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Extended Abstracts

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A new method for low rank tensor completion by fast tri-factorization

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Abstract

One of the new methods of data recovery is the tensor completion problem. Since most of the data such as images and videos and numerical data are in the form of tensors, tensor completion methods for information recovery are of great importance and attractiveness in this field. The purpose of low-rank tensor completion is to recover lost information so that the tensor rank is minimized. So far, various methods have been proposed to solve the tensor completion problem, among which methods based on the nuclear norm are of particular importance due to their convexity. However, methods based on nuclear norm have a high computational complexity, since the calculation of the singular value decomposition in each iteration of the algorithm is needed. For this reason, in order to reduce the computational cost and increase the convergence speed of the answer, the fast tri-factorization method of matrix completion cen be used. In this work, we generalize the fast tri-factorization method of matrix completion to the high order arrays.

Keywords: Matrix Completion, Tensor Completion, Image Recovery, Optimization Problems

Mathematics Subject Classification [2010]: 65F99; 15A69; 58C05

1 Introduction

Matrix completion, the problem of completing missing entries in a data matrix with low-dimentional structure (such as rank), has seen many fruitful approaches and analyses. Matrix and tensor completion methods have many applications in various fields of big data analysis, prediction based on collected data, image processing, and computer vision. Incomplete, distorted, and noisy data has always been a major challenge in the field of big data analysis, especially image processing [4]. Thtoughout this paper, we show the matrix by the capital letters like W and tensors by letter like $\mathbb{T}$.

Definition 1.1. A tensor is a multidimensional array. The dimension of a tensor is called the order (also called the mode). The space of real tensors of order n and size $d_1 \times ... \in d_n$ is denoted as $\mathbb{R}^{d_1 \times ... \in d_n}$. The elements of a tensor $\mathbb{X} \in \mathbb{R}^{d_1 \times ... \in d_n}$ are denoted by $\mathbb{X}_{i_1 ... i_n}$ [2].

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An n-order tensor $\mathbb{X}$ can be matricized in n ways by unfolding it along each of the n modes. The definition for the matricizationof a given tensor is stated below.

Definition 1.2. The mode-k *matricization/unfolding* of a tensor $\mathbb{X} \in \mathbb{R}^{d_1 \times \cdots \in d_n}$, denoted as $X_{(n)}$, is the marix, which is denoted as $\mathbb{X}_{(k)} \in \mathbb{R}^{d_1 \times \Pi_{j \neq k}}$, whose columns are composed of all the vectors obtained from $\mathbb{X}$ by fixing all indices except for the k-th dimension. The mapping $\mathbb{X} \longmapsto \mathbb{X}_k$ is called the mode-k unfolding operator.

The precise order in which fibers are stacked as columns is not important as long as it is consistent. Figure 1 shows the fibers of 3-tensor. *The folding* is the inverse operation of matricization/unfolding.

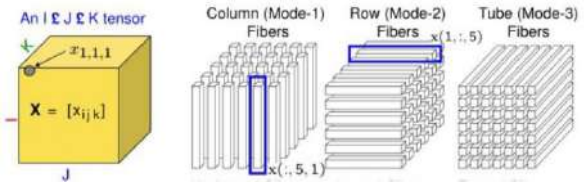

Figure 1: Fibers of a tensor from rank 3.

Definition 1.3. In the general, The *Singular Value Decomposition (SVD)* is a factorization of a real or complex matrix that generalizes the eigen decomposition, which only exists for square normal matrices to any $m \times n$ matrix via an extension or the polar decomposition. In the tensor calculus, similar concepts proposed as follows:

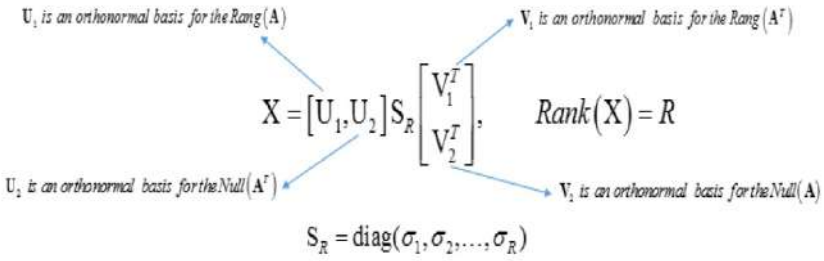

Figure 2: Singular value decomposition.

2 Main results

In this section, we first introduce the tri-factorization matrix completion problem and then we extend to the tensor case. As we will see, the straightforward extension leads to a very complex optimization problem. The matrix completion problem is one of the most important rank minimization applications, in which we would like to recover a low-rank matrix from incomplete samples of its entries. Such a model is applicable in factor analysis collaborative filtering, latent semantic analysis, global positioning and system identification. The general matrix completion problem can be formulated as follows:

$$\begin{cases} \min\limits_{Z} & rank(Z) \\ s.t. & P_\Omega(Z - W) = 0 \end{cases} \tag{1}$$

Where $rank(Z)$ represents the rank of the desired matrix $Z \in \mathbb{R}^{m\times n}$, Ω is the indices set of known entries of W, and $P_\Omega(.)$:

$$P_\Omega(A_{ij}) = \begin{cases} A_{ij} & (i,j) \in \Omega \\ 0 & otherwise \end{cases}$$

It is well known that the ℓ_1 norm or the $\ell_{2,1}$ norm is powerfully capable of inducing sparsity. Also, rank minimization has also been proven to be a strong global constraint and good measure of 2D sparsity. Unfortunately, the optimization problem mentioned above in (1) is generally NP-hard due to the discrete nature of the rank function. This rank minimization problem can be solved by convex relaxation replacing the rank of the matrix with the nuclear norm, also known as the trace norm. Moreover, this relaxation technique achieves a convex optimization problem, which is tractably solved.
In this work, we mainly consider the nuclear (trace) norm minimization model corresponding to low-rank tensor recovery and completion problem. This model can be reformulated as follows:

$$\begin{cases} \min\limits_{Z} & \|Z\|_* \\ s.t. & P_\Omega(Z - W) = 0 \end{cases} \tag{2}$$

where $\|.\|_*$ denotes the nuclear norm of a matrix, that is the sum of its singular values.

Though the nuclear norm minimization problem in (2) is convex, while the size of matrix Z is large, the multiple SVDs computation of this matrix is very expensive. To mitigate SVD computation for nuclear norm minimization problems, is present a novel fast tri-factorization method to approximate nuclear norm minimization problems. Motivated by recent progress in matrix factorization and kernel learning, we first introduce a matrix tri-factorization idea into the nuclear norm regularized framework. We should point out that the idea of using matrix tri-factorization has also appeared in the literature. Then, we perform the matrix tri-factorization $Z = LMR$, where $L \in \mathbb{R}^{m\times r}$ and $R \in \mathbb{R}^{r\times n}(r \ll m, n)$, and convert the original problem into a matrix nuclear norm minimization problem on the small scale matrix $M \in \mathbb{R}^{r\times r}$. Moreover, is presented the specific model to approximately solve the corresponding nuclear norm minimization problem, and develop the iterative schemes under an alternating direction method (ADM) framework to solve the corresponding problem. also analyze the connections between the proposed model and the corresponding nuclear norm minimization problem.
Lemma 1. Let L, R and M be given matrices of compatible dimensions, and suppose both L and R have orthogonal columns and orthogonal rows, respectively, i.e., $L^TL = I$ and $RR^T = I$, then we have : $\|M\|_* = \|LMR\|_*$.
According to the above lemma, it is clear that $\|Z\|_* = \|LMR\|_* = \|M\|_*$. Therefore the nuclear norm minimization problem for the matrix Z is reduced a much smaller matrix M. Substituting $Z = LMR$ into (2) and simplifying, we arrive at the following model for this nuclear norm minimization problem:

$$\begin{cases} \min\limits_{L,M,R} & \|M\|_* \\ s.t. & P_\Omega(LMR - W) = 0 \\ & L^TL = I, RR^T = I. \end{cases}$$

In the following, we will discuss how to solve the proposed model corresponding to this problem. we introduce an auxiliary variable Z and reformulate it as follows:

$$\begin{cases} \min\limits_{Z,L,M,R} & \|M\|_* \\ s.t. & P_\Omega(Z - W) = 0 \\ & Z = LMR \\ & L^TL = I, RR^T = I. \end{cases}$$

And we can get the following partial augmented Lagrange function:

$$\mathbf{L}(Z, L, M, R, Y, \mu) = \|M\|_* + \langle Y, Z - LMR\rangle + \frac{\mu}{2}\|Z - LMR\|_F^2. \tag{3}$$

The optimal solution of the problem (3) with respect to the variables M, Z, L, R and Y is respectively derive by:

$$\begin{aligned} M &= SVT_{\frac{1}{\mu}}(L^T(Z + \tfrac{Y}{\mu})R^T) \\ Z &= LMR - \tfrac{Y}{\mu} + P_\Omega(W - LMR + \tfrac{Y}{\mu}) \\ L &= QR((Z + \tfrac{Y}{\mu})R^T) \\ R &= QR(L^T(Z + \tfrac{Y}{\mu})) \\ Y &= P_\Omega(Y + \mu(Z - LMR)) \end{aligned}$$

Tensor completion is the problem of filling in the missing elements of partially observed tensors. Let $\mathbb{T}$ is the N-modes tensor. For tensor completion problem, we have the follow optimization problem:

$$\begin{cases} \min\limits_{\mathbb{Z}} & rank(\mathbb{Z}) \\ s.t. & P_\Omega(\mathbb{Z} - \mathbb{T}) = 0 \end{cases}$$

By unfolding the tensor along each of its modes, we arrive at N matrix completion problem,i.e.

$$\begin{cases} \min\limits_{Z_i} & rank(Z_i) \\ s.t. & P_\Omega(Z_i - T_i) = 0, \quad i = 1, 2, ..., N. \end{cases}$$

We solve the N preceding optimization problems by using fast tri-factorization method. Then we return the obtained answer matrices to the tensor state. by averaging the obtained tensors in the missing points, we obtain the low-rank tensor completed.
Now we apply the proposed fast tri-factorization method for a picture case: The left hand image of Figure (3) shows the original image. The middle image showes the original image with 90% lost data. The right hand image shows the recovered image by fast tri-factorization method:

Figure 3: Recovery results by the fast tri-factorization method.

3 Conclusion

In this paper, we proposed a fast tri-factorization method to approximate nuclear norm minimization problems. The purpose of low-rank tensor completion is to recover lost information so that the tensor rank is minimized. So far, various methods have been proposed to solve the problem of tensor completion, among which methods based on the nuclear norm are of particular importance due to their convexity. we generalized the fast tri-factorization method of matrix completion to the tensor case.

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Three-term-tensor Sylvester method for a class of third order tensor linear equations

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Abstract

In this paper, we present a direct dense method called three-term-tensor Sylvester, to obtain the solution to a class of third order tensor linear equations. The proposed method investigate process of solution without the explicit use of Kronecker form that is desirable for low rank tensor equations. Numerical experiments illustrate the properties of the considered algorithm.

Keywords: Tensors, Schur decomposition, Generalized Sylvester matrix equation
Mathematics Subject Classification [2010]: 65F10, 65F30, 15A24

1 Introduction

The literatures on tensors, actually about their analysis and the associated approximation methods has grown tremendously, in the last twenty years. Numerous different decompositions of tensor equations have allowed the developments of various problem dependent strategies, see [2–4] and references therein.

In this paper, we are interested in the computation of the unique solution $\mathcal{X} \in \mathbb{R}^{n\times n\times n}$ to the nonsingular system in the following tensor form

$$(\mathbf{H} \otimes A_1 \otimes M_1 + \mathbf{H} \otimes A_2 \otimes \mathbf{M} + H_3 \otimes A_3 \otimes \mathbf{M})\text{vec}(\mathcal{X}) = b_3 \otimes b_2 \otimes b_1, \tag{1}$$

where all coefficient matrices are real and have the same $n \times n$ dimensions. Here $\otimes$ denotes the Kronecker product (to be recalled later) and $\text{vec}(\mathcal{X})$ stacks the components of the tensor $\mathcal{X}$ one after the other. In particular, in (1) two terms share the same matrices, either $\mathbf{M}$ or $\mathbf{H}$ (purposely in bold face), while all other matrices $A_i, i = 1, 2, 3$ and H_3, M_1 have no relation to each other.

The numerical solution to (1) can be given in closed form by unfolding the 3-mode tensor in one of the three directions. In particular, a tensor $\mathcal{X} \in \mathbb{R}^{n_1\times n_2\times n_3}$ can be written using the mode-1 unfolding as in [2]

$$\mathcal{X}_{(1)} = [X_1, X_2, \ldots, X_{n_3}], \quad X_k = \mathcal{X}_{::k} \in \mathbb{R}^{n_1\times n_2}, \quad k = 1, 2, \ldots, n_3$$

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where each X_k is called a frontal slice of the tensor $\mathcal{X}$, and $\mathcal{X}_{(1)}$ is a matrix in $\mathbb{R}^{n_1 \times n_2 n_3}$. Some additional standard notation needs to be recalled. The Kronecker product of two matrices X and Y is defined in the block form as

$$X \otimes Y = \begin{pmatrix} X_{1,1}Y & \cdots & X_{1,n_2}Y \\ \vdots & \ddots & \vdots \\ X_{n_1,1}Y & \cdots & X_{n_1,n_2}Y \end{pmatrix},$$

where $X_{i,j}$ denotes an element of X. Moreover, $\mathrm{vec}(X)$ is the operator stacking all columns of the matrix X one after the other. In the case of third order tensors $\mathcal{X}$, we will apply the $\mathrm{vec}(\mathcal{X})$ operator to the mode-1 unfolding $\mathcal{X}_{(1)}$, that is

$$\mathrm{vec}(\mathcal{X}) := \mathrm{vec}(\mathcal{X}_{(1)}). \tag{2}$$

The reverse operation, for known dimensions of the vector x, will be denoted by $\mathsf{mat}(x, n_1, n_2)$, so that $x = \mathrm{vec}(X)$and $X = \mathsf{mat}(x, n_1, n_2)$. Similarly, $\mathcal{X} = \mathsf{tensor}_{(1)}(x, n_1, n_2, n_3)$ will fold a long vector x into a tensor $\mathcal{X}$ via the mode-1 unfolding.

A standard property of the Kronecker product that will be used repeatedly is the following

$$\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X), \tag{3}$$

where B^T denotes the real transpose of B. The aforementioned equation (3) allows one to go back and forth between the vector and matrix presentation. Other properties used in the sequel are from [1] as

$$i)\,(A \otimes B)^T = A^T \otimes B^T, \quad ii)\,(A \otimes B)(C \otimes D) = (AC \otimes BD), \quad iii)\,(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

Also, Q^* denotes the conjugate transpose of the complex matrix Q, while $H^{-T} = (H^{-1})^T$.

Least approaches try to take *direct* dense methods for low order tensor equations, without the explicit use of the Kronecker product. Here we concentrate on special case of tensor equation (1) motivated from [5], nonetheless appears to be a feasible algebraic formulation of a quite large set of differential problems.

2 Three-term-tensor Sylvester method

In this section, we use the mode-1 unfolding related to the specific location of the repeated matrices $\mathbf{H}$ and $\mathbf{M}$ in (1). The following Theorem provides the unique solution to (1) based on the Schur decompisition and the solution of a generalized Sylvester matrix equation.

Theorem 2.1. *Let* $(\mathbf{M}^{-1}M_1)^T = QRQ^*$ *be the Schur decomposition of* $(\mathbf{M}^{-1}M_1)^T$*, and* $[\gamma_1, \ldots, \gamma_n] := b_1^T \mathbf{M}^{-T} Q$*. Using the mode-1 unfolding, the solution* $\mathcal{X}$ *to* (1) *is given by*

$$\mathcal{X} = \mathsf{tensor}_{(1)}(\mathrm{vec}([Q^{-T}\hat{Z}_1^T, \ldots, Q^{-T}\hat{Z}_n^T]), n, n, n),$$

where for $j = 1, \ldots, n$*, the matrix* $\hat{Z}_j$ *solves the generalized Sylvester equation*

$$A_3 Z H_3^T + (A_2 + R_{j,j}A_1)Z\mathbf{H}^T = b_2\gamma_j b_3^T - A_1\mathsf{mat}([\hat{z}_1, \ldots, \hat{z}_{j-1}]R_{1:j-1,j}, n, n)\mathbf{H}^T,$$

where $R_{j,j}$ *stands for the diagonal entries of the upper triangular matrix* R*, and* $R_{1:j-1,j}$ *denotes the first* $j-1$ *components of* j*-th column of* R*. We set* $\mathsf{mat}([\hat{z}_1, \ldots, \hat{z}_{j-1}]R_{1:j-1,j}, n, n)$ *to be empty array for* $j = 1$*.*

Proof. Using (2) and (3) for the unfolded tensor we have

$$\mathbf{M}\mathcal{X}_{(1)}(\mathbf{H}\otimes A_2 + H_3\otimes A_3)^T + M_1\mathcal{X}_{(1)}(\mathbf{H}\otimes A_1)^T = b_1(b_3\otimes b_2)^T. \tag{4}$$

For later readability, let us transpose both sides of equation (4) and set $Y = \mathcal{X}_{(1)}^T$. Then we have

$$\begin{aligned}
(\mathbf{H}\otimes A_2 + H_3\otimes A_3)Y\mathbf{M}^T + (\mathbf{H}\otimes A_1)YM_1^T &= (b_3\otimes b_2)b_1^T\\
(\mathbf{H}\otimes A_1)^{-1}(\mathbf{H}\otimes A_2 + H_3\otimes A_3)Y + YM_1^T\mathbf{M}^{-T} &= (\mathbf{H}\otimes A_1)^{-1}(b_3\otimes b_2)b_1^T\mathbf{M}^{-T}\\
(I\otimes A_1^{-1}A_2 + \mathbf{H}^{-1}H_3\otimes A_1^{-1}A_3)Y + YM_1^T\mathbf{M}^{-T} &= (\mathbf{H}^{-1}b_3\otimes A_1^{-1}b_2)b_1^T\mathbf{M}^{-T}
\end{aligned}$$

Using $(\mathbf{M}^{-1}M_1)^T = QRQ^*$ and multiplying the equation by Q from the right, we get

$$(I\otimes A_1^{-1}A_2 + \mathbf{H}^{-1}H_3\otimes A_1^{-1}A_3)YQ + YQR = (\mathbf{H}^{-1}b_3\otimes A_1^{-1}b_2)b_1^T\mathbf{M}^{-T}Q$$

Let $YQ = [\hat{z}_1,\dots,\hat{z}_n]$ and $[\gamma_1,\dots,\gamma_n] := b_1^T\mathbf{M}^{-T}Q$. Thanks to the upper triangular form of R, for the first column $\hat{z}_1$ it holds

$$(I\otimes A_1^{-1}A_2 + \mathbf{H}^{-1}H_3\otimes A_1^{-1}A_3)\hat{z}_1 + \hat{z}_1R_{1,1} = (\mathbf{H}^{-1}b_3\otimes A_1^{-1}b_2)\gamma_1$$

For the subsequent columns $j = 2,\dots,n$, taking into account once again the triangular form of R, we set $w_{j-1} = [\hat{z}_1,\dots,\hat{z}_{j-1}]R_{1:j-1,j}$ so that

$$(I\otimes A_1^{-1}A_2 + \mathbf{H}^{-1}H_3\otimes A_1^{-1}A_3)\hat{z}_j + \hat{z}_jR_{j,j} = (\mathbf{H}^{-1}b_3\otimes A_1^{-1}b_2)\gamma_j - w_{j-1}$$

Let us reshape each $\hat{z}_j$ so that $\hat{Z}_j = \mathsf{mat}(\hat{z}_j, n, n)$. For $j = 1$, we can write

$$(A_1^{-1}A_2 + R_{1,1}I)\hat{Z}_1\mathbf{H}^TH_3^{-T} + A_1^{-1}A_3\hat{Z}_1 = A_1^{-1}b_2\gamma_1b_3^TH_3^{-T}$$

Analogously, for $j = 2,\dots,n$ and letting $W_{j-1} = \mathsf{mat}([\hat{z}_1,\dots,\hat{z}_{j-1}]R_{1:j-1,j}, n, n)$, from above issues, we first obtain

$$(A_1^{-1}A_2 + R_{j,j}I)\hat{Z}_j\mathbf{H}^TH_3^{-T} + A_1^{-1}A_3\hat{Z}_j = A_1^{-1}b_2\gamma_jb_3^TH_3^{-T} - W_{j-1}\mathbf{H}^TH_3^{-T}.$$

Multiplying both sides by A_1 (from the left) and by H_3^T (from the right), the result follows. □

Based on the proof of Theorem 2.1, an explicit constructive way is provided to generate the tensor solution, one slice at the time. The complete procedure is described in the algorithm below, in the following called the Three-Term-Tensor Sylvester ($\mathrm{T}^3-\mathrm{SYLV}$) method.

Algorithm 2.2. $\mathrm{T}^3-\mathrm{SYLV}$

1. **Input:** Matrices $A_1, A_2, A_3, M_1, \mathbf{M}, \mathbf{H}, H_3$ of size $n\times n$, vectors b_1, b_2, b_3 of length n.
2. Compute Q and R such that $(\mathbf{M}^{-1}M_1)^T = QRQ^*$ (Schur decomposition)
3. Compute $g = b_1^T\mathbf{M}^{-T}Q$
4. For $k = 1,\dots,n$

   Set $F = A_3^{-1}b_2g_kb_3^T\mathbf{H}^{-T}$

   If $k > 1$, Set $W_{k-1} = \mathsf{mat}([\hat{z}_1,\dots,\hat{z}_{k-1}]R_{1:k-1,k}, n, n)$ and $F = F - A_3^{-1}A_1W_{k-1}$

   Solve $A_3^{-1}(A_2 + R_{k,k}A_1)\hat{Z} + \hat{Z}H_3^T\mathbf{H}^{-T} = F$ to get $\hat{Z}_k$ and $\hat{z}_k = \mathrm{vec}(\hat{Z}_k)$

   end

5. Set $\mathcal{X} = \mathsf{tensor}_{(1)}(\mathrm{vec}(Q[\hat{Z}_1, \ldots, \hat{Z}_n]^T), n, n, n)$
6. **Output:** $\mathcal{X}$ solution to (1)

In practice, using appropriate transformations, the method is a nested Sylvester solver, which treats one slice at the time, and updates the corresponding coefficient matrix and right-hand side F. The solvability of the Sylvester equations is related to that of the original problem, and in particular to the nonsingularity of $\mathcal{A}$. The algorithm relies on the initial Schur decomposition, which provides robust unitary transformations.

Moreover, for each slice, a matrix Sylvester equation needs to be solved, whose solution also involves the Schur decompositions of the coefficient matrices, as a small scale computation studied in [4]. Indeed, if some of the involved matrices are severely ill-conditioned, the solution may lose accuracy.

3 Numerical experiments

In this section, we report some numerical experiments with the $\mathrm{T}^3 - \textsc{Sylv}$ method. All experiments were performed using MATLAB R2015a on an Intel Core i7 Laptop with 8G RAM.

Example 3.1. To test the efficiency of the proposed method, we consider dense matrices with random entries (taken from a uniform distribution in the interval $(0, 1)$, MATLAB function `rand`) of increasing size n. Also, the same is used for the vectors b_1, b_2 and b_3. We stress that the Kronecker form of the problem would involve a dense matrix $\mathcal{A}$ of size $n^3 \times n^3$, which could not even be stored.

Without difficulty, we observe that the method is able to solve a (random) structured dense problem of size $n^3 = 16,777,216$ in about $CPU_{Time} = 33$ seconds. The CPU times in Table 1 show that the computational cost of the method approximately grows between four and ten times as the dimension n doubles. However, going from n to $2n$, the problem dimension in the full space would grow from n^3 to $2^3 n^3$. Hence, the actual cost appears to grow linearly with n^3. Since data are dense, Gauss elimination on $\mathcal{A}$ would instead require $\mathcal{O}((n^3)^3)$ floating point operations.

Table 1: CPU times of $\mathrm{T}^3 - \textsc{Sylv}$ for increasing dimensions of the coefficient matrices, having uniformly distributed random entries

n	CPU Time
8	1.27e-03
16	4.02e-03
32	2.44e-02
64	2.38e-01
128	2.81e+00
256	3.32e+01

4 Conclusion

We have proposed three-term-tensor Sylvester method for solving a class of third order tensor linear equations. The method relied on the Schur decomposition for solving a generalized Sylvester matrix equation to obtain frontal slices od tensor solution, at each time. In fact, the repeated presence of two matrices in the considered third order tensor linear equations make the most of the method, without Kronecker form.

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Convergence region of the generalized accelerated overrelaxation method for double saddle point problems

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Abstract

This paper is devoted to the generalize accelerated overrelaxation iterative method for solving a class of double saddle point problems. Also, we study convergence region of the proposed method and then some numerical results are given to demonstrate the efficiency of the presented method.

Keywords: AOR iterative method, Saddle point problem, convergence analysis
Mathematics Subject Classification [2010]: 65F08,65F10, 65F50

1 Introduction

We consider a class of double saddle point problems as the following large and sparse form

$$\mathcal{A}\mathbf{u} \equiv \begin{pmatrix} A & B & C \\ -B^T & 0 & 0 \\ -C^T & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \equiv \mathbf{b}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite(SPD) matrix, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{n\times p}$ have full column ranks, $x, b_1 \in \mathbb{R}^n$, $y, b_2 \in \mathbb{R}^m$ and $z, b_3 \in \mathbb{R}^p$. For real eigenvalues of A, we use $\lambda_{min}(A)$ and $\lambda_{max}(A)$ to denote the minimum and maximum eigenvalues of A, respectively. Moreover, the notations $\mathrm{Ran}(A)$ and $\rho(A)$ stand for the range and the spectral radius of A, respectively. For given vectors x, y and z of dimension n, m and p, respectively, $\mathbf{u} = (x; y; z)$ will denote a column vector of dimension $n+m+p$. Linear systems of the form (1) arise from mixed finite element approximation of the potential fluid flow problems ; see [1, 2] and the references therein for detailed descriptions of these problems.

The following Proposition given in [2] represents the necessary and sufficient condition of the invertibility of the coefficient matrix $\mathcal{A}$ in (1).

Proposition 1.1. *Let A be a SPD matrix and assume that B and C have full column ranks. Then a necessary and sufficient condition for the invertibility of the matrix $\mathcal{A}$ in (1) is that* $\mathrm{Ran}(B) \cap \mathrm{Ran}(C) = \{0\}$.

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2 Convergence region of DGAOR method

We propose generalized accelerated overrelaxation(AOR) iterative method for solving (1), based on the following splitting

$$\mathcal{A} = \mathcal{D} - \mathcal{E} - \mathcal{F},$$

where

$$\mathcal{D} = \begin{pmatrix} A & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & D \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} 0 & 0 & 0 \\ B^T & 0 & 0 \\ C^T & 0 & 0 \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} 0 & -B & -C \\ 0 & Q & 0 \\ 0 & 0 & D \end{pmatrix}.$$

Here Q and D are SPD matrices. The iteration matrix of the generalized AOR method is defined as

$$\mathcal{M}_{r,\omega} = \begin{pmatrix} (1-\omega)I & -\omega A^{-1}B & -\omega A^{-1}B \\ \omega(1-r)Q^{-1}B^T & I - r\omega Q^{-1}B^T A^{-1}B & 0 \\ \omega(1-r)D^{-1}C^T & 0 & I - r\omega D^{-1}C^T A^{-1}C \end{pmatrix}. \tag{2}$$

Note that if $\omega = 0$, then the proposed generalized AOR method diverges no matter what value the accelerated parameter r takes. In view of this, we will assume $\omega \neq 0$ and so the generalized AOR method to solve double saddle point system (1) (DGAOR) can be defined by the following form

$$\mathbf{u}^{(k+1)} = \mathcal{M}_{r,\omega}\mathbf{u}^{(k)} + \mathbf{c}, \quad k = 0, 1, 2, \ldots, \tag{3}$$

where

$$\mathbf{c} = \begin{pmatrix} A^{-1}b_1 \\ Q^{-1}(rB^T A^{-1}b_1 + b_2) \\ D^{-1}(rC^T A^{-1}b_1 + b_3) \end{pmatrix},$$

and $\mathbf{u}^{(0)} \in \mathbb{R}^{m+n+p}$ is the initial guess.

In the following Lemma, we discuss about $\lambda = 1$ as an eigenvalue of the iteration matrix $\mathcal{M}_{r,\omega}$ of DGAOR method to analyse the spectral properties of the iteration matrix. In fact, we show that $\lambda = 1$ could not be an eigenvalue of $\mathcal{M}_{r,\omega}$.

Lemma 2.1. *If λ is an eigenvalue of the iteration matrix $\mathcal{M}_{r,\omega}$ of DGAOR method corresponding to the eigenvector $\mathbf{u} = (x; y; z)$, then x and z are not equal to zero, simultaneously, and $\lambda \neq 1$.*

Proof. If we let λ be an eigenvalue of $\mathcal{M}_{r,\omega}$ and $\mathbf{u} = (x; y; z)$ be the corresponding eigenvector, then we have

$$(1 - \omega - \lambda)x = \omega A^{-1}(By + Cz), \tag{4}$$

$$(\omega - r + r\lambda)Q^{-1}B^T x = (\lambda - 1)y, \tag{5}$$

$$(\omega - r + r\lambda)D^{-1}C^T x = (\lambda - 1)z. \tag{6}$$

If we set $x = 0$ and $z = 0$, then (4) implies that $By = 0$. Since B has full column rank, so $y = 0$ which is a contradiction. Let $\lambda = 1$, and the associated eigenvector $\mathbf{u} = (x; y; z)$. Then, by equations (4)-(6) we have

$$A^{-1}(By + Cz) = -x, \quad Q^{-1}B^T x = 0, \quad D^{-1}C^T x = z.$$

This is the problem $\mathcal{A}\mathbf{u} = 0$, and by Proposition5 1.1 we have $\mathbf{u} = 0$, which is a contradiction. □

In the following Lemma, we investigate the multiplicity of the eigenvalues of $\mathcal{M}_{r,\omega}$.

Lemma 2.2. *If $r = 1$, then $\lambda = 1 - \omega$ is an eigenvalue of $\mathcal{M}_{r,\omega}$ with multiplicity of at least m. If $r \neq 1$, then $\lambda = 1 - \omega$ is an eigenvalue of $\mathcal{M}_{r,\omega}$ if and only if $n > m$; in this case the multiplicity of λ is $n - m - p$.*

Proof. By definition (2) of $\mathcal{M}_{r,\omega}$, it can be deduced that for $r = 1$, $\lambda = 1 - \omega$ is an eigenvalue of $\mathcal{M}_{r,\omega}$ with multiplicity of at least m. Now, we assume $r \neq 1$. By equations (4)-(6) we have

$$(r-1)x^T BQ^{-1}B^T x = 0,$$

which implies $x = 0$ for $n = m$, and $x \neq 0$ for $n > m$. Thus $\lambda = 1 - \omega$ is an eigenvalue of $\mathcal{M}_{r,\omega}$ with multiplicity of $n - (m + p)$. The latter status comes from the fact that the algebraic multiplicity of an eigenvalue is at least equal to the geometrical multiplicity. □

Corollary 2.3. *Let λ be an eigenvalue of the iteration matrix $\mathcal{M}_{r,\omega}$ and the associated eigenvector is $\mathbf{u} = (x; y; z)$. If $\lambda \neq 1 - \omega$, then $y \neq 0$ or $z \neq 0$.*

In the following Theorem, we give a functional equation between parameters ω and r, and λ as an eigenvalue of $\mathcal{M}_{r,\omega}$ for the next investigation of the convergence region of the parameters in the DGAOR method.

Theorem 2.4. *Let A be a SPD matrix and assume that B and C have full column ranks, such that $\mathrm{Ran}(B) \cap \mathrm{Ran}(C) = \{0\}$. If Q and D are SPD matrices and λ be an eigenvalue of $\mathcal{M}_{r,\omega}$, then for the parameters ω and r, we have*

$$\lambda^2 + ((\omega - 2) + \frac{\gamma + \beta}{\alpha} r\omega)\lambda + (1 - \omega) + \frac{\gamma + \beta}{\alpha}\omega(\omega - r) = 0,$$

where $\alpha = x^* Ax$, $\beta = x^* BQ^{-1}B^T x$, and $\gamma = x^* CD^{-1}C^T x$.

Proof. Equations (4)-(6) give

$$\left(A + \frac{\omega(\omega - r + r\lambda)}{(1 - \omega - \lambda)(1 - \lambda)} CD^{-1}C^T\right) x = \frac{\omega}{1 - \omega - \lambda} By, \quad (\omega - r + r\lambda)Q^{-1}B^T x = (\lambda - 1)y,$$

and therefore by definitions of α, β and γ, we have

$$(1 - \lambda)(1 - \omega - \lambda)\alpha + \omega(\omega - r + r\lambda)(\gamma + \beta) = 0.$$

□

In the following Lemma, we recall a necessary and sufficient condition of both roots of the real quadratic equation to be less than one in modulus.

Lemma 2.5 ([4, Lemma 2.3]). *Both roots of the real quadratic equation $\lambda^2 - b\lambda + c = 0$, are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.*

It follows from Lemma 2.5 that $\rho(\mathcal{M}_{r,\omega}) < 1$ if and only if

$$|1 - \omega + \frac{\gamma + \beta}{\alpha}\omega(\omega - r)| < 1, \tag{7}$$

$$|\omega - 2 + \frac{\gamma + \beta}{\alpha} r\omega| < 1 + 1 - \omega + \frac{\gamma + \beta}{\alpha}\omega(\omega - r). \tag{8}$$

Equations (7) and (8) hold true if we have

$$0 < \omega < 2, \quad \omega - \frac{\alpha}{\gamma + \beta} < r < \frac{\omega}{2}.$$

Consequently, we have the following results.

Theorem 2.6. *Let A be a SPD matrix and assume that B and C have full column ranks, such that $\mathrm{Ran}(B) \cap \mathrm{Ran}(C) = \{0\}$. If $0 < \omega < \min(2, \frac{2\lambda_{min}(A)}{\lambda_{max}(BQ^{-1}B^T) + \lambda_{max}(CD^{-1}C^T)})$ and*

$$\omega - \frac{\lambda_{min}(A)}{\lambda_{max}(BQ^{-1}B^T) + \lambda_{max}(CD^{-1}C^T)} < r < \frac{\omega}{2}$$

then the DGAOR iterative scheme (3) converges to the exact solution of (1).

3 Numerical results

We now describe some numerical experiments were carried out in order to analyze the behaviour of the DGAOR method for different values of the parameter ω and r. The computational study was done in the next problems.

Example 3.1. Let us consider the double saddle point system (1), where the entries of the matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{n\times p}$ are defined as follows

$$A = (a_{ij}) = \begin{cases} i+1, & i = j \\ 1, & |i-j| = 1 \\ 0, & otherwise. \end{cases}, \quad B = (b_{ij}) = \begin{cases} j, & i = n-m+j \\ 0, & otherwise. \end{cases}$$

and

$$C = (c_{ij}) = \begin{cases} j, & i = j \\ 0, & otherwise. \end{cases}$$

For this problem, the condition of Proposition 1.1 are satisfied(especially $\mathrm{Ran}(B)\cap\mathrm{Ran}(C) = \{0\}$), hence $\mathcal{A}$ is nonsingular and the double saddle point problem (1) has a unique solution. The vector $\mathbf{b}$ is chosen so that the components of the exact solution $\mathbf{u}$ of (1) have values equal to 1. We choose the preconditioning matrices $Q = B^TB$ and $D = C^TC$ for the DGAOR method. All runs are started with the initial zero vector and terminated if the current iterations satisfy $ERR = \frac{\|\mathbf{r}^{(k)}\|_2}{\|\mathbf{r}^{(0)}\|_2} \leq 10^{-4}$, or if the prescribed iteration number $k_{\max} = 2000$ is exceeded. Here, we define $\mathbf{r}^{(k)}$ as

$$\mathbf{r}^{(k)} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} A & B & C \\ -B^T & 0 & 0 \\ -C^T & 0 & 0 \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix}.$$

Figure 1 shows ERR in terms of ω and r for $n = 50, m = 30, p = 10$(left) and $n = 200, m = 100, p = 60$(right). In view of Figure 1, we can conclude the fact that the minimum ERR is obtained when the parameters ω and r are near the boundary of the convergence region. From the results reported in Table 1, we can conclude that ERR and computational time are important items to demonstrate the efficiency of the DGAOR method in comparison with the SOR-like method($\omega = r$) [3].

Table 1: CPU time, iteration number and values of the parameters n, m, p, ω and r for DGAOR method

					DGAOR method			SOR-like method		
n	m	p	ω	r	minIT	CPU(s)	ERR	minIT	CPU(s)	ERR
50	30	10	1.2538	0.5769	283	0.0313	9.902e-05	285	0.0625	9.996e-05
80	40	20	1.2538	0.5769	427	0.1719	9.980e-05	429	0.2188	9.978e-05
100	50	40	1.2538	0.5769	528	0.2813	9.918e-05	530	0.3281	9.914e-05
300	150	80	1.2538	0.5769	1571	4.2500	9.983e-05	1573	4.4688	9.983e-05

4 Conclusion

In this paper, a generalization of accelerated overrelaxation (AOR) iterative method for solving a class of double saddle point problems have been proposed, where is denoted by DGAOR. The convergence region of the DGAOR method has been analyzed and numerical experiments were given to demonstrate the efficiency of the DGAOR method in comparison with the SOR-like method.

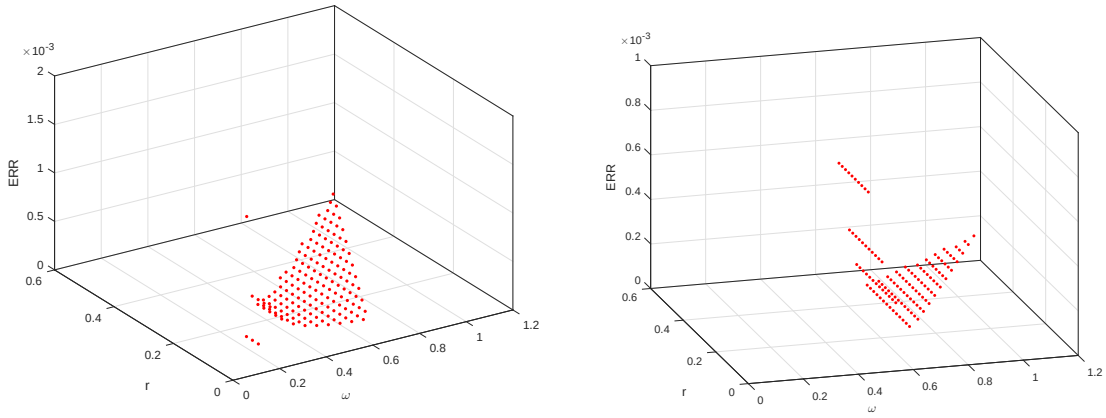

Figure 1: Convergence region for $n = 50, m = 30, p = 10$(left) and $n = 200, m = 100, p = 60$(right)

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An accelerated approach for low rank tensor completion

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Abstract

Tensor completion problem, which is a generalization of the matrix completion problem, is recovering the missing data of a tensor. In many algorithms proposed to complete the tensor, to achieve the answer, the method is executed on all unfolds related to tensor modes. Hence, if a tensor has N modes, each iteration of the algorithm contains N sub-problems, which is equivalent to solving N matrix completion problems. In this paper, to overcome the computational complexity caused by applying the algorithm to each dimension of the tensor, we present an idea in which the problem is implemented on only one tensor unfolding, so its computational complexity decreases in each iteration.

Keywords: Tensor completion problem, n-rank, computational complexity.
Mathematics Subject Classification [2010]: 15A83, 14N07

1 Introduction

Throughout the last decades, matrix representation of data and information has been arisen in many practical problems. In many fields of sciences such as machine learning, control, computer vision, collaborative filtering and dimensionality reduction, missing information is unavoidable in the data collection stage and a main important question is how we can fill these missing entries using known entries. This problem is known as matrix completion problem and since we can fill these unknown entries in different and infinite manners, this problem is ill-posed.
It is proved that by adding low-rankness property, this problem is no longer as an ill-posed problem and has an exact solution. In many cases, the structure of the data we want to recover is (approximately) low rank. So to solve this problem, we assume that the matrix is low rank. Following the data representation as the matrix, tensor, a powerful template for data management and matrix generalization to arbitrary order, was also considered. Data presentation in tensor form, exists in many applications such as neuroscience and chemistry.

Low-rank tensors play an important role in different real-world applications. A significant number of which, can be presented as data recovery problem (completion problem), including image/video inpainting, link-prediction and recommendation systems.

So far, different methods have been proposed to solve the low rank tensor completion problem. A class of these methods is based on unfolding the tensor in each tensor mode. Given the fact that N modes for the tensor will be equivalent to solving the N matrix completion problems,

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obviously, it requires a high memory, cost and computation time.
In this paper, to decrease the required memory and computational complexity, we propose an idea in which instead of solving the problem by using the tensor unfolding in all its modes, we solve the problem in only one unfolding. With this technique, in each iteration, we will need to solve the following problems related to only one tensor mode, and it is expected to create high reduction in the required time and memory.

2 Notations and definitions on tensors

Let $\mathcal{H}$ be the vector space of tensors in $\mathbb{R}^{I_1\times\cdots\times I_n}$. The inner product of $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{H}$ is defined as

$$\langle \mathbf{X}, \mathbf{Y}\rangle = \sum_{i_1=1}^{I_1}\sum_{i_2=1}^{I_2}\cdots\sum_{i_N=1}^{I_N} x_{i_1,i_2,\ldots,i_N} y_{i_1,i_2,\ldots,i_N}, \tag{1}$$

and the Frobenius norm of tensor $\mathbf{X}$ is equal to

$$\|\mathbf{X}\|_F = \sqrt{\sum_{i_1=1}^{I_1}\sum_{i_2=1}^{I_2}\cdots\sum_{i_N=1}^{I_N} x^2_{i_1,i_2,\ldots,i_N}}. \tag{2}$$

A mode-n unfolding of tensor $\mathbf{X}$ is denoted by

$$X_{(n)} = \text{unfold}_n(\mathbf{X}) \in \mathbb{R}^{I_n\times\prod_{i\neq n} I_i}. \tag{3}$$

In this operator, the tensor element $(i_1, i_1, \ldots, i_N)$ is mapped to the matrix element (i_n, j), where

$$j = 1 + \sum_{\substack{k=1\\k\neq n}}^{N}(i_k - 1)J_k \quad \text{with} \quad J_k = \prod_{\substack{m=1\\m\neq n}}^{k-1} I_m.$$

The inverse operator of unfold is defined in such a way that

$$\mathbf{X} = \text{fold}_n X_{(n)}. \tag{4}$$

One of the most important and practical definitions for tensors is the definition of rank. Unlike the matrices, tensors don't have a unique rank definition. Different definitions have been proposed for the concept of tensor rank, one of which is known as n-rank based on Tucker decomposition and is defined for a N-mode tensor $\mathbf{X}$, as [5]

$$\text{n-rank}(\mathbf{X}) = \left(\text{rank}\left(X_{(1)}\right), \ldots, \text{rank}\left(X_{(N)}\right)\right). \tag{5}$$

In this article, we only focus on the n-rank and discuss low rank tensor completion problem based on n-rank definition.

3 Low rank tensor completion problem

The purpose of low rank tensor completion problem (LRTC) is to retrieve the unknown elements of the tensor according to the known entries, assuming the rank of the tensor is low. Under these conditions, the tensor completion problem is expressed as follows [1]

$$\begin{aligned} &\min_{\mathbf{X}\in\mathcal{H}} \text{n-rank}(\mathbf{X})\\ &\text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{T}), \end{aligned} \tag{6}$$

where $\mathbf{T}$ is the observation tensor, $\mathbf{X}$ is low rank approximation of $\mathbf{T}$, Ω is the set of all index pairs in $\mathbf{T}$ that $\mathbf{X}$ shares with $\mathbf{T}$ and $\mathcal{P}_\Omega(.)$ is the sampling operator with zeros at the positions not in Ω. By applying a function f to the problem 6, our interest minimization problem becomes

$$\begin{aligned} &\min_{\mathbf{X}\in\mathcal{H}} \ f(\text{n-rank}(\mathbf{X})) \\ &\text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{T}), \end{aligned} \tag{7}$$

and by putting $f = \|.\|_1$, problem (7) is equivalent to

$$\begin{aligned} &\min \ \sum_{i=1}^{N} \text{rank}(X_{(i)}) \\ &\text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{T}). \end{aligned} \tag{8}$$

Problem (8) is a non-convex, discrete and Np-hard problem and under suitable conditions, problem (8) is formally equivalent to its tightest convex relaxation based on the nuclear norm [1]. The following relaxed form of tensor completion problem is proposed as

$$\begin{aligned} &\min_{\mathbf{X}\in\mathcal{H}} \ \|\mathbf{X}\|_* \\ &\text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{T}). \end{aligned} \tag{9}$$

The nuclear norm for tensors is defined as $\|\mathbf{X}\|_* = \sum_{i=1}^{N} \left\|X_{(i)}\right\|_*$, where $\|A\|_*$ is equal to singular values' sum of the matrix A. By applying this definition to problem (9) we have

$$\begin{aligned} &\min_{\mathbf{X}\in\mathcal{H}} \ \sum_{i=1}^{N} \left\|X_{(i)}\right\|_* \\ &\text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{T}). \end{aligned} \tag{10}$$

There are several methods and algorithms for solving problem (10). In [6] a simple low rank tensor completion algorithm (**SiLRTC**) is proposed to solve (10). The high accuracy low rank tensor completion (**HiLRTC**) is another method proposed in this paper. In [2, 4] two similar methods are proposed based on low rank matrix factorization. In these methods, each mode unfolding is decomposed in two matrices. This idea (**TMac**) is improved in [4] by additional term to objective function in [2]. In [1] the generalization of schatten p-norm for tensor (**SpBCD**) is used as a better surrogate than nuclear norm when $p \to 0$.

By solving N sub-problems in each iteration of all mentioned algorithms, if the dimensions of the tensor are large, they are inefficient and impractical because of their computational cost. In this article, we present an idea to solve this problem, which we will discuss in the next section.

4 Proposed idea

In this article, we apply $\|.\|_\infty$ function on the rank function

$$f(\text{n-rank}(\mathbf{X})) = \|\text{n-rank}(\mathbf{X})\|_\infty = \max_{1\le j\le N} \text{rank}(X_{(j)}). \tag{11}$$

So the tensor completion problem (7) is considered as follows

$$\begin{aligned} &\min_{\mathbf{X}} \max_{1\le j\le N} \ \text{rank}(X_{(j)}) \\ &\qquad \text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{T}). \end{aligned} \tag{12}$$

We show that the optimal value of problem (12) is an upper bound for the optimal value of problem (8).
Let $(\mathbf{X}^*, X^*_{(i)})$ where $\mathbf{X}^* \in \mathcal{H}$ and $X_{(i)} \in \mathbb{R}^{I_i \times \prod_{j\neq i}^{N} I_j}$ be the solution of problem (12). Then we have

$$X^*_{(i)} = \arg\min_{\mathbf{X}} \max_{1\leq j\leq N} \operatorname{rank}(X^*_{(j)}), \tag{13}$$

According to (13) we have

$$\operatorname{rank}(X^*_{(j)}) \leq \operatorname{rank}(X^*_{(i)}), \quad 1 \leq j \leq N,$$
$$\Longrightarrow \min \sum_{j=1}^{N} \operatorname{rank}(X^*_{(j)}) \leq N \times \operatorname{rank}(X^*_{(i)}).$$

Hence the solution of problem (12) is an upper bound of problem (8).

The main problem with (12) is that in practice the rank of the incomplete tensor is unknown to us, so it is impossible to determine in which mode, the matrix resulting from the unfolding of the tensor has the highest rank. To solve this problem, we make a small change in (12), and instead of minimizing the largest rank of tensor unfoldings, we minimize the rank of mode-i unfolding where $I_i = \max\{I_j, j = 1, \ldots, N\}$. With this change, we introduce the new following problem

$$\begin{aligned} &\min_{X_{(i)}} \operatorname{rank}(X_{(i)}) \\ &\text{s.t. } \mathcal{P}_{\Omega}(\mathbf{X}) = \mathcal{P}_{\Omega}(\mathbf{T}). \end{aligned} \tag{14}$$

As mentioned before, problem (14) is formally equivalent to its convex relaxation

$$\begin{aligned} &\min_{X_{(i)}} \left\|X_{(i)}\right\|_* \\ &\text{s.t. } \mathcal{P}_{\Omega}(\mathbf{X}) = \mathcal{P}_{\Omega}(\mathbf{T}) \end{aligned} \tag{15}$$

where $I_i = \max\{I_j, j = 1, \ldots, N\}$, such that they have accurately one unique solution [**?**]. Problem (15) can be solved by every proposed method applied to matrix completion problem and tensor completion method based on matrix unfolding.

5 Numerical results

In this section, we consider the experimental performance of solving problem (10) with four methods **HiLRTC** [6], **SiLRTC** [6], **SpBCD** [1] and **TMac** [2]. We display our accelerated idea implementation with these methods by **HiALRTC**, **SiALRTC**, **ASpBCD** and **ATMac**. We represent some experiments in setting the tensor completion and compare results of applying mentioned algorithms on our proposed problem (15) and problem (10) in terms of computation time and accuracy. The calculations were accomplished on a standard desktop computer by 32 GB of memory, an intel corei7 and 3 GHz processor.

For each test, we extract an index set Ω, uniformly at random, including the prescribed fraction of the total tensor data inputs and show this fraction by Sr such that

$$Sr = \frac{|\Omega|}{\prod_{i=1}^{N} I_j},$$

where $|\Omega|$ shows the cardinal number of the set Ω. The relative error estimation generated using each method is considered as the following function

$$\text{Error} = \frac{\|\mathcal{P}_{\Omega^c}(\mathbf{T} - \mathbf{X})\|_F}{\|\mathcal{P}_{\Omega^c}(\mathbf{T})\|_F}, \tag{16}$$

	$Sr = 0.3$		$Sr = 0.5$		$Sr = 0.8$	
Algorithm	Error	Time(s)	Error	Time(s)	Error	Time(s)
TMac	0.40	1.61	0.12	1.87	0.12	2.00
ATMac	0.14	0.51	0.24	0.64	0.31	0.82
Sp	0.08	2.88	0.06	2.89	0.04	2.84
ASp	0.11	1.05	0.08	0.40	0.07	0.13
Si	0.12	2.90	0.07	2.89	0.04	2.81
ASi	0.13	1.98	0.07	1.98	0.07	1.91
Hi	0.11	2.64	0.07	2.65	0.04	2.62
AHi	0.11	1.57	0.07	1.64	0.04	1.00

Table 1: The comparison results of applying the mentioned methods to (10) and our proposed problem (15), on a $187 \times 270 \times 3$ color image with 3 different values of Sr.

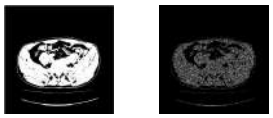

Figure 1: The front slice of an original medical image with 149 layers and its incomplete image with 50 percent of known entries as the input to the problem.

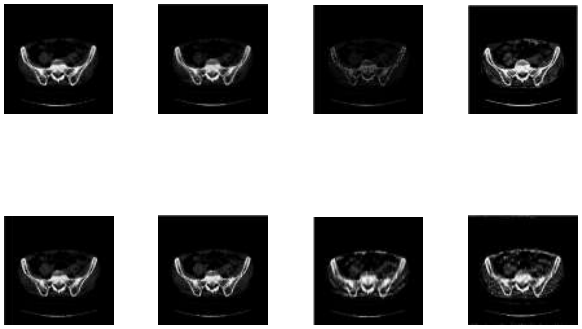

Figure 2: Recovery results of right picture in Figure 1. The first row is retrieved from left to right with algorithms **HiLRTC**, **SiLRTC**, **SpBCD** and **TMac** for solving problem (10) and the second row is recovered from left to right with **HiALRTC**, **SiALRTC**, **ASpBCD** and **ATMac**.

	$Sr = 0.3$		$Sr = 0.5$		$Sr = 0.8$	
Algorithm	Error	Time(s)	Error	Time(s)	Error	Time(s)
TMac	0.05	308.14	0.04	423.02	0.02	429.50
ATMac	0.12	108.84	0.10	104.68	0.07	104.87
Sp	0.07	238.57	0.06	251.88	0.02	242.93
ASp	0.12	16.04	0.10	17.35	0.11	16.14
Si	0.08	230.35	0.05	250.68	0.03	191.80
ASi	0.10	132.58	0.07	147.38	0.05	137.94
Hi	0.08	168.98	0.05	179.87	0.03	171.96
AHi	0.10	41.78	0.07	46.40	0.04	42.91

Table 2: The comparison results of applying the mentioned methods to (10) and our proposed problem (15), on a medical picture with 149 layers shown in Figure 1, with 3 different values of Sr.

where Ω^c is complementary to the corresponding sampling set Ω, $\mathbf{X}$ is the approximation of tensor completion and $\mathbf{T}$ is the original tensor. The error in each step of the algorithms is calculated as follows

$$\mathrm{E}_{tol} = \frac{\left\|\mathcal{P}_{\Omega^c}(\mathbf{X}^{k+1} - \mathbf{X}^k)\right\|_F}{\left\|\mathcal{P}_{\Omega^c}(\mathbf{X}^{k+1})\right\|_F}, \tag{17}$$

where $\mathbf{X^k}$ is the completion of incomplete tensor $\mathbf{W}$ at the kth iteration of the algorithm. The stopping criteria for each algorithm is considered as $\mathrm{E}_{rel} \leq \varepsilon$ where ε shows the desired error threshold and is set to 10^{-4} in our experiments.

For the subsequent experiments we focus on an color image data and an image of the radiological images set from the CPTAC Imaging Special Interest Group with 512×512 and 149 layers shown in Figure 1 and in order to apply the procedures, we removed some entries from the tensor data. Tables 1-2 and Figure 2 inform the comparison results of solving our represented problem (15) and problem (10) by the mentioned algorithms.

Considering the results obtained in Tables 1-2 and the comparison of the computation time of results obtained by mentioned algorithms on problem (10) and accelerated problem (15) show that the retrieval of images with the accelerated idea implementation by mentioned methods considerably reduces the computation time, while the resulted errors by solving problem (15) with mentioned methods are not significantly different from the errors obtained by applying the algorithms on problem (10) and in some cases, the errors are even less than those of results obtaind by solving problem (10). Generally, with a slight change in error and a significant reduction in the calculation time, especially for larger data, it can be said that the accelerated problem (15) significantly improves the complexity of solving problem (10).

6 Conclusion

In this paper, we presented a new idea with the approach of reducing the computational complexity and memory required to solve the tensor completion problem. Another advantage of the proposed idea is the ability to solve the tensor completion problem with various methods provided to solve the matrix completion problem.

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A note on generalized Jordan derivations of triangular rings

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Abstract

In this talk first of all we reveiw some results concerning (generalized) Jordan derivations on triangular ring. Then under some conditions we shall prove that every generalized Jordan derivation associate with Hochschild 2-cocycle α on a triangular ring is a generalized derivation associate with Hochschild 2-cocycle α.

Keywords: generalized derivation, generalized Jordan derivation, Hochschild 2-cocycle, Triangular ring

Mathematics Subject Classification [2010]: 16W25

1 Introduction

Let $\mathcal{R}$ be a ring, an additive mapping $d : \mathcal{R} \longrightarrow \mathcal{R}$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in \mathcal{R}$. Obviously, every derivation is a Jordan derivation. But the inverse is in general not true. The standard problem to find out whether (under some conditions) a Jordan derivation is necessarily a derivation. The first result in this direction was due to Herstein. In 1957, he proved that every Jordan derivation on a 2-torsion free prime ring is a derivation [3]. In 1988, Brešar generalized Herstein's result to Jordan derivations of semprime rings [1].

An additive mapping $f : \mathcal{R} \longrightarrow \mathcal{R}$ is called a generalized derivation(resp. generalized Jordan derivation) if there exists a derivation $d : \mathcal{R} \longrightarrow \mathcal{R}$ such that $f(xy) = f(x)y + xd(y)$, (resp. $f(x^2) = f(x)x + xd(x)$)
for all $x, y \in \mathcal{R}$. We denote it by (f, d). (see Brešar [2])

Ispired by Brešar idea, several outhers have introduced various copies of "generalized" derivations. Recently, Nakajima [4] has introduced a new type of generalized derivations and generalized Jordan derivations associated with Hochschild 2-cocycle in the following way. Let $\mathcal{R}$ be a ring and $\mathcal{M}$ be an $\mathcal{R}$-bimodule and x, y, z be arbitrary elements of $\mathcal{R}$. Let $\alpha : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{M}$ be a biadditive map, that is, an additive map on each components α is called a **Hochschild 2-cocycle** if

$$x\alpha(y, z) - \alpha(xy, z) + \alpha(x, yz) - \alpha(x, y)z = 0$$

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An additive map $f : \mathcal{R} \longrightarrow \mathcal{M}$ is called a generalized derivation if there exists a 2-cocycle α such that

$$f(xy) = f(x)y + xf(y) + \alpha(x, y)$$

and f is called a generalized Jordan derivation if

$$f(x^2) = f(x)x + xf(x) + \alpha(x, x)$$

Examples

(1) Let $d : \mathcal{R} \longrightarrow \mathcal{M}$ be a derivation, if (f, d) is a generalized derivation, then the map $\alpha_1 : \mathcal{R} \times \mathcal{R} \ni (x, y) \mapsto x(d - f)(y) \in \mathcal{M}$ is biadditive and satisfy the 2-cocycle condition. Then f is a generalized derivation associated to α_1.

(2) If $f : \mathcal{R} \longrightarrow \mathcal{M}$ is left multiplier, that is, f is additive and $f(xy) = f(x)y$, then by $f(xy) = f(x)y + xf(y) + x(-f)(y)$, we have 2-cocycle $\alpha_2 : \mathcal{R} \times \mathcal{R} \ni (x, y) \mapsto x(-f)(y) \in \mathcal{M}$ and f is a generalized derivation associated to α_2.

(3) Let f be a (σ, τ)- derivation, that is, σ and τ is a ring homomorphisms of $\mathcal{R}$ and $f(xy) = f(x)\sigma(y) + \tau(x)f(y)$. Then the map $\alpha_3 : \mathcal{R} \times \mathcal{R} \ni (x, y) \mapsto f(x)(\sigma(y) - y) + (\tau(x) - x)f(y) \in \mathcal{M}$ is biadditive and satisfies the 2-cocycle condition. Since

$$f(xy) = f(x)y + xf(y) + \alpha_3(x, y),$$

(σ, τ)- derivation f is a generalized derivation associated to α_3 .

