

# Extended Abstracts

## of

# The 9<sup>th</sup> Seminar on Linear Algebra and its Applications

5-6 July, 2017

University of Tabriz Tabriz, East Azarbayjan, Iran

Mohammad Hossein Jafari and Morteza Faghfouri

## Scientific committee

M. Shahryari	(scientific chair, University of Tabriz),		
Gh. Ebadi	(University of Tabriz),		
H. Kheiri	(University of Tabriz),		
M. H. Jafari	(University of Tabriz),		
M. R. Jabbarzadeh	(University of Tabriz),		
K. Azizi	(University of Tabriz),		
N. Shirmohammadi	(University of Tabriz),		
R. Naghipour	(University of Tabriz),		
S. Shahmorad	(University of Tabriz),		
J. Vakili	(University of Tabriz),		
M. Rajabalipour	(Shahid Bahonar University of Kerman),		
Y. Zamani	(Sahand University of Technology),		
K. Ghanbari	(Sahand University of Technology),		
M. T. Darvish	(Razi University of Kermanshah),		
F. Tootoonian	(Ferdowsi University of Mashad),		
Gh. R. Aghamollayi	(Shahid Bahonar University of Kerman).		

### **Executive Committee**

Gh. Ebadi	(chair of the executive committee),		
H. Jabbari	(University of Tabriz),		
A. Ranjbari	(University of Tabriz),		
H. Vaezi	(University of Tabriz),		
M. Lakestani	(University of Tabriz),		
M. Faghfouri	(University of Tabriz),		
A. R. Madadi	(University of Tabriz),		
Gh. R. Hojjati	(University of Tabriz),		
H. Mousavi	(University of Tabriz),		
P. Sahandi	(University of Tabriz),		
A. Abdi	(University of Tabriz),		
J. Ahmadi	(University of Tabriz),		
D. Ahmadian	(University of Tabriz).		

## Contents

Improving quality of answers in convex feasibility problem Mokhtar Abbasi and Touraj Nikazad1
On the spectra of the graphs obtained by generalized join graph operation Abdollah Alhevaz and Maryam Baghipur5
A sharper bound for the relative operator entropy Mehdi Alinejad and Ismail Nikoufar9
Recent extensions of affine Lie algebras Saeid Azam
Biderivations on triangular Banach algebras S. Barootkoob
Some linear algebra method for counting the number of perfect matchings in graphs Afshin Behmaram
Stabilization of fractional discrete-time systems via output feedback Sakineh Bigom Mirassadi and Hojjat Ahsani Tehrani20
Eigenvalue assignment in fractional two-dimensional Roesser systems Sakineh Bigom Mirassadi and Hojjat Ahsani Tehrani24
Generalization of some quantum information inequalities Ali Dadkhah
Capturing outlines of 2D shapes using B-spline curves and normal equation Alireza Ebrahimi and Ghasem Barid Loghmani31
A characterazation for Moore-Penrose invertible operators on Hilbert $C^*$ -modules Javad Farokhi-Ostad and Alireza Janfada
A solution of operator equation on Hilbert $C^*$ -modules Javad Farokhi-Ostad and Alireza Janfada
Relative reproducing kernels associated with cocycles Saeed Hashemi Sababe and Shahram Najafzadeh
On Jordan centralizers of quaternion rings Hoger Ghahramani, Mohammad Nader Ghosseiri, and Leila Heidaryzadeh
Left-looking approximate inverse preconditioner in block form Amin Rafiei and Samaneh Hosseini Sani51

On the preconditioning of double saddle point matrices with local shift-splitting

Mohammad Mahdi Izadkhah
Parametred iterative refinement for ill-conditioned linear system of equations Mohammad Mahdi Izadkhah
Bernstein polynomials matrices Mostafa Jani, Shahnam Javadi, and Esmail Babolian63
A block version of right-looking A-biconjugation process Amin Rafiei and Leila Kakhki Beydokhti
Some results on polynomial numerical hull of degree $n-1$ Saeed Karami
Recent developments on iterative methods for solving complex symmetric system of linear equations Davod Khojasteh Salkuyeh
Multiplicative splitting iterations for solving Sylvester equation Mohammad Khorsand Zak
Fuzzy congruence relation and fuzzy coset relation on vector spaces Sadegh Khosravi Shoar
Convexity in matrix algebras Mohsen Kian
Linear preservers of slt-majorization on $\mathbb{R}^n$ Ahmad Mohammadhasani and Asma Ilkhanizadeh Manesh
EDMs, Rsr-majorization, and Rp-majorization Ahmad Mohammadhasani and Asma Ilkhanizadeh Manesh93
Stability analysis of Hilfer fractional order differential systems with time delay Vahid Mohammadnezhad, Mostafa Eslami, and Hadi Rezazadeh
A note on linear Boolean algebra Ali Molkhasi
On operator <i>h</i> -convex functions Ali Morassaei
A global regularized conjugate gradient method for solving matrix equation $AXB = C$ Azita Tajaddini and Mehdi Najafi
On the bimodule structure of linear transformations of vector spaces Alireza Najafizadeh
A numerical method for solving fractional-order non-linear finance system Mehran Namjoo, Sadegh Zibaei, and Javaher Langari116

A nonstandard finite difference scheme for advection equation	0
	Ű
The energy and spectrum of non-commuting graphs	1
Maryam Nasiri and Sayyed Heidar Jafari12	4
On a remarkable implication of nearest submatrix	
Alimohammad Nazari and Atiyeh Nezami12	7
Column-action methods in image reconstruction	
Touraj Nikazad	0
Spectral radius of adjacency and Laplacian matrices of fractional powers of graphs	/
Morteza Faghani, Siamak Firouzian, and Mostafa Nouri-Jouybari	6
Homeomorphic images of orthogonal bases	
M. Kebryaee and M. Radjabalipour	0
Some results on invertible hypervector spaces	
Zohreh Nazari, Elham Zangiabadi, and Afsaneh Ranjbar	1
A note on invertible weak hypervector spaces	4
Emain Zangiabadi, Zohren Nazari, and Alsanen Ranjbar14	4
A comparison of two families of global Krylov type methods for matrix equation	
Ghodrat Ebadi and Somaiyeh Rashedi14	7
A fast multipole accelerated meshless technique within a multilevel Newton framework for two-dimensiona	al
nonlinear cubic Schrödinger equations	
Hadi Roohani Ghehsareh	1
Relation between fuzzy continuity and fuzzy boundedness	
Mortaza Sabeli	5
	-
Eigenvalues for tridiagonal 2-toeplitz matrices	~
Maryam Shams Solary	9
On the mutative method in max algebra	
Seyyed Mahmoud Manjegani and Hojr Shokooh Saljooghi16	4
Parameterization of feedback matrix in time-delay systems	
Narges Tahmasbi and Hojjat Ahsani Tehrani	8
Norm mmmiziation controller in fractional systems with delay	2
Tvarges Tanniason and 110jjat Ansani Tennani17.	2
Efron-Stein inequality for operator-valued random matrices	
Ali Talebi and Mohammad Sal Moslehian	6

Noncommutative Etemadi inequality	
Ali Talebi	
On the solutions of operator equations $BXA = B = AXB$ via *-order	
Mahdi Vosough and Mohammad Sal Moslehian	
On the solvability of the matrix inequality $AXA^* + BX^*B^* \ge C$	
Mahdi Vosough and Mohammad Sal Moslehian	
New additive result on Drazin inverse	
Mansour Dana and Ramesh Yousefi	
On Drazin inverse of sum of two operator matrices	
Mansour Dana and Ramesh Yousefi	
Generalized joint rank $-k$ numerical ranges and quantum error correction	
Mohsen Zahraei and Esmaeil Fooladi H	
On the geometry of bounded linear operators	
Ali Zamani	
On the construction of HNN-extensions for dialgebras	
Chia Zargeh	
Rota theorem for finite dimensional Banach spaces	
Rahim Alizadeh	



#### IMPROVING QUALITY OF ANSWERS IN CONVEX FEASIBILITY PROBLEM

MOKHTAR ABBASI AND TOURAJ NIKAZAD

ABSTRACT. Iterative methods for solving Convex Feasibility Problems(CFP) often find a solution which may not be desired enough. In this paper we try to steer solutions from iterative methods toward desired solution by using Superiorization algorithm. The advantages of our method are demonstrated by applying it on an example taken from the field of computerized tomography.

Keywords: Convex Feasibility Problem (CFP); Superiorization; Computerized tomography.

#### 1. INTRODUCTION

Suppose  $C_1, \dots, C_n$  be closed and convex subsets of Hilbert space X such that  $C = \bigcap_{i=1}^n C_i \neq \emptyset$ , finding a point  $x^* \in C$  is called Convex Feasibility Problem (CFP). The CFP has many applications in diverse areas of mathematics and physical science, most notably, computerized tomography. During three past decades, a lot of ierative methods have been introduced by researchers for solving CFP. In most cases a CFP don't have a unique solution and the iterative methods used for it find a solution may not be desire enough from application point of view. In some problems there are prori information about the solution of CFP which can be used to decide about utility of CFP solutions. In recent years, a new algorithmic structure called Superiorization has been developed and successfully applied for solving CFP, specially CFP problems in the field of image reconstruction from projections [1]. Suppose f(x) be a convex function and according to priori information we know for the solution of a CFP, namely  $x^*$ , the value of f(x) is not so large. In summery, The Superiorization algorithm try to steer solutions of iterative methods, for solving CFP, toward those which have smaller f(x) in compared to solutions obtained from original iterative methods (without using Superiorization).

#### 2. PRELEMINARIES AND DEFINITIONS

the  $\Phi$ -class operators family is defined as follow [5]:

**Definition 2.1.** The  $\Phi$ -class consist operators as  $T : D \subseteq \mathbb{R}^n \to D$  which have two following properties:

(I) For every  $z \in \operatorname{Fix} T$  there exist one nonnegative real function  $\phi_z$  such that

- (2.1)  $||z Tx|| \le ||z x|| \phi_z(x) \quad \forall x \in D.$ 
  - (II) If the sequnce  $\{x_m\}$  converges to  $\alpha$  and  $\lim_{m \to \infty} \phi_z(x_m) = 0$  then  $\alpha \in \operatorname{Fix} T$

<sup>2010</sup> Mathematics Subject Classification. 65F10, 65N21.

Speaker: Mokhtar Abbasi.

Many of well known operators for solving CFP are belong to  $\Phi$ -class such as metric projections, subgradient projections, useful part of cutter operators(closed at zero cutter operators). Most of iterative methods for solving CFP problems are made by using composition and convex combination of operators  $T_i$  where  $T_i \in \Phi$  with Fix  $T_i = C_i$ . The  $\Phi$ -class has an important property, closedness under composition and convex combination of its members. the following lemma is a straight result from [5, Proposition 13].

**Lemma 2.2.** Suppose  $T_1, \dots, T_n \in \Phi$  and  $\bigcap_{i=1}^n \operatorname{Fix} T_i \neq \emptyset$  then

(1)  $T = T_n \cdots T_1 \in \Phi$  and  $\operatorname{Fix} T = \bigcap_{i=1}^n \operatorname{Fix} T_i$ (2)  $U(x) = \sum_{i=1}^n w_i T_i(x) \in \Phi$  and  $\operatorname{Fix} U = \bigcap_{i=1}^n \operatorname{Fix} T_i$  where  $\sum_{i=1}^n w_i = 1$  and  $w_i > 0, \forall i = 1, \cdots, n$ 

#### 3. Main results

In this section we consider the consistent linear system Ax = b as a special case of CFP which called Linear Feasibility Problem (LFP). For solving a LFP, we use a method called block EMR. Block EMR is belong to Landweber Type iterations. Suppose the linear system Ax = b partitioned into p blocks such that any equation in linear system should be belong to at least a block. So we have p smaller linear systems as follow:

$$(3.1) A_i x = b_i, i = 1, \cdots, p$$

for any linear system  $A_i x = b_i$  in 3.1, an operator  $T_i$  is defined as follow:

(3.2) 
$$T_i(x) = x + \lambda(x)A_i^T M_i(b_i - A_i x), \qquad i = 1, \cdots, p.$$

where

(3.3) 
$$\lambda(x) = \frac{\|A_i^T M_t (b_i - A_i x)\|_2^2}{\|M_i^{\frac{1}{2}} A_i A_i^T M_i (b_i - A_i x)\|_2^2}$$

and  $\{M_i\}_{i=1}^p$  are symmetric positive definite(SPD) matrices with appropriate size. It's not difficult to see that Fix  $T_i = \{x | A_i x = b_i\}$  and according to [4, Lemma 5.2] we have:

**Lemma 3.1.** The operators  $\{T_i\}_{i=1}^p$  defined by 3.2 and 3.3 belong to  $\Phi$ .

**Corollary 3.2.** Lemma 2.2 and 3.1 lead to that the operator  $T = T_1 T_2 \cdots T_p \in \Phi$ and Fix  $T = \bigcap_{i=1}^n \text{Fix } T_i = \{x | Ax = b\}.$ 

**Theorem 3.3.** Suppose  $\{v_k\}$  be a bounded sequence in  $\mathbb{R}^n$  and  $\{\beta_k\} \subseteq \mathbb{R}^+$  be a summable sequence (i.e.  $\sum_{k=1}^{n} \beta_k < \infty$ ) and  $T \in \Phi$  then the sequence

$$(3.4) x^{k+1} = T(x^k + \beta_k v_k)$$

converges to a point in  $\operatorname{Fix} T$ .

Proof. see [5, Remark 23].

#### 4. Numerical results

Discrete image reconstruction from projections resulted in large, sparse and often undetermined linear systems. When according to priori information, it is known that the reconstructed image is an  $s \times t$  smooth image (the image in which adjacent pixels have similar values), the function  $TV : \mathbb{R}^n \to \mathbb{R}^+$  is defined as follow:

(4.1) 
$$TV(x) = \sum_{k=1}^{s-1} \sum_{l=1}^{t-1} \sqrt{(p_{k+1,l} - p_{k,l})^2 + (p_{k,l+1} - p_{k,l})^2}$$

where  $p_{k,l}$  denotes the value of (i, j)-th pixel for  $k = 1, \dots, s$  and  $l = 1, \dots, t$ . Somehow the TV function measures smoothness of image, the less TV the more smoothness image. From calculus we know any multi-variable function at each point most decreases in the opposite direction of its gradient at that point, we use this fact and by using operator T in 3.2 for solving linear system define a sequence

$$x^{k+1} = T(x^k + \beta_k v_k)$$

in which  $v_k = -\frac{\nabla TV(x^k)}{\|\nabla TV(x^k)\|}$  and  $\beta_k = \beta^k$  where  $0 < \beta < 1$ . According to theorem 3.3 and 3.2 the above iterative method converges to the solution of linear system Ax = b, in addition the term  $\beta_k v_k$  in each step try to steer solution toward a solution with less TV function values. In fact we use a special case of Superiorization algorithm see [3] for general definition of Superiorization algorithm and [6] for utility of Superiorization algorithm in discrete tomography.

In the following, we solve an example of image reconstruction from projection field. To producing data for this example we use Matlab Air-tools package [2]. The image which should be reconstructed is a  $450 \times 450$  pixel image and coefficient matrix A is a  $80550 \times 202500$  which partitioned into 16 equal size blocks. Figure 1 demonstrate relative error  $\left(\frac{\|x^*-x^k\|}{\|x^*\|}\right)$  for EMR method and its superiorization version, as it is seen, after 10 iteration EMR with superiorization hav about 70% better relative error than EMR.



TABLE 1. Summery of results

Images	TV	Relative error	CPU time (in seconds)
Original image	2968	-	-
EMR image	18288	0.1204	83.4
EMR with Superiorization	4023	0.0436	83.8

FIGURE 1. Relative error for EMR method (blue) and EMR with Superiorization (red)

Figure 2 shows original image (middle) and reconstructed image with EMR (left) and reconstructed image with Superiorization (right). Table 1 summarizes results of our example.



FIGURE 2. Recunstructed image with EMR (left), original image (middle), Recunstructed image with EMR and Superiorization (right)

#### References

- Yair Censor, Gabor T Herman, Ming Jiang, Daniel Reem, and Alvaro De Pierro. Superiorization: theory and applications. *Inverse Problems*, 33(4):040301, 2017.
- [2] Per Christian Hansen and Maria Saxild-Hansen. Air toolsa matlab package of algebraic iterative reconstruction methods. *Journal of Computational and Applied Mathematics*, 236(8):2167-2178, 2012.
- [3] Gabor T Herman, Edgar Garduño, Ran Davidi, and Yair Censor. Superiorization: An optimization heuristic for medical physics. *Medical Physics*, 39(9):5532-5546, 2012.
- [4] T Nikazad, M Abbasi, and T Elfving. Error minimizing relaxation strategies in landweber and kaczmarz type iterations. Journal of Inverse and Ill-posed Problems, 2015.
- [5] Touraj Nikazad and Mokhtar Abbasi. A unified treatment of some perturbed fixed point iterative methods with an infinite pool of operators. *Inverse Problems*, 33(4):044002, 2017.
- [6] Touraj Nikazad, Ran Davidi, and Gabor T Herman. Accelerated perturbation-resilient blockiterative projection methods with application to image reconstruction. *Inverse problems*, 28(3):035005, 2012.

Department of Mathematics, University of Qom, PO Box 37161-46611, Qom, Iran  $E\text{-}mail\ address:\ m.abbasi@qom.ac.ir$ 

School of Mathematics, Iran University of Science and Technology, PO Box 16846-13114, Tehran, Iran

E-mail address: tnikazad@iust.ac.ir



#### ON THE SPECTRA OF THE GRAPHS OBTAINED BY GENERALIZED JOIN GRAPH OPERATION

ABDOLLAH ALHEVAZ AND MARYAM BAGHIPUR

ABSTRACT. Let G be a simple connected graph,  $v_i$  its vertex, and the transmission  $Tr_G(v_i)$  is defined to be the sum of the distances from  $v_i$  to all other vertices in G. The distance signless Laplacian matrix of G defined as  $D^Q(G) = Tr(G) + D(G)$ , where D(G) is the distance matrix of G and Tr(G)is the diagonal matrix whose main entries are the vertex transmissions of G. The distance signless Laplacian spectral radius of G, is the spectral radius of  $D^Q(G)$ , which is the largest eigenvalue of  $D^Q(G)$ . In this paper, first we we describe the distance signless Laplacian spectrum of the joined union of some classes of graphs. Then, we report some new bounds for the distance signless Laplacian spectral radius and distance signless Laplacian energy of graphs.

Keywords: Distance signless Laplacian matrix; Joined union; Transmission regular graph; Spectral radius; Energy.

#### 1. INTRODUCTION

Throughout this paper, all graphs considered are undirected, simple and connected. A graph is denoted by G = (V(G), E(G)), where V(G) is its vertex set and E(G) its edge set. The order of G is the number of its vertices and its size is the number of its edges. The set of vertices adjacent to  $v \in V(G)$ , denoted by N(v), refers to the neighborhood of v, and the degree of v means the cardinality of N(v) and denoted by  $deg_G(v)$ . Let G be a connected graph with vertex set V(G). The distance between two vertices  $u, v \in V(G)$  denoted by  $d_G(u, v)$  or  $d_{uv}$ , is defined as the length of a shortest path between u and v in G. The diameter of G is the maximum distance between vertices. The distance matrix of G is denoted by  $D(G) = (d_{v_i v_j})_{v_i, v_j \in V(G)}$ . The transmission  $Tr_G(v)$  of a vertex v is defined to be the sum of the distances from v to all other vertices in G, i.e.,  $Tr_G(v) = \sum_{u \in V(G)} d_G(u, v)$ .

The investigation of matrices related to various graphical structures is a very large and growing area of research. In particular, distance signless Laplacian matrix (spectrum) have attracted serious attention in the literature, since it have many useful applications, see [6] for example of recent results and further references. Let G be a connected graph with  $V(G) = \{v_1, v_2, \ldots, v_n\}$ . For  $1 \le i \le n$ , one can easily see that  $Tr_G(v_i)$  is just the *i*-th row sum of D(G). Let Tr(G) = $diag(Tr_G(v_1), Tr_G(v_2), \ldots, Tr_G(v_n))$  be the diagonal matrix of vertex transmissions of G. The distance signless Laplacian matrix of G is the  $n \times n$  matrix first defined in [1], as  $D^Q(G) = Tr(G) + D(G)$ . For a square matrix M, the collection of its eigenvalues together with their multiplicities is called the *spectrum* of M. In

<sup>2010</sup> Mathematics Subject Classification. Primary 15A42; Secondary 15A18, 05C50. Speaker: Abdollah Alhevaz.

particular, if M is  $D^Q(G)$ , then this collection is called the *distance signless Lapla*cian spectrum of G. The *distance signless Laplacian spectral radius* of G, denoted by  $\rho(G)$ , is the spectral radius of  $D^Q(G)$ , which is the largest eigenvalue of  $D^Q(G)$ .

In [1], Aouchiche and Hansen gave the distance Laplacian characteristic polynomials of some special graphs, and proved that the distance Laplacian eigenvalues do not decrease on deletion of edges. The *transmission* of a connected graph G, denoted by  $\sigma(G)$ , is the sum of distances between all unordered pairs of vertices in G. Clearly,  $\sigma(G) = \frac{1}{2} \sum_{v \in V(G)} Tr_G(v)$ . A graph G is said to be *transmission regular* if  $Tr_G(v)$  is a constant for each  $v \in V(G)$ .

Energy of a graph is a concept defined in 1978 by Ivan Gutman in [2] and originating from theoretical chemistry. Let G be a simple graph of order n with adjacency matrix A(G) having eigenvalues  $\lambda_1, \lambda_2, \ldots, \lambda_n$ . Then the energy of a graph G, denoted by E(G), is defined as  $E(G) = \sum_{i=1}^{n} |\lambda_i|$ . In this paper, we will also want to consider a new kind of energy with respect to distance signless Laplacian matrix, giving a lower and an upper bound for the distance signless Laplacian energy of graphs.

#### 2. Main Results

If  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  are two graphs on disjoint sets of  $n_1$  and  $n_2$  vertices, respectively, then their *union* is the graph  $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ . Their *join* is denoted by  $G_1 \vee G_2$  and consists of  $G_1 \cup G_2$  and all lines joining  $V_1$  and  $V_2$ .

**Definition 2.1.** [5] Let G = (V, E) be a graph of order n and  $G_i = (V_i, E_i)$  be a graph of order  $m_i$ , where i = 1, ..., n. Then, the joined union  $G[G_1, G_2, ..., G_n]$  is the graph H = (W, F) with:

and 
$$W = \bigcup_{i=1}^{n} V_{i}$$
$$F = \bigcup_{i=1}^{n} E_{i} \cup \bigcup_{\{v_{i}, v_{j}\} \in E} V_{i} \times V_{j}.$$

We start our results with the following lemma, which will be useful in the sequel.

**Lemma 2.2.** [6] Let G be a connected graph on n vertices. Then,

$$\rho(G) \ge \frac{4\sigma(G)}{n},$$

and equality holding if and only if G is transmission regular.

**Theorem 2.3.** Let G be a s-regular graph of order n with adjacency eigenvalues  $\mu_1 = s \ge \mu_2 \ge \ldots \ge \mu_n$  and diameter at most 2 and let H be a r-regular graph of order m with adjacency eigenvalues  $\lambda_1 = r \ge \lambda_2 \ge \ldots \ge \lambda_m$ . Then the distance signless Laplacian spectrum of  $G[H, H, \ldots, H]$  consists of eigenvalues  $2nm - \lambda_k - r - sm - 4$  for  $2 \le k \le m$  (n times) also with the eigenvalues of the matrix 2m(J + A(G)) + (2nm - sm - 2r - 4)I that are as 4mn + ms - 2r - 4 and  $2nm + 2m\mu_i - sm - 2r - 4$  for  $2 \le j \le n$ .

**Theorem 2.4.** Let  $G_i$  be an  $r_i$ -regular graph of order  $m_i$  with adjacency eigenvalues  $\lambda_{i1} = r_i \geq \lambda_{i2} \geq \ldots \geq \lambda_{im_i}$ , for i = 1, 2, 3. Then, the distance signless Laplacian spectrum of the joined union  $K_3[G_1, G_2, G_3]$  consists of the eigenvalues

 $m + m_i - \lambda_{ik} - r_i - 4$  for i = 1, 2, 3 and  $k = 2, 3, \ldots, m_i$ , where  $m = \sum_{i=1}^3 m_i$ , also with the eigenvalues of the matrix

$$\begin{bmatrix} m+3m_1-2r_1-4 & m_2 & m_3 \\ m_1 & m+3m_2-2r_2-4 & m_3 \\ m_1 & m_2 & m+3m_3-2r_3-4 \end{bmatrix}$$

**Theorem 2.5.** Let  $G_i$  be an  $r_i$ -regular graph of order  $m_i$  with adjacency eigenvalues  $\lambda_{i1} = r_i \geq \lambda_{i2} \geq \ldots \geq \lambda_{im_i}$ , for i = 1, 2, 3. Then, the distance signless Laplacian spectrum of the joined union  $P_3[G_1, G_2, G_3]$  consists of the eigenvalues  $2m - \lambda_{ik} - r_i - m_2 - 4$  and  $2m + m_2 - \lambda_{2k} - r_2 - 4$  for i = 1, 3 and  $k = 2, 3, \ldots, m_i$ , where  $m = \sum_{i=1}^{3} m_i$ , also with the eigenvalues of the matrix

$$\begin{bmatrix} 2m+2m_1-m_2-2r_1-4 & m_2 & 2m_3 \\ m_1 & m+3m_2-2r_2-4 & m_3 \\ 2m_1 & m_2 & 2m+2m_1-m_2-2r_1-4 \end{bmatrix}.$$

In the following result, we give a lower bound on  $\rho(G)$  in terms of transmission degrees.

**Theorem 2.6.** If the transmission degree sequence of G is  $\{Tr_1, \ldots, Tr_n\}$ , then

$$\rho(G) \ge 4\sigma(G) + \frac{2}{Tr_{\max}^2} \sum_{i < j} (Tr_i^2 - Tr_j^2).$$

**Theorem 2.7.** Let G be a connected graph of order n. Then,

$$\rho(G) \le n(n-1).$$

Next, we establish two lower bounds for the distance signless Laplacian energy of graphs.

**Theorem 2.8.** Let G be a connected graph of order n. Then,

$$E_{D^Q}(G) \ge \frac{2\sigma(G)}{n} + (n-1)(\frac{n|\det D^Q|}{2\sigma(G)})^{\frac{1}{n-1}}.$$

**Theorem 2.9.** If G is a connected graph of order n, then we get  $E_{DQ}(G) \ge \sqrt{2n(n-1)}$ .

#### Acknowledgement

This research was in part supported by a grant from Shahrood University of Technology.

#### References

- M. Aouchiche and P. Hansen, Two Laplacians for the distance matrix of a graph, *Linear Algebra Appl.*, 439 (2013) 21–33.
- I. Gutman, The energy of a graph, Ber. Math. Statist. Sekt. Forschungsz. Graz, 103 (1978), 1–22.
- 3. I. Gutman and B. Zhou, Laplacian energy of a graph, Linear Algebra Appl., 414 (2006) 29–37.
- G. Indulal, I. Gutman and A. Vijayakumar, On distance energy of graphs, MATCH Commun. Math. Comput. Chem., 60 (2008), 461–472.
- D. Stevanović, Large sets of long distance equienergetic graphs, Ars Math. Contemp., 2(1) (2009) 35–40.

R. Xing, B. Zhou and J. Li, On the distance signless Laplacian spectral radius of graphs, *Linear Multilinear Algebra*, 62 (2014), 1377–1387.

Department of Mathematics, Shahrood University of Technology, P.O. Box: 316-3619995161, Shahrood, Iran.

 $E\text{-}mail\ address: \verb"a.alhevaz@shahroodut.ac.ir"$ 

DEPARTMENT OF MATHEMATICS, SHAHROOD UNIVERSITY OF TECHNOLOGY, P.O. BOX: 316-3619995161, SHAHROOD, IRAN.

 $E\text{-}mail\ address: \texttt{maryamb8989@gmail.com}$ 

8



#### A SHARPER BOUND FOR THE RELATIVE OPERATOR ENTROPY

MEHDI ALINEJAD AND ISMAIL NIKOUFAR

ABSTRACT. In this paper, we find some bounds for the Tsallis relative operator entropy. Moreover, we give a sharper bound for the relative operator entropy in compare with the bound introduced by Dragomir.

Keywords: Shannon entropy; Tsallis entropy; Relative operator entropy; Tsallis relative operator entropy; Operator inequality.

#### 1. INTRODUCTION

Tsallis entropy was defined in [10] by

$$S_q(X) = -\sum_x p(x)^q \ln_q p(x)$$

with one parameter q as an extension of Shannon entropy, where q-logarithm is defined by  $\ln_q(x) = \frac{x^{1-q}-1}{1-q}$  for any nonnegative real number q and x in the statistical physics, and p(x) = p(X = x) is the probability distribution of the given random variable X. As  $q \to 1$ , the Tsallis entropy  $S_q(X)$  converges to the Shannon entropy  $-\sum_x p(x) \log p(x)$ . The notion of entropy is essential and fundamental in thermodynamical and statistical physics, information theory, and analytical mathematics such as operator theory and probability theory.

A relative operator entropy of strictly positive operators A, B was introduced in the noncommutative information theory by Fujii and Kamei [4] by

$$S(A|B) := A^{\frac{1}{2}} (\ln A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

When A is positive, one may set  $S(A|B) = \lim_{\epsilon \to 0} S(A + \epsilon I|B)$  if the limit which is taken in the strong operator topology exists.

For two strictly positive operators A, B and  $0 < \lambda \leq 1$ , the Tsallis relative operator entropy introduced by Yanagi et al. [11] and defined by

$$T_{\lambda}(A|B) := \frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\lambda}A^{\frac{1}{2}} - 1}{\lambda}.$$

Clearly, the Tsallis relative operator entropy is a generalization of the relative operator entropy S(A|B), i.e.,

$$\lim_{\lambda \to 0} T_{\lambda}(A, B) = S(A, B).$$

The following inequalities proved in [6, 5]:

(1.1) 
$$T_{-\lambda}(A|B) \le S(A|B) \le T_{\lambda}(A|B),$$

(1.2) 
$$A - AB^{-1}A \le T_{\lambda}(A|B) \le B - A,$$

<sup>2010</sup> Mathematics Subject Classification. 28D20, 47A50. Speaker: Mehdi Alinejad.

where these inequalities were refined in [12] as follows:

$$A - AB^{-1}A \le T_{-\lambda}(A|B) \le S(A|B) \le T_{\lambda}(A|B) \le B - A$$

A fully non-commutative perspective of the one variable function f defined in [3] by setting

$$P_f(A,B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

where A is a strictly positive operator and B is a self-adjoint operator on a finite dimensional Hilbert space  $\mathcal{H}$  with spectra in the closed interval I containing 0.

In this paper, we find some bounds for the Tsallis relative operator entropy. Moreover, we give a sharper bound for the relative operator entropy in compare with the bound introduced by Dragomir. We determined the bounds of the generalized relative operator entropy in [9].

#### 2. TSALLIS RELATIVE OPERATOR ENTROPY

The following theorem proved in [8, Theorem 2.1] for the real valued functions r, s, and k defined on the closed interval  $\mathbb{I}$ .

**Theorem 2.1.** Let r, s, and k be real valued functions on the closed interval  $\mathbb{I}$ . If  $r(t) \leq s(t) \leq k(t)$  for  $t \in \mathbb{I}$ , then

$$P_r(A,B) \le P_s(A,B) \le P_k(A,B)$$

for strictly positive operator A and self-adjoint operator B.

**Remark 2.2.** Dragomir in 2006 [2] proved that if  $\phi : C \to \mathbb{R}$  is a concave function on  $C \subset \mathbb{R}$ , then

$$2r\Big[\phi(\frac{x+y}{2}) - \frac{\phi(x) + \phi(y)}{2}\Big] \le \phi(cx + (1-c)y) - (c\phi(x) + (1-c)\phi(y)) \\\le 2R\Big[\phi(\frac{x+y}{2}) - \frac{\phi(x) + \phi(y)}{2}\Big]$$

for any  $x, y \in \mathbb{R}$  and  $c \in [0, 1]$ , where  $r = \min\{c, 1 - c\}$  and  $R = \max\{c, 1 - c\}$ .

**Theorem 2.3.** Let A and B be two positive operators such that the condition  $mA \leq B \leq MA$  for some m, M with 0 < m < M is valid. Then,

$$2W_{\lambda}(m,M)P_{r}(A,B) \leq T_{\lambda}(A,B) - \frac{m^{\lambda}}{\lambda(M-m)}(MA-B) - \frac{M^{\lambda}}{\lambda(M-m)}(B-mA)$$
$$\leq 2W_{\lambda}(m,M)P_{R}(A,B),$$

where  $0 < \lambda \leq 1$  and

$$W_{\lambda}(m,M) = \frac{(m+M)^{\lambda} - 2^{\lambda}}{2^{\lambda}\lambda} - \frac{m^{\lambda} + M^{\lambda} - 2}{2\lambda},$$
  
$$r(u) = \min\left\{\frac{M-u}{M-m}, \frac{u-m}{M-m}\right\} = \frac{1}{2} - \left|\frac{u - \frac{M+m}{2}}{M-m}\right|,$$
  
$$R(u) = \max\left\{\frac{M-u}{M-m}, \frac{u-m}{M-m}\right\} = \frac{1}{2} + \left|\frac{u - \frac{M+m}{2}}{M-m}\right|.$$

**Theorem 2.4.** Let A and B be two positive operators such that the condition  $mA \leq B \leq MA$  for some m, M with 0 < m < M is valid. Then,

$$0 \le T_{\lambda}(A,B) - \frac{m^{\lambda} - 1}{\lambda(M - m)}(MA - B) - \frac{M^{\lambda} - 1}{\lambda(M - m)}(B - mA)$$
  
$$\le \frac{m^{\lambda - 1} - M^{\lambda - 1}}{M - m}P_f(A,B) \le \frac{1}{4}(M - m)(m^{\lambda - 1} - M^{\lambda - 1})A,$$

where  $0 < \lambda \le 1$  and f(t) = (t - m)(M - t).

**Theorem 2.5.** [1, Theorem 2] Let A and B be two positive operators such that the condition  $mA \leq B \leq MA$  for some m, M with 0 < m < M is valid. Then,

(2.1) 
$$K(\frac{M}{m})P_r(A,B) \le S(A,B) - \frac{\ln m}{M-m}(MA-B) - \frac{\ln M}{M-m}(B-mA) \le K(\frac{M}{m})P_R(A,B),$$

where  $K(h) = \frac{(h+1)^2}{4h}$ , h > 0 is the Kantrovich's constant.

We remark that if  $\lambda$  tends to zero in Theorems 2.3 and 2.4, we obtain Corollaries 2.6 and 2.7, respectively.

**Corollary 2.6.** Let A and B be two positive operators such that the condition  $mA \leq B \leq MA$  for some m, M with 0 < m < M is valid. Then

$$2W_0(m, M)P_r(A, B) \le S(A, B) - \frac{\ln m}{M - m}(MA - B) - \frac{\ln M}{M - m}(B - mA)$$
(2.2) 
$$\le 2W_0(m, M)P_R(A, B),$$

where  $W_0(m, M) = \ln \frac{m+M}{2\sqrt{mM}}$ .

**Corollary 2.7.** Let A and B be two positive operators such that the condition  $mA \leq B \leq MA$  for some m, M with 0 < m < M is valid. Then

$$0 \le S(A,B) - \frac{\ln m}{M-m}(MA-B) - \frac{\ln M}{M-m}(B-mA)$$
$$\le \frac{1}{Mm}P_f(A,B) \le \frac{(M-m)^2}{4mM}A,$$

where f(t) = (t - m)(M - t).

By a simple calculation, we find that  $\frac{(M-m)^2}{4mM} = K(\frac{M}{m}) - 1$  and hence Corollary 2.7 confirms the result appeared in[1, Theorem 3]. Since  $2 \ln x \le x^2$ , x > 0, so for  $x = \frac{M+m}{2\sqrt{mM}}$  we have  $2W_0(m, M) \le K(\frac{M}{m})$ . This shows that our determined upper bound  $2W_0(m, M)$  in (2.2) is sharper than the upper bound  $K(\frac{M}{m})$  determined by Dragomir in (2.1).

#### References

- S. S. DRAGOMIR, Some inequalities for relative operator entropy, Preprint RGMIA Res. Rep. Col l. 18, Art. 145. (2015).
- S. S. DRAGOMIR, Bounds for the normalized Jensen functional, Bull. Austral. Math. Soc., 74, (2006), 471–478.
- A. EBADIAN, I. NIKOUFAR, AND M. ESHAGI GORDJI, Perspectives of matrix convex functions, Proc. Natl. Acad. Sci., 108(18), (2011) 7313–7314.

- J. I. FUJII AND E. KAMEI, Relative operator entropy in noncommutative information theory, Math. Japonica, 34 (1989), 341–348.
- J. I. FUJII AND E. KAMEI, Uhlmann's interpolational method for operator means, Math. Japonica, 34 (1989), 541-547.
- S. FURUICHI, K. YANAGI, K. KURIYAMA, A note on operator inequalities of Tsallis relative operator entropy, Linear Algebra Appl., 407 (2005) 19–31.
- M. OHYA AND D. PETZ, Quantum entropy and its use, Springer-Verlag, Heidelberg, 1993, Second edition, 2004.
- I. NIKOUFAR, On operator inequalities of some relative operator entropies, Adv. Math., 259, (2014), 376–383.
- 9. I. NIKOUFAR AND M. ALINEJAD, Bounds of generalized relative operator entropies, to appear in Math. Ineq. Appl.
- C. TSALLIS, Possible generalization of Bolzmann-Gibbs statistics, J. Stat. Phys., 52 (1988), 479–487.
- K. YANAGI, K. KURIYAMA, AND S. FURUICHI, Generalized Shannon inequalities based on Tsallis relative operator entropy, Linear Alg. Appl., 394 (2005), 109–118.
- L. ZOU, Operator inequalities associated with Tsallis relative operator entropy, Math. Ineq. Appl., 18(2) (2015), 401–406.

Department of Mathematics, Payame Noor University, P.O. Box 19395-3697 Tehran, Iran

E-mail address: alinejad\_mehdi@yahoo.com

Department of Mathematics, Payame Noor University, P.O. Box 19395-3697 Tehran, Iran

*E-mail address*: nikoufar@pnu.ac.ir



#### RECENT EXTENSIONS OF AFFINE LIE ALGEBRAS

SAEID AZAM

ABSTRACT. Lie theory and Linear algebra meet in many places and directions, the well-known meeting-place is the classical matrix groups. In a more technical words, all finite dimensional Lie algebras can be realized as some vector spaces of square matrices. In fact as Ado's theorem states, any finitedimensional Lie algebra over a field of characteristic zero is isomorphic to a Linear Lie algebra, namely it can be viewed as a Lie algebra of square matrices under the commutator bracket. Among Lie algebras, the class of Kac-Moody Lie algebras and in particular its subclass of affine Lie algebras plays a very important role inside and outside of the mathematics, specially in mathematical physics. Since the birth of Kac-Moody theory in 1968, there has been much more progress in the development of the affine Lie theory and its extensions. In this talk we give a survey introduction to the recent developments of the theory, not claiming to be complete. We emphasize on some connections with Linear algebra.

UNIVERSITY OF ISFAHAN AND IPM

Speaker: Saeid Azam.



#### BIDERIVATIONS ON TRIANGULAR BANACH ALGEBRAS

#### S. BAROOTKOOB

ABSTRACT. Let A and B be two approximately unital Banach algebras, X be an approximately unital A-B-module and T be the triangular Banach algebra associated to A, B and X. In this paper we characterize the biderivations and inner biderivations from  $T \times T$  into  $T^*$ . Then we show that the first bicohomology group  $BH^1(T,T^*)$  is equal to  $BH^1(A,A^*) \oplus BH^1(B,B^*)$ .

Keywords: Triangular Banach algebra; Biderivation; Inner biderivation.

#### 1. INTRODUCTION

A derivation from a Banach algebra A to a Banach A-module X is a bounded linear mapping  $d : A \to X$  such that d(ab) = d(a)b + ad(b)  $(a, b \in A)$ . For each  $x \in X$  the mapping  $\delta_x : a \to ax - xa$ ,  $(a \in A)$  is a bounded derivation, called an inner derivation.

A bounded bilinear mapping  $D: A \times A \to X$  is called a biderivation if D is a derivation with respect to both arguments. That is the mappings  $_aD: A \to X$  and  $D_b: A \to X$ , where

$${}_{a}D(b) = D(a,b) = D_{b}(a) \qquad (a,b \in A)$$

are derivations. We denote the space of such biderivations by  $BZ^{1}(A, X)$ .

Let  $x \in Z(A, X) = \{x \in X; ax = xa \ \forall a \in A\}$ . The map  $D_x : A \times A \to X$  that

$$D_x(a,b) = x[a,b] = x(ab - ba) \qquad (a,b \in A)$$

is a basic example of a biderivation which is called an inner biderivation implemented by x. We denote the space of such inner biderivations by  $BN^1(A, X)$ . Also we define the first bicohomology group  $BH^1(A, X)$  as follows,

$$BH^1(A,X) = \frac{BZ^1(A,X)}{BN^1(A,X)}.$$

For more applications of biderivations, see the survey article [6, Section 3]. Some algebraic aspects of biderivations on certain algebras were studied by many authors; see for example [4, 7], where the structures of some biderivations on triangular algebras and generalized matrix algebras are discussed, and particularly the question of whether these biderivations on these algebras are inner, was studied.

Let A and B be Banach algebras and X be an A-B-module. Then the algebra

$$T = \left\{ \left( \begin{array}{cc} a & x \\ 0 & b \end{array} \right); a \in A, x \in X, b \in B \right\}$$

equipped with the usual addition and multiplication of matrix and with the norm

$$\left\| \left( \begin{array}{cc} a & x \\ 0 & b \end{array} \right) \right\| = \|a\| + \|x\| + \|b\|$$

<sup>2010</sup> Mathematics Subject Classification. 46H20, 46H25.

Speaker: S. Barootkoob.

is a Banach algebra which is called triangular Banach algebra associated to X. Then the dual of triangular Banach algebra T is

$$T^* = \left\{ \begin{pmatrix} f & h \\ 0 & g \end{pmatrix}; f \in A^*, h \in X^*, g \in B^* \right\};$$
$$\left( \begin{pmatrix} a & x \\ 0 \end{pmatrix} - f(g) + h(x) + g(h) \right\}$$

where  $\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \begin{pmatrix} \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = f(a) + h(x) + g(b).$ Recall that For every Banach A-module X the dual space X<sup>\*</sup> is a Banach

Recall that For every Banach A-module X the dual space  $X^*$  is a Banach A-module with module structures a.f and f.a that a.f(x) = f(xa) and f.a(x) = f(ax). So  $T^*$  is a T-module with the module actions

$$\left(\begin{array}{cc}a & x\\0 & b\end{array}\right) \cdot \left(\begin{array}{cc}f & h\\0 & g\end{array}\right) = \left(\begin{array}{cc}af+xh & bh\\0 & bg\end{array}\right)$$

and

$$\left(\begin{array}{cc}f&h\\0&g\end{array}\right)\cdot\left(\begin{array}{cc}a&x\\0&b\end{array}\right)=\left(\begin{array}{cc}fa&ha\\0&hx+gb\end{array}\right).$$

A Banach algebra A is called amenable if for each Banach A-module X, the only derivations from A to  $X^*$  are inner derivations. Also A is called weakly amenable if every derivation from A to  $A^*$  is an inner derivation. The concept of amenability was first introduced by Johnson in [9]. Then the concept of weak amenability was introduced by Bade, Curtis and Dales [2] for commutative Banach algebras and then by Johnson [10] for a general Banach algebra.

In this paper we consider A and B as approximately unital Banach algebras with approximate identities  $e_{\alpha}$  and  $E_{\beta}$  respectively, and X as an approximately unital A - B-module that is  $e_{\alpha}x = xe_{\alpha} = E_{\beta}x = xE_{\beta} = x$ , for every  $x \in X$ . Then we characterize the biderivations and inner biderivations from  $T \times T$  to  $T^*$ . In particular we show that  $BH^1(T, T^*) = BH^1(A, A^*) \oplus BH^1(B, B^*)$ .

#### 2. Some biderivations on triangular Banach algebras

Similar to the definitions of amenability and weak amenability of Banach algebras we may define the notion of biamenability and weak biamenability of Banach algebras as follows.

**Definition 2.1.** We say that a Banach algebra A is biamenable if for each A-module X,  $BH^1(A, X^*) = \{0\}$ , and weakly biamenable if  $BH^1(A, A^*) = \{0\}$ .

For examples of some Banach algebras which are biamenable or weakly biamenable see [3].

Noted that despite the apparent similarity between derivations and biderivations and also inner derivations and inner biderivations, there are fundamental differences between these. Especially when a biderivation wants to be an inner bidetivation these differences become more apparent. A part of these differences come from the nature of biderivations which are depend on two components. An other essential part of these differences goes back to the definition of inner biderivations which the implemented elements should be in Z(A, X). According to this the concept of amenability and also weak amenability are different from biamenability and weak biamenability, respectively [3].

For characterizing the biderivations from  $T \times T$  to  $T^*$  we commence with the next lemma which characterize all derivations from T to  $T^*$ .

**Lemma 2.2.** [8, Lemma 3.2] Let  $D: T \to T^*$  be a derivation. Then there exists derivations  $d_1: A \to A^*$  and  $d_2: B \to B^*$  and an element  $\gamma \in X^*$  such that

$$(i) D\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} d_1(a) & \gamma a \\ 0 & 0 \end{pmatrix};$$
$$(ii) D\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & -b\gamma \\ 0 & d_2(b) \end{pmatrix};$$
$$(iii) D\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -x\gamma & 0 \\ 0 & \gamma x \end{pmatrix}.$$

**Theorem 2.3.** The bilinear mapping  $D : T \times T \to T^*$  is a biderivation if and only if there exist biderivations  $d_A : A \times A \to A^*$  and  $d_B : B \times B \to B^*$  such that for each  $a, a' \in A$  and  $b, b' \in B$ ,

$$D(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}) = \begin{pmatrix} d_A(a,a') & 0 \\ 0 & d_B(b,b') \end{pmatrix}.$$

Lemma 2.4.  $Z(T,T^*) = \{ \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}; f \in Z(A,A^*), g \in Z(B,B^*) \}.$ 

**Theorem 2.5.** The biderivation  $D: T \times T \to T^*$  which is defined for each  $a, a' \in A, b, b' \in B$  and  $x, x' \in X$ , by

$$D(\left(\begin{array}{cc}a & x\\ 0 & b\end{array}\right), \left(\begin{array}{cc}a' & x'\\ 0 & b'\end{array}\right)) = \left(\begin{array}{cc}d_A(a,a') & 0\\ 0 & d_B(b,b')\end{array}\right)$$

is an inner biderivation if and only if  $d_A$  and  $d_B$  are inner biderivations.

**Corollary 2.6.** Let A and B be approximately unital Banach algebras, X be an approximately unital A - B-module and T be the triangular Banach algebra associated to A, B and X. Then (i)  $BH^1(T,T^*) = BH^1(A,A^*) \oplus BH^1(B,B^*)$ . (ii) T is weakly biamenable if and only if A and B are weakly biamenable.

#### References

- Gh. Abbaspour, M.S. Moslehian and A. Niknam, Generalized derivations on modules, Bull. Iranian Math. Soc., 32(1) (2006) 21-30.
- [2] W. G. Bade, P. C. Curtis, and H. G. Dales, Amenability and weak amenability for Beurling and Lipschitz algebras, Proc. London Math. Soc. 55 (1987), 359–377.
- [3]S. Barootkoob, H. R. Ebrahimi Vishki<br/>,  $\,n{\rm -weak}$  biamenability of Banach algebras, Preprint.
- [4] D. Benkovič, Biderivations of triangular algebras, Linear Algebra Appl. 431 (2009), 1587-1602.
- [5] M. Brešar, On the distance of the compositions of two derivations to the generalized derivations, Glasgow Math. J., 33(1) (1991) 89-93.
- [6] M. Brešar, Commuting maps: A survey, Taiwanese J. Math. 8 (2004), 361–397.
- [7] Y. Du and Y. Wang, Biderivations of generalized matrix algebras, Linear Algebra. Appl. 438 (2013), 4483–4499.
- [8] B. E. Forrest and L. W. Marcoux, Derivations on triangular Banach algebras Indiana Univ. Math. J., 45 (1996) 441-462.
- [9] B.E. Johnson, Cohomology in Banach algebras, Mem. Amer. Math. Soc. 127 (1972).
- [10] B. E. Johnson, Weak amenability of group algebras, Bull. London Math. Soc. 23 (1991), 281–284.
- [11] M. Mosadeq, Module generalized derivations on triangular Banach algebras Journal of Mahani mathematical research center 2 (1) (2013) 43-52.
- [12] M. Mosadeq, M. Hassani and A. Niknam, $(\sigma, \gamma)$ -Generalized dynamics on modules, J. Dyn. Syst. Geom. Theor., **9**(2) (2011) 171-184.

FACULTY OF BASIC SCIENCES,, UNIVERSITY OF BOJNORD,, P.O. BOX 1339, BOJNORD, IRAN *E-mail address*: s.barutkub@ub.ac.ir



#### SOME LINEAR ALGEBRA METHOD FOR COUNTING THE NUMBER OF PERFECT MATCHINGS IN GRAPHS

AFSHIN BEHMARAM

ABSTRACT. We Apply some linear Algebra method to counting the number of perfect matching or finding upper bounds on perfect matchings in graphs.we describe pfaffian method and transfer matrix method and then apply this method to present some results in counting the number of perfect matching in graphs.also, We apply our results to fullerene graphs.

Keywords: Pfaffian; Determinant enequality; Perfect Matching.

#### 1. INTRODUCTION

Let G = (V, E) be a simple undirected graph. Denote n := |V| the number of vertices; m := |E| the number of edges; d(v) the degree of  $v \in V$ ; Pm(G) the number of perfect matches in G; A(G) the adjacency matrix of G; S(G) is the skew symmetric adjacency matrix of the digraph DG = (V, Arc), where Arc is an orientation of edges E;  $G \times H$  the cartesian product of the graphs G and H;  $K_n$ a complete graph on n vertices,  $K_{r,r}$  a complete bipartite r-regular graph on 2rvertices,  $C_n$  a cycle on n vertices.

A variation of Bregman's inequality yields [3]

(1.1) 
$$Pm(G) \le \prod_{v \in V} (d(v)!)^{\frac{1}{2d(v)}}.$$

Equality holds if and only if G is a union of complete regular bipartite graphs. Since det S(G) is the square of the pfaffian corresponding to S(G) we deduce the well known inequality

(1.2) 
$$\det S(G) \le (Pm(G))^2.$$

A graph is called *pfaffian* if there is an orientation Arc of E such that equality holds in (1.2). It was shown by Kasteleyn [7] that every planar graph is pfaffian. An example of a pfaffian nonplanar graph is  $K_4 \times K_2$ .

Let  $B = [b_{uv}]_{u,v \in V}$  a real symmetric nonnegative definite matrix of order |V|, which is denoted by  $B \succeq 0$ . The generalized Hadamard-Fischer inequality, abbreviated here as H-F inequality, states

(1.3) 
$$\det B[U \cup W] \det B[U \cap W] \le \det B[U] \det B[W] \text{ where } U, W \subset V.$$

Here det  $B[\emptyset] = 1$ . (See for example [5] and references therein.) Assume that  $U \cap W = \emptyset$ . Then the above inequality is equivalent to the Hadamard-Fischer inequality. Furthermore if the left hand-side of (1.3) is positive then equality holds

<sup>2010</sup> Mathematics Subject Classification. 05C30, 05C70.

Speaker: Afshin Behmaram.

if and only if  $B[U \cup W]$  is a block diagonal matrix  $\operatorname{diag}(B[U], B[W])$ . In particular Hadamard's determinat inequality for  $B \succeq 0$  is

(1.4) 
$$\det B \le \prod_{v \in V} b_{vv}.$$

If all diagonal entries of B are positive then equality holds if and only if B is a diagonal matrix.

We apply Hadamard's determinat inequality to give upper bounds for the number of perfect matchings in Pfaffian graphs. We also, derive a number of improvements of our upper bounds using the Hadamard-Fischer inequality mainly for graphs with no 4-cycles.

A fullerene graph is a cubic, planar, 3-connected graph with only pentagonal and hexagonal faces. Classical fullerene graphs have been intensely researched since the discovery of buckminsterfullerene in the fundamental paper [?], which appeared in 1985. This paper gave rise to the whole new area of fullerene science.

A connected 3-regular planar graph G = (V, E) is called an *m*-generalized fullerene [1] if exactly two of its faces are *m*-gons and all other faces are pentagons and/or hexagons. ote that for m = 5, 6 an *m*-generalized fullerene graph is a classical fullerene graph. As for the classical fullerenes it is easy to show that the number of pentagons is fixed, while the number of hexagons is not determined. The smallest *m*-generalized fullerene has 4m vertices and no hexagonal faces. Such graphs are sometimes called *m*-barrels. They have two *m*-gons and 2m pentagons and they can be elongated by inserting  $k \ge 0$  layers of m hexagons between two half-barrels. An **m-barrel fullerene** F(m,k) is obtained from the corresponding barrel by inserting k > 0 layers (or rings) of m hexagons between two halves of the barrel. For m = 5 and m = 6 we obtain classical fullerene nanotubes. Most of the nanotube properties are also preserved by m-barrel fullerene. Note that F(m,k) has n =2m(k+2) vertices. The m-barrels Fullerene are on of the most symmetric fullerene and highly symmetric structure allows for obtaining good bounds and even exact results on the number of perfect matchings in them.see [1] and [2] for some result for the number of perfect matching in fullerene graphs.

#### 2. Main results

The main results of this paper are upper bounds for the prefect matching in pfaffian graphs using Hadamard's determinant inequality and its generalizations. Apply first the Hadamard's determinant inequality to S(G) to deduce that for any pfaffian G one has

(2.1) 
$$\operatorname{perfmat} G \le \prod_{v \in V} d(v)^{\frac{1}{4}}.$$

First we show that a complete bipartite graph  $K_{r,r}$  does not satisfy the above inequality unless r = 1, 2. Next we show that the above upper bound is sharp for the following *d*-regular connected pfaffian graphs. For d = 3 two planar graphs:  $K_4$ and  $C_4 \times K_2$ . For d = 4: a unique planar graph is octahedron (a complement of a perfect match in  $K_6$  and hence planar). Furthermore, if *G* is a planar 3-regular graph having  $3^{\frac{n}{4}}$  perfect matchings then *G* is a disjoint union of  $K_4$  and  $C_4 \times K_2$ . Also, we prove the following improvements of (2.1). **Theorem 2.1.** Let G = (V, E) be a pfaffian connected graph. Denote by G' = (V, E') be the induced graph by G, where  $(u, v) \in E'$  if and only if the distance between u and v in G is 2. Let M' be a match in G'. Assume that G does not have 4-cycles. Then

(2.2) 
$$Pm(G) \le (\prod_{(u,v)\in M'} (d(u)d(v)-1))^{\frac{1}{4}} (\prod_{v\in V\setminus V(M')} d(v))^{\frac{1}{4}}.$$

One can use the following proposition to estimate |M'|.

**Theorem 2.2.** Assume that G = (V, E), n = |V| is a connected graph with a path of length  $l \ge 3$ . Then the induced graph G' given in Theorem 2.1 has a match M' of cardinality  $2|\frac{l}{4}|$  and  $|\frac{l}{2}|$  if l is even or odd respectively.

