Reversible ring.

AMS Mathematics Subject Classification [2010]: 16S50; 16D40; 16D70.

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1. Introduction and preliminaries

Throughout the paper, all rings are associative with identity. Let α be a ring endomorphism of R. We denote $R[[x;\alpha]]$ the skew power series rings whose elements are the power series over R, the addition is defned as usual and the multiplication satisfies in the relation $xa=\alpha(a)x$, for any $a \in R$.
For notations we use $Ni\ell(R)$ and $C_{f(x)}$ for the set of all nilpotent elements of a ring R, the set in R consisting of all the "coefficients" of $f(x)$ where $f(x)$ is a power series, respectively. By $Z_\ell(R)$, $Z_r(R)$ and $Z(R)$, we mean respectively the set of all left zero-divisors of R, the set of all right zero-divisors of R and the set of all zero-divisors of R (i.e., $Z(R)=Z_\ell(R)\cup Z_r(R)$).

A ring R is called *reversible* if $ab=0$ implies $ba=0$, for $a,b \in R$. Note that for the class of reversible rings the set of all left annihilators of any element $a \in R$ coincide with set of its all right annihilators and we denote it by $ann_R(a)$. A ring R is called *semicommutative* if $ab=0$ implies $aRb=0$ for $a,b \in R$. Moreover, a ring is *right* (resp., *left*) *duo* if every right (resp., left) ideal is an ideal. Simple computations show that reversible as well (one-sided) duo rings are semicommutative.

Following [2], a ring R is *right McCoy* if $f(x)g(x)=0$, then $f(x)c=0$ for some non-zero $c \in R$, where $f(x), g(x)$ are non-zero polynomials in $R[x]$. *Left McCoy rings* are

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defined similary and they satisfy similar properties. A ring R is called *McCoy* if it is both left and right McCoy.

The case for formal power series in one variable is much more difficult and by [3], McCoy's theorem fails in general for the case of formal power series ring $R[[x]]$ over a commutative ring R. In fact, as a continuation of works by Gilmer *et al.* in [4], Fields [3, Theorem 5], proved that if R is a commutative Noetherian ring with identity in which $(0) = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is a shortest primary representation of (0) with $\sqrt{Q_i} = P_i$, then $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ is a zero-divisor in $R[[x]]$ if and only if there is a non-zero element $r \in R$ which satisfies $rf(x) = 0$. He also provided an example showing that the condition "R is Noetherian" is not redundant [3, Example 3]. On the other hand, Camillo and Nielsen, constructed an example showing that formal power series rings over an associative noncommutative McCoy ring R, need not be McCoy in general.

Our first main aim in this paper is to provide some rich classes of skew power series McCoy rings. A detailed analysis of our main results is rewarded with the generalization of the main result of [3]. Moreover, we investigate relations between skew power series McCoy property and other standard important ring-theoretic properties such as zip rings, strongly AB rings and rings with Property (A).

2. Annihilators in Noetherian power series rings

Definition 2.1. A ring R is *skew power series-wise McCoy* if whenever any non-zero power series $f(x), g(x) \in R[[x;\alpha]]$ satisfy $f(x)g(x) = 0$, then $f(x)c = 0$ for some non-zero $c \in R$.

As mentioned, McCoy's theorem fails in the skew power series ring $R[[x;\alpha]]$ over either commutative or noncommutative ring R. However, if we assume a "Noetherian" hypothesis on a coefficient ring R, we obtain stronger conditions on the coefficients of power series $f(x) \in R[[x;\alpha]]$. The crucial for some of our results is the following lemma, which might be useful in some other studies.

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Theorem 2.2. *Let R be a reversible right Noetherian ring. If R is an α-compatible ring, then R is a skew right power series-wise McCoy ring.*

Corollary 2.3. *[1, Theorem 2.2] Let R be a reversible right Noetherian ring. Then R is a right power series-wise McCoy ring.*

Theorem 2.4. *Let R be a Noetherian reversible ring. If R is an α-compatible ring, then for each $f(x) = \sum\limits_{i=1}^{\infty} a_i x^i \in R[[x;\alpha]]$, these conditions are equivalent:*

(1) *$f(x)$ is a zero-divisor in $R[[x;\alpha]]$;*
(2) *$f(x) \in P_k[[x;\alpha]]$ for some $1 \leq k \leq n$, with P_k prime;*
(3) *There is a non-zero element $c \in R$ such that $f(x)c = 0$.*

Corollary 2.5. *[1, Theorem 2.3] Let R be a Noetherian reversible ring. Then for each $f(x) = \sum\limits_{i=1}^{\infty} a_i x^i \in R[[x]]$, these conditions are equivalent:*

(1) *$f(x)$ is a zero-divisor in $R[[x]]$;*
(2) *$f(x) \in P_k[[x]]$ for some $1 \leq k \leq n$, with P_k prime;*
(3) *There is a non-zero element $c \in R$ such that $f(x)c = 0$.*

Theorem 2.6. *Let R be a right duo right Noetherian ring. If R is an α-compatible ring, then R is a skew right power series-wise McCoy ring.*

Corollary 2.7. *[1, Theorem 2.4] Let R be a right duo right Noetherian ring. Then R is a right power series-wise McCoy ring.*

Theorem 2.8. *Let R be a Noetherian duo ring. If R is an α-compatible ring, then for each $f(x)=\sum_{i=1}^{\infty} a_i x^i \in R[[x;\alpha]]$, these conditions are equivalent:*

(1) *$f(x)$ is a zero-divisor in $R[[x;\alpha]]$;*
(2) *$f(x)\in P_k[[x;\alpha]]$ for some $1\leq k\leq n$, with P_k prime;*
(3) *There is a non-zero element $c\in R$ such that $f(x)c=0$.*

Corollary 2.9. *[1, Theorem 2.5] Let R be a Noetherian duo ring. Then for each $f(x)=\sum_{i=1}^{\infty} a_i x^i \in R[[x]]$, these conditions are equivalent:*

(1) *$f(x)$ is a zero-divisor in $R[[x]]$;*
(2) *$f(x)\in P_k[[x]]$ for some $1\leq k\leq n$, with P_k prime;*
(3) *There is a non-zero element $c\in R$ such that $f(x)c=0$.*

Proposition 2.10. *Let R be a semicommutative right (left) Noetherian ring. If R is an α-compatible ring, then the ring R is right zip if and only if $R[[x;\alpha]]$ is a zip ring.*

Corollary 2.11. *[1, Proposition 3.1] Let R be a semicommutative right (left) Noetherian ring. Then the ring R is right zip if and only if $R[[x]]$ is a zip ring.*

Corollary 2.12. (a) *Let R be a reversible one-sided Noetherian ring. If R is an α-compatible ring, then the ring R is right zip if and only if $R[[x;\alpha]]$ is a right zip.*

(b) *Let R be a one-sided duo one-sided Noetherian ring. If R is an α-compatible ring, then the ring R is right zip if and only if $R[[x;\alpha]]$ is right zip.*

Theorem 2.13. *Let R be a Noetherian skew power series-wise McCoy ring. Then R is strongly AB if and only if $R[[x;\alpha]]$ is strongly AB.*

Corollary 2.14. *[1, Lemma 3.5] Let R be a Noetherian power series-wise McCoy ring. Then R is strongly AB if and only if $R[[x]]$ is strongly AB.*

Corollary 2.15. (a) *Let R be a reversible Noetherian ring. If R is an α-compatible ring, then R is strongly AB if and only if $R[[x;\alpha]]$ is strongly AB.*

(b) *Let R be a duo Noetherian ring. If R is an α-compatible ring, then R is strongly AB if and only if $R[[x;\alpha]]$ is strongly AB.*

Proposition 2.16. *Let R be a right Noetherian ring such that $R[[x;\alpha]]$ is strongly right AB. If R is an α-compatible ring, then the ring R is right skew power series-wise McCoy.*

Corollary 2.17. *[1, Proposition 3.8] Let R be a right Noetherian ring such that $R[[x]]$ is strongly right AB. Then the ring R is right power series-wise McCoy.*

Corollary 2.18. *Let R be a right Noetherian, α-compatible ring. Then the power series ring $R[[x,\alpha]]$ is strongly right AB if and only if the ring R is skew right power series-wise McCoy and strongly right AB.*

Theorem 2.19. *Let R be a reversible right Noetherian ring and I a right ideal of $R[[x;\alpha]]$. If R is an α-compatible ring and $r_{R[[x;\alpha]]}(I)\neq 0$, then $r_R(I)\neq 0$.*

Corollary 2.20. *Let R be a right Noetherian ring such that the skew power series ring $R[[x;\alpha]]$ is strongly right AB. Then for each subset S of $R[[x;\alpha]]$, if $r_{R[[x;\alpha]]}(S) \neq 0$ then $r_R(S) \neq 0$.*

Definition 2.21. A ring R has *right* (resp., *left*) *Property* (A), if for every finitely generated two-sided ideal $I \subseteq Z_l(R)$ (resp., $I \subseteq Z_r(R)$), there exists non-zero $a \in R$ (resp., $b \in R$) such that $Ia = 0$ (resp., $bI = 0$). So, a ring R has right Property (A) if every finitely generated two-sided ideal consisting entirely of left zero-divisors has a non-zero right annihilator.

Theorem 2.22. *Let R be a right Noetherian right skew power series-wise McCoy ring. If R is an α-compatible ring and R has right Property (A) then so is skew the formal power series ring $R[[x;\alpha]]$.*

Corollary 2.23. *[1, Theorem 3.12] Let R be a right Noetherian right power series-wise McCoy ring. If R has right Property (A) then so is the formal power series ring $R[[x]]$.*

Corollary 2.24. *Let R be a right Noetherian and reversible ring. If R is an α-compatible ring and R has right Property (A) then so is the formal power series ring $R[[x;\alpha]]$.*

Corollary 2.25. *Let R be a right Noetherian and right duo ring. If R is an α-compatible ring and R has right Property (A) then so is the formal power series ring $R[[x;\alpha]]$.*

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Hypergroupoids associated with a metric space

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Abstract. In this paper, we associate some weak commutative hypergroupoids with a given metric space and obtain some results in this respect. We show that these hypergroupoids are weak commutative H_v-groups. Also, we determine some conditions for these hypergroupoids to be hypergroups. Finally, we obtain Γ-H_v-groups by a metric space.

Keywords: hypergroupoid, H_v-group, hypergroup, metric space

AMS Mathematics Subject Classification [2010]: 20N20, 54E35

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1. Introduction

The hyperstructure theory was born in 1934, when Marty introduced the notion of a hypergroup. Since then, many books have been written on this topic (see for instance [1,3,8]). Recently, a book on weak hyperstructure was written by Davvaz and Vougiouklis [4]. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. More exactly, let $P^*(X)$ be the set of all non-empty subsets of a given set X. A *hypergroupoid* is a pair $(X, \circ)$, where X is a non-empty set and $\circ$ is a *hyperoperation*, i.e.,

$$\circ : X \times X \longrightarrow P^*(X), \quad (x, y) \mapsto x \circ y.$$

If $A, B \in P^*(X)$, then we define $A \circ B = \bigcup \{a \circ b \mid a \in A, b \in B\}$, $x \circ B = \{x\} \circ B$ and $A \circ y = A \circ \{y\}$. If $A = \emptyset$ or $B = \emptyset$ we define $A \circ B = \emptyset$.

A hypergroupoid $(X, \circ)$ is called *semihypergroup* if the associative axiom is valid, i.e., $x \circ (y \circ z) = (x \circ y) \circ z$, for all $x, y, z \in X$ and it is called *reproductive* if $x \circ X = X \circ x = X$, for all $x \in X$. A *hypergroup* is a reproductive semihypergroup. A hypergroupoid $(X, \circ)$ is called H_v*-semigroup* if the weak associative axiom is valid, i.e., $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset$, for all $x, y, z \in X$. An H_v*-group* is a reproductive H_v-semigroup. A hypergroupoid $(X, \circ)$ is called (weak) commutative if for all $x, y \in X$ we have

$$(x \circ y \cap y \circ x \neq \emptyset) \; x \circ y = y \circ x.$$

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A commutative hypergroup is called a *join space* if the following implication holds for all elements a, b, c, d of X:

$$a/b \bigcap c/d \neq \emptyset \Longrightarrow a \circ d \bigcap b \circ c \neq \emptyset,$$

where $a/b = \{x \mid a \in x \circ b\}$.

Connections between hypergraphs and hyperstructures were studied by many authors, for example, see [2, 6]. Mirvakili and Manaviyat in [5] obtained commutative quasihypegroups associated with a metric space. In this paper we obtain some other classes of H_v-groups associated with a metric space. These H_v-groups can be non-commutative.

Let X be a non-empty set. Then a function $d : X \times X \to \mathbb{R}$ is said to be a metric on X if it has the following properties for all $x, y, z \in X$:

(M1) $d(x, y) \geq 0$;
(M2) $d(x, y) = 0$ if and only if $x = y$;
(M3) $d(x, y) = d(y, x)$;
(M4) $d(x, y) + d(y, z) \geq d(x, z)$.

The real number $d(x, y)$ is called the distance between x and y, and the set X together with a metric d is called a metric space (X, d) [7].

Given a metric space (X, d) and any real number $r > 0$, the open ball of radius r and center a is the set $B_r(a) \subseteq X$ defined by

$$B_r(a) = \{x \in X | d(x, a) < r\}.$$

2. Main results

Let $\mathcal{X} = (X, d)$ be a metric space and for all $x \in X$ and $r \in \mathbb{R}^+$, $B_r(x)$ be the open ball of radius r and center x.

For every $x, y \in X$ and $r, s \in \mathbb{R}^+$, we define a hyperoperation ${}_r\circ_s$ as follows:

$$x \ {}_r\circ_s \ y = B_r(x) \cup B_s(y), \quad \forall (x, y) \in X^2, \ .$$

If $r = s$ then we show that the hyperoperation ${}_r\circ_s$ by $\circ_r$.

We set ${}_rX_s = (X, {}_r\circ_s)$ and $X_r = (X, \circ_r)$.

By the above notations we have for every $(x, y) \in X^2$:

(1) $x \ {}_r\circ_s \ y \subseteq x \ {}_r\circ_s \ x \cup y \ {}_r\circ_s \ y$;
(2) $\{x, y\} \subseteq x \ {}_r\circ_s \ y$; and
(3) $y \in x \circ_r x \Leftrightarrow x \in y \circ_r y$.
(4) $x \ {}_r\circ_s \ y \cap y \ {}_r\circ_s \ x \neq \emptyset$,

Therefore, we have

Corollary 2.1. *The hypergroupoid ${}_rX_s = (X, {}_r\circ_s)$ is a weak commutative H_v-group.*

Set $\Delta = \sup\{d(x, y) | x, y \in X\}$.

Lemma 2.2. *Let $\mathcal{X} = (X, d)$ be a metric space and $r, s \in \mathbb{R}^+$.*

(1) *If $r \geq \Delta$ or $s \geq \Delta$ then for every $x, y \in X$, $x \ {}_r\circ_s \ y = X$ and this means ${}_rX_s = (X, {}_r\circ_s)$ is a total hypergroup.*
(2) *If $r = s$ or $r \geq \Delta$ or $s \geq \Delta$ then the hypergroupoid ${}_rX_s = (X, {}_r\circ_s)$ is commutative.*

Theorem 2.3. *[5] For every $x, y \in X$, the following statements are satisfied:*

(1) $x \circ_r y = x \circ_r x \cup y \circ_r y$;
(2) $x \in x \circ_r x$; *and*
(3) $y \in x \circ_r x \Leftrightarrow x \in y \circ_r y$.

Theorem 2.3

THEOREM 2.4. *[1] The hypergroupoid X satisfying* (1), (2), (3) *of the Theorem 2.3 also satisfies*

(4) $x \circ_r y \supseteq \{x, y\}$,
(5) $x \circ_r y = y \circ_r x$,
(6) $x \circ_r X = X$,
(7) $(x \circ_r x) \circ_r x = \cup_{z \in x \circ_r x} z \circ_r z$,
(8) $(x \circ_r x) \circ_r (x \circ_r x) = x \circ_r x \circ_r x$.

By (5) and (6) of Theorem 2.4 we obtain:

COROLLARY 2.5. *The hypergroupoid $X_r = (X, \circ_r)$ is a commutative H_v-group.*

The next Theorem shows that necessary an sufficient conditions for associativity of the hyperoperation $\circ_r$.

THEOREM 2.6. *[1] A hypergroupoid $(X, \circ_r)$ satisfying* (1), (2), (3) *of the Theorem 2.3 is a hypergroup if and only if*

$$\forall (a, c) \in X^2, \quad c \circ_r c \circ_r c - c \circ_r c \subseteq a \circ_r a \circ_r a.$$

COROLLARY 2.7. *A hypergroupoid $X_r = (X, \circ_r)$ is a hypergroup if and only if for every $x, y \in X$ we have*

$$B_r^2(x) - B_r(x) \subseteq B_r^2(y),$$

where $B_r^2(x) = \bigcup\limits_{z \in B_r(x)} B_r(z)$.

COROLLARY 2.8. *If for every $x \in X$, $B_r^2(x) = B_r(x)$ then the hypergroupoid $X_r = (X, \circ_r)$ is a hypergroup.*

EXAMPLE 2.9. Let (X, d) be the metric space where d is discrete metric, that is $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise. Then for every $r \in \mathbb{R}$ and for every $x \in X$, we have $B_r^2(x) = B_r(x)$. So, by corollary 2.8, the hypergroupoid $X_r = (X, \circ_r)$ is a hypergroup.

EXAMPLE 2.10. If G is an undirected connected graph, then the set V of vertices of G can be turned into a metric space by defining $d(x, y)$ to be the length of the shortest path connecting the vertices x and y.

The r-ball $B(x, r)$ of center x and radius $r \geq 0$ consists of all vertices of G at distance at most r^- from x: In particular, if $0 \leq r \leq 1$, the ball $B_r(x) = \{x\}$ and if $1 < r \leq 2$, the ball $B_r(x)$ comprises x and $N(x)$, where $N(x)$ is the neighborhood of vertex x in graph G.

Therefore, the hyperoperations $\circ_r$ and ${}_r\circ_s$ coincide with the hyperoperation $\circ$ in [2,6].

THEOREM 2.11. *If the hypergroupoid $(X, {}_r\circ_s)$ is a hypergroup then it is a join space.*

DEFINITION 2.12. Let Γ be a non-empty set of some hyperoperations on X. (X, Γ) is called $\Gamma - H_v$-semigroup, if for all $x, y, z \in X$ and $\alpha, \beta \in \Gamma$, we have

$$x\alpha(y\beta z) \cap (x\alpha y)\beta z \neq \emptyset.$$

Moreover, if for every $\alpha \in \Gamma$, (X, α) is a quasihypergroup, then (X, Γ) is called $\Gamma - H_v$-group,

THEOREM 2.13. *Let $\Gamma \subseteq \{{}_r\circ_s | r, s \in \mathbb{R}^+\}$. Then (X, Γ) is a $\Gamma - H_v$-group.*

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Papers

Part 2: Posters

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Shahid Bahonar University of Kerman
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پنجاه و دومین
کنفرانس ریاضی ایران
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دانشگاه شهید باهنر کرمان
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۸ تا ۱۱ شهریور ۱۴۰۰

Planar iterated line graphs of $\Gamma_I(R)$

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Abstract. Let R be a finite commutative ring and I be a non-zero ideal of R. The *ideal-based zero-divisor graph* of R with respect to the ideal I, denoted by $\Gamma_I(R)$, is the graph on vertices $\{x \in R \setminus I \mid xy \in I$ for some $y \in R \setminus I\}$ where distinct vertices x and y are adjacent if and only if $xy \in I$. In this talk, we study planarity of the iterated line graphs of the ideal-based zero divisor graphs. We give a complete characterization of all these graphs with respect to their planar index.

Keywords: Ideal-based zero divisor graph, Planar index, Iterated line graph.

AMS Mathematics Subject Classification [2010]: 05C10, 13A70

1. Introduction

Finding the relationship between the algebraic structure, using properties of graphs associated to them, has become an interesting topic in the last years. Indeed, it is worthwhile to relate algebraic properties of the rings to the combinatorical properties of the assigned graphs. One of these associated graphs to a commutative ring R is the zero-divisor graph. There are two variations of the zero-divisor graph. This graph was first introduced by Beck, in [**3**], where all the elements of R are considered as the vertices and two vertices x and y are adjacent if and only if $xy = 0$. In a later variant, studied by Anderson and Livingston [**1**] and denoted by $\Gamma(R)$, the vertex set of graph was restricted to $Z(R) \setminus \{0\}$.

The focus of this paper is on a generalization of the zero-divisor graph called the ideal-based zero-divisor graph. Let I be an ideal of R. The *ideal-based zero-divisor graph* of R with respect to the ideal I, denoted by $\Gamma_I(R)$, is the graph on vertices

$$\{x \in R \setminus I \mid xy \in I \text{ for some } y \in R \setminus I\}$$

where distinct vertices x and y are adjacent if and only if $xy \in I$. It is easy to see that if $I = 0$, then $\Gamma_I(R) = \Gamma(R)$. So, this graph, which was defined and studied in [**5**, **6**], is a natural generalization of the zero-divisor graph for commutative rings with nonzero identity.

Let G be a simple graph. *The line graph* $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G, and two vertices of $L(G)$ are adjacent if and only if their

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correspoding edges are incident in G. We denote the kth iterated line graph of G by $L^k(G)$. It is defined recursively via $L^k(G) = L(L^{k-1}(G))$ where $L^0(G) = G$ and $L^1(G) = L(G)$.

In this talk, we study the planarity of the iterated line graphs of the ideal-based zero divisor graphs. We give a complete characterization of ideal-based zero divisor graphs with respect to their planar index.

2. Planar index of the graph $\Gamma_I(R)$

Let G be a graph. Recall that G is *planar* if it can be drawn in the plane without any edge crossings. Kuratowski's theorem states that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

The planar index of G was introduced and studied in [4]. The planar index of G is the smallest k such that $L^k(G)$ is non-planar. We denote the planar index of G by $\gamma(G)$. If for all $k \geqslant 0$, $L^k(G)$ is planar, then we define $\gamma(G) = \infty$.

The planar index of the graph $\Gamma(R)$ was studied in [2]. In the following theorem, we give a characterization of the ideal-based zero divisor graphs with their planar index. To do this, we use the characterization of connected graphs with respect to their planar index which was proved in [4].

Theorem 2.1. *Let R be a finite commutative ring and I be a non-zero ideal of R. Then*

- (a) $\gamma(\Gamma_I(R)) = \infty$ *if and only if one of the following holds*
  - (a_1) *I is a prime ideal of R.*
  - (a_2) *R is isomorphic to a ring with corresponding ideal from Table 1 where $K \cong \mathbb{Z}_2$, Table 2 or Table 3.*
- (b) $\gamma(\Gamma_I(R)) = 1$ *if and only if R is isomorphic to a ring with corresponding ideal from Table 1 where K is a field with at least 3 elements or Tables 5 and 6.*
- (c) $\gamma(\Gamma_I(R)) = 2$ *if and only if R is isomorphic to a ring with corresponding ideal from Table 4.*
- (d) $\gamma(\Gamma_I(R)) = 0$ *otherwise.*

Ring	Ideal
$\mathbb{Z}_4 \times K$	$(2) \times 0$
$\mathbb{Z}_2[X]/(X^2) \times K$	$(x) \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times K$	$\mathbb{Z}_2 \times 0 \times 0$, $0 \times \mathbb{Z}_2 \times 0$ $0 \times 0 \times K$ when $K = \mathbb{Z}_2$

Table 1

Ring	Ideal
$\mathbb{Z}_3 \times \mathbb{Z}_4$	$\mathbb{Z}_3 \times 0$
$\mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2)$	$\mathbb{Z}_3 \times 0$

Table 2

Ring	Ideal
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$	$\mathbb{Z}_2 \times 0$
$\mathbb{Z}_8$	(4)
$\mathbb{Z}_4[X]/(X^2, 2X)$	(x) (2) $(x+2)$
$\mathbb{Z}_4[X]/(2X, X^2-2)$	(2)
$\mathbb{Z}_2[X]/(X^3)$	(x^2)
$\mathbb{Z}_2[X,Y]/(X^2, XY, Y^2)$	(x) (y) $(x+y)$

Table 3

Ring	Ideal
$\mathbb{Z}_2 \times \mathbb{Z}_9$	$\mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_3[X]/(X^2)$	$\mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_8$	$\mathbb{Z}_2 \times (4)$
$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(X^2, 2X)$	$\mathbb{Z}_2 \times I_1$ s.t. $I_1 = (x), (2), (x+2)$
$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2 - 2)$	$\mathbb{Z}_2 \times (2)$
$\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3)$	$\mathbb{Z}_2 \times (x^2)$
$\mathbb{Z}_2 \times \mathbb{Z}_2[X,Y]/(X^2, XY, Y^2)$	$\mathbb{Z}_2 \times I_1$ s.t. $I_1 = (x), (y)$
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\mathbb{Z}_4 \times 0$, $0 \times \mathbb{Z}_4$
$\mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2)$	$\mathbb{Z}_4 \times 0$, $0 \times \mathbb{Z}_2[X]/(X^2)$
$\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)$	$\mathbb{Z}_2[X]/(X^2) \times 0$, $0 \times \mathbb{Z}_2[X]/(X^2)$
$\mathbb{Z}_4 \times \mathbb{F}_4$	$0 \times \mathbb{F}_4$
$\mathbb{Z}_2[X]/(X^2) \times \mathbb{F}_4$	$0 \times \mathbb{F}_4$
$\mathbb{Z}_2[X]/(X^4)$	$\{0, x^2, x^3, x^2 + x^3\}$
$\mathbb{Z}_2[X,Y]/(X^3, XY, Y^2)$	$\{0, x, x^2, x + x^2\}$, $\{0, y, x^2, y + x^2\}$, $\{0, x^2, x + y, x + y + x^2\}$
$\mathbb{Z}_2[X,Y]/(X^2 - Y^2, XY)$	$\{0, x, x^2, x + x^2\}$, $\{0, y, x^2, y + x^2\}$, $\{0, x^2, x + y, x + y + x^2\}$
$\mathbb{Z}_4[X]/(X^2 - 2)$	$\{0, 2, 2x, 2 + 2x\}$
$\mathbb{Z}_4[X]/(X^2 - 2X - 2)$	$\{0, 2, 2x, 2 + 2x\}$
$\mathbb{Z}_2[X,Y,Z]/(X,Y,Z)^2$	$\{0, z, x + y, x + y + z\}$, $\{0, x, y + z, x + y + z\}$ $\{0, x, z, x + z\}$, $\{0, y, z, y + z\}$, $\{0, x, y, x + y\}$, $\{0, x + y, y + z, x + z\}$ $\{0, y, x + z, x + y + z\}$
$\mathbb{Z}_4[X,Y]/(X^2 - 2, XY, Y^2, 2X)$	$\{0, 2, x, 2 + x\}$, $\{0, y, 2, y + 2\}$, $\{0, 2, x + y, x + y + 2\}$
$\mathbb{Z}_4[X,Y]/(X^2 - 2, XY, Y^2 - 2, 2X)$	$\{0, 2, x, 2 + x\}$, $\{0, y, 2, y + 2\}$, $\{0, 2, x + y, x + y + 2\}$
$\mathbb{Z}_4[X,Y]/(X^2, XY - 2, Y^2)$	$\{0, 2, x, 2 + x\}$, $\{0, y, 2, y + 2\}$, $\{0, 2, x + y, x + y + 2\}$
$\mathbb{Z}_4[X]/(X^3, 2X)$	$\{0, 2, x^2, 2 + x^2\}$, $\{0, x, x^2, x + x^2\}$, $\{0, x + 2, x^2, 2 + x + x^2\}$
$\mathbb{Z}_4[X]/(X^2)$	$\{0, 2, 2x, 2 + 2x\}$, $\{0, 2x, x + 2, 3x + 2\}$
$\mathbb{Z}_4[X]/(X^2 - 2X)$	$\{0, x, 2x, 3x\}$, $\{0, 2, 2x, 2x + 2\}$, $\{0, 2x, x + 2, 3x + 2\}$
$\mathbb{Z}_8[X]/(X^2, 2X)$	$\{0, 2, 4, 6\}$, $\{0, 4, x, x + 4\}$, $\{0, 4, x + 2, x + 6\}$
$\mathbb{Z}_8[X]/(X^2, 2X)$	$\{0, 2, 4, 6\}$, $\{0, 4, x, x + 4\}$, $\{0, 4, x + 2, x + 6\}$
$\mathbb{Z}_4[X,Y]/(2,X,Y)^2$	$\{0, 2, x, x + 2\}$, $\{0, 2, y, y + 2\}$, $\{0, 2, x + y, x + y + 2\}$, $\{0, x, y, x + y\}$, $\{0, x, y + 2, x + y + 2\}$, $\{0, y, x + 2, x + y + 2\}$ $\{0, x + y, x + 2, y + 2\}$
$\mathbb{Z}_{16}$	$\{0, 4, 8, 12\}$
$\mathbb{Z}_2[X,Y]/(X^2, Y^2)$	$\{0, x, xy, x + xy\}$, $\{0, y, xy, y + xy\}$, $\{0, xy, x + y, x + y + xy\}$

TABLE 4

3. Conclusion

Let R be a finite commutative ring and I be a non-zero ideal of R. In this paper, we studied the planar index of the graph $\Gamma_I(R)$. We showed that $\gamma(\Gamma_I(R)) \leqslant 2$ or $\gamma(\Gamma_I(R)) = \infty$. Also, we provied the list of all finite commutative rings and their ideals which $\gamma(\Gamma_I(R)) \leqslant 2$.

Ring	Ideal
$\mathbb{Z}_2 \times \mathbb{Z}_8$	$\mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2 - 2)$	$\mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3)$	$\mathbb{Z}_2 \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_2[X, Y]/(X^2, XY, Y^2)$	$\mathbb{Z}_2 \times 0$
$\mathbb{Z}_2[X, Y]/(X^3, XY, Y^2)$	(y), $(y + x^2)$,
$\mathbb{Z}_4[X]/(X^3 - 2, 2X)$	(2)
$\mathbb{Z}_4[X]/(X^2 - 2)$	$(2x)$
$\mathbb{Z}_4[X]/(X^2 - 2X - 2)$	$(2x)$
$\mathbb{Z}_4[X, Y]/(X^2 - 2, XY, Y^2, 2X)$	(y), $(y + 2)$
$\mathbb{Z}_4[X]/(X^3, 2X)$	(2), $(x^2 + 2)$
$\mathbb{Z}_8[X]/(X^2, 2X)$	(x), $(x + 4)$
$\mathbb{Z}_{16}$	(8)

TABLE 5

Ring	Ideal
$\mathbb{Z}_2 \times \mathbb{Z}_8$	$0 \times (4)$
$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2)$	$0 \times (x)$ $0 \times (x)$, $0 \times (x + 2)$
$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2 - 2)$	$0 \times (2)$
$\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3)$	$0 \times (x^2)$
$\mathbb{Z}_2 \times \mathbb{Z}_2[X, Y]/(X^2, XY, Y^2)$	$0 \times (x)$ $0 \times (y)$, $0 \times (x + y)$
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$(2) \times 0$, $0 \times (2)$
$\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)$	$(x) \times 0$, $0 \times (x)$
$\mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2)$	$(2) \times 0$, $0 \times (x)$

TABLE 6

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

On the capacity of sum-networks based on characteristic of the finite field

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Abstract. A sum-network is an instance of a function computation problem over a finite field of information observed at all the sources nodes. In this paper, we consider a family of sum-networks whose network coding capacity is dependent on message alphabet (specifically the characteristic of the finite field) chosen for communication. Our work described construction that improved previous results in this line by demonstrating sum-networks with significantly fewer number of sources and terminals.

Keywords: Network coding, Sum-Network, Coding Capacity, Function Computation.

AMS Mathematics Subject Classification [2010]: 94A24, 94A29, 94B35.

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1. Introduction

Characterizing the capacity of communication networks has been the most important and challenging problem in network information theory. All reported sum-networks are shown to have only certain rational valued coding capacities. For example, it has been shown that for a sum-network having 3 sources and 3 terminals, the coding capacity is either 0, $2/3$ or ≥ 1 [?]. The example sum-networks with 3 sources, n terminals ($n > 3$) and m sources, 3 terminals ($m > 3$) have been shown where coding capacity of those networks is of the form $\frac{k}{k+1}$ for $k \geq 3$ [?]. The paper [?] has also considered general case in which for every positive rational number $\frac{k}{n}$, exists a sum-network which has coding capacity equal to $\frac{k}{n}$. The number of sources and terminals of this network is $(2n-1)+\binom{2n-1}{2}$. The capacity of these sum-networks however was independent of the message alphabet. Recently, the authors have introduced a family of sum-networks based on a structure called BIBD, where have significantly fewer number of sources and terminals and the coding capacity of them is dependent on the message alphabet chosen for communication [?]. The present work is closely related to the last reference. In this paper, we show for any $r \geq 1$, $k \geq 2$, there exists a divisible BIBD structure, and achieve a general coding capacity for general ms/nt sum-networks.

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2. PRELIMINARY

We consider communication over a directed acyclic graph $G = (V, E)$ where, V is the set of nodes and $E \in V \times V$ are the edges denoting the delay-free communication links between them. Subset $S \subset V$ denotes the source and $T \subset V$ denotes the terminal nodes. Each source node $s_i \in S$ generates an independent random process $X_{i1}, X_{i2}, ...$ indexed by time are i.i.d. and each X_{ij} takes values that are uniformly distributed over a finite alphabet $\mathcal{F}$ that is assumed to be a finite field such that $|\mathcal{F}| = q$. The characteristic of $\mathcal{F}$ will be denoted by $ch(\mathcal{F})$. For an edge $e = (i, j) \in E$, we call the node j as the head of the edge and the node i as the tail of edge; and denote them as $head(e)$ and $tail(e)$ respectively. For each node v, the set of incoming edges at the node v is denoted by $In(v) = \{e \in E : head(e) = v\}$ and the set of outgoing edges from the node v is denoted by $Out(v) = \{e \in E : tail(e) = v\}$. A network code is an assignment of encoding functions to each edge in E and a decoding function to each terminal in T. Define a global encoding function that expresses the value transmitted on an edge in terms of the source values: Local encoding function for edge e. $\phi_e : \mathcal{F}^m \to \mathcal{F}^n,\ \ if\ \ tail(e) \in S,\ \ \ \phi_e : \mathcal{F}^{n|In(tail(e))|} \to \mathcal{F}^n,\ \ if\ \ tail(e) \notin S$. Decoding function for the terminal $t_i \in T$. $\psi_{t_i} : \mathcal{F}^{n|In(t_i)|} \to \mathcal{F}^m$. A network code is linear if all the edge and decoding functions are $\mathcal{F}$-linear. A (m, n) fractional network code solution over $\mathcal{F}$ is the sum of m source symbols are communicated to all the terminals in n units of time. The rate of this network code is defined to be m/n. The supremum of all achievable rates is called the capacity of the network.

3. CONSTRUCTION USING BIBDs

Definition 3.1. A $2-(v, k, \lambda)$ balanced incomplete block design (BIBD) is a system $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ which has the following components: *Points*: A set $\mathcal{P}$ of v elements indexed in arbitrary order as $\mathcal{P} = \{p_1, p_2, ..., p_v\}$. A set $\mathcal{B}$ of size b whose elements are k-subsets of $\mathcal{B}$ such that $\mathcal{B} = \{B_1, B_2, ..., B_b\}$. $\mathcal{B}$ satisfies the following regularity property. For $p_i, p_j \in \mathcal{P}$, $i \neq j$,

$$|\{B \in \mathcal{B} : p_i \in B, p_j \in B\}| = \lambda. \tag{1}$$

For such design, we have an incidence matrix A which is a $v \times b$ (0,1)-matrix that records the incidence between points and blocks, i.e.,

$$A(i, j) = \begin{cases} 1 & if\ \ p_i \in B_j, \\ 0 & otherwise. \end{cases} \tag{2}$$

It can be shown that each point is present in a fixed number of blocks (denoted by r). Let $\ \ r = \frac{\lambda(v-1)}{k-1},\ \ \ bk = v.r$. For any $p \in \mathcal{P}$ and $B \in \mathcal{B}$, let

$< p >= \{B \in \mathcal{B} : p \in B\}$, and

$< B >= \cup_{p \in B} < p >= \cup_{p \in B}\{B' \in \mathcal{B} : p \in B'\}$.

Thus we have $| < p > | = r$, for all $p \in \mathcal{P}$.

Example 3.2. Consider the components of a $2-(9, 3, 1)$ design as below: $\mathcal{P} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $\mathcal{B} = \{B_1, B_2, ..., B_6\}$, $B_1 = \{1, 2, 3\}$,

$B_2 = \{4, 5, 6\}$, $B_3 = \{7, 8, 9\}$, $B_4 = \{1, 4, 7\}$, $B_5 = \{2, 5, 8\}$, $B_6 = \{3, 6, 9\}$.

3.1. Sum-network Construction. Consider a indirected simple connected graph $G = (V, E)$, for any BIBD $\mathcal{D}$, where:

(1) Source node set S contains $v + b$ elements corresponding to points and blocks in $\mathcal{D}$. Let $S = \{s_p : p \in \mathcal{P}\} \cup \{s_B : B \in \mathcal{B}\}$.

(2) Terminal node set T contains $\upsilon + b$ elements, in the same manner as S. Let $T = \{t_p : p \in \mathcal{P}\} \cup \{t_B : B \in \mathcal{B}\}$.
(3) Intermediate nodes are 2υ vertices which are elements of $M^H \cup M^T$, where $M^H = \{m_1^h, m_2^h, ..., m_\upsilon^h\}$ and $M^T = \{m_1^t, m_2^t, ..., m_\upsilon^t\}$.
(4) Bottleneck edges are υ unit-capacity edges $e_i = (m_i^t, m_i^h),\ i = 1, 2, ..., \upsilon$. We also make the following connections all e_i: (s_{p_i}, m_i^t) and (s_{B_j}, m_i^t) for all $B_j \in < p_i >$, (m_i^h, t_{p_i}) and (m_i^h, t_{B_j}) for all $B_j \in < p_i >$.
(5) Direct edges. for every $p_i \in \mathcal{P}$ and $B_j \in \mathcal{B}$, we have: (s_{p_l}, t_{p_i}), for all $p_l \notin p_i$, (s_{b_l}, t_{p_i}), for all $B_l \notin < p_i >$, (s_{p_l}, t_{B_j}), for all $p_l \notin B_j$, and (s_{B_l}, t_{B_j}), for all $B_l \notin < B_j >$.

Now, we can express below theorem without proving [**?**].

THEOREM 3.3. *For a $2-(\upsilon, k, 1)$ design $\mathcal{D}$, the coding capacity of the sum-network obtained using the above construction is at most $\frac{k(k-1)}{k(k-1)+\upsilon-1}$ if $ch(\mathcal{F}) \nmid (k-1)$.*

4. UPER BOUND ON THE CAPACITY OF SUM-NETWORK

DEFINITION 4.1. Let $\mathcal{P}$ be a set of υ elements, called points, and $\mathcal{B}$ be a cpllection of subsets of $\mathcal{P}$, called blocks. The pair of $(\mathcal{P}, \mathcal{B})$ is called a (υ_r, b_k)-divisible structure if the following conditions hold.

(1) Each block contains k points.
(2) Each points belongs to r blocks.
(3) Every pair of distinct points of $\mathcal{P}$ belong to at most one block.
(4) All blocks in $\mathcal{B}$ can be partitioned into r parallel classes.

LEMMA 4.2. *For any $r \geq 1$, $k \geq 2$, (that is not nessecary k be a prime power) there exists a (υ_r, b_k)-divisible structure with $\upsilon = k^m$ and m is an arbitrary integer such that $m \geq log_2(r+1)$.*

PROOF. Cosider the $\mathbb{Z}_k$-module, $\mathcal{P} = \mathbb{Z}_k$, where $\mathbb{Z}_k$ is the ring of integers module k. For any $\alpha \in \mathbb{Z}_k$, we use $\alpha(j)$ to denote the jth coordinate of α. For each nonempty $s \subseteq [m]$, let $\alpha_s \in \mathbb{Z}_k^m$ be such that: $\alpha_s(j) = 1$ if $j \in S$ and zero otherwise. Let $B_{s,0} \triangleq \{i.\alpha_s,\ i \in \mathbb{Z}_k\}$. Clearly, $B_{s,0}$ is a subset of $\mathbb{Z}_k^m$ with r elements and $B_{s,0} \cap B_{s',0} = (0, 0, ..., 0)$ for any two distinct nonempty subsets s and s' of $[m]$, $s \neq s'$, $s, s' \neq \varnothing$. Let $\mathcal{B}_s = \{B_{s,l},\ l = 0, 1, 2, ..., k^{m-1} - 1\}$ be the collection of all cosets of $B_{s,0}$. Then $\alpha_1 - \alpha_2 \in B_{s,0}$ for any $l \in \{0, 1, 2, ..., k^{m-1} - 1\}$ and any $\alpha_1, \alpha_2 \in B_{s,l}$. Note that $m \geq log_2(r+1)$ and $[m]$ has $2^m - 1$ nonempty subsets. We can always pick r nonempty subsets of $[m]$, say $s_1, s_2, ..., s_r$. Let $\mathcal{B} = \bigcup_{i=1}^r \mathcal{B}_{s_i}$. We claim that $(\mathcal{P} = \mathbb{Z}_k^m, \mathcal{B})$ is a divisible structure, which can be seen as follows:
Firstly, for each nonempty $s \subseteq [m]$, $\mathcal{B}_s$ is a partition of $\mathcal{P}$, also conditions (1), (2) and (4) of Definition 1 holds. Secondly, if s, s' are two distinct nonempty subsets of $[m]$ and $l, l' \in \{0, 1, 2, ..., 2^m - 1\}$, then we have $|B_{s,l} \cap B_{s',l'}| \leq 1$. If $\alpha_1, \alpha_2 \in B_{s,l} \cap B_{s',l}$, then $\alpha_1 - \alpha_2 \in B_{s,0} \cap B_{s',0} = (0, 0, ..., 0)$. Hence, we have $\alpha_1 = \alpha_2$, as $|B_{s,l} \cap B_{s,l'}| \leq 1$. Moreover, since for each nonempty $s \subseteq \mathcal{B}_s$ is a partition of $\mathcal{P}$, so condition (3) of Definition 1 holds. □

LEMMA 4.3. *For any $k \geq 2$, let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a divisible structure defined in* 4.1. *Then $\mathcal{D}$ clearly is a BIBD.*

PROOF. Straight forward. □

THEOREM 4.4. *For a $2-(v,k,1)$ design $\mathcal{D}$, the coding capacity of the sum-network obtained using the above construction is at most $\frac{k}{k+r}$.*

PROOF. We have $ch(\mathcal{F}) \nmid (k-1)$, so $\frac{m}{n} \leq \frac{k(k-1)}{k(k-1)+v-1} = \frac{v}{v+b} = \frac{k^m}{k^m+k^{m-1}r} = \frac{k}{k+r}$. □

Thus exists a (m,n) fractional network code whose coding capacity just depends on the characteristic of finite field $\mathcal{F}$ and parameter r of BIBD.

EXAMPLE 4.5. Let $k=3$ and $r=2$. Consider the BIBD in 3.2. then we can construct a sum-network with the coding capacity equal to $3/5$. By considering structure in subsection 3.1, the number of sources and terminals of sum-network will be $v+b=15$, while we could construct the other network corresponding to divisible structure with fewer sources and terminals, $v+b=k+r=5$ [**?**].

5. CONCLUSION AND FURTHER RESEARCH

In this paper, for any $k \geq 2$ (that is not nessecary k be a prime power), we define a BIBD on a special finite field and constructe a family of sum-networks whose coding capacity depends on the characteristic of field. These sum-networks are in general, smaller (with fewer sources and terminals) than sum-networks known to achieve the coding capacity in order to previous works. The further research is going to describe general an undirected sum-network on the basic of elements of divisible structure allows us that analyze these (m,n) fractional network code.

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Lyapunov Function for a Compartmental Model in Epidemiology

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Abstract. Mathematical modeling has become an important and useful tool in studying the spread and control of infection disease. The basic reproduction number,is one of the most useful threshold parameters which characterize mathematical problem concerning infections diseases. In this paper,we using Lyapunov techniques for study the stability and asymptotic stability of dynamical systems. and revisit a classical result.

Keywords: Lyapunov function, Infection models, Epidemiology, Global stability

AMS Mathematics Subject Classification [2010]: 37C75, 93A30

1. Introduction

Mathematical modeling has become an important and useful tool in studying the spread and control of infection disease. A major project in deterministic epidemiological modeling of heterogeneous populations is to find conditions for local and global stability of the equilibria and to work out the relations among these stability conditions, the threshold of epidemic take-off, and endemicity, and the basic reproduction.

The basic reproduction number,is one of the most useful threshold parameters which characterize mathematical problem concerning infections diseases.

2. Model formulation

In this section we formulate a epidemic model by the following system of ordinary differential equations. The model is formulated with the state variables, S,E,I,R, that represent respectively, the number of susceptible (S), exposed (E), infected(I), recovered (R) individuals. The model is described by following system of differential equations:

$$\begin{aligned} \dot{S}(t) &= \mu\omega(1-\nu I) - \beta IS - \mu S \\ \dot{E}(t) &= \beta IS - \delta E - \mu E \\ \dot{I}(t) &= q\delta E + \mu\omega\nu I - (\gamma+\mu)I \\ \dot{R}(t) &= (1-q)\delta E + \gamma I + \mu(1-\omega) - \mu R \end{aligned} \tag{1}$$

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In these equations, all the parameters are nonnegative. μ is the birth and death rate, ω is the birth that is unsuccessfully immunized. ν is the proportion of children developing infection born to unimmunized mothers. δ is the rate of transfer from exposed to infections.β is the transmission coefficient.γ is the recovery rate of infected. A proportion q of latent individuals become infected.For simplicity, we normalize the population size to 1.

$$S(t) + E(t) + I(t) + R(t) = 1 \tag{2}$$

Hence, the system (1) becomes

$$\begin{aligned} \dot{S}(t) &= \mu\omega(1 - \nu I) - \beta I S - \mu S \\ \dot{E}(t) &= \beta I S - (\delta - \mu)E \\ \dot{I}(t) &= q\delta E + \mu\omega\nu I - (\gamma + \mu)I \end{aligned} \tag{3}$$

Let

$$\Omega = \{(S, E, I) \in R^3_+ | 0 \leq S + E + I \leq \omega\}$$

Then it is clear that Ω is a positive invariant subset of R^3_+.The system (3) has a disease-free equilibrium (DFE), $P_0 = (S_0, E_0, I_0) = (\omega, 0, 0) \in \Omega$ and has a endemic equilibrium (EE) in Ω, $P^* = (S^*, E^*, I^*) = (\frac{\omega}{R_0}, \frac{\beta\omega I^*}{(\delta+\mu)R_0}, (1 - \frac{1}{R_0})\frac{\mu q\delta\omega}{\mu q\delta\nu\omega-(\delta+\mu)(\mu-\mu\omega\nu})$. The positive EE existes for all $R_0 > 1$. Where the parameter

$$R_0 = \frac{\beta q\delta\omega}{(\delta + \mu)(\gamma + \mu - \mu\omega\nu)} \tag{4}$$

is basic reproduction number got by next generation matrix [5].

3. Stability of equilibria

Theorem 3.1. *For system (3)*

i. *The DEF P_0 of the system (3) always exists, and if $R_0 < 1$, and is globally asymptotically stable and unstable if $R_0 > 1$.*
ii. *If $R_0 > 1$ then the positive EE P^* of the system (3) exists and its locally stable.*

.

PROOF. see [5] □

Let

$$X = \{(S, E, I) \mid S \geq 0, E \geq 0, I \geq 0\}$$

$$X_0 = \{(S, E, I) \in X \mid S \geq 0, E > 0, I > 0\}$$

$$\partial X_0 = X \setminus X_0$$

$$M_\partial = \{(S(0), E(0), I(0)) \in \partial X_0 \mid \Gamma_t(S(0), E(0), I(0)) \in \partial X_0, t \geq 0\}$$

Where $\Gamma_t : X \to X$ is the semiflow defined by (3).

3.1. Global stability of endemic equilibrium. In this section, we prove the global stability of the EE, P^*. Consider the following Lyapunov function:

$$V(t) = S - S^* - S^* ln(\frac{S}{S^*}) + E - E^* - E^* ln(\frac{E}{E^*}) + \frac{\omega}{\delta + \omega}(I - I^* - I^* ln(\frac{I}{I^*}))$$

Differentiating $V(t)$ along the trajectories of model (3) gives

$$\dot{V} = (1 - \frac{S^*}{S})\dot{S} + (1 - \frac{E^*}{E})\dot{E} + \frac{\omega}{\delta + \omega}(1 - \frac{I^*}{I})\dot{I}$$

Substituting the expressions for the derivatives in V, it follows from (3) and using the relation at the steady state EE,$P^* = (S^*, E^*, I^*)$,then

$$\begin{aligned}
\dot{V} =&(1 - \frac{S^*}{S})[\mu\omega(1 - \nu I) - \beta IS - \mu S] + (1 - \frac{E^*}{E})[\beta IS - (\delta - \mu)E] \\
&+ \frac{\omega}{\delta + \omega}(1 - \frac{I^*}{I})[q\delta E + \mu\omega\nu I - (\gamma + \mu)I] = (1 - \frac{S^*}{S})[\frac{\beta I^* S^* - \mu S^*}{\omega\nu I^*}(1 - \nu I) - \beta IS - \mu S] \\
&+ (1 - \frac{E^*}{E})[\beta IS - (\delta - \mu)E] + \frac{\omega}{\delta + \omega}(1 - \frac{I^*}{I})[q\delta E + \frac{\beta I^* S^* - \mu S^*}{\omega\nu I^*}\nu I - (\gamma + \mu)I] \\
=& - \mu\frac{(S^* - S)^2}{S} + 3\beta S^* I^* - \beta S^* I^*(\frac{1}{x} + \frac{xz}{y} + \frac{y}{z}) \leq 0
\end{aligned}$$

Where $\frac{S}{S^*} = x$, $\frac{E}{E^*} = y, \frac{I}{I^*} = z$. There, an application of LaSalle's invariance principle yields that the endemic equilibrium P^* is globally asymptotically stable if $R_0 > 1$. This result is summarized in the following theorem.

Theorem 3.2. *The unique endemic equilibrium P^* of model (3) is globally asymptotically stable in $\partial X_0 = X \setminus X_0$ whenever $R_0 > 1$.*

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An Equivalent Definition of the J-set

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Abstract. Let x be a vector in Banach space X over the field $\mathbb{C}$ and T stands for a bounded linear operator acting on X, then the operator T is called hypercyclic, if the orbit of the vector x under operator T is dense in X. The notion of hypercyclicity was localized by J-sets. We will introduce an equivalent definition of the J-set.

Keywords: Hypercyclic operators, topologically transitive operators, J-class operators.

AMS Mathematics Subject Classification [2010]: Primary; 47A16, Secondary; 37B99, 54H99

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1. Introduction

Assume that X is a Banach space and $T : X \longrightarrow X$ is a continuous linear map. If for every pair of nonempty open subset (U, V) of X, there is $n \in \mathbb{N}$, so that subset $T^n(U) \cap V$ is nonempty, then the operator T is transitive. If the underlying space is considered as a separable Banach space, then transitivity is equivalent to hypercyclicity. To clarify the notion of hypercyclicity, note that if B is a subset of X, then the orbit of the set B under operator T is the set $orb(T, B) = \{T^n x;\ x \in B, n = 0, 1, 2, \cdots\}$. Density of the recent set, given the norm of space X under certain conditions, can lead to exciting results. One of these special conditions is that $B = \{\lambda x;\ \lambda \in \mathbb{C}\}$ for a vector $x \in X$. With these assumptions, if subset $orb(T, B)$ is a dense subset of X, then this operatpr is called a supercyclic operator and vector x is a supercyclic vector for operator T; if B is considered as a single point set $B = \{x\}$ and the subset $orb(T, B)$ is a dense subset of X, then x is called a hypercyclic vector for the hypercyclic operator T. In this case the underlying Banach space X should be separable. Therefore every non-separable Banach space is deleted in hypercyclicity. Two good books for the study on transitivity and hypercyclicity are [2] and [4].

Authors in [3] proposed $J(x)$ for a vector x in Banach space X and $T \in B(X)$ as following:

$$J(x) = \{z \in X; \textit{there exist a sequence } \{z_n\} \subset X \textit{ and a strictly increasing sequence of positive integers } \{m_n\},\ \textit{such that } z_n \longrightarrow x \textit{ and } T^{m_n} z_n \longrightarrow z\}.$$

And

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Definition 1.1. An operator $T \in B(X)$ is called a J-class operator if there exists a non-zero $x \in X$ such that $J(x) = X$. In this case, the vector x is called a J-class vector for T.

they have proved that on a separable Banach space X, an operator T is hypercyclic if and only if, the underlying Banach space X has a dense subset of the J-class vectors for the operator T, and furthermore they showed that there exists a J-class operator on $\ell^\infty(\mathbb{N})$ which is not hypercyclic because $\ell^\infty(\mathbb{N})$ is not separable. In fact, they have tried to introduce the localized hypercyclicity by the notion of J-class, and for a better explanation of this goal, they provided an equivalent definition for the $J(x)$ through the use of open sets. To clarify more, they claimed that;

$$J(x) = \{z \in X; \textit{for every pair of neighborhoods } U,\ V \textit{ of } x,\ z \textit{ respectively, there exists a positive integer } n \textit{ such that } T^n(U) \cap V \neq \phi\}.$$

But unfortunately, the above equivalent is wrong because in the next section, with an example we will show that the above two sets are not equivalent. Then, we will provide an equivalent definition for the set $J(x)$ through the use of open sets, and an application of the equivalent definition of the J-set can be observed in the next section.
More information on the J-class operators can seen in [1], [5] and [6].

2. Main results

Before we state the example, recall that for the bounded sequence $w = \{w_n > 0\}_{n\in\mathbb{N}}$ and the canonical basis $\{e_n\}_{n\in\mathbb{N}}$ of $\ell_p(\mathbb{N})$ the operator B_w on $\ell_p(\mathbb{N})$ for $1 \leq p < \infty$, which is defined by $B_w(e_j) = w_j e_{j-1}$ for $j \geq 2$ and $B_w(e_1) = 0$ is called a unilateral weighted backward shift. It is well known that an equivalent condition for B_w to be hypercyclic on $\ell_p(\mathbb{N})$. In fact, the unilateral weighted shift B_w with the weight sequence $\{w_n\}_{n\in\mathbb{N}}$ is hypercyclic if and only if $\limsup_n (\prod_{i=1}^{n} w_i) = +\infty$, [4].
In the next example, we show that the claim which was proposed in [3, Remark 2.3] is wrong.

Example 2.1. If $T = \frac{1}{2}B$ where B is the backward shift operator on $\ell^2(\mathbb{N})$, the space of square summable sequences, and consider vector $z \in \ell^2(\mathbb{N})$ such that $Tz \neq 0$. Obviously,

$$Tz \in \{y \in X; \textit{for every pair of neighborhoods } U,\ V \textit{ of } z,\ y \textit{ respectively, there exists a positive integer } n \textit{ such that } T^n(U) \cap V \neq \phi\}.$$

but for every strictly increasing sequence of positive integers $\{k_n\}$ and every sequence $\{z_n\} \subset X$, if $z_n \longrightarrow z$ then $T^{k_n} z_n \longrightarrow 0$ and we get $J(z) = \{0\}$.
Therefore

$$J(z) \neq \{y \in X; \textit{for every pair of neighborhoods } U,\ V \textit{ of } z,\ y \textit{ respectively, there exists a positive integer } n \textit{ such that } T^n(U) \cap V \neq \phi\}.$$

Now we are ready to state an equivalent set of the $J(x)$.

Theorem 2.2. *Let $T \in B(X)$ and x be an arbitrary vector in X. Then*

$$J(x) = \{z \in X : \textit{for every pair of neighborhoods } U,\ V \textit{ of } x,\ z \textit{ respectively, and every } N \in \mathbb{N} \textit{ there exists an integer } n > N \textit{ such that } T^n U \cap V \neq \phi\}.$$

Proof. If $y \in J(x)$, then it is obvious that the vector y belongs in the above right side set. So let y be in the above right side set and consider U_x, U_y as two neighborhoods of x and y, respectively. Choose $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U_x$ and $B(y, \varepsilon) \subset U_y$.

Assume that for $n = 1, 2, ..., j$, there exist vectors $x_n \in B(x, \frac{\varepsilon}{n})$ and also there exist integers k_n such that $k_{n-1} < k_n$ and $T^{k_n} x_n \in B(y, \frac{\varepsilon}{n})$. Now consider the open balls $B(x, \frac{\varepsilon}{j+1})$ and $B(y, \frac{\varepsilon}{j+1})$, thus there exists $k_{j+1} \geq k_j + 1$ such that

$$T^{k_{j+1}} B(x, \frac{\varepsilon}{j+1}) \cap B(y, \frac{\varepsilon}{j+1}) \neq \emptyset.$$

Now if we choose $x_{j+1} \in B(x, \frac{\varepsilon}{j+1})$ and rename $T^{k_{j+1}} x_{j+1} \in B(y, \frac{\varepsilon}{j+1})$ by y_{j+1}, then by the induction we can construct the sequences $\{x_n\}$ and $\{k_n\}$ with desired properties or $y \in J(x)$ and the proof is completed. □

As we mentioned in the previous section, the underlying Banach space in hypercyclicity should be separable, however, this restriction is not in the J-class. In other words, in addition to the separable Banach spaces, which are considered in the hypercyclicity, some non-separable Banach spaces such as $\ell^\infty(\mathbb{N})$ support J-class operators, [3, Proposition 5.2]. So we stress that in the following, X denotes only a Banach space, unless emphasis on its separability or non-separability.

Theorem 2.3. *Let $J \in L(X)$ be a J-class operator. Then for every invertible operator, $T \in L(X)$, the operator $T^{-1} J T$ is a J-class operator.*

Proof. To avoid ambiguity, when we use of $J_S(z)$, we mean the $J(z)$ under an operator S. Consider $x \in X$ as a J-class vector for the operator J and fix an $N \in \mathbb{N}$. If $y \in X$ is an arbitrary vector and $U_{T^{-1}x}, V_y$ are two open neighborhoods of $T^{-1}x$ and y, respectively, then by invertibility of T, there exists an integer $n > N$ such that;

$$J^n T(U_{T^{-1}x}) \cap T(V_y) \neq \phi.$$

Therefore $T^{-1} J^n T(U_{T^{-1}x}) \cap V_y \neq \phi$ or equivalently $y \in J_{T^{-1}JT}(T^{-1}x)$. Hence the vector $T^{-1}x$ is a J-class vector for the operator $T^{-1}JT$. □

So the set of all J-class operators on a Banach space X is either empty, or contains many operators of $L(X)$. As you see, the proof was very easily expressed with the help of the equivalent definition for $J(x)$ through the open sets. Now we want to answer the following question;

Why should the J-class vector be non-zero in the definition of J-class operator?
To answer this question, consider the following example in the first step.

Example 2.4. Consider weighted backward shift operator T on $\ell^2(\mathbb{N})$ given by:

$$T(x^1, x^2, \cdots) = (2x^2, \frac{3}{2}x^3, \frac{4}{3}x^4, \cdots).$$

Also let Y be the set of finite sequences with entries $z \in \mathbb{C}$ that $Re(z) \in \mathbb{Q}$, $Im(z) \in \mathbb{Q}$. Since Y is dense in $\ell^2(\mathbb{N})$, so there are strictly increasing sequence $\{2k\}_k$, sequence $\{x_k\} \subset Y$ that

$$x_k = (x^1, 0, x^3, 0, \cdots, x^{2k-1}, 0, 0, \cdots),$$

$x_k \longrightarrow 0$ as $k \longrightarrow \infty$ and $T^{2k} x_k = 0$. Now, for the random member

$$y = (y^1, 0, y^3, 0, \cdots, y^{2m+1}, 0, 0, \cdots) \in Y$$

and $k \geq 1$, we set;

$$w_{2k}(y) = (\underbrace{0, \cdots, 0}_{2k-times}, \frac{y^1}{2k+1}, 0, \frac{3y^3}{2k+3}, 0, \cdots, \frac{(2m+1)y^{2m+1}}{2(k+m)+1}, 0, 0, \cdots).$$

Clearly, for every $k \in \mathbb{N} \cup \{0\}$, $w_{2k}(y)$ belongs to Y and the sequence $\{w_{2k}(y)\}$ is a sequence in $\ell^2(\mathbb{N})$. Since

$$||w_{2k}(y)||^2 = \sum_{j=1}^{2m} |\frac{j}{2k+j} y^j|^2 \leq \frac{4m^2}{(2k+1)^2} ||y||^2,$$

so $w_{2k}(y) \longrightarrow 0$, as $k \longrightarrow \infty$. Note that for $n \geq 1$:

$$T^n(x^1, x^2, x^3, \cdots) = \left((n+1)x^{n+1}, \ \frac{1}{2}(n+2)x^{n+2}, \ \frac{1}{3}(n+3)x^{n+3}, \ \cdots \right),$$

thus

$$T^{2k} w_{2k}(y) = \Bigg((2k+1)\frac{1}{2k+1} y^1, 0, (\frac{2k+3}{3})(\frac{3}{2k+3}) y^3, 0, \cdots, \\ (\frac{2(k+m)+1}{(2m+1)})(\frac{2m+1}{2(k+m)+1}) y^{2m+1}, 0, 0, \cdots \Bigg) = y.$$

Hence all conditions of the J-class Criterion in [1] holds and $J_T(0) = \ell^2(\mathbb{N})$.

Contrary to the obvious example provided in [3], in the above example, we gave a nontrivial operator T such that $J_T(0) = X$. However, it is not a J-class operator. In fact, the unilateral weighted backward shift T on $\ell^2(\mathbb{N})$ with the bounded weight sequence $\{w_n > 0\}_{n \in \mathbb{N}}$ is topological transitive, if, and only if $\limsup_n (\prod_{i=1}^{n} w_i) = +\infty$, therefore the obvious operator T is not local topological transitive anywhere except at zero.

Now the paper is ended with an question on the J-class operators.
It is well known that if an operator T is hypercyclic, then any powers of T is a hypercyclic operator, [4]. We give a similar question as follows;

Question. *If $T \in B(X)$ is a J-class operator, is T^n also a J-class operator operator for every $n > 1$? If so, what is the relation between their J-class vectors?*

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Transcorneal drug delivery enhanced by ultrasound - in silico experiments

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Abstract. In this paper we present an exploratory study on the drug delivery to the anterior segment of the eye through the cornea enhanced by ultrasound. To increase corneal permeability and, consequently, to increase the drug transport, ultrasound is used. The drug delivery is then described by a set of partial differential equation for the propagation of the acoustic pressure waves generated by ultrasound and for the drug concentration in the different corneal layers. Preliminary numerical experiments illustrating the stimulus effects are included.

Keywords: Ultrasound, Mathematical model, Numerical simulation

AMS Mathematics Subject Classification [2010]: 65N06

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1. Introduction

Drug delivery to the eye is a very difficult task due to the eye defenses that protect it from the exterior environment. Traditionally, topical instillation of eye drops is the most used procedure to delivery drugs to treat diseases of the anterior segment of the eye. Due to the eye protection systems that include the reflex blinking, tear film (thin transparent fluid layer), the tear fluid turnover that are responsible for the elimination of 60% of the applied active agent, eye drops are an inefficient drug delivery system ([3]). Even with success, to reach the aqueous chamber, the drug is transported through the cornea that presents lower permeability. To increase corneal permeability and, consequently, to enhance drug transport, ultrasound has been used ([2]).

This paper aims to present an in silico experiment to simulate the drug transport through the cornea when an ultrasound is used to enhance the drug transport through this tissue. A schematic representation of the cornea is presented in Figure 1 that includes three layers: epithelium (Ω_p), stroma (Ω_s) and endothelium (Ω_e). The boundaries and the interface between the corneal layers are also included in this figure. We assume that the drug is in contact with the cornea on the boundary Γ_p and acoustic pressure waves

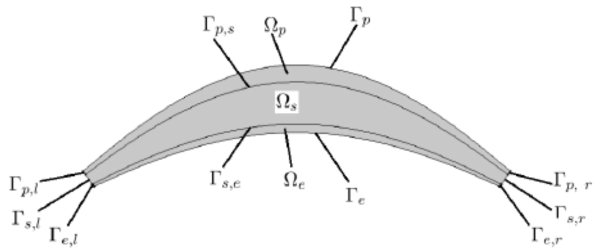

FIGURE 1. A schematic representation of the cornea.

are generated by the transducer placed outside of the cornea and they propagate though the three corneal layers Ω_p, Ω_s and Ω_e. By Ω we represent the cornea region.

In what follows we assume that the propagation of the pressure waves is described by the following wave equation

$$\frac{\partial^2 p_i}{\partial t^2} = c^* \nabla^2 p_i \text{ in } \Omega_i \times (0, T], \tag{1}$$

for $i = p, s$. In (1), c^* denotes the material sound speed. The pressure waves do not change in the endothelium Ω_e. Equation (1) is completed with the initial conditions: $\dfrac{\partial p_i}{\partial t}(0) = p_{i_{00}}$, $p_i(0) = p_{i_0}$, for $i = p, s$; the boundary conditions: the acoustic pressure is assumed to be known on the boundary Γ_p; the other boundaries do not interfere with the acoustic waves propagation, that is, $\nabla p.\eta = 0$ on $\Big(\bigcup_{j=p,s,q=\ell,r} \Gamma_{j,q} \cup \Gamma_{s,e}\Big) \times (0, T]$; and the interface condition: continuity of p and $\nabla p_p.\eta = \nabla p_s.\nu$ on $\Gamma_{p,s} \times (0, T]$. In this last relation, η and ν are the unitary normals on $\Gamma_{p,s}$ exterior to Ω_p and to Ω_s, respectively.

The cornea layers (epithelium and stroma) are composed by intracellular compartments and extracellular micro-domains being the first one composed by cells and the second one by extracellular matrix (ECM) that is mainly composed by collagen homogeneously distributed that is responsible by the transparency and by the organized structure ([1]). The intracellular and extracellular spaces are separated by cell membranes. When the acoustic waves propagate through the cornea, the cells permeability increases and the drug transport between the two media also increases ([2]) according to Fick's law. Let $c_{e,j}$ and $c_{i,j}$ be the ECM and the intracellular drug concentrations in the Ω_j layer. The dynamics of the drug molecules are described by the following equations:

$$\frac{\partial c_{e,j}}{\partial t} = -\nabla . J c_{e,j} + k_j (c_{i,j} - c_{e,j}) \text{ in } \Omega_j \times (0, T], \tag{2}$$

$$\frac{\partial c_{i,j}}{\partial t} + k_j (c_{i,j} - c_{e,j}) = 0 \text{ in } \Omega_j \times (0, T], \tag{3}$$

for $j = p, s$, $Jc_{e,j} = -D_j \nabla c_{e,j}$ is the flux of the extracellular concentration in the epithelium and stroma, respectively, D_j is the effective drug diffusion coefficient in the epithelium and stroma, respectively, given by

$$D_j = D_{j_0} \phi_j, \tag{4}$$

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where D_{j_0}, $j = p, s$, represent the molecular diffusion coefficients in the epithelium and stroma layers, respectively (see ([2, 5])). In (4), ϕ_j, $j = p, s$, denote the porosity of the epithelium and stroma, respectively, which depends on the acoustic pressure by

$$\phi_j = \phi_{j_0} e^{\beta_{jm}(p_j - p_0)}, \tag{5}$$

where p_j, $j = p, s$, are defined by (1), ϕ_{j_0} represents the porosity at a reference pressure p_0 and β_{jm} is constant. In (2 and 3), k_j, $j = p, s$, are the epithelium and stroma permeability coefficient that, according [4], are given by

$$k_j = \frac{\phi_j^3}{c^{**}\tau_j^2 A_j^2} \tag{6}$$

where τ_j is the tortuosity of the medium, A_j is the specific surface area and c^{**} is the Kozeny coefficient.

As mentioned in [6], the endothelium layer does not change when exposed to ultrasound. Consequently, the drug distribution in this corneal layer is defined by

$$\frac{\partial c_e}{\partial t} = -\nabla . Jc_e \text{ in } \Omega_e \times (0, T], \tag{7}$$

where c_e and $Jc_e = -D_e \nabla c_e$ denote the drug concentration and the drug flux in the endothelium layer, respectively. In (7), D_e is the drug diffusion coefficient.

The system of partial differential equations (2)-(7) is complemented with initial condition

$$c_{i,p}(0) = 0, c_{e,p}(0) = 0, c_{i,s}(0) = 0, c_{e,s}(0) = 0, c_e(0) = 0, \tag{8}$$

and boundary conditions

$$\begin{cases} c_{e,p} = c_R, c_{i,p} = 0 \text{ on } \Gamma_p \times (0, T], \\ Jc_{j,p}.\eta = 0 \text{ on } (\cup_{q=\ell,r}\Gamma_{p,q}) \times (0, T], j = e, i, \\ Jc_{j,s}.\eta = 0 \text{ on } (\cup_{q=\ell,r}\Gamma_{s,q}) \times (0, T], j = e, i, \\ Jc_e.\eta = 0 \text{ on } (\cup_{q=\ell,r}\Gamma_{e,q}) \times (0, T], \\ Jc_e.\eta = k_e c_e \text{ on } \Gamma_e \times (0, T], \\ \nabla c_{i,s}.\eta = 0 \text{ on } \Gamma_{s,e} \times (0, T]. \end{cases} \tag{9}$$

The last condition means that intracellular drug in Ω_s does not passes through $\Gamma_{s,e}$ because the endothelium layer is in contact with the stroma through a thick ECM membrane known as Descemets membrane. In (9), c_R is the drug concentration on Γ_p that can be defined by a reservoir in contact with the cornea. In what concerns the intracellular drug concentration, we assume that on this boundary we do not have any intracellular drug. Finally, to complete our mathematical model we need to impose interface conditions

$$\begin{cases} Jc_{j,p}.\eta = k_{p,s}(c_{j,p} - c_{j,s}) \text{ on } \Gamma_{p,s} \times (0, T], j = e, i, \\ Jc_{j,p}.\eta = -Jc_{j,s}.\nu \text{ on } \Gamma_{p,s} \times (0, T], j = e, i, \\ Jc_{e,s}.\eta = k_{s,e}(c_{e,s} - c_e) \text{ on } \Gamma_{s,e} \times (0, T], \\ Jc_{e,s}.\eta = -Jc_e.\nu \text{ on } \Gamma_{s,e} \times (0, T]. \end{cases} \tag{10}$$

In (10), $k_{i,j}$ represents the mass transfer coefficient between the tissues Ω_i and Ω_j.

2. Numerical Simulations

The numerical results presented in what follows were obtained using the commercial tool COMSOL 5.3 Multiphysics. They intend to illustrate qualitatively the effect of the

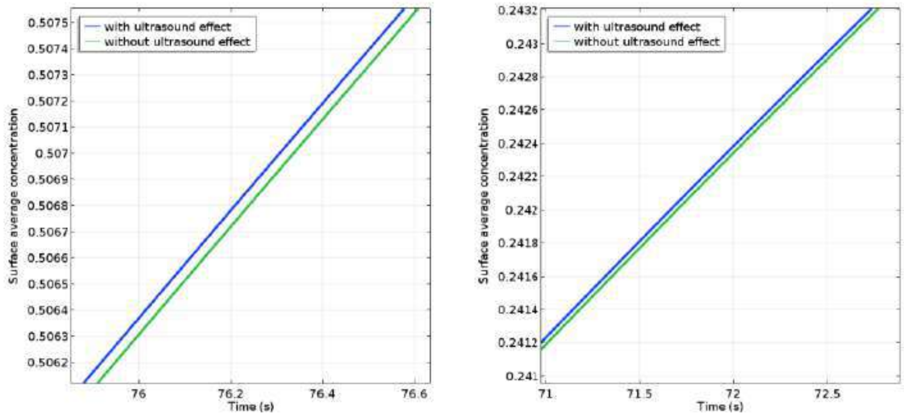

FIGURE 2. The surface average concentration of drug in stroma (left) and anterior champer (right) by comparing with and without ultrasound effect.

ultrasound on the drug transport. The results were obtained fixing the parameters of the model and considering the different values for the acoustic pressure on the boundary Γ_p.