Lemma 1.1. [4, *Lemma*2] . *Let* $(f, \alpha) : \mathcal{R} \longrightarrow \mathcal{M}$ *be a generalized Jordan derivation associate with Hochschild 2-cocycle* α *and* $\mathcal{M}$ *a 2-torsion free module.Then the following relations hold:*

$$\begin{aligned}
&(1) f(xy + yx) = f(x)y + xf(y) + \alpha(x, y) + f(y)x + yf(x) + \alpha(y, x);\\
&(2) f(xyx) = f(x)yx + xf(y)x + xyf(x) + x\alpha(y, x) + \alpha(x, yx);\\
&(3) f(xyz + zyx) = f(x)yz + xf(y)z + xyf(z) + x\alpha(y, z) + \alpha(x, yz)\\
&+ f(z)yx + zf(y)x + zyf(x) + z\alpha(y, x) + \alpha(z, yx).
\end{aligned}$$

In 2006, Zhang and Yu [5] showed that every Jordan derivation on a triangular algebra is a derivation. More precisely, they proved the following result.

Theorem 1.2. [5] *Let* $\mathcal{A}, \mathcal{B}$ *be unital algebras over a 2-torsion free commutative ring* $\mathcal{R}$ *and let* $\mathcal{M}$ *be a unital* $(\mathcal{A}, \mathcal{B})$*-bimodule that is faitful as a left* $\mathcal{A}$*-module and as a right* $\mathcal{B}$*-module. Then every Jordan derivation on the triangular algebra* $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ *is a derivation.*

Recall that a triangular ring $\mathfrak{T} = Tir(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a ring of the form

$$\mathfrak{T} = Tir(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} ; a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

We set

$$\mathfrak{T}_{11} = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} ; a \in \mathcal{A} \right\}$$

,

$$\mathfrak{T}_{12} = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} ; m \in \mathcal{M} \right\}$$

and

$$\mathfrak{T}_{22} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} ; b \in \mathcal{B} \right\}$$

Then we may write $\mathfrak{T} = \mathfrak{T}_{11} \oplus \mathfrak{T}_{12} \oplus \mathfrak{T}_{22}$ and every elemet $\mathcal{A} \in \mathfrak{T}$ can be writen as $\mathcal{A} = A_{11} + A_{12} + A_{22}$.Here, $A_{ij} \in \mathfrak{T}_{ij}, i, j \in \{1, 2\}$.

In this talk under some conditions we shall prove that every generalized Jordan derivation associate with Hochschild 2-cocycle α on a triangular ring is a generalized derivation associate with Hochschild 2-cocycle α.

2 Main results

The following theorem is the main purpose of this paper.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-torsion free semiprime rings and let $\mathcal{M}$ be a faithful non-degenerate $(\mathcal{A}, \mathcal{B})$-bimoudle then every generalized Jordan derivation f associate with Hochschild 2-cocycle α on triangular ring $\mathfrak{T} = Tir(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a generalized derivation associate with Hochschild 2-cocycle α .*

We recall that $(\mathcal{A}, \mathcal{B})$-module $\mathcal{M}$ is called non-degenerate if for all $x \in \mathcal{M}$ $\mathcal{A}x = 0$ implies $x = 0$ and $x\mathcal{B} = 0$ implies $x = 0$.

For the proof of the above theorem we need the following lemmas.

Lemma 2.2. *Let $\alpha : \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ be a Hochschild 2-cocycle. Then for any $A_{11}, B_{11} \in \mathfrak{T}_{11}$, $A_{12}, B_{12}, C_{12} \in \mathfrak{T}_{12}$ and $A_{22}, B_{22} \in \mathfrak{T}_{22}$, we have*

1) $B_{22}\alpha(A_{11}, B_{22}) = \alpha(B_{22}, A_{11})B_{22}$,
2) $B_{11}\alpha(B_{22}, A_{11}) = \alpha(B_{11}, B_{22})A_{11}$,
3) $B_{22}\alpha(B_{12}, A_{11}) = \alpha(B_{22}, B_{12})A_{11}$,
4) $B_{22}\alpha(A_{12}, B_{22}) + \alpha(B_{22}, A_{12}B_{22}) = \alpha(B_{22}, A_{12})B_{22}$,
5) $A_{11}\alpha(B_{22}, A_{12}) = \alpha(A_{11}, B_{22})A_{12}$,
6) $A_{12}\alpha(B_{12}, B_{22}) + \alpha(A_{12}, B_{12}B_{22}) = \alpha(A_{12}, B_{12})B_{22}$,
7) $A_{12}\alpha(B_{12}, A_{11}) = \alpha(A_{12}, B_{12})A_{11}$,
8) $B_{22}\alpha(A_{12}, B_{12}) = \alpha(B_{22}, A_{12})B_{12}$,
9) $A_{11}\alpha(B_{11}, C_{12}) - \alpha(A_{11}B_{11}, C_{12}) + \alpha(A_{11}, B_{11}C_{12}) = \alpha(A_{11}, B_{11})C_{12}$,
10) $C_{12}\alpha(A_{22}, B_{22}) = \alpha(C_{12}A_{22}, B_{22}) - \alpha(C_{12}, A_{22}B_{22}) + \alpha(C_{12}, A_{22})B_{22}$.

Lemma 2.3. *For arbitrary $A_{11} \in \mathfrak{T}_{11}$ and $B_{22} \in \mathfrak{T}_{22}$, we have:*

$$
\begin{aligned}
(i) \quad & [f(B_{22})]_{11}A_{11} = -[\alpha(B_{22}, A_{11})]_{11}, \\
& B_{22}[f(A_{11})]_{22} = -[\alpha(B_{22}, A_{11})]_{22}, \\
& [\alpha(B_{22}, A_{11})]_{12} = 0. \\
(ii) \quad & f(B_{22}A_{11}) = f(B_{22})A_{11} + B_{22}f(A_{11}) + \alpha(B_{22}, A_{11}) \\
(iii) \quad & f(A_{11}B_{22}) = f(A_{11})B_{22} + A_{11}f(B_{22}) + \alpha(A_{11}, B_{22})
\end{aligned}
$$

Lemma 2.4. *For any $B_{12} \in \mathfrak{T}_{12}$, the following is ture:($A_{11} \in \mathfrak{T}_{11}$)*

$$
\begin{aligned}
& [f(B_{12})]_{11}A_{11} = -[\alpha(B_{12}, A_{11})]_{11}, \\
& [\alpha(B_{12}, A_{11})]_{22} = 0, \\
& B_{12}[f(A_{11})]_{22} = -[\alpha(B_{12}, A_{11})]_{12}.
\end{aligned}
$$

Lemma 2.5. *For any $A_{11} \in \mathfrak{T}_{11}$ and $B_{12} \in \mathfrak{T}_{12}$, we have*

$$
\begin{aligned}
& (i) f(B_{12}A_{11}) = f(B_{12})A_{11} + B_{12}f(A_{11}) + \alpha(B_{12}, A_{11}), \\
& (ii) f(A_{11}B_{12}) = f(A_{11})B_{12} + A_{11}f(B_{12}) + \alpha(A_{11}, B_{12}).
\end{aligned}
$$

Lemma 2.6. *For any $A_{12} \in \mathfrak{T}_{12}$, the following is ture:($B_{22} \in \mathfrak{T}_{22}$)*

$$\begin{aligned}&[f(B_{22})]_{11}A_{12} = -[\alpha(B_{22}, A_{12})]_{12},\\ &[\alpha(B_{22}, A_{12})]_{11} = 0,\\ &B_{22}[f(A_{12})]_{22} = -[\alpha(B_{22}, A_{12})]_{22}.\end{aligned}$$

Lemma 2.7. *For any $A_{12} \in \mathfrak{B}_{12}$ and $B_{22} \in \mathfrak{B}_{22}$, we have*

$$\begin{aligned}&(i) f(B_{22}A_{12}) = f(B_{22})A_{12} + B_{22}f(A_{12}) + \alpha(B_{22}, A_{12}),\\ &(ii) f(A_{12}B_{22}) = f(A_{12})B_{22} + A_{12}f(B_{22}) + \alpha(A_{12}, B_{22}).\end{aligned}$$

Lemma 2.8. *For any $A_{12}, B_{12} \in \mathfrak{T}_{12}$, the following is ture:*

$$\begin{aligned}&[f(A_{12})]_{11}B_{12} + A_{12}[f(B_{12})]_{22} = -[\alpha(A_{12}, B_{12})]_{12},\\ &[\alpha(A_{12}, B_{12})]_{11} = 0,\\ &[\alpha(A_{12}, B_{12})]_{22} = 0.\end{aligned}$$

Lemma 2.9. *f is a generalized derivation on $\mathfrak{T}_{12}$.*

Lemma 2.10. *f is a generalized derivation on $\mathfrak{T}_{11}$ and $\mathfrak{T}_{22}$.*

Applying the above lemmas, we have for any $A = A_{11} + A_{12} + A_{22}$ and $B = B_{11} + B_{12} + B_{22}$ in $\mathfrak{T}$.

$$\begin{aligned}f(AB) &= f(\sum_{1=i\le j=2} A_{ij} \sum_{1=k\le l=2} B_{kl})\\ &= f(\sum_{\substack{1=i\le j=2\\ 1=k\le l=2}} A_{ij}B_{kl})\\ &= \sum_{\substack{1=i\le j=2\\ 1=k\le l=2}} f(A_{ij}B_{kl})\\ &= \sum_{\substack{1=i\le j=2\\ 1=k\le l=2}} (f(A_{ij})B_{kl} + A_{ij}f(B_{kl}) + \alpha(A_{ij}, B_{kl}))\\ &= \sum_{1=i\le j=2} f(A_{ij})B + A\sum_{1=k\le l=2} f(B_{kl}) + \sum_{\substack{1=i\le j=2\\ 1=k\le l=2}} \alpha(A_{ij}, B_{kl})\\ &= f(\sum_{1=i\le j=2} A_{ij})B + Af(\sum_{1=k\le l=2} B_{kl}) + \sum_{\substack{1=i\le j=2\\ 1=k\le l=2}} \alpha(A_{ij}, B_{kl})\\ &= f(A)B + Af(B) + \alpha(A, B).\end{aligned}$$

The proof of the Theorem is complete.

3 Conclusion

We extended some results concerning to (generalized) Jordan derivations, to generalized Jordan derivations associated to a Hochschild 2-cocycle.

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$K-$ uniformly Rotund Banach spaces

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Abstract

In this paper we obtain some of the basic geometric properties of $K-$UR spaces. Our purpose is to study the extent to which Banach space properties can be obtained by requiring a uniform behavior for all $n-$dimensional subspaces for some fixed $n \geq 2$.

Keywords: K-uniformly rotund space, Banach space, modulus of K-rotundity, volume of $k+1$ vectors.

Mathematics Subject Classification [2010]: 15A03, 15A15

1 Introduction

Let X be a real Banach space. According to von Neumanns famous geometrical characterizatin X is Hilbert space if and only if for all $x, y \in X$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Thus Hilbert space is distinguished among all real Banach spaces. For an arbitrary space X, one way of measuring the uniformly of the set of two dimensional subspaces is in terms of the real valued modulus of rotundity, i.e. for $\epsilon > 0$

$$\delta_X(\epsilon) = \inf\{2 - \|x+y\| \ : \ \|x\|, \|y\| \leq 1, \ \|x-y\| \geq \epsilon\}.$$

The space is said to be uniformly rotund if for each $\epsilon > 0$ we have $\delta_X(\epsilon) > 0$. Our purpose in this paper is to study the extent to which Banach space properties can be obtained by requiring a uniform behavior for all $n-$dimensional subspaces for some fixed $n \geq 2$. This idea is orginated with Milman [3] who discussed both smoothness and rotundity notation.

2 Main results

Now, we give some Basic definition which used in this paper [1]. Let X be a Banach space. For a nonempty subset A of X, The convex hull and the affine hull of A denote by $con(A)$ and $aff(A)$, respectively, and defined as follows

$$con(A) := \{\sum_{i=1}^{n} \lambda_i x_i : x_i \in A, \lambda_i \geq 0, \quad for \quad i = 1, 2, ..., n \quad \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N}\},$$

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$$aff(A) := \{\sum_{i=1}^{n} \lambda_i x_i : x_i \in A, \lambda_i \in \mathbb{R}, \quad for \quad i = 1, 2, ..., n \quad \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N}\}.$$

The set A is called affine if $aff(A) = A$. One can easily show that every affine set corresponds with a subspace.
Let A be a affine set and Y be its corresponding subspace. Then the dimension of A is defined the same dimension of Y. Further, the dimension of a convex set A is defined as the dimension of the smallest affine set which contains A.
Given $x_1, x_2, ..., x_k \in X$, Sliverman [5], introduced the concept of volume of $k+1$ vectors, denote by $V(x_1, x_2, ..., x_{k+1})$ and defined by

$$V(x_1, x_2, ..., x_{k+1}) = \frac{1}{k!} \sup \left\{ \begin{vmatrix} f_1(x_2 - x_1) & \dots & f_1(x_{k+1} - x_1) \\ f_2(x_2 - x_1) & \dots & f_2(x_{k+1} - x_1) \\ \vdots & \vdots & \vdots \\ f_k(x_2 - x_1) & \dots & f_k(x_{k+1} - x_1) \end{vmatrix} : f_1, ..., f_k \in B_{X^*} \right\}.$$

By one of the consequence of Han-Banach theorem, we get $V(x_1, x_2) = \|x_1 - x_2\|$ for any $x_1, x_2 \in X$.
Note that $V(x_1, x_2, ..., x_{k+1}) = 0$ iff the dimension of the convex hull of $\{x_1, x_2, ..., x_{k+1}\}$ dose not exceed $k - 1$.
Using the notion of volume of $k + 1$ vectors, Sullivan [6] defined the concept of $K-$uniform rotund spaces.

Definition 2.1. [4] Let X be a Banach space and $k \in \mathbb{N}$. Set

$$\mu_X^{(k)} = \sup\{V(x_1, ..., x_{k+1}) : x_1, ..., x_{k+1} \in B_X, i = 1, 2, ..., k + 1\}.$$

The function $\delta_X^{(k)} : [0, \mu_X^{(k)}) \longrightarrow [0, 1]$ is said to the modulus of K-rotundity of X and define by

$$\delta_X^{(k)}(\varepsilon) = \inf \left\{ 1 - \frac{1}{k+1} \| \sum_{i=1}^{k+1} x_i \| : x_1, ..., x_{k+1} \in B_X \quad \text{and} \quad V(x_1, ..., x_{k+1}) \geq \varepsilon \right\},$$

where $\epsilon \in [0, \mu_X^{(k)})$.

The Banach space X is said to be K-uniformly rotund (in short $K - UR$) if $\delta_X^{(k)} > 0$ for any $\epsilon \in (0, \mu_X^{(k)})$.
Note that a Banach space X is said to be $1-$ UR if, for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if x and y are norm-1 vector with $\|x + y\| \geq 2 - \delta(\epsilon)$ then

$$\sup \left\{ \begin{vmatrix} 1 & 1 \\ f(x) & f(y) \end{vmatrix} : f \in B_{X^*} \right\} < \epsilon.$$

By analogy we say that X is $2-$ UR if for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that for all norm-1 $x, y, z \in X$, if $\|x + y + z\| > 3 - \delta(\epsilon)$ then

$$\sup \left\{ \begin{vmatrix} 1 & 1 & 1 \\ f(x) & f(y) & f(z) \\ g(x) & g(y) & g(z) \end{vmatrix} : f, g \in B_{X^*} \right\} < \epsilon.$$

Notice that of $2-$UR the quantity in the brackets can be thought of as twice the area of the triangle with vertices at x, y and z. In geometrical terms, if three points on the surface of $2-$UC space enclose an area $\geq \frac{\epsilon}{2}$ then the centroid of the triangle they determine lies a distance at least $(\frac{\epsilon}{3})$ beneath the surface of the ball. [5]

Theorem 2.2. *[6] if for some k a Banach space X is $K-UR$ then X is $K+1-UR$.*

Proof. Suppose that there are norm-1 sequences $(x_n^{(1)}), (x_n^{(2)}), ..., (x_n^{(k+2)})$ with $\|x_n^{(1)} + x_n^{(2)} + ... + x_n^{(k+2)}\| \to k+2$. Then from the triangle inequality for each j we have

$$\|x_n^{(1)} + ... + x_n^{(j-1)} + x_n^{(j+1)} + ... + x_n^{(k+2)}\| \to k+1.$$

Now, let $f_1, f_2, ..., f_{k+1} \in \mathcal{B}_{X^*}$ and consider the determinant

$$\begin{vmatrix} 1 & 1 & & 1 \\ f_1(x_n^{(1)}) & f_1(x_n^{(2)}) & \dots & f_1(x_n^{(k+2)}) \\ \vdots & \vdots & & \vdots \\ f_{k+1}(x_n^{(1)}) & f_{k+1}(x_n^{(2)}) & \dots & f_{k+1}(x_n^{(k+2)}) \end{vmatrix}$$

expanding in minors along the second row and the fact that X is $K-$UR we conclude that X is $K+1-$UR . □

The converse of above theorem is not true. For example, the Banach space $l^{p,1}(\mathbb{N})$ for $1 < p < \infty$ is $2-$UR but not $1-$UR where $l^{p,1}(\mathbb{N})$ is the $l^p(\mathbb{N})$ space with suitable renorm.

Corollary 2.3. *[2] Let X be a Banach space. Then $\delta_X^{(k)}(\epsilon)$ is continuous on $[0, \mu_X^{(k)})$*

Lemma 2.4. *[4]. Let X be a K-uniformly rotund Banach space for some $k \in \mathbb{N}$. and $x_1, ..., x_{k+1} \in B_X$ such that $V(x_1, ..., x_{k+1}) = \epsilon > 0$ Then*

$$\|t_1x_1 + t_2x_2 + ... + t_{k+1}x_{k+1}\| \leq 1 - (k+1)\min\{t_1, t_2, ..., t_{k+1}\}\delta_X^k(\varepsilon),$$

where $\sum_{i=1}^{k+1} t_i = 1$, $t_i \geq 0$ for $i = 1, 2, ..., k+1$.

Proof. Without loss of generality, we can assume that $t_1 = \min\{t_1, t_2, ..., t_{k+1}\}$.

$$\begin{aligned}
&\|t_1x_1 + t_2x_2 + ... + t_{k+1}x_{k+1}\| \\
&= \|t_1(x_1 + ... + x_{k+1}) + (t_2 - t_1)x_2 + (t_3 - t_1)x_3 + ... + (t_{k+1} - t_1)x_{k+1}\| \\
&\leq (k+1)t_1\|\frac{x_1 + x_2 + ... + x_{k+1}}{k+1}\| + (t_2 - t_1)\|x_2\| + (t_3 - t_1)\|x_3\| + ... + (t_{k+1} - t_1)\|x_{k+1}\| \\
&\leq (k+1)t_1(1 - \delta_X^{(k)}(\epsilon) + t_2 + t_3 + ... + t_{k+1} - kt_1 \\
&= (k+1)t_1 - (k+1)t_1\delta_X^{(k)}(\epsilon) + 1 - (k+1)t_1 \\
&= 1 - (k+1)t_1\delta_X^{(k)}(\epsilon)
\end{aligned}$$

Hence $\|t_1x_1 + t_2x_2 + ... + t_{k+1}x_{k+1}\| \leq 1 - (k+1)\min\{t_1, t_2, ..., t_{k+1}\}\delta_X^k(\varepsilon)$. □

3 Conclusion

Motivated by definition of $K-$UR space we can extend application of uniformly properties of Banach spaces. For example the existence of fixed point problem in the setting of uniformly rotund Banach space extended by Radhakrishnan in [4].

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Numerical ranges of even-order tensor

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Abstract

In this paper, the numerical range of an even-order tensor is defined using the norm of its square matrix unfolding. The basic properties of the numerical range of a matrix, such as compactness and convexity, are proved to hold for the numerical range of an even-order tensor. Also, we introduce normal tensors based on the contraction product. According to the Tucker decomposition, we get the numerical range of a normal tensor. Next, we introduce the singular-value decomposition (SVD) of an even-order tensor. Using this decomposition, we obtain the numerical range of such a tensor.

Keywords: Normal tensors, Numerical range, Toeplitz tensor, Unfolding operators, SVD decomposition, Even-order tensor.

Mathematics Subject Classification [2010]: 15A69, 15A60

1 introduction

For a k^{th}-order l-dimensional tensor A (or $A \in \mathbb{C}^{(k,l)}$), the numerical range is defined by

$$W_{|||.|||}(A) = \bigcap_{\lambda \in \mathbb{C}} \{\mu \in \mathbb{C} :\mid \mu - \lambda \mid \leq \xi . ||| \ A - \lambda I \ |||\},$$

where $I \in \mathbb{C}^{(k,l)}$ is the identity tensor, $|||\ .\ |||$ is a consistent tensor norm on $\mathbb{C}^{(k,l)}$, and ξ is a scalar [2]. One fundamental fact about the numerical range is that $W_{|||.|||}(A)$ is a convex subset of $\mathbb{C}$ that contains the spectrum of A, for every $A \in \mathbb{C}^{(k,l)}$. For high-dimensional problems, the data have an inherent tensor structure, and the difference slices of the data may have some relationships. If we process brain MRI images slice-by-slice, we may lose some information of the tensor structure of the images. Thus, it is necessary to study the tensor eigenvalue problem. The eigenvalues of high-order tensors are used in different fields. For example, medical resonance imaging, diffusion tensor imaging, high-order Markov chains and data mining, positive definiteness of even-order multivariate forms in automatic control, and best rank-one approximation in data analysis. Tensor eigenvalues were introduced by Lim and Qi in 2005.

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Definition 1.1. Let A be a k^{th}-order l-dimensional tensor. If $x \in \mathbb{C}^l \setminus \{0\}$ and $\lambda \in \mathbb{C}$ satisfy

$$Ax^{k-1} = \lambda x^{[k-1]},$$

then we say that λ is an eigenvalue of A, and that x is its corresponding eigenvector. Here,

$$(Ax^{k-1})_i := \sum_{i_2,i_3,\ldots,i_k=1}^{l} a_{i,i_2,i_3,\ldots,i_k} x_{i_2} x_{i_3}, \ldots, x_{i_k},$$

where $1 \leq i \leq l$, $x = (x_1, x_2, \ldots, x_l)^T$, and $x^{[k-1]} = (x_1^{k-1}, x_2^{k-1}, \ldots, x_l^{k-1})$.

It is clear that when $k = 2$, the above definition coincides with the one that defines the eigenvalues and eigenvectors of real matrices. Hence, tensor eigenvalues generalize matrix eigenvalues. According to Definition 1.1, a tensor eigenvalue problem is a nonlinear one which is equivalent to solving a set of multivariate polynomials of variables $x_1, x_2, \ldots, x_n$, and an unknown λ. In general, the tensor eigenvalue problem given by Definition 1.1 is very difficult.

Definition 1.2. Let A be a $2m^{\text{th}}$-order n-dimensional tensor. Furthermore, assume that a nonzero m^{th}-order n-dimensional tensor X and $\lambda \in \mathbb{C}$ satisfy

$$A \cdot X = \lambda X, \tag{1}$$

where

$$(A \cdot X)_{i_1,i_2,\ldots,i_m} = \sum_{k_1,k_2,\ldots,k_m=1}^{n} a_{i_1,i_2,\ldots,i_m,k_1,k_2,\ldots,k_m} x_{k_1,k_2\ldots,k_m},$$

and $1 \leq i_l \leq n, 1 \leq l \leq m$. Then, we refer to λ as an eigenvalue of A, and to X as its corresponding eigentensor.

For $m = 1$, A is a square matrix and Definition 1.2 reduces to the definition of matrix eigenvalues and eigenvectors. In this paper, *eigenvalue* is always meant to be in the sense of Definition 1.2. The unfolding matrix of a tensor is a useful tool for the study of such tensor problems as those concerning eigenvalues and the numerical range.

Recently, *tensor numerical ranges* have been introduced by Ke, Li and Ng [2] on the basis of tensor norms. These have the same properties as those of the numerical ranges of matrices, except the *normality*, *projection*, and *unitary invariance* properties. The numerical ranges contain the eigenvalues. So, computing the numerical range of a tensor may be useful in designing fast algorithms for the calculation of its eigenvalues.

Our idea is to generalize the numerical range of a matrix to the numerical range of an even-order tensor, one that contains the tensor eigenvalues in the sense of Definition 1.2. We define the numerical range of even-order tensors using the even-order tensor unfolding matrix norms. We show that the basic properties of the numerical range of a matrix, such as *compactness* and *convexity*, are valid for the numerical range of an even-order tensor. It is useful to estimate Toeplitz tensor eigenvalues in the process of image restoration. Therefore, numerical ranges of Toeplitz tensors can be used in image processing.

2 Unfolding operations and tensors

Suppose that A is a $2m^{\text{th}}$-order n-dimensional tensor. We can reorder A as a square matrix using the square matrix unfolding of tensors.

Definition 2.1. [1] Let A be a $2m^{\text{th}}$-order n-dimensional tensor. We use $A(i_1, i_2, \ldots, i_m, j_1, j_2, \ldots, j_m)$ to denote the $(i_1, i_2, \ldots, i_m, j_1, j_2, \ldots, j_m)^{\text{th}}$ entry of A. The square matrix unfolding of A with an ordering P is an n^m-by-n^m matrix A_P whose $(k, h)^{\text{th}}$ entry is given by

$$A_P(k, h) = A(i'_1, i'_2, \ldots, i'_m, j'_1, j'_2, \ldots, j'_m),$$

where

$$k = n^{m-1}(i'_1 - 1) + n^{m-2}(i'_2 - 1) + \cdots + n(i'_{m-1} - 1) + i'_m,$$
$$h = n^{m-1}(j'_1 - 1) + n^{m-2}(j'_2 - 1) + \cdots + n(j'_{m-1} - 1) + j'_m,$$

and $1 \leq i'_l, j'_l \leq n, 1 \leq l \leq m$. Here, P is the permutation matrix corresponding to ordering P that satisfies

$$(i'_1, i'_2, \ldots, i'_m) = (i_1, i_2, \ldots, i_m)P, \qquad (j'_1, j'_2, \ldots, j'_m) = (j_1, j_2, \ldots, j_m)P.$$

Example 2.2. [1] Suppose that $A = (a_{i,j,k,l})$ is a 4^{th}-order three-dimensional tensor. Let P be the permutation matrix given by

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We note that $(i'_1, i'_2) = (i_1, i_2)P$. The corresponding square matrix unfolding of A is a 3^2-by-3^2 matrix, namely,

$$A_P = \begin{bmatrix} a_{1111} & a_{1121} & a_{1131} & a_{1112} & a_{1122} & a_{1132} & a_{1113} & a_{1123} & a_{1133} \\ a_{2111} & a_{2121} & a_{2131} & a_{2112} & a_{2122} & a_{2132} & a_{2113} & a_{2123} & a_{2133} \\ a_{3111} & a_{3121} & a_{3131} & a_{3112} & a_{3122} & a_{3132} & a_{3113} & a_{3123} & a_{3133} \\ a_{1211} & a_{1221} & a_{1231} & a_{1212} & a_{1222} & a_{1232} & a_{1213} & a_{1223} & a_{1233} \\ a_{2211} & a_{2221} & a_{2231} & a_{2212} & a_{2222} & a_{2232} & a_{2213} & a_{2223} & a_{2233} \\ a_{3211} & a_{3221} & a_{3231} & a_{3212} & a_{3222} & a_{3232} & a_{3213} & a_{3223} & a_{3233} \\ a_{1311} & a_{1312} & a_{1331} & a_{1312} & a_{1322} & a_{1332} & a_{1313} & a_{1323} & a_{1333} \\ a_{2311} & a_{2321} & a_{2331} & a_{2312} & a_{2322} & a_{2332} & a_{2313} & a_{2323} & a_{2333} \\ a_{3311} & a_{3321} & a_{3331} & a_{3312} & a_{3322} & a_{3332} & a_{3313} & a_{3323} & a_{3333} \end{bmatrix}$$

Given two different orderings P and P', it is interesting to note that A_P and $A_{P'}$ are similar via a permutation matrix. [1] Suppose that P and P' are two different orderings. Then, there exists a permutation matrix $\prod_{P,P'}$ such that

$$\prod_{P,P'} A_P \prod_{P,P'}^T = A_{P'}.$$

Definition 2.3. [1] Let X be an m^{th}-order n-dimensional tensor. The vectorization of X with an ordering P is an n^m-vector x_P whose j^{th} entry $x_P(j)$ is given by

$$x_P(j) = X_{i_1, i_2, \ldots, i_m}, \qquad 1 \leq i_k \leq n, 1 \leq k \leq m,$$

where $j = \sum_{k=1}^{m-1} n^{m-1}(i'_k - 1) + i'_m$ and P is the permutation matrix corresponding to the ordering P:$(i'_1, \ldots, i'_m) = (i_1, \ldots, i_m)P$.

[1] The tensor eigenvalue problem in (1) is equivalent to the matrix eigenvalue system

$$A_P x_P = \lambda x_P.$$

According to Proposition 2, it is possible to calculate the eigenvalues and eigentensors of A by solving the eigenvalue problem of matrix unfolding A_P corresponding to A with ordering P.

Definition 2.4. [1] Let A be a $2m^{\text{th}}$-order n-dimensional tensor. The tensor A^* whose entries are given by $\overline{a}_{j_1,j_2,\ldots,j_m,i_1,i_2,\ldots,i_m}$, for $1 \leq i_k, j_k \leq n$ and $1 \leq k \leq m$, is called the conjugate transpose of A. We say that A is Hermitian if

$$a_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = \overline{a}_{j_1,j_2,\ldots,j_m,i_1,i_2,\ldots,i_m}$$

for $1 \leq i_k, j_k \leq n$ and $1 \leq k \leq m$, that is, $A = A^*$. Note that A is symmetric if $a_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = a_{j_1,j_2,\ldots,j_m,i_1,i_2,\ldots,i_m}$.

We define the *contraction product* of two $2m^{\text{th}}$-order n-dimensional tensors.

Definition 2.5. [1] Let A and B be $2m^{\text{th}}$-order n-dimensional tensors. The contraction product of A and B, denoted by $A \star B$, is a tensor of order $2m$ and dimension n which is defined by

$$(A \star B)_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = \sum_{k_1,k_2,\ldots,k_m=1}^{n} a_{i_1,i_2,\ldots,i_m,k_1,k_2,\ldots,k_m} b_{k_1,k_2,\ldots,k_m,j_1,j_2,\ldots,j_m},$$

where $1 \leq i_k, j_k \leq n$ and $1 \leq k \leq m$.

Also, we can write

$$(A \star B)(:,\ldots,:,j_1,\ldots,j_m) = A \cdot B(:,\ldots,:,j_1,\ldots,j_m),$$

where $1 \leq j_k \leq n, 1 \leq k \leq m$. The following proposition presents the basic properties of the contraction product.

Normal tensors. Let A be a $2m^{\text{th}}$-order n-dimensional tensor. We call A a *normal tensor* if $A \star A^* = A^* \star A$, that is,
$(A \star A^*)_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m}$

$$= \sum_{k_1,k_2,\ldots,k_m=1}^{n} a_{i_1,i_2,\ldots,i_m,k_1,k_2,\ldots,k_m} \overline{a}_{j_1,j_2,\ldots,j_m,k_1,k_2,\ldots,k_m}$$

$$= \sum_{k_1,k_2,\ldots,k_m=1}^{n} \overline{a}_{k_1,k_2,\ldots,k_m,i_1,i_2,\ldots,i_m} a_{k_1,k_2,\ldots,k_m,j_1,j_2,\ldots,j_m}$$

$$= (A^* \star A)_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m},$$

where $1 \leq i_l, j_l \leq n$ and $1 \leq l \leq m$.

Diagonal tensor. A tensor $D = (d_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m}) \in \mathbb{C}^{(2m,n)}$ is called a *diagonal tensor* if $d_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = 0$, when $(i_1, i_2, \ldots, i_m) \neq (j_1, j_2, \ldots, j_m)$.

Unitary tensors. A $2m^{\text{th}}$-order n-dimensional tensor U is said to be a *unitary tensor* if $U \star U^* = U^* \star U = I_E$, where $I_E = (e_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m})$ is the identity tensor in which

$$e_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = \begin{cases} 1, & (i_1, i_2, \ldots, i_m) = (j_1, j_2, \ldots, j_m) \\ 0, & otherwise \end{cases}$$

T-unitary similar tensors. Given $A, B \in \mathbb{C}^{(2m,n)}$, we say that A is *T-unitary similar* to B if

$$A = U^* \star B \star U,$$

where $U \in \mathbb{C}^{(2m,n)}$ is a unitary tensor.

Theorem 2.6. *(Eigenvalue decomposition of normal tensors) Let A be a $2m^{\text{th}}$-order n-dimensional tensor. Then, A is a normal tensor if and only if there is a unitary tensor $V \in \mathbb{C}^{(2m,n)}$ such that*

$$A = V \star D \star V^*,$$

where $D \in \mathbb{C}^{(2m,n)}$ is a diagonal tensor whose entries are the eigenvalues of A. Moreover, the above decomposition can be written as

$$A = \sum_{i=1}^{n^m} \sigma_i V_i \circ V_i^*,$$

where $V_i \in \mathbb{C}^{(m,n)}$ is the eigentensor corresponding to the eigenvalue σ_i.

Tensor norm. We define the *tensor norm* $|||\ .\ |||$ on $\mathbb{C}^{(m,n)}$ as follows. Given X, an m^{th}-order n-dimensional tensor, we define its tensor norm by

$$|||\ X\ ||| = \|x_P\|,$$

where $\|.\|$ is the vector norm.

Below, we provide some examples of tensor norms.
Let $X = (a_{i_1,i_2,\ldots,i_m})$ be an m^{th}-order n-dimensional tensor.

- Tensor F-norm: $|||\ X\ |||_F = \left(\sum_{i_1,i_2,\ldots,i_m=1}^{n} |\ a_{i_1,i_2,\ldots,i_m}\ |^2\right)^{\frac{1}{2}}$.
- Tensor 1-norm: $|||\ X\ |||_1 = \|x_P\|_1$.
- Tensor ∞-norm: $|||\ X\ |||_\infty = \|x_P\|_\infty$.
- Tensor $2-$norm: $|||\ X\ |||_2 = \|x_P\|_2$.

Definition 2.7. If $\|.\|$ is the matrix norm, the function $|||\ .\ |||_P$ defined on $\mathbb{C}^{(2m,n)}$ by

$$|||\ A\ |||_P = \|A_P\|$$

is called the tensor P-norm associated with the permuation matrix P. Also, $|||\ .\ |||_P$ is said to be consistent with the tensor norm on $\mathbb{C}^{(m,n)}$ if

$$|||\ A \cdot X\ ||| \leq |||\ A\ |||_P\ |||\ X\ |||,$$

where $X \in \mathbb{C}^{(m,n)}$ and $|||\ .\ |||$ is the tensor norm.

The following examples are tensor P-norms consistent with the tensor norm $|||\ .\ |||$.
Let $A = (a_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m})$ be a $2m^{\text{th}}$-order n-dimensional tensor.

- Tensor P_F-norm: $|||\ A\ |||_{P_F}=\left(\sum_{k_1,k_2,\ldots,k_m=1}^{n} \mid a_{i_1,i_2,\ldots,i_m,k_1,k_2,\ldots,k_m} \mid^2\right)^{\frac{1}{2}}=$ $\|A_P\|_F$, where A_P is the square marix unfolding of A. This tensor P_F-norm is consistent with the tensor norm $|||\ .\ |||_F$, because

$$|||\ A.X\ |||_F = \left(\sum_{i_1,i_2,\ldots,i_m=1}^{n} \mid \sum_{k_1,k_2,\ldots,k_m=1}^{n} a_{i_1,i_2,\ldots,i_m,k_1,k_2,\ldots,k_m} x_{k_1,k_2,\ldots,k_m} \mid^2 \right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{k_1,k_2,\ldots,k_m=1}^{n} \mid x_{k_1,k_2,\ldots,k_m} \mid^2 \right)^{\frac{1}{2}}$$

$$\left(\sum_{i_1,i_2,\ldots,i_m=1}^{n} \sum_{k_1,k_2,\ldots,k_m=1}^{n} \mid a_{i_1,i_2,\ldots,i_m,k_1,k_2,\ldots,k_m} \mid^2 \right)^{\frac{1}{2}}$$

$$=|||\ X\ |||_F|||\ A\ |||_{P_F}\ .$$

- Tensor P_1-norm: $|||\ A\ |||_{P_1}=\|A_P\|_1$.

- Tensor P_∞-norm: $|||\ A\ |||_{P_\infty}=\|A_P\|_\infty$.

- Tensor P_2-norm: $|||\ A\ |||_{P_2}=\|A_P\|_2$.

It is clear that $|||\ .\ |||_{P_1}, |||\ .\ |||_{P_2}$ and $|||\ .\ |||_{P_\infty}$ are consistent with the tensor norms $|||\ .\ |||_1, |||\ .\ |||_2$ and $|||\ .\ |||_\infty$.

Next, we let $\mathbb{F}$ be a subset of $\mathbb{C}^{(2m,n)}$ such that $A=(a_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m}) \in \mathbb{F}$ if and only if

$$a_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = a_{j_1,j_2,\ldots,j_m,k_1,k_2,\ldots,k_m}$$

for any $i_1, i_2, \ldots, i_m, j_1, j_2, \ldots, j_m$ and $(k_1, k_2, \ldots, k_m)$ being any permutation of $(i_1, \ldots, i_m)$. It is clear $\mathbb{F}$ is a vector space and the set of even-order symmetric tensors is a subset of $\mathbb{F}$.

Definition 2.8. Let $|||\ .\ |||_{max}$ be a norm on $\mathbb{F}$ defined by

$$|||\ A\ |||_{max} = \max_{X\neq 0} \frac{|||\ A.X\ |||}{|||\ X\ |||},$$

where $|||\ .\ |||$ is a tensor norm on $\mathbb{C}^{(m,n)}$ and $|||\ A.X\ ||| = \|A_P x_P\|$.

This norm has the following properties.

- $|||\ I_E\ |||=1$.

- For any $X \in \mathbb{C}^{(m,n)}$,

$$|||\ A \cdot X\ ||| \leq |||\ A\ |||_{max}\ |||\ X\ |||.$$

The following examples are norms of this type.

- $|||\ A\ |||_{max_1} = \max_{X\neq 0} \frac{|||A.X|||_1}{|||X|||_1}$.
- $|||\ A\ |||_{max_2} = \max_{X\neq 0} \frac{|||A.X|||_2}{|||X|||_2}$.
- $|||\ A\ |||_{max_\infty} = \max_{X\neq 0} \frac{|||A.X|||_\infty}{|||X|||_\infty}$.

Definition 2.9. Let A be a $2m^{\text{th}}$-order n-dimensional tensor. The numerical range of A is defined by

$$W_{|||.|||_P}(A) = \bigcap_{\lambda\in\mathbb{C}} \{\mu \in \mathbb{C} :|\ \mu - \lambda\ | \leq |||\ A - \lambda I_E\ |||_P\},$$

where $|||\ .\ |||_P$ is a tensor P-norm consistent with the tensor norm, and I_E is a $2m^{\text{th}}$-order n-dimensional tensor.

Theorem 2.10. *Let A, B and C be $2m^{\text{th}}$-order n-dimensional tensors. If $|||\ .\ |||_P$ is the tensor P-norm consistent with the tensor norm, then the following hold.*

(i) $W_{|||.|||_P}\ (A)$ *is a compact and convex set.*

(ii) $W_{|||.|||_P}\ (A + \alpha I_E) = W_{|||.|||_P}\ (A) + \alpha$, *for any* $\alpha \in \mathbb{C}$.

(iii) $W_{|||.|||_P}\ (\alpha A) = \alpha W_{|||.|||_P}\ (A)$, *for any* $\alpha \in \mathbb{C}$.

(iv) $W_{|||.|||_P}\ (A + B) \subset W_{|||.|||_P}\ (A)\ + W_{|||.|||_P}\ (B)$.

(v) If $A, C \in \mathbb{C}^{(m,n)}$, *then* $W_{|||.|||_{max_2}}\ (A \oplus C) = Co(W_{|||.|||_{max_2}}\ (A) \cup W_{|||.|||_{max_2}}\ (C))$.

Using the CP decomposition, we can find a relationship between numerical ranges and singular values of even-order tensors.

CP decomposition. [3] Let A be a $2m^{\text{th}}$-order n-dimensional tensor. If there exist a positive integer r, scalars α_j for $j \in [r]$, and vectors $x^{(j,i)}$ with $\|x^{(j,i)}\|_2 = 1$ for $i \in [2m]$ and $j \in [r]$ such that

$$A = \sum_{j=1}^{r} \alpha_j x^{(j,1)} \otimes \cdots \otimes x^{(j,2m)} \qquad (2),$$

then (2) is said be a CP decomposition of A. It is easy to see A always admits such a tensor decomposition when r is sufficiently large . The minimal value of r is called the rank of A.

Theorem 2.11. *If* $A \in \mathbb{C}^{(2m,n)}$ *and* $rank(A) = m$, *then*

$$A = U \star D \star V^*,$$

where $U, V, D \in \mathbb{C}^{(2m,n)}$, *and D is the diagonal tensor whose measure of the diagonal entries is equal to the square root of the eigenvalues of* $A \star A^*$.

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A note on Maps preserving the local spectral subspace of skew-product of operators

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Abstract

Let $B(H)$ be the algebra of all bounded linear operators on infinite-dimensional complex Hilbert space H. For $T \in B(H)$ and $\lambda \in \mathbb{C}$, let $H_T(\{\lambda\})$ denotes the local spectral subspace of T associated with $\{\lambda\}$. We show that if an additive map $\varphi : B(H) \longrightarrow B(H)$ has a range containing all operators of rank at most two and satisfies

$$H_{\varphi(T)\varphi(S)^*}(\{\lambda\}) = H_{TS^*}(\{\lambda\})$$

for all $T, S \in B(H)$ and $\lambda \in \mathbb{C}$, then there exist two unitary operators U and V in $B(H)$ such that $\varphi(T) = UTV^*$ for all $T \in B(H)$. Also, we obtain some interesting results in this direction.

Keywords: Local spectrum, Local spectral subspace, Nonlinear preservers, Rank-one operators

Mathematics Subject Classification [2010]: Primary 47A11, Secondary 47A15, 47B48

1 Introduction

Throughout this paper, H and K are two infinite-dimensional complex Hilbert spaces. As usual $B(H, K)$ denotes the space of all bounded linear operators from H into K. When $H = K$ we simply write $B(H)$ instead of $B(H, H)$, and its unit will be denoted by I. The inner product of H or K will be denoted by $\langle , \rangle$ if there is no confusion.
Linear preserver problems, in the most general setting, demand the characterization of linear maps between algebras that leave a certain property, a particular relation, or even a subset invariant. This subject is very old and goes back well over a century to the so-called first linear preserver problem, due to Frobenius [6], that determines linear maps preserving the determinant of matrices. The study of linear and nonlinear local spectra preserver problems attracted the attention of a number of authors. Bourhim and Ransford were the first ones to consider this type of preserver problem, characterizing in [5] additive maps on the algebra of all linear bounded operators on a complex Banach space X that preserve the local spectrum of operators at each vector of X. Their results cleared the way for several authors to describe maps on matrices or operators that preserve local spectrum, local spectral radius, and local inner spectral radius; see, for instance, the last section of the survey article [3] and the references therein. Let $B(X)$ be the algebra of all bounded linear operators on a complex Banach space X and its unit will

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be denoted by I. The local resolvent set, $\rho_T(x)$, of an operator $T \in B(X)$ at some point $x \in X$ is the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood U of λ in $\mathbb{C}$ and a X-valued analytic function $f : U \longrightarrow X$ such that $(\mu I - T)f(\mu) = x$ for all $\mu \in U$. The complement of local resolvent set is called the local spectrum of T at x, denoted by $\sigma_T(x)$, and is obviously a closed subset (possibly empty) of $\sigma(T)$, the spectrum of T. The local spectral radius of T at x is given by $r_T(x) := \limsup_{n\longrightarrow\infty} \|T^n(x)\|^{\frac{1}{n}}$, and coincides with the maximum modulus of $\sigma_T(x)$ provided that T has the single-valued extension property. We recall that an operator $T \in B(X)$ is said to have the single-valued extension property (henceforth abbreviated to SVEP) provided that for every open subset U of $\mathbb{C}$, the equation

$$(\mu I - T)f(\mu) = 0, \quad \forall \mu \in U,$$

has no nontrivial analytic solution f. Every operator $T \in B(X)$ for which the interior of its point spectrum, $\sigma_p(T)$, is empty enjoys this property.

For every subset $F \subseteq \mathbb{C}$ the local spectral subspace $X_T(F)$ is defined by

$$X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}.$$

Clearly, if $F_1 \subseteq F_2$ then $X_T(F_1) \subseteq X_T(F_2)$. The book by P. Aiena [1] provide an excellent exposition as well as a rich bibliography of the local spectral theory.

In [2], H. Benbouziane et al. characterized the forms of surjective weakly continuous maps φ from $B(X)$ into $B(X)$ which satisfy

$$X_{\varphi(T)-\varphi(S)}(\{\lambda\}) = X_{T-S}(\{\lambda\}), \quad (T, S \in B(X), \lambda \in \mathbb{C}).$$

In this paper, we investigate the form of all maps φ on $B(H)$ such that, for every T and S in $B(H)$, the local spectral subspaces of TS^* and $\varphi(T)\varphi(S)^*$ are the same associated with the singleton $\{\lambda\}$.

The first lemma summarizes some known basic properties of the local spectrum.

Lemma 1.1. *[1] Let X be a Banach space and $T \in B(X)$. For every $x, y \in X$ and a scalar $\alpha \in \mathbb{C}$ the following statements hold.*
(a) $\sigma_T(\alpha x) = \sigma_T(x)$ if $\alpha \neq 0$, and $\sigma_{\alpha T}(x) = \alpha\sigma_T(x)$.
(b) If $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then $\sigma_T(x) \subseteq \{\lambda\}$.
(c) If $S \in B(X)$ commutes with T, then $\sigma_T(Sx) \subseteq \sigma_T(x)$.
(d) $\sigma_{T^n}(x) = \{\sigma_T(x)\}^n$ for all $x \in X$ and $n \in \mathbb{N}$.

In the next theorem we collect some of the basic properties of the subspaces $X_T(F)$.

Lemma 1.2. *[1] Let $T \in B(X)$ and $F \subseteq \mathbb{C}$. The following statements hold.*
(i) $X_T(F)$ is a T-hyperinvariant subspace of X.
(ii) $(T - \lambda I)X_T(F) = X_T(F)$ for every $\lambda \in \mathbb{C}\backslash F$.
(iii) $X_T(F) = X_T(F \cap \sigma(T))$.
(iv) If $x \in X$ satisfy $(T - \lambda I)x \in X_T(F)$, then $x \in X_T(F)$.
(v) $ker(T - \lambda I) \subseteq X_T(F)$.
(vi) $X_{\alpha T}(\lambda) = X_T(\frac{\lambda}{\alpha})$ for every $\lambda \in \mathbb{C}$ and non-zero scalar α.

The nonzero local spectrum of $T \in B(H)$ at any $x \in H$ is defined by

$$\sigma_T^*(x) := \begin{cases} \{0\} & if\ \sigma_T(x) = \{0\}, \\ \sigma_T(T) \setminus \{0\} & if\ \sigma_T(x) \neq \{0\}. \end{cases}$$

For any $x, y \in H$, let $x \otimes y$ denote the operator of rank at most one on H defined by

$$(x \otimes y)z = \langle z, y \rangle x, \qquad \forall\ z \in H.$$

Note that every rank one operator in $B(H)$ can be written in this form, and that every finite rank operator $T \in B(H)$ can be written as a finite sum of rank one operators; i.e., $T = \sum_{i=1}^{n} x_i \otimes y_i$ for some $x_i, y_i \in X$ and $i = 1, 2, ..., n$. We denote by $F(H)$ the set of all finite rank operators in $B(H)$ and $F_n(H)$ the set of all operators of rank at most n, n is a positive integer.

The following lemma is an elementary observation that gives the nonzero local spectrum of any rank one operator.

Lemma 1.3. *(See [4]) Let x_0 be a nonzero vector in H. For any $x, y \in H$, we have*

$$\sigma^*_{x\otimes y}(x_0) := \begin{cases} \{0\} & if\ \langle x_0, y\rangle = 0, \\ \langle x, y\rangle & if\ \langle x_0, y\rangle \neq 0. \end{cases}$$

The following theorem, which may be of independent interest, gives a spectral characterization of rank one operators in term of local spectrum.

Theorem 1.4. *(See [4, Theorem 4.1]) For a nonzero vector $x \in H$ and a nonzero operator $R \in B(H)$, the following statements are equivalent.*
(a) R has rank one.
(b) $\sigma^*_{RT}(x)$ contains at most one element for all $T \in B(H)$.
(c) $\sigma^*_{RT}(x)$ contains at most one element for every rank two operator $T \in B(H)$.

The following result characterizes in term of the local spectrum when two operators are the same.

Lemma 1.5. *(See [4, Theorem 3.2]) For a nonzero vector x in H and two operators A and B in $B(H)$, the following statements are equivalent.*
(a) $A = B$.
(b) $\sigma_{AT}(x) = \sigma_{BT}(x)$ for all operators $T \in B(H)$.
(c) $\sigma_{AT}(x) = \sigma_{BT}(x)$ for all rank one operators $T \in B(H)$.
(d) $\sigma^*_{AT}(x) = \sigma^*_{BT}(x)$ for all rank one operators $T \in B(H)$.

2 Main results

We first establish the following lemma.

Lemma 2.1. *Let $T, S \in B(H)$. The following statements are equivalent.*
(1) $T = S$.
(2) $H_{RT^*}(\{\lambda\}) = H_{RS^*}(\{\lambda\})$ *for all $\lambda \in \mathbb{C}$ and $R \in F_1(H)$.*

Proof. We only need to establish implication (2) $\Rightarrow$ (1). □

Proposition 2.2. *If φ_1 and φ_2 are maps from $B(H)$ into $B(H)$ satisfy*

$$H_{\varphi_1(T)\varphi_2(S)^*}(\{\lambda\}) = H_{TS^*}(\{\lambda\}), \quad (T, S \in B(H), \lambda \in \mathbb{C}),$$

then the following statements hold.
(i) φ_2 is injective.
(ii) If the range of φ_1 is contains $F_2(H)$ then φ_2 is homogeneous.

Theorem 2.3. *Let $\varphi : B(H) \to B(H)$ be an additive map such that its range contains $F_2(H)$. If*

$$H_{\varphi(T)\varphi(S)^*}(\{\lambda\}) = H_{TS^*}(\{\lambda\}), \quad (T, S \in B(H), \lambda \in \mathbb{C})$$

then there exist two unitary operators U and V in $B(H)$ such that such that $\varphi(T) = UTV^*$ for all $T \in B(H)$.

Proof. The proof breaks down into several claims.

Claim 1. φ is injective and linear.

Claim 2. φ preserves rank one operators in both directions.

Claim 3. There are bijective linear mappings $A : H \to H$ and $B : H \to H$ such that $\varphi(x \otimes y) = Ax \otimes By$ for all $x, y \in H$.

Claim 4. A and B are bounded unitary operators multiplied by positive scalars α and β such that $\alpha\beta = 1$.

Claim 5. A^* and I are linearly dependent.

Claim 6. φ has the asserted form.

□

From this result, it is easy to deduce a generalization to the case of two different Banach spaces H, K.

Corollary 2.4. *Suppose $P \in B(H, K)$ be a unitary operator. Let φ be an additive map from $B(H)$ onto $B(K)$ which satisfy*

$$K_{\varphi(T)\varphi(S)^*}(\{\lambda\}) = PH_{TS^*}(\{\lambda\}), \quad (T, S \in B(H), \lambda \in \mathbb{C}).$$

Then there exists a unitary operator $Q : K \to H$ such that $\varphi(T) = PTQ$ for all $T \in B(H)$.

Corollary 2.5. *Let $P \in B(H, K)$ be an unitary operator. Let $\varphi : B(H) \to B(H, K)$ be an additive surjective map which satisfy*

$$K_{\varphi(T)\varphi(S)^*}(\{\lambda\}) = PH_{TS^*}(\{\lambda\}), \quad (T, S \in B(H), \lambda \in \mathbb{C}).$$

Then there exists an unitary operator $Q : H \to H$ such that $\varphi(T) = PTQ^*$ for all $T \in B(X)$.

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On reduced minimum modulus preservers

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Abstract

Suppose $\gamma(\cdot)$ denotes the reduced minimum modulus in a Banach algebra. We show that a continuous surjective linear map from a unital Banach algebra $\mathcal{A}$ onto a C^*-algebra $\mathcal{B}$ satisfying $\gamma(\varphi(a)\varphi(b)) = \gamma(ab)$ $(a, b \in \mathcal{A})$, makes $\mathcal{A}$ to be a C^*-algebra $*$-isomorphic to $\mathcal{B}$ up to multiplication by a central invertible element. If $\mathfrak{B}(H)$ is the C^*-algebra of all bounded linear operators on a Hilbert space H, we characterize surjective maps φ on $\mathfrak{B}(H)$ (with no linearity and continuity assumption) preserving the reduced minimum modulus of operator products.

Keywords: reduced minimum modulus, preserver maps, operators on a Hilbert space
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1 Introduction

Throughout the paper all Banach spaces are assumed on the complex field. For a given Banach space X, $\mathcal{S}(X)$ and X^* denote the unit sphere of X and the dual of X, respectively, and $\mathfrak{B}(X)$ denotes the Banach algebra of all bounded linear operators on X. For $T \in \mathfrak{B}(X)$, $R(T)$ and $\ker(T)$ denote the range and the null space of T, respectively. The unit circle in $\mathbb{C}$ will be denoted by $\mathbb{T}$. The *reduced minimum modulus* of an operator $T \in \mathfrak{B}(X)$ is defined by

$$\gamma(T) := \begin{cases} \inf\{\|Tx\| : \operatorname{dist}(x, \ker(T)) \geq 1\} & \text{if } T \neq 0, \\ \infty & \text{if } T = 0. \end{cases} \tag{1}$$

$T \in \mathfrak{B}(X)$, $\gamma(T) > 0$ if and only if $R(T)$ is closed (see [6, Part 10, Chapter II]). It is also defined for Banach algebra elements. Let $\mathcal{A}$ be a Banach algebra. For $a \in \mathcal{A}$, let L_a, R_a and $\sigma(a)$ denote the left and right multiplication operators by a and the spectrum of a, respectively. Harte and Mbekhta [5] considered the *left and right conorm* (*reduced minimum modulus*) of $a \in \mathcal{A}$ as follows

$$\gamma^{left}(a) = \gamma(L_a) = \inf\{\|ax\| : \operatorname{dist}(x, \ker(L_a)) \geq 1\},$$

where $\ker(L_a) = \{x \in \mathcal{A} : ax = 0\}$. Similarly,

$$\gamma^{right}(a) = \gamma(R_a) = \inf\{\|xa\| : \operatorname{dist}(x, \ker(R_a)) \geq 1\},$$

and $\ker(R_a) = \{x \in \mathcal{A} : xa = 0\}$. The *reduced minimum modulus* of $a \in \mathcal{A}$ is defined by $\gamma(a) := \gamma(L_a)$. If $\mathcal{A}$ is a C^*-algebra, $\gamma(L_a) = \gamma(R_a)$ (see [5, Theorem 4]). It is proved that if

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a is invertible then $\gamma(a) = \|a^{-1}\|^{-1}$, see [5]. Suppose $\mathrm{Lan}(\mathcal{A})$ and $\mathrm{Ran}(\mathcal{A})$ denote the left and right annhilators of $\mathcal{A}$, respectively. Obviously, $a \in \mathrm{Lan}(\mathcal{A})$ if and only if $L_a = 0$ if and only if $\gamma(a) = +\infty$. Let $\mathcal{A}$ be a Banach algebra with a right (resp. left) approximate identity (e.g. a C^*-algebra), then $\mathrm{Lan}(\mathcal{A}) = \{0\}$ (resp. $\mathrm{Ran}(\mathcal{A}) = \{0\}$). So, for $a \in \mathcal{A}$, $\gamma(a) = +\infty$ if and only if $a = 0$. Also, if $\mathcal{A}$ is a C^*-algebra, then

$$\gamma(a)^2 = \inf\{\lambda : \lambda \in \sigma(a^*a) \setminus \{0\}\} \tag{2}$$

for all $a \in \mathcal{A}$, [5, Theorem 4]. Consequently, $\gamma(a) = \gamma(a^*a)^{\frac{1}{2}} = \gamma(aa^*)^{\frac{1}{2}} = \gamma(a^*)$. So, $\gamma(a)^2 = \gamma(a^2)$ whenever $a = a^*$. Moreover, if $u, v \in \mathcal{A}$ are unitary elements, then by definition $\gamma(uav) = \gamma(av)$ for all $a \in \mathcal{A}$. However, $\gamma(av) = \gamma(v^*a^*) = \gamma(a^*) = \gamma(a)$. Therefore, $\gamma(uav) = \gamma(a)$ for all $a \in \mathcal{A}$.

Let H be a Hilbert space. We denote by $\mathfrak{R}_1(H)$ the set of all rank one operators on H. We recall that every rank one operator T in $\mathfrak{B}(H)$ is of the form $T = x \otimes y$ for some $x, y \in H$, and $(x \otimes y)^* = y \otimes x$. So, $(x \otimes y)^*(x \otimes y) = (y \otimes x)(x \otimes y) = \|x\|^2 y \otimes y$. Thus, $\sigma((x \otimes y)^*(x \otimes y)) = \{0, \|x\|^2\|y\|^2\}$. Since $\mathfrak{B}(H)$ is a C^*-algebra, we have $\gamma(x \otimes y) = \|x\|\|y\|$.

We study surjective maps preserving the reduced minimum modulus of products. First, we assume that $\mathcal{A}$ is a unital Banach algebra *topologically generated by its idempotents*, that is $\mathcal{A}$ is the norm closure of its subalgebra generated by idempotents. If $\mathcal{B}$ is a C^*-algebra, and $\varphi : \mathcal{A} \to \mathcal{B}$ is a surjective continuous linear map satisfying $\gamma(ab) = \gamma(\varphi(a)\varphi(b))$ for all $a, b \in \mathcal{A}$, then we show that $\mathcal{B}$ is unital, $\varphi(1)$ is a central invertible element in $\mathcal{B}$ and $\varphi(1)^{-1}\varphi$ is an isomorphism. We also show that $\mathcal{A}$ is a C^*-algebra, $*$-isomorphic to $\mathcal{B}$. Then, we assume that H is a complex Hilbert space of dimension greater than 2 and study surjective maps (with no linearity and continuity assumption) on $\mathfrak{B}(H)$ preserving the reduced minimum modulus of operator products. We show that a surjective map ϕ on $\mathfrak{B}(H)$ preserves the reduced minimum modulus of products if and only if ϕ is a linear or conjugate linear $*$-automorphism multiplied by a partial isometry. More precisely, $\phi(T) = U_T\psi(T) = \psi(T)V_T^*$ for all $T \in \mathfrak{B}(H)$, where ψ is a linear or conjugate linear $*$-automorphism and U_T, V_T are partial isometries on $\overline{R(\psi(T))}$ and $\overline{R(\psi(T)^*)}$, respectively. Finally, we study surjections on $\mathfrak{B}(H)$ preserving the reduced minimum modulus of Jordan triple products of operators.