**Theorem 2.3.** Every cubic pfaffian graph with n vertices, which has a perfect match and no 4-cycles, has at most  $8^{\frac{n}{12}}3^{\frac{n}{12}}$  perfect matching.

**Proposition 2.4.**  $Pm(F(3,k)) = 3^{k+2} + 1.$ 

#### Proposition 2.5.

$$Pm(F(4,k)) = 2(2+\sqrt{2})^{k+1} + 2(2-\sqrt{2})^{k+1} + 2^{k+3} + 1.$$

Proposition 2.6.

$$Pm(F(5,k)) = 5^{k+2} + 5\left[\left(\frac{5+\sqrt{5}}{2}\right)^k + \left(\frac{5-\sqrt{5}}{2}\right)^k\right] + 1.$$

#### References

- Afshin Behmaram, Tomislav Doslic, Shmuel Friedland, Matchings in generalized Fullerene, Ars Mathematica contemporanea, Vo11, No2 (2016), pp 301-311
- [2] A. Behmaram, S. Friedland, Upper bounds for perfect matchings in Pfaffian and planar graphs, *Electronic J. Combin.*, 20 (2013) #P64, 1–16.
- [3] N. Alon and S. Friedland, The maximum number of perfect matchings in graphs with a given degree sequence, with N. Alon, *The Electronic Journal of Combinatorics*, 15 (2008), #13, 1-2.
- [4] P.W. Fowler and D.B. Redmond, Symmetry aspects of leapfrog and truncated polyhedra, Match Communications in Mathematical and in Computer Chemistry 33 (1996), 101119.
- [5] S. Friedland and S. Gaubert, Submodular spectral functions of principal submatrices of an hermitian matrix, *Linear Algebra and its Applications*, 2012, available on line, arXiv:1007.3478.
- [6] S. Friedland, E. Krop and K. Markström, On the Number of Matchings in Regular Graphs, *The Electronic Journal of Combinatorics*, 15 (2008), #R110, 1-28, arXiv:0801.2256.
- [7] P.W. Kasteleyn, Graph theory and crystal physics, in *Graph Theory and Theoretical Physics* (ed. by F. Harary), Academic Press, New York 1967. unt, On A-groups, Proc. Cambridge Philos. Soc. 45(1949),24-42.

Faculty of mathematical Sciences, university of Tabriz, Tabriz, Iran  $E\text{-}mail\ address:\ \texttt{behmaram@tabrizu.ac.ir}$ 



#### STABILIZATION OF FRACTIONAL DISCRETE-TIME SYSTEMS VIA OUTPUT FEEDBACK

#### SAKINEH BIGOM MIRASSADI AND HOJJAT AHSANI TEHRANI

ABSTRACT. In this paper, we present a method based on matrix inverse eigenvalue problem to stabilize of unstable fractional discrete-time linear systems practically. The aim is assigning desirable eigenvalues to obtain satisfactory responses by propositional output feedback in not necessarily positive systems. At the end, conclusions are proposed and also the convergence of output vector in the fractional discrete-time linear system to zero is shown by figures in a numerical example.

Keywords: Fractional Discrete-time; Matrix inverse eigenvalue problem; Propositional output feedback.

#### 1. INTRODUCTION

Non-integer derivative has become nowadays a precious tool, currently used in the study of the behaviour of real systems and provides an excellent instrument for the description of memory and hereditary properties of various materials and processes such as viscoelastic systems, chaotic synchronization, electromagnetic systems, electrical circuits theory, fractances, mechatronics systems, signal processing, and chemical mixing. Positivity and stability of positive fractional discrete-time linear systems by state feedbacks have been investigated by some ways like Drazin inverse and Shuffle algorithm and etc. However, we do not deal with some initial conditions like having full row rank matrices in every performed algorithm and finding index of Shuffle and Drazin and we may not devote our method for only positive fractional systems. Stabilizing standard systems by output feedback controller has attracted considerable attention because it is usually not possible or practical to sense all the states and feed them back. In this article, matrix inverse eigenvalue problem is used to investigate practical stability and stabilization of fractional discrete-time linear systems via propositional output feedback.

Consider the fractional discrete-time linear system described by

(1.1) 
$$\begin{cases} \Delta^{\alpha} x_{k+1} = A x_k + B u_k \\ g_k = C x_k, \end{cases}$$

where  $\alpha$  is the fractional order difference of state vector and  $0 < \alpha < 1$ ,  $x_k \in \mathbb{R}^n$ ,  $u_k \in \mathbb{R}^m$ , and  $g_k \in \mathbb{R}^r$  are state, input, and output vectors, the matrices  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ , and  $C \in \mathbb{R}^{r \times n}$  are known constant matrices which  $1 \le m \le n$ , rank(B) = m, rank(C) = r, and also  $k \in \mathbb{Z}^+ = \{0, 1, 2, \cdots\}$ .

**Definition 1.1.** The fractional system (1.1) is called asymptotically stable if and only if  $\lim_{k\to\infty} x_k = 0$  for any  $x_0 \in \mathbb{R}^n$  and  $u_k = 0$ .

<sup>2010</sup> Mathematics Subject Classification. 34A08, 93B52, 93D15.

Speaker: Sakineh Bigom Mirassadi.

**Definition 1.2.** The fractional derivative of the order  $\alpha \in \mathbb{R}^+$  with zero initial point for fractional discrete-time linear systems is defined by  $\Delta^{\alpha} x(t_k) = \Delta^{\alpha} x_k = \sum_{i=0}^{k} (-1)^i {\alpha \choose i} x_{k-i}, 0 < \alpha < 1$  [1].

Using the definition 1.2 we may write the equations (1.1) in the form

(1.2) 
$$\begin{cases} x_{k+1} = A_{\alpha} x_k + \sum_{i=1}^k c_i x_{k-i} + B u_k, \\ g_k = C x_k \end{cases}$$

which

(1.3) 
$$c_i = c_i(\alpha) = (-1)^i \binom{\alpha}{i+1}, i = 1, 2, \cdots, k,$$

and  $A_{\alpha} = A + \alpha I_n$ . Note that the equation (1.2) describes a discrete-time linear system with unlimited delay in state. To make the control of this system possible, we should change it to standard linear system. Although the converted standard linear systems may have large matrices, but stability of them is proved [2].

#### 2. Main results

The coefficients  $c_i$  in (1.3) strongly decrease for increasing *i* when  $0 < \alpha < 1$ . Assuming  $c_i = 0$  for i > L the system (1.2) is converted to the linear system with *L* delays [1]

(2.1) 
$$\begin{cases} x_{k+1} = A_{\alpha} x_k + \sum_{i=1}^{L} c_i x_{k-i} + B u_k, \\ g_k = C x_k \end{cases}$$

We may convert the time delay system (2.1) to the standard system

(2.2) 
$$\begin{cases} X_{k+1} = \bar{A}X_k + \bar{B}U_k \\ G_k = \bar{C}X_k \end{cases}$$

where

$$X_{k} = \begin{bmatrix} x_{k} \\ x_{k-1} \\ x_{k-2} \\ \vdots \\ x_{k-L} \end{bmatrix} \in \mathbb{R}^{\bar{n}}, \quad G_{k} = \begin{bmatrix} g_{k} \\ g_{k-1} \\ g_{k-2} \\ \vdots \\ g_{k-L} \end{bmatrix} \in \mathbb{R}^{r},$$

and  $U_k = u_k \in \mathbb{R}^m$  are state, output, and input vectors,  $\bar{n} = n(L+1)$ , and

$$\bar{A} = \begin{bmatrix} A_{\alpha} & c_1 I & c_2 I & \cdots & c_{L-1} I & c_L I \\ I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix} \in \mathbb{R}^{\bar{n} \times \bar{n}}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{\bar{n} \times m},$$

$$\bar{C} = \begin{bmatrix} C & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{r \times \bar{n}}.$$

**Definition 2.1.** The fractional system (1.1) is called practically stable if and only if the time delay system (2.1) or equivalently the system (2.2) is asymptotically stable [1].

2.1. Eigenvalue assignment with propositional output feedback law. Consider system (2.2) by propositional output feedback law

$$(2.3) U_k = KG_k = K\bar{C}X_k.$$

The aim is to design the propositional output feedback (2.3) which produces a closed-loop system of (2.2) with the satisfactory response by assigning desirable eigenvalues  $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_{\bar{n}}\}$  where  $\lambda_i \in \mathbb{C}, \lambda_i \neq 0$ , and are self-conjugate complex numbers for  $i = 1, 2, \dots, \bar{n}$ . Substitution feedback (2.3) into the equation (2.2) one can rewrite such a standard linear system given by

(2.4) 
$$X_{k+1} = (A + BKC)X_k.$$

**Theorem 2.2.** The standard discrete-time linear system (2.4) is asymptotically stable if and only if eigenvalues of  $(\bar{A} + \bar{B}K\bar{C})$  lie in the unit disk [3].

Consider four linearly independent sets of real *n*-vectors  $\{z_1, z_2, ..., z_p\}, \{z_{p+1}, z_{p+2}, ..., z_{p+q}\}, \{w_1, w_2, ..., w_p\}, \{w_{p+1}, w_{p+2}, ..., w_{p+q}\}$  with  $p + q \leq n$  and a set of complex numbers  $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ . The aim of matrix inverse eigenvalue problem is that to find a real matrix  $\Omega_{n \times n}$  such that  $\Omega z_i = w_i, i = 1, 2, ..., p, \Omega z_j = w_j, j = p + 1, p + 2, ..., p + q$ , and the spectrum of  $\Omega$  be  $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ , where  $\lambda_i, i = 1, ..., n$  are closed under complex conjugation.

Let  $X_r = [z_1, z_2, ..., z_p]$ ,  $X_l = [z_{p+1}, z_{p+2}, ..., z_{p+q}]$ ,  $Y_r = [w_1, w_2, ..., w_p]$ , and  $Y_l = [w_{p+1}, w_{p+2}, ..., w_{p+q}]$ . If the matrix  $\Omega$  of the problem exists, the following consistency condition must be satisfied [4]

**Theorem 2.3.** If the matrix inverse eigenvalue problem satisfies the consistency condition (2.5), then the necessary and sufficient condition for the existence of the matrix  $\Omega$  is that there are vectors  $u_i \in \int_u^i$  and  $v_i \in \int_v^i$ , i = 1, 2, ..., n such that

(2.6) 
$$u_i v_j = \delta_{ij}, \quad i, j = 1, 2, ..., n,$$

where  $\int_{u}^{i}$  and  $\int_{v}^{i}$  are the null spaces  $(\lambda_{i}X_{l}^{t} - Y_{l}^{t})$  and  $(\lambda_{i}X_{r}^{t} - Y_{r}^{t})$  respectively and  $\delta_{ij}$  is the Kronecker delta function. If such  $u_{i}$  exists, then  $\Omega$  can be obtained using the equation  $\Omega = T \operatorname{diag}(\lambda_{1}, \lambda_{2}, ..., \lambda_{n})T^{-1}$ , which  $T = \begin{bmatrix} u_{1} & u_{2} & ... & u_{n} \end{bmatrix}$  [4].

Let the base vectors of  $\int_u^i$  and  $\int_v^i$  be the form of matrices  $S_u^i$  and  $S_v^i$  respectively, then vectors  $u_i$  and  $v_i$ , i = 1, 2, ..., n can be expressed as  $u_i = S_u^i z_i, v_i = S_v^i w_i$ . Thus from equation (2.6) we have  $z_i^t (S_u^i)^t S_v^j w_j = \delta_{ij}, i, j = 1, 2, ..., n$  with  $n^2$  bilinear equations and  $2n^2 - n(p+q)$  unknowns. The number of unknowns may be greater than  $n^2$  because  $p + q \le n$ . Thus we can solve this system using iterative method by converting to 2n linear equations [4].

Consider  $\Omega_{\bar{n}\times\bar{n}} = \bar{A} + \bar{B}K\bar{C}$  as the closed-loop matrix of system (2.2) and  $U_1$ and  $V_1$  as the matrices formed by the base vectors of the null spaces of  $\bar{B}^t$  and  $\bar{C}$ respectively. So we have  $\Omega V_1 = (\bar{A} + \bar{B}K\bar{C})V_1 = \bar{A}V_1$  and  $U_1^t\Omega = U_1^t(\bar{A} + \bar{B}K\bar{C}) = U_1^t\bar{A}$ . Let  $X_l = U_1, X_r = V_1, Y_l = \bar{A}^tU_1$ , and  $Y_r = \bar{A}V_1$ . By theorem 2.3 we can find  $\Omega$ . If such  $\Omega$  exists, the feedback matrix K can be computed through the equation

$$K = \overline{B}^{\dagger}(\Omega - \overline{A})\overline{C}^{\dagger}$$

where  $\bar{B}^{\dagger}$  and  $\bar{C}^{\dagger}$  are the Moore-Penrose generalized inverse of  $\bar{B}$  and  $\bar{C}$  respectively.

**Example 2.4.** Consider the system (2.2) with  $\alpha = 0.3, L = 2$ , the set of eigenvalues  $\Lambda = \{\pm 0.1, \pm 0.2, \pm 0.3, \pm 0.4, 0.5\}$  and following matrices:

$$A = \begin{bmatrix} 1 & 5 & 3 \\ -2 & 1 & 0 \\ 4 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 1 & -1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & -2 \end{bmatrix}.$$

#### 22





FIGURE 1.  $y_i(t)$  in example 2.4

#### 3. Conclusions

Application of matrix inverse eigenvalue problem for practical stability and stabilization of the fractional discrete-time linear system in form of (1.1) has been considered. First, by the definition of fractional order and a new state vector displayed system has been converted to a standard discrete-time linear system (2.2). Second, the matrix inverse eigenvalue problem has been used to assign eigenvalues to the system (2.2) and obtain the output feedback K. Finally, by a numerical example is shown the output vector converges to zero.

#### References

- M. Buslowicz, Stability analysis of continuous-time linear systems consisting of n subsystems with different fractional orders, Bulletin of the Polish Academy of Science: Technical Sciences, Vol. 60(2), pp. 279-284, (2012).
- [2] S. Guermah, S. Djennoune, and M. Bettayeb, A new approach for stability analysis of linear discrete-time fractional order systems, New Trends in Nanotechnology and Fractional Calculus Applications, pp. 151-162, Springer, (2010).
- [3] A. Halanay, and V. Rasvan, Stability and stable oscillations in discrete-time systems, Gordon and Breach Science Publisher, Australia, pp. 21-23, (2000).
- [4] N. Li, An iterative method for pole assignment, Linear Algebra and Its applications, Vol. 23, pp. 77-102, (2001).

PHD STUDENT, SHAHROOD UNIVERSITY OF TECHNOLOGY, SHAHROOD, IRAN *E-mail address*: s.mirassadi@shahroodut.ac.ir

Associate Professor, Shahrood University of Technology, Shahrood, Iran *E-mail address*: hahsani@shahroodut.ac.ir



#### EIGENVALUE ASSIGNMENT IN FRACTIONAL TWO-DIMENSIONAL ROESSER SYSTEMS

SAKINEH BIGOM MIRASSADI AND HOJJAT AHSANI TEHRANI

ABSTRACT. In this paper, a method for stabilization of Roesser two-dimensional model with different orders fractional derivative is introduced. By recalling horizontal and vertical fractional-order definitions and a new state vector our system is converted to a standard Roesser two-dimensional model. Similarity transformation is used to assign desired eigenvalues to closed-loop standard system and stabilize the model.

Keywords: Fractional 2D Roesser model, State feedback, Similarity transformation, Assigning eigenvalues.

#### 1. INTRODUCTION

The early 1970s two-dimensional (2D) models have been attracting the attention of the scientific working in the area of dynamical systems theory. The most popular particularized version of general form is Roesser model. Fractional calculus is a branch of mathematical analysis that studies the possibility of differentiation and integration of arbitrary real or complex orders of the differential operator. At present, the number of applications of fractional calculus rapidly grows. These mathematical phenomena allow us to describe and model a real object more accurately than the classical integer methods. It has played an important role in physics, electrical engineering, control systems, robotics, signal processing, chemical mixing, bioengineering, and so on. Some ways for finding the solution of state equations in positive fractional linear systems were discussed by the use of Drazin inverse and Shuffle algorithm. However, one of the efficient methods for stabilization of the fractional systems is, eigenvalue assignment (EVA) method to modify the dynamic response of linear systems. We do not have some difficulties of positive systems like dealing with long equations of inputs, full row rank matrices in every performed algorithm, and finding index of shuffle and Drazin especially for large systems. In this paper a method for stabilization of fractional 2D Roesser discrete-time model with state feedback is investigated.

Consider the fractional 2D Roesser discrete-time linear model

(1.1) 
$$\begin{bmatrix} \Delta^h_{\alpha} x^h_{i+1,j} \\ \Delta^v_{\beta} x^v_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij},$$

where  $\alpha$  and  $\beta$  are horizontal and vertical fractional orders difference which  $0 < \alpha < 1, 0 < \beta < 1, A_{lk} \in \mathbb{R}^{n_l \times n_k}, k, l = 1, 2, B_1 \in \mathbb{R}^{n_1 \times m}, B_2 \in \mathbb{R}^{n_2 \times m}, x_{ij}^h \in \mathbb{R}^{n_1}, x_{ij}^v \in \mathbb{R}^{n_2}$ , and  $u_{ij} \in \mathbb{R}^m$  are horizontal state, vertical state, and input vectors at the point (i, j) respectively. Also it is assumed that  $1 \le m \le \min\{n_1, n_2\}$ .

<sup>2010</sup> Mathematics Subject Classification. 34A08, 93B52, 93D15.

Speaker: Sakineh Bigom Mirassadi.

Using the horizontal and vertical fractional definitions in [3] we may write the equation (1.1) in the form

(1.2) 
$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & A_{12} \\ A_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} - \begin{bmatrix} \sum_{k=2}^{i+1} c_{\alpha}(k) x_{i-k+1,j}^h \\ \sum_{l=2}^{j+1} c_{\beta}(l) x_{i,j-l+1}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij},$$

where  $\bar{A}_{11} = A_{11} + \alpha I_{n_1}$ ,  $\bar{A}_{22} = A_{22} + \beta I_{n_2}$ . Note that the equations (1.2) describe a 2D discrete-time linear system with unlimited delays in horizontal and vertical states. To make the control of this system possible, we should convert it to standard 2D discrete-time linear system.

The sequences of  $c_{\alpha}(k)$  and  $c_{\beta}(l)$  converge to zero. Assuming  $c_{\alpha}(k) = 0$  for  $k > L_1 + 1$  and  $c_{\beta}(l) = 0$  for  $l > L_2 + 1$  the system (1.2) is converted to a 2D Roesser linear system with  $L_1$  and  $L_2$  delays [1]. So the equation (1.2) takes the form

$$(1.3) \begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & A_{12} \\ A_{21} & \bar{A}_{22} \end{bmatrix} = \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} - \begin{bmatrix} \sum_{k=2}^{L_1+1} c_\alpha(k) x_{i-k+1,j}^h \\ \sum_{l=2}^{L_2+1} c_\beta(l) x_{i,j-l+1}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij}.$$

This system can be rewritten as a standard 2D discrete-time linear system

(1.4) 
$$\begin{bmatrix} X_{i+1,j}^h \\ X_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} X_{ij}^h \\ X_{ij}^v \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} U_{ij},$$

where

$$\begin{split} X_{ij}^{h} &= \begin{bmatrix} x_{ij}^{h} \\ x_{i-1,j}^{h} \\ \vdots \\ x_{i-L_{1}+1,j}^{h} \\ x_{i-L_{1},j}^{h} \end{bmatrix} \in \mathbb{R}^{N_{1}}, X_{ij}^{v} &= \begin{bmatrix} x_{ij}^{v} \\ x_{i,j-1}^{v} \\ \vdots \\ x_{i,j-L_{2}+1}^{v} \end{bmatrix} \in \mathbb{R}^{N_{2}}, \\ \tilde{A}_{11} &= \begin{bmatrix} \tilde{A}_{11} & -c_{\alpha}(2)I_{n_{1}} & \cdots & -c_{\alpha}(L_{1})I_{n_{1}} & -c_{\alpha}(L_{1}+1)I_{n_{1}} \\ I_{n_{1}} & 0 & \cdots & 0 & 0 \\ 0 & I_{n_{1}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{n_{1}} & 0 \end{bmatrix} \in \mathbb{R}^{N_{1} \times N_{1}}, \\ \tilde{A}_{12} &= \begin{bmatrix} A_{12} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_{1}} & 0 \end{bmatrix} \in \mathbb{R}^{N_{1} \times N_{2}}, \\ \tilde{A}_{21} &= \begin{bmatrix} A_{21} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{N_{2} \times N_{1}}, \\ \tilde{A}_{22} &= \begin{bmatrix} \tilde{A}_{22} & -c_{\beta}(2)I_{n_{2}} & \cdots & -c_{\beta}(L_{2})I_{n_{2}} & -c_{\beta}(L_{2}+1)I_{n_{2}} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{n_{2}} & 0 \end{bmatrix} \in \mathbb{R}^{N_{2} \times N_{2}}, \\ \tilde{A}_{22} &= \begin{bmatrix} \tilde{A}_{22} & -c_{\beta}(2)I_{n_{2}} & \cdots & -c_{\beta}(L_{2})I_{n_{2}} & -c_{\beta}(L_{2}+1)I_{n_{2}} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{n_{2}} & 0 \end{bmatrix} \in \mathbb{R}^{N_{2} \times N_{2}}, \\ \tilde{B}_{1} &= \begin{bmatrix} B_{1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{N_{1} \times m}, \\ \tilde{B}_{2} &= \begin{bmatrix} B_{2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{N_{2} \times m}, \\ U_{ij} &= u_{ij} \in \mathbb{R}^{m}, \\ N_{1} &= n_{1}(L_{1}+1), \\ N_{2} &= n_{2}(L_{2}+1), \\ n &= N_{1} + N_{2}. \end{split}$$

**Theorem 1.1.** The fractional 2D discrete-time linear system (1.1) is called practically stable if the time delay system (1.3) or equivalently system (1.4) is asymptotically stable [2].

Consider the standard 2D Roesser model (1.4) with the state feedback

$$U_{ij} = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} X_{ij}^h \\ X_{ij}^v \end{bmatrix},$$

where  $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \in \mathbb{R}^{m \times n}, K_1 \in \mathbb{R}^{m \times N_1}$ , and  $K_2 \in \mathbb{R}^{m \times N_2}$ . Substituting the state feedback K into system (1.4) we may have

(1.5) 
$$\begin{bmatrix} X_{i+1,j}^h \\ X_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} + \tilde{B}_1 K_1 & \tilde{A}_{12} + \tilde{B}_1 K_2 \\ \tilde{A}_{21} + \tilde{B}_2 K_1 & \tilde{A}_{22} + \tilde{B}_2 K_2 \end{bmatrix} \begin{bmatrix} X_{ij}^h \\ X_{ij}^v \end{bmatrix}.$$

The following equivalent system from (1.4) is considered as

$$X_{i+1,j+1} = \tilde{A}X_{ij} + \tilde{B}U_{ij},$$

where

(

$$X_{i+1,j+1} = \begin{bmatrix} X_{i+1,j}^h \\ X_{i,j+1}^v \end{bmatrix}, X_{ij} = \begin{bmatrix} X_{ij}^h \\ X_{ij}^v \end{bmatrix}, \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}.$$

By putting state feedback  $U_{ij} = KX_{ij}$  we obtain  $X_{i+1,j+1} = (A + BK)X_{ij}$ .

**Theorem 1.2.** The standard discrete-time system (1.6) or equivalently (1.5) is asymptotically stable if and only if eigenvalues of  $(\tilde{A} + \tilde{B}K)$  lie in the unit disk [2].

#### 2. Similarity transformation

In this section we describe a method for assigning eigenvalues in closed-loop matrix  $\Gamma = \tilde{A} + \tilde{B}K$  and obtaining state feedback K. First, zero eigenvalues are assigned to this system by F. Then, by the gain F, state feedback K is obtained easily that assign desired eigenvalues to the closed-loop system of (1.6). Consider the state transformation

where T can be obtained by elementary similarity operations as described in [4, 5]. Substituting (2.1) into (1.6) yields  $\tilde{X}_{i+1,j+1} = T^{-1}\tilde{A}T\tilde{X}_{ij} + T^{-1}\tilde{B}U_{ij}$ . It is noted that the transformation matrix T is invertible. In this way,  $N = T^{-1}\tilde{A}T, M = T^{-1}\tilde{B}$  are in a compact canonical form know as vector companion form [4, 5]

$$N = \begin{bmatrix} G_0 \\ I_{n-m} \\ 0 \end{bmatrix}, M = \begin{bmatrix} S_0 \\ 0_{n-m,m} \end{bmatrix}$$

Here  $G_0$  is a  $m \times n$  matrix and  $S_0$  is a  $m \times m$  upper triangular matrix. The state feedback matrix which assigns all the eigenvalues to zero for the transformed pair (M, N) is then chosen as  $\tilde{F} = -S_0^{-1}G_0$ , which results in the primary state feedback matrix for the pair  $(\tilde{B}, \tilde{A})$  defined as  $F = \tilde{F}T^{-1}$ . The transformed closed-loop matrix  $\tilde{\Gamma}_0 = N + M\tilde{F}$  assumes a compact Jordan form with zero eigenvalues

(2.2) 
$$\tilde{\Gamma}_0 = \begin{bmatrix} 0_{m,n} \\ I_{n-m} & 0_{n-m,m} \end{bmatrix}_{n \times m}$$

Simple elementary similarity operations can be used to obtain the matrix  $H_{\lambda}$  from  $\tilde{H} = \tilde{\Gamma}_0 + D$ , where D is a block diagonal matrix of desired eigenvalues  $\Lambda = \{\lambda_1, \dots, \lambda_n\}$  [4] such that

(2.3) 
$$\tilde{H}_{\lambda} = \begin{bmatrix} G_{\lambda} \\ I_{n-m} & , & 0_{n-m,m} \end{bmatrix}$$

Thus the primary feedback matrix K which gives rise to the assignment of eigenvalues  $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  to the system (1.6) becomes  $K = F + S_0^{-1} G_{\lambda} T^{-1}$ .

**Example 2.1.** Consider the system (1.1) with  $\alpha = 0.5, \beta = 0.8, L_1 = L_2 = 2, \Lambda = \{0.1 \pm 0.1i, 0.1, 0.2 \pm 0.2i, 0.2, 0.3 \pm 0.3i, 0.3\}$ , and following matrices

$$A_{11} = \begin{bmatrix} 1 & 2 \\ -0.5 & 1.2 \end{bmatrix}, A_{12} = \begin{bmatrix} 1.5 \\ -1 \end{bmatrix}, A_{21} = \begin{bmatrix} 1.3 & -0.9 \end{bmatrix}, A_{22} = \begin{bmatrix} -0.7 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} -0.4 & 0.5 \\ -0.8 & -1 \end{bmatrix}, B_2 = \begin{bmatrix} -0.3 & 0.9 \end{bmatrix}.$$

The results are as follow which  $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ .

$$K_1 = \begin{bmatrix} 0.85 & 1.89 & 0.24 & -0.15 & -0.02 & 0.07 \\ -0.1 & -0.09 & -1.48 & 8.8 & 1.49 & -0.7 \end{bmatrix}, K_2 = \begin{bmatrix} 0.08 & -0.17 & 0.04 \\ 0.38 & 6.6 & -0.94 \end{bmatrix}.$$



FIGURE 1. Horizontal and vertical state vectors in example 2.1

#### 3. Conclusions

A method for stabilization of fractional 2D Roesser model in form of (1.1) has been considered. First, by the definitions of horizontal and vertical fractional-order and defining new horizontal and vertical state vectors, 2D standard model (1.4)is obtained. Second, desired eigenvalues are assigned by the use of state feedback and a method based on similarity transformation. Finally, the convergence of the horizontal and vertical state vectors to zero are shown.

#### References

- M. Buslowicz, Stability analysis of continuous-time linear systems consisting of n subsystems with different fractional orders, Bulletin of the Polish Academy of Science: Technical Sciences, vol. 60(2), 279–284, (2012).
- [2] A. Halanay, and V. Rasvan, Stability and stable oscillations in discrete-time systems, Gordon and Breach Science Publisher, Australia, 21–23, (2000).
- [3] T. Kaczorek, Stability and stabilization of positive fractional linear systems by state feedbacks, Bulletin of the Polish Academy of Science: Technical Sciences, vol. 58(4), 537–554, (2010).
- [4] S. M. Karbassi, and D. J. Bell, Parametric time-optimal control of linear discrete-time systems by state feedback-part 1: Regular Kronecker invariants, International journal of control, vol. 57, 817–830, (1993).
- [5] S. M. Karbassi, and D. J. Bell, Parametric time-optimal control of linear discrete-time systems by state feedback-part 2: Irregular Kronecker invariants, International journal of control, vol. 57, 831–883, (1993).

PHD STUDENT, SHAHROOD UNIVERSITY OF TECHNOLOGY, SHAHROOD, IRAN *E-mail address*: s.mirassadi@shahroodut.ac.ir

Associate Professor, Shahrood University of Technology, Shahrood, Iran *E-mail address*: hahsani@shahroodut.ac.ir



## GENERALIZATION OF SOME QUANTUM INFORMATION INEQUALITIES

#### ALI DADKHAH

ABSTRACT. We present some generalizations of quantum information inequalities involving tracial positive linear maps between  $C^*$ -algebras. We establish two noncommutative Heisenberg uncertainty relations.

Keywords: Tracial positive linear map; Trace; Covariance, Variance,  $C^*$ -algebra.

#### 1. INTRODUCTION

In quantum measurement theory, the classical expectation value of an observable (self-adjoint element) A in a quantum state (density element)  $\rho$  is expressed by  $\operatorname{Tr}(\rho A)$ . Also, the classical variance for a quantum state  $\rho$  and an observable element A is defined by  $V_{\rho}(A) := \operatorname{Tr}(\rho A^2) - (\operatorname{Tr}(\rho A))^2$ . The Heisenberg uncertainty relation asserts that

(1.1) 
$$V_{\rho}(A)V_{\rho}(B) \ge \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2$$

for a quantum state  $\rho$  and two observables A and B; . It gives a fundamental limit for the measurements of incompatible observables. A further strong result was given by Schrödinger as

(1.2) 
$$V_{\rho}(A)V_{\rho}(B) - |\operatorname{Re}(\operatorname{Cov}_{\rho}(A,B))|^{2} \geq \frac{1}{4}|\operatorname{Tr}(\rho[A,B])|^{2},$$

where [A, B] := AB - BA is the commutator of A, B and the classical covariance is defined by  $\operatorname{Cov}_{\rho}(A) := \operatorname{Tr}(\rho A B) - \operatorname{Tr}(\rho A) \operatorname{Tr}(\rho B)$ .

Let  $\mathbb{B}(\mathcal{H})$  denote the  $C^*$ -algebra of all bounded linear operators on a complex Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  with the unit I. We consider the usual Löwner order  $\leq$  on the real space of self-adjoint operators.

A linear map  $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$  between  $C^*$ -algebras is said to be positive if  $\Phi(A) \geq 0$ whenever  $A \geq 0$ . We say that  $\Phi$  is unital if  $\mathcal{A}, \mathcal{B}$  are unital and  $\Phi$  preserves the unit. A linear map  $\Phi$  is called *n*-positive if the map  $\Phi_n : M_n(\mathcal{A}) \longrightarrow M_n(\mathcal{B})$  defined by  $\Phi_n([a_{ij}]) = [\Phi(a_{ij})]$  is positive, where  $M_n(\mathcal{A})$  stands for the  $C^*$ -algebra of  $n \times n$ matrices with entries in  $\mathcal{A}$ . A map  $\Phi$  is said to be completely positive if it is *n*positive for every  $n \in \mathbb{N}$ . If the range of the positive linear map  $\Phi$  is commutative, then  $\Phi$  is completely positive. It is known (see, e.g., [3] that if  $\Phi$  is a unital positive linear map, then

(1.3) 
$$\Phi(A \sharp B) \le \Phi(A) \sharp \Phi(B).$$

A map  $\Phi$  is called tracial if  $\Phi(AB) = \Phi(BA)$ . The usual trace on the trace class operators acting on a Hilbert space is a tracial positive linear functional.

<sup>2010</sup> Mathematics Subject Classification. Primary 15A60; Secondary 46L05, 47A63. Speaker: Ali Dadkhah.

**Definition 1.1.** Let  $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$  be a tracial positive linear map and  $\rho$  be a  $\Phi$ -density element. Then

 $\operatorname{Cov}_{\rho,\Phi}(A,B) := \Phi(\rho A^*B) - \Phi(\rho A^*)\Phi(\rho B)$  and  $V_{\rho,\Phi}(A) := \operatorname{Cov}_{\rho,\Phi}(A,A)$ , are called the generalized covariance and the generalized variance A, B, respectively.

It is known that for every tracial positive linear map, the matrix

$$\begin{bmatrix} V_{\rho,\Phi}(A) & \operatorname{Cov}_{\rho,\Phi}(B,A) \\ \operatorname{Cov}_{\rho,\Phi}(A,B) & V_{\rho,\Phi}(B) \end{bmatrix}$$

is positive, which is equivalent to

(1.4) 
$$V_{\rho,\Phi}(A) \ge \operatorname{Cov}_{\rho,\Phi}(B,A)(V_{\rho,\Phi}(B))^{-1}\operatorname{Cov}_{\rho,\Phi}(A,B),$$

which is called the variance-covariance inequality.

#### 2. Main results

To achieve our results we need the following lemma.

**Lemma 2.1.** Let  $A > 0, B \ge 0$  be two elements in  $M_n(\mathcal{A})$ . Then the block matrix  $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$  is positive if and only if  $B \ge X^* A^{-1} X$ .

Now we prove our first result.

**Theorem 2.2.** Let  $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$  be a tracial positive linear map between  $C^*$ -algebras and  $\rho \in \mathcal{A}$  be a  $\Phi$ -density element. If  $\Phi(\mathcal{A})$  is a commutative subspace of  $\mathcal{B}$ , then

$$V_{\rho,\Phi}(A)V_{\rho,\Phi}(B) - |\operatorname{Re}(\operatorname{Cov}_{\rho,\Phi}(A,B))|^2 \ge \frac{1}{4}|\Phi(\rho[A,B])|^2$$

for all self-adjoint elements A, B.

If  $\mathcal{A}$  is a  $C^*$ -algebra and  $\mathcal{B}$  is a  $C^*$ -subalgebra of  $\mathcal{A}$ , then a conditional expectation  $\mathcal{E} : \mathcal{A} \longrightarrow \mathcal{B}$  is a positive contractive linear map such that  $\mathcal{E}(BAC) = B\mathcal{E}(A)C$  for every  $A \in \mathcal{A}$  and all  $B, C \in \mathcal{B}$ .

**Corollary 2.3.** Let  $\mathcal{A}$  be a  $C^*$ -algebra and  $\mathcal{B}$  be a  $C^*$ -subalgebra of  $\mathcal{A}$ . If  $\mathcal{E} : \mathcal{A} \longrightarrow \mathcal{B}$  is a tracial conditional expectation, then

$$V_{\rho,\mathcal{E}}(A)V_{\rho,\mathcal{E}}(B) - |\operatorname{Re}(\operatorname{Cov}_{\rho,\mathcal{E}}(A,B))|^2 \ge \frac{1}{4}|\mathcal{E}(\rho[A,B])|^2$$

for all self-adjoint elements  $A, B \in \mathcal{A}$  and each  $\mathcal{E}$ -density element  $\rho \in \mathcal{A}$ .

Now we give a version of Heisenberg's uncertainty relation, in the case that  $\mathcal{B}$  is not a commutative  $C^*$ -algebra. To get this result we need some lemmas.

**Lemma 2.4** (Choi–Tsui). Let  $\mathcal{A}, \mathcal{B}$  be  $C^*$ -algebras such that either one of them is  $W^*$ -algebra or  $\mathcal{B}$  is an injective  $C^*$ -algebra. Let  $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$  be a tracial positive linear map. Then there exist a commutative  $C^*$ -algebra C(X) and tracial positive linear maps  $\phi_1 : \mathcal{A} \longrightarrow C(X)$  and  $\phi_2 : C(X) \longrightarrow \mathcal{B}$  such that  $\Phi = \phi_2 \circ \phi_1$ . Moreover, in case that  $\Phi$  is unital, then  $\phi_1$  and  $\phi_2$  can be chosen to be unital. In particular,  $\Phi$  is completely positive.

**Lemma 2.5.** (Kadison's inequality) If  $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$  is a unital 2-positive linear map between unital  $C^*$ -algebras, then

$$\Phi(|A|^2) \ge |\Phi(A)|^2$$

for every  $A \in \mathcal{A}$ .

In the case that A is a positive element of  $\mathcal{A}$  satisfying  $0 < mI \leq A \leq MI$  for some scalers m < M, by [3, Theorem 1.32], the reverse inequality

(2.1) 
$$\Phi(A^2) \le \frac{(M+m)^2}{4Mm} \Phi(A)^2$$

holds.

**Lemma 2.6.** Let  $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$  be a unital 2-positive linear map between unital  $C^*$ -algebras. If A is a normal element of  $\mathcal{A}$  satisfying  $\operatorname{sp}(A) \subseteq [m, M] \cup [-M, -m]$  for some scalers 0 < m < M, then

(2.2) 
$$|\Phi(A)| \le \sqrt{\frac{(M+m)^2}{4Mm}} \Phi(|A|)$$

In this paper, we denote  $\frac{(M+m)^2}{4Mm}$  for an element  $m \leq A \leq M$  by  $K_{M,m}(A)$ , which is called the Kantorovich constant of the element A.

The next theorem gives a Heisenberg's type uncertainty relation for tracial positive linear maps between  $C^*$ -algebras.

**Theorem 2.7.** Let  $\mathcal{A}, \mathcal{B}$  be  $C^*$ -algebras such that either one of them is  $W^*$ -algebra or  $\mathcal{B}$  is an injective  $C^*$ -algebra. Let  $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$  be a tracial positive linear map and  $\rho \in \mathcal{A}$  be a  $\Phi$ -density element and  $\mathcal{A}, \mathcal{B}$  are self-adjoint elements in  $\mathcal{A}$  such that  $\operatorname{sp}(-i\rho^{\frac{1}{2}}[\mathcal{A}, \mathcal{B}]\rho^{\frac{1}{2}}) \subseteq [m, M]$  for some scalars 0 < m < M, then

$$V_{\rho,\Phi}(A) \sharp V_{\rho,\Phi}(B) \ge \frac{1}{2\sqrt{K_{m,M}(\rho[A,B])}} |\Phi(\rho[A,B])|,$$

where  $K_{[m,M]}(\rho[A,B])$  is the Kantorovich constant of the element  $-i\rho^{\frac{1}{2}}[A,B]\rho^{\frac{1}{2}}$ .

**Corollary 2.8.** For every self-adjoint elements A, B and each density element  $\rho$  we have

$$V_{\rho}(A)V_{\rho}(B) \ge \frac{1}{4} |\mathrm{Tr}(\rho[A, B])|^2$$
.

#### References

- A. Dadkhah, M.S. Moslehian, Grüss type inequalities for positive linear maps on C\*-algebras, Linear Multilinear Algebra, 65 (2017), no. 7, 138–1401.
- [2] A. Dadkhah, M.S. Moslehian, Quantum information inequalities via tracial positive linear maps, J. Math. Anal. Appl. 447 (2017) no. 1, 666–680.
- [3] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Self-adjoint Operators on a Hilbert Space, Element, Zagreb, Croatia, 2005.
- [4] J. S. Matharu and M. S. Moslehian, Gruss inequality for some types of positive linear maps, J. Operator Theory 73 (2015), no. 1, 265–278.
- [5] G. Tóth and D. Petz, Extremal Properties of the Variance and the Quantum Fisher Information, Phys. Rev. A 87, (2013) 0323–24.

Department of Pure Mathematics, Ferdowsi University of Mashhad, Iran,  $E\text{-}mail\ address:\ \texttt{dadkhah61@yahoo.com}$


# CAPTURING OUTLINES OF 2D SHAPES USING B-SPLINE CURVES AND NORMAL EQUATION

## ALIREZA EBRAHIMI AND GHASEM BARID LOGHMANI

ABSTRACT. In this paper, a new technique is proposed to capture the outline of 2D objects. The outline is divided into various segment and curve fitting is then performed over each segment. The presented method employs B-spline curve for curve approximation. The proposed technique approximates each segment using a straight line first. The number of control points of approximating curve is increased until the required results are obtained. In this method, optimal curve approximation is performed using optimal number of control points.

Keywords: Outline capturing; Corner detection; B-spline curve; Curve fitting; Control points.

# 1. INTRODUCTION

Capturing the outline of 2D objects is an important research area in computer aided geometric design (CAGD), computer graphics, as well as vision and imaging. This paper introduce a new curve fitting approach with B-spline curve. The proposed algorithm performs curve approximation using an iterative process. A B-spline curve of a fixed degree like a quadratic B-spline curves is used to approximate the given shape. The segments obtained after corner detection are initially approximated with B-spline curve of lowest number of control points, i.e. 2. The number of control points of approximating curve is then increased iteratively until the error comes under the user specified limit. This ensures optimal fit with least number of control points. This is very efficient and useful for data compression, storage and transmission.

#### 2. Boundary segmentation

Boundary segmentation means dividing the outline of shape into small pieces. It reduces boundarys complexity and simplifies the fitting process. Also, there are natural break points in the outline that we like to maintain as they are. These points are determined by a suitable corner detector. Various algorithms have been proposed by researchers for corner detection. We use the method proposed in [3] due to its efficiency and accuracy. The algorithm comprises of two stages. In the first stage, it finds candidate corners from whole data set while in second stage some extra points are discarded using certain criteria giving only potential corners.

#### 3. CURVE APPROXIMATION

The objective of curve approximation is to fit a curve over each piece of segmented boundary obtained using a corner detector. These curves are then joined together to form the complete shape. In curve fitting we use B-spline curves for its

<sup>2010</sup> Mathematics Subject Classification. 65D17, 65D10.

Speaker: Alireza Ebrahimi.

simple and efficient implementation, which make them highly useful for curve and surface fitting. A B-spline curve of degree p is defined as

(3.1) 
$$C(u) = \sum_{i=0}^{m} N_{i,p}(u) P_i,$$

where  $P_i$  are the control points, and  $N_{i,p}$  are the normalized B-splines defined over the knot vector

`

$$U = \left\{ \underbrace{u_0 = u_1 = \cdots, u_p}_{p+1}, u_{p+1}, \cdots, u_m, \underbrace{u_{m+1} = \cdots = u_{m+p+1}}_{p+1} \right\}$$

The clear advantage of the B-spline curve is that m is not coupled with p. This given us effective control over the shape of the curve together with piecewise polynomial of low degree.

3.1. Least squares curve approximation. Given a set of unorganized data points  $\{Q_i\}_{i=0}^n \subset \mathbb{R}^2$  sampled from the outline of a planar shape, the aim of curve fitting is to find a B-spline curve (3.1) that best approximates the shapes outline. We suppose that knot vector U and a set of precomputed parameters  $t_i, i = 0, ..., n$ , are fixed and therefore not subject to optimization. A B-spline curve of degree p with m + 1 control points is sought that interpolates the end points; i.e.,

$$C(t_0) = P_0 = Q_0, \quad C(t_n) = P_m = Q_n,$$

and approximates the remaining points in the least square sense, that is,

$$\sum_{i=1}^{n-1} |Q_i - C(t_i)|^2$$

is a minimum with respect to independent variables  $P_1, ..., P_{(m-1)}$ . The standard least squares method yields the following system of equations [2].

$$(3.2) (N^T N)P = R,$$

where N is a  $(n-1) \times (m-1)$  matrix

$$\begin{bmatrix} N_{1,p}(t_1) & N_{2,p}(t_1) & \cdots & N_{m-1,p}(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ N_{1,p}(t_{n-1}) & N_{2,p}(t_2) & \cdots & N_{m-1,p}(t_{n-1}) \end{bmatrix},$$

R is vector of (m-1) elements

$$\begin{bmatrix} N_{1,p}(t_1)R_1 + N_{1,p}(t_2)R_2 + \dots + N_{1,p}(t_{n-1})R_{n-1} \\ \vdots \\ N_{m-1,p}(t_1)R_1 + N_{m-1,p}(t_2)R_2 + \dots + N_{m-1,p}(t_{n-1})R_{n-1} \end{bmatrix},$$

with

$$R_i = Q_i - N_{0,p}(t_i)Q_0 - N_{m,p}(t_i)Q_n, \quad i = 1, ..., n - 1,$$

and P is the vector of unknown control points

$$\begin{bmatrix} P_1 \\ \vdots \\ P_{m-1} \end{bmatrix}.$$

In the equation (3.2),  $N^T N$  is symmetric and positive definite, Therefore, we solve the least squares problems by the normal equations and the use of Cholesky factorization [1].

3.2. Iterative process for curve approximation. The strategy for curve approximation is given as follows:

- In the proposed outline capturing system, user specifies the degree of the B-spline curve and the limit for approximation error according to his particular requirements.
- Initially, each segment is approximated using a B-spline curve with two control point i.e. line. If the error obtained is not below the specified limit, a B-spline curve with three control point is then used for approximation. The number of control points of the B-spline curve is iteratively increased until the maximum error comes below the specified limit.

A B-spline curve can be fitted of any degree, but in computer graphic, the degree 2 or 3 are generally found to be sufficient. In this work, we have used quadratic B-spline curve for curve fitting because of their being computationally efficient and having high level of accuracy to approximate 2D shapes.

# 4. Demonstration

The experimental results of presented method are shown in this section. The performance of the proposed algorithm has been tested for the shape of Kanji (Japanese language) character in Figure 1. The original shape to be approximated is given in Figure 1(a). In Figure 1(b) the boundary extracted from this shape is presented. Figure 1(c) displays the corner points which are marked with a square. Curve approximation with the proposed approach is then performed on each of the extracted segments and the result obtained is shown along with the intermediate control points (marked with circles) in Figures 1(d) and 1(e). The error tolerance limit has been kept 3 and 2 for Figures 1(d) and 1(e). The detailed quantitative results for this example are given in Table 1. The algorithm [4] computed this shape with 198 control point which is about 3.96 times more than the proposed algorithm. One reason for higher number of control point is due to suboptimal computation of corner detection and subdivision point by algorithm [4].

Error tolerance	Number of	Maximum	Average	Computation
limit	control point	error	error	$\operatorname{time}$
3	43	2.39	0.65	0.71
2	50	1.98	0.52	0.90
Sarfraz and Razzak [4]	198	1.40	1.13	3.83

TABLE 1. Performance of the proposed algorithm for the shape Kanji character

#### 5. Conclusion

A new outline capturing system is presented in this paper. A curve fitting scheme has been introduced which is based on the B-spline curve. The proposed algorithm performs curve approximation using an iterative process. This algorithm provides



FIGURE 1. Outline capturing of Kanji character. The black and red lines show the original and computed curves respectively.

optimal curve approximation ensuring optimal number of control points. The use of optimum number of control points in the proposed technique is a very useful in vector graphics, font designing, computer supported cartooning, data compression, animations and various other applications of CAD/CAGD.

# References

- [1] B. N. Datta, Numerical linear algebra and applications, Siam, 2010.
- [2] L. Piegl and W. Tiller, The NURBS book, Springer Science and Business Media, 2012.
- [3] M. Sarfraz, A. Masood and M. R. Asim, A new approach to corner detection, Computer Vision and Graphics (2006), 528-533.
- [4] M. Sarfraz and M. Razzak, An algorithm for automatic capturing of the font outlines, Computers and Graphics 26 (2002), 795-804.

YAZD UNIVERSITY E-mail address: a.ebrahimi@stu.yazd.ac.ir

YAZD UNIVERSITY E-mail address: loghmani@yazd.ac.ir



# A CHARACTERAZATION FOR MOORE-PENROSE INVERTIBLE OPERATORS ON HILBERT C\*-MODULES

JAVAD FAROKHI-OSTAD AND ALIREZA JANFADA

ABSTRACT. Recently, the interesting characterization of invertible normal operators has been investigated. In this paper we study and generalize this idea for Moore-Penrose invertible normal operators on Hilbert  $C^*$ -modules. Also, we find some other characterization of this operators using some inequalities related to Corach-Poreta-Recht inequality.

Keywords: EP operators; Hilbert  $C^*$ -module; Moore-Penrose inverse; Operator inequality.

#### 1. INTRODUCTION

Hilbert  $C^*$ -modules form a category in between Banach spaces and Hilbert spaces. The basic idea was to consider module over  $C^*$ -algebra instead of linear space and to allow the inner product to take values in a more general  $C^*$ -algebra than  $\mathbb{C}$ . The structure was first used by Kaplansky [4] in 1952 and more carefully investigated by Rieffel [9] and Paschke [8] later in 1972-73. We give only a brief introduction to the theory of Hilbert  $C^*$ -modules to make our explanations self-contained. For comprehensive accounts we refer to the lecture note of Lance [5]. On the other hand the concept of general inverse and in particular Moore-Penrose inverse very important. The general inverse do not have all properties. In this paper we specialize the investigations to the Moore-Penrose inverse of closed range operators on Hilbert  $C^*$ -modules. The notion of EP operator was extended by Campbell and Meyer for Hilbert Spaces in [1, 2] and so on by Sharifi [10] and Mohammadzadeh Karizaki [3, 6] for Hilbert  $\mathcal{A}$ -modules. In fact the operator  $T \in L(\mathcal{X})$  is called *EP*, if T and T<sup>\*</sup> have the same ranges. Throughout the paper  $\mathcal{A}$  is a C\*-algebra (not necessarily unital). A (right) pre-Hilbert module over a  $C^*$ -algebra  $\mathcal{A}$  is a complex linear space  $\mathcal{X}$ , which is an algebraic right  $\mathcal{A}$ -module and  $\lambda(xa) = (\lambda x)a = x(\lambda a)$  equipped with an  $\mathcal{A}$ -valued inner product  $\langle ., . \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$  satisfying,

- (i)  $\langle x, x \rangle \ge 0$ , and  $\langle x, x \rangle = 0$  iff x = 0,
- (ii)  $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle$ ,
- (iii)  $\langle x, ya \rangle = \langle x, y \rangle a$ ,
- (iv)  $\langle y, x \rangle = \langle x, y \rangle^*$ .

for each  $x, y, z \in \mathcal{X}, \lambda \in \mathbb{C}, a \in \mathcal{A}$ . A pre-Hilbert  $\mathcal{A}$ -module  $\mathcal{X}$  is called a Hilbert  $\mathcal{A}$ -module if it is complete with respect to the norm  $||x|| = ||\langle x, x \rangle||^{\frac{1}{2}}$ . Left Hilbert  $\mathcal{A}$ -modules are defined in a similar way. For example every  $C^*$ -algebra  $\mathcal{A}$  is a Hilbert  $\mathcal{A}$ -module with respect to inner product  $\langle x, y \rangle = x^*y$ , and every Hilbert space is a left Hilbert  $\mathbb{C}$ -module. Suppose that  $\mathcal{X}$  and  $\mathcal{Y}$  are Hilbert  $\mathcal{A}$ -modules. Then,

<sup>2010</sup> Mathematics Subject Classification. 47A53, 15A09.

Speaker: Javad Farokhi-Ostad.

 $\mathcal{L}(\mathcal{X}, \mathcal{Y})$  is the set of all maps  $T : \mathcal{X} \to \mathcal{Y}$  for which there is a map  $T^* : \mathcal{Y} \to \mathcal{X}$ , the so-called adjoint of T such that  $\langle Tx, y \rangle = \langle x, T^*y \rangle$  for each  $x \in \mathcal{X}, y \in \mathcal{Y}$ . It is known that any element T of  $\mathcal{L}(\mathcal{X}, \mathcal{Y})$  must be a bounded linear operator, which is also  $\mathcal{A}$ -linear in the sense that T(xa) = (Tx)a for  $x \in \mathcal{X}$  and  $a \in \mathcal{A}$  [5, Page 8]. We use the notations  $\mathcal{L}(\mathcal{X})$  in place of  $\mathcal{L}(\mathcal{X}, \mathcal{X})$ , and ker( $\cdot$ ) and ran( $\cdot$ ) for the kernel and the range of operators, respectively. The identity operator on  $\mathcal{X}$  is denoted by  $1_{\mathcal{X}}$  or 1 if there is no ambiguity.

Suppose that  $\mathcal{X}$  is a Hilbert  $\mathcal{A}$ -module and  $\mathcal{Y}$  is a closed submodule of  $\mathcal{X}$ . We say that  $\mathcal{Y}$  is orthogonally complemented if  $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Y}^{\perp}$ , where  $\mathcal{Y}^{\perp} := \{y \in \mathcal{X} : \langle x, y \rangle = 0 \text{ for all } x \in \mathcal{Y} \}$  denotes the orthogonal complement of  $\mathcal{Y}$  in  $\mathcal{X}$ .

Throughout this paper  $\mathcal{X}$ ,  $\mathcal{Y}$  and  $\mathcal{Z}$  are Hilbert  $\mathcal{A}$ -modules. Recall that a closed submodule in a Hilbert module is not necessarily orthogonally complemented, however Lance in [5] proved that certain submodules are orthogonally complemented as follows.

**Theorem 1.1.** ([5, Theorem 3.2]) Suppose that  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  has closed range. Then  $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$  has closed range, and

(i) ker(T) is orthogonally complemented in  $\mathcal{X}$ , with  $(\ker(T))^{\perp} = \operatorname{ran}(T^*)$ .

(ii) ran(T) is orthogonally complemented in  $\mathcal{Y}$ , with  $(ran(T))^{\perp} = \ker(T^*)$ .

**Definition 1.2.** Let  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ . The Moore-Penrose inverse  $T^{\dagger}$  of T (if it exists) is an element in  $L(\mathcal{Y}, \mathcal{X})$  which satisfies

Motivated by these conditions,  $T^{\dagger}$  is unique and  $T^{\dagger}T$  and  $TT^{\dagger}$  are orthogonal projections, in the sense that those are selfadjoint idempotent operators. Clearly, T is Moore-Penrose invertible if and only if  $T^*$  is Moore-Penrose invertible, and in this case  $(T^*)^{\dagger} = (T^{\dagger})^*$ . The following theorem is known.

**Theorem 1.3.** ([11, Theorem 2.2]) Suppose that  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ . Then the Moore-Penrose inverse  $T^{\dagger}$  of T exists if and only if T has closed range.

By Definition 1.2, we have

$\operatorname{ran}(\mathbf{T}) = \operatorname{ran}(\mathbf{T}\mathbf{T}^{\dagger}),$	$\operatorname{ran}(\mathrm{T}^{\dagger}) = \operatorname{ran}(\mathrm{T}^{\dagger}\mathrm{T}) = \operatorname{ran}(\mathrm{T}^{*}),$
$\ker(T) = \ker(T^{\dagger}T),$	$\ker(T^{\dagger}) = \ker(T T^{\dagger}) = \ker(T^{\ast}),$

and by Theorem 1.1, we have

$$\mathcal{X} = \ker(T) \oplus \operatorname{ran}(\mathrm{T}^{\dagger}) = \ker(\mathrm{T}^{\dagger}\mathrm{T}) \oplus \operatorname{ran}(\mathrm{T}^{\dagger}\mathrm{T}),$$
$$\mathcal{Y} = \ker(T^{\dagger}) \oplus \operatorname{ran}(\mathrm{T}) = \ker(\mathrm{T}\,\mathrm{T}^{\dagger}) \oplus \operatorname{ran}(\mathrm{T}\,\mathrm{T}^{\dagger}).$$

A matrix form of a bounded adjointable operator  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  can be induced by some natural decompositions of Hilbert  $C^*$ -modules. Indeed, if  $\mathcal{M}$  and  $\mathcal{N}$  are closed orthogonally complemented submodules of  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively, and  $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^{\perp}$ ,  $\mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^{\perp}$ , then T can be written as the following  $2 \times 2$  matrix

(1.1) 
$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$

where,  $T_1 \in \mathcal{L}(\mathcal{M}, \mathcal{N}), T_2 \in \mathcal{L}(\mathcal{M}^{\perp}, \mathcal{N}), T_3 \in \mathcal{L}(\mathcal{M}, \mathcal{N}^{\perp})$  and  $T_4 \in \mathcal{L}(\mathcal{M}^{\perp}, \mathcal{N}^{\perp})$ . Note that  $P_{\mathcal{M}}$  denotes the projection corresponding to  $\mathcal{M}$ .