Figure 2 shows the effect of ultrasound in the drug transport in the cornea. In the left figure we present the mean drug concentration in the stroma with and without the presence of ultrasound. In the right figure we present the corresponding drug mass that enters in the anterior chamber. We observe that the pressure waves generated by the ultrasound increase the drug concentration in the cornea and consequently the drug mass that will be available in the anterior chamber. This figure clearly shows the effect of ultrasound on drug transport through the cornea.

3. Conclusions

In this paper a system of partial differential equations is proposed to describe the drug transport trough the cornea enhanced by ultrasound. The ultrasound induces acoustic pressure waves that increase the corneal porosity ([2]) and then the permeability of corneal tissues ([4]). The in silico experiment confirm that the described scenario leads to an increasing of the drug transport through the corneal tissues leading to an increasing of the drug delivery in the anterior chamber.

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Existence and uniqueness of positive solutions for a class of integro-differential equations

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Abstract. By using Banach contraction principle and Leray-Schauder's nonlinear alternative fixed point theorem, we study the existence and uniqueness of positive solutions for a class of nonlinear integro-differential equations of Caputo type fractional order with three points boundary conditions. Finally, two examples are presented to clarify the applicability of the main results.

Keywords: fractional integro-differential equation, boundary value condition, fixed point theorem, positive solution.

AMS Mathematics Subject Classification [2010]: 34A08, 26A33, 35B33.

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Asymptotic properties of the robust regression based on censored and functional data

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Abstract. In this work, I study a robust version of the local linear regression of the censored scalar response random variable, from a functional random variable X. I construct an estimator by combining both local linear ideas and M-estimation techniques. The main results of this work are the establishment of the almost complete convergence as well as the asymptotic normality for the constructed estimator. These asymptotic results are stated with rate and are proved under a general condition.

Keywords: Local linear method, Robust estimation, censored data.

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1. Introduction

The functional statistic is an important field in statistics, it concerns the modilisation of random variables takes values in infinite-dimensional space. In practice, there is an increasing number of situations coming from different fields of applied sciences in which the data are of functional nature (such as soil science, oceanography, geology, epidemiology, econometrics, forestry, environmental science...). In this work, I'm interested to the nonparametric estimation of the robust regression between a functional variable X and a scalar response variable Y which is not completely observed by using the local linear method.

In robust statistics, robust regression is a form of regression analysis designed to overcome some limitations of traditional parametric and non-parametric methods. Regression analysis seeks to find the relationship between one or more independent variables and a dependent variable. Robust regression methods are designed to be not overly affected by violations of assumptions by the underlying data generating process.
In the literature, several articles devoted to the study of this method. We can refer, among others, to Huber (1964), Härdle (1984), Collomb & Härdle (1986), Boente & Fraiman (1989), Laïb & Ould Saïd (2000), Azzedine & al. (2008) and Cai & Roussas (1992). Recently, Belarbi et al. (2018), Zou et al. (2018), and Abbas et al. (2020).

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In the regression estimation, the local linear adjustment is considered to be greater than the constant local fit. The first results of the local linear method for functional data were obtained by Baillo & Grané (2009), Barrientos-Marin & al. (2010), Berlinet & al (2010) and Zhou & Lin (2016).

For censored variables and in the case of a linear model, Basak (1992), Li and Zheng (2009), Bednarsky (2014); Beran (1981).

Finally, when these two types of incomplete data occur simultaneously in a study, then the model is known as the Left Truncated and Right Censored data point. Under this model, for the linear regression with i.i.d data, see Lai and Ying (1994), Kim and Lai (2000).

2. The model and the estimation

Consider n independent pairs of random variables (X_i, Y_i) for $i = 1, ..., n$ coming from the pair (X, Y). The latter is valued in $\mathcal{F} \times \mathbb{R}$, where $\mathcal{F}$ is a semi-metric space and d denotes a semi-metric. For $x \in \mathcal{F}$ the nonparametric robust regression, denoted by θ_x, is defined as the unique minimizer of

$$\theta_x = \arg\min_{t\in\mathbb{R}} \mathbb{E}\left[\frac{\delta\rho(Y-t)}{\overline{G}(t)}|X=x\right]$$

where $\rho(.)$ is a real-valued Borel function satisfying some regularity conditions to be stated below; and

$$\begin{aligned}\overline{G}_n(t) &= 1 - G_n(t) \\ &= \begin{cases} \Pi_{i=1}^{n}\left(1-\frac{1-\delta_i}{n-i+1}\right)^{\mathbb{1}_{\{Y_i\le t\}}} & \text{if } t \le Y_n \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

The function θ_x is approximated by:

$$\forall\ \mathcal{X} \quad \text{in neighborhood of x} \qquad \theta_{\mathcal{X}} = a + b\beta(\mathcal{X}, x)$$

where a and b are estimated by $\hat{a}$ and $\hat{b}$ are solution of

$$\min_{(a,b)\in\mathbb{R}^2} \sum_{i=1}^{n} \frac{\delta_i}{\overline{G}(t)}\rho\left(Y_i - a - b\beta(X_i, x)\right) K\left(h^{-1}\Delta(x, X_i)\right)$$

Here

- $\beta(.,.)$ is a known function from $\mathcal{F}\times\mathcal{F}$ into $\mathbb{R}$ such that, $\forall\xi \in \mathcal{F}, \beta(\xi,\xi) = 0$,
- K is a kernel function and $h = h_n$ (to simplify the notations) is a sequence of positive real numbers which goes to zero as n goes to infinity, and $d(.,.) = |\Delta(.,.)|$ is a function of $\mathcal{F}\times\mathcal{F}$. A natural estimator of θ_x denoted by $\hat{\theta}_x$.

3. Assumptions

In what follows, when no confusion is possible, we will denote by C and C' some strictly positive generic constants. Moreover, x denotes a fixed point in $\mathcal{F}$, N_x denotes a fixed neighborhood of x. For $i = 1, .., n$, we denote by $K_i = K(h^{-1}\Delta(x, X_i))$, and $\beta_i = \beta(X_i, x)$. Furthermore, we put $\phi_x(r_1, r_2) = \mathbb{P}(r_2 \le |\Delta(x, X)| \le r_1)$ and we assume the following hypotheses :

✓ (H1) $\forall r > 0 \quad \phi_x(r) = \phi_x(-r, r) > 0$ and there exists a function $\chi_x(.)$ such that:

$$\forall t \in (-1,1), \lim_{h \longrightarrow 0} \frac{\phi_x(th,h)}{\phi_x(h)} = \chi_x(t)$$
.

✓ (H2) The function $\rho(.)$ is a strictly convex function, continuously differentiable and has a Lipschitzian derivative $\psi(.)$ such that

$$\mathbb{E}\left[\left|\frac{\delta\psi(Y-t)}{\overline{G}(t)}\right|^p |X=x\right] < C < \infty \text{ almost surely }, p > 2$$
.

✓ (H3) The function $\Gamma_\lambda(x,.) := \mathbb{E}\left[\left|\frac{\delta\psi^\lambda(Y-.)}{\overline{G}^\lambda(t)}\right| |X=x\right]$ is of class $\mathcal{C}^1$ on $[\theta_x - \Delta, \theta_x + \Delta], \Delta > 0$ and $\lambda \in \{1,2\}$, we put $\gamma(x,.) = \frac{d}{dt}\Gamma_1(x,.)$, such that:

(i) $\forall (t_1,t_2) \in [\theta_x - \Delta, \theta_x + \Delta] \times [\theta_x - \Delta, \theta_x + \Delta], \forall (x_1,x_2) \in \mathcal{N}_x \times \mathcal{N}_x,$ and for $(b_1,b_2) > 0$

$$|\Gamma(x_1,t_1) - \Gamma(x_2,t_2)| \leq Cd^{b_1}(x_1,x_2) + |t_1-t_2|^{b_2}.$$

$$|\gamma(x_1,t_1) - \gamma(x_2,t_2)| \leq C'd^{b_1}(x_1,x_2) + |t_1-t_2|^{b_2}$$

(ii)The variable $\delta(x;X)$ is $\sigma(X;x)$-measurable and the two partial derivatives of the function,

$$\Upsilon_x(s,t) = \mathbb{E}[\Gamma(X,t)|\beta(X,x) = s], \text{at } (0;\theta_x) \text{ exist.}$$

✓ (H4) The function $\beta(.,.)$ is such that

$$\forall z \in \mathcal{F}, C|\Delta(x,y)| \leq |\beta(y,x)| \leq C'|\Delta(x,y)|$$

and

$$\sup_{u \in B(x,r)} |\beta(u,x)| - |\Delta(x,u)| = o(r);$$

where $u \in B(x,r) = \{x' \in \mathcal{F}/d(x,x') \leq r\}$

✓ (H5) The kernel K is a positive, differentiable function which is supported with in $(-1,1)$ such that

$$D = \begin{pmatrix} K(1) - \int_{-1}^{1} tK'(t)\Phi_x(t)dt & K(1) - \int_{-1}^{1}(tK(t))'\Phi_x(t)dt \\ K(1) - \int_{-1}^{1}(tK(t))'\Phi_x(t)dt & K(1) - \int_{-1}^{1}(t^2K(t))'\Phi_x(t)dt \end{pmatrix}$$

is a positive definite matrix.

✓ (H6) h is a positive sequel such as

$$\lim_{n \longrightarrow \infty} h = 0 \text{ and } \lim_{n \longrightarrow \infty} \frac{\log n}{n\phi_X(x)} = 0$$
.

4. Result

Theorem 1. Under assumptions (H1)-(H6) and if $\frac{\partial \Gamma}{\partial t}(x, \theta_x) > 0$, we have

$$|\hat{\theta}_x - \theta_x| = O(h^{\min(b_1,b_2)}) + O\left(\sqrt{\frac{\log n}{n\phi_x(h)}}\right) \qquad \text{a.co}$$

Theorem 2. Under assumptions (H1)-(H6) and if $\frac{\partial \Gamma}{\partial t}(x, \theta_x) > 0$ for any $x \in \mathcal{A}$, we get

$$\left(\sqrt{\frac{n\phi_x(h)}{\sigma^2(x)}}\right)\left(\hat{\theta}_x - \theta_x - o(h)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \qquad \text{as } n \longrightarrow \infty$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution,

$\sigma^2(x) = \frac{\Gamma_2(x,\theta_x)}{(\gamma(x,\theta_x))^2} \frac{a_3^2 D_1 - 2a_2 a_3 D_2 + a_2^2 D_3}{(a_1 a_3 - a_2^2)^2}$ and $\{\mathcal{A} = x \in \mathcal{F}, \sigma^2(x) \neq 0\}$

and

$a_j = K(1) - \int_{-1}^{1}(s^{j-1}K(s))'\Phi_x ds$ and $D_j = K(1) - \int_{-1}^{1}(s^{j-1}K(s)^2)'\Phi_x ds$ for $j = 1, 2, 3$

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Biderivations and amenability of a pair of Banach algebras

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Abstract. In this paper, we introduce the concept of amenability for a pair of Banach algebras and we investigate the relation between the existence of derivations on two Banach algebras A and B and the existence of a biderivation on the pair (A, B). Then we study the relation between amenability of a pair of Banach algebras and amenability of the related Banach algebras.

Keywords: biderivation, inner biderivation, biamenability

AMS Mathematics Subject Classification [2010]: 46H20, 46H25

1. Introduction

A derivation from a Banach algebra A to a Banach A-bimodule X is a bounded linear mapping $d : A \to X$ such that

$$d(ab) = d(a)b + ad(b) \quad (a, b \in A).$$

For each $x \in X$ the mapping $\delta_x : a \to ax - xa$, $(a \in A)$ is a bounded derivation, called an inner derivation.
Let X be a Banach A-bimodule. Then X^* is a dual Banach A-bimodule, by defining $a.f$ and $f.a$, for each $a \in A$ and $f \in X^*$ by

$$a.f(x) = f(xa) \quad , \quad f.a(x) = f(ax) \qquad (x \in X).$$

Similarly, the higher duals $X^{(n)}$ can be made into Banach A-bimodules in a natural fashion.

A Banach algebra A is called amenable if for each Banach A-bimodule X, the only derivations from A to X^* are inner derivations. For more details about this notion see [5].

Let A and B be Banach algebras and X be an A-B-bimodule that is X is an A-bimodule and B-bimodule and we have

$$a(xb) = (ax)b \quad , \quad b(xa) = (bx)a \qquad (a \in A, b \in B, x \in X).$$

A bounded bilinear mapping $D : A \times B \to X$ is called a biderivation if D is a derivation with respect to both arguments. That is the mappings ${}_aD : B \to X$ and $D_b : A \to X$ where

$${}_aD(b) = D(a, b) = D_b(a) \qquad (a \in A, b \in B)$$

are derivations. We denote the space of such biderivations by $BZ^1(A \times B, X)$.

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Let $x \in Z(A,X) \cap Z(B,X)$, where

$$Z(A,X) = \{x \in X; ax = xa \quad \forall a \in A\}.$$

The map $D_x : A \times B \to X$ that

$$D_x(a,b) = x[a,b] = (xa)b - (xb)a \qquad (a \in A, b \in B)$$

is a basic example of biderivations and is called an inner biderivation. We denote the space of such inner biderivations by $BN^1(A \times B, X)$.

For more applications of biderivations, see the survey article [3, Section 3]. Some algebraic aspects of biderivations on certain algebras were investigated by many authors; see for example [2,4], where the structures of biderivations on triangular algebras and generalized matrix algebras are discussed, and particularly the question of whether biderivations on these algebras are inner was considered.

Also we define the first bicohomology group $BH^1(A \times B, X)$ as follows,

$$BH^1(A \times B, X) = \frac{BZ^1(A \times B, X)}{BN^1(A \times B, X)}.$$

Obviously $BH^1(A \times B, X) = 0$ if and only if every biderivation from $A \times B$ to X is an inner biderivation. Now we are motivated to define the concept of amenability for a pair of Banach algebras as follows.
We say that the pair (A,B) is amenable if for each A-B-bimodule X, $BH^1(A \times B, X^*) = \{0\}$.

Despite the apparent similarities between (inner) derivations (resp. amenability of Banach algebras) and (inner) biderivations (resp. amenability of a pair of Banach algebras) these concepts also have differences [1]. In this paper, we will examine the similarities of these concepts under certain conditions.

2. Main results

Let A and B be Banach algebras. We say that A and B commute with respect to an A-B-bimodule X if for each $a \in A, b \in B$ and $x \in X$ we have $a(bx) = b(ax)$ and $(xb)a = (xa)b$. Note that if A and B commute with respect to X then they commute with respect to A-B-bimodule X^*.

For example, if we consider X as an A-B-bimodule with module actions zero on A, then A and B commute with respect to this A-B-bimodule. Also if A is commutative then A commutes with itself with respect to A.

Theorem 2.1. *If A and B commute with respect to an A-B-bimodule X and there are nonzero derivations $d : A \to X$ and $d' : B \to X$, then there is a nonzero biderivation from $A \times B$ into X.*

Proposition 2.2. *For each biderivation $D : A \times B \to X$,*

$$D(a,b)[c,d] = [a,b]D(c,d) \qquad (a,c \in A, b,d \in B).$$

Remark 2.3. If $Z(A,X) = \{0\}$ or $Z(B,X) = \{0\}$ then the only inner biderivation $D : A \times B \to X$ is zero. Therefore if X is an A-B-bimodule such that there is a non zero biderivation from $A \times B$ into X^* and $Z(A,X) = \{0\}$, then (A,B) is not amenable.

The following theorem says that amenability of Banach algebras A and B are necessary conditions for amenability of the pair (A,B).

Theorem 2.4. (i) *If $\sigma(B) \neq \emptyset$ and (A,B) is amenable then A is amenable.*

(ii) *If $\sigma(A) \neq \emptyset$ and (A, B) is amenable then B is amenable.*

REMARK 2.5. The converse of this theorem is not true, in general. For example we can show that $(\mathbb{C}, \mathbb{C})$ is not amenable and $\sigma(\mathbb{C}) \neq \emptyset$, but it is amenable.

3. Conclusion

Although the concept of amenability of Banach algebras and amenability of a pair of Banach algebras are almost different. In this paper, it was observed that they are related to each other under certain conditions.

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Stochastic Maximum principle of Markov regime switching forward stochastic differential equations with jumps

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Abstract. In this paper, we study a stochastic maximum principle of Markov regime switching forward stochastic differential equations with jumps-diffusion in infinit horizon. Sufficient and necessary maximum principles for optimal control under partial information are deriven. We illustrate our results by a problem of optimal consumption problem from a cash flow with regime.

Keywords: stochastic maximum principle, optimal control, partial information, regime switching.

AMS Mathematics Subject Classification [2010]: 93E20, 60H10

1. Introduction

The maximum principle is one of the most important methods used to solve optimal control problem, and due to its applications in several fields such as economics, biology and finance, it attracted a large number of researchers. Bismut [2] was the first who studied the stochastic case. Bensoussan [1] used the convexe perturbation method to derive the stochastic maximum principle in local form. In the continuous case, Peng [8] proved the general maximum principle for the stochastic control system by using a second order variational equation and second order adjoint equation to overcome the difficulty appearing along with the nonconvex control domain and control entering the diffusion term, this works was extended in the jumps case by Tang [13].There are many results for other stochastic control systems; we refer the reader to Young and Zhou [14], Wu [16], Peng [10] , Shi and Wu [4], Tao and Wu [12], Hafayed et al The optimal control problem for a Markov regime-switching model has recently received much attention, e.g., see Donnelly [3], Menoukeu [5], Sun et al [11], Zhang et al [15]. In infinite horizon the stochastic maximum principle has been studied by many authors . For example Hadam et al [9], Agram et al [6, 7]. In this work we establish a necessary and sufficient stochastic maximum principle for optimal control within a regime-switching diffusion-jumps model on infinite horizon.

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2. Préliminaries

Let $B(t) = B(t,w)$, $t \geq 0$, $w \in \Omega$ and $\widetilde{\mathcal{N}}(d\zeta, dt) = \mathcal{N}(d\zeta, dt) - v(d\zeta)dt$ be one-dimensional Brownian motion and an independent compensated poisson random measure, respectively, on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$ satisfying the usual conditions we consider a continuous-time, finie-state, observable Markov chain $\{\alpha(t) / t \geq 0\}$. $\{\mathcal{F}\}_{t\geq 0}$ is a right-continuous, P-completed filtration to wich all of the processes defined below, including the Markov chain the Brownian motions, and the poisson random measures, are adapted. Following the convention of Elliott, Aggoun, and Moore, we identify the state space of the chain with a finite state space $\mathcal{S} = \{e_1, ..., e_D\}$,where $D \in \mathbb{N}$, $e_i \in \mathbb{R}^D$, and jthe component of e_i is the Kronecker delta δ_{ij} for each $i, j = 1, 2, ..., D$. the state space $\mathcal{S}$ is called a canonical state space and its use faciliates the mathematics.

We suppose that the chain is homogeneous and irreducible. To specify statistical or probabilistic properties of the chain α. we define the generator $\Lambda = \{\lambda_{ij}\ 1 \leq i \leq j \leq D\}$ of the chain under P. this is also called the rate matrix, or the Q-matrix. Here, for each $i, j = 1, 2, .., D$, λ_{ij} is the constant transition intensity of the chain from state e_i to state e_j at time t. Note that $\lambda_{ij} \geq 0$ for $i \neq j$ and 0. In what follows for each $i, j = 1, 2, .., D$ which $i \neq j$, we suppose that $\lambda_{ij} > 0$, so $\lambda_{ii} < 0$. Elliott, Aggoun, and Moore obtained the following semimartingale dynamics for the chain α :

$$\alpha(t) = \alpha(0) + \Lambda^T \alpha(u)\, du + \mathcal{M}(t)$$

where $\{\mathcal{M}(t) \setminus t \geq 0\}$ is an $\mathbb{R}^D$-valued, $\left(\{\mathcal{F}\}_{t\geq 0}, P\right)$-martingale and y^T denotes the transpose of a matrixe (or, in particular, a victor) .

and

$$\begin{aligned} J^{ij}(t) &= \sum_{0\leq s\leq t} \langle \alpha(s-), e_i\rangle \langle \alpha(s), e_j\rangle \\ &= \int_0^t \langle \alpha(s-), e_i\rangle \langle \Lambda^T \alpha(s), e_j\rangle ds + \int_0^t \langle \alpha(s-), e_i\rangle \langle d\mathcal{M}(s), e_j\rangle ds \\ &= \lambda_{ij} \int_0^t \langle \alpha(s-), e_i\rangle ds + m_{ij}(t), \end{aligned}$$

where $m_{ij} = \{m_{ij}(t) \setminus t \in \tau\}$ with $m_{ij}(t) = \langle \alpha(s-), e_i\rangle \langle d\mathcal{M}(s), e_j\rangle$ is an $\left(\{\mathcal{F}\}_{t\geq 0}, P\right)$-martingale, suppose $\mathcal{N}_i(dz, dt)$, $i = 0, 2, ...D$, are independent Poisson random measures on $(\mathbb{R}^+ \times \mathbb{R}_0, \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}_0)$ under P. Assume that the Poisson random measures $\mathcal{N}_i(dz, dt)$ has the following compensator :

$$\eta^i_\alpha(dz, dt) = \nu^i_{\alpha(t-)}(dz \setminus t)\, \eta(dt) = \langle \alpha(t-), \nu^i(dz \setminus t)\rangle \eta(dt), \tag{1}$$

and

$$\nu(dz \setminus t) = \nu(dz) = (\nu_{e_1}(dz), \nu_{e_2}(dz), .., \nu_{e_D}(dz))^T$$

Furthermore we assume that $\eta(dt) = dt$ and write

$$\begin{aligned} \widetilde{\mathcal{N}}_\alpha(dz, dt) &= \left(\widetilde{\mathcal{N}}^1_\alpha(dz, dt), ..., \widetilde{\mathcal{N}}^D_\alpha(dz, dt)\right)^T \\ &= \left(\mathcal{N}_1(dz, dt) - v^1_\alpha(dz), ..., \mathcal{N}_D(dz, dt) - v^D_\alpha(dz)\right)^T. \end{aligned} \tag{2}$$

Let $X(t) = X^{(u)}(t)$ be a controlled Markov regime-switching jumps-diffusion in $\mathbb{R}$ described by the stochastic differential equation

$$(3)\qquad \begin{cases} dX(t) = & b(t, X(t), u(t), \alpha(t))\,dt + \sigma(t, X(t), u(t), \alpha(t))\,dB(t) \\ & + \int_{\mathbb{R}_0} \eta(t, X(t), u(t), \alpha(t), z)\,\widetilde{\mathcal{N}_\alpha}(dz, dt) \\ & +\gamma(t, X(t), u(t), \alpha(t))\,d\widetilde{\Phi}(t) \qquad 0 \le t \le \infty, \\ X(0) = & x_0. \end{cases}$$

$X(0) = x \in \mathbb{R}$.

Here $b : [0, \infty[\times \mathbb{R} \times U \times \mathcal{S} \times \Omega \to \mathbb{R}$, $\sigma : [0, \infty[\times \mathbb{R} \times U \times \mathcal{S} \times \Omega \to \mathbb{R}$, $\eta : [0, \infty[\times \mathbb{R} \times U \times \mathcal{S} \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$ and $\gamma : [0, \infty[\times \mathbb{R} \times U \times \mathcal{S} \times \Omega \to \mathbb{R}$,

$\widetilde{\mathcal{N}}_\alpha(dz, dt)$ is N-dimentional Markov regime-switching random measures definied by (2) $\widetilde{\Phi}(t) = \left(\widetilde{\Phi}_1, .., \widetilde{\Phi}_D\right)$ whith $\widetilde{\Phi}_j(t)$, $j = 1, 2, .., D, .$ In what follows, we consider the process $\{X(t) \backslash t \in [0, \infty[\}$ as the solution of (3) associated with the control process $\{u(t \backslash t \in [0, \infty[)\}$.

Let $\varepsilon_t \subset \mathcal{F}_t$ be a given subfiltration, representing the information avialable to the controller at time t, $t \ge 0$.

The control process $u(t)$ assumed to be $\{\varepsilon_t\}_{t \ge 0}$ predictable and with value in a convexe set $U \subset \mathbb{R}$. Let $\mathcal{A}_\varepsilon$ be our family of ε_t-predictable controls. Let $\mathcal{R}$ denote the set of functions $r : [0, \infty[\times \mathbb{R}_0 \to \mathbb{R}$ such that

$$(4)\qquad J\left(\overset{*}{x}, e_i, u^*\right) = \underset{u \in \mathcal{A}_\varepsilon}{Sup}\, J(x, e_i, u).$$

Let us define the Hamiltonian $H : [0, \infty[\times \mathbb{R} \times U \times \mathcal{S} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \mathbb{R} \to \mathbb{R}$ by

$$(5)\qquad \begin{aligned} H(t, x, u, e_i, p, q, r, s) \;=\; & f(t, x, u, e_i, w) + pb(t, x, u, e_i, w) + q\sigma(t, x, u, e_i, w) \\ & + \int_{\mathbb{R}_0} \eta(t, x, u, e_i, z, w)\, r(t, z)\, \nu_i(dz) \\ & + \sum_{j=1}^{D} \gamma^j(t, x, u, e_i, w)\, s_j(t)\, \lambda_{ij}. \end{aligned}$$

The adjoint equation in the unknown $\mathcal{F}_t$-predictable processes $(p(t), q(t), r(t, z), s(t))$ is

$$(6)\qquad \begin{aligned} dp(t) \;=\; & -\frac{\partial}{\partial x} H(t, X(t), u(t), \alpha(t), p(t), q(t), r(t, .), s(t))\,dt \\ & +q(t)\,dB(t) + \int_{\mathbb{R}_0} r(t, z)\,\widetilde{\mathcal{N}}_\alpha(dz, dt) + s(t)\,d\widetilde{\Phi}(t) \end{aligned}$$

LaTeX

3. Optimal control with partial information and infinite horizon

THEOREM 3.1 (Sufficient Infinite Horizon Maximum Principle). *Let $\overset{*}{u} \in \mathcal{A}_\varepsilon$ and let $\left(\overset{*}{p}(t), \overset{*}{q}(t), \overset{*}{r}(t, z), \overset{*}{s}(t)\right)$ be an associated solution to Eq (6). Assume that for all $u \in \mathcal{A}_\varepsilon$ the following terminal condition holds :*

$$(7)\qquad 0 \le E\left[\overline{\lim_{t \to \infty}}\left[\overset{*}{p}(t)\left(X(t) - \overset{*}{X}(t)\right)\right]\right] < \infty$$

Hoveover, assume that $H\left(t,x,u,e_i,\overset{*}{p}(t),\overset{*}{q}(t),\overset{*}{r}(t,.),\overset{*}{s}(t)\right)$ is concave in x and u and

$$\begin{aligned} &E\left[H\left(t,\overset{*}{X}(t),\overset{*}{u}(t),\overset{*}{e_i},\overset{*}{p}(t),\overset{*}{q}(t),\overset{*}{r}(t,.),\overset{*}{s}(t)\right)\backslash\varepsilon_t\right] \\ &= \max_{u\in U}E\left[H\left(t,\overset{*}{X}(t),u,\overset{*}{e_i},\overset{*}{p}(t),\overset{*}{q}(t),\overset{*}{r}(t,.),\overset{*}{s}(t)\right)\backslash\varepsilon_t\right] \end{aligned} \tag{8}$$

In addition we assume that for all $T=\infty$

(9)

$$E\left[\int_0^\infty\left\{\left(\overset{*}{X}(t)-X^u(t)\right)^2\left(\overset{*}{q}(t)^2+\int_{\mathbb{R}_0}\overset{*}{r}(t,z)\,\nu_\alpha(dz)+\sum_{j=1}^D\left(\left(\overset{*}{s}\right)^j\right)^2(t)\,\lambda_j(t)\right)\right\}dt\right]<\infty$$

and

$$E\left[\int_0^\infty\overset{*}{p}(t)^2\left\{\left(\sigma(t)-\overset{*}{\sigma}(t)\right)^2+\int_{\mathbb{R}_0}\left(\eta(t,z)-\overset{*}{\eta}(t,z)\right)^2\nu_\alpha(dz)+\sum_{j=1}^D\left(\gamma^j-\overset{*}{\gamma}{}^j\right)^2\lambda_j(t)\right\}dt\right]<\infty$$

$$E\left[\left|\frac{\partial}{\partial u}H\left(t,\overset{*}{X}(t-),\overset{*}{u}(t),\alpha(t-),\overset{*}{p}(t-),\overset{*}{q}(t),\overset{*}{r}(t,.),\overset{*}{s}(t)\right)\right|^2\right]<\infty, \tag{10}$$

and that

$$E\left[\int_0^\infty\left|H\left(t,X(t-),u(t),\alpha(t-),\overset{*}{p}(t-),\overset{*}{q}(t),\overset{*}{r}(t,.),\overset{*}{s}(t)\right)\right|\right]<\infty \tag{11}$$

for all u.

Then we have that $\overset{*}{u}(t)$ is optimal.

4. Optimal control with partial information and infinite horizon

Theorem 4.1 (necessary maximum principle). *Let $\overset{*}{u}\in\mathcal{A}_\varepsilon$ be an optimal control of problem (6) subject to the controlled system (3) and let $\left(\overset{*}{p}(t),\ \overset{*}{q}(t),\ \overset{*}{r}(t),\ \overset{*}{s}(t)\right)$ be the unique solution of (3.2). Moreover, let us assume that, and*

$$\lim_{T\to\infty}E\left[\overset{*}{p}(T)\,\overset{*}{\varepsilon}(T)\right]=0. \tag{25}$$

Then the following assertions are equivalent:

(i): *For all bounded $\beta\in\mathcal{A}_\varepsilon$, $*$*

$$\frac{d}{ds}J\left(\overset{*}{u}+s\beta\right)\Big|_{s=0}=0.$$

(ii): *For all $t\in[0,\infty[$,*

$$E\left[\frac{\partial H}{\partial u}\left(t,\overset{*}{X}(t),\ u,\ \overset{*}{p}(t),\ \overset{*}{q}(t),\ \overset{*}{r}(t,.),\ \overset{*}{s}(t)\right)\backslash\varepsilon_t\right]_{u=\overset{*}{u}(t)}=0\ a.s.$$

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Integral pseudo MTL-algebras

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Abstract. Pseudo MTL-algebras or weak pseudo BL-algebras are noncommutative structures arise from pseudo t-norm, namely, pseudo BL-algebras without the pseudo-divisibility condition. In this paper, we introduce the special classes of pseudo MTL-algebras and we call it integral pseudo MTL-algebras. Also, integral filters of pseudo MTL-algebras are defined and studied.

Keywords: (pseudo) BL-algebra, integral (pseudo) MTL-algebra, integral filter.

AMS Mathematics Subject Classification [2010]: 13D45, 39B42

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1. Introduction

Basic fuzzy logic (BL from now on) is the many-valued residutated logic introduced by Hajek in [5] to cope with the logic of continues t-norms and their residua. Pseudo MTL-algebras were define in [4] under the name weak BL-algebras in order to obtain a structurer on $[0,1]$, since there is not pseudo BL-algebra on $[0,1]$. So, pseudo MTL-algebras are non commutative fuzzy structure which arise from pseudo t-norm, namely, pseudo BL-algebras without the pseudo divisibility condition. In 2013, R.A. Borzooei et al. in [1] introduced the concepts of integral filters and integral BL-algebras and extracted important information and results about the relationship between these concepts with local and perfect BL-algebras. In this paper, we introduce the notion of special classes of pseudo MTL-algebras and we call it integral pseudo MTL-algebra and provide properties and examples of it. Moreover, we define integral filters and we prove that apseudo MTL-algebras are integral if and only if $\{1\}$ is integral filter. Also, we show that in finite pseudo MTL-algebras, integral filters coincide with perfect filters.

2. Preliminaries

Definition 2.1. ([2]) A pseudo MTL-algebra is an algebra $(A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ of the type $(2,2,2,2,2,0,0)$ satisfying the following conditions:

(M1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;
(M2) $(A, \odot, 1)$ is a monoid;
(M3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$; iff $y \leq x \rightsquigarrow z$ for any $x, y, z \in A$;

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(M4) $(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$ (pseudo-prelinearity).

We will refer to $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ by its universe A.

Remark 2.2. (1) If additionally for $x, y \in A$, satisfies the axiom:
(M5) $(x \to y) \odot x = x \odot (x \rightsquigarrow y) = x \wedge y$ (pseudo-divisibility)
then A is a pseudo BL-algebra.
(2) If A satisfies the conditions (M1), (M2), (M3) and (M5), then it is a bounded divisible residuated lattice.
A is called commutative if the operation $\odot$ is commutative. In this case $\to = \rightsquigarrow$ and thus, a commutative pseudo MTL-algebra is a MTL-algebra. A totally ordered (linear ordered) pseudo MTL-algebra is called chain. In this study, we will agree that the operations $\wedge, \vee, \odot$ have higher priority than the operations $\to, \rightsquigarrow$.

Proposition 2.3. ([2–4]) *In any pseudo MTL-algebra A the following hold:*

(c1) $x \to (y \to z) = (x \odot y) \to z$ and $x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z$;
(c2) $x \le y$ iff $x \to y$ iff $x \rightsquigarrow y = 1$;
(c3) $x \to x = x \rightsquigarrow x = 1$ and $x \to 1 = x \rightsquigarrow 1 = 1$;
(c4) $0 \to x = 0 \rightsquigarrow x = 1$;
(c5) $x \odot 0 = 0 \odot x = 0$;
(c6) $1^- = 1^\sim = 0$ and $0^- = 0^\sim = 1$;
(c7) $x^- \odot x = 0$ and $x \odot x^\sim = 0$;
(c8) $x \le y^-$ iff $x \odot y = 0$ and $x \le y^\sim$ iff $y \odot x = 0$;
(c9) $x \to y^- = (x \odot y)^-$ and $x \rightsquigarrow y^\sim = (y \odot x)^\sim$;
(c10) $x \le y^-$ iff $y \le x^\sim$.

Definition 2.4. ([2]) (a) A nonempty subset $F \subseteq A$ is called filter of A if satisfies the following axioms:

(F1) if $x, y \in F$, then $x \odot y \in F$;
(F2) if $x \in F$, $y \in A$, $x \le y$, then $y \in F$.

(b) A proper normal filter P of a A is called primary if for all $x, y \in A$, $((x \odot y)^n)^\sim \in P$ for some $n \in \mathbb{N} \cup \{0\}$ implies $(x^m)^\sim \in P$ or $(y^m)^\sim \in P$ for some $m \in \mathbb{N} \cup \{0\}$.
(c) A filter of A is maximal (ultra filter) if it is proper and it is not contained in any other proper filter;
(d) A proper filter P on A is called perfect filter if for all$x \in A$, $(x^n)^- \in P$ and $(x^n)^\sim \in P$ for some $n \in \mathbb{N} \cup \{0\}$ iff $((x^-)^m)^- \notin P$ or $((x^\sim)^m)^\sim \notin P$ for all $m \in \mathbb{N} \cup \{0\}$.

Proposition 2.5. ([2]) *Let A be a pseudo MTL-algebra, and let P be a proper normal filter of A. Then the following are equivalent:*

(i) *P is perfect filter of A;*
(ii) *A/P is a perfect pseudo MTL-algebra.*

Definition 2.6. ([2]) A pseudo MTL-algebra is called local if it has a unique ultera filter.

Definition 2.7. ([2]) A pseudo MTL-algebra A is called perfect perfect if it satifies the following conditions:

(i) A is local good pseudo MTL-algebra;
(ii) for any $x \in A$, $ord(x) < \infty$ iff $ord(x^-) = \infty$ iff $ord(x^\sim) = \infty$.

3. Integral pseudo MTL-algebras

In this section we study a class of pseudo MTL-algebras that called integral pseudo MTL-algebras and we give some of properties.

DEFINITION 3.1. A is called an integral pseudo MTL-algebra, if $x \odot y = 0$, then $x = 0$ or $y = 0$ for all $x, y \in A$.

EXAMPLE 3.2. ([2]) (a) Let's consider $A = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$ and the operations $\odot$, $\rightarrow$, $\rightsquigarrow$ given by the following tables:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	a	b	b
c	0	a	a	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	b	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

Then $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is an integral pseudo MTL-algebra.

(b) Let's consider $A = \{0, a, b, c, 1\}$ with $0 < a < b, c < 1$, but b, c are incomparable, and the operations $\odot$, $\rightarrow$, $\rightsquigarrow$ given by the following tables:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	a	0	a
b	0	0	b	0	b
c	0	a	b	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	c	1	c	1
c	b	b	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	c	1
c	0	b	b	c	1
1	0	a	b	c	1

Then $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a pseudo MTL-algebra, but not an integral pseudo MTL-algebra, since $b \odot a = 0$, for $a, b \neq 0$ and $b \odot c = 0$, for $b, c \neq 0$.

THEOREM 3.3. *Let P be a filter of A and A/P be an integral pseudo MTL-algebra. Then P is a primary filter of A.*

PROOF. Assume that $(x \odot y)^- \in P$ and $(x \odot y)^\sim \in P$, for $x, y \in A$. Then $(x \odot y) \rightarrow 0 \in P$ and $(x \odot y) \rightsquigarrow 0 \in P$. Using (c4), we get $0 \rightarrow (x \odot y) = 1 \in P$ and $0 \rightsquigarrow (x \odot y) = 1 \in P$. Hence $[x \odot y] = [0]$, and so $[x] \cdot [y] = [x \odot y] = [0]$. Since A/P is an integral pseudo MTL-algebra, we have $[x] = [0]$ or $[y] = [0]$. Hence $x^- = x \rightarrow 0 \in P$, $x^\sim = x \rightsquigarrow 0 \in P$ or $y^- = y \rightarrow 0 \in P$, $y^\sim = y \rightsquigarrow 0 \in P$. Thus, P is a primary filter of A. $\square$

THEOREM 3.4. *Let A be an integral pseudo MTL-algebra. Then*

(i) *A is local pseudo MTL-algebra and $B(A) = \{0, 1\}$;*
(ii) *$M(A) = A \setminus \{0\}$;*
(iii) *A is perfect pseudo MTL-algebra and $ord(x) = \infty$, where $0 \neq x \in A$;*
(iv) *if A is an Boolean algebra, then $A = \{0, 1\}$.*

COROLLARY 3.5. *Let P be a primary filter of A and $[x] \cdot [y] = [0]$, for some $[x], [y] \in A/P$. Then $[x]$ or $[y]$ is nilpotent.*

4. Integral filters of pseudo MTL-algebras

Definition 4.1. A proper filter P of A is called integral filter, if for all $x, y \in A$, $(x \odot y)^- \in P$ implies $x^- \in P$ or $y^- \in P$ and $(x \odot y)^\sim \in P$ implies $x^\sim \in P$ or $y^\sim \in P$.

Example 4.2. Consider the pseudo MTL-algerba in Example 3.2, the set of integral filters of A is $F(A) = \{\{1\}, \{c, 1\}, \{a, b, c, 1\}, A\}$.

Proposition 4.3. *Every integral filter is a primary filter.*

Theorem 4.4. *Let $F \subseteq G$, where F, G be filters of A and F be an integral filters of A. Then G is an integral filter, too.*

Theorem 4.5. *Let P be a proper filter of A. Then P is an integral filter if and only if A/P is an integral pseudo MTL-algebra.*

Theorem 4.6. *The following statements are equivalent:*

(i) *$\{1\}$ is an integral filter of A;*
(ii) *any filter of A is an integral filter;*
(iii) *A is an integral pseudo MTL-algebra.*

Theorem 4.7. *Let P be an integral filter of A. Then P is a perfect filter of A.*

Corollary 4.8. *Let A be finite. Then A is an integral pseudo MTL-algebra if and only if $MV(A)$ is an integral pseudo MTL-algebra, where*

$$MV(A) = \{x \in A : x^{\sim\sim} = x^{--} = x\}.$$

Proof. Assume A be an integral pseudo MTL-algebra and $X \odot y = 0$, for some $x, y \in MV(A)$. Since $MV(A) \subseteq A$, then $x = 0$ or $y = 0$, and so $MV(A)$ is an integral pseudo MTL-algebra. Conversely, let $MV(A)$ is an integral pseudo MTL-algebra. Since A is finite pseudo MTL-algebra, using Theorem 3.4(iii), $MV(A)$ is a finite prefect MTL-algebra. Therefore, by Theorem 3.4 (i), $MV(A) = \{0, 1\}$. Now, let $x \odot y = 0$ for some $x, y \in A$. Then by (c8), $x \leq y^-$, and so $y^- \in MV(A)$. Hence $y^- = 0$ or $Y^- = 1$ and $y = 1$ oy $y = 0$. If $y = 1$, since $x \odot y = 0$, then $x = 0$ and otherwise $y = 0$. Thus, A is a pseudo MTL-algebra. □

5. Conclusion

The results of this paper will be devoted to study the local and perfect pseudo MTL-algebras. Also, we introduced the notions of integral pseudo MTL-algebras and integral filters, and proved that these filters are perfect and primary, but there is still an open problem: under what suitable conditions the convers of Proposition 4.3 holds. This issue is an our agenda in future research.

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۸ تا ۱۱ شهریور ۱۴۰۰

Existence fixed point in orthogonal b-metric spaces

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Abstract. Very recently, many authors extended orthogonal metric spaces and discussed on fixed points for several various contractive mappings in spaces. In this article we extend the notion of orthogonal metric space to orthogonal b-metric space. We obtain several fixed point results concerning this mapping in the framework of new spaces, which is called orthogonal b-metric spaces. Also, all main results, new definitions and theorems are supported by some interesting example.

Keywords: fixed point, O-set, orthogonal b-metric space

AMS Mathematics Subject Classification [2010]: 47H10; 54H25

1. Introduction

After the first results of Banach in 1922, many authors studied on this theory in various spaces. One of these spaces is a b-metric space defined by Bakhtin [1] .

Definition 1.1. [1] Let $X \neq \emptyset$ and $s \geq 1$ be a real number. Assume that a mapping $d : X \times X \to [0, \infty)$ for every $x, y, z \in X$ satisfies in the following relations:
(d_1) $d(x, y) = 0$ iff $x = y$;
(d_2) $d(x, y) = d(y, x)$;
(d_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
Then d is named a b-metric on X and (X, d) is named a b-metric space.

In 2008, Eshaghi et al. [2] defined the idea of orthogonal sets and orthogonal metric spaces. In the sequel, we consider some definitions and notations about these concepts.

Definition 1.2. [2] Assume $X \neq \emptyset$ and consider a binary relation $\perp$ on $X \times X$ by

$$\exists a_0;\ ((\forall b, b \perp a_0) \quad \text{or} \quad (\forall b, a_0 \perp b)).$$

Then X is named an orthogonal set. Also, a_0 is named an orthogonal element.

Example 1.3. [3] Let $X = [2, \infty)$ and consider $e \perp f$ if $e \leq f$ for every $e, f \in X$. Then, by considering $a_0 = 2$, $(X, \perp)$ is an O-set.

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Definition 1.4. [2] Consider an O-set $(X, \perp)$ with a sequence $\{a_n\}$ therein. Then $\{a_n\}$ is named orthogonal sequence (or same O-sequence) whenever

$$((\forall n \in \mathbb{N}; a_n \perp a_{n+1}) \quad \text{or} \quad (\forall n \in \mathbb{N}; a_{n+1} \perp a_n)).$$

Analogously, a Cauchy sequence $\{a_n\}$ is named a Cauchy O-sequence whenever

$$((\forall n \in \mathbb{N}; a_n \perp a_{n+1}) \quad \text{or} \quad (\forall n \in \mathbb{N}; a_{n+1} \perp a_n)).$$

Now, if we consider a b-metric instead a metric, then we can rewrite the following definitions that were previously introduced by other researchers. Consider the $(X, \perp)$ and a b-metric d on X with a real number $s \geq 1$. The triple $(X, \perp, d)$ is named an orthogonal b-metric space.

Definition 1.5. [2] The triple $(X, \perp, d)$ is named a complete O-b-metric space (O-complete) whenever each Cauchy O-sequence converges in X.

Definition 1.6. [2] Assume that $(X, \perp, d)$ is an orthogonal b-metric space and T is a self-mapping on X. T is named an orthogonal preserving ($\perp$-preserving) whenever $g \perp l$ implies $T(g) \perp T(l)$ for all $g, l \in X$.

Definition 1.7. [4] Assume that $(X, \perp, d)$ is an orthogonal b-metric space and T is a self-mapping on X. Then T is named orthogonal continuous ($\perp$-continuous) in $a \in X$ if for all O-sequences $\{a_n\}$ in X so that $a_n \longrightarrow a$ then $T(a_n) \longrightarrow T(a)$. Also, T is $\perp$-continuous on X if T is $\perp$-continuous for all $a \in X$.

2. Main results

Here, we express several fixed point results in an orthogonal complete b-metric space.

Theorem 2.1. *Assume that $(X, \perp, d)$ is an orthogonal complete b-metric space. Also, suppose that $T : X \longrightarrow X$ is a $\perp$-preserving and O-continuous mapping. Moreover, assume that there exist $\alpha, \beta, \gamma \geq 0$ with $\alpha s + \beta(1+s) + \gamma(s^2+s) < 1$ provided that*

$$d(Ta, Tb) \leq \alpha d(a, b) + \beta[d(a, Ta) + d(b, Tb)] + \gamma[d(a, Tb) + d(b, Ta)] \tag{1}$$

for every $a, b \in X$, where $a \perp b$. Then T has a unique fixed point $a' \in X$ and $T^n a \longrightarrow a'$ for each $a \in X$.

Proof. Suppose a_0 is an orthogonal element in X so that

$$((\forall b \in X; a_0 \perp b) \quad \text{or} \quad (\forall b \in X; b \perp a_0)).$$

Consider a sequence $\{a_n\}$ by $a_n = T(a_{n-1}) = T^n a_0$. By using the property $\perp$-preserving of T, $\{a_n\}$ is an O-sequence, i.e.,

$$((\forall n \in \mathbb{N}; a_n \perp a_{n+1}) \quad \text{or} \quad (\forall n \in \mathbb{N}; a_{n+1} \perp a_n)).$$

Now, set $a = a_{n-1}$ and $b = a_n$ in (1). Then, for any $n \in \mathbb{N}$, we get

$$\begin{aligned}
d(a_n, a_{n+1}) = d(Ta_{n-1}, Ta_n) &\leq \alpha d(a_{n-1}, a_n) + \beta[d(a_{n-1}, Ta_{n-1}) + d(a_n, Ta_n)] \\
&\quad + \gamma[d(a_{n-1}, Ta_n) + d(a_n, Ta_{n-1})] \\
&= \alpha d(a_{n-1}, a_n) + \beta[d(a_{n-1}, a_n) + d(a_n, a_{n+1})] \\
&\quad + \gamma[d(a_{n-1}, a_{n+1}) + d(a_n, a_n)] \\
&\leq \alpha d(a_{n-1}, a_n) + \beta[d(a_{n-1}, a_n) + d(a_n, a_{n+1})] \\
&\quad + \gamma s[d(a_{n-1}, a_n) + d(a_n, a_{n+1})]
\end{aligned}$$

$$\leq (\alpha+\beta+\gamma s)d(a_{n-1},a_n)+(\beta+\gamma s)d(a_n,a_{n+1})]. \tag{2}$$

Now, (2) implies that $d(a_n,a_{n+1}) \leq \lambda d(a_{n-1},a_n)$ for each $n \in \mathbb{N}$, where $\lambda = \frac{\alpha+\beta+\gamma s}{1-\beta-\gamma s} < \dfrac{1}{s}$. By continuing this process, we get $d(a_n,a_{n+1}) \leq \lambda^n d(a_0,a_1)$ for all $n \in \mathbb{N}$. Now, let $m,n \in \mathbb{N}$ with $m > n$. Then, we get

$$d(a_n,a_m) \leq (\frac{s\lambda^n}{1-\lambda s})d(a_0,a_1) \longrightarrow 0 \qquad \text{as} \qquad n \longrightarrow \infty,$$

which implies that $\{a_n\}$ is a Cauchy O-sequence in orthogonal O-complete b-metric space X. Thus, $\{a_n\}$ converges to element $a' \in X$. Now, since T is O-continuous and $Ta_n \longrightarrow Ta'$, we have

$$d(a',Ta') = \lim_{n\longrightarrow\infty} d(a_{n+1},Ta') = \lim_{n\longrightarrow\infty} d(Ta_n,Ta') = d(Ta',Ta') = 0.$$

Thus, a' is a fixed point for T. Now, we demonstrate a' is unique. Assume b' is another fixed point of T. Then, we get

$$(a_0 \perp a' \;\text{ and }\; a_0 \perp b') \quad \text{or} \quad (a' \perp a_0 \;\text{ and }\; b' \perp a_0).$$

Since T is a $\perp$-preserving mapping, we obtain

$$(T^n a_0 \perp a' \;\text{ and }\; T^n a_0 \perp b') \quad \text{or} \quad (a' \perp T^n a_0 \;\text{ and }\; b' \perp T^n a_0)$$

for any $n \in \mathbb{N}$. Using (1), we obtain

$$d(a_n,a') = d(T^n a_0, T^n a') \leq \lambda^n d(a_0,a'),$$
$$d(a_n,b') = d(T^n a_0, T^n b') \leq \lambda^n d(a_0,b').$$

Now, $d(a',b') \leq sd(a',a_n) + sd(a_n,b')$ implies that $a' = b'$; i.e., T has a unique fixed point. □

Corollary 2.2. *Assume that $(X,\perp,d)$ is an orthogonal complete b-metric space. Also, suppose that $T : X \longrightarrow X$ is a $\perp$-preserving and O-continuous mapping. Moreover, assume that there exists $\gamma \geq 0$ with $\gamma \in [0, \frac{1}{s^2+s})$ so that*

$$d(Ta,Tb) \leq \gamma[d(a,Tb)+d(b,Ta)]$$

for every $a,b \in X$, where $a \perp b$. Then T has a unique fixed point $a' \in X$ and $T^n a \longrightarrow a'$ for any point $a \in X$.

Proof. Set $\alpha = \beta = 0$ in (1) and apply Theorem 2.1. □

Corollary 2.3. *Assume that $(X,\perp,d)$ is an orthogonal complete b-metric space. Also, suppose that $T : X \longrightarrow X$ is a $\perp$-preserving and O-continuous mapping. Moreover, assume that there exists $\beta \geq 0$ with $\beta \in [0, \frac{1}{1+s})$ so that*

$$d(Ta,Tb) \leq \beta[d(a,Ta)+d(b,Tb)]$$

for each $a,b \in X$, where $a \perp b$. Then T has a unique fixed point $a' \in X$ and $T^n a \longrightarrow a'$ for each $a \in X$.

Proof. Set $\alpha = \gamma = 0$ in (1) and consider Theorem 2.1. □

Corollary 2.4. *Let $(X,\perp,d)$ be an orthogonal complete b-metric space. Also, suppose that $T : X \longrightarrow X$ be a $\perp$-preserving and O-continuous mapping. Moreover, assume that there exist $\alpha,\beta,\gamma \geq 0$ so that*

$$d(Ta,Tb) \leq \alpha d(a,b) + \beta d(a,Ta) + \gamma d(b,Tb)$$

for every $a, b \in X$ with $a \perp b$, where $\alpha s + \beta s + \gamma < 1$. Then T has a unique fixed point $a' \in X$ and $T^n a \longrightarrow a'$ for every $x \in X$.

Example 2.5. Set $X = [0, 12]$ and define $d : X \times X \to [0, \infty)$ by $d(a, b) = |a - b|^2$ for each $a, b \in X$. Consider the binary relation $\perp$ on X by $a \perp b$ if $ab \leq (a \vee b)$, where $a \vee b = a$ *or* b. Then $(X, d, \perp)$ is an O-complete b-metric space with $s = 2$. Consider the mapping $T : X \to X$ by

$$Ta = \begin{cases} \frac{a}{3} & 0 \leq a \leq 3, \\ 0 & 3 < a \leq 12 \end{cases} \qquad \big(a \in [0, 12]\big).$$

Let $a \perp b$ and $\alpha = \frac{1}{16}$, $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{24}$ in (1). Without loss of generality, we may consider $ab \leq b$. Now, we have

- if $a = 0$ and $0 \leq b \leq 3$, then $Ta = 0$ and $Tb = \frac{b}{3}$, and

$$\begin{aligned} d(Ta, Tb) &= \frac{b^2}{9} \leq \frac{1}{16} b^2 + \frac{1}{4} \cdot \frac{4b^2}{9} + \frac{1}{24} \cdot \frac{10b^2}{9} \\ &= \alpha d(a, b) + \beta\big(d(a, Ta) + d(b, Tb)\big) + \gamma\big(d(a, Tb) + d(b, Ta)\big), \end{aligned}$$

- if $a = 0$ and $3 \leq b \leq 12$, then $Ta = Tb = 0$, and

$$\begin{aligned} d(Ta, Tb) &= 0 \leq \frac{1}{16} b^2 + \frac{1}{4} b^2 + \frac{1}{24} b^2 \\ &= \alpha d(a, b) + \beta\big(d(a, Ta) + d(b, Tb)\big) + \gamma\big(d(a, Tb) + d(b, Ta)\big), \end{aligned}$$

- if $0 \leq b \leq 1$ and $0 \leq a \leq 3$, then $Ta = \frac{a}{3}$ and $Tb = \frac{b}{3}$, and

$$\begin{aligned} d(Ta, Tb) &= \frac{1}{9}|a - b|^2 \leq \frac{1}{9}(a^2 + b^2) = \frac{1}{4} \cdot (\frac{4a^2}{9} + \frac{4b^2}{9}) \\ &\leq \frac{1}{16}|a - b|^2 + \frac{1}{4} \cdot (\frac{4a^2}{9} + \frac{4b^2}{9}) + \frac{1}{24} \cdot (|a - \frac{b}{3}|^2 + |y - \frac{a}{3}|^2) \\ &= \alpha d(a, b) + \beta\big(d(a, Ta) + d(b, Tb)\big) + \gamma\big(d(a, Tb) + d(b, Ta)\big), \end{aligned}$$

- if $0 \leq b \leq 1$ and $3 < a \leq 12$, then $Ta = 0$ and $Tb = \frac{b}{3}$, and so

$$\begin{aligned} d(Ta, Tb) &= \frac{b^2}{9} \leq \frac{1}{16}|a - b|^2 + \frac{1}{4} \cdot (a^2 + \frac{4b^2}{9}) + \frac{1}{24} \cdot (|a - \frac{b}{3}|^2 + b^2) \\ &= \alpha d(a, b) + \beta\big(d(a, Ta) + d(b, Tb)\big) + \gamma\big(d(a, Tb) + d(b, Ta)\big). \end{aligned}$$

Thus, the relation (1) is valid. So, all hypotheses of Theorem 2.1 are held. Consequently, T has a unique fixed point $a = 0$ in X.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

A Stochastic Model for Zika Virus Transmission

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Abstract. In this paper we develop and analyze a mathematical model for the transmission of Zika virus. Firstly we construct stochastic environment because of parameters random essence, and introduce Zika epidemic model in stochastic form. Moreover, the equilibria of the system is considered. Finally, disease-free equilibrium point of the model and biologically feasible region for this dynamical system are presented.

Keywords: Zika virus, stochastic modeling, stochastic differential equation, transmission simulation

AMS Mathematics Subject Classification [2010]: 92C60, 93A30, 60H10

1. Introduction

The mathematical modeling has been used in various sciences, including engineering, chemistry, biology, mathematical finance and physics [1–4]. In particular, stochastic systems are very useful tools for modeling in various fields such as physics, chemistry, biology, mathematical finance and other sciences [1,3]. Application of mathematical models to study mosquito related diseases have been studied by several researchers [4–6]. Zika is a flavivirus transmitted to humans through either the bite of infected Aedes mosquitoes or sexual intercourse with infected individuals. In this paper, we present a mathematical model based on two modes of transmission. In mathematical models of Zika virus transmission it is assumed that the virus is usually transmitted from mosquitoes to humans, while according to WHO, in addition to the transmission through mosquitoes, Zika virus is transmitted through infected blood as well as through sexual contact with an infected person. Mathematical models for transmission dynamics of mosquito-borne diseases can be useful in providing better insights into the behaviour of this disease. The models have played great roles in influencing the decision making processes regarding intervention strategies for preventing and controlling the insurgence of mosquito-borne diseases [4–6]. In this work, we establish a model for investigation of the Zika transmission based on stochastic version of the SIR model with additional degree of realism. This paper is organized as follows. In Section 2, preliminaries and notations are presented. In Section 3, the model of transmission of Zika virus and the process of the implementation stochastic form are presented.

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2. Preliminaries

Let $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. We also let $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_i > 0,\ 1 \leq i \leq d\}$, $d-$dimensional stochastic differential equation can be expressed as follows:

$$dX(t) = f(t, X(t))dt + g(t, X(t))dB(t), \ \ t \geq t_0, \tag{1}$$

with initial value $X(t_0) = X_0$, where $f(t, x)$ is a function in $\mathbb{R}^d$ defined on $[t_0, +\infty) \times \mathbb{R}^d$, $g(t, x)$ is a $d \times m$ matrix and $B(t)$ is an $m-$dimensional standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$. We define the differential operator L associated with Equation (1) by,

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(t, x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} [g^T(t, x)g(t, x)]_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

If L acts on function V in $C^{2\times 1}(\mathbb{R}^d \times [t_0, \infty))$, The generalized Itô formula implies that

$$dV(t, x) = LV(t, x)dt + V_x(t, x)g(t, x)dB(t).$$

3. Stochastic Zika Virus Model

In this work, we consider a mathematical model for Zika virus transmission dynamics. This model consists of nine differential equations. In this section, we formulate a stochastic mathematical model for Zika virus transmission based on three populations. In human population, we use a sex-structured mathematical model. Based on their disease status, sexually active male (N_M) and female (NF) individuals are grouped into different disjoint classes. Classes of susceptible male and female individuals are denoted by S_M and S_F , respectively. The classes I_M and I_F stand for infectious male and female individuals, respectively. We grouped recovered male and female individuals from Zika virus in R_M and R_F classes,respectively. Thus,

$$N_H = N_M + N_F \quad , \quad N_M = S_M + I_M + R_M \quad , \quad N_F = S_F + I_F + R_F.$$

In the mosquito population (N_V), the susceptible, latent and infected classes of Zika virus are denoted by S_V, E_V and I_V ,respectively. Hence, we have

$$N_V = S_V + E_V + I_V.$$

Also, the model parameters and their definitions are given in Tab. 1.

TABLE 1. Description of parameters

β_1	Sexual transmission rate from infectious males to susceptible female individuals
β_2	Mosquito-to-human transmission rate
β_3	Human-to-mosquito transmission rate
β_4	Sexual transmission rate from infectious females to susceptible male individuals
γ	Human recovery rate
μ	Mosquito death rate
Λ	Mosquito recruitment rate
α	Mosquito transition rate from latent to infectious class

To describe the mechanism of the spread of Zika virus, we consider the deterministic mathematical model as follows [5]:

$$\begin{aligned}
dS_M &= -(\beta_2 I_v + \beta_4 I_F)S_M dt\\
dI_M &= ((\beta_2 I_v + \beta_4 I_F)S_M - \gamma I_M)dt\\
dR_M &= \gamma I_M dt\\
dS_F &= -(\beta_1 I_M + \beta_2 I_v)S_F dt\\
dI_F &= ((\beta_1 I_M + \beta_2 I_v)S_F - \gamma I_F)dt\\
dR_F &= \gamma I_F dt\\
dS_v &= (\Lambda - \beta_3(I_M + I_F)S_v - MS_v)dt\\
dE_v &= (\beta_3(I_M + I_F)S_v - (\alpha + M))dt\\
dI_v &= (\alpha E_v - MI_v)dt
\end{aligned}$$

We can provide an additional degree of realism by defining the white noise and Brownian motion and introduce a stochastic model. Therefore, we implement this idea by replacing random parameters

$$\begin{cases}\beta_1 \to \beta_1 + \sigma_1 \dot{B}_1(t)\\ \beta_2 \to \beta_2 + \sigma_2 \dot{B}_2(t)\\ \beta_3 \to \beta_3 + \sigma_3 \dot{B}_3(t)\\ \beta_4 \to \beta_4 + \sigma_4 \dot{B}_4(t)\end{cases} \qquad \begin{cases}\gamma \to \gamma + \sigma_5 \dot{B}_5(t)\\ M \to M + \sigma_6 \dot{B}_6(t)\\ \Lambda \to \Lambda + \sigma_7 \dot{B}_7(t)\\ \alpha \to \alpha + \sigma_8 \dot{B}_8(t)\end{cases}$$

where $B_i(t)$ and σ_i, $i = 1, 2, \ldots, 6$ are the Brownian motions and the intensities of the white noises, respectively. These parameters are selected for implementation of stochastic environment because of their random essence. So, we present the following modified model with stochastic approach for Zika virus:

(2)

$$\begin{array}{rcl}
ds_M &=& -(\beta_2 I_v + \beta_4 I_F)S_M dt - \sigma_2 I_v S_M dB_2(t) - \sigma_4 I_F S_M dB_4(t),\\
dI_M &=& ((\beta_2 I_v + \beta_4 I_F)S_M - \gamma I_M)dt + \sigma_2 I_v S_M dB_2(t) + \sigma_4 I_F S_M dB_4(t) - \sigma_5 I_M dB_5(t),\\
dR_M &=& \gamma I_M dt + \sigma_5 I_M dB_5(t),\\
dS_F &=& -(\beta_1 I_M + \beta_2 I_v)S_F dt - \sigma_1 I_M S_F dB_1(t) - \sigma_2 I_v S_F dB_2(t),\\
dI_F &=& ((\beta_1 I_M + \beta_2 I_v)S_F - \gamma I_F)dt + \sigma_1 I_M S_F dB_1(t) + \sigma_2 I_v S_F dB_2(t) - \sigma_5 I_F dB_5(t),\\
dR_F &=& \gamma I_F dt + \sigma_5 I_F dB_5(t),\\
dS_v &=& (\Lambda - \beta_3(I_M + I_F)S_v - MS_v)dt + \sigma_7 dB_7(t) - \sigma_3(I_M + I_F)S_v dB_3(t),\\
&& -\sigma_6 S_v dB_6(t),\\
dE_v &=& (\beta_3(I_M + I_F)S_v - (\alpha + M)dt + \sigma_3(I_M + I_F)S_v dB_3(t) - \sigma_8 E_v dB_8(t),\\
&& -\sigma_6 E_v dB_6(t),\\
dI_v &=& (\alpha E_v - MI_v)dt + \sigma_8 E_v dB_8(t) - \sigma_6 I_v dB_6(t).
\end{array}$$

THEOREM 3.1. *Zika virus model (2) is a dynamical system on the biologically feasible region*

$$\begin{array}{rcl}\Omega = \{(S_M, I_M, R_M, S_F, I_F, R_F, S_v, E_v, I_v) \in \mathbb{R}^9_+ : 0 &\le& S_M + I_M + R_M + S_F + I_F + R_F\\ && = N_H = const\end{array}$$

and

$$0 \le S_v + E_v + I_v = N_v \le \Lambda/\mu.\}$$

PROOF. The proof is presented in [5]. □

Theorem 3.2. *The disease-free equilibrium point of the model is obtained from the system*

$$\begin{aligned}
\frac{S_M}{1+(\beta_2 I_v+\beta_4 I_F)\phi} &= S_M \\
\frac{(\beta_2 I_v+\beta_4 I_F)\phi S_M + I_M}{1+\gamma\phi} &= I_M \\
\gamma\phi I_M + R_M R_M \\
\frac{S_F}{1+(\beta_1 I_M+\beta_2 I_v)\phi} &= S_F \\
\frac{(\beta_1 I_M+\beta_2 I_v)\phi S_F + I_F}{1+\gamma\phi} &= I_F \\
\gamma\phi I_F + R_F &= R_F \\
\frac{\Lambda\phi + S_v}{1+(\beta_3(I_M+I_F)+\mu)\phi} &= S_v \\
\frac{\beta_3(I_M+I_F)\phi S_v + E_v}{1+(\alpha+\mu)\phi} &= E_v \\
\frac{\alpha\phi E_v + I_v}{1+\mu\phi} &= I_v
\end{aligned}$$

and gives

$$(S_M^k, I_M^k, R_M^k, S_F^k, I_F^k, R_F^k, S_v^k, E_v^k, I_v^k) = (S_{M_0}, 0, 0, S_{F_0}, 0, 0, \frac{\Lambda}{\mu}, 0, 0).$$

4. Conclusion

In this paper, we have analyzed a model for simulating transmissibility of the Zika virus. We use the white noise and Brownian motion to construct the corresponding stochastic model for the transmission of the Zika virus. We established a stochastic model for Zika virus with additional degree of realism. Finally, disease-free equilibrium point of the model and biologically feasible region for this dynamical system is presented.

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Geometric relationships on hom-Lie color algebra extensions

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Abstract. In this paper we introduce notions of hom-Lie color algebra and investigate some basic results on it. A hom-Lie color algebra is a quadruple $(\mathfrak{L}, [.,.], \epsilon, \alpha)$ with some property. We study (non-Abelian) extensions of a given hom-Lie color algebra and characterize an extension of a hom-Lie color algebra $\mathfrak{L}$ by another hom-Lie color algebra $\mathfrak{h}$ and discuss the case where $\mathfrak{h}$ has no center.

Keywords: Hom-Lie algebra, Hom-Lie color algebra, Extensions of hom-Lie color algebra

AMS Mathematics Subject Classification [2010]: 17B56, 17B75, 17B40

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1. Introduction

The notion of Lie algebra is one of the important concepts of modern mathematics and mathematical physics. The general theory of Lie algebras leads to a rich assortment of important explicit examples of geometric objects. There are different kinds of generalizations of Lie algebras, such as hom-Lie algebras, hom-Lie superalgebras, color hom-Lie algebras, etc. [1–3, 5] The notion of hom-Lie algebra was introduced by Hartwig et al. [6] as part of study of deformations of the Witt and the Virasoro algebras. In a hom-Lie algebra, the jacobi identity is twisted by a linear map, called the hom-Jacobi identity. Recently, many results of hom-Lie algebras were generalized to hom-Lie color algebras which are the natural generalizations of Lie algebras and Lie superalgebras, by Larsson and Silvestrov [6, 7]. The main objects that this paper deals with are hom-Lie color algebras. Therefore at the first, hom-Lie algebras, hom-Lie color algebras and some of their useful related definitions are presented. At first let us to recall some basic concepts from [2, 4, 8].

Definition 1.1. [6, 7] A hom-Lie algebra is a triple $(\mathfrak{L}, [.,.], \alpha)$, where $\mathfrak{L}$ is a linear space equipped with a skew-symmetric bilinear map $[.,.] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ and a linear map $\alpha : \mathfrak{L} \to \mathfrak{L}$ such that

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0,$$

for all x, y, z in $\mathfrak{L}$, that is called hom-Jacobi identity.

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- A hom-Lie algebra is called a multiplicative hom-Lie algebra, if α is an algebraic morphism, i.e. for any $x, y \in \mathfrak{L}$,

$$\alpha([x, y]) = [\alpha(x), \alpha(y)].$$

- We call a hom-Lie algebra regular, if α is an automorphism.

Let $(\mathfrak{L}, [.,.], \alpha)$ be multiplicative hom-Lie algebra. Denote by α^s the s-times composition of α by itself, for any nonnegative integer s, i.e.

$$\alpha^s = \alpha \circ ... \circ \alpha \quad (s - times),$$

where we denote $\alpha^0 = Id$ and $\alpha^1 = \alpha$. For a regular hom-Lie algebra $\mathfrak{L}$, let

$$\alpha^{-s} = \alpha^{-1} \circ ... \circ \alpha^{-1} \quad (s - times).$$

A linear sub-vector space $I \subseteq \mathfrak{L}$ is a hom-Lie subalgebra of $(\mathfrak{L}, [.,.], \alpha)$, if $\alpha(I) \subseteq I$ and I is closed under the bracket operation, i.e., $[I, I] \subseteq I$.

Definition 1.2. [1, 4] Let Γ is a commutative group which in what follows will be referred to as thr grading group, a commutation factor on Γ with values in the multiplicative group $K \setminus \{0\}$ of a field K of characteristic 0 is a map $\epsilon : \Gamma \times \Gamma \to K \setminus \{0\}$ satisfying three properties:

1. $\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \gamma)\epsilon(\beta, \gamma)$,
2. $\epsilon(\alpha, \gamma + \beta) = \epsilon(\alpha, \gamma)\epsilon(\alpha, \beta)$,
3. $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = 1$.

A Γ-graded ϵ-Lie algebra (or a color Lie algebra) is a Γ-graded linear space $X = \bigoplus_{\gamma\in\Gamma} X_\gamma$, with a bilinear multiplication (bracket) $[.,.] : X \times X \to X$ satisfying the following properties:

1. **Grading axiom:** $[X_\alpha, X_\beta] \subseteq X_{\alpha+\beta}$,
2. **Graded skew-symmetry:** $[a, b] = -\epsilon(\alpha, \beta)[b, a]$,
3. **Generalized Jacobi identity:**

$$\epsilon(\gamma, \alpha)[a, [b, c]] + \epsilon(\alpha, \beta)[b, [c, a]] + \epsilon(\beta, \gamma)[c, [a, b]] = 0,$$

for all $a \in X_\alpha$, $b \in X_\beta$, $c \in X_\gamma$ and $\alpha, \beta, \gamma \in \Gamma$.

The elements of X_γ are called homogeneous of degree γ, for all $\gamma \in \Gamma$. Suppose $\mathfrak{L} = \bigoplus_{\gamma\in\Gamma} \mathfrak{L}_\gamma$ and $\mathfrak{h} = \bigoplus_{\gamma\in\Gamma} \mathfrak{h}_\gamma$ be Γ-graded linear spaces. A linear mapping $f : \mathfrak{L} \to \mathfrak{h}$ is said to be graded of degree $\mu \in \Gamma$ if $f(\mathfrak{L}_\gamma) = \mathfrak{h}_{\gamma+\mu}$, for all $\gamma \in \Gamma$. A linear mapping $f : \mathfrak{L} \to \mathfrak{h}$ is said to be graded of degree zero if $f(\mathfrak{L}_\gamma) = \mathfrak{h}_\gamma$, holds for all $\gamma \in \Gamma$. Sometimes such f is said to be even.

Now, we introduce the notions of hom-Lie color algebras as a special class of color quasi-Lie algebras.