2 Preserving reduced minimum modulus of products on Banach algebras

If $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras with $\mathrm{Lan}(\mathcal{B}) = \{0\}$ and $\varphi : \mathcal{A} \to \mathcal{B}$ is a map satisfying

$$\gamma(\varphi(a)\varphi(b)) = \gamma(ab) \quad (a, b \in \mathcal{A}), \tag{3}$$

then obviously, for $a, b \in \mathcal{A}$, $ab = 0 \Rightarrow \varphi(a)\varphi(b) = 0$. Thus preserving zero product plays an important role in our arguments.

Lemma 2.1. *Let $\mathcal{A}$ be a unital Banach algebra which is topologically generated by its idempotents. Suppose $\mathcal{B}$ is a C^*-algebra and $\varphi : \mathcal{A} \to \mathcal{B}$ is a surjective continuous linear map preserving zero products in both directions. Then,*

(i) *φ is injective.*

(ii) *$\mathcal{B}$ is unital, $\varphi(1)$ is a central invertible element and $\varphi(1)^{-1}\varphi$ is an isomorphism.*

Proof. It is proved that φ is injective and [2, Lemma 2.1 and Theorem 2.2] leads to (ii). □

Remark 2.2. We recall that if $\mathcal{A}$ is a Banach algebra and Φ is an isomorphism from $\mathcal{A}$ onto a C^*-algebra $\mathcal{B}$ (which is automatically continuous), then it follows from defining $a^* = \Phi^{-1}(\Phi(a)^*)$ and $\|a\|_0 = \|\Phi(a)\|$ $(a \in \mathcal{A})$ that $(\mathcal{A}, *, \|\cdot\|_0)$ is a C^*-algebra. Being isomorphic to $\mathcal{B}$, the Banach algebra $\mathcal{A}$ is semisimple and so $\|\cdot\|_0$ is equivalent to the original norm on $\mathcal{A}$, and the two norms coincide if and only if Φ is an isometry.

In the following theorem we show that if $\mathcal{A}$ and $\mathcal{B}$ satisfy the conditions of Lemma 2.1 and if $\varphi : \mathcal{A} \to \mathcal{B}$ is a continuous surjective linear map satisfying (3), then $\mathcal{A}$ is a C^*-algebra with respect to its original norm, which is $*$-isomorphic to $\mathcal{B}$.

Theorem 2.3. *Let $\mathcal{A}$ be a unital Banach algebra which is topologically generated by its idempotents. Suppose $\mathcal{B}$ is a C^*-algebra and $\varphi : \mathcal{A} \to \mathcal{B}$ is a surjective continuous linear map satisfying*

$$\gamma(\varphi(a)\varphi(b)) = \gamma(ab). \quad (a, b \in \mathcal{A})$$

Then, $\mathcal{A}$ is a C^*-algebra which is $*$-isomorphic to $\mathcal{B}$.

Proof. Since φ satisfies (3), it preserves zero products in both directions. So by Lemma 2.1, φ is injective, $\mathcal{B}$ is unital, $\varphi(1)$ is a central invertible element and $\Psi : \mathcal{A} \to \mathcal{B}$ defined by $\Psi(a) = \varphi(1)^{-1}\varphi(a)$ $(a \in \mathcal{A})$ is an isomorphism. By using [1, Lemma 4.1], one can see that $\mathcal{A}$ is a C^*-algebra and Ψ is a $*$-isomorphism. □

3 Maps preserving reduced minimum modulus of operator products

Let H be a complex Hilbert space of dimension ≥ 3 and let $\mathcal{U}(H)$ denote the set of unitaries on H. In the sequel, we describe a surjective (with no linearity and continuity assumption) map $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ satisfying

$$\gamma\big(\phi(T)\phi(S)\big) = \gamma(TS) \quad (T, S \in \mathfrak{B}(H)). \tag{4}$$

As $\mathfrak{B}(H)$ is a C^*-algebra, (4) implies that ϕ preserves zero product. In addition, $\gamma(p) = \inf\{\lambda : \lambda \in \sigma(p^*p) \setminus \{0\}\}\big)^{\frac{1}{2}} = 1$ for all projections $p \in \mathfrak{B}(H)$. Particularly, $\gamma(.)$ is constant on the set of all rank one projections. So, we have the same characterization as in [3, Theorem 2.3] on $\mathfrak{R}_1(H)$. Hence by a similar discussion leading to [3, Theorem 3.2], we see that a surjective map ϕ on $\mathfrak{B}(H)$ satisfies (4) if and only if there exist a unitary or an anti-unitary U_0 in $\mathfrak{B}(H)$ and functions $h_1, h_2 : \mathfrak{B}(H) \to \mathcal{U}(H)$ satisfying $h_1(T)T = Th_2(T)$ for all $T \in \mathfrak{B}(H)$, such that

$$\phi(T) = U_0 h_1(T) T U_0^* = U_0 T h_2(T) U_0^*, \tag{5}$$

for all $T \in \mathfrak{B}(H)$.

Here by using properties of γ, we are going to find further necessary and sufficient conditions for ϕ to satisfy (4). To prove our main results we need the following Lemma.

Lemma 3.1. *Let $A, B \in \mathfrak{B}(H)$. Then the following statements are equivalent.*

(i) $\gamma(AT) = \gamma(BT)$ *for all* $T \in \mathfrak{B}(H)$.

(ii) $\gamma(AT) = \gamma(BT)$ *for all* $T \in \mathfrak{R}_1(H)$.

(iii) $|A| = |B|$.

The following statements are also equivalent.

(i)′ $\gamma(TA) = \gamma(TB)$ *for all* $T \in \mathfrak{B}(H)$.

(ii)′ $\gamma(TA) = \gamma(TB)$ *for all* $T \in \mathfrak{R}_1(H)$.

(iii)′ $|A^*| = |B^*|$.

We can state the following proposition by applying (5) and Lemma 3.1.

Proposition 3.2. *Let H be a complex Hilbert space with $dimH \geq 3$, and $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ a surjective map. Then ϕ satisfies (4) if and only if there exists a linear or conjugate linear $*$-automorphism $\psi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ such that $|\phi(T)| = |\psi(T)|$ and $|\phi(T)^*| = |\psi(T)^*|$ for all $T \in \mathfrak{B}(H)$.*

The following Lemma and Proposition 3.2 leads to the next Theorem.

Lemma 3.3. *Let $T \in \mathfrak{B}(H)$ and U be a partial isometry on $\overline{R(T)}$. Then, $\gamma(UTS) = \gamma(TS)$ for all $S \in \mathfrak{B}(H)$.*

Theorem 3.4. *Let H be a complex Hilbert space with $dimH \geq 3$, and $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ a surjective map. Then ϕ satisfies (4) if and only if*

$$\phi(T) = U_T\psi(T) = \psi(T)V_T^* \quad (T \in \mathfrak{B}(H)),$$

where ψ is a linear or conjugate linear $*$-automorphism on $\mathfrak{B}(H)$ and for each $T \in \mathfrak{B}(H)$, U_T , V_T are partial isometries on $\overline{R(\psi(T))}$, $\overline{R(\psi(T)^*)}$, respectively. As a consequence, there is a unitary or anti-unitary operator U on H such that $\phi(T) = U_TUTU^* = UTU^*V_T^*$.

We recall that the Jordan triple product of operators T, S is TST. In the sequel, we consider a surjective maps ϕ on $\mathfrak{B}(H)$ satisfying

$$\gamma(\phi(T)\phi(S)\phi(T)) = \gamma(TST) \quad (T, S \in \mathfrak{B}(H)). \tag{6}$$

It is easily seen that such a map preserves zero Jordan triple product in both directions, that is

$$TST = 0 \Longleftrightarrow \phi(T)\phi(S)\phi(T) = 0. \tag{7}$$

We apply the characterization of maps satisfying (7) [4] to find a finer characterization for maps satisfying (6).

Remark 3.5. (1) Applying [4, Theorem 2.2], we conclude that if H is infinite dimensional and a surjection ϕ on $\mathfrak{B}(H)$ satisfies (7), then there is a function $\mu : \mathfrak{B}(H) \to \mathbb{C} \setminus \{0\}$ and a bounded invertible linear or conjugate linear operator $A : H \to H$ such that either

(a) $\phi(T) = \mu(T)ATA^{-1} \quad (T \in \mathfrak{B}(H))$, or,

(b) $\phi(T) = \mu(T)AT^\star A^{-1} \quad (T \in \mathfrak{B}(H))$.

Here $T^\star$ denotes the Banach space adjoint of $T \in \mathfrak{B}(H)$. If J is the conjugate linear isomorphism from H onto its dual H^*, then it is easily seen that $T^\star = JT^*J^{-1}$, for all $T \in \mathfrak{B}(H)$. Therefore,

$$\phi(T) = \mu(T)AJT^*J^{-1}A^{-1} \quad (T \in \mathfrak{B}(H)).$$

Clearly, AJ is linear or conjugate linear depending on A is conjugate linear or linear, respectively. Renaming AJ into A, we arrive at

(b)′ $\phi(T) = \mu(T)AT^*A^{-1}$, for all $T \in \mathfrak{B}(H)$,

where A is a linear or conjugate linear invertible operator.

(2) Suppose that $H = \mathbb{C}^n$, $n \geq 3$, and that $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a surjective map satisfying (7). Applying [4, Theorem 2.1] shows that there exist an invertible matrix $S \in M_n(\mathbb{C})$, a field automorphism $f_0 : \mathbb{C} \to \mathbb{C}$, and a scalar function $\mu : M_n(\mathbb{C}) \to \mathbb{C} \setminus \{0\}$ such that one of the following holds:

(c) $\phi(A) = \mu(A)Sf(A)S^{-1}$ $(A \in M_n(\mathbb{C}))$, or

(d) $\phi(A) = \mu(A)Sf(A)^{tr}S^{-1}$ $(A \in M_n(\mathbb{C}))$, where $f([a_{ij}]) = [f_0(a_{ij})]$.

In the two following theorems, we show that if a surjective map ϕ on $\mathfrak{B}(H)$ satisfies (6), then the invertible operators A and S in Remark 3.5(1)-(2) can be replaced by unitaries and moreover, $|\mu| = 1$. As a consequence, ϕ is norm preserving.

Theorem 3.6. *Let H be an infinite dimensional complex Hilbert space. A surjective map $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ satisfies (6) if and only if there exist a function $\mu : \mathfrak{B}(H) \to \mathbb{T}$ and a unitary or anti-unitary operator U on H such that either $\phi(T) = \mu(T)UTU^*$ or $\phi(T) = \mu(T)UT^*U^*$, for all $T \in \mathfrak{B}(H)$.*

The proof of the following theorem follows the same line as the proof of [4, Theorem 4.1].

Theorem 3.7. *Suppose $n \geq 3$, then $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfies (6) if and only if there exists a unitary matrix U and a function $\mu : M_n(\mathbb{C}) \to \mathbb{T}$ such that either*

(i) $\phi(A) = \mu(A)Uf(A)U^*$, *or*

(ii) $\phi(A) = \mu(A)U(f(A))^{tr}U^*$

for all $A = [a_{ij}] \in M_n(\mathbb{C})$. We have $f([a_{ij}]) = [f_0(a_{ij})]$ where, $f_0 : \mathbb{C} \to \mathbb{C}$ is the identity or the complex conjugate on $\mathbb{C}$.

4 Conclusion

A continuous linear map φ preserving the reduced minimum modules of products from a Banach algebra $\mathcal{A}$ topologically generated by its idempotents onto a C^*-algebra, makes $\mathcal{A}$ to be a C^*-algebra. Surjective maps on $\mathfrak{B}(H)$ preserving the reduced minimum modulus of products are $*$-isomorphisms multiplied by partial isometries and hence are isometries.

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A new preconditioning technique for SOR algorithm for solving multi-linear systems

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Abstract

Preconditioning techniques are useful procedures in increasing the rate of convergence of iterative methods and in some cases in eliminating possible stagnation in solving multi-linear systems with nonsingular $\mathcal{M}$-tensors. In this paper, we propose a novel preconditioner obtained by minimizing the norm of the iteration tensor. We also consider a preconditioned SOR iterative method for solving tensor equations whose coefficient tensor is an $\mathcal{M}$-tensor. Numerical examples, and comparison results are given to show the efciency of the preconditioner.

Keywords: Preconditioner, Multi-linear systems, $\mathcal{M}$-tensor, SOR method

Mathematics Subject Classification [2010]: 65F08, 65F10

1 Introduction

Tensor equations have many applications in engineering, and scientic computing [2], such as evolutionary game dynamics [5], partial differential equations, and data mining [1], and image processing [3].

Consider the following tensor equation of the form

$$\mathcal{A}x^{m-1} = b, \tag{1}$$

where $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is an order m dimension n tensor, x, and b are vectors in $\mathbb{C}^n$. The tensor-vector product is a vector where the entries are defined by

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2 i_3 \cdots i_n = 1}^{n} a_{i i_2 i_3 \cdots i_n} x_{i_2} x_{i_3} \cdots x_{i_n}, \qquad i = 1, 2, \cdots, n,$$

where x_i denotes the ith component of x. It can be seen that multi-linear systems are made up of a series of non-linear equations.

Many theoretical analyses, and algorithms were presented for solving (1). It is proofed that (1) will have a unique positive solution if $\mathcal{A}$ be a nonsingular $\mathcal{M}$-tensor, and b be a positive vector. In addition, some conditions were presented for the existence and uniqueness of the solution of (1).

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The role of the preconditioning technique is clear in solving linear and multi-linear systems, which can improve the convergence rate of the method if a suitable preconditioner is chosen. Lots of efficient preconditioners were proposed to solve linear systems. Although a little research introduced preconditioned methods for solving multi-linear systems. Li et al. [6] proposed the preconditioned tensor splitting method for solving the following preconditioned multi-linear systems (1):

$$P\mathcal{A}x^{m-1} = Pb,$$

where P is a preconditioner and the iterative scheme is as follows:

$$x_k = (M(\mathcal{E}_P)^{-1}\mathcal{F}_P x_{k-1}^{m-1} + M(\mathcal{E}_P)^{-1}Pb)^{[\frac{1}{m-1}]}, \qquad k = 1, 2, \cdots,$$

so that $P\mathcal{A} = \mathcal{E}_P - \mathcal{F}_P$ is a tensor splitting of $P\mathcal{A}$. A modified preconditioned Gauss-Seidel method was proposed [4].

In this paper, we proposed a diagonal preconditioner for the SOR method to solve multi-linear systems. We apply the new preconditioned SOR method to some Numerical examples and compare the new method to the original SOR method. Numerical experiments and comparison results confirm the power of the preconditioner.

This paper is organized as follows. In Section 2, a new preconditioner is introduced, and the preconditioned SOR method is constructed. Section 3 consist of some numerical examples which demonstrate the efficiency of the presented preconditioned iterative method. The final section consists of conclusion.

2 Main results

In this section, we propose a diagonal preconditioner by minimizing the Frobenius norm of $\mathcal{I}_m - P\mathcal{A}$. Consider the following multi-linear system:

$$\mathcal{A}x^{m-1} = b, \tag{2}$$

where $\mathcal{A} \in \mathbb{R}^{[3,n]}$, x and b are vectors in $\mathbb{C}^n$. Applying a nonsingular matrix P as a preconditioner, we have

$$P\mathcal{A}x^{m-1} = Pb. \tag{3}$$

We can add x^{m-1} to both sides of (3). So

$$x^{m-1} = (\mathcal{I}_m - P\mathcal{A})x^{m-1} + Pb,$$

which implies

$$x_{k+1}^{m-1} = (\mathcal{I}_m - P\mathcal{A})x_k^{m-1} + Pb.$$

When we choose $P = \alpha I$, a constant diagonal matrix will appear. So the iteration transforms to the Richardson iteration.

$$x_{k+1}^{m-1} = (\mathcal{I}_m - \alpha\mathcal{A})x_k^{m-1} + \alpha b,$$

where α is the value of the diagonal entries.

We use the Frobenius norm, and we want to make a diagonal preconditioning matrix P. We denote the set of a diagonal matrix of size n by Δ_n. The optimization problem will be as

$$\min_{P\in\Delta_n} \|\mathcal{I}_m - P\mathcal{A}\|_F, \qquad or \qquad \min_{P\in\Delta_n} \|\mathcal{I}_m - P\mathcal{A}\|_F^2.$$

Finding the entries of P, we define the preconditioner by

$$P = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix},$$

then

$$\min_{\alpha_1,\alpha_2,\cdots,\alpha_n} \|\mathcal{I}_m - P\mathcal{A}\|_F.$$

The tensor $P\mathcal{A}$ is a tensor where the ith row of each frontal slice of $\mathcal{A}$ is multiplied by α_i, because P is diagonal. We call it row scaling.

By the definition of the Frobenius norm and t-product, we have

$$\begin{aligned} \|\mathcal{I}_m - P\mathcal{A}\|_F^2 &= tr\{[(\mathcal{I}_m - P\mathcal{A}) * (\mathcal{I}_m - P\mathcal{A})^T]_{(:,:,1)}\} \\ &= tr\{[\mathcal{I}_m * \mathcal{I}_m^T - \mathcal{I}_m * (P\mathcal{A})^T - (P\mathcal{A}) * \mathcal{I}_m + (P\mathcal{A}) * (P\mathcal{A})^T]_{(:,:,1)}\} \\ &= 1 - 2tr(\mathcal{I}_m * (P\mathcal{A})^T)_{(:,:,1)} + tr((P\mathcal{A}) * (P\mathcal{A})^T)_{(:,:,1)} \end{aligned}$$

where

$$tr(\mathcal{I}_m * (P\mathcal{A})^T)_{(:,:,1)} = \sum_{i=1}^{n} \alpha_i a_{iii},$$

and

$$tr((P\mathcal{A}) * (P\mathcal{A})^T)_{(:,:,1)} = \sum_{i=1}^{n} \alpha_i^2 \|\mathcal{A}_{(i,:,:)}\|_2^2.$$

Therefore

$$\|\mathcal{I}_m - P\mathcal{A}\|_F^2 = 1 - 2\sum_{i=1}^{n} \alpha_i a_{iii} + \sum_{i=1}^{n} \alpha_i^2 \|\mathcal{A}_{(i,:,:)}\|_2^2.$$

$\|\mathcal{I}_m - P\mathcal{A}\|_F^2$ is a convex function in the α_i, and we can find the minimize. We can set the partial derivatives with respect to $\alpha_i,\ i = 1, 2, \cdots, n$ equal to zero. Thus

$$\frac{d}{d\alpha_i}\|\mathcal{I}_m - P\mathcal{A}\|_F^2 = -2\sum_{i=1}^{n} a_{iii} + 2\sum_{i=1}^{n} \alpha_i \|\mathcal{A}_{(i,:,:)}\|_2^2 = 0.$$

Since α_i is the only varibale of each equation, then

$$\alpha_i = \frac{a_{iii}}{\|\mathcal{A}_{(i,:,:)}\|_2^2}, \quad i = 1, 2, \cdots, n.$$

Accordingly, the diagonal preconditioner which is optimal in the Frobenius norm can be defined as follows

$$P = \begin{pmatrix} \frac{a_{111}}{\|\mathcal{A}_{(1,:,:)}\|_2^2} & & & \\ & \frac{a_{222}}{\|\mathcal{A}_{(2,:,:)}\|_2^2} & & \\ & & \ddots & \\ & & & \frac{a_{nnn}}{\|\mathcal{A}_{(n,:,:)}\|_2^2} \end{pmatrix}.$$

We only consider that the coefcient tensor of (1) is a strong $\mathcal{M}$-tensor. Without loss of generality, we assume that each diagonal entry of the tensor $\mathcal{A}$ is 1. We have the following preconditioned multi-linear system with our new preconditioner:

$$\hat{\mathcal{A}}x^{m-1} = \hat{b},$$

where $\hat{\mathcal{A}} = P\mathcal{A}$ and $\hat{b} = Pb$. Consider

$$\hat{\mathcal{A}} = \hat{\mathcal{D}} - \hat{\mathcal{L}} - \hat{\mathcal{F}}, \quad or \quad \hat{\mathcal{A}} = \hat{\mathcal{I}}_m - \hat{\mathcal{L}} - \hat{\mathcal{F}},$$

with $\hat{\mathcal{D}} = \hat{D}\mathcal{I}_m$, $\hat{\mathcal{L}} = \hat{L}\mathcal{I}_m$, where D is the positive diagonal matrix, $-L$ is the strictly lower triangle matrix of $M(\hat{\mathcal{A}})$. We take the preconditioned SOR method as:

$$x_k = (\mathcal{T}_p x_{k-1}^{m-1} + q_p)^{[\frac{1}{m-1}]}, \qquad k = 1, 2, \cdots,$$

where

$$\begin{aligned}
\mathcal{T}_p &= M(\hat{\mathcal{E}}_p)^{-1}\hat{\mathcal{F}}_p, \\
\hat{\mathcal{E}}_p &= \frac{1}{\omega}(\hat{\mathcal{D}} - \omega\hat{\mathcal{L}}), \\
\hat{\mathcal{F}}_p &= (1-\omega)\hat{\mathcal{D}} - \omega\hat{\mathcal{F}}, \\
q_p &= M(\hat{\mathcal{E}}_p)^{-1}b.
\end{aligned}$$

Theorem 2.1. *Let $\mathcal{A} \in \mathbb{R}^{[3,n]}$ be a strong $\mathcal{M}$-tensor. Then for the new preconditioner P, $\hat{\mathcal{A}} = P\mathcal{A}$ is a strong $\mathcal{M}$-tensor.*

3 Numerical results

n this section, we use some numerical experiments to show the effectiveness and superiority of the preconditioned SOR method. The stopping criterion $\| \mathcal{A}x^{m-1} - b \| \leq 10^{-10}$ is used and a maximum of 1000 iterations is allowed. In all the examples, we take the starting vector x_0, and the right hand-side vector b equal to $ones(n, 1)$. Finding the optimal parameter ω, we search from 0.01 to 2 in the interval of 0.01. All the examples were executed in double precision in MATLAB R2014a.

We show the number of iterations by "*Iter*", the norm of $\mathcal{A}x_k^{m-1} - b$ (x_k is the kth approximate solution) in seconds by "*Error*" and the CPU time in third by "*time*" for the preconditioned SOR (PSOR) and the SOR methods, respectively.

The product $\mathcal{A}x^{m-1}$ denoted in (1) can be computed by transforming into the following matrix-vector product:

$$\mathcal{A}x^{m-1} = \mathcal{A}\underbrace{(x \otimes x \otimes \cdots \otimes x)}_{m-1},$$

where $\otimes$ shows the Kronecker product. Also the matrix-tensor product $B\mathcal{A}$ is defined in (2).

Example 3.1. Consider $\mathcal{B} \in \mathbb{R}^{[3,10]}$ as a nonnegetive tensor with

$$b_{ijk} = \mid tan(i + j + k) \mid .$$

We have $\rho(\mathcal{B}) = 1450.3$. Thus, $\mathcal{A} = 1500\mathcal{I} - \mathcal{B}$ is a symmetric nonsingular $\mathcal{M}$-tensor.

We take $b = ones(10, 1)$ and initial vector $x_0 = ones(10, 1)$. For different ω, we compare the presented PSOR method and SOR method for solving a nonsingular tensor equation. The

results are shown in Table 1. We show the results for different amounts of ω. In this example, we could understand that the $\omega = 1$ is the optimal value of this parameter, which means that preconditioned Jacobian method performs better than the preconditioned SOR method. The comparison results demonstrate that the preconditioned method could be more efcient than the original method.

Table 1: *Numerical results of Example 3.1.*

	PSOR			SOR		
ω	Iter	Error	time	Iter	Error	time
0.25	86	9.32e-11	0.019	112	8.42e-11	0.028
0.5	37	5.76e-11	0.017	47	9.00e-11	0.032
0.75	19	6.11e-11	0.016	24	9.70e-11	0.024
1	6	2.16e-11	0.016	7	2.24e-11	0.028
1.25	19	3.49e-11	0.017	24	4.83e-11	0.021
1.5	36	7.49e-11	0.018	47	5.26e-11	0.021
1.75	85	8.79e-11	0.021	110	9.87e-11	0.024

Example 3.2. In this example, we consider $\mathcal{B} \in \mathbb{R}^{[3,n]}$ as a nonnegative tensor with

$$b_{ijk} = \mid sin(i+j+k) \mid .$$

and solve the 3rd-order $\mathcal{M}$-tensor system $\mathcal{A}x^2 = b$ where $\mathcal{A} = s\mathcal{I} - \mathcal{B}$. We can set $s = n^2$, since

$$\rho(\mathcal{B}) \leq \max_{1\leq i\leq n} \sum_{j,k=1}^{n} b_{ijk} \leq \max_{1\leq i\leq n} \sum_{j,k=1}^{n} 1 = n^2.$$

Hence $\mathcal{A}$ is a strong $\mathcal{M}$-tensor.

We apply preconditioned and un-preconditioned SOR methods to solve (1) with different amounts of n. The numerical results are reported in Table 2 which confirm that the preconditioned SOR method performs better in both CPU times and iterative steps than the SOR method.

Table 2: *Numerical results of Example 3.2 with* $\omega_{opt} = 1.5$.

	PSOR			*SOR*		
n	Iter	Error	time	Iter	Error	time
100	32	7.87e-11	0.148	44	6.09e-11	0.202
200	33	5.27e-11	1.125	46	7.41e-11	1.270
300	33	6.70e-11	4.423	47	9.61e-11	5.238
400	33	7.87e-11	29.262	48	9.15e-11	62.974

Figure 1 illustrates the relationship between the number of iterations and the norm $\mathcal{A}x^{m-1} - b$ and confirms the efficiency of the preconditioner in reducing the error.

4 Conclusion

In this paper, we proposed a diagonal preconditioner by minimizing the Frobenius norm of $\mathcal{I}_m - P\mathcal{A}$ to solve multi-linear systems. We apply the SOR method and PSOR method to solve

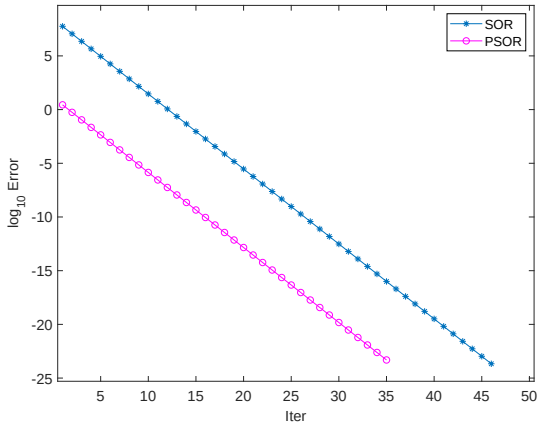

Figure 1: Performance of the SOR and PSOR in reducing residual norm for Example 3.2.

some numerical examples. Analyzing the comparison results shows that the preconditioner improves the method especially in reducing the number of iterations and CPU time.

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Maps preserving the λ-Lie product of operators

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Abstract

Let $\mathcal{A}$ be a standard operator algebra on a Banach space $\mathcal{X}$ with $\dim \mathcal{X} \geq 2$. In this paper, we characterize the forms of additive maps on $\mathcal{A}$ which strongly preserve the square zero of λ-Lie product of operators, i.e., if $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ is an additive map which satisfies

$$[A, B]_\lambda^2 = 0 \Rightarrow [\phi(A), B]_\lambda^2 = 0,$$

for every $A, B \in \mathcal{A}$ and for a scalar number λ with $\lambda \neq -1$, then it is shown that there exists a function $\sigma : \mathcal{A} \to \mathbb{C}$ such that $\phi(A) = \sigma(A)A$, for every $A \in \mathcal{A}$.

Keywords: Preserver problem, Standard operator algebra, λ-Lie product, Lie product.
Mathematics Subject Classification [2010]: 46J10, 47B48

1 Introduction

In last decade, Many mathematician research on the preserving problems. Specially, maps preserving a certain property of products of elements are considered. We point to some of them close to our purpose.

Let $\mathcal{A}$ be a Banach algebra, $A, B \in \mathcal{A}$ and λ be a scalar. $AB + \lambda BA$ is said to be the λ-Lie product of A and B and is denoted by $[A, B]_\lambda$. λ-Lie product is said to be Jordan product and Lie product, whenever $\lambda = 1$ and $\lambda = -1$, respectively. Lie product is denoted by $[A, B]$, too. Moreover, triple Jordan product A and B is defined as ABA. These products play rather important role in mathematical physics.

In [3], authors consider the maps strongly preserve the η-Lie product, that is $\phi(A)\phi(P) + \eta\phi(P)\phi(A) = AP + \eta PA$, for every A, some idempotent P and some scalar η.

Let $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on a Banach space $\mathcal{X}$. Recall that a standard operator algebra on X is a norm closed subalgebra of $\mathcal{B}(\mathcal{X})$ which contains the identity and all finite rank operators. In [2], authors characterize the form of unital surjective maps on $\mathcal{B}(\mathcal{X})$ preserving the nonzero idempotency of product of operators, in both directions. Also in [4], authors characterize the form of linear surjective maps on $\mathcal{B}(\mathcal{X})$ preserving the nonzero idempotency of either products of operators or triple Jordan products of operators. In [1], Authors characterize the form of linear surjective maps on $\mathcal{B}(\mathcal{X})$ preserving the nonzero idempotency of Jordan products of operators.

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We say that a map $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ strongly preserves the square zero of λ-Lie product of operators whenever

$$[A, B]_\lambda^2 = 0 \Rightarrow [\phi(A), B]_\lambda^2 = 0,$$

for every $A, B \in \mathcal{A}$.

In this paper, we characterize the forms of additive maps which strongly preserve the square zero of λ-Lie products of operators. Our main result is the following theorem.

Theorem 1.1. *Assume that $\mathcal{A}$ is a standard operator algebra on a Banach space $\mathcal{X}$ with $\dim \mathcal{X} \geq 2$. Let $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ be an additive map which satisfies*

$$[A, B]_\lambda^2 = 0 \Rightarrow [\phi(A), B]_\lambda^2 = 0,$$

for every $A, B \in \mathcal{A}$ and for a scalar number λ with $\lambda \neq -1$. Then there exists a function $\sigma : \mathcal{A} \to \mathbb{C}$ such that $\phi(A) = \sigma(A)A$, for every $A \in \mathcal{A}$.

2 Proofs

First we recall some notations. We denote by $\mathcal{X}^*$, the dual space of $\mathcal{X}$. For every nonzero $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, the symbol $x \otimes f$ stands for the rank one linear operator on $\mathcal{X}$ defined by $(x \otimes f)y = f(y)x$ for any $y \in \mathcal{X}$. Note that every rank one operator in $\mathcal{B}(\mathcal{X})$ can be written in this way. We denote by $\mathcal{F}_1(\mathcal{X})$ the set of all rank one operators in $\mathcal{B}(\mathcal{X})$. The rank one operator $x \otimes f$ is idempotent if and only if $f(x) = 1$ and is nilpotent if and only if $f(x) = 0$.

To prove the main theorem, we must first prove the following results.

Proposition 2.1. *Let $A \in \mathcal{A}$, $x \in \mathcal{X}$, $f \in \mathcal{X}^*$ such that $f(x) \neq 0$ and $\lambda \neq 0, -1$. Then $[A, x \otimes f]_\lambda^2 = 0$ if and only if one of the following statements occurs:*
(i) $Axf(Ax) = -\lambda xf(A^2x)$ and $Axf(x) = -\lambda xf(Ax)$.
(ii) $fA = 0$.

In the following lemmas, assume $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ is a map which satisfies

$$[A, B]_\lambda^2 = 0 \Rightarrow [\phi(A), B]_\lambda^2 = 0,$$

for every $A, B \in \mathcal{A}$ and for a scalar number λ with $\lambda \neq 0, -1$.

Lemma 2.2. $\ker A \subseteq \ker \phi(A)$, *for every $A \in \mathcal{A}$.*

Next assume that ϕ is additive.

Lemma 2.3. $\phi(A) = 0$ *or* $\phi(A) = \kappa(A)A$, *for every rank one operator A, where $\kappa : \mathcal{A} \to \mathbb{C}$ is a function.*

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g-Fusion Frames in Hilbert Spaces

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Abstract

In this paper, we investigate the notion of g-fusion frames in Hilbert spaces. Then, we present sufficient conditions for g-fusion frames in terms of g-frames in Hilbert spaces. We extend some of the recent results of standard g-frames and fusion frames to g-fusion frames.

Keywords: Hilbert space, Frame, Generalized Frames, G-fusion Frame.

Mathematics Subject Classification [2010]: $42C15$, $46B20$

1 Introduction

Frames for Hilbert space were first introduced by Duffin and Schaeffer [3] in 1952. Daubechies, Grossmann and Meyer [2] reintroduced frames, in 1986 [2] and considered from then. Frame theory has applications in signal processing, image processing, data compression and sampling theory.

Orthonormal bases are special case of frames in Hilbert space. Any element in Hilbert can be present as an infinite linear combination, not necessary unique, of the frame element.

Some new type and generalization of frame were introduced by researcher such as fusion frames, g-frames, woven frames, etc. Frame of subspaces or fusion frames are a generalization of frames which were introduced by Cassaza and Kutyniok [1] in 2003. Generalized frames or in abbreviation g-frames were introduced by Sun [6] in 2006. Most recently, g-fusion frames in Hilbert space were introduced by Sadri et.al. [5].

In this paper, motivated and inspired by the above-mentioned works we introduce the concept of g-fusion frame. This frame includes g-frames and fusion frames. We extend some of the recent results of standard g-frames and fusion frames to g-fusion frames.

The paper is organized as follows: Section 2 contains the basic definitions about fusion frames, g-frames, g-fusion frames. Section 3 belongs to the g-fusion frames and investigating their structuers.

2 Main results

As a preliminary of frames, at the first, we mention fusion frames. Also we review g-frames, g-fusion frames and woven frames . Through of this paper, $\mathcal{I}$ is the indexing set where it can be finite or infinity countable set. Also, $\mathcal{H}$ and $\mathcal{H}_i$ are separable Hilbert spaces and $B(\mathcal{H},\mathcal{H}_i)$ is the collection of all the bounded linear operators of $\mathcal{H}$ into $\mathcal{H}_i$. If $\mathcal{H}=\mathcal{H}_i$, then $B(\mathcal{H},\mathcal{H})$ will be denoted by $B(\mathcal{H})$ and P is the orthogonal projection.

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2.1 Fusion frames

In 2003, a new type of generalization of frames was introduced by Cassaza and Kutyniok to the science world that today we know as fusion frames. In this section, we briefly recall some basic notations, definitions and some important properties of fusion frames that are useful for our study.

Definition 2.1. Let $\{v_i\}_{i\in\mathcal{I}}$ be a family of real weights such that $v_i > 0$ for all $i \in \mathcal{I}$. A family of closed subspaces $\{W_i\}_{i\in\mathcal{I}}$ of a Hilbert space $\mathcal{H}$ is called a fusion frame (or frame of subspaces) for $\mathcal{H}$ with respect to weights $\{v_i\}_{i\in\mathcal{I}}$, if there exist constants $C, D > 0$ such that

$$C||f||^2 \leq \sum_{i\in\mathcal{I}} v_i^2 ||P_{W_i}(f)||^2 \leq D||f||^2, \qquad \forall f \in \mathcal{H}, \tag{1}$$

where P_{W_i} is the orthogonal projection of $\mathcal{H}$ to W_i. The constants C and D are called the lower and upper fusion frame bounds, respectively. If the right inequality in (1) holds, the family of subspace $\{W_i\}_{i\in\mathcal{I}}$ is called a Bessel sequence of subspaces with respect to $\{v_i\}_{i\in\mathcal{I}}$ with Bessel bound D. Also is called tight fusion frame with respect to $\{v_i\}_{i\in\mathcal{I}}$, if $C = D$ and is called Parseval fusion frame, if $C = D = 1$. We say $\{W_i\}_{i\in\mathcal{I}}$ an orthogonal fusion basis for $\mathcal{H}$, if $\mathcal{H} = \bigoplus_{i\in\mathcal{I}} W_i$.

Definition 2.2. The fusion frame $\{W_i\}_{i\in\mathcal{I}}$ with respect to some family of weights is called a Riesz decomposition of $\mathcal{H}$, if for every $f \in \mathcal{H}$, there is a unique choice of $f_i \in W_i$ so that $f = \sum_{i\in\mathcal{I}} f_i$.

For each family of subspaces $\{W_i\}_{i\in\mathcal{I}}$ of $\mathcal{H}$, the representation space:

$$\left(\sum_{i\in\mathcal{I}} \oplus W_i\right)_{\ell^2} = \left\{\{f_i\}_{i\in\mathcal{I}} | f_i \in W_i \ \ and \ \ \sum_{i\in\mathcal{I}} ||f_i||^2 < \infty\right\},$$

with inner product

$$\langle \{f_i\}_{i\in\mathcal{I}}, \{g_i\}_{i\in\mathcal{I}} \rangle = \sum_{i\in\mathcal{I}} \langle f_i, g_i \rangle,$$

is a Hilbert space. This space is needed in the studying of fusion systems.

Definition 2.3. Let $\{W_i\}_{i\in\mathcal{I}}$ be a fusion frame family for $\mathcal{H}$ with respect to $\{v_i\}_{i\in\mathcal{I}}$. Then the analysis operator for $\{W_i\}_{i\in\mathcal{I}}$ with weights $\{v_i\}_{i\in\mathcal{I}}$ is defined by:

$$U_{W,v} : \mathcal{H} \to \left(\sum_{i\in\mathcal{I}} \oplus W_i\right)_{\ell^2}, \qquad U_{W,v}(f) = \{v_i P_{W_i}(f)\}_{i\in\mathcal{I}}.$$

The adjoint of $U_{W,v}$ is called the synthesis operator, we denote $T_{W,v} = U^*_{W,v}$.

By elementary calculation, we have

$$T_{W,v} : \left(\sum_{i\in\mathcal{I}} \oplus W_i\right)_{\ell^2} \to \mathcal{H}, \qquad T_{W,v}(\{f_i\}_{i\in\mathcal{I}}) = \sum_{i\in\mathcal{I}} v_i P_{W_i} f_i.$$

Like discrete frames, the fusion frame operator for $\{W_i\}_{i\in\mathcal{I}}$ with respect to $\{v_i\}_{i\in\mathcal{I}}$ is the composition of analysis and synthesis operators,

$$S_{W,v} : \mathcal{H} \to \mathcal{H}, \qquad S_{W,v}(f) = T_{W,v}U_{W,v}(f) = \sum_{i\in I} v_i^2 P_{W_i}(f), \quad \forall f \in \mathcal{H}.$$

The following theorem presents the equivalence conditions between the fusion frames and their operators.

Theorem 2.4. *Let $\{W_i\}_{i\in\mathcal{I}}$ be a family of subspaces in $\mathcal{H}$ and $\{v_i\}_{i\in\mathcal{I}}$ be a family of weights. Then the following conditions are equivalent:*

(i) The family $\{W_i\}_{i\in\mathcal{I}}$ is a fusion frame with respect to $\{v_i\}_{i\in\mathcal{I}}$,

(ii) The synthesis operator $T_{W,v}$ is bounded, linear and onto,

(iii) The analysis operator $U_{W,v}$ is a (possibly into) isomorphism.

2.2 Generalized frames

Sun [6] introduced g-frames which are generalized frames and include ordinary frames and many recent generalizations of frames.

Definition 2.5. Let $\{\mathcal{H}_i\}_{i\in\mathcal{I}}$ be a family of Hilbert spaces. We call $\Lambda = \{\Lambda_i \in B(\mathcal{H},\mathcal{H}_i),\ i\in\mathcal{I}\}$ a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathcal{I}}$, or simply, a g-frame for H, if there exist two positive constants C, D such that

$$C||f||^2 \leq \sum_{i\in\mathcal{I}} ||\Lambda_i f||^2 \leq D||f||^2, \qquad \forall f\in\mathcal{H}. \tag{2}$$

The positive numbers C and D are called the lower and upper g-frame bounds, respectively. We call Λ a tight g-frame, if $C = D$ and we call it a Parseval g-frame, if $C = D = 1$. If only the second inequality holds, we call it g-Bessel sequence. If Λ is a g-frame, then the g-frame operator S_Λ is defined by

$$S_\Lambda f = \sum_{i\in\mathcal{I}} \Lambda_i^*\Lambda_i f, \qquad f\in\mathcal{H},$$

which is a bounded, positive and invertible operator such that

$$CI \leq S_\Lambda \leq DI,$$

and for each $f\in\mathcal{H}$, we have

$$f = S_\Lambda S_\Lambda^{-1} f = S_\Lambda^{-1} S_\Lambda f = \sum_{i\in\mathcal{I}} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f = \sum_{i\in\mathcal{I}} \Lambda_i^*\Lambda_i S_\Lambda^{-1} f.$$

The canonical dual g-frame for Λ is defined by $\{\Lambda_i S_\Lambda^{-1}\}_{i\in\mathcal{I}}$ with bounds $\frac{1}{D}, \frac{1}{C}$. In other words, $\{\Lambda_i S_\Lambda^{-1}\}_{i\in\mathcal{I}}$ and $\{\Lambda_i\}_{i\in\mathcal{I}}$ are dual g-frames with respect to each other.

It is easy to show that by letting $\mathcal{H}_i = W_i$, $\Lambda_i = P_{W_i}$ and $v_i = 1$, a fusion frame is a g-frame.

3 Generalized Fusion frames

Generalized fusion frames (g-fusion frames) in Hilbert space were introduced by Sadri et.al. [5].

Let

$$\left(\sum_{i\in\mathcal{I}}\oplus\mathcal{H}_i\right)_{\ell^2} = \left\{\{f_i\}_{i\in\mathcal{I}} | f_i\in\mathcal{H}_i \ \text{ and } \ \sum_{i\in\mathcal{I}} ||f_i||^2 < \infty\right\},$$

with the inner product defined by

$$\langle\{f_i\}_{i\in\mathcal{I}}, \{g_i\}_{i\in\mathcal{I}}\rangle = \sum_{i\in\mathcal{I}} \langle f_i, g_i\rangle,$$

is a Hilbert space.

Definition 3.1. Let $W = \{W_i\}_{i\in\mathcal{I}}$ be a family of closed subspaces of $\mathcal{H}$, $\{v_i\}_{i\in\mathcal{I}}$ be a family of weights, i.e. $v_i > 0$ and $\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)$ for all $i \in \mathcal{I}$. We say $\Lambda := (\Lambda_i, W_i, v_i)$ is a generalized fusion frame (or g-fusion frame) for $\mathcal{H}$, if there exists $0 < A \leq B < \infty$ such that for each $f \in \mathcal{H}$

$$A||f||^2 \leq \sum_{i\in\mathcal{I}} v_i^2 ||\Lambda_i P_{W_i} f||^2 \leq B||f||^2. \tag{3}$$

We call Λ a Parseval g-fusion frame, if $A = B = 1$. When the right hand of (3) holds, Λ is called a g-fusion Bessel sequence for $\mathcal{H}$ with bound B. If $\mathcal{H}_i = \mathcal{H}$ for all $i \in \mathcal{I}$ and $\Lambda_i = I_{\mathcal{H}}$, then we get the fusion frame (W_i, v_i) for $\mathcal{H}$. Throughout this paper, Λ will be a triple (Λ_i, W_i, v_i) with $i \in \mathcal{I}$ unless otherwise stated.

Definition 3.2. Let Λ be a g-fusion frame for $\mathcal{H}$. Then, the analysis operator for Λ is defined by

$$U_\Lambda : \mathcal{H} \to \left(\sum_{i\in\mathcal{I}} \oplus \mathcal{H}_i\right)_{\ell^2}, \qquad U_\Lambda(f) = \{v_i \Lambda_i P_{W_i}(f)\}_{i\in\mathcal{I}}.$$

The adjoint of U_Λ is called the synthesis operator, we denote $T_\Lambda = U_\Lambda^*$.

By the elementary calculation, we have

$$T_\Lambda : \left(\sum_{i\in\mathcal{I}} \oplus \mathcal{H}_i\right)_{\ell^2} \to \mathcal{H}, \qquad T_\Lambda(\{f_i\}_{i\in\mathcal{I}}) = \sum_{i\in\mathcal{I}} v_i P_{W_i} \Lambda_i^* f_i.$$

The g-fusion frame operator Λ is the composition of analysis and synthesis operators,

$$S_\Lambda : \mathcal{H} \to \mathcal{H}, \qquad S_\Lambda f = T_\Lambda U_\Lambda(f) = \sum_{i\in\mathcal{I}} v_i^2 P_{W_i} \Lambda_i^* \Lambda_i P_{W_i} f.$$

We have

$$\langle S_\Lambda f, f\rangle = \sum_{i\in\mathcal{I}} v_i^2 ||\Lambda_i P_{W_i} f||^2.$$

Therefore

$$AI \leq S_\Lambda \leq BI.$$

This means that S_Λ is bounded, positive and invertible operator (with adjoint inverse). So, we have the reconstruction formula for any $f \in \mathcal{H}$

$$f = \sum_{i\in\mathcal{I}} v_i^2 P_{W_i} \Lambda_i^* \Lambda_i P_{W_i} S_\Lambda^{-1} f = \sum_{i\in\mathcal{I}} v_i^2 S_\Lambda^{-1} P_{W_i} \Lambda_i^* \Lambda_i P_{W_i} f.$$

The following theorem give the equivalence conditions between the g-fusion frames and their operators.

Theorem 3.3. *[5] Let Λ be the triple (Λ_i, W_i, v_i) with $i \in \mathcal{I}$. Then the following conditions are equivalent:*

(i) Λ is a g-fusion frame for $\mathcal{H}$,

(ii) The synthesis operator T_Λ is bounded, linear and onto,

(iii) The analysis operator S_Λ is well-defined, bounded, surjectire.

Lemma 3.4. *Let $\Lambda = (\Lambda_i, W_i, v_i)$ be a g-fusion frame with universal bounds A, B, and $Q \in B(\mathcal{H})$ be bounded below by $m > 0$, i.e.*
$m\|f\| \leq \|Qf\|$ for every $f \in \mathcal{H}$. Then $\Lambda = (\Lambda_i Q, W_i, v_i)$ is a g-fusion frame with bounds Am^2, and $B\|Q\|^2$.

Proof. for each $f \in \mathcal{H}$ we have

$$S_{\Lambda Q} f = \sum_{i\in\mathcal{I}} v_i^2 P_{W_i} Q^* \Lambda_i^* \Lambda_i Q P_{W_i} f = Q^*(\sum_{i\in\mathcal{I}} v_i^2 P_{W_i} \Lambda_i^* \Lambda_i P_{W_i}) Q f = Q^*(S_\Lambda) Q f$$

where $S_\Lambda = \sum_{i\in\mathcal{I}} v_i^2 P_{W_i} \Lambda_i^* \Lambda_i P_{W_i}$.
Since $m^2.I \leq Q^*Q, \quad A.I \leq S_\Lambda \leq B.I$, then

$$Am^2.I \leq A.Q^*Q \leq S_{\Lambda Q} \leq B.Q^*Q \leq B\|Q\|^2.I,$$

and we have the result. □

Corollary 3.5. *Let $\{\Lambda_i\}_{i\in\mathcal{I}}$ be a g-frame, and $Q \in B(\mathcal{H})$ be bounded below. Then $\{\Lambda_i Q\}_{i\in\mathcal{I}}$ is a g-frame.*

Corollary 3.6. *Let $\{W_i\}_{i\in\mathcal{I}}$ be a fusion frame family for $\mathcal{H}$ with respect to $\{v_i\}_{i\in\mathcal{I}}$ and $Q \in B(\mathcal{H})$ be bounded below. Then $\{QW_i\}_{i\in\mathcal{I}}$ is a fusion frame with respect to $\{v_i\}_{i\in\mathcal{I}}$.*

[4, Proposition 2.1] leads us to the following result.

Theorem 3.7. *Let $\Lambda = (\Lambda_i, W_i, v_i)$ be a g-fusion frame. Let $V = \{V_i\}_{i\in\mathcal{I}}$ be a family of closed subspaces of $\mathcal{H}$, and $\Gamma_i \in B(\mathcal{H}, \mathcal{H}_i)$ for all $i \in \mathcal{I}$. If $U : \mathcal{H} \to \mathcal{H}$ defined by*

$$U(f) = \sum_{i\in\mathcal{I}} v_i^2 (P_{V_i} \Gamma_i^* \Gamma_i P_{V_i}(f) - P_{W_i} \Lambda_i^* \Lambda_i P_{W_i}(f)) \qquad (f \in \mathcal{H})$$

is a compact operator. Then $\Gamma := (\Gamma_i, V_i, v_i)$ is g-fusion frame for $\overline{span}\{\Gamma_i^*(V_i)\}_{i\in\mathcal{I}}$.

Proof. Let A and B be the g-fusion frame bounds for (Λ_i, W_i, v_i). we have

$$\|S_\Lambda\| = \sup_{\|f\|\leq 1} |\langle S_\Lambda f, f\rangle| = \sup_{\|f\|\leq 1} \sum_{i\in\mathcal{I}} v_i^2 ||\Lambda_i P_{W_i} f||^2 \leq B.$$

A simple calculation shows that U is a self-adjoint operator on $\mathcal{H}$. So if $T : \mathcal{H} \to \mathcal{H}$ is defined by $T = S_\Lambda + U$, then T is a bounded, linear, self-adjoint operator. Therefore, we have

$$\|T\| = \sup_{\|f\|\leq 1} |\langle Tf, f\rangle| = \sup_{\|f\|\leq 1} \sum_{i\in\mathcal{I}} v_i^2 ||\Gamma_i P_{V_i} f||^2$$

and

$$\sum_{i\in\mathcal{I}} v_i^2 ||\Gamma_i P_{V_i} f||^2 \leq \|T\|\|f\|^2 \leq (\|S_\Lambda\| + \|U\|)\|f\|^2 \leq (B + \|U\|)\|f\|^2 \qquad (f \in \mathcal{H}) \qquad (*).$$

Now we obtain a lower bound for (Γ_i, V_i, v_i). Since U is a compact operator on $\mathcal{H}$, so US_Λ^{-1} is also a compact operator. Therefore the operator $US_\Lambda^{-1} + id_\mathcal{H}$ has closed range. As result $T = (US_\Lambda^{-1} + id_\mathcal{H})S_\Lambda$ is closed range. We consider T as an operator on the closed subspace $\overline{span}\{\Gamma_i^*(V_i)\}_{i\in\mathcal{I}}$. Now we show that T is injective. If $f \in \overline{span}\{\Gamma_i^*(V_i)\}_{i\in\mathcal{I}}$ and $T(f) = 0$, then

$$\sum_{i\in\mathcal{I}} v_i^2 ||\Gamma_i P_{V_i} f||^2 = \langle Tf, f\rangle = 0,$$

so $f = 0$. Also we have

$$Range(T) = (N(T^*))^\perp = (N(T))^\perp = \overline{span}\{\Gamma_i^*(V_i)\}_{i\in\mathcal{I}}.$$

Hence T is surjective and for every $f \in \overline{span}\{\Gamma_i^*(V_i)\}_{i\in\mathcal{I}}$ we have

$$\|T^{-1}\|^{-1}\|f\| \leq \|T(f)\| \leq \|T\|\|f\|.$$

Now by using the Cauchy-Schwartz inequality and $(*)$, we have

$$\begin{aligned}
\|T(f)\|^4 &= (\langle\sum_{i\in\mathcal{I}} v_i^2 P_{V_i}\Gamma_i^*\Gamma_i P_{V_i}(f), T(f)\rangle)^2 = (\langle\sum_{i\in\mathcal{I}} v_i^2\Gamma_i P_{V_i}(f), \Gamma_i P_{V_i}(T(f))\rangle)^2 \\
&\leq (\sum_{i\in\mathcal{I}} v_i^2\|\Gamma_i P_{V_i}(f)\|\|\Gamma_i P_{V_i}(T(f))\|)^2 \\
&\leq (\sum_{i\in\mathcal{I}} v_i^2\|\Gamma_i P_{V_i}(f)\|^2(\sum_{i\in\mathcal{I}} v_i^2\|\Gamma_i P_{V_i}(T(f))\|^2) \\
&\leq (B+\|U\|)\|Tf\|^2(\sum_{i\in\mathcal{I}} v_i^2\|\Gamma_i P_{V_i}(f)\|^2).
\end{aligned}$$

Therefore,

$$\|Tf\|^2 \leq (B+\|U\|)(\sum_{i\in\mathcal{I}} v_i^2\|\Gamma_i P_{V_i}(f)\|^2),$$

Thus,

$$\sum_{i\in\mathcal{I}} v_i^2\|\Gamma_i P_{V_i}(f)\|^2 \geq (B+\|U\|)^{-1}\|Tf\|^2 \geq (B+\|U\|)^{-1}\|T^{-1}\|^{-2}\|f\|^2.$$

□

4 Conclusion

We conclud that if $\Lambda = (\Lambda_i, W_i, v_i)$ is a g-fusion frame , $Q \in B(\mathcal{H})$ a bounded below operator and $m > 0$. Then $\Lambda = (\Lambda_i Q, W_i, v_i)$ is a g-fusion frame. Also if $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame, and $Q \in B(\mathcal{H})$ be bounded below, then $\{\Lambda_i Q\}_{i\in\mathcal{I}}$ is a g-frame.

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Decomposability of multivariate majorization

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Abstract

Let $x, y \in \mathbb{R}^n$. We use the notation $x \prec y$ when x is multivariate majorized by y. We say that $x \prec y$ is decomposable at k $(1 \leq k < n)$ if $x \prec y$ has a coincidence at k and $y_k \neq y_{k+1}$. Corresponding to this majorization we have a doubly stochastic matrix D such that $x = Dy$. Levow proved that if $x \prec y$ is decomposable at some k $(1 \leq k < n)$ then D is of the form $D_1 \oplus D_2$ where D_1 and D_2 are doubly stochastic matrices, this paper presents the converse of this theorem.

Keywords: Decomposable, Doubly stochastic matrix, Majorization

Mathematics Subject Classification [2010]: 15A03 and 15A51

1 Introduction

Let $\mathbf{M}_n$ be the set of all real matrices of order n. A matrix $D \in \mathbf{M}_n$ of nonnegative real numbers for which the sums of the entries in each row and each column are all one is said to be *doubly stochastic*. We denote the set of all doubly stochastic matrices of order n by Ω_n.
Let $\mathbb{R}^n$ be the set of all $n-$tuples $(x_1, \ldots, x_n)$ of real numbers. Also, let $x^{\downarrow}$ be the vectors obtained by rearranging the coordinates of $x \in \mathbb{R}^n$ in the decreasing order. Thus, if $x^{\downarrow} = (x_1^{\downarrow}, \ldots, x_n^{\downarrow})$, then $x_1^{\downarrow} \geq \cdots \geq x_n^{\downarrow}$.
Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. We say that x is *majorized* (*multivariate majorized*) by y, in symbols $x \prec y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}, \qquad 1 \leq k \leq n,$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

For further information about majorization, we refer the reader to [1–5].

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2 Main results

The following concepts are defined by Levow in [4].
Let $x, y \in \mathbb{R}^n$, $x \prec y$, and

$$\delta_k = \sum_{i=1}^{k} (y_i^{\downarrow} - x_i^{\downarrow}), 1 \leq k \leq n-1, \tag{1}$$

then $\delta_k \geq 0$.
If $\delta_k = 0$, we say that $x \prec y$ has a *coincidence* at k. If $x \prec y$ has a coincidence at k and $y_k \neq y_{k+1}$, we say that $x \prec y$ is *decomposable* at k.

Theorem 2.1 ([5, A.4]). *Let $x, y \in \mathbb{R}^n$, then $x \prec y$ if and only if $x = Dy$ for some doubly stochastic matrix $D \in \boldsymbol{M}_n$*

If $x \prec y$ and $y = (D_1 \oplus D_2)x$, where $D_1 \in \Omega_k$ and $D_2 \in \Omega_{n-k}$, then it is clear from the theorem of [2] that $x \prec y$ has a coincidence at k. Also, if $x \prec y$ has a coincidence at k, it is clear from the Theorem2.1 that $D = D_1 \oplus D_2$ where $D_1 \in \Omega_k$ and $D_2 \in \Omega_{n-k}$ such that $x = Dy$. In the Theorem2.2 Levow proves that if $x = Dy$, the decomposability of $x \prec y$ at k guarantees that D is a direct sum of matrices in Ω_k and Ω_{n-k}.

Theorem 2.2 ([4, Theorem 2]). *Suppose that $x \prec y$ is decomposable at k and that $y = Dx$ then there are matrices $D_1 \in \Omega_k$ and $D_2 \in \Omega_{n-k}$ such that $D = D_1 \oplus D_2$.*

In [4], Levow proved that decomposability of $x \prec y$ implies that if $x = Dy, D \in \Omega_n$, then D is the direct sum of two doubly stochastic matrices. Here, we study majorization and related doubly stochastic matrices. We also present that decomposability of $x \prec y$ is a necessary condition for D to be a direct sum of $D_1 \oplus D_2$, where $x = Dy$ and $D_1, D_2 \in \Omega_n$.

Theorem 2.3. *Let $x, y \in \mathbb{R}_+^n$ with $x \prec y$. If there exists some k $(1 \leq k \leq n)$ that for every $D \in \boldsymbol{M}_n$ such that $x = Dy$, we have $D = D_1 \oplus D_2$ where $D_1, D_2 \in \Omega_k$ and $D_2 \in \Omega_{n-k}$, then $x \prec y$ is decomposable at k.*

Proof. Let $x \prec y$. Then $x = Dy$ for some doubly stochastic matrix D. The hypothesis ensures that $D = D_1 \oplus D_2$ where $D \in \Omega_k$ and $D_2, \in \Omega_{n-k}$.