In fact  $T_1 = P_{\mathcal{N}}TP_{\mathcal{M}}$ ,  $T_2 = P_{\mathcal{N}}T(1 - P_{\mathcal{M}})$   $T_3 = (1 - P_{\mathcal{N}})TP_{\mathcal{M}}$  and  $T_4 = (1 - P_{\mathcal{N}})T(1 - P_{\mathcal{M}})$ .

Recall that if  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  has closed range, then  $TT^{\dagger} = P_{\operatorname{ran}(T)}$  and  $T^{\dagger}T = P_{\operatorname{ran}(T^*)}$ .

**Lemma 1.4.** ([6, Theorem 2.2]) Suppose that  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  has closed range. Then T has the following matrix decomposition with respect to the orthogonal decompositions of closed submodules  $\mathcal{X} = \operatorname{ran}(T^*) \oplus \ker(T)$  and  $\mathcal{Y} = \operatorname{ran}(T) \oplus \ker(T^*)$ 

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(\mathrm{T}^*) \\ \ker(T) \end{bmatrix} \mapsto \begin{bmatrix} \operatorname{ran}(\mathrm{T}) \\ \ker(T^*) \end{bmatrix}$$

where  $T_1$  is invertible. Moreover,

$$T^{\dagger} = \begin{bmatrix} T_1^{-1} & 0\\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(T)\\ \ker(T^*) \end{bmatrix} \mapsto \begin{bmatrix} \operatorname{ran}(T^*)\\ \ker(T) \end{bmatrix}$$

An operator T with closed range is called EP if  $\mathcal{N}(\mathcal{T}) = \mathcal{N}(\mathcal{T}^*)$ . It is easy to see that,

$$Tis \ EP \Leftrightarrow \mathcal{R}(\mathcal{T}) = \mathcal{R}(\mathcal{T}^*) \Leftrightarrow TT^{\dagger} = T^{\dagger}T.$$

# 2. Main Results

In this section, some characterization of Moore-Penrose invertible normal operators on Hilbert  $C^*$ -modules has been founded. Also, some conditions which is state that, the product of operators is EP has been presented.

**Definition 2.1.** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be Hilbert  $\mathcal{A}$ -modules. An operator  $V \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  is a partial isometry if for each  $x \in (\ker V)^{\perp}$ ; ||Vx|| = ||x||.

Similar to [7, Theorem 2.3.4], each  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  with closed range has a polar decomposition T = V|T|, where  $V \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  is a partial isometry,  $|T| = (\underline{T^*T})^{\frac{1}{2}}$ ,  $\ker(V) = \ker(T)$ ,  $\operatorname{ran}(V) = \overline{\operatorname{ran}(T)}$ ,  $\ker(V^*) = \ker(T^*)$ ,  $\operatorname{ran}(V^*) = \overline{\operatorname{ran}(V^*)} = \operatorname{ran}(|T|)$  and  $V^*T = |T|$ .

Suppose that  $\mathcal{X}$  and  $\mathcal{Y}$  are Hilbert  $\mathcal{A}$ -modules and  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  is an operator with a polar decomposition T = V|T|. Since  $V \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  is a partial isometry, ran(V) = ran(T) = ranT. Then V has a closed range and  $V^{\dagger}$  exists. By [5, Page 30] and the uniqueness of Moore-Penrose inverse,  $V^* = V^{\dagger}$ . Utilizing polar decomposition, we have ran(|T|) = ran( $T^*$ ), so ran(|T|) is closed. Since V has closed range, (ker V)<sup> $\perp$ </sup> = ran( $V^*$ ) = ran(|T|) = ran(|T|). It is clear that, the reverse order law hold for polar decomposition. We use the above mentioned to proof of main theorem of this section.

**Theorem 2.2.** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be Hilbert  $\mathcal{A}$ -modules and  $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  have closed ranges. Then  $TT^* = T^{\dagger}T$  if and only if  $||T^*ST^{\dagger}|| = ||T^{\dagger}TST^{\dagger}||$ 

**Theorem 2.3.** Let  $\mathcal{X}$  be Hilbert  $\mathcal{A}$ -module and  $T \in \mathcal{L}(\mathcal{X})$  has closed ranges. Then the following statements are equivalent:

- (ii) T commute with  $T^* + T^{\dagger}$ ,
- (iii)  $T^{\dagger}$  commute with  $T + T^*$ .

<sup>(</sup>i) T is normal,

**Theorem 2.4.** Let  $\mathcal{X}$  be Hilbert  $\mathcal{A}$ -module and  $T \in \mathcal{L}(\mathcal{X})$  has closed ranges. Then the following statements are equivalent:

- (i) T is normal,
- (ii)  $TT^{\dagger}T^{*}TTT^{\dagger} = TT^{*},$
- (iii)  $T^{\dagger}TT^{*}TT^{\dagger}T = TT^{*}$ .

**Theorem 2.5.** Let  $\mathcal{X}$  be Hilbert  $\mathcal{A}$ -module and  $T \in \mathcal{L}(\mathcal{X})$  has closed ranges and *EP* operator, then the following statements are equivalent:

- (i) T is normal,
- (ii)  $\forall S \in \mathcal{L}(\mathcal{X}), \qquad ||TST^{\dagger}|| + ||T^{\dagger}TS|| = ||T^*ST^{\dagger}|| + ||T^{\dagger}ST^*||,$

(iii)  $\forall S \in \mathcal{L}(\mathcal{X}), \qquad ||TST^{\dagger}|| + ||T^{\dagger}TS|| \ge 2||TT^{\dagger}ST^{\dagger}T'||.$ 

**Theorem 2.6.** Let  $T, S \in L(\mathcal{X})$  are EP operators with closed ranges. Then TS is EP if and only if  $(1 - SS^{\dagger})(TS) = 0$ .

**Theorem 2.7.** Let  $T, S \in L(\mathcal{X})$ . If S is EP and  $(TS)^{\dagger} = S^{\dagger}T^{\dagger}$ , then TS is EP.

# References

- S. Campbell and C.D. Meyer, Continuity properties of the Drazin pseudo inverse, Linear Algebra and Its Applications, 10 (1975) 77-83.
- [2] S. Campbell and C.D. Meyer, EP operators and generalized inverses, Canadian Math. Bull., 18 (1975) 327-333.
- [3] J. Farokhi-ostad and M. Mohammadzadeh Karizaki, The reverse order law for EP modular operators, J. Math. Computer Sci. 16 (2016), 412-418.
- [4] I. Kaplansky, Algebras of type I, Ann. Math., 56 (1952), 460-472.
- [5] E.C. Lance, Hilbert C\*-Modules, LMS Lecture Note Series 210, Cambridge Univ. Press, 1995.
- [6] M. Mohammadzadeh Karizaki and Dragan S. Djordjevic, Commuting C\* modular operators, Aequationes Mathematicae (2016) 1-12.
- [7] G. J. Murphy, C\*-algebras and operator theory, Academic Press, 1990.
- [8] W. Paschke, Inner product modules over B\*-algebras, Trans. Amer. Math. Soc., 182 (1973), 443-468.
- [9] M. Rieffel, Morita equivalence for  $C^*$ -algebra and  $W^*$ -algebra, J. Pure Allpied Algebra, 5 (1974), 51-96.
- [10] K. Sharifi and B. Ahmadi Bonakdar, The reverse order law for Moore-Penrose inverses of operators on Hilbert C\*-modules, Bull. Iran. Math. Soc., 42 (2016) 53- 60.
- Q. Xu, L. Sheng, Positive semi-definite matrices of adjointable operators on Hilbert C\*modules, Linear Algebra Appl. 428 (2008), 992–1000.

DEPARTMENT OF MATHEMATICS, FACULTY OF BASIC SCIENCES, BIRJAND UNIVERSITY OF TECHNOLOGY, BIRJAND, IRAN.

 ${\it E-mail\ address:\ j.farokhi@birjandut.ac.ir.\ javadfarrokhi90@gmail.com}$ 

DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICS AND STATISTICS, UNIVERSITY OF BIRJAND, BIRJAND, IRAN.

E-mail address: ajanfada@birjand.ac.ir



# A SOLUTION OF OPERATOR EQUATION ON HILBERT $C^*$ -MODULES

## JAVAD FAROKHI-OSTAD AND ALIREZA JANFADA

ABSTRACT. In this paper, we study the modular operator equations on Hilbert  $C^*$ -modules. Also, by using some block operator matrix techniques, we find the sufficient and necessary conditions of solvability for such equations. Furtheremore, the explicit solution of ploynomial modular operator equations has been founded.

Keywords: Modular operator equation; Moore-Penrose inverse; Hilbert  $C^*$ -module.

#### 1. INTRODUCTION

The operator equations plays important roles in control of system and fault detection. It is special forms include Lyapanov and Sylvester equations. These equations are solved in some fields such as matrix equation, or over a finite field, and so on as operator equations on Hilbert spaces. For details refer to other work in formal [7] [1]. In this paper, we should solved such modular operators equations on Hilbert C<sup>\*</sup>-modules. The equation  $TXS^* - SX^*T^* = A$  was studied by Yuan [16] for finite matrices and Xu et al. [15] generalized the results to Hilbert  $C^*$ -modules, under the condition that ran(S) is contained in ran(T). When T equals an identity matrix or identity operator, this equation reduces to  $XS^* - SX^* = A$ , which was studied by Braden [1] for finite matrices, and Djordjevic [2] for the Hilbert space operators. In this paper, by using block operator matrix techniques and properties of the Moore-Penrose inverse, we present a solution X for adjointable Hilbert module operators than those with closed ranges as  $AXA^* = B$ . Furthermore, we solve the operator equation  $A^*X - X^*A = B$ . This paper is organized as follows. In the remainder of this section, some preliminaries is given, which are used in the following sections. Section 2 the main theorem of our work is appeared, which is present the solution of modular operator equations by a simple technique of matrix forms of that operators. Hilbert  $C^*$ -modules form a category between Banach spaces and Hilbert spaces. The basic idea was to consider module over  $C^*$ -algebra instead of linear space and to allow the inner product to take values in a more general  $C^*$ -algebra than  $\mathbb{C}$ . The structure was first used by Kaplansky [5] in 1952 and more carefully investigated by Rieffel [13] and Paschke [12] later in 1972–73. We give only a brief introduction to the theory of Hilbert  $C^*$ -modules to make our explanations self-contained. For comprehensive accounts we refer to the lecture note of Lance [6].

We now give the definition of Hilbert  $C^*$ -modules.

**Definition 1.1.** Let  $\mathcal{A}$  be a  $C^*$ -algebra (not necessarily unital). A (right) pre-Hilbert module over the  $C^*$ -algebra  $\mathcal{A}$  is a complex linear space  $\mathcal{X}$ , which is an

<sup>2010</sup> Mathematics Subject Classification. Primary 47A62, Secondary 15A24; 46L08. Speaker: Javad Farokhi-Ostad.

algebraic right  $\mathcal{A}$ -module and  $\lambda(xa) = (\lambda x)a = x(\lambda a)$  equipped with an  $\mathcal{A}$ -valued inner product  $\langle ., . \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$  satisfying, (i)  $\langle x, x \rangle \geq 0$ , and  $\langle x, x \rangle = 0$  iff x = 0, (ii)  $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle$ , (iii)  $\langle x, ya \rangle = \langle x, y \rangle a$ , (iv)  $\langle y, x \rangle = \langle x, y \rangle^*$ . for each  $x, y, z \in \mathcal{X}$ ,  $\lambda \in \mathbb{C}$  and  $a \in \mathcal{A}$ .

**Remark 1.2.** Left pre-Hilbert C<sup>\*</sup>-modules are defined similarly.

If  $\mathcal{X}$  satisfies all the conditions for an inner-product  $\mathcal{A}$ -module except for the second part of (i), then we call  $\mathcal{X}$  a semi-inner-product  $\mathcal{A}$ -module. For such modules there is a useful version of the Cauchy-Schwarz inequality:

**Theorem 1.3.** If  $\mathcal{X}$  is a semi-inner-product  $\mathcal{A}$ -module and  $x, y \in \mathcal{X}$ , then

 $\langle y, x \rangle \langle x, y \rangle \le \| \langle x, x \rangle \| \langle y, y \rangle.$ 

For  $x \in \mathcal{X}$ , put

$$||x|| = ||\langle x, x \rangle||^{\frac{1}{2}}.$$

Then  $\|\langle x, y \rangle\|^2 = \|\langle x, y \rangle^* \langle x, y \rangle\| = \|\langle y, x \rangle \langle x, y \rangle\| \le \|\langle x, x \rangle\| \| \langle y, y \rangle\| = \|x\|^2 \|y\|^2$ . Hence

 $\|\langle x, y \rangle\| \le \|x\| \|y\|$ 

It follows that  $\langle ., . \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$  is continuous. Obviously  $\|.\|$  is a norm on  $\mathcal{X}$ .

**Definition 1.4.** A pre-Hilbert  $C^*$ -module  $\mathcal{X}$  is called a Hilbert  $C^*$ -module if  $\mathcal{X}$  equipped with the above norm is complete. Note that completion of a pre-Hilbert  $\mathcal{A}$ -module, is a Hilbert  $C^*$ -module.

We have  $0 \le \langle xa, xa \rangle = a^* \langle x, x \rangle a \le a^* ||x||^2 a$ . Taking norm we get

$$\|\langle xa, xa \rangle\| \le \|a^*\|x\|^2 a\|\|xa\|^2 \le \|x\|^2\|a\|^2 \text{or} \|xa\| \le \|x\|\|a\|.$$

Hence  $\mathcal{X}$  is a normed  $\mathcal{A}$ -module. There is also an  $\mathcal{A}$ -valued norm  $| \cdot |$  on  $\mathcal{X}$  given by  $|x| = \langle x, x \rangle^{1/2}$ .

Note that submodules in Hilbert  $C^*$ -modules need not be complementable in general; however, Lance [6] proved that certain submodules are orthogonally complemented as follows.

Throughout this paper,  $\mathcal{X}$  and  $\mathcal{Y}$  are Hilbert  $\mathcal{A}$ -modules. We use the notations ker(·) and ran(·) for the kernel and the range of operators, respectively. The identity operator on  $\mathcal{X}$  is denoted by  $1_{\mathcal{X}}$  or 1 if there is no ambiguity. The reader is referred to [3, 4, 6] and the references cited therein for more details in Hilbert  $C^*$ -module.

**Theorem 1.5.** (see [6, Theorem 3.2]) Let  $\mathcal{X}$ ,  $\mathcal{Y}$  be Hilbert  $\mathcal{A}$ -modules and  $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  have closed range. Then

- $\ker(A)$  is orthogonally complemented in  $\mathcal{X}$ , with complement  $\operatorname{ran}(A^*)$ .
- ran(A) is orthogonally complemented in  $\mathcal{Y}$ , with complement ker( $A^*$ ).
- The map  $A^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$  has closed range.

**Remark 1.6.** The proof of [6, Theorem 3.2] indicates that if  $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  with closed range, then  $\operatorname{ran}(AA^*) = \operatorname{ran}(A)$ , and thus  $\operatorname{ran}(AA^*)$  is also closed. On the other hand, if  $\operatorname{ran}(AA^*)$  is closed, then

$$\mathcal{Y} = \operatorname{ran}(AA^*) \oplus \ker(AA^*) = \operatorname{ran}(TAA^*) \oplus \ker(A^*) \subseteq \operatorname{ran}(A) \oplus \ker(A^*) \subseteq \mathcal{Y},$$

so that  $ran(A) \oplus ker(A^*) = \mathcal{Y}$ , which implies that ran(A) is closed. Replacing A by  $A^*$ , we conclude that

 $\operatorname{ran}(A) \text{ is closed } \iff \operatorname{ran}(A^*) \text{ is closed } \iff \operatorname{ran}(AA^*) \text{ is closed } \iff \operatorname{ran}(A^*A) \text{ is closed.}$ 

A generalized inverse of  $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  is an operator  $A^{\times} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$  such that

(1.1) 
$$AA^{\times}A = A \text{ and } A^{\times}AA^{\times} = A^{\times}.$$

It is known that a bounded adjointable operator A has generalized inverse if and only if ran(A) is closed.

**Definition 1.7.** Let  $TA \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ . The Moore-Penrose inverse  $A^{\dagger}$  of A (if it exists) is an element of  $\mathcal{L}(\mathcal{Y}, \mathcal{X})$  which satisfies:

(a)  $A A^{\dagger} A = A$ , (b)  $A^{\dagger} A A^{\dagger} = A^{\dagger}$ , (c)  $(A A^{\dagger})^* = A A^{\dagger}$ , (d)  $(A^{\dagger} A)^* = A^{\dagger} A$ .

Clearly, A is Moore-Penrose invertible if and only if  $A^*$  is Moore-Penrose invertible, and in this case  $(A^*)^{\dagger} = (A^{\dagger})^*$  and  $A^{\dagger}A$  and  $AA^{\dagger}$  are orthogonal projections.

By Theorem 1.5, we know that

$$\mathcal{X} = \ker(A) \oplus \operatorname{ran}(A^{\dagger}) = \ker(A^{\dagger}A) \oplus \operatorname{ran}(A^{\dagger}A)$$
$$\mathcal{Y} = \ker(A^{\dagger}) \oplus \operatorname{ran}(A) = \ker(A A^{\dagger}) \oplus \operatorname{ran}(A A^{\dagger}).$$

Xu and Sheng in the Theorem 1.8 showed that a bounded adjointable operator between two Hilbert  $\mathcal{A}$ -modules admits a bounded Moore-Penrose inverse if and only if it has closed range.

**Theorem 1.8.** Let  $\mathcal{X}$ ,  $\mathcal{Y}$  be Hilbert  $\mathcal{A}$ -modules and  $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ . Then the Moore-Penrose inverse  $A^{\dagger}$  of A exists if and only if A has closed range.

## 2. Main results

In this section, by using some block operator matrix techniques, we find the sufficient and necessary conditions of solvability for operator equations on Hilbert  $C^*$ -modules. Furthermore, we solve the operator equation  $AXA^* = B$ .

**Theorem 2.1.** Let  $\mathcal{X}$ ,  $\mathcal{Y}$  and  $\mathcal{Z}$  are Hilbert  $\mathcal{A}$ -modules and  $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ ,  $B \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$  has closed range, then

AX = B has solution if and only if  $\lambda AA^* - BB^* \ge 0$ , for some  $\lambda > 0$ .

**Lemma 2.2.** Let  $\mathcal{X}$ ,  $\mathcal{Y}$  and  $\mathcal{Z}$  are Hilbert  $\mathcal{A}$ -modules and  $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ , then  $AA^*$  and  $A^*A$  are positive.

**Theorem 2.3.** Let  $\mathcal{X}$ ,  $\mathcal{Y}$  and  $\mathcal{Z}$  are Hilbert  $\mathcal{A}$ -modules and  $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  has closed range, then AX = B has a positive solution if and only if ran(B)  $\subseteq$  ran(A) and  $BA^* \geq 0$ .

**Theorem 2.4.** Let  $\mathcal{X}$ ,  $\mathcal{Y}$  and  $\mathcal{Z}$  are Hilbert  $\mathcal{A}$ -modules and  $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  has closed ranke, then  $AXA^* = B$  has a solution if and only if  $AA^{\dagger}B(A^*)^{\dagger}(A^*) = B$ .

**Theorem 2.5.** Let  $\mathcal{X}$ ,  $\mathcal{Y}$  and  $\mathcal{Z}$  are Hilbert  $\mathcal{A}$ -modules and  $A \in \mathcal{L}(\mathcal{X})$  has closed range and  $B \in \mathcal{L}(\mathcal{Y})$ , then

 $A^*X - X^*A = Bhas solution if and only if B = -B^*$ .

**Lemma 2.6.** Let  $A \in \mathcal{L}(\mathcal{X})$  has closed range. If  $AX - XA^* = B$  has solution, then for a positive integer n we have  $A^nX - X(A^*)^n = \sum_{i=1}^n A^{n-i}B(A^*)^{i-1}$ .

**Theorem 2.7.** Let  $A \in \mathcal{L}(\mathcal{X})$  has closed range. If  $p(x) = x^n + a_1 x^{n-1} + ... a_1 x + a_0$  is a ploynomial over ran(A), then

$$P(A)X = B_n + a_1B_{n-1} + \dots + a_{n-1}B_1,$$

where  $B_k = \sum_{i=1}^n A^{k-i} B(A^*)^{i-1}$ .

#### References

- [1] H. Braden, The equations  $A^T X \pm X^T A = B$ , Siam J. Matrix Anal. Appl. **20** (1998), 295–302.
- [2] D. S. Djordjevic, Explicit solution of the operator equation A\*X + X\*A = B, J. Comput. Appl. Math. 200 (2007), 701–704.
- [3] J. Farokhi-ostad and M. Mohammadzadeh Karizaki, The reverse order law for EP modular operators, J. Math. Computer Sci. 16 (2016) 412–418.
- [4] J. Farokhi-ostad and A. R. Janfada, Products Of EP Operators On Hilbert C\*-Modules, Sahand Communications in Math. Anal., (2017) (To appear).
- [5] I. Kaplansky, Algebras of type I, Ann. Math., 56 (1952), 460–472.
- [6] E.C. Lance, Hilbert C\*-Modules, LMS Lecture Note Series 210, Cambridge Univ. Press, 1995.
- [7] M. Mohammadzadeh Karizaki and Dragan S. Djordjevic, Commuting C\* modular operators, Aequationes Mathematicae 90 (6) (2016) 1103–1114.
- [8] M. Mohammadzadeh Karizaki, M. Hassani and M. Amyari, Moore-Penrose inverse of product operators in Hilbert C<sup>\*</sup>-modules, Filomat, 13 (2016) 3397–3402.
- M. Mohammadzadeh Karizaki, M. Hassani, M. Amyari and M. Khosravi, Operator matrix of Moore-Penrose inverse operators on Hilbert C<sup>\*</sup>-modules, Colloq. Math. 140 (2015) 171–182.
- [10] M. Mohammadzadeh Karizaki, M. Hassani, The solutions to some operator equations in Hilbert  $C^*$ -module, Journal of Linear and Topological Algebra (To appear)
- [11] M. Mohammadzadeh Karizaki, M. Hassani, Solutions to some solvable operator equations in Hilbert C\*-module, The Journal of Mathematics and Computer Science (To appear)
- [12] W. Paschke, Inner product modules over B\*-algebras, Trans. Amer. Math. Soc., 182 (1973), 443-468.
- [13] M. Rieffel, Morita equivalence for C\*-algebra and W\*-algebra, J. Pure Allpied Algebra, 5 (1974), 51–96.
- [14] Q. Xu, L. Sheng, Positive semi-definite matrices of adjointable operators on Hilbert C\*modules, *Linear Algebra Appl.* 428 (2008), 992–1000.
- [15] Q. Xu, L. Sheng, Y. Gu, The solutions to some operator equations, *Linear Algebra Appl.* 429 (2008), 1997–2024.
- [16] Y. Yuan, Solvability for a class of matrix equation and its applications, J, Nanjing Univ. (Math. Biquart.) 18 (2001), 221–227.

DEPARTMENT OF MATHEMATICS, FACULTY OF BASIC SCIENCES, BIRJAND UNIVERSITY OF TECHNOLOGY, BIRJAND, IRAN.

E-mail address: j.farokhi@birjandut.ac.ir. javadfarrokhi90@gmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICS AND STATISTICS, UNIVERSITY OF BIRJAND, BIRJAND, IRAN.

*E-mail address*: ajanfada@birjand.ac.ir



# RELATIVE REPRODUCING KERNELS ASSOCIATED WITH COCYCLES

## SAEED HASHEMI SABABE AND SHAHRAM NAJAFZADEH

ABSTRACT. In this paper we speak about reproducing kernels whose ranges are subsets of a  $C^*$ -algebra or a Hilbert  $C^*$ -module. In particular, we show how such a reproducing kernel can naturally be expressed in terms of operators on a Hilbert  $C^*$ -module. We focus on relative reproducing kernels and extend this concept to such spaces associated with cocycles.

Keywords: Reproducing kernels;  $C^*$ -modules; Cocyles.

## 1. INTRODUCTION

The theory of reproducing kernels is fundamental theory in recent decades and has widely application in mathematics. Special application of reproducing kernel theory appears inside the domain of physics and provides an extension in basis of stochastic processes and signal processing. Bergman kernels, the harmonic kernels, and the Szego kernels are important examples of reproducing kernels. Related to these spaces, Alpay produced a new constructer, relative reproducing kernels which is a wider concept and has some application in electric circuits. Hashemi Sababe and his co-authors tried to express and develop this concept in [1, 2, 3].

The purpose of this paper is to discuss positive definite kernels on semigroups, reproducing kernels on locally compact spaces, and their decompositions via Hilbert  $C^*$ -modules. Unitary representations of a group  $\mathcal{G}$  acting on a space S in which the Hilbert  $C^*$ -module is an operator-valued function space on S and the representation is a representation associated by a cocycle on  $S \subset G$  is studied before by Heo [4]. In the following of his works, we try to extend relative reproducing property to reproducing kernels associated with cocycles.

# 2. Preliminaries

We start with fundamental definitions and theorems. Most interesting properties of  $C^*$ -Hilbert modules and relative reproducing kernels is worked before and we refer readers to [1, 2, 3, 4] for more details.

Throughout this article, S and  $\mathcal{A}$  denote a nonempty set and a  $C^*$ -algebra, respectively. We denote by X a self-dual Hilbert  $\mathcal{A}$ -module of  $\mathcal{A}$ -valued functions on S such that each valuation  $f \mapsto f(s) \in \mathcal{A}$  is continuous and linear. Then, for each  $s \in S$  and  $f \in X$  there exists an element  $g_s \in X$  such that

$$f(s) = \langle g_s, f \rangle_X.$$

The corresponding reproducing kernel  $\kappa: S \times S \to \mathcal{A}$  is given by

$$\kappa(s,t) = \langle g_s, g_t \rangle_X \in \mathcal{A}.$$

<sup>2010</sup> Mathematics Subject Classification. 47B32, 47A70.

Speaker: Saeed Hashemi Sababe.

For every  $t \in S$  the function  $g_t \in X$  will be denoted by  $\kappa(\cdot, t)$ .

**Theorem 2.1** (Theorem 3.2 [4]). If a kernel  $\kappa : S \times S \to A$  is positive definite, then there exists a Hilbert A-module X of A-valued functions on S such that  $\kappa$  is the reproducing kernel of X.

**Definition 2.2.** Let  $\mathcal{B}$  be a  $C^*$ -algebra. A kernel  $\kappa : S \times S \to \mathcal{B}$  is positive definite if for every  $n \in \mathbb{N}$  and  $s_1, \ldots, s_n \in S$  the matrix  $[\kappa(s_i, s_j)]_{i,j=1}^n$  is positive in  $M_n(\mathcal{B})$ . It follows from the definition that a kernel  $\kappa : S \times S \to \mathcal{B}$  is positive definite if and only if for all  $s_1, \ldots, s_n \in S$  and  $b_1, \ldots, b_n \in \mathcal{B}$ , the sum  $\sum_{i,j=1}^n b_i^* \kappa(s_i, s_j) b_j$  is positive in  $\mathcal{B}$ . If a kernel  $\kappa$  from  $S \times S$  into  $L_{\mathcal{A}}(X)$  can be written in the form

$$\kappa(s,t) = \upsilon(s)^* \upsilon(t) \quad for \ any \ s,t \in S,$$

where v is a map from S to  $L_{\mathcal{A}}(X, X_v)$  for some right Hilbert  $\mathcal{A}$ -module  $X_v$ , then  $\kappa$  is automatically positive definite. Such a map v is called the Kolmogorov decomposition for  $\kappa$ .

We continue our work with an important example.

**Example 2.3.** Let  $\mathcal{G}$  be a locally compact group with a left Haar measure  $\mu$ ,  $\mathcal{A}$  a unital  $C^*$ -algebra, and X a Hilbert  $\mathcal{A}$ -module. Assume that  $\mathcal{G}$  is unimodular. We denote by  $\mathcal{K}(\mathcal{G}, \mathcal{A})$  the right  $\mathcal{A}$ -module of continuous functions from  $\mathcal{G}$  to  $\mathcal{A}$  with compact supports. For all f, g in  $\mathcal{K}(\mathcal{G}, \mathcal{A})$ , we define an  $\mathcal{A}$ -valued map  $\langle \cdot, \cdot \rangle$  by

$$\langle f,g\rangle = \int_G f(a)^* g(a) d\mu(a).$$

We denote by  $L^2(\mathcal{G}, \mathcal{A})$  its completion. Let  $\pi : \mathcal{G} \to L_{\mathcal{A}}(X)$  be a strongly continuous unitary representation and let  $x_0 \in X$  be fixed. Let  $\theta$  be the  $\mathcal{A}$ -module map from the Hilbert  $\mathcal{A}$ -module X into the space of  $\mathcal{A}$ -valued continuous maps defined by

$$[\theta(x)](a) = \langle \pi(a)x_0, x \rangle_X \quad x \in X, a \in \mathcal{G}$$

It is not hard to see that  $[\theta(\pi(a)x)](b) = [\theta(x)](a^{-1}b)$  for all  $x \in X$  and all  $a, b \in G$ . Let Y denote the image  $\theta(X)$  of X under  $\theta$ . Assume that  $\langle x_0, x_0 \rangle = I$  and  $x_0$ is a cyclic unit vector for the representation  $\pi$ , that is, the closed span of the set  $\{\pi(a)x_0\alpha : a \in G, \alpha \in A\}$  is the whole of X. Then the map  $\theta : X \to Y$  is an A-module isomorphism, so that the right A-module Y can be given the structure of a Hilbert A-module isometric to X. Indeed, for all  $x, y \in X$ , we have

$$\langle \theta(x), \theta(y) \rangle_Y = \int_{\mathcal{G}} [\theta(x)](a)^* [\theta(y)](a) d\mu(a)$$
$$= \int_{\mathcal{G}} \langle \pi(a)x_0, x \rangle_X^* \langle \pi(a)x_0, y \rangle_X d\mu(a) = \langle x, y \rangle_X$$

If  $\mathcal{G}$  is a transformation group on S acting on the right, then there is a canonical action  $\pi$  of  $\mathcal{G}$  on the space of  $\mathcal{A}$ -valued functions according to the formula

$$(\pi(a)f)(s) = f(sa) \quad a \in \mathcal{G}, s \in S,$$

where  $f: S \to \mathcal{A}$  is a function. If X is a self-dual Hilbert  $\mathcal{A}$ -module of  $\mathcal{A}$ -valued functions on S with a unitary representation  $\pi$  of  $\mathcal{G}$  on X, that is,  $\pi(a)f \in X$  and  $\|\pi(a)f\| = \|f\|$  for all  $a \in \mathcal{G}$  and  $f \in X$ , then X is called a  $(\mathcal{G}, \pi)$ -Hilbert  $\mathcal{A}$ -module.

**Definition 2.4.** Let X be a Hilbert A-module of A-valued functions on S such that each evaluation  $f \mapsto f(s)$  is continuous and  $\kappa$  is the reproducing kernel of X. Let  $\mathcal{G}$ be a transformation group on S acting on the right. We denote by  $\mathcal{Z}(\mathcal{A})$  the center of  $\mathcal{A}$ . A continuous function  $\alpha : S \times \mathcal{G} \to \mathcal{Z}(\mathcal{A}) \setminus \{0\}$  is called a cocycle if

$$\alpha(s, ab) = \alpha(s, a)\alpha(sa, b)$$

for all  $a, b \in \mathcal{G}$  and  $s \in S$ . For each cocyle  $\alpha$ , there is an action  $\pi_{\alpha}$  of a group  $\mathcal{G}$  on the space of  $\mathcal{A}$ -valued functions on S such that

$$(\pi_{\alpha}(a)f)(s) = f(s,a)\alpha(s,a).$$

Let  $\alpha : S \times \mathcal{G} \to L_{\mathcal{A}}(X)$  be a cocycle. A kernel  $\kappa : S \times S \to L_{\mathcal{A}}(X)$  is a  $\mathcal{G}$ -kernel associated with  $\kappa$  if for all  $s, t \in S$  and all  $a \in \mathcal{G}$ 

$$\kappa(s,t) = \alpha(s,a)\kappa(sa,ta)\alpha(t,a)^*.$$

#### 3. Main results

Heo defines reproducing kernel associated with cocycles in [4]. In following we extend this notion to relative reproducing kernel associated with cocycles.

**Definition 3.1.** Let S be a non empty set, A a C<sup>\*</sup>-algebra and X a Hilbert Amodule such that for any  $s, t \in S$  and  $f \in L_{\mathcal{X}}$  we have  $f \to f(s) - f(t)$  is continuous. A function  $\mathcal{M}_{s,t} : S \times S \to A$  is called a relative kernel if

$$f(s) - f(t) = \langle f, \mathcal{M}_{s,t} \rangle \quad \forall f \in L_{\mathcal{A}}(X), s, t \in S.$$

**Theorem 3.2.** For any strictly negative definite kernel  $\mathcal{M} : S \times S \to L_{\mathcal{A}}(X)$  there exists a unique Hilbert  $\mathcal{A}$ -module  $\mathcal{Y}_{\mathcal{M}}$  of continuous functions  $f : S \to X$  such that

- $\mathcal{Y}^0_{\mathcal{M}}$  is a dense submodule of  $\mathcal{Y}_{\mathcal{M}}$ .
- For each  $s \in S$ , the evaluation map  $e_s$  is continuous from  $\mathcal{Y}_{\mathcal{M}}$  to X.

**Definition 3.3.** Let  $\alpha : S \times \mathcal{G} \to L_{\mathcal{A}}(X)$  be a cocycle. A relative kernel  $\mathcal{M} : S \times S \to L_{\mathcal{A}}(X)$  is a relative  $\mathcal{G}$ -kernel associated with  $\alpha$  if for all  $s, t \in S$  and all  $a \in \mathcal{G}$ ,

$$\mathcal{M}(s,t) = \alpha(s,a)\mathcal{M}(sa,ta)\alpha(t,a)^*.$$

**Theorem 3.4.** For any relative  $\mathcal{G}$ -kernel  $\mathcal{M}$  associated with a cocycle  $\alpha$ , the equation

$$[\pi(a)f](s) = \alpha(s,a)f(s,a),$$

defines a unitary representation  $\pi$  of  $\mathcal{G}$  on the Hilbert  $\mathcal{A}$ -module  $\mathcal{Y}_{\mathcal{M}}$ .

**Theorem 3.5.** Let  $\mathcal{M}$  be a relative  $\mathcal{G}$ -kernel associated with a cocycle  $\alpha$  and let A be an invertible operator in  $L_{\mathcal{A}}(X)$ . The kernel  $\mathcal{M}_A$  defined by  $\mathcal{M}_A(s,t) = A^* \mathcal{M}(s,t)A$  is a positive definite relative  $\mathcal{G}$ -kernel associated with a cocycle  $\alpha_A$  given by

$$\alpha_A(s,a) = A^* \alpha(s,a) (A^{-1})^* \quad s \in S, a \in \mathcal{G}.$$

In following, we take an important example which leads us to application of above theorems.

**Example 3.6.** Let  $X = \mathcal{G}$  be a unimodular locally compact group and suppose that  $\varphi : \mathcal{G} \to \mathbb{C}$  is a continuous hermitian positive function. It is well known that  $\varphi$  defines a reproducing kernel Hilbert space. Let  $C_0(\mathcal{G})$  denote the continuous functions of compact support on  $\mathcal{G}$ . Define

$$\varphi(\mathcal{G}) = \{ f * \varphi | f \in C_0(\mathcal{G}) \}$$

where

$$(f * \varphi)(x) = \int f(y)\varphi(y^{-1}x)dy.$$

Let  $\varphi(f) = f * \varphi$  and define an inner product on  $\varphi(\mathcal{G})$  by

$$\langle \varphi(f), \varphi(g) \rangle = \int \varphi(y^{-1}x) \overline{f(x)}g(y) dx dy$$

where dx is Haar measure on  $\mathcal{G}$ . Let  $\mathcal{H}(\varphi)$  be the completion of  $\varphi(\mathcal{G})$  in the norm comes from above inner product. It is easy to see that in such a way,  $\mathcal{H}(\varphi)$  is a relative reproducing Hilbert module. Representation associated to  $\mathcal{H}(\varphi)$  is defined as

$$[\pi_x(\varphi)f](y) = \varphi(f)(x^{-1}y).$$

Relative reproducing kernels have important application in several fields. One of the most important field is graphs on  $C^*$  Hilbert modules. It is known that every weighted graph induces a relative reproducing kernel Hilbert space, and an associated graph Laplacian. The converse is also holds. It means for a given relative reproducing kernel  $C^*$ -Hilbert module X on a set S, it is then possible in a canonical way to construct a weighted graph G such that S is the set of vertices in G, and such that its energy Hilbert space coincides with X itself.

#### Acknowledgement

A main idea of this research was carried out while the third seminar of operator theory and its application, mashhad, 2017. The first author is grateful to the invited speaker of the seminar, professor Un Cig Ji for his wonderful suggestions.

#### References

- S. Hashemi Sababe, A. Ebadian and Sh. Najafzadeh, On Generalized reproducing kernel Hilbert spaces, Bulletin of Korean mathematical society, to apear.
- [2] S. Hashemi Sababe, A. Ebadian, A Course in the Theory of Groups, Sahand Communications in Mathematical Analysis (SCMA), 6 (2017).
- [3] S. Hashemi Sababe, M. Yazdi, Generalised reproducing property on noncommutative spaces, proc. Third seminar of operator theory and its application, Mashhad, Iran, (2017).
- [4] J. Heo, Reproducing kernel Hilbert C\*- modules and kernels associated with cocycles, Journal of Mathematical Physics, 49 (2008).

Department of Mathematics, Payame Noor University (PNU), P.O. Box, 19395-3697, Tehran, Iran.

E-mail address: Hashemi\_1365@yahoo.com

DEPARTMENT OF MATHEMATICS, PAYAME NOOR UNIVERSITY (PNU), P.O. Box, 19395-3697, Tehran, Iran.

E-mail address: Najafzadeh1234@yahoo.ie



#### ON JORDAN CENTRALIZERS OF QUATERNION RINGS

HOGER GHAHRAMANI, MOHAMMAD NADER GHOSSEIRI, AND LEILA HEIDARYZADEH

ABSTRACT. Let S be a unital ring in which 2 is invertible, and let R = H(S) be the quaternion ring over S. The main results of this paper is a characterization of Jordan centralizers of R and determining the conditions under which every Jordan centralizer of R is a centralizer.

Keywords: Quaternion ring; Centralizer; Jordan centralizer.

#### 1. INTRODUCTION

Let R be a ring and Z(R) be the center of R. For each x, y in R, denote the Jordan product of x, y by  $x \circ y = xy + yx$ . Recall that R is prime if for any  $a, b \in R$ , aRb = 0 implies that a = 0 or b = 0, and R is called semiprime if aRa = 0 with  $a \in R$  implies a = 0. An additive map  $\varphi : R \longrightarrow R$  is said to be a centralizer (or multiplier) if  $\varphi(xy) = \varphi(x)y = x\varphi(y)$  for each  $x, y \in R$ . In case R has an identity 1,  $\varphi$  is a centralizer if and only if  $\varphi(x) = \varphi(1)x$  for each  $x \in R$ , where  $\varphi(1) \in Z(R)$ . We define a Jordan centralizer to be an additive mapping  $\varphi$  on R that satisfies  $\varphi(x \circ y) = \varphi(x) \circ y = x \circ \varphi(y)$  for all  $x, y \in R$ . Clearly, each centralizer is a Jordan centralizer, but the converse is, in general, not true (see [5], Example 2.6). An additive mapping  $f : R \longrightarrow R$  is called a generalized derivation if there exists an additive map  $d : R \longrightarrow R$  such that f(xy) = f(x)y + xf(y) for all  $x, y \in R$ . The mapping  $I_a : R \longrightarrow R$  given by  $I_a(x) = [x, a]$  for all  $x \in R$  is called the inner derivation induced by a. Also, a generalized inner derivation is an additive mapping  $\varphi(x) = mx + xn$ , for some  $m, n \in R$ .

There has been a great interest in the study of Jordan derivations of rings or algebras as well as other operator algebras in the last decades. In most cases, each Jordan derivation is a derivation. On the other hand, it is obvious that centralizers are generalized derivations. Hence there exists a natural connection between generalized derivations and centralizers. But, what is natural and intresting is to find some conditions under which a Jordan centralizer is a centralizer. Zalar [11] showed that any Jordan centralizer on a 2-torsion free semiprime ring is a centralizer. Vukman [10] proved that an additive map  $\varphi : R \longrightarrow R$ , where R is a 2-torsion free semiprime ring, with the property that  $2\varphi(xyx) = \varphi(x)yx + xy\varphi(x)$ for all  $x, y \in R$ , is a centralizer. Benkovič et al. [3] proved that if there exists an additive mapping  $\varphi : R \longrightarrow R$ , where R is a prime ring with suitable characteristic restrictions satisfying the relation  $2\varphi(x^{n+1}) = \varphi(x)x^n + x^n\varphi(x)$  for all  $x \in R$  and some fixed integer n, then  $\varphi$  is a centralizer. Ghahramani [4] has shown that every Jordan centralizer of 2-torsion free triangular algebra through zero product is a

<sup>2010</sup> Mathematics Subject Classification. 16W25.

Speaker: Leila Heidaryzadeh.

centralizer. We refer the reader to [3, 4, 5, 10, 11] and references therein for results concerning centralizers on rings and algebras.

In this paper, motivated by the results mentioned above, we characterize Jordan centralizers of quaternion rings and consider some sufficient conditions on these rings so that every Jordan centralizer is a centralizer. Among the reasons for studying the mappings on quaternion rings, are the recently published books and papers [1, 2, 6, 7, 9] in which the authors have considered the important roles of quaternion algebras in different sciences such as differential geometry, analysis, mechanics and quantum fields. One of the latest work in this contex is done by Ghahramani et al. [6] in which the authors have described the superderivations of quaternion rings considered as  $\mathbb{Z}_2$ -graded rings. They have also considered the quaternion ring as a suitable context for answering some questions on superalgebras.

#### 2. Main results

**Definition 2.1.** Let S be a ring with identity. Set

$$H(S) = \{s_0 + s_1i + s_2j + s_3k : s_i \in S\} = S \oplus Si \oplus Sj \oplus Sk,$$

where  $i^2 = j^2 = k^2 = ijk = -1$ , and ij = -ji. Then, with the componentwise addition and multiplication subject to the given relations and the convensions that i, j, k commute with S elementwise, H(S) is a ring called quaternion ring over S, which is a generalization of the Hamilton's division ring of real quaternions  $\mathbb{H} = H(\mathbb{R})$ . In general, H(S) is not commutative unless S is commutative with char(S) = 2.

**Remark 2.2.** It is worthwhile to mension that in some special cases, it turns out that the quaternion ring R = H(S) is (isomorphic to) a 2 × 2 full matrix ring. For instance, assume that there are elements a and  $f \in R$  such that  $a^2 = f^2 = 0$ and af + fa = 1. Then by Theorem 17.10 in [8], R is isomorphic to a 2 × 2 full matrix ring. In such cases then any characterization of a mapping on R has a similar characterization when this map is considered on the associated matrix ring and conversely.

From now on, we assume that S is a ring with identity such that  $\frac{1}{2} \in S$  and R = H(S).

**Lemma 2.3.** Let S and R be as above. Then Z(R) = Z(S).

The structure of ideals of quaternion rings is similar to matrix rings:

**Theorem 2.4.** I is an ideal of R iff there exists an ideal A of S such that

$$I = H(A) = A \oplus Ai \oplus Aj \oplus Ak.$$

**Lemma 2.5.** For every ideals I, J of R we have H(I)H(J) = H(IJ).

**Theorem 2.6.** Let S and R be as above. Then R is a prime (resp. semiprime) ring iff so is S.

**Theorem 2.7.** Let  $\varphi : R \longrightarrow R$  be a Jordan centralizer. Then  $\varphi$  is a generalized inner derivation. Inparticular, if the ring S is commutative then  $\varphi$  is a centralizer.

*Proof.* Let  $\varphi$  be a Jordan centralizer on R, and assume that  $\varphi(i) = a + bi + cj + dk$ ,  $\varphi(j) = a' + b'i + c'j + d'k$ , for some suitable coefficients in S. Recalling that  $\varphi(x \circ y) = x\varphi(y) + \varphi(y)x = y\varphi(x) + \varphi(x)y$  for all  $x, y \in R$ , applying  $\varphi$  on ij + ji = 0we have 2a'i - 2b' = 0 = 2aj - 2c. Therefore, a' = b' = 0 = c = a. Hence,  $\varphi(i) = bi + dk, \varphi(j) = c'j + d'k$ .

Also, since  $-\varphi(1) = \varphi(i^2) = \varphi(j^2) = \varphi(k^2)$ , if  $\varphi(1) = a'' + b''i + c''j + d''k$ , for some  $a'', b'', c'', d'' \in S$ , then applying  $\varphi$  on ii + ii = -2, we get b'' = c'' = d'' = 0, a'' = b, and so  $\varphi(1) = b$ .

Similarly, applying  $\varphi$  on jj + jj = -2, we find that c' = b, so  $\varphi(i) = bi + dk$ ,  $\varphi(j) = bj + d'k$ .

Now, put  $\varphi(k) = a''' + b'''i + c'''j + d'''k$ . Applying  $\varphi$  on ik + ki = 0, we have a''' = b''' = 0, d = 0. Hence,  $\varphi(i) = bi$ ,  $\varphi(k) = c'''j + d'''k$ . Applying  $\varphi$  on jk + kj = 0 and kk + kk = -2, we observe that

$$\varphi(i) = bi, \quad \varphi(j) = bj \quad \text{and} \quad \varphi(k) = bk.$$

Consequently, we can find  $\varphi(s), \varphi(si), \varphi(sj), \varphi(sk)$  for each  $s \in S$ , by applying  $\varphi$  on the identities 2s = s1 + 1s, 2si = si + is, 2sj = sj + js and 2sk = sk + ks, respectively. We obtain

$$\varphi(s) = \frac{1}{2}(s \circ b), \varphi(si) = \frac{1}{2}(s \circ b)i, \varphi(sj) = \frac{1}{2}(s \circ b)j, \varphi(sk) = \frac{1}{2}(s \circ b)k,$$

where  $b = \varphi(1)$ .

Now, let  $t=x+yi+zj+wk\in R$  be arbitray. Using the above equations we find that

$$\varphi(t) = \frac{1}{2}(t\varphi(1) + \varphi(1)t),$$

concluding that  $\varphi$  is a generalized inner derivation. Finally, if S is commutative, by Lemma 2.1,  $\varphi(1) \in Z(R)$  and so,  $\varphi(t) = \varphi(1)t$ , proving that  $\varphi$  is a centralizer.  $\Box$ 

**Corollary 2.8.** Assume that the ring S is semiprime. Then every Jordan centralizer of R is a centralizer.

### References

- S.L. Adler, Quaternionic quantum mechanics and quantum fields, Oxford University Press Inc, New York, 1995.
- [2] O.P. Agrawal, Hamilton operators and dual-number-quaternions in spatial kinematics, Mech. Mach. Theory, Vol. 22, no. 6 (1987), 569-575.
- [3] D. Benkovič, D. Eremita, J. Vukman, A characterization of the centroid of a prime ring, Studia Sci. Math. Hungar. 45 (2008), 379-394.
- [4] H. Ghahramani, Characterizing Jordan centralizers and Jordan generalized derivations on triangular rings through zero products, arXiv: 1312.6958v2 [math.RA] 2 Jan 2014.
- [5] H. Ghahramani, On centralizers of Banach algebras, Bull. Malays. Math. Sci. Soc. in press.
- [6] H. Ghahramani, M.N. Ghosseiri and S. Safari, Some questions concerning superderivations on Z<sub>2</sub>-graded rings, Aequationes Math. in press.
- [7] M. Jafari, Y. Yayli, Generalized quaternions and their algebraic properties, Fac. Sci. Ank. Series A1 Volume 64, Number 1 (2015), 15-27.
- [8] T.Y. Lam, Lectures on modules and rings, Springer-Verlage New York Berlin Heidelberg, 1999.
- [9] J. Voight, The arithmetic of quaternion algebras, University of Vermont, Burlington (2014).
- [10] J. Vukman, Centralizers on semiprime rings, Comment. Math. Univ. Carolinae, 42 (2001), 237-245.

[11] B. Zalar, On centralizers of semiprime rings, Comment. Math. Univ. Carolinae, 32 (1991), 609-614.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KURDISTAN, P. O. BOX 416, SANANDAJ, IRAN. *E-mail address*: mnghosseiri@yahoo.com; mnghosseiri@uok.ac.ir

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KURDISTAN, P. O. BOX 416, SANANDAJ, IRAN. *E-mail address*: heidaryzadehleila@yahoo.com; l.heidaryzadeh@sci.uok.ac.ir

50



# LEFT-LOOKING APPROXIMATE INVERSE PRECONDITIONER IN BLOCK FORM

#### AMIN RAFIEI AND SAMANEH HOSSEINI SANI

ABSTRACT. In this paper we present a block version of left-looking AINV preconditioner. In the numerical tests, we compare the quality of the plain and block versions of this preconditioner.

Keywords: AINV preconditioner; GMRES method.

#### 1. INTRODUCTION

Consider the linear system

$$Ax = b$$

where  $A \in \mathbb{R}^{n \times n}$  is real, nonsymmetric and invertible matrix and  $x, b \in \mathbb{R}^{n \times 1}$ . The left-looking A-biconjugation algorithm can be used to factorize  $A^{-1}$  in the following form

$$A^{-1} = ZD^{-1}W^T.$$

In this factorization Z and W are unit upper triangular matrices and D is a diagonal matrix [1]. If we apply dropping in this algorithm, then the left-looking AINV will be obtained. In [2], we have presented a block format of left-looking A-biconjugation algorithm. In this case, Z and W factors are agin unit upper triangular while D is a block diagonal matrix. The diagonal blocks of D are of order  $1 \times 1$  or  $2 \times 2$ . Applying the dropping strategy in this block algorithm will give us a block version of left-looking AINV preconditioner. In this paper, we will compare the quality of the block and plain left-looking AINV preconditioners at reducing the number of iterations of the GMRES [4] Krylov subspace method.

#### 2. Algorithms of plain and block left-looking AINV preconditioners

In this section, we have presented three algorithms. Algorithm 1 and 2 are used to compute the block version of left-looking AINV preconditioner. We call Algorithm 2 inside Algorithm 1 to construct a column of matrices Z and W.

<sup>2010</sup> Mathematics Subject Classification. 65F10, 65F08. Speaker: Samaneh Hosseini Sani.

#### Algorithm 1 (A block format of left-looking AINV preconditioner)

**Input:**  $A \in \mathbb{R}^{n \times n}$  a nonsymmetric matrix,  $\tau_w, \tau_z \in (0, 1)$  are the drop tolerance parameters for W and Z. **Output:**  $A^{-1} \approx ZD^{-1}W^T$ , Z and W are unit upper triangular and D is a block diagonal matrix.

1.  $logic_z = logic_w = true$  $\begin{array}{l} 2. \ status(i) = 0, \ 1 \leq i \leq n \\ 3. \ Z = [z_1^{(0)}, z_2^{(0)}, ..., z_n^{(0)}] = I_{n \times n}, W = [w_1^{(0)}, w_2^{(0)}, ..., w_n^{(0)}] = I_{n \times n}, D = 0 \in \mathbb{R}^{n \times n} \end{array}$ 4. i = 15. while i < n do 6. 7. 8. 9. if  $logic_z$  then  $call \ column - const(Z, \tau_z, A, D, i, status)$ else $logic_z = true$ 10. 11.end if end if call  $column - const(Z, \tau_z, A, D, i + 1, status)$  $S_{ii}^{(i-1)} = e_i^T A z_i^{(i-1)}$ for j = i + 1 to n do  $S_{ji}^{(i-1)} = e_j^T A z_i^{(i-1)}$ ,  $S_{j,i+1}^{(i-1)} = e_j^T A z_{i+1}^{(i-1)}$ 12.13.14.15.end for 16. if logic\_w then 17. $column-const(W, \tau_w, A^T, D^T, i, status)$  $18. \\
19.$  $\mathbf{else}$  $logic_w = true$  $\frac{20}{21}$ . 
$$\begin{split} & \text{end if} \\ & \text{call column - const}(W, \tau_w, A^T, D^T, i+1, status) \\ & S_{ij}^{(i-1)} = (w_i^{(i-1)})^T Ae_j, \quad j \geq i+1 \\ & S_{i+1,j}^{(i-1)} = (w_{i+1}^{(i-1)})^T Ae_j, \quad j \geq i+2 \\ & v_i = max\{\frac{1}{|s_{ij}^{(i-1)}|}\sum_{j=i+1}^{n}|S_{ii}^{(i-1)}|, \frac{1}{|s_{ii}^{(i-1)}|}\sum_{j=i+1}^{n}|S_{ji}^{(i-1)}| \\ & w_i^1 = \sum_{j=i+2}^{n}||\left[\frac{S_{i-1}^{(i-1)}}{S_{i+1,i}^{(i-1)}}S_{i+1,i+1}^{(i-1)}\right]^{-1}\left(\frac{S_{ij}^{(i-1)}}{S_{i+1,j}^{(i-1)}}\right)||_{\infty} \\ & w_i^2 = \sum_{j=i+2}^{n}||\left(S_{ij}^{(i-1)} S_{i+1,j}^{(i-1)}\right)\left[\frac{S_{ii}^{(i-1)}}{S_{i+1,i}^{(i-1)}}S_{i+1,i+1}^{(i-1)}\right]^{-1}||_{\infty} \\ & w_i = max\{w_i^1 w_i^2\} \end{split}$$
end if 22. 23.24.25.26. $\begin{array}{l} w_i \,=\, max\{w_i^1,w_i^2\} \\ {\rm if} \,\, v_i < w_i \,\, {\rm then} \\ D_{ii} \,=\, S_{ii}^{(i-1)} \end{array}$ 27.28.29.  $z_{i+1}^{(i)} = z_{i+1}^{(i-1)} - \left(\frac{e_i^T A z_{(i+1)}^{(i-1)}}{D_{i_i}}\right) z_i^{(i-1)}.$  For all  $l \leq i$  apply dropping rule to  $z_{l,i+1}^{(i)}$  if its absolute value is less than  $\tau_z$  $\dots \quad \text{arrowpting rule to } z_{l,i+1}^{(i)} \text{ if its absolute value} \\ w_{i+1}^{(i)} = w_{i+1}^{(i-1)} - (\frac{(w_{i+1}^{(i-1)})^T A e_i}{D_{ii}}) w_i^{(i-1)}. \text{ For all } l \leq i \text{ apply dropping rule to } w_{l,i+1}^{(i)} \text{ if its absolute value is less than } \pi_w \\ logic_z = false, \quad logic_w = false \\ status(i) = 1 \\ i = i+1 \end{cases}$ 30. 31. $\frac{32}{33}$ .  $\frac{34}{35}$ . i = i + 1 $\begin{array}{l} D_{i:i+1,i:i+1} = \begin{bmatrix} S_{ii}^{(i-1)} & S_{ii+1}^{(i-1)} \\ S_{ii+1}^{(i-1)} & S_{i+1i}^{(i-1)} \end{bmatrix} \\ z_{i+1}^{(i)} = z_{i+1}^{(i-1)}, \ w_{i+1}^{(i)} = w_{i+1}^{(i-1)} \\ status(i) = 2 \\ i = i+2 \end{array}$ else 36. 37. 38. 39. i = i + 240. end if 41. end while 42. if status(n-1) = 0 then 43.call  $column - const(Z, \tau_z, A, D, n, status)$ 44. call column - const( $W, \tau_w, A^T, D^T, n, status$ )  $D_{n,n} = e_n^T A z_n^{(n-1)}$ 45.46. end if 47. if status(n-1) = 1 then  $48. \qquad D_{n,n} = e_n^T A z_n^{(n-1)}$ 49. end if50. Return  $Z = [z_1^{(0)}, z_2^{(1)}, ..., z_n^{(n-1)}], W = [w_1^{(0)}, w_2^{(1)}, ..., w_n^{(n-1)}]$  and D

Algorithm 3 gives the plain left-looking AINV preconditioner. This algorithm was first introduced by Benzi and Tuma in reference [1].