Definition 1.3. A hom-Lie color algebra is a quadruple $(\mathfrak{L}, [.,.], \epsilon, \alpha)$ consisting of a Γ-graded linear space $\mathfrak{L} = \bigoplus_{\gamma\in\Gamma} \mathfrak{L}_\gamma$, a bi-character ϵ, a graded bilinear mapping $[.,.] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ (i.e., $[\mathfrak{L}_a, \mathfrak{L}_b] \subseteq \mathfrak{L}_{a+b}$, for all $a, b \in \Gamma$) and a graded homomorphism $\alpha : \mathfrak{L} \to \mathfrak{L}$ of grading degree zero ($\alpha(\mathfrak{L}_\gamma) \subseteq \mathfrak{L}_\gamma$, for all $\gamma \in \Gamma$) such that for homogeneous elements $x, y, z \in \mathfrak{L}$ we have

1. **-ϵ-skew symmetry:** $[x, y] = -\epsilon(x, y)[y, x]$,
2. **ϵ-hom-Jacobi identity:**

$$\epsilon(z, x)[\alpha(x), [y, z]] + \epsilon(x, y)[\alpha(y), [z, x]] + \epsilon(y, z)[\alpha(z), [x, y]] = 0.$$

We call $(\mathfrak{L}, [.,.], \epsilon, \alpha)$ a multiplicative hom-Lie color algebra, if α is a morphism of color Lie algebras, i.e., $\alpha \circ [.,.] = [.,.] \circ \alpha^{\otimes 2}$.

Definition 1.4. Let $(\mathfrak{L}, [.,.], \epsilon, \alpha)$ be a hom-Lie color algebra. For any nonnegetive integer s, a linera map $D : \mathfrak{L} \to \mathfrak{L}$ of degree d is called a homogeneous α^s-derivations of the multiplicative hom-Lie color algebra $(\mathfrak{L}, [.,.], \epsilon, \alpha)$ if

(1) $D(\mathfrak{L}_\gamma) \subseteq \mathfrak{L}_{\gamma+d}$,
(2) $[D, \alpha] = 0$, i.e., $D \circ \alpha = \alpha \circ D$,
(3) $D([x, y]) = [D(x), \alpha^s(y)] + \epsilon(d, x)[\alpha^s(x), D(y)]$ for all $x, y \in \mathfrak{L}$.

Denoted by $Der^{\gamma}_{\alpha^s}(\mathfrak{L})$ the set of all homogeneous α^s-derivation of the multiplicative hom-Lie color algebra $(\mathfrak{L}, [.,.], \epsilon, \alpha)$. The space $Der(\mathfrak{L}) = \bigoplus_{s \geq 0} Der_{\alpha^s}(\mathfrak{L})$, provided with the color-commutator and the following linear map

$$\hat{\alpha} : Der(\mathfrak{L}) \to Der(\mathfrak{L}), \quad \hat{\alpha}(D) = D \circ \alpha,$$

is color hom-Lie algebra. For any $x \in \mathfrak{L}$ satisfying $\alpha(x) = x$, define $ad_s(x) : \mathfrak{L} \to \mathfrak{L}$ by $ad_s(x)(y) = [\alpha^s(y), x]$ for all $y \in \mathfrak{L}$.

Definition 1.5. Let $\mathfrak{L}, \mathfrak{h}$ be two hom-Lie color algebras, $\mathfrak{e}$ is called an extension of the hom-Lie color algebra $\mathfrak{L}$ by $\mathfrak{h}$ if there exists a short exact sequence

$$0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{L} \to 0$$

of hom-Lie color algebras and their morphisms.

Tow extensions

$$0 \longrightarrow \mathfrak{h} \xrightarrow{i_t} \mathfrak{e}_t \xrightarrow{p_t} \mathfrak{L} \longrightarrow 0 \quad (t = 1, 2)$$

are equivalent if there is an isomorphism $f : \mathfrak{e}_1 \to \mathfrak{e}_2$ such that $f \circ i_1 = i_2$ and $p_2 \circ f = p_1$. Let there exists an extension

$$0 \longrightarrow \mathfrak{h} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{L} \longrightarrow 0$$

and $q : \mathfrak{L} \to \mathfrak{e}$ be a even degree graded linear map such that $p \circ q = Id_{\mathfrak{L}}$. We define $\psi : \mathfrak{L} \to Der_{\alpha^s(\mathfrak{h})}$, such that

$$\psi_x(y) = [\alpha^{s-1}(s(x)), y], \tag{1}$$

and $\rho : \bigwedge^2_{\Gamma-graded} \mathfrak{L} \to \mathfrak{h}$, such that

$$\rho(x, y) = [q(x), q(y)] - q([x, y]). \tag{2}$$

Lemma 1.6. *Tha maps ψ and ρ defined in (1) and (2) satisfy*

$$[\psi_x, \psi_y] - \psi_{[x,y]} = ad_{s-1}(\rho(x, y)),$$

$$\sum_{cyclic\{x,y,z\}} \epsilon(x, z)(\psi_x(\rho(y, z)) - \rho([x, y], z)) = 0.$$

Now, by using the above lemma, the following main theorem is obtained.

Theorem 1.7. *Suppose $\mathfrak{L}, \mathfrak{h}$ be tow hom-Lie color algebras. The short exact sequences of the form*

$$0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{L} \to 0$$

are in the one-to-one correspondence with the data of the following form. An even linear map $\psi : \mathfrak{L} \to Der_{\alpha^s(\mathfrak{h})}$ *and a graded even skew symmetric bilinear map* $\rho : \bigwedge^2_{\Gamma-graded} \mathfrak{L} \to \mathfrak{h}$ *such that*

$$[\psi_x, \psi_y] - \psi_{[x,y]} = ad_{s-1}(\rho(x,y)),$$
$$\sum_{cyclic\{x,y,z\}} \epsilon(x,z)(\psi_x(\rho(y,z)) - \rho([x,y],z)) = 0.$$

The extension that corresponds to ψ *and* ρ *is the vector space* $\mathfrak{e} = \mathfrak{h} \oplus \mathfrak{L}$ *which hom-Lie color algebra structure is given by*

$$[y_1 + q(x_1), y_2 + q(x_2)]_{\mathfrak{e}} = ([y_1, y_2]_{\mathfrak{h}} + \psi_{x_1} y_2 - \epsilon(x_2, y_1)\psi_{x_2} y_1 + \rho(x_1, x_2)) + [x_1, x_2]_{\mathfrak{L}}$$

and its short exact sequence is

$$0 \longrightarrow \mathfrak{h} \xrightarrow{i} \mathfrak{h} \oplus \mathfrak{L} = \mathfrak{e} \xrightarrow{p} \mathfrak{L} \longrightarrow 0.$$

Two data (ψ, ρ) *and* (ψ', ρ') *are equivalent if there exists a even linear map* $f : \mathfrak{L} \to \mathfrak{h}$ *such that* $\psi'_x = \psi_x + ad^{\mathfrak{h}}_{s-1}(f(x))$ *and*

$$\rho'(x,y) = \rho(x,y) + \psi_x(f(y)) - \epsilon(x,y)\psi_y(f(x)) - f([x,y]) + [f(x), f(y)].$$

Thus the corresponding equivalence will be

$$\mathfrak{e} = \mathfrak{h} \oplus \mathfrak{L} \to \mathfrak{e}' = \mathfrak{h} \oplus \mathfrak{L}$$
$$y + x \mapsto y - f(x) + x.$$

In the special case of above theorem, we have the following corollary.

Corollary 1.8. *Suppose* $\mathfrak{L}, \mathfrak{h}$ *be two hom-Lie color algebras and* $Z(\mathfrak{h}) = 0$. *Then the extensions of* $\mathfrak{L}$ *by* $\mathfrak{h}$ *is in one-to-one correspondence with isomorphisms of the form*

$$\hat{\psi} : \mathfrak{L} \to out(\mathfrak{h}) = \frac{Der_{\alpha^s}(\mathfrak{h})}{Inn_{\alpha^s}(\mathfrak{h})}.$$

Acknowledgement

This research is supported by Grant No. 99GRC1M82582 Shiraz University, Shiraz, Iran.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Woven Frame and Riesz Basis in Hilbert C^*- Modules

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Abstract. In this paper we introduce modular frame, woven modular frame in Hilbert C^*-modules. And we study some definitions and basic properties of Hilbert C^*-modules and woven frames , Riesz basis in Hilbert C^*-modules.Under what conditions can a sequence be turned into a modular frame in Hilbert C^*-modules? Also we show that every woven modular Riesz basis is a module frame.

Keywords: frame, woven frame, modular frame, C^*-Modules

AMS Mathematics Subject Classification [2010]: Primary 46L99; Secondary 42C15, 46H25.

1. Introduction

Hilbert space frames were originally introduced by Duffin and Schaeffer to deal with some problems in non-harmonic Fourier analysis [5]. Frames can be viewed as redundant bases which are generalizations of Riesz bases [1–4]. This redundancy property sometimes is extremely important in applications such as signal and image processing, data compression and sampling theory. Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. Frames for Hilbert spaces have natural analogues for Hilbert C^*-modules. These frames are called Hilbert C^*-modular frames or just simply modular frames. Modular frames are not trivial generalizations of Hilbert space frames due to the complex structure of C^*-algebras. It is well known that the theory of Hilbert C^*-modules is quite different from that of Hilbert spaces. For example, we know that, any closed linear subspace in a Hilbert space has an orthogonal complement. But this is no longer true in Hilbert C^*-module setting since not every closed submodule of a Hilbert C^*-module is complemented. Moreover, the Riesz representation theorem for continuous functionals on Hilbert spaces does not hold in Hilbert C^*-modules, and so there exist nonadjointable bounded linear operators on Hilbert C^*-modules [1, 2]. Therefore it is expected that problems about frames in HilbertC^*-modules are more complicated than those in Hilbert spaces. While some of the results about frames in Hilbert spaces can be easily extended

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to Hilbert C^*-modular frames, many others cannot be obtained by simply modifying the approaches used in Hilbert spaces case.

2. Woven Modular Riesz Basis

In this section, first we recall some definitions and basic properties of Hilbert C^*-Modules and p-woven frame and g-frame in Hilbert C^*- Modules. Throughout this note A is a unital C^*-algebra and H, K_i are finitely or countably generated Hilbert A-modules. For each $i \in I$, $L(H, K_i)$ will denote the set of all adjointable A-linear maps from H to K_i. We also define

$$\ell^2(A) := \{a = (a_i) \in A : \sum_{i\in I} a_i^* a_i \text{ is norm convergent in } A\}$$

Definition 2.1. A pre-Hilbert A-module is a left A-module H equipped with an A-valued inner product $\langle .,. \rangle : H \times H \longrightarrow A$, such that
(i) $\langle x, x\rangle \geq 0$ for all $x \in H$ and $\langle x, x\rangle = 0$ if and only if $x = 0$,
$(ii)\langle x, y\rangle = \langle y, x\rangle^*$ for all $x, y \in H$,
(iii) $\langle ax + y, z\rangle = a\langle x, z\rangle + \langle y, z\rangle$ for all $a \in A$ and $x, y, z \in H$.

We assume that the linear operations of A and H are compatible.i.e. $\lambda(ax) = (\lambda a)x$ for every $\lambda \in \mathbb{C}$, $a \in A$ and $x \in H$. For every $x \in H$, we define

$$\| x \| = \| \langle x, x\rangle \|^{\frac{1}{2}} \qquad \text{and} \qquad |x| = \langle x, x\rangle^{\frac{1}{2}}.$$

If the pre-Hilbert A-module $(H, \langle ., .\rangle)$ is complete with respect to $\| \, . \, \|$, it is called a Hilbert A-module or a Hilbert C^*-modules over A. In this paper we focus on finitely and countably generated Hilbert C^*-modules over unital C^*-algebra A. A Hilbert A-module H is (algebraically) finitely generated if there exists a finite subset $\{x_1, x_2, ..., x_m\}$ of H such that every element $x \in H$ can be expressed as an A-linear combination $x = \sum_{i=1}^m a_i x_i, a_i \in A$. A Hilbert A-module H is countably generated if there exists a countable set of generators. We now recall the definitions of frames and Riesz bases in Hilbert C^*-modules as follows.

Definition 2.2. Let H be a Hilbert A-module. A family $\{x_i : i \in I\}$ of elements of H is a (standard) frame for H, if there exits constants $0 < C \leq D < \infty$, such that for all $x \in H$,

$$C\langle x, x\rangle \leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq D\langle x, x\rangle. \qquad (1)$$

Where the sum in the middle of the inequality convergent in norm for $x \in H$.
The numbers C and D are called frame bounds, If $C = D = \lambda$, it is called a λ-tight frame and when $C = D = 1$, it is called a Parseval frame. $\{x_i : i \in I\}$ is said to be a Bessel sequence if only the right-hand side inequality is required.If the sum of (1) is convergent in norm, the frame is called standard.

According to what Arambasic and Khosravi proved, the above definition is equivalent to, $C \| x \|^2 \leq \| \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \| \leq D \| x \|^2$, A sequence $\{x_i : i \in I\}$ is said to be a Riesz basis of H if it is a frame and a generating set with the additional property that A-linear combinations $\sum_{i\in S} a_i x_i$ with coefficients $\{a_i : i \in S\} \subseteq A$ and $S \subseteq I$ are equal to zero if and only if in particular every summand $a_i x_i$ equal zero for $i \in S$. Note that we can also define the analysis operator, synthesis operator and frame operator for modular frame as follows. Suppose that $\{x_i : i \in I\}$ is a frame of a finitely or countably generated Hilbert A-module H over a unital C^*-algebra A. The operator $T : H \to \ell^2(A)$ defined by $Tx = \{\langle x, x_i\rangle\}_{i\in I}$, is called the analysis operator. The adjoint operator $T^* : \ell^2(A) \to H$ is given by $T^*\{a_i\}_{i\in I} = \sum_{i\in I} a_i x_i$. T^* is called pre-frame operator or the synthesis operator. By composing T and T^*, we obtain the frame operator $S : H \to H$,

$$Sx = T^*Tx = \sum_{i\in I}\langle x, x_i\rangle x_i, \qquad (1)$$

is a frame operator for H. That is $S \in End_A^*(H)$, positive and invertible. Where $End_A^*(H)$ is the set of adjointable A-linear maps on H.
The frame $\{S^{-1}x_i : i \in I\}$ is said to be the canonical dual frame of $\{x_i : i \in I\}$.

REMARK 2.3. If A be a unital C^*-algebra, H be a finitely or countably generated Hilbert A-module and $\{x_i : i \in I\}$ be Parseval frame(not necessarily standard) of H, then the reconstruction formula $x = \sum_{i\in I}\langle x, x_i\rangle x_i$, holds for every $x \in H$. Also from equation (1) we see that $x = \sum_{i\in I}\langle x, S^{-1}x_i\rangle x_i$, is vailid for every $x \in H$. Moreover, if $\{x_i : i \in I\}$ be standard frame, then there exists a unique operator $S \in End_A^*(H)$ such that $x = \sum_{i\in I}\langle x, Sx_i\rangle x_i$.

THEOREM 2.4. *Let* $\{x_i : i \in I\}$ *be a modular frame with frame bounds* C, D. *Let* $\{y_i : i \in I\} \subseteq H$ *and assume that there exist constants* $\lambda_1, \lambda_2, \mu \geq 0$ *such that* $max\{\lambda_1 + \frac{\mu}{\sqrt{C}}, \lambda_2\} < 1$ *and*

$$\| \sum_{i\in I} a_i(x_i - y_i) \| \leq \lambda_1 \| \sum_{i\in I} a_i x_i \| + \lambda_2 \| \sum_{i\in I} a_i y_i \| + \mu(\sum_{i\in I} \| a_i \|^2)^{\frac{1}{2}},$$

for all $a_i \in \ell^2(A)$. *Then* $\{y_i : i \in I\}$ *is a modular frame with bounds*

$$A(1 - \frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{C}}}{1+\lambda_2})^2 \qquad , \qquad D(1 + \frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{D}}}{1-\lambda_2})^2$$

PROOF. Since $\{x_i : i \in I\}$ is a frame, we can define a bounded $A-$linear operator

$$T^* : \ell^2(A) \to H, T^*\{a_i\}_{i\in I} = \sum_{i\in I} a_i x_i.$$

$$\| T^*\{a_i\}_{i\in I} \|^2 = \| \langle T^*\{a_i\}_{i\in I}, T^*\{a_i\}_{i\in I}\rangle \| = \| \langle \sum_{i\in I} a_i x_i, \sum_{i\in I} a_i x_i\rangle \|$$

$$= \| \sum_{i\in I} a_i\langle x_i, x_i\rangle a_i^* \| \leq (\sum_{i\in I} \| a_i a_i^* \|^2)^{\frac{1}{2}} (\sum_{i\in I} \| \langle x_i, x_i\rangle \|^2)^{\frac{1}{2}} \leq \sqrt{D} \| \{a_i\}_{i\in I} \| .$$

□

DEFINITION 2.5. Two frames $\{x_i : i \in I\}$ and $\{y_i : i \in I\}$ for a Hilbert C^*-module H are woven modular frame for H. If for each $\sigma \subseteq I$, the $\{x_i : i \in \sigma\} \cup \{y_i : i \in \sigma^c\}$ is a modular frame for H.

COROLLARY 2.6. *Let* $\{x_i : i \in I\}$ *be a modular frame and* $Q \in L(H)$ *be bounded below. Then* $\{Qx_i : i \in I\}$ *is a modular frames.*

THEOREM 2.7. *Let* $\{x_i : i \in I\}$, $\{y_i : i \in I\}$ *be woven modular frame with frame bounds* C, D *and* $Q \in L(H)$, *be bounded below by* $m > 0$ *i.e.* $m \| x \| \leq \| Qx \|$ *for every* $x \in H$. *Then* $\{Qx_i : i \in I\}$, $\{Qy_i : i \in I\}$ *are woven modular frame with bounds* $Cm^2, D \| Q \|^2$.

PROOF. Let $S, S^{'}$ be frame operator $\{x_i : i \in I\}$, $\{y_i : i \in I\}$, respectively. For each $\sigma \subseteq I, x \in H$ we have

$$S_\sigma^Q x + S_{\sigma^c}^{'Q} x = \sum_{i\in\sigma}\langle x, Qx_i\rangle Qx_i + \sum_{i\in\sigma^c}\langle x, Qy_i\rangle Qy_i$$

by $A-$linear

$$Q(\sum_{i\in\sigma}\langle x, Qx_i\rangle x_i + \sum_{i\in\sigma^c}\langle x, Qy_i\rangle y_i) = Q(S_\sigma Q^* x + S^{'}_{\sigma^c} Q^* x) =$$

$$Q(S_\sigma + S^{'}_{\sigma^c})Q^* x,$$

therefore $S^{Q}_{\sigma} + S'^{Q}_{\sigma^c} = Q(S_\sigma + S'_{\sigma^c})Q^*$. Since $m^2.I \le Q^*Q, C.I \le S_\sigma + S'_{\sigma^c} \le D.I$, then

$$Cm^2.I \le CQ^*Q \le S_\sigma + S'_{\sigma^c} \le DQ^*Q \le D \parallel Qx \parallel^2 .I,$$

and we have the result. □

Definition 2.8. Let H be a Hilbert A-module. We say that $\{x_i^j : i \in I\}$ for $j = 1, 2, ..., m$ is a woven frame, if sequence $\{x_i^j : i \in I\}$ for $j = 1, 2, ..., m$ be Bessel and there exists a partition $p = \{\sigma_1, \sigma_2, ..., \sigma_m\}$ of I such that $\bigcup_{j=1}^m \{x_i^j : i \in \sigma_j\}$ is a frame.

Theorem 2.9. *Let A be a unital C^*-algebra with identity 1_A, and H be a finitely or countably generated Hilbert A-module. Every woven modular Riesz basis is a module frame.*

Proof. Let $\{x_i^j : i \in I\}$ for $j = 1, 2, ..., m$ be a woven modular Riesz basis for H corresponding to $p = \{\sigma_1, \sigma_2, ..., \sigma_m\}$. Hence $\{x_i^j : i \in I\}$ is a Bessel sequence and $\{x_i^j : i \in \sigma_j, j = 1, 2, ..., m\}$ is an A-modular Riesz basis. Hence there exist an othonormal basis $\{e_i^j : j = 1, 2, ..., m, i \in \sigma_j\}$ and an invertible operator $U^j \in L(\ell^2(A), H)$ such that $x_i^j = U(e_i^j) \qquad (i \in \sigma_j, j = 1, 2, ..., m)$.
We also know that $(U^{-1})^* : \ell^2(A) \to H$ is adjointable and invertable with $(U^*)^{-1} = (U^{-1})^*$ and $\{(U^*)^{-1}(e_i^j) : j = 1, 2, ..., m, i \in \sigma_j\}$ is the unique dual of $\{x_i^j : i \in \sigma_j, j = 1, 2, ..., m\}$ See[], and $\{(U^*)^{-1}(e_i^j) : j = 1, 2, ..., m, i \in \sigma_j\} = \{(U^*)^{-1}U^{-1}(x_i^j) : j = 1, 2, ..., m, i \in \sigma_j\}$. A Riesz basis. Now we define $y_i^j = (U^*)^{-1}(e_i^j)$ for each $i \in I$ and $j = 1, 2, ..., m$ we show that every $\{y_i^j : i \in I\}$ is a Bessel modular sequence. For every $x \in H$

$$\sum_{i\in I} |\langle x, y_i^j\rangle|^2 = \sum_{i\in I} |\langle x, (U^*)^{-1}U^{-1}(x_i^j)\rangle|^2 = \sum_{i\in I} |\langle (U^{-1})^*U^{-1}x, x_i^j\rangle|^2 \le$$

$$B_j\langle (U^{-1})^*U^{-1}x, (U^{-1})^*U^{-1}x\rangle \le B_j \parallel (U^{-1})^* \parallel^2 \parallel U^{-1} \parallel^4 \langle x, x\rangle.$$

Where B_j is a Bessel modular bound of $\{x_i^j : i \in I\}$ and $\{y_i^j : i \in I\}$ is a Bessel modular sequence. □

Corollary 2.10. *Let $\{x_i^j : i \in I\}$ and $\{y_i^j : i \in I\}$ for $j = 1, 2, ..., m$ be a pair of p-woven modular dual Riesz beses, corresponding to partition $p = \{\sigma_1, \sigma_2, ..., \sigma_m\}$. Then*

$$x = \sum_{j=1}^m \sum_{i\in\sigma_j} \langle x, x_i^j\rangle y_i^j = \sum_{j=1}^m \sum_{i\in\sigma_j} \langle x, y_i^j\rangle x_i^j \qquad (x \in H)$$

Acknowledgement

Acknowledgements could be placed at the end of the text but before the references.

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New inequalities for entropy and Tsallis entropy of two accretive operators

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Abstract. The following important inequality for the relative operator inequality have been proven by Raissoli et al.

$$\mathcal{R}(S(A|B)) \geq S(\mathcal{R}A|\mathcal{R}B).$$

We give a reverse inequality to above mentioned inequality under some conditions. we also present some new inequality for sector matrices involving the relative operator entropy.

Keywords: accretive operators, entropy, Tsallis entropy.

AMS Mathematics Subject Classification [2010]: 15A45, 15A60, 47A63

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1. Introduction

If $A,\ B$ are two strictly positive operators in $B(H)$, and $0 \leq \lambda \leq 1$ is a real number, then the relative operator entropy $S(A|B)$ is defined by

$$S(A|B) = A^{\frac{1}{2}} \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$$

Moreover, the Tsallis relative operator entropy is defined by

$$\mathcal{T}_\lambda(A|B) := \frac{A\sharp_\lambda B - A}{\lambda}.$$

It is known that [3, Theorem 5.18]

$$\lim_{\lambda \to 0} \mathcal{T}_\lambda(A|B) = S(A|B).$$

For more details, we refer the reader to [5].
In what follows, let $\mathbb{M}_n(\mathbb{C})$ be the space of all $n \times n$ complex matrices. An operator $A \in B(H)$ is called accretive if in its Cartesian (or Toeplitz) decomposition, $A = \mathcal{R}z + i\mathcal{I}z$,

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$\mathcal{R}z$ is positive, where $\mathcal{R}z = \frac{A+A^*}{2}$, $\mathcal{I}z = \frac{A-A^*}{2}$. We refer the reader to [1,2] as a sample of articles treating this topic. The numerical rang of $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

When talking about accretive matrices, we need to introduce sectorial matrices. A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is said to be sectorial if $W(A) \subset S_\alpha$ for some $0 \leq \alpha < \frac{\pi}{2}$, where S_α denote the sector regions in the complex plane as follows:

$$S_\alpha = \{z \in \mathbb{C} : \mathcal{R}z \geq 0, |\mathcal{I}z| \leq (\mathcal{R}z)\tan\alpha\}.$$

Raissouli et al. [4] defined the weighted geometric mean for two accretive operator $A, B \in B(H)$ by

$$\begin{aligned} A\sharp_\lambda B := &\frac{\sin\lambda\pi}{\pi}\int_0^\infty t^{\lambda-1}\left(A + tB^{-1}\right)^{-1} dt \\ = &\frac{\sin\lambda\pi}{\pi}\int_0^\infty t^{\lambda-1}A(B+tA)^{-1}Bdt. \end{aligned}$$

where $\lambda \in (0,1)$. Furthermore, the Tsallis relative operator entropy of A and B is defined by

$$\mathcal{T}_\lambda(A|B) = \frac{\sin\lambda\pi}{\lambda\pi}\int_0^1 \left(\frac{t}{1-t}\right)^\lambda \left(\frac{A!_tB - A}{t}\right) dt,$$

and

$$S(A|B) = \int_0^1 \frac{A!_tB - A}{t} dt.$$

2. Main results

Raïssouli et al. [4] proved that if $A, B \in B(H)$ are accretive, then

$$\mathcal{R}\left(\mathcal{T}_\lambda(A|B)\right) \geq \mathcal{T}_\lambda(\mathcal{R}A \mid \mathcal{R}B) \tag{1}$$

and

$$\mathcal{R}\left(S(A|B)\right) \geq S(\mathcal{R}A|\mathcal{R}B). \tag{2}$$

where $\lambda \in (0,1)$. In this paper, we obtain a reverse of (2) and some other inequalities.

Theorem 2.1. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subset S_\alpha$ and $0 < \lambda \leq 1$. If there exists a $m > 1$ such that $m\mathcal{R}A \leq \mathcal{R}B$, then there is a $\beta > 1 + \frac{\tan^2\alpha}{\lambda}$ such that*

$$\frac{(\beta-1)\lambda}{(\beta-1)\lambda - \tan^2\alpha} \leq m,$$

and

$$\mathcal{R}(S(A \mid B)) \leq \beta S(\mathcal{R}A \mid \mathcal{R}B).$$

Proof. for $< t \leq 1$ we have

$$\frac{(\beta-1)\lambda}{(\beta-1)\lambda - \tan^2\alpha} \to 1 \ as \ \beta \to \infty$$

therefore

$$\frac{(\beta-1)\lambda}{(\beta-1)\lambda - \tan^2\alpha}\mathcal{R}A \leq m\mathcal{R}A \leq \mathcal{R}B$$

This implies that

$$\begin{aligned}
&\mathcal{R}A!_\lambda \left(\frac{(\beta-1)t}{(\beta-1)t-\tan^2\alpha} \right) \mathcal{R}A \leq \mathcal{R}A!_\lambda \mathcal{R}B \\
&\Rightarrow \frac{(\beta-1)}{\beta-\sec^2\alpha} \mathcal{R}A \leq \mathcal{R}A!_\lambda \mathcal{R}B \\
&\Rightarrow \beta-\sec^2\alpha(\mathcal{R}A!_\lambda \mathcal{R}B) \geq (\beta-1)\mathcal{R}A \\
&\Rightarrow \sec^2\alpha(\mathcal{R}A!_\lambda \mathcal{R}B) - \mathcal{R}A \leq \beta(\mathcal{R}A!_\lambda \mathcal{R}B) - \beta\mathcal{R}A
\end{aligned}$$

therefore

$$\begin{aligned}
\mathcal{R}S(A \mid B) &= \int_0^1 \frac{\mathcal{R}(A!_\lambda B) - \mathcal{R}A}{t} dt \\
&\leq \int_0^1 \frac{\sec^2\alpha(\mathcal{R}A!_\lambda \mathcal{R}B) - \mathcal{R}A}{t} dt \\
&\leq \int_0^1 \frac{\sec^2\alpha(\mathcal{R}A!_\lambda \mathcal{R}B) - \mathcal{R}A}{t} dt \\
&= \beta \int_0^1 \frac{\mathcal{R}A!_\lambda \mathcal{R}B) - \mathcal{R}A}{t} dt \\
&= \beta S(\mathcal{R}A \mid \mathcal{R}B).
\end{aligned}$$

□

THEOREM 2.2. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subset S_\alpha$, if there exists a $m > 1$ such that $m\mathcal{R}A \leq \mathcal{R}B$. Then, for every positive unital linear map ϕ,*

$$\mathcal{R}\phi(S(A \mid B)) \leq \beta \mathcal{R}S(\phi(A) \mid \phi(B)).$$

PROOF. by Theorem 2.1 we have

$$\begin{aligned}
\mathcal{R}\phi(S(A \mid B)) &= \phi(\mathcal{R}S(A \mid B)) \\
&\leq \beta\mathcal{R}S(\phi(A) \mid \phi(B)) \\
&\leq \beta S(\phi(\mathcal{R}A) \mid \phi(\mathcal{R}B)) \\
&= \beta S(\mathcal{R}\phi(A) \mid \mathcal{R}\phi(B)) \\
&\leq \beta\mathcal{R}S(\phi(A) \mid \phi(B)).
\end{aligned}$$

□

THEOREM 2.3. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subset S_\alpha$ and $0 < \lambda \leq 1$, if there exists a $m > 1$ such that $m\mathcal{R}A \leq \mathcal{R}B$, then*

$$\mathcal{R}(\mathcal{T}_\lambda(A_1 \mid B_1) + \mathcal{T}_\lambda(A_2 \mid B_2)) \leq \beta\mathcal{R}\mathcal{T}_\lambda(A_1 + A_2 \mid B_1 + B_2).$$

so

$$\beta\mathcal{R}S(A_1 + A_2 \mid B_1 + B_2) \geq \mathcal{R}(S(A_1 \mid B_1) + S(A_2 \mid B_2))$$

PROOF. by Theorem 2.1 we have

$$\begin{aligned}
\mathcal{R}\mathcal{T}_\lambda(A_1 + A_2 \mid B_1 + B_2) &\geq \mathcal{T}_\lambda(\mathcal{R}(A_1 + A_2) \mid \mathcal{R}(B_1 + B_2)) \\
&= \mathcal{T}_\lambda(\mathcal{R}A_1 + \mathcal{R}A_2 \mid \mathcal{R}B_1 + \mathcal{R}B_2) \\
&= \frac{1}{\lambda}((\mathcal{R}A_1 + \mathcal{R}A_2)\sharp_\lambda(\mathcal{R}B_1 + \mathcal{R}B_2) - (\mathcal{R}A_1 + \mathcal{R}A_2))
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{\lambda}(\mathcal{R}A_1 \sharp_\lambda \mathcal{R}B_1 - \mathcal{R}A_1 + \mathcal{R}A_2 \sharp_\lambda \mathcal{R}B_2 - \mathcal{R}A_2) \\
&= \mathcal{T}_\lambda(\mathcal{R}A_1 \mathcal{R}B_1) + \mathcal{T}_\lambda(\mathcal{R}A_2 \mathcal{R}B_2) \\
&\geq \beta^{-1}\mathcal{R}\mathcal{T}_\lambda(A_1 \mid B_1) + \beta^{-1}\mathcal{R}\mathcal{T}_\lambda(A_2 \mid B_2) \\
&= \beta^{-1}\mathcal{R}(\mathcal{T}_\lambda(A_1 \mid B_1) + \mathcal{T}_\lambda(A_2 \mid B_2))
\end{aligned}
$$

□

THEOREM 2.4. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subset S_\alpha$ and $0 < \lambda \leq 1$. If there exists a $m > 1$ such that $m\mathcal{R}A \leq \mathcal{R}B$, then*

$$\beta\lambda\mathcal{R}\mathcal{T}_\lambda(A \mid B) \leq \mathcal{R}(A \sharp_\lambda B).$$

PROOF. By Theorem 2.1 we have

$$
\begin{aligned}
\mathcal{R}\mathcal{T}_\lambda(A \mid B) &\leq \beta^{-1}\mathcal{T}_\lambda(\mathcal{R}A \mid \mathcal{R}B) \\
&\leq \frac{1}{\beta\lambda}(\mathcal{R}A \sharp_\lambda \mathcal{R}B) \\
&\leq \frac{1}{\beta\lambda}\mathcal{R}(A \sharp_\lambda B).
\end{aligned}
$$

□

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On the solvability of three-pair networks

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Abstract. This paper presents a property to diagnose the solvability of a class of three-pair networks using region decomposition method. The proposed property considers basic region graph which the topological structure of it is more simple than the original network.

Keywords: three-pair networks, region decomposition, solvability.

AMS Mathematics Subject Classification [2010]: 68Q06, 68M10, 94A05.

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1. Introduction

A three-pair network is considered as a directed, acyclic, finite graph $G = (V, E, S, T)$, where V is the vertex set, E is the edge set, $S = \{s_1, s_2, s_3\}$ is source set and $T = \{t_1, t_2, t_3\}$ is sink set. Each source s_i generates a message $X_i \in F$ and each sink t_i wants to obtain the message X_i, where F is a finite field. We assume that each link has the unit capacity and can carry one symbol in each use. If each source s_i can send a unit rate of information flow to t_i, for each $i \in \{1, 2, 3\}$, then, the three-pair network is solvable. It is assumed that each link $e \in E$ has the unit capacity which means $C(e) = 1$, where $C(e)$ is the capacity of edge e. We assume that each source s_i has an imaginary incoming link, called X_i source link.

By [1], Cai and Han presented a sufficient and necessary condition to diagnose the solvability of three-pair networks with common bottleneck links. Fragouli and Soljanin proposed a method called information flow decomposition which reduces the complexity of network code design for single session multicast networks [3]. For non-multicast networks, namely three-pair networks, there exists a few results. In [4–6], the region decomposition method is presented to diagnose the solvability of non-multicast networks.

In this paper, we consider a class of three-pair networks and present a property to characterize the solvability of them using region decomposition method. Our approach concentrates on a network with a topological structure more simple than the original network.

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1.1. Three-pair networks. In this part, we present some properties of three-pair networks. First, we state the following definition:

Definition 1.1. [1].

1. A communication network with $S = \{s_i\}$ and $T = \{t_j\}$ is called a point to point network and is denoted by $G_{i,j} = (V, E, s_i, t_j)$.
2. Consider the point to point network $G_{i,j}$. If $V = A \cup (V \setminus A)$ be a vertex partition such that $s_i \in A$ and $t_j \in V \setminus A$, then the $s_i - t_j$ cut C is the collection of all edges from A to $V \setminus A$. The capacity of C is defined as $\sum_{e \in C} C(e)$. The minimum cut of $G_{i,j}$ is a cut with minimum capacity.
3. For the point to point network $G_{i,j}$, the $A_{i,j}$-set is defined as the union of all minimum cuts of $G_{i,j}$.

In this paper, we consider a class of the three-pair networks with

$$A(1, 2, 3) = A_{1,1} \cap A_{2,2} \cap A_{3,3} \neq \emptyset,$$

and, for each distinct $i, j \in \{1, 2, 3\}$, we have

$$A_{i,j} \cap A(1, 2, 3) = \emptyset.$$

The network G' shown in Fig. 1(a) is an example of such networks that first presented by Dougherty et al. [2]. The following definition presents a property to characterize the solvability of G'.

Definition 1.2. [2]. For a network containing messages a, b, c and labeled edges w, x, y, z, we say that a code over an alphabet $\mathcal{A}$ has property P if there exist permutations $\pi_1, \ldots, \pi_6$ of $\mathcal{A}$ and a mapping $\oplus : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that $(\mathcal{A}, \oplus)$ is an Abelian group and

$$\begin{aligned} w &= \pi_4(\pi_1(a) \oplus \pi_2(b)), \\ x &= \pi_5(\pi_1(a) \oplus \pi_3(c)), \\ y &= \pi_6(\pi_2(b) \oplus \pi_3(c)), \\ z &= \pi_1(a) \oplus \pi_2(b) \oplus \pi_3(c). \end{aligned}$$

Proposition 1.3. *[2]. A code over an alphabet $\mathcal{A}$ is a solution for network G' if and only if it satisfies Property P.*

By Proposition 1.3, network G' shown in Fig. 1(a) is solvable.

1.2. Region decomposition method. In this section, we briefly introduce some definitions and notations about the region decomposition of a given graph.

Definition 1.4. [4]. Let R be a non-empty subset of E. It is called a region of G if there is an $e_l \in R$ such that for any $e \in R \setminus \{e_l\}$, R contains an incoming link of e. If E is partitioned into mutually disjoint regions, say $R_1, R_2, \cdots, R_N$, then, set $D = \{R_1, R_2, \cdots, R_N\}$ ia a *region decomposition* of G.

The edge e_l in Definition 1.4 is called the *leader* of R and is denoted as $e_l = lead(R)$. A region R is called the X_i source region (or a source region for short) if lead(R) is the X_i source link.

Definition 1.5. [4]. Let D be a region decomposition of G. The region graph of D is a directed, simple graph with vertex set D and edge set ε_D, where ε_D is the set of all ordered pairs (R', R) such that R' contains an incoming link of lead(R).

We use $RG(D)$ to denote the region graph of D. Let D^{**} be a region decomposition of G. D^{**} is called a basic region decomposition of G if
(1) For any $R \in D^{**}$ and any $e \in R \setminus \{lead(R)\}$, $In(e) \subseteq R$.
(2) Each non-source region R in D^{**} has at least two parents in $RG(D^{**})$.
Notations D^{**} and $RG(D^{**})$ are used to denote the basic region decomposition and basic region graph of G, respectively. Fig. 1(b) shows the basic region graph of network G' depicted in Fig. 1(a). The following theorem plays an important role in this paper.

THEOREM 1.6. *[4]. Let D^{**} be the basic region decomposition of G. Then G is solvable if and only if $RG(D^{**})$ is feasible.*

DEFINITION 1.7. [6]. Let θ be a subset of D^{**}. The super region $reg(\theta)$ is a subset of D^{**} which is defined recursively as follows:
(1) $\theta \subseteq reg(\theta)$.
(2) For all $R \in D^{**}$, if $In(R) \subseteq reg(\theta)$, then $R \in reg(\theta)$.

Define $reg^{\circ}(\theta) = reg(\theta) \setminus \theta$. Moreover, if $\theta = \{R_1, \cdots, R_k\}$, then denote $reg(\theta) = reg(R_1, \cdots, R_k)$ and $reg^{\circ}(\theta) = reg^{\circ}(R_1, \cdots, R_k)$.

REMARK 1.8. [4]. Let G be a three-pair network and $RG(D^{**})$ be the region graph of it. If $\widetilde{C} = \{d_R \in F^3; R \in D^{**}\}$ be a linear code of $RG(D^{**})$ and $\emptyset \neq \theta \subseteq D^{**}$, then $d_R \in < d_{R'}; R' \in \theta >$, for all $R \in reg(\theta)$.

2. Main results

The following theorem plays an important role in the obtained result.

THEOREM 2.1. *Let $\widetilde{C}$ be a linear code on $RG(D^{**})$, then $\widetilde{C}$ is a solution for $RG(D^{**})$ if $RG(D^{**})$ has property $\mathcal{I}'$, where is denoted as follows:*
($\mathcal{I}'$): Network $RG(D^{**})$ contains regions R^*, R_1, R_2 and R_3 such that $R^* \in reg^{\circ}(S_1, S_2, S_3)$, $R_1 \in reg^{\circ}(S_2, S_3)$, $R_2 \in reg^{\circ}(S_1, S_3)$ and $R_3 \in reg^{\circ}(S_1, S_2)$, where S_i is a source region, for $i \in \{1, 2, 3\}$.

PROOF. Supposing that $\widetilde{C}$ is a linear code on $RG(D^{**})$ and $RG(D^{**})$ has property $\mathcal{I}'$. Then, by Remark 1.8, we conclude that

$$lead(R^*) \in < a, b, c >, \ lead(R_1) \in < b, c >, \ lead(R_2) \in < a, c >,$$

and $lead(R_3) \in < a, b >$, where a, b and c are generated at source regions S_1, S_2 and S_3, respectively. So, by letting $z = lead(R^*)$, $w = lead(R_3)$, $x = lead(R_2)$ and $y = lead(R_1)$ and by Definition 1.2, we conclude that $RG(D^{**})$ has Property P. Thus, by Proposition 1.3, $RG(D^{**})$ is solvable. □

COROLLARY 2.2. *Consider three-pair network G'. If the basic region graph of G' satisfies Property $\mathcal{I}'$, then G' is solvable.*

PROOF. The basic region graph of G' satisfies Property $\mathcal{I}'$, by Theorem 2.1, $RG(D^{**})$ is solvable. Then, by Theorem 1.6, the network G' is solvable. □

3. Figures

4. Conclusion

This paper presents a property to diagnose the solvability of a class of three-pair networks using region decomposition method. This property concentrates on a network with a topological structure more simple than original network.

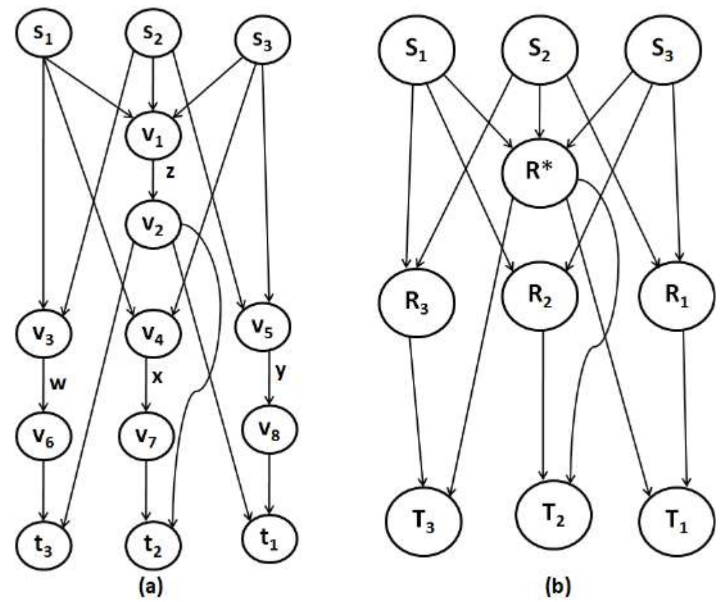

FIGURE 1. (a) The network G' with Property P. (b) The basic region graph of G' with Property $\mathcal{I}'$.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
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۸ تا ۱۱ شهریور ۱۴۰۰

Best proximity point for $(\alpha - \psi)$-contractive mapping

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Abstract. **In this paper we consider the class of $\alpha - \psi$-contractive mapping, and prove that the existence of best proximity point results for such mapping, can be concluded from the corresponding result in fixed point theory**.

Keywords: Best proximity point, Fixed point, $\alpha - \psi$-contractive mapping, α-admissible, Non-self-mapping

AMS Mathematics Subject Classification [2010]: 54H25, 47H10

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1. Introduction

Let (X, d) be a metric space and (A, B) be a pair of nonempty subsets of X. Consider a non-self-mapping $T : A \to B$. An element $x^\star \in A$ is said to be a best proximity point for the mapping T if $d(x^\star, Tx^\star) = d(A, B)$, where $d(A, B) := inf\{d(x, y) : x \in A, y \in B\}$. If the mapping under consideration is a self-mapping, we note that this best proximity point reduces to a fixed point.

In 2013, M.Jleli, B.Samet [2] introduced a new concept of $\alpha - \psi$-contractive type mapping. recently, M.Gabeleh, J.Markin [4] introduced the notion of p-proximal contraction and surveyed best proximity point theorems for such class of non self mappings in metric spaces. Existence, uniqueness and convergence of best proximity points for various classes of non-self mappings can be considered as generalizations of fixed point theorems which has recently attracted the attention of many authors. Clearly, the interest for best proximity points is real when the mapping under investigation has no fixed-point. Fixed point theory is one of the most powerful tools in nonlinear analysis, there are a lot of results on this topic.

In this paper we show that the main conclusion of [2] is a straightforward consequence of the same fixed point result. To this end, we need the following notions and notations, which will be used in the sequel.

Definition 1.1 ([2]). Let A and B be two nonempty subsets of a metric space (X, d). We denote by A_0 and B_0 the following sets:
$A_0 = \{x \in A : d(x, y) = d(A, B), \text{for some } y \in B\}$,

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$B_0 = \{y \in B : d(x, y) = d(A, B), \text{for some } x \in A\}$,
where $d(A, B) := inf\{d(x, y) : x \in A, y \in B\}$.

Definition 1.2 ([3]). Let (A,B) be a pair of nonempty subset of a metric space (X, d) with $A_0 \neq \varnothing$. Then the pair (A, B) is said to have the P- property if and only if

$$\left.\begin{aligned} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{aligned}\right\} \Rightarrow d(x_1, x_2) = d(y_1, y_2) \tag{1}$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Definition 1.3 ([2]). Let $T : A \to B$ and $\alpha : A \times A \to [0, +\infty)$. We say that T is α-proxsimal admissible if

$$\left.\begin{aligned} \alpha(x_1, x_2) \geq 1 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{aligned}\right\} \Rightarrow \alpha(u_1, u_2) \geq 1 \tag{2}$$

for all $x_1, x_2, u_1, u_2 \in A$.

Lemma 1.4. *For every function $\psi : [0, +\infty) \to [0, +\infty)$ the following hold: if ψ is nondecreasing, then for each $t > 0$, $\lim_{n \to +\infty} \psi^n(t) = 0$ implies $\psi(t) < t$.*

Definition 1.5 ([1]). Let (X, d) be a metric space. A self mapping $T : X \to X$ is said to be an $\alpha - \psi$ contractive mapping, if there exist two function $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that $\alpha(x, y)d(Tx, Ty) \leq \psi d(x, y)$, $\forall x, y \in X$.

Definition 1.6 ([1]). Let $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ be two functions. The map T is an α-admissible mapping if, for all $x, y \in X$, $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$.

Theorem 1.7 ([1]). *Let (X, d) be a complete metric space and $T : X \to X$ be an $\alpha - \psi$-contractive mapping satisfying the following conditions:*
(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.
Then T has a fixed point, that is, there exists $x^\star \in X$ such that $Tx^\star = x^\star$.

Theorem 1.8 ([2]). *Let (A, B) be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0, +\infty)$ and $\psi \in \Psi$. Suppose that $T : A \to B$ is a non-self-mapping satisfying the following condition:*
(i) $T(A_0) \subset B_0$ and (A, B) satisfies the P- property;
(ii) T is α- proximal admissible;
(iii) there exists an element x_0 and x_1 in A_0 such that $d(x_1, Tx_1) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$
(iv) T is a continuous $(\alpha - \psi)$- proximal contraction.
Then, there exists an alement $x^\star \in A_0$, such that $d(x^\star, Tx^\star) = d(A, B)$.

Theorem 1.9. *Theorem 1.8 is a straightforward consequence of Theorem 1.7*

Proof. From condition (iii) there exist elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$. since T satisfies the P- property, $x_1 \in A$ is unique. To see that x_1 is unique assume there is a point $x_2 \in A_0$, such that $d(x_2, Tx_0) = d(A, B)$
then $d(x_1, x_2) = d(Tx_0, Tx_0)$.
This implies that $x_1 = x_2$.

Put $x_1 = S(x_0)$. Hence $S : A_0 \to A_0$ is a mapping satisfies $d(Sx_0, Tx_0) = d(A, B)$. for all $x \in A_0$. We now claim that the $S : A_0 \to A_0$ is $\alpha - \psi$ contractive mapping. Indeed, for any $x, y \in A_0$, we have

$$\left.\begin{aligned} d(Sx, Tx) = d(A, B) \\ d(Sy, Ty) = d(A, B) \end{aligned}\right\} \Rightarrow \alpha d(Sx, Sy) = d(Tx, Ty), \tag{3}$$

and again by using the fact that T is $\alpha - \psi$- proximal contraction we conclud that $\alpha(x, y)d(Sx, Sy) = \alpha(x, y)d(Tx, Ty) \leq \psi d(x, y)$. This yields S is $\alpha - \psi$- proximal contraction.
Since T is α- proximal admissible, we have

$$\left.\begin{aligned} \alpha(x, y) \geq 1 \\ d(Sx, Tx) = dist(A, B) \\ d(Sy, Sy) = d(A, B) \end{aligned}\right\} \Rightarrow \alpha(Sx, Sy) \geq 1, \tag{4}$$

for all $x, y \in A$. This yields S is α-admissible. By (iii) there exists an element x_0 and x_1 in A_0 such that $d(x_1, Tx_1) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$. $S : A_0 \to A_0$ is a mapping and since T is a continuous and according P- property, we have $d(Sx_0, Tx_0) = dist(A, B)$, $\alpha(x_0, sx_)) \geq 1$. It now follows from Theorem 1.7 that the mapping S has a unique fixed point z in the complete metric space A_0. Put $z = S(z)$ and therefore,

$$d(z, Tz) = d(Sz, Tz) = d(A, B),$$

which ensures that T has a unique best proximity point. $\square$

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Faculty of Mathematics and Computer
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Kerman, Iran
30 August - 02 September 2021

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Lower and upper bounds for varentropy and varextropy

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Abstract. In this paper, we provide bounds for the variance of a function of random variables in terms of Chernoff-type inequalities. In fact, we obtain bounds for the varentropy and varextropy of random variables, in terms of measures of reliability and information theory. Moreover, we obtain an equivalence relation for cumulative residual entropy via covariance identity.

Keywords: entropy, extropy, varentropy, varextropy, Chernoff inequality

AMS Mathematics Subject Classification [2010]: 60E15

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1. Introduction

During the last four decades, several papers have been prepared regarding upper bounds for functions of the random variables, based on the Chernoff type inequality. Let Z be a standard normal random variable and $g : \mathbb{R} \to \mathbb{R}$ any absolutely continuous function with derivative g', such that $\mathrm{E}[g'(Z)] < \infty$. [1], using Hermite polynomials, proved that,

$$\mathrm{Var}[g(Z)] \leq \mathrm{E}[g'(Z)]^2. \tag{1}$$

The equality in (1) holds if and only if g is a linear function. [2] and [3] obtained upper and lower bounds for the variance of functions of arbitrary random variables. [4] established, if there are functions $h(x)$ and $z(x)$ such that

$$\mathrm{Cov}(h(X), g(X)) = \mathrm{E}(z(X)g'(X)), \tag{2}$$

holds for every differentiable g, then $h(x)$, $z(x)$ and the density f are related through

$$z(x) = \frac{1}{f(x)} \int_a^x (\mathrm{E}[h(X)] - h(t))f(t)dt. \tag{3}$$

[5] showed that, under the conditions of identity (2), for every absolutely continuous function $h(x)$ with $h'(x) > 0$,

$$\mathrm{Var}[g(X)] \geq \frac{\mathrm{E}^2[z(X)g'(X)]}{\mathrm{E}[z(X)h'(X)]}, \tag{4}$$

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with equality if and only if $g(x) = c_1 h(x) + c_2$, where

$$z(x)f(x) = \int_a^x (\mathrm{E}[h(X)] - h(t))f(t)dt.$$

They, also established under the preceding conditions, for every absolutely continuous function $h(x)$ with $h'(x) > 0$ the following inequality:

$$\mathrm{Var}[g(X)] \leq \mathrm{E}\left[\frac{z(X)}{h'(X)}(g'(X))^2\right], \tag{5}$$

holds with equality if and only if $g(x) = c_1 h(x) + c_2$.

Let X be a continuous random variable with density function f, shannon entropy of X is defined by

$$H(X) = -\int_{-\infty}^{\infty} f(x) \ln f(x) dx. \tag{6}$$

In continuous case, $H(X)$ is also referred to as the differential entropy.

It should be noted that according to, the variance entropy (varentropy) of a random variable X is defined as

$$\mathrm{Var}(-\ln f(X)) = \int_{-\infty}^{\infty} f(x)[\ln f(x)]^2 dx - \left[\int_{-\infty}^{\infty} f(x) \ln f(x) dx\right]^2. \tag{7}$$

A uncertainty measure, the cumulative residual entropy (CRE), is defined through

$$\mathscr{E}(X) = -\int_0^{\infty} \overline{F}(x) \ln \overline{F}(x) dx. \tag{8}$$

The notion of entropy is recently entwined with a complementary dual measure, designated as extropy, by [6]. The extropy of the random variable X is defined as:

$$J(X) = -\frac{1}{2}\int_{-\infty}^{\infty} f^2(x) dx. \tag{9}$$

In some situations, one may have two random variables with the same extropy; then, this problem leads to the well-known question “Which of the extropies is the most appropriate criterion for measuring the uncertainty?”. For example, the extropy values of standard uniform and an exponential distribution with the parameter 2 are both equal to $-\frac{1}{2}$. This question motivates one to investigate the variance of $-\frac{1}{2}f(X)$, which is called varextropy. Varextropy measure indicates how the information content is scattered around the extropy.

The varextropy can be defined as

$$VJ(X) := Var[-\frac{1}{2}f(X)] = \frac{1}{4}E[f^2(X)] - J^2(X). \tag{10}$$

2. Main results

In this section, we give bounds for the variance of the function of given random variables in text in terms of measurses of reliability and information theory.

In the following, we first give a few examples to illustrate the varextropy for random variables from some distributions.

Example 2.1. i) If X follows the Gamma distribution with probability density function

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \ x > 0,$$

then, a direct computation yields

$$VJ(X) = \frac{\beta^2}{4[\Gamma(\alpha)]^3}\left\{\frac{\Gamma(3\alpha-2)}{3^{3\alpha-2}} - \frac{[\Gamma(2\alpha-1)]^2}{\Gamma(\alpha)2^{4\alpha-2}}\right\}.$$

ii) If X follows Cramer distribution with density function

$$f(x) = \frac{\theta}{2(1+\theta|x|)^2}, \quad -\infty < x < \infty, \ \theta > 0,$$

then we have

$$VJ(X) = \frac{\theta^2}{180},$$

and

$$\mathrm{Var}[-\ln f(X)] = 4.$$

In this case, $\mathrm{Var}[-\ln f(X)]$ does not depend on the parameter and $VJ(X) < \mathrm{Var}[-\ln f(X)]$ for $\theta < 12\sqrt{5}$.

In the following, a lower bound for $VJ(X)$ based on Chernoff's inequality is given.

Proposition 2.2. *Let X be an absolutely continuous non-negative random variable. If g is an absolutely continuous function with derivative g' then*

$$\mathrm{Var}[g(X)] \geq 12\,\mathrm{E}^2[z(X)g'(X)]. \tag{11}$$

The equality holds if and only if $g(x)$ is a linear function of $F(x)$.

Proof. By using inequality (4), let $h(x) = F(x)$, then $h'(x) = f(x)$ and

$$z(x)f(x) = \int_0^x (E[F(X)] - F(t))f(t)dt = \int_0^x (\frac{1}{2} - F(t))f(t)dt = \frac{1}{2}F(x)(1-F(x)). \tag{12}$$

Now since

$$E[z(X)h'(X)] = \frac{1}{2}\int_0^\infty f(x)F(x)(1-F(x))dx = \frac{1}{12},$$

thus

$$\mathrm{Var}[g(X)] \geq 12E^2(z(X)g'(X)). \tag{13}$$

□

Remark 2.3. In the Proposition 2.2, if $g(x) = -\frac{1}{2}f(x)$ then, we obtain a lower bound for varextropy as follows:

$$VJ(X) \geq \frac{3}{4}\left(\int_0^\infty F(x)(1-F(x))f'(x)dx\right)^2. \tag{14}$$

The equality holds if and only if F has two-parameter exponential distribution.

In inequality (14), if $f'(x) = c = constant$, then

$$VJ(X) \geq \frac{3c^2}{4}(\xi_2(X))^2,$$

where $\xi_2(X) = \int_0^\infty(\overline{F}(x) - \overline{F}^2(x))dx$.

Lemma 2.4. *If $Y = \phi(X)$ is an increasing differentiable function, then*

$$\xi_\alpha(Y) = \frac{1}{\alpha-1}\int_{\phi(0)}^\infty (\overline{F}(x) - (\overline{F}(x))^\alpha)\phi'(x)dx.$$

Now, in inequality (14), by using Lemma (2.4) we can conclude that if f is an increasing function and $f(0)=0$ then $VJ(X)\geq \frac{3}{4}(\xi_2(Y))^2$ where $Y=f(X)$.

EXAMPLE 2.5. If X follows a power distribution with parameter $\alpha>1$, i.e., $f(x)=\alpha x^{\alpha-1}$, $x\in(0,1)$, then, we have

$$\int_0^1 (x^\alpha - x^{2\alpha})\alpha(\alpha-1)x^{\alpha-2}dx = \frac{\alpha^2(\alpha-1)}{(2\alpha-1)(3\alpha-1)},$$

hence $VJ(X)\geq \frac{3}{4}\frac{\alpha^4(\alpha-1)^2}{(2\alpha-1)^2(3\alpha-1)^2}$.

Now, in equality (2), for a non-negative random variable X, if we take $h(x)=\Lambda(x)=-\ln\overline{F}(x)$ and $g(x)=x$, then

$$Cov(X,\Lambda(X)) = E[z(X)]. \tag{15}$$

On the other hand, since

$$z(x)f(x)=\int_0^x (E[\Lambda(X)]-\Lambda(t))dt = -\overline{F}(x)\ln\overline{F}(x),$$

hence

$$Cov(X,\Lambda(X)) = -\int_0^\infty \overline{F}(x)\ln\overline{F}(x)dx = \mathscr{E}(X).$$

Moreover, it can be shown that $\mathscr{E}(X)=E[m(X)]$, therefore $Cov(X,\Lambda(X))=E[m(X)]$.

PROPOSITION 2.6. *Let X be an absolutely continuous non-negative random variable. Then*

$$VJ(X)=\mathrm{Var}\left[-\frac{1}{2}f(X)\right]\leq \frac{1}{8}E\left[\overline{F}(X)(1-\overline{F}(X))\eta^2(X)\right], \tag{16}$$

where $\eta(x)=\frac{-f'(x)}{f(x)}$ is eta function.

The equality holds if and only if F has two-parameter exponential distribution.

PROOF. The proof similar to that of Proposition 2.2. $\square$

EXAMPLE 2.7. Let X have a Rayleigh distribution with the probability density function $f_X(x)=2xe^{-x^2}$, $x>0$. Then by using inequality (16), we obtain

$$VJ(X)\leq \int_0^\infty 2xe^{-2x^2}(1-e^{-x^2})(1/x-2x)^2dx = \ln(\frac{3}{2})-\frac{1}{9}.$$

PROPOSITION 2.8. *Let X be an absolutely continuous non-negative random variable. Then*

$$Var[-\ln f(X)]\leq \frac{1}{2}E\left[\frac{(\eta(X))^2}{r(X)\tilde{r}(X)}\right]. \tag{17}$$

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A variational approach for some singular elliptic problems

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Abstract. By using variational methods (a critical point result for differentiable functionals), we establish the existence of infinitely many weak solutions for some singular elliptic problems involving a p-Laplace operator, subject to Dirichlet boundary conditions in a smooth bounded domain in $\mathbb{R}^N$. A concrete example is presented to illustrate the main result.

Keywords: singular problem, p-Laplace operator, variational methods, critical point

AMS Mathematics Subject Classification [2010]: 35J35, 35J60

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1. Introduction

We present some recent results on some variational problems. More precisely, we deal with the following nonlinear Dirichlet boundary-value problem

$$
\begin{cases} -\Delta_p u + \dfrac{|u|^{q-2}u}{|x|^q} = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}
$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplace operator, Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ containing the origin and with smooth boundary $\partial\Omega$, $1 < q < N < p$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function.

Denote by X the space $W_0^{1,p}(\Omega)$ endowed with the norm

$$
\|u\| := \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p}.
$$

Also, let $\|\cdot\|_1$ denotes the usual norm of $L^1(\Omega)$; i.e.,

$$
\|u\|_1 := \int_\Omega |u(x)| dx.
$$

We recall classical Hardy's inequality, which says that

$$
\int_\Omega \frac{|u(x)|^q}{|x|^q} dx \leq \frac{1}{H} \int_\Omega |\nabla u(x)|^q dx, \qquad (\forall u \in X), \tag{2}
$$

*Speaker. Email address: hadjian83@gmail.com, a.hadjian@ub.ac.ir where $H := (\frac{N-q}{q})^q$; see, for instance, the paper [2].

Let us define $F(x,\xi) := \int_0^\xi f(x,t)dt$, for every (x,ξ) in $\Omega \times \mathbb{R}$. Moreover we introduce the functional $I_\lambda : X \to \mathbb{R}$ associated with (1),

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u),$$

for every $u \in X$, where

$$\Phi(u) := \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx + \frac{1}{q}\int_\Omega \frac{|u(x)|^q}{|x|^q}dx, \qquad \Psi(u) := \int_\Omega F(x,u(x))dx,$$

for every $u \in X$. By standard arguments, one has that Φ is well defined (by Hardy's inequality), Gâteaux differentiable and sequentially weakly lower semicontinuous, and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$ given by

$$\Phi'(u)(v) = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx + \int_\Omega \frac{|u(x)|^{q-2}}{|x|^q}u(x)v(x)dx,$$

for every $v \in X$ and clearly Φ is coercive. It is easy to prove that Φ is strongly continuous. On the other hand, standard arguments show that Ψ is well defined and continuously Gâteaux differentiable functional whose Gâteaux derivative

$$\Psi'(u)(v) = \int_\Omega f(x,u(x))dx,$$

for every $v \in X$, is a compact operator from X to the dual X^*.

Fixing the real parameter λ, a function $u : \Omega \to \mathbb{R}$ is said to be a weak solution of (1) if $u \in X$ and

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx + \int_\Omega \frac{|u(x)|^{q-2}}{|x|^q}u(x)v(x)dx - \lambda\int_\Omega f(x,u(x))v(x)dx = 0,$$

for every $v \in X$. Hence, the critical points of I_λ are exactly the weak solutions of (1).

Our main tool to investigate the existence of infinitely many solutions for the problem (1) is the classical Ricceri's variational principle [3, Theorem 2.5], which we now recall.

Theorem 1.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly continuous. For every $r > \inf_X \Phi$, put*

$$\varphi(r) := \inf_{\Phi(u)<r} \frac{\left(\sup_{\Phi(v)<r}\Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r\to+\infty}\varphi(r), \qquad \text{and} \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+}\varphi(r).$$

Then, if $\gamma < +\infty$, then for each $\lambda \in]0, 1/\gamma[$, the following alternative holds: either

(b$_1$) *I_λ possesses a global minimum, or*

(b$_2$) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n\to+\infty}\Phi(u_n) = +\infty.$$

2. Main results

Put

$$k := \sup_{u \in X,\, u \neq 0} \left(\frac{\max_{x \in \bar{\Omega}} |u(x)|}{\|u\|} \right). \tag{3}$$

Since the embedding $X \hookrightarrow C(\bar{\Omega})$ is compact, one has $k < +\infty$. Fix $x_0 \in \Omega$ and $D > 0$ such that $B(x_0, D) \subset \Omega$ and $\overline{B(x_0, D)}$ not containing the origin, where $B(x_0, D)$ denotes the ball with center at x_0 and radius D.

Put

$$\omega := \frac{1}{p} \left[\left(\frac{2}{D}\right)^p m \left(D^N - \left(\frac{D}{2}\right)^N \right) \right], \tag{4}$$

$$\theta := \frac{1}{q} \left[\left(\frac{2}{D}\right)^q m \left(D^N - \left(\frac{D}{2}\right)^N \right) \right], \tag{5}$$

and

$$\alpha := \int_{B(x_0, \frac{D}{2})} \frac{1}{|x|^q} dx, \qquad \beta := \left(\frac{2}{D}\right)^q \int_{B(x_0, D) \setminus B(x_0, \frac{D}{2})} \frac{(D - |x - x_0|)^q}{|x|^q} dx,$$

where $m := \frac{\pi^{N/2}}{\Gamma(1 + \frac{N}{2})}$. Here Γ is the Gamma function defined by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1} e^{-z} dz \qquad (\forall t > 0).$$

Put

$$A := \liminf_{\xi \to +\infty} \frac{\|l_\xi\|_1}{\xi^{p-1}},$$

and

$$B := \limsup_{\xi \to +\infty} \frac{\int_{B(x_0, \frac{D}{2})} F(x, \xi) dx}{\xi^p},$$

where $l_\xi \in L^1(\Omega)$ satisfies condition (C_3) on $f(x,t)$ for every $\xi > 0$.

Our main result is the following.

Theorem 2.1. *Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function such that*

(i) *$F(x,t) \geq 0$ for every $(x,t) \in \Omega \times \mathbb{R}^+$;*

(ii) *$A < \frac{1}{p\omega k^p} B$, where k and ω are given by (3) and (4), respectively.*

Then, for every $\lambda \in \Lambda := \left] \frac{\omega}{B}, \frac{1}{pk^p A} \right[$, the problem (1) admits a sequence of weak solutions which is unbounded in X.

Sketch of Proof. Fix $\lambda \in \left] \frac{\omega}{B}, \frac{1}{pk^p A} \right[$. Our aim is to apply Theorem 1.1 with $X := W_0^{1,p}(\Omega)$ and where Φ and Ψ are the functionals introduced in Section 1. As seen before, the functionals Φ and Ψ satisfy the regularity assumptions requested in Theorem 1.1. Now, we look on the existence of critical points of the functional $I_\lambda := \Phi - \lambda\Psi$ in X. To this end, we first prove that $\gamma < +\infty$. Further, we can establish that the functional I_λ is unbounded from below.

Therefore, owing to Theorem 1.1, the functional I_λ admits an unbounded sequence $\{u_n\} \subset X$ of critical points. Then, the problem (1) admits a sequence of weak solutions which is unbounded in X.

Corollary 2.2. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Assume that condition* (i) *of Theorem 2.1 holds. Further, require that*

(iii) $A < \frac{1}{pk^p}$ *and* $B > \omega$, *where k and ω are given by* (3) *and* (4), *respectively.*

Then the following problem

$$\begin{cases} -\Delta_p u + \dfrac{|u|^{q-2}u}{|x|^q} = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

admits a sequences of weak solutions which is unbounded in X.

Example 2.3. Let $r > 0$ be a real number and $\{t_n\}$, $\{s_n\}$ be two strictly increasing sequences of real numbers that defined by induction $t_1 = r$, $s_1 = 2r$ and for $n \geq 1$,

$$t_{2n} = \left(2^{2n+1} - 1\right) t_{2n-1}, \quad t_{2n+1} = \left(2 - \frac{1}{2^{2n+1}}\right) t_{2n},$$

$$s_{2n} = \frac{t_{2n}}{2^n} = \left(2 - \frac{1}{2^{2n}}\right) s_{2n-1}, \quad s_{2n+1} = 2^{n+1} t_{2n+1} = \left(2^{2n+2} - 1\right) s_{2n}.$$

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x,t) := \begin{cases} 2\varphi(x)t, & (x,t) \in \Omega \times [0, t_1], \\ \varphi(x)\left(s_{n-1} + \frac{s_n - s_{n-1}}{t_n - t_{n-1}}(t - t_{n-1})\right), & (x,t) \in \Omega \times [t_{n-1}, t_n] \text{ for some } n > 1, \end{cases}$$

where $\varphi : \Omega \to \mathbb{R}$ is a positive continuous function with $0 < m \leq \varphi(x) \leq M$. Then f is an L^1-Carathéodory function and since $f(x,t)$ is strictly increasing with respect to t argument at every $x \in \Omega$, the function $l_\xi(x) := f(x,\xi)$ satisfies in condition (C_3) on f; i.e.,

$$\sup_{|t| \leq \xi} |f(x,t)| \leq l_\xi(x), \quad \text{for a.e. } x \in \Omega.$$

We have

$$\limsup_{\xi \to +\infty} \frac{\int_{B(x_0, \frac{D}{2})} F(x,\xi)dx}{\xi^{\frac{7}{3}}} = +\infty, \qquad \liminf_{\xi \to +\infty} \frac{\|l_\xi\|_1}{\xi^{\frac{4}{3}}} = 0,$$

for every $x_0 \in \Omega$ and $D > 0$ such that $B(x_0, D) \subset \Omega$ and $\overline{B(x_0, D)}$ not containing the origin, where Ω is a bounded domain in $\mathbb{R}^2$ containing the origin and with smooth boundary $\partial\Omega$. Hence, by Theorem 2.1, for every $\lambda \in]0, +\infty[$, the following problem

$$\begin{cases} -\Delta_{\frac{7}{3}} u + \dfrac{|u|^{-\frac{1}{2}}u}{|x|^{\frac{3}{2}}} = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

possesses an unbounded sequence of weak solutions in $W_0^{1,\frac{7}{3}}(\Omega)$.

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Rglt-Majorization on $\mathbf{M}_{n,m}$ and its Linear Preservers

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Keywords: G-row stochastic matrix, (Strong) linear preserver, Rglt-majorization.
AMS Mathematics Subject Classification [2010]: 15A04, 15A21.

Abstract. Let $\mathbf{M}_{n,m}$ be the set of all n-by-m real matrices. A matrix R is called generalized row stochastic (*g-row stochastic*) if the sum of entries on every row of R is one. For A, $B \in \mathbf{M}_{n,m}$, it is said that A is *rglt-majorized* by B, and we write $A \prec_{rglt} B$, if there exists an m-by-m lower triangular g-row stochastic matrix R so that $A = BR$. In this paper, the concept right lower triangular generalized row stochastic majorization, or rglt- majorization, is investigated and then the linear preservers and strong linear preservers of this concept are characterized on $\mathbb{R}_n$ and $\mathbf{M}_{n,m}$.

1. Introduction

Majorization is one of the vital topics in mathematics and statistics. It plays a basic role in matrix theory. For instance, majorization relation among eigenvalues and singular values of matrices produce a lot of norm inequalities. It was intensively studied in various directions; see, e.g., [1]- [7]. One of the directions is the study of linear functions that preserve or strongly preserve right matrix majorization ; see, e.g., [3] and [5].

Some of our notations and symbols are explained as the following: $\mathcal{R}_m^{gut}$ ($\mathcal{R}_m^{glt}$) for the collection of all m-by-m upper (lower) triangular g-row stochastic matrices; e for the column vector with all entries equal to one;E for the m-by-m matrix with all of the entries of the last column equal to one and the other entries equal to zero; E^* for the m-by-m matrix with all of the entries of the first column equal to one and the other entries equal to zero;$\mathbb{N}_k$ for the set $\{1, \dots, k\} \subset \mathbb{N}$; $\mathcal{P}_n$ for the set of all n-by-n permutation matrices; $tr(x)$ for the summation of all components of a vector x in $\mathbb{R}_n$; P_n for the n-by-n backward identity matrix; $[T]$ for the matrix representation of a linear function $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ with respect to the standard basis; $[T]_i$ for the i^{th} column of the matrix representation of a linear function T; r_i for the summation of all entries of i^{th} row of $[T]$; span(S) for the set of all linear combinations of the elements of S.

Let $\sim$ be a relation on $\mathbf{M}_{n,m}$. A linear function $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ is said to be a linear preserver (or strong linear preserver) of $\sim$, if $TX \sim TY$ whenever $X \sim Y$ (or $TX \sim TY$

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if and only if $X \sim Y$). Let $X, Y \in \mathbf{M}_{n,m}$. The matrix X is said to be *rgut-majorized* by Y (in symbol $X \prec_{rgut} Y$) if $X = YR$, for some $R \in \mathcal{R}_m^{gut}$.

2. Main results

We intend to find all (strong) linear preservers of rglt-majorization on $\mathbb{R}_n$ and $\mathbf{M}_{n,m}$, too.

Definition 2.1. Let $x, y \in \mathbb{R}_n$. We say that x rglt-majorized by y (in symbol $x \prec_{rglt} y$) if $x = yR$, for some $R \in \mathcal{R}_n^{glt}$.

We bring the following propositions without proof.