The relation $x = Dy$ ensures that $(x_1, \ldots, x_k) = D_1(y_1, \ldots, y_k)$ where $D_1 \in \Omega_k$, and so $(x_1, \ldots, x_k) \prec (y_1, \ldots, y_k)$. It follows that $\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i$. Therefore, $x \prec y$ has a coincidence at k.
Now, we claim that $y_k \neq y_{k+1}$. If not; $y_k = y_{k+1}$. We will constract a matrix D' such that $x = D'y$, but D' is not as a direct sum of two doubly stochastic matrices.
Set $D = [D^1 D^2 \ldots D^n]$, where D^i is the ith column of the matrix D. Now, define $D' = [D^1 \ldots D^{k-1} D^{k+1} D^k D^{k+2} \ldots D^n]$. We observe that $x = D'y$. As $y_k = y_{k+1}$ and D has the form given in the hypothesis, we see that D' has the same form which we wanted to create. It is a contradiction. Hence $y_k \neq y_{k+1}$, and so $x \prec y$ is decomposable at k. □

3 Conclusion

Decomposability is some conditions on the majorization relation $\prec$ on vectors. Here we prove the converse of a theorem of Levow. Examining the decomposability condition on other majorization relation can be subject of future research works.

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Generalized conjugate gradient method for solving multilinear systems

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Abstract

Let $\mathcal{L}$ be a real linear operator with a positive definite symmetric part $\mathcal{M}$. In certain applications, several problems of the form $\mathcal{M} \star_N \mathcal{Y} =$ can be solved with less human or computational effort than the original equation $\mathcal{L} \star_N = \mathcal{F}$. In this paper, the generalized conjugate gradient method of Concus and Golub [Lecture Notes in Economics and Mathematical Systems 134, Springer-Verlag, New York, 1976] and Widlund [SIAM J. Numer. Anal., 15 (1978), pp. 801-812] is extended for solving some tensor equations via Einstein product. An example is also provided to show the efficiency of the proposed method. Finally, some concluding remarks are given.

Keywords: Generalized conjugate gradient method, Tensor, Multilinear systems

Mathematics Subject Classification [2010]: 15A10, 15A69, 15A72

1 Introduction

The generalized conjugate gradient method of Concus and Golub [2] and Widlund [3] is an iterative method for solving a system of linear equations $\mathbf{Ax} = \mathbf{b}$ when the coefficient matrix $\mathbf{A}$ is real and has positive definite symmetric part $\mathbf{M} = (\mathbf{A} + \mathbf{A}^{\mathsf{T}})/2$. This method can be summarized as follows:

Algorithm 1. Generalized Conjugate Gradient (GCG)

1. Let $\mathbf{x}_0$ be given and set $\mathbf{x}_{-1} = 0$
2. For $j = 0, 1, ...$ until convergence solve $\mathbf{Mv}_j = \mathbf{b} - \mathbf{Ax}_j$ and compute $\rho_j =< \mathbf{Mv}_j, \mathbf{v}_j >$
3. If $j = 0$ set $\omega_{j+1} = 1$ else compute $\omega_{j+1} = [1 + \frac{\rho_j}{\rho_{j-1}} \frac{1}{\omega_j}]^{-1}$
4. Compute $\mathbf{x}_{j+1} = \mathbf{x}_{j-1} + \omega_{j+1}(\mathbf{v}_j + \mathbf{x}_j - \mathbf{x}_{j-1})$,

where $< \mathbf{y}, \mathbf{x} >$ denotes the Euclidean inner product.

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Herein, from [4] some definitions and notations are collected. Tensors are written in calligraphic font, e.g., $\mathcal{A}$. Let N be a positive integer, an order N tensor $\mathcal{A} = (a_{i_1\dots i_N}) = (\mathcal{A})_{i_1\dots i_N} (1 \leq i_j \leq I_j,\ j = 1, 2, ..., N)$ is a multidimensional array with $I (I = I_1 ... I_N)$ entries. Each entry of $\mathcal{A}$ is denoted by $a_{i_1 i_2 \dots i_N}$. $\mathcal{O}$ with all entries zero denotes the zero tensor. With this definition of tensors, matrices are tensors of order two where signified by bolded capital letters, e.g., $\mathbf{A}$. As usual, $\mathbb{R}$ and $\mathbb{C}$ denotes the real and complex number field, respectively. Let $\mathbb{R}^{I_1\times\dots\times I_N}$ and $\mathbb{C}^{I_1\times\dots\times I_N}$ be the set of order N, dimension $I_1 \times I_2 \times ... \times I_N$ tensors over $\mathbb{R}$ and $\mathbb{C}$, respectively. Let N, M, L be the positive integers, $\mathcal{A} \in \mathbb{C}^{I_1\times\dots\times I_N\times K_1\times\dots\times K_M}$ and $\mathcal{B} \in \mathbb{C}^{K_1\times\dots\times K_M\times J_1\times\dots\times J_L}$. The Einstein product of $\mathcal{A}$ and $\mathcal{B}$ is defined by the operation $\star_M$ via

$$(\mathcal{A} \star_M \mathcal{B})_{i_1\dots i_N j_1\dots j_L} = \sum_{k_M=1}^{K_M} \dots \sum_{k_1=1}^{K_1} a_{i_1\dots i_N k_1\dots k_M} b_{k_1\dots k_M j_1\dots j_L}.$$

Let $\mathcal{A} = (a_{i_1\dots i_N j_1\dots j_N}) \in \mathbb{C}^{I_1\times\dots\times I_N\times I_1\times\dots\times I_N}$, then $\mathcal{A}^{i+1} = \mathcal{A} \star_N \mathcal{A}^i,\ i = 1, 2, ...$ Let $\mathcal{B} = (b_{i_1\dots i_M j_1\dots j_N}) \in \mathbb{C}^{J_1\times\dots\times J_M\times I_1\times\dots\times I_N}$ be the conjugate transpose of $\mathcal{A}$, where $b_{i_1\dots i_M j_1\dots j_N} = \bar{a}_{j_1\dots j_M i_1\dots i_N}$. The tensor $\mathcal{B}$ is denoted by $\mathcal{A}^*$. When $b_{i_1\dots i_M j_1\dots j_N} = a_{j_1\dots j_M i_1\dots i_N}$, $\mathcal{B}$ is called the transpose of $\mathcal{A}$, denoted by $\mathcal{A}^\mathsf{T}$. Trace of $\mathcal{A}$ is defined by $tr(\mathcal{A}) = \sum_{i_N=1}^{I_N} \dots \sum_{i_1=1}^{I_1} a_{i_1\dots i_N i_1\dots i_N}$. Inner product of two tensors $\mathcal{X}, \mathcal{Y} \in \mathbb{C}^{I_1\times\dots\times I_N\times J_1\times\dots\times J_M}$ is defined by

$$< \mathcal{X}, \mathcal{Y} >= tr(\mathcal{Y}^* \star_N \mathcal{X}) = \sum_{j_M=1}^{J_M} \dots \sum_{j_1=1}^{J_1} \sum_{i_N=1}^{I_N} \dots \sum_{i_1=1}^{I_1} x_{i_1\dots i_N j_1\dots j_M} \bar{y}_{j_1\dots j_M i_1\dots i_N},$$

so the tensor norm that generated by this inner product is

$$||\mathcal{X}|| = \sqrt{< \mathcal{X}, \mathcal{X} >} = \sqrt{\sum_{j_M=1}^{J_M} \dots \sum_{j_1=1}^{J_1} \sum_{i_N=1}^{I_N} \dots \sum_{i_1=1}^{I_1} |x_{i_1\dots i_N j_1\dots j_M}|^2},$$

which is the tensor Frobenius norm. $\mathcal{A} \in \mathbb{R}^{I_1\times\dots\times I_N\times I_1\times\dots\times I_N}$ is said to be a diagonal tensor if $a_{i_1\dots i_N j_1\dots j_N} = 0$ for $i_l \neq j_l$ and $l = 1, ..., N$. A diagonal tensor $\mathcal{I} \in \mathbb{R}^{I_1\times\dots\times I_N\times I_1\times\dots\times I_N}$ is an identity tensor if $i_{i_1\dots i_N j_1\dots j_N} = \prod_{k=1}^{N} \delta_{i_k j_k}$, where $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$

2 Main results

In this paper we extend the GCG algorithm, named GCG-BTF, for solving the following tensor equations (or multilinear systems) via Einstein product:

$$\mathcal{A} \star_N \mathcal{X} = \mathcal{B}, \tag{1}$$

where $\mathcal{A} \in \mathbb{R}^{I_1\times\dots\times I_N\times I_1\times\dots\times I_N}$ and $\mathcal{X}, \mathcal{B} \in \mathbb{R}^{I_1\times\dots\times I_N}$.

Algorithm 2. Generalized Conjugate Gradient Based Tensor Form (GCG-BTF)

1. Let $\mathcal{X}_0$ be given and set $\mathcal{X}_{-1} = \mathcal{O}$.
2. For $j = 0, 1, ...$ until convergence solve $\mathcal{M} \star_N \mathcal{V}_j = \mathcal{B} - \mathcal{A} \star_N \mathcal{X}_j$ and compute $\rho_j =< \mathcal{M} \star_N \mathcal{V}_j, \mathcal{V}_j >$
3. If $j = 0$ set $\omega_{j+1} = 1$ else compute $\omega_{j+1} = [1 + \frac{\rho_j}{\rho_{j-1}} \frac{1}{\omega_j}]^{-1}$

4. Compute $\mathcal{X} = \mathcal{X}_{j-1} + \omega_{j+1}(\mathcal{V}_j + \mathcal{X}_j - \mathcal{X}_{j-1})$.

Let $\mathcal{A} = \mathcal{M} - \mathcal{N}$, whence $-\mathcal{N} = (\mathcal{A} - \mathcal{A}^{\mathsf{T}})/2$ is the skew-symmetric part of $\mathcal{A}$, and let $\mathcal{K} = \mathcal{M}^{-1} \star_N \mathcal{N}$. Then the iterate $\mathcal{X}_j$ can be characterized as the unique element in the affine Krylov-type subspace

$$\mathcal{X}_0 + Span\{\mathcal{V}_0, \mathcal{K} \star_N \mathcal{V}_0, \mathcal{K}^2 \star_N \mathcal{V}_0, ..., \mathcal{K}^{j-1} \star_N \mathcal{V}_0\} \equiv \mathcal{X}_0 + \varphi_j,$$

satisfying the Galerkin condition

$$< \mathcal{Z}, \mathcal{A} \star_N \mathcal{E}_j >= 0 \qquad for\ all \quad \mathcal{Z} \in \varphi_j, \tag{2}$$

where $\mathcal{E}_j = \mathcal{X} - \mathcal{X}_j$. Moreover, it can be shown that

$$\mathcal{X}_j = \mathcal{X} + P_j(\mathcal{K}) \star_N \mathcal{E}_0, \tag{3}$$

where $P_j(\mu)$ is an even (odd) polynomial of degree at most j for j even (odd) and $P_j(1) = 1$.

Notation. $< \mathcal{Y}, \mathcal{Z} >_{\mathcal{M}}$ denotes the $\mathcal{M}$-inner product $< \mathcal{M} \star_N \mathcal{Y}, \mathcal{Z} >$ and $||\mathcal{Z}||_{\mathcal{M}}$ denotes the corresponding norm. Note that

$$\begin{aligned} < \mathcal{K} \star_N \mathcal{Y}, \mathcal{Z} >_{\mathcal{M}} &=< \mathcal{N} \star_N \mathcal{Y}, \mathcal{Z} >= - < \mathcal{Y}, \mathcal{N} \star_N \mathcal{Z} > \\ &= - < \mathcal{M} \star_N \mathcal{Y}, \mathcal{M}^{-1} \star_N \mathcal{N} \star_N \mathcal{Z} >= - < \mathcal{Y}, \mathcal{K} \star_N \mathcal{Z} >_{\mathcal{M}}, \end{aligned}$$

so that $\mathcal{K}$ is skew-symmetric with respect to $< ., . >_{\mathcal{M}}$ and $< \mathcal{K} \star_N \mathcal{Z}, \mathcal{Z} >_{\mathcal{M}} = 0$ for all $\mathcal{Z}$.

We note that the Krylov-type sequence is completely specified by its first element $\mathcal{V}_0$. We have worked exclusively with $\mathcal{V}_0 = \mathcal{M}^{-1} \star_N \mathcal{R}_0$, where $\mathcal{R}_j = \mathcal{B} - \mathcal{A} \star_N \mathcal{X}_j$. This is a very natural choice especially when the norm of the operator $\mathcal{K}$ is small.

Remark. It can be shown that the iterate $\mathcal{X}_j$ generated by the GCG-BTF method is the best approximation to $\mathcal{X}$ with respect to a certain j-dimensional affine subspace, but not with respect to the affine Krylov-type subspace $\mathcal{X}_0 + \varphi_j$ (unless $\mathcal{X}_j = \mathcal{X}$).

Error bounds. It is not difficult to see that, use the best approximation property of the iterates $\mathcal{X}_j$, the error bound for the GCG-BTF method is as follows:

$$||\mathcal{X}_j - \mathcal{X}||_{\mathcal{M}} \leq ||Q_j(\mathcal{K}) \star_N (\mathcal{X}_0 - \mathcal{X})||_{\mathcal{M}},$$

for any real polynomial $Q_j(\mu)$ of degree at most j satisfying $Q_j(1) = 1$ and $Q_j(-1) = (-1)^j$.

3 Numerical results

In this section, we give a numerical example to show the performance of the proposed algorithm. All tests were carried out in double precision with a MATLAB code and initial tensor $\mathcal{X}_0 = \mathcal{O}$, when the computer specifications are Microsoft Windows 10 Intel(R), Core(TM)i7-7500U, CPU 2.70 GHz, with 8 GB of RAM. All used codes came from the MATLAB tensor toolbox developed by Bader and Kolda [1]. We compared the proposed methods with CG-BTF, CGS-BTF and Bi-CGSTAB-BTF algorithms, where the stopping criterion is $\|\mathcal{R}_j\| < 10^{-8}$.

We consider two-dimensional (2D) Poisson problem

$$\begin{aligned} -\nabla^2 v = f, \quad in\ \Omega = [0,1] \times [0,1], \\ v = 0, \quad on\ \partial\Omega, \end{aligned} \tag{4}$$

where f is a known function,

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}, \tag{5}$$

and v is unknown function.

Several problems in physics and mechanics are modeled by (4), where v represents, for example, temperature, electromagnetic potential, or displacement of an elastic membrane fixed at the boundary. We compute an approximation of the unknown function $v(x, y)$ in (4). The mesh points are obtained by discretizing the unit square domain with step sizes, Δx in the x-direction and Δy in the y-direction. If we assume that $\Delta x = \Delta y = h = \dfrac{1}{n+1}$, after the standard central difference approximations, we obtain the difference formula

$$4v_{ij} - v_{i-1j} - v_{i+1j} - v_{ij-1} - v_{ij+1} = h^2 f_{ij}, \qquad i, j = 1, 2, \cdots, n. \tag{6}$$

The higher order tensor representation of the 2D discretized Poisson problem (4) is

$$\mathcal{A}_n \star_2 \mathcal{V} = \mathcal{F}, \tag{7}$$

where $\mathcal{A}_n \in \mathbb{R}^{n\times n\times n\times n}$ and $\mathcal{V}, \mathcal{F} \in \mathbb{R}^{n\times n}$ are discretized on the unit square. The nonzeros entries of the tensor block $(\mathcal{A}_n^{(2,4)})_{k=\alpha,l=\beta}$ are in the following five-point stencil

$$\begin{cases} (\mathcal{A}^{(2,4)}_{n_{\alpha,\beta}})_{\alpha,\beta} = \dfrac{4}{h^2}, \\ (\mathcal{A}^{(2,4)}_{n_{\alpha,\beta}})_{\alpha-1,\beta} = (\mathcal{A}^{(2,4)}_{n_{\alpha,\beta}})_{\alpha,\beta-1} = \dfrac{-1}{h^2}, \\ (\mathcal{A}^{(2,4)}_{n_{\alpha,\beta}})_{\alpha+1,\beta} = (\mathcal{A}^{(2,4)}_{n_{\alpha,\beta}})_{\alpha,\beta+1} = \dfrac{-1}{h^2}, \end{cases} \tag{8}$$

for $\alpha, \beta = 2, ..., n-1$ and $\mathcal{F} = 10 * tenrand(n, n) \in \mathbb{R}^{n\times n}$. The numerical results are depicted in Figure 1 for $n = 30$, where in GCG-BTF method, we choose $\mathcal{X}_1 = tenones(n, n)$ and the obtained approximation of the inversion of $\mathcal{M}^{-1}$ using RAPID algorithm [4] is obtained for solving $\mathcal{M} \star_2 \mathcal{V}_j = \mathcal{B} - \mathcal{A} \star_2 \mathcal{X}_j$.

Figure 1: Residual curves.

4 Conclusion

In this paper, the generalized conjugate gradient method is extended for solving tensor equation $\mathcal{A}\star_N\mathcal{X}=\mathcal{B}$. The proposed numerical example provided the efficiency of the GCG-BTF method.

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Generalized inverse of matrices

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Abstract

In this paper, some relations between Drazin, Moore-Penrose inverses with *DMP*, *CMP* generalized inverses are studied.

Keywords: Moore Penrose inverse, *Drazin inverse*, *CMP inverse*, *DMP inverse*
Mathematics Subject Classification [2010]: 15A09, 15A03

1 Introduction

Let $\mathbb{C}^{m\times n}$ be the set of all $m \times n$ complex matrices and assume A^*, $\mathcal{R}(A)$ and rank(A) denote the conjugate transpose, column space and rank of $A \in \mathbb{C}^{m\times n}$, respectively. For $A \in \mathbb{C}^{n\times n}$, the smallest nonnegative integer m defined by the condition rank(A^m)=rank(A^{m+1}) is called the index of A and is denoted by ind(A).
The Drazin inverse of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $A^D = X \in \mathbb{C}^{n\times n}$ such that

$$A^{m+1}X = A^m, \quad XAX = X, \quad AX = XA,$$

where $m = ind(A)$. If $ind(A) = 1$, then A^D is the group inverse of A, which is denoted by $A^{\#}$. The basic theory and various applications of the Drazin inverse can be found in the monographs [1,6].
The Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$ is the unique matrix $A^\dagger = X \in \mathbb{C}^{n\times m}$ which satisfies the Penrose equations

$$AXA = A, \quad XAX = X, \quad (AX)^* = XA, \quad (XA)^* = XA.$$

If X satisfies the equation $AXA = A$, then X is called a g-inverse of A. A matrix X is an outer inverse of A, if $XAX = X$ holds. An important feature of the Moore-Penrose inverse is that it can be used to represent orthogonal projectors. For instance, $P_A = AA^\dagger$ and $Q_A = A^\dagger A$ are the orthogonal projectors onto $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively.

In [5, Theorem 2.2.21] it has been proven that A can be written as the sum of two matrices A_1 and A_2 i.e. $A = A_1 + A_2$, where

- $rank(A_1) = rank(A_1^2)$ i.e. $\text{ind}(A_1) \leq 1$,
- A_2 is nilpotent,

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- $A_1A_2 = A_2A_1 = 0$.

The matrices A_1 and A_2 are called the core and nilpotent parts of A respectively, and this decomposition is unique.
The concept of *DMP inverse* of A was introduced in [4]. In this case, the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$XAX = X, \quad XA = A^DA, \quad \textit{and} \quad A^mX = A^mA^\dagger,$$

is called the *DMP inverse* of A and is denoted by $A^{D,\dagger}$. Moreover, it was proved that $A^{D,\dagger} = A^DAA^\dagger$. Dually, it is easy to see that the dual *DMP inverse* is given by $A^{\dagger,D} = A^\dagger AA^D$.
The *CMP inverse* for a complex matrix was introduced by [2]. The *CMP inverse* of A is a matrix $X \in \mathbb{C}^{n\times n}$ such that the following equations hold:

$$XAX = X, \quad AXA = A_1, \quad AX = A_1A^\dagger, \quad XA = A^\dagger A_1.$$

Such matrix X is unique and denoted by $A^{c\dagger} = A^\dagger A_1 A^\dagger$.

2 Main results

In this section some relations between Drazin, inverses with *DMP*, *CMP* generalized inverses are studied. Easy calculations show the following lemma which will be helpful throughout the paper.

Lemma 2.1. *Let $A \in \mathbb{C}^{n\times n}$ and let $A = A_1 + A_2$ be the core-nilpotent decomposition. Then*

1. $A^DA_1 = A_1A^D$,
2. A^D *is a g-inverse of* A_1,
3. A^D *is an outer inverse of* A.

Theorem 2.2. *Let $A \in \mathbb{C}^{n\times n}$. Then $X = A^D$ is a solution of the following equations.*

$$A^{c,\dagger}XA^{c,\dagger} = A^{c,\dagger}XA^{D,\dagger} = A^{\dagger,D}XA^{D,\dagger} = A^\dagger A^D A^\dagger.$$

Proof. 1. We have

$$\begin{aligned}
A^{c,\dagger}XA^{c,\dagger} &= A^{c,\dagger}A^DA^\dagger AA^DAA^\dagger = A^{c,\dagger}(A^D)^2AA^\dagger AA^DAA^\dagger \\
&= A^{c,\dagger}(A^D)^2AA^DAA^\dagger = A^{c,\dagger}A^DA^DAA^\dagger = A^{c,\dagger}A^DA^{D,\dagger} \\
&= A^\dagger AA^DAA^\dagger A^DA^{D,\dagger} = A^\dagger AA^DAA^\dagger A(A^D)^2A^{D,\dagger} \\
&= A^\dagger AA^DA(A^D)^2A^{D,\dagger} = A^\dagger AA^DA^DA^{D,\dagger} = A^{\dagger,D}A^DA^{D,\dagger} \\
&= A^\dagger AA^DA^DA^DAA^\dagger = A^\dagger A^DA^DAA^\dagger = A^\dagger A^DA^\dagger
\end{aligned}$$

□

Corollary 2.3. *Let $A \in \mathbb{C}^{n\times n}$. Then $X = A^D$ is a solution of the following equations.*

$$A^{D,\dagger}XA^{\dagger,D} = X^2A^{D,\dagger}AX = X^2A^{\dagger,D} = (A^D)^3.$$

The *Hartwig-Spindelbock decomposition* [3, Corollary 6] of any matrix $A \in \mathbb{C}^{n\times n}$ of rank r is given by

$$A = U \begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix} U^*, \tag{1}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = diag(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_t I_{r_t})$ is a diagonal matrix of the nonzero singular values of A, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$, $K \in \mathbb{C}^{r\times r}$ and $L \in \mathbb{C}^{r\times(n-r)}$ satisfy

$$KK^* + LL^* = I_r.$$

The Drazin inverse and the Moore-penrose inverse of A are as follows [4].

$$A^D = U \begin{pmatrix} (\Sigma K)^D & ((\Sigma K)^D)^2 \Sigma L \\ 0 & 0 \end{pmatrix} U^*, \quad \text{and} \quad A^\dagger = U \begin{pmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{pmatrix} U^*. \tag{2}$$

By using $A_1 = AA^DA$, we obtain the following

$$A_1 = U \begin{pmatrix} \Sigma K(\Sigma K)^D \Sigma K & \Sigma K(\Sigma K)^D \Sigma L \\ 0 & 0 \end{pmatrix} U^*. \tag{3}$$

By using Hartwig-Spindelbock decomposition, we obtain the following theorem.

Theorem 2.4. *Let $A \in \mathbb{C}^{n\times n}$ be as in (1). Then*

1. *$A_1 = A^\dagger$ if and only if $(\Sigma K)_1 = K^*\Sigma^{-1}$ and $L = 0$,*
2. *If $A^D = A_1$ is, then $(\Sigma K)_1 = (\Sigma K)^D$,*
3. *$A_1 = A^{D,\dagger}$ if and only if $(\Sigma K)^D = \left((\Sigma K)^D\right)^3$ and $(\Sigma K)^D \Sigma L = 0$,*
4. *$A_1 = A$ if and only if $\Sigma K(\Sigma K)^D = I_r$,*

where $(\Sigma K)_1 = (\Sigma K)(\Sigma K)^D(\Sigma K)$.

Proof. 1. Let $A_1 = A^\dagger$. Then by using (2) and (3), we have

$$(\Sigma K)(\Sigma K)^D(\Sigma K) = K^*\Sigma^{-1}, \qquad (\Sigma K)(\Sigma K)^D \Sigma L = 0, \qquad L^*\Sigma^{-1} = 0.$$

Therefore, $A_1 = A^\dagger$ if and only if $(\Sigma K)(\Sigma K)^D(\Sigma K) = K^*\Sigma^{-1}$ and $L = 0$.

2. Let $A^D = A_1$. Then by using (2) and (3), we have

$$(\Sigma K)(\Sigma K)^D(\Sigma K) = (\Sigma K)^D, \qquad (\Sigma K)(\Sigma K)^D(\Sigma L) = ((\Sigma K)^D)^2(\Sigma L). \tag{4}$$

The quation (4) is equivalent to the following equation

$$(\Sigma K)(\Sigma K)^D(\Sigma K) = ((\Sigma K)^D)^2(\Sigma K), \qquad (\Sigma K)(\Sigma K)^D(\Sigma L) = ((\Sigma K)^D)^2(\Sigma L). \tag{5}$$

Right-multiplying both of these equalities (5) by K^*K and L^*K, respectively and using $KK^* + LL^* = I_r$, we get $(\Sigma K)(\Sigma K)^D(\Sigma K) = (\Sigma K)^D$. Therefore $(\Sigma K)_1 = (\Sigma K)^D$.

3. Let $A_1 = A^{D,\dagger}$. Then by using (3) and [4, Theorem 2.5], we have

$$(\Sigma K)(\Sigma K)^D(\Sigma K) = (\Sigma K)^D, \qquad (\Sigma K)(\Sigma K)^D \Sigma L = 0. \tag{6}$$

Pre-multiplying both of these equalities (6) by $\left((\Sigma K)^D\right)^2$ and $(\Sigma K)^D$, respectively. Therefore, $A_1 = A^{D,\dagger}$ if and only if $(\Sigma K)^D = \left((\Sigma K)^D\right)^3$ and $(\Sigma K)^D \Sigma L = 0$.

4. Suppose that $A_1 = A$. Then by using (1) and (3), we have

$$(\Sigma K)(\Sigma K)^D(\Sigma K) = (\Sigma K), \qquad (\Sigma K)(\Sigma K)^D \Sigma L = \Sigma L. \tag{7}$$

Rigth-multiplying both of these equalities (7) by $K^*\Sigma^{-1}$ and $L^*\Sigma^{-1}$, respectively and using $KK^* + LL^* = I_r$, the equality $A_1 = A$ holds if and only if $\Sigma K(\Sigma K)^D = I_r$.

□

Theorem 2.5. *Let $A \in \mathbb{C}^{n\times n}$ be as in (1) and $P_A = AA^\dagger$. Then*

(a) $P_A A_1 = A_1 P_A$ if and only if $(\Sigma K)^D \Sigma L = 0$,

(b) $A_1(I_n - P_A) = (I_n - P_A)A_1$ if and only if $(\Sigma K)^D \Sigma L = 0$,

(c) $A^{c\dagger}(I_n - P_A) = (I_n - P_A)A^{c\dagger}$ if and only if $L^*K(\Sigma K)^D = 0$.

Proof. (a). Let $A \in \mathbb{C}^{n\times n}$. By using (1), (2) and (3), we have

$$P_A A_1 = U \begin{pmatrix} \Sigma K(\Sigma K)^D \Sigma K & \Sigma K(\Sigma K)^D \Sigma L \\ 0 & 0 \end{pmatrix} U^*, \tag{8}$$

$$A_1 P_A = U \begin{pmatrix} \Sigma K(\Sigma K)^D \Sigma K & 0 \\ 0 & 0 \end{pmatrix} U^*. \tag{9}$$

By (8) and (9), the equality $P_A A_1 = A_1 P_A$ holds if and only if $\Sigma K(\Sigma K)^D \Sigma L = 0$.
Pre-multiplying the equation $\Sigma K(\Sigma K)^D \Sigma L = 0$ by $(\Sigma K)^D$ and using $(\Sigma K)^D = (\Sigma K)^D(\Sigma K)(\Sigma K)^D$, we get $(\Sigma K)^D \Sigma L = 0$.
Therefore, $P_A A_1 = A_1 P_A$ if and only if $(\Sigma K)^D \Sigma L = 0$.
By (1), (2) and (3), (b) holds.
By [2, P.3(7)], (1) and (2), (c) holds.

□

3 Conclusion

In this paper, by using *DMP*, *CMP* generalized inverses, we obtain some equations. The relation between Drazin inverse and these equations are studied.

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Some topological properties of nonlinear positive mappings between C^*-algebras

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Abstract

It is known that every positive linear map between unital C^*-algebras is norm-norm continuous. However, this is not necessarily true in the setting of nonlinear positive maps between C^*-algebras, in general. In this talk, we present some conditions to provide the continuity of nonlinear positive maps between C^*-algebras with respect to the strong and norm operator topologies. These results imply the automatic continuity of a large family of nonlinear positive maps between C^*-algebras. Moreover, we show that in some classes of nonlinear positive maps the continuity property implies the operator monotonicity.

Keywords: nonlinear positive map; n-monotone map; continuity; strong operator topology.
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1 Introduction

Let us denote by $\mathbb{B}(\mathscr{H})$ the algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. In the case when $\mathscr{H} = \mathbb{C}^n$, we identify $\mathbb{B}(\mathbb{C}^n)$ with the matrix algebra of $n \times n$ complex matrices $M_n(\mathbb{C})$. Here we consider the usual Löwner order $\leq$ on the real space of self-adjoint operators. An operator A is said to be strictly positive (denoted by $A > 0$) if it is a positive invertible operator. Due to the Gelfand–Naimark–Segal theorem, we may assume that any C^*-algebra is a closed C^*-subalgebra of $\mathbb{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. We use $\mathscr{A}, \mathscr{B}, \cdots$ to denote C^*-algebras and $\mathscr{A}_+$ and $\mathscr{A}_{++}$ to denote the sets of all positive and positive invertible elements of $\mathscr{A}$, respectively. To shorten notation, we denote the relative strong operator weak operator topologies on any subset of $\mathbb{B}(\mathscr{H})$ by SOT and WOT, respectively.
Let $\Phi : \mathscr{A} \to \mathscr{B}$ be a (not necessarily linear) map between C^*-algebras. Then Φ is said to be $*$-map or self-adjoint if it is $*$-preserving i.e. $\Phi(A^*) = \Phi(A)^*$ and it is called positive if Φ maps the cone of positive elements of $\mathscr{A}$ to $\mathscr{B}_+$. If $\Phi(\mathscr{A}_{++}) \subset \mathscr{B}_{++}$, then Φ is called strictly positive. We say that Φ is unital if $\mathscr{A}, \mathscr{B}$ are unital and Φ preserves the unit. We simply denote both units of $\mathscr{A}$ and $\mathscr{B}$ by I. A map Φ is called n-positive if the map $\Phi_n : M_n(\mathscr{A}) \to M_n(\mathscr{B})$ defined by $\Phi_n([a_{ij}]) = [\Phi(a_{ij})]$ is positive, where $M_n(\mathscr{A})$ is the C^*-algebra of $n \times n$ matrices with entries in $\mathscr{A}$. If Φ is n-positive for all $n \in \mathbb{N}$, then it is called a completely positive map. We say a positive map $\Phi : \mathscr{A} \to \mathscr{B}$ is n-momotone, whenever the map Φ_n is monotone on $M_n(\mathscr{A})_+$ in the sense that if $[A_{ij}], [B_{ij}] \in M_n(\mathscr{A})_+$, then

$$[A_{ij}] \geq [B_{ij}] \Longrightarrow \Phi_n([A_{ij}]) \geq \Phi_n([B_{ij}]).$$

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2 Main results

We start our work by giving some examples of n-positive maps and n-monotone maps.

Example 2.1. 1. If $\mathscr{A}$ is a C^*-algebra, then the operator norm of $\mathscr{A}$ is a 2-positive map from $\mathscr{A}$ to $\mathbb{C}$, which is not 2-monotone in general.

2. Every $2n$-positive map is n-monotone.

3. For every $1 \leq p < 2$, the power function $\Phi_p : \mathbb{C} \to \mathbb{C}$ defined by $\Phi_p(x) = |x|^p$ is 2-monotone. Note that Φ_p is 3-positive but not 4-positive.

4. Every positive semidefinite matrix $P \in M_n(\mathbb{C})$ induces a map $\phi_P : M_n(\mathbb{C}) \to \mathbb{C}$ defined by $\phi_P(A) = |\mathrm{tr}(AP)|$, which is a 3-positive semi–norm on $M_n(\mathbb{C})$. Moreover, ϕ is 2-monotone.

5. Every positive linear functional $\varphi : \mathscr{A} \longrightarrow \mathbb{C}$ on a C^*-algebra induces a nonlinear 3-positive and 2-monotone map $\Phi : \mathscr{A} \longrightarrow \mathbb{C}$ given by $\Phi(A) = |\varphi(A)|$.

6. The positive map $\Psi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{H})$ given by

$$\Psi(X) = \begin{cases} X, & X \neq -I, \\ 0, & X = -I, \end{cases}$$

is a 2-positive and 1-monotone map. However, Φ is neither 3-positive nor 2-monotone.

In the study of the continuity (with respect to various topologies) of a map between C^*-algebras a natural question is, what is the contribution of the positivity and the linearity? The exitance of several example of linear maps which are discontinuous is konwn. It is easy to present an example of discontinuous positive map. Indeed, consider the map $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{C}$ given by

$$\Phi(X) = \begin{cases} \|X\| & \|X\| \leq I, \\ 0 & \text{otherwise}, \end{cases}.$$

Then Φ is a discontinuous unital positive map. Moreover, The map Ψ (is defined in Example 2.1-6) is a discontinuous 2-positive map. Hence, the 2-positivity property is not enough to get the continuity of a map, in general. However, we can deduce a type of continuity with respect to SOT for 2-positive maps as follows.
If $\mathscr{A}$ and $\mathscr{B}$ are von Neumann algebras, then a positive map $\Phi : \mathscr{A}_+ \to \mathscr{B}_+$ is called *normal* if for every bounded increasing net $\{A_\alpha\}_\alpha \subseteq \mathscr{A}_+$, it holds that

$$\Phi(\sup_\alpha A_\alpha) = \sup_\alpha \Phi(A_\alpha).$$

A map $\Phi : \mathscr{A}_+ \to \mathscr{B}_+$ is said to be *weakly normal* if for every norm convergent increasing net $\{A_\alpha\}_\alpha \subseteq \mathscr{A}_+$, the equlity

$$\Phi(\sup_\alpha A_\alpha) = \sup_\alpha \Phi(A_\alpha)$$

holds.
The first result reads as follows.

Theorem 2.2. *If $\Phi : \mathscr{A} \to \mathscr{B}$ is a 2-positive map between von Neumann algebras, then it is weakly normal.*

In [2, Example 2.3], authors show that if $\mathscr{H}$ is an infinite-dimensional separable Hilbert space, then the operator norm $\|\cdot\| : \mathbb{B}(\mathscr{H}) \to \mathbb{C}$ is an example of a 2-positive map, which is not normal.
We extend our work and give some results about the continuity of 3-positive maps.

Theorem 2.3. *Let $\Phi : \mathscr{A} \to \mathscr{B}$ be a 3-positive map between unital C^*-algebras. Consider a topology $\tau \in \{\|\cdot\|, \mathrm{SOT}, \mathrm{WOT}\}$ on $\mathscr{A}$. Then the following conditions are equivalent:*

1. *Φ is norm-τ-continuous,*
2. *$\Phi\left(\frac{1}{n}I\right) \xrightarrow{\tau} \Phi(0)$,*
3. *$\Phi\left(Z + \frac{1}{n}I\right) \xrightarrow{\tau} \Phi(Z)$ for some $Z \in \mathscr{A}_{++}$.*

Now we have the following result.

Proposition 2.4. *If $\Phi : \mathscr{A} \to \mathscr{B}$ is a 3-positive map between unital C^*-algebras, then*

$$\mathrm{SOT}-\lim_n \Phi\left(\frac{1}{n}I\right) = \Phi(0).$$

We notice that the norm continuity of the sequence $\{\Phi\left(\frac{1}{n}I\right)\}$ is still an open problem.
According to Theorem 2.3 and Proposition 2.4, we get the following corollary.

Corollary 2.5. *All 3-positive maps between unital C^*-algebras are norm-SOT continuous. In addition, if a 3-positive map is norm-norm continuous at a positive invertable operator, then it is norm-norm continuous on its domain.*

The map Ψ (is defined in Example 2.1-6) is a 2-positive map which is not norm-SOT continuous. Hence, the 3-positivity condition in the above results is necessary.
Let $\Phi : \mathscr{A} \to \mathscr{B}$ be a 3-positive map between unital C^*-algebras. In the case, when $\mathscr{B}$ is finite-dimensional, Corollary 2.5 ensures that Φ is norm continuous. In particular, every 3-positive map $\Phi : \mathscr{A} \to M_n(\mathbb{C})$ is norm-continuous.
Now we aim to investigate the continuity of n-monotone mappings between C^*-algebras. The next result shows that the 2-monotonicity provides suitable conditions to ensure the continuity of a nonlinear positive map.

Theorem 2.6. *[2, Theorem 3.2] If $\Phi : \mathscr{A} \to \mathscr{B}$ is a 2-monotone map between unital C^*-algebras, then Φ is norm-continuous.*

The map Ψ (is defined in Example 2.1-6) is a 1-monotone map which is not norm continuous. Hence, the 2-monotonicity condition in Theorem 2.6 is necessary.
It is known that every 4-positive map is 2-monotone; see [2, Section 4]. Therefore, all 4-positive maps between unital C^*-algebras are norm-continuous.
We aim to prove that in some cases a continuity condition for positive maps provides a monotonicity property for them.

Definition 2.7. [1, Definition IX.5.6] A map $\varphi : M_n(\mathbb{C}) \to \mathbb{C}$ is called a Lieb function if it satisfies the following two conditions:

- (Monotonicity) $\varphi(A) \geq \varphi(B) \geq 0$ if $A \geq B \geq 0$;
- (Cauchy–Schwarz) $\varphi(A^*A)\varphi(B^*B) \geq |\varphi(A^*B)|^2$ for every $A, B \in M_n(\mathbb{C})$.

An extension of the above definition was presented and investigated in [3] as follows:

Definition 2.8. A map $\Phi : \mathscr{A} \to \mathscr{B}$ between C^*-algebras is said to be a Lieb map if it has the following properties:

- (Monotonicity) $\Phi(A) \geq \Phi(B)$ if $A \geq B \geq 0$;
- (Cauchy–Schwarz) $\begin{bmatrix} \Phi(A^*A) & \Phi(A^*B) \\ \Phi(A^*B)^* & \Phi(B^*B) \end{bmatrix} \geq 0$ for all $A, B \in \mathscr{A}$.

The final result reads as follows.

Theorem 2.9. *Let $\mathscr{A}$ be a von Neumann algebra and let $\mathscr{B}$ be a unital C^*-algebra. If $\Phi : \mathscr{A} \to \mathscr{B}$ is an* SOT-SOT*-continuous map, then the Cauchy–Schwarz condition in Definitions 2.7 and 2.8 imply the monotonicity conditions.*

Corollary 2.10. *In the definition of* SOT-SOT*-continuous Lieb maps as well as in the definition of continuous Lieb functions, the monotonicity condition is redundant.*

3 Conclusion

The automatic continuity of positive linear maps between unital C^*-algebras can be extended to the framwork of nonlinear positive mappings if they are n-positive or m-monotone for sufficiently large $n, m \in \mathbb{N}$. Moreover, it seems that the n-monotonicity provides suitable conditions to imply the continuity of a nonlinear positive map. In addition, in many cases the reverse of the above facts hold. More precisely, the n-positivity and m-monotonicity of nonlinear positive maps can be deduced from a continuity property and a Cauchy–Schwarz type condition for such maps.

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Linear Algebra and Quadratic Reciprocity Law

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Abstract

The quadratic reciprocity law was proved by Gauss who produced six different proofs of this law in his life. Although there are different proofs of this well-known theorem, we adopt concepts from linear algebra to calculate the Gauss sum and give a different proof of this theorem.

Keywords: Gauss sum, quadratic reciprocity, trace, equivalence.
Mathematics Subject Classification [2010]: 11A15, 11L05.

1 Introduction

Let p be an odd prime and a an integer. The Legendre symbol $(\frac{a}{p})$ is defined to be 1 if the quadratic equation $x^2 \equiv a(\mod p)$ has a solution, otherwise $(\frac{a}{p}) = -1$. If $(\frac{a}{p}) = 1$, then a is called a quadratic residue modulo 1, otherwise a non-quadratic residue modulo 1.

Let p and q be district odd primes, then the quadratic reciprocity law states that

$$(\frac{p}{q})(\frac{q}{p}) = (-1)^{\frac{(p-1)(q-1)}{4}}$$

This law first proved by Gauss in 1801 [3], while he was only 19 years old, but he proved six different proof of this law in his life time. Up to present time many different proofs of this law has been published whose number exceeds 150, see [1]. The proofs use number theory, trigonometry, character theory, etc. For an elementary proof see [4]. In this paper we present a proof that uses linear algebra.

2 Preliminaries

Let A be an abelian group. If the composition law in A is written addively a character of A is a function $\chi : A \longrightarrow \mathbb{C}^\times$ such that $\chi(x+y) = \chi(x)\chi(y)$, for all $x, y \in A$, If the law of composition in A is written multiplicatively $\chi(xy) = \chi(x)\chi(y)$, where $\mathbb{C}^\times$ denotes the non-zero complex numbers under multiplication.

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Let p be an odd prime. Then $\chi : \mathbb{Z}_p^\times \longrightarrow \mathbb{C}$ defined by $\chi(a) = (\frac{a}{p})$, $a \in \mathbb{Z}_p^\times$, is a character because of the property of the Legendre symbol $(\frac{ab}{p}) = (\frac{a}{p})(\frac{b}{p})$. Let ξ be a primitive pth of unity, i. e. $\xi = e^{\frac{2\pi i}{p}}$. We let $e(x) = e^{2\pi i x}$. The additive group of $\mathbb{Z}_p$ is generated by 1, then the function $\chi : \mathbb{Z}_p \longrightarrow \mathbb{C}^\times$ defined by $\chi(1) = \xi$, extended by $\chi(k) = \xi^k$, is a character of $\mathbb{Z}_p$. All characters of $\mathbb{Z}_p$ are of this type. If $1 \leq r \leq p$, then $\chi_r : \mathbb{Z}_p \longrightarrow \mathbb{C}^\times$ defined by $\chi_r(s) = \xi^{rs}$, $1 \leq s \leq p$, is a character of $\mathbb{Z}_p$ and all the r characters of $\mathbb{Z}_p$ are of this form.

The character table of $\mathbb{Z}_p$ is a $p \times p$ matrix $X = (\xi^{rs})_{1 \leq r,s \leq p}$. If we use the orthogonality relation on the character table of X, [2], we obtain

$$\overline{X}^t X = \begin{bmatrix} p & & & & \\ & p & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & \ddots & \\ & & & & p \end{bmatrix} = pI_p$$

where $\overline{X}$ is the matrix obtained from X by conjugation of entries and t denotes transpose.

If we take the determinate we obtain:

$|\det X|^2 = p^p$. It is known that either $\det X$ is real or pure imaginary. Therefore $\det X = p^{\frac{p}{2}}$ or $\det X = ip^{\frac{p}{2}}$.

3 Main Result

It we take the trace of X we obtain $tr\ X = \sum_{r=1}^{p} \xi^{r^2} = \sum_{r=1}^{p} (\frac{r}{p})\xi^r$.

Corollary 3.1.

$$tr\ X = \begin{cases} \sqrt{p}, & if\ p \equiv 1 (mod 4), \\ i\sqrt{p}, & if\ p \equiv 3 (mod 4). \end{cases}$$

Proof. With respect tour ordering of the rows and columns of the matrix X we have:

$$X = \begin{bmatrix} \xi & \xi^2 & \xi^3 & \dots & \xi^{p-1} & 1 \\ \xi^2 & \xi^4 & \xi^6 & \dots & \xi^{2(p-1)} & 1 \\ \vdots & & \vdots & & & \vdots \\ \xi^{p-1} & \xi^{2(p-1)} & \xi^{3(p-1)} & \dots & \xi^{(p-1)^2} & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$$

We compute the matrix X^2. If $X^2 = (c_{ij})_{1 \leq i,j \leq p}$, then

$$c_{ij} = \sum_{k=1}^{p} \xi^{rk}\xi^{ks} = \sum_{k=1}^{p} \xi^{k(r+s)} = \begin{cases} p, & if\ p \mid r+s, \\ 0, & otherwise. \end{cases}$$

Therefore

$$X^2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & p & 0 \\ 0 & 0 & \cdots & p & 0 & 0 \\ \vdots & & & \vdots & & \vdots \\ p & 0 & & \cdots & & 0 \\ 0 & 0 & & \cdots & & p \end{bmatrix} = PS$$

where

$$S = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & & & \vdots & & \vdots \\ 1 & 0 & & \cdots & 0 & 0 \\ 0 & 0 & & \cdots & 0 & 1 \end{bmatrix}$$

is the permutation matrix corresponding to the permutation

$$(1 \quad p-1)(2 \quad p-2)\cdots(\frac{p-1}{2} \quad \frac{p+1}{2})(p)$$

of the symmetric group $\mathbb{S}_p$. Therefore we can calculate the eigenvalues of X to be $\pm\sqrt{p}$, $\pm i\sqrt{p}$. Further calculators reveal the trace of X as indicated in the Corollary. □

Definition 3.2. Using the formula for $tr\ X$ we define $G_p(a) = \sum_{r=1}^{p} (\frac{r}{p})\xi^{r^2 a}$, where $(a,p) = 1$ and ξ is a primitive pth root of unity in $\mathbb{C}$.

It is clear that $G_p(1) = tr\ X = \sqrt{p}$ or $i\sqrt{p}$. We set $t(p) = 1$ if $p \equiv 1(mod\ 4)$ and $t(p) = i$ if $p \equiv 3(mod\ 4)$, hence $tr\ X = t(p)\sqrt{p}$.

Lemma 3.3. *Let p and q be distinct odd prime numbers. Then $G_p(q)G_q(p) = G_{pq}(1)$.*

Proof. We have $G_p(q) = \sum_{r=1}^{p} e^{\frac{2\pi i r^2 q}{p}}$ and $G_q(p) = \sum_{s=1}^{q} e^{\frac{2\pi i s^2 p}{q}}$. Therefore

$$G_p(q)G_q(p) = \sum_{r=1}^{p} e^{\frac{2\pi i r^2 q}{p}} \sum_{s=1}^{q} e^{\frac{2\pi i s^2 p}{q}} = \sum_{r=1}^{p}\sum_{s=1}^{q} e^{\frac{2\pi i}{pq}(r^2q^2+s^2p^2)}$$
$$= \sum_{r,s} e^{\frac{2\pi i}{pq}(rq+sp)^2} = G_{pq}(1)$$

This is because $rq + sp$ forms a complete residue classes moduls pq. □

Theorem 3.4 (quadratic reciprocity law)**.** *Let p and q be district odd primes, then:*

$$(\frac{p}{q})(\frac{q}{p}) = (-1)^{\frac{(p-1)(q-1)}{4}}$$

Proof. By convention $G_p(1) = t(p)\sqrt{p}$. Thus,

$$t(pq)\sqrt{pq} = G_{pq}(1) = G_p(q)G_q(p) = (\frac{q}{p})(\frac{p}{q})G_p(1)G_q(1) = (\frac{q}{p})(\frac{p}{q})t(p)t(q)\sqrt{pq}$$

Therefore $(\frac{q}{p})(\frac{p}{q})t(p)t(q) = t(pq)$, implying $(\frac{p}{q})(\frac{q}{p}) = (-1)^{\frac{(p-1)(q-1)}{4}}$. □

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Pseudoconvex Spectral Functions

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Abstract

In this paper, we consider generalized directional derivative and Clarke subdifferential of spectral function and we show that Pseudoconvexity of spectral $f \circ \lambda$ is inherited from the Pseudoconvexity of corresponding symmetric function f and vice versa.

Keywords: Spectral function, Convex analysis, Pseudoconvexity
Mathematics Subject Classification [2010]: 15A18, 49J52, 47A75

1 Introduction and Preliminaries

There has been a growing interest in the variational analysis of spectral functions. This growing trend is due to spectral functions that have important applications to some fundamental problems in applied mathematics such as semi-definite programming and engineering problems (see [2, 3], and references therein).
A function F defined on $\mathcal{S}_n$ is called spectral if

$$F(U^T AU) = F(A), \ \forall \ A \in \mathcal{S}_n, \ \forall \ U \in \mathcal{O}_n,$$

where $\mathcal{S}_n$ is the vector space of all $n \times n$ real symmetric matrices and $\mathcal{O}_n$ is the group of all real orthogonal matrices.
One can easily see [2] that every spectral function is the composition of a symmetric function f defined on $\mathbb{R}^n$ and the eigenvalue function $\lambda : \mathcal{S}_n \longrightarrow \mathbb{R}^n$, i.e.,

$$F(A) = (f \circ \lambda)(A), \ \forall \ A \in \mathcal{S}_n.$$

Hence there exists a one-to-one correspondence between the spectral functions F defined on $\mathcal{S}_n$ and the symmetric functions f defined on $\mathbb{R}^n$. In recent years a lot of research shows that the properties of F are inherited from the properties of f, and vice versa [2, 3].
We present some definitions and theorems for nonconvex locally Lipschitz continuous functions, for more details see [1].
Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a locally Lipschitz continuous functions at $x \in \mathbb{R}^n$. The generalized diretional derivative of f at x in directional of $d \in \mathbb{R}^n$ is defined by

$$f^\circ(x, d) = \limsup_{y \to x, t \downarrow 0} \frac{f(y + td) - f(y)}{t}$$

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Also the Clarke subdifferential of f at x is the set $\partial_C f(x)$ of vector $\xi \in \mathbb{R}^n$ such that

$$\partial_C f(x) = \{\xi \in \mathbb{R}^n : f^\circ(x,d) \geqslant \langle \xi, d\rangle, \quad \forall d \in \mathbb{R}^n\}$$

The following theorem present that generalized directional derivative can be calculated from Clarke subdifferential.

Theorem 1.1. ([1],Theorem 3.4) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a locally Lipschitz continuous functions at $x \in \mathbb{R}^n$. Then

$$f^\circ(x,d) = \max\{\langle \xi, d\rangle : \xi \in \partial_C f(x)\}, \qquad \forall d \in \mathbb{R}^n. \tag{1}$$

It is well known that if f is a convex funtion then the generalized directional derivative equals to directional derivative of f and Clarke subdifferential coincides with classical subdifferential of f.

Definition 1.1. A continuously differentiable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is pseudoconvex, if for all $x, y \in \mathbb{R}^n$

$$f(y) < f(x) \quad \Longrightarrow \quad \langle \nabla f(x), y - x\rangle < 0$$

Definition 1.2. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is generalized pseudoconvex, if it is locally Lipschitz continuous and for all $x, y \in \mathbb{R}^n$

$$f(y) < f(x) \quad \Longrightarrow \quad f^\circ(x, y - x) < 0$$

It is clear that a locally Lipschitz continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is generalized pseudoconvex if and only if for all $x, y \in \mathbb{R}^n$

$$f^\circ(x, y - x) \geqslant 0 \quad \Longrightarrow \quad f(y) \geqslant f(x).$$

Let $\mathcal{S}_n$ be the vector space of all $n \times n$ real symmetric matrices. We denote by $\mathcal{O}_n$ the group of all real orthogonal matrices. We endow $\mathcal{S}_n$ with the trace inner product [2]:

$$\langle A, B\rangle := tr(AB), \ \forall \ A, B \in \mathcal{S}_n.$$

This inner product induces the Frobenius norm [2], i.e., $\|A\|_F = \sqrt{tr(A^2)}$. For any $x \in \mathbb{R}^n$, we denote by the symbol $Diag(x)$ the $n \times n$ matrix with components of x on its diagonal and with zero off the diagonal.
Define the eigenvalue function $\lambda : \mathcal{S}_n \longrightarrow \mathbb{R}^n$ by $\lambda(A) := (\lambda_1(A), \lambda_2(A), \cdots, \lambda_n(A))$ for each $A \in \mathcal{S}_n$, where $\lambda_1(A), \lambda_2(A), \cdots, \lambda_n(A)$ are the eigenvalues of A and ordered in a non-increasing order, i.e., $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. The following theorem due to von Neumann plays a central role in the spectral variation analysis.

Theorem 1.2. [2] For any $A, B \in \mathcal{S}_n$, we have

$$\|\lambda(A) - \lambda(B)\| \leq \|A - B\|_F,$$

and

$$\langle A, B\rangle \leq \langle \lambda(A), \lambda(B)\rangle. \tag{2}$$

Every $A \in \mathcal{S}_n$ admits a spectral decomposition of the form $A = UDiag(\lambda(A))U^T$ for some $U \in \mathcal{O}_n$. For each $A \in \mathcal{S}_n$, define the set of all orthogonal matrices giving the ordered spectral decomposition of A by

$$\mathcal{O}_A := \{U \in \mathcal{O}_n : U^T AU = Diag(\lambda(A))\}.$$

It is clear that $\mathcal{O}_A$ is non-empty for each $A \in \mathcal{S}_n$.
A function $F : \mathcal{S}_n \longrightarrow \bar{\mathbb{R}}$ is called spectral if F is $\mathcal{O}_n$-invariant, i.e.,

$$F(U^T AU) = F(A), \ \forall \ A \in dom(F), \ \forall \ U \in \mathcal{O}_n.$$

It is not difficult to see [2] that any spectral function F defined on $\mathcal{S}_n$ can be written as a composition $f \circ \lambda$ for some symmetric function f defined on $\mathbb{R}^n$ (a function $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ is called symmetric if $f(x) = f(Px)$ for all permutation matrices P and for all $x \in \mathbb{R}^n$). For instance, the Frobenius norm is a spectral function defined on $\mathcal{S}_n$ associated with the standard Euclidean norm on $\mathbb{R}^n$.
The following theorems present some properties of differentiablity and subdifferential of spectral functions.

Theorem 1.3. [3] Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a symmetric function. Then the spectral function $f \circ \lambda$ is differentiable at the X if and only if f is differentiable at the vector $\lambda(X)$. In this case the gradient of $f \circ \lambda$ at X is

$$\nabla(f \circ \lambda)(X) = U^T(Diag(\nabla f(\lambda(X))))U$$

for any orthogonal matrix $U \in \mathcal{O}_X$

Theorem 1.4. [3] Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a symmetric function. If f is differentiable at the point μ, then the spectral function $f \circ \lambda$ is differentiable at the matrix $Diag(\mu)$ with

$$\nabla(f \circ \lambda)(Diag(\mu)) = Diag(\nabla f(\mu))$$

Theorem 1.5. [3] Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a symmetric and locally Lipschitz function. Then, for $X \in \mathcal{S}_n$ we have

$$\partial_C(f \circ \lambda)(X) = \{U^T(Diag(\gamma))U : \gamma \in \partial_C f(\lambda(X)), U \in \mathcal{O}_X\}$$

2 Main results

Now, in the following we give the main results.

Lemma 2.1. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a symmetric function. If $\xi \in \partial_C f(x)$, then $Diag(\xi) \in \partial_C(f \circ \lambda)(Diag(x))$.

Theorem 2.1. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a symmetric function. Then f is pseudoconvex if and only if $f \circ \lambda$ is pseudoconvex.

Proof. First, suppose that f is pseudoconvex. Let $A, B \in \mathcal{S}_n$ be such that $f \circ \lambda(A) \leqslant f \circ \lambda(B)$. We must show that $\langle \nabla(f \circ \lambda)(B), A - B\rangle < 0$. In view of theorm 1.3 and (2), for some $U \in \mathcal{O}_B$ we have

$$\begin{aligned}
\langle \nabla(f \circ \lambda)(B), A - B\rangle &= \langle U^T(Diag(\nabla f(\lambda(B)))U, B - A\rangle \\
&= \langle Diag(\nabla f(\lambda(B)), UAU^T\rangle - \langle Diag(\nabla f(\lambda(B)), UBU^T\rangle \\
&\leqslant \langle \lambda(Diag(\nabla f(\lambda(B)))), \lambda(UAU^T)\rangle - \langle Diag(\nabla f(\lambda(B)))), Diag(\lambda(B))\rangle \\
&= \langle \nabla f(\lambda(B)), \lambda(A) - \lambda(B)\rangle < 0
\end{aligned}$$

Note that the last inequality follows from pseudoconvexity of f, hence $f \circ \lambda$ is pseudoconvex. For the converse, suppose that $x, y \in \mathbb{R}^n$ be such that $f(x) < f(y)$. Hence $(f \circ \lambda)(Diag(x)) < (f \circ \lambda)((Diag(y))$. Since $f \circ \lambda$ is pseudoconvex, we conclude

$$\langle \nabla(f \circ \lambda)(Diag(y)), Diag(x) - Diag(y)\rangle < 0$$

In view of theorem 1.4, we have

$$\langle \nabla f(y), x-y\rangle = \langle Diag(\nabla f(y)), Diag(x-y)\rangle = \langle \nabla (f\circ\lambda)(Diag(y)), Diag(x)-Diag(y)\rangle < 0$$

Which complete the proof. □

Theorem 2.2. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a symmetric and locally Lipschitz continuous function. Then f is generalized pseudoconvex if and only if $f\circ\lambda$ is generalized pseudoconvex.

Proof. Suppose that f is generalized pseudoconvex. Let $A, B \in \mathcal{S}_n$ be such that $f^\circ(A, B-A) \geqslant 0$. In view of theorem 1.1, there exist $X \in \partial_C(f\circ\lambda)(A)$ such that $\langle X, B-A\rangle \geqslant 0$. Now, From theorem 1.5 there exist $\gamma \in \partial_C f(\lambda(X))$ and $U \in \mathcal{O}_A$ such that $X = U^T(Diag(\gamma))U$. Therefore

$$\begin{aligned} 0 \leqslant \langle U^T(Diag(\gamma))U, B-A\rangle &= \langle Diag(\gamma), UBU^T - \lambda(A)\rangle \\ &\leqslant \langle \gamma, \lambda(B)-\lambda(A)\rangle \\ &\leqslant f^\circ(\lambda(A), \lambda(B)-\lambda(A)). \end{aligned}$$

Since f is generalized pseudoconvex, we obtain $f(\lambda(B)) \geqslant f(\lambda(A))$ and hence $f\circ\lambda$ is generalized pseudoconvex.
For the converse, let $x, y \in \mathbb{R}^n$ be such that $f^\circ(x, y-x) \geqslant 0$. In view of theorem 1.1, there exist $\xi \in \partial_C f(x)$ such that $\langle \xi, y-x\rangle \geqslant 0$. From lemma 2.1 and theorem 1.1, we have

$$0 \leqslant \langle \xi, y-x\rangle = \langle Diag(\xi), Diag(y)-Diag(x)\rangle \leqslant (f\circ\lambda)^\circ(Diag(x), Diag(y)-Diag(x)).$$

Since $f\circ\lambda$ is generalized pseudoconvex, we obtain $(f\circ\lambda)(Diag(x)) \leqslant (f\circ\lambda)(Diag(y))$. Hence $f(x) \leqslant f(y)$ and the proof is complete. □

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Improved weighted sum method for solving multiobjective quadratically constrained quadratic programming

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Abstract

Most of existing methods solving multiobjective quadratic programming with convex functions and linear constraints. In this paper, the improved weighted sum method is used to solve this problem with both convex and nonconvex quadratic functions. An algorithm is proposed, which converges to a set of efficient solutions.