Algorithm 2 (Column construction of a matrix)

 $\begin{array}{l} \hline Column\_Const(Z, \tau_z, A, D, i, status) \\ \textbf{Input:} \ Z = [z_1^{(0)}, z_2^{(1)}, \cdots, z_{i-1}^{(i-2)}, z_i^{(0)}, \cdots, z_n^{(0)}] \in \mathbb{R}^{n \times n}, \ \tau_z \in (0,1) \ \text{is the drop tolerance for } Z, \end{array}$  $A \in \mathbb{R}^{n \times n}$  a nonsymmetric matrix,  $D \in \mathbb{R}^{n \times n}$  a block diagonal matrix, *i* is an integer, status is an integer array **Output**: updated Z1. j = 12. while  $j \leq i - 1$  do  $\begin{array}{l} \underset{k = j + status(j) - 1}{\text{if } status(j) = 1 \text{ then}} \\ z_i^{(k)} = z_i^{(j-1)} - z_j^{(j-1)} \times \frac{1}{D_{jj}} \times A_{j,:} \times z_i^{(j-1)} \end{array}$ 3. 4. 5. $\begin{array}{l} j = j + 1 \\ \textbf{else if } status(j) = 2 \textbf{ then} \\ z_i^{(k)} = z_i^{(j-1)} - [z_j^{(j-1)} \ z_{j+1}^{(j)}] \times [D(j:j+1,j:j+1)]^{-1} \times A_{j:j+1,:} \times z_i^{(j-1)} \\ j = j+2 \end{array}$ 6. 7. 8. 9 10. end if Consider  $z_i^{(k)} = (z_{l_i}^{(k)})$ . For all l, apply dropping rule to  $z_{l_i}^{(k)}$  if its absolute value is less than  $\tau_z$ 11. 12. end while 13. Return Z

# Algorithm 3 (Left-looking AINV preconditioner)

#### 3. Numerical tests

In this section, we have reported the results of numerical experiments. We have implemented 3 algorithms in Matlab. We have selected 4 matrices from [3]. Then, we constructed the artificial linear systems  $A[1, \dots, 1]^T = b$ . These systems have been solved by the GMRES(15) Krylov subspace method. The command GMRES in Matlab provides us this method. The matrix information and the convergence results of the GMRES(15) can be found in Table 1. In this table, n and nnz are the dimension and number of nonzero entries of the matrix. iter(1) and iter(2) are the number of external and internal iterations of GMRES(15), respectively. Time is the iteration time which is in seconds.

TABLE 1. matrix properties and results of GMRES(15)

Matrix properties			GMRES(15)			
name	n	nnz	iter(1)	iter(2)	Time	
sherman4	1104	3786	37	12	0.305	
orsirr-2	886	5970	397	9	1.629	
sherman1	1000	2375	132	15	0.559	
cdde1	961	4681	9	2	0.035	

In Table 2, we have set  $\tau_z = \tau_w = 0.1$  and computed both the plain and the block left-looking AINV preconditioners. The notations LLAINV(0.1) and BLLAINV(0.1) refer to these two cases. Then, these two preconditioners have been used as the right preconditioner for linear systems. After that, the preconditioned systems have been solved by the GMRES(15) method. The results of these tests can be found in Table 2. In this table, *ptime* is the preconditioning time which is in seconds and *density* is defined as

$$density = \frac{nnz(Z) + nnz(W)}{nnz(A)},$$

where nnz(Z), nnz(W) and nnz(A) are the number of nonzero entries of matrices Z, W and A, respectively. In this table, iter(1) and iter(2) have the same definition as in Table 1 and Ttime is the summation of preconditioning time and the iteration time of the GMRES(15) method.

TABLE 2. properties of the preconditioners and results of GMRES(15) to solve the preconditioned systems

	LLAINV(0.1)+GMRES(15)				BLLAINV(0.1)+GMRES(15)					
matrix	ptime	density	iter(1)	iter(2)	Ttime	ptime	density	iter(1)	iter(2)	Ttime
sherman4	40.84	1.331	7	13	41.0025	5641.11	4.079	7	3	5641.44
orsirr-2	26.05	0.919	4	10	26.1109	3321.73	3.458	4	9	3321.79
sherman1	29.61	1.350	1	6	29.621	3589.79	1.551	1	6	3589.8
cdde1	34.41	1.558	20	14	34.7793	4550.61	15.341	9	1	4550.76

The results in Table 2 indicate that for matrices sherman4, orsirr-2 and cdde1, the block left-looking AINV preconditioner is more effective than the plain leftlooking AINV at reducing the number of iterations of GMRES(15) method. For matrix sherman1, both preconditioners make the GMRES(15) method convergent in the same number of iterations. Comparing the preconditioning time of both preconditioners show that the block left-looking AINV needs more time to be constructed. By analyzing the number of iterations in Tables 1 and 2 one may come to conclusion that both preconditioners are effective tools at reducing the number of iterations of GMRES(15) method.

#### References

- M.Benzi and M.Tuma, A sparse approximate inverse preconditioner for nonsymmetric linear systems, SIAM J.Sci. Comput. 19(3), (1988) 968-994
- [2] A.Rafiei, M.Bollhofer and T.Huckle, A block factorization format for the inverse of a nonsymmetric matrix, 47<sup>th</sup> Annual Iranian Mathematics conference (AIMC47), Kharazmi University, (2016).
- [3] T. Davis, The SuiteSparse Matrix Collection, http://www.cise.ufl.edu/research/sparse/matrices.
- [4] Iterative methods for sparse linear systems, SIAM, Philadelphia. 2nd edition (2003).

DEPARTMENT OF APPLIED MATHEMATICS, HAKIM SABZEVARI UNIVERSITY, IRAN *E-mail address*: rafiei.am@gmail.com, a.rafiei@hsu.ac.ir

Department of applied mathematics, Hakim Sabzevari University, Iran E-mail address: samanehhoseini@yahoo.com

#### 54



# ON THE PRECONDITIONING OF DOUBLE SADDLE POINT MATRICES WITH LOCAL SHIFT-SPLITTING

### MOHAMMAD MAHDI IZADKHAH

ABSTRACT. In this paper, the author proposes a local shift-splitting precontioner for the double saddle piont matrices. Some properties of the local shiftsplitting precontioned double saddle piont matrix are studied. Finally, numerical experiments of a model Stokes problem are presented to show the effectiveness of the proposed preconditioner.

Keywords: Double saddle point problem; Preconditioner; Local shift-splitting; Symmetric positive definite.

#### 1. INTRODUCTION

We consider the solution of the system of linear equations with the following block  $3\times 3$  structure

(1.1) 
$$\mathcal{A}u \equiv \begin{bmatrix} A & B^T & C^T \\ -B & 0 & 0 \\ -C & 0 & D \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \equiv b,$$

where  $A \in \mathbb{R}^{n \times n}$  and  $D \in \mathbb{R}^{p \times p}$  are symmetric positive definite(SPD),  $B \in \mathbb{R}^{m \times n}$ with rank $(B) = m < n, C \in \mathbb{R}^{p \times n}$ ,  $x, b_1 \in \mathbb{R}^n, y, b_2 \in \mathbb{R}^m$  and  $z, b_3 \in \mathbb{R}^p$ . Throughout the paper we assume that  $n \ge m + p$ . The linear system (1.1) is called double saddle point problem. Linear systems of the form (1.1) arise frequently from mixed and mixed-hybrid formulations of second-order elliptic equations [5]. The solvibity conditions and numerical methods have been investigated in [1, 6].

**Proposition 1.1.** Assume that A and D are symmetric positive definite(SPD). Then matrix  $\mathcal{A}$  is invertible if and only if  $B^T$  has full column rank.

Both Uzawa-type stationary methods and block preconditioned Krylov subspace methods are discussed in [1] for double saddle point problem (1.1).

# 2. Main results

Based on the iteration methods studied in [2, 3], a local shift-spliting of the double saddle point matrix  $\mathcal{A}$  can be constructed as follows

(2.1) 
$$\mathcal{A} = \mathcal{P}_{LSS} - \mathcal{Q}_{LSS} = \frac{1}{2} \begin{bmatrix} A & B^T & C^T \\ -B & \alpha I & 0 \\ -C & 0 & D \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -A & -B^T & -C^T \\ B & \alpha I & 0 \\ C & 0 & -D \end{bmatrix},$$

where  $\alpha > 0$  is a constant and I is the identity matrix with appropriate dimension.

Speaker: Mohammad Mahdi Izadkhah.

<sup>2010</sup> Mathematics Subject Classification. 65F10, 65F50, 65N22.

We can do the following matrix factorization for  $\mathcal{P}_{LSS}$ 

(2.2) 
$$\mathcal{P}_{LSS} = \frac{1}{2} \begin{bmatrix} I & \frac{1}{\alpha} B^T & C^T D^{-1} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} S & 0 & 0 \\ 0 & \alpha I & 0 \\ 0 & 0 & D \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -\frac{1}{\alpha} B & I & 0 \\ -D^{-1} C & 0 & I \end{bmatrix},$$

where  $S = A + \frac{1}{\alpha}B^TB + C^TD^{-1}C$ . In the following, we turn to study some properties of the preconditioned matrix  $\mathcal{P}_{LSS}^{-1}\mathcal{A}$ .

**Theorem 2.1.** Assume that  $A \in \mathbb{R}^{n \times n}$  and  $D \in \mathbb{R}^{p \times p}$  are symmetric positive definite(SPD),  $B \in \mathbb{R}^{m \times n}$  with rank $(B) = m < n, C \in \mathbb{R}^{p \times n}$ . Let  $\alpha$  be a positive constant. The local shift-splitting preconditioner  $\mathcal{P}_{LSS}$  defined as in (2.1). Then the preconditioned matrix  $\mathcal{P}_{LSS}$  has an eigenvalue 2 with multiplicity n + p, the remaining eigenvalues are  $\frac{2\sigma_i^2}{\alpha + \sigma_i^2}$  for i = 1, 2, ..., m, where  $\sigma_i$  is the positive singular values of the matrix  $BA^{-\frac{1}{2}}$ .

*Proof.* By using the factorization in (2.3), we have

(2.3) 
$$\mathcal{P}_{LSS}^{-1}\mathcal{A} = \begin{bmatrix} 2I & 2S^{-1}B^T & 0\\ 0 & \frac{2}{\alpha}BS^{-1}B^T & 0\\ 0 & 2D^{-1}CS^{-1}B^T & 2I \end{bmatrix}$$

Consider the block diagonal matrix

$$P = \left[ \begin{array}{ccc} A^{\frac{1}{2}} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & D^{\frac{1}{2}} \end{array} \right]$$

Then the preconditioned matrix  $\mathcal{P}_{LSS}^{-1}\mathcal{A}$  is similar to the following matrix  $\tilde{P}$ 

$$\tilde{P} = P \mathcal{P}_{LSS}^{-1} \mathcal{A} P^{-1} = \begin{bmatrix} 2I & 2S_1^{-1} A^{-\frac{1}{2}} B^T & 0\\ 0 & \frac{2}{\alpha} B^{-\frac{1}{2}} S_1^{-1} A^{-\frac{1}{2}} B^T & 0\\ 0 & 2D^{-\frac{1}{2}} C A^{-\frac{1}{2}} S_2^{-1} A^{-\frac{1}{2}} B^T & 2I \end{bmatrix},$$

where  $S_1 = I + \frac{1}{\alpha} A^{-\frac{1}{2}} B^T B A^{-\frac{1}{2}}$  and  $S_2 = I + \frac{1}{\alpha} A^{-\frac{1}{2}} B^T B A^{-\frac{1}{2}} + A^{-\frac{1}{2}} C^T D^{-\frac{1}{2}} D^{-\frac{1}{2}} C A^{-\frac{1}{2}}$ . Let  $\sigma_i$  for  $i = 1, 2, \ldots, m$  and  $\delta_j$  for  $j = 1, 2, \ldots, p$  are the positive sigular values of the matrices  $BA^{-\frac{1}{2}}$  and  $D^{-\frac{1}{2}} C A^{-\frac{1}{2}}$ , respectively. Consider the singular value decomposition of the matrix  $BA^{-\frac{1}{2}} = U_1^T \tilde{\Sigma}_1 V_1 = U_1^T [\Sigma_1 \quad 0] V_1$  and  $D^{-\frac{1}{2}} C A^{-\frac{1}{2}} = U_2^T \tilde{\Sigma}_2 V_2 = U_2^T [\Sigma_2 \quad 0] V_2$ , where  $U_1 \in \mathbb{R}^{m \times m}$ ,  $V_1 \in \mathbb{R}^{n \times n}$ ,  $U_2 \in \mathbb{R}^{p \times p}$  and  $V_2 \in \mathbb{R}^{n \times n}$  are orthogonal matrices. Also  $\Sigma_1 = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m)$  and  $\Sigma_2 = \text{diag}(\delta_1, \delta_2, \ldots, \delta_p)$  are diagonal matrices with  $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m > 0$  and  $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_p > 0$  being the nonzero singular values of  $BA^{-\frac{1}{2}}$  and  $D^{-\frac{1}{2}}CA^{-\frac{1}{2}}$ , respectively.

Then  $\tilde{P}$  can be written as

$$\begin{bmatrix} 2I & 2(I + \frac{1}{\alpha}V_1^T \tilde{\Sigma}_1^T \tilde{\Sigma}_1 V_1)^{-1} V_1^T \tilde{\Sigma}_1^T U_1 & 0\\ 0 & \frac{2}{\alpha}U_1^T \tilde{\Sigma}_1 V_1 (I + \frac{1}{\alpha}V_1^T \tilde{\Sigma}_1^T \tilde{\Sigma}_1 V_1)^{-1} V_1^T \tilde{\Sigma}_1^T U_1 & 0\\ 0 & 2U_2^T \tilde{\Sigma}_2 V_2 (I + \frac{1}{\alpha}V_1^T \tilde{\Sigma}_1^T \tilde{\Sigma}_1 V_1 + V_2^T \tilde{\Sigma}_2^T \tilde{\Sigma}_2 V_2)^{-1} V_1^T \tilde{\Sigma}_1^T U_1 & 2I \end{bmatrix}.$$

Define

$$Q = \left[ \begin{array}{rrrr} V_1 & 0 & 0 \\ 0 & U_1 & 0 \\ 0 & 0 & U_2 \end{array} \right].$$

56

Then Q is an orthogonal matrix and it holds that

$$\hat{P} = Q\tilde{P}Q^T = \begin{bmatrix} 2I & 0 & 2(I + \frac{1}{\alpha}\Sigma_1^2)^{-1}\Sigma_1 & 0 \\ 0 & 2I & 0 & 0 \\ 0 & 0 & \frac{2}{\alpha}\Sigma_1(I + \frac{1}{\alpha}\Sigma_1^2)^{-1}\Sigma_1 & 0 \\ 0 & 0 & 2\tilde{\Sigma}_2(V_1V_2^T + \frac{1}{\alpha}V_1\tilde{\Sigma}_1^T\tilde{\Sigma}_1V_2^T + V_1V_2^T\tilde{\Sigma}_2^T\tilde{\Sigma}_2)^{-1}\tilde{\Sigma}_1 & 2I \end{bmatrix}.$$

It is easy to check that the matrix  $\hat{P}$  has an eigenvalue 2 with multiplicity n + p, the remaining eigenvalues are of the form  $\frac{2\sigma_i^2}{\alpha + \sigma_i^2}$  for  $i = 1, 2, \ldots, m$ .

**Theorem 2.2.** Let the local shift-splitting preconditioner  $\mathcal{P}_{LSS}$  be defined as in (2.1), then the degree of the minimal polynomial of the preconditioned matrix  $\mathcal{P}_{LSS}^{-1}\mathcal{A}$  is at most m + 1. Thus, the dimension of the Krylove subspace  $\mathcal{K}(\mathcal{P}_{LSS}^{-1}\mathcal{A}, b)$  is at most m + 1.

*Proof.* We know that the preconditioned matrix  $\mathcal{P}_{LSS}^{-1}\mathcal{A}$  takes the form

(2.4) 
$$\mathcal{P}_{LSS}^{-1}\mathcal{A} = \begin{bmatrix} 2I & \Theta_2 & 0\\ 0 & \Theta_1 & 0\\ 0 & \Theta_3 & 2I \end{bmatrix}$$

where  $\Theta_1 = \frac{2}{\alpha}BS^{-1}B^T \in \mathbb{R}^{m \times m}, \Theta_2 = 2S^{-1}B^T \in \mathbb{R}^{m \times m}$  and  $\Theta_3 = 2D^{-1}CS^{-1}B^T \in \mathbb{R}^{m \times m}$ . Let  $\mu_i$  for i = 1, 2, ..., m be the eigenvalues of  $\Theta_1$ . Note that  $\mu_i$  for i = 1, 2, ..., m are also the eigenvalue of the preconditioned matrix  $\mathcal{P}_{LSS}^{-1}\mathcal{A}$ . Then from (2.4), we obtain the characteristic polynomial of the preconditioned matrix  $\mathcal{P}_{LSS}^{-1}\mathcal{A}$  is

$$\Phi(\lambda) = (\lambda - 2)^{n+p} \prod_{i=1}^{m} (\lambda - \mu_i).$$

Consider the polynomial  $\Psi(\lambda) = (\lambda - 2) \prod_{i=1}^{m} (\lambda - \mu_i)$  of degree m+1, and expanding the polynomial  $\Psi(\mathcal{P}_{LSS}^{-1}\mathcal{A})$ , we obtain

$$(\mathcal{P}_{LSS}^{-1}\mathcal{A} - 2I)\prod_{i=1}^{m}((\mathcal{P}_{LSS}^{-1}\mathcal{A} - \mu_{i}I) = \begin{bmatrix} 0 & \Theta_{2}\prod_{i=1}^{m}(\Theta_{1} - \mu_{i}I) & 0\\ 0 & (\Theta_{1} - 2I)\prod_{i=1}^{m}(\Theta_{1} - \mu_{i}I) & 0\\ 0 & \Theta_{3}\prod_{i=1}^{m}(\Theta_{1} - \mu_{i}I) & 0 \end{bmatrix}.$$

Since  $\mu_i$  for i = 1, 2, ..., m are the eigenvalues of  $\Theta_1$ , by Cayley-Hamilton theorem, we have

$$\prod_{i=1}^{m} (\Theta_1 - \mu_i I) = 0.$$

Therefor, from [4], we know that the degree of the minimal polynomial is equal to the dimension of the corresponding Krylov subspace  $\mathcal{K}(\mathcal{P}_{LSS}^{-1}\mathcal{A}, b)$ . This completes the proof.

#### 3. Numerical experiments

**Example 3.1.** By the finite difference scheme of the Stokes problem, the submatrices of the coefficient matrix in the double saddle point problems have the folloinwing form

$$A = \begin{bmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{bmatrix} \in \mathbb{R}^{2q^2 \times 2q^2}, \quad B^T = \begin{bmatrix} I \otimes F \\ F \otimes I \end{bmatrix} \in \mathbb{R}^{2q^2 \times q^2},$$
$$D = I \otimes T + T \otimes I \in \mathbb{R}^{q^2 \times 2q^2}, \qquad C^T = \begin{bmatrix} I \otimes F \\ F \otimes I \end{bmatrix} \in \mathbb{R}^{2q^2 \times q^2},$$



FIGURE 1. Eigenvalue distribution for Example 3.1 with  $32 \times 32$  grids for  $\nu = 0.1$ 



FIGURE 2. Eigenvalue distribution for Example 3.1 with  $40 \times 40$  grids for  $\nu = 0.01$ 

where

$$T = \frac{\nu}{h^2} \operatorname{tridiag}(-1, 2, -1) \in \mathbb{R}^{q \times q}, \quad F = \frac{1}{h} \operatorname{tridiag}(-1, 1, 0) \in \mathbb{R}^{q \times q},$$

with  $\otimes$  being the Kronecker product symbol and  $h = \frac{1}{q+1}$  the discretization mesh size.

From Figure 1 and Figure 2, we see that most eigenvalues of the local shiftsplitting preconditioned matrix  $\mathcal{P}_{LSS}^{-1}\mathcal{A}$  are 2. These results are in good agreement with the theorical ones in Theorem 2.1 and 2.2.

#### References

- F. Panjeh Ali Beik and M. Benzi, Iterative methods for double saddle point systems, Technical Report, DOI: 10.13140/RG.2.2.33918.38722, March 2017.
- [2] Z.-Z. Bai, J.-F. Yin and Y.-F. Su, A shift-splitting preconditioner for non-Hermitian positive definite matrices, J. Comput. Math. 24(2006) 539-552.
- [3] Y. Cao, J. Du and Q. Niu, Shift-splitting preconditioners for saddle point problems, J. Comput. Appl. Math. 272(2014) 239-250.
- [4] Y. Saad, Iterative Methods for Sparse Linear Systems, Second Edition, Society for Industrial and Applied Mathematics, Philadelphia, 2003.
- [5] D. Boffi, F. Brezzi and M. Fortin, Mixed Element Methods and Applications, Springer Ser. Comput. Math. Springer-Verlag, New York, 2013.
- [6] B. Morini, V. Simoncini, and M. Tani, Spectral estimates for unreduced symmetric KKT systems arising from Interior Point methods, Numer. Linear Algebra Appl., 23 (2016), 776-800.

DEPARTMENT OF COMPUTER SCIENCE, FACULTY OF COMPUTER & INDUSTRIAL ENGINEERING, BIRJAND UNIVERSITY OF TECHNOLOGY, BIRJAND, IRAN

*E-mail address*: izadkhah@birjandut.ac.ir



The 9<sup>th</sup> Seminar on Linear Algebra and its Applications July 5-6, 2017, University of Tabriz, Tabriz, Iran

# PARAMETRED ITERATIVE REFINEMENT FOR ILL-CONDITIONED LINEAR SYSTEM OF EQUATIONS

MOHAMMAD MAHDI IZADKHAH

ABSTRACT. In this paper, the author present a parametered variant of generalized two-step iterative refinement procedure to solve ill-conditioned system of linear equations. Numerical examples are examined to demonstrate the accuracy and efficiency of the proposed algorithms for solving ill-conditioned system of linear equations.

Keywords: Iterative method; Iterative refinement; Symmetric positive definite; Ill-conditioned problems.

# 1. INTRODUCTION

Consider the following linear system of equations

where  $A \in \mathbb{R}^{n \times n}$  is a known symmetric positive definite matrix,  $b \in \mathbb{R}^n$  is given and  $x \in \mathbb{R}^n$  is the unknown vector to be determined. Throughout this paper, it is assumed that A is ill-conditioned. that is  $\kappa(A) = ||A|| ||A^{-1}||$  is very large where  $\kappa(A)$  is referred to the condition number of A and  $|| \cdot ||$  is the spectral matrix norm. It is well-known that the obtained solutions computed by iterative methods are not

# Algorithm 1 Wu's iterative refinement

1: Set  $x_0 = 0$ , choose parameters u > 0 and compute the Cholesky factor Lof uI + A2: begin 3: for k = 0, 1, 2, ..., until convergence do 4:  $r_k = b - Ax_k$ 5: Solve  $LL^Ty_k = r_k$  for  $y_k$ 6:  $x_{k+1} = x_k + y_k$ 7: end 8: end

sufficiently accurate when A is ill-conditioned [6]. To overcome this difficulty, so far, some iterative refinements have been offered in the literature. For instanse, Wilkinson's iterative refinement has been proposed in [1, 2]. Wu's iterative refinement has been given in [3], presented in Algorithm 1. Recently, Two-step method

<sup>2010</sup> Mathematics Subject Classification. 65F10, 65F50, 65N22.

Speaker: Mohammad Mahdi Izadkhah.

(TSM) [4] and generalized Two-step method (GTSM)[5] have been proposed to solve ill-conditioned system of linear equations.

## 2. Main results

The current paper deals with a parametered variant of generalized Two-step method [5]. The decomposition A = M - N is called a splitting, if M and N belong to  $\mathbb{R}^{n \times n}$  and M is nonsingular.

We split A into  $(M + \omega I) - (N + \omega I)$ , where M is symmetric positive definite matrix and N is a symmetric matrix, and  $\omega \ge 0$ . Consider Equation (1.1), we set

(2.1) 
$$(M + \omega I)x = (N + (\omega + \beta)I)x - \beta x + b$$

where  $\omega \geq 0$  and  $\beta \in \mathbb{R}$ .

Using the Equation (2.1), one may derive the following recursive formula

(2.2) 
$$(M + \omega I)x_{k+1} = (N + (\omega + \beta)I)x_k - \beta x_{k-1} + b$$

By some strightforward computations from Equation (2.2), we have

 $x_{k+1} = (M + \omega I)^{-1} (r_k + \beta (x_k - x_{k-1})) + x_k,$ 

where  $r = b - Ax_k$ . therefore, the (k + 1)th approximate solution can obtained by

$$y_k = (M + \omega I)^{-1} (r_k + \beta (x_k - x_{k-1}))$$

(2.3) 
$$x_{k+1} = x_k + y_k$$

**Remark 2.1.** If we consider  $M = A + \alpha I$  and  $N = \alpha I$ , then for  $\omega \neq 0$  and  $\beta \neq 0$  the iterative method (2.3) reduces to the method given in [4]. Furthermore, for  $\beta = 0$ , Equation (2.3) is equivalent to the method proposed in [3]. Also, we have the presented iterative method in [5], if  $\omega = 0$ .

# Algorithm 2 Parametered iterative refinemment

1: Set  $x_0 = 0$ , choose parameters  $\omega$  and  $\beta$  and the splitting  $A = (M + \omega I) - (N + \omega I)$  such that the assumptions of Theorem 2.2 hold **2:** Compute the Cholesky factor L of  $M + \omega I$ **3:** Compute  $x_1$  by Algorithm 1 4: begin 5: for  $k = 1, 2, \ldots$  until convergence do  $r_k = b - Ax_k$ 6: Solve  $LL^T y_k = r_k + \beta (x_k - x_{k-1})$  for  $y_k$ 7: 8:  $x_{k+1} = x_k + y_k$ 9: end 10: end

In the sequel, we present Theorem 2.3 in [5], which supplies a sufcient condition which guarantees the convergence of the iterative scheme (2.3).

**Theorem 2.2.** Let A = M - N be a splitting for A such that M is a symmetric positive definite and N is a symmetric matrix. If

(2.4) 
$$\max\left\{\left(\lambda_{\min}(N) + \frac{1}{2}\lambda_{\min}(A)\right), -\lambda_{\min}(M)\right\} < \beta < \lambda_{\min}(M)$$

then the iterative method (2.3) is convergent to the exact solution of Equation (1.1).

#### 3. Numerical experiments

In this section, some numerical experiments are given to show the efficiency of the PGTSM and comparing it with the Wilkinson [1, 2], Wu [3], TSM [4] and GTSM [5]. For this end, we give three examples being from [4, 5]. The stopping criterion

$$||x_{m+1} - x_m||_2 < 5 \times 10^{-6}$$

is always used, and the maximum number of iterations is 100,000. In what follows, the matrix D is a diagonal matrix obtained by considering the diagonal part of the coefficient matrix. In Algorithm GTSM in [5], we set  $M = \alpha D + A$ . Also, for all of examples, we set  $u = \alpha = 10^{-5}$ ,  $\beta = 10^{-6}$  and  $\omega = 10^{-6}$ . We see the results of the numerical experiments in Table 1, Table 2, and Table 3 for different values of n. For each method, the number of iterations for the convergence and the relative error  $RE = \frac{\|x_m - x^*\|_2}{\|x^*\|_2}$  are given. † shows that the method fails.

**Example 3.1.** consider the well-known ill-conditioned system of linear equations (1.1) where  $A = [a_{ij}]$  is the Hilbert matrix of order n, i.e.  $a_{ij} = \frac{1}{i+j-1}$ , and the right-hand side  $b = (b_1, b_2, \ldots, b_n)^T$  is constructed such that  $b_i = \sum_{j=1}^n a_{ij} \times j$  for  $i = 1, 2, \ldots, n$ , that is,  $x^* = (1, 2, \ldots, n)^T$  is the exact solution of the

		n	
method	20	50	90
Wilkinson $[1, 2]$	ť	ť	†
	-	-	-
Algorithm 1 [3]	2080	4225	11201
	2.80e-04	2.48e-04	1.50e-04
TSM[4]	1941	4101	10620
	2.78e-04	2.42e-04	1.47e-04
GTSM [5]	t	t	†
	-	-	-
Algorithm $2$	177	110	426
	1.38e-05	8.39e-09	3.77e-06

TABLE 1. The numerical results for Example 3.1

**Example 3.2.** In this instance, we focus on the ill-conditioned linear system of equations (1.1) where  $A = [a_{ij}] \in \mathbb{R}^{n \times n}$  with

$$u_{ij} = \begin{cases} 1 & i \neq j \\ 1+p^2 & i=j. \end{cases}$$

The right-hand side  $b = (b_1, b_2, \ldots, b_n)^T$  is constructed such that  $x^* = (1, 2, \ldots, n)^T$  is the exact solution (1.1). As described in [4], the spectral condition number of A is  $\kappa(A) = (n + p^2)/p^2$ . We set  $p = 5 \times 10^{-4}$  which results  $\kappa(A) = 1 + 4n \times 10^6$ .

**Example 3.3.** Consider the matrix  $A = B^T B$ , where  $B = [b_{ij}]$  is a symmetric matrix of order n with  $b_{ij} = i/j$ , for  $i \ge j$ . The right-hand side  $b = (b_1, b_2, \ldots, b_n)^T$  is chosen in such a way that  $b_i = \sum_{j=1}^n a_{ij} \times j$ ,  $i = 1, 2, \ldots, n$ . Obviously, the exact solution is  $x^* = (1, 2, \ldots, n)^T$ 

		n	
method	150	170	200
Wilkinson $[1, 2]$	†	Ť	†
	-	-	-
Algorithm 1 [3]	598	606	619
	1.96e-07	1.61e-07	1.25e-07
$\mathrm{TSM}[4]$	541	549	557
	1.76e-07	1.44e-07	1.23e-07
GTSM [5]	541	549	559
	1.78e-07	1.42e-07	1.17e-07
Algorithm $2$	598	603	614
	1.88e-07	1.65e-07	1.33e-07

TABLE 2. The numerical results for Example 3.2

TABLE 3. The numerical results for Example 3.3

	n					
method	400	425	450	500		
Wilkinson [1, 2]	44	169	16707	†		
	3.16e-09	2.73e-09	4.23e-09	-		
Algorithm 1 [3]	12	348	8001	t		
	3.87e-09	4.63e-09	4.90e-09	-		
TSM[4]	99	364	10983	†		
	2.89e-09	2.63e-09	4.15e-09	-		
GTSM [5]	4	4	4	4		
	1.17e-08	1.03e-08	1.03e-08	7.80e-09		
Algorithm 2	4	4	4	4		
	1.17e-08	1.03e-08	1.02e-08	7.80e-09		

## References

- R.S. Martin, G. Peters, and J.H. Wilkinson, Symmetric decompositions of a positive definite matrix, Numer. Math. 7 (1965), 362-383.
- [2] Iterative refinement of the solution of a positive definite system of equations, Numer. Math. 8(1966), 203-216
- [3] X. Wu, R. Shao, and Y. Zhu, New iterative improvement of solution for an ill-conditioned system of equations based on a linear dynamic system, Comput. Math. Appl. 44(2002), 1109-1116
- [4] D.K. Salkuyeh and A. Fahim, A new iterative refinement of the solution of ill-conditioned linear system of equations, Int. Comput. Math. 88(5) (2011), 950-956
- [5] F. Panjeh Ali Beik, S. Ahmadi-Asl and A. Ameri, On the iterative refinement of the solution of ill-conditioned linear system of equations, Int. Comput. Math., (2017), DOI: 10.1080/00207160.2017.1290436.
- [6] Y. Saad, Iterative Methods for Sparse Linear Systems, PWS Press, New York, 1995.

DEPARTMENT OF COMPUTER SCIENCE, FACULTY OF COMPUTER & INDUSTRIAL ENGINEERING, BIRJAND UNIVERSITY OF TECHNOLOGY, BIRJAND, IRAN

 $E\text{-}mail\ address: \texttt{izadkhah@birjandut.ac.ir}$ 



The 9<sup>th</sup> Seminar on Linear Algebra and its Applications July 5-6, 2017, University of Tabriz, Tabriz, Iran

# BERNSTEIN POLYNOMIALS MATRICES

MOSTAFA JANI, SHAHNAM JAVADI, AND ESMAIL BABOLIAN

ABSTRACT. Matrix transformation from Bernstein to some other polynomial basis to other basis functions is ill-conditioned. In this paper, we present a new compact formula for higher derivatives of Bernstein polynomials. Then, we introduce the associated exact operational matrices without using any matrix transformation. Then, we present a stable spectral Galerkin method based on Bernstein polynomial basis. The accuracy and stability of the method is proved and supported by some numerical experiments on fractional advectiondispersion equations. It is seen that the method has a spectral order of convergence.

Keywords: Bernstein matrices.

#### 1. INTRODUCTION

It is common in literature that a basis transformation, commonly to the power basis  $\{1, x, \ldots, x^N\}$ , is used in order to derive the operational matrices for the derivatives and integrals of Bernstein polynomials. The developed numerical method for differential and integral equations based on those matrices leads to a linear system whose coefficient matrix is neither banded nor sparse. It is also worth to note that the explicit conversion between the Bernstein basis and the power basis is exponentially ill-conditioned [5].

In this paper, we directly derive narrow-banded operational matrices for the derivatives of Bernstein polynomials without using any basis transformation and we show that it leads to less computational cost and less round-off errors. As an application, we propose a numerical scheme for the time fractional advection-dispersion equation (FAD). We use the Bernstein polynomials as the trial functions and the dual Bernstein polynomials as the test functions for the Petrov-Galerkin variational formulation. We then show that under some reasonable assumptions, the derived linear system has a unique solution.

#### 2. Main results

**Definition 2.1.** The Bernstein polynomials of degree  $N \ge 0$  on [a, b] are defined by

(2.1) 
$$B_{i,N}(x) = \frac{\binom{N}{i}(x-a)^{i}(b-x)^{N-i}}{(b-a)^{N}}, \qquad 0 \le i \le N.$$

We adopt the convention  $B_{i,N}(x) \equiv 0$  for i < 0 and i > N. The set  $B_N = \{B_{i,N}(x) : i = 0, ..., N\}$  is a basis for  $\mathbb{P}_N$ , the space of polynomials of degree not exceeding N.

<sup>2010</sup> Mathematics Subject Classification. 41A10.

Speaker: Mostafa Jani.

for  $N \ge 1$ , the basis has the following degree elevation property:

$$B_{i,N-1}(x) = \frac{1}{N} \left[ (N-i) B_{i,N}(x) + (i+1) B_{i+1,N}(x) \right], \quad 0 \le i \le N.$$

This relation may be recursively used to get the following general formula [4].

**Lemma 2.2.** Let *i*, *j* and *N* be nonnegative integers,  $j \leq N$  and  $i \leq N - j$ . Then,

(2.2) 
$$B_{i,N-j}(x) = \binom{N-j}{i} \sum_{r=i}^{j+i} \frac{\binom{j}{r-i}}{\binom{N}{r}} B_{r,N}(x)$$

The derivatives of the Bernstein polynomials satisfy the following recurrence relation [4].

$$B'_{i,N}(x) = \frac{N}{b-a} \left( B_{i-1,N-1}(x) - B_{i,N-1}(x) \right), \quad 0 \le i \le N.$$

Using the Leibniz's rule we have, (see [3]).

**Lemma 2.3.** Let N and p be nonnegative integers and  $p \leq N$ . Then, for  $0 \leq i \leq N$ ,

(2.3) 
$$B_{i,N}^{(p)}(x) = c_{p,N} \sum_{k=\max(0,i+p-N)}^{\min(i,p)} (-1)^k \binom{p}{k} B_{i-k,N-p}(x),$$

where

(2.4) 
$$c_{p,N} = \frac{(-1)^p N!}{(b-a)^p (N-p)!}.$$

**Theorem 2.4.** citejuttler The elements of the dual basis  $B_N^* = \{B_{i,N}^*(x) : i = 0, ..., N\}$  associated with the Bernstein basis  $B_N$  on [a, b] are given by

(2.5) 
$$B_{i,N}^{\star}(x) = \sum_{j=0}^{N} d_{i,j} B_{j,N}(x), \quad 0 \le i \le N,$$

with the coefficients

$$d_{i,j} = \frac{(-1)^{i+j}}{(b-a)\binom{N}{i}\binom{N}{j}} \sum_{r=0}^{\min(i,j)} (2r+1)\binom{N+r+1}{N-i}\binom{N-r}{N-i}\binom{N+r+1}{N-j}\binom{N-r}{N-j},$$

for i, j = 0, 1, ..., N. Two sets  $B_N$  and  $B_N^{\star}$  form a biorthogonal system, i.e.

(2.6) 
$$\int_{a}^{b} B_{i,N}(x) B_{j,N}^{\star}(x) dx = \delta_{ij},$$

for i, j = 0, 1, ..., N.

**Theorem 2.5.** Let N be any nonnegative integer and  $0 \le i \le N$ . Then, for  $p \le N$ , we have the following at most (2p + 1) term relation

(2.7) 
$$B_{i,N}^{(p)}(x) = c_{p,N} \sum_{j=\max(0,i-p)}^{\min(N,i+p)} \omega_{i,j} B_{j,N}(x), \quad 0 \le i \le N,$$

64

where  $c_{p,N}$  is defined as (2.4) and

(2.8) 
$$\omega_{i,j} = \sum_{k=\max(0,i-j)}^{\min(p,i-j+p)} \mu_{i,k,j},$$
$$\mu_{i,k,j} = (-1)^k \frac{\binom{p}{k} \binom{N-p}{i-k} \binom{p}{j-i+k}}{\binom{N}{j}}.$$

Remark 2.6. For p = 1 and p = 2, the relation (2.7) reads as (2.9)  $B'_{i,N}(x) = \frac{1}{b-a}((N-i+1)B_{i-1,N}(x) - (N-2i)B_{i,N}(x) - (i+1)B_{i+1,N}(x)),$ (2.10)  $B''_{i,N}(x) = \frac{1}{(b-a)^2}((N-i+2)(N-i+1)B_{i-2,N}(x) - 2(N-i+1)(N-2i+1)B_{i-1,N}(x) + (N^2 - 6Ni + 6i^2 - N)B_{i,N}(x) + 2(i+1)(N-2i-1)B_{i+1,N}(x) + (i+2)(i+1)B_{i+2,N}(x)),$ 

for 
$$i = 0, 1, ..., N$$

Let  $\phi = [B_{i,N}(x) : i = 0, 1, ..., N]^T$ . By using (2.7), we have

$$\frac{d^p}{dx^p}\phi = \mathbf{D}_p\phi, \quad p \ge 1,$$

where  $\mathbf{D}_p$  is a (2p+1)-diagonal, or (p, p)-banded, matrix expressed as



The above matrix, written according to Theorem 2.5, avoids matrix multiplications  $\mathbf{D}_p = \mathbf{D}_1^p$ .

**Remark 2.7.** We especially note that  $\mathbf{D}_0 = \mathbf{I}$ , the identity matrix,  $\mathbf{D}_1$  is the following tridiagonal matrix:

$$\mathbf{D}_{1} = \frac{1}{b-a} \begin{bmatrix} -N & -1 & 0 & \cdots & 0\\ N & 2-N & -2 & 0 & \vdots \\ 0 & N-1 & 4-N & -3 & 0 \\ & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & 2 & N-2 & -N \\ 0 & \cdots & 0 & 1 & N \end{bmatrix},$$

and  $\mathbf{D}_2$  is a pentadiagonal matrix with elements given by

$$(\mathbf{D}_2)_{i,j} = \frac{1}{(b-a)^2} \times \begin{cases} (N-i+2)(N-i+1), & i-j=-2, \\ -2(N-i+1)(N-2i+1), & i-j=-1, \\ N^2 - 6Ni + 6i^2 - N, & i-j=0, \\ 2(i+1)(N-2i-1), & i-j=1, \\ (i+2)(i+1), & i-j=2, \\ 0, & |i-j| > 2. \end{cases}$$

**Remark 2.8.** For  $\mathbf{D}_1$ , it is seen that sum of the elements in each column is zero. This holds for  $\mathbf{D}_p$ ,  $p \ge 1$ . To see this, let  $0 \le j \le N$  and p > 1. Then,

$$\sum_{i=0}^{N} (\mathbf{D}_{p})_{i,j} = \sum_{i=0}^{N} (\mathbf{D}_{1}\mathbf{D}_{p-1})_{i,j} = \sum_{i=0}^{N} \sum_{k=0}^{N} (\mathbf{D}_{1})_{i,k} (\mathbf{D}_{p-1})_{k,j}$$
$$= \sum_{k=0}^{N} (\mathbf{D}_{p-1})_{k,j} \sum_{i=0}^{N} (\mathbf{D}_{1})_{i,k} = 0.$$

#### References

- Jani, M., Babolian, E., Javadi, S., Bhatta, D.: Banded operational matrices for Bernstein polynomials and application to the fractional advection-dispersion equation. Numer Algorithms doi:10.1007/s11075-016-0229-1 (2016)
- [2] S. Javadi, M. Jani, E. Babolian, A numerical scheme for space-time fractional advectiondispersion equation. Int. J. Nonlinear Anal. Appl 7, 331-343 (2016)
- [3] Doha, E.H., Bhrawy, A.H., Saker, M.A.: On the derivatives of Bernstein polynomials: An application for the solution of high even-order differential equations. Boundary Value Problems 2011, 1-16 (2011) doi:10.1155/2011/829543
- [4] Farouki, R.T., Rajan, V.T.: Algorithms for polynomials in Bernstein form. Comput. Aided Geom. Des 5, 1-26 (1988)
- [5] Farouki, R.T.: On the stability of transformations between power and Bernstein polynomial forms. Comput. Aided Geom. Des 8, 29-36 (1991)

DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICAL SCIENCES AND COMPUTER, KHARAZMI UNIVERSITY, TEHRAN, IRAN

E-mail address: mostafa.jani@gmail.com

E-mail address: javadi@khu.ac.ir

 $E\text{-}mail \ address: \texttt{babolian@khu.ac.ir}$


# A BLOCK VERSION OF RIGHT-LOOKING A-BICONJUGATION PROCESS

#### AMIN RAFIEI AND LEILA KAKHKI BEYDOKHTI

ABSTRACT. In this paper, we present a block algorithm to factorize the inverse of a nonsymmetric and real matrix. This factorization has two unit upper triangular factors and a block diagonal matrix.

Keywords: Right-looking A-biconjugation process.

## 1. INTRODUCTION

Consider a nonsymmetric, real, square and invertible matrix A. In [1], the authors have used an algorithm called right-looking A-biconjugation to factorize the inverse of matrix A as the following

(1.1) 
$$A^{-1} = Z D^{-1} W^T,$$

where Z and W are unit upper triangular matrices and D is a diagonal matrix. Suppose that matrix A has the factorization

$$A = LDU,$$

where L and  $U^T$  are unit upper triangular matrices and D is a diagonal matrix. This factorization can be computed by the Gaussian elimination process [3]. At step i of this process, relation



holds. In (1.2), the submatrix  $(S^{(i-1)})_{j,k\geq i}$  is termed the Schur-Complement matrix. In [2], the author has presented a block version of this algorithm. In this

<sup>2010</sup> Mathematics Subject Classification. 65F10, 65F08.

Speaker: Leila Kakhki Beydokhti.

**Input**:  $A \in \mathbb{R}^{n \times n}$  a nonsymmetric matrix.

**Output:**  $A = ZD^{-1}W^T$ . Z and W are unit upper triangular and D is a block diagonal.

1.  $status(i) = 0, 1 \le i \le n$ 2. i = 13. while i < n do 4. for j = i to n do  $S_{ij}^{(i-1)} = (w_i^{(i-1)})^T A z_j^{(i-1)}$  $S_{i+1,j}^{(i-1)} = (w_{i+1}^{(i-1)})^T A z_j^{(i-1)}$ 5.6. 7. end for  $\begin{array}{l} \text{for } j = i+2 \text{ to } n \text{ do} \\ S_{ji}^{(i-1)} = (w_j^{(i-1)})^T A z_i^{(i-1)} \\ S_{j,i+1}^{(i-1)} = (w_j^{(i-1)})^T A z_{i+1}^{(i-1)} \end{array}$ 8. 9. 10. 11. end for  $\begin{array}{l} \text{end for} \\ v_i = max \{ \frac{1}{|S_{ii}^{(i-1)}|} \sum_{j=i+1}^{n} |S_{ij}^{(i-1)}|, \frac{1}{|S_{ii}^{(i-1)}|} \sum_{j=i+1}^{n} |S_{ij}^{(i-1)}| \} \\ w_i^1 = \sum_{j=i+2}^{n} || \begin{bmatrix} S_{ii}^{(i-1)} & S_{i+1}^{(i-1)} \\ S_{i+1,i}^{(i-1)} & S_{i+1,i+1}^{(i-1)} \end{bmatrix}^{-1} \begin{pmatrix} S_{ij}^{(i-1)} \\ S_{i+1,j}^{(i-1)} \\ S_{i+1,j}^{(i-1)} \end{pmatrix} ||_{\infty} \\ w_i^2 = \sum_{j=i+2}^{n} || \begin{pmatrix} S_{ji}^{(i-1)} & S_{j,i+1}^{(i-1)} \\ S_{j,i+1}^{(i-1)} \end{pmatrix} \begin{bmatrix} S_{i-1}^{(i-1)} & S_{i+1,i+1}^{(i-1)} \\ S_{i+1,i}^{(i-1)} & S_{i+1,i+1}^{(i-1)} \end{bmatrix}^{-1} ||_{\infty} \end{array}$ 12.13.14. $w_i = \max\{w_i^1, w_i^2\}$ 15.16.if  $v_i < w_i$  then  $D_{ii} = S_{ii}^{(i-1)}$ 17.status(i) = 118. Column-Const (Z, A, D, i, status)Column-Const  $(W, A^T, D^T, i, status)$ 19.20.21.i = i + 122.end if 23.if  $v_i \geq w_i$  then  $D_{i:i+1,i:i+1} = \begin{bmatrix} S_{ii}^{(i-1)} \\ S_{i+1,i}^{(i-1)} \end{bmatrix}$  $S_{i,i+1}^{(i-1)} \\ S_{i+1,i+1}^{(i-1)}$ 24.  $\begin{aligned} z_{i+1}^{(i)} &= z_{i+1}^{(i-1)} \\ w_{i+1}^{(i)} &= w_{i+1}^{(i-1)} \end{aligned}$ 25.26. 27.status(i) = 228.Column - Const(Z, A, D, i, status) $Column - Const(W, A^T, D^T, i, status)$ 29.30. i = i + 2end if 31. 32. end while 33. if status(n-1) = 0 || status(n-1) = 1 then 34.  $D_{nn} = (w_n^{(i-1)})^T A z_n^{(i-1)}$ 35. end if 36. Return  $Z = [z_1^{(0)}, z_2^{(1)}, \cdots, z_n^{(n-1)}], W = [w_1^{(0)}, w_2^{(1)}, \cdots, w_n^{(n-1)}]$  and D

block format, L and  $U^T$  are again unit upper triangular and D is a block diagonal matrix. The diagonal blocks of D are of order  $1 \times 1$  or  $2 \times 2$ . In this paper we use the method in [2] and present a block format for the right-looking A-biconjugation process.

### 2. Block algorithm to factorize $A^{-1}$

Algorithm 1 is the block version of A-biconjugation process. It computes the factorization in (1.1). Z and W are unit upper triangular while D is a block diagonal matrix with blocks of order  $1 \times 1$  or  $2 \times 2$ . Here we explian step *i* of this algorithm. Before this step, we first initialize the array *status*. Then *i* is set to 1 and we enter the internal *while* loop. In lines 4-11 we obtain the first two rows and columns of the Schur-Complement matrix  $(S^{(i-1)})_{i,k>i}$  in (1.2). In line 12, the

Algorithm 2 (Column construction)

1. Column - Const(Z, A, D, i, status)2. if status(i) = 1 then for j = i + 1 to n do  $z_j^{(i)} = z_j^{(i-1)} - z_i^{(i-1)} \frac{1}{D_{ii}} A_{i,:} z_j^{(i-1)}$ 3. 4. 5.end for 6. end if 7. if status(i) = 2 then 8. for j = i + 2 to n do 
$$\begin{split} & k = i + status(i) - 1 \\ & z_j^{(k)} = z_j^{(i-1)} - [z_i^{(i-1)} \ z_{i+1}^{(i)}] D_{i:i+1,i:i+1}^{-1} A_{i:i+1} z_j^{(i-1)} \end{split}$$
9. 10.11. end for 12. end if

parameter  $v_i$  is defined. Then, in lines 13 and 14 we compute the two parameters  $w_i^1$  and  $w_i^2$  and in line 15,  $w_i$  is considered as the maximum of these two values. After that, we check in line 16 if  $v_i < w_i$ . In this case,  $D_{ii}$  is considered as  $S_{ii}^{(i-1)}$ , status(i) is set to 1 and we call Algorithm 2 in lines 19 and 20 to update the columns (i + 1)st to n of matrices Z and W. At the end, i is incremented by 1. If  $v_i \ge w_i$ , then in lines 24-30 we define a  $2 \times 2$  block diagonal, the final (i + 1)st columns of W and Z are introduced, status(i) is set to 2 and we call Algorithm 2 twice in lines 28 and 29. Finally, i is incremented by 2. After the *while* loop, if status(n-1) is equal to 0 or 1, then the last  $1 \times 1$  diagonal entry is computed.

#### 3. Numerical example

We have implemented Algorithms  $1 \ {\rm and} \ 2$  in MATLAB. Consider the nonsymmetric and invertible matrix A as

	3	3	1	1	4	2	2	
	3	2	2	1	4	3	3	
	4	4	2	2	2	3	1	
A =	4	4	1	2	1	1	2	
	3	3	4	4	2	1	4	
	2	3	1	2	2	2	1	
	4	3	1	1	3	2	4	

If we provide this matrix as the input argument of Algorithm 1, then the computed  $Z,\,D$  and W are

	Г	1.0000	0	-1.3333	1.0000	-8.0000	0.1154	-2.0385	٦
		0	1.0000	1.0000	-1.0000	5.0000	-1.1923	2.7308	
		0	0	1.0000	-1.0000	5.0000	-2.1923	1.7308	
Z =		0	0	0	1.0000	0	2.6538	-2.8846	
		0	0	0	0	1.0000	0.1923	-0.7308	
		0	0	0	0	0	1.0000	0	
	L	0	0	0	0	0	0	1.0000	
		[ 3.0000	3.0000	0	0	0	0	0 -	1
		3.0000	2.0000	0	0	0	0	0	
		0	0	0.6667	0	0	0	0	
D :	_	0	0	0	1.0000	-6.0000	0	0	
		0	0	0	0	13.0000	0	0	
		0	0	0	0	0	2.1538	-0.3846	
		0	0	0	0	0	-0.0769	0.6923	

	1.0000	0	-1.3333	-2.0000	5.0000	-1.3077	0.1538
	0	1.0000	0	0	0	1.0000	-1.0000
	0	0	1.0000	0.5000	-4.5000	0.0769	0.4615
W =	0	0	0	1.0000	0	0	-1.0000
	0	0	0	0	1.0000	-0.4615	0.2308
	0	0	0	0	0	1.0000	0
	0	0	0	0	0	0	1.0000

## References

- M. Benzi and M. Tuma, A sparse approximate inverse preconditioner for nonsymmetric linear systems, SIAM J. Sci. comput. 19(3), (1998) 968-994.
- [2] Ch. Kruschel, Losen von positive definiten, unsymmetrischen Matrizen mit Matching-Methoden am Beispiel von Konvektion-Diffusionsgleichungen, Bachelor of Sciences thesis, Technische Universitat Braunschweig, (2009).
- [3] Y. Saad, Iterative Methods for Sparse Linear Systems, SIAM, Philadelphia. 2nd edition (2003).

 $\label{eq:constraint} \begin{array}{l} \text{Department of applied mathematics, Hakim Sabzevari University, Sabzevar, Iran $E-mail address: rafiei.am@gmail.com, a.rafiei@hsu.ac.ir, amin.rafiei@tu-bs.de} \end{array}$ 

Department of applied mathematics, Hakim Sabzevari University, Sabzevar, Iran  $E\text{-}mail \ address: l.kakhki2971@gmail.com$ 



# SOME RESULTS ON POLYNOMIAL NUMERICAL HULL OF DEGREE n-1

SAEED KARAMI

ABSTRACT. Let  $J_k(\lambda)$  be the  $k \times k$  Jordan block with eigenvalue  $\lambda$  and let N be an  $m \times m$  normal matrix. In this paper we study the polynomial numerical hull of order n-1. In particular for  $A = J_{n_1}(\lambda_1) \oplus \ldots \oplus J_{n_k}(\lambda_k) \oplus N$ , we show that if  $\sigma(N)$  is neither co-linear nor co-circular, then  $V^{n-1}(A)$  has at most one point more than  $\sigma(A)$ , where  $n = n_1 + \ldots + n_k + m$ .

Keywords: Polynomial numerical hull; Jordan block; Direct Sum.

## 1. INTRODUCTION

Let  $M_n$  be the set of all  $n \times n$  complex matrices. Polynomial numerical hull of degree k for a matrix  $A \in M_n$ , is a useful tool for analysis of convergence behavior of Krylov subspace methods for solving linear systems, which was introduced by Nevanlinna in [1] and further studied by Greenbaum in [8]. It is defined as

(1.1) 
$$V^{k}(A) = \{z \in \mathbb{C} : |p(z)| \leq ||p(A)||, \forall p \in \mathcal{P}_{k}\},\$$

where  $\mathcal{P}_k$  is the set of complex polynomials with degree at most k. Polynomial numerical hulls give more information than the spectrum alone can provide about the behavior of the matrix under the action of polynomials and other functions. If k = 1, then  $V^1(A)$  coincides with the classical numerical range of matrices i.e,  $V^1(A) = W(A) = \{x^*Ax : x \in \mathbb{C}^n, ||x|| = 1\}$  [1]. In the following lemma we summarize useful properties of polynomial numerical hulls.