Proposition 2.2. *Let A, $B \in \mathcal{R}_n^{glt}$. Then*
(a) $AB \in \mathcal{R}_n^{glt}$.
(b) If A is invertible, then $A^{-1} \in \mathcal{R}_n^{glt}$.

Assume that $T : \mathbb{R}_n \to \mathbb{R}_m$ be a linear function. Define $\tau : \mathbb{R}_n \to \mathbb{R}_m$ by $\tau(x) = T(xP_n)P_m$.

Proposition 2.3. *Let x, $y \in \mathbb{R}_n$. Then $x \prec_{rgut} y$ if and only if $xP_n \prec_{rglt} yP_n$. Also, $xP_n \prec_{rgut} yP_n$ if and only if $x \prec_{rglt} y$.*

Proposition 2.4. *Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}_n$. Then $x \prec_{rglt} y$ if and only if $tr(x) = tr(y)$ and $x_i \in \operatorname{span}\{0, y_i, \ldots, y_n\}$, for all $2 \leq i \leq n$.*

Now, we assert some prerequisites for introducing the main results of this section.

Proposition 2.5. *Let $T : \mathbb{R}_n \to \mathbb{R}_m$ be a linear function. Then T preserves $\prec_{rgut}$ if and only if τ preserves $\prec_{rglt}$.*
Also, T preserves $\prec_{rglt}$ if and only if τ preserves $\prec_{rgut}$.

Define

$$\mathcal{A}_j(t_j) := \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 \\ \alpha_1^j & \alpha_2^j & \ldots & \alpha_{t_j}^j \\ * & * & \ldots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \ldots & * \end{pmatrix} \in \mathbf{M}_{n,t_j}, \tag{1}$$

where $j \geq 1$, $\alpha_{t_j}^j \neq 0$, $\begin{pmatrix} \alpha_1^j & \alpha_2^j & \ldots & \alpha_{t_j}^j \end{pmatrix}$ is the $n-j+1^{th}$ row of $\mathcal{A}_j(t_j)$, and $*$ is a real number.

Also, define

$$\mathcal{B}_1(k_1) := \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_{k_1} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1 & \alpha_2 & \ldots & \alpha_{k_1} \end{pmatrix} \in \mathbf{M}_{n,k_1}, \tag{2}$$

where $\alpha_{k_1} \neq 0$, and

$$(3)\qquad \mathcal{B}_j(k_j) := \begin{pmatrix} \alpha_1^j & \alpha_2^j & \dots & \alpha_{k_j}^j \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^j & \alpha_2^j & \dots & \alpha_{k_j}^j \\ \beta_1^j & \beta_2^j & \dots & \beta_{k_j}^j \\ * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & * \end{pmatrix} \in \mathbf{M}_{n,k_j},$$

where $j \geq 2$, $\begin{pmatrix} \beta_1^j & \beta_2^j & \dots & \beta_{k_j}^j \end{pmatrix}$ is the $n - j + 2^{th}$ row of $\mathcal{B}_j(k_j)$, and $\alpha_i^j \neq \beta_i^j$, for each $i \in \mathbb{N}_{k_j}$.

The following theorem characterizes structure of the linear functions $T : \mathbb{R}_n \to \mathbb{R}_m$ preserving rglt-majorization. The proofs are long. We have to give it up.

Theorem 2.6. *Let $T : \mathbb{R}_n \to \mathbb{R}_m$ be a linear function. Then T preserves $\prec_{rglt}$ if and only if $r_1 = \cdots = r_n$ and there exists a permutation matrix $P \in \mathcal{P}_m$ so that one of the following conditions occurs.*
a) $[T] = 0$,
b) $[T] = \begin{pmatrix} [T]_1 & * & \mathcal{B}_{n-1}(k_{n-1}) & \dots & \mathcal{B}_1(k_1) \end{pmatrix} P$,
c) $[T] = \begin{pmatrix} [T]_1 & \mathcal{B}_l(k_1) & \dots & \mathcal{B}_1(k_1) \end{pmatrix} P$,
d) $[T] = \begin{pmatrix} [T]_1 & * & \mathcal{A}_n(t_n) & \dots & \mathcal{A}_1(t_1) \end{pmatrix} P$,
where $\mathcal{B}_1(k_1)$, $\mathcal{B}_j(k_j)$ $(j \geq 2)$, and $\mathcal{A}_j(t_j)$ $(j \in \mathbb{N}_n)$ are the same as in (2), (3), (1), respectively, in (b) $\sum_{j=1}^{n-1} k_j \leq m - 1$ and in (c) $\sum_{j=1}^{l} k_j = m - 1$.
e) $[T] = \begin{pmatrix} \mathcal{B} & 0 & \dots & 0 \\ * & \mathcal{A}_k(k_k)' & \dots & \mathcal{A}_1(t_1)' \end{pmatrix} P$, *where $\mathcal{A}_j(t_j)' \in \boldsymbol{M}_{k,t_j}$ $(j \in \mathbb{N}_k)$ is the same as in (1), and $\mathcal{B} \in \boldsymbol{M}_{n-k,m-\sum_{j=1}^{k} t_j}$ can be the zero matrix, or one of the forms (b) or (c).*

Theorem 2.7. *Let $T : \mathbb{R}_n \to \mathbb{R}_n$ be a linear function. Then T strongly preserves $\prec_{rglt}$ if and only if $[T] = \alpha A$, for some $\alpha \in \mathbb{R} \setminus \{0\}$ and an invertible matrix $A \in \mathcal{R}_n^{glt}$.*

The following theorem characterizes the strong linear preservers of $\prec_{rglt}$ on $\mathbf{M}_{n,m}$. We come to this without proof.

Theorem 2.8. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear function. Then T strongly preserves $\prec_{rglt}$ if and only if $TX = RXA + SXE^*$, for some $R, S \in \boldsymbol{M}_n$ and an invertible matrix $A \in \mathcal{R}_m^{glt}$ so that $R(R + S)$ is invertible.*

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30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Linear Maps on Algebra of Operators in Hilbert C^*-Modules Characterized by Action on Zero Products

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ABSTRACT. Let M be a Hilbert C^*-module on a commutative C^*-algebra with unit element e. In this paper we are going to characterize the linear maps δ, τ from $End_A(M)$ into itself, satisfying $S\tau(T) + \delta(S)T = 0$ whenever $S, T \in End_A(M)$ and $ST = 0$ and we try to express some results.

Keywords: Hilbert C^*-modules, generalized derivations

AMS Mathematics Subject Classification [2010]: 46L08, 47B47

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1. Introduction

One of the lines of study in recent years is to investigate the behavior of linear mapping such as derivatives and generalized derivation and the like, under certain conditions such as zero-product elements. One of the research in this field was conducted by Jing et al. Chebotar et al. continued studying in this context in 2004 and they considered prime rings with special conditions and were examined derivations at the zero-product elements. Then, in 2007, Brešar investigated the behavior of derivations, homomorphisms, and multipliers at zero-product elements. In 2009, Jing spoke with different conceptions about derivation at the zero-product elements in the algebra of all bounded linear operators on Hilbert space. Motivated from these researches, Benkovič, Dominik, and Mateja Grašič worked on generalized derivations on a unital algebra having nontrivial idempotent in their essay, [2]. After that, in 2019, Ghahramani et al. [1], described the linear mappings in standard operator algebras of Banach space that satisfying generalized derivation equation in zero-product elements. Our goal in this article is to have a brief investigation of generalized derivations at zero-product elements on algebra of operators in Hilbert C^*-Modules.

Let A be an algebra and M be an $A-$bimodule. Generalized derivation is defined as a linear map $\delta : A \to M$ along with corresponding linear mapping $\kappa : A \to M$ that for all $x, y \in A$ applies to

$$\delta(xy) = \delta(x)y + x\kappa(y). \tag{1}$$

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Derivation that is defined as a linear mapping δ satisfying

$$\delta(xy) = \delta(x)y + x\delta(y), \tag{2}$$

for all $x, y \in A$, is an example of generalized derivation. Also left multipliers and right multipliers that is defined as a linear mapping ρ in unital algebra A, satisfying

$$\rho(xy) = \rho(x)y, \tag{3}$$

and

$$\rho(xy) = x\rho(y), \tag{4}$$

respectively, for all $x, y \in A$, are examples of generalized derivations. When we look at the behavior of these mappings at zero-product elements, it means that the equations (1), (2), (3), and (4) applies whenever $xy = 0$.

Hilbert C^*-modules are Banach spaces that generalizes Hilbert spaces by replacing the complex number field with an arbitrary C^*-algebra. The important role of Hilbert C^*-modules is in the theory of operator algebras. In this paper, we are going to characterize the linear maps that satisfying (1) at zero product elements on algebra of operators in Hilbert C^*-modules. The second section is devoted to our main results which are proved by several lemmas. In the following, at first, we would review some properties of Hilbert C^*- modules that we need in the next section.

Let A be a C^*-algebra. A pre-Hilbert $A-$module is a left $A-$module M equipped with a mapping $\langle ., . \rangle : M \times M \to A$ with the following properties:

(1) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$, for $x \in M$;
(2) $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$, for $x, y, , z \in M$ and $\lambda \in \mathbb{C}$;
(3) $\langle ax, y \rangle = a\langle x, y \rangle$, for $x, y \in M$ and $a \in A$;
(4) $\langle x, y \rangle = \langle y, x \rangle^*$, for $x, y \in M$.

The mapping $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ defines a norm on M. A pre-Hilbert $A-$module is called Hilbert C^*-module over A or in short Hilbert $A-$module, if it is complete with respect to this norm.

An $A-$linear mapping is a linear mapping that satisfied to $T(ax) = aT(x)$, for any $a \in A$ and $x \in M$. An operator on a Hilbert $A-$module M is defined as a bounded $A-$linear mapping from M into itself. Let us following [5], use $End_A(M)$ to denote the set of all operators on Hilbert C^*-module M. It is well known that $End_A(M)$ is a Banach algebra. Let $T \in End_A(M)$. Then T is said adjointable, if there exists an operator $S \in End_A(M)$ such that $\langle Tx, y \rangle = \langle x, Sy \rangle$ for all $x, y \in M$. Denote S by T^*. We denote by $End^*_A(M)$ the set of all adjointable operator in $End_A(M)$.

By [5], M' indicate the set of all $A-$linear mappings from M to A and we know that by $(f.a)(x) := a^*f(x)$ and $(\lambda.f) := \overline{\lambda}f(x)$, where $\lambda \in \mathbf{C}$ and $a \in A$, M' is a right $A-$module.

The operator $\theta_{x,f}$ is defined by $\theta_{x,f}(y) = f(y)x$, where $x \in M$ and $f \in M'$. The closed linear span of these operators denoted by $K(M)$ and called the space of compact operators. Let us define a mapping $\hat{x}$ as follow: $\hat{x}(y) = \langle y, x \rangle$, for $x, y \in M$. Clearly, $\hat{x} \in M'$, for all $x \in M$. We have

$$\theta_{x,\hat{y}}z = \hat{y}(z)x = \langle z, y \rangle x, \qquad x, y, z \in M.$$

For convenience, throughout this article, we use $\theta_{x,y}$ rather than $\theta_{x,\hat{y}}$.

By the above definitions, we remind the following lemma from [5].

Lemma 1.1. *[5] Let A be a C^*-algebra and M be a Hilbert C^*-module over A. Suppose $T \in End_A(M)$. Then for any $x, y, z, w \in M$, the following statements satisfy:*

(1) $\theta_{x,\hat{y}}T = \theta_{x,\hat{y}oT}$;
(2) $T\theta_{x,\hat{y}} = \theta_{Tx,\hat{y}}$;
(3) *If in addition A be a commutative C^*-algebra, we have $\theta_{x,\hat{y}}\theta_{z,\hat{w}} = \langle z, y\rangle\theta_{x,\hat{w}}$ and for any $a \in A$, $\theta_{ax,\hat{y}} = a\theta_{x,\hat{y}}$.*

Let M be an $A-$bimodule over algebra A. Recall that inner derivation for some fixed $m \in M$, is defined as a linear mapping $d_m(a) = ma - am$, for all $a \in A$. Clearly, d_m is a derivation.

Theorem 1.2. *[4] Let M be a Hilbert $A-$module over a unital C^*-algebra A with the property that there exists x_0 in M and f_0 in M' such that $f_0(x_0) = e_A$. Then every derivation on $End_A(M)$ is an inner derivation.*

Now we are going to review the following lemma from [3] that we need in the next section.

Lemma 1.3. *[3] Suppose A be a commutative C^*-algebra with unit e_A, and M be a Hilbert $A-$module over A such that $f_0(x_0) = e_A$, for some $x_0 \in M$ and $f_0 \in M'$. Let we denote the linear span of the set of $\{\theta_{x,f_0} : x \in M\}$ by $\mathfrak{L}$, and the linear span of the set of $\{\theta_{x_0,f} : f \in M'\}$ by $\mathfrak{R}$. Then the following statements hold:*

(1) *θ_{x_0,f_0} is an idempotent;*
(2) *$\mathfrak{L}$ is a left ideal of $End_A(M)$, and $\mathfrak{R}$ is a right ideal of $End_A(M)$;*
(3) *$\mathfrak{L}$ is a left separating set of $End_A(M)$ and $\mathfrak{R}$ is a right separating set of $End_A(M)$.(We recall that the left separating set of $End_A(M)$ namely, for any $T \in End_A(M)$, $T\mathfrak{L} = 0$ implies that $T = 0$, and the right separating set of $End_A(M)$ means that for any $T \in End_A(M)$, $\mathfrak{R}T = 0$ implies that $T = 0$).*

2. main results

In this section, the main results of this research are expressed.

Theorem 2.1. *Let M be a Hilbert C^*-module on a unital commutative C^*-algebra A and there exists x_0, y_0 in M such that $\langle x_0, y_0\rangle = e_A$, the unit element of A. Let δ and τ are $A-$linear maps from $End_A(M)$ to itself satisfying*

$$ST = 0 \Longrightarrow S\tau(T) + \delta(S)T = 0, \qquad S, T \in End_A(M).$$

Then, there exists $R, G, K \in End_A(M)$ such that

$$\delta(S) = SK - RS, \quad \tau(S) = SG - KS, \quad \forall S \in End_A(M).$$

The following lemmas will help us to make the proof of theorem 2.1

Lemma 2.2. *Suppose the assumptions of theorem 2.1 are valid. For all $S, T \in End_A(M)$, we have*

$$\delta(ST) = S\delta(T) + \delta(S)T - S\delta(I)T$$

Lemma 2.3. *Suppose the assumptions of theorem 2.1 are valid. For all $S, T \in End_A(M)$, we have*

$$\tau(ST) = S\tau(T) + \tau(S)T - S\tau(I)T.$$

Lemma 2.4. *Suppose the assumptions of theorem 2.1 are valid. For all* $S \in End_A(M)$, *we have*

$$\tau(S) - S\tau(I) = \delta(S) - \delta(I)S$$

Corollary 2.5. *Let* M *be a Hilbert* C^*-*module on commutative* C^*-*algebra* A *with unit element* e_A, *and there exists* x_0, y_0 *in* M *such that* $\langle x_0, y_0 \rangle = e_A$. *Let* ρ *be an* $A-$*linear map from* $End_A(M)$ *to itself.*

(1) ρ *satisfies*

$$ST = 0 \Longrightarrow S\rho(T) = 0, \qquad (S, T \in End_A(M)),$$

if and only if for some $D \in End_A(M)$, $\rho(S) = SD$, $(S \in End_A(M))$.

(2) ρ *satisfies*

$$ST = 0 \Longrightarrow \rho(S)T = 0, \qquad (S, T \in End_A(M)),$$

if and only if for some $D \in End_A(M)$, $\rho(S) = DS$, $(S \in End_A(M))$.

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Interior inverse problems for a Sturm-Liouville operator with an impulse

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ABSTRACT. This paper deals with the spectral boundary value problems for the Sturm-Liouville operator having an impulse. We prove the uniqueness of the potential and the coefficients of the boundary conditions by the interior point method.

Keywords: Interior inverse problem, Sturm-Liouville operator, Impulse, Spectrum

AMS Mathematics Subject Classification [2010]: 34A55, 34B24

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1. Introduction

An inverse spectral problem is a determination of the operator from information about the spectrum. These kinds of problems arise in various problems of geophysics, optics, etc. [1]. Inverse problems for the Sturm-Liouville operator with an impulse have been investigated in many works and have been studied by various methods [5]. In this work, we would like to study the interior inverse problem for this operator with the spectral boundary condition. The interior inverse problem can be seen in the works [3, 4].

Consider the boundary valve problem $L = L(q, r, h_0, h_1)$ of the form

$$-y'' + q(x)y = \lambda r(x)y, \quad x \in (0, \pi), \tag{1}$$

$$U(y) := y'(0) - (h_1\rho + h_0)y(0) = 0, \quad V(y) := y'(\pi) = 0. \tag{2}$$

Here the weight function $r(x) = 1,\ x < \frac{\pi}{2}$ and $r(x) = \alpha^2,\ x > \frac{\pi}{2}$ for $\alpha > 1$. The function $q(x)$ is real-valued in $L^2(0, \pi)$. The parameters h_0, h_1 are real and $\lambda = \rho^2$ is a spectral parameter.

In this paper, it is shown that the potential function and the coefficients of the boundary conditions can be uniquely determined by one spectrum and a set of values of eigenfunctions at the point $x = \frac{\pi}{2}$.

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2. Results

Suppose that $y(x,\rho)$ be the solution of the equation (1) satisfying the initial conditions $y(0,\rho)=1$ and $y'(0,\rho)=h_1\rho+h_0$. For sufficiently large ρ and uniformly in x, one has (see [5, 6])

$$y(x,\rho)=\cos\rho x+h_1\sin\rho x+O\left(\frac{1}{\rho}\exp(\tau x)\right),\qquad x<\frac{\pi}{2},\tag{3}$$

$$\begin{aligned} y(x,\rho) &= \frac{\alpha+1}{2\alpha}(\cos\rho\alpha(x)+h_1\sin\rho\alpha(x)) \\ &\quad+\frac{\alpha-1}{2\alpha}\left(\cos\rho\left(2\alpha\left(\frac{\pi}{2}\right)-\alpha(x)\right)+h_1\sin\rho\left(2\alpha\left(\frac{\pi}{2}\right)-\alpha(x)\right)\right) \\ &\quad+O\left(\frac{1}{\rho}\exp(\tau\alpha(x))\right),\qquad x>\frac{\pi}{2}, \end{aligned}\tag{4}$$

where $\alpha(x)=\int_0^x\sqrt{r(t)}dt$ and $\tau=|\Im\rho|$.

Define $\Delta(\rho):=V(y(x,\rho))$. The functions $y(x,\rho)$ and $\Delta(\rho)$ are entire in ρ, and the zeros λ_n of the characteristic function $\Delta(\rho)$ coincide with the eigenvalues of L [2]. Using (4), we have for $|\rho|\to\infty$,

$$\begin{aligned} \Delta(\rho) &= \frac{\rho\sqrt{1+h_1^2}}{2}\left((\alpha-1)\sin\left(\rho(1-\alpha)\frac{\pi}{2}-\varepsilon\right)-(\alpha+1)\sin\left(\rho(1+\alpha)\frac{\pi}{2}-\varepsilon\right)\right) \\ &\quad+O\left(\exp\left(\tau(\alpha+1)\frac{\pi}{2}\right)\right), \end{aligned}\tag{5}$$

where $\varepsilon=\frac{1}{2i}ln\frac{i-h_1}{i+h_1}$.

The eigenvalues ρ_n satisfy the following asymptotic form for sufficiently large n,

$$\rho_n=\frac{2}{\alpha+1}\left(n+\frac{\varepsilon}{\pi}\right)+O(n^{-1}).\tag{6}$$

Now we can present the main result of this paper. Together with $L=L(q,r,h_0,h_1)$, we consider a boundary value problem $\widetilde{L}=L(\widetilde{q},r,\widetilde{h}_0,\widetilde{h}_1)$ of the same form L but with different coefficients. We agree that if a certain symbol δ denotes an object related to L, then $\widetilde{\delta}$ will denote an analogous object related to $\widetilde{\delta}$.

THEOREM 2.1. *If for any $n\in\mathbb{N}$,*

$$\lambda_n=\widetilde{\lambda}_n,\quad \frac{y_n(\frac{\pi}{2},\rho)}{y_n'(\frac{\pi}{2},\rho)}=\frac{\widetilde{y}_n(\frac{\pi}{2},\rho)}{\widetilde{y}_n'(\frac{\pi}{2},\rho)},$$

then $q(x)=\widetilde{q}(x)$ a.e. on $(0,\pi)$ and $h_0=\widetilde{h}_0,\ h_1=\widetilde{h}_1$.

PROOF. Let $y(x,\rho)$ be the solution of the equation (1) and $\widetilde{y}(x,\rho)$ be the solution of the equation (1) with tilde. If we multiply (1) by $\widetilde{y}(x,\rho)$ and the corresponding equation with tilde by $y(x,\rho)$, and subtract, and then integrate on $\left(0,\frac{\pi}{2}\right)$, we will have

$$\begin{aligned} H(\rho) &:= \int_0^{\frac{\pi}{2}}(q(x)-\widetilde{q}(x))y(x)\widetilde{y}(x)dx+(\widetilde{h}_1-h_1)\rho+(\widetilde{h}_0-h_0) \\ &= y'\left(\frac{\pi}{2}\right)\widetilde{y}\left(\frac{\pi}{2}\right)-y\left(\frac{\pi}{2}\right)\widetilde{y}'\left(\frac{\pi}{2}\right). \end{aligned}\tag{7}$$

Taking the assumption of the theorem, it follows that $H(\rho_n)=0$. Now we have to prove that $H(\rho)=0$ for $\rho\neq\rho_n$.

Define the entire function

$$\phi(\rho)=\frac{H(\rho)}{\Delta(\rho)}. \tag{8}$$

From the integral form of the solution

$$y(x,\rho)=\cos\rho x+h_1\sin\rho x+\int_0^x A(x,t)\cos\rho t dt+\int_0^x B(x,t)\sin\rho t dt,\quad x<\frac{\pi}{2}, \tag{9}$$

for bounded functions $A(x,t)$ and $B(x,t)$ [2], (5), (7) and (8), we can give that $\phi(\rho)=0$ for large enough ρ. So, we can infer that $H(\rho)=0$ for all ρ and then

$$\int_0^{\frac{\pi}{2}}\big(q(x)-\widetilde{q}(x)\big)y(x)\widetilde{y}(x)dx+(\widetilde{h}_1-h_1)\rho+(\widetilde{h}_0-h_0)=0. \tag{10}$$

By taking the relations (9) and (10), the Riemann-Lebesgue lemma and the completeness of the functions " cos " and " sin " [2], we result that $q(x)=\widetilde{q}(x)$ a.e on $(0,\frac{\pi}{2})$ and $h_0=\widetilde{h}_0$, $h_1=\widetilde{h}_1$.

We can prove the problem on $(\frac{\pi}{2},\pi)$, by repeating the proof for the supplementary problem $\widehat{L}$,

$$-y''+q_1(x)y=\lambda r_1(x)y,\quad x\in(0,\pi),$$
$$U(y):=y'(0)=0,\quad V(y):=y'(\pi)+(h_1\rho+h_0)y(\pi)=0,$$

where $q_1(x)=q(\pi-x)$ and $r_1(x)=r(\pi-x)$. The proof is completed. $\square$

Example 2.2. Define $r(x)=1$ for $x<\frac{\pi}{2}$ and $r(x)=\alpha^2$ for $x>\frac{\pi}{2}$, and then consider the boundary value problem L,

$$\begin{aligned}&-y''+q(x)y=\lambda r(x)y,\quad x\in(0,\pi),\\ &U(y):=y'(0)-(h_1\rho+h_0)y(0)=0,\\ &V(y):=y'(\pi)=0.\end{aligned}$$

The eigenvalues λ_n for the boundary value problem L have the form

$$\lambda_n=\frac{4}{(\alpha+1)^2}\left(n^2+\frac{2\varepsilon n}{\pi}+\frac{\varepsilon^2}{\pi^2}\right)+O(n^{-1}),$$

for sufficiently large n. Also consider the boundary value problem $\widetilde{L}$,

$$\begin{aligned}&-y''=\lambda r(x)y,\quad x\in(0,\pi),\\ &U(y):=y'(0)=0,\\ &V(y):=y'(\pi)=0.\end{aligned}$$

A direct computation yields that the eigenvalues $\widetilde{\lambda}_n$ for the boundary value problem $\widetilde{L}$ satisfy

$$\widetilde{\lambda}_n=\frac{4n^2}{(\alpha+1)^2},$$

for sufficiently large n. Let $\lambda_n=\widetilde{\lambda}_n$ and the Wronskian of the functions $y_n(x,\rho)$ and $\widetilde{y}_n(x,\rho)$ at the point $x=\frac{\pi}{2}$ be equal to zero, i.e.,

$$\frac{y_n(\frac{\pi}{2},\rho)}{y_n'(\frac{\pi}{2},\rho)}=\frac{\widetilde{y}_n(\frac{\pi}{2},\rho)}{\widetilde{y}_n'(\frac{\pi}{2},\rho)},$$

then on the base of Theorem 2.1, we will have $q(x)=0$ and $h_0=h_1=0$.

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Parametric type of the Jensen-Fisher information divergence

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Abstract. In this work, we first consider the Fisher information and then propose Jensen-Fisher information type of parameter and Jensen-chi square divergence measures. Then, we provide some connections between these measures with some known informational measures such as chi-square divergence, Kullback-Leibler, Jeffreys and Jensen-Shannon divergences.

Keywords: Fisher information, Jensen-Fisher information, Chi-square divergence, Jensen-Shannon Entropy

AMS Mathematics Subject Classification [2010]: 18A32, 18F20, 05C65

1. Introduction

Several different criteria have been introduced for measuring the uncertainty of a probabilistic model. Shannon entropy and Fisher information are the most important information measures that have been used rather extensively in this regard. The starting point of information theory is the Shannon entropy, introduced in the pioneering work of Shannon [7], based on a study of systems described by probability density (or mass) functions. Nearly two decades earlier, Fisher [1] had proposed another information, describing the interior properties of a probabilistic model, that has become vital to likelihood-based inferential methods. Fisher information as well as Shannon entropy are very important and fundamental criteria in statistical inference, physics, thermodynamics, information theory and some other disciplines. Complex systems can be described by means of their behavior (Shannon) and their architecture (Fisher) information. For more details, see Kharazmi and Balakrishnan ([4], [5]) and Zegers [8]. We now briefly introduce some informational measures that will be used in the sequel.

The Fisher information of a random variable X, or its PDF $f(x;\theta)$, about the parameter θ is defined as

$$I(\theta) = I(f_\theta) = \int \left[\frac{\partial \log f(x;\theta)}{\partial\theta}\right]^2 f(x;\theta)dx. \tag{1}$$

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It is assumed that θ lies in an open interval in the real line and that $f(x;\theta) > 0$ for all values of θ in the parameter space and is differentiable with respect to θ.

Let X and Y be two continuous random variables with absolutely continuous density functions f and g, respectively. Then, the Jensen-Fisher divergence between f and g, for $0 \leq \alpha \leq 1$, is defined as

$$JF(f,g;\alpha) = \alpha I(f) + (1-\alpha)I(g) - I(\alpha f + (1-\alpha)g),$$

where $I(f)$ is Fisher information for the own density function f. Fore more details, see Sánchez-Moreno et al. [6].

Let X be an absolutely continuous random variable with PDF f. Then, the Shannon entropy of X (or density f) is defined as

$$H(X) = H(f) = -\int_X f(x)\log f(x)dx,$$

where log denotes the natural logarithm; see Shannon [7]. As mentioned earlier, we suppress X for integration with respect to dx throughout the paper, unless a distinction needs to be made.

Next, let X and Y be two continuous random variables with absolutely continuous density functions f and g, respectively. Then, the Kullback-Leibler distance between X and Y (or f and g) is defined as

$$KL(f,g) = \int f(x)\log\frac{f(x)}{g(x)}dx.$$

The Kullback-Leibler (KL) discrimination between Y and X is defined analogously. For more details, see Kullback and Leibler (1951).

Lin [3] introduced Jensen-Shannon (JS) entropy divergence for probability vectors based on Jensen inequality and concavity of Shannon entropy. Let X and Y be two continuous random variables with absolutely continuous density functions f and g, respectively. Then, the JS divergence between f and g, for $0 \leq \alpha \leq 1$, is defined as

$$JS(f,g;\alpha) = H(\alpha f + (1-\alpha)g) - \Big\{\alpha H(f) + (1-\alpha)H(g)\Big\}. \tag{2}$$

This measure is a symmetric version of KL divergence and possesses the property that

$$JS(f,g;\alpha) = \alpha KL\big(f, \alpha f + (1-\alpha)g\big) + (1-\alpha)KL\big(g, \alpha f + (1-\alpha)g\big).$$

Another important diversity measure between two density functions f and g is the well-known chi-square divergence, defined as

$$\chi^2(f,g) = \int \frac{(f(x)-g(x))^2}{g(x)}dx.$$

In the same way, we can define $\chi^2(g,f)$.

The purpose of this work is two-fold. The first part is to define two Jensen type information measures based on Fisher information type of parameter) and chi-square divergence measures. In the second part, we establish the connections between these new divergence measures and the well-known Jensen-Shannon divergence.

2. Jensen-Fisher information type of parameter and Jensen-χ^2 divergence measures

2.1. Let X and Y be two continuous random variables with absolutely continuous density functions $f(x;\theta)$ and $g(x;\theta)$, respectively. Then, the Jensen-Fisher information about the parameter θ, for any $0<\alpha<1$, is defined as

$$JF(f_\theta, g_\theta; \alpha) = \alpha I(f_\theta) + (1-\alpha)I(g_\theta) - I\big(\alpha f_\theta + (1-\alpha)g_\theta\big), \tag{3}$$

where

$$I(f_\theta) = \int \left[\frac{\partial \log f(x;\theta)}{\partial\theta}\right]^2 f(x;\theta)dx. \tag{4}$$

Consider three arithmetic mixture distributions with density functions as

$$f_\Lambda(x) = \Lambda f_0(x) + (1-\Lambda)f_1(x), \tag{5}$$

$$g_\Lambda(x) = \Lambda g_0(x) + (1-\Lambda)g_1(x) \tag{6}$$

and

$$h_{\alpha,\Lambda}(x) = \alpha f_\Lambda(x) + (1-\alpha)g_\Lambda(x). \tag{7}$$

Now, the Jensen-Fisher information about mixing parameter Λ for $0\le\alpha\le1$, is given by

$$JF(f_\Lambda, g_\Lambda; \alpha) = \alpha I(f_\Lambda) + (1-\alpha)I(g_\Lambda) - I(h_{\alpha,\Lambda}), \tag{8}$$

where

$$I(f_\Lambda) = \int \left(\frac{\partial}{\partial\Lambda}\log f_\Lambda(x)\right)^2 f_\Lambda(x)dx,\ I(g_\Lambda) = \int \left(\frac{\partial}{\partial\Lambda}\log g_\Lambda(x)\right)^2 g_\Lambda(x)dx$$

and

$$I(h_{\alpha,\Lambda}) = \int \left(\frac{\partial}{\partial\Lambda}\log h_{\alpha,\Lambda}(x)\right)^2 h_{\alpha,\Lambda}(x)dx.$$

2.2. Consider the density functions f_0, f_1, g_0 and g_1, and for $\Lambda\in(0,1)$, let f_Λ and g_Λ be as in (5) and (6), respectively. Then, the Jensen-χ^2 divergence (JCD) measure for $\alpha\in(0,1)$ is defined by

$$JC(f_\Lambda, g_\Lambda; \alpha) = \alpha\chi^2(f_\Lambda, f_1) + (1-\alpha)\chi^2(g_\Lambda, g_1) - \chi^2(h_{\alpha,\Lambda}, \alpha f_1 + (1-\alpha)g_1). \tag{9}$$

3. Connections between JF and JCD information divergences

The following theorem investigates the connection between Jensen-Fisher information in (8) Jensen-χ^2 divergence in (9).

3.1. *The Jensen-Fisher information in (8) can be given based on Jensen-χ^2 divergence as*

$$JF(f_\Lambda, g_\Lambda; \alpha) = \frac{1}{\Lambda^2}JC(f_\Lambda, g_\Lambda; \alpha).$$

3.2. *The Jensen-χ^2 divergence in (9) is non-negative.*

3.3. *For any $s > r$, we have*

$$(10)\quad (s-r)\int_r^s JF(f_\Lambda, g_\Lambda, \alpha)d\Lambda \;=\; \alpha J(f_s, f_r) + (1-\alpha)J(g_s, g_r) - J(h_{\alpha,s}, h_{\alpha,r})$$

where

$$h_{\alpha,s}(x) = \alpha f_s(x) + (1-\alpha)g_s(x),\;\; h_{\alpha,r}(x) = \alpha f_r(x) + (1-\alpha)g_r(x)$$

and $J(.,.)$ is Jeffrey's divergence between two densities.

3.4. From Theorem 3.3, it is easy to see that the Jeffrey's divergence is jointly convex.

The following lemma shows the connection between Jensen-Fisher in (8) and Jensen-Shannon information measure in (2).

3.5. *The connection between JF and JS information divergences is given by*

$$\begin{aligned}\int_0^1 JF(f_\Lambda, g_\Lambda, \alpha)d\Lambda \;&=\; JS(f_0, g_0; \alpha) + JS(f_1, g_1; \alpha)\\ &+\; \alpha(K(f_0, f_1) + K(f_1, f_0)) + (1-\alpha)(K(g_0, g_1) + K(g_1, g_0))\\ &-\; [K(m_\alpha, n_\alpha) + K(n_\alpha, m_\alpha)],\end{aligned}$$

where $K(f,g)$ is Kerridge's inaccuracy measure $K(f,g) = H(f) + KL(f,g)$ and

$$m_\alpha(x) = \alpha f_0(x) + (1-\alpha)g_0(x),$$

$$n_\alpha(x) = \alpha f_1(x) + (1-\alpha)g_1(x).$$

4. Conclusion

In this paper, we have introduced two Jensen type information measures in terms of Fisher information, chi-square divergence and Kullback-Leibler divergence measures. Further, we have considered three arithmetic mixture density function and then derived the connections between these new information measures with some well-known divergence measures such as Jensen-Shannon entropy, Kerridge's inaccuracy and Jensen-Jeffrey's divergence measures.

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Positive solutions for a class of nonlinear fractional integro-differential equation involving Caputo-Hadamard fractional derivative

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Abstract. In this work, we prove the existence and uniqueness of positive solutions for a boundary value problem of nonlinear fractional integro-differential equations involving Caputo-Hadamard fractional derivative with integral boundary conditions. The technique used to prove our results depends on the upper and lower solution, the Schauder fixed point theorem and the Banach contraction principle. An example is given which illustrate the effectiveness of the theoretical results.

Keywords: Fractional integro-differential equations, positive solutions, upper and lower solutions, fixed point theorem.

AMS Mathematics Subject Classification [2010]: 34A08, 34A12, 34B18

1. Introduction

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences,practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1,2] and the references therein.
Inspired and motivated by the works mentioned in [1, 2]. In this work, we used the upper and lower solution method, Schauder fixed point theorem and Banach contraction principle to obtain the existence and uniqueness of a positive solution for the following fractional differential equation with integral boundary conditions

$$(1)\qquad \begin{cases} {}^{C}_{H}\mathfrak{D}^{\alpha}u(t) = f\left(t, u\left(t\right)\right) + {}^{H}\mathfrak{I}^{\beta}g(t, u(t)),\ t \in (1, T]\,, \\ u(1) = \lambda \int_{1}^{T} k(s) u\left(s\right) ds + d,\ \lambda \geq 0,\ \ d > 0 \end{cases}$$

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where ${}^{C}_{H}\mathfrak{D}^{\alpha}$ is the Caputo-Hadamard fractional derivative of order $0 < \alpha \leq 1$, and ${}^{H}\mathfrak{J}^{\beta}$ is the Hadamard fractional integral of order $\beta \in (0,1)$, $f : [1,T] \times [0,\infty) \to [0,\infty)$, $g : [1,T] \times [0,\infty) \to [0,\infty)$ and $k : [1,T] \to [0,\infty)$ are given continuous functions. g is non-decreasing on u.
In what follows, we present some essential ideas of fractional calculus and fixed point theorems that prerequisite in our analysis.

Let $J = [1,T]$. Denote by $C(J)$ the Banach space of all continuous functions defined on J endowed with the norm $\|u\| = \sup\{|u(t)| : t \in J\}$, and A a nonempty closed subset of $C(J)$ defined as

$$A = \{u(t) \in C(J) : u(t) \geq 0, t \in J\}.$$

Definition 1.1. [3] The Hadamard fractional integral of order $\alpha > 0$ for a continuous function $u : [1,\infty) \to \mathbb{R}$ is defined as

$${}^{H}\mathfrak{J}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} u(s) \frac{ds}{s}, \ \alpha > 0.$$

where Γ denotes the Gamma function.

Definition 1.2. [2] The Caputo-Hadamard fractional derivative of order $\alpha > 0$ for a function $u \in C^n([1,\infty), \mathbb{R})$ is defined as

$${}^{C}_{H}\mathfrak{D}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-a-1} \delta^n u(s) \frac{ds}{s}, \ n-1 < \alpha < n,$$

where $\delta^n = \left(t\frac{d}{dt}\right)^{(n)}$, $n \in \mathbb{N}$.

Lemma 1.3. *[2] Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ and $u \in C^n(J)$. Then the Caputo-Hadamard fractional differential equation ${}^{C}_{H}\mathfrak{D}^{\alpha}u(t) = 0$, has a solution*

$$u(t) = \sum_{k=0}^{n-1} c_k (\log t)^k,$$

and the following formula holds:

$${}^{H}\mathfrak{J}^{\alpha}\left({}^{C}_{H}\mathfrak{D}^{\alpha}x(t)\right) = u(t) - \sum_{k=0}^{n-1} c_k (\log t)^k,$$

where $c_k \in \mathbb{R}$, $k = 0, 1, ..., n-1$.

Theorem 1.4. *(Schauder's fixed point theorem [4]) Let Ω be a nonempty closed convex subset of a Banach space S and $\Phi : \Omega \to \Omega$ be a continuous compact operator. Then has a fixed point in Ω.*

Theorem 1.5. *(Banach contraction principle [4]) Let Ω be a non-empty closed subset of a Banach space $(S, \|.\|)$, then any contraction mapping Φ of Ω into itself has a unique fixed point.*

Definition 1.6. A function $u \in C^1(J)$ is said to be a solution of problem (1) if u satisfies the equation ${}^{C}_{H}\mathfrak{D}^{\alpha}u(t) = f(t, u(t)) +^{H} \mathfrak{J}^{\beta} g(t, u(t))$ for all $t \in J$ with integral boundary conditions $u(1) = \lambda \int_1^T k(s) u(s) ds + d$.

Definition 1.7. A function $u \in C^1(J)$ is called a positive solution of problem (1) if $u(t) \geq 0$ for all $t \in J$ and u satisfies the problem (1).

Definition 1.8. Let $a, b \in \mathbb{R}^+$, and $b > a$: For any $u \in [a, b]$, we define the upper-control function $U(t,u) = \sup_{a \leq \rho \leq u} f(t, \rho)$, and lower-control function $L(t,u)$ $=\inf_{u \leq \rho \leq b} f(t, \rho)$. Obviously, $U(t,u)$ and $L(t,u)$ are monotonous non-decreasing on $[a, b]$ and $L(t,u) \leq f(t,u) \leq U(t,u)$.

2. Main results

In this section, we shall give existence and uniqueness results of problem (1) and prove it. Before starting and proving the main result, we introduce the following lemma:

Lemma 2.1. *Let $u \in C(J)$, u' exists, then u is a solution of problem (1) if and only if u is a solution of the integral equation*

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} + \frac{1}{\Gamma(\alpha+\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+\beta-1} g(s, u(s)) \frac{ds}{s}$$
$$+\lambda \int_1^T k(s)\, u(s)\, ds + d,\ t \in J. \tag{2}$$

Proof. Following the same steps in [1, Lemma 3], we obtain the integral equation defined in (2). □

To transform the problem (1) into a fixed point problem, we define the operator $\Phi : A \to A$ by

$$(\Phi u)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} + \frac{1}{\Gamma(\alpha+\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+\beta-1} g(s, u(s)) \frac{ds}{s}$$
$$+\lambda \int_1^T k(s)\, u(s)\, ds + d,\ t \in J. \tag{3}$$

Clearly, the solution of (1) is as a fixed point of the operator Φ.

To obtain our findings, we need the following assumptions

(H1) Let $u^*, u_* \in A$ such that $a \leq u_*(t) \leq u^*(t) \leq b$ and

$$\begin{cases} {}^C_H\mathfrak{D}^\alpha u^*(t) - {}^H\mathfrak{I}^\beta g(t, u^*(t)) \geq U(t, u^*(t)),\ \forall t \in J, \\ {}^C_H\mathfrak{D}^\alpha u_*(t) - {}^H\mathfrak{I}^\beta g(t, u_*(t)) \leq L(t, u_*(t)),\ \forall t \in J, \end{cases}$$

(H2) For $t \in J$, and $u, v \in [0, \infty)$, there exist two positives number l_f and l_k such that

$$\begin{aligned} |f(t,u) - f(t,v)| &\leq l_f |u - v|, \\ |g(s,u) - g(s,v)| &\leq l_g |u - v|. \end{aligned}$$

The function $u^*$and u_* are respectively called the pair of upper and lower solution for problem (1).

The first result is based on the Schauder fixed point theorem.

Theorem 2.2. *Assume that (H1) is satisfied, then problem(1) has at least one positive solution.*

Proof. Let $\Omega = \{u \in A : u_*(t) \leq u(t) \leq u^*(t),\ t \in J\}$ endowed with the norm $\|u\| = \max_{t \in J} |u(t)|$, then we have $\|u\| \leq b$. Hence, Ω is convex bounded and closed subset of the Banach space $C(J)$. The proof will be given in the following steps.

Step 1. The continuity of f,g implies the continuity of the operator Φ defined by (3).

Step 2. We prove that the operator Φ is compact.

Step 3. We show that $\Phi(\Omega) \subset \Omega$. for any $u \in \Omega$.
Clearly, all the hypotheses of the Schauder fixed point theorem are satisfied. Thus the operator Φ has at least one fixed point $u \in \Omega$, which is a positive solution of (1). $\square$

The second result is based on the Banach contraction principle.

THEOREM 2.3. *Assume that (H2) is satisfied and*

$$\left(\frac{l_f (\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{l_g (\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \lambda k^* (T-1) \right) < 1. \tag{4}$$

Then problem (1) has a unique positive solution.

PROOF. From Theorem (2.2), it follows that problem (1) has at least one positive solution. Hence, we need only to prove that the operator defined in (3) is a contraction in Ω. In fact, for each $u, v \in \Omega$, we have

$$|(\Phi u)(t) - (\Phi v)(t)| \leq \left(\frac{l_f (\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{l_g (\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \lambda k^* (T-1) \right) \|u-v\|.$$

From (4), Φ is a contraction. As a result of Banach's fixed point theorem, Φ has a unique fixed point that is the corresponding unique positive solution of the problem (1). $\square$

3. Example

As an application of our result, we consider the following fractional integro-differential equation with integral boundary condition

$$\begin{cases} {}_H^C\mathfrak{D}^{\frac{1}{2}} u(t) = \frac{\sin(t)}{\exp(t^2-1)+7}\left(\frac{|u|}{|u|+1}\right) + \frac{1}{4} I^{\frac{1}{4}} \frac{u(t)}{\exp(t-1)}, \ t \in (1, e], \\ u(1) = \frac{1}{6}\int_1^e \frac{1}{t} u(s)\, ds + \frac{1}{10}. \end{cases} \tag{5}$$

Here, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$, $\lambda = \frac{1}{6}$, $d = \frac{1}{10}$, $f(t, u(t)) = \frac{\sin(t)}{\exp(t^2-1)+7}\left(\frac{|u|}{|u|+1}\right)$, $g(t,u) = \frac{u}{4\exp(t-1)}$ and $k(t) = \frac{1}{t}$.

Since f, g and k are continuous positive functions, g is non-decreasing on u and

$$\frac{l_f (\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{l_g (\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \lambda k^* (T-1) \simeq 0.7 < 1.$$

Then, by Theorem (2.3), the problem (5) has a unique positive solution.

4. Conclusion

We can conclude that the main results of this work have been successfully achieved, that is, through the Banach fixed point theorem and the Schauder fixed point theorem combined with the method of upper and lower solution, we scrutinized the existence and uniqueness of positive solutions for a fractional integro-differential equation with integral boundary conditions involving Caputo-Hadamard fractional derivative.

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Faculty of Mathematics and Computer
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Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Rings with specific Lie ideals

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Abstract. We show that if I is a non-central Lie ideal of a ring R with $\mathrm{Char}(R) \neq 2$, such that all of its non-zero elements are invertible, then R is a division ring. We prove that if R is an F-central algebra and I is a Lie ideal without zero divisor such that the set of multiplicative cosets $\{aF \mid a \in I\}$ is of finite cardinality, then either R is a field or I is central. We show the only non-central Lie ideal without zero divisor of a non-commutative central F-algebra R with $\mathrm{Char}(R) \neq 2$ and radical over the center is $[R, R]$, the additive commutator subgroup of R.

Keywords: Division ring, Lie ideal, Quaternion algebra.

AMS Mathematics Subject Classification [2010]: 16K40, 17A35

1. Introduction

Throughout this paper R is a unitary ring with center $Z(R)$ and F is a field. For a pair of elements a, b of R we denote by $[a, b] = ab - ba$ the Lie product of a and b. Also elements of R with such a representation as $ab - ba$ for two elements a, b in R are called additive commutators. For two subsets A and B let $A \setminus B = \{a \in A \mid a \notin B\}$. An additive subgroup I of R is said to be a Lie ideal of R if $[r, a] \in I$ for every $r \in R$ and $a \in I$. Also, for subsets A, B of R we denote by $[A, B]$ the additive subgroup of R generated by all $[a, b]$ with $a \in A$ and $b \in B$. By this notation $[R, R]$ is called the additive commutator subgroup of R. An element $a \in R$ is said to be radical over the center or $Z(R)$ if there exists a positive integer $n = n(a)$ such that $a^n \in Z(R)$. A subset $S \subseteq R$ is said to be radical over $Z(R)$, if each element of S is radical over $Z(R)$. We denote the characteristic of R by $\mathrm{Char}(R)$. For a subset $S \subseteq R$, the centralizer of S in R is defined by $C_R(S) = \{r \in R \mid rs = sr \ for\ all\ s \in S\}$. An element a of R is called a zero divisor if there exists a non-zero $b \in R$ such that either $ab = 0$ or $ba = 0$. We say the ring R is an F-central algebra if R is an algebra over F and $Z(R) = F$. A derivation on R is an additive group homomorphism $d : R \longrightarrow R$ satisfying $d(r_1 r_2) = (d(r_1))r_2 + r_1(d(r_2))$.

In Theorem 1.1, stated below, Bergen et al. [3] proved that a unitary ring R with a derivation such that all of its non-zero images are invertible, except for a special case, either is a division ring D or $M_2(D)$, the ring of 2×2 matrices over a division ring D. Bergen and Carini gave a generalization of this result to the case of a Lie ideal. More precisely, a semiprime unitary ring R with a derivation such that all of its non-zero images

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over a non-central Lie ideal are invertible, either is a division ring D or $M_2(D)$ for a division ring D.

In this paper, we study the rings with specific Lie ideals. Our study begins with two theorems having a similar idea as above. We prove that when the non-zero additive commutators or the non-zero elements of a Lie ideal I of a ring R are invertible then R is a division ring. As a consequence we present a commutativity condition over a ring. We show that if I is a Lie ideal of an F-central algebra R without zero divisor such that the set of multiplicative cosets $\{aF \mid a \in I\}$ is of finite cardinality, then either R is a field or I is central. We prove that any division ring D with $\mathrm{Char}(D) \neq 2$ which contains a non-central Lie ideal I without zero divisor and radical over the center is isomorphic to a generalized quaternion algebra and $I = [D, D]$. At last we prove that when R is an F-central algebra and $\mathrm{Char}(R) \neq 2$ and I is a Lie ideal without zero divisor, if the residual additive group $(\frac{I+F}{F}, +)$ is of finite cardinality, then I is central. First, we recall the following theorems.

Theorem 1.1. [3] *Let R be a ring with 1 and $d \neq 0$ a derivation of R such that, for each $x \in R$, $d(x) = 0$ or $d(x)$ is invertible in R. Then R is either*

(i) *a division ring D, or*
(ii) *$M_2(D)$ for some division ring D or*
(iii) *$D[x]/(x^2)$, for some division ring D, where $charD = 2$, $d(D) = 0$ and $d(x) = 1 + ax$ for some a in the center $Z(D)$ of D.*

Furthermore, if $2R \neq 0$ then $R = M_2(D)$ is possible if and only if D does not contain all quadratic extensions of $Z(D)$, the center of D; equivalently if and only if some element in $Z(D)$ is not a square in D.

Theorem 1.2. [6] *Let D be a division ring with center F, such that $(xy - yx)^{n(x,y)} \in F$, $n(x, y) \geq 1$ for all $x, y \in D$, then $dim_F(D) \leq 4$.*

Theorem 1.3. [5, p. 5] *Let D be a division algebra with center F and $Char(D) \neq 2$ and let I be a Lie ideal of D. Then either $I \subseteq F$ or $[D, D] \subseteq I$.*

2. Main results

We show that the invertibility condition on some special subsets or substructures of a ring may imply the invertibility of all none-zero elements of the ring. In particular, we show that a ring with all its non-zero additive commutators invertible is a division ring. This is the content of the following theorem, which is really an easy consequence of Theorem 1.1 above.

Theorem 2.1. [1] Let R be a non-commutative ring with center $Z(R)$, and with all its non-zero additive commutators invertible. Then R is a division ring.

Clearly $[R, R]$ is a Lie ideal in any ring R, containing all additive commutators. When R is a division ring with $\mathrm{Char}(R) \neq 2$, then by Theorem 1.3 all non-central Lie ideals contain $[R, R]$, but in general there is not a clear relation between a Lie ideal and $[R, R]$. So one may ask what would be the case when the same condition, as above theorem, is valid on a Lie ideal of a ring. In the following theorem we show that only a division ring may contains a Lie ideal such that all of its non-zero elements are invertible.

Theorem 2.2. [1] Let R be a ring and $\mathrm{Char}(R) \neq 2$. If I is a non-central Lie ideal of R, all of whose non-zero elements are invertible, then R is a division ring.

We use the previous theorem to present a commutativity condition in terms of Lie ideals without zero divisor.

Theorem 2.3. [1] Let F be a field and R be an F-central algebra with a Lie ideal I without zero divisor. If the set of multiplicative cosets $\{aF \mid a \in I\}$ is of finite cardinality, then either R is a field or I is central.

To present our next result, we need to recall the following:
Let F be a field with $\mathrm{Char}(F) \neq 2$. By [2] when R is a finite dimensional F-algebra, then $[R, R]$ is a hyperplane in R. Consider the generalized quaternion algebra

$$D = \left(\frac{a, b}{F}\right) = \{\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \mid \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in F\},$$

where $i^2 = a$, $j^2 = b$, $ij = k$ and $a, b \in F$ [4, p. 136]. Then one can easily show that $[D, D] = span(\{i, j, k\}) = \{\alpha_1 i + \alpha_2 j + \alpha_3 k \mid \alpha_1, \alpha_2, \alpha_3 \in F\}$. In what follows we show that only generalized quaternion algebras D may contain non-central radical Lie ideal I without zero divisor and in this case $I = [D, D]$.

Theorem 2.4. [1] Let R be a non-commutative central F-algebra and $\mathrm{Char}(R) \neq 2$. If I is a non-central Lie ideal of R without zero divisor and radical over F, then $I = [R, R]$ and R is a generalized quaternion algebra.

We need the following technical Lemma to give our next result.

Lemma 2.5. [1] Let R be a ring and $a, y \in R$, such that a is not zero divisor. If $ay \in Z(R)$, then $ay = ya$.

Theorem 2.6. [1] Let R be a central F-algebra with $\mathrm{Char}(R) \neq 2$ and let I be a Lie ideal of R without zero divisor. If the residual additive group $(\frac{I+F}{F}, +)$ is of finite cardinality, then I is central.

3. Conclusion

In this paper, we study the rings with specific Lie ideals. We prove that when the non-zero additive commutators or the non-zero elements of a Lie ideal I of a ring R are invertible then R is a division ring. As a consequence we present a commutativity condition over a ring.

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A CIP method for solving PDEs problem on an unbounded domain by using artificial boundary conditions

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Abstract. In this paper, we propose a basis set approach by the Constrained Interpolation Profile (CIP) method for the solution of partial differential equations on an unbounded domain.Two exact artificial boundary conditions are introduced to reduce the original problem into an initial boundary value problem with a finite computational domain. We present a three stage numerical scheme Laplace transform in time variable, the CIP method and Talbot's method for numerical inversion of the Laplace transformation. Efficiency of the scheme is demonstrated by numerical results of sample problem.

Keywords: CIP method, unbounded domain, artificial boundary conditions, Laplace transform, Talbot's method

AMS Mathematics Subject Classification [2010]: 35R15

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1. Introduction

In this paper, we consider the numerical solution of partial differential equations(PDEs) problem on an unbounded domain. When numerically solving a differential equation defined on an unbounded domain, it is necessary to consider a finite sub-domain and to use artificial boundary conditions in such a way that the solutions in the finite sub-domain approximate the original solution [2]. If the approximation is exact, the transfer is called exact and the corresponding artificial boundary condition is called exact or transparent. The purpose of this paper is to establish a systematic and simple method to find the solution of the PDEs with any desired accuracy by generalizing the concept of the CIP method [5]. This method has been successfully applied to various complex fluid flow problems, covering both compressible and incompressible flow, such as shock wave generation, laser-induced evaporation, and elastic-plastic flows [5]. In comparison with finite element and finite difference methods, our method is at least simple for programming and transforms the differential integral equation to an algebraical equation.

Here, we consider an algorithm consist of three parts. The first part consists of Laplace transform method in time variable, converting the problem to a one parametric Sturm-Liouville equation with Robin's boundary conditions. The CIP method is used in the second part. The basis function has compact support, and is composed of simple polynomials that are easily extendable to any desired higher-order accuracy. The last part utilizes

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Talbot's method [4] for numerical inversion of the Laplace transformation and generates solution for the transformed problem. The difficulties associated with this inverse problem are due to its intrinsic ill-posedness, and the derivation of error estimates is the first step towards the development of a reliable inversion algorithm.

The organization of this paper is as follows. In section 2, we introduce and study one dimensional time dependent PDEs problem with transparent boundary conditions and it's corresponding transformed equations. Section 3, briefly describes the CIP method for approximating numerical solution of PDEs . Section 4 is devoted to presentation of numerical example to show the effectiveness of our approach.

2. Construction and study of the model

In this paper, we extended the mathematical model of [3], where the governing equation is nonhomogenous and this is the major part of our work. Here, we study the problem of the numerical approximation of a dispersive wave $u(x,t)$, the solution of the PDEs in an unbounded domain. More concretely, we consider the following linear equation:

$$\alpha\frac{\partial u}{\partial t}(x,t) = \beta\frac{\partial^2 u}{\partial x^2}(x,t) + \gamma u(x,t) + f(x,t), \quad \forall (x,t) \in \Omega, \tag{1}$$

$$u(x,0) = u^0(x) = g(x), \quad \forall x \in R,$$

where $\Omega = \{(x,t)| -\infty < x < +\infty, 0 < t \leqslant T\}, u^0(x)$ is the initial data given on R, the unknown function $u(x,t)$ is a function on Ω , $f(x,t)$ is known and α, β, γ are constants .

For computing the numerical solutions of such a whole space problem, we consider a finite subdomain and impose an artificial condition. Let us split the initial domain Ω into three regions. First introduce two artificial boundaries:

$$\Gamma_0 = \{(x,t)|\ x = 0,\ 0 < t \leqslant T\}, \quad \Gamma_1 = \{(x,t)|\ x = 1,\ 0 < t \leqslant T\}.$$

Then the domain Ω is divided into two unbounded parts:

$$\Omega_0 = \{(x,t)|\ -\infty < x \leqslant 0,\ 0 < t \leqslant T\}, \quad \Omega_1 = \{(x,t)|\ 1 \leqslant x < +\infty,\ 0 < t \leqslant T\},$$

and one bounded part: $\Omega^c = \{(x,t)|\ 0 < x < 1,\ 0 < t \leqslant T.$

Suppose that u^0 is compact support with: $supp\{u^0\} \subset [0,1]$. We consider the restriction of the solution of problem (1) on the domain Ω^c. Transparent boundary conditions for this problem are non-local in t and read:

(2)

$$\frac{\partial u}{\partial x}(0,t) = k_0\frac{d}{dt}\int_0^t \frac{u(0,\lambda)}{\sqrt{t-\lambda}}d\lambda + f_0(t),\ on\ \Gamma_0, \quad \frac{\partial u}{\partial x}(1,t) = k_1\frac{d}{dt}\int_0^t \frac{u(1,\lambda)}{\sqrt{t-\lambda}}d\lambda + f_1(t),\ on\ \Gamma_1,$$

where $f_0(t), f_1(t)$ are known and k_0, k_1 are constants. This initial boundary value problem is well-posed and its solution coincides with the solution of the original problem (1) restricted to $\overline{\Omega}^c$ [1]. Here, we focus on model for boundaries. We consider $u(0,0) = u(1,0) = 0$ and using Leibniz differentiation formula, we can rewrite Eqs (2) as follows:

(3)

$$\frac{\partial u}{\partial x}(0,t) = k_0\int_0^t \frac{d}{d\lambda}\{u(0,\lambda)\}\frac{d\lambda}{\sqrt{t-\lambda}} + f_0(t), \quad \frac{\partial u}{\partial x}(1,t) = k_1\int_0^t \frac{d}{d\lambda}\{u(1,\lambda)\}\frac{d\lambda}{\sqrt{t-\lambda}} + f_1(t).$$

We apply the Laplace transform to Eqs. (1), (3), we have:

$$\alpha s U(x,s) - \alpha g(x) = \beta\frac{\partial^2 U}{\partial x^2}(x,s) + \gamma U(x,s) + F(x,s), \quad \forall (x,s) \in \Omega^e, \tag{4}$$

$$\frac{\partial U}{\partial x}(0,s) = k_0 s\sqrt{\frac{\pi}{s}}.U(0,s) + F_0(s), \qquad \delta_0 < s < \infty, \tag{5}$$

$$\frac{\partial U}{\partial x}(1,s) = k_1 s\sqrt{\frac{\pi}{s}}.U(1,s) + F_1(s), \qquad \delta_0 < s < \infty. \tag{6}$$

where U, F, F_0, F_1 are Laplace transform of u, f, f_0, f_1 respectively and

$$\Omega^e = \{(x,s)| \quad 0 < x < 1, \quad \delta_0 < s < \infty\}.$$

3. Interpolation in the CIP method

For discrete, we consider $\triangle x = \frac{1}{N}$, where N is a positive integer and $\triangle x$ is spatial size [5]. The one dimensional nodal points are defined as $x_i = i\triangle x, \;\; i = 0, 1, 2, \ldots, N$. We consider the basis set as $u(x,t) = \sum_{i=0}^{N} r_i(t)\phi_i(x)$. Laplace transform of $u(x,t)$ is $U(x,s) = \sum_{i=0}^{N} R_i(s)\phi_i(x)$, where R is Laplace transform of r. Substituting $U(x,s)$ into Eq.(4), and multiplying $< \phi_i|$ from the left, we obtain the following simultaneous ordinary differential equation:

$$\alpha s(R_{i-1}(s) + 4R_i(s) + R_{i+1}(s))\frac{\triangle x}{6} - \alpha < \phi_i, g(x) >= \frac{\beta}{\triangle x}(R_{i-1}(s) - 2R_i(s) + R_{i+1}(s)) + \gamma\frac{\triangle x}{6}(R_{i-1}(s) + 4R_i(s) + R_{i+1}(s)) + < \phi_i, F(x,s) > .$$

After simplifying, we have:

$$\xi(s)R_{i-1}(s) + \delta(s)R_i(s) + \xi(s)R_{i+1}(s) = \alpha < \phi_i, g(x) > + < \phi_i, F(x,s) >, \tag{7}$$

where: $\xi(s) = \alpha s\frac{\triangle(x)}{6} - \frac{\beta}{\triangle(x)} - \gamma\frac{\triangle(x)}{6}$, $\delta(s) = 4\alpha s\frac{\triangle(x)}{6} + 2\frac{\beta}{\triangle(x)} - 4\gamma\frac{\triangle(x)}{6}$.
Substituting $U(x,s)$ into Eq.(5), then by multiplying $< \phi_0|$ from the left , we have:

$$4\mu(s)R_0(s) + (1+\mu(s))R_1(s) = 2 < \phi_0, F_0(s) >, \;\; \mu(s) = -\frac{k_0 s\triangle(x)}{3}\sqrt{\frac{\pi}{s}}. \tag{8}$$

Similarly, substituting $U(x,s)$ into Eq.(6), then by multiplying $< \phi_N|$ from the left , we have:

$$(1+\vartheta(s))R_{N-1}(s) + 4\vartheta(s)R_N(s) = -2 < \phi_N, F_1(s) >, \;\; \vartheta(s) = \frac{k_1 s\triangle(x)}{3}\sqrt{\frac{\pi}{s}}. \tag{9}$$

If we introduce the state vector $R(s) = (R_0(s), R_1(s), ..., R_N(s))^T$, and matrices:

$$W = \begin{pmatrix} 4\mu(s) & (1+\mu(s)) & 0 & . & . & . & 0 & 0 & 0 \\ \xi(s) & \delta(s) & \xi(s) & 0 & . & . & . & . & . \\ 0 & \xi(s) & \delta(s) & \xi(s) & 0 & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & 0 & \xi(s) & \delta(s) & \xi(s) & 0 \\ . & . & . & . & . & 0 & \xi(s) & \delta(s) & \xi(s) \\ 0 & 0 & 0 & . & . & . & 0 & (1+\vartheta(s)) & 4\vartheta(s) \end{pmatrix}$$

$B = (2 < \phi_0, F_0(s) >, \alpha < \phi_1, g(x) > + < \phi_1, F(x,s) >, ..., \alpha < \phi_{N-1}, g(x) > + < \phi_{N-1}, F(x,s) >, -2 < \phi_N, F_1(s) >)^T$

then, we obtain the corresponding system of (7), (8) and (9) which involves $(N+1)$ equations and $(N+1)$ unknowns $WR(s) = B$. This system can be solved by using LU decomposition methods. The next step of our numerical scheme consists of approximation inversion of Laplace transform with Talbot's method [4].

4. Numerical results

EXAMPLE 4.1. Consider the following problem:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + u(x,t) + x^8[t^{\frac{5}{2}}x^3 + t^2(x^2 - x) + \sqrt{t}(x^3 - x^2)],\ 0 < x < 1\ , 0 < t < 1,$$

$$\frac{\partial u}{\partial x}(0,t) = \int_0^t \frac{du(0,\lambda)}{d\lambda}.\frac{1}{\sqrt{t-\lambda}}d\lambda + tsint,\quad \frac{\partial u}{\partial x}(1,t) = 2\int_0^t \frac{du(1,\lambda)}{d\lambda}.\frac{1}{\sqrt{t-\lambda}}d\lambda + cost,\quad 0 < t < 1,$$

$$u(x,0) = 0, \qquad 0 \leqslant x \leqslant 1.$$

The set of nodal points are $R_M = \{(x_i, t_j) | 0 \leq i, j \leq M\}$, where $(x_i, t_j) \in (0,1) \times (0,1)$ are two dimensional random nodes. We take number of nodes N. In Talbot's method, there are some geometrical parameters υ, λ, δ and accuracy parameter n, that are different values. Numerical results of example 4.1 are given in Table 1 and Figure 1 , where $(x,t) \in [0.01, 0.8] \times [0.01, 0.8]$:

TABLE 1. Numerical results of example

$No.$	N	n	υ	λ	δ	$Max.Res$
1	4	4	10^{-1}	10^{-1}	0.99	4.3494×10^{-2}
2	4	4	10^{-1}	10^{-2}	0.99	5.9076×10^{-3}
3	8	4	10^{-1}	10^{-1}	0.99	7.5615×10^{-3}
4	8	4	10^{-1}	10^{-2}	0.99	5.9071×10^{-3}
5	14	4	10^{-1}	10^{-1}	0.99	6.4413×10^{-3}

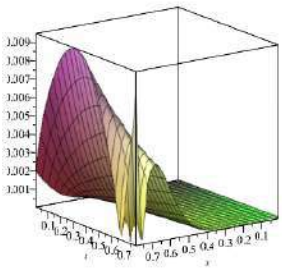

FIGURE 1. In this figure $N = 8$, $n = 4$, $\upsilon = 10^{-1}$, $\lambda = 10^{-2}$, $\delta = 0.99$.

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A new asymmetric distribution generated by Laplace distribution

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Abstract. In this paper, a new method is proposed for generating families of continuous distributions. It is used for the classic Laplace distribution and a new class of asymmetric continuous distributions is introduced. Some mathematical properties of the new distribution are provided. In particular, rth moment, variance, skewness, kurtosis are derived. Also, the asymptotic distribution of the extreme order statistics are investigated. Using the maximum likelihood estimation method the proposed distribution is fitted to the set of real data and by AIC and BIC criteria the goodness of fitting the proposed distribution is demonstrated.

Keywords: Classic Laplace distribution, Two-sided Laplace transform, Asymmetric distribution.

AMS Mathematics Subject Classification [2010]: 60E05, 62E10.

1. Introduction

Statistical distributions are commonly applied to describe real world phenomena. The usefulness of statistical distributions has motivated researchers seeking and developing new and more flexible distributions. Many generalized classes of distributions have been developed and applied to describe various phenomena.

One of the earliest distributions in probability theory is the Laplace distribution that its probability density function (pdf) and cumulative distribution function (cdf) are given by

$$f(x) = \frac{1}{2\sigma}\exp(-\frac{|x-\theta|}{\sigma}),\ F(x) = \begin{cases} \frac{1}{2}\exp(\frac{x-\theta}{\sigma}) & if \quad x \le \theta \\ 1-\frac{1}{2}\exp(\frac{\theta-x}{\sigma}) & if \quad x > \theta \end{cases} \tag{1}$$

where $\theta \in \mathbb{R}$ and $\sigma > 0$. This distribution is very popular in many areas of science and engineering and often used for modeling phenomena with heavier than normal tails. See [3] for an overview.

In the last several decades, many studies have been published with extensions and applications of the Laplace distribution and various form of skewed Laplace distributions have appeared in the literature. Yu and Zhang [5] have proposed a three parameter

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asymmetric Laplace distribution. Corderio and Lemonte [1] have proposed the so called beta Laplace distribution as an extension of the Laplace distribution. Kozubowski and Nadarajah [2] provided a comprehensive review of the known Laplace distributions along with their properties and applications.

In this article, we first present a new technique to generate families of continuous probability distributions. This is used for the classic Laplace distribution and the new skew family is proposed, called the new skew Laplace (NSL) distribution. This distribution is convenient for modeling asymmetric data. The article is outlined as follows. In section 2, we introduce the NSL distribution and obtain some statistical properties. In section 3, we investigate the asymptotic distribution of the extreme order statistics. In section 4, we test the validity of the proposed distribution by considering a real data set and compare the values of Akaike information criterion (AIC) and Bayesian information criterion (BIC) with the values of some other distributions. The article ends with some conclusions.

2. Definition and basic properties

In this section, we first describe the methodology for generating new distributions. Then, we present the new skew Laplace distribution and some of its more important properties.

THEOREM 2.1. *Let X be a continuous random variable with pdf g, cdf G and support $\mathbb{R}$, if we have*

$$\lim_{x \to -\infty} \exp(-\lambda x) G(x) = 0 \tag{2}$$

for each $\lambda \in \mathcal{D} \subseteq \mathbb{R}^+$, then the following function

$$f(x) = \frac{\lambda \exp(-\lambda x) G(x)}{\mathcal{L}(\lambda)} \tag{3}$$

is a density function, where $\mathcal{L}(\lambda)$ is the two-sided Laplace of $g(x)$ defined as

$$\mathcal{L}(\lambda) = \int_{-\infty}^{\infty} \exp(-\lambda x) g(x) dx. \tag{4}$$

Now, we introduce a new class of skew Laplace distribution by taking g and G in Theorem 2.1 to be the pdf and cdf of the classic Laplace distribution given by (1) with $\sigma = 1$. The corresponding pdf is

$$f(x) = \begin{cases} \frac{\lambda(1-\lambda^2)}{2} \exp(-\lambda(x-\theta)) \exp(x-\theta) & if \quad x \leq \theta \\ \\ \lambda(1-\lambda^2) \exp(-\lambda(x-\theta)) \{1 - \frac{1}{2} \exp(\theta - x)\} & if \quad x > \theta \end{cases} \tag{5}$$

where $\theta \in \mathbb{R}$ and $0 < \lambda < 1$. If the pdf of a random variable is given by (5), it is denoted by $X \sim NSL(\theta, \lambda)$ and it has the following cdf

$$F(x) = \begin{cases} \frac{\lambda(1+\lambda)}{2} \exp(-\lambda(x-\theta)) \exp(x-\theta) & if \quad x \leq \theta \\ \\ 1 - (1-\lambda^2) \exp(-\lambda(x-\theta)) \{1 - \frac{\lambda}{2(1+\lambda)} \exp(\theta - x)\} & if \quad x > \theta \end{cases} \tag{6}$$

Figure 1 shows shape of the pdf (5) for $\lambda = 0.1, 0.3, 0.5, 0.7, 0.9$ and $\theta = 0$. The main feature of the new skew Laplace distribution in (5) is that the parameter λ control skewness and kurtosis.

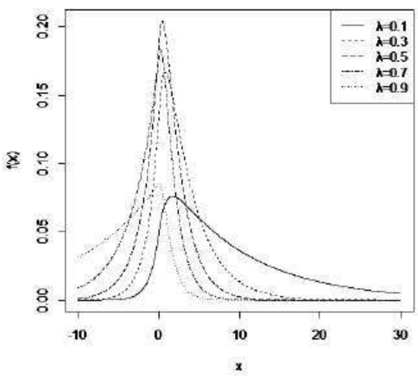

FIGURE 1. Density function of the new skew Laplace distribution with $\theta = 0$ and various values of λ.