Keywords: Multiobjective programming, Weighted sum method, Quadratic programming, Linear relaxation, Convex and concave envelop.

Mathematics Subject Classification [2010]: 58E17, 90C20, 47N10.

1 Introduction

We consider the following problem:

$$\begin{array}{lll} \min f(x) = (f_1(x), ..., f_p(x)) & & \\ s.t. & f_k(x) \geqslant 0, & k = p+1, ..., m, \\ & x \in [a, b], & \end{array} \tag{1}$$

where a, b are n dimensional vectors of nonnegative real numbers and $x \in [a, b]$ means that $a_i \leqslant x_i \leqslant b_i$ for all $i = 1, ..., n$. Each $f_k : \mathbb{R}^n \to \mathbb{R}$ is a quadratic function in the form of

$$f_k(x) = x^t H^k x + c_k^t x + d_k, \qquad k = 1, ..., p, \tag{2}$$

which $H^1, ..., H^p$ are real and symmetric $n \times n$ matrixes, $c_1, ..., c_p \in \mathbb{R}^n$ and $d_1, ..., d_p \in \mathbb{R}$.

Problem (1) is a multiobjective quadratically constrained quadratic programming (MQCQP) problem. If $p = 1$, it involves a single objective and we use the term SQCQP instead of MQCQP. In either case, the feasible set of problem (1) is denoted by $X := \{x \in \mathbb{R}^n | f_k(x) \geqq 0, k = p+1, ..., m, x \in [a, b]\}$ and the set $Y := \{y \in \mathbb{R}^p | y = f(x), x \in X\}$ called the image of X under f in the objective space.

MQCQP and SQCQP have been applied in many fields of science, including engineering, economics and etc (see, for example [3, 6]). At first we introduce some basic notations and definitions from [2, 5, 6]. Throughout the paper, $\mathbb{R}^n$ denotes the n dimensional Euclidean space. If $x, y \in \mathbb{R}^n$ then $x \leqq y (x < y)$ if and only if $x_i \leqslant y_i (x_i < y_i), \forall i = 1, ..., n$. In addition, $x \leq y$ means that $x \leqq y$ and $x \neq y$. We will denote by $\mathbb{R}^n_{\leqq}$ the set $\{x \in \mathbb{R}^n | x \geqq 0\}$.

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Definition 1.1. ([2]) Consider an MQCQP problem. The feasible solution $\hat{x} \in X$ is called efficient (weak efficient) if there is no another $x \in X$ such that $f(x) \leq f(\hat{x})(f(x) < f(\hat{x}))$. If $\hat{x} \in X$ is efficient (weak efficient) then $\hat{y} = f(\hat{x})$ is called a nondominated (weak nondominated) point. The set of all efficient solutions and nondominated points are called the efficient set and efficient frontier, respectively

The sets of weakly efficient solutions and efficient solutions are denoted by X_{wE} and X_E, respectively.

Definition 1.2. ([6]) A symmetric $n \times n$ matrix H is called

- Positive definite if and only if $x^tHx > 0$ for all $x \in \mathbb{R}^n$ and $x \neq 0$.
- Positive semidefinite if and only if $x^tHx \geqslant 0$ for all $x \in \mathbb{R}^n$.

Proposition 1.3. *([6]) Let C be a convex subset of $\mathbb{R}^n$ and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be twice continuously differentiable over $\mathbb{R}^n$.*

- *If $\bigtriangledown^2 f(x)$ (Hessian of f) is positive semidefinite for all $x \in C$, then f is convex over C.*
- *If $\bigtriangledown^2 f(x)$ is positive definite for all $x \in C$, then f is strictly convex over C.*

Corollary 1.4. *Consider the quadratic function $f(x) = x^tHx+c^tx+d$, where H is a symmetric $n \times n$ matrix, $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. Then, f is convex if the Hessian matrix H is positive semidefinite. Moreover, f is strictly convex if H is positive definite.*

Definition 1.5. ([6]) A function $h : \mathbb{R}^n_{\leqq} \longrightarrow \mathbb{R}$ is called an increasing function if $h(x) \leqslant h(y)$ for $x \leqq y$. It is a *d.m* (difference of monotonic) function if $h(x) = h^+(x) - h^-(x)$, where h^+ and h^- are increasing functions.

Remark 1.6. Each quadratic function can be represented as a difference of two quadratic functions with nonnegative coefficients. So, every quadratic function is a *d.m* function.

Definition 1.7. ([5]) Let X be a convex and compact subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$. The convex envelop of the function f over X is denoted by $Vex_X f$ and for all $x \in X$ is defined as

$$Vex_X f(x) = \sup\{g(x) : \text{g is convex on X}, \ g(y) \leqslant f(y), \quad \forall y \in X\}$$

Definition 1.8. ([5]) Let X be a convex and compact subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$. The concave envelop of the function f over X is denoted by $Cav_X f$ and for all $x \in X$ is defined as

$$Cav_X f(x) = \inf\{g(x) : \text{g is convex on X}, \ f(y) \leqslant g(y), \quad \forall y \in X\}$$

Theorem 1.9. *([1]) The convex envelop and concave envelop of the two dimensional bilinear function $f(x, y) = xy$ on the hyperrectangle $R = \{(x, y) \in \mathbb{R}^2 : \ell \leqslant x \leqslant u, m \leqslant y \leqslant M\}$ are respectively*

$$Vex_R(xy) = \max\{\ell y + mx - \ell m, uy + Mx - uM\},$$

$$Cav_R(xy) = \min\{\ell y + Mx - \ell M, uy + mx - um\}.$$

2 Improved weighted sum method

One of the well-known scalarization techniques in solving multiobjective optimization problems is the weighted sum method:

$$\min_{x\in X} \sum_{k=1}^{p} w_k f_k(x), \tag{3}$$

where $w = (w_1, w_2, \cdots, w_p) \in \mathbb{R}^p_{\geqq}$. For the scalarization model (3), the following results can be obtained.

Theorem 2.1. *([2]) If $\hat{x}$ is an optimal solution of (3) (and $w > 0$), then $\hat{x}$ is a weakly efficient (an efficient) solution of MQCQP (1).*

To get more results, the feasible set X is restricted by additional constraints such that each objective function is bounded from above. So, the improved weighted sum method is proposed ([4]):

$$\begin{aligned} \min_{x\in X} \quad & \sum_{k=1}^{p} w_k f_k(x) \\ & f_k(x) \leqslant \epsilon_k, \qquad k = 1, ..., p, \end{aligned} \tag{4}$$

where $\epsilon = (\epsilon_1, ..., \epsilon_p)^t$ is an arbitrary vector in $\mathbb{R}^p$. The vector ϵ contains parameters $\epsilon_1, ..., \epsilon_p$, that can be determined by the decision-maker or an expert. Theoretically, these parameters can be any arbitrary value. Theorem 2.1 is still hold for any $\epsilon \in \mathbb{R}^p$. Also, we have

Theorem 2.2. *([4]) If $\hat{x}$ is an efficient (weakly efficient) solution of MQCQP (1). Then, there exist $w > 0$ ($w \geqq 0$) and $\epsilon \in \mathbb{R}^p$, such that $\hat{x}$ is an optimal solution of (4).*

3 Main Results

Assume that $\hat{x}$ be a feasible solution of the MQCQP (1). Consider the scalarization problem (4) and set $\epsilon_k = f_k(\hat{x})$ for $k = 1, ..., p$. In this case, If $\hat{x}$ be an optimal solution of (4), then by Theorem 2.1, $\hat{x}$ is a (weakly) efficient solution of the problem (1). Therefore, to solve problem (1), at first, the scalarization problem (4) corresponding to the problem (1) is written as follows:

$$\begin{aligned} \min \quad & \sum_{k=1}^{p} w_k f_k(x) \\ s.t. \quad & f_k(\hat{x}) - f_k(x) \geqslant 0, \quad k = 1, ..., p, \\ & f_k(x) \geqslant 0, \qquad\quad k = p+1, ..., m, \\ & x \in [a, b], \end{aligned} \tag{5}$$

An approach to find approximate solutions of the SQCQP (5) is to solve a linear relaxation of this problem. Here, we use a linear relaxation of problem (5), which is based on the convex and concave envelops of the bilinear terms in the quadratic functions $f_k(x)$. We denote this

linear relaxation by $LP(a,b,\hat{x})$.

$$
\begin{array}{lll}
\min & \sum_{k=1}^{p} w_k f_k(x) & \\
s.t. & t_j^k \geqslant a_j H_j^k x + m_j^k x_j - a_j m_j^k, & j=1,...,n, \quad k=1,...,p, \\
& t_j^k \geqslant b_j H_j^k x + M_j^k x_j - b_j M_j^k, & j=1,...,n, \quad k=1,...,p, \\
& t_j^k \leqslant a_j H_j^k x + M_j^k x_j - a_j M_j^k, & j=1,...,n, \quad k=p+1,...,m, \\
& t_j^k \leqslant b_j H_j^k x + m_j^k x_j - b_j m_j^k, & j=1,...,n, \quad k=p+1,...,m, \\
& \sum_{j=1}^{n} t_j^k + c_k^t x + d_k \leqslant f_k(\hat{x}), & k=1,...,p, \\
& \sum_{j=1}^{n} t_j^k + c_k^t x + d_k \geqslant 0, & k=p+1,...,m, \\
& x \in [a,b], &
\end{array}
\tag{6}
$$

where t_j^k is the corresponding variable to the convex (concave) envelop of the bilinear function $x_j y_j^k$ such that $y_j^k = H_j^k x$ and H_j^k is the j-th row of the matrix H^k. Also m_j^k and M_j^k are the minimum and maximum of the linear function $H_j^k x$ on the interval $[a,b]$, respectively. Therefore,

$$
m_j^k = \min\{H_j^k x : x \in [a,b]\} = \sum_{q=1}^{n} \min\{H_{jq}^k a_q, H_{jq}^k b_q\},
$$

$$
M_j^k = \max\{H_j^k x : x \in [a,b]\} = \sum_{q=1}^{n} \max\{H_{jq}^k a_q, H_{jq}^k b_q\},
$$

wherein, H_{jq}^k is element (j,q) of the matrix H^k ([1]).
The next theorem shows that an optimal objective value of the linear programming problem (6) is a lower bound to an optimal objective value of the quadratic programming problem (5). At first, we prove the following lemma.

Lemma 3.1. *Assume that $\bar{x}$ be a feasible solution to the quadratic problem (5). Then there exists a vector $\bar{t} = (\bar{t}_1^1, ..., \bar{t}_n^m)$ such that $(\bar{x}, \bar{t})$ is a feasible solution of the linear problem (6).*

Theorem 3.2. *Assume $(x^\star, t^{\star 1}_1, ..., t^{\star m}_n)$ be the optimal solution of the linear problem (6) and $\bar{x}$ be the optimal solution of the quadratic problem (5). Then*

(i) $\sum_{k=1}^{p} w_k f_k(x^\star) \leqslant \sum_{k=1}^{p} w_k f_k(\bar{x})$;

(ii) If $x^\star$ is a feasible solution for the quadratic problem (5), then
$\sum_{k=1}^{p} w_k f_k(x^\star) = \sum_{k=1}^{p} w_k f_k(\bar{x})$.

4 Proposed Algorithm

In the following, we propose an algorithm to solve problem (1) when $[a,b] \subseteq \mathbb{R}^2_{\geqq}$. At first, we divide the box $[a,b]$ into smaller sub boxes. Then, for each sub box, we solve the linear problem (6) to find a set of approximate (weakly) efficient solutions of the quadratic problem (1). By repeating this procedure and removing the non efficient solutions of this set at each iteration of the algorithm, we will have a better approximation of the efficient solutions set of problem (1).

Algorithm 4.1.

- **Input** $f = (f_1 = f_1^+ - f_1^-, ..., f_p = f_p^+ - f_p^-), w = (w_1, ..., w_p) \in \mathbb{R}^p_{\geqq}, a = (a_1, a_2) \in \mathbb{R}^2_{\geqq}, b = (b_1, b_2) \in \mathbb{R}^2_{\geqq}$, positive integer m and positive real number Δ.

- **Initialization** $t := 1, [a^t, b^t] := [a, b], \mathcal{X}_E^{t-1} := \emptyset$.

- **Step 1** Divide rectangular $[a^t, b^t]$ into $(tm)^2$ subrectangular $[\ell_{ij}^t, u_{ij}^t]$ such that

$$\ell_{ij}^t = (a_1^t + (j-1)s_1^t,\ a_2^t + (i-1)s_2^t), \quad \text{and} \quad u_{ij}^t = (a_1^t + js_1^t,\ a_2^t + is_2^t),$$

  for $i, j = 1, ..., tm$ and $s_r^t := \frac{b_r^t - a_r^t}{tm}$ for $r = 1, 2$.

- **Step 2** For each subrectangular $[\ell_{ij}^t, u_{ij}^t]$ for $i, j = 1, ..., tm$, solve the linear problem (6), where $[a, b] = [a^t, b^t]$ and $\hat{x}$ is an arbitrary point in $[\ell_{ij}^t, u_{ij}^t]$. Set $\mathcal{A} := \mathcal{A} \cup \{\bar{x}\}$, where $(\bar{x}, \bar{t}_1^1, ..., \bar{t}_n^m)$ is the optimal solution of (6).

- **Step 3** Construct the set $\mathcal{X}_E^t$ which is obtained by removing the non efficient points of problem (1) from $\mathcal{X}_E^{t-1} \cup \mathcal{A}$. Set, $\mathcal{X}_E := \mathcal{X}_E^t$.

- **Step 4** If $\frac{\|b^t - a^t\|}{tm} > \Delta$
  Set $t := t + 1, a^t := (a_1^t, a_2^t)$ and $b^t := (b_1^t, b_2^t)$ where $a_i^t = \min\limits_{\bar{x} \in \mathcal{X}_E^t} \bar{x}_i$ and $b_i^t = \max\limits_{\bar{x} \in \mathcal{X}_E^t} \bar{x}_i$, for $i = 1, 2$ then goto Step 1, else stop.
  end if.

- **Output** The sets $\mathcal{X}_E$ and $\mathcal{Y}_E := f(\mathcal{X}_E)$ as a discrete approximations of efficient set and efficient frointer set of problem (1), respectively.

In the sequel we prove that Algorithm 4.1 is convergent.

Theorem 4.2. *For each $\Delta > 0$, Algorithm 4.1 terminates after a finite number of iterations.*

The following example show the performance of the algorithm.

Example 4.3. Consider the following biobjective quadratic programming problem ([3]):

$$\begin{array}{ll} \min & (f_1(x), f_2(x)) \\ s.t. & -2x_1 - x_2 + 3 \leqslant 0, \\ & -x_1 - 2x_2 + 3 \leqslant 0, \\ & -2x_1 + 3x_2 - 3 \leqslant 0, \\ & x \in [(0.5, 0.5), (3, 3)], \end{array}$$

where $f_1(x) = 0.5(5x_1^2 + x_2^2)$ and $f_2(x) = 0.5(x_1^2 + 5x_2^2)$. By [3], the efficient set is two line segments between the points of $\{(\frac{3}{4}, \frac{3}{2}), (1, 1)\}$ and $\{(1, 1), (\frac{5}{3}, \frac{2}{3})\}$. Figure 1 shows the output of Algorithm 4.1 with $a = (0.5, 0.5), b = (3, 3), m = 7$ and $\Delta = 0.07$, in feasible space X and objective space Y.

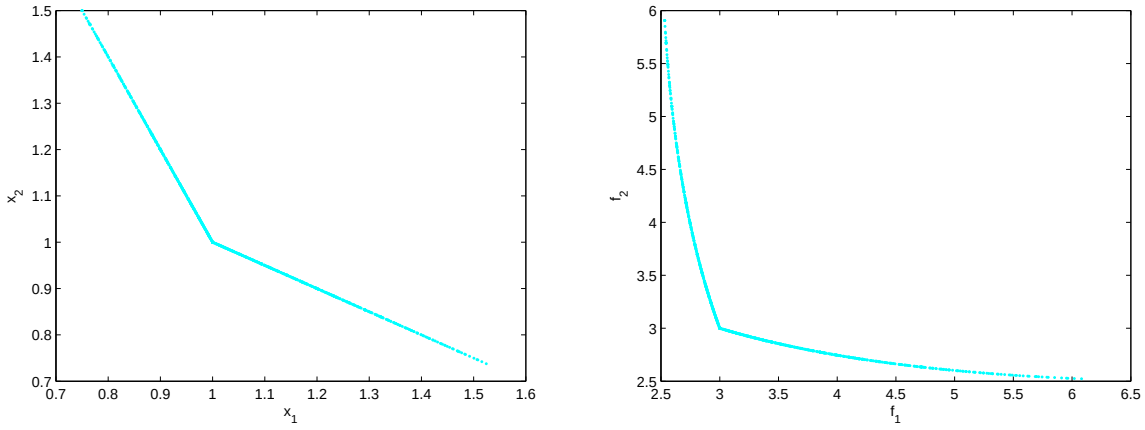

Figure 1: The sets $\mathcal{X}_E$ and $\mathcal{Y}_E$ for example 4.3.

5 Conclusion

While most of existing method for solving multiobjective quadratic problems consider convex objective functions and linear constraints, by the new version of the weighted sum method we solve this problem when the objective functions are either convex or nonconvex and constraints are in both linear and quadratic form. In fact, we convert problem (1) to an SQCQP by the improved weighted sum scalarization. A linear relaxation of SQCQP is extracted which calculate a lower bound for the optimal objective value of SQCQP on a given box. An algorithm is proposed to solve multiobjective quadratic problem with quadratic constraints when $[a, b] \subseteq \mathbb{R}^2_{\overset{\leq}{=}}$. It terminates after a finite number of iterations.

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Partitioning the stocks of a portfolio by k-medoids clustering approach

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Abstract

Machine learning is mainly used in practice because of the existence of large set of data. The target of this article is to study partitioning a large set of stocks inside a portfolio by the simple yet efficient k-medoids procedure. An algorithm is developed for this purpose. The unsupervised model is capable to receive financial returns and to illustrate the most and least risky clusters of stocks to manage the risk.

Keywords: Clustering; efficient portfolio; risk measure; financial returns; k-medoids
Mathematics Subject Classification [2010]: 91C20; 91G70

1 Introduction

Recently, there has been a proliferation of machine learning (ML) techniques and growing interest in their applications in finance, where they have been applied to sentiment analysis of news, trend analysis, portfolio optimization, and risk modeling for supporting investment management, see [4] for a general literature review and reading about several applications.

ML - as a sub-field of artificial intelligence (AI) - uses statistical methods to train machines from a given data set. After 'learning', the systems produce optimized models that express the data in the best way and restrict the potential biases, and further enabling better assessments and making appropriate decisions. Thus, such models are also broadly employed for predictions. In fact, clustering analysis and classification as data mining techniques can be employed for prediction of future outcome, [1, Chapter 22].

This paper explores the potential of ML to enhance the investment process. In this paper, an algorithm based on ML is discussed for grouping large portfolio of risky stocks specially when the numbers of stocks is high. Precisely, an application of ML in financial mathematics is discussed, see the book [1, Chapter 16] for some background.

We provide an ML-based clustering approach to narrow down the list of risky stocks in a portfolio. The advantages of this study comprising:

- Considering many stocks in a portfolio, then the taxonomy analysis gives us several different groups of stocks which have the most similarities and dissimilarity in the preceding trading day.

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- A risk measures is employed on each cluster of financial data to obtain the level entailed risk.
- The groups of data having the most and least risky performance are clarified and paid attention in upcoming trades.

The rest of this work is unfolded as follows. Section 2 is devoted to recall the definitions of a risk measure in finance that will be used later in work. Then, in Section 3, some discussions about the k-medoids are reminded and then a new algorithm for partitioning financial returns is brought forward as the novelty of this work. Next, the results are furnished in Section 4 along some notes. A conclusion is drawn in Section 5.

2 A risk measure

It is known that the volatility or the standard deviation (S.D.) of returns were adopted as a measure for risk in the seminal work by Markowitz in [6]. Having the pre-determined confidence level α, the risk measure of value-at-risk (VaR) can be defined as comes next:

$$\mathrm{VaR}_\alpha(X) := \inf\{z \in \mathbb{R} | F_X(z) \geq \alpha\}, \tag{1}$$

wherein X is a random variate, and $F_X(\cdot)$ is the cumulative distribution function (CDF). To discuss about (1), we recall that there is occasionally a chance of an adverse market movement that can lead a high loss. Note that VaR does not satisfy the sub-additivity property, and hence it is not coherent.

3 k-medoids procedure with VaR

At the presence of a large set of financial data, it will be observed that the regression-type methods which are classical statistical tools cannot anymore be employed to tackle problems. In fact, reasonable CPU times and prediction ability too make ML a promising approach than the traditional regression-like methods, [3, Chapter 1].

One of the fundamental clustering procedures is the k-medoids problem which is somewhat similar to k-means approach. These approaches break the large data sets into several groups (considered to be partitioned) and try to minimize the distance between nodes labeled to be in a cluster and a point designated as the center of the specified group, [5].

The algorithm of k-medoids works by choosing the true actual data nodes as centers (which we call sometimes as exemplars or medoids), unlike the standard k-means algorithm at which the group's center is not necessarily one of the input data nodes (it is the average between the groups' nodes.) This permits for greater interpretability of the cluster centers than in k-means. Furthermore, k-medoids could be employed with arbitrary dissimilarity measures, whereas k-means in general needs the Euclidean distance for efficient solutions.

Noting that due to the fact that k-medoids does the minimization on a sum of pairwise dissimilarities in lieu of a sum of squared Euclidean distances, this method is better for noisy environment and outliers than k-means.

3.1 An algorithm

To increase the reliability of the clustering algorithm, only one price for each stock (ticker) is not considered and in fact all the adjusted open-high-low-closed (OHLC) prices are used. The values are different to raw prices. This leads to the point that list of data containing 4D arrays.

In addition, since the construction of an efficient portfolio is mainly based on financial returns, here they are used and computed as follows:

$$\text{Returns}(\text{OHLC}_i) = \frac{\log\ \text{p}_{\text{OHLC}}(2021-12-20)}{\log\ \text{p}_{\text{OHLC}}(2021-12-21)}, \tag{2}$$

for $i = 1, 2, \ldots, n$, where n is the total number of stocks and log stands for the natural logarithm. In this work, to check the generalizability of the clustering approach, we assume to have a portfolio of S&P500, which includes the 500 large companies (with 505 stocks) listed on stock exchanges in the United States. To show how the first 10 stocks of such a portfolio of companies acts along time, we compare their price trends in Figure 1.

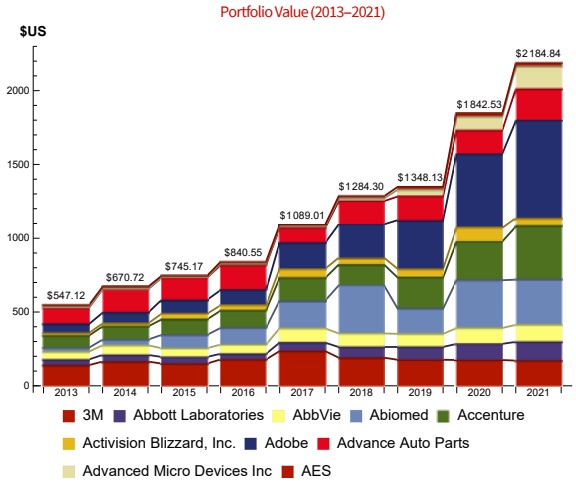

Figure 1: Price trends for the first 10 stocks of S&P500 over time.

The proposed algorithm of this work in order to tackle large portfolio by putting the stocks having the highest and lowest risk in similar groups are given in what follows.

Algorithm 3.1. Clustering large portfolio of stocks based on the k-medoids method and VaR.

1. Consider a large portfolio having stocks. And extract the financial returns corresponding to the stocks based upon two successive trading day. Here we used (2).

2. The missing data must be identified and the corresponding ticker from the list of stocks must be excluded. Here, only the data for one stock (position 303 out 505) is missing based on the trading days. Noting that the missing data can be tackled in different ways, see [3, Chapter 1].

3. Find the clusters on the large set of financial returns by k-medoids method. The number of partitions k must be given along with the distance function. Here, we use the squared Euclidean distance whenever required.

4. Find the size of each cluster. Besides, since most statistical features are required for decision making on each cluster, now flatten the data and compute their, mean, standard deviation, skewness, and kurtosis. This answers that does each cluster follow the normal distribution? Or does it have skewness with fat tails?

5. Compute the VaR measure for each cluster using (1). The strategy is to employ such values of VaR in forecasting the risk and trading in the next working day. The highest the

VaR value, the riskiest the group of stocks is. A negative value for VaR shows that the portfolio leads to profit with a high probability.

6. Make the decision via the obtained riskiest stocks. If the number of riskiest stocks is still high, one may repeat the procedure with higher values of k.

4 Computational results

In this section, the results of employing Algorithm 3.1 are furnished. The computational simulations are given for one-day 99% VaR in Table 1 and Figure 2 when $k = 6$. Noting that we set the seed of the random number generator to 1234 whenever needed. This means that due to randomness in employing unsupervised algorithms of clustering such as k-means and k-medoids, the results might change a bit each time, though the procedure of getting the most risky clusters would be similar. In implementations, we have used the command `FindCluster[]` in Mathematica 12 to call k-medoids algorithm properly.

Based on Figure 2-right, the second and the fifth clusters contain the riskiest stocks. The statistical properties for each cluster in this case based on our proposed algorithm are furnished in Table 1. Results reveal that $73 + 21$ stocks are in risky groups and none of the cluster follow normal distribution.

Table 1: The information of clusters using $k = 6$ and Algorithm 3.1.

	Cluster 1	Cluster 2	Cluster 3	Cluster 4	Cluster 5	Cluster 6
Size	97	73	228	84	21	1
Norm	0.174	0.554	0.402	0.471	0.601	0.216
Mean	-0.001	0.031	0.011	0.021	0.063	-0.096
S.D.	0.011	0.011	0.009	0.016	0.021	0.057
Skewness	-1.201	0.412	-0.841	-0.130	0.621	0.136
Kurtosis	5.275	3.329	4.967	4.009	4.831	1.335

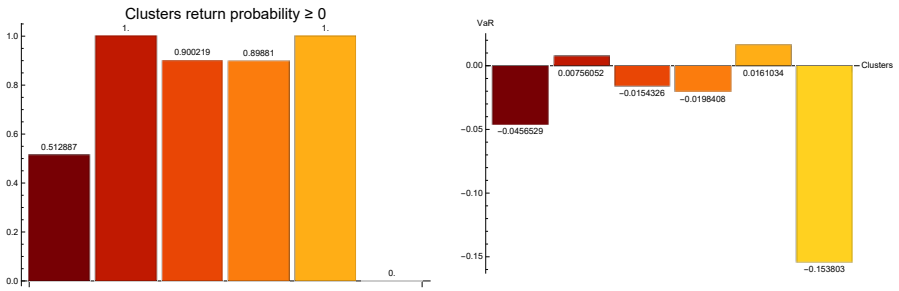

Figure 2: Probability of cluster returns being positive in left and the VaR measure values to assess the risk of each cluster in right.

4.1 Higher k

When the number of riskiest stocks is still high for the trader to make appropriate decisions, we may use a higher k to cluster the financial returns as much as possible. Following this, we may choose $k = 50$ as an instance. Selecting this, first yields to more number of clusters. However, for our real data from S&P500, only 16 clusters will be given. Results are given in Table 2. This reveals that Algorithm 3.1 does not necessarily put the items into useless groups and k is just an upper bound for the number of clusters.

Table 2: The information of clusters using $k = 50$ and Algorithm 3.1.

Cluster	1	2	3	4	5	6	7	8
Size	34	40	41	46	32	78	66	30
Norm	0.049	0.099	0.421	0.150	0.225	0.332	0.170	0.177
Mean	0.003	-0.	0.030	0.008	0.019	0.016	0.01	0.016
S.D.	0.005	0.01	0.015	0.009	0.006	0.010	0.006	0.006
Skewness	-0.086	-0.422	0.057	-0.197	-0.033	-0.575	-0.446	0.239
Kurtosis	2.892	3.797	2.788	2.482	2.883	2.580	3.904	3.532
Cluster	9	10	11	12	13	14	15	16
Size	33	13	32	24	11	16	7	1
Norm	0.538	0.209	0.299	0.153	0.142	0.193	0.445	0.216
Mean	0.046	0.018	0.026	0.002	-0.019	0.019	0.083	-0.096
S.D.	0.015	0.024	0.009	0.017	0.015	0.016	0.018	0.057
Skewness	0.285	-0.160	-0.301	-0.307	-0.241	-0.045	1.639	0.137
Kurtosis	2.838	2.965	2.725	2.058	2.432	1.960	6.053	1.336

Results in this case show that the most risky clusters are 5,8,9,11 and 15. Figure 3 is provided to show the associated VaR values. The highest VaR belongs to cluster 15 consisting of 7 stocks only. Hence, the riskiest stocks based on this higher VaR value can be then obtained as follows:

```
"NYSE:CCL", "NASDAQ:CTXS", "NYSE:DVN",
"NASDAQ:EXPE", "NYSE:LYV", "NASDAQ:MU", "NYSE:NCLH"
```

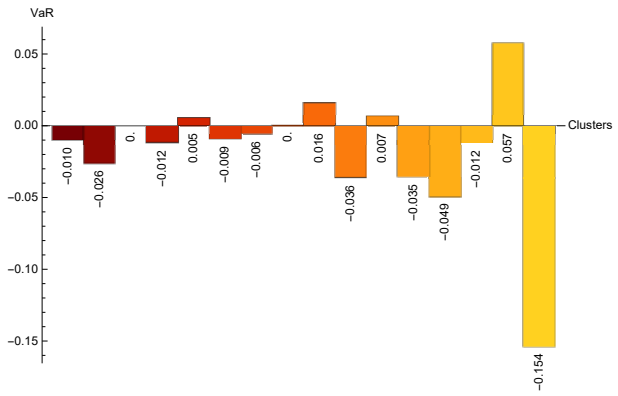

Figure 3: The VaR risk measure when $k = 50$ based on Algorithm 3.1.

4.2 Comparisons

Here a query may arise that applying a clustering technique for partitioning portfolio without comparison to the existing techniques does not bring novelty to the work. To respond this, it is pointed out that authors in the work [2] employed k-medoids or similar algorithms for partitioning while in this work we construct a k-medoids-VaR approach which works based on a risk measures. In addition, here k is the maximum number of clusters and a comparison along the existing techniques is done in Table 3 for the case $k = 50$. The time is reported on seconds. The results show that due to the type of our financial data and some other restrictions, some of the clustering techniques cannot even be used. k-means also relies on the choice of the initial seeds too much and thus the only competitor for k-medoids-VaR could be the "Optimize" method. This method also yields in many groups having positive VaR values which lead to difficulty in choosing the most risky stocks.

Table 3: Comparison of the existing clustering techniques for the experiment in Subsection 4.1.

Algorithm	Description	CPU time	Number of clusters
Agglomerate	Find clustering hierarchically	0.40	50
Optimize	Find clustering by local optimization	0.03	50
DBSCAN	Density-based spatial clustering of applications with noise	Cannot be used	-
Gaussian Mixture	Variational Gaussian mixture algorithm	Cannot be used	-
Jarvis Patrick	Jarvis-Patrick clustering algorithm	Cannot be used	-
KMeans	k-means clustering algorithm	0.12	16
KMedoids	Partitioning around medoids	0.15	16
Mean Shift	Mean-shift clustering algorithm	Cannot be used	-
Neighborhood Contraction	Displace examples toward high-density region	Cannot be used	-
Spanning Tree	Minimum spanning tree-based clustering algorithm	Cannot be used	-
Spectral	Spectral clustering algorithm	0.94	1

5 Conclusion

We have proposed an algorithm based on unsupervised k-medoids clustering approach in the category of ML models for unlabeled classifications of financial returns. The algorithm works on multi-dimensional data of any sizes and only gets an upper bound for k to work. The proposed procedure is able to pin down the riskiest stocks existing in the most risky clusters using a risk measure. Note that, the procedure does not say anything about asset allocation. If a trader wants to do such a thing, some nonlinear optimizations based on the well-known Markowitz portfolio construction can be done to find the allocations of assets.

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Perron-Frobenius theory and max-plus algebra on tensors

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Abstract

In this paper we generalize the max plus algebra system of real matrices to the class of real tensors and derive its fundamental properties. Also we define the weakly finite tensor and the new version of the Perron Frobenius theory for some class of weakly finite tensors is given.

Keywords: weakly finite tensor, Max plus algebra, Eigenvalue, Perron Frobenius Theorem.
Mathematics Subject Classification [2010]: Primary: 15A18; Secondary: 15A69, 74B99.

In max plus algebra we work with the max plus semi-ring which is the set $\Re_{\max} = \Re \cup \{-\infty\}$ together with operations $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$. The additive and multiplicative identities are taken to be $\varepsilon = -\infty$ and $e = 0$ respectively . Max plus algebra is one of many idempotent semirings which have been considered in various fields of mathematics. Max plus algebra is becoming more popular not only because its operations are associative, commutative and distributive as in conventional algebra but because it takes systems that are non-linear in conventional algebra and makes them linear. It has found applications in many areas such as combinatorics, optimization, mathematical physics and algebraic geometry. It is also used in control theory, machine scheduling, discrete event processes, manufacturing systems, telecommunication networks, parallel processing systems and traffic control.

Tensors are increasingly ubiquitous in various areas of applied, computational, and industrial mathematics and have wide applications in data analysis and mining, information science, signal/image processing, computational biology, and so on. A tensor can be regarded as a higher order generalization of a matrix, which takes the form

$$\mathbb{A} = (a_{i_1,\ldots,i_m}), \ a_{i_1,\ldots,i_m} \in \Re, \ 1 \leq i_1, \ldots, i_m \leq n,$$

where $\Re$ is a real field. Such a multi-array $\mathbb{A}$ is said to be an mth order n-dimensional square real tensor with n^m entries $a_{i_1,\ldots,i_m}$. In this regard, a vector is a first order tensor and a matrix is a second order tensor. Nonnegative tensors have attracted more and more attention because they share some intrinsic properties with those of the nonnegative matrices. One of those properties is the Perron Frobenius theorem on eigenvalues. In [2], Chang et al. generalized the Perron Frobenius theorem for nonnegative matrices to irreducible nonnegative tensors. In [3], Friedland et al. generalized the Perron Frobenius theorem to weakly irreducible nonnegative tensors. Further generalization of the Perron Frobenius theorem to nonnegative tensors can be found in [4].

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Now, the question arises is it possible to define max plus algebra system for nonnegative tensors as a generalization of max algebra on nonnegative matrices? in this paper we show the answer is affirmative.

We first add a comment on the notation that is used. Vectors are written as $(x, y, \ldots)$, matrices correspond to $(A, B, \ldots)$ and tensors are written as $(\mathbb{A}, \mathbb{B}, \ldots)$. The entry with row index i and column index j in a matrix A, i.e. $(A)_{ij}$ is symbolized by a_{ij} (also $(\mathbb{A})_{i_1 i_2 \cdots i_m} = a_{i_1 i_2 \cdots i_m}$).

$\Re$ and $\mathcal{C}$ represents the real and complex field, respectively. For each nonnegative integer n, denote $[n] = \{1, 2, \ldots, n\}$. $\Re^n_+$ $\left(\Re^n_{++}\right)$ denotes the cone $\{x \in \Re^n : x_i \geq (>)\ 0,\ i = 1, \ldots, n\}$.

1 Preliminaries

1.1 Max plus algebra system

In this subsection we give the basic definition of the max plus algebra. For the proofs and more information about max plus algebra the reader is referred to [1, 6]. If $a, b \in \Re_{max}$, then we set $a \oplus b = \max(a, b)$, and $a \otimes b = a + b$. For example,

$$(-1) \oplus 2 = \max(-1, 2) = 2 = \max(2, -1) = 2 \oplus (-1),$$
$$7 \otimes 3 = 7 + 3 = 10 = 3 + 7 = 3 \otimes 7.$$

By max plus algebra we understand the analogue of linear algebra developed for the pair of operations $(\oplus, \otimes)$, extended to matrices and vectors as in conventional linear algebra. That is, for vectors $x = (x_i)$, $y = (y_i)$ in $\Re^n_{max}$ and $c \in \bar{\Re}_{\max}$ the vectors $x \oplus y = (\max\{x_i, y_i\})$ and $c \otimes x = (c \otimes x_i)$ are defined entrywise. The sum $A \oplus B$ of two matrices is defined analogously. If A, B, C are matrices of compatible sizes with entries from $\Re_{max}$, we write $C = A \oplus B$ if $c_{ij} = max(a_{ij}, b_{ij})$ for all $i, j \in [n]$.

If $A = (a_{ik}) \in M_n(\bar{\Re}_{\max})$, then the map $x \in \Re_{max} \Rightarrow A \otimes x \in \Re_{max}$, where $(A \otimes x)_i = \max_k (a_{ik} + x_k)$, $i \in [n]$, is linear in the sense given above, namely for all $x, y \in \Re^n_{max}$, $c \in \bar{\Re}_{\max}$

$$A \otimes (x \oplus y) = (A \otimes x) \oplus (A \otimes y), A \otimes (c \otimes x) = c \otimes (A \otimes x).$$

Also we write $C = (c_{il}) = A \otimes B$, if $c_{il} = \max_k (a_{ik} + b_{kl})$, for all $i, l \in [n]$ and $c \otimes A = A \otimes c = (c \otimes a_{ij})$ for $c \in \bar{\Re}_{\max}$.

1.2 Basic definition of tensor

In this subsection, we will cover some fundamental notions and properties on tensors. We denote the set of all mth order n-dimensional tensors by $\Re^{[m,n]}_{\max}$ such that all entries belong to $\bar{\Re}_{\max}$. For a vector $x = (x_1, \ldots, x_n)^T \in \Re^n$, let $\mathbb{A}x^{m-1}$ be a vector in $\Re^n$ whose ith component is defined as the following [5]:

$$\left(\mathbb{A}x^{m-1}\right)_i = \sum_{i_2, \ldots, i_m = 1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}, \tag{1}$$

and let $x^{[m]} = (x_1^m, \ldots, x_n^m)^T$.

Definition 1.1. Let $\mathbb{A}$ (and $\mathbb{B}$) be an order $m \geq 2$ (and order $k \geq 1$), dimension n tensor, respectively. The product $\mathbb{A}\mathbb{B}$ is defined to be the following tensor $\mathbb{C}$ of order $(m-1)(k-1)+1$ and dimension n:

$$c_{i\alpha_1 \cdots \alpha_{m-1}} = \sum_{i_2, \ldots, i_m = 1}^{n} a_{i i_2 \cdots i_m} b_{i_2 \alpha_1} \cdots b_{i_m \alpha_{m-1}},$$

where $(i \in [n], \alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1})$.

It is easy to check from the definition that $I_n\mathbb{A} = \mathbb{A} = \mathbb{A}I_n$, where I_n is the identity matrix of order n. When $k = 1$ and $\mathbb{B} = x \in \mathcal{C}^n$ is a vector of dimension n, then $(m-1)(k-1)+1 = 1$. Thus $\mathbb{A}\mathbb{B} = \mathbb{A}x$ is still a vector of dimension n, and we have

$$(\mathbb{A}x)_i = (\mathbb{A}\mathbb{B})_i = c_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m} = \left(\mathbb{A}x^{m-1}\right)_i,$$

Thus we have $\mathbb{A}x^{m-1} = \mathbb{A}x$. So the first application of the tensor product defined above is that now $\mathbb{A}x^{m-1}$ can be simply written as $\mathbb{A}x$.

Definition 1.2. A tensor $\mathbb{A} \in \Re^{[m,n]}$ is called reducible, if there exists a nonempty proper index subset $I \subset \{1,\ldots,n\}$ such that

$$a_{i_1,\ldots,i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2,\ldots,i_m \notin I,$$

If $\mathbb{A}$ is not reducible, then we call $\mathbb{A}$ irreducible.

2 Main results

In this section we define the max plus algebra system on tensors and our interest will be in describing the analogue of the Perron Frobenius theory for this new system, referred to as the max version of the theory.

Definition 2.1. The max plus algebraic addition ($\oplus$) and multiplication ($\otimes$) are defined as follows:
(i). Suppose that $\mathbb{A}, \mathbb{B}$ are mth order n dimensional tensors with entries from $\Re_{\max}$ then we have $\mathbb{A} \oplus \mathbb{B}$ is mth order n dimensional tensor and

$$(\mathbb{A} \oplus \mathbb{B})_{i_1\cdots i_m} = a_{i_1\cdots i_m} \oplus b_{i_1\cdots i_m} = \max\left(a_{i_1\cdots i_m}, b_{i_1\cdots i_m}\right). \tag{2}$$

(ii). Suppose that $\mathbb{A} \in \Re_{max}^{[m,n]}$ and $\mathbb{B} \in \Re_{max}^{[k,n]}$ where $m \geq 2, k \geq 1$ then we have $\mathbb{A} \otimes \mathbb{B} \in \Re_{max}^{[(m-1)(k-1)+1,n]}$ and

$$\begin{aligned}(\mathbb{A} \otimes \mathbb{B})_{i\alpha_1\cdots\alpha_{m-1}} &= \bigoplus_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} \otimes b_{i_2\alpha_1} \otimes \cdots \otimes b_{i_m\otimes\alpha_{m-1}} \\ &= \max_{1\leq i_2,\ldots,i_m\leq n} \left\{a_{ii_2\cdots i_m} + b_{i_2\alpha_1} + \cdots + b_{i_m\alpha_{m-1}}\right\},\end{aligned} \tag{3}$$

where $i \in \{1,\ldots,n\}$, $\alpha_1,\ldots,\alpha_{m-1} \in [n]^{k-1}$. In particular for $x \in \Re_{max}^n$ we have

$$(\mathbb{A} \otimes x)_i = \max_{1\leq i_2\cdots i_m\leq n} \left\{a_{ii_2\cdots i_m} + x_{i_2} + \cdots + x_{i_m}\right\}.$$

Definition 2.2. Let $\mathbb{A} \in \Re_{max}^{[m,n]}$. Then we define $\lambda \in \Re_{max}$ to be the eigenvalue of $\mathbb{A}$ with eigenvector $x \in \Re_{max}^n$, where at least one entry is not $-\infty$, provided λ and x satisfy the max plus equation

$$\mathbb{A} \otimes x = \lambda \otimes x^{[m-1]} = \lambda + (m-1)\, x. \tag{4}$$

We refer to (λ, x) as an eigenpair for $\mathbb{A}$. When we find a particular eigenvector, any max-plus scalar multiple of it is also an eigenvector. When we refer to a unique eigenvector, we include the scalar max-plus multiples in the uniqueness.

Definition 2.3. An eigenvector is called a finite eigenvector if it has all finite entries.

The definition above also allows the eigenvalue to be equal to ε and the eigenvector to have entries equal to ε. Consider the following lemma.

Lemma 2.4. *Let $\mathbb{A} \in \Re_{max}^{[m,n]}$. Then ε is an eigenvalue of $\mathbb{A}$ if and only if $\mathbb{A}$ has a face of all ε entries.*

Theorem 2.5. *Let $\mathbb{A} \in \Re_{max}^{[m,n]}$. if $\mathbb{A}$ is irreducible then ε is not an eigenvalue of $\mathbb{A}$.*

Lemma 2.6. *Let $\mathbb{A} \in \Re_{max}^{[m,n]}$ with eigenpair (λ, x) and $\alpha \in \Re$. Then $(\lambda, \alpha \otimes x)$ is also an eigenpair of $\mathbb{A}$.*

Lemma 2.7. *Let $\mathbb{A} \in \Re_{max}^{[m,n]}$ with at least one cycle, then*
(i). *$\mathbb{A}$ has finite eigenvalue k if and only if $-k \otimes \mathbb{A}$ has eigenvalue 0.*
(ii). *$\mu(A) = m$ if and only if $\mu(-m \otimes A) = 0$.*

Definition 2.8. A tensor $\mathbb{A} \in \Re_{max}^{[m,n]}$ is essentially finite tensor iff for any $i, j \in [n]$, $a_{ij\cdots j} > -\infty$ holds. Also an essentially finite tensor is irreducible.

Definition 2.9. A tensor $\mathbb{A} \in \Re_{max}^{[m,n]}$ is called weakly finite tensor if for any $i \neq j \in [n]$, $a_{ij\cdots j} > -\infty$. An essentially finite tensor is weakly finite tensor.

Theorem 2.10. *Let $\mathbb{A} \in \Re_{max}^{[m,n]}$ be a weakly finite tensor. Then $\mu(A)$ is the unique finite eigenvalue of $\mathbb{A}$.*

Definition 2.11. We define NC to be the set of all $\mathbb{A} \in \Re_{max}^{[m,n]}$ such that for it, there exist $x \neq 0$, $x \in \Re_{+}^{n}$ and $\lambda > 0$ such that $\mathbb{A} \otimes x = \lambda \otimes x^{[m-1]}$ and $\left\{(i,j) : a_{ij\cdots j} x_j^{m-1} = \lambda x_i^{m-1},\ 1 \leq i, j \leq n\right\}$ has at least a circuit.

Lemma 2.12. *Let $\mathbb{A}$ be essentially positive tensor such that belong to NC, Then $\lambda = \mu(\mathbb{A})$.*

Theorem 2.13. *Let $\mathbb{A}$ be weakly finite tensor such that belong to NC. there exist $\mu(\mathbb{A})$ and a corresponding vector x, where at least one entry is not $-\infty$ such that*

$$\max_{1 \leq i_2, \ldots, i_m \leq n} \{a_{ii_2\cdots i_m} + x_{i_2} + \cdots + x_{i_m}\} = \mu(\mathbb{A}) + (m-1)\, x_i,\ \ i = 1, 2, \ldots, n.$$

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Tridiagonal 3-Toeplitz Matrices

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Abstract

This paper gives an answer by presenting of eigenvalues for real tridiagonal 3-Toeplitz matrices of different order. It surveys the central results of the theory by finding roots of a combination of Chebyshev polynomials of the second kind. This answer solves the eigenproblem for integer powers of such matrices.

Keywords: 3-Toeplitz matrix, Chebyshev Polynomials, Eigenvalue

Mathematics Subject Classification [2010]: 15A03, 15A18, 65F15

1 Introduction

Toeplitz matrices are frequently used in many fields of science and engineering such as solving the inverse of a matrix, systems of linear equations, problems in the field of sound propagation, the stability of difference approximations to differential equations, etc. k-Toeplitz matrices are tridiagonal matrices of the form $A = [a_{i,j}]_{i,j=1}^{n}$ (with $n \geq k$) such that $a_{i+k,k+j} = a_{i,j}$ $(i, j = 1, 2, ..., n-k)$, so that they are k-periodic along the diagonals parallel to the main diagonal.
A Toeplitz matrix is a k-Toeplitz matrix when $k = 1$.
In this paper, we focus on the case of the form

$$A_n = \begin{pmatrix} a_1 & b_1 & & & & & & \\ c_1 & a_2 & b_2 & & & & & \\ & c_2 & a_3 & b_3 & & & & \\ & & c_3 & a_1 & b_1 & & & \\ & & & c_1 & a_2 & b_2 & & \\ & & & & c_2 & a_3 & b_3 & \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix} \in \mathbb{R}^{n\times n}. \tag{1}$$

This matrix is an $n \times n$ real tridiagonal 3-Toeplitz matrices. The description of some explicit expressions for eigenvalues of a tridiagonal 3-Toeplitz matrices is the main topic of this note.

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2 Main results

The first key idea of our work is the expressions for eigenvalues of a tridiagonal 3-Toeplitz matrices by the following theorem:

Theorem 2.1. *(Marcella'n and Petronilho [4]). Let A_n, $n = 3, 4, 5, ...$, be the irreducible tridiagonal 3-Toeplitz matrix given by (1), where b_1, b_2, b_3, c_1, c_2 and c_3 are positive numbers. Define the sequence $\{S_n\}_{n\geq 0}$ of orthogonal polynomials associated with the matrices A_n as*

$$S_{3k}(x) = (b_1b_2b_3)^{-k}\{P_k(\pi_3(x)) + b_3c_3(x-a_2)P_{k-1}(\pi_3(x))\}, \tag{2}$$

$$S_{3k+1}(x) = b_1^{-1}(b_1b_2b_3)^{-k}\{(x-a_1)P_k(\pi_3(x)) + b_1c_1b_3c_3P_{k-1}(\pi_3(x))\}, \tag{3}$$

$$S_{3k+2}(x) = (b_1b_2)^{-1}(b_1b_2b_3)^{-k}(x-\xi_1)(x-\xi_2)P_k(\pi_3(x)),\ \ k = 0, 1, \ldots, \tag{4}$$

where ξ_1 and ξ_2 are the roots of the polynomial

$$(x-a_1)(x-a_2) - b_1c_1, \tag{5}$$

and

$$\pi_3(x) := \begin{vmatrix} x-a_1 & 1 & 1 \\ b_1c_1 & x-a_2 & 1 \\ b_3c_3 & b_2c_2 & x-a_3 \end{vmatrix}. \tag{6}$$

Then the eigenvalues $\lambda_{n,m}$ of A_n are the zeros of S_n, and the corresponding eigenvectors $\mathbf{v_{n,m}}$ are given by

$$\mathbf{v_{n,m}} = \begin{pmatrix} S_0(\lambda_{n,m}) \\ S_1(\lambda_{n,m}) \\ \vdots \\ S_{n-1}(\lambda_{n,m}) \end{pmatrix}, \quad m = 1, 2, \ldots, n. \tag{7}$$

Define

$$P_n(x) = (b_1b_2b_3c_1c_2c_3)^{n/2}U_n\left(\frac{x - b_1c_1 - b_2c_2 - b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right),\ n = 0, 1, 2, \ldots, \tag{8}$$

where $U_n(x)$ is the Chebyshev polynomial of degree n of the second kind with $n \in \mathbb{N} \cup \{-1, 0\}$.
All Chebyshev polynomials, among them $U_n(x)$, satisfy the three-term recurrence relations [3]:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad (U_{-1}(x) = 0,\ U_0(x) = 1,\ U_1(x) = 2x).$$

The sequence $\{S_k\}_k$ is an orthogonal polynomial sequence corresponding to a positive definite case. So, the zeros are simple and interlace [3, 5], i.e., if $\{x_{k,j}\}_{j=1}^k$ denotes the zeros of the polynomial S_k, then $x_{k,j} < x_{k-1,j} < x_{k,j+1},\ \ j = 1, 2, ..., k-1$.
When $n = 3k+2$, from Equation (4), the eigenvalues $\lambda_{3k+2,m}$ of A_{3k+2} $(m = 1, 2, ..., 3k+2)$ are $\lambda_{3k+2,1} = \xi_1,\ \lambda_{3k+2,2} = \xi_2$ in the solutions of the cubic equations

$$Q(\lambda) := \pi_3(\lambda) - \left[b_1c_1 + b_2c_2 + b_3c_3 + 2\sqrt{b_1b_2b_3c_1c_2c_3}\cos\frac{i\pi}{k+1}\right] = 0,\ i = 1, \ldots, k. \tag{9}$$

From (6)

$$\begin{aligned}\pi_3(\lambda) = (\lambda-a_1)(\lambda-a_2)(\lambda-a_2) - (b_1c_1 + b_2c_2 + b_3c_3)(\lambda-a_3) \\ +b_2c_2(a_1-a_3) + b_3c_3(a_2-a_3) + b_1c_1 + b_2c_2 + b_3c_3,\end{aligned} \tag{10}$$

and from Shengjin formulas are given in [6], we compute the roots of the cubic Equation (9). The coefficients $Q(\lambda)$ of Equation (9) are

$q_1 = 1,\ q_2 = -(a_1 + a_2 + a_3),\ q_3 = a_1a_2 + a_2a_3 + a_1a_3 - b_1c_1 - b_2c_2 - b_3c_3,$
$q_4 = a_3b_1c_1 + a_1b_2c_2 + a_2b_3c_3 - a_1a_2a_3 - 2\sqrt{b_1b_2b_3c_1c_2c_3}\cos\frac{i\pi}{k+1}.$

Let

$\Delta_1 = q_2^2 - 3q_1q_3,\ \Delta_2 = q_2q_3 - 9q_1q_4,\ \Delta_3 = q_3^2 - 3q_2q_4,\ \Delta_4 = \Delta_2^2 - 4\Delta_1\Delta_3.$

Then we have

(1) If $\Delta_1 = \Delta_2 = 0$, $Q(\lambda)$ has only one real triple root;
(2) If $\Delta > 0$, $Q(\lambda)$ has one real root and a pair of conjugate imaginary roots;
(3) If $\Delta = 0$, $Q(\lambda)$ has three real roots: one simple and the other double;
(4) If $\Delta < 0$, $Q(\lambda)$ has three different real roots.

The corresponding eigenvectors $\mathbf{v_{n,m}}$ are given by (7).

When $n = 3k + 1$, in Equation (3), the eigenvalues $\lambda_{3k+1,m}$ of A_{3k+1}
$(m = 1, 2, ..., 3k + 1)$ are the roots x of $S_{3k+1}(x)$ satisfy the equation

$$b_1^{-1}(b_1b_2b_3)^{-k}\{(x - a_1)P_k(\pi_3(x)) + b_1c_1b_3c_3P_{k-1}(\pi_3(x))\} = 0. \tag{11}$$

With following (8) in Equation (11), we have $s = \frac{\sqrt{b_1b_3c_1c_3}}{\sqrt{b_2c_2}}$.
If x is not a common root of $U_{n-1}\left(\frac{\pi_3(x)-b_1c_1-b_2c_2-b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)$ and $a_1 - x$, then we conclude

$$\frac{U_n\left(\frac{\pi_3(x)-b_1c_1-b_2c_2-b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)}{U_{n-1}\left(\frac{\pi_3(x)-b_1c_1-b_2c_2-b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)} = \frac{s}{a_1 - x}. \tag{12}$$

Note 1. Let $\eta_0 < \xi_1 < \eta_1 < \xi_2 < \ldots < \eta_{i-1} < \xi_i < \eta_i < \xi_{i+1} < \ldots < \eta_{n-1} < \xi_n < \eta_n$ with $\eta_0 = -\infty,\ \eta_n = \infty$, where $\xi_1,\ \xi_2, \ldots,\ \xi_n$ are the roots of $U_n(x)$, and $\eta_1,\ \eta_2, \ldots,\ \eta_{n-1}$ are the roots of $U_{n-1}(x)$ in Equation (12). Let $U_n(x)/U_{n-1}(x) = p_{n,n-1}(x)$, $n \geq 1$ and $p_{0,-1}(x) = 1$.
Next we denote $g(x) = s/(a_1 - x)$ that here $s = \frac{\sqrt{b_1b_3c_1c_3}}{\sqrt{b_2c_2}}$.

Theorem 2.2. *If $s > 0$, for some i in Equation (12) and Note 1, then there are precisely two additional roots, exactly one lying in each of the intervals*

$$(\eta_{i-1}, a_1) \text{ and } (a_1, \eta_i).$$

If $s < 0$, then one or two additional roots of Equation (12) can be zero, in the interval (η_{i-1}, η_i). Finally, the next elseif $s = 0$, the problem is solved easily by finding roots of $U_n(x)$.

Note that, here $s > 0$. Then by the results of Theorem 2.2, the function (12) has the same roots as

$$h(x) \equiv (a_1 - x)U_n\left(\frac{\pi_3(x) - b_1c_1 - b_2c_2 - b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right) - sU_{n-1}\left(\frac{\pi_3(x) - b_1c_1 - b_2c_2 - b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right), \tag{13}$$

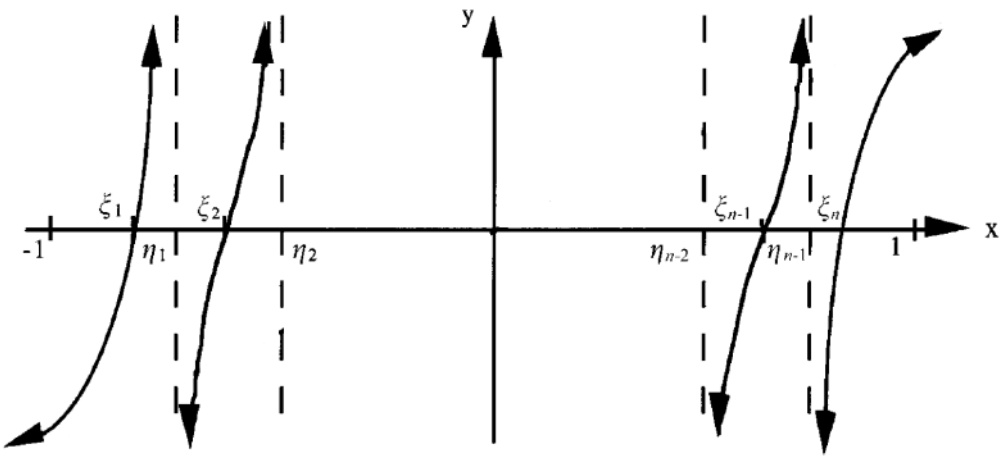

Figure 1: $p_{n,n-1}(x)$

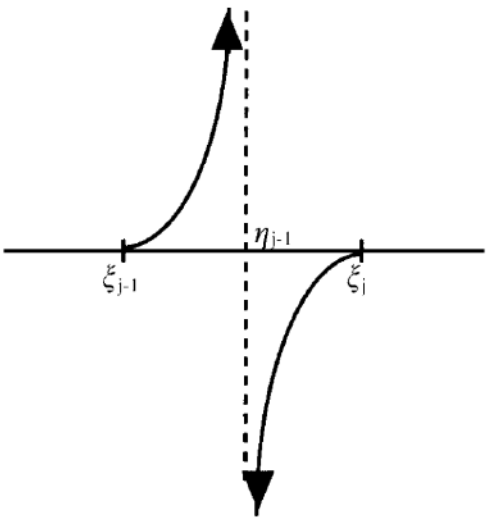

Figure 2: $p_{n,n-1}(x)$

The graph of $p_{n,n-1}(x)$ is shown in Fig. 1. Also Fig. 2 shows $p_{n,n-1}(x)$ in the interval (ξ_{j-1}, ξ_j).
For more details see [5].
Now, we have the roots of the following function by to approximate function $h(x)$ by Chebyshev interpolation for every interval (η_{i-1}, η_i), $i = 1, \ldots, n$, then use Chebyshev companion matrix to find roots.
To increase the accuracy, we can apply Chebfun for this work [3]. Chebfun finds roots with a global rootfinding capability by a method that shows in [5].