**Lemma 1.1** ([8, 4]). Let  $A \in M_n$ . Then the following properties hold for polynomial numerical hulls of order k:

- (1)  $\sigma(A) = V^n(A) \subseteq V^{k+1}(A) \subseteq V^k(A) \subseteq V^1(A) = W(A)$  for all k = 2, ..., n-1.
- (2)  $V^{k}(A)$  is not convex in general, but is polynomially convex of degree k and  $pco_{k}(\partial V^{k}(A)) = V^{k}(A).$
- (3)  $V^k(.)$  is invariant under unitary similarity transformations.
- (4)  $V^k(\alpha A + \beta I) = \alpha V^k(A) + \beta$  for any scalars  $\alpha, \beta \in \mathbb{C}$ .
- (5) Let A be a Hermitian matrix. Then  $V^2(A) = \sigma(A)$ .

**Theorem 1.2.** [7] Let  $A \in M_n$ . Then

(1.2) 
$$V^k(A) = \{ z \in \mathbb{C} : p(z) \in W(p(A)), \ \forall \ p \in \mathcal{P}_k \}.$$

<sup>2010</sup> Mathematics Subject Classification. 15A60, 65F15. Speaker: Saeed Karami.

We denote the  $n \times n$  Jordan block with eigenvalue  $\lambda$  by

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & \cdots & 0 & 0\\ 0 & \lambda & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \lambda & 1\\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}.$$

In [9], it has been shown that the polynomial numerical hulls of order k < n for a Jordan block  $J_n(\lambda)$  is a disk about  $\lambda$  and the radius of the disk has been obtained when k = n - 1. Also in [4] the polynomial numerical hulls of normal matrices has been studied, specially in [5] the polynomial numerical hull of order two for all normal matrices and matrices whose squares are Hermitian have been characterized.

In this paper, by using the results in [3], [9] and [5], we study polynomial numerical hulls of particular nonnormal matrice.

#### 2. Polynomial numerical hull of order n-1

Polynomial numerical hull of degree k provides a convenient lower bound on the norm of polynomials of degree k or less; i.e.,  $||p(A)|| \ge \max_{z \in V^k(A)} |p(z)|$ . Since any primary matrix function f(A) such as  $e^{tA}$  can be written as a polynomial of degree at most n-1 in A [9], we can write

$$||f(A)|| = ||p_f(A)|| \ge \max_{z \in V^{n-1}(A)} |p_f(z)|.$$

Therefore characterizing of the set  $V^{n-1}(A)$  is a valuable tools for studding norm behavior of matrix functions. In this section we study  $V^{n-1}(A)$  for some special matrices. First we state the following theorem for normal case.

**Theorem 2.1.** [5] Let  $A \in M_n$  be a normal matrix with  $n \ge 3$ . Then  $V^{n-1}(A)$  has at most n + 1 points.

In [6], we generelized the result of Theorem 2.1 for matrices which are direct sum of a Jordan block and a normal matrix. We say that the points  $z_1, \ldots$ ,  $z_m$  are co-circular if there exists a circle passing through  $z_1, \ldots, z_m$ . For every three non co-linear points we know that there exists a circle passing through all of these three points.

**Theorem 2.2.** [6] Let  $A = J_k(\lambda) \oplus N$  where N is a normal matrix with  $\sigma(N) = \{z_1, \ldots, z_m\}$  and let n = k + m. If the points  $z_1, \ldots, z_m$  are neither co-linear nor co-circular, then  $V^{n-1}(A)$  has at most one point more than  $\sigma(A)$ .

In the following theorem we generalize the result of Theorem 2.2 for matrice which are direct sum of a normal matrix and several Jordan blocks with different eigenvalues.

**Theorem 2.3.** Let  $A = J_{n_1}(\lambda_1) \oplus \ldots \oplus J_{n_k}(\lambda_k) \oplus \text{diag}\{z_1, \ldots, z_m\}$ , and  $n = n_1 + \ldots + n_k + m$ . If the points  $z_1, \ldots, z_m$  are neither co-linear nor co-circular, then  $V^{n-1}(A)$  has at most one point more than  $\sigma(A)$ .

*Proof.* If there exist  $1 \leq i \leq k$  and  $\leq j \leq m$  such that  $z_j = \lambda_i$  or there exist  $i \neq j$  such that  $z_i = z_j$  or  $\lambda_i = \lambda_j$ , then by [4, Lemma 1.2] and part (1) of Lemma 1.1 we obtain that  $V^{n-1}(A) = \sigma(A)$  and the result holds, so we assume  $\lambda_j, z_i, j = 1, \ldots, k$ 

and  $i = 1, \ldots, m$  are distinct complex numbers. Since  $z_1, \ldots, z_m$  are neither colinear nor co-circular we conclude that  $m \ge 4$ . For  $j = 1, \ldots, m$  let

(2.1) 
$$Q_j(z) = \frac{\prod_{i=1, i \neq j}^m (z - z_i)}{\prod_{i=1, i \neq j}^m (z_j - z_i)}$$

and  $P_j(z) = \frac{(z-\lambda_1)^{n_1}\dots(z-\lambda_k)^{n_k}}{(z_j-\lambda_1)^{n_1}\dots(z_j-\lambda_k)^{n_k}}Q_j(z).$ Note that  $P_j(z)$  is a polynomial of degree n-1. Let  $z_0 = x_0 + iy_0 \in V^{n-1}(A) - U^{n-1}(A)$  $\sigma(A)$ . Since  $P_j(z_i) = \delta_{ij}$  and  $(J_k(\lambda) - \lambda I_k)^k = 0_k$  by equation (1.2) we obtain that  $P_j(z_0) \in W(P_j(A)) = [0,1]$ . Whereas  $P_j(z) = 0$  if and only if  $z \in \sigma(A)$ , we conclude that  $0 < P_j(z_0) \le 1$ . For  $j \ne 1$  easy calculation shows that

$$\frac{P_j(z_0)}{P_1(z_0)} = \frac{\gamma_1(z_0 - z_1)}{\gamma_j(z_0 - z_j)},$$

where  $\gamma_j = (z_j - \lambda)^k \prod_{i=1, i \neq j}^m (z_j - z_i), j = 1, ..., m$ . Suppose  $z_j = a_j + ib_j$  and  $\gamma_j = \alpha_j + i\beta_j, j = 1, ..., m$ . By the above equation we obtain that

$$P_j(z_0)[((x_0 - a_j) + i(y_0 - b_j))(\alpha_j + i\beta_j)] = P_1(z_0)[((x_0 - a_1) + i(y_0 - b_1))(\alpha_1 + i\beta_1)].$$

Since the real and imaginary parts of the above equation are equal, easy calculation shows that

(2.2) 
$$\begin{aligned} & (\alpha_1\beta_j - \alpha_j\beta_1)(x_0 - a_1)(x_0 - a_j) + (\alpha_1\beta_j - \alpha_j\beta_1)(y_0 - b_1)(y_0 - b_j) \\ & + (\alpha_1\alpha_j + \beta_1\beta_j)(x_0 - a_1)(y_0 - b_j) - (\alpha_1\alpha_j + \beta_1\beta_j)(x_0 - a_j)(y_0 - b_1) = 0. \end{aligned}$$

We can rewrite the left hand side of equation (2.2) as a two variable function in the following form:

(2.3)  
$$f_{j}(x,y) = (\alpha_{1}\beta_{j} - \alpha_{j}\beta_{1})(x^{2} + y^{2}) + ((b_{1} - b_{j})(\alpha_{1}\alpha_{j} + \beta_{1}\beta_{j}) - (a_{1} + a_{j})(\alpha_{1}\beta_{j} - \alpha_{j}\beta_{1}))x + ((a_{j} - a_{1})(\alpha_{1}\alpha_{j} + \beta_{1}\beta_{j}) - (b_{1} + b_{j})(\alpha_{1}\beta_{j} - \alpha_{j}\beta_{1}))y + ((a_{1}a_{j} + b_{1}b_{j})(\alpha_{1}\beta_{j} - \alpha_{j}\beta_{1}) + (a_{1}b_{j} - a_{j}b_{1})(\alpha_{1}\alpha_{j} + \beta_{1}\beta_{j})),$$

where  $f_j(x_0, y_0) = 0$ . For j = 2, ..., m let  $C_j := \{z = x + iy : f_j(x, y) = 0\}$ . According to (2.2) any  $z_0 \in V^{n-1}(A) - \sigma(A)$  must lies on  $C_j, j = 2, \ldots, m$  and hence  $V^{n-1}(A) - \sigma(A) \subseteq \bigcap_{j=2}^{m} C_j$ . By (2.3), it is easy to see that the sets  $C_j$ ,  $j = 2, \ldots, m$ are circles or lines and by (2.2) we conclude that  $C_j$  passes through the points  $z_1$ and  $z_j$ . Thus  $z_1 \in \bigcap_{j=2}^m C_j$ . Whereas  $z_1, \ldots, z_m$  are neither co-linear nor co-circular, there exist  $2 \leq i < j \leq m$  such that  $C_i$  and  $C_j$  are not identical. Since  $C_i \cap C_j$  has at most two points which one of them is  $z_1 \in \sigma(A)$ , we conclude that  $V^{n-1}(A) - \sigma(A)$ has at most one point. Thus  $V^{n-1}(A)$  has at most one point more than  $\sigma(A)$ .

**Remark 2.4.** Note that for a general matrix the polynomial numerical hull of order n-1 may has infinite many points! For instance in [9] it is shown that the polynomial numerical hull of order n-1 for a Jordan block is a circular disk around its eigenvalue with positive radius  $r_{n-1,n}$ , where

$$1 - \frac{\log(2n)}{n} \le r_{n-1,n} \le 1 - \frac{\log(2n)}{n} + \frac{\log(\log(2n))}{n}$$

**Example 2.5.** Let  $A = J_5(0)$ . Then  $V^4(A)$  is a circular disk with the radius  $r_{4,5}$ , where  $0.5395 < r_{4,5} < 0.7063$ .

#### References

- [1] O. Nevalinna Convergence of iterations for linear equation. Basel: Birkhauser, 1993.
- [2] R. A. Horn and C. R. Johnson *Topics in Matrix Analysis*. Cambridge University Press, 1994.
- [3] V. Faber, W. Joubert, E. Knill and T. Manteuffel Minimal Residual Method Stronger than Polynomial Preconditioning. SIAM J. Matrix Anal. Appl., 17:707–729, 1996.
- [4] Ch. Davis and A. Salemi On polynomial numerical hulls of normal matrices. *Linear Algebra Appl.*, 383:151–161, 2004.
- [5] Chandler Davis, Chi-Kwong Li and Abbas Salemi Polynomial numerical hulls of matrices. Linear Algebra Appl., 428:137–153, 2008.
- [6] Saeed Karami and Abbas Salemi Polynomial Numerical Hulls of the direct sum of a normal matrix and a Jordan Block. *Linear Algebra Appl.*, 465:143–160, 2015.
- [7] E. B. Davies Spectral Bounds Using Higher-Order Numerical Ranges. LMS-J-COMPUT-MATH, 8:17–45, 2005.
- [8] Anne Greenbaum Generalizations of the field of values useful in the study of polynomial functions of a matrix. *Linear Algebra Appl.*, 347:233–249, 2002.
- [9] Vance Faber, Anne Greenbaum and Donald E. Marshall The polynomial numerical hulls of Jordan blocks and related matrices. *Linear Algebra Appl.*, 374:231–246, 2003.
- [10] James Burke and Anne Greenbaum Characterizations of the polynomial numerical hull of degree k. Linear Algebra Appl., 419:37–47, 2006.

INSTITUTE FOR ADVANCED STUDIES IN BASIC SCIENCES (IASBS), ZANJAN, IRAN. *E-mail address*: s.karami@iasbs.ac.ir



# RECENT DEVELOPMENTS ON ITERATIVE METHODS FOR SOLVING COMPLEX SYMMETRIC SYSTEM OF LINEAR EQUATIONS

DAVOD KHOJASTEH SALKUYEH

ABSTRACT. We review recent iterative methods for solving the complex system of linear equations (W+iT)x = b, where the matrix  $W \in \mathbb{R}^{n \times n}$  and  $T \in \mathbb{R}^{n \times n}$  are symmetric.

Keywords: Complex symmetric; MHSS; SCSP; TSCSP; TTSCSP.

#### 1. INTRODUCTION

We are concerned with the iteration solution of the system of linear equations

$$(1.1) Au = (W + iT)u = b,$$

where  $W, T \in \mathbb{R}^{n \times n}$ , u = x + iy and b = p + iq, such that the vectors x, y, pand q are in  $\mathbb{R}^n$  and  $i = \sqrt{-1}$ . We assume that W and T are symmetric. Several iterative methods have been presented for solving the system (1.1) in the literature. Based upon the Hermitian and skew-Hermitian splitting (HSS) of the matrix A, Bai et al. in [3] introduced the Hermitian/skew-Hermitian splitting (HSS) method to solve non-Hermitian positive definite system of linear equations. Next, Bai et al. presented a modified version of the HSS iterative method say MHSS [1] to solve the systems of the form (1.1). Then, a preconditioned version of the MHSS iteration method, called PMHSS, was presented by Bai et al. in [2].

Let

be the Hermitian/Skew-Hermitian (HS) splitting of the matrix A, where

$$H = \frac{1}{2}(A + A^{H}) = W, \quad S = \frac{1}{2}(A - A^{H}) = iT,$$

with  $A^H$  being the conjugate transpose of A. Let also  $V \in \mathbb{R}^{n \times n}$  be a symmetric positive definite. Then, the PMHSS iteration method can be described as follows.

The PMHSS method: Let  $u^{(0)} \in \mathbb{C}^n$  be an initial guess. For k = 0, 1, 2, ...,until  $\{u^{(k)}\}$  converges, compute  $u^{(k+1)}$  according to the following sequence:

(1.3) 
$$\begin{cases} (\alpha V + W)u^{(k+\frac{1}{2})} = (\alpha V - iT)u^{(k)} + b, \\ (\alpha V + T)u^{(k+1)} = (\alpha V + iW)u^{(k+\frac{1}{2})} - ib, \end{cases}$$

where  $\alpha$  is a given positive constant.

When the matrix V is equal to the identity matrix, then the PMHSS iteration method reduces to MHSS. When W and V are symmetric positive definite and

<sup>2010</sup> Mathematics Subject Classification. 65F10, 65F50.

Speaker: Davod Khojasteh Salkuyeh.

T is symmetric positive semidefinite, it has been proved that the PMHSS iteration converges unconditionally to the unique solution of the complex symmetric system (1.1) for any initial guess (see [2]). Numerical implementation presented in [2] show that a Krylov subspace iteration method such as generalized minimal residual (GMRES) [5] in conjunction with the resulting PMHSS preconditioner is very efficient to solve the system (1.1). In particular, both the PMHSS iteration method and the MHSS-preconditioned GMRES show meshsize-independent and parameter-insensitive convergence behaviour (see [2]).

Recently, Hezari et al. in [4] presented the Scale-Splitting (SCSP) iteration method to solve (1.1) which can be described as follows.

The SCSP iteration method: Let  $u^{(0)} \in \mathbb{C}^n$  be an initial guess. For  $k = 0, 1, \ldots$ , until  $\{u^{(k)}\}$  converges, compute  $u^{(k+1)}$  according to the following sequence

(1.4) 
$$(\alpha W + T)u^{(k+1)} = i(W - \alpha T)u^{(k)} + (\alpha - i)b_{i},$$

where  $\alpha$  is a given positive constant.

At each iteration of the SCSP iteration method, it is required to solve a linear system with coefficient matrix  $\alpha W + T$ . In [4] it was proved that if W and T are symmetric positive semidefinite matrices satisfying  $\operatorname{null}(W) \cap \operatorname{null}(T) = \{0\}$ , then the SCSP iteration method is convergent provided that

$$\begin{cases} \frac{1-\mu_{\min}}{1+\mu_{\min}} < \alpha < \frac{1+\mu_{\max}}{\mu_{\max}-1}, & \text{for } \mu_{\max} > 1, \\\\ \frac{1-\mu_{\min}}{1+\mu_{\min}} < \alpha, & \text{for } \mu_{\max} \le 1, \end{cases}$$

where  $\mu_{\min}$  and  $\mu_{\max}$  are the smallest and largest generalized eigenvalues of the matrix pair (W, T), respectively.

In the next section we present a two-step version of the SCSP iteration method to solve the system (1.1).

## 2. Two-step SCSP

Let  $\alpha > 0$ . Using the idea of the SCSP method, we multiply both sides of Eq. (1.1) by  $\alpha - i$  to get the equivalent system

(2.1) 
$$(\alpha - i)Au = (\alpha - i)b,$$

where  $i = \sqrt{-1}$ . Then, we split the coefficient matrix of the system (2.1) as

$$(\alpha - i)A = (\alpha W + T) - i(W - \alpha T).$$

Using this splitting, we rewrite the system (2.1) as the fixed-point equation

(2.2) 
$$(\alpha W + T)u = i(W - \alpha T)u + (\alpha - i)b.$$

On the other hand, for  $\beta > 0$  we multiply both sides of Eq. (2.1) by  $1 - \beta i$  to obtain the equivalent system

$$(1 - \beta i)Au = (1 - \beta i)b$$

which yields the fixed point equation

(2.3) 
$$(W + \beta T)u = i(\beta W - T)u + (1 - \beta i)b.$$

From Eqs. (2.2) and (2.3), we now state the Two-parameter Two-step SCSP (TTSCSP) algorithm as follows [7].

The TTSCSP iteration method: Let  $u^{(0)} \in \mathbb{C}^n$  be an initial guess. For  $k = 0, 1, 2, \ldots$ , until  $\{u^{(k)}\}$  converges, compute  $u^{(k+1)}$  according to the following sequence

(2.4) 
$$\begin{cases} (\alpha W + T)u^{(k+\frac{1}{2})} = i(W - \alpha T)u^{(k)} + (\alpha - i)b, \\ (W + \beta T)u^{(k+1)} = i(\beta W - T)u^{(k+\frac{1}{2})} + (1 - \beta i)b, \end{cases}$$

where  $\alpha, \beta > 0$ .

Computing  $u^{(k+\frac{1}{2})}$  from the first equation in (2.4) and substituting it in second equation of (2.4) gives the one-step form of the TTSCSP iteration method as

(2.5) 
$$u^{(k+1)} = \mathcal{G}_{\alpha,\beta} u^{(k)} + \mathcal{C}_{\alpha,\beta},$$

where the iteration matrix of method is given by

$$\mathcal{G}_{\alpha,\beta} = (W + \beta T)^{-1} (T - \beta W) (\alpha W + T)^{-1} (W - \alpha T),$$

and

$$\mathcal{C}_{\alpha,\beta} = (\alpha + \beta)(W + \beta T)^{-1}(W - iT)(\alpha W + T)^{-1}b$$

When  $\beta = \alpha$ , the TTSCSP iteration method reduces to the Two-step SCSP (TSCSP) method and in this case iteration matrix of the method is denoted by  $\mathcal{G}_{\alpha}$  (see [7, 8]). Converges analysis of the TSCSP and the TTSCSP methods along with the optimal values of the parameters  $\alpha$  and  $\beta$  are presented as follows.

**Theorem 2.1.** Let  $W \in \mathbb{R}^{n \times n}$  be symmetric positive definite,  $T \in \mathbb{R}^{n \times n}$  be symmetric positive semidefinite and

$$0 \le \mu_1 \le \dots \le \mu_r < 1 \le \mu_{r+1} \le \dots \le \mu_n,$$

be the eigenvalues of  $S = W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$ . Then, the TTSCSP iteration method is convergent, i.e.,  $\rho(\mathcal{G}_{\alpha,\beta}) < 1$ , provided that the parameters  $\alpha$  and  $\beta$  satisfy

$$\frac{1-\mu_1}{1+\mu_1} < \alpha < \frac{\mu_n+1}{\mu_n-1} \quad and \quad \frac{\mu_n-1}{\mu_n+1} < \beta < \frac{1+\mu_1}{1-\mu_1}$$

**Remark 2.2.** If  $0 \le \mu_1 \le \cdots \le \mu_n \le 1$ , then the sufficient conditions for the convergence of the TTSCSP iteration method given in Theorem 2.1 reduce to

$$\alpha > \frac{1-\mu_1}{1+\mu_1} \quad and \quad \beta < \frac{1+\mu_1}{1-\mu_1}$$

and if  $1 \leq \mu_1 \leq \cdots \leq \mu_n$ , then the sufficient convergence conditions reduce to

$$\alpha < rac{\mu_n+1}{\mu_n-1} \quad and \quad \beta > rac{\mu_n-1}{\mu_n+1}.$$

**Theorem 2.3.** Let all the assumptions of Theorem 2.1 hold. Then,

$$\rho(\mathcal{G}_{\alpha,\beta}) \leq \max_{\mu_j \in \sigma(S)} \left| \frac{\mu_j - \beta}{1 + \beta \mu_j} \right| \cdot \max_{\mu_j \in \sigma(S)} \left| \frac{1 - \alpha \mu_j}{\alpha + \mu_j} \right| =: \sigma(\alpha, \beta).$$

Moreover, if we define

$$(\alpha^*, \beta^*) = \operatorname{argmin}_{\beta > 0} \sigma(\alpha, \beta),$$

then

$$\alpha^* = \frac{\gamma + \sqrt{\gamma^2 + \eta^2}}{\eta} \quad \text{and} \quad \beta^* = \frac{1}{\alpha^*},$$

where  $\eta = \mu_1 + \mu_n$  and  $\gamma = 1 - \mu_1 \mu_n$ .

**Theorem 2.4.** Assume that the matrices  $W \in \mathbb{R}^{n \times n}$  and  $T \in \mathbb{R}^{n \times n}$  are symmetric positive definite and  $\beta = \alpha > 0$ . Then,  $\rho(\mathcal{G}_{\alpha}) < 1$ . That is, the TSCSP iteration method is unconditionally convergent for every  $\alpha > 0$ .

**Theorem 2.5.** Let  $W \in \mathbb{R}^{n \times n}$  and  $T \in \mathbb{R}^{n \times n}$  be symmetric positive definite matrices. Let also  $\mu_i$ , i = 1, 2, ..., n be the eigenvalues of  $S = W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$ . (a) If  $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq 1$  or  $1 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$  set  $\gamma = \mu_1$  and

(a) If  $0 < \mu_1 \le \mu_2 \le \cdots \le \mu_n \le 1$  of  $1 \le \mu_1 \le \mu_2 \le \cdots \le \mu_n$  set  $\gamma = \mu_1$  and  $\delta = \mu_n$ .

(b) If  $0 < \mu_1 < \cdots \leq \mu_k \leq 1 \leq \mu_{k+1} \leq \cdots \leq \mu_n$ , then if  $\mu_1 \mu_n \geq 1$ , set  $\gamma = \mu_{k+1}$  and  $\delta = \mu_n$ , otherwise  $\gamma = \mu_1$  and  $\delta = \mu_k$ .

Then, the optimal values of  $\alpha$  in the TSCSP iteration method are given by

$$\alpha_{opt}^{\pm} = \arg\min_{\alpha>0} \rho(\mathcal{G}_{\alpha}) = \frac{1}{2} \left( \eta \pm \sqrt{\eta^2 - 4} \right),$$

where

$$\eta = \sqrt{\frac{(1+\gamma^2)(1+\delta^2)}{\gamma\delta}}$$

Moreover

$$\rho\left(\mathcal{G}_{\alpha_{opt}^{\pm}}\right) = \left|\frac{\delta^2 - \eta\delta + 1}{\delta^2 + \eta\delta + 1}\right|.$$

Our numerical results show that the TTSCSP method (as well as the TSCSP method) outperforms the MHSS, PHMSS and SCSP iteration methods.

#### References

- Z.-Z. Bai, M. Benzi and F. Chen, Modified HSS iterative methods for a class of complex symmetric linear systems, Computing 87 (2010), 93–111.
- [2] Z.-Z. Bai, M. Benzi and F. Chen, On preconditioned MHSS iteration methods for complex symmetric linear systems, Numer. Algor 56 (2010), 93-317.
- [3] Z.-Z. Bai, G.H. Golub and M.K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, SIAM. J. Matrix Anal. Appl. 24 (2003), 603– 626.
- [4] D. Hezari, D.K. Salkuyeh and V. Edalatpour, A new iterative method for solving a class of complex symmetric linear of equathions, Numerical Algorithm 73 (2016), 927-955.
- [5] Y. Saad and M. H. Schultz, GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems, SIAM Journal on Scientific and Statistical Computing 7 (1986), 856–869.
- [6] D.K. Salkuyeh, Two-step scale-splitting method for solving complex symmetric system of linear equations, submitted, 2017, arXiv:1705.02468.
- [7] D.K. Salkuyeh and T. Salimi Siahkalaei, Two-parameter TSCSP method for solving complex symmetric system of linear equations, submitted, 2017, arXiv:1705.02464.
- [8] Z. Zheng, F.-L. Huang and Y.-C. Peng, Double-step scale splitting iteration method for a class o complex symmetric linear systems, Appl. Math. Lett. 73 (2017), 91–97.

FACULTY OF MATHEMATICAL SCIENCES, UNIVERSITY OF GUILAN, RASHT, IRAN *E-mail address*: khojasteh@guilan.ac.ir

78



## MULTIPLICATIVE SPLITTING ITERATIONS FOR SOLVING SYLVESTER EQUATION

MOHAMMAD KHORSAND ZAK

ABSTRACT. We present a multiplicative splitting iteration method for solving large sparse continuous Sylvester equation, in which both coefficient matrices are (non-Hermitian) positive semi-definite, and at least one of them is positive definite. Convergence conditions of this method are studied and numerical experiments show the efficiency of this method.

Keywords: Sylvester equation; Multiplicative splitting; Iterative methods.

#### 1. INTRODUCTION

The continuous Sylvester equation is possibly the most broadly employed linear matrix equation, and is given as

$$AX + XB = C,$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{m \times m}$  and  $C \in \mathbb{R}^{n \times m}$  are defined matrices and  $X \in \mathbb{R}^{n \times m}$  is an unknown matrix. The continuous Sylvester equation (1.1) has a unique solution if and only if A and -B have no common eigenvalues. The matrix equation (1.1) is mathematically equivalent to the linear system of equations

(1.2) 
$$\mathcal{A}x = c$$

where the matrix  $\mathcal{A}$  is of dimension  $nm \times nm$  and is given by

(1.3) 
$$\mathcal{A} = I_m \otimes A + B^T \otimes I_n,$$

where  $\otimes$  denotes the Kronecker product  $(A \otimes B = [a_{ij}B])$  and

$$c = vec(C) = (c_{11}, c_{21}, \cdots, c_{n1}, c_{12}, c_{22}, \cdots, c_{n2}, \cdots, c_{nn})^T$$
  
$$x = vec(X) = (x_{11}, x_{21}, \cdots, x_{n1}, x_{12}, x_{22}, \cdots, x_{n2}, \cdots, x_{nn})^T.$$

Of course, this is a numerically poor way to determine the solution X of the Sylvester equation (1.1), as the linear system of equations (1.2) is costly to solve and can be ill-conditioned.

## 2. Main results

Let  $A = M_i - N_i$  and  $B = P_i - Q_i$ , (i = 1, 2) be two splittings of the matrices A and B, such that  $M_i$  and  $P_i$ , (i = 1, 2) are symmetric positive definite. The continuous Sylvester equation (1.1) can be equivalently written as the multiplicative splitting matrix equations

$$\begin{cases} M_1U + UP_1 = N_1X + XQ_1 + C \\ M_2X + XP_2 = N_2U + UQ_2 + C \end{cases}$$

<sup>2010</sup> Mathematics Subject Classification. 15A24, 15A30, 15A69, 65F10. Speaker: Mohammad Khorsand Zak.

Under the assumption that  $M_i$  and  $P_i$ , (i = 1, 2) are symmetric positive definite, we easily know that there is no common eigenvalues between the matrices  $M_i$  and  $-P_i$ , (i = 1, 2), so that this two multiplicative splitting matrix equations have unique solutions for all given right hand side matrices.

Now, based on the above observations, we can establish the following multiplicative splitting iterations (MSI) for solving the continuous Sylvester equation (1.1). **MSI method for Sylvester equation** 

Given an initial guess  $X^{(0)} \in \mathbb{R}^{m \times n}$ , For  $k = 1, 2, \cdots$  until convergence, do Solve  $M_1 U^{(k+1)} + U^{(k+1)} P_1 = N_1 X^{(k)} + X^{(k)} Q_1 + C$ Solve  $M_2 X^{(k+1)} + X^{(k+1)} P_2 = N_2 U^{(k+1)} + U^{(k+1)} Q_2 + C$ 

end

**Lemma 2.1.** [1] Let  $B, C \in \mathbb{R}^{n \times n}$  be two Hermitian matrices. Then BC = CB if and only if B and C have a common set of orthonormal eigenvectors.

**Lemma 2.2.** [3] Suppose that  $\mathcal{A} = \mathcal{M} - \mathcal{N}$  is a splitting such that  $\mathcal{M}$  is symmetric positive definite, with  $\mathcal{M} = I_m \otimes M + P^T \otimes I_n$  and  $\mathcal{N} = I_m \otimes N + Q^T \otimes I_n$ . If

$$\theta^3 \frac{\max |\lambda(N)| + \max |\lambda(Q)|}{\lambda_{\min}(M) + \lambda_{\min}(P)} < 1,$$

where  $\theta = \sqrt{\frac{\lambda_{\max}(M) + \lambda_{\max}(P)}{\lambda_{\min}(M) + \lambda_{\min}(P)}}$ , then  $||\mathcal{M}^{-1}\mathcal{N}||_{\mathcal{M}} < 1$ .

**Theorem 2.3.** Let  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{m \times m}$  and suppose that there is no common eigenvalue between A and -B. Consider two splittings  $A = M_i - N_i$  and  $B = P_i - Q_i$  (i = 1, 2) such that  $M_i$  and  $P_i$ , (i = 1, 2) are symmetric positive definite. Denote by  $A = \mathcal{M}_i - \mathcal{N}_i$  (i = 1, 2) with  $\mathcal{M}_i = I_m \otimes M_i + P_i^T \otimes I_n$  and  $\mathcal{N}_i = I_m \otimes N_i + Q_i^T \otimes I_n$  (i = 1, 2), and assume that  $\mathcal{M}_1 \mathcal{A}^{-1} \mathcal{M}_2 = \mathcal{M}_2 \mathcal{A}^{-1} \mathcal{M}_1$ . Then the MSI method is convergent if  $\varrho_1 \varrho_2 < 1$ , where

$$\varrho_i = \theta_i^3 \frac{\max|\lambda(N_i)| + \max|\lambda(Q_i)|}{\lambda_{\min}(M_i) + \lambda_{\min}(P_i)}, \text{ and } \theta_i = \sqrt{\frac{\lambda_{\max}(M_i) + \lambda_{\max}(P_i)}{\lambda_{\min}(M_i) + \lambda_{\min}(P_i)}}, (i = 1, 2).$$

*Proof.* By making use of the Kronecker product, we can rewrite the above described MSI method in the following matrix-vector form:

$$\begin{cases} (I_m \otimes M_1 + P_1^T \otimes I_n)u^{(k+1)} = (I_m \otimes N_1 + Q_1^T \otimes I_n)x^{(k)} + c \\ (I_m \otimes M_2 + P_2^T \otimes I_n)x^{(k+1)} = (I_m \otimes N_2 + Q_2^T \otimes I_n)u^{(k+1)} + c \end{cases}$$

which can be arranged equivalently as

$$\begin{cases} \mathcal{M}_1 u^{(k+1)} = \mathcal{N}_1 x^{(k)} + c \\ \mathcal{M}_2 x^{(k+1)} = \mathcal{N}_2 u^{(k+1)} + c \end{cases}$$

which can be obtained the following iteration method

(2.1) 
$$\begin{cases} u^{(k+1)} = \mathcal{M}_1^{-1} \mathcal{N}_1 x^{(k)} + \mathcal{M}_1^{-1} c \\ x^{(k+1)} = \mathcal{M}_2^{-1} \mathcal{N}_2 u^{(k+1)} + \mathcal{M}_2^{-1} c \end{cases}$$

Evidently, the above iteration scheme is the MSI-method [1] for solving system of linear equations (1.2) with  $\mathcal{A} = \mathcal{M}_i - \mathcal{N}_i$  (i = 1, 2). The MSI iteration (2.1) can

be neatly expressed as a stationary fixed-point iteration as follows,

$$c^{(k+1)} = \mathcal{T}x^{(k)} + \mathcal{G}c$$

with  $\mathcal{T} = \mathcal{M}_2^{-1} \mathcal{N}_2 \mathcal{M}_1^{-1} \mathcal{N}_2$  and  $\mathcal{G} = \mathcal{M}_2^{-1} \mathcal{N}_2 \mathcal{M}_1^{-1} + \mathcal{M}_1^{-1}$ . Because  $\mathcal{M}_1 \mathcal{A}^{-1} \mathcal{M}_2 = \mathcal{M}_2 \mathcal{A}^{-1} \mathcal{M}_1$  is equivalent to that the two matrices  $\mathcal{M}_1 \mathcal{A}^{-1}$ 

Because  $\mathcal{M}_1 \mathcal{A}^{-1} \mathcal{M}_2 = \mathcal{M}_2 \mathcal{A}^{-1} \mathcal{M}_1$  is equivalent to that the two matrices  $\mathcal{M}_1 \mathcal{A}^{-1}$ and  $\mathcal{M}_2 \mathcal{A}^{-1}$  are commutative, according to Lemma 2.1 we know that  $\mathcal{M}_1 \mathcal{A}^{-1}$  and  $\mathcal{M}_2 \mathcal{A}^{-1}$  have a common set of orthonormal eigenvectors. That is say, there exists a unitary matrix  $\mathcal{Q} \in \mathbb{R}^{nm \times nm}$  and two diagonal matrices  $\Lambda_i = diag(\lambda_1^{(i)}, \lambda_2^{(i)}, \cdots, \lambda_{nm}^{(i)}), i =$ 1, 2, such that  $\mathcal{Q}\mathcal{M}_i^{-1}\mathcal{A}\mathcal{Q}^* = \Lambda_i, i = 1, 2$ . Noticing that

$$\begin{aligned} \mathcal{T} &= \mathcal{M}_2^{-1} \mathcal{N}_2 \mathcal{M}_1^{-1} \mathcal{N}_1 \\ &= \mathcal{M}_2^{-1} (\mathcal{M}_2 - \mathcal{A}) \mathcal{M}_1^{-1} (\mathcal{M}_1 - \mathcal{A}) \\ &= (I - \mathcal{M}_2^{-1} \mathcal{A}) (I - \mathcal{M}_1^{-1} \mathcal{A}) \\ &= (\mathcal{Q}^* \mathcal{Q} - \mathcal{Q}^* \mathcal{Q} \mathcal{M}_2^{-1} \mathcal{A} \mathcal{Q}^* \mathcal{Q}) (\mathcal{Q}^* \mathcal{Q} - \mathcal{Q}^* \mathcal{Q} \mathcal{M}_1 \mathcal{A} \mathcal{Q}^* \mathcal{Q}) \\ &= (\mathcal{Q}^* \mathcal{Q} - \mathcal{Q}^* \Lambda_2 \mathcal{Q}) (\mathcal{Q}^* \mathcal{Q} - \mathcal{Q}^* \Lambda_1 \mathcal{Q}) \\ &= \mathcal{Q}^* (I - \Lambda_2) \mathcal{Q}^* \mathcal{Q} (I - \Lambda_1) \mathcal{Q} \\ &= \mathcal{Q}^* (I - \Lambda_2) (I - \Lambda_1) \mathcal{Q} \end{aligned}$$

so we have

$$\begin{aligned}
\rho(\mathcal{T}) &\leq \max_{1 \leq i, j \leq nm} |(1 - \lambda_i^{(2)})(1 - \lambda_j^{(1)})| \\
&\leq \max_{1 \leq i \leq nm} |(1 - \lambda_i^{(2)})| \max_{1 \leq j \leq nm} |(1 - \lambda_j^{(1)})| \\
&= \rho(I - \mathcal{M}_2^{-1}\mathcal{A})\rho(I - \mathcal{M}_1^{-1}\mathcal{A}) \\
&= \rho(\mathcal{M}_2^{-1}\mathcal{N}_2)\rho(\mathcal{M}_1^{-1}\mathcal{N}_1) \\
&\leq ||\mathcal{M}_2^{-1}\mathcal{N}_2||_{\mathcal{M}_2} ||\mathcal{M}_1^{-1}\mathcal{N}_1||_{\mathcal{M}_1}
\end{aligned}$$

Therefore, by Lemma 2.2 we have

$$\rho(\mathcal{T}) \le \theta_1^3 \frac{\max|\lambda(N_1)| + \max|\lambda(Q_1)|}{\lambda_{\min}(M_1) + \lambda_{\min}(P_1)} \theta_2^3 \frac{\max|\lambda(N_2)| + \max|\lambda(Q_2)|}{\lambda_{\min}(M_2) + \lambda_{\min}(P_2)} = \varrho_1 \varrho_2$$

and this completes the prove.

#### 3. Numerical results

When both coefficient matrices A and B, in Sylvester equation (1.1) are (nonsymmetric) positive semi-definite, and at least one of them is positive definite; we can choose Hermitian and skew-Hermitian (HS) splittings of matrices A and B, in the first equation in MSI method, and the splitting of Jacobi iterations [4] for matrices A and B, in the second equation in MSI method.

All numerical experiments presented in this section were computed in double precision with a number of MATLAB codes. All iterations are started from the zero matrix for initial  $X^{(0)}$  and terminated when the current iterate satisfies  $\frac{||R^{(k)}||_F}{||R^{(0)}||_F} \leq 10^{-10}$ , where  $R^{(k)} = C - AX^{(k)} - X^{(k)}B$  is the residual of the *k*th iterate. Also we use the tolerance  $\varepsilon = 0.01$  for inner iterations in corresponding methods. For each experiment we report the number of iterations or the number of total outer iteration steps and CPU time (in parentheses), and compare the MSI method with HSS and IHSS iterative methods [2].

**Example 3.1.** For this example, we use the matrices

$$A = B = M + 2rN + \frac{100}{(n+1)^2}I,$$

where M = tridiag(-1, 2, -1), N = tridiag(0.5, 0, -0.5) and r = 0.01 [2]. We apply the iteration methods to this problem with different dimensions. The results are given in Table 1. The pair (n, m) in the first row of the table, represents the dimension of matrices A and B, respectively.

Method	(16, 16)	(32, 32)	(64, 64)	(128, 128)	(256, 256)
MSI	5(0.015)	6(0.062)	6(0.432)	8(3.463)	10(21.450)
HSS	34(0.040)	63(0.276)	118(2.060)	220(16.407)	407(191.796)
IHSS	30(0.019)	59(0.073)	104(0.499)	203(3.823)	379(43.251)

TABLE 1. Results of the Example 1

From the results presented in the Table 1, it can be seen that the MSI method is more efficient than the other methods.

## References

- Z. Z. Bai, On the convergence of additive and multiplicative splitting iterations for systems of linear equations, J. Comput. Appl. Math., 154 (2003) 195-214.
- [2] Z.-Z. Bai, On Hermitian and skew-Hermitian splitting iteration methods for continuous Sylvester equations, J. Comput. Math., 29:2 (2011) 185-198.
- [3] M. Khorsand Zak and F. Toutounian, Nested splitting CG-like iterative method for solving the continuous Sylvester equation and preconditioning, Adv. Comput. Math., 40 (2014) 865-880.
- [4] Y. Saad, Iterative Methods for Sparse Linear Systems, Second edition, SIAM, Philadelphia, 2003.

Department of Applied Mathematics, Aligoudarz Branch, Islamic Azad University, Aligoudarz, Iran

E-mail address: m.khorsandzak@gmail.com



## FUZZY CONGRUENCE RELATION AND FUZZY COSET RELATION ON VECTOR SPACES

SADEGH KHOSRAVI SHOAR

ABSTRACT. In this work we introduce and characterize the concept of a fuzzy congruence relations on a vector space and find some properties of it. We define the concept of a fuzzy coset relation and sum operation between two fuzzy coset relations and show that the set of all fuzzy coset relations on a vector space is a vector space.

Keywords: Fuzzy congruence relation; Fuzzy coset relation; Fuzzy relation; Vector spaces.

#### 1. INTRODUCTION

The systematic generalization of crisp concepts to the fuzzy case has proven to be an important theoretical tool for the development of new methods of reasoning under uncertainty, imprecision and lack of information. Regarding the generalization level, it is important to note that the definition of fuzzy sets originally presented as mappings with codomain [0 1], was soon replaced by more general structures, for instance a complete lattice, as in the L-fuzzy sets introduced by Goguen [2]. Katsaras and Lin in [3] introduced the notion of a fuzzy subspace of a vector space and obtained some fundamental results pertaining to this notion. Subsequently, Das [1] redefined a fuzzy subspace with respect to a t-norm and showed that most of the results established for fuzzy subspaces with respect to the t-norm 'min' are valid for t-norms of a more general nature. N. P. Mukherjee in [5] introduced and characterized the concepts of fuzzy normal subgroups and fuzzy cosets and showed that the level subgroups of a fuzzy normal subgroup are normal. By the similar way R. Kumar [4] researched that these subject in vector spaces and introduced the concepts of fuzzy vector spaces and fuzzy cosests. In this paper we define the concept of a fuzzy congruence relation on a vector space and characterize its properties. We present an operation  $\oplus$  between two fuzzy relations on a vector space and show that the set of all fuzzy congruence relations on a vector space with operation  $\oplus$  is a commutative semi-group. We also introduce the concept of a fuzzy coset relation on a vector space and prove that the set of all fuzzy coset relations on a vector space with operation  $\oplus$  and scalar operation that mentioned in this paper is a vector space.

### 2. Preliminaries

Throughout the paper, we assume that V is a vector space over F, the filed of real numbers, unless otherwise specified.

<sup>2010</sup> Mathematics Subject Classification. 15A03, 08A79. Speaker: Sadegh Khosravi Shoar.

**Definition 2.1.** [10] A mapping  $\mu : L \to [0,1]$  is called a fuzzy subset in L.

**Definition 2.2.** [3] Let  $\mu$  be a fuzzy subset of V. Then (i) The set  $\mu_t = \{x \in V \mid \mu(x) \ge t\}, t \in [0, 1]$ , is called level subset of  $\mu$ . (ii)  $\mu$  is called a fuzzy subspace of V if  $\mu(av + bu) \ge \mu(v) \land \mu(u)$ , where  $u, v \in V$  and  $a, b \in F$ .

**Definition 2.3.** [8, 9] Let X be a non-empty set. A fuzzy relation on X is a map  $\eta$  from  $X \times X$  to the unit interval [0, 1], and R(X) will denote the set of all fuzzy relations on X.

**Definition 2.4.** [8] Let  $\varphi, \psi \in R(X)$ . Then (i)  $\varphi \subseteq \psi$  if and only if  $\forall x, y \in X, \ \varphi(x, y) \leq \psi(x, y)$ . (ii)  $(\varphi \cup \psi)(x, y) = \varphi(x, y) \lor \psi(x, y)$ . (iii)  $(\varphi \cap \psi)(x, y) = \varphi(x, y) \land \psi(x, y)$ . (iii)  $\varphi^{-1}(x, y) = \varphi(y, x)$ . (v)  $(\varphi \circ \psi)(x, y) = \bigvee_{z \in X} (\varphi(x, z) \land \psi(z, y))$ .

**Definition 2.5.** [6, 7] A fuzzy relation  $\rho$  on X is called a fuzzy equivalence relation on X if

(i)  $\rho(x,x) = 1$  for all  $x \in X$  (reflexive). (ii) $\rho(x,y) = \rho(y,x)$  for all  $x, y \in X$  (symmetric). (iii)  $\rho \circ \rho \leq \rho$  (transitive).

#### 3. Main results

**Definition 3.1.** A fuzzy equivalence relation  $\rho$  on V is called a fuzzy congruence relation on V if

 $\rho(rx+z,sy+t) \geq \rho(x,y) \wedge \rho(z,t) \text{ for all } x,y,z,t \in V \text{ and } r,s \in F$ 

**Example 3.2.** Let  $V = Z_4$  and  $F = \{0, 1\}$ . Then fuzzy relation

$$\rho(x,y) = \begin{cases}
1 & if \ x = y \\
.5 & if \ (x,y) \in \{(2,3), (3,2)\} \\
.3 & otherwise
\end{cases}$$

is a fuzzy equivalence relation on V, but it is not a fuzzy congruence relation on V. Since  $\rho(2+2,3+2) \not\ge \rho(2,3) \land \rho(2,2)$ .

**Lemma 3.3.** Let  $\rho$  be a fuzzy congruence relation on V. Then (i)  $\rho(rx, sy) \ge \rho(x, y)$  for all  $r, s \in F$  and  $\rho(rx, sy) = \rho(x, y)$  for all  $r, s \in F - \{0\}$ . (ii) If  $\rho(x, y) < \rho(z, t)$ , then  $\rho(x, y) = \rho(rx + z, sy + t)$  for all  $r, s \in F - \{0\}$ .

**Theorem 3.4.** Let  $\rho$  be a fuzzy relation on V. Then  $\alpha$  is a fuzzy congruence relation if and only if for all  $t \in [0, 1]$ ,  $\alpha_t = \{x - y \in V | \alpha(x, y) \ge t\}$  is a subspace of V.

**Example 3.5.** Let W be a subspace of V, that  $W \neq V$ . Then

$$\rho(x,y) = \begin{cases} 1 & if \ x - y \in W \\ t & otherwise \end{cases}$$

where  $t \in [0, 1)$ , is a fuzzy congruence relation on V.

**Definition 3.6.** Let  $\alpha$  and  $\beta$  be two fuzzy relations on V. Then the composition  $\alpha \oplus \beta$  is defined as follows:

$$(\alpha \oplus \beta)(x,y) = \bigvee_{\substack{x=ra+sc\\y=kb+ld}} \{\alpha(a,b) \land \beta(c,d)\}(a,b,c,d \in V \text{ and } r,s,k,l \in F)$$

**Theorem 3.7.** Let FR(V) be the family of all fuzzy relations on V. Then  $(FR(V), \oplus)$  is a commutative semigroup.

**Theorem 3.8.** Let FC(V) be the family of all fuzzy congruence relations on V. Then  $(FC(V), \oplus)$  is a commutative semigroup.

**Definition 3.9.** Let  $\rho$  be a fuzzy congruence relation on V. For  $a, b \in V$ , the fuzzy relation  $\rho^{(a,b)}$  on V which is defined by  $\rho^{(a,b)}(x,y) = \rho(a-x,b-y)$ , for all  $x, y \in V$ , is called a fuzzy coset relation determined by a, b and  $\rho$ .

**Lemma 3.10.** Let  $\rho^{(a,b)}$  and  $\rho^{(c,d)}$  be fuzzy coset relations on V. Then

 $\rho^{(a,b)} = \rho^{(c,d)}$  if and only if  $\rho(a-c,b-d) = 1$ 

We define the operations of addition and scalar multiplication on fuzzy coset relations as follows:

$$(\rho^{(a,b)} \oplus \rho^{(c,d)})(x,y) = \bigvee_{\substack{x=x_1+x_2\\y=y_1+y_2}} \{\rho^{(a,b)}(x_1,y_1) \land \rho^{(c,d)}(x_2,y_2)\}$$
$$k\rho^{(a,b)} = \rho^{(ka,kb) \ (1)}$$

**Proposition 3.11.** Let  $\rho^{(a,b)}$  and  $\rho^{(c,d)}$  be two fuzzy coset relations on V. Then  $(\rho^{(a,b)} \oplus \rho^{(c,d)})(x,y) = \rho^{(a+c,b+d)}(x,y)$  (2)

**Theorem 3.12.** Let  $(V \times V)_{\rho} = \{\rho^{(a,b)} | a, b \in V\}$ . Then  $(V \times V)_{\rho}$  is a vector space over F.

**Theorem 3.13.** Let  $\rho$  be a fuzzy congruence relation on V. Then the map  $\varphi$ :  $V \times V \longrightarrow (V \times V)_{\rho}$  defined by  $\varphi(a, b) = \rho^{(a,b)}, \forall a, b \in V$  is an onto linear function with  $Ker\varphi = I_V$ .

**Corollary 3.14.** Let  $\rho$  be a fuzzy congruence relation on V. Then

- $(i) \quad (V \times V)/I_V \cong (V \times V)_{\rho}.$
- (ii)  $\dim(V \times V)/I_V = \dim(V \times V)_{\rho}$ .

### Acknowledgement

The authors would like to thank the referees for their valuable suggestions and comments.

#### References

- [1] P. Das, Fuzzy vector spaces under triangular norms, Fuzzy Sets Syst. 25(1988) 73-85
- [2] J. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl, 18:145-174, 1967.
- [3] A. K. Katsaras, D. B. Liu, Fuzzy vector spaces and fuzzy topological vector spaces, J. Math. Anal. Appl. 58 135-146 (1977).
- [4] R. Kumar, Fuzzy vector spaces and fuzzy cosets, Fuzzy Sets Syst. 45 (1992) 109-116.

- [5] N. P. Mukherjee Fuzzy normal subgroups and fuzzy cosets, Information Sciences 34, 225-239 (1984)
- [6] V. Murali, Fuzzy equivalence relations, Fuzzy Sets and Systems, 30 (2)(1989)155-163.
- [7] V. Murali, Fuzzy congruences, Fuzzy Sets and Systems, 41(3) (1991) 359-369.
- [8] M. Samhan, Fuzzy congruence in semigroups, Inform. Sci. 74 (1993) 165-175. Systems, 95(2) (1998) 249-253.
- Y. Tan, Fuzzy congruences on a regular semigroup, Fuzzy Sets and Systems, 117(3) (2001)399-408.
- [10] L. A. Zadeh, Fuzzy sets. Information and Control, 8 (1965) 338-353.

Department of Mathematics, Fasa University, Fasa, Iran  $E\text{-}mail\ address:$  khosravi.shoar@fasau.ac.ir

86



The 9<sup>th</sup> Seminar on Linear Algebra and its Applications July 5-6, 2017, University of Tabriz, Tabriz, Iran

#### CONVEXITY IN MATRIX ALGEBRAS

MOHSEN KIAN

ABSTRACT. We have a glimpse into the notion of convexity in matrix algebras. Convexity of matrix sets and matrix functions and the relations between them have been investigated.

Keywords: Matrix convex set; Matrix convex function.

#### 1. INTRODUCTION

Assume that  $\mathbb{M}_n$  is the algebra of all  $n \times n$  matrices with complex entries and I is the identity matrix. A Hermitian matrix X is said to be positive (denoted by  $X \ge 0$ ) if all of its eigenvalues are non-negative. For two Hermitian matrix X, Y, we say that  $X \leq Y$  whenever  $Y - X \geq 0$ .

A Hermitian matrix X enjoys the spectral representation, say

$$X = \sum_{j=1}^{n} \lambda_j P_j,$$

where  $\lambda_j$ 's are eigenvalues of X and  $P_j$ 's are projections with  $\sum_{j=1}^n P_j = I$ . If f is a real-valued continuous function whose domain contains  $\lambda_j$   $(j = 1, \dots, n)$ , then the Hermitian matrix f(X) can be defined as  $f(X) = \sum_{j=1}^{n} f(\lambda_j) P_j$ . Let  $(E, \langle \cdot, \cdot \rangle_k)$  be a left Hilbert  $\mathbb{M}_k$ -module and  $(E, \langle \cdot, \cdot \rangle_n)$  be a right Hilbert

 $\mathbb{M}_n$ -module satisfying

$$\langle x, y \rangle_k z = x \langle y, z \rangle_n, \qquad (x, y, z \in E).$$

Then E is called a Hilbert  $\mathbb{M}_k$ - $\mathbb{M}_n$ -bimodule.

### 2. Main results

A subset  $\mathcal{K}$  of  $\mathbb{M}_n$  is called matrix convex if  $X_i \in \mathcal{K}, C_i \in \mathbb{M}_n$   $(i = 1, \dots, m)$ and  $\sum_{i=1}^{m} C_i^* C_i = I$  implies that  $\sum_{i=1}^{m} C_i^* X_i C_i \in \mathcal{K}$ . This can be regarded as a non-commutative convexity. For  $t \in [0, 1]$ , with

 $C_1 = \sqrt{t}$  and  $C_2 = \sqrt{1-t}$ , it is evident that every matrix convex set is convex in the usual sense. However, the converse is not valid in general. For example, a single point set  $\{A\}$  is convex, while it is not matrix convex unless  $A = \alpha I$  for some  $\alpha \in \mathbb{C}$ . Typical examples of matrix convex sets are

- (1)  $\{T \in \mathbb{M}_n : 0 \le T \le I\};$
- (2)  $\{T \in \mathbb{M}_n; \|T\| \le M\}$  for a fix scalar M > 0;
- (3)  $\{T \in \mathbb{M}_n : \omega(T) \leq r\}$ , where  $\omega(T)$  is the numerical radius of T.

<sup>2010</sup> Mathematics Subject Classification. 47A30, 47A63. Speaker: Mohsen Kian.

A continuous function  $f: J \to \mathbb{R}$  is said to be matrix convex if

$$f(tX + (1-t)Y) \le tf(X) + (1-t)f(Y)$$

for all Hermitian matrices X, Y with eigenvalues in J and all  $t \in [0, 1]$ . f is called matrix concave if -f is matrix convex. The so-called Löwner-Heinz theorem asserts that the power function  $f(x) = x^p$  is matrix convex (concave) if and only if  $p \in [-1, 0] \cup [1, 2]$  ( $p \in [0, 1]$ ). For a continuous real function  $f : J \to \mathbb{R}$ , we define the matrix epigraph of f by

$$\operatorname{ME}(f) := \{ (X, Y) \in sp(J) \times \mathbb{M}_n; \ f(X) \le Y \}.$$

The matrix convex sets and matrix convex functions are related via the following theorem.

**Theorem 2.1.** A continuous function  $f : J \to \mathbb{R}$  is matrix convex if and only if ME(f) is matrix convex.

Moreover, the next theorem gives a way of constructing matrix convex sets.

**Theorem 2.2.** Assume that f, g > 0 are matrix concave functions on (0, 1). The set

$$\mathcal{U}_{f,g} = \left\{ X; \quad \left[ \begin{array}{cc} f(A) & X \\ X^* & g(A) \end{array} \right] \ge 0, \quad for \ some \ \ 0 \le A \le I \right\}$$

is matrix convex.

Another notion of convexity is the absolute convexity of sets in bimodules.

**Definition 2.3.** Let E be a Hilbert  $\mathbb{M}_k$ - $\mathbb{M}_n$ -bimodule. A subset  $\mathcal{K}$  of E is called  $\mathbb{M}_k$ - $\mathbb{M}_n$ -convex if

$$\sum_{i=1}^{n} A_i A_i^* = I_k, \quad \sum_{i=1}^{n} B_i^* B_i = I_n \implies \sum_{i=1}^{n} A_i X_i B_i \in \mathcal{K}$$

for all  $A_i \in \mathbb{M}_k$ ,  $B_i \in \mathbb{M}_n$  and  $X_i \in \mathcal{K}$ .