PROPOSITION 2.2. *Let $X \sim NSL(\theta, \lambda)$, then the rth moment of X is*

$$E(X^r) = \begin{cases} \dfrac{\lambda(1-\lambda^2)}{2(1-\lambda)^{r+1}} \displaystyle\sum_{i=0}^{r} \frac{r!}{i!}(-1)^i(\theta(1-\lambda))^i + \frac{(1-\lambda^2)}{\lambda^r} \sum_{i=0}^{r} \frac{r!}{i!}(\lambda\theta)^i & \\ - \dfrac{\lambda(1-\lambda^2)}{2(1+\lambda)^{r+1}} \displaystyle\sum_{i=0}^{r} \frac{r!}{i!}(\theta(1+\lambda))^i & \text{for } r \text{ even,} \\ & \\ \dfrac{\lambda(1-\lambda^2)}{2(1-\lambda)^{r+1}} \displaystyle\sum_{i=0}^{r} \frac{r!}{i!}(-1)^{i-1}(\theta(1-\lambda))^i + \frac{(1-\lambda^2)}{\lambda^r} \sum_{i=0}^{r} \frac{r!}{i!}(\lambda\theta)^i & \\ - \dfrac{\lambda(1-\lambda^2)}{2(1+\lambda)^{r+1}} \displaystyle\sum_{i=0}^{r} \frac{r!}{i!}(\theta(1+\lambda))^i & \text{for } r \text{ odd.} \end{cases}$$

Using the above proposition, the expectation, variance, skewness and the kurtosis of X are

$$E(X) = \theta + \frac{1-3\lambda^2}{\lambda(1-\lambda^2)}, \; Skewness(X) = \frac{2\,(1-3\lambda^2-3\lambda^4-3\lambda^6)}{(3\lambda^4+1)^{\frac{3}{2}}}$$

$$Var(X) = \frac{3\lambda^4+1}{\lambda^2(1-\lambda^2)^2}, \; Kurtosis(X) = \frac{3\,(3-8\lambda^2+22\lambda^4+16\lambda^6+15\lambda^8)}{(3\lambda^4+1)^2}$$

3. Asymptotic Distribution of Extremes

In this section, the asymptotic distribution of extreme values $M_n = max(X_1, \ldots, X_n)$ and $m_n = min(X_1, \ldots, X_n)$ are investigated. Let $X_1, X_2, \ldots$ be independent random variables with probability density and cumulative distribution function (5) and (6), respectively. It can be easily shown that

$$\lim_{t\to\infty} \frac{1-F(t+\frac{x}{\lambda})}{1-F(t)} = \exp(-x) \tag{7}$$

then by Theorem 1.6.2 in [4] , the asymptotic distribution of M_n normalized belong to Type I Extreme Value distribution, i.e. for suitable normalizing constant $a_n > 0$ and b_n, we have

$$P(a_n(M_n - b_n) \leq x) \to \exp(-\exp(-x)) \; as \; n \to \infty$$

Using Corollary 1.6.3 in [4], the form of the normalizing constants can be determined, for instance $a_n = \lambda$ and $b_n = \mu + \frac{\ln n(1-\lambda^2)}{\lambda}$. As a similar method, since

$$\lim_{t\to\infty} \frac{F(-t - \frac{x}{1-\lambda})}{F(-t)} = \exp(-x) \tag{8}$$

then, for suitable constants $c_n > 0$ and d_n, we have

$$P(c_n(m_n - d_n) \leq x) \to 1 - \exp(-\exp(x)) \; as \; n \to \infty.$$

4. Application

As an application, a real data set was considered, which consisted of physical and blood measurements of 202 athletes at the Australian Institute of Sport. (This data set is available in the `DAAG` package in `R` software, where it is named `ais`). We considered the body mass index (BMI) of athletes. Using maximum likelihood estimation method (MLE), we fitted three distributions, classic Laplace (CL), new skew Laplace (NSL) and skewed Laplace distribution introduced by Aryal and Nadarajah (ANSL), [2], to the data and as can be seen in the below tabel, the NSL distribution has the best fitness.

Table 1. Comparison of fit of the three distributions for BMI data.

Distribution	Estimated values (MLE)	AIC	BIC
CL	$\hat{\theta} = 22.702$, $\hat{\sigma} = 2.122$	992.158	998.776
NSL	$\hat{\theta} = 21.436$, $\hat{\lambda} = 0.403$	**986.113**	**992.7302**
ANSL	$\hat{\phi} = 22.951$, $\hat{\lambda} = 47.021$	1673.964	1680.581

5. Conclusion

In this study, a new method for generating families of continuous distributions is proposed. Based on the proposed method a new skew Laplace distribution is introduced. Some of its basic properties are investigated. From the computation, it is examined that the proposed distribution provides best fitting to the data set under consideration in terms of the criteria, AIC and BIC. This study focuse on the case that the basic distribution in Theorem 2.1 is the classic Laplace distribution. Different basic distributions may be suggested.

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Kerman, Iran
30 August - 02 September 2021

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کنفرانس ریاضی ایران
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دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Some fixed point theorems for weak contraction in Q-fuzzy metric spaces

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Abstract. In this paper, we define weak contraction mappings in generalized fuzzy metric space. We also prove several common fixed point theorems for mappings in generalized fuzzy metric space.

Keywords: Fixed point; Generalized fuzzy metric; weak contraction; Common fixed point.

AMS Mathematics Subject Classification [2010]: 47H10, 54H25, 54A40

1. Introduction

Investigation of the existence and uniqueness of fixed points of certain mappings in the framework of metric spaces is one of the centers of interests in nonlinear functional analysis [1]. Fixed point theory has a wide application in almost all fields of quantitative sciences such as economics, biology, physics, chemistry, computer science and many branches of engineering. In 1965, Zadeh [12] introduced the concept of fuzzy sets. In 1975, Kramosil and Michalek [8] put forward a new concept of fuzzy metric space. Then, in 1994, George and Veeramani [6] revised the notion of fuzzy metric space with the help of continuous t- norm. Grabiec [5] initiated the study of fixed point theory in fuzzy metric space. In 2000, Gregori and Sapena [7] introduced new kind of contractive mappings in modified fuzzy metric spaces and proved a fuzzy version of Banach contraction principle. In 1992, Dhage ([2], [3], [4]) in his Ph. D. thesis introduced a new class of generalized metric space called D-metric spaces. In 2003, Mustafa and Sims [9] showed that most of the results claimed concerning of such spaces are invalid. Then they introduced a generalization of metric spaces (X, d), which are called G-metric spaces [10]. In 2010, Sun and Yang [11] introduced the concept of Q-fuzzy metric space, as a generalization of fuzzy metric space, and proved two common fixed point theorems for four mappings. In this paper, we define ψ-weak contraction mappings in generalized fuzzy metric space. We also prove several common fixed point theorems for mappings in generalized fuzzy metric space. Our results in this paper improve and generalize know results due to Gregori and Sapena [7]. We recall some definitions and propositionositionerties for G-metric spaces given by Mustafa

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and Sims. Fuzzification of generalized metric space was studied by Sun and Yang [11]. They defined the Q-fuzzy metric space as follows:

Definition 1.1 ([11]). A 3-tuple $(X, Q, *)$ is called a Q-fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and Q is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions, for each $x, y, z, a \in X$ and $t, s > 0$:

(1) $Q(x, x, y, t) > 0$ and $Q(x, x, y, t) \geq Q(x, y, z, t)$ for all $x, y, z \in X$ with $z \neq y$,
(2) $Q(x, y, z, t) = 1$ if and only if $x = y = z$,
(3) $Q(x, y, z, t) = Q(P(x, y, z), t)$, (symmetry) where P is a permutation function,
(4) $Q(x, y, z, t + s) \geq Q(x, a, a, t) * Q(a, y, z, s)$,
(5) $Q(x, y, z, .) : (0, \infty) \to [0, 1]$ is continuous.

Example 1.2 ([11]). Let X is a nonempty set and G is the G-metric on X. Denote $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t > 0$:

$$Q(x, y, z, t) = \frac{t}{t + G(x, y, z)}$$

Then $(X, Q, *)$ is a Q-fuzzy metric.

Let $(X, Q, *)$ be a Q-fuzzy metric space. For $t > 0$, the open ball $B_Q(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by:

$$B_Q(x, r, t) = \{y \in X : Q(x, y, y, t) > 1 - r\}.$$

Definition 1.3 ([11]). A sequence $\{x_n\}$ in X converges to x if and only if $Q(x_n, x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$.

Definition 1.4 ([11]). A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that $Q(x_m, x_n, x_n, t) > 1 - \epsilon$ for each $n, m \geq n_0$.

Remark 1.5 ([11]). Let $(X, Q, *)$ be a Q-fuzzy metric space. Then if

a) there exists a $k \in (0, 1)$ such that:

$$Q(y_{n+2}, y_{n+1}, y_{n+1}, kt) \geq Q(y_{n+1}, y_n, y_n, t)$$

for each $t > 0$ and $n \in \mathbb{N}$. Then $\{y_n\}$ is a Cauchy sequence in X.

b) there exists a $k \in (0, 1)$ such that:

$$Q(x, y, z, kt) \geq Q(x, y, z, t)$$

for each $t > 0$ and $n \in \mathbb{N}$. Then $x = y = z$.

Definition 1.6 ([11]). Let $(X, Q, *)$ be a Q-fuzzy metric space. Then Q is a continuous function on $X^3 \times (0, \infty)$.

Definition 1.7 ([11]). Let f and g be self mappings on a Q- fuzzy metric space $(X, Q, *)$. Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is, $fx = gx$ implies that $fgx = gfx$.

Definition 1.8 ([11]). Let f and g be self mappings on a Q-fuzzy metric space $(X, Q, *)$. The pair (f, g) is said to be compatible if

$$\lim_{n\to\infty} Q\big(fgx_n, gfx_n, gfx_n, t\big) = 1$$

whenever $\{x_n\}$ is a sequence in X suth that:

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$$

for some $z \in X$.

The fuzzy Banach contraction theorem was studied by Grabiec in 1988. The fuzzy contractive was introduced by Gregori and Sapena [7] in 2000.

Definition 1.9 ([7]). Let $(X, M, *)$ be a fuzzy metric space. The mapping $f : X \to X$ is said a fuzzy contractive if there exists $0 < k < 1$ such that

$$\frac{1}{M(fx, fy, t)} - 1 \leq k\Big(\frac{1}{M(x, y, t)} - 1\Big)$$

for each $x, y \in X$ and $t > 0$. (k is called the contractive constant of f).

2. Main Results

We introduce fuzzy contractive mapping in generalized fuzzy metric space (Q-fuzzy metric space). We also prove several common fixed point theorems for mappings in generalized fuzzy metric space.

Definition 2.1. Let $(X, Q, *)$ be a Q-fuzzy metric space. We will say the mapping $f : X \to X$ is Q-fuzzy contractive if there exists $0 < k < 1$ such that

$$\frac{1}{Q(fx, fy, fy, t)} - 1 \leq k\Big(\frac{1}{Q(x, y, y, t)} - 1\Big)$$

for each $x, y \in X$ and $t > 0$. (k is called the contractive constant of f).

Definition 2.2. Let $(X, Q, *)$ be a Q-fuzzy metric space. We will say that the sequence $\{x_n\}$ in X is Q-fuzzy contractive if there exists $0 < k < 1$ such that

$$\frac{1}{Q(x_{n+1}, x_{n+2}, x_{n+2}, t)} - 1 \leq k\Big(\frac{1}{Q(x_n, x_{n+1}, x_{n+1}, t)} - 1\Big)$$

for all $t > 0$, $n \in \mathbb{N}$.

Now we introduce the notion of ψ-weak contractivity of a mapping T with respect to a self mapping f on a Q-fuzzy metric space X as follows:

Definition 2.3. Let $(X, Q, *)$ be a Q-fuzzy metric space and $f : X \to X$ be a mapping. The mapping $T : X \to X$ is called a ψ-weak contraction with respect to f if there exists a function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ such that

$$\frac{1}{Q(Tx, Ty, Ty, t)} - 1 \leq \Big(\frac{1}{Q(fx, fy, fy, t)} - 1\Big) - \psi\Big(\frac{1}{Q(fx, fy, fy, t)} - 1\Big)$$

for every $x, y \in X$ and each $t > 0$. If f is an identity mapping on X, then T is called a ψ-weak contraction.

Theorem 2.4. *Let $(X, Q, *)$ be a Q-fuzzy metric space and $T : X \to X$ be a ψ-weak contraction with respect to self mapping f on X. If the range of f contains the range of T and $f(X)$ is a complete subspace of X, then f and T have coincidence point in X provided that ψ is a continuous mapping.*

Example 2.5. Let $X = [0, 1]$ with the usual G-metric defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}.$$

Let $*$ be the continuous t-norm defined by $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. For all $x, y, z \in X$ and $t \in (0, \infty)$ define Q-fuzzy metric by

$$Q(x, y, z, t) = \frac{t}{t + G(x, y, z)}.$$

Define $\psi : [0, \infty) \to [0, \infty)$ as $\psi(t) = \frac{1}{c}t$, $Tx = ax$, $a \neq 0$ and $fx = cx + b$, $c > 0$, $a \neq b$, $b \neq 0, 1$ and $c - 1 \geq a$. Now, we have

$$\begin{aligned}
\Big(\frac{1}{Q(fx, fy, fy, t)} - 1\Big) - \psi\Big(\frac{1}{Q(fx, fy, fy, t)} - 1\Big) &= (c-1)\frac{|x-y|}{t} \\
&\geq a\Big(\frac{|x-y|}{t}\Big) \\
&= \frac{1}{Q(Tx, Ty, Ty, t)} - 1.
\end{aligned}$$

Therefore T satisfies all the conditions of Theorem 2.4. Moreover f and T have a coincidence point.

Corollary 2.6. *Let $(X, Q, *)$ be a complete Q-fuzzy metric space. $f : X \to X$ be a ψ-weak contraction. If ψ is continuous then f has a unique fixed point.*

Corollary 2.7. *Let $(X, Q, *)$ be a Q-fuzzy metric space. $f : X \to X$ be a mapping satisfying*

$$\frac{1}{Q(fx, fy, fy, t)} - 1 \leq k\Big(\frac{1}{Q(x, y, y, t)} - 1\Big)$$

for each $x, y \in X, t > 0$ and $k \in (0, 1)$. Then f has a unique fixed point.

Proof. We get $\psi(r) = (1-k)r$ for all $r > 0$ in corollary 2.6. □

Theorem 2.8. *Let $(X, Q, *)$ be a Q-fuzzy metric space and $T : X \to X$ be a ψ-weak contraction with respect to self mapping f on X. If the range of f contains the range of T and $f(X)$ is a complete subspace of X, then f and T have a common fixed point in X provided that ψ is a continuous and the pair of mappings (T, f) is weakly compatible.*

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RD-injectivity from a different perspective

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Abstract. We study the notion of RD-injectivity from a different perspective. A module M is said to be *A-RD-subinjective* if for every RD-extension B of A, every homomorphism from A to M can be extended to a homomorphism from B to M. The *RD-subinjectivity domain* of a module M, $\underline{RDI}^{-1}(M)$, is defined to be the collection of all modules A such that M is A-RD-subinjective.

Keywords: RD-injective module; RD-projective module; A-RD-subinjective module.

AMS Mathematics Subject Classification [2010]: 16D80; 16D10; 16D50

1. Introduction

In [1], Aydoğdu and López-Permouth studied the notion of subinjectivity. Namely, a module M is called *A-subinjective* if for every extension B of A, every homomorphism from A to M can be extended to a homomorphism from B to M. For a module M, the *subinjectivity domain* of M, $\underline{I}^{-1}(M)$, is defined to be the collection of all modules A such that M is A-subinjective. In contrast to the notion of pure-injectivity, López-Permouth et al. studied in [5] the notion of pure-subinjectivity. Namely, a module M is called *A-pure-subinjective* if for every pure extension B of A, every homomorphism from A to M can be extended to a homomorphism from B to M. For a module M, the *pure-subinjectivity domain* of M, $\underline{PI}^{-1}(M)$, is defined to be the collection of all modules A such that M is A-pure-subinjective. Clearly, the subinjectivity domain $\underline{I}^{-1}(M)$ of a module M is contained in $\underline{PI}^{-1}(M)$.

The concept of relatively divisible (RD) submodule, with the related notions of RD-projective and RD-injective module, was introduced by Warfield [6] in 1969. The goal of this paper is to initiate the study of an alternative perspective on the analysis of the RD-injectivity of a module. In contrast to the well-known notion of RD-injectivity, we introduce the notion of RD-subinjectivity. Namely, a module M is said to be *A-RD-subinjective* if for every RD-extension B of A, every homomorphism from A to M can be extended to a homomorphism from B to M. For every module M, the *RD-subinjectivity domain* of M consists those modules A such that M is A-RD-subinjective.

Throughout this paper, R denotes an associative ring with identity and all modules will be assumed to be unitary. In what follows $\mathrm{E}(M)$ and $\mathrm{RDE}(M)$ denote the injective

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hull and the RD-injective hull of a module M, respectively. A cyclic right R-module C_R is called cyclically presented if $M_R \cong R/aR$ for some $a \in R$.

2. Main Results

Recall that an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of right R-modules is said to be *pure exact* (resp., *RD-exact*) if the induced homomorphism

$$\mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, C)$$

is surjective for any finitely presented (resp., cyclically presented) right R-module M. A submodule A of a right R-module B is called a *pure submodule* (resp., *RD-submodule*) if the exact sequence

$$0 \longrightarrow A \hookrightarrow B \longrightarrow B/A \longrightarrow 0$$

is pure (resp., RD-exact). An R-module M is said to be *pure-injective* (resp., *RD-injective*) if it is injective with respect to pure exact (resp., RD-exact) sequences. Also, an R-module M is said to be *pure-projective* (resp., *RD-projective*) if it is projective with respect to pure exact (resp., RD-exact) sequences (see [2], [3] and [6]).

Definition 2.1. Given modules M and A, we say that M is *A-RD-subinjective* if for every RD-extension B of A and every homomorphism $\varphi : A \to M$, there exists a homomorphism $\phi : B \to M$ such that $\phi|_A = \varphi$. The *RD-subinjectivity domain* of a module M, $\underline{RDI}^{-1}(M)$, is defined to be the collection of all modules A such that M is A-RD-subinjective.

Lemma 2.2. *An RD-monomorphism $f : A \to B$ extends to a splitting monomorphism $g : \mathrm{RDE}(A) \to \mathrm{RDE}(B)$.*

Proof. See [4, Theorem 1.8]. □

Proposition 2.3. *Let A_1, A_2 and M be right R-module. Then $A_1 \oplus A_2 \in \underline{RDI}^{-1}(M)$ if and only if $A_i \in \underline{RDI}^{-1}(M)$ for $i = 1, 2$.*

Proof. Assume that $A_1 \oplus A_2 \in \underline{RDI}^{-1}(M)$ and $f : A_1 \to M$ a homomorphism. Since A_1 is an RD-submodule of $A_1 \oplus A_2$, by Lemma 2.2, $\mathrm{RDE}(A_1)$ is a direct summand of $RDE(A_1 \oplus A_2)$. Suppose that $\pi_{A_1} : A_1 \oplus A_2 \to A_1$ and $\iota_{A_1 \oplus A_2} : A_1 \oplus A_2 \to RDE(A_1 \oplus A_2)$ are the natural projection and inclusion, respectively. Then there exists a homomorphism $g : RDE(A_1 \oplus A_2) \to M$ such that $g\iota_{A_1 \oplus A_2} = f\pi_{A_1}$. It is easy to check that $g\iota_{RDE(A_1)}\iota'_{A_1} = f$, where $\iota_{RDE(A_1)} : RDE(A_1) \to RDE(A_1 \oplus A_2)$ and $\iota'_{A_1} : A_1 \to RDE(A_1)$ are the inclusions. Thus, $A_1 \in \underline{RDI}^{-1}(M)$. Similarly, $A_2 \in \underline{RDI}^{-1}(M)$, as required. Now, suppose that $f : A_1 \oplus A_2 \to M$ is a homomorphism, $\iota_{A_1 \oplus A_2}$ and ι_{A_j} denote the inclusions $A_1 \oplus A_2 \hookrightarrow RDE(A_1 \oplus A_2)$ and $A_j \hookrightarrow A_1 \oplus A_2$, respectively, for each $j = 1, 2$. Since $A_j \in \underline{RDI}^{-1}(M)$, there exist $g_j : RED(A_j) \to M$ with $g_j\iota'_{A_j} = f\iota_{N_j}$ for $j = 1, 2$. Since $RDE(A_1)$ and $RDE(A_2)$ are direct summands of $RDE(A_1 \oplus A_2)$, consider the homomorphism $g_1\pi_{RDE(A_1)} + g_2\pi_{RDE(A_2)} : RDE(A_1 \oplus A_2) \to M$ where $\pi_{RDE(A_1)}$ and $\pi_{RDE(A_2)}$ are natural projections. Therefore,

$$(g_1\pi_{RDE(A_1)} + g_2\pi_{RDE(A_2)})\iota_{A_1 \oplus A_2} = f$$

and so M is $(A_1 \oplus A_2)$-RD-subinjective. □

Lemma 2.4. *A module P is RD-projective if and only if it has the projective property relative to every RD-exact sequence $0 \to K \to M \to L \to 0$ of modules with M RD-injective.*

Proposition 2.5. *For a right R-module A, consider the following conditions:*

(1) *A is RD-projective.*
(2) *Every RD-quotient of an A-RD-subinjective module is A-RD-subinjective.*
(3) *Every RD-quotient of an RD-injective module is A-RD-subinjective.*

Then (1)⇒(2)⇒(3). *If $RDE(A)$ is RD-projective, then* (3)⇒(1).

Proof. (1)⇒(2) Assume that M be an A-RD-subinjective module, K an RD-submodule of M and $f : A \to M/K$ a homomorphism. Suppose $\pi : M \to M/K$ denote the natural projection. By the RD-projectivity of A, there exists a homomorphism $g : A \to M$ such that $f = \pi g$. Since M is A-RD-subinjective, $g = h\iota_A$ for some homomorphism $h : RDE(A) \to M$ where $\iota_A : A \to RDE(A)$ is the inclusion. Therefore, $\pi h\iota_A = f$ and so M/K is A-RD-subinjective.

(2)⇒(3) is clear.

(3)⇒(1) Assume that M be an RD-injective module and consider an RD-exact sequence $0 \to K \hookrightarrow M \to M/K \to 0$ where K is an RD-submodule of M and $\pi : M \to M/K$ is projection. Let $f : A \to M/K$ be a homomorphism. By hypothesis, $A \in \underline{RDI}^{-1}(M/K)$. Then $g\iota = f$ for some $g : RDE(A) \to M/K$ where $\iota : A \to RDE(A)$ is the inclusion. By the RD-projectivity of $RDE(A)$, there exists $h : RDE(A) \to M$ such that $\pi h = g$. Hence, we have $\pi h\iota = f$. Therefore, by Lemma 2.4, A is RD-projective. □

Proposition 2.6. *For a right R-module A, consider the following conditions:*

(1) *A is projective.*
(2) *Every quotient of an A-RD-subinjective module is A-RD-subinjective.*
(3) *Every quotient of an injective module is A-RD-subinjective.*

Then (1)⇒(2)⇒(3). *If $RDE(A)$ is projective, then* (3)⇒(1).

Proof. The proof similar to the proof of Proposition 2.5. □

Corollary 2.7. *For a right R-module M, the following statements are equivalent:*

(1) *M is C-RD-subinjective, for each cyclically presented right R-module C.*
(2) *Every finitely generated RD-projective right R-module belongs to $\underline{RDI}^{-1}(M/K)$ for each RD-submodule K of M.*

Proof. (1) ⇒ (2) Assume that $C \in \underline{RDI}^{-1}(M)$, for each cyclically presented right R-module C and P is finitely generated RD-projective right R-module. Thus, by [6, Corollary 1], P is isomorphic to a direct summand of $N_R := \oplus_{i=1}^{n} R/r_iR$ where $r_i \in R$ $(1 \leq i \leq n)$. So, by Proposition 2.3, $N \in \underline{RDI}^{-1}(M)$ and so $P \in \underline{RDI}^{-1}(M)$. Therefore, Proposition 2.5 allows us to conclude.

(2) ⇒ (1) is clear, since 0 is an RD-submodule of M and every cyclically presented module is RD-projective. □

Similar to the proof of Corollary 2.7, we have:

Corollary 2.8. *For a right R-module M, the following statements are equivalent:*

(1) *M is R-RD-subinjective.*
(2) *Every finitely generated projective right R-module belongs to $\underline{RDI}^{-1}(M/K)$ for each submodule K of M.*
(3) *Every finitely generated projective right R-module belongs to $\underline{RDI}^{-1}(M/K)$ for each RD-submodule K of M.*

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Some results on Generalized module maps and ideal submodules of Finsler modules

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Abstract. Let E and F be Finsler modules over C^*-algebras A and B, respectively. In this talk, we state the notion of ideal submodules and unitary operators on Finsler modules and investigate some features of them. In particular, we show that for a $*$-homomorphism $\varphi : A \to B$, a surjective φ-module map $T : E \to F$ is a unitary operator if and only if φ is a bijection and T is an isometry. Finally, introducing the concepts of bi-generalized unitary equivalence and the induced Finsler modules, we show that the Finsler B-module F and the induced Finsler A-module E are unitary equivalent.

Keywords: (Full) Finsler modules; Hilbert C^*-module; unitary operator.

AMS Mathematics Subject Classification [2010]: 46L08

1. Introduction

A (left) Hilbert C^*-module over a C^*-algebra A is an algebraic left A-module E equipped with an A-valued inner product $< ., . >$ which is A-linear in the first and conjugate linear in the second variable such that E is Banach space with respect to the norm $\|x\| = \| < x, x > \|^{\frac{1}{2}}$. The Hilbert A-module E is called *full* if $A_E := span\{< x, y >: x, y \in E\}$ is dense in A. Note that A_E is an ideal in E, called the *range ideal* of E. We denote by $< E, E >$ the closure of A_E and call it the *support* of E. Therefore, E is a full Hilbert A-module if $< E, E >$ is equal to A (see [3]).

Let E and F be Hilbert modules over C^*-algebra B, $\varphi : A \to B$ a linear isomorphism and $T : E \to F$ a bijective linear operator. Consider the module action $a.x := T^{-1}(\varphi(a).T(x))$ on E. Then, E via the inner product $<< x, z >>:= \varphi^{-1}(< T(x), T(z) >)$ is a Hilbert A-module called, *the Hilbert A-module induced by* (φ, T) or briefly *the induced Hilbert A-module*. We denote the so-called induced Hilbert A-module by $E^{(\varphi,T)}$.

The induced Hilbert C^*-modules was introduced in [4].

Finsler modules over C^*-algebras are generalization of Hilbert C^*-modules. This concept is introduced by N.C.Phillips and N.Weaver, [6]. Let A be a C^*-algebra and A^+ be the set of all positive elements of A. Suppose that E is complex linear space which is a left A-module (and $\lambda(ax) = (\lambda a)x = a(\lambda x)$ where $\lambda \in C, a \in A$ and $x \in E$) and the a map $\rho_A : E \to A^+$ satisfies the following conditions.

(i) The map $\|.\|_E : x \to \|\rho_A(x)\|$ is a norm on E and

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(ii) $\rho_A(ax)^2 = a\rho_A(x)^2a^*$ for all $a \in A$ and $x \in E$.

Then, E is called a *Finsler module over C^*-algebra A* or *pre-Finsler A-module.* If $(E, \|.\|_E)$ is complete, then it is called a *Finsler A-module.* A Finsler A-module is said to be *full* if the linear span $\{\rho_A(x)^2 : x \in E\}$ denoted by $< \rho_A(E) >$ is dense in A.

As an example, suppose A is a C^*-algebra and take $E := A$. Then, A with respect to its product as the usual action, is a left A-module. Additionally, A equipped with the map $\rho_A : A \to A^+$ defined by $\rho_A(a) := (a.a^*)^{\frac{1}{2}}$ is a Finsler A-module. Now, for every $a \in A^+$, there exists a unique element $b \in A$ (in fact, $b \in A^+$) such that $a^2 = b.b^*$ and hence, $a = (b.b^*)^{\frac{1}{2}} = \rho_A(b)$. Consequently, E is a full Finsler module.

Let E be a Finsler A-module and I be an ideal in A (throughout this paper by an ideal we always mean a closed two-sided ideal). The *associated ideal submodule* E_I is defined by the closed linear span of the action of I on E. That is

$$E_I := \overline{span\ \{ax : \ a \in I, \ x \in E\}}.$$

Clearly, E_I is a closed submodule of E and by the Hewitt-Cohen factorization theorem, it is easy to show that $E_I = \{ax : \ a \in I, \ x \in E\}$. Also, it is known [1] that $E_I = \{x \in E : \ \rho_A(x) \in I\}$. It is notable that E_I can be regarded as a Finsler module over I via $\rho_I(ax) := \rho_A(ax)$ since $\rho_A(ax)^2 = a\rho_A(x)^2a^* \in I$ whenever $a \in I$ and $x \in E$. Denote by $\pi : A \to \frac{A}{I}$ and $q : E \to \frac{E}{E_I}$ the quotient maps. Then, $\frac{E}{E_I}$ is an $\frac{A}{I}$-module equipped with $\pi(a)q(x) := q(ax)$. Moreover, it is a Finsler module over $\frac{A}{I}$ with the via $\rho_{\frac{A}{I}}(q(x)) := \pi(\rho_A(x))$. So, we obtain a natural Finsler module structure on the quotient of the Finsler module E over the ideal submodule E_I (see [6, Lemma 12]).

From now to the end of this section, E, F and G are assumed to be Finsler modules over C^*-algebras A, B and C, respectively.

Definition 1.1. Let $\varphi : A \to B$ be a linear $*$-homomorphism. A linear map $T : E \to F$ is called a *φ-module map* if $T(ax) = \varphi(a)T(x)$ for all $a \in A$, $x \in E$. More generally, a linear map $T : E \to F$ is a *generalized module map* if there exists a linear homomorphism $\varphi : A \to B$ such that T is a φ-module map.

A φ-module map $T : E \to F$ is called a *φ-homomorphism* if $\rho_B(T(x)) = \varphi(\rho_A(x))$ for all $a \in A$, $x \in E$. More generally, a linear map $T : E \to F$ is said to be a *generalized homomorphism* if there exists a linear homomorphism $\varphi : A \to B$ such that T is a φ-homomorphism.

The concept of φ-morphism was first introduced by Bakic and Guljas in 2002. Amyari and Niknam generalized this notion over Finsler module and called it φ-homomorphism. Recently, the author studied bimorphisms as a class of bi-generalized module maps on Hilbert bimodules and investigated their basic properties. We refer the reader to [1, 2, 5] for more details.

Following [1], we call a linear map $T : E \to F$ a *unitary operator* if there exists an injective homomorphism $\varphi : A \to B$ of C^*-algebras such that T is a surjective φ-morphism.

Let $T : E \to F$ be a φ-morphism. It is known from [1] that, if φ is an injection, then T is an isometry. Thus, each unitary operator of Finsler modules is an isometry. In particular, if F is a full Finsler B-module, then φ is surjective and so it is an isomorphism of C^*-algebras.

2. Main results

THEOREM 2.1. *Let I be an ideal in A and J be an ideal in B. Suppose that $\varphi : A \to B$ is a linear $*$-homomorphism and $T : E \to F$ is a φ-morphism. Then, $M_{ker\varphi} = kerT$, $T(E_I) = ImT_{\varphi(I)}$ and $T^{-1}(F_J) = E_{\varphi^{-1}(J)}$.*

LEMMA 2.2. *Let E be a full Finsler A-module and Let I be an ideal in A. Then, $\frac{E}{E_I}$ is a full Finsler $\frac{A}{I}$-module.*

THEOREM 2.3. *Suppose that E and F are Finsler B-module, $\varphi : A \to B$ is a linear isomorphism of C^*-algebras and $T : E \to F$ be a bijective linear operator. Define the module action $a.x := T^{-1}(\varphi(a).T(x))$ on E. Then, E equipped with $\rho_A : E \to A^+$ defined by $\rho_A(x) := \varphi^{-1}(\rho_B(T(x))$ $(x \in E)$ can also be regarded as a Finsler A-module. Moreover, if J is an ideal in B, then $\frac{E}{E_{\varphi^{-1}(J)}}$ is a Finsler $\frac{A}{\varphi^{-1}(J)}$-module.*

We call the above alternative Finsler A-module E *the Finsler A-module induced by* (φ, T) or briefly *the induced Finsler A-module* and denote it by $E^{(\varphi,T)}$.

The three next results present some algebraic properties of the induced Finsler modules.

LEMMA 2.4. *Let $E^{(\varphi,T)}$ be the induced Finsler A-module. Then, T is a φ-morphism. More precisely, T is a unitary operator.*

THEOREM 2.5. *F is a full Finsler B-module if and only if the induced Finsler A-module $E^{(\varphi,T)}$ is full. In particular, if J is an ideal in B and F is a full Finsler B-module, then $\frac{E^{(\varphi,T)}}{E^{(\varphi,T)}_{\varphi^{-1}(J)}}$ is a full Finsler $\frac{A}{\varphi^{-1}(J)}$-module, where φ is the $*$-isomorphism in Theorem 2.2.*

LEMMA 2.6. *Let F be a full Finsler B-module and $a \in A$. If $ax = 0$ for every $x \in E^{(\varphi,T)}$, then $a = 0$.*

Now, we are ready to present the following main result which establishes a converse of Theorem 2.3.

THEOREM 2.7. *Let E be a full Finsler A-module, F be a Finsler B-module and $T : E \to F$ be a bijective linear operator. If there exists a map $\varphi : A \to B$ such that $a.x = T^{-1}(\varphi(a).T(x))$ and $\varphi(\rho_A(x)) = \rho_B(T(x))$ $(a \in A,\ x \in E)$, then φ is a $*$-isomorphism of C^*-algebras if and only if F is full.*

Applying the previous theorem together with Lemma 2.2, we have the following result.

COROLLARY 2.8. *Let E be a full Finsler A-module and F be a full Finsler B-module and $T : E \to F$ be a bijective linear operator. If there exists a map $\varphi : A \to B$ such that $T(a.x) = \varphi(a).T(x)$ and $\varphi(\rho_A(x)) = \rho_B(T(x))$ $(a \in A,\ x \in E)$, then $\frac{F}{F_{\varphi(I)}}$ is a full Finsler $\frac{B}{\varphi(I)}$-module.*

Applying the Theorem 2.7, we now present a necessary and sufficient condition for a linear map between two full Finsler modules to be a unitary operator.

THEOREM 2.9. *Let E be a full Finsler A-module and F be a full Finsler B-module and $T : E \to F$ be a linear operator. Then, T is a unitary operator if and only if T is bijective and there exists a map $\varphi : A \to B$ such that $T(a.x) = \varphi(a).T(x)$ and $\varphi(\rho_A(x)) = \rho_B(T(x))$ for all $a \in A$ and $x \in E$.*

In Theorem 2.7, if $F = E$ and $T := I_E$, then we obtain the next corollary.

Corollary 2.10. *Let E be a full Finsler A-module and a Finsler B-module and there exists a map $\varphi : A \to B$ such that $a.x = \varphi(a).x$ and $\varphi(\rho_A(x)) = \rho_B(x)$ $(a \in A,\ x \in E)$. Then, φ is a $*$-isomorphism of C^*-algebras if and only if the Finsler B-module E is full.*

The following is an immediate consequence of Corollary 2.10.

Corollary 2.11. *Let E be both a full Finsler A-module and a full Finsler B-module and there exists a map $\varphi : A \to B$ such that $a.x = \varphi(a).x$ and $\varphi(\rho_A(x)) = \rho_B(x)$ $(a \in A,$ $x \in E)$. Then, the topology on E induced by ρ_A and ρ_B are equivalent.*

Before we state the second characterization theorem, we need the following useful lemma which can be found in [**3**].

Lemma 2.12. *Suppose that b and c are positive elements of C^*-algebra $\mathcal{A}$ such that $\|ac\| = \|ab\|$ for all $a \in \mathcal{A}$. Then $c = b$.*

Theorem 2.13. *Let E be a full Finsler A-module and F be a full Finsler B-module and $\varphi : A \to B$ is a linear $*$-homomorphism of C^*-algebras and $T : M \to N$ be a surjective φ-module map. Then, T is a unitary operator if and only if φ is a bijection and T is a bi-isometry.*

Definition 2.14. Two Finsler modules E and F are said to be *unitary equivalent* if there is a unitary operator from E to F

Proposition 2.15. *Unitary equivalence in the set of full Finsler modules is an equivalence relation.*

Before we state the next theorem, we need the following useful lemma which can be found in [**1**].

Lemma 2.16. *Let E and F be Finsler modules over C^*-algebras A and B, respectively. Let $\varphi : A \to B$ be a $*$-homomorphism of C^*-algebras and let $T : E \to F$ be a φ-homomorphism of Finsler modules. If F is a full Finsler B-module and T is surjective, then φ is a surjection.*

Applying Theorem 2.3 together with the previous lemma we have the last result.

Theorem 2.17. *The Finsler B-module F and the induced Finsler A-module $E^{(\varphi,T)}$ are unitary equivalent. Conversely, if E and F are full unitary equivalent Finsler modules over C^*-algebras A and B, respectively, then A and B are isomorphic C^*-algebras.*

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On (para)uniform MV-algebras

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Abstract. In this paper, the notions of (para) uniform MV-algebras are defined and continuity of the operations of the uniform MV-algebras are studied. Also, some uniform topologies are obtained by ideals. Then, it is proved that an MV-algebra with induced topology by congruence relation on ideals turn into a topological MV-algebra.

Keywords: MV-algebra, Ideal, (Para)uniform MV-algebra, Uniform topology.

AMS Mathematics Subject Classification [2010]: 54E15, 06D35, 11F23

1. Introduction

Topology and algebra, play complementary roles. In recent decades, many mathematicians have investigated some topological properties of some classes of algebraic structures which are endowed whith topology. Undoubtedly, MV-algebras are one of the most important structures of logical systems. The concept of MV-algebra was introduced by chang [1]. The study of MV-algebras endowed with a topology has exprienced a tremendous growth over the recent years. For example, Najafi et al. [4] introduced (semi, quasi)topological MV-algebras and investigated some of their properties. In recent years, the structures of Logic systems which are equipped with uniformity and uniform topology have also been discussed. See [3].
This article is organized as follows: In Section 2, we define the notions of (para)uniform MV-algebras and study uniform continuity of their operations. In Section 3, we obtain some uniform topologies on MV-algebras by ideals. Also, we use the congruence relations on ideals to construct topological MV-algebras.

2. (Para)Uniform MV-algebras

In this section, we first collect the relevant definitions from MV-algebra and uniform space theories to make this paper easy to read. Then we define the notions of (para)uniform MV-algebras and study the uniform continuity of the operations of uniform MV-algebras.

Definition 2.1. [1] An MV-algebra is an algebra $(A, \oplus, *, 0)$ of type (2,1,0) such that for any $x, y \in A$:
(MV_1) $(A, \oplus, 0)$ is an abelian monoid, (MV_2) $x \oplus 0^* = 0^*$,
(MV_3) $(x^*)^* = x$, (MV_4) $(x^* \oplus y)^* \oplus y = (x \oplus y^*)^* \oplus x$.

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Definition 2.2. [1] In MV-algebra A for any $x, y \in A$, we define:
(MV_5) $1 := 0^*$, (MV_6) $x \odot y := (x^* \oplus y^*)^*$, (MV_7) $x \ominus y := x \odot y^*$,
(MV_8) $x \to y := (x \odot y^*)^*$, (MV_9) $x \rightsquigarrow y := (x \oplus y^*)^*$, (MV_{10}) $x \vee y = y \oplus (x \ominus y)$

Definition 2.3. [1] Let A be an MV-algebra. A none-empty subset I of A is called an *ideal* if it satisfies the following conditions:
(i) for any $x, y \in I$, $x \oplus y \in I$, (ii) if $x \in I$ and $y \leq x$, then $y \in I$.

Definition 2.4. [2] Let X be a non-empty set. A uniformity on X is a non-empty family $\mathcal{U}$ of subsets of $X \times X$ with the following properties:
(U_1) If $U \in \mathcal{U}$, then $\triangle = \{(x, x) : x \in X\} \subseteq \mathcal{U}$,
(U_2) If $U \in \mathcal{U}$, then $U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\} \in \mathcal{U}$,
(U_3) If $U \in \mathcal{U}$, then $V \circ V \subseteq U$ for some $V \in \mathcal{U}$, where
$V \circ V = \{(x, y) \in X \times X : \exists z \in X \ s.t \ (x, z), (z, y) \in V\}$,
(U_4) If $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$,
(U_5) If $U \in \mathcal{U}$ and $U \subseteq V$, then $V \in \mathcal{U}$.

Let $(X,\mathcal{U})$ be a uniform space. $\mathcal{T}_{\mathcal{U}} = \{G \subseteq X : \forall x \in G \ \exists U \in \mathcal{U} \ s.t \ U[x] \subseteq G\}$, where $U[x] = \{y \in X : (x, y) \in U\}$, is a topology on X which called *uniform topology*.

Definition 2.5. Let A be an MV-algebra and $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be uniformities on A. $(A,\mathcal{U} \times \mathcal{V}, \mathcal{W})$ is called:
(i) *topological* MV-algebra, if the operations $\oplus$ and $*$ are continuous.
(ii) *parauniform* MV-algebra, if the map $\oplus : (A \times A, \mathcal{U} \times \mathcal{V}) \longrightarrow (A, \mathcal{W})$ is *uniformly continuous*.
(iii) *uniform* MV-algebra, if it is a parauniform MV-algebra and the maps $(A, *, \mathcal{U})$, $(A, *, \mathcal{V})$ and $(A, *, \mathcal{W})$ are uniformly continuous.

Example 2.6. Let $A = [0, 1]$ be the standard MV-algebra, where $x \oplus y = (x + y) \wedge 1$ and $x^* = 1 - x$ (i) The family $\{V_a\}_{0<a<1}$, where $V_a = \triangle \cup \{(x, y) : x < a, y < a\}$, is a base for a uniformity $\mathcal{V}$ on A. (ii) The family $\{U_\varepsilon\}_{\varepsilon>0}$, where $U_\varepsilon = \{(x, y) :| x - y |< \varepsilon\}$, is a base for a uniformity $\mathcal{U}$ on A. (iii) $\mathcal{D} = \{D : \triangle \in D \subseteq A \times A\}$ is a uniformity on A which is called *discrete uniformity*. (iv) $\mathcal{C} = \{A \times A\}$ is the *indiscrete uniformity* on A. (v) $\mathcal{W} = \{W : \triangle \cup \bigtriangledown \in W \subseteq A \times A\}$, where $\bigtriangledown = \{(x, x^*) : x \in A\}$, is a uniformity on A.
Obviously, $(A, *, \mathcal{U}(\mathcal{C}, \mathcal{D}, \mathcal{W}))$ are uniformly continuous, but $* : (A, \mathcal{V}) \longrightarrow (A, \mathcal{V})$ isn't uniformly continuous. To show this, let $V_a \in \mathcal{V}$ and there exists $V_b \in \mathcal{V}$ such that ${V_b}^* \subseteq V_a$. For any $(x, y) \in V_b$, $1 - b \leq x^*, y^* < a$. Therefore, $V_b = \{(x, y) : 1 - a \leq x, y < b\}$, which is a contradiction.
Now let $\oplus : (A \times A, \mathcal{D} \times \mathcal{W}) \longrightarrow (A,\mathcal{U})$. We show that $(A, \mathcal{D} \times \mathcal{W},\mathcal{U})$ isn't a parauniform MV-algebra. To do this, let $U_\varepsilon = U_{1/8} \in \mathcal{U}$ and $(x, y) \in U_{1/8}$. Suppose that there exist $D \in \mathcal{D}$ and $W \in \mathcal{W}$ such that $D \oplus W \subseteq U_{1/8}$. Since $(0, 0) \in \triangle \subseteq D$ and $(1/3, 2/3) \in \bigtriangledown \subseteq W$, we conclude that $(1/3, 2/3) \in D \oplus W$. On the other hand, $| 2/3 - 1/3 |= 1/3 > 1/8$, which is a contradiction.
Let $\oplus : (A \times A, \mathcal{D} \times \mathcal{V}) \longrightarrow (A,\mathcal{U})$. We prove that $(A, \mathcal{D} \times \mathcal{V},\mathcal{U})$ is a parauniform MV-algebra. Let $U_\varepsilon \in \mathcal{U}$. We find $D \in \mathcal{D}$ and $V \in \mathcal{V}$ such that $D \oplus V \subseteq U_\varepsilon$. We put $D = \triangle$ and $V = V_{\varepsilon/2}$. Let $(x, x) \in \triangle$ and $(y, z) \in V_{\varepsilon/2}$. Thus, $(x, x) \oplus (y, z) = (x \oplus y, x \oplus z) \in \triangle \oplus V_{\varepsilon/2}$. We need to calculate $| (x \oplus y) - (x \oplus z) |$. There exist three states: (1) $x + y < 1$ and $x + z < 1$. (2) $x + y > 1$ and $x + z > 1$. Clearly, in these cases $| (x \oplus y) - (x \oplus z) |< \varepsilon/2$. (3) $x + y > 1$ and $x + z < 1$ or $x + y < 1$ and $x + z > 1$. Let $x + y > 1$ and $x + z < 1$. Then, $y > z$. Suppose that $y - z = \alpha$, where $\alpha < \varepsilon/2$. $| (x \oplus y) - (x \oplus z) |=| 1 - (x + z) |=| \alpha |< \varepsilon/2$. Therefore, $(x, x) \oplus (y, z) \in U_\varepsilon$.

Theorem 2.7. *Let* $(A, \mathcal{U} \times \mathcal{V}, \mathcal{W})$ *be a uniform MV-algebra. Then the operations* $\odot, \ominus, \rightarrow, \rightsquigarrow$ *are uniformly continuous.*

Proof. We prove one case. The others can be proved similarly. Let $(A, \mathcal{U} \times \mathcal{V}, \mathcal{W})$ be a uniform MV-algebra and $W \in \mathcal{W}$. Since $(A, *, \mathcal{W})$ is uniformly continuous, there exists $f \in \mathcal{W}$ such that $f^* \subseteq W$. Hence, $f = (f^*)^* \subseteq W^*$. Therefore, by (U_5), $W^* \in \mathcal{W}$. Also, there exist $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $U \oplus V \subseteq W^*$. Because the maps $(A, *, \mathcal{U})$ and $(A, *, \mathcal{V})$ are uniformly continuous, there exist $R \in \mathcal{U}$ and $K \in \mathcal{V}$ such that $R^* \subseteq U$ and $K^* \subseteq V$. Thus, $R \odot K = (R^* \oplus K^*)^* \subseteq (U \oplus V)^* \subseteq (W^*)^* = W$. □

Proposition 2.8. *Let* A *be an MV-algebra and* $\mathcal{U}, \mathcal{V}, \mathcal{W}$ *be uniformities on* A *such that* $\mathcal{U} \subseteq \mathcal{W}$. *If* $(A, \mathcal{U} \times \mathcal{V}, \mathcal{W})$ *is a uniform MV-algebra, then the operations* $\vee$ *and* $\wedge$ *are uniformly continuous.*

Proof. Let A be an MV-algebra and $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be uniformities on A and $W \in \mathcal{W}$. Since $(A, \mathcal{U} \times \mathcal{V}, \mathcal{W})$ is a uniform MV-algebra, there exist $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $U \oplus V \subseteq W$. By Theorem 2.7, there exist $F \in \mathcal{U}$ and $R \in \mathcal{V}$ such that $F \ominus R \subseteq U$. Hence, $(F \ominus R) \oplus V \subseteq W$. By (U_4), there exist $K \in \mathcal{V}$ such that $K \subseteq R \cap V$. Therefore, $F \vee K = (F \ominus K) \oplus K \subseteq W$. Since $x \wedge y = (x^* \vee y^*)^*$, we can conclude that $\wedge$ is also uniformly continuous. □

Theorem 2.9. *Let* $\mathcal{U}$ *be a uniformity on MV-algebra* A *and* $x, y \in A$. *Then*
(i) $(A, \mathcal{U} \times \mathcal{U}, \mathcal{U})$ *is a uniform MV-algebra if and only if the map* $F : A \times A \longrightarrow A$ *defined by* $F(x, y) = x \oplus y^*$ *is uniformly continuous.*
(ii) *There exists a uniformly continuous map* $f : A \longrightarrow A$ *such that* $f(x) = y$.

Proof. The necessity is obvious, we prove the sufficiency. Let the function F be uniformly continuous and $W \in \mathcal{U}$. Thus, the map $K = F\mid_{\{0\} \times U}$ is uniformly continuous. Hence, there exists $V \in \mathcal{U}$ such that $\{0\} \oplus V^* = V^* \subseteq W$ which implies that the map $(A, *, \mathcal{U})$ is uniformly continuous. Since the map F is uniformly continuous there exist $U, V \in \mathcal{U}$ such that $U \oplus V^* \subseteq W$. Because $(A, *, \mathcal{U})$ is uniformly continuous, there exists $R \in \mathcal{U}$ such that $R^* \subseteq V$. Then, $R = (R^*)^* \subseteq V^*$ and so $U \oplus R \subseteq W$.
(ii) Let $x, y \in A$. Since $(A, \mathcal{U} \times \mathcal{U}, \mathcal{U})$ is a uniform MV-algebra, $T_y : A \longrightarrow A$ and $R_x : A \longrightarrow A$ defined by $T_y(a) = a \oplus y$ and $R_x(a) = a \ominus x$ are uniformly continuous. Hence, $f = T_y \circ R_x$ is uniformly continuous and $f(x) = T_y \circ R_x(x) = T_y(x \ominus x) = T_y(0) = 0 \oplus y = y$. □

3. Uniform topology on MV-algebras

In this section, we construct some uniform topologies on MV-algebras by ideals.

Proposition 3.1. *Let* A *be an MV-algebra and* $\mathcal{I}$ *be a family of ideals of* A *which is closed under intersection. Also, let* $I[x] = \{y \in A : y \neq 1, x \ominus y \in I\}$. *If there exist* $I, J \in \mathcal{I}$ *such that* $I[x] \cap J[y] = \emptyset$, *for* $x \neq y$, *then* $\mathcal{B} = \{I[x] : I \in \mathcal{T}, x \in A\} \cup \{1\}$ *is a base for a uniform topology* $\mathcal{T}$ *on* A.

Proof. Let $\mathcal{B} = \{I[x] : I \in \mathcal{T}, x \in A\} \cup \{1\}$. Clearly, $A \subseteq \cup\mathcal{B}$. Let $I_1[x_1], I_2[x_2] \in \mathcal{B}$ and $a \in I_1[x_1] \cap I_2[x_2]$. We put $I = I_1 \cap I_2 \in \mathcal{I}$ and claim $I[a] \subseteq I_1[x_1] \cap I_2[x_2]$. If $z \in I[a]$, then $x_i \ominus z \in I_i[x_i]$, for $i = 1, 2$, because $(x_i \ominus z) \leq (x_i \ominus a) \oplus (a \ominus z)$. Thus, $\mathcal{B}$ is a base for a topology $\mathcal{T}$ on A. We show that $(A, \mathcal{T})$ is a compact space. Let $\{U_i : i \in I\}$ be an open covering of A. Since $0 \in A$, for some $i \in I, 0 \in U_i$. As $\mathcal{B}$ is a base for $\mathcal{T}$ there exists $I \in \mathcal{I}$ and $x \in A$ such that $0 \in I[x]$. Consequently, $x \in I$. Let $y \neq 1$ is an arbitrary element of A. Since $x \ominus y \leq x$ and I is an ideal, we conclude that $x \ominus y \in I$ and so $y \in I[x]$. On the other hand, $1 \in U_j$, for some $j \in I$. Therefore, $A \subseteq U_i \cup U_j$ which implies that A is compact. Since $(A, \mathcal{T})$ is a compact Hausdorff space, $\mathcal{T}$ is a uniform topology [2]. □

Theorem 3.2. *Let $(A,\mathcal{T})$ be a topological MV-algebra. If*
(i) *there exists a family $\mathcal{I}$ of open ideals which is closed under intersection and contains a nontrivial J,*
(ii) *for any $x \neq y$ there exists $I \in \mathcal{I}$ such that $x \notin I$ or $y \notin I$,*
then $\mathcal{B} = \{\frac{x}{I} : I \in \mathcal{I}, x \in A\}$ is a base for a nontrivial uniform topology $\mathcal{V}$ on A which is coarser than $\mathcal{T}$ and $(A,\mathcal{V})$ is a topological MV-algebra.

Proof. It is easy to prove that the set $\mathcal{B}$ is a base for the topology $\mathcal{V} = \{V \subseteq A : \forall x \in V\ \exists I \in \mathcal{I}\ s.t\ \frac{x}{I} \subseteq V\}$ and $\mathcal{V}$ is a nontrivial topology on A. We prove that $\mathcal{V}$ is coarser than $\mathcal{T}$. Let $x \in V \in \mathcal{V}$. Then for some $I \in \mathcal{I}, x \in \frac{x}{I} \subseteq V$. Since $(A,\mathcal{T})$ is a topological MV-algebra, there exists $U \in \mathcal{T}$ such that $x \in U$, $x \ominus U \subseteq I$ and $U \ominus x \subseteq I$. We claim that $U \subseteq V$. Let $z \in U$. Since $x \ominus z \in I$ and $z \ominus x \in I$, then $z \in \frac{x}{I} \subseteq V$. It is obvious that $(A,\mathcal{V})$ is a normal and T_1 space and so it is a uniform topology. Since $\frac{x}{I} \oplus \frac{y}{I} = \frac{x \oplus y}{I}$ and $\frac{x^*}{I} = \frac{x}{I}$, $(A,\mathcal{V})$ is a topological MV-algebra. □

Theorem 3.3. *Let A be an MV-algebra and $\mathcal{I}$ be a family of ideals which is closed under intersection. (i) There exist a uniformity $\mathcal{U}_\mathcal{I}$ on A such that $(A,\mathcal{U}_\mathcal{I} \times \mathcal{U}_\mathcal{I},\mathcal{U}_\mathcal{I})$ is a uniform MV-algebra. (ii) If $\{0\} \in \mathcal{I}$, then $\mathcal{U}_\mathcal{I}$ is a discrete uniformity.*

Proof. One can show that the relation $x \overset{I}{\equiv} y \Longleftrightarrow x \ominus y \in I, y \ominus x \in I$ is a congruence relation on A, for any ideal $I \in A$. We put $U_I = \{(x,y) : x \overset{I}{\equiv} y\}$. It is easy to prove that $\mathcal{B}$ is a base for a uniformiy $\mathcal{U}_\mathcal{I}$ and $(A,\mathcal{U}_\mathcal{I} \times \mathcal{U}_\mathcal{I},\mathcal{U}_\mathcal{I})$ is a uniform MV-algebra. Also, $B = \{U_I[x] : U_I \in \mathcal{B}, x \in A\}$ is a base for the uniform topology $\mathcal{T}_\mathcal{I} = \{G \subseteq X : \forall x \in G\ \exists U_I \in \mathcal{B}\ s.t\ U_I[x] \subseteq G\}$ on A and $(A,\mathcal{T}_\mathcal{I})$ is a topological MV-algebra. (ii) If $I = \{0\} \in \mathcal{T}$, then $U_I = \{(x,y) : x \overset{I}{\equiv} y\} = \{(x,y) \in A \times A : x \ominus y = y \ominus x = 0\} = \{(x,x) : x \in A\}$. Therefore, $\mathcal{U}_\mathcal{I}$ is a discrete uniformity and $\mathcal{T}_I$ is a discrete topology. □

Example 3.4. Let $A = \{0,a,b,1\}$, where $0 < a,b < 1$. consider the following tables:

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$*$	0	a	b	1
	1	b	a	0

$(A,\oplus,*,0)$ is an MV-algebra. $\{0\},\{0,a\},\{0,b\}$ and A are the ideals of A. Also, $U_{\{0\}} = \triangle$, $U_{\{0,a\}} = \triangle \cup \{(1,b),(b,1)\}$, $U_{\{0,b\}} = \triangle \cup \{(1,a),(a,1)\}, U_A = A \times A$. If $I = \{0,a\}$, then $U_I[0] = U_I[a] = \{0,a\}, U_I[b] = U_I[1] = \{1,b\}$ and $\mathcal{T}_I = \{\emptyset, A, \{0,a\},\{b,1\}\}$. By Theorem 3.3, $(A,\mathcal{T}_I)$ is a topological MV-algebra.

4. Conclusion

This paper shows that uniform MV-algebras can be constructed by various uniformities. Also, there exists uniformities on MV-algebras which turn them into a topological MV-algebras. Properties of these uniformities can be considered for feature research.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

On a generalization of classical notion of the ring

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Abstract. This talk is about some basic facts and new results on Γ-rings, as a generalized notion of the classical ring, which have been obtained over the past years. In particular, under the condition that the given Γ-ring is semi-prime some properties associated to Γ-rings imply each other. Moreover, some outlines about new researches of the subject under discussion are given.

Keywords: multiplication, ring, Γ-ring.

AMS Mathematics Subject Classification [2010]: 16N60,16W25,16Y99.

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1. Introduction

The notion of a Γ-ring was first introduced by Nobuo Nobusawa [6] which generalized extensively the classical concept of the ring. The extensions and generalizations of various important results in the theory of classical rings to the theory of Γ-rings have been attracted a wider attentions as an emerging area of research to the modern algebraists to enrich the world of algebra. All over the world, many prominent mathematicians have worked out on this interesting area of research to determine many basic properties of Γ-rings and have executed more productive and creative results of Γ-rings in the last few decades. In this talk some progresses which have been achieved over the past years for Γ-rings are reviewed. In particular, we recall some results from [1–5]. Finally, some outlines concerning to the sequel of this subject are given.

2. Main results

Definition 2.1. Let M and Γ be additive abelian groups. Then, M is called a Γ-ring if there exists a function $\mathrm{M} \times \Gamma \times \mathrm{M} \to \mathrm{M}$ (with $(m, \gamma, n) \to m\gamma n \in \mathrm{M}$) which satisfies $m\gamma(n + p) = m\gamma n + m\gamma p$, $(m + n)\gamma p = m\gamma p + n\gamma p$, $m(\gamma + \nu)n = m\gamma n + m\nu n$ and $m\gamma(n\mu p) = (m\gamma n)\mu p$ for all $m, n, p \in \mathrm{M}$ and all $\gamma, \mu \in \Gamma$.

Example 2.2. Every ring can be considered as a Γ-ring. Indeed, If S is a ring and R is a two-sided ideal of S, take $\Gamma = S$ and let $(r_1, s, r_2) = r_1 s r_2$ where the latter is the product in S.

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Definition 2.3. Let M be a Γ-ring. Then an additive subgroup U of M is called a left (right) ideal of M if $M\Gamma U \subset U (U\Gamma M \subset U)$. If U is both a left and a right ideal, then we say U is an ideal of M.

Definition 2.4. Let M be a Γ-ring. Then, M is said to be prime if $a\Gamma M\Gamma b = 0$ with $a, b \in M$ implies that $a = 0$ or $b = 0$. Moreover, M is called semi-prime if $a\Gamma M\Gamma a = 0$ with $a \in M$ implies that $a = 0$.

Definition 2.5. Let M be a Γ-ring. Then M is said to be commutative if $x\alpha y = y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. The set $Z(M)$ consists of all $x \in M$ such that $x\alpha y = y\alpha x$ for all $y \in M$ and $\alpha \in \Gamma$ is called the center of the Γ-ring M. If M is a Γ-ring, then $[x, y]_\alpha = x\alpha y - y\alpha x$ is known as the commutator of x and y with respect to α, where $x, y \in M$ and $\alpha \in \Gamma$. A map $f : M \to M$ is said to be commuting on a left ideal J of M if $[f(x), x]_\alpha = 0$ for all $x \in J$ and $\alpha \in \Gamma$. Moreover, f is said to be centralizing on J if $[f(x), x]_\alpha \in Z(M)$ for all $x \in J$ and $\alpha \in \Gamma$. An additive map $T : M \to M$ is a left centralizer if $T(x\alpha y) = T(x)\alpha y$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover, T is called a right centralizer if $T(x\alpha y) = x\alpha T(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. A centralizer is an additive map which is both a left and a right centralizer. On the other hand, T is called Jordan left centralizer if $T(\alpha x) = T(x)\alpha x$ for all $x \in M$ and $\alpha \in \Gamma$. Furthermore, T is called Jordan right centralizer if $T(\alpha x) = x\alpha T(x)$ for all $x \in M$ and $\alpha \in \Gamma$. An additive map $T : M \to M$ is called a Jordan centralizer if $T(x\alpha x) = T(x)\alpha x + x\alpha T(x)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$.

Clearly, every (left) centralizer is a Jordan (left) centralizer but the converse is not true in general.

Example 2.6. Let M be a Γ-ring. Define $M_1 = \{(x, x) : x \in M\}$ and $\Gamma_1 = \{(\alpha, \alpha) : \alpha \in \Gamma\}$. We define addition and multiplication on M_1 as:

$$(x_1, x_1) + (x_2, x_2) = (x_1 + x_2, x_1 + x_2), (x_1, x_1)(\alpha, \alpha)(x_2, x_2) = (x_1\alpha x_2, x_1\alpha x_2)$$

for every $x_1, x_2 \in M$ and $\alpha \in \Gamma$. Then, M_1 is a Γ-ring. Let $d_1 : M \to M$ be a left centralizing map and $d_2 : M \to M$ be a right centralizing and commutating map. Let $T : M_1 \to M_1$ be the additive map defined by $T(x, x) = (d_1(x), d_2(x))$. Let $(x, x) = a \in M_1$ and $(\alpha, \alpha) = \gamma \in \Gamma_1$. We have $T(a\gamma a) = T(a)\gamma a$. Hence T is a Jordan left centralizer, which is not a left centralizer. Moreover, we put $(x, x) = a, (y, y) = b \in M$ and $(\alpha, \alpha) = \gamma \in \Gamma_1$. We have $T(a\gamma b + b\gamma a) = T(a)\gamma b + T(b)\gamma a$. Hence T is a Jordan centralizer which is not a centralizer.

Definition 2.7. Let M be a Γ-ring. Then M is said to be a 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$.

Now, in view of Theorem 3.1 in [1], we have:

Theorem 2.8. *Every Jordan centralizer of a 2-torsion free semi-prime Γ-ring M satisfying $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ is a centralizer.*

Definition 2.9. Let M be a 2-torsion free semi-prime Γ-ring and let θ be an endomorphism of M. An additive map $T : M \to M$ is a left θ-centralizer if $T(x\alpha y) = T(x)\alpha\theta(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover, T is called a right centralizer if $T(x\alpha y) = \theta(x)\alpha T(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. If T is a left and a right θ-centralizer, then it is called a θ-centralizer.

Moreover, we get the following results in view of Theorem 2.1 and Theorem 2.2 in [2] and [3], respectively:

Theorem 2.10. *Let M be a 2-torsion free semi-prime Γ-ring satisfying $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Moreover, let θ be an endomorphism of M. If $T : M \to M$ is an additive map such that $2T(x\alpha y\beta x) = T(x)\alpha\theta(y)\beta\theta(x) + \theta(x)\alpha\theta(y)\beta T(x)$ for all pairs $x, y \in M$ and $\alpha, \beta \in \Gamma$, then T is a θ-centralizer.*

Theorem 2.11. *Let M be a 2-torsion free semi-prime Γ-ring with identity element 1 satisfying $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Let $T : M \to M$ be an additive map such that $T(x\alpha x\beta x) = x\alpha T(x)\beta x$ for all $x \in M$ and $\alpha, \beta \in \Gamma$. Then T is a centralizer.*

Definition 2.12. Let $D : M \to M$ be an additive map. Then D is called a derivation if $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover, D is called a Jordan derivation if $D(x\alpha x) = D(x)\alpha x + x\alpha D(x)$ for all $x \in M$ and $\alpha \in \Gamma$. Moreover, an additive map $f : M \to M$ is said to be a generalized derivation on M if $f(x\alpha y) = f(x)\alpha y + x\alpha D(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$, D a derivation on M.

Now we have the following results as Theorem 3.1 and Theorem 3.2 in [4]:

Theorem 2.13. *Let M be a prime Γ-ring satisfying $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ and D a nonzero derivation on M. If f is a generalized derivation on a left ideal J of M such that f is commuting on J, then M is commutative.*

Theorem 2.14. *Let M be a prime Γ-ring satisfying $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ and J a left ideal of M with $J \cap Z(M) = 0$. If f is a generalized derivation on M with associated non-zero derivation D such that f is commuting on J, then M is commutative.*

Definition 2.15. Let D be a derivation on M. Then the additive map $F : M \to M$ is called a generalized derivation if $F(x\alpha y) = F(x)\alpha y + x\alpha D(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover, an additive map I is called an involution if $II(x) = x$, $I(x + y) = I(x) + I(y)$ and $I(x\alpha y) = I(y)\alpha I(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 2.16. Let M be a Γ-ring with involution I. An additive map $D : M \to M$ is called an I-derivation if $D(x\alpha y) = D(x)\alpha I(y) + x\alpha D(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover, D is called a reverse I-derivation if $D(x\alpha y) = D(y)\alpha I(x) + y\alpha D(x)$ for all $x, y \in M$ and $\alpha \in \Gamma$. An additive map $T : M \to M$ is called a left (right) I-centralizer if $T(x\alpha y) = T(x)\alpha I(y)$ (resp. $T(x\alpha y) = I(x)\alpha T(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. An additive map $F : M \to M$ is called a generalized I-derivation if $F(x\alpha y) = F(x)\alpha I(y) + x\alpha D(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$, D an I-derivation on M. An additive map $F : M \to M$ is called a generalized reverse I-derivation if $F(x\alpha y) = F(y)\alpha I(x) + y\alpha D(x)$ for all $x, y \in M$ and $\alpha \in \Gamma$, D a reverse I-derivation on M.

Now, Theorem 2.6 in [5] imply that :

Theorem 2.17. *Let M be a semi-prime Γ-ring with involution I and D a reverse I-derivation on M. If F is a generalized reverse I-derivation on M, then $[D(x), y]_\alpha = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$.*

As corollaries to this result we get corollaries 2.7 and 2.8 in [5], respectively:

Corollary 2.18. *Let M be a non-commutative prime Γ-ring with involution I and D a reverse I-derivation on M. If F is a generalized reverse I-derivation on M, then F is a reverse left I-centralizer on M.*

Corollary 2.19. *Suppose that M is a semi-prime Γ-ring with involution I. If D is a reverse I-derivation on M, then D maps M into $Z(M)$.*

3. Conclusion

Some results related to notions such as (left) centralizer, Jordan (left) centralizer, θ-centralizer, derivation, generalized derivation, generalized I-derivation and I-centralizer for an involution I are deduced for Γ-rings under the hypotheses that the given Γ-ring is a prime or semi-prime ring. In particular, it is studied that under what conditions the above-mentioned notions imply each other. It is natural to investigate about such results under weaker hypotheses like completely prime, weakly prime, weakly completely prime Γ-rings. It should be pointed out that the research for such problems is under investigation by the author and not published yet.

Acknowledgement

This work was supported by a grant from Payame Noor University of I. R. of Iran. The author would like to thank Payame Noor University for the support during the preparation of this work.

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Kerman, Iran
30 August - 02 September 2021

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کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

The generalize of the absolute central automorphisms group

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Abstract. In this paper, by a definition we generlize the absolute central automorphisms and represent their properties.

Keywords: absolute central automorphism, n-absolute central automorphism, autonilpotent group, n-autonilpotent group.

AMS Mathematics Subject Classification [2010]: 20D45, 20D15, 20E36.

1. Introduction

The concepts of absolute central subgroup, n-th absolute central subgroup, absolute central automorphism and autonilpotent groups have been the idea of many researchers articles and they are very important in the discussion of the automorphisms. In this paper p denotes a prime number. Let G be a group. Let us denote by G', Z(G), $\phi(G)$, exp(G), Hom(G, H), Aut(G) and Inn(G), respectively the commutator subgroup, the centre, Frattini subgroup, the exponent, the group of homomorphisms of G into an abelian group H, the full automorphism group and the inner automorphisms. The absolute centre of a group G, denoted by L(G), is defined as

$$L(G) = \{g \in G \mid g^{-1}\alpha(g) = [g, \alpha] = 1,\ \forall \alpha \in Aut(G)\}.$$

An automorphism α of G is called an absolute central automorphism if $x^{-1}\alpha(x) \in L(G)$ for each $x \in G$. The set of all absolute central automorphisms of G is denoted by $Aut_l(G)$. The nth-absolute centre is defined as

$$L_n(G) = \{g \in G \mid [g, \alpha_1, \alpha_2, \ldots, \alpha_n] = 1,\ \forall \alpha_1, \alpha_2, \ldots, \alpha_n \in Aut(G)\}.$$

A group G is called autonilpotent of class at most n if $L_n(G) = G$, for some natural number n. Also, for each natural number m and n, we have

$$L_n^m(G) = \{g \in G \mid [g, \alpha_1^m, \alpha_2^m, \ldots, \alpha_n^m] = 1,\ \forall \alpha_1, \alpha_2, \ldots, \alpha_n \in Aut(G)\}.$$

We call a group G m-autonilpotent group of class at most c if there exists some integer c such that $L_c^m(G) = G$. For each natural number m, every autonilpotent group is a m-autonilpotent group, but reverse is not necessary true in general. For more information on these definitions and their properties, we refer the reader to [1] and [4]- [7].

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2. Main results

In this section, after a new definition, we give our main results about the generalize of the absolute central automorphisms group.

DEFINITION 2.1. An automorphisms $\alpha = \alpha_1^m \ldots \alpha_n^m \in Aut(G)$ is called n-absolute central automorphism if

$$\forall m \in \mathbb{N},\ \forall g \in G:\ [g, \alpha_1^m, \ldots, \alpha_n^m] = g^{-1}\alpha_1^m \ldots \alpha_n^m(g) \in L_n^m(G).$$

The set of all n-absolute central automorphisms of G is denoted by $Aut_{nl}(G)$.

THEOREM 2.2. *$Aut_{nl}(G)$ is a non-trivial and normal subgroup of $Aut(G)$.*

PROOF. For every arbitrary group G, the identity automorphism is a n-absolute central automorphism of G, therefore $\emptyset \neq Aut_{nl}(G)$. The continue is done by a similar way of $Aut_l(G) \trianglelefteq Aut(G)$ proof. $\square$

EXAMPLE 2.3. Let G be a group.

a) If $L_n^m(G) = \langle 1 \rangle$, then $Aut_{nl}(G) = \langle 1 \rangle$.
b) If G be an autonilpotent or m-autonilpotent group of class c, then $L_c^m(G) = G$, hence every arbitrary automorphism of this group is a c-absolute central automorphism of G, i.e. $Aut_{cl}(G) = Aut(G)$.

THEOREM 2.4. *Let G be a group. Then $Aut_{nl}(G) \cong Hom\big(G/L_n^m(G), L_n^m(G)\big)$.*

PROOF. The proof is similar to the proof of [6, Lemma 2.6], except that wherever $L(G) \leqslant Z(G)$ is used, we use the fact that every automorphism of G acts trivially on $L_n^m(G)$. $\square$

According to the above theorem, it follow that $Aut_{nl}(G)$ is abelian.

LEMMA 2.5. *Let G be a non-abelian nilpotent p-group of class 2 for which $G/L_n^m(G)$ is abelian. Then*

$$|Hom\big(G/Z(G), L_n^m(G)\big)| \geq |G/Z(G)|p^{r(s-1)}$$

where $r = rank\big(G/Z(G)\big)$ and $s = rank\big(L_n^m(G)\big)$.

PROOF. The proof is similar to the proof of [6, Lemma 2.8]. $\square$

LEMMA 2.6. *Let G be a finite group, then $G/L_n^m(G)$ is abelian if and only if $Inn(G) \leqslant Aut_{nl}(G)$.*

PROOF. The proof is similar to the proof of [6, Lemma 2.7]. $\square$

COROLLARY 2.7. *Let G be a finite p-group. If $L_n^m(G) = \phi(G)$, then $Inn(G) \leqslant Aut_{nl}(G)$.*

PROOF. Because G is a p-group, $L_n^m(G) = \phi(G) = G'G^p$, therefore $G' \leqslant L_n^m(G)$. thus $G/L_n^m(G)$ is abelian, so the previous lemma gives the result. $\square$

LEMMA 2.8. *Let G be an arbitrary group, then $C_{Aut_{nl}(G)}\big(Z(G)\big) \cong Hom\big(G/Z(G), L_n^m(G)\big)$.*

PROOF. The proof is similar to the proof of theorem 2.4. $\square$

COROLLARY 2.9. *Suppose G is a non-abelian p-group for which $G/L_n^m(G)$ is abelian. Then*

$$|C_{Aut_{nl}(G)}\big(Z(G)\big)| \geq |G/Z(G)|p^{r(s-1)}$$

where $r = rank\big(G/Z(G)\big)$ and $s = rank\big(L_n^m(G)\big)$.