When $n = 3k$, from Equation (2), the eigenvalues $\lambda_{3k,m}$ of A_{3k} $(m = 1, 2, ..., 3k)$ are the roots x of $S_{3k}(x)$ satisfy the equation

$$(b_1b_2b_3)^{-k}\{P_k(\pi_3(x)) + b_3c_3(x - a_2)P_{k-1}(\pi_3(x))\} = 0. \tag{14}$$

If x is not a common root of $U_n\left(\frac{\pi_3(x)-b_1c_1-b_2c_2-b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)$ and $a_2 - x$, then we conclude

$$\frac{U_{n-1}\left(\frac{\pi_3(x)-b_1c_1-b_2c_2-b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)}{U_n\left(\frac{\pi_3(x)-b_1c_1-b_2c_2-b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)} = \frac{\sqrt{b_1b_2c_1c_2}}{\sqrt{b_3c_3}(a_2 - x)}. \tag{15}$$

Here, suppose $U_{n-1}(x)/U_n(x) = p_{n-1,n}(x)$, $n \geq 1$ and $g(x) = s/(a_2 - x)$ that $s = \frac{\sqrt{b_1b_2c_1c_2}}{\sqrt{b_3c_3}}$.
Where $\xi_1 < \eta_1 < \xi_2 < \ldots < \eta_{i-1} < \xi_i < \eta_i < \xi_{i+1} < \ldots < \eta_{n-1} < \xi_n$.
$\xi_1,\ \xi_2, \ldots,\ \xi_n$ are the roots of $U_n(x)$ and $\eta_1,\ \eta_2, \ldots,\ \eta_{n-1}$ are the roots of $U_{n-1}(x)$.
Now, we apply Theorem 2.2 for finding eigenvalues of matrix (1) when $n = 3k$.

We show another way to the problem concerning the study of the eigenvalues of the sequences of matrices dened by (1), based on some results in [1,2]. We will study the case when the order $n = 3k$ of the matrix A_n in (1). Then A_n is the block Toeplitz matrix

$$A_n = \begin{pmatrix} B_0 & B_1 & & & & & & \\ B_{-1} & B_0 & B_1 & & & & & \\ & B_{-1} & B_0 & B_1 & & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & & \ddots & \ddots & B_1 \\ & & & & & & B_{-1} & B_0 \end{pmatrix},$$

generated by the 3×3 matrix valued polynomial

$$f(x) := B_0 + B_1e^{ix} + B_{-1}e^{-ix}$$

with

$$B_0 = \begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & a_2 & b_2 \\ 0 & c_2 & a_3 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_3 & 0 & 0 \end{pmatrix}, \quad B_{-1} = \begin{pmatrix} 0 & 0 & c_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

However, from Theorem 2.1, we know $b_1,\ b_2,\ b_3,\ c_1, c_2$ and c_3 are positive numbers and so it is well-known that, under such conditions, A_n is similar to the block Toeplitz matrix $\hat{A}_n$ by diagonal transformations, that is generated by the 3×3 matrix valued polynomial

$$\hat{f}(x) := \hat{B}_0 + \hat{B}_1e^{ix} + \hat{B}_{-1}e^{-ix}$$

with

$$\hat{B}_0 = \begin{pmatrix} a_1 & \sqrt{b_1c_1} & 0 \\ \sqrt{b_1c_1} & a_2 & \sqrt{b_2c_2} \\ 0 & \sqrt{b_2c_2} & a_3 \end{pmatrix}, \quad \hat{B}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{b_3c_3} & 0 & 0 \end{pmatrix}, \quad \hat{B}_{-1} = \begin{pmatrix} 0 & 0 & \sqrt{b_3c_3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

There are some papers for the Evaluation of the Eigenvalues of a Banded Toeplitz Block Matrix, such as [1, 2].

3 Conclusion

In this note we have considered a novel analysis review of spectral problem involving a tridiagonal 3-Toeplitz matrix for the cases $n = 3k+2$, $n = 3k+1$ and $n = 3k$ with some details on explicitly or implicitly tools.

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Some results on G-Drazin inverse of matrices over a Banach algebra

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Abstract

In this paper we prove that some conditions of the definition of g-Drazin inverse can be obtained from other conditions and so we have new results on g-Drazin inverse of matrices by applying this definition. We also investigate the g-Drazin inverse of an antitriangular matrix and obtain some necessary and sufficient conditions for a matrix over a Banach algebra to have a g-Drazin inverse.

Keywords: Banach algebra, Drazin inverse, g-Drazin inverse, quasinilpotent
Mathematics Subject Classification [2010]: , 15A09, 32A65

1 Introduction

Throughout the paper, $\mathcal{A}$ denotes the complex Banach algebra with an identity. We use $\mathbb{C}$ and $\mathbb{C}^{n\times n}$ to denote the field of all complex numbers and the Banach algebra of all $n \times n$ complex matrices, respectively. $\mathbb{N}$ stands for the set of all natural numbers. Let $\mathcal{A}$ be a Banach algebra. An element a in $\mathcal{A}$ has g-Drazin inverse (i.e., generalized Drazin inverse) provided that there exists some $b \in \mathcal{A}$ such that $b = bab, ab = ba$ *and* $a - a^2b \in \mathcal{A}^{qnil}$. Here, $\mathcal{A}^{qnil}$ is the set of all quasinilpotents in $\mathcal{A}$. Evidently, $\mathcal{A}^{qnil} = \{x \in \mathcal{A} \mid \lim_{n\to\infty} \sqrt[n]{\| x^n \|} = 0\}$. The preceding b is unique, if it exists, and we denote it by a^d. If we replace the quasinilpotent by the nilpotent in $\mathcal{A}$, b is called the Drazin inverse of a, and denote it by a^D. The g-Drazin inverse plays an important role in matrix and operator theory. We drop the regular condition for the g-Drazin invertibility of the definition. We then thereby prove that an element a in a Banach algebra $\mathcal{A}$ has g-Drazin inverse if and only if there exist an idempotent e, a unit u and a quasinilpotent w which commute each other such that $a = eu + w$.

Let E, F be block complex matrices. The representation of the Drazin inverse of the 2×2 block complex $\begin{pmatrix} E & F \\ I & 0 \end{pmatrix}$ is attractive. It was proved that the solutions to singular systems of differential equations is determined by the Drazin invertibility of such block complex matrix. In 2005, Castro-González and Dopazo gave the representations of the Drazin inverse for a class of operator matrix $\begin{pmatrix} I & I \\ F & 0 \end{pmatrix}$ (see [2, Theorem 3.5]). In 2011, Bu et al. investigate the g-Drazin inverse of the preceding block complex matrix M under the condition $EF = FE$ (see [**?**]). Let X be a Banach space. In 2013, Xu et al. investigated the general setting of bounded linear

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operators on Banach spaces and presented an expression of $\begin{pmatrix} E & F \\ 1 & 0 \end{pmatrix} \in \mathcal{B}(X \oplus X)$ under the condition that $E, F \in \mathcal{B}(X)^D$ and $EF = FE$ (see [6, Theorem 3.8]). For a complex block matrix, the Drazin and g-Drazin inverse coincide with each other. Evidently, we also extend these known results to a wider case $EF = \lambda FE$ for a nonzero complex number λ. This thereby solves a wider kind of singular differential equations. We apply the results for certain anti-triangular matrices over a Banach algebra and provide some necessary and sufficient conditions for such matrices to be g-Drazin invertible. The g-Drazin inverse of a matrix has various applications in singular differential equations, Markov chains and iterative methods.

2 Main results

The aim of this section is to provide a new characterization of g-Drazin inverse in a Banach algebra. We recall that an element $a \in \mathcal{A}$ has strongly Drazin inverse if it is the sum of an idempotent and a quasinilpotent that commute (see [3]).

Lemma 2.1. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) *$a \in \mathcal{A}$ has strongly g-Drazin inverse.*

(2) *$a - a^2 \in \mathcal{A}^{qnil}$.*

Proof. See [3, Lemma 2.2]. □

We come now to the demonstration for which this paper has been developed.

Theorem 2.2. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) *$a \in \mathcal{A}^d$.*

(2) *There exists some $x \in comm(a)$ such that $a - a^2x \in \mathcal{A}^{qnil}$.*

Proof. (1) $\Rightarrow$ (2) This is obvious by choosing $x = a^d$.

(2) $\Rightarrow$ (1) By hypothesis, there exists some $x \in comm(a)$ such that $a - a^2x \in \mathcal{A}^{qnil}$. Set $z = xax$. Then $z \in comm(a)$. We check that

$$\begin{array}{rcl} a - a^2z & = & a - axaxa \\ & = & (1 + ax)(a - a^2x) \\ & \in & \mathcal{A}^{qnil}, \\ z - z^2a & = & xax - xaxaxax \\ & = & x(a - a^2x)x + xax(a - a^2x)x \\ & \in & \mathcal{A}^{qnil}. \end{array}$$

$$az - (az)^2 = (a - a^2z)z \in \mathcal{A}^{qnil}.$$

Then we have idempotent $e \in comm^2(az)$ such that $az - e \in \mathcal{A}^{qnil}$. We easily check that

$$(a + 1 - az)((z + 1 - az) = 1 + (a - a^2z)(1 - z) + (z - z^2a).$$

Hence,

$$\begin{array}{rcl} a + 1 - e & = & (a + 1 - az) + (az - e) \in \mathcal{A}^{-1}, \\ a(1 - e) & = & (a - a^2z) + a(az - e) \in \mathcal{A}^{qnil}. \end{array}$$

Since $a \in comm(az)$, we have $ea = ae$. That is, $a \in \mathcal{A}$ is quasipolar. Therefore $a \in \mathcal{A}^d$, by [4, Theorem 4.2]. □

Now we apply Theorem 2.2 to matrices over a Banach algebra and deduce necessary and sufficient conditions for the existence of the g- Drazin inverse for a class of 2×2 block antitriangular matrices.

Lemma 2.3. *Let* $M = \begin{pmatrix} 1 & 1 \\ a & 0 \end{pmatrix} \in M_2(\mathcal{A})$. *Then*

(1) *For any* $n \in \mathbb{N}$, $M^n = \begin{pmatrix} U(n) & U(n-1) \\ U(n-1)a & U(n-2)a \end{pmatrix}$, *where* $U(m) = \sum_{i=0}^{[\frac{m}{2}]} \begin{pmatrix} m-i \\ i \end{pmatrix} a^i, m \geq 0; U(-1) = 0$.

(2) $U(n) - U(n-1) = U(n-2)a$ *for any* $n \in \mathbb{N}$.

Proof. See [1, Proposition 3.7]. □

Lemma 2.4. *Let* $a \in \mathcal{A}$. *Then the following are equivalent:*

(1) $a \in \mathcal{A}^d$.

(2) $\begin{pmatrix} 1 & 1 \\ a & 0 \end{pmatrix} \in M_2(\mathcal{A})^d$.

By the above Lemma we extend the main result of [5, Theorem 2.6] for the g-Drazin inverse.

Theorem 2.5. *Let* $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in M_2(\mathcal{A})$. *If* $a^2 = a \in \mathcal{A}$ *and* $ab = b$, *then the following are equivalent:*

(1) $M \in M_2(\mathcal{A})^d$.

(2) $bc \in \mathcal{A}^d$.

Corollary 2.6. *Let* $M = \begin{pmatrix} a & a \\ b & 0 \end{pmatrix} \in M_2(\mathcal{A})$. *If* $a^2 = a \in \mathcal{A}$, *then the following are equivalent:*

(1) $M \in M_2(\mathcal{A})^d$.

(2) $ab \in \mathcal{A}^d$.

The following Lemma helps us to prove our main result on the Drazin invertiblity of special block matrices.

Lemma 2.7. *Let* $b \in \mathcal{A}$ *be nilpotent of index* r. *Then* $M = \begin{pmatrix} 1 & b \\ 1 & 0 \end{pmatrix} \in M_2(\mathcal{A})^D$. *In this case,*

$$M^D = \begin{pmatrix} x_1 & x_2 b \\ x_2 & x_1 - x_2 \end{pmatrix},$$

where

$$x_1 = \sum_{i=0}^{r-1} (-1)^i \begin{pmatrix} 2i \\ i \end{pmatrix} b^i, x_2 = \sum_{i=0}^{r-1} (-1)^i \begin{pmatrix} 2i+1 \\ i \end{pmatrix} b^i.$$

Proof. See [1, Theorem 3.2]. □

The following theorem helps us to solve the wider class of singular differential equations.

Theorem 2.8.

Let $E, F \in \mathcal{B}(X)$ have Drazin inverses, and let $M = \begin{pmatrix} E & F \\ 1 & 0 \end{pmatrix} \in \mathcal{B}(X \oplus X)$. If $EF = \lambda FE$, then M has Drazin inverse and

$$
\begin{aligned}
&M^D = Q^\pi P^D + \sum_{i=0}^{\infty} (Q^D)^{i+1} P^i P^\pi, \\
&P = \begin{pmatrix} EF^\pi & FF^\pi \\ F^\pi & 0 \end{pmatrix}, Q = \begin{pmatrix} EFF^D & F^2F^D \\ FF^D & 0 \end{pmatrix}, \\
&Q^D = \begin{pmatrix} 0 & FF^D \\ F^d & -EF^d \end{pmatrix}, P^D = S^D + (S^D)^2 R; \\
&R = \begin{pmatrix} 0 & 0 \\ -FF^D & 0 \end{pmatrix}, S = \begin{pmatrix} EF^\pi & FF^\pi \\ I & 0 \end{pmatrix}; S^D = \sum_{i=0}^{\infty} H^i (G^D)^{i+1}, \\
&G = \begin{pmatrix} E^2E^DF^\pi & EE^DFF^\pi \\ EE^DF^\pi & 0 \end{pmatrix}, H = \begin{pmatrix} EE^\pi F^\pi & E^\pi FF^\pi \\ E^\pi F^\pi & 0 \end{pmatrix}, \\
&G^D = E^D F^\pi \begin{pmatrix} I & 0 \\ 0 & E^d F^\pi \end{pmatrix} (K^D)^2 \begin{pmatrix} I & E^D FF^\pi \\ I & 0 \end{pmatrix}, \\
&K^D = \begin{pmatrix} X_1 & X_2 (E^D)^2 FF^\pi \\ X_2 & X_1 - X_2 \end{pmatrix}, \\
&X_1 = \sum_{i=0}^{\infty} (-1)^i \begin{pmatrix} 2i \\ i \end{pmatrix} (E^D)^2 F^i F^\pi, \\
&X_2 = \sum_{i=0}^{\infty} (-1)^i \begin{pmatrix} 2i+1 \\ i \end{pmatrix} (E^D)^2 F^i F^\pi.
\end{aligned}
$$

3 Conclusion

In this paper we find a new characterization of G-Drazin inverse in Banach algebras. This provides some necessary and sufficient conditions for a block square matrix to have g-Drazin inverse. We also investigate the Drazin invertiblity of some special operator matrices which help to solve the wider class of the systems of linear differntial equations.

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A martingale inequality in tracial von Neumann algebras

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Abstract

In this talk, using the Lieb–Araki concavity, we obtain a noncommutative version of Freedman's inequality for martingales, which gives an upper bound for the tail probabilities of a supermartingale in the setting of von Neumann algebras.

Keywords: Noncommutative probability space; trace; Freedman inequality; martingale.
2020 *Mathematics Subject Classification*: 47A63, 47A08, 15A60.

1 Introduction

Freedman's inequality [2] asserts that if $\{X_n : X_0 = 0, n \geq 0\}$ is a martingale in the classical probability space $(\mathbb{P}, \Omega, \mathcal{F})$ with martingale difference sequence $\{D_n : D_0 = 0, n \geq 0\}$ such that $D_n = X_n - X_{n-1} \leq 1$ for all n, then, for all $c \geq 0$ and $h > 0$, it holds that

$$\text{Prob}\big(X_n \geq c \text{ and } Y_n \leq h \text{ for some } n \geq 0\big) \leq \left(\frac{h}{c+h}\right)^{c+h} e^c,$$

in which $Y_n = \sum_{k=1}^{n} \mathbb{E}_{k-1}(D_k^2)$ is predictable quadratic variation, where $\mathbb{E}_{k-1}$ denotes the conditional expectation onto the $k-1$'st σ-algebra in the underlying filter. Applying the Lieb concavity theotem, an extension of Freedman's inequality is provided by Tropp [6] to the case of matrix martingales.

In this talk, we are inspired by some ideas in the commutative case and Tropp's result to provide a Freedman-type inequality in the framework of noncommutative probability spaces.

A von Neumann algebra $\mathfrak{M}$ on a Hilbert space $\mathcal{H}$ with unit element 1 equipped with a normal faithful tracial state $\tau : \mathcal{M} \to \mathbb{C}$ is called a noncommutative probability space. We denote by $\leq$ the usual order on the self-adjoint part $\mathcal{M}^{sa}$ of $\mathcal{M}$. For each self-adjoint operator $a \in \mathcal{M}$, there exists a unique spectral measure E as a σ-additive mapping with respect to the strong operator topology from the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ into the set of all orthogonal projections such that for every Borel function $f : \sigma(a) \to \mathbb{C}$ the operator $f(a)$ is defined by $f(a) = \int f(\lambda) dE(\lambda)$, in which $\sigma(a)$ is the spectrum of a, in particular, $\mathbf{1}_B(x) = \int_B dE(\lambda) = E(B)$.

Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. Then the conditional expectation $\mathcal{E}_{\mathcal{N}}$ of $\mathcal{M}$ with respect to $\mathcal{N}$ is a normal positive contractive projection $\mathcal{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ satisfying the following properties:
(i) $\mathcal{E}_{\mathcal{N}}(axb) = a\mathcal{E}_{\mathcal{N}}(x)b$ for any $x \in \mathcal{M}$ and $a, b \in \mathcal{N}$;

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(ii) $\tau \circ \mathcal{E}_{\mathcal{N}} = \tau$.
Moreover, $\mathcal{E}_{\mathcal{N}}$ is the unique mapping satisfying (i) and (ii).

A filtration of $\mathcal{M}$ is an increasing sequence $(\mathcal{M}_j, \mathcal{E}_j)_{0\leq j\leq n}$ of von Neumann subalgebras of $\mathcal{M}$ together with the conditional expectations $\mathcal{E}_j$ of $\mathcal{M}$ with respect to $\mathcal{M}_j$ such that $\bigcup_j \mathcal{M}_j$ is w^*–dense in $\mathcal{M}$.

A sequence $(a_j)_{j\geq 0}$ in $\mathcal{M}$) is called a martingale (supermartingale, resp.) with respect to the filtration $(\mathcal{M}_j)_{0\leq j\leq n}$ if $a_j \in \mathcal{M}_j$ and $\mathcal{E}_j(a_{j+1}) = a_j$ ($\mathcal{E}_j(a_{j+1}) \leq a_j$, resp.) for every $j \geq 0$. Put $da_j = x_j - x_{j-1}$ ($j \geq 0$) with the convention that $a_{-1} = 0$. Then $da = (da_j)_{j\geq 0}$ is called the martingale difference of (a_j). The reader is referred to [5] for more information.

Given a family of projections $(p_\lambda)_{\lambda\in\Lambda}$ of $\mathcal{M}$, we denote by $\bigwedge_{\lambda\in\Lambda} p_\lambda$ the projection from $\mathcal{H}$ onto the closed subspace $\bigcap_{\lambda\in\Lambda} p_\lambda(\mathcal{H})$.

2 Main results

In the sequal, we assume that $(a_n)_{n\geq 0}$ is a self-adjoint martingale in $\mathcal{M}$ with respect to a filtration $(\mathcal{M}_n, \mathcal{E}_n)_{n\geq 0}$ with $x_0 = 0$ such that

$$d_n \leq 1 \quad \text{for all} \quad n \geq 1.$$

Put $b_0 = 0$ and $b_n = \sum_{k=1}^{n} \mathcal{E}_{k-1}(d_k^2)$, and for any positive number t, define

$$u_n^{(t)} := \exp\left\{ta_n - (e^t - 1 - t)b_n\right\}.$$

A generalization of the Lieb concavity [3], is proved by Araki [1] in the setting of von Neumann algebras as follows.

Theorem 2.1 (Lieb–Araki concavity [1]). *Let $b \in \mathcal{M}$ be a self-adjoint operator. Then the function*

$$\phi : a \mapsto \tau\left(\exp(b + \log(a))\right)$$

is concave on the strictly positive part of $\mathcal{M}$.

Remark 2.2. The Jensen's inequality [4, Theorem A] states that if α is a unital positive map on $\mathcal{M}$ and f is a real concave function on $[0,\infty)$, then for any self-adjoint element $a \in \mathcal{M}$, it holds that $\tau\left(\alpha(f(a))\right) \leq \tau\left(f(\alpha(a))\right)$, where $f(a)$ is defined by the functional calculus. Now, let $b \in \mathcal{M}$ be self-adjoint and $a \in \mathcal{M}$ be a strictly positive operator. Define the continuous function $g : (0,\infty) \to (0,\infty)$ by $g(t) = \tau\left(\exp(b + \log(t + a))\right)$. It follows from the Lieb–Araki concavity that g is concave. Let $\mathcal{E}_{\mathcal{N}}$ be any conditional expectation $\mathcal{E}_{\mathcal{N}}$ corresponding to a von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$. Applying the Jensen's inequality, one may deduce that

$$\tau\left(\exp\left\{b + \log a\right\}\right) \leq \tau\left(\exp\left\{b + \log \mathcal{E}_{\mathcal{N}}(a)\right\}\right)$$

In the following result, we show that the sequence $(u_n^{(t)})_{n\geq 0}$ is trace-decreasing.

Proposition 2.3. *If the sequence $(a_n)_{n\geq 0}$, fixed as in the begining of section 2, is a positive martingale in $\mathcal{M}$, then $\tau(u_{n+1}^{(t)}) \leq \tau(u_n^{(t)})$ for all $n \geq 0$. Moreover, $\tau(u_n^{(t)}) \leq 1$ for all $n \geq 0$.*

Freedman's original proof and Tropp's approach are based on stopped martingales. However, an applicable version of noncommutative stopped martingales is not available. We present a noncommutative version of the Freedman inequality under some mild conditions as follows.

Theorem 2.4. *Let $\alpha \geq 0$ and $\beta > 0$ be real numbers and $(u_n^{(t_0)})_{n\geq 0}$ is a supermartingale in $\mathcal{M}$, where $t_0 = \log\left(\frac{\alpha+\beta}{\beta}\right)$. Then there is a sequence $(e_n)_{n\geq 1}$ of mutually orthogonal projections such that*

$$\sup_{n\geq 1} \frac{\tau\left(\mathbf{1}_{[\alpha,\infty)}(a_n) \wedge \mathbf{1}_{[0,\beta]}(b_n)\right)}{2^{n-1}} \leq \tau\left(\sum_{n=1}^{\infty} e_n\right) \leq \left(\frac{\beta}{\alpha+\beta}\right)^{\alpha+\beta} e^{\alpha}.$$

In what follows, we give an example [5] of the assumption that $u_n^{(t)}$ is a supermartingale. We use the software MATLAB for computations, not a proof.

Example 2.5. Let us consider the von Neumann algebra $\mathcal{M} = \mathbb{M}_2(\mathbb{C})$ of all 2×2 complex matrices with the identity I_2. Let $\tau := \frac{1}{2}\mathrm{tr}$ be the normalized trace on $\mathcal{M}$. Denote by $\mathcal{N}$ the subalgebra of diagonal matrices. Then

$$\mathcal{E}_{\mathcal{N}}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

Definer the filtration $(\mathcal{M}_n, \mathcal{E}_n)_{n\geq 1}$ by

$$\mathcal{M}_0 = \mathbb{C}I_2,\ \mathcal{E}_0(x) = \tau(x)I_2,\ \mathcal{M}_1 = \mathcal{N},\ \mathcal{E}_1 = \mathcal{E}_{\mathcal{N}},\ \text{and}\ \ \mathcal{M}_n = \mathcal{M},\ \mathcal{E}_n = \mathrm{id}_{\mathcal{M}}\ (n \geq 2).$$

If we set

$$a_0 := 0,\ a_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ a_2 := \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix},\quad \text{and}\quad a_n := a_2 \quad \text{for every } n \geq 2,$$

then clearly $(a_n)_{n\geq 0}$ is a self-adjoint martingale and $a_1a_2 \neq a_2a_1$. In addition,

$$d_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \leq 1,\ d_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \leq 1,\ \text{and}\ d_n = 0 \leq 1\ \ (n \geq 3),$$

is the corresponding difference sequence. Moreover,

$$\begin{aligned} b_0 &= 0, \\ b_1 &= \mathcal{E}_0(d_1^2) = \tau\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) I_2 = I_2 \\ b_2 &= \mathcal{E}_0(d_1^2) + \mathcal{E}_1(d_2^2) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ b_n &= b_2\ (n \geq 3). \end{aligned}$$

Set $t = 2$ and $\lambda = e^2 - 3$, where e is Euler's constant, we have

$$\begin{aligned} u_0^{(2)} &= \exp\{2a_0 - \lambda b_0\} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ u_1^{(2)} &= \exp\{2a_1 - \lambda b_1\} = \begin{pmatrix} e^{5-e^2} & 0 \\ 0 & e^{2-e^2} \end{pmatrix} \simeq \begin{pmatrix} 0.0917 & 0 \\ 0 & 0.0045 \end{pmatrix} \\ u_2^{(2)} &= \exp\{2a_2 - \lambda b_2\} = \exp\begin{pmatrix} 8-2e^2 & 2i \\ -2i & 4-2e^2 \end{pmatrix} \simeq \begin{pmatrix} 0.0022 & 0.0009i \\ -0.0009i & 0.0004 \end{pmatrix} \\ u_n^{(2)} &= u_2^{(2)} \quad (n \geq 3). \end{aligned}$$

We have

$$\mathcal{E}_0(u_1^{(2)}) = 0.0962 I_2 \leq u_0^{(2)}$$
$$\text{and} \quad \mathcal{E}_1\left(u_2^{(2)}\right) \simeq \begin{pmatrix} 0.0022 & 0 \\ 0 & 0.0004 \end{pmatrix} \leq \begin{pmatrix} 0.0917 & 0 \\ 0 & 0.0045 \end{pmatrix} = u_1^{(2)}.$$

Therefore, $(u_n^{(1)})_{n\geq 0}$ is a supermartingale.

One may conclude the classical Freedman inequality as follows.

Corollary 2.6. *Let $\{X_n : X_0 = 0, n \geq 0\}$ be a commutative martingale of bounded random variables. If the martingale difference sequence $\{D_n : D_0 = 0, n \geq 0\}$ satisfies $D_n \leq 1 \quad (n \geq 0)$, then for every $c \geq 0$ and $h > 0$, it holds that*

$$\text{Prob}\,(X_n \geq c \text{ and } Y_n \leq h \text{ for some } n \geq 0) \leq \left(\frac{h}{c+h}\right)^{c+h} e^c, \tag{1}$$

in which $Y_n = \sum_{k=1}^{n} \mathbb{E}_{k-1}(D_k^2)$.

Proof. In the commutative case, the projection $\sum_{n=1}^{\infty} e_n = \bigvee_{n=1}^{\infty} e_n$, appeared in Theorem 2.4, is the indicator variable of

$$\begin{aligned} A &= \{\omega : X_n(\omega) \geq c \text{ and } Y_n(\omega) \leq h \text{ for some } n \geq 0\} \\ &= \bigcup_{n=1}^{\infty} \left(\{\omega : X_n(\omega) \geq c\} \cap \{\omega : Y_n(\omega) \leq h\} \right), \end{aligned}$$

where $e_n := \chi_{\{X_n \geq c\}} \chi_{\{Y_n \leq h\}}$. □

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Construction of sensing matrices based on Rudin-Shapiro polynomials

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Abstract

For compressed sensing (CS) applications, it is significant to construct deterministic measurement matrices with good sparse recovery performance. In this article we present a method to obtain a class sensing matrices, by using Rudin-Shapiro polynomials.

Keywords: Rudin-Shapiro polynomials, Equiangular tight frames, compressive sensing, sparse recovery, Steiner system

Mathematics Subject Classification [2010]: 65F10, 15A29, 94A12, 42C40, 42A05

1 Introduction

Compressive sensing (CS), also known as compressive sampling, has received considerable research of interest in various applications due to its superior capability to recovery a sparse signal from a much smaller number of measurements than its original dimension. Mathematically speaking, given a measurement matrix (sensing matrix) $A \in R^{m\times N}$ with $m \ll N$, and given a measurement vector $\mathbf{y} = A\mathbf{x} \in R^m$ associated with an s-sparse vector $\mathbf{x} \in R^N$ (a vector that has at most s nonzero entries), we want to access this vector in a numerically tractable way. For solving CS problems, there are several classes of algorithms that have been used in applications, such as l_1 minimization algorithms, greedy algorithms, Iterative thresholding/shrinkage algorithms and combinatorial algorithms. In addition, a number of variants of the greedy pursuit algorithms have also been proposed by various authors, e.g., Orthogonal Matching Pursuit (OMP), Compressive Sampling Matching Pursuit (CoSaMP) and Subspace Pursuit (SP) (see [5]). Sensing matrix design is one important topic in compressive sensing. In this paper we consider the matrix sensing problem and present a class of measurements matrices, named Rudin-Shapiro equiangular tight frames (ETFs).

A family of vectors $F = \{f_i\}_{i=1}^N$ is a *frame* for a real M-dimensional Hilbert space $\mathbb{H}^M$ if there are constants $0 < A \leq B < \infty$ so that for all $f \in \mathbb{H}^M$

$$A\|f\|^2 \leq \sum_{i=1}^{N} |\langle f, f_i\rangle|^2 \leq B\|f\|^2.$$

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The corresponding *frame operator* is $FF^* = \sum_{n=1}^{N} f_n f_n^*$, where f_n^* denotes the linear functional that maps a given $f \in \mathbb{H}^M$ to the scalar $\langle f, f_n \rangle$.

The sequence F is said to be a *tight frame* if there exists $A > 0$ such that $FF^* = AI$. Meanwhile, F is *equiangular* if $\|f_n\| = 1$ for all n and if there exists $\alpha \geq 0$ such that $|\langle f_n, f_m \rangle| = \alpha$ for all $n \neq m$.

Many approaches to constructing ETFs have focused on the special case in which every entry of F is a root of unity [6]. In this article, we provide a construction of ETFs. A version of the ETF construction method we present here, was employed by M. Fickus and et. al. [4] (see also [1, 3] and references therein). To do this, at first we construct a class of discrete wavelet transforms, by applying the Rudin-Shapiro polynomials. These transformations as matrix representation, are orthogonal matrices. We apply these matrices with Steiner systems to present a class of ETFs, we named it Rudin-Shapiro ETFs..

This paper is organized as follows. In section 2 we introduce the classical Rudin-Shapiro polynomials to construct a family of discrete wavelet transforms, named Rudin-Shapiro DWT. In section 3, we construct tight frames with the introduced DWTs and finally the numerical simulation results are presented.

2 Classical Rudin-Shapiro polynomials

In this section we introduce a special type of trigonometric polynomials (as a "pair") called Rudin-Shapiro polynomials that will be used to construct a sequence of low-pass filters. They were introduced by H. S. Shapiro in his study of the magnitude for certain trigonometric sums. The Shapiro result was rediscovered by Rudin and published in 1959, and is now known as the Rudin-Shapiro polynomials. These polynomials have been used by many authors (see for example [2] and references therein).
If $P_0 \equiv 1$ and $Q_0 \equiv 1$, for $\xi \in [0, 1)$ and for all $n \geq 0$, we define the *Rudin-Shapiro polynomials* recursively by

$$P_{n+1}(\xi) = P_n(\xi) + e^{i2\pi 2^n \xi} Q_n(\xi), \tag{2.1}$$
$$Q_{n+1}(\xi) = P_n(\xi) - e^{i2\pi 2^n \xi} Q_n(\xi). \tag{2.2}$$

It can easily be verify that the coefficients of both P_n and Q_n are $+1$ or -1. If the sequence $\{\alpha_k\}_{k=0}^{\infty}$ in $\{-1, 1\}$ is so that

$$P_n(\xi) = \sum_{k=0}^{2^n - 1} \alpha_k e^{2\pi i k \xi},$$

then $\alpha_0 = 1$ and for $k > 0$, we have $\alpha_{2k} = \alpha_k$ and $\alpha_{2k+1} = (-1)^k \alpha_k$. The following results can be found in [2].

Theorem 2.1. For any integer $n \geq 0$ and $0 \leq k \leq 2^{2n+1} - 1$, let

$$\alpha_n(k) = \frac{1}{2^{n+1}} \widehat{P}_{2n+1}(k) \text{ and } \beta_n(k) = (-1)^k \alpha_n(2^{2n+1} - 1 - k).$$

Then the functions φ_n and ψ_n which satisfy the following two-scale equations are father and mother wavelets, respectively,

$$\varphi_n(x) = \sqrt{2} \sum_{k=0}^{2^{2n+1}-1} \alpha_n(k) \varphi_n(2x - k), \tag{2.3}$$

$$\psi_n(x) = \sqrt{2}\sum_{k=0}^{2^{2n+1}-1}\beta_n(k)\phi_n(2x-k). \tag{2.4}$$

In the case of $n = 0, \varphi_0(x) = \varphi_0(2x) + \varphi_0(2x-1)$, which implies that $\varphi_0 = \chi_{[0,1)}$ and $\psi_0(x) = \varphi_0(2x) - \varphi_0(2x-1)$.
Now let $h(i)$ be a signal and $\alpha_n(k), \beta_n(k)$ be as in the previous theorem. Then the approximation operator H_n and the detail operator G_n corresponding to $\alpha_n(k)$ are defined by

$$(H_n h)(k) = \sum_i h(i)\overline{\alpha_n(i-2k)}, \quad (G_n h)(k) = \sum_i h(i)\overline{\beta_n(i-2k)}.$$

Therefore the matrix representation of these operators are as follows:

$$H_n = \begin{pmatrix} \alpha_n(0) & \alpha_n(1) & \alpha_n(2) & \dots & \alpha_n(M-2) & \alpha_n(M-1) \\ \alpha_n(M-2) & \alpha_n(M-1) & \alpha_n(0) & \dots & \alpha_n(M-4) & \alpha_n(M-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_n(2) & \alpha_n(3) & \alpha_n(4) & \dots & \alpha_n(0) & \alpha_n(1) \end{pmatrix}, \tag{2.5}$$

$$G_n = \begin{pmatrix} \beta_n(0) & \beta_n(1) & \beta_n(2) & \dots & \beta_n(M-2) & \beta_n(M-1) \\ \beta_n(M-2) & \beta_n(M-1) & \beta_n(0) & \dots & \beta_n(M-4) & \beta_n(M-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_n(2) & \beta_n(3) & \beta_n(4) & \dots & \beta_n(0) & \beta_n(1) \end{pmatrix} \tag{2.6}$$

Putting $M = 2^{2n+1}$, the Rudin-Shapiro discrete wavelet transform (DWT) is defined by

$$\mathbf{W}_n = \frac{1}{\sqrt{M}}\begin{pmatrix} H_n \\ G_n \end{pmatrix}$$

3 Constructing ETF

In [4], the authors have provided a new method for constructing equiangular tight frames (ETFs). The construction is based on a tensor-like combination of a Steiner system and a regular simplex.

Steiner system has been studied for over a century. In short, a Steiner system with parameters k, ν, written $(2, k, \nu)$-*Steiner system*, is an ν-element set V together with a set of order $\frac{\nu(\nu-1)}{k(k-1)}$ contains k-element subsets of V (called blocks) with the property that each 2-element subset of V is contained in exactly one block.

Here we employ DWT and Steiner system to construct tight frames.

Theorem 3.1. Every $(2, k, \nu)$-Steiner system generates a tight frame consisting of $N = \nu(1 + \frac{\nu-1}{k-1})$ vectors in $M = \frac{\nu(\nu-1)}{k(k-1)}$ dimensional space with redundancy $\frac{M}{N} = k(1 + \frac{k-1}{\nu-1})$ and density $\frac{k}{\nu} = (\frac{N-1}{M(N-M)})^{\frac{1}{2}}$.

Specifically, a $(\frac{\nu(\nu-1)}{k(k-1)}) \times \nu(1 + \frac{\nu-1}{k-1})$ tight frame matrix F may be constructed as follows:

(1) Let A^T be the $\frac{\nu(\nu-1)}{k(k-1)} \times \nu$ transpose of the adjacency matrix of a $(2, k, \nu)$-Steiner system;

(2) Let W be any $(1 + \frac{\nu-1}{k-1}) \times (1 + \frac{\nu-1}{k-1})$ matrix of corresponding to a DWT;

(3) For given $1 \le j \le \nu$, let F_j be the $\frac{\nu(\nu-1)}{k(k-1)} \times (1 + \frac{\nu-1}{k-1})$ matrix obtained from the j-th column of A^T by replacing each of the one-valued entries with a distinct rows of W, and every zero-valued entry with a row of zeros;

(4) $F = [F_1 F_2 \cdots F_\nu]$

In theorem (3.1) if W is a Hadamard matrix then the tight frame that we construct, will be equiangular [4]. For every positive integer n, $\mathbf{W}_n$ that constructed as a multiple of a Hadamard matrix. In fact, $2^{\frac{2n+1}{2}}\mathbf{W}_n$ is a Hadamard matrix, so we can put it instead of W in theorem(3.1) and get an equiangular tight frame.

For every $(2, k, \nu)$-Steiner system such that $\frac{\nu-1}{k-1} + 1 = 2^{2n+1}$ for some positive and integer n, let A^T be $\frac{\nu(\nu-1)}{k(k-1)} \times \nu$ transpose of the adjacency matrix of a $(2, k, \nu)$-Steiner system, and for each $j = 1, ..., \nu$ let F_j be a $\frac{\nu(\nu-1)}{k(k-1)} \times 2^{2n+1}$ matrix obtained from the j-th column of A^T by replacing each of the one-valued entries with a distinct row of the Hadamard matrix which is multiple of Golay wavelet transform. Then the columns of the $\frac{\nu(\nu-1)}{k(k-1)} \times \nu 2^{2n+1}$ matrix $F = [F_1 ... F_\nu]$, which have orthogonal rows and unit norm columns whose inner products have constant modulus α, provide a ETFs.

Example 3.1. The Ruding-Shapiro discrete wavelet transform corresponding to $n = 1$ is

$$\mathbf{W} = \frac{1}{\sqrt{8}} \begin{pmatrix} -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \end{pmatrix}$$

Let A^T be 28×8 transpose of the adjacency matrix of $(2, 2, 8)$-Steiner system, and for each $j = 1, ..., 8$ let F_j be 28×8 matrix obtained from the jth column of D^T by replacing each of the one-valued entries with a distinct row of $\sqrt{8}W$, and every zero-valued entry with a row of zeros. Then

$$F = \frac{4}{\sqrt{14}} \begin{bmatrix} F_1 & \dots & F_8 \end{bmatrix}$$

is an equiangular tight frame consisting of 64 vectors in $\mathbb{R}^{28}$ with redundancy $\frac{16}{7}$ and density $\frac{1}{4}$.

Also suppose A^T is a 35×35 transpose of the adjacency matrix of $(2, 3, 15)$-Steiner system, and for each $j = 1, ..., 15$ let F_j be 35×8 matrix obtained from the jth column of A^T by replacing each of the one-valued entries with a distinct row of $\sqrt{8}W$, and every zero-valued entry with a row of zeros. Then

$$F = \frac{4}{\sqrt{14}} \begin{bmatrix} F_1 & \dots & F_{15} \end{bmatrix}$$

is an equiangular tight frame consisting of 120 vectors in $\mathbb{R}^{35}$ with redundancy $\frac{24}{7}$ and density $\frac{1}{5}$.

4 Experimental Results

In Figure 1, we compare OMP and CoSaMP algorithms with both Rudin-Shapiro ETF and Gaussian measurement matrices $A \in R^{m \times n}$. The measurement signal is given by $y = Ax$. Reconstruction performance is quantified by the relative error, which is defined by

$$\text{relative error } = \frac{\|\tilde{x} - x\|_2}{\|x\|_2},$$

where $\tilde{x}$ is the reconstructed signal matrix and x is the original one.

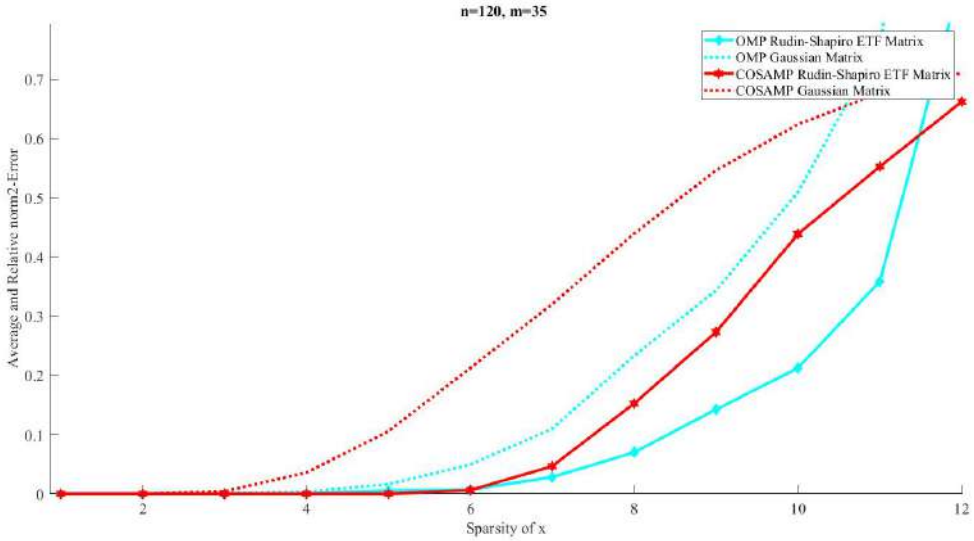

Figure 1: Plots of $\|x-x_0\|/\|x_0\|$ as a function for OMP and CoSaMP. These methods have the advantage at recovering Gaussian sparse vectors with Rudin-Shapiro ETF and Gaussian sensing matrices. The results are average of 100 runs.

The experiments illustrate how the relative error of each algorithm changes along the sparsity. We set $n = 120$ and $m = 35$. Let s (sparsity level) changes from 1 to 12 for measurement matrices. For each sparsity value ss, the algorithms are tested for 100 trials.

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Approximate solution of linear singular fractional integro-differential equation with the radial basis function

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Abstract

In this paper, we present a new method for the numerical solution of the linear singular fractional integro- differential equations. The main purpose of this work is to apply shifted Chebyshev polynomial and radial basis function for solving the linear singular fractional integro-differential equation and obtain an approximate solution. Furthermore, the efficiency of the proposed method will be considered some numerical examples.

Keywords: Linear singular fractional integro-differential equation, Collocation method, Chebyshev polynomial, Radial basis functions, Error estimation

Mathematics Subject Classification [2010]: 45D05, 65L60

1 Introduction

In recent one decade, fractional differential equations have been worked by many authors; such as Rawashdeh in [2] approximate the solution of the fractional equations with using of the collocation spline method, and Momani in [1] for integro-differential equation obtained local and global existence and uniqueness solution. In this paper, we use collocation method based on shifted Chebyshev polynomial and radial basis function to obtain an approximate solution for linear singular fractional integro-differential equation as follow:

$$D^{\alpha}u(x) + \lambda \int_0^x (x-t)^{\beta-1}u(t)dt = f(x), \qquad 1 < \alpha \leq 2, \quad 0 < \beta < 1, \quad x \in [a,b], \tag{1}$$

with boundary condition

$$u(a) = p, \quad u(b) = q, \qquad \Omega = [a,b], \tag{2}$$

where D^{α} denotes the fractional differential operator of order α and given by

$$D^{\alpha}u(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1}u^{(n)}(t)dt, \qquad n-1 < \alpha \leq n, \tag{3}$$

where $\alpha > 0$ is the order of the derivative and n is the smallest integer than α.

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1.1 Chebyshev polynomials

The Chebyshev polynomials with the interval of orthogonality $[-1, 1]$ are defined as

$$T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x), \quad \text{and} \quad T_0(x) = 1, \quad T_1(x) = x. \tag{4}$$

In order to use these polynomials on the interval $x \in [a, b]$ we define the so-called shifted Chebyshev polynomials by introducing the change of variable $z = \frac{2x-(b+a)}{b-a}$. Let the shifted Chebyshev polynomials $T_i(\frac{2x-(b+a)}{b-a})$ be denoted by $\phi_i(x)$. Then $\phi_i(x)$ can be obtained as follows

$$\phi_{i+1}(x) = 2\big(\frac{2x-(b+a)}{b-a}\big)\phi_i(x) - \phi_{i-1}(x), \quad \text{and} \quad \phi_0(x) = 1, \quad \phi_1(x) = \frac{2x-(b+a)}{b-a}. \tag{5}$$

1.2 Radial Basis Functions

Radial Basis Function methods have been introduced for interpolation of scattered data. Some well-known RBFs are listed in Table 1. RBF spaces are generated by the shifts of a radial function $\phi_j(\cdot) = \phi(\| \cdot - x_j \|)$, where $\phi : \mathbb{R}_+ \to \mathbb{R}$ is a given, continuous univariate function, and $\{x_j\}$ are some nodes in the domain of the problem. Let the set $X = \{x_j\}_{j=1}^M$, where M is the number of data points. Given data $\{x_j, u(x_j)\}_{j=1}^M$, the interpolant is schemed as follows

$$s(x) = \sum_{j=1}^{M} \lambda_j \phi(\| x - x_j \|), \qquad x \in \mathbb{R}^d,$$

where the λ_j are real coefficients that satisfy the interpolation conditions $s|_X = u|_X$, i.e. $s(x_i) = u(x_i)$ for $i = 1, \ldots, M$, which result in the following linear system of equations:

$$\Phi\Lambda = \mathbf{u},$$

where $\Lambda = [\lambda_1, \ldots, \lambda_M]^T, \mathbf{u} = [u_1, \ldots, u_M]^T$ and $\Phi = (\phi(\| x_k - x_j \|))$ is the coefficient matrix.

Table 1: Some well-known functions that generate RBFs.

Name of function	Definition
Gaussian (GA)	$\phi(r) = e^{\frac{-r^2}{2c^2}}$
Hardy multiquadrics(MQ)	$\phi(r) = \sqrt{r^2 + c^2}$
Inverse multiquadrics(IMQ)	$\phi(r) = \frac{1}{\sqrt{r^2+c^2}}$
Inverse quadric(IQ)	$\phi(r) = \frac{1}{r^2+c^2}$

The matrix Φ has been shown to be positive definite (and therefore, nonsingular) for distinct interpolation points for GA, IMQ and IQ by Schoenbergs Theorem [7]. Additionally, by using the Micchelli Theorem [6] we can show that Φ is invertible for distinct sets of the scattered points in the case of MQ. For the existence, uniqueness and convergence proofs the interested readers are referred to [3, 4].

Although the matrix Φ is nonsingular in the above cases, usually it is very ill-conditioned, i.e. the condition number of

$$\kappa_s(\Phi) = \| \Phi \|_s \| \Phi^{-1} \|_s, \qquad s = 1, 2, \ldots$$

is a very large number. Therefore, a small perturbation in initial data may produce a large amount of perturbation in the solution. Thus we have to use more precision arithmetic than

the standard floating-point arithmetic in our computation. For a fixed number of interpolation points the condition number of Φ depends on the shape parameter c, support of the RBFs and minimum separation distance of interpolation points. Furthermore, the condition number grows with M for definite values of shape parameter c. In practice, the shape parameter c must be adjusted to the number of interpolating points in order to produce an interpolation matrix which is well conditioned enough to be inverted in finite precision arithmetic [5].

2 Implementation of the collocation method

Now we use the technique of collocation method to find an approximate solution of (1). For this,

$$u(x) = \sum_{i=0}^{m} a_i \phi_i(x), \quad x \in [a, b]. \tag{6}$$

Substituting (6) into (1) we obtain

$$D^{\alpha}\left(\sum_{i=1}^{m} a_i \phi_i(x)\right) + \lambda \int_0^x (x-t)^{\beta-1}\left[\sum_{i=1}^{m} a_i \phi_i(t)\right] dt = f(x) \tag{7}$$

by simplifying, we have

$$\sum_{i=1}^{m} a_i \left(D^{\alpha}\big(\phi_i(x)\big) + \lambda \int_0^x (x-t)^{\beta-1}\big(\phi_i(t)\big) dt\right) = f(x) \tag{8}$$

Hence the residual equation is defined as

$$R(x) = \sum_{i=0}^{m} a_i \left(D^{\alpha}\big(\phi_i(x)\big) + \lambda \int_0^x (x-t)^{\beta-1}\big(\phi_i(t)\big) dt\right) - f(x). \tag{9}$$

The unknown coefficients a_i are defined by selecting several collocation points x_j so that $R(x_j) = 0$ for j from 0 to m. In this study the collocation points are evenly selected from the space $[0, 1]$ that

$$x_j = a + jh, \qquad j = 0, 1, \ldots, m, \qquad h = \frac{b-a}{m}. \tag{10}$$

Thus, this integral equation (8) can be converted to a system of linear equations $AX = B$ where

$$\begin{aligned} A &= \left(D^{\alpha}\big(\phi_i(x_j)\big) + \lambda \int_0^{x_j} (x_j - t)^{\beta-1}\big(\phi_i(t)\big) dt\right)_{i,j}, \quad i, j = 0, 1, \ldots, m, \\ B &= \Big[f(x_0), f(x_1), \ldots, f(x_m)\Big]^T, \quad j = 0, 1, \ldots, m, \\ X &= \Big[a_0, a_1, \ldots, a_m\Big]^T, \end{aligned} \tag{11}$$

Then by updating at the boundary condition, we have

$$\begin{aligned} &A[1, j] = \phi_j(a), && A[m+1, j] = \phi_j(b), \quad j = 0, 1, \ldots, m, \\ &B(1) = p, && B(m+1) = q. \end{aligned} \tag{12}$$

By solving the above system we obtain the values of the unknown coefficients and the approximate solution of (1).

3 Numerical Examples

In this section, some numerical examples of linear singular fractional Integro-differential equations are presented to illustrate the above results.

Example 3.1. Consider the following linear singular fractional integro-differential equation:

$$D^{\frac{3}{2}}u(x) + 3\int_0^x (x-t)^{-\frac{1}{2}}\, u(t)dt = \frac{4}{5}\,\frac{\sqrt{x}\,(4\,x^2\sqrt{\pi}+5)}{\sqrt{\pi}} \qquad 0 \le x \le 2,$$

with the boundary condition

$$u(0) = 0, \qquad u(2) = 4.$$

The exact solution is $u(x) = x^2$.
For $M = 5$ and using polynomials of Chebyshev and radial basis functions, the results obtained are presented in Figures 1.

x	Exact solution	$\frac{\text{Chebyshev Method}}{M=5}$	Error of Chebyshev Method	$\frac{RBFmethod}{M=10}$	Error of RBF Method
0.1	0.0100000	0.0100000	2.88×10^{-48}	0.0099999	3.90×10^{-16}
0.3	0.0900000	0.0900000	2.79×10^{-48}	0.0089999	1.09×10^{-15}
0.5	0.2500000	0.2500000	2.38×10^{-48}	0.2499999	1.59×10^{-15}
0.7	0.4900000	0.4900000	1.74×10^{-48}	0.4899999	1.76×10^{-15}
0.9	0.8100000	0.8100000	9.10×10^{-49}	0.8099999	1.57×10^{-15}
1.1	1.2100000	1.2099999	1.00×10^{-49}	1.2099999	1.05×10^{-15}
1.3	1.6900000	1.6899999	1.10×10^{-48}	1.6899999	3.17×10^{-16}
1.5	2.2500000	2.2499999	2.00×10^{-48}	2.2500000	4.87×10^{-16}
1.7	2.8900000	2.8899999	2.10×10^{-48}	2.8900000	1.18×10^{-15}
1.9	3.6100000	3.6099999	7.00×10^{-49}	3.6100000	1.56×10^{-15}

Table 2: The comparison between exact and numerical solutions for Example 3.1.

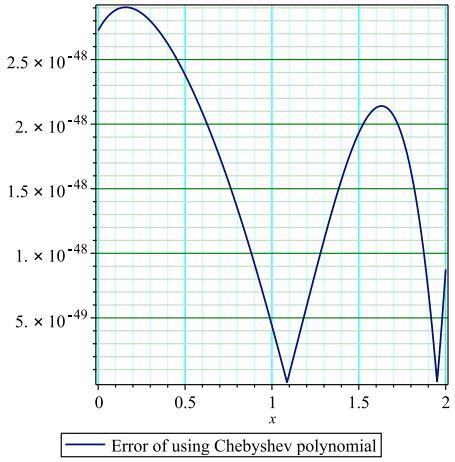
Example 3.2. In this example, we consider Consider the following linear complex Volterra integro-differential equation:

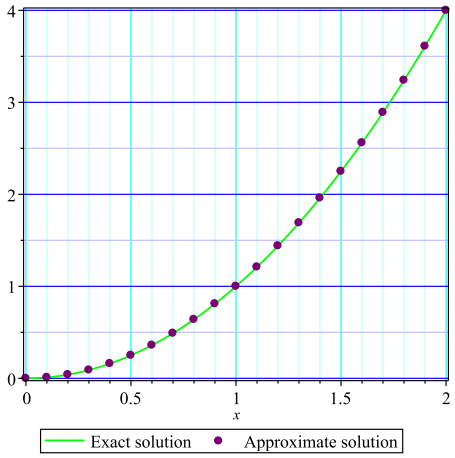
$$D^{\frac{9}{5}}u(x) + 3\int_0^x (x-t)^{-\frac{1}{5}}\, u(t)dt = -\frac{5\,x^{\frac{4}{5}}\left(2660\,\Gamma\left(\frac{4}{5}\right)x^{\frac{2}{5}}\sin\left(\frac{\pi}{5}\right) + 125\,x^3\pi - 399\,\pi\right)}{532\,\pi} \qquad 0 \le x \le 2,$$

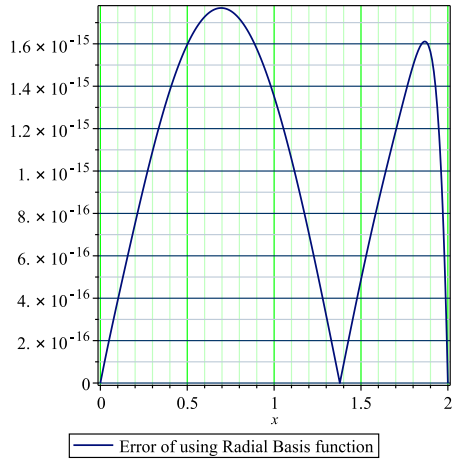
with the boundary condition $u(0) = 1, u(2) = -7$. The exact solution is $u(x) = 1 - x^3$.
For $M = 5$ and using polynomials of Chebyshev and radial basis functions, the results obtained are presented in Figures 2.

4 Conclusions

In this paper, we used of method collocation method based on shifted Chebyshev polynomial and radial basis function to obtain an approximate solution for one of the most important kind of singular integral equation (Abel integral equations) that derived directly from a concrete problem of mechanics or physics. The main purpose of this work is to apply shifted Chebyshev polynomial and radial basis function for solving linear singular fractional integro-differential equation and obtain an approximate solution. We use the technique of collocation method to



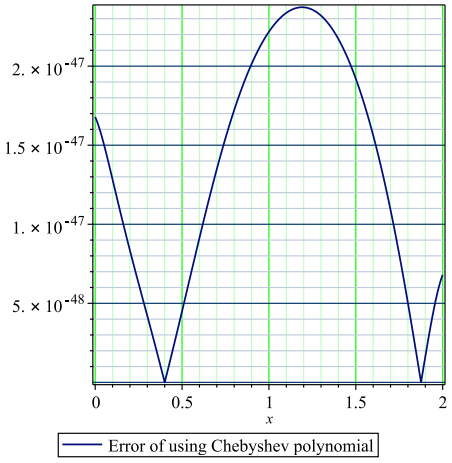


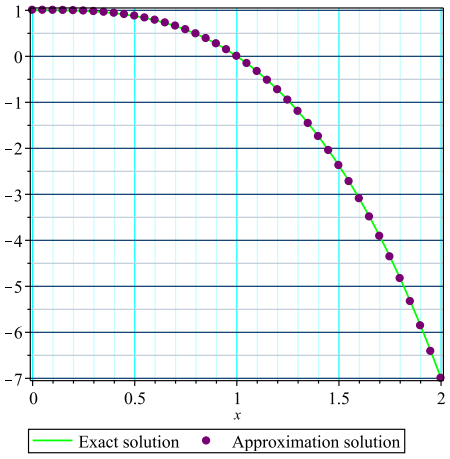


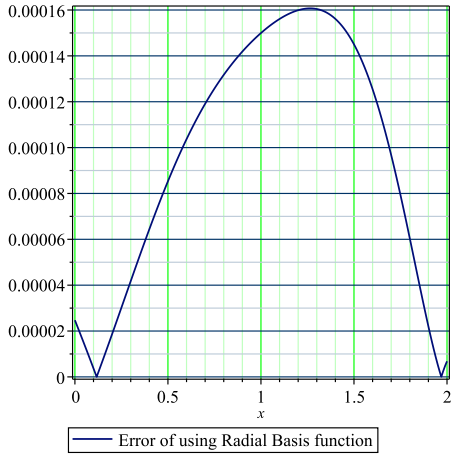
x	Exact solution	Chebyshev Method M=5	Error of Chebyshev Method	$RBFmethod$ M=10	Error of RBF Method
0.1	0.9990000	0.9990000	1.28×10^{-47}	0.9989963	3.66×10^{-6}
0.3	0.9730000	0.9730000	4.20×10^{-48}	0.9730417	4.17×10^{-5}
0.5	0.8750000	0.8749999	4.49×10^{-48}	0.8750853	8.53×10^{-5}
0.7	0.6570000	0.6569999	1.13×10^{-47}	0.6571191	1.19×10^{-4}
0.9	0.2710000	0.2709999	2.01×10^{-47}	0.2711418	1.41×10^{-4}
1.1	−0.3310000	−0.3310000	2.33×10^{-47}	−0.33084390	1.56×10^{-4}
1.3	−1.1970000	−1.1970000	2.33×10^{-47}	−1.19683955	1.60×10^{-4}
1.5	−2.3750000	−2.3750000	1.91×10^{-47}	−2.37485496	1.45×10^{-4}
1.7	−3.9130000	−3.9130000	1.10×10^{-47}	−3.91290326	9.67×10^{-5}
1.9	−5.8590000	−5.8589999	1.70×10^{-48}	−5.85897798	2.20×10^{-5}

Table 3: The comparison between exact and numerical solutions for Example 3.2.

find an approximate solution, and with using of the system of linear equations , we can obtain the approximate solution. In section Numerical Examples, we have solved two problems considered the results obtained are presented in table 2, 3 and figure 1 and 2, the comparison of results confirms the better accuracy with this method.