**Example 2.4.** Let  $\Gamma$  be an index set. Define  $\Omega$  to be the set

$$\Omega = \left\{ (X_{\alpha})_{\alpha \in \Gamma} \middle| X_{\alpha} \in \mathbb{M}_n, \sum_{\alpha \in \Gamma} X_{\alpha}^* X_{\alpha} \text{ norm converges in } \mathbb{M}_n \right\}.$$

Then  $\Omega$  is a Hilbert  $\mathbb{M}_n$ - $\mathbb{M}_n$ -bimodule. If r is a positive real number, then the subset S of  $\Omega$  defined by

$$\mathcal{S} = \{ (X_{\alpha})_{\alpha \in \Gamma} \in \Omega \mid 0 \le X_{\alpha}^* X_{\alpha} \le r, \ \alpha \in \Gamma \}$$

is  $\mathbb{M}_n$ - $\mathbb{M}_n$ -convex.

**Example 2.5.** Consider  $\mathcal{M}_2(\mathbb{C})$  as a  $\mathcal{M}_2(\mathbb{C})$ - $\mathcal{M}_2(\mathbb{C})$ -bimodule. Let  $\alpha$  be a fixed scalar and I be the identity matrix. It is clear that the set  $\mathcal{S} = \{\alpha I\}$  is a  $C^*$ -convex subset of  $\mathcal{M}_2(\mathbb{C})$ . However, it is not  $\mathcal{M}_2(\mathbb{C})$ - $\mathcal{M}_2(\mathbb{C})$ -convex. Put

$$A = \begin{pmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix} \quad and \quad B = \begin{pmatrix} \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix}.$$

Then  $AA^* = I = B^*B$ , while  $A(\alpha I)B = \alpha AB \notin S$ .

**Theorem 2.6.** Consider  $\mathbb{M}_{n,k}$  as a  $\mathbb{M}_k$ - $\mathbb{M}_n$ -bimodule. If M is a positive scalar, then the set  $\mathcal{K} := \{T \in \mathbb{M}_{n,k}, \|T\| \leq M\}$  is  $\mathbb{M}_k$ - $\mathbb{M}_n$ -convex.

#### References

- [1] M. Kian,  $C^{\ast}\mbox{-}\mbox{Convexity of Norm Unit Balls, J. Math. Anal. Appl., 445 (2017), 1417-1427.$
- [2] M. Kian, Epigraph of Operator Functions, Quaes. Math. 39 (2016), 587–594.
- [3] M. Kian and M. Dehghani, A noncommutative convexity in C\*-bimodules, Surv. Math. Appl., 12 (2017), 7–21.
- [4] P. B. Morenz, The structure of C\*-convex sets, Canad. Math. J. Math. 46 (1994), 1007–1026.

Department of Mathematics, University of Bojnord, Bojnord, Iran  $E\text{-}mail \ address: kian@ub.ac.ir$ 



#### LINEAR PRESERVERS OF SLT-MAJORIZATION ON $\mathbb{R}^n$

AHMAD MOHAMMADHASANI AND ASMA ILKHANIZADEH MANESH

ABSTRACT. A matrix R is called *row substochastic* if it has nonnegative entries, and the sum of entries on every row of R is less than or equal to one. For  $x, y \in \mathbb{R}^n$ , it is said that x is *slt-majorized* by y (denoted by  $x \prec_{slt} y$ ) if there exists an *n*-by-n lower triangular row substochastic matrix R such that x = Ry. Here, the structure of all linear functions  $T : \mathbb{R}^n \to \mathbb{R}^n$  preserving (resp. strongly preserving) slt-majorization with additional condition  $Te_n \neq 0$  (resp. with no condition) will be characterized.

Keywords: Row substochastic; SLT-majorization; (Strong) linear preservers.

#### 1. INTRODUCTION

The study of linear preserver problems is one of the action research topics in matrix theory and linear algebra. See the references. In this article, we work on some kind of majorization and characterize all linear functions  $T : \mathbb{R}^n \to \mathbb{R}^n$  preserving slt-majorization with additional condition  $Te_n \neq 0$  and strong preserving slt-majorization.

The following will be fixed throughout the article.  $\mathbb{R}^n$  denotes the set of all n-by-1 real vectors.  $\{e_1, \ldots, e_n\}$  denotes the standard basis of  $\mathbb{R}^n$ .  $\mathbb{N}_k$  denotes the set  $\{1, \ldots, k\} \subset \mathbb{N}$ .  $A^t$  denotes the transpose of a given matrix  $A \in \mathbb{R}^n$ .  $P_n$  denotes the backward identity. [T] denotes the matrix representation of a linear function  $T : \mathbb{R}^n \to \mathbb{R}^n$  with respect to the standard basis.  $\mathcal{C}(\mathcal{A})$  denotes the set  $\{\sum_{i=1}^m \lambda_i a_i \mid m \in \mathbb{N}, \ \lambda_i \geq 0, \ \sum_{i=1}^m \lambda_i \leq 1, \ a_i \in A, \ \forall i \in \mathbb{N}_m\}$ , where  $A \subseteq \mathbb{R}^n$ .  $x^{\downarrow} = (x_1^{\downarrow}, \ldots, x_n^{\downarrow})^t$  denotes the decreasing rearrangement of a vector  $x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$ . This means  $x_1 \geq \ldots \geq x_n$ .  $x^{\uparrow} = (x_1^{\uparrow}, \ldots, x_n^{\uparrow})^t$  denotes the increasing rearrangement of a vector  $x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$ . This means  $x_1 \leq \ldots \leq x_n$ .

Let  $\sim$  be a relation on  $\mathbb{R}^n$ . A linear function  $T : \mathbb{R}^n \to \mathbb{R}^n$  is said to be a linear preserver (or strong linear preserver) of  $\sim$ , if  $Tx \sim Ty$  whenever  $x \sim y$  (or  $Tx \sim Ty$  if and only if  $x \sim y$ ).

For  $x, y \in \mathbb{R}^n$ , it is said that x is *sut-majorized* by y (denoted by  $x \prec_{sut} y$ ) if there exists an *n*-by-*n* upper triangular row substochastic matrix R such that x = Ry.

In [2], the author found the (strong) linear preservers of  $\prec_{sut}$  on  $\mathbb{R}^n$  with additional condition  $Te_1 \neq 0$  (with no condition), respectively, as follows.

<sup>2010</sup> Mathematics Subject Classification. Primary 15A04,15A21; Secondary 15A51. Speaker: Ahmad Mohammadhasani.

**Theorem 1.1.** Let  $T : \mathbb{R}^n \to \mathbb{R}^n$  be a linear function. Assume that  $[T] = [a_{ij}]$ and  $Te_1 \neq 0$ . Then T preserves  $\prec_{sut}$  if and only if

$$[T] = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0\\ 0 & a_{22} & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix},$$

and the vector  $(0, a_{11}, \ldots, a_{nn})^t$  is monotone.

**Theorem 1.2.** A linear function  $T : \mathbb{R}^n \to \mathbb{R}^n$  strongly preserves  $\prec_{sut}$  if and only if  $[T] = \alpha I_n$ , for some  $\alpha \in \mathbb{R} \setminus \{0\}$ .

We wish to find all (strong) linear preservers of slt-majorization on  $\mathbb{R}^n$ .

### 2. Main results

In this section, we focus on the lower triangular row substochastic matrices and introduce a new type of majorization. Then we characterize all linear functions T:  $\mathbb{R}^n \to \mathbb{R}^n$  (strongly) preserving  $\prec_{slt}$ .

**Definition 2.1.** Let  $x, y \in \mathbb{R}^n$ . We say that x slt-majorized by y (in symbol  $x \prec_{slt} y$ ) if x = Ry, for some lower triangular row substochastic  $R \in \mathbb{R}^n$ .

**Proposition 2.2.** Let  $x, y \in \mathbb{R}^n$ . Then  $x \prec_{sut} y$  if and only if  $P_n x \prec_{slt} P_n y$ . Also,  $P_n x \prec_{sut} P_n y$  if and only if  $x \prec_{slt} y$ .

*Proof.* First, suppose that  $x \prec_{sut} y$ . So x = Ry, for some upper triangular row substochastic  $R \in \mathbb{R}^n$ . It implies that  $P_n x = (P_n R P_n)(P_n y)$ . As  $P_n R P_n$  is a lower triangular row substochastic, we see that  $P_n x \prec_{slt} P_n y$ .

Next, assume that  $P_n x \prec_{slt} P_n y$ . This ensures that there exists some lower triangular row substochastic matrix R such that  $P_n x = R(P_n y)$ , and then  $x = (P_n R P_n)y$ . Thus,  $x \prec_{slt} y$ .

Now,  $P_n x \prec_{sut} P_n y$  if and only if  $P_n(P_n x) \prec_{slt} P_n(P_n y)$  if and only if  $x \prec_{slt} y$ .

For continuing we introduce the following function. Assume that  $T : \mathbb{R}^n \to \mathbb{R}^n$  be a linear function. Define  $\tau : \mathbb{R}^n \to \mathbb{R}^n$  by  $\tau(x) = P_n T(P_n x)$ .

This proposition provides a criterion for slt-majorization on  $\mathbb{R}^n$ .

**Proposition 2.3.** Let  $x = (x_1, \ldots, x_n)^t$ ,  $y = (y_1, \ldots, y_n)^t \in \mathbb{R}^n$ . Then  $x \prec_{slt} y$  if and only if  $x_i \in \mathcal{C}\{y_1, \ldots, y_i\}$ , for all  $i \in \mathbb{N}_n$ .

*Proof.* Proposition 2.2 ensures that  $x \prec_{slt} y$  if and only if  $P_n x \prec_{sut} P_n y$ . By applying [2], Proposition 2.3,  $P_n x \prec_{sut} P_n y$  is equivalent to  $x_i \in \mathcal{C}\{y_1, \ldots, y_i\}$ , for all  $i \in \mathbb{N}_n$ .

**Proposition 2.4.** Let  $T : \mathbb{R}^n \to \mathbb{R}^n$  be a linear function. Then T preserves  $\prec_{sut}$  if and only if  $\tau$  preserves  $\prec_{slt}$ .

Also, T preserves  $\prec_{slt}$  if and only if  $\tau$  preserves  $\prec_{sut}$ .

*Proof.* If T preserves  $\prec_{sut}$ ; Let  $x, y \in \mathbb{R}^n$ , and let  $x \prec_{slt} y$ . Proposition 2.2 shows that  $P_n x \prec_{slt} P_n y$ , and then  $T(P_n x) \prec_{sut} T(P_n y)$ . So  $\tau(x) \prec_{slt} \tau(y)$ . Therefore,  $\tau$  preserves  $\prec_{slt}$ . One can prove the rest of the proof easily.

The following theorem characterizes structure of the linear functions  $T : \mathbb{R}^n \to \mathbb{R}^n$  preserving slt-majorization with additional condition  $Te_n \neq 0$ . Note that the vector  $x = (x_1, \ldots, x_n)^t$  is monotone if  $x = (x_1^{\uparrow}, \ldots, x_n^{\uparrow})^t$  or  $x = (x_1^{\downarrow}, \ldots, x_n^{\downarrow})^t$ .

**Theorem 2.5.** Let  $T : \mathbb{R}^n \to \mathbb{R}^n$  be a linear function. Assume that  $[T] = [a_{ij}]$ and  $Te_n \neq 0$ . Then T preserves  $\prec_{slt}$  if and only if

$$[T] = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0\\ 0 & a_{22} & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix}$$

and the vector  $(0, a_{nn}, \ldots, a_{11})^t$  is monotone.

*Proof.* T preserves  $\prec_{slt}$ , with the condition  $Te_n \neq 0$ , if and only if  $\tau$  preserves  $\prec_{sut}$ , with the condition  $\tau e_1 \neq 0$ , if and only if, because of Theorem 1.1,

$$[\tau] = \begin{pmatrix} a_{nn} & 0 & 0 & \dots & 0\\ 0 & a_{n-1n-1} & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0 & a_{11} \end{pmatrix},$$

and the vector  $(0, a_{nn}, \ldots, a_{11})^t$  is monotone, if and only if, as  $[\tau] = P_n[T]P_n$ ,

$$[T] = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix}$$

and the vector  $(0, a_{nn}, \ldots, a_{11})^t$  is monotone.

**Lemma 2.6.** Let  $T : \mathbb{R}^n \to \mathbb{R}^n$  be a linear function that strongly preserves  $\prec_{slt}$ . Then T is invertible.

**Theorem 2.7.** A linear function  $T : \mathbb{R}^n \to \mathbb{R}^n$  strongly preserves  $\prec_{slt}$  if and only if  $[T] = \alpha I_n$ , for some  $\alpha \in \mathbb{R} \setminus \{0\}$ .

*Proof.* T strongly preserves  $\prec_{slt}$  if and only if T and  $T^{-1}$  preserve  $\prec_{slt}$  if and only if  $\tau$  and  $\tau^{-1}$  preserve  $\prec_{sut}$  if and only if  $\tau$  strongly preserves  $\prec_{sut}$  if and only if, because of Theorem 1.2,  $[\tau] = \alpha I_n$ , for some  $\alpha \in \mathbb{R} \setminus \{0\}$  if and only if  $[T] = \alpha I_n$ , for some  $\alpha \in \mathbb{R} \setminus \{0\}$ .

#### References

- A.M. Hasani and M. Radjabalipour, On linear preservers of (right) matrix majorization, Linear Algebra and its Applications, 423 (2007) 255-261.
- [2] A. Ilkhanizadeh Manesh, Linear functions preserving sut-majorization on  $\mathbb{R}^n$ , Iranian Journal of Mathematical Sciences and Informatics, **11(2)** 111-118 (2016).
- [3] A. Ilkhanizadeh Manesh, Right gut-Majorization on M<sub>n,m</sub>, Electronic Journal of Linear Algebra, **31(1)** :13-26, (2016).

Department of Mathematics, Sirjan University of technology, Sirjan, Iran.  $E\text{-}mail\ address:$  a.mohammadhasani53@gmail.com

DEPARTMENT OF MATHEMATICS, VALI-E-ASR UNIVERSITY OF RAFSANJAN, P.O. BOX: 7713936417, RAFSANJAN, IRAN.

E-mail address: a.ilkhani@vru.ac.ir

92



#### EDMS, RSR-MAJORIZATION, AND RP-MAJORIZATION

AHMAD MOHAMMADHASANI AND ASMA ILKHANIZADEH MANESH

ABSTRACT. Let  $D_1$  and  $D_2$  be two Euclidean distance matrices (EDMs) with corresponding positive semidefinite matrices  $B_1$  and  $B_2$  respectively. Suppose that  $\lambda(A) = (\lambda_i(A))_{i=1}^n$  is the vector of eigenvalues of a matrix A such that  $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$ . Here the relation between the eigenvalues of EDMs and those of the corresponding positive semidefinite matrices respect to  $\prec_{rsr}$  and  $\prec_{rp}$  on  $\mathbb{R}_2$  will be investigated.

Keywords: Euclidean distance matrices; Rp-majorization; Rsr-majorization.

#### 1. INTRODUCTION

An  $n \times n$  nonnegative and symmetric matrix  $D = (d_{ij}^2)$  with zero diagonal elements is called a predistance matrix. A predistance matrix D is called Euclidean or a Euclidean distance matrix (EDM) if there exist a positive integer r and a set of n points  $\{p_1, \ldots, p_n\}$  such that  $p_1, \ldots, p_n \in \mathbb{R}^r$  and  $d_{ij}^2 = \|p_i - p_j\|^2$   $(i, j = 1, \ldots, n)$ , where  $\|.\|$  denotes the usual Euclidean norm.

The smallest value of r that satisfies the above condition is called the embedding dimension. As is well known, a predistance matrix D is Euclidean if and only if the matrix  $B = \frac{-1}{2}PDP$  with  $P = I_n - \frac{1}{n}ee^t$ , where  $I_n$  is the  $n \times n$  identity matrix, and e is the vector of all ones, which is positive semidefinite matrix.

Let  $\Lambda_n$  be the set of  $n \times n$  EDMs, and  $\Omega_n(e)$  be the set of  $n \times n$  positive semidefinite matrices B such that Be = 0. Then the linear mapping  $\tau : \Lambda_n \to \Omega_n(e)$  defined by  $\tau(D) = \frac{-1}{2}PDP$  is invertible, and its inverse mapping, say  $\kappa : \Omega_n(e) \to \Lambda_n$ is given by  $\kappa(B) = be^t + eb^t - 2B$  with b = diag(B), where diag(B) is the vector consisting of the diagonal elements of B.

For general reference on this topic see, e.g., [1]. One can see some types of majorization in [2]-[5].

In this paper, the relation between the eigenvalues of EDMs and those of the corresponding positive semidefinite matrices respect to  $\prec_{rsr}$  and  $\prec_{rp}$  on  $\mathbb{R}_2$  will be investigated.

An nonnegative matrix R is called row stochastic if the sum of entries of each row of R is equal to one. A matrix is said to be doubly substochastic if it has nonnegative components and each row and each column sum is at most 1.

For  $x, y \in \mathbb{R}_n$ , we say that x rsr-majorized by y (write as  $x \prec_{rsr} y$ ) if for some symmetric row stochastic matrix R with all its main diagonal entries equal, x = yR. For  $x, y \in \mathbb{R}_n$ , it is said that x rp-majorized by y and denoted by  $x \prec_{rp} y$  if there exists a predistance doubly substochastic matrix P such that x = yP.

<sup>2010</sup> Mathematics Subject Classification. Primary 15A04,15A21; Secondary 15A51. Speaker: Ahmad Mohammadhasani.

The following notation will be fixed throughout the paper.

The summation of all components of a vector x is denoted by tr(x).

The set  $\{1, \ldots, k\} \subset \mathbb{N}$  is denoted by  $\mathbb{N}_k$ .

The collection of all  $n \times n$  symmetric row stochastic matrices with all its main diagonal entries equal is denoted by  $\mathcal{R}_n^{sr}$ .

 $\mathcal{D}_n^{ps}$  for the collection of all  $n \times n$  predistance doubly substochastic matrices.

The set  $\{\sum_{i=1}^{m} \lambda_i a_i \mid m \in \mathbb{N}, \lambda_i \ge 0, \sum_{i=1}^{m} \lambda_i = 1, a_i \in A, i \in \mathbb{N}_m\}$ , where  $A \subseteq \mathbb{R}_n$ , is denoted by Co(A).

## 2. Main results

The following proposition gives an equivalent condition for rsr-majorization on  $\mathbb{R}_2$ .

**Proposition 2.1.** Let  $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}_2$ . Then  $x \prec_{rsr} y$  if and only if  $x_i \in Co\{y_1, y_2\}$   $(i \in \mathbb{N}_2)$  and tr(x) = tr(y).

*Proof.* First, suppose that  $x \prec_{rsr} y$ . Then there exists  $R \in \mathcal{R}_2^{sr}$  such that x = yR. We see that  $R = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ , for some  $a, b \ge 0$  such that a + b = 1. It is seen that  $x_i \in Co\{y_1, y_2\}$   $(i \in \mathbb{N}_2)$  and tr(x) = tr(y).

Next, assume that  $x_i \in Co\{y_1, y_2\}$   $(i \in \mathbb{N}_2)$  and tr(x) = tr(y). So  $x_1 = \alpha y_1 + (1-\alpha)y_2$  and  $x_2 = \beta y_1 + (1-\beta)y_2$ , for some  $0 \le \alpha, \beta \le 1$ . Since tr(x) = tr(y), we deduce  $(1-\alpha-\beta)(y_1-y_2) = 0$ . If  $y_1 \ne y_2$ ; Then  $\alpha + \beta = 1$ , and put  $R = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$ . It is clear that  $R \in \mathcal{R}_2^{sr}$  and x = yR. Therefore,  $x \prec_{rsr} y$ . If  $y_1 = y_2$ ; Put  $R = I_2$  and see x = yR. Hence  $x \prec_{rsr} y$ .

The following proposition provides a criterion for rp-majorization on  $\mathbb{R}_2$ .

**Proposition 2.2.** Let  $x = (x_1, x_2)$ ,  $y = (y_1, y_2) \in \mathbb{R}_2$ . Then  $x \prec_{rp} y$  if and only if  $x_1 \in Co\{0, y_2\}$ ,  $x_2 \in Co\{0, y_1\}$ , and  $x_1y_1 = x_2y_2$ .

*Proof.* If  $x \prec_{rp} y$ , then there exists  $P \in \mathcal{D}_2^{ps}$  such that x = yP. We observe that  $P = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$ , for some  $0 \leq \alpha \leq 1$ . So  $x_1 \in Co\{0, y_2\}, x_2 \in Co\{0, y_1\}$ , and  $x_1y_1 = x_2y_2$ .

Next, assume that  $x_1 \in Co\{0, y_2\}$ ,  $x_2 \in Co\{0, y_1\}$ , and  $x_1y_1 = x_2y_2$ . Then there exist  $\alpha, \beta$   $(0 \le \alpha, \beta \le 1)$  such that  $x_1 = \alpha y_2$  and  $x_2 = \beta y_1$ . If  $y_1y_2 \ne 0$ , as  $x_1y_1 = x_2y_2$ , then  $\alpha = \beta$ . Put  $P = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$ . If  $y_1 = 0$ , then set  $P = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$ , and if  $y_2 = 0$ , then put  $P = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}$ . We see in each case  $p \in \mathcal{D}_2^{ps}$  and x = yP. So  $x \prec_p y$ .

Till the end of this paper, the relation between the eigenvalues of EDMs and those of the corresponding positive semidefinite matrices respect to  $\prec_{rsr}$  and  $\prec_{rp}$  on  $\mathbb{R}_2$  will be specify.

**Theorem 2.3.** Let  $B, \widetilde{B} \in \Omega_2(e)$ , and let  $D = \kappa(B)$  and  $\widetilde{D} = \kappa(\widetilde{B})$ . Then

- (a)  $\lambda(B) \prec_{rsr} \lambda(\widetilde{B}) \Rightarrow \lambda(D) \prec_{rsr} \lambda(\widetilde{D}), \text{ but } \lambda(D) \prec_{rsr} \lambda(\widetilde{D}) \neq \lambda(B) \prec_{rsr} \lambda(\widetilde{B});$
- (b)  $\lambda(B) \prec_{rp} \lambda(\widetilde{B}) \Leftrightarrow \lambda(D) \prec_{rp} \lambda(\widetilde{D}).$

94

Proof. Since  $B, \widetilde{B} \in \Omega_2(e)$ , there exist  $\alpha, \beta \geq 0$  such that  $B = \begin{pmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{pmatrix}$ , and  $\widetilde{B} = \begin{pmatrix} \beta & -\beta \\ -\beta & \beta \end{pmatrix}$ , and  $\{0, 2\alpha\}$  and  $\{0, 2\beta\}$  are the set of eigenvalues of B and  $\widetilde{B}$ , respectively. By the definition of  $\kappa$ ,  $D = \begin{pmatrix} 0 & 4\alpha \\ 4\alpha & 0 \end{pmatrix}$  and  $\widetilde{D} = \begin{pmatrix} 0 & 4\beta \\ 4\beta & 0 \end{pmatrix}$ . So  $\{-4\alpha, 4\alpha\}$  and  $\{-4\beta, 4\beta\}$  are the set of eigenvalues of D and  $\widetilde{D}$ , respectively.

(a) : Suppose that  $\lambda(B) \prec_{rsr} \lambda(\widetilde{B})$ . Then, by Proposition 2.1,  $\alpha = \beta$ , that is,  $\lambda(B) = \lambda(\widetilde{B})$ . So, since  $\lambda(D) \prec_{rsr} \lambda(\widetilde{D})$  if and only if  $\alpha \leq \beta$ , we can conclude that  $\lambda(D) \prec_{rsr} \lambda(\widetilde{D})$  if  $\lambda(B) \prec_{rsr} \lambda(\widetilde{B})$ .

 $\lambda(D) \prec_{rsr} \lambda(\widetilde{D}) \text{ if } \lambda(B) \prec_{rsr} \lambda(\widetilde{B}).$ For the next part, consider  $B = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}$  and  $\widetilde{B} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ . Hence  $D = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ and  $\widetilde{D} = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$ . We see  $\lambda(D) \prec_{rsr} \lambda(\widetilde{D})$ , but  $\lambda(B) \not\prec_{rsr} \lambda(\widetilde{B})$ .

(b) : By Proposition 2.2,  $\lambda(B) \prec_{rp} \lambda(\widetilde{B})$  if and only if  $\alpha = 0$  if and only if  $\lambda(D) \prec_{rp} \lambda(\widetilde{D})$ .

## References

- A. Y. Alfakih, On the eigenvalues of Euclidean distance matrices, Journal of Computational and Applied Mathematics, 27 (2008) 237-250.
- [2] A. Armandnejad and A. Ilkhanizadeh Manesh, Gut-majorization on  $\mathbf{M}_{n,m}$  and its linear preservers, Electronic Journal of Linear Algebra, **23** (2012) 646-654.
- [3] A. M. Hasani and M. Radjabalipour, The structure of linear operators strongly preserving majorizations of matrices, Electronic Journal of Linear Algebra, 15 (2006) 260-268.
- [4] A. Ilkhanizadeh Manesh, Right gut-Majorization on  $\mathbf{M}_{n,m}$ , Electronic Journal of Linear Algebra, **31** (2016) 646-654.
- [5] A. Ilkhanizadeh Manesh and A. Armandnejad, Ut-Majorization on R<sup>n</sup> and its Linear Preservers, Operator Theory: Advances and Applications, 242 (2014) 253-259.

Department of Mathematics, Sirjan University of technology, Sirjan, Iran. E-mail address: a.mohammadhasani53@gmail.com

DEPARTMENT OF MATHEMATICS, VALI-E-ASR UNIVERSITY OF RAFSANJAN, P.O. BOX: 7713936417, RAFSANJAN, IRAN.

E-mail address: a.ilkhani@vru.ac.ir



## STABILITY ANALYSIS OF HILFER FRACTIONAL ORDER DIFFERENTIAL SYSTEMS WITH TIME DELAY

#### VAHID MOHAMMADNEZHAD, MOSTAFA ESLAMI, AND HADI REZAZADEH

ABSTRACT. In this paper, we first state some sufficient conditions for asymptotic stability of the linear Hilfer fractional delay differential systems . Regarding this result, we will derive asymptotic stability for the non-linear Hilfer fractional delay system. Finally, we give one example of Hilfer fractional Van der Pol's Oscillator system with delay to illustrate the effectiveness of our theoretical results.

Keywords: Asymptotic stability; Hilfer fractional derivative; Van der Pol's Oscillator.

#### 1. INTRODUCTION

In the last decades, many researchers have more interests in the fractional stability of linear systems with time delay on the basis of real problems. For example, Deng et al. examined the stability of linear fractional differential equations with time delays in 2007 and presented results which confirmed determined the asymptotically stable region of the system [1]. Similarly, this article has introduced linear Hilfer fractional delay differential systems and has the evaluated the asymptotic stability of this type of fractional differential system using Laplace transform property. Hilfer fractional derivative is an interpolation from Riemann-Liouville fractional derivative and Caputo fractional derivative which its concept initially appeared in the theoretical modeling of broadband dielectric relaxation spectroscopy for glasses [2]. Recently new results have been obtained for this type of derivative. For example, Furati et al. proved existence and uniqueness of global solutions in the space of weighted continuous functions for a class of nonlinear fractional differential equations involving Hilfer fractional derivative. In [3], exact solution of the space-time fractional diffusion equations with Hilfer fractional derivative is expressed in terms of the Mittag-Leffler function and Fax H-function. In addition, Rezazadeh et al. studied the stability and asymptotic stability of the zero solution of the linear Hilfer fractional differential system by using the Laplace transform, the asymptotic expansion of the Mittag-Leffler function and the Gronwall inequality [4].

**Definition 1.1.** The Hilfer fractional derivative of order  $\alpha$  and type  $\beta$  for an absolutely integrable function f(t) is defined by [2] (1.1)

$$\left(_{0^{+}}D_{t}^{\alpha,\beta}f\right)(t) = \left(_{0^{+}}I_{t}^{\beta(1-\alpha)}\frac{d}{dt}_{0^{+}}I_{t}^{(1-\beta)(1-\alpha)}f\right)(t), \ 0 < \alpha < 1, \ 0 \le \beta \le 1.$$

2010 Mathematics Subject Classification. 34M03, 35B35.

Speaker: Vahid Mohammadnezhad.

**Lemma 1.2.** If  $f(t) \in AC[0,b]$  the Hilfer fractional derivative (1.1) coincides with the Caputo fractional derivative of order  $\alpha \in (0,1)$ , and the regularized Hilfer fractional derivative can be written as

(1.2) 
$$\binom{C}{0^+} D_t^{\alpha} f(t) = \left( {}_{0^+} D_t^{\alpha,\beta} f(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(0^+) \right).$$

# 2. Stability analysis of linear Hilfer fractional delay differential system

In this Section, based on the Laplace transform and the Final Value Theorem, we present the sufficient conditions of asymptotic stability for the linear Hilfer fractional delay differential system as follows (2.1)

$$\begin{cases} {}_{0^{+}}D_{t}^{\alpha_{1},\beta}x_{1}(t) = a_{11}x_{1}(t) + \dots + a_{1n}x_{n}(t) + b_{11}x_{1}(t-\tau) + \dots + b_{1n}x_{n}(t-\tau), \\ {}_{0^{+}}D_{t}^{\alpha_{2},\beta}x_{2}(t) = a_{21}x_{1}(t) + \dots + a_{2n}x_{n}(t) + b_{21}x_{1}(t-\tau) + \dots + b_{2n}x_{n}(t-\tau), \\ {}_{\vdots} \\ {}_{0^{+}}D_{t}^{\alpha_{n},\beta}x_{n}(t) = a_{n1}x_{1}(t) + \dots + a_{nn}x_{n}(t) + b_{n1}x_{1}(t-\tau) + \dots + b_{nn}x_{n}(t-\tau) \end{cases}$$

the initial values is given by

 ${}_{0^+}I_t^{1-\gamma_i}x_i(t) = \varphi_i(t), \quad \gamma_i = \alpha_i + \beta - \alpha_i\beta, \ 0 < \alpha_i < 1, \quad 0 \le \beta \le 1, \quad 1 \le i \le n,$ and, for subsequent discussion, we shall set

$$_{0^+}I_t^{1-\gamma_i}x_i(0^+) = x_{i0}, \quad 1 \le i \le n.$$

In this system,  $A = (a_{ij})_{n \times n}$ ,  $B = (b_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$  are the system matrices,  $x_i(t), x_i(t-\tau) \in \mathbb{R}$  denote the state vectors and  $\varphi_i(t) \in C^0[-\tau, 0]$  are initial values.

**Definition 2.1.** The zero solution of linear Hilfer fractional differential system (2.1) is said to be stable if for any initial value  $x_0$ , there exists an  $\epsilon > 0$  such that  $||x(t)|| \le \epsilon$  for all t > 0. The zero solution is said to be asymptotically stable if, in addition to being stable,  $||x(t)|| \to 0$  as  $t \to \infty$ .

Now, we express the main theorem for checking the stability of system (2.1) by applying the Laplace transforms on both sides of this system.

## Theorem 2.2. If all roots of

$$det \begin{pmatrix} s^{\alpha_1} - a_{11} - b_{11}e^{-s\tau} & -a_{12} - b_{12}e^{-s\tau} & \dots & -a_{1n} - b_{1n}e^{-s\tau} \\ -a_{21} - b_{21}e^{-s\tau} & s^{\alpha_2} - a_{22} - b_{22}e^{-s\tau} & \dots & -a_{2n} - b_{2n}e^{-s\tau} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} - b_{n1}e^{-s\tau} & -a_{n2} - b_{n2}e^{-s\tau} & \dots & s^{\alpha_n} - a_{nn} - b_{nn}e^{-s\tau} \end{pmatrix} = 0,$$

have negative real parts, then the zero solution system of (2.1) is asymptotically stable.

Now, we will discuss the stability of a non-linear Hilfer fractional delay differential system, which can be described by

(2.2) 
$$_{0+}D_t^{\boldsymbol{\alpha},\beta}X(t) = F(X(t), X(t-\tau)), \quad t > 0, \quad 0 \le \beta \le 1,$$

with the initial value  $_{0^+}I_t^{1-\gamma}X(0^+) = X_0 = (x_{10}, \ldots, x_{n0})^T, \gamma = (\gamma_1, \ldots, \gamma_n),$  $\mathbf{1} = (1, \ldots, 1)$  and  $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_n]$  such that  $0 < \alpha_i < 1, \gamma_i = \alpha_i + \beta - \alpha_i\beta$ , for  $i = 1, \ldots, n$ , where

$$F(X(t), X(t-\tau)) = \begin{pmatrix} f_1(x_1(t), \dots, x_n(t), x_1(t-\tau), \dots, x_n(t-\tau)) \\ f_2(x_1(t), \dots, x_n(t), x_1(t-\tau), \dots, x_n(t-\tau)) \\ \vdots \\ f_n(x_1(t), \dots, x_n(t), x_1(t-\tau), \dots, x_n(t-\tau)) \end{pmatrix},$$
  
and  $X(t) = (x_1(t), \dots, x_n(t))^T, X(t-\tau) = (x_1(t-\tau), \dots, x_n(t-\tau))^T \in \mathbb{R}^n.$ 

**Definition 2.3.**  $\hat{X} = (\hat{x}_1, \dots, \hat{x}_n)$  is an equilibrium point of system (2.2) if  $f_i(\hat{x}_1, \dots, \hat{x}_n, \hat{x}_1, \dots, \hat{x}_n) = 0$ , for  $i = 1, \dots, n$ .

**Theorem 2.4.** Let  $\hat{X}$  is the equilibrium of system (2.2),  $J = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1,...,n}$ , is the Jacobian matrix at the point  $\hat{X}$  with respect to X and  $J_{\tau} = \left(\frac{\partial f_i}{\partial x_{j\tau}}\right)_{i,j=1,...,n}$ , is the Jacobian matrix at the point  $\hat{X}$  with respect to  $X(t-\tau)$ , where for j = 1,...,n, denoting  $x_j = x_j(t)$  and  $x_{j\tau} = x_j(t-\tau)$ . If all roots of

(2.3) 
$$\det \left( I_{\alpha} - J - e^{-s \tau} J_{\tau} \right) = 0,$$

where

$$I_{\alpha} = \begin{pmatrix} s^{\alpha_1} & 0 & \dots & 0 \\ 0 & s^{\alpha_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s^{\alpha_n} \end{pmatrix}$$

,

have negative real parts, then the point  $\hat{X}$  of system (2.2) is asymptotically stable.

#### 3. Illustrative examples

This Section will list one example to show the effectiveness of our new criteria for asymptotic stability of Hilfer fractional order systems. The Hilfer fractional Van der Pol's Oscillator with delay can be written in the form

(3.1) 
$$\begin{cases} {}_{0^{+}}D_{t}^{\alpha_{1},\beta}x_{1}(t) = x_{2}(t), \\ {}_{0^{+}}D_{t}^{\alpha_{2},\beta}x_{2}(t) = -x_{1}(t) + \varepsilon kx_{1}(t-\tau) - \varepsilon x_{2}(t) \left(x_{1}^{2}(t) - 1\right), \end{cases}$$

where  $0 \leq \beta \leq 1$  with the initial value  $_{0^+}I_t^{1-\gamma_1}x_1(0^+) = x_{10}$  and  $_{0^+}I_t^{1-\gamma_2}x_2(0^+) = x_{20}$  such that  $0 < \alpha_i < 1$ ,  $\gamma_i = \alpha_i + \beta - \alpha_i\beta$ , for i = 1, 2. The system (3.1) has an equilibrium point E = (0, 0) and  $\varepsilon > 0$  is the control parameter,  $\tau$  is a positive quantity representing the delay and k is the feedback gain. The characteristic equation at an equilibrium point  $(\hat{x}_1, \hat{x}_2)$  turns out to be

(3.2) 
$$det \begin{pmatrix} s^{\alpha_1} & -1 \\ 1 - k\varepsilon e^{-s\tau} + 2\varepsilon \hat{x}_1 \hat{x}_2 & s^{\alpha_2} + \varepsilon \left( \hat{x}_1^2(t) - 1 \right) \end{pmatrix} = 0.$$

The characteristic equation of the system (3.1) evaluated at the equilibrium E is

$$s^{\alpha_1 + \alpha_2} - \varepsilon s^{\alpha_1} - k\varepsilon e^{-s\tau} + 1 = 0.$$

Let us investigate the Hilfer fractional Van der Pol's Oscillator (3.1) with parameters  $\varepsilon = 0.1, k = 5, \beta = 0.5$  and  $(\alpha_1, \alpha_2) = (1, 0.95)$  and consider various values of  $\tau$ . The results are given in Table 1.

TABLE 1. Stability of the equilibrium point of the (3.1) with different  $\tau$ .

au	approximation roots	state stability
0.03	$0.014 \pm i \ 0.702$	unstable
0.15	$-57.259 \pm i \ 1.350$	asymptotically stable
1.5	$-0.849 \pm i \ 0.121$	asymptotically stable

To verify the efficiency of the obtained results in Table 1 , the numerical solution for the Hilfer fractional Van der Pol's Oscillator with delay have been computed. Simulations results are displayed in Figure 1.



FIGURE 1. The numerical approximations of system (3.1) when  $\tau = 0.03$  with  $(\alpha_1, \alpha_2) = (1, 0.95)$ ,  $\varepsilon = 0.1$  and k = 5. The equilibrium E = (0, 0) is unstable.

### References

- W. DENG, C. LI AND J. LU, Stability analysis of linear fractional differential system with multiple time delays, Nonlinear Dynam. 48(2007), 409–416.
- [2] R. HILFER, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
- [3] H.M. SRIVASTAVA, Z. TOMOVSKI, Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel, Appl. Math. Comput. 211(2009), 198–210.
- [4] H. REZAZADEH, H. AMINIKHAH AND A. REFAHI SHEIKHANI, Stability Analysis of Hilfer Fractional Differential Systems, Math. Commun. 21(2016), 45–64.

DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICAL SCIENCES, UNIVERSITY OF MAZAN-DARAN, BABOLSAR, IRAN

 $E\text{-}mail\ address:\ \texttt{mfvahid70@gmail.com}$ 

DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICAL SCIENCES, UNIVERSITY OF MAZANDARAN, BABOLSAR, IRAN

E-mail address: mostafa.eslami@umz.ac.ir

FACULTY OF ENGINEERING TECHNOLOGY, AMOL UNIVERSITY OF SPECIAL MODERN TECHNOLOGIES, AMOL, IRAN.

E-mail address: h.rezazadeh@ausmt.ac.ir



The 9<sup>th</sup> Seminar on Linear Algebra and its Applications July 5-6, 2017, University of Tabriz, Tabriz, Iran

## A NOTE ON LINEAR BOOLEAN ALGEBRA

ALI MOLKHASI

ABSTRACT. The aim of the present paper is to prove that linear span of the set of components of an Archimedean Riesz space with unit 1 is a linear Boolean algebra.

Keywords: Archimedean Riesz space; Boolean algebras; Linear Boolean algebras.

#### 1. INTRODUCTION

In this paper k is a field of characteristic zero. Let S and V be the category of sets and the category of k-vector spaces, respectively. By  $\overline{x}$  the free k-vector space generated by  $x \in S$ . Let

# $\overline{(\ )}: S \longrightarrow \mathbf{V}$

be the monoidal functor, that x into  $\overline{x}$  and sends  $g: x \longrightarrow y$  the linear transformation  $\overline{g}: \overline{x} \longrightarrow \overline{y}$  which value is g(a) if  $a \in x$ . Recall the monoidal structure on  $\mathbf{V}$  is the tensor product  $\otimes$ . For example, if x is a monoid, then x carries the structure of an associative algebra. Thus associative algebras is the linear analogue of monoid. Boolean algebras [4, 6] has been known at least since 1854 and constitute a cornerstone of modern mathematics and is the algebra of two-valued logic with only sentential connectives, or equivalently of algebras of sets under union and complementation.

In this paper we study the linear Boolean algebras and Archimedean Riesz space. It is proved that linear span of the set of components of an Archimedean Riesz space with unit 1 is a linear Boolean algebra.

### 2. Main results

We now turn to one of the most important topics in the theory of lattices that of Boolean algebras. A lattice L is distributive when it satisfies  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$  for all  $x, y, z \in L$ . In a bounded distributive lattice, a is a complement of b iff  $a \wedge b = 0$  and  $a \vee b = 1$ . Let  $a \in [b, c]$ ; x is a relative complement of a in [b, c] iff  $a \wedge x = b$  and  $a \vee x = c$ . A complement lattice is a bounded lattice in which every element has a complement. A relatively complemented lattice is a lattice in which every element has a relative complement in any interval containing it. A Boolean lattice is a complemented distributive lattice. A Boolean algebra is Boolean lattice in which 0, 1, and ' are considered to be operation. In any Boolean algebra A, a' is the largest element x such that  $x \wedge a = 0$ . More generally,  $a \wedge x \leq b$ if and only if  $a \wedge x \wedge b' = 0$ , that is  $(a \wedge b') \wedge x = 0$  or  $x \leq (a \wedge b')' = b \vee a'$ .

Tensor products of vector spaces are well-known, but it exists in many other categories equipped with a forgetful functor to **Set** [1]. Examples of such categories

<sup>2010</sup> Mathematics Subject Classification. 06E99, 06A06.

Speaker: Ali Molkhasi.

that matters to us are boolean algebra, distributive lattices, semilattices with zero, etc. Given A, B and X, three object of the same category, a bimorphism from  $A \times B$  to X is a set theoretic map  $f : A \times B \longrightarrow X$  such that for all  $a \in A$  and for all  $b \in B$ , the mappings  $f(a, \_) : B \longrightarrow X$  and  $f(\_, b) : A \longrightarrow X$  are morphisms. Being given an object X of the category and a bimorphism  $i : A \times B \longrightarrow X$ , we say that X is a tensor product of  $A \times B$  if for every object C and every bimorphism  $f : A \times B \longrightarrow C$ , there exist a unique morphism  $h : X \longrightarrow C$ , such that  $f = h \circ i$ . Tensor products are unique up to isomorphisms and they are denoted by  $A \otimes B$ . The bimorphism i is not surjective but its image generates  $A \otimes B$ , thus we call generating elements (of  $A \otimes B$ ) those coming from  $A \otimes B$  and we will write  $i(a, b) = a \otimes b$ .

**Definition 2.1.** A linear Boolean algebra is a k-vector space V together with the data:

1. Linear maps  $\cup : V \otimes V \longrightarrow V$ ,  $\cap : V \otimes V \longrightarrow V$ , and  $c : V \longleftarrow V$  called a union, an intersection and a complement, respectively.

2. Linear maps  $T: k \longrightarrow V, E: k \longrightarrow V$  called the empty map and the total map, respectively.

3. Linear map  $\triangle : V \longrightarrow V \otimes V$  called a coproduct.

4. Linear map  $ev: V \longrightarrow k$  called the evaluation map.

The axioms below must hold:

- $\bullet \cup = \cup \circ S, \ \cap = \cap \circ S.$
- $\bullet \cup \circ (\cup \otimes I) = \cup \circ (I \otimes \cup),$
- $\cap \circ (\cap \otimes I) = \cap \circ (I \otimes \cup).$
- $\cap \circ (I \otimes \cup) = \cup \circ (\cap \otimes \cap) \circ (I \otimes S \otimes I) \circ (\triangle \otimes I \otimes I),$
- $\cup \circ (I \otimes \cap) = \cap \circ (\cup \otimes \cup) \circ (I \otimes S \otimes I) \circ (\triangle \otimes I \otimes I).$
- $\bullet \cap \circ (I \otimes \cup) \circ (\triangle I) = I \otimes ev,$
- $\cup \circ (I \otimes \cap) \circ (\triangle I) = I \otimes ev.$
- $\cup \circ (I \otimes E) = I, \ \cap \circ (I \otimes T) = I,$
- $\cap \circ (I \otimes c) \circ \triangle = E \circ ev,$
- $\cup \circ (I \otimes c) \circ \triangle = T \circ ev.$
- $(\triangle \otimes I) \circ \triangle = (I \otimes \triangle) \circ \triangle.$
- $\bullet \ S \circ \bigtriangleup = \bigtriangleup.$

A partially ordered vector space is a poset  $(E, \leq)$  where E is a real vector space such that the following conditions are satisfied: if  $x, y, x \in E$  and  $\lambda > 0$  then  $x + z \leq y + z$  and  $\lambda x \leq \lambda y$  whenever  $x \leq y$ .

**Definition 2.2.** *E* is a Riesz space if it is a partially ordered vector space and, at the same time, a lattice under its ordering.

**Definition 2.3.** Suppose E is an Archimedean Riesz space with unit 1. If  $e \in E$  and  $e \wedge (1 - e) = 0$ , then e is said a component.

For example, 0 and 1 in C([0, 1]) are the only components. Also, the set of components of E contain in [0, 1] (the set of elements x in E with  $0 \le x \le 1$ ). On the other had, We know that the set of components of an Archimedean Riesz space with unit 1 is a Boolean algebra and the intersection, the union and the complement are the linear extensions of the corresponding maps on the Boolean algebra. So we have the following theorem:

**Theorem 2.4.** If E is an Archimedean Riesz space with unit 1, then linear span of the set of components of E is a linear Boolean algebra.

## References

- [1] Z. Shmuely, The tensor product of distributive lattices, Algebra Universalis, 9 (1979), 281-296.
- [2] M. Hall, The theory of groups, Chelsea Publishing Company, New York, 1976.
- [3] M. Suzuki, Group algebra II, Springer-Verlag-New York, 1986.
- [4] R. Sikorski, Boolean algebras, Springer-Verlag, Berlin, 1964.
- [5] D. Taunt, On A-groups, Proc. Cambridge Philos. Soc., 45 (1949), 24-42.
- [6] J. Whitesitt, Boolean algebra and its applications, Addison-Wesley Publishing, Massachusetts, 1961.

FARHANGIAN UNIVERSITY, IRAN *E-mail address:* molkhasi@cfu.ac.ir


The 9<sup>th</sup> Seminar on Linear Algebra and its Applications July 5-6, 2017, University of Tabriz, Tabriz, Iran

### **ON OPERATOR** *h*-CONVEX FUNCTIONS

#### ALI MORASSAEI

ABSTRACT. In this paper we define h-convex function for operators on a Hilbert space and then present many example. In continuation, we state new results for h-convex functions.

Keywords: *h*-convex function; Operator inequality; Hilbert space.

#### 1. INTRODUCTION

Assume that I be an interval in  $\mathbb{R}$ . Let us recall definitions of some special classes of functions.

We say that [4]  $f : I \to \mathbb{R}$  is a Godunova-Levin function or that f belongs to the class Q(I) if f is non-negative and for all  $x, y \in I$  and  $\alpha \in (0, 1)$  we have

$$f(\alpha x + (1 - \alpha)y) \le \frac{f(x)}{\alpha} + \frac{f(y)}{1 - \alpha}$$

For  $s \in (0, 1]$ , a function  $f: [0, \infty) \to [0, \infty)$  is said to be s-convex function, if

$$f(\alpha x + (1 - \alpha)y) \le \alpha^s f(x) + (1 - \alpha)^s f(y)$$

for every  $x, y \in [0, \infty)$  and  $\alpha \in [0, 1]$ , see [1]. Also, we say that  $f : I \to [0, \infty)$  is a *P*-function [2], if for all  $x, y \in I$  and  $\alpha \in [0, 1]$  we have

$$f(\alpha x + (1 - \alpha)y) \le f(x) + f(y).$$

In [5], Varošanec defined the h-convex function as follows: Let  $h: J \subseteq \mathbb{R} \to \mathbb{R}$  be an non-negative function,  $h \neq 0$ . We say that  $f: I \to \mathbb{R}$  is an h-convex function, if f is non-negative and for all  $x, y \in I$ ,  $\alpha \in (0, 1)$  we have

(1.1) 
$$f(\alpha x + (1-\alpha)y) \le h(\alpha)f(x) + h(1-\alpha)f(y).$$

If inequality (3.2) is reversed, then f is said to be h-concave.

# 2. Preliminaries

In what follows we assume that  $\mathcal{H}$  and  $\mathcal{K}$  are Hilbert spaces,  $\mathbb{B}(\mathcal{H})$  and  $\mathbb{B}(\mathcal{K})$  are  $C^*$ -algebras of all bounded linear operators on the appropriate Hilbert space with identities  $I_{\mathcal{H}}$  and  $I_{\mathcal{K}}$ ,  $\mathbb{B}_h(\mathcal{H})$  denote the algebra of all self-adjoint operators in  $\mathbb{B}(\mathcal{H})$ . In the case when dim  $\mathcal{H} = n$ , we identify  $\mathbb{B}(\mathcal{H})$  with the full matrix algebra  $\mathcal{M}_n(\mathbb{C})$  of all  $n \times n$  matrices with entries in the complex field. An operator  $A \in \mathbb{B}_h(\mathcal{H})$  is called *positive* (positive-semidefinite for matrices) if  $\langle Ax, x \rangle \geq 0$  holds for every  $x \in \mathcal{H}$  and then we write  $A \geq 0$ . For  $A, B \in \mathbb{B}_h(\mathcal{H})$ , we say  $A \leq B$  if  $B - A \geq 0$ . We write A > 0 and say A is *strictly positive operator*, if A is a positive invertible operator. Let f be a continuous real valued function defined on an interval J. The

<sup>2010</sup> Mathematics Subject Classification. 47A63.

Speaker: Ali Morassaei.

function f is called *operator decreasing* if  $B \leq A$  implies  $f(A) \leq f(B)$  for all A, B with spectra in J. A function f is said to be *operator concave* on J if

$$\lambda f(A) + (1 - \lambda)f(B) \le f(\lambda A + (1 - \lambda)B)$$

for all  $A, B \in \mathbb{B}_{h}(\mathcal{H})$  with spectra in J and all  $\lambda \in [0, 1]$ . A map  $\Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K})$ is called *positive* if  $\Phi(A) \geq 0$  whenever  $A \geq 0$  and is said to be *normalized* if  $\Phi(I_{\mathcal{H}}) = I_{\mathcal{K}}$ . We denote by  $\mathbf{P}[\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{K})]$  the set of all positive linear maps  $\Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K})$  and by  $\mathbf{P}_{N}[\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{K})]$  the set of all normalized positive linear maps  $\Phi \in \mathbf{P}[\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{K})]$ . If  $\Phi \in \mathbf{P}_{N}[\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{K})]$  and f is an operator concave function on an interval J, then

(2.1) 
$$f(\Phi(A)) \ge \Phi(f(A))$$
 (Davis-Choi-Jensen's inequality)

for every self-adjoint operator A on  $\mathcal{H}$ , whose spectrum is contained in J, see [3].

## 3. Main results

In this section, we present the definition of operator h-convex function for operators acting on a Hilbert space.

**Definition 3.1.** Let  $h: J \subseteq \mathbb{R} \to \mathbb{R}$  be an non-negative function,  $h \neq 0$ . We say that  $f: I \to \mathbb{R}$  is an operator h-midconvex function, if f is non-negative and for all  $A, B \in \mathbb{B}(\mathcal{H})$  with  $\sigma(A), \sigma(B) \subseteq I$ , we have

(3.1) 
$$f\left(\frac{A+B}{2}\right) \le h\left(\frac{1}{2}\right)\left(f(A) + f(B)\right).$$

If inequality (3.2) is reversed, then f is said to be operator h-midconcave.

**Definition 3.2.** Let  $h: J \subseteq \mathbb{R} \to \mathbb{R}$  be an non-negative function,  $h \neq 0$ . We say that  $f: I \to \mathbb{R}$  is an operator h-convex function, if f is non-negative and for all  $A, B \in \mathbb{B}(\mathcal{H})$  with  $\sigma(A), \sigma(B) \subseteq I$  and  $\lambda \in (0, 1)$ , we have

(3.2) 
$$f(\lambda A + (1-\lambda)B) \le h(\lambda)f(A) + h(1-\lambda)f(B).$$

If inequality (3.2) is reversed, then f is said to be operator h-concave.

A function  $h: J \to \mathbb{R}$  is said to be a supermultiplicative function if

(3.3) 
$$h(xy) \ge h(x)h(y),$$

for all  $x, y \in J$  [5].

If inequality (3.3) is reversed, then h is said to be a submultiplicative function. If the equality holds in (3.3), then h is said to be a multiplicative function.

**Example 3.3.** [5] Consider the function  $h: [0, +\infty) \to \mathbb{R}$  by  $h(x) = (c+x)^{p-1}$ . If c = 0, then the function h is multiplicative. If  $c \ge 1$ , then for  $p \in (0, 1)$  the function h is supermultiplicative and for p > 1 the function h is submultiplicative.

**Example 3.4.** Assume that h is a function on  $[0, \infty)$  such that  $h(t) \ge t$  and  $f(t) = t^2$  on an interval  $I \subseteq \mathbb{R}$ . Then f is operator h-midconvex function. To

prove this,

$$h\left(\frac{1}{2}\right)\left(A^{2}+B^{2}\right) - \left(\frac{A+B}{2}\right)^{2}$$
  
=  $h\left(\frac{1}{2}\right)\left(A^{2}+B^{2}\right) - \frac{A^{2}+AB+BA+B^{2}}{4}$   
=  $\frac{\left(4h(1/2)-1\right)A^{2}-AB-BA+\left(4h(1/2)-1\right)B^{2}}{4}$   
 $\geq \frac{1}{4}\left(A^{2}-AB-BA-B^{2}\right)$   
=  $\frac{1}{4}(A-B)^{2} \geq 0$ .

**Theorem 3.5.** Let  $w_1, w_2, \dots, w_n$  be positive real numbers  $(n \ge 2)$  such that  $\sum_{j=1}^n w_j = 1$ . If h is nonnegative supermutiplicative function and if f is h-convex function on an interval  $I \subseteq \mathbb{R}, A_1, \dots, A_n$  are selfadjoint operators in  $\mathbb{B}(\mathcal{H})$  such that  $\sigma(A_j) \subseteq I$  and  $\Phi \in \mathbf{P}_N[\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{K})]$ , then

(3.4) 
$$f\left(\sum_{j=1}^{n} w_j \Phi(A_j)\right) \le \sum_{j=1}^{n} h(w_j) f(\Phi(A_j)).$$

If h is submultiplicative and f is operator h-concave on I, then inequality (3.4) is reversed.

*Proof.* We prove this theorem by induction on n. If n = 2, then inequality (3.4) is equivalent to inequality (3.2) with  $\lambda = w_1$  and  $1 - \lambda = w_2$ . Assume that inequality (3.4) holds for n - 1. Then for n, we have

$$\begin{split} f\left(\sum_{j=1}^{n} w_{j} \Phi(A_{j})\right) &= f\left(w_{n} \Phi(A_{n}) + \sum_{j=1}^{n-1} w_{j} \Phi(A_{j})\right) \\ &= f\left(w_{n} \Phi(A_{n}) + (w_{1} + \dots + w_{n-1}) \sum_{j=1}^{n-1} \frac{w_{j}}{w_{1} + \dots + w_{n-1}} \Phi(A_{j})\right) \\ &\leq h(w_{n}) f(\Phi(A_{n})) + h(w_{1} + \dots + w_{n-1}) f\left(\sum_{j=1}^{n-1} \frac{w_{j}}{w_{1} + \dots + w_{n-1}} \Phi(A_{j})\right) \\ &\leq h(w_{n}) f(\Phi(A_{n})) + h(w_{1} + \dots + w_{n-1}) \sum_{j=1}^{n-1} h\left(\frac{w_{j}}{w_{1} + \dots + w_{n-1}}\right) f(\Phi(A_{j})) \\ &\leq h(w_{n}) f(\Phi(A_{n})) + \sum_{j=1}^{n-1} h(w_{j}) f(\Phi(A_{j})) \\ &= \sum_{j=1}^{n} h(w_{j}) f(\Phi(A_{j})) \,. \end{split}$$

#### References

- S.S. Dragomir, J. Pečarić and L.E. Persson, Some inequalities of Hadamard type, Soochow J. Math. 21 (1995) 335–341.