Proof. Easily, it follows from lemmas 2.5 and 2.8. $\square$

Corollary 2.10. *Let G be a p-group such that $G/L_n^m(G)$ is abelian. If $L_n^m(G)$ is elementary abelian and $s = rank\big(L_n^m(G)\big)$, then*

a) *$Aut_{nl}(G)$ is elementary abelian of order p^{ks} that $k = rank\big(G/L_n^m(G)\big)$.*
b) *$C_{Aut_{nl}(G)}\big(Z(G)\big)$ is elementary abelian of order p^{rs} that $r = rank\big(G/L_n^m(G)\big)$.*

Proof. It follows from basis theorem for finite abelian groups, theorem 2.2 and lemma 2.8. $\square$

Corollary 2.11. *Let G be a non-abelian p-group such that $G/L_n^m(G)$ is abelian. If $Aut_{nl}(G)$ is elementary abelian, then $exp(G') = p$.*

Proof. The proof is similar to the proof of [2, Lemma 6]. $\square$

Theorem 2.12. *Suppose G is a non-abelian finite p-group for which $G/L_n^m(G)$ is abelian. Then*

$$|Aut_{nl}(G) : Inn(G)| \geq p^{r(s-1)}$$

where $r = rank\big(G/Z(G)\big)$ and $s = rank\big(L_n^m(G)\big)$.

Proof. The proof is similar to the proof of [6, Theorem 3.1]. $\square$

Proposition 2.13. *For every finite group G, if $Aut_{nl}(G) = Inn(G)$, then $exp\big(G/Z(G)\big) \leq exp\big(L_n^m(G)\big)$.*

Proof. Since $Aut_{nl}(G) = Inn(G)$, so

$$Hom\big(G/L_n^m(G), L_n^m(G)\big) \cong G/Z(G).$$

Therefore $exp\big(G/Z(G)\big) \leq exp\big(L_n^m(G)\big)$, because otherwise

$$|Aut_{nl}(G)| > |G/Z(G)| = |Inn(G)|$$

which is a contradiction. $\square$

Proposition 2.14. *Let G be a non-abelian p-group(p odd) such that $Aut_{nl}(G) = Inn(G)$, then $exp\big(G/Z(G)\big) \leq exp\big(L_n^m(G)\big)$.*

Proof. The proof is similar to the proof of [3, Theorem 3.2]. $\square$

Lemma 2.15. *Let G be a nilpotent p-group of class 2 such that $exp\big(G/Z(G)\big) \leq exp\big(L_n^m(G)\big)$, then*

$$|Hom\big(G/Z(G), L_n^m(G)\big)| \geq |G/Z(G)| p^{r(s-1)}$$

where $r = rank\big(G/Z(G)\big)$ and $s = rank\big(L_n^m(G)\big)$.

Proof. The proof is similar to the proof of [3, Lemma 2.6]. $\square$

Corollary 2.16. *Let G be a nilpotent p-group of class 2 such that $exp\big(G/Z(G)\big) \leq exp\big(L_n^m(G)\big)$, then*

$$|Aut_{nl}(G)| \geq |G/Z(G)| p^{r(s-1)}$$

where $r = rank\big(G/Z(G)\big)$ and $s = rank\big(L_n^m(G)\big)$.

Proof. The proof is similar to the proof of [3, Lemma 2.7]. $\square$

Lemma 2.17. *Let G be a non-abelian p-group. If $Aut_{nl}(G) = Inn(G)$, then $G/L_n^m(G)$ is cyclic.*

PROOF. The proof is similar to the proof of [3, Lemma 3.1]. □

THEOREM 2.18. *Let G be a non-abelian p-group, then*

a) *If* $C_{Aut_{nl}(G)}\big(Z(G)\big) = Inn(G)$, *then* $G' \leqslant L_n^m(G)$ *and* $L_n^m(G)$ *is cyclic.*
b) *If G be a nilpotent group of class 2,* $G' \leqslant L_n^m(G)$ *and* $L_n^m(G)$ *is cyclic, then* $C_{Aut_{nl}(G)}\big(Z(G)\big) = Inn(G)$.

PROOF. The proof is similar to the proof of [3, Theorem 3.3]. □

THEOREM 2.19. *Let G be a non-abelian p-group. If* $G' \leqslant L_n^m(G)$, $L_n^m(G)$ *is cyclic and* $Z(G) = L_n^m(G)G^{p^k}$ *where* $p^k = exp\big(L_n^m(G)\big)$, *then* $Aut_{nl}(G) = Inn(G)$.

PROOF. The proof is similar to the proof of [3, Theorem 3.4]. □

The reverse of above theorem is hold if G be a nilpotent group of class 2 and $L_n^m(G) \leqslant Z(G)$.

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On high-order stability of spacelike hypersurfaces in Lorentz-Minkowski space

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Abstract. In this paper, we study the kth stability of spacelike hypersurfaces in the Lorentz space $\mathbb{L}^{n+1}$. The stability of order k (briefly, k-stability) is a natural extension of the ordinary stability. The k-stability is defined based on the linearized operator L_k as an extension of the Laplace operator (i.e. $L_0 = \Delta$). We give sufficient conditions for a bounded domain in a k-maximal hypersurface of the Lorentz-Minkowski space to be k-stable. Especially, in the case $k = 1$, the Gauss- Kronecker curvature of 1-stable hypersurfaces has to be null on a special submanifold.

Keywords: k-maximal hypersurface, k-Stability, Minkowski space.

AMS Mathematics Subject Classification [2010]: 53C42, 53C20, 53B30

1. Introduction

The stability of hypersurfaces of Riemannian space forms as a well-known topic in differential geometry has been started by Simons in [7] and followed by others. In this paper, we study the spacelike hypersurfaces of Lorentzian space, which have important role in relativity ([3]). It is well-known that a maximal spacelike entire graph in the Minkowski space $\mathbb{L}^{n+1}$ is a linear hyperplane. We extend the notion of stability and the results to k-maximal spacelike hypersurfaces of $\mathbb{L}^{n+1}$.

1.1. Preliminaries. Now, we recall some basic preliminaries from [5]. By $\mathbb{R}^m_p$, we mean the vector space $\mathbb{R}^m$ with metric $< x, y >:= -\Sigma_{i=1}^p x_i y_i + \Sigma_{j>p} x_j y_j$. Especially, $\mathbb{R}^m_0 = \mathbb{R}^m$, and $\mathbb{R}^m_1$ is the Lorentz-Minkowski space. For $r > 0$ and $q = 0, 1$, $\mathbb{S}^{n+1}_q(r) = \{y \in \mathbb{R}^{n+2}_q | < y, y >= r^2\}$ denotes the sphere (for $q = 0$) and de Sitter space (for $q = 1$) of radius r and curvature $1/r^2$, and $\mathbb{H}^{n+1}_q(-r) = \{y \in \mathbb{R}^{n+2}_{q+1} | < y, y >= -r^2\}$ denotes the hyperbolic space (for $q = 0$) and anti-de Sitter space (for $q = 1$) of radius r and curvature $-1/r^2$. The simply connected space form $\tilde{M}^{n+1}_q(c)$ of curvature c and index q is $\mathbb{R}^{n+1}_q$ for $c = 0$, $\mathbb{S}^{n+1}_q = \mathbb{S}^{n+1}_q(1)$ for $c = 1$ and $\mathbb{H}^{n+1}_q = \mathbb{H}^{n+1}_q(-1)$ for $c = -1$. When $q = 0$, we take a component of H^{n+1}_0. The Weingarten formula for a spacelike hypersurface $x : M^n \to \tilde{M}^{n+1}_q(c)$ is $\bar{\nabla}_V W = \nabla_V W - \epsilon < SV, W > \mathbf{N}$, for $V, W \in \chi(M)$ where,

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$\epsilon = 2q-1$, $q \in \{0,1\}$ and S is the shape operator of M associated to a unit normal vector field $\mathbf{N}$ on M with $<\mathbf{N},\mathbf{N}>= -\epsilon$. Since M is spacelike, S can be diagonalized. Denote its eigenvalues (the principal curvatures of M) by the functions $\kappa_1, ..., \kappa_n$ on M, define the elementary symmetric function as $s_j := \sum_{1\le i_1<...<i_j\le n} \kappa_{i_1}...\kappa_{i_j}$ and the j-th mean curvature of M by $\binom{n}{j}H_j = (-\epsilon)^j s_j$. The hypersurface M^n in $\mathbb{R}^{n+1}_p$ is called *j-minimal* if its $(j+1)$th mean curvature H_{j+1} is identically zero.

In particular, $H_1 = -\epsilon(1/n)tr(S)$ and $\mathbf{H} = H_1\mathbf{N}$ are respectively the mean curvature and the mean curvature vector of M. The relation between the scalar curvature of M and H_2 as $tr(Ric) = n(n-1)(-\epsilon H_2)$. In general, H_j is extrinsic (respectively, intrinsic) when j is an odd (respectively, an even) number, since the sign of H_j depends on the chosen orientation only in the odd case.

For an spacelike hypersurface $x : M^n_p \to \mathbb{R}^{n+1}_1$ we introduce, as (4) in [5], the Newton transformations $P_j : \chi(M) \to \chi(M)$, associated with the shape operator S, inductively by

$$P_0 = I, P_j = (-1)^j s_j I + S \circ P_{j-1}(j = 1, ..., n),$$

where I is the identity on $\chi(M)$. It can be seen that P_j has an explicit formula,

$$P_j = (-1)^j \Sigma_{l=0}^{j} (-1)^l s_{j-l} S^l = \sum_{l=0}^{j} \binom{n}{j-l} H_{j-l} S^l$$

, where, $H_0 = 1$ and $S^0 = I$. According to the characteristic polynomial of S, $Q_S(t) = det(tI - S) = \sum_{l=0}^{n}(-1)^{n-l}s_{n-l}t^l$, the Cayley-Hamilton theorem gives $P_n = 0$.

Now, we define the notion of variation, the linearized operator L_k and the concept of k-stability.

Definition 1.1. Let $x : M^n_p \to \mathbb{R}^{n+1}_1(c)$ be a compact connected orientable hypersurface isometrically immersed into an standard Riemannian or Lorentzian space form, $c \in \{-1,0,1\}$ and $q \in \{0,1\}$. A map $X : (-\epsilon,\epsilon) \times M^n \to \mathbb{R}^{n+1}_1(c)$ is called a *variation* of M^n if it satisfies the following properties:

(1) For each $t \in (-\epsilon,\epsilon)$ the map $X_t : M^n \to \tilde{M}^{n+1}_q(c)$ by rule $X_t(p) := X(t,p)$, is an immersion.

(2) $X_0 = x$ and for every $t \in (-\epsilon,\epsilon)$, $X_t|_b d(M) = x|_b d(M)$.

Definition 1.2. The *linearized operator* of the $(k+1)$-th mean curvature of M, $L_k : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is defined by the formula $L_k(f) := tr(P_k \circ \nabla^2 f)$, where $\nabla^2 f$ is given by $<\nabla^2 f(X), Y >= Hess(f)(X,Y)$.

Among many interesting properties of L_k, we point that for a normal variation of M with variational field $\frac{dX_t}{dt}(t)|_{t=1} = f\mathbf{N}$, we have the equality

$\frac{d}{dt}s_{k+1}|_{t=1} = L_k f + (s_1 s_{k+1} - (k+2)s_{k+2})f$,

where L_k is the principal part of the linearized operator associated to s_{k+1}. For convenience, we define the operator J_k as:

$$J_k := L_k + (s_1 s_{k+1} - (k+2)s_{k+2})I,$$

as well as a bilinear symmetric form B_k can be defined by $B_k(f,g) := -\int_M g J_k f dM$.

Definition 1.3. Let $x : M^n \to \mathbb{L}^{n+1}$ be as in Definition 1.1 with condition that H_{k+1} is constant. M^n is called *k-stable* if $B_k(f,f) \ge 0$ for all $f \in \mathcal{C}^\infty_c(M)$.

The kth area functional $A_k : (-\epsilon,\epsilon) \to \mathbb{R}$ associated to a variation X is given by $A_k(t) := \int_M F_k(t) dM_t$, where $F_k(t)$ is recursively defined by $F_0(t) \equiv 1$, $F_1(t) := -s_1(t)$

and for $2 \leq k \leq n-1$, $F_k(t) := (-1)^r s_k(t) - \frac{n-k+1}{k-1} F_{k-2}(t)$. In the case $k = 0$, the functional A_0 is the classical area functional. If $s_{k+1} = 0$, there is a function $f : M \to \mathbb{R}$ supported in a compact domain $K \subset M$, such that the second variation of $A_k(t)$ is as:

$$A_k''(t) = (k+1) \int_M [L_k f + tr(P_k) f - tr(S^2 \circ P_k) f] f dM_t.$$

Remark 1.4. We want to point out that the first variation formula (Proposition 1.6) were first proved by R. Reilly. We wish to study spacelike immersions $x : M^n \to \mathbb{L}^{n+1}$ that maximize A_k for all volume-preserving variations X of x. The above discussion shows that M^n must have constant $(k+1)$th mean curvature and, for such an M^n, it leads us naturally to compute the second variation of A_k. So, if $x : M^n \to \mathbb{L}^{n+1}$ be a closed hypersurface having constant $(k+1)$th mean curvature, one can see that, x is k-stable if $A_k''(0) \geq 0$ for all volume-preserving variation of x.

1.2. Auxiliary lemmas.

Proposition 1.5. *([2]) Let $x : M^n \to \mathbb{L}^{n+1}$ (where $n \geq 2$) be a connected spacelike hypersurface of the Minkowski space $\mathbb{L}^{n+1}$, $\kappa_1, ..., \kappa_n$ be the principal curvatures of M_p^n and H_k be the kth mean curvature of M. Then we have:*

(i) For $0 < k < n$, $H_k^2 \geq H_{k-1} H_{k+1}$. If $k = 1$ or if $k > 1$ and $H_{k+1} \neq 0$, then the equality happens if and only if $\kappa_1 = ... = \kappa_n$;

(ii) $H_1 \geq (H_2)^{1/2} \geq ... \geq (H_k)^{1/k}$ if $H_i > 0$ for $i = 1, ..., k$;

Let $e_1, ..., e_n$ be a local orthonormal tangent frame on M that diagonalizes S and P_js as $Se_i = \kappa_i e_i$ and $P_j e_i = \mu_{i,j} e_i$, for $i = 1, 2, ..., n$, where $\mu_{i,j} = (-1)^j \Sigma_{i_1 < ... < i_j, i_l \neq i} \kappa_{i_1} ... \kappa_{i_j}$, (for $j = 0, 1, ..., n-1$). Using this and the useful identity

$$\kappa_i \mu_{i,j} = \mu_{i,j+1} - (-1)^{j+1} s_{j+1} = \mu_{i,j+1} - \tbinom{n}{j+1} H_{j+1}, \tag{1}$$

and the notation $c_j = (n-j)\binom{n}{j} = (j+1)\binom{n}{j+1}$, the following properties of P_k may be obtained easily:

$$tr(P_j) = (-1)^j (n-j) s_j = c_j H_j, \tag{2}$$

$$tr(S \circ P_j) = (-1)^j (j+1) s_{j+1} = -c_j H_{j+1}, \tag{3}$$

$$tr(S^2 \circ P_j) = \tbinom{n}{j+1} [n H_1 H_{j+1} - (n-j-1) H_{j+2}], \tag{4}$$

$$tr(P_j \circ \nabla_X S) = -\tbinom{n}{j+1} < grad(H_{j+1}), X > . \ (X \in \chi(M)) \tag{5}$$

Proposition 1.6. *Let $x : M^n \to \mathbb{L}^{n+1}$ (where $n \geq 2$) be a connected spacelike hypersurface isometrically immersed into the Minkowski space and $\{e_1, ..., e_n\}$ and $\kappa_1, ..., \kappa_n$ be as in Proposition 1.5 and P_k be the kth Newton transformation. If at a point $p \in M$, $H_k(p) = 0$ and $H_{k+1}(p) \neq 0$, then P_{k-1} is definite at p.*

2. Main results

In [1], the sufficient conditions for a regular domain on a k-minimal hypersurface of the Euclidean space are given to be k-stable. Their result assumes that the quotient $\frac{|H_n|}{||\sqrt{P_k} S||^2}$ is constant. In this case the hypersurface is said to be k-special.

Theorem 2.1. *Let $x : M^n \to \mathbb{L}^{n+1}$ be an oriented k-special hypersurface with $H_{k+1} = 0$ and $H_n \neq 0$. Then there exist some open submanifolds of M^n which are k-stable.*

Theorem 2.2. *Let $x : M^n \to \mathbb{L}^{n+1}$ be a complete non-compact oriented k-stable k-maximal hypersurface in $\mathbb{L}^{n+1}$, then M^n has only one end.*

Remark 2.3. In the Riemannian case, the proof of last theorem relies on the Sobolev ineqality for minimal submanifolds due to Michael and Simon [4] and the Liouville theorem for harmonic maps due to Schoen and Yau [6]. One crucial step in the proof is to show the existence of a non-trivial bounded harmonic function with finite Dirichlet energy in case the minimal hypersurface has more than one end. This is done by using the Sobolev inequality together with a choice of cut-off functions based on the fact that the minimal submanifold has more than one end. We remark that our cut-off function actually has noncompact support.

Theorem 2.4. *Let $x : M^n \to \mathbb{L}^{n+1}$ be a complete noncompact spacelike hypersurface which has at least two ends of infinite volume. In each one of two following cases there exists a non-constant bounded harmonic function with finite Dirichlet energy:*
(a) the Sobolev inequality holds on M;
(b) the first eigenvalue of M is positive.

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Existence of weak solutions for a dispersive wave equation with strong damping, nonlinear boundary source term and interior logarithmic nonlinearity

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Abstract. In this talk, we are dealing with an initial - boundary value problem for a class of dispersive wave equations with strong damping, nonlinear boundary source term and interior logarithmic nonlinearity. We prove the existence of local weak solutions by using the Galerkin approximation method and the Banach fixed point theorem.

Keywords: wave equation, logarithmic nonlinearity, existence

AMS Mathematics Subject Classification [2010]: 35A01, 35B45, 35D30

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1. Introduction

In this paper we consider the following initial-boundary value problem

$$(1)\qquad \begin{cases} u_{tt} - \Delta u - \Delta u_{tt} - \Delta u_t = u \ln |u|^k, \quad x \in \Omega, \ t \in (0,T), \\ u = 0, \qquad x \in \Gamma_0, \quad t \in (0,T), \\ \partial_\nu u_{tt} + \partial_\nu u + \partial_\nu u_t = u|u|^{p-2}, \qquad x \in \Gamma_1, \quad t \in (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases}$$

where Ω is an open bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, Γ_0 and Γ_1 are closed with positive measures, ν denotes the outward normal derivative to the boundary, $k, T > 0$ and p satisfies some assumptions to be specified later.

Logarithmic nonlinearity is of much interest in many areas of physics such as cosmology, optics and etc (for example see [1] and the references therein). There are numerous results about the existence, asymptotic stability and instability of solutions for wave equations with logarithmic nonlinearities. In this connection and to have an overview of previous studies we refer to the recent work by Liu [3].

Di and Shang in [2] investigated the following fourth order wave equation

$$(2)\qquad u_{tt} - \Delta u + \Delta^2 u - \omega(\Delta u_{tt} + \Delta u_t) + |u_t|^{r-1} u_t = f(u), \qquad x \in \Omega, \quad t > 0,$$

with initial and boundary conditions. Under the presence of dispersion-dissipation effect $(\omega > 0)$, the authors obtained existence of global solutions for $f(u) = u|u|^{p-2}$ by combination of the Galerkin approximation method and monotonicity-compactness arguments.

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Later, Chen and Xu in [4] considered (2) with $f(u) = u \ln |u|$. They established global existence by applying the potential well method under subcritical initial energy and when the initial energy is equal to critical initial energy level. Motivated with these studies, in this article we consider a dispersive-dissipative wave equation in (1) with a logarithmic nonlinearity in the domain and a polynomial source on the boundary. To the best of our knowledge, there is no study in the literature in which both types of logarithmic and polynomial nonlinearities act in a wave equation with dispersive-dissipative structure. In what follows in this article we will investigate existence of local weak solutions by employing the Galerkin approximation method and the Banach fixed point theorem.

To this end , first we introduce the space

$$V = \{w \in H^1(\Omega);\ w = 0 \ \text{ on } \ \Gamma_0\},$$

with the norm $\|.\|_V = \|\nabla .\|_2$. In the sequel we use the Sobolev imbedding $V \hookrightarrow L^s(\Omega)$ (with the best constant B_Ω for $2 \le s \le \frac{2N}{N-2}$ if $N \ge 3$, and $s = +\infty$ if $N = 1, 2$) and the Sobolev imbedding trace $V \hookrightarrow L^s(\Gamma_1)$ (with the best constant B_{Γ_1} for $2 \le s \le \frac{2(N-1)}{N-2}$ if $N \ge 3$, and $s = +\infty$ if $N = 1, 2$).
For the exponent p we assume

$$2 < p \le \bar{p}, \quad \text{where} \quad \bar{p} = \begin{cases} 2 + \frac{1}{N-2} & \text{if} \quad N > 2, \\ +\infty \quad \text{if} & N = 1, 2. \end{cases} \tag{3}$$

Next, in the following definition, we introduce the weak solutions associated to the initial-boundary value problem (1).

Definition 1.1. Let $\{u_0, u_1\} \subset V$. By a weak solution of problem (1) we mean a function u such that

$$u \in L^\infty(0,T;V), \quad u_t \in L^\infty(0,T;V) \cap L^\infty(0,T;L^2(\Omega)), \quad u_{tt} \in L^2(0,T;V),$$

which satisfies

$$\begin{aligned}(u_t - u_1, \phi) + &(\nabla u_t - \nabla u_1, \nabla\phi) + (\nabla u - \nabla u_0, \nabla\phi) + \int_0^t (\nabla u(s), \nabla\phi) ds \\ &= \int_0^t \int_\Omega u(s) \ln |u(s)|^k \phi dx ds + \int_0^t \int_{\Gamma_1} u(s)|u(s)|^{p-2} \phi dx ds,\end{aligned}$$

for any $\phi \in V$.

2. Local Weak Solutions

In this section we will discuss on existence of local weak solutions of the problem (1). Let us to define the following initial-boundary value problem:

$$\begin{cases} z_{tt} - \Delta z - \Delta z_{tt} - \Delta z_t = u \ln |u|^k, & x \in \Omega, \quad t \in (0,T), \\ z(x,t) = 0, \qquad x \in \Gamma_0, \quad t \in (0,T), \\ \partial_\nu(z_{tt}) + \partial_\nu(z) + \partial_\nu(z_t) = u|u|^{p-2}, & x \in \Gamma_1, \quad t \in (0,T), \\ z(x,0) = u_0(x), \qquad z_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{4}$$

Let us to introduce the space

$$\mathcal{V} = \big\{ u \mid u \in L^\infty(0,T;V),\ u_t \in L^\infty(0,T;V) \cap L^\infty(0,T;L^2(\Omega)) \big\}.$$

Lemma 2.1. *Suppose that* $\{u_0, u_1\} \subset V$, (3) *holds and* $u \in \mathcal{V}$. *Then the problem* (4), *for any* $T > 0$, *has a unique solution* z *in the class*

$$z \in L^\infty(0,T;V), \qquad z_t \in L^\infty(0,T;L^2(\Omega)) \cap L^\infty(0,T;V),$$
$$z_{tt} \in L^2(0,T;L^2(\Omega)) \cap L^2(0,T;V).$$

Proof. Suppose that $\{\phi_h(x)\}$, $(h = 1,2,...)$, is a basis for V and

$$z_n(t) = \sum_{h=1}^{n} \beta_n^h(t)\phi_h, \qquad n = 1,2,3,...$$

where $\{z_n\}$ satisfies

$$\begin{aligned} &((z_n)_{tt}, \phi_h) + \big(\nabla z_n, \nabla\phi_h\big) + \big(\nabla(z_n)_{tt}, \nabla\phi_h\big) \\ &\qquad + \big(\nabla(z_n)_t, \nabla\phi_h\big) = \int_{\Gamma_1} |u|^{p-2}u\phi_h d\Gamma + \int_\Omega u \ln|u|^k \phi_h dx, \end{aligned} \tag{5}$$

with initial conditions

$$\begin{aligned} z_n(0) = u_{0n}(x,0) \to u_0(x) \qquad &\text{in} \quad V, \\ (z_n)_t(0) = u_{1n}(x,0) \to u_1(x) \qquad &\text{in} \quad V. \end{aligned} \tag{6}$$

By theory of ordinary differential equations, the problem (5)-(6) has a unique solution on some interval $[0,T_n)$ $(0 < T_n < T)$. To find a priori estimates we multiply (5) by $(\beta_n^h)'(t)$ and $(\beta_n^h)''(t)$ separately and summing the results from $h = 1$ to $h = n$, integrating over $(0,t)$ and using Sobolev imbedding and trace Sobolev imbedding we get

$$\begin{cases} \{z_n\} \text{ is uniformly bounded in } L^\infty(0,T;V), \\ \{(z_n)_t\} \text{ is uniformly bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^\infty(0,T;V), \\ \{(z_n)_{tt}\} \text{ is uniformly bounded in } L^2(0,T;L^2(\Omega)) \cap L^2(0,T;V). \end{cases}$$

On the other hand, by integrating (5) over $(0,t)$, we have

$$\begin{aligned} &((z_n)_t, \phi_h) + (\nabla(z_n)_t, \nabla\phi_h) + (\nabla z_n, \nabla\phi_h) + \int_0^t (\nabla z_n(s), \nabla\phi_h)ds \\ &\quad = ((z_n)_t(0), \phi_h) + (\nabla(z_n)_t(0), \nabla\phi_h) + (\nabla z_n(0), \nabla\phi_h) \\ &\qquad + \int_0^t \int_{\Gamma_1} |u(s)|^{p-2}u(s)\phi_h d\Gamma ds + \int_0^t \int_\Omega u(s)\ln|u(s)|^k \phi_h dx ds. \end{aligned} \tag{7}$$

Thus, by extracting appropriate subsequences and then letting $n \to +\infty$ in (7) we can see that z satisfies the equation with the asserted regularities. It is easy to check that the solution of (4) is unique and so we omit the proof of uniqueness. □

Theorem 2.2. *Suppose that* $\{u_0, u_1\} \subset V$ *and* (3) *holds. Then, there exists a unique local solution* $u \in \mathcal{V}_T$ *of* (1) *for some* $T > 0$.

Proof. We consider the space $\mathcal{V}$ equipped with the following norm

$$\|v\|_{\mathcal{V}} = \left(\max_{0\le t\le T}\Big\{\|\nabla v(t)\|_2^2 + \|\nabla v_t(t)\|_2^2 \ : \ v \in \mathcal{V}\Big\}\right)^{\frac{1}{2}}.$$

We define

$$\mathcal{B}_R = \Big\{v \in \mathcal{V} \mid v(x,0) = u_0(x),\ v_t(x,0) = u_1(x),\ \ \|v\|_{\mathcal{V}} \le R\Big\}, \quad T > 0, \quad R > 0.$$

For any $u \in \mathcal{B}_R$, by Lemma 2.1, we set the mapping $\Psi(u) = z$ where z is a solution of problem (4). Then, we show that there exist positive constants T and R such that Ψ is a contraction mapping from $\mathcal{B}_R$ into itself. Multiplying the equation in (4) by z_t, integrating over $\Omega \times (0,t)$, and using again the Sobolev imbeddings for nonlinearities, we obtain

$$\|\nabla z(t)\|_2^2 + \|\nabla z_t(t)\|_2^2 \leq \left(L + C^2(R)\frac{T}{2}\right) + \frac{1}{2}\int_0^t \left(\|\nabla z(s)\|_2^2 + \|\nabla z_t(s)\|_2^2\right) ds, \tag{8}$$

where $L = \|u_0(x)\|_2^2 + \|\nabla u_0(x)\|_2^2 + \|\nabla u_1(x)\|_2^2$ and

$$C(R) = 2k\left(B^2 e^{-2}|\Omega| + \left(\frac{n-2}{2}\right)^2 B^{\frac{4(n-1)}{n-2}} R^{\frac{2n}{n-2}}\right)^{\frac{1}{2}} + 2B_\Omega B_{\Gamma_1}^{p-1} R^{p-1}.$$

Therefore, applying Gronwall's inequality on (8), we arrive at

$$\|\nabla z(t)\|_2^2 + \|\nabla z_t(t)\|_2^2 \leq \left(L + C^2(R)\frac{T}{2}\right) e^{\frac{T}{2}}.$$

Now, we let $R := 2L$. Then, we can choose T sufficiently small such that

$$\left(\frac{R}{2} + C^2(R)\frac{T}{2}\right) e^{\frac{T}{2}} < R,$$

which immediately results $\|\Psi(u)\|_{\mathcal{V}} < R$ and so $\Psi(u) \in \mathcal{B}_R$. To show that Ψ is a contraction mapping suppose that $\{u_1, u_2\} \subset \mathcal{B}_R$ and $\{z_1, z_2\}$ denotes corresponding solutions that is $z_1 = \Psi(u_1)$ and $z_2 = \Psi(u_2)$. Define $\omega := z_1 - z_2$. Then we can show that

$$\|\nabla \omega(t)\|_2^2 + \|\nabla \omega_t(t)\|_2^2 \leq K^2 T^2 \|u_1 - u_2\|_{\mathcal{V}}^2,$$

for some positive constant K. By choosing $T < K^{-1}$, the above inequality leads us to

$$\|\Psi(u_1) - \Psi(u_2)\|_{\mathcal{V}} \leq \alpha \|u_1 - u_2\|_{\mathcal{V}}, \quad \text{with} \quad 0 < \alpha < 1.$$

□

3. Conclusion

In this work we proved the existence of weak solutions for the problem (1) with the regularities in Definition 1.1. Our future focus on the problem (1) is to study long time behavior and blow up properties of solutions. It is well known that the logarithmic nonlinearity alone is not strong enough to cause the solutions blow up in finite time. But, because of the polynomial source it seems that solutions can blow up in finite time. However, presence of logarithmic term brings some kind of new challenges.

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Two major concepts in fractal calculus: Staircase function $S_C^\alpha(x)$ and Characteristic function $\chi_C(x)$

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Abstract. In this article, we study the Cantor set, the integral staircase function, and the characteristic function. These concepts are required to define smooth and differentiable structures on fractals. First, the unique properties of the Cantor set are presented. Then, we show how to draw the Cantor function interactively in the Jupyter notebook environment.

Keywords: Fractal, Cantor set, staircase function, characteristic function

AMS Mathematics Subject Classification [2010]: 28A80

1. Introduction

In ordinary calculus, we deal with discontinuity, lack of continuity in some points or intervals. There are also some situations where a derivative of a function fails to exist. Thus, discontinuity and non-differentiability are two common problems in ordinary calculus. On the other hand, we observe fractals [1, 2], which are continuous or discontinuous and usually nowhere differentiable. Moreover, fractals are often so irregular that defining smooth, differentiable structures on them seems very difficult. To study fractals some remarkable approaches have been used. They include fractal geometry, analysis on fractals, Harmonic analysis on fractals and in the past few years fractal calculus [3, 4]. In [3, 4] a new calculus based on fractal subsets of the real line is formulated which involves an integral of order $\alpha, 0 < \alpha < 1$, called F^α-integral and a derivative of order $\alpha, 0 < \alpha < 1$, called F^α-derivative. This enables us to differentiate functions, like the Cantor staircase, "changing" only on a fractal set. The F^α-derivative is local, unlike the classical fractional derivative. The main concepts in fractal calculus are flag function, mass function, ***integral staircase function*** $S_C^\alpha(x)$, set of change of a staircase function, compact set, α-perfect set, and ***characteristic function*** $\chi_C(x)$ [5, 6]. Cantor or staircase function is the strange continuous function on the unit interval, whose derivative is "zero" almost everywhere, but it somehow magically rises from 0 to 1.

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If you plot this function, you get something called the "Devil's Staircase". It is related to the standard Cantor set in the following way. This function is constant at all the removed intervals from the standard Cantor set. For instance if x is in $[1/3, 2/3]$, then $f(x) = 1/2$. If x is in $[1/9, 2/9]$, then $f(x) = 1/4$; if x is in $[7/9, 8/9]$, then $f(x) = 3/4$.

If one plots this function, then he will see that it is not differentiable at the Cantor set points but has zero derivatives everywhere else. However, since a Cantor set has a measure zero, this function has zero derivatives practically everywhere and only "rises" on Cantor set points!.

1.1. Properties of Cantor set. If we look at the construction of the Cantor set with a new perspective, by viewing the numbers in base 3, we will be able to see exactly which points remain (see Fig.1). Cantor set is defined as

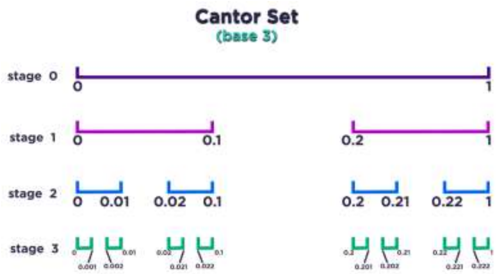

FIGURE 1. Cantor set in base 3.

(1) The Cantor set is uncountable.

$$\begin{aligned} C_0 &= [0,1], \\ C_1 &= [0,1/3] \cup [2/3,1], \\ C_2 &= [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1], \\ \vdots &= \vdots \\ C &= \bigcap_{n=0}^{\infty} C_n. \end{aligned}$$

(2) Length of the Cantor set is zero.

$$\lim_{n\to\infty} (2/3)^n = 0.$$

The fractal dimension for the Cantor set is $\alpha = \ln 2/\ln 3 = 0.63$.

2. Main results

Plotting Cantor function requires introducing the following functions (Fig.2) and combining them. Although the floor function $\lfloor x \rfloor$ is discontinuous, combining them can lead to a continuous function.

$$g(x) = \frac{\lfloor 3x \rfloor}{2} - \frac{4\lfloor x \rfloor}{2} + \frac{3\lfloor \frac{x}{3} \rfloor}{2}). \tag{1}$$

$$f(x) = \cos(3x \arccos(-0.5)) + 0.5 \tag{2}$$

$$k(x) = \frac{\lfloor 3x \rfloor}{2} - \frac{\lfloor x \rfloor}{2} + \left(\frac{3}{2}x - \frac{3\lfloor 3x \rfloor}{4} + \frac{3}{4}\lfloor x \rfloor\right)\left(\frac{f(x)}{2\left|f(x)\right|} + 0.5\right) \tag{3}$$

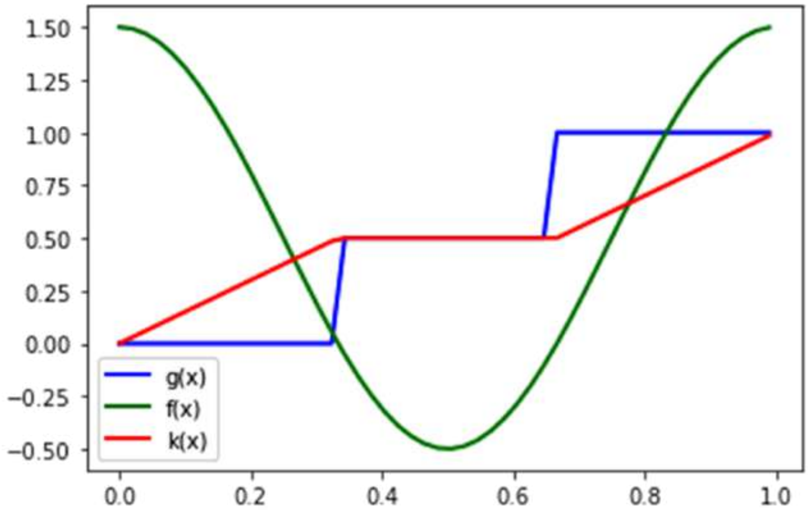

FIGURE 2. The three basic functions to make Cantor function.

$$z(x) = k(x) - \frac{3}{2}\lfloor x \rfloor + \frac{3\lfloor \frac{x}{3} \rfloor}{2} \tag{4}$$

The convenient function, not defined by multiple sub-functions, that we use to draw the characteristic function (Fig.3) is :

$$h(l, x) = \prod_{j=1}^{l}\left(\frac{\sin\left(3^j \pi x\right)}{2\left|\sin\left(3^j \pi x\right)\right|} + \frac{1}{2}\right) \tag{5}$$

Finally, by combining the above four functions, the following staircase function is created (Fig.4):

$$Cantor function = \left(\left(\sum_{i=0}^{n} \frac{g\left(3^i x\right) h(i, x)}{2^i}\right) + \frac{z\left(3^{(n+1)} x\right) h(n+1, x)}{2^{(n+1)}}\right)\{0 < x < 1\} \tag{6}$$

3. Conclusion

Defining, analytically, staircase and characteristic function in python (jupyter notebook) can help us to plot them interactively. This tool enables us to calculate fractal differentiation and fractal integration of a function in the interval [0,1].

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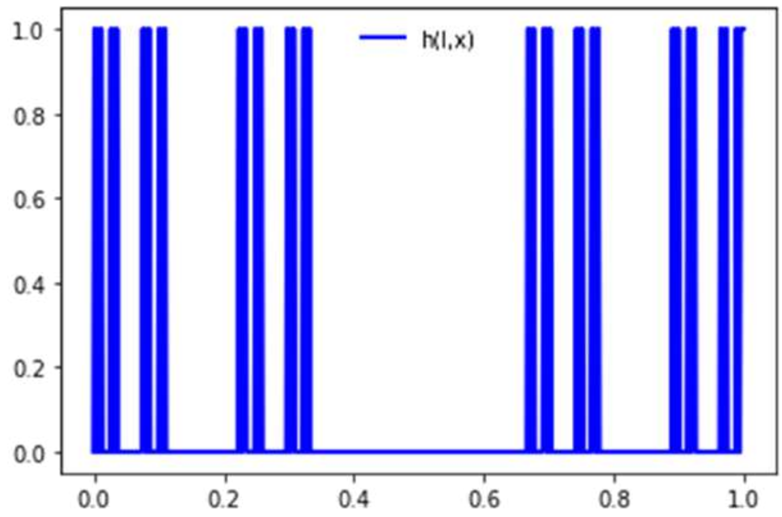

FIGURE 3. Characteristic function.

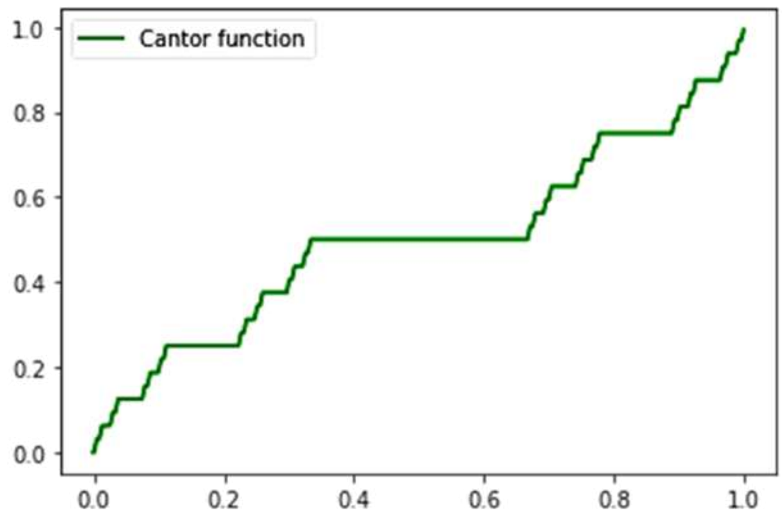

FIGURE 4. Cantor or Staircase function.

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A Characterization of Values (l, m) Such That $(l, m) \in \sum(q)$

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Abstract. A length ml, index l quasi-cyclic code can be viewed as a cyclic code of length m over the field $\mathbb{F}_{q^l}$ via a basis of the extension $\mathbb{F}_{q^l}/\mathbb{F}_q$. Let $\sum(q)$ be the set of all (l, m) values for one-generator length ml, index l quasi-cyclic codes C for which it is impossible to have an $\mathbb{F}_{q^l}$-linear image $\phi_\beta(C)$, for any choice of the polynomial basis β. In this paper we characterize values (l, m) such that $(l, m) \in \sum(q)$.

Keywords: Cyclic code, Quasi-cyclic code, Additive cyclic code, Linear code.
AMS Mathematics Subject Classification [2010]: 94B15, 94B05.

1. Introduction

Throughout this paper, q is a prime power, $\mathbb{F}_q$ denotes the finite field with q elements, m and l are positive integers such that $l > 1$ and $gcd(q, m) = 1$. A length ml, index l quasi-cyclic code is defined to be an $\mathbb{F}_q$-linear code in $\mathbb{F}_q^{ml}$ which is closed under T^l, where T is the shift operator defined by $T(c_0, c_1, \ldots, c_{ml-1}) = (c_{ml-1}, c_0, \ldots, c_{ml-2})$.

A length ml, index l quasi-cyclic code C over $\mathbb{F}_q$ can be viewed as an $R(m, q)$-submodule of $R(m, q)^l$, where $R(m, q) = \mathbb{F}_q[x]/\langle x^m - 1\rangle$. Using a polynomial basis β of $\mathbb{F}_{q^l}/\mathbb{F}_q$ and the map ϕ_β defined in [2], we map the quasi-cyclic code C to $R(m, q^l) = \mathbb{F}_{q^l}[x]/\langle x^m - 1\rangle$. We denote this image by $\phi_\beta(C)$ and it becomes an $R(m, q)$-submodule of $R(m, q^l)$. Equivalently, $\phi_\beta(C)$ is an $\mathbb{F}_q$-linear cyclic code of length m over $\mathbb{F}_{q^l}$. Such codes are called additive cyclic codes [1].

In [4,5], the following question was posed: when is the image under a basis extension of a quasi-cyclic code $\mathbb{F}_{q^l}$-linear, hence a classical cyclic code? In [2], the authors answer this question and characterize quasi-cyclic codes with an $\mathbb{F}_{q^l}$-linear image in $R(m, q^l)$. This characterization is particularly simple in the case of a one-generator quasi-cyclic code. They also characterize the (l, m) values for one-generator quasi-cyclic codes for which it is impossible to have an $\mathbb{F}_q$-linear image for any choice of the polynomial basis of $\mathbb{F}_{q^l}/\mathbb{F}_q$. But these conclusions should be checked for each case as in multiple steps and for some (l, m) values, these conclusions and characterizations is very intricate

In this paper, asing the characterization in [2], we give a more simple characterization to list the (l, m) values for one-generator quasi-cyclic codes for which it is impossible to have an $\mathbb{F}_{q^l}$-linear image for any choice of the polynomial basis of $\mathbb{F}_{q^l}/\mathbb{F}_q$.

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2. Main results

Throughout this section, l, m are positive integers such that $l > 1$ and $gcd(q, m) = 1$.

Let $\sum(q)$ be the set of all (l, m) values for one-generator length ml, index l quasi-cyclic codes C for which it is impossible to have an $\mathbb{F}_{q^l}$-linear image $\phi_\beta(C)$, for any choice of the polynomial basis β.

Let $x^m - 1 = f_1(x) \dots f_s(x)$ be the decomposition of $x^m - 1$ into irreducible polynomials of $\mathbb{F}_q[x]$. Suppose that $degf_i(x) = t_i$ $(1 \le i \le s)$. Put $T_q(m) = \{t_1, \dots, t_s\}$. Hence $R(m, q) \cong \mathbb{F}_{q^{t_1}} \oplus \dots \oplus \mathbb{F}_{q^{t_s}}$.

Theorem 2.1. *Let l, m be positive integers such that $l > 1$, $gcd(q, m) = 1$ and $T_q(m) = \{t_1, \dots, t_s\}$. Then $(l, m) \in \sum(q)$ if and only if for every i $(1 \le i \le s)$, $l \not| t_i$.*

Proof: At first we assume that for every i $(1 \le i \le s)$, $l \not| t_i$. Suppose that there exist a length ml, index l one-generator quasi-cyclic code C and a polynomial basis β of $\mathbb{F}_{q^l}/\mathbb{F}_q$ such that $\phi_\beta(C)$ is $\mathbb{F}_{q^l}$-linear. Since C is a one-generator quasi-cyclic code by [2, Theorem 4.1(ii)], $C = 0$ and hence $(l, m) \in \sum(q)$. Conversely we assume that there exists i $(1 \le i \le s)$ such that $l | t_i$, Let $\beta = \{1, \alpha, \dots, \alpha^{l-1}\}$ be a basis of $\mathbb{F}_{q^l}/\mathbb{F}_q$ and $\mathbb{F}_{q^l} = \mathbb{F}_q[x]/\langle f_\alpha(x)\rangle$ such that $f_\alpha(x) \in \mathbb{F}_q[x]$ is irreducible, $degf_\alpha(x) = l$ and $f_\alpha(\alpha) = 0$. Asing [2, section 2], we have $\mathbb{E}_i = \mathbb{F}_q[\xi_i]$, where $\xi_i = \xi^{u_i}$ and $\xi^m = 1$. Since $[\mathbb{E}_i : \mathbb{F}_q] = t_i$ and $l | t_i$, there exists $d \in \mathbb{N}$ such that $t_i = ld$. Hence there exists an irreducible polynomial $\lambda(x) \in \mathbb{F}_{q^l}[x]$ of degree d such that $\mathbb{E}_i \cong \mathbb{F}_{q^l}[x]/\langle \lambda(x)\rangle$. So there exists a root δ of $\lambda(x)$ such that $\mathbb{E}_i \cong \mathbb{F}_{q^l}[\delta]$ and hence $\mathbb{F}_{q^l}$ is embedded in $\mathbb{E}_i$. Now since $\alpha \in \mathbb{F}_{q^l}$, there exists $\omega \in \mathbb{E}_i$ such that $f_\alpha(\omega) = 0$ and hence $x - \omega | f_\alpha(x)$ in $\mathbb{E}_i[x]$. In this case, by the observations in [2, section 4], we have $b_i = l$ and $d_i = 1$. Now we constitute $\mathbb{E}_i$-subspace W_i^1 for $f_{\alpha,1}(x) = x - \omega$. Hence we have $W_i^1 = \{u \in \mathbb{E}_i^l \mid (M_\alpha - \omega I)u = 0\}$. By the observations above [2, Theorem 4.1], $dim_{\mathbb{E}_i} W_i^1 = degf_{\alpha,1}(x) = deg(x - \omega) = 1$ and so there exist $g_k(x) \in F_q[x]$ $(0 \le k \le l-1)$ such that $W_i^1 = \langle g_0(\omega), \dots, g_{l-1}(\omega)\rangle$. Since $\omega \in \mathbb{E}_i$, $g_k(\omega) \in \mathbb{E}_i = \mathbb{F}_q[\xi_i]$ $(0 \le k \le l-1)$ and hence for every k $(0 \le k \le l-1)$, there exist $h_k(x) \in \mathbb{F}_q[x]$ such that $g_k(\omega) = h_k(\xi_i)$ $(0 \le k \le l-1)$. Now let $\theta(x) = \prod_{i \ne j=1}^s f_j(x)$. We have $gcd(f_i(x), \theta(x)) = 1$. Since $\mathbb{F}_q[x]$ is a PID, there exist $\psi(x), \psi'(x) \in \mathbb{F}_q[x]$ such that $\psi(x)f_i(x) + \psi'(x)\theta(x) = 1$. Set $c_k(x) = \theta(x)\psi'(x)h_k(x)$. Now we have $C_j = \langle c_0(\xi_j), \dots, c_i(\xi_j), \dots, c_{l-1}(\xi_j)\rangle$ $(1 \le j \le s)$. Let $j \ne i$ and $1 \le j \le s$. Since $f_j(\xi_j) = 0$, $\theta(\xi_j) = 0$ and hence $c_k(\xi_j) = \theta(\xi_j)\psi'(\xi_j)h_k(\xi_j) = 0$ $(0 \le k \le l-1)$. Therefore for every $j \ne i$ and $1 \le j \le s$, $C_j = 0$. Let $j = i$. Since $f_i(\xi_i) = 0$, $c_k(\xi_i) = \theta(\xi_i)\psi'(\xi_i)h_k(\xi_i) = (1 - f_i(\xi_i)\psi(\xi_i))h_k(\xi_i) = h_k(\xi_i) = g_k(\omega)$ $(0 \le k \le l-1)$. Therefore $C_i = W_i^1$ and so $C = C_1 \oplus \dots \oplus C_{i-1} \oplus C_i \oplus C_{i+1} \oplus \dots \oplus C_{l-1} = 0 \oplus \dots \oplus 0 \oplus W_i^1 \oplus 0 \oplus \dots \oplus 0 = W_i^1$. Then by the observation above [2, Theorem 4.1], C is $\mathbb{F}_{q^l}$-linear and so $(l, m) \notin \sum(q)$.

Set $D_q(m) = \{1 \ne l \in \mathbb{N} \mid l | t_i, \text{ for some } i \ (1 \le i \le s)\}$.

In the following tables, for the convenience of the reader, we list the set $T_3(m)$ for 43 values of m with $gcd(m, 3) = 1$.

m	$T_3(m)$	m	$T_3(m)$	m	$T_3(m)$	m	$T_3(m)$
2	$\{1\}$	13	$\{1,3\}$	23	$\{1,11\}$	34	$\{1,16\}$
4	$\{1,2\}$	14	$\{1,6\}$	25	$\{1,4,20\}$	35	$\{1,4,6,12\}$
5	$\{1,4\}$	16	$\{1,2,4\}$	26	$\{1,3\}$	37	$\{1,18\}$
7	$\{1,6\}$	17	$\{1,16\}$	28	$\{1,2,6\}$	38	$\{1,18\}$
8	$\{1,2\}$	19	$\{1,18\}$	29	$\{1,28\}$	40	$\{1,2,4\}$
10	$\{1,4\}$	20	$\{1,2,4\}$	31	$\{1,30\}$	41	$\{1,8\}$
11	$\{1,5\}$	22	$\{1,5\}$	32	$\{1,2,4,8\}$	43	$\{1,42\}$

.

Example 2.2. Let $\Omega = \bigcup\{D_3(m)|m \leqslant 43, gcd(m,3) = 1\} = \{2,3,4,5,6,7,8,9,10, 11,12,14,15,16,18,20,21,28,30,42\}$. By Theorem 2.1, for every number $1 \neq l \notin \Omega$ and every $m \leqslant 43$ with $gcd(m,3) = 1$, we have $(l,m) \in \sum(3)$. In the following table, for every $l \in \Omega$, we list m values up to 43 with $gcd(m,3) = 1$ such that $(l,m) \in \sum(3)$.

l	m
2	2,11,13,22,23,26
3	2,4,5,8,10,11,16,17,20,22,23,25,29,32,34,40,41
4	2,4,7,8,11,13,14,19,22,23,26,28,31,37,38,42
5	2,4,5,7,8,10,13,14,16,17,19,20,23,26,28,29,32,34,35,37,38,40,41,43
6	2,4,5,8,10,11,13,16,17,20,22,23,25,26,29,32,34,40,41
7	2,4,5,7,8,10,11,13,14,16,17,19,20,22,23,25,26,28,31,32,34,35,37,38,40,41
8	2,4,5,7,8,10,11,13,14,16,19,20,22,23,25,26,28,29,31,35,37,38,40,43
9	2,4,5,7,8,10,11,13,14,16,17,20,22,23,25,26,28,29,31,32,34,35,40,41,43
10	2,4,5,7,8,10,11,13,14,16,17,19,20,22,23,26,28,29,32,34,35,37,38,40,41,43
11	2,4,5,7,8,10,11,13,14,16,17,19,20,22,25,26,28,29,31,32,34,35,37,38,40,41,43
12	2,4,5,7,8,10,11,13,14,16,17,19,20,22,23,25,26,28,29,31,32,34,37,38,40,41,43
14	2,4,5,7,8,10,11,13,14,16,17,19,20,22,23,25,26,28,31,32,34,35,37,38,40,41
15	2,4,5,7,8,10,11,13,14,16,17,19,20,22,23,25,26,28,29,32,34,35,37,38,40,41,43
16	2,4,5,7,8,10,11,13,14,16,19,20,22,23,25,26,28,29,31,32,35,37,38,40,41,43
18	2,4,5,7,8,10,11,13,14,16,17,20,22,23,25,26,28,29,31,32,34,35,40,41,43
20	2,4,5,7,8,10,11,13,14,16,17,19,20,22,23,26,28,29,31,32,34,35,37,38,40,41,43
21	2,4,5,7,8,10,11,13,14,16,17,19,20,22,23,25,26,28,29,31,32,34,35,37,38,40,41
28	2,4,5,7,8,10,11,13,14,16,17,19,20,22,23,25,26,28,31,32,34,35,37,38,40,41,43
30	2,4,5,7,8,10,11,13,14,16,17,19,20,22,23,25,26,28,29,32,34,35,37,38,40,41,43
42	2,4,5,7,8,10,11,13,14,16,17,19,20,22,23,25,26,28,29,31,32,34,35,37,38,40,41

.

Now we give a list of some (l,m) values such that without the decomposition of $R(m,q)$ to field extensions of $\mathbb{F}_q$, we can realize if $(l,m) \in \sum(q)$ or not.

Proposition 2.3. *Let $m|q^n - 1$, for some $n \geq 1$ and n be minimal. Then $D_q(m) = \{1 \neq l \in \mathbb{N} \mid l \mid n\}$.*

Proof: Let $\{s_1, \ldots, s_k\}$ be a complete set of representatives of cyclotomic cosets of q modulo m [3, Definition 3.4.5], and $d = \dfrac{q^n - 1}{m}$. Suppose that $D_{ds_i} (1 \leq i \leq k)$ is the ds_i-th cyclotomic cosets of q modulo $q^n - 1$. Put $|D_{ds_i}| = t_i$ $(1 \leq i \leq k)$. By [3, Theorem 3.4.11 and Remark 3.4.9(i)], we have $T_q(m) = \{t_1, \ldots, t_k\}$. It is easy to see that $t_i | n$ $(1 \leq i \leq k)$

and so $D_q(m) \subseteq \{1 \neq l \in \mathbb{N} \mid l|n\}$. Conversely, let $1 \neq l|n$. We will prove that $l \in D_q(m)$. Without loss of generality, assume that $s_1 = 0$ and $s_2 = 1$. Then $|D_{ds_2}| = |D_d| = t_2$. Since $t_2|n$, $t_2 \leq n$. We will prove $t_2 = n$. By [3, Definition 3.4.5], t_2 is the minimal number such that $dq^{t_2} \equiv d(\text{mod } q^n - 1)$. Since $q^n - 1 = md$, $m|q^{t_2} - 1$ and since n is minimal, $t_2 = n$. Then $n \in T_q(m)$. Now by the definition of $D_q(m)$, $l \in D_q(m)$ and the proof is complete.

Corollary 2.4. *Let l, m be positive integers such that $l > 1$, $m|q^n - 1$, for some $n \geq 1$ and n be minimal. Then $(l, m) \in \sum(q)$ if and only if $l \not| n$.*

Proof: It follows by Theorem 2.1 and Proposition 2.3.

Example 2.5. i) Let $m = 1023$. Since $1023 = 2^{10} - 1$, by Corollary 2.4, $(l, 1023) \in \sum(2)$ if and only if $l \notin \{2, 5, 10\}$.

Let $m = 51$. We have $51|2^8 - 1$ and 8 is minimal. By Corollary 2.4, $(l, 51) \in \sum(2)$ if and only if $l \notin \{2, 4, 8\}$.

Let $A(q) = \{m \in \mathbb{N} \mid m \mid q^{m-1} - 1 \text{ and } m \nmid q^n - 1, \text{ for all } n < m - 1\}$.

Corollary 2.6. *Let $k \in A(q)$ and $m \equiv k(\text{mod } qk)$. Then for every l, such that $l|k-1$, $(l, m) \notin \sum(q)$.*

Proof: Let $k \in A(q)$ and $m \equiv k(\text{mod } qk)$. So there exists $t \in \mathbb{N}$ such that $m = qtk+k$. Hence $x^m - 1 = (x - 1)(x^{qtk+k-1} + x^{qtk+k-2} + \cdots + x^2 + x + 1)$. Let $f(x) = x^{k-1} + \cdots + x^2 + x + 1$. Since $k \in A(q)$, the cyclotomic cosets of q modulo k are $C_0 = \{0\}$ and $C_1 = \{1, q, \ldots, q^{k-2}\}$. So $\{0, 1\}$ is a complete set of representatives of cyclotomic cosets of q modulo k. Therefore by [3, Corollary 3.4.12], the number of monic irreducible factors of $x^k - 1$ over $\mathbb{F}_q$ is equal to 2. Then $x^k - 1 = (x - 1)f(x)$ and so $f(x) \in \mathbb{F}_q[x]$ is irreducible. Let $g(x) = x^{qtk+k-1} + x^{qtk+k-2} + \cdots + x^2 + x + 1$ and $h(x) = x^{qtk} + x^{(qt-1)k} + \cdots + x^{2k} + x^k + 1$. We have $g(x) = f(x)h(x)$ and so $f(x)|g(x)$. Hence $f(x)|x^m - 1$. Now since $degf(x) = k-1$, $k - 1 \in T_q(m)$ and the proof follows by Theorem 2.1.

3. Conclusion

In this paper for every length ml, index l quasi-cyclic code C, only by knowing the set $T_q(m)$, we can realize if that $(l, m) \in \sum(q)$ or not. Also we give a list of some (l, m) values such that without knowing the set $T_q(m)$, we can realize if $(l, m) \in \sum(q)$ or not.

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Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

(C, C')-Controlled g-Fusion Frames in Hilbert Spaces

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Abstract. In this paper we develop a theory based on g-fusion frames on Hilbert spaces, which provides exactly the frameworks not only to model new frames on Hilbert spaces but also for deriving robust operators. In particular, we can define analysis, synthesis and frame operators with representation space compatible for (C, C')-Controlled g-fusion frames, which even yield a reconstruction formula.

Keywords: G-fusion frame, Controlled fusion frame, Controlled g-fusion frame

AMS Mathematics Subject Classification [2010]: 42C15; 42C40, 41A58

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1. Introduction

Frames, as a generalization of the bases in Hilbert spaces, were first introduced by Duffin and Schaeffer [2] during their study of nonharmonic Fourier series in 1952.

Throughout this paper H and K are separable Hilbert spaces, $\{H_j\}_{j\in\mathbb{I}}$ is a sequence of Hilbert spaces and $I \subseteq \mathbb{Z}$. We denote by $\mathcal{B}(H, K)$ the set of all the bounded and linear operators from H to K. If $H = K$, then $\mathcal{B}(H, H)$ will be denoted as $\mathcal{B}(H)$. Also, $GL(H)$ is called the set of all bounded linear operators which have bounded inverses on H. It is easy to check that if $C, C' \in GL(H)$, then C^*, C^{-1} and CC' are in $GL(H)$. Assume that Id_H is the identity operator on H and π_W is the orthogonal projection from H onto a closed subspace $V \subseteq H$.

Definition 1.1. [3] Let $W := \{W_i\}_{i\in\mathbb{I}}$ be a family of closed subspaces of H, $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights i.e. $v_i > 0$ for all $i \in \mathbb{I}$ and $\Lambda_i \in \mathcal{B}(H, H_i)$. We say $\Lambda := (W_i, \Lambda_i, v_i)$ is a g-fusion frame for H if there exist constants $0 < A \leq B < \infty$ such that for all $f \in H$

$$A\,\|f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2\,\|\Lambda_i \pi_{W_i} f\|^2 \leq B\|f\|^2.$$

We call Λ a Parseval g-fusion frame if $A = B = 1$. When the right hand of above inequality holds, Λ is called a g-fusion Bessel sequence for H with bound B. We define the space

$$\mathscr{H}_2 := (\sum_{j\in\mathbb{J}} \bigoplus H_j)_{l_2}$$

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by:

$$\mathscr{H}_2 = \big\{\{f_j\}_{j\in\mathbb{J}} \ : \ f_j \in H_j, \ \sum_{j\in\mathbb{J}} \|f_j\|^2 < \infty\big\}.$$

with the inner product defined by

$$\langle\{f_j\},\{g_j\}\rangle = \sum_{j\in\mathbb{J}}\langle f_j, g_j\rangle.$$

2. Main results

DEFINITION 2.1. Let $W := \{W_i\}_{i\in\mathbb{I}}$ be a family of closed subspaces of H and $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights i.e. $v_i > 0$ for all $i \in \mathbb{I}$. Let $\{H_i\}_{i\in\mathbb{I}}$ be a sequence of Hilbert spaces, $C, C' \in GL(H)$ and $\Lambda_i \in \mathcal{B}(H, H_i)$. $\Lambda_{CC'} := (W_i, \Lambda_i, v_i)$ is a (C, C')-controlled g-fusion frame (briefly CC'-GF) for H if there exist constants $0 < A \le B < \infty$ such that for all $f \in H$

$$A\|f\|^2 \le \sum_{i\in\mathbb{I}} v_i^2\langle\Lambda_i\pi_{W_i}C'f, \Lambda_i\pi_{W_i}Cf\rangle \le B\|f\|^2.$$

We call $\Lambda_{CC'}$ is a Parseval CC'-GF if $A = B = 1$. If only the second Inequality is required, We call $\Lambda_{CC'}$ is a (C, C')-Controlled Bessel g-fusion sequence (briefly CC'-GBS) with bound B If we assume $\Lambda_{CC'}$ is a CC'-GF for H and $C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'$ is a positive operator for each $i \in \mathbb{I}$, then $C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C' = C'^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C$ and therefore

$$A\|f\|^2 \le \sum_{i\in\mathbb{I}} v_i^2 \left\|(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{1/2}f\right\|^2 \le B\|f\|^2.$$

Let

$$\mathcal{K}^2_{\Lambda_j} := \{v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f \ : \ f \in H\} \subset (\bigoplus_{i\in\mathbb{I}} H)_{l^2}.$$

It is easy to check that $\mathcal{K}^2_{\Lambda_j}$ is a closed subspace. We can define the *controlled analysis operator* T^*_Λ by

$$T^*_\Lambda : H \to \mathcal{K}^2_{\Lambda_j},$$
$$T^*_\Lambda f = \{v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\}_{i\in\mathbb{I}}.$$

It is easy to check that the controlled analysis operator is bounded linear operator. Thus, $T_\Lambda := (T^*_\Lambda)^*$ is well-defined and bounded and the *controlled synthesis operator* T_Λ can be defined by

$$T_\Lambda : \mathcal{K}^2_{\Lambda_j} \to H,$$
$$T_\Lambda\Big(v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\Big) = \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f.$$

Now, we can define the CC'-GF operator $S_{CC'}$ on H by

$$S_{CC'}f := T_\Lambda T^*_\Lambda f = \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f.$$

We can write for each $f \in H$

$$\langle S_{CC'}f, f\rangle = \sum_{i\in\mathbb{I}} v_i^2\langle C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f, f\rangle$$

$$= \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle,$$

therefore, we get

$$AId_H \leq S_{CC'} \leq BId_H.$$

THEOREM 2.2. *[4] $\Lambda_{CC'}$ be a CC'-GBS for H with bound B if and only if the operator*

$$T_\Lambda : \mathcal{K}^2_{\Lambda_j} \to H,$$

$$T_\Lambda(v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f) = \sum_{i\in\mathbb{I}} v_i^2 C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C' f.$$

is well -defined and bounded operator with $\|T_\Lambda\| \leq \sqrt{B}$.

PROOF. The necessary condition follows from the definition of CC'-GBS. We only need to prove that the sufficient condition holds. Let T_Λ be well-defined and bounded operator with $\|T_\Lambda\| \leq \sqrt{B}$. For any $f \in H$, we have

$$\begin{aligned}
\sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle &= \sum_{i\in\mathbb{I}} v_i^2 \langle C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C' f, f\rangle \\
&= \langle T_\Lambda\big(v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f\big), f\rangle \\
&\leq \|T_\Lambda\| \left\|(v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f\right\| \|f\|.
\end{aligned}$$

But

$$\left\|(v_i(C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C')^{\frac{1}{2}} f\right\|^2 = \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle.$$

It follows that

$$\sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle \leq B \|f\|^2$$

and this means that $\Lambda_{CC'}$ is a CC'-GBS for H. □

THEOREM 2.3. *[4] Let $C \in GL^+(H)$. $\Lambda := (W_i, \Lambda_i, v_i)$ is a g-fusion frame for H if and only if Λ is a CC-GF.*

PROOF. Suppose that Λ is a CC-GF with Bounds A and B for H. for each $f \in \mathbb{H}$, we obtain

$$\begin{aligned}
A\|f\|^2 &= A\left\|CC^{-1}f\right\|^2 \\
&\leq A\|C\|^2 . \left\|C^{-1}f\right\|^2 \\
&\leq \|C\|^2 \sum_{i\in\mathbb{I}} v_i^2 \left\|\Lambda_i \pi_{W_i} CC^{-1} f\right\|^2 \\
&= \|C\|^2 \sum_{i\in\mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} f\|^2
\end{aligned}$$

Hence

$$A\|C\|^{-2}\|f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} f\|^2 .$$

On the other hand, for any $f \in H$, we have

$$\begin{aligned}\sum_{i\in\mathbb{I}} v_i^2 \left\|\Lambda_i \pi_{W_i} f\right\|^2 &= \sum_{i\in\mathbb{I}} v_i^2 \left\|\Lambda_i \pi_{W_i} C C^{-1} f\right\|^2 \\ &\leq B \left\|C^{-1} f\right\|^2 \\ &\leq B \left\|C^{-1}\right\|^2 . \left\|f\right\|^2\end{aligned}$$

Thus, Λ is a g-fusion frame for H with bounds $A\left\|C\right\|^{-2}$,$B\left\|C^{-1}\right\|^2$.

Conversely, assume that Λ is a g-fusion frame for H with bounds A, B. Then, for each $f \in H$ we get

$$\sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C f, \Lambda_i \pi_{W_i} C f\rangle = \sum_{i\in\mathbb{I}} v_i^2 \left\|\Lambda_i \pi_{W_i} C f\right\|^2 \leq B \left\|C\right\|^2 \left\|f\right\|^2 .$$

For the lower bound, we can write for any $f \in H$,

$$\begin{aligned}A\left\|f\right\|^2 &= A \left\|C^{-1} C f\right\|^2 \\ &\leq A \left\|C^{-1}\right\|^2 \left\|C f\right\|^2 \\ &\leq \left\|C^{-1}\right\|^2 \sum_{i\in\mathbb{I}} v_i^2 \left\|\Lambda_i \pi_{W_i} C f\right\|^2 .\end{aligned}$$

Therefore, Λ is a CC-GF for H with bounds $A\left\|C^{-1}\right\|^{-2}, B\left\|C^{-1}\right\|^{-2}$. □

Acknowledgement

We gratefully thank the referee for carefully reading the paper and for the suggestions that greatly improved the presentation of the paper.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
52nd Annual Iranian Mathematics Conference

On C-epiretractable acts

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Abstract. Let S be a monoid. A right $S-$act A is said to be C-epiretractable if every cyclic subact B of A, is a homomorphic image of A. Our objective is to give some examples and characterizations of C-epiretractable acts by properties of underlying sets and investigate the relation between these properties with some other properties such as injectivity, projectivity and flatness.

Keywords: right S-act, C-epiretractable , generators

AMS Mathematics Subject Classification [2010]: 20M30, 20M50

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1. Introduction and Preliminaries

Throughout this article, unless otherwise stated, S will denote a monoid and a right S-act A_S (or act A for short) is a unitary S-act. Retractablity of modules over rings was introduced in [2] and many papers on this notion have been published since then. In the category of $S-$acts (**Act**-S) the concept of retractablity was introduced by Khosravi in [1]. As in [1], an S-act A is called *retractable* if for any subcat B of A, $Hom(A, B) \neq \phi$. Here we introduce a generalization of retractable acts. The right S-act A is said to be *C-epireractable* if any cyclic subact of A is a homomorphic image of A, i.e., for any $a \in A$, there exists an epimorphism $f : A \longrightarrow aS$. In this paper the notion of C-epiretractable acts is investigated and some classifications of monoids are given when C-epiretractablity of acts implies some other properties. Another related generalization of this notion in **Act**-S, is defined in [4] as follows. A right S-act is called *fully idempotent* if for any subact B of A, $B = \bigcup_{\alpha\in Hom(A,B)} \alpha(B)$. It is easy to see the following strict implications.

being a fully idempotent act $\Rightarrow$ C-epiretractablity $\Rightarrow$ retractablity
Recall from [3, Page 146] that the trace of an S-act B in an S-act A is defined by

$$Tr(B, A) := \bigcup_{\varphi\in Hom(B,A)} \varphi(B).$$

Also by [3, Theorem 2.3.16] an S-act G is a generator if and only if $Tr(G, S) = S_S$. An S-act A is called *injective* if for any S-act B, any subact C of B and any homomorphism $f : C \longrightarrow A$, there exists a homomorphism $\bar{f} : B \longrightarrow A$ such that $\bar{f} \mid_C = f$. Some

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*Speaker. Email address: m.rooeintan@yahoo.com weak forms of injectivity are studied in [3]. We denote in short "weake injectivity", "finitely generated weak injectivity" and "principal weak injectivity" by w-injectivity, fg-w injectivity and pw-injectivity respectively. Moreover the one element act is denoted by $\Theta = \{\theta\}$. For an S-act A, by $E(A)$, we mean the injective envelop of A. we refer the reader to [3] for a thorough account on the preliminaries.

2. Main results

Definition 2.1. Over a semigroup S, an S-act A is said to be *C-epireractable*, if for any cyclic subact C of A, there exists an epimorphism $f : A \longrightarrow C$.

Clearly every monoid S is C-epiretractable and by a routine argument we can see that over a commutative monoid S any cyclic S-act is C-epiretractable. Also every generator in the category of right $S-$acts is C-epiretractable. In particular, $A \times S$ is C-epiretractable for any S-act A. The following proposition contains some properties of C-epiretractable acts which are needed in the sequel. Recall that a subcat B of an S-act A is called fully invariant if $f(B) \subseteq B$ for any homomorphism $f : A \longrightarrow A$ (for more details see [5]). Also by A/B we mean the Rees factor act A/ρ_B of A by a subact B.

Proposition 2.2. *The following statements hold over a monoid S:*
(i) *A right S-act A is C-epiretractable if and only if Tr(A,B)=B for any subact B of A.*
(ii) *For a C-epiretractable act A, if B is a fully invariant subact, A/B is C-epiretractable. In particular A/AI is C-epiretractable for every right ideal I of S.*
(iii) *If $\{A_i\}_{i\in I}$ is a family of C-epiretractable acts which $Hom(A_i, A_j) \neq \emptyset$ for any $i, j \in I$, then $\coprod_{i\in I} A_i$ is C-epiretractable. In particular every free S-act is C-epiretractable.*

Note that in general C-epiretractable acts are not preseved under product and coproduct. For instance, see [4, Example 2.4].

Proposition 2.3. *Let S be a left reversible monoid. If every injective act is C-epiretractable, then every projective act is injective.*

Now we give some classifications of monoids when C-epiretractable acts imply other properties of $S-$acts.

Suppose (P) is a property on $S-$acts which is transferable from products to their components. Regarding the fact that $A \times S$ is C-epiretractable for any S-act A, we conclude that all acts have property (P) if and only if all C-epiretractable acts have (P). Consequently the next corollary is clear.

Proposition 2.4. *Over a monoid S all acts are injective (resp. w-injective, fg-w injective, pw-injective, divisible) if and only if all C-epiretractable acts are injective (resp. w-injective, fg-w injective, pw-injective, divisible).*

Proposition 2.5. *The following conditions are equivalent over a monoid S.*
(i) *All acts are projective.*
(ii) *All C-epiretractable acts are projective.*
(iii) $S = \{1\}$.