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Solvable ultra-groups

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Abstract

An ultra-group ${}_HM$ is a new structure algebra depends on a group G and its subgroup H. In the present paper we will investigate some identities characterizition of the solvable ultra-group and then investigate jordan-holder theorem in ultra-group.

Keywords: Transversal, Ultra-groups, Solvable.
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1 Introduction

A pair of (A, B) of subsets of a group G is called transversal if the the equality $ab = a'b'$ implies $a = a'$ and $b = b'$, where $a, a' \in A$, $b, b' \in B$. This notation was introduced by Kurosh in [1] which is the base of the concept of an ultra-group. This definition can be generalized for subgroups. It is not hard to deduce that a pair (H, M) of subgroups of G is transversal if and only if $H \cap M = \{e\}$. Moreover, for a subgroup H and a subset M of G we conclude that the pair (H, M) is a transversal if and only if $M \cap Hg$ contains at most one element, for all $g \in G$.
A subset M of G is called (right unitary) complementary set with respect to subgroup H, if for any elements $m \in M$ and $h \in H$ there exists unique elements $h' \in H$ and $m' \in M$ such that $mh = h'm'$ and $e \in M$. We denote h' and m' by ${}^mh$ and m^h, respectively.Similarly for any elements $m_1, m_2 \in M$ there exist unique elements $[m_1, m_2] \in M$ and $(m_1, m_2) \in H$ such that $m_1m_2 = (m_1, m_2)[m_1, m_2]$. For every element $a \in M$, there exists a^{-1} belongs to G. As $G = HM$, there is $a^{(-1)} \in H$ and $a^{[-1]} \in M$ such that $a^{-1} = a^{(-1)}a^{[-1]}$. Now we are ready to define an ultra-group. A (right) ultra-group ${}_HM$ is a complementary set of H over group G with a binary operation $\alpha :_H M \times_H M \to_H M$ and unary operation $\beta_h :_H M \to_H M$ defined by $\alpha((m_1, m_2)) := [m_1, m_2]$ and $\beta_h(m) := m^h$ for all $h \in H$.
A (left) ultra-group M_H is defined similarly via (left unitary) complementary set with respect to subgroup H and unary operation ${}_h\beta : M_H \to M_H$ defined by ${}_h\beta(m) := {}^hm$ for all $h \in H$.
From now on, unless specified otherwise, ultra-group means right ultra-group.
Moreover, one may consider the properties of the ultra-groups in justifying the associate theorems for the groups. All the notations in this paper is standard, we may refer the reader to see [2, 5, 6] for more details.

Definition 1.1. Let ${}_HM$ be an ultra-group of subgroup H of a group G. A subset $S \subseteq {}_HM$ which contains the identity element of the group G, is called a subultra-group of ${}_HM$, if S is closed under operation α and β_h. This is denoted by $S <_H M$.

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Definition 1.2. An ultra-group ${}_HM$ is called abelian, if for all elements a, b in ${}_HM$, $[a, b] = [b, a]$.

Proposition 1.3. *[5] Let ${}_HM$ be an ultra-group of subgroup H over the group G. Then we have the following properties:*

(i) $a^{hh'} = (a^h)^{h'}$,

(ii) $[a, b]^h = [a^{(bh)}, b^h]$,

(iii) $[[a, b], c] = [a^{(b,c)}, [b, c]]$,

(iv) $e^h = e, \quad a^e = a$,

(v) $[e, a] = a = [a, e]$,

(vi) $[a^{[-1]}, a] = e = [a^{(a^{(-1)})}, a^{[-1]}]$,

for $a, b, c \in M$ and $h, h' \in H$.

Lemma 1.4. *[5] Let S be a subultra-group of ultra-group ${}_HM$ of H over the group G and $a, b \in {}_HM$. Then the following conditions are equivalent.*

(i) $a \in [S, b]$,

(ii) $[S, a] = [S, b]$,

(iii) $[a^{(b^{(-1)})}, b^{[-1]}] \in S$.

Definition 1.5. Suppose ${}_{H_i}M_i$ is ultra-group of H_i over group G_i, $i = 1, 2$. A function $f : {}_{H_1}M_1 \longrightarrow {}_{H_2}M_2$ is an ultra-group homomorphism provided that for all $m, m_1, m_2 \in {}_{H_1}M_1$ and $h \in H_1$.

(i) $f([m_1, m_2]) = [f(m_1), f(m_2)]$,

(ii) $(f(m))^{\varphi(h)} = f(m^h)$, where φ is a group homomorphism between two subgroups H_1 and H_2.

Definition 1.6. A subultra-group N of ${}_HM$ is called normal if $[a, [N, b]] = [N, [a, b]]$, for all $a, b \in {}_HM$.

According to the definition, every ultra-group such ${}_HM$, has subultra-group $\{e\}$ and ultra-group ${}_HM$ is normal subultra-group when the left cancellation low be established for ${}_HM$.

Theorem 1.7. *(First isomorphism theorem for ultra-groups) Let f be a surjective ultra-group homomorphism between two ultra-groups ${}_{H_1}M_1$ and ${}_{H_2}M_2$ and θ a congruence over ${}_{H_1}M_1$ such that $\theta \subseteq \mathrm{Ker}(f)$. If $\pi : {}_{H_1}M_1 \longrightarrow {}_{H_1}M_1/\theta$, then there exists a homomorphism g satisfying $g\pi = f$.*

Theorem 1.8. *(Second isomorphism theorem of ultra-groups) [5] If N', N are normal subultra-groups of ultra-group ${}_HM$ such that $N \subseteq N'$, then*

$$\frac{\frac{{}_HM}{N}}{\frac{N'}{N}} \cong \frac{{}_HM}{N'}.$$

2 solvable ultra-groups

Similar to what was proved for the solvable groups, we obtain for ultra-groups and charactrization of solvable ultragroups as well as the famous theorem jordan-holder and Zassenhaus lemma.

Definition 2.1. A subnormal series of an ultra-group ${}_HM$ is a chain of subultra-groups ${}_HM = M_0 > M_1 > \ldots > M_n$ such that M_{i+1} is normal subultra-group in M_i for $0 \leq i \leq n$. The factors of the series are the quotiont ultra-groups of stric inclusions(or alternatively, the number of nonidedtity factor).

Definition 2.2. A subnormal series of ultragroups such that M_i is normal in ${}_HM$ for all i is said to be normal series of ultra-groups.

Every ultra-group ${}_HM$ has normal series such as ${}_HM > \{e\}$ since $\{e\}$ is normal subultra-group of ${}_HM$.

Lemma 2.3. *A subnormal series of ultra-groups need not be normal series of ultra-groups.*

Proof. Let $D_8 = \langle a, b | a^4 = b^2 = e, \quad (ab)^2 = e \rangle$ and $H = \{e\}$. The series $\{e\} < \{e, b\} < \{e, b, a^2, a^2b\} < D_8$ is subnormal series that it isn't normal series since $\{e, b\} \not\lhd D_8$. □

Definition 2.4. Let ${}_HM = M_0 > M_1 > \ldots > M_n$ be subnormal series of ultra-groups. A one-step refinement of this series is any series of the form ${}_HM = M_0 > M_1 > \ldots > M_i > N > M_{i+1} >_n$ or ${}_HM = M_0 > M_1 > \ldots > M_i > N > \ldots M_n > N$ where N is normal subulta-group of M_i and (if $i < n$) M_{i+1} is normal in N.

Definition 2.5. A refinement of subnormal series S is any subnormal series of ultra-groups obtaind from S by a finite sequence of one-step refinement.

Definition 2.6. An ultra-group ${}_HM$ is called simple if has just the normal subultra-group $\{e\}$.

Definition 2.7. A subnormal series ${}_HM = M_0 > M_1 > \ldots > M_n = e$ of ultra-group ${}_HM$ is composition series of ultra-groups if each factor $\dfrac{M_i}{M_{i+1}}$ is simple.

Definition 2.8. A subnormal series ${}_HM = M_0 > M_1 > \ldots > M_n = e$ of ultra-group ${}_HM$ is solvable series if each factor $\dfrac{M_i}{M_{i+1}}$ is abelian.

Definition 2.9. A maximal subultra-group S of an ultra-group ${}_HM$ is a proper subultra-group, such that no proper subultra-group K contains S strictly. Similary, a normal subultra-group N of ${}_HM$ is said to be a maximal proper normal subultra-group of ${}_HM$ if $N <_H M$ and there is no normal subultra-group K of ${}_HM$ such that $N < K <_H M$.

If N is a normal subultra-group of an ultra-group ${}_HM$, then every subultra-group of $\dfrac{{}_HM}{N}$ is the form $\dfrac{K}{N}$ where K is subultra-group of ${}_HM$ containing N. In the other hand $\dfrac{K}{N}$ is normal subultra group of $\dfrac{{}_HM}{N}$ if and only if K is normal subultra-group of ${}_HM$. Thus, when ${}_HM \neq N$, $\dfrac{{}_HM}{N}$ is simple if and only if N is maximal in the set of all normal subultra groups L of ${}_HM$ with $L \neq_H M$. Such an ultra-group N is called a maximal subultra-group of ${}_HM$.

Theorem 2.10. *Let ${}_HM$ be an ultra-group of subgroup H over group G then we have the following properties:*
i) Every finite ultra-group ${}_HM$ has composition series.
ii) Every rifinement of a solvable series of ultra-group is a solvable series .
iii) A subnormal series of ultra-groups is a composition series if and only if it has no proper refinements.

Definition 2.11. Two subnormal series S and T of an ultra-group ${}_HM$ are equvalent if there is a one to one correspondence between the nontrivial factors of S and the nontrivial factors of T such that corresponding factors are isomorphic ultra-groups.

Two subnormal series need not have the same number of terms in order to be equivalent, but they have the same length (that is, the same number of nontrivial factors). Clearly, equivalence of subnormal series is an equivalence relation.

Lemma 2.12. *If S is a composition series of ${}_HM$, then any refinement of S is equvalent to S.*

In this section, we show that, similar Poropositions Zassenhaus and Schreier, Jordan-holder in group theory, it also holds in ultra-groups. Befor proving the Zassenhaus lemma we first prove some required lemma.

Lemma 2.13. *For every $a, b, c \in {}_HM$ and every subultra-group K of ${}_HM$ if $[a, b] = c$ and $a, c \in K$ then $b \in K$.*

Theorem 2.14. *If K, N is subultra-groups of ultra-group ${}_HM$ such that $N \lhd {}_HM$ then $N \cap K \lhd K$*

The next lemma is quite technical. It's value will be immediately apparent in the proof of theorem 2.16.

Lemma 2.15. ***(Zassenhaus)***
Let K^*, N^*, N, K be subultra-groups of ultra-group ${}_HM$ such that K^* is normal in K and N^* is normal in N. Then we have the following properties:
i) $[N^*, (N \cap K^*)] \lhd [N^*, (N \cap K)]$
ii) $[K^*, (N^* \cap K)] \lhd [K^*, (N \cap K)]$
iii) $\frac{[N^*,((N\cap K)]}{[N^*,(N\cap K^*)]} \cong \frac{[K^*,(N\cap K)]}{[K^*,(N^*\cap K)]}$

Theorem 2.16. ***(Schreier)***
Any two subnormal (resp. normal) series of an ultra-group ${}_HM$ have subnormal(resp. normal) refinements are equivalent.

Theorem 2.17. ***(Jorden-Holder)***
Any two composition series of an ultra-group ${}_HM$ are equivalent. There for every ultra-group having a composition series determines a unique list of simple ultra-groups.

Proof. Since composition series are subnormal series, any two composition series have equivalent refinement by theorem Schrier. But every refinement of a composition series S is equivalent to S by lemma 2.12. It follows that every two composition series are equivalent. □

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On inessential operators in Hilbert C^*-modules

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Abstract

In this paper, we study inessential modular operators $\mathcal{I}(\mathcal{X},\mathcal{Y})$ beetwen finite-dimensional Hilbert C^*-modules. Also, in this case, we show that the class of inessential modular operators presents a perfect symmetry with respect to the dim(ker(T)) and co-dim(ran(T)).

Keywords: Modular operator, Hilbert C^*-modules, Inessential modular operator
Mathematics Subject Classification [2010]: 46L99, 47A06, 47A55.

1 Introduction and Preliminaries

The Atkinson characterization of Fredholm operators establishes that a bounded operator on a Banach space X is a Fredholm operator precisely when it is invertible in $L(X)$ modulo the ideal $F(X)$ of all finite-dimensional operators. The ideal $F(X)$ is the socle of the semi-simple Banach algebra $L(X)$. This suggests how to extend Fredholm theory to the more abstract framework of Banach algebras. A natural way of defining a Fredholm element of a Banach algebra A is that this is an element of A invertible modulo a fixed ideal J. However, the results on the elentary stractures suggest that in order to obtain a deeper Fredholm theory for Banach algebras which reflects more closely the classical Fredholm operator theory we need to assume that the ideal J is the socle, or more generally that J is inessential. In [1], the ideal of inessential operators $I(E)$ on a complex Banach space E as the largest ideal of the class $A(E)$(of all bounded linear operators A) having the property that the restrictions $A|_M$ of A on any closed infinite-dimensional invariant subspace M have been characterized.

Recently, the novel results have been obtained for inessential operators on Hilbert C^*-modules. we show that the class of inessential modular operators presents a perfect symmetry with respect to the dim(ker(T)) and co-dim(ran(T)).

Let us briefly recall the basic information about Hilbert C^*-modules and modular operators, which can be found in [3]– [5].

Throughout this paper, $\mathcal{A}$ will denote a C*-algebra. An inner-product $\mathcal{A}$-module is a linear space $\mathcal{X}$ which is a right $\mathcal{A}$-module, together with a map $(x,y)\mapsto\langle x,y\rangle : \mathcal{X}\times\mathcal{X}\to\mathcal{A}$ such that for any $x,y,z\in\mathcal{X}$, $\alpha,\beta\in\mathbb{C}$ and $a\in\mathcal{A}$, the following conditions hold:

(i) $\langle x,\alpha y+\beta z\rangle=\alpha\langle x,y\rangle+\beta\langle x,z\rangle$;

(ii) $\langle x,ya\rangle=\langle x,y\rangle a$;

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(iii) $\langle y, x\rangle = \langle x, y\rangle^*$;

(iv) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0 \Longleftrightarrow x = 0$.

An inner-product $\mathcal{A}$-module $\mathcal{X}$ which is complete with respect to the induced norm $\|x\| = \sqrt{\|\langle x, x\rangle\|}$ for any $x \in \mathcal{X}$ is called a (right) Hilbert $\mathcal{A}$-module. A closed submodule $\mathcal{M}$ of a Hilbert $\mathcal{A}$-module $\mathcal{X}$ is said to be orthogonally complemented if $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, where

$$\mathcal{M}^{\perp} = \{x \in \mathcal{X} : \langle x, y\rangle = 0 \text{ for any } \ y \in \mathcal{M}\}.$$

Now, suppose that $\mathcal{X}$ and $\mathcal{Y}$ are two Hilbert $\mathcal{A}$-modules, let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ be the set of operators $T : \mathcal{X} \to \mathcal{Y}$ for which there is an operator $T^* : \mathcal{Y} \to \mathcal{X}$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for any } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}.$$

It is known that any element $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(xa) = (Tx)a$, for $x \in \mathcal{X}$ and $a \in \mathcal{A}$. We call $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, the set of adjointable operators from $\mathcal{X}$ to $\mathcal{Y}$. For any $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, the null and the range space of T are denoted by $\ker(T)$ and ran(T), respectively. In the case $\mathcal{X} = \mathcal{Y}$, $\mathcal{L}(\mathcal{X}, \mathcal{X})$ which is abbreviated to $\mathcal{L}(\mathcal{X})$, is a C^*-algebra. Let $\mathcal{L}(\mathcal{X})_{sa}$ be the set of self-adjoint elements and $\mathcal{L}(\mathcal{X})_+$ be the set of positive elements in $\mathcal{L}(\mathcal{X})$, respectively. In addition, we reserve the notion of $K(\mathcal{X})$ for the set of all compact modular operators on $\mathcal{X}$ (for more details and informations refer to [6]). The identity operator on $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$ or 1 if there is no ambiguity.

Lemma 1.1. *[4] Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and $T \in \mathcal{L}(\mathcal{X})$. Then $T \in \mathcal{L}(\mathcal{X})_+$ if and only if $\langle Tx, x\rangle \geq 0$ for all x in $\mathcal{X}$.*

Theorem 1.2. *[4] Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then*

(i) $\ker(T)$ *is orthogonally complemented in $\mathcal{X}$, with complement* $\text{ran(T}^*\text{)}$.

(ii) ran(T) *is orthogonally complemented in $\mathcal{Y}$, with complement* $\ker(T^*)$.

(iii) The map $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ has closed range.

Definition 1.3. Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The Moore-Penrose inverse $T^{\dagger}$ of T is an element in $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ which satisfies:

(i) $TT^{\dagger}T = T$,

(ii) $T^{\dagger}TT^{\dagger} = T^{\dagger}$,

(iii) $(TT^{\dagger})^* = TT^{\dagger}$,

(iv) $(T^{\dagger}T)^* = T^{\dagger}T$.

If there exists this $T^{\dagger}$, we say that T is MP invertible. Motivated by these conditions $T^{\dagger}$ is unique and $T^{\dagger}T$ and $TT^{\dagger}$ are orthogonal projections, in the sense that they are selfadjoint and idempotent operators. Clearly, T is Moore-Penrose invertible(briefly MP invertible) if and only if T^* is MP invertible, and in this case $(T^*)^{\dagger} = (T^{\dagger})^*$.

Further information about Hilbert C^*-modules and modular operators on them may be found in [4] and [5] and refrences therein.

A matrix form of a bounded adjointable operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ can be induced by some natural decompositions of Hilbert C^*-modules. Indeed, if $\mathcal{M}$ and $\mathcal{N}$ are closed orthogonally

complemented submodules of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, $\mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^{\perp}$, then T can be written as the following 2×2 matrix

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

where, $T_1 = P_{\mathcal{N}} T P_{\mathcal{M}} \in \mathcal{L}(\mathcal{M}, \mathcal{N})$, $T_2 = P_{\mathcal{N}} T (1 - P_{\mathcal{M}}) \in \mathcal{L}(\mathcal{M}^{\perp}, \mathcal{N})$, $T_3 = (1 - P_{\mathcal{N}}) T P_{\mathcal{M}} \in \mathcal{L}(\mathcal{M}, \mathcal{N}^{\perp})$ and $T_4 = (1 - P_{\mathcal{N}}) T (1 - P_{\mathcal{M}}) \in \mathcal{L}(\mathcal{M}^{\perp}, \mathcal{N}^{\perp})$ and $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ denote the projections corresponding to $\mathcal{M}$ and $\mathcal{N}$, respectively.

From here onwards, $\mathcal{A}$ be an unital C^*-algebra and $\mathcal{X}, \mathcal{Y}$ are finite-dimensional Hilbert $\mathcal{A}$-modules. Meanwhile, some further researchers on the finite-dimensional Hilbert $\mathcal{A}$-modules may be found in [2].

Definition 1.4. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite-dimensional Hilbert $\mathcal{A}$-modules and $K(\mathcal{X})$ be the set of all compact modular operators on $\mathcal{X}$, then

(i) $\Phi_+(\mathcal{X}, \mathcal{Y}) = \{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) | T \text{ is MP-invertible and } \ker(T) \text{be finite-dimensional}\}$

(ii) $\Phi_-(\mathcal{X}, \mathcal{Y}) = \{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) | T \text{ is MP-invertible and co-dim(ran(T))be finite-dimensional}\}$

(iii) $\Phi_l(\mathcal{X}, \mathcal{Y}) = \{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) | \exists S \in L(\mathcal{Y}, \mathcal{X}) \ni (I_{\mathcal{X}} - ST) \in K(\mathcal{X})\}$

(iv) $\Phi_r(\mathcal{X}, \mathcal{Y}) = \{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) | \exists S \in L(\mathcal{Y}, \mathcal{X}) \ni (I_{\mathcal{Y}} - TS) \in K(\mathcal{X})\}$.

2 Main Results

The perturbation classes of modular operators are often called classes of admissible perturbations. We shall see that the class of inessential operators is the class of all admissible perturbations, since it is the biggest perturbation class of the semi-group of all Fredholm operators. These results will be established in the general framework of operators acting between two finite-dimensional Hilbert C^*-modules. Moreover, we shall see that every modular operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ may be characterized either in terms of $\dim(\ker(T))$ or, alternatively, in terms of the co-dim(ran(T)).

Theorem 2.1. *Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be an operator on a Hilbert C^*-modules. Then the following assertions are equivalent:*

(i) T is inessential;

(ii) $\ker(I_{\mathcal{X}} - ST)$ be finite-dimensional, for all $S \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$;

(iii) $\ker(I_{\mathcal{Y}} - TU)$ be finite-dimensional, for all $U \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$.

Theorem 2.2. *Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $S \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ be two operators on a Hilbert C^*-modules. Then the following assertions are equivalent:*

(i) T is inessential;

(ii) co-dim$\overline{(\mathrm{ran}(\mathrm{I}_{\mathcal{X}} - \mathrm{ST}))}$ be finite-dimensional, for all $S \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$;

(iii) co-dim$\overline{(\mathrm{ran}(\mathrm{I}_{\mathcal{Y}} - \mathrm{TU}))}$ be finite-dimensional, for all $U \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$.

Lemma 2.3. *Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be an operator on a Hilbert C^*-modules. Then he following equalities hold:*

(i) $\dim(\ker(I_{\mathcal{X}} - ST)) = \dim(\ker(I_{\mathcal{Y}} - TS))$;

(ii) $co\text{-}dim(\mathrm{ran}(\mathrm{I}_{\mathcal{X}} - \mathrm{ST})) = co\text{-}dim(\mathrm{ran}(\mathrm{I}_{\mathcal{Y}} - \mathrm{TS}))$;

(iii) $co\text{-}dim\overline{(\mathrm{ran}(\mathrm{I}_{\mathcal{X}} - \mathrm{ST}))} = co\text{-}dim\overline{(\mathrm{ran}(\mathrm{I}_{\mathcal{Y}} - \mathrm{TS}))}$.

Proposition 2.4. *$T \in I(\mathcal{X}, \mathcal{Y})$ if and only if $I_{\mathcal{Y}} - TS \in \Phi(\mathcal{Y})$.*

Proposition 2.5. *If $T^* \in \mathcal{L}(\mathcal{Y}^*, \mathcal{X}^*)$, then $T \in I(\mathcal{X}, \mathcal{Y})$.*

Corollary 2.6. *$\mathcal{L}(\mathcal{X}, \mathcal{Y}) = I(\mathcal{X}, \mathcal{Y})$ if and only if $\mathcal{L}(\mathcal{Y}, \mathcal{X}) = I(\mathcal{Y}, \mathcal{X})$.*

Theorem 2.7. *For every Hilbert C^*-module $\mathcal{X}$, we have $I(\mathcal{X}) = \mathcal{P}\Phi_r(\mathcal{X}) = \mathcal{P}\Phi_l(\mathcal{X}) = \mathcal{P}\Phi(\mathcal{X})$.*

Let $\mathcal{C}$ be the class of operators between Hilbert C^*-modules, its perturbation class $\mathcal{PC}$ is defined as the class of all operators F such that $T+F \in \mathcal{C}$ for every $T \in \mathcal{C}$. This definition is not intrinsic, in the sense that determining whether an operator belongs to $\mathcal{PC}$ involves studying its behaviour with respect to every operator in $\mathcal{C}$.

Theorem 2.8. *Let $\mathcal{X}$ and $\mathcal{C}$ be the class of operators between Hilbert C^*-modules. If $I(\mathcal{X}) = \mathcal{P}\Phi_r(\mathcal{X}) = \mathcal{P}\Phi_l(\mathcal{X}) = \mathcal{P}\Phi(\mathcal{X})$.*

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n-Root of Matrices

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Abstract

Assume that $A \in M_m(C)$, when C is an algebraically closed field. Also, consider that n is an arbitrary natural number. Here we are going to find a matrix $B \in M_m(C)$ with a construction method such that $B^n = A$.
Assume that $C \in M_m(\mathbb{H})$. Also, consider that t is an arbitrary natural number. We can find a matrix $D \in M_m(\mathbb{H})$ such that $D^t = C$.

Keywords: Root of matrices, Division algebra, Field, Radicable group.
Mathematics Subject Classification [2010]: 16K20, 12E99, 20H25

1 Introduction

In group theory, a multiplicative group G is called radicable if for every $a \in G$ and $n \in \mathbb{N}$, there exists an element $b \in G$ such that $b^n = a$. The adjective divisible is reserved for abelian G. . A division algebra D is called radicable if the unit group D^* is radicable. In particular, when D^* is abelian, the notion of divisibility and radicability coincide and D is called divisible (otherwise indivisible).

But, the structure of nonabelian radicable groups, in general, is unknown. The most extensive previous result in this direction was proved by Mahdavi-Hezavehi and Motiee. They determined in [3] and [4] the class of all radicable F-central division algebras D. In fact, for an indivisible field F, they showed that the following statements are equivalent:

1. D is radicable;
2. D contains a divisible subfield K/F;
3. D is isomorphic to $(\frac{-1,-1}{F})$ (the ordinary quaternion division algebra) and $F(\sqrt{-1})$ is divisible.

In addition, G. ten Have (cf. [2]) gives a constructive way of finding roots in $M_m(K)$, where K is an arbitrary subfield of $\mathbb{C}$.

Given an algebraically closed field C, let $J_m(\lambda)$ denote the Jordan block of size $m \times m$ corresponding to eigenvalue $\lambda \in C$. When $\lambda \neq 0$ and the field C has characteristic zero, the Jordan form of $J_m(\lambda)^n$ is just also $J_m(\lambda^n)$. The above result holds when the field C has nonzero

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characteristic p and $m < p$. When $m \geqslant p$, things get messier. For instance, over $\mathbb{F}_2$, consider the case $m = 3$, when $n = 2$ and

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

So,

$$A^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, A^2 in this case has Jordan blocks of sizes smaller than 3.

Let K be a subfield of C. Then a matrix $A \in M_m(K)$ has a K-normal form $C_A = \bigoplus_{i=1}^{t} d_i C(f_i^{k_i})$, where the $f_i^{k_i}(x)$ are the elementary divisors in $K[x]$ of A. By the definition of elementary divisor the $f_i(x)$ are irreducible, and by using the aforementioned multiplicity d_i we can assume that no two of the $f_i^{k_i}(x)$ are the same. Note that, when m_i is the degree of $f_i(x)$, the order m of the matrix A is equal to $\sum_{i=1}^{t} d_i k_i m_i$. Now, for every i, take an $\alpha_i = {\alpha_i}^{(1)}$ which satisfies $f_i(\alpha_i) = 0$, and for $j = 1, \cdots, n$, take the n distinct values γ_{ij} such that $\gamma_{ij}^n = \alpha_i$. Let n_{ij} be given by $n_{ij} = [K(\gamma_{ij}) : K(\alpha_i)]$ for $i = 1, \cdots, t$ and $j = 1, \cdots, n$. In [1], it is proved that for a Given an algebraically closed field C, let K be a subfield of C, and let $A \in M_m(K)$ be nonsingular. Then, with notation as above,

1. If $\mathrm{Char}(C) = 0$, then A has an n-th root in $M_m(K)$ if and only if there are nonnegative integers $b_{i1}, \ldots, b_{in}$, such that $d_i = b_{i1}n_{i1} + \cdots + b_{in}n_{in}$ for $i = 1, \ldots, t$.

2. If $\mathrm{Char}(C) = p > 0$, $m < p$ and $p \nmid n$, then A has an n-th root in $M_m(K)$ if and only if there are nonnegative integers $b_{i1}, \ldots, b_{in}$, such that $d_i = b_{i1}n_{i1} + \cdots + b_{in}n_{in}$ for $i = 1, \ldots, t$.

In this article we will try to present a method to find the roots of a matrix in $M_m(\mathbb{C})$.

2 Main results

In the introduction, a brief overview of some of the steps taken to examine the root of a matrix is given. As we have seen, there are various theorems about the conditions under which the root of a matrix exists. In this section, we are going to present a constructive method for finding the $n-$roots of members of the ring $M_m(\mathbb{C})$. As mentioned in key theorem of Reference [1], every member of $M_m(\mathbb{C})$ has a $n-$roots, for any natural number n.

Main result (Constructive method). Assume that $A \in M_m(C)$, when C is an algebraically closed field. Also, consider that n is an arbitrary natural number. Here we are going to find a matrix $B \in M_m(C)$ with a construction method such that $B^n = A$.

We know that for any real number x, $(e^{\frac{1}{n}lnx})^n = x$. Now it suffices to generalize this concept to matrices.Since C is an algebraically closed field, then A has a Jordan canonical form. Without loss of genreality, it is enough to reduce the problem to Jordan form. matrices. Therefore, we ay assume that:

$$A = \begin{bmatrix} a & 0 & . & . \\ 1 & a & 0 & . \\ . & . & . & . \\ . & 0 & 1 & a \end{bmatrix},$$

when a is an arbitrary element in C. Thus, $A = aI_m + N$, when $N \in M_m(C)$ such that $N^m = 0$. To simplify the calculations, we set $a = 1$, and thus Thus, $A = I_m + N$, when $N \in M_m(C)$ such that $N^m = 0$. Now, we define:

$$ln(A) = ln(I_m + N) = \sum_{i=1}^{m-1} \frac{(-1)^{i-1}}{i} N^i.$$

Now, set $B = (e^{\frac{1}{n} ln(A)})$. Then $B^n = A$, as we desired.

Assume that $\mathbb{H}$ is real quaternion division algebra over $\mathbb{R}$. By main result in [5], any $A \in M_m(\mathbb{H})$ has a canonical Jordan form in $M_{2m}(\mathbb{C})$. Of course, if we pay attention to the type of calculations used above and by the same argument as above, for any $A \in M_m(\mathbb{H})$ and $n \in \mathbb{N}$, we can find $B \in M_m(\mathbb{H})$ such that $B^n = A$.

3 Conclusion

Assume that $A \in M_m(\mathbb{H})$. Also, consider that n is an arbitrary natural number. We can find a matrix $B \in M_m(\mathbb{H})$ such that $B^n = A$.

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Solving optimal control problems of a linear oscillator with the differential equation

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Abstract

A computational strategy for solving optimal control problems (OCPs) is displayed. This technique is based on the Bezier curve strategy. Thus, an optimal control problem (OCP) with pantograph delays converts to an optimization problem, which can be solved easily. Numerical examples are given to exhibit the applicability and efficiency of the technique.

Keywords: Bezier curve, Optimal control problems
Mathematics Subject Classification [2010]: 65K10, 26A33

1 Introduction

OCPs have an important role in some areas including engineering economics and finance. A computational strategy for solving OCP is developed by Wu, et al. [6] which is obtained by a switched dynamical system with time delay. Kharatishidi [3] approached this problem by extending the Pontryagin's maximum principle to time delay systems (TDS). The actual solution involves a two-point boundary value problem in which advances and delays are stated. In addition, this solution does not yield a feedback controller. OCP with time delay has been considered by Oguztoreli [5] who achieving several findings concerning bang-bang controls which are parallel to those of LaSalle [4] for non delay systems. For a time invariant system with an infinite upper limit in the performance measure, An optimal regulator for a linear system with multiple states, input delays and a quadratic criterion is presented in some papers. The optimal regulator equations were achieving by reducing original problem to the linear quadratic regulator design for a system without delays. In this paper, we will solve OCP by Bezier curve.
The outline of this paper is as follows: In Section 2, Bezier curve technique is introduced. Also, a remark is stated. Some examples are given in Section 3. Section 4 is dedicated the conclusion.

2 Bezier curve technique

Our system is utilizing Bezier curves to approximate the solutions $x_i(t)$ and $u(t)$ where $x_i(t)$ and $u(t)$ are given below. Define the Bezier polynomials of degree n over the interval $[t_0, t_f]$ as follows:

$$x_i(t) = \sum_{r=0}^{n} a_r^i B_{r,n}(\frac{t-t_0}{h}), \tag{1}$$

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$$u(t) = \sum_{r=0}^{n} b_r B_{r,n}(\frac{t-t_0}{h}) \tag{2}$$

where $h = t_f - t_0$, and

$$B_{r,n}(\frac{t-t_0}{h}) = \binom{n}{r} \frac{1}{h^n}(t_f - t)^{n-r}(t-t_0)^r,$$

is the Bernstein polynomial of degree n over the interval $[t_0, t_f]$, a_r^i and b_r are the control points. By substituting $x_i(t)$ and $u(t)$ in the optimal control of a direct oscillator with the differential equation, one may solve these problems by Maple 15.
Ghomanjani et al. [1] demonstrated the convergence of this technique when $n \to \infty$.

Remark 2.1. The optimal control of a linear oscillator with the differential equation will be considered

$$u(t) \quad = \quad \ddot{x}(t) + w^2 x(t), \quad t \in [-T, 0] \tag{3}$$

when T is specified. Eq. (3) is equivalent to the following state equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -w^2 x_1(t) + u(t), \\ x_1(-T) &= x_0, \quad x_2(-T) = \dot{x}_0, \\ x_1(0) &= 0, \quad x_2(0) = 0. \end{aligned}$$

with the following cost function

$$J \quad = \quad \frac{1}{2}\int_{-T}^{0} u^2(t)\, dt,$$

when it has the following exact solution (see [2]):

$$\begin{aligned} x_1(t) &= \frac{1}{2w^2}[Awt\sin(wt) + B(\sin(wt) - wt\cos(wt))], \\ x_2(t) &= \frac{1}{2w}[A(wt\sin(wt) + wt\cos(wt)) + Bwt\sin(wt)], \\ u(t) &= A\cos(wt) + B\sin(wt), \\ J &= \frac{1}{8w}[2wT(A^2+B^2) + (A^2-B^2)\sin(2wT) - 4AB\sin^2(wT)], \end{aligned}$$

where

$$\begin{aligned} A &= \frac{2w[x_0 w^2 T\sin(wT) - \dot{x}_0(wT\cos(wT) - \sin(wT))]}{w^2T^2 - \sin^2(wT)} \\ B &= \frac{2w^2[\dot{x}_0 T\sin(wT) + x_0(wT\cos(wT) + \sin(wT))]}{w^2T^2 - \sin^2(wT)} \end{aligned}$$

3 Numerical applications

Now, some numerical examples are explained.

Example 3.1. Consider the following problem

$$\begin{aligned} &\min J = \frac{1}{2}\int_0^2 u^2(t)\ dt, \\ &s.t.\ \ u(t) = \ddot{x}(t) + \dot{x}(t), \\ &\quad x(0) = 0,\ x(2) = 5,\ \dot{x}(0) = 0,\ \dot{x}(2) = 2, \end{aligned}$$

when the exact solutions for $x(t)$ and $u(t)$ are

$$\begin{aligned} x(t) &= -6.103 + 7.289t + 6.696e^{-t} - 0.593e^{t}, \\ u(t) &= 7.289 - 1.186e^{t}, \end{aligned}$$

utilizing the proposed strategy with $n = 10$, one may have $J = 16.75072576$ where the exact value is $J_{exact} = 16.74543860$, and

$$\begin{aligned} x_{approx}(t) &= -1.215249320t^3 - 0.6028389216 * 10^{-1}t^5 + 0.7871472800 * 10^{-2}t^6 \\ &- 0.9888591200 * 10^{-3}t^7 - 0.5385276000 * 10^{-4}t^8 + 0.3124650000 * 10^{-4}t^9 \\ &- 0.4281570000 * 10^{-5}t^10 + .2541508128t^4 + 3.052409559t^2 \\ &- 1.659151229 * 10^{-307} \\ u_{approx}(t) &= -1.186676818t - .1890746400t^3 + 0.5697011680 * 10^{-2}t^5 \\ &- 0.9937861840 * 10^{-2}t^6 + 0.1818916840 * 10^{-2}t^7 - 0.1041147500 * 10^{-3}t^8 \\ &- 0.4281581000 * 10^{-4}t^9 - 0.6527546688 * 10^{-1}t^4 - .5959381133t^2 \\ &+ 6.104819121 \end{aligned}$$

The graphs of approximated and exact solution $x(t)$ and $u(t)$ are plotted respectively in Figs. 1, 2.

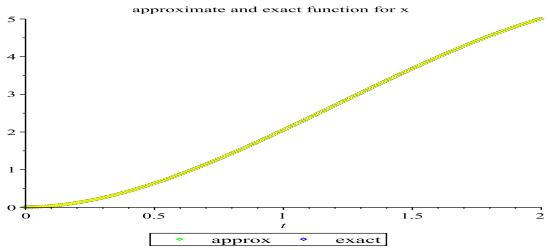

Figure 1: The graph of approximated and exact solution for Example 3.1

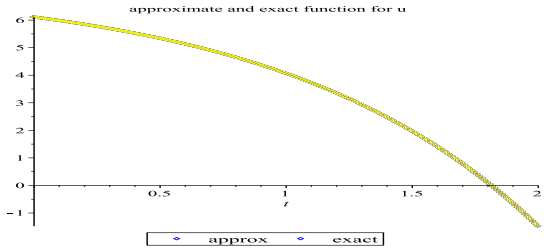

Figure 2: The graph of approximated and exact solution for Example 3.1

Example 3.2. Consider the following problem

$$\begin{aligned}
&\min J = \frac{1}{2}\int_{-2}^{0} u^2(t)\ dt,\\
&s.t.\ \ u(t) = \ddot{x}(t) + x(t),\\
&\quad x(-2) = 0.5,\ x(0) = 0,\ \dot{x}(-2) = -0.5,\ \dot{x}(0) = 0,
\end{aligned}$$

utilizing the proposed technique with $n = 10$, one may have $J = 0.1848585418$ where the exact value is $J_{exact} = 0.1848585422$ and

$$\begin{aligned}
x_{approx}(t) &= 7.062378300 * 10^{-7} * t^{10} + 0.6874780960 * 10^{-5}t^8 + 0.1194692496 * 10^{-3}t^6\\
&+ 0.9304003280 * 10^{-5}t^9 - 0.3102224362 * 10^{-3}t^7 + 0.9160017711 * 10^{-2}t^5\\
&- 0.9147355857 * 10^{-1}t^3 - 0.2017277233 * 10^{-2}t^4 + 0.1213363708 * 10^{-1}t^2,\\
u_{approx}(t) &= 7.062378300 * 10^{-7}t^{10} + 0.7043560663 * 10^{-4}t^8 + 0.5044622766 * 10^{-3}t^6\\
&- .5488413512t + 0.9304003280 * 10^{-5}t^9 + 0.3596649426 * 10^{-3}t^7\\
&- 0.3869309236 * 10^{-2}t^5 + 0.9172680103 * 10^{-1}t^3 + 0.1566814558 * 10^{-2}t^4\\
&- 0.1207368883 * 10^{-1}t^2 + 0.2426727416 * 10^{-1}.
\end{aligned}$$

The graphs of approximated and exact solution $x(t)$ and $u(t)$ are plotted respectively in Figs. 3, 4.

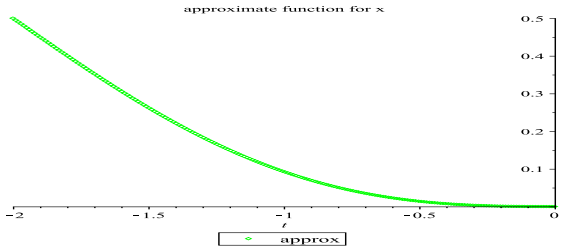

Figure 3: The graph of approximated solution for Example 3.2

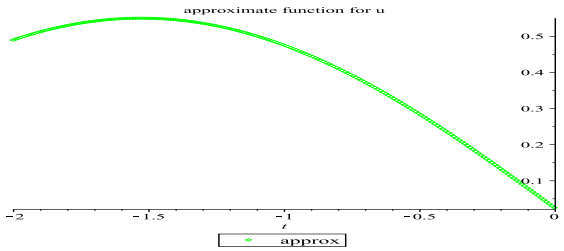

Figure 4: The graph of approximated solution for Example 3.2

4 Conclusion

This paper presents a numerical technique for solving non linear OCPs by Bezier curve technique. The efficiency of the method was obtained by some numerical examples.

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Extremal transmission irregular trees with respect to Wiener index

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Abstract

Let G be a graph. G is called transmission irregular (shortly TI graph) graph if no two row sum are the same in the distance matrix of G. A family of TI trees with three branches are presented and extremal TI trees with respect to the Wiener index are determined.

Keywords: Distance matrix, Wiener index, transmission irregular graph;
Mathematics Subject Classification [2010]: 05C35, 05C12

1 Introduction

throughout the paper we consider only simple connected graphs. Let $G(V,E)$ be a graph of order n with vertex set $V(G) = \{v_1, v_2 \cdots v_n\}$. Degree of vertex v, $deg(v)$ is the number of vertices adjacent to v. Adjacency matrix of G is a square matrix of order n whose ij-th entry A_{ij} is 1 if vertices v_i and v_j are adjacent else $A_{ij} = 0$. A k-regular graph is a graph whose all vertices have same degree k. In the other words sum of entries of each column or row equals k. Let $\Delta(G)$ denotes a diagonal matrix whose i-th entry is $deg(v_i)$. Laplacian matrix of G, $L(G)$ definesd as $L(G) = \Delta(G) - A(G)$. Let $\lambda_1 \geq \lambda_2 \cdots \lambda_n$ be eigenvalues of $L(G)$. It is well known fact that $\lambda_n = 0$ for all graphs and for connected graphs $\lambda_{n-1} > 0$ that is named algebraic connectivity. The theory of Laplacian spectra of graphs has been extensively studied. Readers reffered to [9,11] for a review. Let v and w be two vertices of G. distance $d(u,v)$ is the length of shortest path connecting u to v. Distance matrix of G, $D(G)$ is a square matrix of order n with $D_{ij} = d(v_i, v_j)$. Transmission of v, $Tr_G(v)$ is sum of distances between v and other vertices of G. Abviously sum of entries $i-$th row or column of distance matrix equals transmission of v_i. Let Γ be set of all graphs. A topological index, I is function from Γ to real numbers such that if G and H are two isomorphic graph then $I(G) = I(H)$. A well known topological index based on distance in graph is Wiener index introduced as sum of all distances between pair vertices of a graph.

$$W(G) = \sum_{\{u,v\} \subset V(G)} d(u,v).$$

The Wiener index can be presented as

$$W(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} D_{ij} = \frac{1}{2} \sum_{v \in V(G)} Tr(v)$$

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An interesting result with respect to Wiener index and Laplacian spectrum on trees was communicated independently in [8, 10–12] as follows:

Theorem 1.1. *Let T be a tree of order n with Laplacian spectrum $\lambda_1 \geq \lambda_2 \cdots \lambda_n$ then*

$$W(T) = \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{\lambda_i}$$

Wiener complexity of G, $C_w(G)$ is the number of different vertex transmissions of G. A graph with $C_w(G) = 1$ is called transmission regular graph. Also a graph is called tarnsmission irregular graph if each two vertices have different transmission. In the other words $C_w(G) = n$ where n denotes the order of G. It is well knwon fact proved by Moor and Moser [13] that allmost all graph are of diameter 2. Moreover we have

Lemma 1.2. *if G is a graph of diameter at most 2, then G is regular if and only if G is transmission regular.*

Theorem 1.3. *If G is a k-regular graph of diameter at most 2. Then*

$$\begin{aligned} trace(A^2) &= nk, \\ trace(D^2) &= n(4n - 3k - 4) = 2W(G) + 2n(n - k - 1). \end{aligned}$$

2 Transmission irregular graphs

An automorphism of a graph preserves the distance function. Hence, if u and v are vertices of a graph G such that $\alpha(u) = v$ holds for some $\alpha \in Aut(G)$, then $Tr(u) = Tr(v)$. It follows that a transmission irregular graph is asymmetric and, as it well known, almost all graphs are asymmetric [7]. On the other hand, the fraction of transmission irregular graphs among asymmetric graphs is small as the next result asserts.

Theorem 2.1. *Almost all graphs are not transmission irregular.*

Recently transmission irregular graphs has been interesting and several infinite family of such graphs were constructed. For instance TI starlike trees with three branches characterized in [1,2]. Moreover Dobrynin constructed several infinite family of TI trees of even order [3] 2-connected TI graphs [4,5] and 3-connected TI graphs [6]. The TI star like $T_{n_1 n_2 n_3}$, which $1 \leq n_1 \leq n_2 \leq n_3$ introduced in [1] has vertex set $\{u\} \cup \{x_1, x_2, \cdots x_{n_1}\} \cup \{y_1, y_2, \cdots y_{n_2}\} \cup \{z_1, z_2, \cdots z_{n_3}\}$ and the edge set$\{ux_1, x_1x_2, \cdots x_{n_1-1}x_{n_1}\} \cup \{uy_1, y_1y_2, \cdots y_{n_2-1}y_{n_2}\} \cup \{uz_1, z_1z_2, \cdots z_{n_3-1}z_{n_3}\}$ TI star like

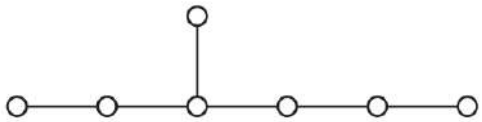

Figure 1: TI tree $T_{1,2,3}$

with three branches were determined in [1] as follows.

Theorem 2.2. *If $1 = n_1 < n_2 < n_3$ then T_{1,n_2,n_3} is transmission regular graph if and only if $n_3 = n_2 + 1$ and $n_2 \notin \{\frac{k^2-1}{2}, \frac{k^2-2}{2}\}$ for some integer $k \geq 3$*

We conclude the section with the following result that in a way support the theorem 2.2.

Theorem 2.3. *If a graph G has three vertices of the same degree, then not both G and $\bar{G}$ are transmission irregular.*

Amog asymmetric trees on same order n, trees $T_{1,2,n-4}$ get the maximum Wiener index.

Theorem 2.4. *Let T be a tree of order n. Then*

$$W(T) \leq \frac{1}{6}(n^3 - 13n + 48)$$

with equality holding if and only if $T = T_{1,2,n-4}$.

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New inequalities on the weighted geometric mean of accreative matrices through non-standard domains

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Abstract

In this article we have proved some of the inequalities for accreative matrices through non-standard domains which have recently proved these inequalities for standard domains accreative matrices.

Keywords: accreative matrices, Hermitian matrices, matrix mean, inequality
Mathematics Subject Classification [2010]: 15A02, 15B48, 47A63

1 Introduction

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices. For Hermitian matrices $A, B \in \mathbb{M}_n$, we write that $A \geqslant 0$ if A is positive semidefinite, i.e. if $\langle Ax, x\rangle \geqslant 0$ for all vectors $x \in \mathbb{C}^n$. We also write $A > 0$ if A is positive definite, i.e. if $\langle Ax, x\rangle > 0$ for all vectors $x \in \mathbb{C}^n$, and $A \geqslant B$ if $A - B \geqslant 0$.

A matrix $A \in \mathbb{M}_n$ is called accretive if in its Cartesian (or Toeplitz) decomposition, $A = \mathcal{R}z + i\mathcal{I}z$, $\mathcal{R}z$ is positive definite, where $\mathcal{R}z = \frac{A+A^*}{2}$, $\mathcal{I}z = \frac{A-A^*}{2}$.
Later, Raissouli et. al. [4] defined the following weigheted geometric mean of two accretive matrices $A, B \in \mathbb{M}_n$,

$$A\sharp_\nu B = \frac{\sin \nu\pi}{\pi} \int_0^1 t^{\nu-1}(A^{-1} + tB^{-1})^{-1} \frac{dt}{t}$$

Raissouli et al. in [4] showed that if $A, B \in \mathbb{M}_n$ are accretive and $r \in (0, 1)$. Then

$$A\sharp_r B = B\sharp_{1-r} A \tag{1}$$

and

$$(\alpha A)\sharp_r(\beta B) = (\alpha\sharp_r\beta)(A\sharp_r B) \tag{2}$$

Bedrani et al. [2] showed if $A, B \in \mathbb{M}_n$ be accretive and $r \in (1, 2)$, then

$$A\sharp_r B = B(A\sharp_{2-r} B)^{-1}B \tag{3}$$

and if $r \in (-1, 0)$, then

$$A\sharp_r B = A(A\sharp_{-r} B)^{-1}A \tag{4}$$

For operator mean of accretive operators, we have the following result.

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Lemma 1.1. *[2] Let $A \in \mathbb{M}_n$ be accretive and $r \in (1,2)$. Then*

$$A\sharp_r B = \int_0^1 \left((1-s)B^{-1} + sB^{-1}AB^{-1}\right)^{-1} d\mu(s),$$
$$\text{where } \; d\mu(s) = \frac{\sin(r-1)\pi}{\pi} \frac{s^{r-2}}{(1-s)^{r-1}} ds.$$

Lemma 1.2. *[2] Let $A \in \mathbb{M}_n$ be accretive and $r \in (-1,0)$. Then*

$$A\sharp_r B = \int_0^1 \left((1-s)A^{-1}BA^{-1} + sA^{-1}\right)^{-1} d\nu(s),$$
$$\text{where } \; d\nu(s) = \frac{\sin(r+1)\pi}{\pi} \frac{s^{r}}{(1-s)^{r+1}} ds.$$

Lemma 1.3. *[2] Let $A \in \mathbb{M}_n$ be accretive and $r \in (-1,0) \cup (1,2)$. Then*

$$\mathcal{R}(A\sharp_r B) \leq \mathcal{R}A\sharp_r \mathcal{R}B$$

2 Main results

If $A \in \mathbb{M}_n$ are positive semidefinite and $r \in (0,1)$ is a real number, then equality $A\sharp_r B = B\sharp_{1-r}A$ is know. Raissouli et al. in [4] showed that if $A \in \mathbb{M}_n$ are accretive and $r \in (0,1)$, we still have $A\sharp_r B = B\sharp_{1-r}A$. In this section, I want to show that if $A \in \mathbb{M}_n$ are accretive and $r \in (-1,0) \cup (1,2)$, we also have this equality.

Proposition 2.1. *Let $A \in \mathbb{M}_n$ be accretive and $r \in (-1,0) \cup (1,2)$. Then*

$$A\sharp_r B = B\sharp_{1-r}A$$

Proof. If $r \in (-1,0)$, then

$$\begin{aligned} A\sharp_r B &= \int_0^1 \left((1-s)A^{-1}BA^{-1} + sA^{-1}\right)^{-1} d\nu(s) && \text{(by Lemma 1.2)} \\ &= \int_0^1 \left((1-s)A^{-1}BA^{-1} + sA^{-1}\right)^{-1} d\mu(s) \\ &= B\sharp_{1-r}A && \text{(by Lemma 1.1).} \end{aligned}$$

□

Proposition 2.2. *Let $A \in \mathbb{M}_n$ be accretive and $r \in (1,2)$. Then*

$$(\alpha A)\sharp_r B = \alpha^{r-1}(A\sharp_r B).$$

Proof. If $r \in (1,2)$, then $2-r \in (0,1)$ and we have

$$\begin{aligned} \alpha A\sharp_r B &= B(\alpha A\sharp_{2-r}B)^{-1}B && \text{(by (3))} \\ &= B\left(\alpha^{r-1}(A\sharp_{2-r}B)\right)^{-1}B && \text{(by (2))} \\ &= \alpha^{r-1}B(A\sharp_{2-r}B)^{-1}B \\ &= \alpha^{r-1}(A\sharp_r B) && \text{(by (3)),} \end{aligned}$$

□

Proposition 2.3. *Let $A \in \mathbb{M}_n$ be accretive and $r \in (-1,0)$. Then*

$$(\alpha A)\sharp_r B = \alpha^{r-1}(A\sharp_r B).$$

Proof. If $r \in (-1,0)$, then $-r \in (0,1)$ and we have

$$\begin{aligned}
\alpha A\sharp_r B &= \alpha A(\alpha A\sharp_{-r} B)^{-1}\alpha A && \text{(by (4))}\\
&= \alpha^2 A(\alpha A\sharp_{-r} B)^{-1}A\\
&= \alpha^2 A\left((\alpha^{1+r}A\sharp_{-r}B)\right)^{-1}A && \text{(by (2))}\\
&= \alpha^2\alpha^{-1-r}A(A\sharp_{-r}B)^{-1}A\\
&= \alpha^{1-r}A(A\sharp_{-r}B)^{-1}A\\
&= \alpha^{1-r}(A\sharp_r B) && \text{(by (4))}
\end{aligned}$$

□

Next, we present following theorem that is an analogue of [4, Proposition 4.1].

Theorem 2.4. *Let $A \in \mathbb{M}_n$ be accretive and $r \in (-1,0)\cup(1,2)$. Then*

$$(\alpha A)\sharp_r(\beta B) = (\alpha\sharp_r\beta)A\sharp_r B.$$

Proof. Let $r \in (1,2)$. Then

$$\begin{aligned}
(\alpha A)\sharp_r(\beta B) &= \alpha^{r-1}(A\sharp_r\beta B) && \text{(by Proposition 2.2)}\\
&= \alpha^{r-1}(\beta B\sharp_{1-r}A) && \text{(by Proposition 2.1)}\\
&= \alpha^{r-1}\beta^r(B\sharp_{1-r}A) && \text{(by Proposition 2.3)}\\
&= \alpha^{r-1}\beta^r(A\sharp_r B) && \text{(by Proposition 2.1).}
\end{aligned}$$

If $r \in (-1,0)$, then $1-r \in (1,2)$ so proof is complete. □

Remark 2.5. Bakherad and Moslehian in [1] proved that if $a,b>0$ and $r \notin [0,1]$, then

$$ra + (1-r)b \le a^r b^{1-r}, \tag{5}$$

so if $a,b>0$ and $r \in (-1,0)\cup(1,2)$. Then

$$a!_r b = ((1-r)a^{-1} + rb^{-1})^{-1} \ge (a^{-r}b^{r-1})^{-1} = a^{1-r}b^r. \tag{6}$$

Therefore if $A \in \mathbb{M}_n$ be accretive and $r \in (-1,0)\cup(1,2)$ by applied (5) and (6) we have

$$\mathcal{R}A\sharp_r\mathcal{R}B \le \mathcal{R}A!_r\mathcal{R}B,$$

finally by Lemma 1.3 we will have

$$\mathcal{R}(A\sharp_r B) \le \mathcal{R}A\sharp_r\mathcal{R}B \le \mathcal{R}A!_r\mathcal{R}B. \tag{7}$$

Remark 2.6. Fujii [3] proved that if $A,B \in \mathbb{M}_n^+$, then

$$\Phi(A\sharp_r B) \ge \Phi(A)\sharp_r\Phi(B), \quad r \in (-1,0),$$

it holds, for any positive unital linear map Φ.
This inequality also holds for $r \in (1,2)$, because by Proposition 2.1 if $r \in (1,2)$ we have

$$\Phi(A\sharp_r B) = \Phi(B\sharp_{1-r}A) \ge \Phi(B)\sharp_{1-r}\Phi(A) = \Phi(A)\sharp_r\Phi(B).$$

Therefore if $A, B \in \mathbb{M}_n^+$ and $r \in (-1,0) \cup (1,2)$, then

$$\Phi(A\sharp_r B) \geq \Phi(A)\sharp_r \Phi(B),$$

in particular, with $C \geq 0$ and $\Phi(X) = Tr(XC)$, one has

$$Tr(C.(A\sharp_r B)) \geq Tr(CA)\sharp_r Tr(CB)$$

from which it follows that for every unitarily invariant norm $\|.\|$ we get

$$\|A\sharp_r B\| \geq \|A\|\sharp_r \|B\|. \tag{8}$$

As an application of Theorem 2.4, we have the following inequality.

Theorem 2.7. *Let $A \in \mathbb{M}_n$ be accretive and $r \in (-1,0) \cup (1,2)$. Then*

$$\sum_{k=1}^{n} \langle (\Re(A\sharp_r B))^{-1} x_k, x_k\rangle \geq \left(\sum_{k=1}^{n} \langle (\Re A)^{-1} x_k, x_k\rangle\right) \sharp_r \left(\sum_{k=1}^{n} \langle (\Re B)^{-1} x_k, x_k\rangle\right),$$

for any family of vectors $(x_k)_{k=1}^n \in H$.

Proof. Using Theorem 2.4, (7) and a method similar to the proof of [4, Theorem 4.2] is proved. □

As a consequence of the Theorem 2.7, we have the following theorem.

Theorem 2.8. *Let $A \in \mathbb{M}_n$ be accretive and $r \in (-1,0) \cup (1,2)$. Then*

$$\| (\Re(A\sharp_r B))^{-1} \| \geq \|(\Re A)^{-1}\|^{1-r} \|(\Re B)^{-1}\|^r.$$

Proof. Taking the supremum over $\|x\| = 1$ of the latter inequality implies

$$\begin{aligned}
\| (\Re(A\sharp_r B))^{-1} \| &= \sup\langle \Re(A\sharp_r B)x, x\rangle \\
&\geq \sup\langle (\Re A)^{-1} x_k, x_k\rangle \sharp_r \langle (\Re B)^{-1} x_k, x_k\rangle && \text{(by Theorem 2.7)} \\
&= \|(\Re A)^{-1} \sharp_r (\Re B)^{-1}\| \\
&\geq \|(\Re A)^{-1}\| \sharp_r \|(\Re B)^{-1}\| && \text{(by (8))} \\
&= \|(\Re A)^{-1}\|^{1-r} \|(\Re B)^{-1}\|^r.
\end{aligned}$$

This completes the proof. □

3 Conclusion

In this paper, we have shown some of the inequalities that exist for accreative matrices that included the arithmetic mean of the standard domain in the opposite direction to the non-standard domain.

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Eigencone of Nonnegative Irreducible Matrix in Max Algebra

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Abstract

The aim of this paper is to calculate the eigencone of nonnegative irreducible matrix in max algebra with new method. In this method we use elements of critical graph and study some of spectral properties.

Keywords: Max algebra, max eigenvalue, max eigenvector, critical graph
Mathematics Subject Classification [2010]: 15A18, 15A80, 15B48

1 Introduction

Max algebra is the set of nonnegative real numbers $\mathbb{R}_+$, equipped with the operations of addition $a \oplus b = \max(a, b)$ and classical multiplication ab. As in standard arithmetic, the operations of addition and multiplication are associative and commutative, and multiplication is distributive over addition. Matrix and polynomial operations are defined similarly to their standard counterparts, with the max operation replacing the standard summation.