- [3] T. Furuta, J. Mićić Hot, J.E. Pečarić and Y. Seo, Mond-Pečarić method in operator inequalities, Element, Zagreb, 2005.
- [4] E.K. Godunova and V.I. Levin, Neravenstva dlja funkcii širokogo klassa, soderžaščego vypuklye, monotonnye i nekotorye drugie vidy funkcii, in: Vyčislitel. Mat. i. Mat. Fiz. Mežvuzov. Sb. Nauč. Trudov, MGPI, Moskva, (1985) 138–142.
- [5] S. Varošanec, On *h*-convexity, J. Math. Anal. Appl., 326 (2007), 303-311.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES, UNIVERSITY OF ZANJAN, UNIVERSITY BLVD., ZANJAN 45371-38791, IRAN

*E-mail address*: morassaei@znu.ac.ir



# A GLOBAL REGULARIZED CONJUGATE GRADIENT METHOD FOR SOLVING MATRIX EQUATION AXB = C

AZITA TAJADDINI AND MEHDI NAJAFI

ABSTRACT. In this paper, we present a global regularized conjugate gradient iteration method for solving a matrix equation with symmetric positive definite coefficient matrices. This method is actually inner-outer iterations, which employs a global conjugate gradient-like method as inner iteration to approximate each outer iterates. The convergence properties of this method are discussed.

Keywords: Conjugate gradient method; Regularized conjugate gradient method; Regularization; Matrix equations.

### 1. INTRODUCTION

Consider the matrix equation

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{s \times s}$ ,  $C \in \mathbb{R}^{n \times m}$  and the matrices A and B are symmetric positive definite (SPD). The linear matrix equation (1.1) can be written as the following  $nm \times nm$  linear system

(1.2) 
$$(B \otimes A) \operatorname{vec}(X) = \operatorname{vec}(C),$$

where  $\operatorname{vec}(X)$  is the vector of  $\mathbb{R}^{nm}$  obtained by stacking the columns of the nm matrix X. Of course, this is a numerically poor way to determine the solution X of the matrix equation (1.1), as the linear system of equations (1.2) is costly to solve and can be ill-conditioned. The RCG method [1] was proposed for solving linear system Ax = b. In this paper, we extend this method for solving matrix equation (1.1) that is called the global regularized conjugate gradient (GL-RCG) method. In the GL-RCG method, the matrix equation (1.1) is first regularized by reasonably shifting and contracting the spectrum of the coefficient matrix  $B \otimes A$ , and its solution is then approximated successively by a sequence of regularized matrix equations. At each step of iteration, the regularized matrix equation itself iteratively solved by th GL-CG method. Hence, the GL-RCG method is actually an inner/outer iterative method [3].

Throughout this paper, we use the following notations. For two matrices Y and Z in  $\mathbb{R}^{n \times m}$ , we define the inner product  $\langle X, Y \rangle_F = \operatorname{tr}(X^T Y)$ , where  $\operatorname{tr}(Z)$  denotes the trace of the square matrix Z. Also  $X^T$  denotes the transpose of the matrix X. The associated norm is the well-known Frobenius norm denoted by  $\|\cdot\|_F$ . A system of vectors (matrices) of  $\mathbb{R}^{n \times m}$  is said to be F-orthonormal if it is orthonormal with respect to  $\langle \cdot, \cdot \rangle_F$ .  $A \otimes B = [a_{i,j}B]$  denotes the Kronecker product of the matrices A and B.

<sup>2010</sup> Mathematics Subject Classification. 65F10, 65F50, 65N12.

Speaker: Mehdi Najafi.

# 2. The GL-RCG method for AXB = C

In this section, we explain the global regularized conjugate gradient method for solving (1.1).

Suppose that function f is defined as follows

$$f(x,y) = (\nu_1 + x)(\nu_2 + y), \text{ where } \nu_1, \nu_2 \ge 0.$$

Evidently,  $f(B; A) = (\nu_1 I + B) \otimes (\nu_2 I + A)$ . Therefore,

$$f(B;A) = \nu_1 \nu_2 I + \nu_2 (B \otimes I) + \nu_1 (I \otimes A) + (B \otimes A).$$

The fuction f maps the spectrum  $B \otimes A$  onto a new set  $f(\sigma(B \otimes A))$ . If we choose  $\nu_1 > 0, \nu_2 > 0$ , then we have the following relationship

$$\begin{aligned} \kappa_2\left(f(B;A)\right) &= \frac{\lambda_{\max}\left(f(B;A)\right)}{\lambda_{\min}(f(B;A))} = \frac{\lambda_{\max}(\nu_1 I + B) \cdot \lambda_{\max}(\nu_2 I + A)}{\lambda_{\min}(\nu_1 I + B) \cdot \lambda_{\min}(\nu_2 I + A)} \\ &= \frac{\nu_1 + \lambda_{\max}(B)}{\nu_1 + \lambda_{\min}(B)} \cdot \frac{\nu_2 + \lambda_{\max}(A)}{\nu_2 + \lambda_{\min}(A)} \\ &\leq \frac{\lambda_{\max}(B)}{\lambda_{\min}(B)} \cdot \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \\ &= \kappa_2(A) \cdot \kappa_2(B) \\ &= \kappa_2(B \otimes A). \end{aligned}$$

Now, we can write the matrix equation (1.1) as

$$(\nu_2 I + A)X(\nu_1 I + B) = \nu_1 \nu_2 X + \nu_2 X B + \nu_1 A X + A X B, \ \nu_1 > 0, \nu_2 > 0.$$

The basic idea of the global regularized conjugate gradient method is as follows: Let an initial guess  $X_0 \in \mathbb{R}^{n \times s}$  is given. Suppose that we have obtained approximations  $X^{(0)}, \ldots, X^{(k)}$  to the solution  $X^*$  of the matrix equation (1.1), then the next approximation  $X^{(k+1)}$  to  $X^*$  is computed through solving the matrix equations

(2.1) 
$$(\nu_2 I + A) X (\nu_1 I + B) = \nu_1 \nu_2 X^{(k)} + \nu_2 X^{(k)} B + \nu_1 A X^{(k)} + C,$$

iteratively with GL-CG method. This method is described in Algorithm 1. Algorithm 1 The GL-RCG method for the matrix equation AXB = C

1:	$A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{s \times s}$ , initial guess $X^{(0)} \in \mathbb{R}^{n \times s}$ and iteration parameters $\nu_1 > 0, \nu_2 > 0$ . The lar				
	admissible number of iteration steps $k_{\max}$ and the stoping tolerance $\varepsilon$ of GL-RCG ( $\varepsilon_{\text{GL-RCG}}$ ). The				
	admissible number of iteration steps $\ell_{\max}$ and the stoping tolerance $\varepsilon$ of GL-CG ( $\varepsilon_{GL-CG}$ ).				
2:	Compute $R^{(0)} = C - AX^{(0)}B$ .				
3:	Set $X^{(0,0)} = X^{(0)}$ .				
4:	For $k = 0, 1,, k_{\max}$ Do:				
5:	Compute $\tilde{C} = \nu_1 \nu_2 X^{(0,k)} + \nu_2 X^{(0,k)} B + \nu_1 A X^{(0,k)} + C.$				
6:	Compute $R^{(0,k)} = \tilde{C} - (\nu_2 I + A) X^{(0,k)} (\nu_1 I + B).$				
7:	Set $P^{(0)} = R^{(0,k)}$ .				
8:	For $\ell = 0, 1, \dots, \ell_{\max}$ Do:				
9:	Compute $W^{(\ell)} = (\nu_2 I + A) P^{(\ell)} (\nu_1 I + B).$				
10.	Compute $\alpha_{\ell} = \langle R^{(\ell,k)}, R^{(\ell,k)} \rangle_F$				
10.	Compute $u_{\ell} = \frac{1}{\langle W^{(\ell)}, R^{(\ell,k)} \rangle_F}$ .				
11:	Compute $X^{(\ell+1,k)} = X^{(\ell,k)} + \alpha_{\ell} P^{(\ell)}$ .				
12:	Compute $R^{(\ell+1,k)} = R^{(\ell,k)} - \alpha_\ell W^{(\ell)}$ .				
13:	If $  R^{(\ell+1,k)}  _F \le \varepsilon_{\text{ol.col}}   R^{(0,k)}  _F$ go to 19.				
14	$\langle R^{(\ell+1,k)}, R^{(\ell+1,k)} \rangle_F$				
14:	Compute $\beta_{\ell} = \frac{1}{\langle R^{(\ell,k)}, R^{(\ell,k)} \rangle_{F}}$ .				
15:	Compute $P^{(\ell+1)} = R^{(\ell+1,k)} + \beta_{\ell} P^{(\ell)}$ .				
16:	End do				
17:	Set $X^{(k+1)} = X^{(\ell+1,k)}$ . (1.11)				
18:	Compute $R^{(k+1)} = C - AX^{(k+1)}B$ .				
19:	If $\ R^{(k+1)}\ _F \le \varepsilon_{\text{GL-RCG}} \ R^{(0)}\ _F$ Stop.				
20:	Set $X^{(0,k+1)} = X^{(k+1)}$ .				
21:	End do				

# 3. Convergence Analysis

In this section, we describe the convergence analysis of the global reqularized conjugate gradient method. In the next, we need the following lemmas.

**Lemma 3.1.** ([2]). Let  $A \in \mathbb{R}^{n \times n}$  be the SPD matrix. Then for any  $z \in \mathbb{R}^n$ ,

$$\left|A^{\frac{1}{2}}z\right|_{2} = \|z\|_{A}, \qquad \sqrt{\lambda_{\min}(A)}\|z\|_{A} \le \|Az\|_{2} \le \sqrt{\lambda_{\max}(A)}\|z\|_{A}.$$

**Lemma 3.2.** ([4]). Let  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{m \times m}$  be the SPD matrices. Then for any  $X \in \mathbb{R}^{n \times m}$ ,

(3.1) 
$$\|\operatorname{vec}(X)\|_{B\otimes A}^2 = \|X\|_{(A,B)}^2$$

Now, we can demonstrate a precise estimate about the speed convergence of the GL-RCG method.

**Theorem 3.3.** Let  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{m \times m}$  are SPD matrices and  $\nu_1 > 0, \nu_2 > 0$ . If  $X^{(0)}$  be an initial guess for the GL-RCG method, and if we apply  $m^{(k)}$  steps of GL-CG iterations to obtain the next approximation  $X^{(k+1)} = X^{(m_k,k)}$  to the solution of the matrix equation (1.1), then we have

a) 
$$\frac{\|X^{(k+1)} - X^{(*,k)}\|_{(\nu_{2}I+A,\nu_{1}I+B)}}{\|X^{(0,k)} - X^{(*,k)}\|_{(\nu_{2}I+A,\nu_{1}I+B)}} \leq \min_{\substack{p_{m_{k}} \in \mathbb{P}_{m_{k}} \ \mu \in \sigma(\mathscr{B}(\nu_{1}))\\p_{m_{k}}(0,0) = 1 \ \lambda \in \sigma(\mathscr{A}(\nu_{2}))}} \max_{\substack{p_{m_{k}} \in \mathbb{P}_{m_{k}} \ \mu \in \sigma(\mathscr{B}(\nu_{1}))\\p_{m_{k}}(0,0) = 1 \ \lambda \in \sigma(\mathscr{A}(\nu_{2}))}} \max_{\substack{p_{m_{k}} \in \mathbb{P}_{m_{k}} \ \mu \in \sigma(\mathscr{B}(\nu_{1}))\\p_{m_{k}}(0,0) = 1 \ \lambda \in \sigma(\mathscr{B}(\nu_{1}))}} \max_{\substack{p_{m_{k}} \in \mathbb{P}_{m_{k}} \ \mu \in \sigma(\mathscr{B}(\nu_{1}))\\p_{m_{k}}(0,0) = 1 \ \lambda \in \sigma(\mathscr{A}(\nu_{2}))}} |p_{m_{k}}(\mu,\lambda)|$$

where  $\mathscr{A}(\nu_2) = \nu_2 I + A$  and  $\mathscr{B}(\nu_1) = \nu_1 I + B$ . **Proof.** Assume that  $D^{(0,k)} = -X^{(0,k)} + X^{(*,k)}$  then

$$R^{(0,k)} = \nu_1 \nu_2 X^{(k)} + \nu_2 X^{(k)} B + \nu_1 A X^{(k)} + C - (\nu_2 I + A) X^{(0,k)} (\nu_1 I + B)$$
  
=  $(\nu_2 I + A) D^{(0,k)} (\nu_1 I + B).$ 

Since

$$\begin{aligned} X^{(k+1)} \in \mathcal{G}K_m\left(\mathscr{A}(\nu_2), R^{(0,k)}, \mathscr{B}(\nu_1)\right), \\ \mathcal{G}K_m\left(\mathscr{A}(\nu_2), R^{(0,k)}, \mathscr{B}(\nu_1)\right) &= \operatorname{span}\left\{R^{(0,k)}, \mathscr{A}(\nu_2)R^{(0,k)}\mathscr{B}(\nu_1), \dots, \mathscr{A}(\nu_2)^{m-1}R^{(0,k)}\mathscr{B}(\nu_1)^{m-1}\right\}. \end{aligned}$$
  
Then there exits  $\alpha_i \in \mathbb{R}, i = 1, 2, \dots, k$ , such that

$$X^{(*,k)} - X^{(k+1)} = D^{(0,k)} - \sum_{i=0}^{m_k-1} \alpha_i \mathscr{A}^{i+1}(\nu_2) D^{(0,k)} \mathscr{B}^{i+1}(\nu_1).$$

Hence

$$\operatorname{vec}\left(X^{(*,k)} - X^{(k+1)}\right) = \operatorname{vec}\left(D^{(0,k)}\right) - \sum_{i=0}^{m_k-1} \alpha_i \left(\mathscr{B}(\nu_1) \otimes \mathscr{A}(\nu_2)\right)^{i+1} \operatorname{vec}\left(D^{(0,k)}\right)$$

$$(3.2) \qquad \qquad = p_{m_k}(\mathscr{B}(\nu_1); \mathscr{A}(\nu_2)) \operatorname{vec}\left(D^{(0,k)}\right),$$

where  $p_{m_k}(x, y) = 1 - \sum_{i=0}^{m_k-1} \alpha_i x^{i+1} y^{i+1}$ ,  $p_{m_k}(0, 0) = 1$ . Let  $U^T \mathscr{B}(\nu_1) U = \Lambda$  and  $V^T \mathscr{A}(\nu_2) V = D$  with  $U^T U = I$  and  $V^T V = I$ , where D and  $\Lambda$  are the diagonal matrices whose elements are the eigenvalues  $\mu_1, \ldots, \mu_n$  of  $\mathscr{A}(\nu_2)$  and the eigenvalues  $\lambda_1, \ldots, \lambda_m$  of  $\mathscr{B}(\nu_1)$ , respectively. Then

$$p_{m_k}\left(\mathscr{B}(\nu_1);\mathscr{A}(\nu_2)\right) = p_{m_k}\left(U\Lambda U^T; VDV^T\right)$$
$$= \left\{\sum_{i=0}^{m_k-1} \alpha_i \left(U\Lambda U^T \otimes VDV^T\right)^{i+1}\right\}$$
$$= I - \sum_{i=0}^{m_k-1} \alpha_i (U \otimes V)(\Lambda \otimes D)^{i+1} (U^T \otimes V^T)$$
$$(3.3) = (U \otimes V)p_{m_k}(\Lambda, D)(U^T \otimes V^T).$$
Substituting (2.2) are set

Substituting (3.3) into (3.2), we get

$$\operatorname{vec}\left(X^{(*,k)} - X^{(k+1)}\right) = \left(U \otimes V\right) p_{m_k}\left(\Lambda, D\right) \left(U^T \otimes V^T\right) \operatorname{vec}\left(D^{(0,k)}\right).$$

Then it can be shown that a holds. In the following, we prove the statement b.  

$$\begin{split} \left\| R^{(k+1)}(\nu_1,\nu_2) \right\|_F &= \left\| C + \nu_1 \nu_2 X^{(k)} + \nu_2 X^{(k)} B + \nu_1 A X^{(k)} - (\nu_2 I + A) X^{(k+1)}(\nu_1 I + B) \right\|_F \\ &= \left\| (\nu_2 I + A) (X^{(*,k)} - X^{(k+1)})(\nu_1 I + B) \right\|_F \\ &= \left\| \operatorname{vec} \left( (\nu_2 I + A) (X^{(*,k)} - X^{(k+1)})(\nu_1 I + B) \right) \right\|_2 \\ &= \left\| (\nu_1 I + B) \otimes (\nu_2 I + A) \operatorname{vec} \left( (\nu_2 I + A) (X^{(*,k)} - X^{(k+1)}) \right) \right\|_2, \end{split}$$

by lemma 3.1 and lemma 3.2 we have

A GLOBAL REGULARIZED CONJUGATE GRADIENT METHOD FOR SOLVING  $\ldots$  111

$$\leq \sqrt{\lambda_{\max}(\mathscr{B}(\nu_1) \otimes \mathscr{A}(\nu_2))} \cdot \left\| \operatorname{vec} \left( (\nu_2 I + A)(X^{(*,k)} - X^{(k+1)}) \right) \right\|_{\mathscr{B}(\nu_1) \otimes \mathscr{A}(\nu_2))} \\ = \sqrt{\lambda_{\max}(\mathscr{B}(\nu_1)) \cdot \lambda_{\max}(\mathscr{A}(\nu_2))} \cdot \left\| (\nu_2 I + A)(X^{(*,k)} - X^{(k+1)}) \right\|_{(\mathscr{A}(\nu_2), \mathscr{B}(\nu_1))} \\ \leq \sqrt{\lambda_{\max}(\mathscr{B}(\nu_1)) \cdot \lambda_{\max}(\mathscr{A}(\nu_2))} \cdot \min_{\substack{p_{m_k} \in \mathbb{P}_{m_k}}} \max_{\substack{\mu \in \sigma(\mathscr{B}(\nu_1)) \\ p_{m_k}(0,0) = 1}} |p_{m_k}(\mu, \lambda)| \cdot \left\| (X^{(0,k)} - X^{(*,k)}) \right\|_{(\mathscr{A}(\nu_2), \mathscr{B}(\nu_1))} \\ \leq \sqrt{\kappa_2(\mathscr{B}(\nu_1))} \cdot \sqrt{\kappa_2(\mathscr{A}(\nu_2))} \cdot \min_{\substack{p_{m_k} \in \mathbb{P}_{m_k}}} \max_{\substack{\mu \in \sigma(\mathscr{B}(\nu_1)) \\ p_{m_k}(0,0) = 1}} |p_{m_k}(\mu, \lambda)| \cdot \left\| \operatorname{vec} \left( \mathscr{A}(\nu_1) D^{(0,k)} \mathscr{B}(\nu_1) \right) \right\|_2 \\ = \sqrt{\kappa_2(\mathscr{B}(\nu_1))} \cdot \sqrt{\kappa_2(\mathscr{A}(\nu_2))} \cdot \min_{\substack{p_{m_k} \in \mathbb{P}_{m_k}}} \max_{\substack{\mu \in \sigma(\mathscr{B}(\nu_1)) \\ p_{m_k}(0,0) = 1}} |p_{m_k}(\mu, \lambda)| \cdot \|R^{(k,0)}\|_F. \end{aligned}$$

# References

- 1. Bai Z. Z. and Zhang S. L., A regularized conjugate gradient method for symmetric positive definite system of linear equations, J. Comput. Math. 20(2002), 437-448.
- 2. Kelley C. T., Iterative Methods for Linear and Nonlinear Equations, *SIAM, Philadelphia*, 1995.
- 3. Khorsand Zak M. and Toutounian F., Nested splitting conjugate gradient method for matrix equation AXB = C and preconditioning, *Comput. Math. Appl.* **66**(2013), 269-278.
- 4. Moghadam M. M., Rivaz A., Tajaddini A. and Saberi Movahed F., Convergence analysis of the global FOM and GMRES methods for solving matrix equations AXB = C with SPD coefficients, *Bull. Iran Math. Soc.* **11**(2015), 981-1001.

DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF MATHEMATICS & COMPUTER SCIENCES, SHAHID BAHONAR UNIVERSITY OF KERMAN, KERMAN, IRAN. *E-mail address:* atajadini@uk.ac.ir

DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF MATHEMATICS & COMPUTER SCIENCES, SHAHID BAHONAR UNIVERSITY OF KERMAN, KERMAN, IRAN.

E-mail address: m.najafi@math.uk.ac.ir



# ON THE BIMODULE STRUCTURE OF LINEAR TRANSFORMATIONS OF VECTOR SPACES

ALIREZA NAJAFIZADEH

ABSTRACT. Let D be a division ring and let V, W be right vector spaces over D. Let L(V, W) be the (L(W), L(V))-bimodule of all right linear transformations from V into W. In this talk, some basic properties of the bi-submodules of L(V, W) are presented.

Keywords: Linear transformation; Bimodule; Division.

#### 1. INTRODUCTION

Throughout this note, D denotes a division ring, V and W right vector spaces over D, and L(V, W) the set of all right linear transformations  $T: V \to W$  such that T(x+y) = Tx + Ty and  $T(x\lambda) = (Tx)\lambda$  for all  $x, y \in V$  and  $\lambda \in D$ . The set L(V,W) forms an abelian group under the addition of linear transformations. In the case that V = W, we use the symbol L(V) to denote L(V, W). Clearly, L(V)forms a ring under the addition and multiplication of linear transformations which are, respectively, defined by (T+S)(x) = Tx + Sx and (TS)(x) = T(Sx). Moreover, L(V, W) is a right L(V)-module via the multiplication of linear transformations. By the image and the kernel of a family  $F \subseteq L(V, W)$ , denoted by Im(F) and Ker(F), respectively, we mean  $\langle \{Tx: T \in F, x \in V\} \rangle$  and  $\bigcap_{T \in F} KerT$ . The Coimage and Cokernel of the family F, denoted by Coim(F) and Coker(F), respectively, are defined as V/KerF and W/Im(F). As usual, we use the symbol V' for L(V, D). Also, when V is a right vector space, V' is a left vector space over D endowed with the addition and scalar multiplication defined by (f + g)(x) = f(x) + g(x) and  $(\lambda f)(x) := \lambda f(x)$  for all  $x \in V$  and  $\lambda \in D$ . The second dual of V, denoted by V'', is the dual of V'. For a collection C of vectors in a right vector space V over D,  $\langle C \rangle$  is used to denote the right linear subspace spanned by C. For a subset S of V, we define  $S^{\perp} = \{f \in V' : f(S) = 0\}$ . Clearly,  $S^{\perp}$  is a subspace of V'. For  $T \in L(V, W)$ ,  $T' \in L(W', V')$  denotes the adjoint of T which is defined by (T'f)(v) = f(Tv)where  $f \in W', v \in V$ . We refer the reader to [1, 2, 3, 5] for general references on rings, modules, and linear algebra over division rings.

## 2. Main results

**Proposition 2.1.** Let V, W and Z be vector spaces over a division ring D. Then, the followings hold.

(1) Let  $S \in L(V, W)$  and  $T \in L(V, Z)$ . Then, KerS = KerT exactly if there exists a one-to-one linear transformation  $P \in L(W, Z)$  or  $Q \in L(Z, W)$  such that T = PS or S = QT depending on whether  $dimCoker(S) \leq dimCoker(T)$  or  $dimCoker(T) \leq dimCoker(S)$ , respectively.

<sup>2010</sup> Mathematics Subject Classification. 15A04, 15A99, 16D99. Speaker: Alireza Najafizadeh.

- (2) Let  $S \in L(V, W)$  and  $T \in L(V, Z)$ . Then,  $KerS \subseteq KerT$  exactly if there exists a  $P \in L(W, Z)$  such that T = PS.
- (3) Let  $S_1, S_2, \ldots, S_n \in L(V, W)$  and  $T \in L(V, Z)$ . Then,  $\bigcap_{i=1}^n KerS_i \subseteq KerT$ exactly if there exist  $P_1, P_2, \ldots, P_n \in L(W, Z)$  such that  $T = P_1S_1 + \ldots + P_nS_n$ .

*Proof.* See [4, Proposition 1.1].

**Corollary 2.2.** Let V, W and Z be vector spaces over a division ring D. Let  $S \in L(V, W)$  and  $T \in L(V, Z)$ . If dimCoker(S) = dimCoker(T), then KerS = KerT exactly if there exists an invertible linear transformation  $P \in L(W, Z)$  such that T = PS.

*Proof.* Follows from part (i) of Proposition 2.1.

**Proposition 2.3.** Let V, W and Z be vector spaces over a division ring D. Then, the followings hold.

- (1) Let  $S \in L(V, W)$  and  $T \in L(Z, W)$ . Then, Im(S) = Im(T) exactly if there exists a surjective linear transformation  $P \in L(Z, V)$  or  $Q \in L(V, Z)$  such that T = SP or S = TQ depending on whether dimKerS  $\leq$  dimKerT or dimKerT  $\leq$  dimKerS, respectively.
- (2) Let  $S \in L(V, W)$  and  $T \in L(Z, W)$ . Then,  $Im(S) \subseteq Im(T)$  exactly if there exists a  $P \in L(V, Z)$  such that S = TP.
- (3) Let  $S \in L(V, W)$  and  $T_1, T_2, \ldots, T_n \in L(Z, W)$ . Then,  $Im(S) \subseteq Im(\{T_i\}_{i=1}^n)$ exactly if there exist  $P_1, P_2, \ldots, P_n \in L(V, Z)$  such that  $S = T_1P_1 + \ldots + T_nP_n$ .

*Proof.* See [4, Proposition 1.2].

**Corollary 2.4.** Let V, W and Z be vector spaces over a division ring D. Let  $S \in L(V, W)$  and  $T \in L(Z, W)$ . If  $\dim Ker(S) = \dim Ker(T)$ , then Im(S) = Im(T) exactly if there exists an invertible linear transformation  $P \in L(Z, V)$  such that T = SP.

*Proof.* Follows from part (i) of Proposition 2.3.

n

**Proposition 2.5.** Let V and W be two vector spaces over a division ring D and  $C \in L(V, W)$ . Then the followings hold.

(1)  $(ImC)^{\perp} = Ker(C')$ , where  $C' = \{T' : T \in C\}$ . In other words,

$$\langle \bigcup_{T \in C} TV \rangle^{\perp} = \bigcap_{T \in C} KerT'.$$

(2) If 
$$C = \{T_i\}_{i=1}^n$$
, where  $n \in \mathbb{N}$ , then  $(KerC)^{\perp} = Im(C')$ . In other words,

$$(\bigcap_{i=1} KerT_i)^{\perp} = \langle \bigcup_{T \in C} T'_i W' \rangle.$$

*Proof.* See [4, Theorem 4.1].

**Corollary 2.6.** Let V and W be two vector spaces over a division ring D and  $C \in L(V, W)$ .

(1) If V is finite dimensional, then  $(KerC)^{\perp} = Im(C')$ . In other words,

$$(\bigcap_{i=1}^{n} KerT_{i})^{\perp} = \langle \bigcup_{T \in C} T_{i}'W' \rangle.$$

(2) If n = dimCoIm(C) is finite, then there exist some  $T_i \in C$ , where  $1 \le i \le m \le n$  such that

$$(\bigcap_{i=1}^{m} KerT_i)^{\perp} = Im(C').$$

*Proof.* Follows from Proposition 2.5.

**Corollary 2.7.** Let V and W be vector spaces over a division ring D. Then the followings hold.

- (1) Let  $S_1, S_2, \ldots, S_n \in L(V, W)$  and  $T \in L(V, Z)$ . Then,  $\bigcap_{i=1}^n KerS_i \subseteq KerT$ exactly if  $Im(T') \subseteq Im(\{S'_i\}_{i=1}^n)$  exactly if there exist  $P_1, P_2, \ldots, P_n \in L(W, Z)$  such that  $T = P_1S_1 + \ldots + P_nS_n$ .
- (2) Let  $S \in L(V, W)$  and  $T_1, T_2, \ldots, T_n \in L(Z, W)$ . Then,  $Im(S) \subseteq Im(\{T_i\}_{i=1}^n)$ exactly if  $\bigcap_{i=1}^n KerT'_i \subseteq KerS'$  exactly if there exist  $P_i \in L(V, Z)$  such that and  $S = T_1P_1 + \ldots + T_nP_n$ .

*Proof.* Follows from Proposition 2.1, 2.3 and 2.5.

**Theorem 2.8.** Let V and W be vector spaces over a division ring D and I be a right submodule of L(V, W). If I finitely generated or W is finite-dimensional, then  $I = \{T \in L(V, W) : TV \subseteq Im(I)\}$ . Moreover, if  $dimV \ge dimW$  then every such right submodule is principal. In particular, if V = W, then every finitely generated one-sided ideal of L(V) is principal.

*Proof.* See [4, Theorem 1.5].

**Theorem 2.9.** Let V and W be vector spaces over a division ring D and I be a right submodule of L(V,W) such that  $Im(I) = Im(\{T_i\}_{i=1}^n)$ , where  $n \in \mathbb{N}$  and  $T_i \in I$  for i = 1, 2, ..., n. Then I is finitely generated. Moreover, if  $dimV \ge dimW$ then I is principal.

*Proof.* See [4, Theorem 1.6].

We observe that if V = W, the transformations Ti can be chosen to be idempotents. And such submodules (ideals) of L(V), i.e., those one-sided ideals whose images or kernels are the same as those of a finite subset of the ideals, are principal.

**Lemma 2.10.** Let V and W be vector spaces over a division ring D and  $M \subseteq V'$ a finite-dimensional subspace of V' such that  $\dim W \ge \dim M$ . Then, there exists a  $T \in L(V, W)$  such that Im(T') = M.

*Proof.* See [4, Lemma 1.7].

**Theorem 2.11.** Let V and W be vector spaces over a division ring D and let I be a right submodule of L(V,W). If the image of I is finite-dimensional, then I is finitely generated, and hence  $I = \{T \in L(V,W) : TV \subseteq Im(I)\}$ . Moreover, if  $\dim V \ge \dim W$ , then I is principal.

*Proof.* See [4, Theorem 1.8].

**Theorem 2.12.** Let V and W be vector spaces over a division ring D and I be a right submodule of L(V, W). Then,  $I \cap F(V, W) = \{T \in F(V, W) : Im(T) \subseteq Im(I)\}$ .

Proof. See [4, Theorem 1.9].

**Corollary 2.13.** Let V, W, X and Y be vector spaces over a division ring D. Then the followings hold.

- (1) Let  $S \in L(V,W)$  and  $T \in L(X,Y)$ . Then,  $rank(S) \leq rank(T)$  exactly if there exist  $P \in L(Y,W)$  and  $Q \in L(V,X)$  such that S = PTQ. Moreover, if  $S \in L(V,W)$  and  $T_1, T_2, \ldots, T_n \in L(X,Y)$ , then  $rank(S) \leq \sum_{i=1}^n rank(T_i)$ exactly if there exist  $P_1, P_2, \ldots, P_n \in L(Y,W)$  and  $Q_1, Q_2, \ldots, Q_n \in L(V,X)$ such that  $S = \sum_{i=1}^n P_i T_i Q_i$ .
- (2) Let  $S, T \in L(V, W)$ . Then  $rank(S) \leq rank(T)$  exactly if there exist  $P \in L(W)$  and  $Q \in L(V)$  such that S = PTQ. Moreover, if  $S, T_1, T_2, \ldots, T_n \in L(V, W)$  then,  $rank(S) \leq \sum_{i=1}^n rank(T_i)$  exactly if there exist  $P_1, P_2, \ldots, P_n \in L(W)$  and  $Q_1, Q_2, \ldots, Q_n \in L(V)$  such that  $S = \sum_{i=1}^n P_i T_i Q_i$ .

Proof. See [4, Corollary 1.10].

In view of the Rank-Nullity Theorem, it follows from part (ii) of the Corollary 2.13, that if V is finite-dimensional and  $S, T \in L(V, W)$ , then  $dimKerS \ge dimKerT$  exactly if there exist  $P \in L(W)$  and  $Q \in L(V)$  such that S = PTQ.

It is straightforward to check that every nonzero bi-submodule of L(V, W) includes all rank-one, and hence all finite-rank transformations. Therefore, if V or W is finite-dimensional, the trivial bi-submodules of L(V, W), are the only bi-submodules of L(V, W). In view of the preceding corollary, inspired by [2, Theorem IX.5] or [3, Theorem IV.17.1], the following characterization of nontrivial bi-submodules of L(V, W) are obtained, whenever V and W are both infinite dimensional.

**Corollary 2.14.** Let V and W be infinite-dimensional vector spaces over a division ring D. Then the nontrivial bi-submodules of L(V, W) are of the form

$$\{T \in L(V, W) : rank(T) < e\},\$$

for some unique infinite cardinal number  $e \leq \min\{\dim V, \dim W\}$ .

Proof. See [4, Corollary 1.11].

**Acknowledgement** This research was supported by a grant from Payame Noor University of I. R. of Iran. The author would like to thank Payame Noor University for the financial support during the preparation of this research.

#### References

- [1] T.W. Hungerford, Algebra, Springer-Verlag, New York, 1974.
- [2] N. Jacobson, Lectures in Abstract Algebra II: Linear Algebra, van Nostrand, Princeton, 1953.
- [3] N. Jacobson, Structure of Rings, American Mathematical Society, Providence, 1964.
- [4] M. Rahimi-Alangi, B. R. Yahaghi, On modules of linear transformations, Linear Algebra and its Applications, 445 (2014), 127-137.
- [5] L.H. Rowen, Ring Theory Vol. I, Pure and Applied Mathematics 127, Academic Press, 1988..

DEPARTMENT OF MATHEMATICS, PAYAME NOOR UNIVERSITY, I. R. OF IRAN *E-mail address*: najafizadeh@pnu.ac.ir

115



# A NUMERICAL METHOD FOR SOLVING FRACTIONAL-ORDER NON-LINEAR FINANCE SYSTEM

#### MEHRAN NAMJOO, SADEGH ZIBAEI, AND JAVAHER LANGARI

ABSTRACT. In this paper, we introduce a fractional–order non–linear finance chaotic system. Stability analysis of the fractional–order system is studied using the fractional Routh–Hurwitz criteria. The nonstandard finite difference (NSFD) scheme is implemented to study the dynamic behaviors in the novel fractional–order chaotic system. The lowest order for the system to remain chaotic is found via numerical simulation.

Keywords: Grünwald–Letnikov derivative; Stability; Fractional calculus; Nonstandard finite difference scheme.

# 1. INTRODUCTION

In the recent years there is increasing interest in fractional calculus which deals with integration or differentiation of arbitrary orders. The list of applications of fractional calculus has been ever growing and includes control theory, viscoelasticity, diffusion, turbulence, biology, economics, electromagnetism and many other physical processes [1, 2]. The interest in the study of fractional–order nonlinear systems lies in the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties, which are not taken into account in the classical integer–order models. This paper is organized as follows. In next section, we give some basic definitions and properties of the Grünwald–Letnikov (GL) approximation. In section 3, we introduce a fractional–order finance chaotic system and will discuss the stability analysis of fractional system. In section 4, numerical results show that the NSFD approach is easy to be implemented and accurate when applied to fractional–order finance system.

### 2. Grünwald–Letnikov fractional derivative and NSFD schemes

Derivatives of fractional-order have been introduced in several ways. In this paper we consider Grünwald-Letnikov (GL) approach. The GL method of approximation for the one-dimensional fractional derivative takes the following form [2]

(2.1) 
$$D^{\alpha}x(t) = f(t, x(t)), \quad x(0) = x_0, \quad t \in [0, t_f],$$

$$D^{\alpha}x(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{\left\lfloor \frac{t}{h} \right\rfloor} (-1)^j \binom{\alpha}{j} x(t-jh),$$

<sup>2010</sup> Mathematics Subject Classification. 37M05, 34A08, 34H10. Speaker: Mehran Namjoo.

where  $0 < \alpha \leq 1$ ,  $D^{\alpha}$  denotes the fractional derivative, h is the stepsize and  $\left[\frac{t}{h}\right]$  represents the integer part of  $\frac{t}{h}$ . Therefore, Eq. (2.1) is discretized in the next form

$$\sum_{j=0}^{n} c_j^{\alpha} x_{n-j} = f(t_n, x_n), \qquad n = 1, 2, 3, \dots$$

where  $t_n = nh$  and  $c_j^{\alpha}$  are the GL coefficients defined as

$$c_j^{\alpha} = (1 - \frac{1 + \alpha}{j})c_{j-1}^{\alpha}, \qquad c_0^{\alpha} = h^{-\alpha}, \qquad j = 1, 2, 3, \dots$$

The nonstandard discretization technique is a general scheme where we replace the stepsize h by a function  $\phi(h)$ . Examples of function  $\phi(h)$  are

$$h, \quad \sin(h), \quad 1 - e^{-h}, \quad \frac{1 - e^{-\lambda h}}{\lambda},$$

and so forth see [4, 5]. Moreover, linear and nonlinear terms should be represented by nonlocal discrete representations on the discrete computational lattice. For example

$$xy \approx 2x_{n+1}y_n - x_{n+1}y_{n+1}, \qquad x^2 \approx x_{n+1}x_n, \qquad x^3 \approx (\frac{x_{n+1} + x_{n-1}}{2})x_n^2$$

By applying this technique and using the GL discretization method, it yields the following relations

$$x_{n+1} = c_0^{-\alpha} \bigg( -\sum_{j=1}^{n+1} c_j^{\alpha} x_{n+1-j} + f(t_{n+1}, x_{n+1}) \bigg), \qquad n = 0, 1, \dots$$

where  $c_0^{\alpha} = \phi(h)^{-\alpha}$ .

# 3. The novel fractional-order finance chaotic system

Consider a fractional-order generalization of the novel finance system [3]. In this system, the integer-order derivatives are replaced by fractional-order derivatives, as follows

(3.1) 
$$D^{\alpha_1} x = z + (y - a)x D^{\alpha_2} y = 1 - by - x^2, D^{\alpha_3} z = -x - cz.$$

where  $0 < \alpha_i \leq 1$ , for i = 1, 2, 3 and where variable x represents the interest rate in the model, variable y represents the investment demand and variable z is the price exponent. The parameter a is the saving, b is the per investment cost and c is the elasticity of demands of commercials. If  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$  then the finance system is called commensurate otherwise incommensurate. This system is equivalent to the classical integer–order novel system when  $\alpha = 1$ , which is chaotic at a = 0.9, b = 0.2and c = 1.2. In order to analyze the stability of the system, stability theorems on fractional–order systems and fractional Routh–Hurwitz stability conditions for fractional–order differential equations are introduced [4]. 3.1. Stability analysis of the fractional-order finance system. We assume  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ , and the fractional system Eqs. (3.1) has the equilibrium points:

 $E_1 = (0, 5, 0), E_2 = (0.80829, 1.73333, -0.67358), E_3 = (-0.80829, 1.73333, 0.67358).$ 

The Jacobian matrix J of the system Eqs. (3.1) at the equilibrium point  $E^* = (x^*, y^*, z^*)$  is computed as

(3.2) 
$$J(E^*) = \begin{pmatrix} y^* - a & x^* & 1\\ -2x^* & -b & 0\\ -1 & 0 & -c \end{pmatrix}.$$

The existence and local stability conditions of these equilibrium points are as follows.

**Theorem 3.1.** For the parameters a = 0.9, b = 0.2 and c = 1.2, the equilibrium point  $E_1$  of system Eqs. (3.1) is unstable for any  $\alpha \in (0, 1)$ .

*Proof.* When a = 0.9, b = 0.2 and c = 1.2, the eigenvalues corresponding to the equilibrium point  $E_1$  are  $\lambda_1 = 3.90408$ ,  $\lambda_2 = -1.00408$  and  $\lambda_3 = -0.2$ . Therefore, based on Routh-Hurwitz stability conditions [4], the equilibrium point  $E_1$  is unstable for any  $\alpha \in (0, 1)$ .

**Theorem 3.2.** When the parameters a = 0.9, b = 0.2 and c = 1.2, if  $\alpha < 0.92$ , then equilibrium points  $E_2$  and  $E_3$  of system Eqs. (3.1) are stable.

*Proof.* When a = 0.9, b = 0.2 and c = 1.2, substituting coordinate of  $E_2$  or  $E_3$  into the Jacobian matrix (3.2), we can get the characteristic polynomial as follows

(3.3) 
$$\lambda^3 + 0.56667\lambda^2 + 1.3\lambda + 1.568 = 0.$$

The eigenvalues of Eq. (3.3) are  $\lambda_1 = -0.91977$ ,  $\lambda_2 = 0.17655 + 1.29368i$  and  $\lambda_3 = 0.17655 - 1.29368i$ . Here  $\lambda_1 = -0.91977$  is a negative real number, and the arguments of  $\lambda_2 = 0.17655 + 1.29368i$ ,  $\lambda_3 = 0.17655 - 1.29368i$  are 1.43516 and -1.43516, respectively. Hence, under the assumptions

$$|\arg(\lambda_{2,3})| = 1.43516 > \frac{\pi\alpha}{2},$$

then based on Routh–Hurwitz stability conditions and the stability theory of linear incommensurate [4], if  $\alpha < 0.91$ , then system Eqs. (3.1) is stable at the equilibrium points  $E_2$  and  $E_3$ , although the real part of the eigenvalues are positive.

## 4. SIMULATION AND RESULTS

Analytical studies always remain incomplete without numerical verification of the results. In this section, numerical results from the implementation of NSFD scheme for the fractional-order finance system are presented. Using NSFD scheme, when  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ , the simulation results demonstrate that the lowest order for the system (3.1) to remain chaotic is  $\alpha = 0.91$ . The approximate solutions are displayed in Fig. (1) for  $\alpha_1 = 0.9$ ,  $\alpha_2 = 0.85$ ,  $\alpha_3 = 0.8$  with the initial conditions (x(0), y(0), z(0)) = (2, 1, -1) and stepsize h = 0.1.



FIGURE 1. Phase trajectory of the equilibrium point  $E_3$  when  $\alpha_1 = 0.90$ ,  $\alpha_2 = 0.85$  and  $\alpha_3 = 0.80$ .

## CONCLUSION

In this paper, using NSFD scheme the lowest order for the fractional–order finance system to remain chaotic is found. Furthermore, we have studied the local stability of the equilibrium points using the fractional Routh–Hurwitz conditions. Simulation results have illustrated the effectiveness of the proposed controlled method.

## References

- [1] Hilfer, R. (2000), Applications of fractional calculus in physics, World Scientific Singapore.
- [2] Podlubny, I. (1999), Fractional Differential Equations, New York.
- [3] M. Yang, G. Cai, Chaos Control of a Non-linear Finance System. Journal of Uncertain Systems 5, 4, (2011), 263-270.
- [4] S. Zibaei and M. Namjoo, A Nonstandard Finite Difference Scheme for Solving Three–Species Food Chain with Fractional–Order Lotka-Volterra Model. Iran. J. Numer. Anal. Optim. 6, 1 (2016) 53–78.
- [5] S. Zibaei and M. Namjoo, A NSFD scheme for Lotka–Volterra food web model. Iran. J. Sci. Technol. Trans. A Sci. 38 (2014), 399–414.

DEPARTMENT OF MATHEMATICS, SCHOOL OF MATHEMATICAL SCIENCES, VALI-E-ASR UNIVERSITY OF RAFSANJAN, RAFSANJAN, IRAN.

E-mail address: namjoo@vru.ac.ir

DEPARTMENT OF MATHEMATICS, FACULTY OF BASIC SCIENCES, UNIVERSITY OF BOJNOURD, BOJNOURD, IRAN.

E-mail address: z.sadegh133@gmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF BASIC SCIENCES, UNIVERSITY OF BOJNOURD, BOJNOURD, IRAN.

E-mail address: javaher\_ langari61@yahoo.com



# A NONSTANDARD FINITE DIFFERENCE SCHEME FOR ADVECTION EQUATION

MEHRAN NAMJOO

ABSTRACT. In this paper, numerical solution of the advection equation is presented based on the nonstandard finite difference (NSFD) scheme. At first two exact finite difference schemes for the advection equation are obtained. Afterwards, an NSFD scheme is presented for this equation. The stability of this scheme is discussed. The numerical results obtained by the NSFD scheme is compared with the exact solution and standard finite difference (SFD) scheme to verify the accuracy and efficiency of the NSFD scheme.

Keywords: Nonstandard finite difference scheme; Stability; Advection equation.

## 1. INTRODUCTION

Partial differential equations (PDEs) play an important role in the various fields of physical science and engineering. One of the important PDE which appears in various applications, is the advection equation. In this paper we consider the advection of the form

(1.1) 
$$u_t + \beta_1 u_x = \lambda u - \lambda u^{\alpha},$$

where the coefficients  $\beta_1$ ,  $\lambda$  and  $\alpha$  are nonnegative constants. NSFD schemes which introduced by Mickens to solve ordinary differential equations (ODEs) and PDEs [2], have been proved to be one of the most efficient methods in recent years. Instead of classical methods, NSFD schemes can alternatively be used to obtain more qualitative results and remove numerical instabilities. These schemes are developed for compensating the weaknesses, such as numerical instabilities that may be caused by SFD methods [3, 4]. In this paper, we try to construct an NSFD scheme that could preserve the qualitative behaviour of the solution (1.1).

## 2. Analysis of exact finite difference schemes

In this section, we obtain two exact finite difference schemes for the advection equation (1.1). Along the direction of characteristics [1], equation (1.1) can be translated into a group of ODEs as follows

$$x'(t) = \beta_1, \quad w'(t) = \lambda w(t) - \lambda w^{\alpha}(t), \quad x(t_0) = x_0, \quad w(t_0) = h(x_0),$$

where w(t) = u(x(t), t) and  $x(t) = \beta_1(t - t_0) + x_0$  denotes a characteristic of (1.1). For  $w(t_0) > 0$  and  $\alpha > 1$ , the analytical solution

(2.1) 
$$w'(t) = \lambda w(t) - \lambda w^{\alpha}(t),$$

is

$$w(t) = \left[e^{(1-\alpha)\lambda(t-t_0)}((w^{1-\alpha}(t_0)-1)-1)\right]^{\frac{1}{1-\alpha}}.$$

2010 Mathematics Subject Classification. 39A14, 65M06.

Speaker: Mehran Namjoo.

Let  $\Delta x$  and  $\Delta t$  denote the space and time step sizes, respectively. Set  $x_j = j\Delta x$ for  $j \in Z$  and  $t_n = n\Delta t$  for  $n \in N$ . Define  $w_{n+1} = w(t_{n+1})$  and  $w_n = w(t_n)$ .

**Lemma 2.1.** For  $\alpha > 1$  and the initial value  $w(t_0) > 0$ (2.2)

$$\frac{w_{n+1} - w_n}{\phi_1(\Delta t)} = \frac{\lambda w_n - \lambda w_n^{\alpha}}{\int_0^1 (1 + s(\frac{w_{n+1}}{w_n} - 1))^{-\alpha} ds}, \quad \frac{w_{n+1} - w_n}{\phi_2(\Delta t)} = \frac{\lambda w_{n+1} - \lambda w_{n+1}^{\alpha}}{\int_0^1 (1 + s(\frac{w_n}{w_{n+1}} - 1))^{-\alpha} ds},$$

are two exact finite difference schemes for equation (2.1), where

$$\phi_1(\Delta t) = \frac{1 - e^{-(\alpha - 1)\lambda\Delta t}}{(\alpha - 1)\lambda}, \quad \phi_2(\Delta t) = \frac{e^{(\alpha - 1)\lambda\Delta t - 1}}{(\alpha - 1)\lambda}$$

Let  $u_m^n = u(x_m, t_n)$ , where u(x, t) denotes the solution of (1.1). Under a relation between the step sizes

(2.3) 
$$\Delta x = \beta_1 \Delta x$$

 $u_m^{n+1}$  and  $u_{m-1}^n$  are on the same characteristic  $x(t) - x_{m-1} = \beta_1(t - t_n)$ . Then, with  $w(t) = u(x(t), t), \ w_n = u_{m-1}^n$  implies  $w_{n+1} = u_m^{n+1}$ .

**Theorem 2.2.** In the case the initial value h(x) > 0, under the condition (2.3)

(2.4) 
$$\frac{u_m^{n+1} - u_m^n}{\phi_1(\Delta t)} + \frac{u_m^n - u_{m-1}^n}{\phi_1(\frac{\Delta x}{\beta_1})} = \frac{\lambda u_{m-1}^n - \lambda (u_{m-1}^n)^{\alpha}}{\int_0^1 (1 + s(\frac{u_m^{n+1}}{u_{m-1}^n} - 1))^{-\alpha} ds}$$

and

$$\frac{u_m^{n+1} - u_m^n}{\phi_2(\Delta t)} + \frac{u_m^n - u_{m-1}^n}{\phi_2(\frac{\Delta x}{\beta_1})} = \frac{\lambda u_m^{n+1} - \lambda (u_m^{n+1})^{\alpha}}{\int_0^1 (1 + s(\frac{u_m^n - 1}{u_m^{n+1}} - 1))^{-\alpha} ds},$$

are two exact schemes of advection equation (1.1) with  $\alpha > 1$ , where  $\varphi_1(\Delta t) = \frac{1-e^{-(\alpha-1)\lambda\Delta t}}{\alpha-1}$  and  $\varphi_2(\Delta t) = \frac{e^{(\alpha-1)\lambda\Delta t}-1}{(\alpha-1)\lambda}$ .

## 3. Analysis of NSFD scheme

In this section, we present an NSFD scheme for the advection equation (1.1). Exact scheme (2.2) can be written as

(3.1) 
$$\frac{w_{n+1} - w_n}{\phi_1(\Delta t)} = \frac{\lambda w_n}{\int_0^1 (1 + s(\frac{w_{n+1}}{w_n} - 1))^{-\alpha} ds} - \frac{\lambda w_n^{\alpha}}{\int_0^1 (1 + s(\frac{w_{n+1}}{w_n} - 1))^{-\alpha} ds}$$

For the integral in the first term on the right hand (3.1), the following approximation is used

(3.2) 
$$\int_0^1 (1 + s(\frac{w_{n+1}}{w_n} - 1))^{-\alpha} ds \approx 1.$$

Next, consider the integral in the second term. For  $\frac{w_{n+1}}{w_n} \neq 1$ , set  $s_0 = \frac{\left(\frac{w_{n+1}}{w_n}\right)^{\frac{1}{\alpha}} - 1}{\left(\frac{w_{n+1}}{w_n} - 1\right)}$ . Let  $\frac{w_{n+1}}{w_n} > 0$ , then for  $\alpha > 1$ , we can deduce  $0 < s_0 < 1$ . Thus the second integral could be approximated by

(3.3) 
$$\int_0^1 (1+s(\frac{w_{n+1}}{w_n}-1))^{-\alpha} ds \approx (1+s_0(\frac{w_{n+1}}{w_n}-1))^{-\alpha} \cdot 1 = \frac{w_{n+1}}{w_n}$$

By (3.2) and (3.3), we obtain the following NSFD scheme for (2.1)

(3.4) 
$$\frac{w_{n+1} - w_n}{\varphi(\Delta t)} = \lambda w_n - \lambda w_n^{\alpha - 1} w_{n+1},$$

where  $\varphi(\Delta t) = \Delta t + O(\Delta t^2)$ .

**Lemma 3.1.** Let  $\alpha > 1$  and the denominator function  $\varphi(\Delta t)$  of NSFD scheme (3.4) satisfying

$$\begin{split} \varphi(\Delta t) &> 0, \qquad 1 < \alpha \leq 2, \\ 0 < \varphi(\Delta t) \leq \frac{1}{(\alpha - 2)} \lambda, \qquad \alpha > 2 \end{split}$$

then the solution has the following properties.

(i) The fixed point 0 is locally asymptotically unstable, while the fixed point 1 is asymptotically stable.

(ii) If the initial value  $0 < w_0 < 1$ , the solution  $w_n$  increases monotonously to the fixed point 1 as  $n \to \infty$ .

With the NSFD (3.4), an NSFD scheme for (1.1) is provided as

(3.5) 
$$\frac{u_m^{n+1} - u_m^n}{\varphi(\Delta t)} + \frac{u_m^n - u_{m-1}^n}{\varphi(\frac{\Delta x}{\beta_1})} = \lambda u_{m-1}^n - \lambda (u_{m-1}^n)^{\alpha - 1} u_m^{n+1},$$

where the denominator function satisfies

(3.6) 
$$\varphi(\Delta t) = \Delta t + O(\Delta t^2).$$

**Theorem 3.2.** For  $\alpha > 1$ , consider NSFD scheme (3.5) with the denominator function  $\varphi(\Delta t)$  given by (3.6) satisfying

$$\varphi(\Delta t) > 0,$$
  $1 < \alpha \le 2,$   
 $0 < \varphi(\Delta t) \le \frac{1}{(\alpha - 2)}\lambda,$   $\alpha > 2.$ 

Then with the relation (2.3) between the step sizes, the solution of (3.5) has the following properties.

(i) The fixed point 0 is locally asymptotically unstable, while the fixed point 1 is locally asymptotically stable.

(ii) If  $0 < u_m^n < 1$ ,  $u_m^k$  increase monotonously to the fixed point 1 as  $k \to \infty$ .

# 4. Numerical results

**Example 4.1.** Consider the advection equation (1.1) with  $\beta_1 = 2$ ,  $\lambda = 10$  and  $\alpha = \frac{9}{4}$ . The initial value is given as  $u(x,0) = e^{-0.01x^2}$ . Let  $\Delta x = 2\Delta t = 0.4$ . The condition in Theorem 3.2 about the NSFD scheme (3.5) is  $\varphi(\Delta t) < \frac{2}{5}$ , and thus, we may choose  $\varphi(\Delta t) = \frac{1-e^{-\frac{5}{2}\Delta t}}{\frac{5}{2}}$  in scheme (3.5). In Figure 1, the NSFD scheme (3.5) is compared with the exact solution and the following SFD scheme

(4.1) 
$$\frac{u_m^{n+1} - u_m^n}{\Delta t} + 2\frac{u_m^n - u_{m-1}^n}{\Delta x} = 10[u_{m-1}^n - (u_{m-1}^n)^{\frac{9}{4}}].$$



A NONSTANDARD FINITE DIFFERENCE SCHEME FOR ADVECTION EQUATION 123

FIGURE 1. Example 4.1 with  $\Delta t=0.2$ ,  $\Delta x=0.4$ .

### 5. Conclusion

In this paper, we present two exact finite difference schemes for advection equation. The proposed denominator functions depend on  $\Delta x$ ,  $\Delta t$  and NSFD scheme for the advection equation can be constructed by using the method in Mickens paper. The results show that the NSFD scheme meet the properties that physically relevant solutions should have. By comparison, the presented NSFD scheme is also found to be accurate and effective.

### References

- R. Larsson, V. Thomee, Partial differential equations with numerical methods. Springer, Berlin, 2003.
- [2] R. Mickens, Calculation of denominator functions for nonstandard finite-difference schemes for differential equations satisfying a positivity condition, Numer. Methods Partial Differential Equations. 23 (2007), 672–691.
- [3] M. Namjoo, S. Zibaei, Numerical solutions of FitzHugh-Nagumo equation by exact finitedifference and NSFD schemes. Comp. Appl. Math. (2016), 1–17, DOI 10.1007/s40314-016-0406-9.
- [4] S. Zibaei, M. Namjoo, A NSFD scheme for Lotka-Volterra food web model, Iran. J. Sci. Technol. Trans. A Sci. 38 (2014), 399–414.