Proof. (ii)$\Rightarrow$ (i) Since the right S-act Θ is projective, S contains a left zero. Thus for every S-act A, the projectivity of $A \times S$ implies projectivity of A. (i)$\Leftrightarrow$ (iii) See [3, section 4.11]. □

Proposition 2.6. *Suppose S is a monoid. All acts are weakly flat(falt) if and only if all C-epiretractable acts are weakly flat(falt).*

Proof. Note that the right S-act Θ is weakly flat if and only if S is right reversible (see [3, 3.11.2]). Now since by assumption $A \times S$ is weakly flat for any right S-act A, by [6, Theorem 3.3] the result follows. The proof for flat acts is similar. $\square$

Proposition 2.7. *The following conditions are equivalent over a monoid S:*
(i) *All acts are principally weakly flat.*
(ii) *All C-epiretractable acts are principally weakly flat.*
(iii) *S is a regular monoid.*

A right S-act A satisfies Condition (P) if $as = a's'$ for $a, a' \in A$, $s, s' \in S$, implies the existence of $a'' \in A$, $u, v \in S$ such that $a = a''u$, $a' = a''v$, $us = vs'$. If I is a proper right ideal of S and $i \in I$, then considering the structure of $S \sqcup^I S, (1, x)i = (1, y)i = i$. But there is no element $a'' \in S \sqcup^I S$ such that $(1, x) = a''u$ and $(1, y) = a''v$ for any $u, v \in S$. Thus $S \sqcup^I S$ does not satisfy condition (P). Accordingly the following proposition can be proved using [3, Theorem 4.9.10].

Theorem 2.8. *The following conditions are equivalent over a monoid S:*
(i) *All acts satisfy condition (P).*
(ii) *All C-epiretractable acts satisfy condition (P).*
(iii) *S is a group.*

A right S-act A satisfies Condition (E) if $as = as'$ for $a \in A_S$, $s, s' \in S$, implies that there exist $a' \in A$, $u \in S$ such that $a = a'u$, $us = us'$. By a straightforward argument, we can see that the right S-act Θ satiesfies condition (E) if and only if S is left collapsible.

Theorem 2.9. *The following conditions are equivalent over a monoid S:*
(i) *All acts satisfy condition (E).*
(ii) *All C-epiretractable acts satisfy condition (E).*

Proof. (ii)$\Rightarrow$ (i) Since the right S-act Θ satisfies condition (E), S is left collapsible. Now since $A \times S$ satisfies condition (E) for any S-act A, by [6, Theorem 3.5], we have the result. $\square$

Theorem 2.10. *The following conditions are equivalent over a monoid S:*
(i) *All acts are torsion free.*
(ii) *All C-epiretractable acts are torsion free.*
(iii) *Every right cancellable element of S is right invertible.*

Theorem 2.11. *The following conditions are equivalent over a monoid S:*
(i) *All acts are free.*
(ii) *All C-epiretractable acts are free.*
(iii) *$S = \{1\}$.*

Recall that any cofree S-act can be considerd of the form $X^S = \{f \mid f : S \longrightarrow X\}$ where X is a non-empty set and $(fs)t = f(st)$ for any $s, t \in S$. Thus if the right S-act S_S is cofree then for some set $X, \mid S \mid = \mid X^S \mid$ which implies that $S = \{1\}$. Now we have the following proposition.

Theorem 2.12. *The following conditions are equivalent over a monoid S:*
(i) *All acts are cofree.*
(ii) *All C-epiretractable acts are cofree.*
(iii) *$S = \{1\}$.*

3. Conclusion

In this paper over a monoid S, we introduce a class of right S-acts namely C-epiretractabe acts which are useful to characterize monoids over which all acts are projrctive, injective(w-injective, fg-w injective, pw-injective, divisible), flat (weakly falt, principally weakly flat).

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کنفرانس ریاضی ایران
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دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

On concircular transformations in Finsler geometry

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Abstract. A geodesic circle in Finsler geometry is a natural extension of that in a Euclidean space. In this paper, we characterize Finsler manifolds admitting a concircular transformation such that the difference of the two Ricci tensors is a constant multiple of the metric. We characterize a concircular transformation with some PDEs on the tangent bundle, and then we obtain the solution.

Keywords: Geodesic circle, Concircular transformation, Ricci tensor, Scalar curvature

AMS Mathematics Subject Classification [2010]: 53B40, 53C60

1. Introduction

A geodesic circle in an Euclidean space is a straight line or a circle with finite positive radius, which can be generalized naturally to Riemannian or Finsler geometry. Firstly, In 1940, Yano introduced concircular transformations on Riemannian manifolds [7]. Exactly, a geodesic circle in a Riemannian manifold, as well as in a Finsler space, is a curve with constant first Frenet curvature and zero second one. In other words, a geodesic circle is a torsion free curve with constant curvature. A concircular transformation on a Riemannian manifold is a conformal transformation which preserves geodesic circles ([2], [7]). Many researchers have developed the theory of concircular transformations to different contents. In 1970, Vogel showed that every concircular transformation on a Riemannian manifold is conformal. This notion has been extended to Finsler geometry by Agrawal and Izumi [1]. Also, a similar result is proved by Bidabad and shen in 2012 [4]. That is, every transformation which preserves geodesic circles has to be conformal. So, by the modified definition, a diffeomorphism φ, between two Finsler manifolds (M, F) and $(\bar{M}, \bar{F})$, is said to be concircular if it maps geodesic circles to geodesic circles. Also, two Finsler metrics defined on a manifold are said to be concircular if they have the same geodesic circles.

In a coordinate system, For a Finsler metric $F = F(x, y)$ on a manifold M, the fundamental metric tensor g_{ij}(while g^{ij} is its inverse), the Cartan torsion C^i_{jk} and the mean Cartan torsion I_i (respectively) will be denoted as follow:

$$g_{ij} := \dot{\partial}_i \dot{\partial}_j (\frac{F^2}{2}), \quad 2C_{ijk} := \dot{\partial}_k g_{ij}, \quad I_i := g^{jk} C_{ijk} = C^r_{ir}, \quad (\dot{\partial}_i = \frac{\partial}{\partial y^i}). \tag{1}$$

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Clearly, a Finsler metric will be a Riemannian metric if its Cartan torsion or mean Cartan torsion be null [3]. In this paper, we consider concircular transformations on a Finsler manifold, where the difference of whose Ricci tensors are a constant multiple of the Finsler metric $\bar{F}$. A geodesic circle in an Euclidean space is a straight line or a circle with finite positive radius, which can be generalized naturally to Riemannian or Finsler geometry. Firstly, In 1940, Yano introduced concircular transformations on Riemannian manifolds [7]. Exactly, a geodesic circle in a Riemannian manifold, as well as in a Finsler space, is a curve with constant first Frenet curvature and zero second one. In other words, a geodesic circle is a torsion free curve with constant curvature. A concircular transformation on a Riemannian manifold is a conformal transformation which preserves geodesic circles ([2], [7]). Many researchers have developed the theory of concircular transformations to different contents. In 1970, Vogel showed that every concircular transformation on a Riemannian manifold is conformal. This notion has been extended to Finsler geometry by Agrawal and Izumi [1]. Also, a similar result is proved by Bidabad and shen in 2012 [4]. That is, every transformation which preserves geodesic circles has to be conformal. So, by the modified definition, a diffeomorphism φ, between two Finsler manifolds (M, F) and $(\bar{M}, \bar{F})$, is said to be concircular if it maps geodesic circles to geodesic circles. Also, two Finsler metrics defined on a manifold are said to be concircular if they have the same geodesic circles.

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Clearly, a Finsler metric will be a Riemannian metric if its Cartan torsion or mean Cartan torsion be null. In this paper, we consider concircular transformations on a Finsler manifold, where the difference of whose Ricci tensors are a constant multiple of the Finsler metric $\bar{F}$.

2. Preliminary

Let M be an $n-$dimensional manifold of class C^∞. We denote by $\pi : TM \to M$ the bundle of tangent vectors and by $\pi_0 : TM_0 \to M$ the fiber bundle of non-zero tangent vectors. A Finsler structure on M is a function $F : TM \to [0, \infty)$, with the following properties:

I) F is differentiable (C^∞) on TM_0;

II) $F(x, y)$ is positively homogeneous of degree one in y, i.e. $F(x, \lambda y) = \lambda F(x, y), \forall\lambda > 0$, where we denote an element of TM by (x, y).

III) The Hessian matrix of $\frac{F^2}{2}$ is positive definite on TM_0; $g_{ij} := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$.
A Finsler manifold (M, g) is a pair of a differential manifold M and a tensor field $g = (g_{ij})$ on TM which defined by a Finsler structure F. The spray of a Finsler structure F is a vector field on TM as:

$$G = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i} \tag{3}$$

and G^i be the geodesic coefficients of F, which are defined by

(4) $$G^i = \frac{1}{4}g^{il}\{F^2_{x^m y^l}y^m - F^2_{x^l}\}$$

where $g_{ij}(x,y) := \frac{1}{2}[F^2]_{y^i y^j}(x,y)$ and $(g^{ij}) := (g_{ij})^{-1}$.
The geodesics of F are characterized by the second order differential equation:

$$\frac{d^2c^i}{dt^2} + 2G^i(c(t),\dot{c}(t)) = 0.$$

The Riemann curvature $R_y : T_pM \to T_pM$ is a linear transformations on tangent spaces, which is defined by

(5) $$R_y = R^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k$$

(6) $$R^i{}_k := 2\frac{\partial G^i}{\partial x^i} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

For a two-dimensional plane $P \subset T_pM$ and $y \in T_pM \setminus \{0\}$ such that $P = span\{y,u\}$, the pair $\{P,y\}$ is called a flag in T_pM. The flag curvature $K(P,y)$ is defined by

$$K(P,y) := \frac{g_y(u,R_y(u))}{g_y(y,y)g_y(u,u) - g_y(y,u)^2}.$$

We say that F is of scalar curvature if for any $y \in T_pM \setminus \{0\}$ the flag curvature $K(P,y) = \lambda(y)$ is independent of P containing y. This is equivalent to the following condition in a local coordinate system (x^i,y^i) in TM:

$$R^i_k = \lambda F^2\{\delta^i_k - F^{-1}F_{y^k}y^i\}.$$

If λ is a constant, then F is said to be of constant curvature.
The Ricci curvature is the trace of the Riemann curvature, i.e.,

(7) $$Ric(y) := R^m_m(y)$$

and $R(y) := \frac{1}{n-1}Ric(y)$ is called the Ricci scalar.

Let $\bar{F}$ and F be two Finsler metrics on an n-dimensional manifold M. There is a relation between the geodesic coefficients $\bar{G}^i$ and G^i as follows:

(8) $$\bar{G}^i = G^i + \frac{\bar{F}_{;}ky^k}{2\bar{F}}y^i + \frac{\bar{F}}{2}\bar{g}^{il}\{\bar{F}_{;k.l}y^k - \bar{F}_{;l}\}.$$

If $\bar{F} = e^{c(x)}F$ then we have

$$\bar{G}^i = G^i + (c_k y^k)y^i - \frac{F^2}{2}c^i,$$

where $c^i = g^{il}c_l$.

Example 2.1. Examples of geodesic circles are small circles on the sphere. It is not required that a geodesic circle is a closed curve. It might be something like a spiral even if the length is infinite.

Lemma 2.2. *(Yano [7]): c is a geodesic circle in Riemannian space if and only if $\dddot{c}$ is a scalar multiple of $\dot{c}$. In this case necessarily $\dddot{c} = -\langle\ddot{c},\ddot{c}\rangle.\dot{c}$.*

LEMMA 2.3. *Let $\bar{F}$ and F be two conformally related Finsler metrics on a same manifold M with $\bar{F} = u^{-1}F$. Then we have*

- *$\bar{F}$ and F are concircular if and only if*

$$u_{i|j} = \lambda g_{ij}, \quad u^r C^k_{ri} = 0, \quad (u_i := u_{x^i}, u^i := g^{ir} u_r), \tag{9}$$

where $\lambda = \lambda(x)$ is a scalar function on M and the symbol $|$ means the horizontal covariant derivative of Cartan (or Chern) connection.

3. Main results

THEOREM 3.1. *Let (M, F) be a Finsler metric and admitting a concircular transformation $\bar{F} = u^{-1}F$ satisfying*

$$Ric_{\bar{g}} - Ric_g = (n-1)c\bar{F}^2 \tag{10}$$

where c is a constant and $u = u(x)$ is a scalar function on M, Then we have:

1. *If c is zero we have the solution in (22).*

2. *For any c we have the solution in (23).*

PROOF OF THEOREM 3.1. We give a brief introduction for some basic points needed here. It is known that if two sprays $\bar{G}^i$ and G^i satisfy equation $\bar{G}^i = G^i + H^i$, then their Reimann curvature tensors satisfy

$$\bar{R}^i_k = R^i_k + 2H^i_{;k} - y^m H^i_{;m.k} + 2H^m H^i_{.m.k} - H^i_{.m} H^m_{.k} \tag{11}$$

where the symbol " ; " denotes the horizontal covariant derivative of Berwald connection of G^i.
Now, since $\bar{F} = u^{-1}F$, the sprays $\bar{G}^i$ and G^i (related to $\bar{F}$ and F, respectively) satisfy the following equalities:

$$\bar{G}^i = G^i - \frac{1}{u}u_0 y^i + \frac{1}{2u}F^2 u^i, \; \bar{G}^i_j = G^i_j - \frac{1}{u}(u_j y^i + u_0 \delta^i_j - y_j u^i + F^2 C^i_{jr} u^r) \tag{12}$$

by taking H^i

$$H^i = -\frac{1}{u}u_0 y^i + \frac{1}{2u}F^2 u^i \tag{13}$$

and plugging (13) into (11), we obtain

$$\begin{aligned} \bar{R}^i_k = R^i_k &+ \frac{u u_{0;0} - (u_m u^m)F^2}{u^2}\delta^i_k + \frac{1}{u}F^2 u^i_{;k} + \frac{u^m u_m}{u^2} y^i y^k - \frac{1}{u}(y^i u_{k;0} + y_k u^i_{;0}) \\ &- \frac{u^m u^r}{u^2}F^2(y^i C_{kmr} + y_k C^i_{mr}) + \frac{1}{u^2}F^2(u u^r_{;0} - 3u_0 u^r)C^i_{kr} + \frac{1}{u}F^2 u^r C^i_{kr;0} \\ &+ \frac{u^r u^m}{u^2}F^4(C^i_{pr}C^p_{km} - C^i_{mr.k}). \end{aligned} \tag{14}$$

Then by (14), the Ricci curvatures $\bar{Ric} := \bar{R}^m_m$ and $Ric := R^m_m$ are related by

$$\begin{aligned} \bar{Ric} = Ric &+ \frac{n-2}{u}u_{0;0} + \frac{1}{u^2}[u u^m_{;m} - (n-1)u^m u_m + u I^r u_{r;0} + u^r(u I_{r;0} - 3u_0 I_r)]F^2 \\ &- \frac{1}{u^2}u^r u^m (C^i_{jm} C^j_{ir} - 2I^i C_{imr} + I_{m.r})F^4. \end{aligned} \tag{15}$$

Now suppose F and $\bar{F}$ are concircular. Then by Lemma 2.3, we have

$$u_{i|j} = \lambda g_{ij}, \; u^r C^k_{ri} = 0, (u_i := u_{x^i}, \; u^i := g^{ir} u_r), \tag{16}$$

where $\lambda = \lambda(x)$ is a scalar function on M and the symbol $|$ means the horizontal covariant derivative of Cartan (or Chern) connection of F. Plugging (16) into (14) and (15), we respectively have

$$\bar{R}^i_k = R^i_k + u^{-2}(2\lambda u - u_m u^m)(F^2\delta^i_k - y^i y_k), \tag{17}$$

$$Ric_{\bar{g}} = Ric_g + (n-1)u^{-2}(2\lambda u - u_m u^m)F^2. \tag{18}$$

Substituting (10) into (18) yields

$$u^m u_m - 2\lambda(x)u(x) + c \quad = 0 \tag{19}$$

where $\|\nabla u\|^2 := u^m u_m = g_{im}u^i u^m$.

$$\|\nabla u\|^2 - 2\lambda(x)u(x) + c = 0 \tag{20}$$

We have along the unit speed geodesic $\gamma(t)$ in the direction of $grad\, u = u'.\frac{\partial}{\partial t}$. Hence we get

$$(u')^2 - 2\lambda(x)u(x) + c = 0 \tag{21}$$

Case I: $c = 0$. From (21) we can obtain the solution of u in the following

$$u(x) = \frac{1}{2}(\int \sqrt{\lambda(x)}dx + const)^2. \tag{22}$$

Case II: for any c we have

$$u(x) = \frac{1}{2}\lambda(x + const)^2 + \frac{c}{(2\lambda)}. \tag{23}$$

Therefore the general solution of u is

$$u(x) = \frac{1}{2}(\int \sqrt{\lambda(x)}dx + const)^2 + af(x) \tag{24}$$

where a is a constant. □

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Kerman, Iran
30 August - 02 September 2021

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کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Generalized harmonic analysis on the Amalgam spaces

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Abstract. In this paper, we deal with the amalgam spaces in the following way. Let G be a locally compact group and $p, q > 0$. In this paper, we investigate some inclusions and important properties of the amalgam space $L^{\pi}_{(p,q)}(G)$. Also in this work, we investigate the property that when $f * g$ exists, for all $f, g \in L^{\pi}_{(p,q)}(G)$, for the case that G an IN-group.

Keywords: Amalgam spaces, uniform partition, IN-group

AMS Mathematics Subject Classification [2010]: Primary 47J30; Secondary 30H05, 46A18.

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1. Introduction

The first version of the amalgam spaces has been introduced on the real line as

$$(L^p, l^q)(\mathcal{R}) := \left\{ f \in L^p_{loc}(\mathcal{R}) : \left[\sum_{n \in \mathcal{Z}} \left(\int_n^{n+1} |f|^p dx \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} < \infty \right\},$$

where $1 \leq p, q < \infty$ and $L^p_{loc}(\mathcal{R})$ is the space consisting of all locally integrable functions on $\mathcal{R}$; see [3]. Then Bertrandias et al [1] generalized this definition for some special abelian locally compact groups G.

In 1979, Stewart [4] extended the definition of Bertrandias to locally compact abelian group G.
In 1980, Busby and Smith [2] generalized the definition of Stewart for an arbitrary locally compact group G, not necessarily abelian. This definition is based on a so-called $U - V$ uniform partition π of G, where U and V are relatively compact open neighborhoods of identity with $U \subseteq V$. These spaces were denoted by $L^{\pi}_{(p,q)}(G)$; see [2] for more information about these spaces.
An important aspect of amalgam space theory is that spaces $L^p(G)$ are their particular cases. For example if G is compact, then $L^{\pi}_{(p,q)}(G)$ reduces to $L^p(G)$. Also if G is discrete then, $L^{\pi}_{(p,q)}(G) = \ell^q(G)$. This provides the possibility of extending the known results in

*Speaker. Email address: Sabzali.navid5@gmail.com the theory of spaces $L^p(G)$ to amalgam spaces. For measurable functions f and g on G, the convolution multiplication

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)d\lambda(y)$$

is defined at each point $x \in G$ if the function $y \mapsto f(y)g(y^{-1}x)$ is $\lambda-$integrable. Then $f*g$ is said to exist if $(f*g)(x)$ exists for almost all $x \in G$ a.e. Moreover $(f*g)$ exists locally almost every where (l.a.e) if $(f*g)$ exists a.e on any compact subset $K \subseteq G$.

2. Preliminaries

In this section we provide some required definitions, preliminaries and known results; see [2] and [5] for more information.

Throughout the work G is a locally compact group with the fixed left Haar measure λ. Let U and V be two relatively compact open neighborhoods of the identity element of G, such that $\overline{U} \subseteq V$. The partition π of G is called Borel if it consists of disjoint Borel subsets of G. A Borel partition π is called $(U-V)-$uniform if for each E in π there is $x \in G$ such that $xU \subseteq E \subseteq xV$. A Borel partition is called uniform if it is a $(U-V)-$uniform partition, for some U and V with $\overline{U} \subseteq V$. By [2, Proposition 3.3], for a locally compact group G, uniform partitions exist in abundance. In fact for every symmetric relatively compact open neighborhood of the identity U, there exists a $(U-U^2)-$uniform partition of G. Recall that a relatively compact neighborhood of identity is called invariant if $xW = Wx$, for each $x \in G$. It is easily verified that for any such a this neighborhood of identity W, if π_1 is a $(W-W^2)-$uniform partition of G, then

$$\pi_1^{-1} = \{E^{-1} : E \in \pi\}$$

is also a $(W-W^2)-$uniform partition of G.

Let π be an $U-V$ uniform partition of G. The amalgam space of L^p and ℓ^q $(p,q>0)$ is the space, consisting of all functions on G which are locally L^p and have l^q behavior at infinity in the sense that the L^p- norms over certain compact subsets of G form an ℓ^q-sequence. In fact a complex-valued function f on G is called locally L^p and globally l^q relative to Borel partition π, if for each E in π

$$f_E = f\chi_E \in L^p(E),$$

and

$$\|f\|^{\pi}_{(p,q)} = \left(\sum_{E\in\pi} \|f_E\|_p^q\right)^{\frac{1}{q}} < \infty.$$

The space consisting of all such a these functions will be denoted by $L^{\pi}_{(p,q)}(G)$. By [2] and [2] $L^{\pi}_{(p,q)}(G)$ is a Banach space, for $p,q \geq 1$.

In [2, Proposition 3.8], it has been proved that if $p,q>1$, then for all different uniform partitions π_1 and π_2 of G, there are $M_1(\pi_1,\pi_2), M_2(\pi_1,\pi_2) > 0$ such that

$$M_1(\pi_1,\pi_2)\|f\|^{\pi_2}_{(p,q)} \leq \|f\|^{\pi_1}_{(p,q)} \leq M_2(\pi_1,\pi_2)\|f\|^{\pi_1}_{(p,q)} \tag{1}$$

for all $\lambda-$measurable functions f on G.

3. Main results

We begin our study of amalgam spaces with some important inclusion relations.

THEOREM 3.1. *Let G be a locally compact group and $p \geq 1$, $q > 1$. Then G is compact if and only if $L^{\pi}_{(p,q)}(G) \subseteq L^1(G)$.*

PROOF. If G is compact, then $L^{\pi}_{(p,q)}(G) = L^p(G) \subseteq L^1(G)$. Now suppose that G is not compact. Then there exist the infinite families $\{x_{\alpha_n}\}$ of G and $\{E_{\alpha_n}\}_{n=1}^{\infty}$ of π such that

$$x_{\alpha_n} U \subseteq E_{\alpha_n} \subseteq x_{\alpha_n} V$$

for all $n \geq 1$. So

$$0 < \lambda(U) \leq \lambda(E_{\alpha_n}) \leq \lambda(V) \qquad (n \in \mathbb{N}).$$

Define

$$f := \sum_{n=1}^{\infty} \frac{1}{n\lambda(E_{\alpha_n})} \chi_{x_{\alpha_n} U}.$$

Thus for each $n \in \mathbb{N}$

$$\|f_{E_{\alpha_n}}\|_p = \left\| \frac{\chi_{x_{\alpha_n} E_{\alpha_n}}}{n\lambda(E_{\alpha_n})} \right\|_p \leq \frac{1}{n(\lambda(U))^{1-\frac{1}{p}}}.$$

This implies that

$$(\|f\|^{\pi}_{(p,q)})^q = \sum_{n=1}^{\infty} \|f_{E_{\alpha_n}}\|_p^q \leq \frac{1}{\lambda(U)^{q(1-\frac{1}{p})}} \sum_{n=1}^{\infty} \frac{1}{n^q}$$

and so $f \in L^{\pi}_{(p,q)}(G)$ by $q > 1$. But

$$\int f(x) d\lambda(x) = \sum_{n=1}^{\infty} \frac{1}{n} \int \frac{\chi_{\alpha_n U}(x)}{\lambda(E_{\alpha_n})} d\lambda(x) \geq \frac{\lambda(U)}{\lambda(V)} \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

which implies that $f \notin L^1(G)$. This completes the proof. □

Now similar to the proof of Theorem 2.1, the following result is immidiate.

THEOREM 3.2. *Let G be a locally compact group, $p > 1$, $q > 1$. Then G is discrete if and only if $L^1(G) \subseteq L^{\pi}_{(p,q)}(G)$.*

THEOREM 3.3. *Let G be a locally compact group and $0 < p, q < \infty$. If either p or q is less than 1, then $L^{\pi}_{(p,q)}(G)$ is a topological vector space.*

PROOF. For show this claim, it is sufficient to show that $\|.\|^{\pi}_{(p,q)}$ is a quasinorm.
Let $f, g \in L^{\pi}_{(p,q)}(G)$ with $f_\alpha = f\chi_{E_\alpha}$ and $g_\alpha = g\chi_{E_\alpha}$ for any $\alpha \in I$.
If $1 \leq p < \infty$ and $0 < q < 1$, we have

$$\begin{aligned}(\|f+g\|^{\pi}_{(p,q)})^q &= (\textstyle\sum_{\alpha\in J} \|(f+g)_\alpha\|_p^q) \leq \textstyle\sum_{\alpha\in J} (\|f_\alpha\|_p + \|g_\alpha\|_p)^q \\ &\leq \textstyle\sum_{\alpha\in J} \|f_\alpha\|_p^q + \textstyle\sum_{\alpha\in J} \|g_\alpha\|_p^q \leq (\|f\|^{\pi}_{(p,q)})^q + (\|g\|_{(p,q)})^q.\end{aligned}$$

So

$$(\|f+g\|^{\pi}_{(p,q)})^q \leq 2^{\frac{1}{q}-1} (\|f\|^{\pi}_{(p,q)} + \|g\|^{\pi}_{(p,q)}).$$

□

The Amalgame spaces have been studied for $p \geq 1, q \geq 1$ and in this case they known that $L^{\pi}_{(p,q)}(G)$ is Banach space. However, if p or q is less than 1, it is a topological vector space by using the following inequality.

THEOREM 3.4. *Let π and π' be respectively $U-V$ and $U'-V'$ uniform partitions of a locally compact group G. Moreover, let $0 < p, q < \infty$. Then, there exists $M > 0$ such that*

$$\|f\|^{\pi'}_{(p,q)} \leq M\|f\|^{\pi}_{(p,q)} \tag{2}$$

for any $f \in L^{\pi}_{(p,q)}(G)$.

COROLLARY 3.5. *Let G ba locally compact group and $0 < p, q < \infty$. Then f exists l.a.e for any $f \in L^{\pi}_{(p,q)}(G)$.*

Several authors have studied the existence of convolution on various function spaces. they proved an important conjecture related to L^p-spaces, called L^p-conjecture. In fact he proved that for $1 < p < \infty$, $f * g$ exists and belongs to $L^p(G)$ for all $f, g \in L^p(G)$ if and only if G is compact. This subject have been studied more recently. The main aim of the present work is investigating the property that when $f * g$ exists, for all $f, g \in L^{\pi}_{(p,q)}(G)$, for the case that IN-group (we say that a locally compact group G is an IN-group if there is a compact neighborhood W of the identity such that $xW = Wx$ for any $x \in G$).We summarize all important results that we can easily obtain according to the above results in the next theorem.

THEOREM 3.6. *Let G be a IN-group. Then we have the following statements.*

(i) *If $p \geq 1, q > 2$, then G is compact if and only if for all functions $f, g \in L^{\pi}_{(p,q)}(G)$, $f * g$ exists l.a.e.*

(ii) *If $0 < p < 1, 0 < q < 1$, then G is discrete if and only if for all functions $f, g \in L^{\pi}_{(p,q)}(G)$, $f * g$ exists l.a.e.*

(iii) *If $p \geq 1, 1 \leq q \leq 2$, then for all functions $f, g \in L^{\pi}_{(p,q)}(G)$, $f * g$ exists.*

(iv) *If $0 < p < 1, q > 1$ and $f * g$ exists l.a.e, for all functions $f, g \in L^{\pi}_{(p,q)}(G)$, then G is discrete.*

(v) *If $0 < p < 1$ and $f * g$ exists l.a.e for all functions $f, g \in L^{\pi}_{(p,1)}(G)$, then G is discrete.*

(vi) *If $0 < q < 1$, then $f * g$ exists l.a.e, for all functions $f, g \in L^{\pi}_{(1,q)}(G)$.*

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Characterization of $L_2(27)$ and $L_2(32)$ by the Number of Sylow Subgroups

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Abstract. Let G be a finite group with trivial center and $n_p(G)$ be the number of Sylow subgroups of G. Put $L = L_2(q)$, where $q \in \{27, 32\}$, and suppose that $n_p(G) = n_p(L)$, for every prime $p \in \pi(G)$. In this paper we show that $G \cong L$, if $q = 27$, and $G \cong L$ or $\mathrm{Aut}(L)$, if $q = 32$.

Keywords: projective special linear group, sylow subgroup, characterization

AMS Mathematics Subject Classification [2010]: 20D06, 20D20

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1. Introduction

Let G be a finite group, n be a natural number and denote by $\pi(n)$ the set of all prime divisors of n. By $\mathrm{Syl_p}(G)$ for every $p \in \pi(G)$, we denote the set of Sylow p-subgroups of G, and $n_p(G) = |\mathrm{Syl_p}(G)|$. A finite group G is called characterizable by the order of normalizer of its Sylow subgroups, if $S \cong G$ for every finite group S with $|N_S(P)| = |N_G(Q)|$, for every $P \in Syl_p(S)$ and $Q \in Syl_p(G)$. This type of characterization is done for some simple groups such as Alternating groups, $L_n(q)$ [2], $L_2(p^k)$ [1] and $U_n(q)$ [3].

Let S be one of the above simple groups. If $n_p(G) = n_p(S)$ for every prime p and $|G| = |S|$, it is clear that $G \cong S$. in [4] the assumption of $|G| = |S|$ was replaced with $Z(G) = 1$ and the authors proved the following theorem for some simple groups $S = L_2(q)$.

Theorem 1.1. *Let G be a finite group with trivial center such that $n_p(G) = n_p(L_2(q)$, for every prime $p \in \pi(G)$, where $q \in \{16, 17, 19, 23, 25\}$. Then if $q \in \{16, 17, 19\}$, $G \cong L_2(q)$ and if $q \in \{23, 25\}$, $L_2(q) \leqslant G \leqslant \mathrm{Aut}(L_2(q))$.*

Also in [5], the following theorem was proved:

Theorem 1.2. *Let G be a finite group with trivial center such that $n_p(G) = n_p(L_2(29)$, for every prime $p \in \pi(G)$. Then $G \cong L_2(29)$.*

In continue, we investigated the simple groups $L_2(27)$ and $L_2(32)$, and we show that in this problem, $G \cong L$, for $L = L_2(27)$, and $G \cong L$ or $\mathrm{Aut}(L)$, for $L = L_2(32)$.

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2. Preliminaries

Lemma 2.1. *Let G be a finite group and M be a normal subgroup of G. Then both Sylow p-number $n_p(M)$ and the Sylow p-number $n_p(G/M)$ of the quotient G/M divide the Sylow p-number $n_p(G)$ of G and moreover $n_p(M)n_p(G/M)|n_p(G)$.*

Definition 2.2. A finite simple group is called a simple K_n-group if its order is divisible by exactly n distinct primes.

Theorem 2.3. *If G is a simple K_3-group, then G is isomorphic to one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ and $U_4(2)$.*

Lemma 2.4. *Let S be a simple K_4-group. Then S is isomorphic to one of the following groups:*

(1) $L_2(r)$ with r a prime satisfying $r^2 - 1 = 2^a 3^b u^c, where a \geq 1, b \geq 1, c \geq 1, u > 3$ a prime.

(2) $L_2(2^m)$ with $2^m - 1 = u$ and $2^m + 1 = 3t$, where $m \geq 1$, u and t are primes and $t > 3$.

(3) $L_2(3^m)$ with $3^m - 1 = 2u$ and $3^m + 1 = 4t$, where $m \geq 1$, u and t are primes.

(4) $L_2(24), L_2(52), L_2(72), L_2(34), L_2(35)$.

(5) $M_{11}, M_{12}, J_2, A_7, A_8, A_9, A_{10}, L_3(4), L_3(5), L_3(7), L_3(8), L_3(17), S_4(4), S_4(5), S_4(7), S_4(9), U_3(4), U_3(5), U_3(7), U_3(8), U_3(9), L_4(3), S_6(2), O_8^+(2), G_2(3), U_4(3), U_5(2),^3 D_4(2), {}^2F_4(2), Sz(8), Sz(32)$.

Lemma 2.5. *Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1 \cdots p_r\}$ and h_m be the number of π-Hall subgroups of G. Then $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$, satisfies the following conditions for all $i \in \{1, 2, \cdots, s\}$:*

1. *$q_i^{\beta_i} \equiv 1$ (mod p_j), for some p_j.*
2. *The order of some chief factor of G is divisible by $q_i^{\beta_i}$.*

Lemma 2.6. *Let $n \geq 2$ and $q = p^f$. Then*

1. $\mathrm{Out}(PSL(n, q)) \cong \mathbb{Z}_{(n,q-1)} : \mathbb{Z}_f : \mathbb{Z}_2$ *if $n \geq 2$;*
2. $\mathrm{Out}(PSL(2, q)) \cong \mathbb{Z}_{(2,q-1)} \times \mathbb{Z}_f$.

Note that by Lemma 2.6, $\mathrm{Out}(PSL(2, 27)) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, and $\mathrm{Out}(PSL(2, 32)) \cong \mathbb{Z}_5$.

Definition 2.7. A group G is said to be an almost simple group related to S if and only if $S \trianglelefteq G \leq \mathrm{Aut}(S)$ for some non-abelian simple group S.

Remark 2.8. For some simple and almost simple groups G, we present the number of some p-Sylow subgroups, which we have calculated in GAP.

1. $n_2(L_2(27)) = 819 = 3^2 \cdot 7 \cdot 13$, $n_3(L_2(27)) = 28 = 2^2 \cdot 7$, $n_7(L_2(27)) = 351 = 3^3 \cdot 13$, $n_{13}(L_2(27)) = 378 = 2 \cdot 3^3 \cdot 7$, $n_2(PGL_2(27)) = 2457 = 3^3 \cdot 7 \cdot 13$, $n_3(P\Sigma L_2(27)) = 364 = 2^2 \cdot 7 \cdot 13$, $n_2(\mathrm{Aut}(L_2(27))) = 2457 = 3^3 \cdot 7 \cdot 13$.

2. $n_2(A_5) = 5$, $n_2(A_6) = 45$, $n_2(A_7) = 315$, $n_2(A_8) = 315$, $n_2(A_9) = 2835$, $n_2(A_{10}) = 14175$, $n_7(L_2(7)) = 8$, $n_7(L_2(8)) = 36$, $n_2(L_2(17)) = 153 = 3^2 \cdot 17$, $n_2(L_3(3)) = 351 = 3^3 \cdot 13$, $n_2(U_3(3)) = 189 = 3^3 \cdot 7$, $n_2(U_4(2)) = 135$, $n_3(L_2(13)) = 91 = 7 \cdot 13$, $n_2(G_2(3)) = 66339 = 3^6 \cdot 7 \cdot 13$.

3. $n_2(L_2(32)) = 33$, $n_3(L_2(32)) = 496 = 2^4 \cdot 31$, $n_{11}(L_2(32)) = 496 = 2^4 \cdot 31$, $n_{31}(L_2(32)) = 528 = 2^4 \cdot 3 \cdot 11$.

4. $n_2(\mathrm{Aut}(L_2(32))) = 33$, $n_3(\mathrm{Aut}(L_2(32))) = 496 = 2^4 \cdot 31$, $n_{11}(\mathrm{Aut}(L_2(32))) = 496 = 2^4 \cdot 31$, $n_{31}(\mathrm{Aut}(L_2(32))) = 528 = 2^4 \cdot 3 \cdot 11$.

3. Main result

Theorem 3.1. *Let G be a finite group with trivial center. Put $L = L_2(q)$, where $q \in \{27, 32\}$, and suppose that $n_p(G) = n_p(L)$, for every prime $p \in \pi(G)$. Then $G \cong L$, for $q = 27$, and $G \cong L$ or* $\mathrm{Aut}(L)$, *for $q = 32$.*

Proof. We prove the theorem in two separate parts:

Part 1: $q = 27$.

We break the proof of each part into four steps:

Step 1. $|L_2(27)| = 2^2 {\cdot} 3^3 {\cdot} 7 {\cdot} 13$, then $\pi(G) = \{2, 3, 7, 13\}$ and $|G| = 2^{\alpha_1} {\cdot} 3^{\alpha_2} {\cdot} 7^{\alpha_3} {\cdot} 13^{\alpha_4}$, for some integer α_i's. Put $m = 13^{\alpha_4}$, therefore in Lemma 2.5, $h_m = n_{13}(G) = n_{13}(L_2(27)) = 2 \cdot 3^3 \cdot 7$, by Remark 2.8. Then Lemma 2.5 implies that $7 \equiv 1 \pmod{13}$, that is a contradiction. Hence G is non-solvable. So G has a normal series $1 \trianglelefteq N \trianglelefteq H \trianglelefteq G$ where N is a maximal normal solvable subgroup of G and H/N is a non-abelian simple group or H/N is a direct product of isomorphic non-abelian simple groups. In fact $H/N \cong S_1 \times S_2 \times ... \times S_r$, where S_1 is a non-solvable simple group and $S_1 \cong S_2 \cong ... \cong S_r$.

Step 2. Suppose that H/N is a simple K_3-group. Use Lemma 2.3. If $H/N \cong A_5$, Lemma 2.1 and Remark 2.8 imply that $5|819$, that is a contradiction. If $H/N \cong A_6$, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ or $U_4(2)$, by Remark 2.8, similar to the above we get a contradiction. Let H/N be a simple K_4-group. Use Lemma 2.4. If $H/N \cong L_2(r)$, By [6], Table 1, we conclude that $H/N \cong L_2(13)$, because $\pi(H/N) = \{2, 3, 7, 13\}$. Lemma 2.1 and Remark 2.8 imply that $91|28$, that is a contradiction. In other cases, except $H/N \cong L_2(3^m)$, $G_2(3)$ and ${}^3D_4(2)$, By [6], Table 1, and also considering $\pi(H/N) = \{2, 3, 7, 13\}$, we get a contradiction too. If $H/N \cong G_2(3)$, then Lemma 2.1 and Remark 2.8 imply that $66339|819$, which is impossible. If $H/N \cong^3 D_4(2)$, Since $L_2(8) \leq^3 D_4(2)$ (see Atlas of finite groups, page 89), similar to the above, $36|351$, which is impossible. So $H/N \cong L_2(3^m)$, then by [6], Table 1, and also considering $\pi(H/N) = \{2, 3, 7, 13\}$, we conclude that, $H/N \cong L_2(27)$.

Step 3. Let $K = \{x \in G \mid xN \in C_{G/N}(H/N)\}$, then $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$. Hence $L_2(27) \leqslant G/K \leqslant \mathrm{Aut}(L_2(27))$. Since $\mathrm{Out}(L_2(27)) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, hence $G/K \cong L_2(27)$, $L_2(27) \cdot 2 \cong PGL_2(27)$, $L_2(27) \cdot 3 \cong P\Sigma L_2(27)$ or $L_2(27) \cdot 6 \cong \mathrm{Aut}(L_2(27))$.

Step 4. If $G/K \cong L_2(27)$, then $K = N$. Suppose that $K \neq N$. By Lemma 2.1 we conclude that $n_p(K) = 1$ for every prime $p \in \pi(G)$. Then K is a nilpotent subgroup of G. Since $C_{\overline{G}}(\overline{H}) \cong K/N$ and N is a maximal normal solvable subgroup of G, then K ia a non-solvable normal subgroup of G, which is a contradiction. Hence $K = N$ and then $G/N \cong L_2(27)$. Also it was shown that $N = 1$ and therefore $G \cong L_2(27)$.

If $G/K \cong PGL_2(27)$ or $P\Sigma L_2(27)$, then by Lemma 2.1 and Remark 2.8 we conclude that $2457|819$ and $364|28$, respectively, which are impossible.

If $G/K \cong \mathrm{Aut}(L_2(27))$, similar to the above $2457|819$, that is a contradiction.

Part 2: $q = 32$.

Step 1. $|L_2(32)| = 2^5 \cdot 3 \cdot 11 \cdot 31$, then $\pi(G) = \{2, 3, 11, 31\}$ and $|G| = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 11^{\alpha_3} \cdot 31^{\alpha_4}$, for some integer α_i's. Put $m = 31^{\alpha_4}$, therefore in Lemma 2.5, $h_m = n_{31}(G) = n_{31}(L_2(32)) = 2^4 \cdot 3 \cdot 11$, by Remark 2.8. Then Lemma 2.5 implies that $11 \equiv 1 \pmod{31}$, that is a contradiction. Hence G is non-solvable. So G has a normal series $1 \trianglelefteq N \trianglelefteq H \trianglelefteq G$ where N is a maximal normal solvable subgroup of G and H/N is a non-abelian simple group or H/N is a direct product of isomorphic non-abelian simple groups. In fact $H/N \cong S_1 \times S_2 \times ... \times S_r$, where S_1 is a non-solvable simple group and $S_1 \cong S_2 \cong ... \cong S_r$.

Step 2. Suppose that H/N is a simple K_3-group. By Use of Lemma 2.1 and Remark 2.8 similarly to Part 1-Step 1, we get a contradiction. Let H/N be a simple K_4-group. Use Lemma 2.4. In all cases, except $H/N \cong L_2(2^m)$, By [6], Table 1, and also considering $\pi(H/N) = \{2, 3, 11, 31\}$, we get a contradiction, also If $H/N \cong L_2(2^m)$, we conclude that $H/N \cong L_2(32)$.

Step 3. Let $K = \{x \in G \mid xN \in C_{G/N}(H/N)\}$, then $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$. Hence $L_2(32) \leqslant G/K \leqslant \mathrm{Aut}(L_2(32))$. Since $\mathrm{Out}(L_2(27)) \cong \mathbb{Z}_5$, hence $G/K \cong L_2(32)$ or $L_2(32) \cdot 5 \cong \mathrm{Aut}(L_2(32))$.

Step 4. If $G/K \cong L_2(32)$, then similar to Part 1-Step 4, we can conclude $K = N$. Then $G/N \cong L_2(32)$. Also it was shown that $N = 1$ and therefore $G \cong L_2(32)$.

If $G/K \cong \mathrm{Aut}(L_2(32))$, similarly we obtain $G \cong \mathrm{Aut}(L_2(32))$ (note that in this case, $n_p(G/K) = n_p(G)$, for every $p \in \pi(G)$, see Remark 2.8-4).

Here the proof of Theorem 3.1 is completed. $\square$

Acknowledgement

The authors would like to thank the referee.

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دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Strongly p-limited completely continuous spaces

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Abstract. By introducing strongly p-limited completely continuous subspaces of the space of operator ideals, we will give some characterizations of this concept in terms of limited p-convergent of all its evaluation operators related to that subspace. In particular, when X^* or Y has the p-Gelfand-Phillips property (in short, p-GPP), we give a characterization of p-GPP of $\mathcal{M}$ of a closed subspace $\mathcal{M} \subset K(X,Y)$ in terms of strong p- limited complete continuity of $\mathcal{M}$.

Keywords: weakly p-summable set, p- Gelfand-Phillips property, operator ideals, limited p-convergent

AMS Mathematics Subject Classification [2010]: 47L05; 47L20; 46B28; 46B99

1. Introduction

A subset A of a Banach space X is called limited, if every weak* null sequence (x_n^*) in X^* converges uniformly on A, that is

$$\lim_{n\to\infty} \sup_{a\in A} |\langle a, x_n^*\rangle| = 0.$$

Every relatively compact subset of X is limited, but the converse of this assertion, in general, is false. If every limited subset of a Banach space X is relatively compact, then X has the Gelfand-Phillips (GP) property. For example, the classical Banach spaces c_0 and ℓ_1 have the Gpp and every separable Banach space, every Schur space (i.e., weak and norm convergence of sequences in X coincide), spaces with their duals containing no copy of ℓ_1, such as reflexive spaces, have the same property.
A sequence (x_n) in a Banach spacee X is called weakly p-summable with $1 \le p < \infty$, if for each $x^* \in X^*$, the sequence $(\langle x_n, x^*\rangle) \in \ell_p$ and a sequence (x_n) in X is said to be weakly p-convergent to $x \in X$ if the sequence $(x_n - x) \in \ell_p^{weak}(X)$, where $\ell_p^{weak}(X)$ denoted the space of all weakly p-summable sequences in X [2].
The weakly ∞-convergent sequences are simply the weakly convergent sequences. A bounded set K in a Banach space is said to be relatively weakly p-compact, $1 < p < \infty$, if every sequence in K has a weakly p-convergent subsequence. If the limit point of each weakly p-convergent subsequence is in K, then we call K a weakly p-compact set. An operator $T \in L(X;Y)$ is limited p-convergent if it transfers limited and weakly p-summable

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sequences into norm null ones. we denote the space of all limited p-convergent operators from X into Y by $Clp(X,Y)$. A Banach space X has the p-GP property ($1 \leq p < \infty$) if every limited and weakly p-summable sequence in X is norm null. In other words, if $1 \leq p < \infty$, X has the p-GP property if and only every limited and weakly p-compact subset of X is compact. Also one note that every GP space has the p-GP property [3]. Here, we introduce the concept of strongly p-limited completely continuous subspaces of the space of operator ideals and we give a characterization of the p-GP property of $\mathcal{M}$ of a closed subspace $\mathcal{M} \subset K(X,Y)$ in terms of strong p-limited complete continuity of $\mathcal{M}$.

2. strongly p-limited completely continuous subspaces

For a subspace $\mathcal{U}(X,Y)$, one can find the evaluation operators related to $\mathcal{M}$ by ϕ_x and ψ_{y^*}, where $\phi_x(T) = Tx$ and $\psi_{y^*}(T) = T^*y^*$ for $x \in X$, $y^* \in Y^*$ and $T \in \mathcal{M}$.
We recall that an operator $T : X \to Y$ between two Banach spaces is limited completely continuous (abb. *lcc*) if T carries weakly null and limited sequences in X to norm null ones, and the class of *lcc* operators is denoted by $Lcc(X,Y)$. [4]
Also subalgebra $\mathcal{A}$ of $\mathcal{U}(X)$ is limited completely continuous if for each $S \in \mathcal{A}$, the left and right multiplications L_S and R_S are limited completely continuous on $\mathcal{A}$, where $L_S(T) = ST$ and $R_S(T) = TS$ [1].
Here we give a similar definition of limited completely continuous subalgebras of $\mathcal{U}(X)$ by using limited p-converging operators instead of lcc operators.

Definition 2.1. A linear subalgebra $\mathcal{A}$ of $\mathcal{U}(X)$ is p-limited completely continuous if for each $S \in \mathcal{A}$, the left and right multiplications L_S and R_S are limited p-convergent operators on $\mathcal{A}$.

Now, we give a refinement of this concept for subspaces of $\mathcal{U}(X,Y)$:

Definition 2.2. A linear subspace $\mathcal{M} \subset \mathcal{U}(X,Y)$ is called strongly p-limited completely continuous in $K(X,Y)$ (resp., $\mathcal{U}(X,Y)$) if for all Banach spaces W and Z and all compact operators $R : Y \to W$ and $S : Z \to X$, the left and right multiplication operators L_R and R_S as operators from $\mathcal{M}$ into $K(X,W)$ and $K(Z,Y)$ (resp., $U(X,W)$ and $U(Z,Y)$) respectively, are limited p-convergent.

In the following theorem we present a wide class of subspaces of $L(X,Y)$ with strong limited complete continuity. At first we need a following lemma:

Lemma 2.3. *A subset $H \subset K(X,Y)$ is relatively compact iff:*
(a) *$H(B_X)$ is relatively compact,*
(b) *$H^*(y^*)$ is relatively compact for all $y^* \in Y^*$,*
or
(a) *$H(x)$ is relatively compact for all $x \in X$,*
(b) *$H^*(B_{Y^*})$ is relatively compact.*

Proposition 2.4. *A bounded linear operator $T : X \to Y$ is p-limited completely continuous if and only if for every limited weakly p-compact set A in X, $T(A)$ is relatively compact.*

Theorem 2.5. *Let $\mathcal{M} \subset L(X,Y)$ be a linear subspace such that all of evaluation operators ϕ_x and ψ_{y^*} are limited p-convergent. Then $\mathcal{M}$ is strongly p-limited completely continuous in $K(X,Y)$.*

As a corollary, we extend Theorem 2.5 to some classes of operator ideals. We recall that an operator ideal $\mathcal{U}$ is closed if its components $\mathcal{U}(X, Y)$ are closed in $L(X, Y)$.

Corollary 2.6. *Let $\mathcal{U}$ be a closed operator ideal and $\mathcal{M}$ be a linear subspace of $\mathcal{U}(X, Y)$ such that all of evaluation operators ϕ_x and ψ_{y^*} are limited p-convergent. Then $\mathcal{M}$ is strongly p-limited completely continuous in $\mathcal{U}(X, Y)$.*

Now we will prove that the converse of the above result is also valid in every operator ideal $\mathcal{U}$.

Theorem 2.7. *Let $\mathcal{M}$ be a linear subspace of $\mathcal{U}(X, Y)$ such that for some Banach spaces W and Z, the operators $L_R : \mathcal{M} \to \mathcal{U}(X, W)$ and $R_S : \mathcal{M} \to \mathcal{U}(Z, Y)$ are limited p-convergent for all finite-rank operators $R : Y \to W$ and $S : Z \to X$. Then all of evaluation operators ϕ_x and ψ_{y^*} are limited p- convergent.*

Corollary 2.8. *Let $\mathcal{U}$ be a closed operator ideal and $\mathcal{M}$ be a linear subspace of $\mathcal{U}(X, Y)$. Then the following assertions are equivalent:*

(a) *all of evaluation operators ϕ_x and ψ_{y^*} are limited p-convergent,*
(b) *$\mathcal{M}$ is strongly p-limited completely continuous in $\mathcal{U}(X, Y)$,*
(c) *$\mathcal{M}$ is strongly p-limited completely continuous in $K(X, Y)$,*
(d) *for some Banach spaces W and Z, the operators $L_R : \mathcal{M} \to \mathcal{U}(X, W)$ and $R_S : \mathcal{M} \to \mathcal{U}(Z, Y)$ are limited p- convergent for all finite-rank operators $R : Y \to W$ and $S : Z \to X$.*

Corollary 2.9. *Let $\mathcal{M}$ be a linear subspace of $K(X, Y)$. Then the following assertions are equivalent:*

(a) *all evaluation operators ϕ_x and ψ_{y^*} are limited p- convergent,*
(b) *for all Banach spaces W and Z and all limited p- convergent operators $R : Y \to W$ and $S : Z \to X$, the operators L_R and R_S are limited p- convergent.*

The following theorem, gives a relationship between p- GP property and strongly p-limited completely continuous subspaces of operator ideals.

We note that if a Banach space X has the p-GP property, then each limited operator on X is limited p-convergent. So Every linear subspace $\mathcal{M}$ of $L(X, Y)$ with the p-GP property is strongly p-limited completely continuous.
Finally we show that under some conditions the converse is true. At first we need two following theorems.

Theorem 2.10. *Let $\mathcal{M} \subset K(X, Y)$ be a linear subspace. If X^* has the p-GP property and ϕ_x is limited p-convergent, then $\mathcal{M}$ has the p-GP property.*

Theorem 2.11. *Let $\mathcal{M} \subset K(X, Y)$ be a linear subspace. If Y has the p-GP property and ψ_{y^*} is limited p- convergent, then $\mathcal{M}$ has the p-GP property.*

Theorem 2.12. *Let $\mathcal{M} \subset L(X, Y)$ be strongly p-limited completely continuous. If X^* or Y has the p-GP property, then $\mathcal{M}$ has the p-GP property.*

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A computational method for second order boundary value problems by wavelets

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Abstract. This paper proposed a method based on operational matrices of Chelyshkov wavelets for solving a boundary value problem of second order and transforms it to a system of linear equations. The integration operational matrices play an important role to obtaining a linear system of algebraic equations. Numerical examples are given to demonstrate applicability of this method.

Keywords: Chelyshkov wavelets, Operational matrix, Chelyshkov polynomials, Ordinary Differential Equations

AMS Mathematics Subject Classification [2010]: 15A60, 46N40, 47N40

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1. Introduction

Differential equations form the backbone of various science and engineering problems viz. structural mechanics, image processing, control theory, stationary analysis of circuits, etc. Many practical problems give rise to second order differential equations. For instance, in structural mechanics the governing equation of motion [5]

$$m\,\frac{d^2y}{dx^2} + c\,\frac{dy}{dx} + k\,y = f(x) \qquad y(0) = \alpha\ , \quad y(L) = \beta.$$

is expressed in the form of differential equation with respect to the rate of change in time.

Since analytical methods are not adequate for finding accurate solutions to most differential equations, numerical methods are required. Wavelet method has been proven to be an efficient tool in analyzing dynamic systems and differential equations arising in other science and engineering problems [1, 2].

Chelyshkov Wavelets (ChWs), $\psi_{n,m}(x) = \psi(k, n, m, x)$, are defined on the interval $[0, L)$ by [3, 4, 6]:

$$\psi_{n,m}(t) = \begin{cases} \sqrt{2^k(2m+1)}\,P_m(2^k\frac{t}{L} - n), & \frac{n}{2^k}L \le t < \frac{n+1}{2^k}L, \\ 0, & \text{otherwise.} \end{cases}$$

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where $P_m(t)$ is the Chelyshkov polynomial, which is defined as follows:

$$P_m(t) := \rho_{m,M}(t) = \sum_{j=0}^{M-m} a_{j,m}\, t^{m+j} \;, \quad m = 0, 1, \ldots, M, \tag{1}$$

where:

$$a_{j,m} = (-1)^j \binom{M-m}{j} \binom{M+m+j+1}{M-m}.$$

These polynomials are orthogonal over the interval $[0, 1]$ with respect to the weight function $w(t) = 1$. According to the definition (1) it is obvious that for a fixed integer M, the polynomials $P_m(t)\,, \quad m = 0, 1, \ldots, M$ are polynomials exactly of degree M.

The ChWs $\{\psi_{n,m}(t) \,|\, n = 0, 1, \ldots 2^k - 1,\ m = 0, 1, \ldots M\}$ forms an orthonormal basis for $L^2[0, L]$. By using the orthogonality of ChWs, any function $f(t) \in L^2[0, L]$ can be expanded in terms of ChWs as:

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m}\, \psi_{n,m}(t), \tag{2}$$

where $c_{n,m} = \langle f(t), \psi_{n,m} \rangle = \int_0^L f(t)\, \psi_{n,m}(t)\, dt$. If the infinite series in Eq. (2) is truncated, then it can be written as:

$$f(t) \simeq \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} c_{n,m}\, \psi_{n,m}(t) = \mathbf{C}^T \mathbf{\Psi}(t),$$

where $\mathbf{C}$ and $\mathbf{\Psi}$ are $\hat{m} = 2^k(M + 1)$-vectors, given by:

$$\begin{aligned} \mathbf{C}^T &= [c_{0,0}, c_{0,1}, \ldots, c_{0,M}, c_{1,0}, \ldots, c_{1,M}, \ldots, c_{2^k-1,0} \ldots, c_{2^k-1,M}] \\ &= [c_1, c_2, \ldots, c_{\hat{m}}], \end{aligned} \tag{3}$$

$$\begin{aligned} \mathbf{\Psi}(t)^T &= [\psi_{0,0}(t), \ldots, \psi_{0,M}(t), \psi_{1,0}(t), \ldots, \psi_{1,M}(t), \ldots, \psi_{2^k-1,0}(t), \ldots, \psi_{2^k-1,M}(t)] \\ &= [\psi_1(t), \psi_2(t), \ldots, \psi_{\hat{m}}(t)]. \end{aligned} \tag{4}$$

The purpose of this paper is introducing an operational method for solving the following differential equation with boundary conditions, by using ChWs:

$$\alpha y''(x) + \beta y'(x) + \gamma y(x) = r(x) \;, y(0) = y_0 \;, \quad y(L) = y_L \;, \tag{5}$$

where y_0, y_L, α, β and γ are given constants and $r(x)$ is a known function.

2. Main results

2.1. Operational Matrix of Integration for ChW Vector $\mathbf{\Psi(t)}$. Let $\mathbf{\Psi}(\mathbf{t})$ be the ChW vector of size $\hat{m} = 2^k(M + 1)$ defined in (4). The integral for vector $\mathbf{\Psi}(t)$ can be approximated by:

$$\int_0^x \mathbf{\Psi}(t)\, dt \simeq \mathbf{P}\, \mathbf{\Psi}(x), \tag{6}$$

where $\mathbf{P} = [p_{i,j}]$ is an $\hat{m} \times \hat{m}$ matrix, known as the operational integral matrix for the ChW, defined by:

$$p_{ij} = \langle \int_0^x \psi_i(t)\, dt, \psi_j(x) \rangle.$$

In a similar way and by some calculations and simplifications we will get

$$\int_0^x \int_0^t \mathbf{\Psi}(z)\, dz\, dt \simeq \mathbf{P}^2\, \mathbf{\Psi}(x), \tag{7}$$

2.2. Solving the Equation. We start by considering an approximation for $y''(x)$ in terms of Chws i.e.

$$y''(x) \simeq \mathbf{C}^T\, \mathbf{\Psi}(x) \tag{8}$$

where the vector $\mathbf{C}$ is unknown currently. Then we will have

$$y'(x) \simeq \mathbf{C}^T\, \mathbf{P}\, \mathbf{\Psi}(x) + y'(0) \tag{9}$$

and thereby

$$y(x) \simeq \mathbf{C}^T\, \mathbf{P}^2\, \mathbf{\Psi}(x) + y'(0)\, x + y_0\ , \tag{10}$$

where $y'(0)$ is not given and must be determine. By substituting the boundary condition $y(L) = y_L$ in Eq. (10) we obtain

$$y_L = \mathbf{C}^T\, \mathbf{P}^2\, \mathbf{\Psi}(L) + y'(0)\, L + y_0\ ,$$

and this gives us

$$y'(0) = \frac{y_L - y_0}{L} - \frac{\mathbf{C}^T \mathbf{P}^2 \mathbf{\Psi}(L)}{L}.$$

Substituting $y'(0)$ in Eqns. (9) and (10) we have

$$\begin{aligned} y'(x) &= \mathbf{C}^T \mathbf{P} \mathbf{\Psi}(x) - \frac{\mathbf{C}^T \mathbf{P}^2 \mathbf{\Psi}(L)}{L} + \frac{y_L - y_0}{L}\ , \\ y(x) &= \mathbf{C}^T \mathbf{P}^2 \mathbf{\Psi}(x) - \frac{\mathbf{C}^T \mathbf{P}^2 \mathbf{\Psi}(L)}{L}\, x + \frac{y_L - y_0}{L}\, x + y_0. \end{aligned}$$

For simplicity and ease of calculations we define $y_{L0} := y_L - y_0$, $F := \mathbf{P}^2 \mathbf{\Psi}(L)$, $1 := \mathbf{E}^T \mathbf{\Psi}(x)$, $x := \mathbf{B}^T \mathbf{\Psi}(x)$ and $r(x) \simeq \mathbf{R}^T \mathbf{\Psi}(x)$ and substitute these abbreviations together with Eqs. (8), (9) and (10) in Eq. (5) and simplifying we will have:

$$\left(\alpha \mathbf{I} + \beta \mathbf{P} + \gamma \mathbf{P}^2 - \frac{1}{L} \mathbf{F}(\beta \mathbf{E}^T + \gamma \mathbf{B}^T)\right)^T \mathbf{C} = \left(\mathbf{R} - \frac{y_{L0}}{L}(\beta \mathbf{E} + \gamma \mathbf{B}) - \gamma y_L \mathbf{E}\right) \tag{11}$$

Solving this system of linear equations will be obtained $\mathbf{C}$ and leads us the approximate solution of Eq. (5) as:

$$y(x) \simeq \mathbf{C}^T \mathbf{P}^2 \mathbf{\Psi}(x) + \frac{1}{L}\left(y_{L0} - \mathbf{C}^T \mathbf{F}\right) x + y_L \tag{12}$$

3. Numerical results

In this section, an example presented to verify the capability and efficiency of the proposed method.

Example. Consider the following ordinary differential equation

$$y''(x) - 2y'(x) + 3y(x) = e^{-x}\left(6x^2 - 14x + 4e^x(x-1)\sin(x) - 4e^x(x-1)\cos(x) + 24\right)$$

with boundary conditions

$$y(-1) = 5e + 2\sin(1)\ , \qquad y(3) = \frac{9}{e^3} + 6\sin(3)\ ,$$

which has the exact solution

$$y(x) = e^{-x}\ \left(x^2 - x + 3\right) + 2x\ \sin(x)$$

Figure (1) shows the absolute error of the approximated solution produced by the presented method with $M = 4$ and $k = 5$. As one can see the accuracy of the method is acceptable in many problems.

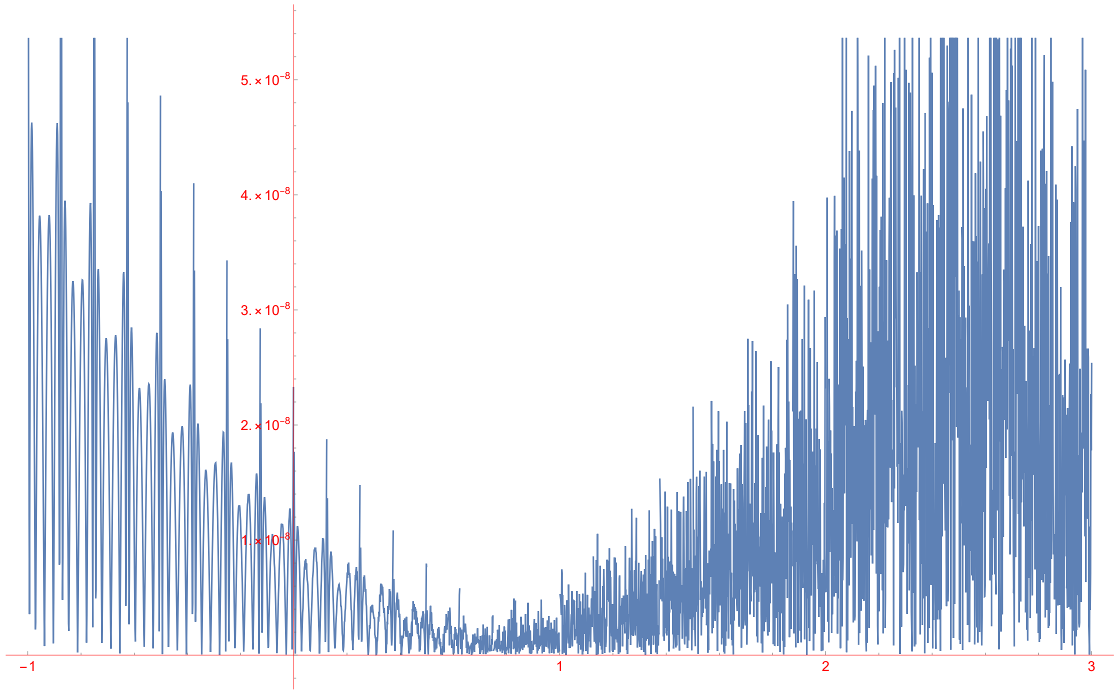

FIGURE 1. Error function of Example for $M = 4$ and $k = 5$

4. Conclusion

As seen, in the presented method, using wavelets transforms solving of a differential equation to solve a linear system of equations with desired accuracy.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Results on the monomial operations and normally torsion-freeness of monomial ideals

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Abstract. An ideal I in a commutative Noetherian ring R is called normally torsion-free if $\mathrm{Ass}_R(R/I^k) \subseteq \mathrm{Ass}_R(R/I)$ for all positive integers k. In this paper, by using some monomial operators such as expansion, weighted, monomial multiple, monomial localization, contraction, and deletion, we introduce several methods for constructing new normally torsion-free monomial ideals based on the monomial ideals which have normally torsion-freeness.

Keywords: normally torsion-free ideals, monomial operators, associated primes

AMS Mathematics Subject Classification [2010]: 13B25, 13F20

1. Introduction

Let R be a commutative Noetherian ring and I be an ideal of R. A prime ideal $\mathfrak{p} \subset R$ is an *associated prime* of I if there exists an element v in R such that $\mathfrak{p} = (I :_R v)$. The *set of associated primes* of I, denoted by $\mathrm{Ass}_R(R/I)$, is the set of all prime ideals associated to I. We will be interested in the sets $\mathrm{Ass}_R(R/I^k)$ when k varies. Brodmann [2] proved that the sequence $\{\mathrm{Ass}_R(R/I^k)\}_{k\geq 1}$ of associated prime ideals is stationary for large k. In fact, there exists a positive integer k_0 such that $\mathrm{Ass}_R(R/I^k) = \mathrm{Ass}_R(R/I^{k_0})$ for all integers $k \geq k_0$. The least such integer k_0 is called the *index of stability* of I and $\mathrm{Ass}_R(R/I^{k_0})$ is called the *stable set of associated prime ideals to I*.

Many problems arise in the context of Brodmann's theorem. One of them is related to normally torsion-free ideals. An ideal I in a commutative Noetherian ring R is called *normally torsion-free* if $\mathrm{Ass}_R(R/I^k) \subseteq \mathrm{Ass}_R(R/I)$ for all positive integers k (see [4, Definition 1.4.5]). A few examples of normally torsion-free monomial ideals appear from graph theory. Already, it has been proved that a finite simple graph G is bipartite if and only if its edge ideal is normally torsion-free. Moreover, it is well-known that the cover ideals of bipartite graphs are normally torsion-free. In addition, it has been verified that every transversal polymatroidal ideal is normally torsion-free. However, normally torsion-free square-free monomial ideals have been studied, but little is known for the normally torsion-free monomial ideals which are not square-free.

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The main motivation in this paper is to introduce several methods for constructing new normally torsion-free non-square-free monomial ideals based on the monomial ideals which have normally torsion-freeness.

2. Main results

Proposition 2.1. *[6, Proposition 2.1] Let I be an ideal in a commutative Noetherian ring S. Then there exists a positive integer t such that I^t is a normally torsion-free ideal.*

Lemma 2.2. *[6, Lemma 2.2] Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$. If I is normally torsion-free, then, for all positive integers s, I^s is normally torsion-free.*

Lemma 2.3. *[6, Lemma 2.4] Let I be a monomial ideal of $R = K[x_1, \ldots, x_n]$ such that $I = I_1R + I_2R$, where $G(I_1) \subseteq R_1 = K[x_1, \ldots, x_m]$ and $G(I_2) \subseteq R_2 = K[x_{m+1}, \ldots, x_n]$ for some positive integer m. Then $I^s = \bigcap_{i=1}^{s}(I_1^i + I_2^{s+1-i})$ for all $s \in \mathbb{N}$.*

Theorem 2.4. *[6, Theorem 2.5] Let I be a monomial ideal in $R = K[x_1, \ldots, x_n]$ such that $I = I_1R+I_2R$, where $G(I_1) \subseteq R_1 = K[x_1, \ldots, x_m]$ and $G(I_2) \subseteq R_2 = K[x_{m+1}, \ldots, x_n]$ for some positive integer m. If I_1 and I_2 are normally torsion-free, then I is so.*

From now, we introduce several methods for constructing new classes of monomial ideals which have normally torsion-freeness. Thus we need to recall the definition of the expansion operator on monomial ideals, which has been stated in [1].

Let K be a field and $R = K[x_1, \ldots, x_n]$ be the polynomial ring over a field K in the variables $x_1, \ldots, x_n$. Fix an ordered n-tuple $(i_1, \ldots, i_n)$ of positive integers, and consider the polynomial ring $R^{(i_1,\ldots,i_n)}$ over K in the variables

$$x_{11}, \ldots, x_{1i_1}, x_{21}, \ldots, x_{2i_2}, \ldots, x_{n1}, \ldots, x_{ni_n}.$$

Let $\mathfrak{p}_j$ be the monomial prime ideal $(x_{j1}, x_{j2}, \ldots, x_{ji_j}) \subseteq R^{(i_1,\ldots,i_n)}$ for all $j = 1, \ldots, n$. Attached to each monomial ideal $I \subset R$ a set of monomial generators $\{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$, where $\mathbf{x}^{\mathbf{a}_i} = {x_1}^{a_i(1)} \cdots {x_n}^{a_i(n)}$ and $a_i(j)$ denotes the jth component of the vector $\mathbf{a}_i = (a_i(1), \ldots, a_i(n))$ for all $i = 1, \ldots, m$. We define the *expansion of I with respect to the n-tuple* $(i_1, \ldots, i_n)$, denoted by $I^{(i_1,\ldots,i_n)}$, to be the monomial ideal

$$I^{(i_1,\ldots,i_n)} = \sum_{i=1}^{m} \mathfrak{p}_1^{a_i(1)} \cdots \mathfrak{p}_n^{a_i(n)} \subseteq R^{(i_1,\ldots,i_n)}.$$

We simply write R^* and I^*, respectively, rather than $R^{(i_1,\ldots,i_n)}$ and $I^{(i_1,\ldots,i_n)}$.

For example, consider $R = K[x_1, x_2, x_3]$ and the ordered 3-tuple $(1, 3, 2)$. Then we have $\mathfrak{p}_1 = (x_{11})$, $\mathfrak{p}_2 = (x_{21}, x_{22}, x_{23})$, and $\mathfrak{p}_3 = (x_{31}, x_{32})$. So for the monomial ideal $I = (x_1x_2, x_3^2)$, the ideal $I^* \subseteq K[x_{11}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}]$ is $\mathfrak{p}_1\mathfrak{p}_2 + \mathfrak{p}_3^2$, namely

$$I^* = (x_{11}x_{21}, x_{11}x_{22}, x_{11}x_{23}, x_{31}^2, x_{31}x_{32}, x_{32}^2).$$

Theorem 2.5. *[6, Theorem 3.3] Let I be a monomial ideal of R. Then I is normally torsion-free if and only if I^* is.*

Here, we state the second method for constructing new normally torsion-free monomial ideals. To achieve this, one requires to recall the following definition.

Definition 2.6. [6, Definition 3.4] A weight over a polynomial ring $R = K[x_1, \ldots, x_n]$ is a function $W : \{x_1, \ldots, x_n\} \to \mathbb{N}$, and $w_i = W(x_i)$ is called the *weight* of the variable x_i. Given a monomial ideal I and a weight W, we define the *weighted ideal*, denoted by

I_W, to be the ideal generated by $\{h(u) : u \in G(I)\}$, where h is the unique homomorphism $h : R \to R$ given by $h(x_i) = x_i^{w_i}$. For a monomial $u \in R$, we denote $h(u) = u_W$.

For instance, consider the monomial ideal $I = (x_1^2x_2x_3^6, x_2^3x_4x_5^4)$ in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$. Furthermore, let $W : \{x_1, x_2, x_3, x_4, x_5\} \to \mathbb{N}$ be a weight over R with $W(x_1) = 2$, $W(x_2) = 4$, $W(x_3) = 2$, $W(x_4) = 3$, and $W(x_5) = 1$. Thus, the weighted ideal I_W is given by $I_W = (x_1^4x_2^4x_3^{12}, x_2^{12}x_4^3x_5^4)$.