In particular, for two $n \times n$ nonnegative matrices A and B the max algebra product $A \otimes B$ is defined by

$$(A \otimes B)_{ij} = \max_{l=1,\dots,n} a_{il} b_{lj}$$

for all $i, j = 1, \dots, n$ (Recall that a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, If for $1 \le i, j \le n$, $a_{ij} \ge 0$ then A is said to be a *nonnegative* matrix and we write $A \ge 0$). The mth power in max algebra of A is denoted by $A^k_{\otimes}$. More precisely,

$$[A^k_{\otimes}]_{ij} = \max_{i_1,\dots,i_{k-1}\in\{1,\dots,n\}} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j}$$

for all $i, j = 1, \dots, n$ (and thus $[A^k_{\otimes}]_{ij}$ equals the heaviest path from i to j in a suitably defined graph associated to A. Let $A = [a_{ij}] \in M_n(\mathbb{R}_+)$. The weighted directed graph associated with A is denoted by $D(A) = (V, E)$, where A has vertex set $V = \{1, 2, \dots, n\}$ and edge (i, j) from i to j with weight a_{ij} if and only if $a_{ij} > 0$. A circuit of length k is a sequence of k edges $(i_1, i_2), \dots, (i_k, i_1)$, where $i_1, i_2, \dots, i_k$ are distinct (a circuit (i, i) of length one which is called a *loop* [4]). The k^{th} positive root of *circuit product* $a_{i_1,i_2} \dots a_{i_k,i_1}$ is called a *circuit geometric mean* of matrix A. The maximum circuit geometric mean in D(A) is denoted by $\mu(A)$. It is known that $\mu(A)$ is the largest max-eigenvalue of A. A circuit with circuit geometric mean equal to $\mu(A)$ is called a *critical circuit* with symbol $\mathcal{C}(A)$, and vertices on $\mathcal{C}(A)$ are called *critical vertices*.

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The maximum circuit geometric mean in D(A) is denoted by $\mu(A)$. It is known that $\mu(A)$ is the largest max-eigenvalue of A. A circuit with circuit geometric mean equal to $\mu(A)$ is called a *critical circuit*, and vertices on critical circuits are called *critical vertices*.

Definition 1.1. The *critical matrix* of A, denoted by $A^c = [a_{ij}^c]$, is formed from the principal submatrix of A on the rows and columns corresponding to critical vertices by setting a_{ij}^c, where

$$a_{ij}{}^c = \begin{cases} a_{ij} & \text{if } (i,j) \text{ is in a critical circuit ,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus the critical graph $D(A^c)$ has vertex set V^c of all critical vertices.

A matrix $A = [a_{ij}]$ is said to have a *strongly connected* (SC) property, if for every pair of distinct p, q with $1 \leq p, q \leq n$ there is a sequence of distinct integers $k_1 = p, k_2, \ldots, k_{m-1}, k_m = q$, $1 \leq m \leq n$, such that all of the matrix entries $a_{k_1k_2}, a_{k_2k_3}, \ldots, a_{k_{n-1}k_n}$ are nonzero. A directed graph $D(A)$ is *strongly connected* if between every pair of distinct nodes p_i, p_j in $D(A)$ there is a directed path of finite length that begins at p_i and ends p_j.

Theorem 1.2. *[4] Let $A \in M_n, n \geq 2$. The following are equivalent:*

1. *A is irreducible.*
2. *$D(A)$is strongly connected.*
3. *A has SC property.*

The role of irreducible matrices is very important in this paper. Therefore, we will first recall these matrices.

Definition 1.3. Matrix $A \in M_n(\mathbb{R}_+)$ is called reducible matrix if either $n = 1$ and $A = 0$ or if $n \geq 2$, there is a permutation matrix $P \in M_n$, and there is an integer r with $1 \leq r \leq n-1$ such that

$$P^T AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where $B \in M_r(\mathbb{R}_+)$, $D \in M_{n-r}(\mathbb{R}_+)$, $C \in M_{r\times(n-r)}(\mathbb{R}_+)$, and $0 \in M_{(n-r)\times r}(\mathbb{R})$ is a zero matrix. Matrix $A \in M_n(\mathbb{R}_+)$ is said to be irreducible if it is not reducible.

2 Main result

The problem of eigenvalue in Max algebra is expressed as follows.

$$A \otimes x = \lambda x$$

The set of max eigenvalue of A denoted by $\sigma_\otimes(A)$ and the greatest such eigenvalue will be denoted by $\mu(A)$ and called the *principal max eigenvalue* of A. We have prove new explicit asymptotic formulae in [5] between max eigenvalues in max-algebra and classical distinguished eigenvalues of nonnegative matrices, which are useful tools for transferring results between both settings.The set of max eigenvector of x associated with λ, with the zero vector adjoined to it, is called *max eigencone*. It is denoted by $V_\oplus(A, \lambda)$. General A may have several max eigenvalues, but if A is irreducible A has only unique eigenvalue with non-negative eigenvector. In this case the unique max eigenvalue of A is also called the max Perron root. We refer to the next result as the max version of the Perron-Frobenius theorem.

Theorem 2.1. *[1] Let A be an $n \times n$ nonnegative irreducible matrix. Then there exists a positive vector x such that $A \otimes x = \mu(A)x$.*

The set $S \subset \mathbb{R}_+^n$ is called a max-algebraic cone if $\alpha u \oplus \beta v \in S$ for every $u, v \in S$ and $\alpha, \beta \in \mathbb{R}_+$. A vector $v = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}_+^n$ is called a max-combination of S if

$$v = \overset{\oplus}{\sum_{x \in S}} \alpha_x x, \quad \alpha_x \in \mathbb{R}_+,$$

where only a finite number of α_x are non-zero. The set of all max-combinations of S is denoted by $\mathrm{span}(S)$. We set $\mathrm{span}(\emptyset) = \{0\}$. It is clear that $\mathrm{span}(S)$ is a max-algebraic cone. If $\mathrm{span}(S) = T$, then S is called the set of generators for T.

A vector $v \in S$ is called an *extremal* in S if $v = u \oplus w$ for $u, w \in S$ implies $v = u$ or $v = w$. Clearly, if $v \in S$ is an extremal in S and $\alpha \in \mathbb{R}_+$, then αv is also extremal in S. The set S is called *dependent* if v is a max-combination of $S - \{v\}$ for some $v \in S$. Otherwise S is independent. Let $S, T \subseteq \mathbb{R}_+^n$. The set S is called a basis of T if it is an independent set of generators for T. In the article [6] that is being prepared, by presenting an algorithm have calculated the max eigenvalues and max eigenvectors for a nonnegative matrix A by its determinant. It was shown that we may have more than one critical circuits related to $\mu(A)$. In this case the max eigenvector corresponding to $\mu(A)$ is not unique, in fact we have more than one max independent max eigenvectors. Such max eigenvectors are called *principal* max eigenvectors (see e.g. [2, Chapters 3 and 4]). $P_{me}(A, \mu(A))$ shall denote the set of all principal max eigenvectors. It was shown in [6] corresponding each critical circuits $\mathcal{C}_s(A)$ of A there exists matrix A^* such that $\mu(A)$ is an eigenvalue of A^*.

Example 2.2. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Then we have $detA = a_{11}a_{22} - a_{12}a_{21} = 1 \times 1 - 2 \times 2$, so we have three critical circuits $a_{11}, a_{22}, a_{12}a_{21}$. Therefore since A is irreducible, we have only one max eigenvalue which is

$$\mu(A) = \max\{a_{11}, a_{22}.\sqrt{a_{12}a_{21}}\} = 2,$$

and we have $A^c = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$. To calculate max eigenvector x, using equation $A^*x = \mu(A)x$ which implies $x = (1, 1)^T$.

Unlike in nonnegative algebra, there is an explicit description of $V_{\oplus}(A, \mu(A))$. For a nonnegative matrix A the Kleene star matrix A is defined

$$\Delta(A) = I \oplus A \oplus A^{\otimes 2} \oplus A^{\otimes 3} \ldots . \tag{1}$$

Series (1) converges if and only if $\mu(A) \leq 1$, in which case $\Delta(A) = I \oplus A \oplus A^{\otimes 2} \oplus \cdots A^{\otimes(n-1)}$. Note that if $\mu(A) \neq 0$, then $\mu(\frac{1}{\mu(A)}A) = 1$, hence $\Delta(\frac{1}{\mu(A)}A)$ always converges [2]. We are interested to find relation between columns of $\Delta(A)$ and $P_{me}(A, \mu(A))$. In the above example, we have

$$\Delta(\frac{1}{2}A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Observe that both columns are equal max principal eigenvector x.

Example 2.3. Let

$$A = \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Then

$$\mu(A) = a_{11} = a_{22} = \sqrt{a_{12}a_{21}} = 1.$$

The principal max eigenvector of A is

$$x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The Kleene star matrix A is

$$\Delta(A) = I \oplus A = \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Observe that each column of $\Delta(A)$ is a multiple of x (principal max-eigenvector of A).

Example 2.4. Let

$$A = \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}.$$

Then

$$\mu(A) = \sqrt{a_{12}a_{21}} = a_{22} = a_{33} = 1.$$

Therefore we have two disjoint critical circuits which implies have two principal max eigenvectors

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ \frac{1}{2} \end{pmatrix}, \quad x_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

corresponding to $\mu(A) = 1$.

$$\Delta(A) = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix},$$

As we see, columns of $\Delta(A)$ are max eigenvectors. First and second columns of A are equal to x_1 and third columns is equal to x_2.

Example 2.5. Let

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then $r = 1$ and

$$\mu(A) = M_1 = a_{11} = 1.$$

Thus

$$A_1^* = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

The principal max eigenvector of A is

$$x_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

which is also max-eigenvector (eigenvector) of A_1^*.

$$\Delta(A) = I \oplus A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Observe that only first column of $\Delta(A)$ is a max-eigenvector of A which multiple of x_1 (principal max-eigenvector of A).

We can write the following proposition as a result of Theorem 6.2 from [3].

Proposition 2.6. *Let $A \in \mathbb{R}_+^{n\times n}$ be an irreducible matrix with $\mu(A) = 1$. Then $V_\oplus(A, \mu(A))$ is generated by columns of $\Delta(A)$ that are in $V_\oplus(A, \mu(A))$.*

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Decomposition of Invariant Derivations as the Tensor Sum Form

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Abstract

Let A and B be C^*-algebras over a field $\mathbb{F}$, we show that every $\mathbb{F}$-invariant derivation δ of $A \otimes B$ can be represented as tensor sum $\delta = \Delta \otimes id + id \otimes \nabla$ where id stands for the identity operator, Δ and ∇ are derivations on A, B, respectively.

Keywords: Derivation,Tensor Sum, Invariant Subspace.

Mathematics Subject Classification [2010]: 15A03, 15A23, 15B36 (At least one and at most three codes)

1 Introduction

Let A and B are C^*-algebras over $\mathbb{F}$. If A, B do not have identities, denote by $A \oplus \mathbb{F}$, $B \oplus \mathbb{F}$ the C^*-algebras obtained by adjoining an identity to A, B, respectively. Since an arbitrary C^*-norm on $A \otimes B$ can be extended to a C^*-norm on $(A \oplus \mathbb{F}) \otimes (B \oplus \mathbb{F})$, the restriction of the spatial C^*-norm on $(A \oplus \mathbb{F}) \otimes (B \oplus \mathbb{F})$ to $A \otimes B$ is the spatial norm on $A \otimes B$. Thus we may assume that A and B have identity elements. The linear span of elements of the form $(x \otimes \mu)(\lambda \otimes y)$, where $x \in A$, $y \in B$, and $\mu, \lambda \in \mathbb{F}$, with $(x \otimes \mu)(\lambda \otimes y) = \lambda x \otimes \mu y$ is equal to $A \otimes B$. Then $A \otimes \mathbb{F}$ and $\mathbb{F} \otimes B$ are embeddable in the tensor product $A \otimes B$. Tensor sum of operators can be thought of as an extension to infinite dimensional spaces of the traditional Kronecker sum of matrices on finite dimensional spaces [1,3]. The goal of this paper is to prove that every invariant derivation of $A \otimes B$ is the tensor sum of the derivations. From the nature of this result, and the relative simplicity of its proof, one would expect that it is known; however, we have not been able to find it in the literature among related results. Let us now outline the contents of this paper. The main topic of the paper is contained in section 2, and we proved the main theorem about an invariant derivation on a tensor product. In the end of paper we explore corollares, examples and more results of tensor sum. Niknam [4] proved in 1993 the operator $\Delta \otimes id + id \otimes \nabla$ is a $*-$derivation on $A \otimes B$, where δ, ∇ are $*-$derivations over A and B, respectively. In this paper we show that if δ is a derivation on tensor product $A \otimes B$ with invariant property has the following form.

$$\Delta \otimes id + id \otimes \nabla.$$

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In the next section we provide all lemmas and proof their. Let A be a C^*-algebra. Then a $*$-derivation δ of A means a linear mapping from A into A such that $\delta(xy) = \delta(x)y + x\delta(y)$, $\delta(x^*) = \delta(x)^*$, for every $x, y \in A$.
By $Der(A)$ we denote the set of all derivations of A. Further, for every $u \in A$ we define ad $u : A \longrightarrow A$ by ad $u(x) = ux - xu$. Note that ad $u \in Der(A)$ and such a derivation is said to be an inner derivation [4].
If A and B are normed spaces, and $u \in A \otimes B$, then there exist linearly independent sets $\{x_i\}$, $\{y_i\}$ such that $u = \sum_{i=1}^{n} x_i \otimes y_i$, see [2].

Lemma 1.1. *Let A be a normed space over a field $\mathbb{F}$. Then each element u of $A \otimes \mathbb{F}$ may be written uniquely in the form $u = x_u \otimes 1$, where $x_u \in A$. In particular, $\|u\| = \|x_u\|$.*

The following lemma will be needed in the proof of the main result.

Lemma 1.2. *Let A be C^*-algebra over a field $\mathbb{F}$, let δ be a $*$-derivation of $A \otimes \mathbb{F}$. Then there exists a $*$-derivation Δ of A such that for every $x \in A$ we have*

$$\delta(x \otimes 1) = \Delta(x) \otimes 1. \tag{1}$$

Furthermore, $\|\Delta\| \leq \|\delta\|$. Inparticular, if δ is closable, so is Δ.

A similar result same as the above lemma holds for a derivation δ of $\mathbb{F} \otimes B$, where B is a C^*-algebra over $\mathbb{F}$.
For example, if we put $y = 1$ in ad $u(x \otimes y) = \Delta(x) \otimes y$ for every $u \in A \otimes \mathbb{F}$, then ad $u = \Delta \otimes id$ is a derivation of $A \otimes \mathbb{F}$. If in addition δ is a derivation of $A \otimes \mathbb{F}$, then $\alpha\delta\alpha^{-1}$ is a derivation of A, where α is the isomorphism from $A \otimes \mathbb{F}$ onto A.

2 Main result

In this section, we peresent our main theorem. A mapping δ of tensor product $A \otimes B$ is called $\mathbb{F}$-invariant if $A \otimes \mathbb{F}$ and $\mathbb{F} \otimes B$ are invariant under δ [6].

Theorem 2.1. *Let A and B be C^*-algebras over $\mathbb{F}$. Then every $\mathbb{F}$-invariant derivation δ of $A \otimes B$ can be written as*

$$\delta = \Delta \otimes id + id \otimes \nabla,$$

where $\Delta \in Der(A)$, $\nabla \in Der(B)$.

3 corollaries

Corollary 3.1. *Let A, B be C^*-algebras. If every derivation of A and B are inner then every derivation of $A \otimes B$ is inner.*

An element z in $A \otimes B$ is said to be tensor sumable if there are x, y in A and B, respectively such that $z = x \otimes 1 + 1 \otimes y$.

Example 3.2.

The matrix $Z = \begin{pmatrix} 0 & 2 & 1 & 2 & 0 & 0 \\ -1 & 0 & 5 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 4 & 2 & 1 \\ 0 & 0 & 0 & -1 & 4 & 5 \\ 0 & 0 & 0 & -1 & 0 & 3 \end{pmatrix}$ is an element of $M_{2\times 2}(\mathbb{R}) \otimes M_{3\times 3}(\mathbb{R})$, where $M_{n\times n}(\mathbb{R})$ is the set of all $n \times n$ matrices over $\mathbb{R}$, hear $n = 2, 3$. If $X = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 5 \\ -1 & 0 & 0 \end{pmatrix}$, then $Z = X \otimes I_3 + I_2 \otimes Y$ is a tensor sumable in $M_{2\times 2}(\mathbb{R}) \otimes M_{3\times 3}(\mathbb{R})$.

Corollary 3.3. *If A, B are non commutative, then every inner derivation ad z of $A \otimes B$ where z is a tensor sumable, can be written as tensor sum of inner derivations.*

Corollary 3.4. *Let A, B be C^*-algebras. If Δ and ∇ are inner derivations over, A, B, respectively. Then $\Delta \otimes id + id \otimes \nabla = adz$ for some tensor sumable $z \in A \otimes B$.*

A derivation δ on $A \otimes B$ is called tensor sumable if there exist two derivations Δ, ∇ over A and B, respectively such that $\delta = \Delta \otimes id + id \otimes \nabla$ and we write $\delta = \Delta \boxplus \nabla$. Also, the tensor difference of Δ, ∇ is denoted by $\Delta \boxminus \nabla$ and it is definition as follows:

$$\Delta \boxminus \nabla = \Delta \otimes id - id \otimes \nabla$$

Theorem 3.5. *Let A and B are C^*-algebras over $\mathbb{F}$. Then for every $\alpha, \beta \in \mathbb{F}$, $\Delta_1, \Delta_2 \in Der(A)$ and $\nabla_1, \nabla_2 \in Der(B)$,*
(i) $\alpha\beta(\beta^{-1}\Delta_1 \boxplus \nabla_1\alpha^{-1}) = \alpha\Delta_1 \boxplus \nabla_1\beta$ where α, β are non zero,
(ii) $\Delta_1 \boxplus \nabla_1 + \Delta_2 \boxplus \nabla_2 = \Delta_1 \boxplus \nabla_2 + \Delta_2 \boxplus \nabla_1$,
(iii) $\alpha(\Delta_1 \boxplus \nabla_1)\beta = \alpha\Delta_1 \boxplus \nabla_1\beta$,
(iv) $\Delta_1 \boxplus \Delta_1 = \Delta_1 \boxplus id + id \boxplus \Delta_1 - id \boxplus id$,
(v) $\Delta_1 \boxplus \nabla_1 = \Delta_1 \otimes \nabla_1$ iff $\Delta_1 \otimes id$ is a quasi-inverse of $id \otimes \nabla_1$,
(vi) $\|\Delta_1 \boxplus \nabla_1\| = \|\Delta_1\|\|\nabla_1\|$ if and only if $\Delta_1 \otimes id$ is a quasi-inverse of $id \otimes \Delta_1$,
(vii) If $\Delta \neq \lambda id$ for every non zero scalar $\lambda \in \mathbb{F}$, then $\Delta \boxplus \nabla \neq 0$,
(viii) $-(\Delta_1 \boxplus \nabla_1) = -\Delta_1 \boxplus -\nabla_1$.
(ix) If Δ_1 and ∇_1 are normal, so is $\Delta \boxplus \nabla$.

Theorem 3.6. *Let Δ, ∇ be $*$-derivations on C^*-algebras A, B respectively. Then $\Delta \boxminus \nabla$ is a $*$-derivation on $A \otimes B$, and $(\Delta \boxplus i\nabla)^* = \Delta^* \boxminus i\nabla^*$. Furthermore,*

$$(\Delta \boxplus \nabla)(\Delta \boxminus \nabla) = \Delta^2 \boxminus \nabla^2$$

4 Conclusion

Let us suppose that δ be an $\mathbb{F}$-invariant derivation on $A \otimes B$ and restrictions of δ to $A \otimes \mathbb{F}$, $\mathbb{F} \otimes B$ be δ_1, δ_2, respectively. Since $A \otimes \mathbb{F}$ and $\mathbb{F} \otimes B$ are isomorphic to A, B, respectively, then there exist isomorphisms maps $\alpha : A \otimes \mathbb{F} \longrightarrow A$ given by $\alpha(x \otimes r) = rx$ and $\beta : \mathbb{F} \otimes B \longrightarrow B$ given by $\beta(s \otimes y) = sy$. Take $\Delta = \alpha\delta_1\alpha^{-1}$, and $\nabla = \beta\delta_2\beta^{-1}$, therefore $\delta = \Delta \otimes id + id \otimes \nabla$. then Δ and ∇ are derivation, for if $x, x' \in A$, then the liniearity of Δ follows immediately that $\alpha\delta_1\alpha^{-1}$ is a linear operator. It is enough to show that Δ satisfies Leibniz rule. To this

$$\Delta(xx') = (\alpha\delta_1\alpha^{-1})(xx') = (\alpha\delta_1)(\alpha^{-1}(x).\alpha^{-1}(x'))$$

$$
\begin{aligned}
&= \alpha(\delta_1(\alpha^{-1}(x))\alpha^{-1}(x') + \alpha^{-1}(x)\delta_1(\alpha^{-1}(x'))) \\
&= \alpha(\delta_1(x \otimes 1)(x' \otimes 1) + (x \otimes 1)\delta_1(x' \otimes 1)) \\
&= \alpha\delta_1(x \otimes 1)\alpha(x' \otimes 1) + \alpha(x \otimes 1)\alpha\delta_1(x' \otimes 1) \\
&= \alpha\delta_1\alpha^{-1}(x)x' + x\alpha\delta_1\alpha^{-1}(x') \\
&= \Delta(x)x' + x\Delta(x').
\end{aligned}
$$

Similarly, we can show that ∇ is a derivation on B. Now we prove $\delta = \Delta \otimes id + id \otimes \nabla$. Let $x \otimes y$ be an arbitrary elemen in $A \otimes B$ we have

$$
\begin{aligned}
(\Delta \otimes id + id \otimes \nabla)(x \otimes y) &= \Delta(x) \otimes y + x \otimes \nabla(y) \\
&= \alpha\delta_1\alpha^{-1}(x) \otimes y + x \otimes \beta\delta_2\beta^{-1}(y) \\
&= \alpha\delta_1(x \otimes 1) \otimes y + x \otimes \beta\delta_2(1 \otimes y) \\
&= \alpha(a \otimes 1) \otimes y + x \otimes \beta(1 \otimes b),
\end{aligned}
$$

where $\delta_1(x \otimes 1) = a \otimes 1$ and $\delta_2(1 \otimes y) = 1 \otimes b$ (such elements exist for invariance of δ). Hence

$$
\begin{aligned}
\alpha(a \otimes 1) \otimes y + x \otimes \beta(1 \otimes b) &= a \otimes y + x \otimes b \\
&= (a \otimes 1)(1 \otimes y) + (x \otimes 1)(1 \otimes b) \\
&= \delta(x \otimes 1)(1 \otimes y) + (x \otimes 1)\delta(1 \otimes y) \\
&= \delta((x \otimes 1)(1 \otimes y)) \\
&= \delta(x \otimes y).
\end{aligned}
$$

Thus, $\delta = \Delta \otimes id + id \otimes \nabla$.

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Numerical simulation of a Coronavirus disease model with using an efficient nonstandard finite difference scheme

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Abstract

In this manuscript, we introduce a new nonstandard finite difference (NSFD) scheme to approximate solution of the coronavirus disease 2019 (COVID–19) model. In the beginning, the positivity and boundedness of solution of the COVID–19 model are discussed. The stability analysis of the equilibrium points the proposed COVID–19 model are then analyzed. Lastly, to ascertain the efficacy and accuracy of the suggested NSFD scheme, some numerical results are provided.

Keywords: COVID–19 model, Nonstandard finite difference scheme, Positivity, Boundedness, Stability.

Mathematics Subject Classification [2010]: 34D05, 92D30.

1 Introduction

Mathematical modelling plays a basic role in predicting and controlling present and future epidemics. Some patients with pneumonia of unidentified cause appeared in some medical institutions in December 2019 which happened in China. The World Health Organization (WHO) has announced the outbreak COVID–19 as a pandemic on March 2019. As of the end of April 2020, more than 2 millions COVID–19 cases and 200 thousand deaths have been reported from more than 200 countries. Medicine is continuously evolving in terms of refining, revising and discovering new knowledge about COVID–19. To bock the spread of the virus, there are some strategies such as citywide lock down, traffic halt, community management and social distance that have been adapted by the governments some countries in the world. In many cases, mathematical modelling of the COVID–19 can be described by a nonlinear system of ordinary differential equations (ODEs), (see [1, 3] for more details). A very few numbers of nonlinear ODEs can be solved by an analytical solution. Most of these ODEs cannot be solved by the well-known analytical method suitably. For this reason, various numerical methods were discussed to solve such ODEs. In this work, in order to approximate the solution of the COVID–19 model, we will

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construct an efficient NSFD scheme which is positive. A sensible model for the COVID–19 at time t can be described by the following initial value problem

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = A - dS - \beta SI - \gamma, \\ \frac{\mathrm{d}I}{\mathrm{d}t} = \beta SI - dI, \\ \frac{\mathrm{d}R}{\mathrm{d}t} = \gamma - dR, \\ S(0) = S_0,\ I(0) = I_0,\ R(0) = R_0. \end{cases} \tag{1}$$

In this model, $S(t)$ is the number of the susceptible individuals at time t, $I(t)$ stands the infective individuals at time t and $R(t)$ denotes the recovered individuals at time t. Here, the death rates of the susceptible, the infective and the recovered are the same which is denoted by d. Also, the birth rate of the susceptible is A and the susceptible become the infective at a rate βSI where β is the contact rate. Moreover, the susceptible individuals become the recovered by the constant rate γ which is assumed that $A > \gamma$. This paper is structured as follows. Positivity and boundedness the solution of model (1) are proved in Section 2. Stability analysis of the equilibrium points of proposed COVID–19 model are investigated in Section 3. Section 4, is devoted to the study of an efficient NSFD scheme for the numerical solution of proposed COVID–19 model. Finally, numerical results are given in Section 5.

2 Positivity and boundedness

In this part, we want to prove positivity and boundedness of the solution model (1).

Theorem 2.1. *If $A > \gamma$ and $S(0),\ I(0),\ R(0) > 0$, then for all $t \geq 0$, $S(t),\ I(t)$ and $R(t) > 0$.*

Proof. Since the SR-coordinate plane is invariant under the flows of system, hence for all $t \geq 0$, $I(t) > 0$. Let $C = \{t \geq 0 | S(t) < 0\}$ and $D = \{t \geq 0 | R(t) < 0\}$. We will show that $C = \emptyset$. Suppose that $C \neq \emptyset$ and $C_0 = \inf(C)$, therefore $S(C_0) = 0$. Since $S(0) > 0$, thus $C_0 > 0$. By assumption, $C_0 = \inf(C)$ it follows that $S(t) \geq 0$, for all $t \in [0, C_0]$. This implies that from the third equation of the system (1), $S'(C_0) = A - \gamma > 0$. Hence, there exists $\varepsilon > 0$, such that $S'(t) > 0$, for all $t \in (C_0 - \varepsilon,\ C_0 + \varepsilon)$. Therefore, for all $t \in (C_0, C_0 + \varepsilon)$, $S(t) > S(C_0) = 0$, which contradicts $C_0 = \inf(C)$. By a similar argument, we can show that $R(t) \geq 0$, for all $t \geq 0$. □

In order to prove the boundedness of solution model (1), we first state the following proposition.

Proposition 2.2. *Let $K(t) : [0, +\infty) \longrightarrow \mathbb{R}$ be a derivative function such that $K(t) \geq 0$ for all $t \geq 0$. If $\alpha > 0$, $\beta \in \mathbb{R}$, such that $K'(t) + \alpha K(t) \leq \beta$, for every $t \geq 0$, then $K(t) \leq K(0) + \frac{\beta}{\alpha}$.*

Lemma 2.3. *If $A \geq \gamma$, then for all $t \geq 0$, we have $S(t) + I(t) + R(t) \leq S(0) + I(0) + R(0) + \frac{A}{d}$.*

Proof. Define $K(t) = \frac{1}{A}(S(t) + I(t) + R(t))$, hence $K'(t) + dK(t) \leq 1$. It follows from Proposition 2.2 that $K(t) \geq K(0) + \frac{1}{d}$. This establishes the desired result. □

3 Stability analysis of the COVID–19 model

The equilibrium points of the model (1) are given by $E_1 = (\frac{A-\gamma}{d}, 0, \frac{\gamma}{d})$, $E_2 = (\frac{d}{\beta}, \frac{A - \frac{d^2}{\beta} - \gamma}{d}, \frac{\gamma}{d})$.

Theorem 3.1. *The system (1) is*

(i) locally asymptotically stable around E_1 *if* $\gamma < A < \frac{d^2}{\beta}$.

(ii) locally asymptotically stable around E_2 *if* $\frac{\beta}{d}(A-\gamma) - d > 0$.

Proof. The variational matrix of system (1) corresponding to any arbitrary equilibrium point (S^*, I^*, R^*) can be expressed as

$$J(S^*, I^*, R^*) = \begin{pmatrix} -d-\beta I^* & -\beta S^* & 0 \\ \beta I^* & \beta S^* - d & 0 \\ 0 & 0 & -d \end{pmatrix}.$$

At equilibrium point E_1, The variational matrix is

$$J(E_1) = \begin{pmatrix} -d & \frac{-\beta}{d}(A-\gamma) & 0 \\ 0 & \frac{\beta}{d}(A-\gamma) - d & 0 \\ 0 & 0 & -d \end{pmatrix}.$$

The corresponding eigenvalues are $\lambda_1 = -d$, $\lambda_2 = \frac{\beta}{d}(A-\gamma) - d$ and $\lambda_3 = -d$. Therefore, the equilibrium point E_1 is locally asymptotically stable if and only if $\frac{\beta}{d}(A-\gamma) - d < 0$. At the equilibrium point E_2, The variational matrix is

$$J(E_2) = \begin{pmatrix} -d-\beta I_1 & -d & 0 \\ \beta I_1 & 0 & 0 \\ 0 & 0 & -d \end{pmatrix},$$

where $I_1 = \frac{1}{d}(A - \frac{d^2}{\beta} - \gamma)$. The characteristic equation the above matrix is $P(\lambda) = (\lambda+d)(\lambda^2 + (d+\beta I_1)\lambda + d\beta I_1)$. One eigenvalue of the above Jacobian matrix is $\lambda_1 = -d$. Observe that if $I_1 > 0$ then all of roots the polynomial $P^*(\lambda) = \lambda^2 + (d+\beta I_1)\lambda + d\beta I_1$, are negative. Hence the equilibrium point E_2 is locally asymptotically stable. □

4 A new NSFD scheme for the COVID–19 model

The NSFD schemes were firstly introduced by Mickens. In order to introduce the general aspect of a NSFD scheme consider the following initial value problem

$$X'(t) = f(X(t)), \quad X(t_0) = X_0. \tag{2}$$

Suppose that a discretization $t_k = kh$, is given. A NSFD scheme for the problem (2) is constructed by the following two steps (see for instance [2]).

(i) The first order derivative in the initial value problem (2) at time step $t = t_k$ is replaced by a discrete form $X'(t_k) \approx \frac{X_{k+1}-X_k}{\phi(h)}$, where X_k is an approximation of $X(t_k)$ and the denominator function $\phi(h)$ satisfies the condition $\phi(h) = h + O(h^2)$ with $0 < \phi(h) < 1$.

(ii) The linear and nonlinear terms in the initial value problem (2) can be replaced by nonlocal discrete approximations.

Based on the Mickens rules, a NSFD scheme for the COVID–19 model (1) can be written as

$$\begin{cases} \frac{S_{k+1}-S_k}{\phi} = A - \gamma - dS_{k+1} - \beta S_{k+1} I_k, \\ \frac{I_{k+1}-I_k}{\phi} = \beta S_{k+1} I_k - dI, \\ \frac{R_{k+1}-R_k}{\phi} = \gamma - dR_{k+1}, \end{cases} \tag{3}$$

where $\phi(h) = \frac{e^{dh}-1}{d}$. A simple computation shows that

$$\begin{cases} S_{k+1} = \dfrac{(A-\gamma)\phi + S_k}{1 + d\phi + \beta\phi I_k}, \\ I_{k+1} = \dfrac{(1+\beta\phi S_k)I_k}{1+d\phi}, \\ R_{k+1} = \dfrac{\gamma\phi + R_k}{1+d\phi}. \end{cases} \tag{4}$$

Proposition 4.1. *If S_0, I_0 and $R_0 > 0$, then for all stepsize h, the values S_k, I_k and R_k are always positive.*

5 Simulation results

In this part, the numerical solutions of the proposed NSFD scheme on the two cases are presented. At the first simulation, we choose the parameter values $A = 0.5$, $d = 0.3$, $\beta = 0.5$ and $\gamma = 0.21$ with the initial condition $S_0 = 25$, $I_0 = 30$ and $R_0 = 20$ for simulating time 200 and stepsize $h = 0.4$. Figure 1 confirms that the NSFD scheme (4) converges to the equilibrium point $E = (0.6, 0.366, 0.7)$. In Figure 2, we plot the behaviour of the NSFD scheme (4) for the parameter values $A = 0.5$, $d = 0.7$, $\beta = 0.5$ and $\gamma = 0.21$ with choosing stepsize $h = 2$ and the initial condition $S_0 = 25$, $I_0 = 30$ and $R_0 = 20$. The Figure 2 shows that (S_k, I_k, R_k) approaches the equilibrium point $E = (0.41, 0, 0.3)$. The results show that the numerical solutions of the proposed NSFD schemes preserves the main properties of the COVID–19 model such as, positivity and stability, even for large stepsize h.

6 Concluding remarks

In this work, we studied an efficient NSFD scheme for numerical solutions for the COVID–19 model. We portrayed the simulation results in Figures 1–2, which indicate the new NSFD scheme preserved the positivity and stability properties of the COVID–19 model, even for choosing the large stepsize h. As a future research work, we can focus on the fractional–order COVID–19 model and obtain an efficient NSFD scheme which preserves the positivity and stability properties of the fractional–order COVID–19 model.

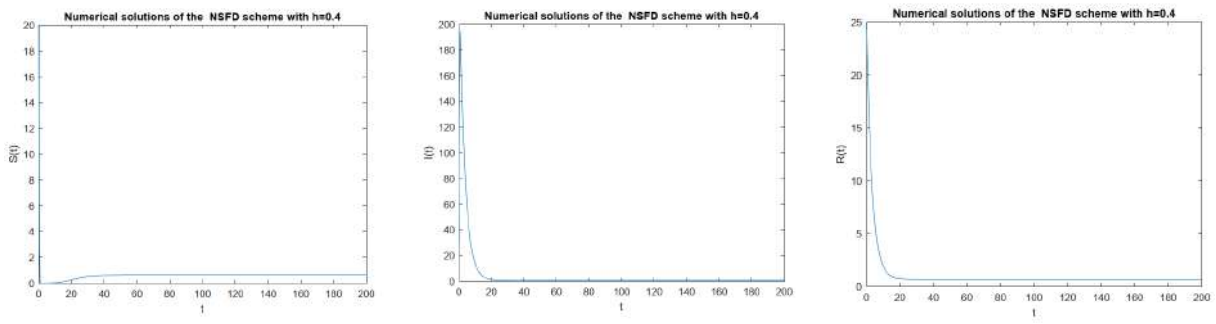

Figure 1: Numerical simulation with $h = 0.4$ for the NSFD scheme (3).

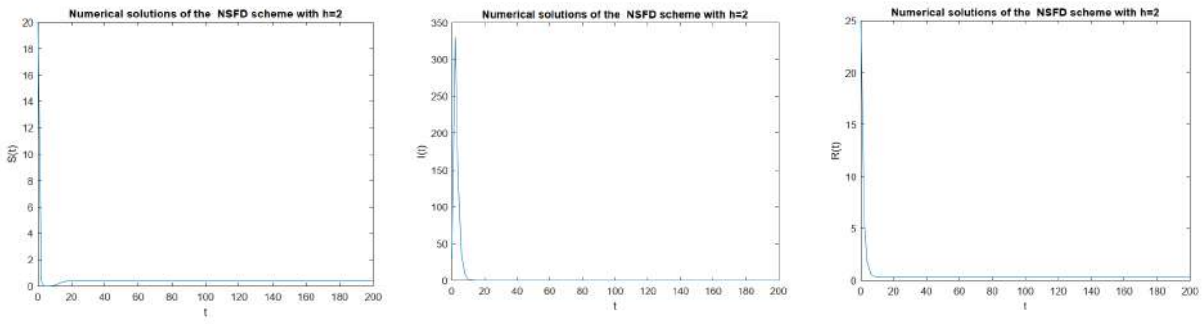

Figure 2: Numerical simulation with $h = 2$ for the NSFD scheme (3).

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Poster Presentations

Multiplicatively local spectrum preserving maps on Matrices

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Abstract

Let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices, and fix a nonzero vector $x_0 \in \mathbb{C}^n$. We show that if two maps φ_1 and φ_2 from $M_n(\mathbb{C})$ into itself satisfies

$$\sigma_{\varphi_1(T)\varphi_2(S)^*}(x_0) = \sigma_{TS^*}(x_0) \qquad (T, S \in M_n(\mathbb{C})),$$

if and only if there exist two unitary matrices $A, B \in M_n(\mathbb{C})$ such that $\varphi_1(T) = ATB$ for all $T \in M_n(\mathbb{C})$ and $\varphi_2(T) = B^*TA^*$ for all $T \notin M_{n,x_0}$.

Keywords: Nonlinear preserver, local spectrum, matrix

Mathematics Subject Classification [2010]: Primary 47B49, Secondary 47A11

1 Introduction

Nonlinear preserver problems, in the most general setting, demand the characterization of maps between algebras that leave a certain property, a particular relation, or even a subset invariant without assuming in advance algebraic conditions such as linearity, additivity or multiplicity. More recently, there has been an upsurge of interest in linear and nonlinear local spectra preserver problems. Bourhim and Ransford were the first ones to consider this type of preserver problem, characterizing in [4] additive maps on the algebra of all linear bounded operators on a complex Banach space X that preserve the local spectrum of operators at each vector of X. Their results cleared the way for several authors to describe maps on matrices or operators that preserve local spectrum, local spectral radius, and local inner spectral radius; see, for instance, the last section of the survey article [3] and the references therein. Gonzalez and Mbekhta in [6] described the linear maps from $M_n(\mathbb{C})$ to itself which preserve the local spectrum at a fixed nonzero vector $x_0 \in \mathbb{C}^n$. In [2], nonlinear maps on $M_n(\mathbb{C})$ preserving the local spectrum of the product and the triple product of operators have been characterized. The aim of this paper is to characterize the form of all maps φ_1 and φ_2 (not supposed to be surjective or even linear) on $M_n(\mathbb{C})$ that for every T and S in $M_n(\mathbb{C})$, the local spectral subspaces of TS^* and $\varphi_1(T)\varphi_2(S)^*$ are the same.

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2 Main results

Throughout this paper, let $M_n(\mathbb{C})$ denote as usual the algebra of all $n \times n$ complex matrices, and let T^{tr} and T^* denote the transpose and adjoint of any matrix $T \in M_n(\mathbb{C})$. For any vector $x_0 \in \mathbb{C}^n$, let $M_{n,x_0}(\mathbb{C})$ be the collection of all matrices in $M_n(\mathbb{C})$ vanishing at x_0.
Let $B(X)$ be the algebra of all bounded linear operators on a complex Banach space X, and its unit will be denoted by I. The local resolvent set, $\rho_T(x)$, of an operator $T \in B(X)$ at some point $x \in X$ is the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood U of λ in $\mathbb{C}$ and a X-valued analytic function $f : U \longrightarrow X$ such that $(\mu I - T)f(\mu) = x$ for all $\mu \in U$. The complement of local resolvent set is called the local spectrum of T at x, denoted by $\sigma_T(x)$. The local resolvent set, $\rho_T(x)$, of an operator $T \in B(X)$ at some point $x \in X$ is the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood U of λ in $\mathbb{C}$ and a X-valued analytic function $f : U \longrightarrow X$ such that $(\mu I - T)f(\mu) = x$ for all $\mu \in U$. The complement of local resolvent set is called the local spectrum of T at x, denoted by $\sigma_T(x)$. The local spectral radius of T at x is given by $r_T(x) := \limsup_{n \longrightarrow \infty} \|T^n(x)\|^{\frac{1}{n}}$, and coincides with the maximum modulus of $\sigma_T(x)$ provided that T has the single-valued extension property. Recall that an operator $T \in B(X)$ is said to have the single-valued extension property (henceforth abbreviated to SVEP) if, for every open subset U of $\mathbb{C}$, there exists no nonzero analytic solution, $f : U \longrightarrow X$, of the equation

$$(\mu I - T)f(\mu) = 0, \quad \forall\ \mu \in U.$$

Every operator $T \in B(X)$ for which the interior of its point spectrum, $\sigma_p(T)$, is empty enjoys this property. For more information about these notions one can see the books [1].

In the case X is a finite dimensional space, we have a good description of the concepts involved in local spectral theory; see for instance [6].

Remark. [6]. Let $T \in M_n(\mathbb{C})$ be an $n \times n$ matrix. Let $\lambda_1, \lambda_2, ..., \lambda_r$ be the distinct eigenvalues of T and denote by $E_1, E_2, ..., E_r$ the corresponding root spaces. We have $\mathbb{C}^n = E_1 \oplus E_2 \oplus ... \oplus E_r$ and $T = T_1 \oplus T_2 \oplus ... \oplus T_r$ where T_i is the restriction of T to E_i. It follows that for every $x \in \mathbb{C}^n$,

$$\sigma_T(x) = \bigcup \{\sigma_{T_i}(P_i x)\} = \{\lambda_i : 1 \leq i \leq r, P_i(x) \neq 0\}$$

where $P_i : \mathbb{C}^n \to E_i$ is the canonical projection. Therefore,

$$r_T(x) = max\{r_{T_i}(P_i x) : 1 \leq i \leq r\} = max\{|\lambda_i| : 1 \leq i \leq r, with\ P_i(x) \neq 0\}$$

For a finite set K and a fixed nonzero vector $x_0 \in \mathbb{C}^n$ we will denote by $|K|$ the number of elements of K, and by M_{x_0} the set defined by

$$M_{x_0} := \{T \in M_n(\mathbb{C}) : |\sigma(T)| = n, \quad \{T(x_0)^k\}_{k=1}^n is\ a\ basis\ of\ \mathbb{C}^n\}.$$

According to [6] Proposition 6, M_{x_0} is an open dense subset of $M_n(\mathbb{C})$ and $\sigma(T) = \sigma_T(x_0)$ for all $T \in M_{x_0}$.

Gonzalez and Mbekhta [6] characterized linear maps on $M_n(\mathbb{C})$ that preserving the local spectrum at only a fixed nonzero vector $x_0 \in \mathbb{C}^n$.

Theorem 2.1. *[6] Let $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map and $x_0 \in \mathbb{C}^n$ be a fixed nonzero vector. Then φ preserves the local spectrum at x_0 if and only if there exists an invertible $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\varphi(T) = ATA^{-1}$ for every $T \in M_n(\mathbb{C})$.*

Bendaoud et al. [2] in the following theorem characterized nonlinear maps on $M_n(\mathbb{C})$ that preserve the local spectrum of the product of matrices at a fixed nonzero vector.

Theorem 2.2. *[2] Let x_0 be a fixed nonzero vector in $\mathbb{C}^n$. A map φ from $M_n(\mathbb{C})$ into itself satisfies*

$$\sigma_{\varphi(T)\varphi(S)}(x_0) = \sigma_{TS}(x_0), \qquad (T, S \in M_n(\mathbb{C}))$$

if and only if there exist a scalar $\varepsilon = \pm 1$ and an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$, and $\varphi(T) = \varepsilon ATA^{-1}$ for all $T \in M_n(\mathbb{C})$.

Theorem 2.3. *Two maps φ_1 and φ_2 on $M_n(\mathbb{C})$ satisfy*

$$\sigma_{\varphi_1(T)\varphi_2(S)^*}(\{\lambda\}) = \sigma_{TS^*}(\{\lambda\}) \qquad (T, S \in M_n(\mathbb{C}),\ \lambda \in \mathbb{C})$$

if and only if φ_2 maps $M_{n,x_0}(\mathbb{C})$ into itself and there are two unitary matrices A and B in $M_n(\mathbb{C})$ such that

$$\varphi_1(T) = ATB, \quad (T \in M_n(\mathbb{C})),$$

and

$$\varphi_2(T) = B^*TA^*, \quad (T \notin M_{n,x_0}(\mathbb{C})).$$

Proof. First, we show that the restriction of both φ_1 and φ_2 on GL_n are linear and bijective, where, GL_n denote the collection of all invertible matrices in $M_n(\mathbb{C})$. Therefore, following the same argument as the one in the proof of Assertion 1 and Assertion 2 of [5, Theorem 2.1], one concludes that there are two bijective linear maps L_1 and L_2 on $M_n(\mathbb{C})$ such that $\varphi_i = L_i$ on GL_n for $i = 1, 2$.
Next, we show that there are two unitary matrices A and B in $M_n(\mathbb{C})$ such that either

$$L_1(T) = ATB, \quad \text{and} \quad L_2(T) = B^*TA^* \quad (T \in M_n(\mathbb{C})), \qquad (1)$$

or

$$L_1(T) = AT^{tr}B \quad \text{and} \quad L_2(T) = B^*T^{tr}A^* \quad (T \in M_n(\mathbb{C})) \qquad (2).$$

We prove that (2) cannot occur, and thus L_1 and L_2 are of the form (1).
Finally, we show that φ_1 and φ_2 have the asserted forms. □

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On the ring of linear transformations of vector spaces

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Abstract

Let V be a right vector space over a division ring D and $L(V)$ be the ring of all right linear transformations on V. In this talk, some results related to the characterization and properties of one-sided ideals of the ring $L(V)$ which have been obtained over the past decades are reviewed. Moreover, some outlines for the continuation of the subject under investigation is presented.

Keywords: linear transformation, ideal, reducible

Mathematics Subject Classification [2010]: 15A33, 15A30, 16S50

1 Introduction

Throughout this talk, D denotes a division ring, V and W right vector spaces over D, and $L(V,W)$ the set of all right linear transformations $T : V \to W$ such that $T(x+y) = Tx+Ty$ and $T(x\lambda) = (Tx)\lambda$ for all $x, y \in V$ and $\lambda \in D$. If we consider the left linear transformations $T : V \to W$, then $T(x+y) = Tx+Ty$ and $T(\lambda x) = \lambda T(x)$ for all $x, y \in V$ and $\lambda \in D$. When $V = W$, we use the symbol $L(V)$ to denote $L(V,V)$. The set $L(V)$ forms a ring under the addition and multiplication of linear transformations defined respectively as $(T+S)(x) = Tx+Sx$ and $(TS)(x) = T(Sx)$. A subspace M is called invariant for a collection F in $L(V)$ if $T(M) \subseteq M$ for all $T \in F$. A collection F of linear transformations in $L(V)$ is called reducible if $F = \{0\}$ or it has a non-trivial invariant subspace and irreducible otherwise. A collection F of linear transformations in $L(V)$ is called simultaneously triangularizable or simply triangularizable if there exists a maximal chain of the subspaces of V each of which is invariant under the collection F. In the case that V is finite-dimensional, this is equivalent to saying that there exists a basis for the vector space V relative to which all matrices in the family are upper triangular. The dual space of V denoted by V^* is defined to be $L(V,D)$, where D is regarded as a one-dimensional vector space over itself. The members of V^* are called linear functionals on V. If V is a right vector space, then V^* is a left vector space over D endowed with the addition and the scalar multiplication defined by $(f+g)(x) = f(x)+g(x)$ and $(\alpha f)(x) = \alpha f(x)$ for all $x \in V$ and $\alpha \in D$. The second dual of V, denoted by V^{**} is the dual of V^*. Let $\{x_i\}_{i\in I}$ be an independent subset of V. Then, there exists an independent subset $\{f^i\}_{i\in I}$ of linear functionals on V satisfying $f^i(x_j) = \delta_{ij}$, where $i, j \in I$ and δ denotes the Kronecker delta. Every such independent subset

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of V^* is called a dual independent subset with respect to $\{x_i\}_{i\in I}$. For a collection C of vectors in a right vector space V over D, the symbol $\langle C\rangle$ is used to denote the right linear subspace spanned by C. For $T \in L(V,W)$ the symbol $T^* \in L(W^*,V^*)$ denotes the adjoint of T which is defined by $(T^*f)(v) = f(Tv)$, where $f \in W^*$ and $v \in V$. In this talk some specific one-sided ideals of the ring $L(V)$ in terms of their rank-one idempotents are characterized. Moreover, some characterizations of a division ring D in terms of the one-sided ideals of $M_n(D)$ are given. The results are reviewed from [2]; however the research is in progress by the authors which have led to some results and have not been published yet. The improvements relevant to this subject have been done over the past decades by some authors such as [1–4].

2 Main results

An important subset of $L(V,W)$ is the class of rank-one linear transformations. Every rank-one linear transformation is of the form $x\otimes f$ for some $x \in W$ and $f \in V^*$, where $(x\otimes f)(y) = xf(y)$ for all $y \in V$. For a family $F \subseteq L(V)$, the image of the family F denoted by $im(F)$ is the subspace of V generated by $\{T(x) : T \in F, x \in V\}$. Moreover, the kernel of this family is denoted by $\ker(F)$ which is defined as $\cap_{T\in F}\ker(T)$. The co-image and co-kernel of the family F, denoted by $coim(F)$ and $coker(F)$ respectively, are defined as $V/\ker F$ and $V/im(F)$. The following theorem characterizes all right ideals in $L(V)$ with finite-dimensional image. The left ideals in $L(V)$ with finite-dimensional co-image are characterized as well.

Theorem 2.1. *Let D be a division ring and V be a right or left vector space over D. If I is a non-zero right ideal in $L(V)$ with finite-dimensional image, then there exist $x_i \in V$ and $f_i \in V^*$ for some i with $(1 \le i \le r)$ which are dual to each other such that $I = x_1\otimes f_1L(V)+\ldots+x_r\otimes f_rL(V) = TL(V)$, where $r = dimim(I)$. Moreover, in the case that I is a non-zero left ideal in $L(V)$ with finite-dimensional co-image, we have $I = L(V)x_1\otimes f_1+\ldots+L(V)x_r\otimes f_r = L(V)T$, where $r = dimcoim(I)$. In both cases, $T = x_1\otimes f_1+\ldots+x_r\otimes f_r$ is an idempotent in I. Therefore, every right (resp. left) ideal of $L(V)$ whose image (resp. co-image) is finite-dimensional is principal; in fact the right (resp. left) ideal is generated by a finite-rank idempotent whose rank is equal to the dimension of the image (resp. co-image) of the right (resp. left) ideal.*

Corollary 2.2. *Let D be a division ring and V be a right or left vector space over D. Let $T \in L(V)$ be an arbitrary operator. Then the following are equivalent.*

1. *$rank(T) = r$.*
2. *The right ideal of $L(V)$ generated by T is equal to $TL(V) = x_1\otimes f_1L(V)+\ldots+x_r\otimes f_rL(V)$, where $\{x_i\}_{1\le i\le r}$ is a basis for $im(T)$ and f_i's are dual to x_i's $(1 \le i \le r)$.*
3. *The left ideal of $L(V)$ generated by T is equal to $L(V)T = L(V)x_1\otimes f_1+\ldots+L(V)x_r\otimes f_r$, where $\{x_i+\ker T\}_{1\le i\le r}$ is a basis for $coim(T)$ and f_i's are dual to x_i's $(1 \le i \le r)$.*

We observe that if the space V is finite-dimensional in Theorem 2.1, then it is enough to present the proof of the assertion for right ideals. The proof for left ideals follows then by taking adjoints.

Theorem 2.3. *Let D be a division ring and V be a right or left vector space over D. Let I be a non-zero right ideal of $L(V)$ containing a linear transformation T whose rank $r \in \mathbb{N}$ is maximal among all elements of I. Then, there exist $x_i \in V$ and $f_i \in V^*(1 \le i \le r)$ which are dual to each other such that $I = TL(V) = x_1\otimes f_1L(V)+\ldots+x_r\otimes f_rL(V)$. Moreover, in the case that I is a non-zero left ideal of $L(V)$ containing a linear transformation T whose rank $r \in \mathbb{N}$ is maximal among all elements of I. Then, $I = L(V)T = L(V)x_1\otimes f_1+\ldots+L(V)x_r\otimes f_r$. Moreover, each*

x_i can be chosen to be in the range or in the complement of the kernel of T if I is a right or left ideal respectively. In particular, if V is finite-dimensional, then the above holds for all right and left ideals of $L(V)$.

If the image (resp. co-image) of the right (resp. left) ideal I is the whole space, then we may characterize all one-sided ideals of $L(V)$ that are irreducible.

Theorem 2.4. *Let V be a vector space over a division ring D and I be a right ideal in the ring $L(V)$. Then the following are equivalent.*

1. *I includes all finite-rank transformations in $L(V)$.*
2. *I is irreducible.*
3. *$im(I) = V$.*

Therefore, when V is finite-dimensional, then $I = L(V)$ if and only if the right ideal I is irreducible if and only if $imI = V$.

Theorem 2.5. *Let V be a vector space over a division ring D and I be a left ideal in the ring $L(V)$. Then the following are equivalent.*

1. *I includes all finite-rank transformations in $L(V)$.*
2. *I is irreducible.*
3. *$coimI = V$, which is equivalent to $\ker I = 0$.*

Therefore, when V is finite-dimensional, then $I = L(V)$ if and only if the left ideal I is irreducible if and only if $coimI = V$.

Proof. Clearly, (1) implies (2). Moreover, we can get easily (3) from (2). Therefore, it suffices to prove (3) $\Rightarrow$ (1). Let I be a right ideal in $L(V)$ whose image is V. To prove the assertion, in view of Theorem 2.1, it suffices to show that I contains all rank-one linear transformations. Let $x \otimes f$ be an arbitrary rank-one linear transformation, where $x \in V$ and $f \in V^*$. It follows from the hypothesis that there are $A_j \in I$ and $y_j \in V$ $(1 \leq j \leq m)$ such that $x = \sum_{1 \leq j \leq m} A_j y_j$. We have:

$$x \otimes f = \Big(\sum_{1 \leq j \leq m} A_j y_j \Big) \otimes f = \sum_{1 \leq j \leq m} A_j (y_j \otimes f),$$

implying that $x \otimes f \in I$ because each A_j belong to I and I is a right ideal in $L(V)$. Now suppose that I is a left ideal in $L(V)$ whose kernel is zero. Again, in view of Theorem 2.1, it suffices to show that $x \otimes f \in I$ for all $x \in V$ and $f \in V^*$. To do this, let $x \otimes f \in L(V)$ be an arbitrary element. We have $V^* = \{f \in V^* : f(0) = 0\} = \{f \in V^* : f(\cap_{A \in I} \ker A) = 0\} = \langle \cup_{A \in I} A^* V^* \rangle$. This implies that for $f \in V^*$, there are $A_i \in I$ and $f_i \in V^*$ for $(1 \leq i \leq m)$ and a positive integer m such that $f = \sum_{1 \leq i \leq m} A_i^* f_i = \sum_{1 \leq i \leq m} f_i A_i$. Therefore, we can write $x \otimes f = x \otimes (\sum_{1 \leq i \leq m} f_i A_i) = \sum_{1 \leq i \leq m} (x \otimes f_i) A_i$ implying that $x \otimes f \in I$, for each A_i belongs to I and I is a left ideal in $L(V)$. □

The following result characterizes all one-sided ideals of $L(V)$ that are triangularizable. The assertion is true for both right and left ideals.

Theorem 2.6. *Let V be a vector space over a division ring D and I a non-zero right ideal in the ring $L(V)$. Then the following are equivalent.*

1. *I is triangularizable.*

2. *I is generated by a rank-one idempotent elements.*

3. *I consists of all linear transformations of rank at most one.*

4. *The rank of $TS - ST$ is at most one for all $T, S \in I$.*

In particular, a linear transformation $T \in L(V)$ has rank one if and only if one of the one-sided ideals generated by T is triangularizable.

Corollary 2.7. *Let D be a division ring, and $n \in \mathbb{N}$. Then the following hold.*

1. *The ideal $M_n(D)$ is the only irreducible left (resp. right) ideal in $M_n(D)$.*

2. *The only triangularizable one-sided ideals of $M_n(D)$ are those of the form $AM_n(D)$ or $M_n(D)A$ for some rank-one idempotent $A \in M_n(D)$.*

The next results characterize a division ring D in terms of the one-sided ideals of $M_n(D)$ up to similarity. First, we state a lemma.

Lemma 2.8. *Let D be a ring such that $D^2 \neq 0$. Then the following are equivalent.*

1. *The ring D is a division ring.*

2. *Zero is the only proper left ideal in D.*

3. *Zero is the only proper right ideal in D.*

4. *Zero is the only proper two-sided ideal in D and that D has the property that for all $x, y \in D$ there exists a $z \in D$ such that $xy = zx$.*

5. *The only left ideals in $M_n(D)$ are those collections whose columns are 0^n or D_n.*

6. *The only right ideals in $M_n(D)$ are those collections whose rows are 0_n or D_n.*

Moreover, if D is a division ring and I a left (resp. right) ideal in $M_n(D)$, then every two nonzero columns (resp. rows) of the ideal I are either linked or they are independent.

Theorem 2.9. *Let D be a unital ring. Then the following are equivalent.*

1. *The ring D is a division ring.*

2. *The only left ideals in $M_n(D)$ are those that consist of all matrices in $M_n(D)$ whose first r columns are completely arbitrary and whose last $n - r$ columns are zero, where $r \leq n$ is a nonnegative integer depending on the ideal.*

3. *The only right ideals in $M_n(D)$ are those that consist of all matrices in $M_n(D)$ whose first r rows are completely arbitrary and whose last $n - r$ rows are zero, where $r \leq n$ is a nonnegative integer depending on the ideal.*

If D is a division ring, then the integer r in (1) (resp. in (2)) above is the dimension of the co-image (resp. the image) of the left (resp. the right) ideal.

Corollary 2.10. *Let D be a division ring, n a positive integer, and $A \in M_n(D)$. Then the following are equivalent.*

1. *The rank of A is equal to r.*

2. *The left ideal generated by A in $M_n(D)$ is the left ideal that consists of all matrices in$M_n(D)$ whose first r columns are completely arbitrary and whose last $n - r$ columns are zero.*

3. *The right ideal generated by A in $M_n(D)$ is the right ideal that consists of all matrices in $M_n(D)$ whose first r rows are completely arbitrary and whose last nr rows are zero.*

3 Conclusion

In this talk, the ring $L(V)$ of all right linear transformations on a right vector space V over a division ring D is closely studied. In particular, some specific one-sided ideals of the ring $L(V)$ in terms of their rank-one idempotents are characterized. Moreover, some characterizations of a division ring D in terms of the one-sided ideals of $M_n(D)$ are given. The research for the subject under investigation, in particular for some specific kinds of rings of linear transformations is in progress by the authors which have led to some results and have not been published yet.

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