Theorem 2.7. *[6, Theorem 3.10] Let I be a monomial ideal of R, and W a weight over R. Then I is normally torsion-free if and only if I_W is.*

Lemma 2.8. *[6, Lemma 3.12] Let I be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ with $G(I) = \{u_1, \ldots, u_m\}$, and $h = x_{j_1}^{b_1} \cdots x_{j_s}^{b_s}$ with $j_1, \ldots, j_s \in \{1, \ldots, n\}$ be a monomial in R. Then I is normally torsion-free if and only if hI is normally torsion-free.*

Now, we recall the definition of the monomial localization of a monomial ideal with respect to a monomial prime ideal as has been introduced in [5]. Let I be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K. We also denote by $V^*(I)$ the set of monomial prime ideals containing I. Let $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_r})$ be a monomial prime ideal. The *monomial localization* of I with respect to $\mathfrak{p}$, denoted by $I(\mathfrak{p})$, is the ideal in the polynomial ring $R(\mathfrak{p}) = K[x_{i_1}, \ldots, x_{i_r}]$ which is obtained from I by applying the K-algebra homomorphism $R \to R(\mathfrak{p})$ with $x_j \mapsto 1$ for all $x_j \notin \{x_{i_1}, \ldots, x_{i_r}\}$.

Theorem 2.9. *[6, Theorem 3.15] Let I be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$, and $\mathfrak{p} \in V^*(I)$. If I is normally torsion-free, then $I(\mathfrak{p})$ is so.*

To state the subsequent results, we need the definition of the contraction operator as has been given in [3, P. 303]. Let I be a monomial ideal in $R = K[x_1, \ldots, x_n]$ with $G(I) = \{u_1, \ldots, u_m\}$. For some $1 \leq j \leq n$, recall that the *contraction* I/x_j is obtained by setting $x_j = 1$ in u_i for each $i = 1, \ldots, m$. Note that the contraction I/x_j is exactly the monomial localization of I with respect to $\mathfrak{p} = \mathfrak{m} \setminus \{x_j\}$, where $\mathfrak{m} = (x_1, \ldots, x_n)$ is the graded maximal ideal of $R = K[x_1, \ldots, x_n]$.

Theorem 2.10. *[6, Theorem 3.19] Let I be a monomial ideal in $R = K[x_1, \ldots, x_n]$, and $1 \leq j \leq n$. If I is normally torsion-free, then I/x_j is so.*

To express the next result, we require the definition of the deletion operator, as has been given in [3, P. 303]. Let I be a monomial ideal in $R = K[x_1, \ldots, x_n]$ with $G(I) = \{u_1, \ldots, u_m\}$. For some $1 \leq j \leq n$, the *deletion* $I \setminus x_j$ is formed by putting $x_j = 0$ in u_i for each $i = 1, \ldots, m$.

Theorem 2.11. *[6, Theorem 3.21] Let I be a square-free monomial ideal in $R = K[x_1, \ldots, x_n]$, and $1 \leq j \leq n$. If I is normally torsion-free, then $I \setminus x_j$ is so.*

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Entropy for multi-valued mapps

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Abstract. In mathematics, a multi-valued function, also called set-valued function, is similar to a function, but may associate several values to each input. The complexity of multi-valued mapps is usually measured by the topological entropy. The aim of this paper is to give new definition of topological entropy (or just entropy) for multi-valued maps. Some related properties are also presented.

Keywords: entropy, multi-valued functions, spanning set, separated set

AMS Mathematics Subject Classification [2010]: 37B40, 26E25, 37A35

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1. Introduction

Definition 1.1. Let (X, d) be a metric space. We denote by 2^X the power set of X, i.e., the family of all subsets of X .

Let $F : X \longrightarrow 2^X$ be a multi-valued function, a function $f : X \longrightarrow X$ is called selection of F if $f(x) \in F(x)$, for all $x \in X$.

A multi-valued function $F : X \longrightarrow 2^X$ is called finite if *card* $F(x) < \infty$, for every $x \in X$.

Define $F^0(x) := \{x\}$ and $F^n : X \longrightarrow 2^X$ by

$$F^n(x) := \bigcup_{y \in F^{n-1}(x)} F(y),$$

for every $n \in \mathbb{N}$.

1.1. Mail results.

Lemma 1.2. *Let (X, d) be a metric space. Then $d_n : X \times X \longrightarrow [0, +\infty)$ defined by*

$$d_n(x, y) := \sup_{0 \le i \le n-1} \{d(F^i(x), F^i(y))\}$$

is a metric on X, where $d(A, B) := \sup_{a \in A, b \in B}\{d(a, b)\}$.

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Proof. We only prove the triangle inequality. Let $x, y, z \in X$ and $a \in \bigcup_{0\leq i\leq n-1} F^i(x), b \in \bigcup_{0\leq i\leq n-1} F^i(y)$ and $c \in \bigcup_{0\leq i\leq n-1} F^i(z)$ are arbitrary. Then

$$d(a, c) \leq d(a, b) + d(b, c) \leq d_n(x, y) + d_n(y, z).$$

Hence, $d_n(x, z) \leq d_n(x, y) + d_n(y, z)$. □

Example 1.3. Suppose that $\mathbb{C}$ is the set of complex numbers and $F : \mathbb{C} \longrightarrow \mathbb{C}$ is defined by $F(z) := \sqrt{z}$ then $d(\sqrt{1}, \sqrt{4}) = 3$ and $d(\sqrt{1}, \sqrt{-4}) = \sqrt{5}$.

Suppose that $A \subseteq X$ is a subset, the subset $F^{-1}(A)$ is defined by

$$F^{-1}(A) := \{x \in X : F(x) \subseteq A\}.$$

$F : X \longrightarrow 2^X$ is said to be continuous if $F^{-1}(\mathcal{U})$ is an open set, for every open set $\mathcal{U}$, and it is uniformly continuous if for every number $\epsilon > 0$ there exists $\delta > 0$ such that if $a \in F(x)$, $b \in f(y)$ and $d(x, y) < \delta$ then $d(a, b) < \epsilon$.

Definition 1.4. For a natural number n, $\epsilon > 0$, and a compact subset K of X we say that a subset G of X is an (n, ϵ)-span of K with respect to G, if it satisfies the following properties:
If $x \in K$, then there is $y \in G$ such that $d_n(x, y) \leq \epsilon$ (i.e.,

$$x \in \bigcap_{i=0}^{n-1} F^{-i}(\bigcap_{a\in F^i(y)} \overline{B_\epsilon(a)}).)$$

So,

$$K \subseteq \bigcup_{y\in G} \bigcap_{i=0}^{n-1} F^{-i}(\bigcap_{a\in F^i(y)} \overline{B_\epsilon(a)}).$$

Definition 1.5. If n is a natural number, $\epsilon > 0$ and K is a compact subset of X then we denote the smallest cardinality of any (n, ϵ)-spanning set of K with respect to F by $r_n(\epsilon, K)$. (When we need to emphasise on F we shall write $r_n(\epsilon, K, F)$).

Since K is compact, then $r_n(\epsilon, K) < \infty$. and it is clear that $r_n(., K)$ is a non-increasing map on $(0, +\infty)$.
If K a compact subset of X, and $\epsilon > 0$ then $r(\epsilon, K, F) = \limsup_{n\to\infty} \frac{\log r_n(\epsilon,K)}{n}$, we also denote $r(\epsilon, K, F)$ by $r(\epsilon, K, F, d)$.
The value of $r'_n(\epsilon, K)$ could be ∞, and $r'_n(., K)$ is a non-increasing map on $(0, +\infty)$.

Definition 1.6. Let $h(F, K) = \lim_{\epsilon\to 0} r(\epsilon, K, F)$, where K is a compact subset X. Then the topological entropy of F is $h(F) = \sup_K h(F, K)$, where the supremum is taken over the collection of all compact subsets of X. We sometimes write $h_d(F)$ instead of $h(F)$ to emphasis the dependence on d.

Lemma 1.7. *If $F : X \longrightarrow 2^X$ is a multi-valued function and $f : X \longrightarrow X$ is a selection of F, then $h_d(f) \leq h_d(F)$.*

Proof. Let $F : X \longrightarrow 2^X$ be a multi-valued function, and $f : X \longrightarrow X$ is a selection of F and K be an open subset of X. Since

$$\sup_{01\leq i\leq n-1} \{d(f^i(x), f^i(y))\} \leq \sup_{0\leq i\leq n-1} \{d(F^i(x), F^i(y))\},$$

any (n,ϵ)-spanning set G of K with respect to F is an (n,ϵ)-spanning set of K with respect to f. Therefore $r_n(\epsilon,K,f) \leq r_n(\epsilon,K,F)$. Thus $r(\epsilon,K,f) \leq r(\epsilon,K,F)$ and $h(K,f) \leq h(K,F)$. So $h_d(f) \leq h_d(F)$. $\square$

Now we shall give an equivalent definition. In this definition we use of the idea of separated sets which are dual to spanning sets.
For a natural number n, $\epsilon > 0$, and a compact subset K of X, a subset E of X is called an (n,ϵ)-separated of K with respect to F, if it satisfies the following property:
If $x, y \in E$, and $x \neq y$, then $d_n(x,y) > \epsilon$.

If n is a natural number, $\epsilon > 0$ and K is a compact subset of X then $s_n(\epsilon,K)$ denotes the largest cardinality of any (n,ϵ)-separated subset of K with respect to on F. (When we need to emphasis F we shall write $s_n(\epsilon,K,F)$).

Theorem 1.8. $r_n(\epsilon,K,F) \leq s_n(\epsilon,K,F) \leq r_n(\frac{\epsilon}{2},K,F)$.

Proof. If E is an (n,ϵ)-separated subset of K with the maximal cardinality then E is an (n,ϵ)-spanning set for K, because if $x \in K$ then there is $y \in E$ such that $d_n(x,y) \leq \epsilon$. Therefore $r_n(\epsilon,K,F) \leq s_n(\epsilon,K,F)$. To show the other inequality suppose E is an (n,ϵ)-separated subset of K and G is an $(n,\frac{\epsilon}{2})$-spanning set for K. Define $\Phi : E \longrightarrow G$ by choosing, for for each $x \in E$, some point $\Phi(x) \in G$ with $d_n(x,\Phi(x)) \leq \frac{\epsilon}{2}$. Then Φ is injective, because if $d_n(x,y) \leq \frac{\epsilon}{2}$ and $d_n(z,x) \leq \frac{\epsilon}{2}$ then $d_n(y,z) \leq \epsilon$, which is a contradiction. Therefore the cardinality of E is not greater than the cardinality of G. Hence $s_n(\epsilon,K,F) \leq r_n(\frac{\epsilon}{2},K,F)$. $\square$

Since K is compact, then $s_n(\epsilon,K) < \infty$. and it is obvious that $s_n(.,K)$ is non-increasing map on $(0,+\infty)$. Theorem 2.1 also implies that $s_n(\epsilon,K,T) < \infty$. If K is a compact subset of X, $\epsilon > 0$ then we define $s(\epsilon,K,T)$ by $\limsup_{n\to\infty} \frac{\log s_n(\epsilon,K)}{n}$.

Also we write $s(\epsilon,K,F,d)$ if we need to emphasis to the metric d. The value of $s_n(\epsilon,K)$ could be ∞, and it is obvious that $s_n(.,K)$ is a non-increasing map of $(0,+\infty)$.

Corollary 1.9. *We have*

(1) $r(\epsilon,K,F) \leq s(\epsilon,K,F) \leq r(\frac{\epsilon}{2},K,F)$,
(2) $s(.,K)$ *is non-increasing map of* $(0,+\infty)$,
(3) $h(F,K) = \lim_{\epsilon\to 0} s(\epsilon,K,F)$, $h(F) = \sup_K h(F,K) = \lim_{\epsilon\to 0} s(\epsilon,K,F)$, *where the supremum is taken over the collection of all compact subset of* X.

Theorem 1.10. *If* (X,d) *is a metric space,* F *is uniformly continuous and* m *is a natural number then* $h(F^m) = mh(F)$.

Proof. Let C be a (ϵ,mn)-span K with respect F. So for every each $x \in K$ there is $y \in C$ such that

$$\max_{1\leq i\leq mn-1} \max_{a\in F^i(x),b\in F^i(y)} \{d(a,b)\} < \epsilon.$$

Hence for each $x \in K$ there is $y \in C$ such that

$$\max_{1\leq i\leq n-1} \max_{a\in F^{mi}(x),b\in F^{mi}(y)} \{d(a,b)\} < \epsilon.$$

Thus C is a (ϵ,n)-span K with respect F^m. Hence

$$r_n(\epsilon,K,F^m) \leq r_{mn}(\epsilon,K,F),$$

and we have

$$\tfrac{1}{n}\log r_n(\epsilon,K,F^m) \leq \tfrac{m}{mn}\log r_{mn}(\epsilon,K,F).$$

So $h(F^m) \leq mh(F)$.
If F is uniformly continuous then for given $\epsilon > 0$ there is $\delta > 0$ such that if $a \in F(x)$, $b \in f(y)$ and $d(x, y) < \delta$ then $d(a, b) < \epsilon$. Therefore a (δ, n)-spanning set for K with respect to F^m is an (ϵ, mn)-spanning set for K with respect to F. So

$$r_n(\delta, K, F^m) \geq r_{mn}(\epsilon, K, F).$$

Hence

$$r(\delta, K, F^m) \geq mr(\epsilon, K, F).$$

So $h(F^m) \geq mh(F)$. Thus $h(F^m) = mh(F)$. □

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پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
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۸ تا ۱۱ شهریور ۱۴۰۰

Stability and Hopf Bifurcation of an Autonomous Chaotic System via Time-delayed Feedback Control Method

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Abstract. This paper is concerned with the control of chaos in a chaotic dynamical system which is investigated by time-delayed feedback control technique. By designing appropriate feedback strength and delay, the chaotic system is controlled to be stable, or stable bifurcating periodic solutions emerge near an unstable equilibrium. Therefore, regarding the delay of the system as a bifurcation parameter and analyzing the characteristic equation of the corresponding linearized system, stability and the existence of Hopf bifurcation are theoretically proved. Furthermore, some numerical simulations are provided to examine the analytical results.

Keywords: Chaotic system, Chaos control, Time-delayed feedback, Stability, Hopf bifurcation.

AMS Mathematics Subject Classification [2010]: 18A32, 18F20, 05C65 (at least 1 and at most 3)

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1. Introduction

In recent years, the trend of analyzing and understanding chaos has been developed to a new phase of controlling and utilizing the chaotic systems. Thus, the topics of chaos and chaotic control are growing rapidly in many different fields such as biological systems, ecological and chemical systems, and so on [4, 6]. Delayed feedback control (DFC) is a convenient and powerful tool to stabilize unstable periodic orbits(UPO's) or control of unstable steady states. The method is based on a feedback of the difference between the current state and the delayed state [2–4]. The delay time is set to correspond to the period of the desired unstable periodic orbit (UPO) and the feedback term vanishes after the UPO is stabilized [2, 3].

In this paper, we consider the following time delayed feedback control system

$$\begin{aligned}
\dot{x}(t) &= \beta x(t) - y^2(t),\\
\dot{y}(t) &= \mu\big(z(t) - y(t)\big),\\
\dot{z}(t) &= x(t)y(t) + (\alpha - \mu)y(t) + \alpha z(t) + K\big(z(t) - z(t-\tau)\big),
\end{aligned} \tag{1}$$

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where, $K \in \mathbb{R}$ is the feedback strength which represents the intensity of control per unit of time.
The organization of this paper is as follows. In Section 2, we first focus on the stability and local Hopf bifurcation and determine the ranges of delay τ in which the equilibrium point of the chaotic system can be controlled to be stable. Then, we obtain some sufficient conditions such that Hopf bifurcations occur. In Section 3, numerical simulations for a set of parameters proposed in [5] are performed to illustrate the obtained analytical results. The brief conclusions are finally given in Section 4.

2. Main results

In this section, we investigate the effect of delay on the dynamic behavior of system (1). We discuss stability analysis of the system at the equilibrium point $E^*(x^*, y^*, z^*)$ for which $x^* = -(2\alpha - \mu) = \mu - 2\alpha$, $y^* = \sqrt{\beta(\mu - 2\alpha)}$ and $z^* = \sqrt{\beta(\mu - 2\alpha)}$. This equilibrium point is feasible if $(\mathrm{H}_1): \ \beta(\mu - 2\alpha) > 0$ holds. Under the transformation $X = x - x^*, Y = y - y^*, Z = z - z^*$ and hypothesis (H_1), we linearize the system as follows:

$$
\begin{cases}
\frac{dX}{dt} = \beta X(t) - 2y^* Y(t), \\
\frac{dY}{dt} = -\mu Y(t) + \mu Z(t), \\
\frac{dZ}{dt} = y^* X(t) + \big(x^* + (\alpha - \mu)\big) Y(t) + (\alpha + K) Z(t) - K Z(t - \tau).
\end{cases}
\tag{2}
$$

Then, the characteristic equation can be described by

$$
\triangle(\lambda, \tau) = \lambda^3 + m_2 \lambda^2 + m_1 \lambda + m_0 + (n_2 \lambda^2 + n_1 \lambda + n_0) e^{-\lambda \tau} = 0,
\tag{3}
$$

where

$$
m_2 = -K - \alpha + \mu - \beta, \qquad m_1 = \mu^2 + (-K - x^* - 2\alpha - \beta)\mu + (\alpha + K)\beta
$$

$$
m_0 = \big(-\beta\mu + (K + x^* + 2\alpha)\beta + 2y^* \big)\mu \quad n_2 = K, \quad n_1 = -(\beta K - K\mu), \quad n_0 = -K\mu\beta.
$$

When $\tau = 0$, Eq. (3) becomes

$$
\triangle(\lambda) = \lambda^3 + (m_2 + n_2)\lambda^2 + (m_1 + n_1)\lambda + m_0 + n_0 = 0.
\tag{4}
$$

Hence, E^* becomes asymptotically stable if

$$
(\mathrm{H}_2) \qquad m_2 + n_2 > 0, \qquad m_0 + n_0 > 0, \qquad (m_2 + n_2)(m_1 + n_1) > m_0 + n_0.
$$

holds. For Hopf bifurcation analysis, let $\lambda = \mathrm{i}\omega$ $(\omega > 0)$ be a root of Eq. (3), then we

FIGURE 1. Phase portrait of the system (1) for $\tau = 0$ or $K = 0$.

obtain

$$
\omega^6 + (m_2^2 - n_2^2 - 2m_1)\omega^4 + (m_1^2 - n_1^2 - 2m_2 m_0 + 2n_2 n_0)\omega^2 + m_0^2 - n_0^2 = 0.
\tag{5}
$$

Let $\omega^2 = z$, then from (5), we get

$$
z^3 + pz^2 + qz + r = 0.
\tag{6}
$$

where

$$p = m_2^2 - n_2^2 - 2m_1, \qquad q = m_1^2 - n_1^2 - 2m_2m_0 + 2n_2n_0, \qquad r = m_0^2 - n_0^2.$$

Suppose $h(z) = z^3 + pz^2 + qz + r$ and $h'(z) = 3z^2 + 2pz + q$. Since $\lim_{z\longrightarrow+\infty} h(z) = +\infty$ and $h(0) = r = m_0^2 - n_0^2 < 0$, then (6) has at least one positive real root and we can state the following results.

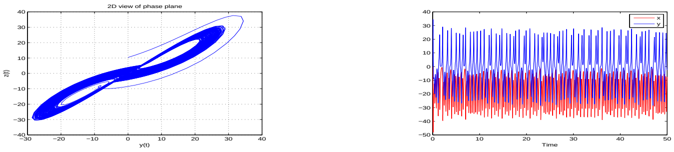

FIGURE 2. Chaos still exists for $K = -7.75$ and $\tau = 0.034$.

LEMMA 2.1. *For the polynomial Eq. (6), we have the following results.*

- **(i):** *If* $(\mathrm{H}_3): \ r > 0 \ \ \& \ \ \Delta = p^2 - 3q < 0$, *then Eq. (6) has no positive roots, i.e., the necessary condition for this eqution to have positive real roots is* $\Delta \geq 0$*;*
- **(ii):** *If and only if* $\Delta > 0$, $z_1^* = \frac{-p+\sqrt{\Delta}}{4} > 0$ *and* $h(z_1^*) \leq 0$, *then Eq. (6) have positive real roots. More precisely, if the condition* $(\mathrm{H}_4): \ r > 0, \ z_1^* > 0, \ h(z_1^*) < 0$ *holds, then (6) has two positive roots,* z_1 *and* z_2*.*

LEMMA 2.2. *When* $\tau = \tau_k^{(j)}$ $(k = 1, 2, 3; j = 0, 1, 2, \cdots)$, *i.e.*

$$\tau_k^{(j)} = \begin{cases} \frac{1}{\omega_k}[\arccos(P) + 2j\pi], & Q \geq 0 \\ \frac{1}{\omega_k}[2\pi - \arccos(P) + 2j\pi], & Q < 0, \end{cases} \tag{7}$$

where

$$Q = \sin(\omega_k\tau_k) = \frac{\big(n_2\omega_k^4 + (m_2n_1 - m_1n_2 - n_0)\omega_k^2 + m_0n_1 + n_0m_1\big)\omega_k}{n_2\omega_k^4 + (n_1^2 - 2n_0n_2)\omega_k^2 + n_0^2},$$

$$P = \cos(\omega_k\tau_k) = \frac{(n_1 - m_2n_2)\omega_k^4 + (m_0n_2 - m_1n_1 + m_2n_0)\omega_k^2 - m_0n_0}{n_2^2\omega_k^4(n_1^2 - 2n_2n_0)\omega_k^2 + n_0^2},$$

and if (H_4) *in Lemma 2.1 holds, then Eq. (3) has a pair of complex conjugate pure imaginary roots* $\pm \mathrm{i}\omega_0$, *and all other roots have nonzero real parts.*

Furthermore, let $\lambda(\tau) = \eta(\tau) + \mathrm{i}\omega(\tau)$ be a root of (3) near $\tau = \tau_k^{(j)}$ satisfying $\eta(\tau_k^{(j)}) = 0$ and $\omega(\tau_k^{(j)}) = \omega_k$, then, the following transversality condition holds.

LEMMA 2.3. *If* $h'(z_k) \neq 0$ *and* (H_3) *in Lemma 2.1 holds, then* $\frac{d\left[\mathrm{Re}\lambda\left(\tau_k^{(j)}\right)\right]}{d\tau} \neq 0$ *and* $h'(z_k)$ *have the same sign.*

Define $\tau_0 = \tau_{k_0} = \min_{1\leq k\leq 3}\{\tau_k\}, \omega_0 = \omega_{k_0}, z_0 = \omega_0^2$, according to the derived Lemmas 2.1, 2.2, 2.3, we can conclude the existence of Hopf bifurcation as described in the following theorem.

THEOREM 2.4. *For system (1), the following results hold.*

- **(i):** *If the condition* (H_3) *in Lemma (2.1) holds, then* E^* *is asymptotically stable for all* $\tau > 0$.

(ii): *If $\Delta = p^2 - 3q > 0$, $h'(z_0) \neq 0$ and if there exists only one positive real root, then there exists a positive number τ_0 such that the equilibrium E^* is locally asymptotically stable when $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$. Moreover, system (1) undergoes a Hopf bifurcation at E^* when $\tau = \tau_0$.*

(iii): *If (H_4) in Lemma (2.1) and $h'(z_k) \neq 0$ hold, then there is a positive integer m such that E^* is stable when $\tau \in [0, \tau_0^+) \cup (\tau_0^-, \tau_1^+) \cup \cdots (\tau_{m-1}^-, \tau_m^+)$ and unstable when $\tau \in [\tau_0^+, \tau_0^-) \cup (\tau_1^+, \tau_1^-) \cup \cdots (\tau_{m-1}^+, \tau_{m-1}^-) \cup (\tau_m^+, \infty)$. Moreover, the system (1) undergoes a Hopf bifurcation at E^* when $\tau = \tau_k^{(j)}$ for $k = 1, 2$; $j = 0, 1, 2, \cdots$.*

3. Numerical simulations

For the parameters $\beta = -10, \alpha = 37, \mu = 55$, we get $E^*(-19, 13.784, 13.784)$. Also, for the purpose of controlling chaos, we consider $K > -1.72$, especially $K \in (-1.72, -13.75)$. Thus when $\tau = 0$ or $K = 0$, system (1) is chaotic (see Fig. 1).

The graphical results with initial value $(0.1, 0.1, 10.38)$ show that when $\tau < 0.048$, the

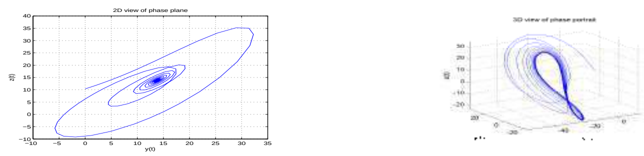

FIGURE 3. (LHS) Stable equilibrium for $K = -7.75$ and $\tau = 0.034$. (RHS) Stable periodic solution for $K = -7.75$ and $\tau = 0.231$.

equilibrium E^* still displays chaotic behavior (see Fig. 2). When $\tau \in (0.048, 0.13]$, E^* is asymptotically stable. Fig. 3 shows a phase trajectories of system (1) for $\tau = 0.12$. For the critical value $\tau = \tau_1^{(0)} = \tau_0 = 0.213$, the system undergoes a Hopf bifurcation and a periodic solution emerges around E^*. Thus for $\tau = 0.231$, a limit cycle appears which is showed in Fig. 3. By increasing of τ, stability of E^* changes and the system regains its complex dynamical behavior, and becomes chaotic when $\tau > 0.36$.

4. Conclusion

In this study, the time-delayed feedback control method is used to stabilize UPOs and unstable equilibrium point of the chaotic system. We investigated stability and Hopf bifurcation both analytically and numerically.

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021
پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference
دانشگاه شهید باهنر کرمان
دانشکده ریاضی و کامپیوتر
۸ تا ۱۱ شهریور ۱۴۰۰

Fuzzy Quasi-uniformities by Entourage

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Abstract. Uniform structures on the fuzzy spaces are defined using different set of axioms and basic terms.In this paper we present another characterization of fuzzy uniformities in the style of Weil that we call it $\mathbb{T}$ - Weil uniformity. We formulated and investigated a definition of entourage uniformity alternative to that one of Hutton. It is expressed in terms of coproduct of fuzzy spaces. We have showed that every fuzzy topological space is $\mathbb{T}$ - Weil quasi- uniformizable.

Keywords: $\mathbb{T}$-fuzzy topological space, $\mathbb{T}$ - Weil uniformity, $\mathbb{T}$ - Weil quasi- uniformizable

AMS Mathematics Subject Classification [2010]: 54A40,54E15

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1. Introduction

Fuzzy topological structures is just a kind of mathematics developed on fuzzy sets. In mathematics literature, extensive study of fuzzy topological spaces and fuzzy uniform structures is done by various researchers. Uniformity in fuzzy topology was studied by three authors, B. Hutton [2] , U. Höhle [1] , and R. Lowen [5] . The approaches of U. Höhle and R. Lowen underpinned by power sets of the form $I^{X\times X}$ or $\mathbb{T}^{X\times X}$ and B. Hutton approach is based on exponential power sets of the form $(\mathbb{T}^X)^{\mathbb{T}^X}$. In this paper we present another characterization of fuzzy uniformities in the style of Weil that we call it $\mathbb{T}$ - Weil uniformity. We formulated and investigated a definition of entourage uniformity alternative to that one of Hutton. It is expressed in terms of coproduct of fuzzy spaces. We have showed that every fuzzy topological space is $\mathbb{T}$ - Weil quasi- uniformizable.

Keywords: $\mathbb{T}$-fuzzy topological space, $\mathbb{T}$ - Weil uniformity, $\mathbb{T}$ - Weil quasi- uniformizable.

2. fuzzy topological space

Definition 2.1. A frame $\mathbb{T}$is a complete lattice satisfying the distribution law $x \wedge \vee(A) = \vee\{x \wedge a | a \in A\}$,the bottom resp top of $\mathbb{T}$ will be denoted by 0 resp 1.

Definition 2.2. [4] Let X be a nonempty ordinary set, $\delta \subset \mathbb{T}^X, \delta$ is called a $\mathbb{T}$-fuzzy topology on X, and $(\mathbb{T}^X, \delta)$ is called an $\mathbb{T}$-fuzzy topological space or$\mathbb{T}$-fts for short, if δ satisfies the following three conditions:

(LFT1) $\underline{0}, \underline{1} \in \delta$

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(LFT2) $\forall \mathcal{A} \subset \delta,\ \vee\mathcal{A} \in \delta$
(LFT3) $\forall U, V \in \delta, U \wedge V \in \delta$.

Definition 2.3. [4] Let $(X,\delta),(Y,\mu)$ be $\mathbb{T}$-fts, $f^{\rightarrow} : X \to Y$ an $\mathbb{T}$-fuzzy mapping. We say $f^{\rightarrow}$ is an $\mathbb{T}$-fuzzy continuous mapping from $(\mathbb{T}^X,\delta)$ to $(\mathbb{T}^Y,\mu)$ or call $f^{\rightarrow}$ continuous for short, if its $\mathbb{T}$-fuzzy reverse mapping $f^{\leftarrow} : \mathbb{T}^Y \to \mathbb{T}^X$ maps every open subset in $(\mathbb{T}^Y,\mu)$ as an open one in $(\mathbb{T}^X,\delta)$ i.e. $\forall\ V \in \mu, f^{\leftarrow}(V) \in \delta$.

Definition 2.4. [4] Let X be a nonempty ordinary set, $i, c : \mathbb{T}^X \to \mathbb{T}^X$ mapping on $\mathbb{T}^X$.

i is called an interior operator on $\mathbb{T}^X$, if it fulfills the following conditions :
(I01) $i(\underline{1}) = \underline{1}$
(I02) $\forall\ A \in \mathbb{T}^X, i(A) \leq A$
(I03) $\forall\ A, B \in \mathbb{T}^X, i(A \wedge B) = i(A) \wedge i(B)$
(I04) $\forall\ A \in \mathbb{T}^X, i(i(A)) = i(A)$.

For an interior operator i on $\mathbb{T}^X$, define the $\mathbb{T}$-fuzzy topology generated by i as $\delta = \{A \in \mathbb{T}^X | i(A) = A\}$.

c is called a closure operator on $\mathbb{T}^X$, if it fulfills the following conditions :
(C01) $c(\underline{0}) = \underline{0}$
(C02) $\forall\ A \in L^X, A \leq c(A)$
(C03) $\forall\ A, B \in \mathbb{T}^X, c(A \vee B) = c(A) \vee c(B)$
(C04) $\forall\ A \in \mathbb{T}^X, c(c(A)) = c(A)$.

For a closure operator c on $\mathbb{T}^X$, define the $\mathbb{T}$-fuzzy topology generated by c as $\delta = \{A \in \mathbb{T}^X | c(A') = A'\}$.

3. $\mathbb{T}$-Weil Uniform Spaces

Definition 3.1. [3] (Binary coproducts of $\mathbb{T}^{X_i}, (i \in \{1,2\})$:The coproduct of $\mathbb{T}^{X_1}$ and $\mathbb{T}^{X_2}$ is a C-ideal of $\mathbb{T}^{X_1} \times \mathbb{T}^{X_2}$ as a down-set $A \subseteq \mathbb{T}^{X_1} \times \mathbb{T}^{X_2}$ satisfying $\{f\} \times S \subseteq A \Rightarrow (f, \vee S) \in A$ and

$$S \times \{g\} \subseteq A \Rightarrow (\vee S, g) \in A.$$

the set of all C-ideals of $\mathbb{T}^{X_1} \times \mathbb{T}^{X_2}$ is $\mathbb{T}^{X_1} \oplus \mathbb{T}^{X_2}$.

For any frame homomorphism $\overrightarrow{f_i}$: $\mathbb{T}^{X_i} \to \mathbb{T}^{X_i{}'}, (i \in \{1,2\})$, $\overrightarrow{f_1} \oplus \overrightarrow{f_2}$ for the unique morphism from $\mathbb{T}^{X_1} \oplus \mathbb{T}^{X_2}$ to $\mathbb{T}^{X_1{}'} \oplus \mathbb{T}^{X_2{}'}$define as follow: $\overrightarrow{f_1} \oplus \overrightarrow{f_2}\ (\bigvee_{\gamma\in\Gamma}(f_\gamma \oplus g_\gamma)) = \bigvee_{\gamma\in\Gamma}(\overrightarrow{f_1}\ (f_\gamma) \oplus \overrightarrow{f_1}\ (g_\gamma))$.

Definition 3.2. $E \in \mathbb{T}^X \oplus \mathbb{T}^X$ is $\mathbb{T}$-Weil entourage of $\mathbb{T}^X$ if and only if $\{\mu \in \mathbb{T}^X | (\mu,\mu) \in E\}$ is a cover of $\mathbb{T}^X$. That is $\vee\{\mu \in \mathbb{T}^X | (\mu,\mu) \in E\} = \underline{1}$. The collection $\mathbb{T} - W\ Ent(X)$ of all $\mathbb{T}$-Weil entourage of $\mathbb{T}^X$ may be partially ordered by inclusion.

Proposition 3.3. *Let E be a $\mathbb{T}$-Weil entourage. Then*
(a) for any $f \in \mathbb{T}^X, f \leq st(f, E)$
(b) $E^n \subseteq E^{n+1}$ for every natural n.
(c) For any down set A of $\mathbb{T}^X \times \mathbb{T}^X, A \subseteq (EoA) \cap (AoE)$.
(d) for any $f \in \mathbb{T}^X, st(st(f, F), F) \leq st(f, F^2)$.

Definition 3.4. Let X be a nonempty set and $\mathcal{E} \subset \mathbb{T}wEnt(X)$. We say $(X, \mathcal{E})$ is a $\mathbb{T}$-Weil quasi uniformity on X if it satisfies the following conditions :
($\mathbb{T}-$WQU$_1$) $\mathcal{E}$ is a filter of $(\mathbb{T} - W\ Ent(X), \subseteq)$,
($\mathbb{T}-$WQU$_2$) For each $E \in \mathcal{E}$ there is $F \in \mathcal{E}$ such that $FoF \subseteq E$.

The pair $(X, \mathcal{E})$ is said to be a $\mathbb{T}$-Weil quasi-uniform space.

A $\mathbb{T}$-Weil quasi uniform space $(X, \mathcal{E})$ is called a $\mathbb{T}$-Weil uniform space if it satisfies

($\mathbb{T}-\mathrm{WU}_3$) for any $E \in \mathcal{E}, E^{-1}$ is also in $\mathcal{E}$.

It is useful to note that the symmetric $\mathbb{T}$- Weil entourages E of $\mathcal{E}$ form a basis for $\mathcal{E}$. In fact, if $E \in \mathcal{E}$ then $E^{-1} \in \mathcal{E}$ so $E \cap E^{-1}$ is a symmetric $\mathbb{T}$-Weil entourage of $\mathcal{E}$ contained in E.

Definition 3.5. Let $(X, \mathcal{E}), (X', \mathcal{E}')$ be two $\mathbb{T}$-Weil uniform spaces. A mapping $f : X \to X'$ is said to be uniformly homomorphic if $(\overrightarrow{f} \oplus \overrightarrow{f})(E) \in \mathcal{E}'$ whenever $E \in \mathcal{E}$.

We will denote by $\mathbb{T}$-W Unif the category whose objects are $\mathbb{T}$-Weil uniform spaces and morphisms are uniformly homomorphism mappings.

4. Relationships between $\mathbb{T}$-Weil quasi uniformity and $\mathbb{T}$-valued space

Theorem 4.1. *Let $(X, \mathcal{E})$ be a $\mathbb{T}$-Weil quasi uniform space. Mapping $i : \mathbb{T}^X \to \mathbb{T}^X$ be defined as follows: $\forall\ A \in \mathbb{T}^X\ i(A) = \vee\{C \in \mathbb{T}^X | \exists E \in \mathcal{E},\ st(C, E) \leq A\}$. Then i is an interior operator on $\mathbb{T}^X$.*

Proof. (I01) Since $st(C, E) \leq \underline{1}$ for every $E \in \mathcal{E}$, so $i(\underline{1}) = \underline{1}$.

(I02) $i(A) = \vee\{C \in \mathbb{T}^X | \exists E \in \mathcal{E}, st(C, E) \leq A\}$ since $C \leq st(C, E)$ for all $C \in \mathbb{T}^X$, $i(A) \leq A$.

(I03) We need to prove $i(A) \wedge i(B) \leq i(A \wedge B)$, for arbitrary $A, B \in \mathbb{T}^X$. In fact, since for arbitrary $E, F \in \mathcal{E}$ and arbitrary $A, B, C, D \in \mathbb{T}^X$ such that $st(C, E) \leq A$ and $st(D, F) \leq B$, we have $st(C \wedge D, E \cap F) \leq st(C, E) \wedge st(D, F) \leq A \wedge B$. So $i(A) \wedge i(B) = \vee\{C \wedge D | C, D \in L^X, \exists E, F \in \mathcal{E}, st(C, E) \leq A, st(D, E) \leq B\} \leq \vee\{C \wedge D | C, D \in \mathbb{T}^X, \exists E, F \in \mathcal{E}\ st(C \wedge D, E \cap F) \leq A \wedge B\} = i(A \wedge B)$.

(I04) By (I02) we have $i(i(A)) \leq i(A)$. We want to show that $i(A) \leq i(i(A)).Let C \in \mathbb{T}^X, E \in \mathcal{E}, st(C, E) \leq A$. By ($\mathrm{WQE}_2$) $\exists F \in \mathcal{E}, FoF \subset E$ .Then $st(C, FoF) < st(C, E)$and, by Proposition 3.7(d)we have $st(st(C, F), F) < st(C, FoF) \leq A$ .Then $st(C, F) \leq i(A)$, $C \leq i(i(A))$ and ,so

$$i(A) = \vee\{C \in L^X | \exists E \in \mathcal{E}\ st(C, E) \leq A\} \leq i(i(A)).$$

□

Definition 4.2. Let $\mathcal{E}$ be an $\mathbb{T}$- Weil quasi uniformity on X ,the interior operator defined in 4.1 is called the interior operator on $\mathbb{T}^X$ generated by $\mathbb{T}$- Weil quasi uniformity $\mathcal{E}$. The $\mathbb{T}$-fuzzy space generated by the $\mathbb{T}$-Weil quasi-uniformity $\mathcal{E}$, denoted by $\delta(\mathcal{E}), (X, \delta(\mathcal{E}))$ is called the $\mathbb{T}$-Top corresponding to $(X, \mathcal{E})$.

Theorem 4.3. *Let $(X, \mathcal{E})$ be a $\mathbb{T}$-Weil quasi-uniform space, mapping $c : \mathbb{T}^X \to \mathbb{T}^X$ be defined as $\forall\ A \in \mathbb{T}^X, c(A) = \wedge\{st(A, E) | E \in \mathcal{E}\}$. Then c is a closure operator on $\mathbb{T}^X$.*

Proof. (C01) Since for every $E \in \mathcal{E}\ st(\underline{0}, E) = \underline{0}$,so $c(\underline{0}) = \underline{0}$.

(C02) Since $A \leq st(A, E)$ for every $E \in \mathcal{E}$ then $A \leq c(A)$.

(C03) We need only to prove $c(A \vee B) \leq c(A) \vee c(B)$ for arbitrary $A, B \in \mathbb{T}^X$. It is trivial,

$st(A \vee B, E_1 \cap E_2) \leq st(A, E_1) \vee st(B, E_2)$. Suppose $e \in \mathbb{T}^X$ such that $e \not\leq c(A) \vee c(B)$,then $e \not\leq c(A), e \not\leq c(B)$,then there exist $E, F \in \mathcal{E}$ such that $e \not\leq st(A, E), e \not\leq st(B, F)$ .Then $e \not\leq st(A \vee B, E \cap F), e \not\leq c(A \vee B)$,so $c(A \vee B) \leq c(A) \vee c(B)$. (C04) For every $A \in \mathbb{T}^X$, we have $st(st(C, F), F)) < st(C, FoF)$. For every $A \in \mathbb{T}^X, E \in \mathcal{E},\ c(A) = \wedge\{st(A, E) | E \in \mathcal{E}\}$. By ($\mathrm{WQE}_2$), there exist $F \in \mathcal{E}$ such that $FoF \subset E$, $c(c(A)) \leq$

$st(st(A,F),F) \leq st(A,FoF) \leq st(A,E)$ .Then $c(c(A) \leq c(A)$ and by (C02) $c(A) \leq c(c(A))$ and, so $c(c(A)) = c(A)$. □

Definition 4.4. Let $\mathcal{E}$ be an $\mathbb{T}$-Weil quasi-uniformity on X. The closure operator defined in 4.3 is called the closure operator on $\mathbb{T}^X$ generated by the $\mathbb{T}-$Weil quasi-uniformity $\mathcal{E}$.

Theorems 4.1, 4.3 shows that every $\mathbb{T}-$Weil quasi- uniformity can generates a $\mathbb{T}$-valued space, but the unexpected result is that its converse is also true.

Remark 4.5. Let (X,δ) be a $\mathbb{T}$-valued space.For every $U \in \delta$, define a self mapping f_U on $\mathbb{T}^X$ as follows:

$$\forall\ A \in \mathbb{T}^X,$$
$$f_U(A) = \begin{cases} \underline{1} & A \not\leq U \\ U & \underline{0} \neq A \leq U \\ \underline{0} & A = \underline{0} \end{cases}$$

It is easy to find that f_U is value increasing, $f_U(\vee\mathcal{A}) = \vee_{A\in\mathcal{A}} f_U(A)$, $f_U o f_U = f_U$. Let $\mathcal{D} = \{f \in \Omega(L^X) | \exists\ \mathcal{A} \in [\delta]^{<w},\ f \geq \wedge_{U\in\mathcal{A}} f_U\}$ then for all $g, f \in \mathcal{D}$ there exist $h \in \mathcal{D}$ such that$h \leq g \wedge f$. (1) Take $\mathcal{A} \in [\delta]^{<w}$ such that $f \geq \wedge_{U\in\mathcal{A}} f_U = \wedge_{U\in\mathcal{A}}(f_U o f_U)$ since for every $V \in \mathcal{A}$ $f_V o f_V \geq (\wedge_{U\in\mathcal{A}} f_U)0(\wedge_{U\in\mathcal{A}} f_U)$ so take $g = \wedge_{U\in\mathcal{A}} f_U$ we have $g \in \mathcal{D}$ and $gog \leq f$. (2)

Theorem 4.6. *Let (X,δ) be a $\mathbb{T}$-valued space.Define $E_f = \cup\{\alpha \oplus \alpha | \alpha \in U_f\}$such that U_f be the cover of all f-small elements of $\mathbb{T}^X$ .Then $\mathcal{E}_\delta = \{E_f | f \in \mathcal{D}\}$is a $\mathbb{T}$-Weil quasi uniformity on X.*

Proof. $\mathbb{T}-$WQU$_1$) It is obviously satisfied by (1). $\mathbb{T}-$WQU$_2$) Let $E_e \in \mathcal{E}\delta$ we can take $f \in \mathcal{D}$ such that $f^3 \leq e$. By Lemma 4.4.3, we have $E_f o E_f = (\cup_{\alpha\in U_f} \alpha \oplus \alpha) o (\cup_{\alpha\in U_f} \alpha \oplus \alpha)$. Let $(a,c) \in E_f o E_f$ then $(a,b) \leq (\alpha,\alpha)$ and $(b,c) \leq (\beta,\beta)$ where $\alpha, \beta \in U_f$ then $a < \alpha < st(\alpha, E_f), c < B < st(\alpha, E_f)$ we prove $st(\alpha, E_f)$ is e-small.

Let $\lambda \wedge st(\alpha, E_f) \neq \underline{0}, (\gamma,\gamma) \in E_f$ with $\gamma \wedge \alpha \neq \underline{0}$ and $\gamma \wedge \lambda \neq \underline{0}$ then α, γ is f-smallness then $\gamma < f(\lambda), \alpha < f(\gamma)$ then $\alpha < f^2(\lambda)$. Therefore, for every $(\gamma', \gamma') \in E_f$ such that $\gamma' \wedge \alpha \neq \underline{0}$ we have $\gamma' \leq f(\alpha) < f^3(\lambda) \leq e(\lambda)$. Then $st(\alpha, E_f)$ is e-small then $st(\alpha, E_f) \oplus st(\alpha, E_f) \in E_e$ so $(a,c) \in E_e$. By $\mathbb{T}-$WQU$_1$and $\mathbb{T}-$WQU$_2$,$\mathcal{E}_\delta$is $\mathbb{T}-$Weil quasi-uniformity on X. □

Corollary 4.7. *by theorem [4.6] can be restated as "every L-fts is fuzzy weil quasi-uniformizable'.*

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Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
30 August - 02 September 2021

پنجاه و دومین
کنفرانس ریاضی ایران
52nd Annual Iranian Mathematics Conference

دانشگاه شهید باهنر کرمان
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۸ تا ۱۱ شهریور ۱۴۰۰

The application of Hybrid methods between Tensor and Manifold theories in the Image Processing

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Abstract. Tensors as vector fields structures and manifolds as great geometrical-topological structures have many applications in the fields of big data, such as tensorial analysis methods, geometric and topological data analysis. Based on the types of norms, metrics, and scalable structures that have been defined on the data space, different methods could be defined for various data analysis purposes. Nowadays, the hybrid methods between tensorial algorithms and manifold learning (MaL) methods have been attracted some attention. In image and signal processing, from image recovery to face recognition, these methods have appeared very excellent. According to our experiments by **MATLAB R2020b**, the hybrid algorithms are powerful other than algorithms based on the universal popular parameters for comparing.

Keywords: Image Processing, Manifolds, Manifold Learning, Tensors, Tensor Completion

AMS Mathematics Subject Classification [2010]: 15A69, 58J60, 68T07

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1. Introduction

Matrix and tensor completion methods have many applications in various fields of big data analysis, prediction based on collected data, image processing, and computer vision. Incomplete, distorted, and noisy data has always been a major challenge in the field of big data analysis, especially image processing [5]. This problem appears in digital images as a variety of noise and image distortion. Matrix and tensor completion methods have the ability to compensate to a significant degree (up to 90 percent distortion) [10]. On the other hand, manifold learning methods based on a manifold theory with the ability to reduce the dimension and eliminate noise and outliers data, significantly increase computational efficiency [4]. The use of hybrid methods has become very common today. By combining these large mathematical structures in geometry (manifolds) and algebra (tensors), advanced and precise methods can be achieved that, while having high efficiency with significant detection and recovery rates, also have high computational efficiency [6].

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2. Notations and Preliminaries

In this section, we briefly state some preliminaries for tensor calculus and tensor completion. For more details and information, please refer to [5], and [10].

Definition 2.1. A tensor is a multidimensional array, The dimensionality of it is described as its order. An Nth-order tensor is an N-way array, also known as N-dimensional or N-mode tensor, denoted by X. We use the term order to refer to the dimensionality of a tensor (e.g., Nth-order tensor), and the term mode to describe operations on a specific dimension (e.g., mode-n product) [1]. We denote the set of all n-dimensional tensors of order m by $T_{m,n}$. For tensor A, if all of $a_{i_1,\dots,i_n}$ are invariant under any permutation of indices, then A is called a symmetric tensor. We show the set of all real n-dimensional symmetric tensors of order m with $S_{m,n}$.

Definition 2.2. The inner product of two tensors X and Y of the same size is defined as $< X, Y >$. Unless otherwise specified, we treat it as dot product defined as follows [9]:

$$< X, Y > := \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \cdots \sum_{i_N=1}^{I_N} X_{i_1,i_2,\cdots,i_N} y_{i_1,i_2,\cdots,i_N} \tag{1}$$

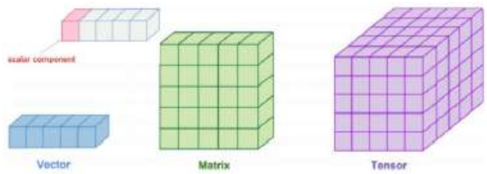

Figure 1. The representation of tensors.

Definition 2.3. Generalized from matrix Frobenius norm, the F-norm of a tensor X is defined as [5]:

$$||X||_F := \sqrt{< X, X >} = \sqrt{\sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \cdots \sum_{i_N=1}^{I_N} X^2_{i_1,i_2,\cdots,i_N}} \tag{2}$$

Definition 2.4. The well-known optimization problem for matrix completion as follows:

$$\begin{aligned} &Min_X : \frac{1}{2}||X - M||^2_\Omega \\ &s.t. rank(X) \leq r, \end{aligned}$$

where $X, M \in \mathbb{R}^{p\times q}$, and the elements of M in the set Ω are geven while the remaining are missing. We aim to use a low rank matrix X to approximate the missing elements [2].

Definition 2.5. The tensors is the generalization of the matrix concept. Given a low-rank tensor T with missing entries, the goal of completing it can be formulated as the following optimization problem [3]:

$$\begin{aligned} &Min_X : \frac{1}{2}||X - Y||^2_F \\ &s.t. ||X|| \leq c \end{aligned}$$

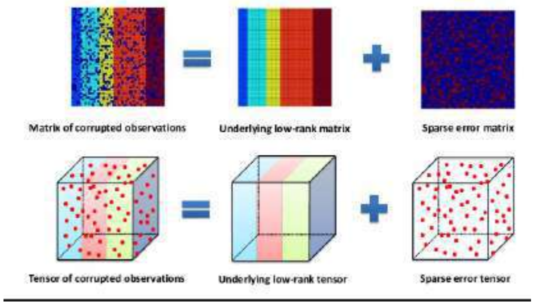

FIGURE 2. The comparision scheme of matrix completion vs tensor completion.

$$Y_\Omega = T_\Omega$$

where X, Y, T are n-mode tensors with identical size in each mode. Figure 2 shows the comparision between matrix and tensor completion problems.

DEFINITION 2.6. A manifold is a Hausdorff topological space which looks locally like a finite-dimensional Cartesian space $\mathbb{R}^n$, a topological space in which case one speaks of a manifold of dimension n or n-fold, but possibly an infinite-dimensional topological vector space, in which case one has an infinite-dimensional manifold [7]. The circle and torus are simple 1-and 2-dimensional manifolds. The topological manifold of M is called smooth (differentiable) if M has continuous differentials. In fact, the topological manifold C^0 is continuous and the topological manifold whose derivatives of any order is continuous order, called C^∞ or smooth.

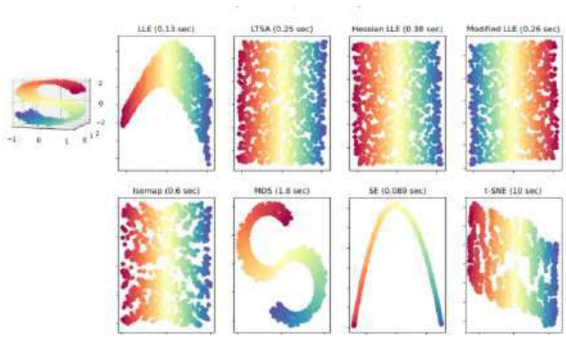

FIGURE 3. The comparision scheme of some manifold learning methods with 1000 points.

3. Main Results

Dimensionality reduction is the transformation of high-dimensional data into a meaningful representation of reduced dimensionality [4]. Ideally, the reduced representation should have a dimensionality that corresponds to the intrinsic dimensionality of the data. The intrinsic dimensionality of data is the minimum number of parameters needed to account for the observed properties of the data [6]. Traditionally, dimensionality reduction was performed using linear techniques such as Principal Components Analysis (PCA), factor analysis, classical scaling, and t-SNE [4]. However, these linear techniques cannot adequately handle complex nonlinear data. Thus, Manifold Learning methods have emerged. Manifold learning is an approach to non-linear dimensionality reduction. Algorithms for this task are based on the idea that the dimensionality of many data sets is only artificially high, For more details, see figure 3 [6]. In less than 20 years, with the development of dimensionality reduction methods, manifold's theory has been widely used in the field of artificial intelligence and has led to the discovery of a new concept called manifold learning. The main idea is that the dimension of the data set or space is artificially high and with appropriate geometric methods, a low-dimensional manifold can be achieved that contains valuable and important information of the original data space (Whitney Theorem). This embedded manifold is called the Whitney manifold. The main goal of manifold learning methods is to reduce the dimension and increase computational efficiency. The concept of a tensor is also presented in the form of a tensor field, so the combination of tensor methods and manifold learning methods in recent years is very much in the spotlight and promises the emergence of faster and more efficient methods for processing all types of big data, especially high-resolution images. The format of digital images and videos has been changed. In the field of applications of manifold learning methods, we can mention handwriting manifold learning through LLE or Isompe methods (in general, Isompe is one of the most basic methods for manifold learning, which can be considered as MDS and PCA expansion while maintaining geodesic distances between points). Application in image processing in the stages of recovery and recognition in medical images such as brain MRI, face recognition, and high ability to reconstruct human face images is also one of the important applications.

4. Conclusion

In this paper, we have investigated new computational methods that lead to advanced hybrid algorithms for the registration, reconstruction, recovery, and recognition of objects and human images (face detections). These methods integrate and combine two powerful objects in mathematics i.e. tensors and manifolds. According to our studies and experiments, in addition to computational savings due to reduced dimensions, these methods have high detection and recognition rates more than 90 percent and even up to 99 percent in some cases. On the other hand, these methods have suitable computational costs and more efficient than other methods. The combination of conventional linear dimensional reduction methods such as PCA and LDA with tensors and the development of new algorithms such as MPCA and MLDA is a testament to this claim. In any type of problem, depending on the case study conditions such as type of images or data (structured, semi-structured, or unstructured), by choosing the appropriate tensor analysis method, multiplication and metric, the type of optimization method depending on equal or unequal constraints of the problem, or convexity or concavity, the best method, and algorithm for achieved a better result, should be selected. Finally, the hybrid between tensors and manifolds methods result in efficient and hopeful methods for big data analysis, especially digital images.

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Paper

Talk (continued)

پنجاه و دومین کنفرانس ریاضی ایران
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۸ تا ۱۱ شهریور ۱۴۰۰

A global optimal solution for the weighted power mean programming problem constrained with fuzzy relational equalities

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Abstract. In this paper, a non-linear programming problem is investigated in which the objective function is defined by the weighted power mean and the feasible region is formed as a special type of fuzzy relational equalities. In this type of fuzzy relational equalities, fuzzy composition is considered as the weighted power mean operator. Some theoretical properties of the feasible region are described. Based on the structural properties of the problem, it is proved that the maximum solution of the feasible region is the unique optimal solution for the problem and finally, an example is presented to illustrate the proposed method.

Keywords: Fuzzy relational equalities, mean operators, weighted power mean, fuzzy compositions.

AMS Mathematics Subject Classification [2010]: 90C26

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1. Introduction

The Resolution of fuzzy relational equations (FRE) with max-min composition was first studied by Sanchez [6]. Besides, Sanchez developed the application of FRE in medical diagnosis in biotechnology. In addition to the preceding application, FRE theory has been applied in many fields, including fuzzy control, discrete dynamic systems, prediction of fuzzy systems, fuzzy decision making, fuzzy pattern recognition, fuzzy clustering, image compression and reconstruction, fuzzy information retrieval, and so on. The solution set of FRE is often a non-convex set that is completely determined by one maximum solution and a finite number of minimal solutions [1]. This non-convexity property is significant bottleneck making major contribution to the increase of complexity in problems that are related to FRE, especially in the optimization problems subjected to a system of fuzzy relations. The optimization problems with general nonlinear objective functions and FRE or FRI constraints were studied in [2-5]. In general, a heuristic algorithm was applied to deal with this kind of problems. However, some fuzzy relation nonlinear optimization problems could be solved by some specific method. In this paper, we study the

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following non-linear optimization problem in which the objective function is defined as the weighted power mean function and the constraints are formed as the fuzzy relational equalities:

$$
\begin{aligned}
& \max \ \textstyle\sum_{j=1}^{n}(w c_j^p + (1-w)x_j^p)^{1/p} \\
& A \odot x = b \\
& x \in [0,1]^n
\end{aligned} \tag{1}
$$

where $I = \{1,2,...,m\}$, $J = \{1,2,...,n\}$, $0 \le w \le 1$ and $p > 0$. Also, $A = (a_{ij})_{m\times n}$, $0 \le a_{ij} \le 1$ ($\forall i \in I$ and $\forall j \in J$), is a fuzzy matrix, $b = (b_i)_{m\times 1}$, $0 \le b_i \le 1$ ($\forall i \in I$), is an m–dimensional fuzzy vector, $0 \le c_j \le 1$ ($\forall j \in J$) are fuzzy cost coefficients, and "$\odot$" is the max-weighted power mean composition, that is, $x \odot y = (wx^p + (1-w)y^p)^{1/p}$. If a_i is the i'th row of matrix A, then the constraints can be expressed as $a_i \odot x = b_i (i \in I)$, where $a_i \odot x = \max_{j\in J}\{a_{ij} \odot x_j\}$ and $a_{ij} \odot x_j = (w a_{ij}^p + (1-w)x_j^p)^{1/p}$.

2. Unique optimal solution for problem (1)

In this section, it is proved that problem (1) has indeed the unique optimal solution. Denote $S(A,b) = \{x \in [0,1]^n \ : \ A \odot x = b\}$, that is, set $S(A,b)$ represents the solution set of system (1). Similar to the general mathematical programming problem, a vector x satisfying the constraints in (1), i.e., $x \in S(A,b)$, is said to be a feasible solution of problem (1).

Definition 1. A solution $\bar{x} \in S(A,b)$ is said to be the maximum solution of $S(A,b)$ when $x \le \bar{x}$ for all $x \in S(A,b)$.

Lemma 1. Let $i \in I$, $j_0 \in J$ and $a_{ij_0} > \frac{b_i}{\sqrt[p]{w}}$. Then, $a_i \odot x > b_i$.

Proof. Since the operator $\odot$ is an increasing function on $[0,1]^2$ in both variables, we note that $a_{ij_0} \odot x_j > \left(b_i \big/ \sqrt[p]{w}\right) \odot x_j = (b_i^p + (1-w)x_{j_0}^p)^{1/p} \ge b_i$. Thus, for each $x \in [0,1]^n$ we have $a_i \odot x = \max_{j\in J}\{a_{ij} \odot x_j\} \ge a_{ij_0} \odot x_{j_0} > b_i$. □

Lemma 2. Let $a_{ij_0} \le \frac{b_i}{\sqrt[p]{w}}$ for some $i \in I$ and $j_0 \in J$. If $b_i^p \ge 1-w$ and $a_{ij_0} < [(b_i^p + w - 1)/w]^{1/p}$, then $a_{ij_0} \odot x_{j_0} < b_i$, $\forall x_{j_0} \in [0,1]$.

Proof. Since $b_i^p \ge 1-w$, then $[(b_i^p + w - 1)/w]^{1/p} \ge 0$. Now, the result follows from the relations $a_{ij_0} \odot x_{j_0} < [(b_i^p + w - 1)/w]^{1/p} \odot 1 = b_i$.□

Lemma 3. Let $a_{ij_0} \le \frac{b_i}{\sqrt[p]{w}}$ for some $i \in I$ and $j_0 \in J$. Also, suppose that either $b_i^p < 1-w$ or $a_{ij_0} \ge [(b_i^p + w - 1)/w]^{1/p}$. Then, $x_{j_0} = [(b_i^p - w a_{ij_0}^p)/(1-w)]^{1/p}$ is the unique solution to the equality $a_{ij_0} \odot x_{j_0} = b_i$.

Proof. It is easy to verify that $a_{ij_0} \odot x_{j_0} = b_i$. Now, Since the operator $\odot$ is an increasing function , we have $a_{ij_0} \odot x_j > b_i$ if $x_j > [(b_i^p - w a_{ij_0}^p)/(1-w)]^{1/p}$ and $a_{ij_0} \odot x_j < b_i$ if $x_j < [(b_i^p - w a_{ij_0}^p)/(1-w)]^{1/p}$. □

Definition 2. A solution For an arbitrary fixed $i \in I$, let $J^{-}(i) = \{j \in J \ : \ a_{ij} > b_i/\sqrt[p]{w}\}$. Additionally, define $J^{\infty}(i) = \left\{j \in J \ : \ b_i^p \geq 1 - w \ , \ a_{ij} < [(b_i^p + w - 1)/w]^{1/p}\right\}$ and $J(i) = J - \{J^{-}(i) \bigcup J^{\infty}(i)\}$.

According to Lemmas 1-3, the following corollary is directly attained. This corollary characterizes all the feasible solutions to the equation $a_i \odot x = b_i$.

Corollary 1. x' is a solution for $a_i \odot x = b_i$ if and only if $J^{-}(i) = \emptyset$, $J(i) \neq \emptyset$ and
(a) $x'_j \in [0,1]$, $\forall j \in J^{\infty}(i)$.
(b) $x'_j \leq [(b_i^p - w a_{ij}^p)/(1-w)]^{1/p}$, $\forall j \in J(i)$.
(c) There exist at least some $j_0 \in J(i)$ such that $x'_{j_0} = [(b_i^p - w a_{ij_0}^p)/(1-w)]^{1/p}$.

Definition 3. Let $\bar{x}(i) \in [0,1]^n$ such that

$$\bar{x}(i)_j = \begin{cases} [(b_i^p - w a_{ij}^p)/(1-w)]^{1/p} & , if \ j \in J(i) \\ 1 & , if \ j \in J^{\infty}(i) \end{cases}$$

Theorem 1. $\bar{x}(i)$ is the unique maximum solution to the equation $a_i \odot x = b_i$.
Proof. Based on Corollary 1, $\bar{x}(i)$ satisfies $a_i \odot x = b_i$. Suppose that x' is a feasible solution for $a_i \odot x = b_i$. So, from Corollary 1, $x'_j \leq [(b_i^p - w a_{ij}^p)/(1-w)]^{1/p}$, $\forall j \in J(i)$, and $x'_j \leq 1$, $\forall j \in J^{\infty}(i)$. Therefore, $x'_j \leq \bar{x}(i)_j$, $\forall j \in J$. □

Definition 4. Let $\bar{x}(i)$ be as in Definition 2, $\forall i \in I$. We define $\bar{x} = \min\limits_{i \in I} \{\bar{x}(i)\}$.
Theorem 2. $\bar{x}$ is the maximum solution of $S(A,b)$.
Proof. By contradiction, suppose that $x' \in S(A,b)$ and $\bar{x}_j < x'_j$ for some $j \in J$. Based on Definitions 3 and 4, either $\bar{x}_j = 1$ or $\bar{x}_j = [(b_i^p - w a_{ij}^p)/(1-w)]^{1/p}$ for some $i \in I$ such that $j \in J(i)$. Therefore, since $x'_j \in [0,1]$, then we must have $[(b_i^p - w a_{ij}^p)/(1-w)]^{1/p} < x'_j$ for some $i \in I$ such that $j \in J(i)$. But, in this case, Corollary 1(b) implies that x' violates $a_i \odot x = b_i$, i.e., $x' \notin S(A,b)$. □

Theorem 3. $\bar{x}$ is the global optimal solution for problem (1).
Proof. The result follows from Theorem 2 and the fact that operator $\odot$ is an increasing function on $[0,1]^2$. □

3. Numerical results

Consider the following non-linear programming problem (1):

$$\min \ Z = -0.6582x_1 - 0.029x_2 + 0.6277x_3 - 0.3x_4 + 0.0157x_5 - 0.4737x_6 + 0.2926x_7$$

$$\begin{bmatrix} 0.6763 & 0.8969 & 0.8403 & 0.3000 & 0.0710 & 0.0758 & 0.3529 \\ 0.3362 & 0.2721 & 0.1956 & 0.3396 & 0.0101 & 0.2557 & 0.1193 \\ 0.1637 & 0.5426 & 0.2534 & 0.3701 & 0.4916 & 0.5761 & 0.2454 \\ 0.5161 & 0.1330 & 0.9090 & 0.1477 & 0.3827 & 0.7212 & 0.2452 \\ 0.2319 & 0.8371 & 0.1275 & 0.8609 & 0.5201 & 0.6163 & 0.0654 \end{bmatrix} \odot x = \begin{bmatrix} 0.8657 \\ 0.6520 \\ 0.6926 \\ 0.8833 \\ 0.8350 \end{bmatrix}$$

$$x \in [0,1]^7$$

where $w = 3/4$ and $p = 3$. Moreover, $c_1 = -7.6582$,$c_2 = -2.029$,$c_3 = 6.6277$,$c_4 = -6.3$,$c_5 = 0.0157$,$c_6 = -7.4737$ and $c_7 = 7.2926$. For each $i \in I$, we have $J^-(i) = \emptyset$. Also, $J(1) = \{2,3\}$, $J(2) = \{1,4\}$, $J(3) = \{2,5,6\}$, $J(4) = \{3\}$ and $J(5) = \{2,4\}$. According to Definition 3, the maximum solutions of $a_i \odot x = b_i$, $\forall i \in I$, are attained as follows:
$\bar{x}(1) = [1,\ 0.7552\ ,\ 0.9341\ ,1,1,1,1]$, $\bar{x}(2) = [0.9982\ ,1,1,\ 0.9970\ ,1,1,1]$,
$\bar{x}(3) = [1\ ,\ 0.9471\ ,\ 1\ ,\ 1\ ,\ 0.9908\ ,\ 0.9107\ ,\ 1]$, $\bar{x}(4) = [1\ ,\ 1\ ,\ 0.7955\ ,\ 1\ ,\ 1\ ,\ 1\ ,\ 1]$ and $\bar{x}(5) = [1\ ,\ 0.8286\ ,\ 1\ ,\ 0.7456\ ,\ 1\ ,\ 1\ ,\ 1]$. Hence, by Definition 4 and Theorem 3, the global optimal solution is obtained as follows:

$$\bar{x} = [0.9982\ ,\ 0.7552\ ,\ 0.7955\ ,\ 0.7456\ ,\ 0.9908\ ,\ 0.9107\ ,\ 1].$$

4. Conclusion

In this paper, we introduced a non-linear weighted power mean optimization model constrained with fuzzy relational equalities. These fuzzy equalities were defined by the weighted power mean operator. It is proved that the maximum solution of the feasible region is the unique global solution of the problem.

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Paper

Poster (continued)

A method to construct a homomorphism on Hilbert modules

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Abstract. Let A be a C^*-algebra, $\varphi : A \to A$ be a linear homomorphism and e be an element in a Hilbert A-module M satisfying $\|e\| = 1$. In this paper, we demonstrate an algebraic structure on M, denoted by π_e under which (M, π_e) is a Banach algebra. Moreover, applying the concepts of φ-module map and φ-morphism, we establish a necessary and sufficient condition for a φ-module map (resp. φ-morphism) on M to be a homomorphism on (M, π_e).

Keywords: Full Hilbert C^*-module; generalized module map.

AMS Mathematics Subject Classification [2010]: Primary: 47L08, Secondary: 47C05.

1. Introduction

A *left pre-Hilbert module* M over C^*-algebra A is a left A-module equipped with an A-valued inner product denoted by $< ., . >$ from $M \times E$ to M such that for every $x, y, z \in M$, $\lambda \in \mathbb{C}$ and $a \in A$, satisfies the following conditions:

(i) $< x, x > \geq 0$;
(ii) $< x, x >= 0$ if and only if $x = 0$;
(iii) $< x + \lambda y, z >=< x, z > +\lambda < y, z >$;
(iv) $< x, y >=< y, x >^*$;
(v) $< ax, y >= a. < x, y >$.

A left pre-Hilbert A-module M is called a *left Hilbert A-module* or *left Hilbert C^*-module over C^*-algebra* A, if it is complete with respect to the norm $\|x\| = \| < x, x > \|^{\frac{1}{2}}$.

The Hilbert A-module M is called *full* if $A_M := span\{< x, y >: \quad x, y \in M\}$ is dense in A. Note that A_M is an ideal in A, called the *range ideal* of M. We denote by $< M, M >$ the closure of A_M and call it the *support* of M. Therefore, M is a full Hilbert A-module if $< M, M >$ is equal to A.

As an example, suppose that A is a C^*-algebra and take $M := A$. Then, A with respect to its product as the usual action, is a left A-module. Additionally, A equipped with the inner product $< a, b >= ab^*$ is a Hilbert$A$-module. Additionally, if $a \in A$ and $\{e_\lambda\}$ is an approximate identity for A, then $\lim_\lambda < a, e_\lambda >= \lim_\lambda ae^*_\lambda = a$. Hence $< M, M >= A$ and therefore M is a full Hilbert A-module.

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Hilbert C^*-modules are a generalization of Hilbert spaces, but there are some differences between these two classes. For more information on Hilbert C^*-modules the reader is referred to [2].

In the remainder of this section, we introduce two important classes of operators between Hilbert modules.

DEFINITION 1.1. Let A be a C^*-algebra and $\varphi : A \to A$ be a linear homomorphism.

(i) A linear mapping $T : M \to M$ is called a *φ-module map* if $T(ax) = \varphi(a)T(x)$ for all $a \in A$, $x \in M$.

(ii) A linear mapping $T : M \to M$ is called a *φ-morphism* if $< T(x), T(y) >= \varphi(< x, y >)$, for all $x, y \in M$.

It is easy to see that if T is a φ-morphism, then T is necessarily a linear operator and a φ-module map. Also applying the polarization identity one immediately conclude that T is a φ-morphism if and only if $< T(x), T(x) >= \varphi(< x, x >)$, for all $x \in M$. The notion of φ-morphism was introduced by Bakic and Guljas in 2002. The reader is referred to [1] for more details.

REMARK 1.2. Let $T : M \to M$ be a $\varphi-$ morphism. It is known from [1], that T is a contraction and ImT is a closed subspace of M. Also it is a Hilbert C^*-module over C^*-algebra $Im\varphi$ such that $< ImT, ImT >= \varphi(< M, M >)$.

EXAMPLE 1.3. Let A be a C^*-algebra and consider A as a Hilbert A-module. Suppose that $\varphi : A \to A$ is a linear $*$-endomorphism and u is an arbitrary unitary element in A. Define $T : A \to A$ by $T(a) := \varphi(a)u$. Then, T is a φ-morphism. Moreover, T is a Surjection if and only if φ is an epimorphism.

We end this section with the following lemma which can be found in [3].

LEMMA 1.4. *Let M be a full Hilbert A-module and $a \in A$. Then, $ax = 0$ for all $x \in M$ iff $a = 0$.*

2. A method to construct a homomorphism on Hilbert modules

Throughout this section, A is a C^*-algebra. The following theorem manifests an algebraic structure on a Hilbert A-module M under which M is a Banach algebra.

THEOREM 2.1. *Suppose that M is a left full Hilbert A-module. If there exists an element e $\in M$ such that $\|e\| = 1$, then the mapping $\pi : M \times M \to M$ defined by $\pi_e(x, y) :=< x, e > .y$ is a product on M which making it into a Banach algebra. We denote by (M, π_e) the aforementioned Banach algebra [4].*

THEOREM 2.2. *Let $\varphi : A \to A$ be a linear homomorphism, M be a full Hilbert A-module and let $T : M \to M$ be a surjective φ-module map. Then T is a homomorphism on (M, π_e) if and only if $\varphi(< x, e >) =< T(x), e >$ for all $x \in M$.*

THEOREM 2.3. *Let $\varphi : A \to A$ be a linear homomorphism, M be a full Hilbert A-module and let $T : M \to M$ be a surjective φ-morphism. Then T is a homomorphism on (M, π_e) if and only if $T(e) = e$.*

COROLLARY 2.4. *Let e be an element in A such that $\|e\| = 1$ and $\varphi : A \to A$ be a surjective linear homomorphism. Then φ is a homomorphism on (A, π_e) if and only if $\varphi(e^*) = e^*$.*

EXAMPLE 2.5. Let e be a unitary element of A and $\varphi : A \to A$ be the inner automorphism $\varphi(a) := eae^*$. Then, φ is a surjection satisfying $\varphi(e^*) = e^*$ and by the previous corollary φ is a homomorphism on (A, π_e).

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