DEPARTMENT OF MATHEMATICS, VALI-E-ASR UNIVERSITY OF RAFSANJAN, RAFSANJAN, IRAN. *E-mail address*: namjoo@vru.ac.ir



# THE ENERGY AND SPECTRUM OF NON-COMMUTING GRAPHS

MARYAM NASIRI AND SAYYED HEIDAR JAFARI

ABSTRACT. Let G be a non-abelian group and Z(G) be the center of G. The non-commuting graph  $\Gamma(G)$  of G is a graph with vertex set G - Z(G) in which two vertices x and y are joined if and only if  $xy \neq yx$ . In this paper we calculate the energy, Laplacian energy and spectrum of non-commuting graph of dihedral group  $D_{2n}$ . Also we obtain the characteristic polynomial and energy of non-commuting graph of GL(2, q).

Keywords: Non-commuting graph; Energy of a graph; Spectrum.

### 1. INTRODUCTION

Let G be a non-abelian finite group and Z(G) be its center. The non-commuting graph  $\Gamma(G)$  of G is a graph whose vertex set is G - Z(G) and two vertices x and y are joined if and only if  $xy \neq yx$ . Note that if G is an abelian, then  $\Gamma(G)$  is the null graph. The adjacency matrix of graph  $\Gamma$  is the (0,1) matrix A indexed by the vertex set  $V(\Gamma)$  of  $\Gamma$ , where  $A_{xy} = 1$  when there is an edge from x to y in  $\Gamma$  and  $A_{xy} = 0$  otherwise. The characteristic polynomial of A, denoted by  $P_A(x)$ , is the polynomial defined by  $P_A(x) = det(xI - A)$  where I denotes the identity matrix. The spectrum of a finite graph  $\Gamma$  is by definition the spectrum of the adjacency matrix A, that is, its set of eigenvalues together with their multiplicities. Assume that  $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_t$  are t distinct eigenvalues of  $\Gamma$  with the corresponding multiplicities  $k_1, k_2, \ldots, k_t$ . We denote by

$$spec(\Gamma) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_t \\ k_1 & k_2 & \dots & k_t \end{pmatrix}.$$

Let  $\Gamma$  be an undirected graph without loops. The Laplacian matrix of  $\Gamma$  is the matrix L indexed by the vertex set of  $\Gamma$ , with zero row sums. If D is the diagonal matrix, indexed by the vertex set of  $\Gamma$  such that  $D_{xx}$  is the degree of x then L = D - A. The energy of a graph  $\Gamma$ , denoted by  $E(\Gamma)$ , is defined as

$$E(\Gamma) = \sum_{i=1}^{n} |\lambda_i| \, .$$

where  $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$  are the eigenvalues of the adjacency matrix of  $\Gamma$ . This concept was introduced by Gutman and is intensively studied in chemistry, since it can be used to approximate the total  $\pi$ -electron energy of a molecule(see, e.g.[3,4]). Let  $\Gamma$  be a graph with n vertices and m edges. Let  $\mu_1, \mu_2, \ldots, \mu_n$  be the Laplacian

<sup>2010</sup> Mathematics Subject Classification. 20D99, 05C50.

Speaker: Maryam Nasiri.

eigenvalues of  $\Gamma$ . The Laplacian energy of a graph  $\Gamma$ , is defined as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|.$$

# 2. Main results

In this section, we calculate the energy and Laplacian energy of non-commuting graph of dihedral group  $D_{2n}$  and also the characteristic polynomial and energy of GL(2,q).

# Theorem 2.1.

(1) If n is even, then

$$spec(\Gamma_{D_{2n}}) = \begin{pmatrix} -2 & 0 & \frac{(n-2)-\sqrt{5n^2-12n+4}}{2} & \frac{(n-2)+\sqrt{5n^2-12n+4}}{2} \\ \\ \frac{n}{2} - 1 & \frac{3n}{2} - 3 & 1 & 1 \end{pmatrix}$$

(2) If n is odd, then

$$spec(\Gamma_{D_{2n}}) = \begin{pmatrix} -1 & 0 & \frac{(n-1)-\sqrt{5n^2-6n+1}}{2} & \frac{(n-1)+\sqrt{5n^2-6n+1}}{2} \\ & & & \\ (n-1) & (n-2) & 1 & & 1 \end{pmatrix}.$$

Corollary 2.2.

$$E(\Gamma_{D_{2n}}) = \begin{cases} (n-2) + \sqrt{5n^2 - 12n + 4} & \text{if } n \text{ is even} \\ (n-1) + \sqrt{5n^2 - 6n + 1} & \text{if } n \text{ is odd.} \end{cases}$$

In Table 1, the energies of some non-commuting graphs of dihedral groups is given.

groups	characteristic polynomial	eigenvalues	energy
$D_6$	$(-x)(-x-1)^2(x^2-2x-6)$	$(0)_1, (-1)_2, (1+\sqrt{7}), (1-\sqrt{7})$	$2 + 2\sqrt{7}$
$D_8$	$(-x)^{3}(-x+4)(-x-2)^{2}$	$(0)_3, (4)_1, (-2)_2$	8
$D_{10}$	$(-x)^{3}(-x-1)^{4}(x^{2}-4x-20)$	$(0)_3, (-1)_4, (2-2\sqrt{6}), (2+2\sqrt{6})$	$4 + 4\sqrt{6}$
$D_{12}$	$(-x)^{6}(-x-2)^{2}(x^{2}-4x-24)$	$(0)_6, (-2)_2, (2+2\sqrt{7}), (2-2\sqrt{7})$	$4 + 4\sqrt{7}$
$D_{16}$	$(-x)^9(-x-2)^3(x^2-6x-48)$	$(0)_9, (-2)_3, (3+\sqrt{57}), (3-\sqrt{57})$	$6 + 2\sqrt{57}$

TABLE 1. The characteristic polynomial , eigenvalues and energy of some graphs.

**Theorem 2.3.** Let G = GL(2,q), where  $q = p^n > 2$  (p is prime). Then

$$P_{\Gamma_G}(x) = x^{(n-t)} \left[ x^3 + \left( -q^4 + q^3 + 4q^2 - 6q + 2 \right) x^2 + \left( -2q^6 + 6q^5 - q^4 - 13q^3 + 15q^2 - 5q \right) x - (q-1)^4 q^2 (q-2)(q+1) \right] (x + (q-1)^2)^q (x + q(q-1))^{\frac{q^2 - q - 2}{2}} (x + (q-1)(q-2))^{\frac{q^2 + q - 2}{2}}$$

, where  $t = q^2 + q + 1$ .

Corollary 2.4.  $E(\Gamma_{GL(2,q)}) = |2A - \frac{b}{3}| + 2|A + \frac{b}{3}| + q(q-1)^2 + \frac{q^2 - q - 2}{2}q(q-1) + \frac{q^2 + q - 2}{2}(q-1)(q-2).$ 

# Theorem 2.5.

(1) If n is even, then

$$spec(L(\Gamma_{D_{2n}})) = \begin{pmatrix} 2n-2 & 2n-4 & n & 0\\ \frac{n}{2} & \frac{n}{2} & n-3 & 1 \end{pmatrix}$$

(2) If n is odd, then

$$spec(L(\Gamma_{D_{2n}})) = \begin{pmatrix} 0 & n & 2n-1 \\ 1 & n-2 & n \end{pmatrix}.$$

Corollary 2.6.

$$LE(\Gamma_{D_{2n}}) = \begin{cases} \frac{2n(n^2 - 4n + 6)}{2n - 2} \text{ if } n \text{ is even} \\ 3n(n-1) \text{ if } n \text{ is odd.} \end{cases}$$

In Table 2, the Laplacian characteristic polynomial and eigenvalues of noncommuting graphs of some dihedral groups is given.

groups	characteristic polynomial	eigenvalues	energy
$D_8$	$x(x-4)^3(x-6)^2$	$(0)_1, (4)_3, (6)_2$	8
$D_{10}$	$(-x)(x-5)^3(9-x)^5$	$(0)_1, (5)_3, (9)_5$	60
$D_{12}$	$x(x-6)^3(x-8)_3(x-10)_3$	$(0)_1, (6)_3, (8)_3, (10)_3$	$\frac{108}{5}$
$D_{14}$	$(-x)(x-7)^5(13-x)^7$	$(0)_1, (7)_5, (13)_7$	126
$D_{16}$	$x(x-8)^5(x-12)^4(x-14)^4$	$(0)_1, (8)_5, (12)_4, (14)_4$	$\frac{304}{7}$

TABLE 2. The characteristic polynomial, eigenvalues and Laplacian energy of some graphs.

## References

- 1. A. Abdollahi, S. Akbari and H. R. Maimani, Non-commuting graph of a group, J.Algebra 298(2006),no .2,468-492.
- D. Cvetković, M. Doob, H. Sachs, Spectra of graphs -Theory and application, Academic press, Newyork, 1980.
- 3. I. Gutman, O. E. Polansky, Mathematical consepts in organic chemistry, springer, Berlin, 1986.
- 4. I.Gutman, The energy of a graph: old and new results, in: A. Betten, A. Kohnert, R. Laue and A. Wassermann(Eds)-verlag, Berlin, 2001, pp 196-211.

Department of Mathematics, Shahrood University of Technology, Shahrood, Iran, P.O. Box: 316-3619995116

E-mail address: nasirimaryam.63@gmail.com

Department of Mathematics, Shahrood University of Technology, Shahrood, Iran, P.O. Box: 316-3619995116

E-mail address: shjafari550gmail.com



# ON A REMARKABLE IMPLICATION OF NEAREST SUBMATRIX

ALIMOHAMMAD NAZARI AND ATIYEH NEZAMI

ABSTRACT. In this paper for a given normal block matrix  $G_D = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ that  $A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times m}, C \in \mathbb{C}^{m \times n}, D \in \mathbb{C}^{m \times m}$  and  $\operatorname{Rank}\{\operatorname{G_D}\} < n + m - 1$  and given eigenvalues of  $\mathcal{M} = D - CA^{-1}B$  as  $\lambda_1, \lambda_2, \cdots, \lambda_m$ , we find a remarkable formula for distance set of matrix  $G_{D_0}(\operatorname{same to } G_D \operatorname{instead} \operatorname{of} D$  we set matrix X) to  $G_D$  as  $\rho(G_D, G_{D_0}) = \left(\frac{|\lambda_m| + |\lambda_{m-1}|}{2}\right)^{\frac{1}{2}}$  where  $G_{D_0}$  is a normal matrix.

Keywords: Eigenvalues; Normal matrix; Distance norm.

# 1. INTRODUCTION

In [1] Gracia and Valesco for a block matrix  $G_D = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  found the block matrix  $G_{D_0} = \begin{pmatrix} A & B \\ C & D_0 \end{pmatrix}$  with two eigenvalues zero that be the closest matrix to  $G_D$ . Nazari and Ikramov in [2] given an explicit formula for the distance  $\rho_2(A, L)$  in case of a normal matrix A, that  $\rho_2(A, L)$  is distance from A to the set L of matrices with a multiple zero eigenvalue. If  $G_D$  be normal matrix then we can obtained the similar formula in [2] for the distance  $\rho_2(G_D, \mathcal{L})$ , where  $\mathcal{L}$  is set of block normal matrices  $G_D$  that have multiple eigenvalue zero.

2. Main results

Assume that  $\mathcal{M}$  and  $\mathcal{N} \in \mathbb{C}^{m \times m}$  such that

$$(2.1) \qquad \qquad \mathcal{M} = D - CA^{-1}B$$

(2.2) 
$$\mathcal{N} = I_m + CA^{-2}B$$

and  $\gamma \in \mathbb{R}$  and

(2.3) 
$$P(\gamma) = \begin{pmatrix} \mathcal{M} & \gamma \mathcal{N} \\ 0 & \mathcal{M} \end{pmatrix}, \qquad p(\gamma) = \sigma_{2m-1}(P(\gamma)),$$

where  $\sigma(P(\gamma))$  is singular values of  $P(\gamma)$  that considered decreasing order. Now we assume that  $\mathcal{M}$  be a normal  $m \times m$  matrix, whose eigenvalue decomposition is

(2.4) 
$$\mathcal{M} = Q\Lambda Q^*,$$

where Q is a unitary matrix of eigenvectors and

(2.5) 
$$\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_m).$$

<sup>2010</sup> Mathematics Subject Classification. 15A18, 15A60, 93B10. Speaker: Alimohammad Nazari.

Let us apply the unitary similarity transformation

$$P_0(\gamma) \to R_0(\gamma) = U^* P_0(\gamma) U$$

to block matrix (2.3), where

$$U = Q \oplus Q,$$

that the symbol  $\oplus$  is denoted direct sum of matrices. We have

(2.6) 
$$R_0(\gamma) = \begin{pmatrix} \Lambda & \gamma \mathcal{N} \\ 0 & \Lambda \end{pmatrix}.$$

By a symmetric permutation of rows and columns, matrix (2.6) with some of calculations we can be obtained a direct sum as following

(2.7) 
$$S_0(\gamma) = \Gamma_1 \oplus \ldots \oplus \Gamma_m,$$

where

$$\Gamma_i = \begin{pmatrix} \lambda_i & \gamma \\ 0 & \lambda_i \end{pmatrix}, \qquad i = 1, 2, \dots, m.$$

Since matrices (2.6) and (2.7) are unitarily similar, by invarient property of norm-2 related to unitary matrices, we have

(2.8) 
$$\rho_2(G_D, \mathcal{L}) = \max_{\gamma \ge 0} \sigma_{2(m)-1}(S_0(\gamma)).$$

Now similar to [2] we study the behavior of the singular values  $\mu_1$  and  $\mu_2$  ( $\mu_1 \ge \mu_2$ ) of the matrix

$$\Gamma = \left(\begin{array}{cc} a & \gamma \\ 0 & a \end{array}\right), \qquad a \in \mathbb{C}$$

as functions of the nonnegative parameter  $\gamma$ .

The numbers  $\mu_1$  and  $\mu_2$  squared are the roots of the quadratic equation

(2.9) 
$$v^2 - (2|a|^2 + \gamma^2)v + |a|^4 = 0$$

The larger root of Eq. (2.9) is given by the formula

(2.10) 
$$v_1 = \mu_1^2 = |a|^2 + \frac{\gamma^2}{2} + \sqrt{|a|^2\gamma^2 + \frac{\gamma^4}{4}}.$$

Thus,  $\mu_1$  monotonically increases as a function of  $\gamma$ ; moreover,  $\mu_1 \to +\infty$  as  $\gamma \to +\infty$ . On the other hand, since

$$\mu_1 \mu_2 = |a|^2 = \text{const}$$

we have that  $\mu_2$  is monotonically decreasing to zero.

Returning to matrix (2.7), consider two possible cases.

Case 1:  $|\lambda_{m-1}| = |\lambda_m|$ .

Let  $\xi_1$  and  $\xi_2$  be the singular values of  $\Gamma_{m-1}$ , and  $\eta_1$  and  $\eta_2$  be the singular values of  $\Gamma_m$ . It is obvious that  $\xi_1 = \eta_1, \xi_2 = \eta_2$ . The analysis in above implies that, when  $\gamma > 0$ ,

$$\xi_2 < |\lambda_m|.$$

Hence,

$$\sigma_{2(m)-1}(S_0(\gamma)) < |\lambda_m|.$$

Since  $P_0(0)$  satisfies the relation

$$\sigma_{2(m)-1}(P_0(0)) = |\lambda_m|,$$

we see that, in the case under consideration, the maximum in formula for  $\rho_2(G_D, \mathcal{L})$ is attained at  $\gamma = 0$ . One can easily indicate the matrix  $L_0 \in \mathcal{L}$  closest to  $G_D$ :

$$L_0 = Q\Lambda_0 Q^*$$

where

$$\Lambda_0 = \operatorname{diag}(\lambda_1, \dots, \lambda_{m-2}, 0, 0).$$

*Case 2*:  $|\lambda_{m-1}| > |\lambda_m|$ .

We retain the former meaning of the notation  $\xi_1, \xi_2, \eta_1, \eta_2$ . According to above, the function  $\xi_2(\gamma)$  monotonically decreases to zero as  $\gamma \to +\infty$ , whereas  $\eta_1(\gamma)$ monotonically increases to infinity. Since

$$\gamma_1(0) = |\lambda_m| < \xi_2(0) = |\lambda_{m-1}|,$$

there exists a uniquely defined  $\gamma^* > 0$  such that

(2.11) 
$$\eta_1(\gamma^*) = \xi_2(\gamma^*)$$

It follows that

$$\sigma_{2(m)-1}(P_0(\gamma)) = \begin{cases} \eta_1(\gamma), & 0 \le \gamma \le \gamma^*, \\ \xi_2(\gamma), & \gamma \ge \gamma^* \end{cases}$$

and (2.12)

$$\rho_2(G_D, \mathcal{L}) = \sigma_{2(m)-1}(P_0(\gamma^*)).$$

Thus, we have that, in Case 2,

$$\lambda_m | = \sigma_{2(m)-1}(P_0(0)) < \rho_2(G_D, \mathcal{L}) < |\lambda_{m-1}| = \rho_2(G_D, \mathcal{K})$$

Meanwhile,  $\mathcal{K}$  is a subset of  $\mathcal{L}$ . We arrive at a remarkable conclusion:

Let  $\mathcal{M}$  be a normal  $m \times m$  matrix with the eigenvalues  $\lambda_i$ . If

$$|\lambda_{m-1}| > |\lambda_m|,$$

then the matrix with a multiple zero eigenvalue closest (with respect to the 2-norm) to  $G_D$  has rank m + n - 1 and hence is nondiagonalizable.

In fact, we are able to give an explicit formula for the distance  $\rho_2(G_D, \mathcal{L})$  in case of a normal matrix  $G_D$ .

**Theorem 2.1.** Let  $G_D$  be a normal  $(n+m) \times (n+m)$  matrix that  $\mathcal{M} = D - CA^{-1}B$  has the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$ .

(2.13) 
$$\rho_2(G_D, \mathcal{L}) = \left(\frac{|\lambda_{m-1}|^2 + |\lambda_m|^2}{2}\right)^{1/2}$$

#### References

- Juan-Miguel Gracia, Francisco E. Velasco, Nearesr Southeast Submatrix that makes multiple a prescribed eigenvalue. Part 1. Linear Algebra and its Applications 430(2009)1196-1215.
- [2] Kh.D. Ikramov, A.M. Nazari, On a remarkable implication of the Malyshev formula, Dokl. Akad. Nauk., 385 (2002) 599-600.

DEPARTMENT OF MATHEMATICS, ARAK UNIVERSITY, P.O. BOX 38156-8-8349, ARAK, IRAN; *E-mail address: a-nazari@araku.ac.ir* 

E-mail address: atiyeh.nezami@gmail.com



The 9<sup>th</sup> Seminar on Linear Algebra and its Applications July 5-6, 2017, University of Tabriz, Tabriz, Iran

# COLUMN-ACTION METHODS IN IMAGE RECONSTRUCTION

#### TOURAJ NIKAZAD

ABSTRACT. We give a fast review of algebraic reconstruction methods which are based on row partitioning (called row action methods). Furthermore, we represent their different types as simultaneous, sequential and block versions and give their convergence analysis. We also discuss the semi-convergence phenomenon and the regularization property of the methods. We next give the column versions of algebraic reconstruction methods which are interesting alternatives to their row-version counterparts: they converge to a least squares solution, and they provide a basis for saving computational work by skipping small updates. At the end, we introduce the flexible versions of both row and column iterative methods.

Keywords: Row and column-block methods; Block iterative methods; Semi-convergence; Relaxation parameter; Quotient convergence.

## 1. INTRODUCTION

Ill-posed sets of linear equations typically arise when discretizing certain types of integral transforms. A well known example is image reconstruction, which can be modeled using the Radon transform. After expanding the solution into a finite series of basis functions a large, sparse and ill-conditioned linear system occurs. We consider the solution of such systems, i.e.,

(1.1)

Ax = b

where  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ .

Our main application is in the field of image reconstruction from projections [?]. We work with a medical test phantom and with a reconstruction problem in electron microscopy. In both of our examples a planar cross section of the object is considered, and the distribution of some physical parameter (the X-ray attenuation in medicine and the Coulomb potential in electron microscopy) in the cross section has to be reconstructed from estimates of its line integrals for a finite number m of lines in the cross section; we use  $b_i$  to represent the estimated line integral for the *i*-th line. The unknown function of two variables has real values and is called the picture. A fundamental model for solving this task is provided by the series-expansion approach (e.g., Herman [?] or Censor [?]), which we formulate here as follows. A Cartesian grid of square picture-elements, called pixels, is introduced into the region of interest so that it covers the whole picture that has to be reconstructed. The pixels are numbered in some agreed manner, say from 1 (top left corner pixel) to n (bottom right corner pixel); see Figure 1.

<sup>2010</sup> Mathematics Subject Classification. 65F10, 65R32. Speaker: Touraj Nikazad.



FIGURE 1. The fully discretized model of the image reconstruction problem.



FIGURE 2. (right to left) sequential method, simultaneous method and sequential block-iterative method.

Let  $||x|| = \sqrt{x^T x}$  denote the 2-norm, and  $||x||_M = \sqrt{x^T M x}$  denote a weighted Euclidean norm. Also, let  $M^{1/2}$  denote the square root of M, and  $\rho(Q)$  denote the spectral radius of Q. For a  $m \times n$  matrix W, we use N(W) and R(W) to denote the null space and range of W, respectively. The orthogonal projection from  $\mathbb{R}^n$  onto N(W) is denoted by P(W). Here  $x_M(A, b)$  denotes a solution of min  $||Ax - b||_M$ with the minimal Euclidean norm.

#### 2. Row-iterative methods

The Algebraic Reconstruction Technique (ART) is a fully sequential method, and has a long history and rich literature. Originally it was proposed by Kaczmarz [?], and independently, for use in image reconstruction by [?]. The vector of unknowns is up-dated at each equation of the system, after which the next equation is addressed. In the simultaneous algorithms the current iterate is first projected on all sets to obtain intermediate points, and then the next iterate is made by an averaging process, as convex combination, of intermediate points. We next explain blockiterative methods. The basic idea of a block-iterative algorithm is to partition Initially the iteration vectors approach a regularized solution while continuing the iteration often leads to iteration vectors corrupted by noise. This phenomenon is called semi-convergence by Natterer [?, p. 89]; for analysis of the phenomenon, see, e.g., [?, ?, ?, ?, ?, ?]. The typical overall error behavior is shown in Figure 3.

2.1. Block methods. Let  $B = \{1, \dots, m\}$  be an index set which is partitioned into T nonempty subset  $B_t$  such that

$$(2.1) B = \bigcup_{1 \le t \le T} B_t$$

Here, the subsets  $\{B_t\}_{t=1}^T$  are not necessarily disjoint and each of them consists of  $m(t) \ge 1$  indices, e.g.,

(2.2) 
$$B_t = \{i_1^t, \cdots, i_{m(t)}^t\}.$$

Let

(2.3) 
$$A_t = \begin{pmatrix} a^{i_1^t} \\ \vdots \\ a^{i_{m(t)}^t} \end{pmatrix}$$

(2.4) 
$$b_t = \begin{pmatrix} b^{i_1^t} \\ \vdots \\ b^{i_{m(t)}^t} \end{pmatrix}$$

be the corresponding blocks of A and b with respect to the partitioning (2.1-2.2), respectively. Here  $a^i$  and  $b^i$  are the *i*-th row of A and b, respectively.

The SeqBI method is defined as follows

(2.5) 
$$x_{k+1} = x_k + \lambda_k A_{[k]}^T M_{[k]}(b_{[k]} - A_{[k]} x_k), \ k = 0, 1, 2, \cdots$$

where  $\{\lambda_k\}_{k=0}^{\infty}$  denote positive relaxation parameters and  $\{M_t\}_{t=1}^T$  are given SPD matrices. Also  $x_0 \in \mathbb{R}^n$  is an arbitrary initial vector and  $[k] \triangleq k \pmod{T} + 1$ . An iterative step moves from  $x_k$  to  $x_{k+1}$  and one cycle is defined as one pass through



FIGURE 3. Semi-convergence phenomenon.

all data. With T = 1 there is just one block so  $M_t = M \in \mathbb{R}^{m \times m}$ , and the method is called fully simultaneous iteration. On the other end when p = m each block consists of a single row so  $M_t \in \mathbb{R}$ , t = 1, 2, ..., m, and we get a fully sequential iteration.

The convergence theorems of (2.5) are given in [16, Theorem II.1, Remark A.11], see also [?]. The iterative reconstruction algorithms obtain proper and desired results even if the measured data contain noises [13, 10, 11] or there are insufficient data [15, 14]. This class includes several well-known iterative algorithms such as algebraic reconstruction technique (ART), also called Kaczmarz's method [17], and Cimmino's method [7], which is a simultaneous block iterative version of the ART. The simultaneous algebraic reconstruction technique (SART) was proposed as a refinement and superior implementation of the ART [1]. The methods DROP [4] and CAV [6] were presented to improve the rate of convergence using the sparsity of the matrix A. The block version of the CAV algorithm, called BICAV, was proposed in [5].

## 3. Column-iterative methods

Column-oriented algorithms have not been explored much in the literature. The studies in [3, 9] explain how to base an iterative reconstruction algorithm on columns. Furthermore, a column-based reconstruction method using nonnegativity constraints and a two-parameter algorithm based on a block-column-partitioning is considered in [21, 2], respectively. We have to mention that their analysis is not applicable to our block-column method.

In a recent paper [12], the authors studied a stationary column-oriented version of algebraic iterative methods which is called block-column-iteration (BCI). The BCI method is an iterative reconstruction algorithm based on column-partitioning of A rather than row-partitioning. Against row-action methods, the column-action method updates a part (bunch) of the approximate solution in each iteration. Furthermore, their numerical tests show that the column-action method provides a basis for saving computational work. Under mild assumptions, they show that the sequence generated by the BCI algorithm converges to a least squares solution of (1.1).

Let A be partitioned into q disjoint block-columns  $\{A_t\}_{t=1}^q$ , where

$$A_t \in \mathbb{R}^{m \times n_t}, \ \sum_{t=1}^q n_t = n,$$

and accordingly the vector x be partitioned as

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{pmatrix}, \quad x_t \in \mathbb{R}^{n_t}$$

for t = 1, ..., q. Furthermore, we consider  $\{\omega_t\}_{t=1}^q$  and  $\{M_t\}_{t=1}^q \in \mathbb{R}^{n_t \times n_t}$  as the sets of positive relaxation parameters and symmetric positive definite matrices (weight matrices), respectively. The BCI method is the following algorithm, see [12].

**Algorithm 3.1.** Initialization:  $x^0 \in \mathbb{R}^n$  is arbitrary, and  $r^{0,1} = b - Ax^0$  $k = 0, 1, 2, \dots$  (cycles or outer iterations)  $t = 1, 2, \ldots, q$  (inner iterations)

$$\begin{aligned} x_t^{k+1} &= x_t^k + \omega_t M_t A_t^T r^{k,t} \\ r^{k,t+1} &= b - \sum_{i=1}^t A_i x_i^{k+1} - \sum_{i=t+1}^q A_i x_i^k \end{aligned}$$

 $r^{k+1,1} = r^{k,q+1}$ 

Let a cycle denote one pass through all blocks, i.e., one outer iteration. Since BCI uses a single block in every inner iteration, it takes q inner iterations to complete a cycle. Some special choices for  $M_t$  are given in [12, section 3.3].

The convergence analysis of BCI algorithm is given as follows.

**Theorem 3.2.** [12] The iterates of BCI algorithm converge to a least squares solution of (1.1) if for any  $0 < \epsilon < 2$  it holds

$$\omega_t \in \left[\epsilon, (2-\epsilon)/\rho(A_t M_t A_t^T)\right], \quad t = 1, 2, \dots, q$$

As it is seen in the BCI method, the blocks  $A_t$ , the relaxation parameters  $\omega_t$ and the weight matrices  $M_t$  are constant in all cycles. The first open question on BCI method is: If the weight matrices are updated in each iteration (cycle), i.e. flexible version, then does the new method still remain convergent? What about the relaxation parameters, could it be updated them in each iteration (cycle)? Also, one may consider other possibilities as involving constraints, considering noisy data and studying semiconvergence analysis and finding optimal relaxation parameters and proper weight matrices. To the extent of our knowledge, the mentioned problems have been partially studied for row-action methods by recent research works. To involve the convex constraints for the row-action methods, see [19, section 6] and [18, section 4.1]. To control semi-convergence phenomenon, we refer to [13, 10, 11, 20]. Furthermore, the research works [8] and [19] introduce the optimal relaxation parameter in different senses.

#### References

- Anders H Andersen and Avinash C Kak. Simultaneous algebraic reconstruction technique (sart): a superior implementation of the art algorithm. Ultrasonic imaging, 6(1):81–94, 1984.
- [2] Z-Z Bai and C-H Jin. Column-decomposed relaxation methods for the overdetermined systems of linear equations. *International journal of applied mathematics*, 13(1):71–82, 2003.
- [3] Åke Björck and Tommy Elfving. Accelerated projection methods for computing pseudoinverse solutions of systems of linear equations. *BIT Numerical Mathematics*, 19(2):145–163, 1979.
- [4] Yair Censor, Tommy Elfving, Gabor T Herman, and Touraj Nikazad. On diagonally relaxed orthogonal projection methods. SIAM Journal on Scientific Computing, 30(1):473–504, 2008.
- [5] Yair Censor, Dan Gordon, and Rachel Gordon. Bicav: A block-iterative parallel algorithm for sparse systems with pixel-related weighting. *Medical Imaging, IEEE Transactions on*, 20(10):1050–1060, 2001.
- [6] Yair Censor, Dan Gordon, and Rachel Gordon. Component averaging: An efficient iterative parallel algorithm for large and sparse unstructured problems. *Parallel computing*, 27(6):777– 808, 2001.
- [7] Gianfranco Cimmino. Calcolo approssimato per le soluzioni dei sistemi di equazioni lineari. Istituto per le applicazioni del calcolo, 1938.
- [8] A. R. De Pierro. Methodos de projeção para a resolução de sistemas gerais de equações algébricas lineares. Doctotal thesis, UFRJ, Cidade Universitaria, 1981. Rio de Janerio, Brasil.
- [9] Tommy Elfving. Block-iterative methods for consistent and inconsistent linear equations. *Numerische Mathematik*, 35(1):1–12, 1980.

- [10] Tommy Elfving, Per Christian Hansen, and Touraj Nikazad. Semiconvergence and relaxation parameters for projected SIRT algorithms. SIAM Journal on Scientific Computing, 34(4):A2000–A2017, 2012.
- [11] Tommy Elfving, Per Christian Hansen, and Touraj Nikazad. Semi-convergence properties of Kaczmarz's method. *Inverse Problems*, 30(5):055007, 2014.
- [12] Tommy Elfving, Per Christian Hansen, and Touraj Nikazad. Convergence analysis for columnaction methods in image reconstruction. *Numerical Algorithms*, pages 1–20, 2016.
- [13] Tommy Elfving, Touraj Nikazad, and Per Christian Hansen. Semi-convergence and relaxation parameters for a class of SIRT algorithms. *Electronic Transactions on Numerical Analysis*, 37:321–336, 2010.
- [14] E Garduño, GT Herman, and R Davidi. Reconstruction from a few projections by  $\ell_1$ -minimization of the haar transform. *Inverse problems*, 27(5):055006, 2011.
- [15] Gabor T Herman and Ran Davidi. Image reconstruction from a small number of projections. Inverse problems, 24(4):045011, 2008.
- [16] Ming Jiang and Ge Wang. Convergence studies on iterative algorithms for image reconstruction. Medical Imaging, IEEE Transactions on, 22(5):569–579, 2003.
- [17] Stefan Kaczmarz. Angenäherte auflösung von systemen linearer gleichungen. Bulletin International de l' Académie Polonaise des Sciences et des Lettres, 35:355–357, 1937.
- [18] T. Nikazad and M. Abbasi. A unified treatment of some perturbed fixed point iterative methods with an infinite pool of operators. *Inverse problems, Special issue on superiorization* theory and applications, 2016, to be appeared.
- [19] T. Nikazad, M. Abbasi, and T. Elfving. Error minimizing relaxation strategies in landweber and kaczmarz type iterations, doi: 10.1515/jiip-2015-0082. J. Inverse Ill-Posed Probl., 2015.
- [20] T Nikazad and M Karimpour. Controlling noise error in block iterative methods. Numerical Algorithms, pages 1–19, 2016.
- [21] David W Watt. Column-relaxed algebraic reconstruction algorithm for tomography with noisy data. Applied optics, 33(20):4420–4427, 1994.

School of Mathematics, Iran University of Science and Technology, P.O. Box: 16846-13114, Tehran , Iran

E-mail address: tnikazad@iust.ac.ir



# SPECTRAL RADIUS OF ADJACENCY AND LAPLACIAN MATRICES OF FRACTIONAL POWERS OF GRAPHS

MORTEZA FAGHANI, SIAMAK FIROUZIAN, AND MOSTAFA NOURI-JOUYBARI

ABSTRACT. For any  $k \in \mathbb{N}$ , the k-subdivision of a graph G is a simple graph  $G^{\frac{1}{k}}$ , which is constructed by replacing each edge of G with a path of length k. The *m*-th power of the *n*-subdivision of G has been introduced as a fractional power of G, denoted by  $G^{\frac{m}{n}}$ . In this paper, we present some results related to the spectral radius in terms of adjacency and Laplacian matrices of fractional powers of given graphs.

Keywords: Eigenvalue; Adjacency Matrix; Laplacian Matrix.

## 1. INTRODUCTION

Suppose that G = (V, E) is a graph. For any  $\gamma \in \mathbb{N}$ , the  $\gamma$ -power of the graph G, denoted by  $G^{\gamma}$ , is a graph on the same vertex set as G such that two vertices are adjacent in G if and only if their distance in G is at most  $\gamma$ . or equivalently,  $E(G^{\gamma}) = \{xy : 1 \leq d_G(x, y) \leq \gamma\}$ . The  $\gamma$ -subdivision of G, denoted by  $G^{\frac{1}{\gamma}}$ , is constructed by replacing each edge ab of G with a path of length  $\gamma$ . In this terminology,  $G^{\frac{1}{1}}$  is isomorphic to G.

In [1], Henning has introduced the fractional powers of a graph. The fractional power  $G^{\frac{m}{n}}$  is then defined to be the *m*-power of the *n*-subdivision of *G*; that is,  $G^{\frac{m}{n}} = (G^{\frac{1}{n}})^m$ . In [4] Iradmusa investigate domination number and independent domination number of fractional powers of graphs. In [3] he inquire the chromatic number and clique number of fractional power of graphs.

Suppose A(G) stands for the adjacency matrix of G. Bring into mind that A(G) is given as a symmetric (0, 1)-matrix of order G, say it n, where  $[A(G)]_{uv} = 1$  if and only if the vertex u is adjacent to the vertex v. Moreover, let us consider  $\lambda_{max}(A(G))$  (briefly,  $\lambda_{max}(G)$ ) to indicate the largest eigenvalue of the matrix A(G), and we will  $\lambda_{min}(A(G))$  (briefly,  $\lambda_{min}(G)$ ) denote the smallest eigenvalue, similarly. In [5] Lovasz state some upper and lower bound about  $\lambda_{max}(G)$  and  $\lambda_{min}(G)$ . The Laplacian of the graph is defined as the  $n \times n$  matrix  $L(G) = (L_{ij})$  in which  $L_{ij} = d_i$  if i = j and  $L_{ij} = -A_{ij}$  if  $i \neq j$ . Here  $d_i$  denotes the degree of node i. In the case of weighted graphs, we define  $d_i = \sum_j A_{ij}$ . So L(G) = D(G) - A(G), where D(G) is the diagonal matrix of the degrees of G. In [2] have state some upper and lower bound about  $\lambda_{max}(L(G))$  and  $\lambda_{min}(L(G))$ .

In this paper, we present some results related to the  $\lambda_{max}$  and  $\lambda_{min}$ . of adjacency and laplacian matrices of fractional powers of graphs.

 $<sup>2010\</sup> Mathematics\ Subject\ Classification.\ 15B99,\ 05C50.$ 

Speaker: Mostafa Nouri-Jouybari.

#### 2. Main results

In this section we state some results related to the eigenvalues of adjacency and laplacian matrices of fractional powers of graphs.

**Lemma 2.1.** Let the graph G be an arbitrary connected graph. Then for  $k \in \mathbb{N}$ , we have:

(2.1) 
$$\lambda_{max}(A(G^{\frac{1}{k}})) \ge \lambda_{max}(A(G)) \ge \lambda_{min}(A(G)) \ge \lambda_{min}(A(G^{\frac{1}{k}}))$$

*Proof.* Comparing the structures of  $A(G^{\frac{1}{k}})$  and A(G) for given graph G with order n and |E(G)| = m, we have for any  $y \in \mathbb{R}^n$  that:

(2.2) 
$$\mathbf{y}^T A(G) \mathbf{y} = \begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix}^T A(G^{\frac{1}{k}}) \begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix}$$

Now, for the certain **y** as an eigenvector of A(G) of eigenvalue  $\lambda_{max}(A(G))$  we obtain:

T

(2.3) 
$$\lambda_{max}(A(G)) = \frac{\mathbf{y}^T A(G) \mathbf{y}}{\mathbf{y}^T \mathbf{y}} = \frac{\begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix}^T A(G^{\frac{1}{k}}) \begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix}}{\begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix}^T \begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix}} \leq max_{\mathbf{x} \in \mathbb{R}^{n+(k-1)m}} \frac{\mathbf{x}^T A(G^{\frac{1}{k}}) \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \lambda_{max}(A(G^{\frac{1}{k}})).$$

Similarly, the same argument works for the smallest eigenvalues and the consequence follows.  $\hfill \Box$ 

**Remark 2.2.** We note that this Lemma is also valid by replacing  $A(G^{\frac{1}{k}})$  and A(G) with any symmetric matrix A and its sub-matrix A(S), respectively.

Lemma 2.3. Suppose the matrix A with its sub-matrices given by:

is a symmetric matrix. Then:

(2.5) 
$$\lambda_{max}(A) + \lambda_{min}(A) \le \lambda_{max}(B) + \lambda_{max}(D)$$

*Proof.* In order to prove the claim, consider  $\mathbf{x}$  as an eigenvector of A of eigenvalue  $\lambda_{max}(A)$ . To facilitate the forthcoming process, let us also assume that  $\mathbf{x}$  is a unit vector. Take it as  $\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$  by applying the same partition as (2.4) for A. Related to the sub-vectors  $\mathbf{x}_1$  and  $\mathbf{x}_2$  we have the following cases. We first consider the case in which neither  $\mathbf{x}_1$  nor  $\mathbf{x}_2$  is an all-zero vector. In this case, we put:

(2.6) 
$$\mathbf{y} = \begin{pmatrix} \frac{\|\mathbf{x}_2\|}{\|\mathbf{x}_1\|} \mathbf{x}_1 \\ -\frac{\|\mathbf{x}_1\|}{\|\mathbf{x}_2\|} \mathbf{x}_2 \end{pmatrix}$$

One can verify that **y** is also a unit vector, so:

(2.7) 
$$\mathbf{y}^T A \mathbf{y} \ge \lambda_{min}(A)$$

Then, we derive:

(2.8)

$$\begin{split} \lambda_{max}(A) + \lambda_{min}(A) &\leq \mathbf{x}^{T} A \mathbf{x} + \mathbf{y}^{T} A \mathbf{y} \\ \mathbf{x}_{1}^{T} B \mathbf{x}_{1} + \mathbf{x}_{1}^{T} C \mathbf{x}_{2} + \mathbf{x}_{2}^{T} C^{T} \mathbf{x}_{1} + \mathbf{x}_{2}^{T} D \mathbf{x}_{2} \\ &+ \frac{\parallel \mathbf{x}_{2} \parallel^{2}}{\parallel \mathbf{x}_{1} \parallel^{2}} \mathbf{x}_{1}^{T} B \mathbf{x}_{1} - \mathbf{x}_{1}^{T} C \mathbf{x}_{2} - \mathbf{x}_{2}^{T} C^{T} \mathbf{x}_{1} + \frac{\parallel \mathbf{x}_{1} \parallel^{2}}{\parallel \mathbf{x}_{2} \parallel^{2}} \mathbf{x}_{2}^{T} D \mathbf{x}_{2} \\ &= \mathbf{x}_{1}^{T} B \mathbf{x}_{1} + \mathbf{x}_{2}^{T} D \mathbf{x}_{2} + \frac{\parallel \mathbf{x}_{2} \parallel^{2}}{\parallel \mathbf{x}_{1} \parallel^{2}} \mathbf{x}_{1}^{T} B \mathbf{x}_{1} + \frac{\parallel \mathbf{x}_{1} \parallel^{2}}{\parallel \mathbf{x}_{2} \parallel^{2}} \mathbf{x}_{2}^{T} D \mathbf{x}_{2} \\ &= \left(1 + \frac{\parallel \mathbf{x}_{2} \parallel^{2}}{\parallel \mathbf{x}_{1} \parallel^{2}}\right) \mathbf{x}_{1}^{T} B \mathbf{x}_{1} + \left(1 + \frac{\parallel \mathbf{x}_{1} \parallel^{2}}{\parallel \mathbf{x}_{2} \parallel^{2}}\right) \mathbf{x}_{2}^{T} D \mathbf{x}_{2} \\ &\leq \lambda_{max}(B) \left(\parallel \mathbf{x}_{1} \parallel^{2} + \parallel \mathbf{x}_{2} \parallel^{2}\right) + \lambda_{max}(D) \left(\parallel \mathbf{x}_{1} \parallel^{2} + \parallel \mathbf{x}_{2} \parallel^{2}\right) \\ &= \lambda_{max}(B) + \lambda_{max}(D) \end{split}$$

since  $\mathbf{x}$  has been taken as a unit vector.

We now turn our attention to the case in which  $\| \mathbf{x}_2 \| = 0$  (or  $\| \mathbf{x}_1 \|$ , which is actually the same case). Remark 2.2 together with (2.1) implies that  $\lambda_{max}(B) \leq \lambda_{max}(A)$ . Therefore, it must be the case that  $\mathbf{x}_1$  is an eigenvector of eigenvalue  $\lambda_{max}(A)$  of  $\lambda_{max}(B)$ , and thus  $\lambda_{max}(A) = \lambda_{max}(B)$  To terminate the proof, also observe that the recent remark and (2.1) make us to conclude that:

(2.9) 
$$\lambda_{max}(D) \ge \lambda_{min}(D) \ge \lambda_{min}(A)$$

which mixing to the recent equality, completes the proof.

**Theorem 2.4.** Let us assume that G is an arbitrary graph with adjacent matrix 
$$A(G)$$
. Then:

(2.10) 
$$\lambda_{max}(A(G)) = \lambda_{max}(A(G^{\frac{1}{2}})) \quad and \quad \lambda_{min}(A(G^{\frac{1}{2}})) = 0$$

*Proof.* Consider the adjacent matrix of graph  $G^{\frac{1}{2}}$  given by

(2.11) 
$$A(G^{\frac{1}{2}}) = \begin{pmatrix} A(G) & C \\ C^T & B \end{pmatrix}$$

Suppose that  $x_{e_1}, x_{e_2}, \ldots, x_{e_m}$  are the vertex of  $V(G^{\frac{1}{2}}) \setminus V(G)$  where the indices  $e_1, e_2, \ldots, e_m$  correspond to the edges of G. Then, an arrangement of vertices shows that the row of sub-matrix  $C^T$  and the column of sub-matrix C can be  $x_{e_1}, x_{e_2}, \ldots, x_{e_m}$  which yields the sub-matrix  $B = (b_{x_{e_i}x_{e_j}})$ . Now, as we know that  $x_{e_i}x_{e_j} \notin E(G^{\frac{1}{2}})$  for any  $1 \leq i \neq j \leq m$ , then B = 0. On the other hand, from the linear algebra and from the fact that all eigenvalue of nilpotent matrices are zero, we obtain  $\lambda_{max}(B) = 0$ . This together with (2.1) and (2.3) supplies the following:

$$\lambda_{max}(A(G)) \le \lambda_{max}(A(G^{\frac{1}{2}})) \le \lambda_{max}(A(G^{\frac{1}{2}})) + \lambda_{min}(A(G^{\frac{1}{2}})) \le \lambda_{max}(A(G))$$

This means that  $\lambda_{max}(A(G)) = \lambda_{max}(A(G^{\frac{1}{2}}))$  and  $\lambda_{min}(A(G^{\frac{1}{2}})) = 0$ 

138
#### References

- Henning M. A., Distance domination in graphs, Domination in graphs, Textbooks Pure Appl. Math. 209(1998)321- 349.
- [2] Hoffman A.J., On eigenvalues and colorings of graphs, Graph Theory and its Applications, Academic Press, New York (1970), 79-91.
- [3] Iradmusa M. N., On colorings of graph fractional powers, Discrete Math. 310(2010)1551-1556.
- [4] Iradmusa M. N. , Domination number of graph fractional powers, Bull. Iranian Math. Soc.40(2014) 1479-1489.
- [5] Lov'asz L., Eigenvalues of graphs (2007).

Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-3697 Tehran, Iran

 $E\text{-}mail\ address:\ {\tt mo_faghan@yahoo.com}$ 

Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-3697 Tehran, Iran

E-mail address: siamfirouzian@pnu.ac.ir

Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-3697 Tehran, Iran

*E-mail address*: mostafa.umz@gmail.com



## HOMEOMORPHIC IMAGES OF ORTHOGONAL BASES

M. KEBRYAEE AND M. RADJABALIPOUR

ABSTRACT. Necessary and sufficient conditions are obtained for a sequence  $\{x_j : j \in \mathbb{J}\}$  in a Hilbert space to be, up to the elimination of a finite subset of  $\mathbb{J}$ , the linear homeomorphic image of an orthogonal basis of some Hilbert space K. This extends a similar result for orthonormal bases due to Holub [J.R. Holub, Pre-frame operators, Besselian frames, and near-Riesz bases in Hilbert spaces, Pro. Amer. Math. Soc., 122(3): 779-785, 1994]. The proofs given here are based on simple linear algebra techniques.

SHAHID BAHONAR UNIVERSITY OF KERMAN

Speaker: M. Radjabalipour.



## SOME RESULTS ON INVERTIBLE HYPERVECTOR SPACES

ZOHREH NAZARI, ELHAM ZANGIABADI, AND AFSANEH RANJBAR

ABSTRACT. In this paper, the concept of invertible hypervector space is introduced and some of properties are investigated.

Keywords: Hypervector space; Invertible hypervector space.

## 1. INTRODUCTION

Hyperstructures theory was initially introduced in 1934 by Marty, [5]. He defined hypergroups that are a generalization of groups and studied some properties and applications of them in non-commutative groups, rational functions and algebraic functions.

Several authors studied various aspects of hyperstructures [1, 2, 3, 4].

In 1998, the notion of hypervector spaces was initially introduced by Tallini [9]. She investigated some properties of them [8]. Some properties of hypervector spaces are studied in [6, 7].

In this paper we introduce the notion of invertible hypervector space and we study some results on it. Let us recall some definitions which are useful in our results.

**Definition 1.1.** A hypervector space over a field K a quadruplet  $(V, +, \circ, K)$  is an abelian group and

$$p: K \times V \to p^*(V)$$

is a mapping of  $K \times V$  into the power set of V (deprived of the empty set), such that the following properties holds:

 $(1.1) (a+b) \circ x \subseteq (a \circ x) + (b \circ x), \ \forall a, b \in K, \ \forall x \in V,$ 

$$(1.2) a \circ (x+y) \subseteq (a \circ x) + (a \circ y), \ \forall a \in K, \ \forall x, y \in V$$

- $(1.3) a \circ (b \circ x) = (ab) \circ x, \ \forall a, b \in K, \ \forall \ x \in V,$
- $(1.4) x \in 1 \circ x, \ \forall x \in V.$
- (1.5)  $a \circ (-x) = -a \circ x, \forall a \in K, \forall x \in V.$

If in (1) the equality holds, the hypervector space is called strongly left distributive. If in (2) the equality holds, the hypervector space is called strongly right distributive. The distributive hypervector spaces were studied in [9].

**Example 1.2.** Let  $V = \mathbb{R}^{\nvDash}$ . Define

$$\circ: \mathbb{R} \times \mathbb{R}^{\mu} \longrightarrow \mathbb{R}^{\mu}$$
$$a \circ (x, y) = ax \times \mathbb{R},$$

Then  $(\mathbb{R}^{\neq}, +, \circ)$  is a strongly distributive hypervector space.

<sup>2010</sup> Mathematics Subject Classification. 20N20, 22A30.

Speaker: Afsaneh Ranjbar.

**Definition 1.3.** Let V be a hypervector space over the field K;  $H \subseteq V$  is a subspace of V, if

$$H \neq \phi, \ H - H \subseteq H, \ a \circ H \subseteq H, \ \forall a \in K.$$

**Definition 1.4.** Let  $(V, +, \circ, K)$  and  $(V', +', \circ', K)$  be two hypervector spaces over the same field K. A mapping

$$f:V\to V'$$

is called a homomorphism between V and V', if  $\forall a \in K, \ \forall x, y \in V$ :

(1.6) 
$$f(x+y) = f(x) + f(y),$$

(1.7) 
$$f(a \circ x) \subseteq a \circ' f(x).$$

If in (6) the equality holds, then f is called strong homomorphisms.

**Theorem 1.5.** ([10]) Let V and W be hypervector spaces. If H is a subspace of V, then f(H) is a subspace of W. In particular Imf is a subspace of W, conversely if H' is a subspace of W, then  $f^{-1}(H')$  is a subspace of V.

## 2. Main results

In this section first we define the concept of a hypervector space and then we study some properties of it.

**Definition 2.1.** A hypervector space  $(V, +, \circ, K)$  is called an invertible hypervector space, if for all  $a, b \in K$  and  $x, y \in V$  it satisfies in the following conditions:

1)  $1 \circ x = \{x\}$ 2)  $x \in a \circ y \Longrightarrow y \in a^{-1} \circ x.$ 

**Definition 2.2.** Let  $(V, +, \circ)$  be a hypervector space and  $\rho$  be an equivalence relation on V. If A, B are non-empty subset of V, then  $A\overline{\rho}B$  means that  $\forall a \in A, \exists b \in B \text{ such that } a\rho b \text{ and } \forall b' \in B, \exists a' \in A \text{ such that } a'\rho b'.$ 

**Definition 2.3.** Let  $(V, +, \circ)$  be a hypervector space and  $\rho$  be an equivalence relation on V. The equivalence relation  $\rho$  is called regular if for all  $r \in K$ , from  $a\rho b$ , it follows that  $(r \circ a)\overline{\rho}(r \circ b)$ .

**Theorem 2.4.** Let  $(V, +, \circ)$  be an invertible hypervector space,  $\rho$  be an equivalence relation on V and  $V/\rho = \{[v] : v \in V\}$ . Then  $V/\rho$  is an invertible hypervector space with respect to actions

 $\begin{array}{l} \oplus: V/\rho \times V/\rho \to V/\rho, \\ where \ ([x], [y]) \mapsto [x] \oplus [y] = [x+y], \ \forall x, y \in V. \ and \\ \odot: K \times V/\rho \to V/\rho \\ k \odot [x] = \{[z] \ : \ z \in k \circ x\}, \end{array}$ 

if and only if  $\rho$  is regular.

*Proof.* First we check that the mappings  $\oplus$  and  $\odot$  are well-defined on  $V/\rho$ . Simply, we see that  $\oplus$  is well-defined. It is enough to show that  $\odot$  is well-defined. Let (r, [x]) = (r', [x']). So r = r' and [x] = [x']. Now we check that  $r \odot [x] = r \odot [x']$ . We have  $x\rho x'$ . Since  $\rho$  is regular it follows that  $(rox)\overline{\rho}(rox')$ . Hence, for all  $z \in r \circ x$ , there exists  $z_1 \in r \circ x'$  such that  $z\rho z_1$ , which means that  $[z] = [z_1]$ . It follows that  $r \odot [x] \subseteq r \odot [x']$  and similarly we obtain the converse inclusion.

Now we show that  $(V/\rho, \oplus, \odot)$  is an invertible hypervector space. Simply we see

that  $(V/\rho, \oplus)$  is an abelian group. It is enough to show that  $\odot$  has the properties of the Definitions 1.1 and 2.1. For all  $a, b \in K$  we have

$$(a+b) \odot [x] = \{[z] : z \in (a+b) \circ x\}$$
$$\subseteq \{[z] : z \in a \circ x + b \circ x\}$$
$$= \{[c] \oplus [d] \mid c \in a \circ x, d \in b \circ x\}$$
$$= a \odot [x] \oplus b \odot [x].$$

Also the properties (ii), (iii), (iv) in Definition 1.1 are easily obtained. It is enough to check that the properties of Definition 2.1. So we have

1)  $1 \odot [x] = \{[z] : z \in 1 \circ x\} = [x].$ 2)  $[y] \in a \circ [x] = \{[z] : z \in a \circ x\} \Rightarrow y \in a \circ x \Rightarrow x \in a^{-1} \circ y \Rightarrow [x] \in a^{-1} \odot [y].$ 

**Theorem 2.5.** Let  $(V, +, \circ)$  and  $(V', +', \circ')$  be two hypervector spaces,  $f : V \to V'$  be a strong homomorphism and H be a subspace of V. Then f(H) is an invertible hypervector space. In particularly Im(f) is an invertible hypervector space.

*Proof.* One can easily check that f(H) is a hypervector space. It is enough to show that f(H) is invetible hypervector space. So for all  $y = f(x) \in f(H)$  we have

1)  $1 \circ' y = 1 \circ' f(x) = f(1 \circ x) = f(x)$ .

2)  $f(x) \in r \circ' f(z) = f(r \circ y) \Rightarrow x \in r \circ y \Rightarrow y \in r^{-1} \circ x \Rightarrow f(y) \in f(r^{-1} \circ x)$ 

since f is strong homomorphism so  $f(y) \in r^{-1} \circ' f(x)$ . It completes the proof.  $\Box$ 

## References

- R. A. Borzoei, A. Hasankhani and H. Rezaei, Some results on canonical, cyclic hypergroups and join spaces, Ita. J. Pure Appl. Math., 11 (2002), 77-87.
- [2] P. Corsini, Prolegomena of Hypergroup Theory, Second edition, Aviani Editore, 1993.
- [3] I. Cristea and M. Stefanescu, Hypergroups and n-ary relations, European J. Combin., 31 (2010) 780-789.
- $\label{eq:alpha} [4] \ B. \ Davvaz, Polygroups and permutation polygroups, Math. Balkanica (N.S.), {\bf 14} (2000) \ 41-58.$
- [5] F. Marty. Sur nue generalization de la notion do group, 8<sup>th</sup> congress of the Scandinavic Mathematics, Stockholm, (1934), 45-49.
   [6] B. Baia, M. Vormaur, Normal Murray and the superior fraction for the second statement of Mathematical Sciences.
- [6] P. Raja, M. Vaezpour, Normed Hypervector spaces, Iranian Journal of Mathematical Sciences and Informatics, 2 (2007) 35-44.
- [7] A. Taghavi, R. Hosseinzadeh, Operators on Weak Hypervector Spaces. Ratio Mathematica, 22 (2012) 37-43.
- [8] M. S. Tallini, Weak hypervector space and norms in such spaces, Algebraaic Hyper Structurs and Applications, Jast, Rumania, Hadronic Press, (1994), 199-206.
- [9] M.S. Tallini, Hypervector spaces, Proceedings of the forth international congress of algebraic hyperstructures and applications, Xanthi, Greece, (1990) 167-174.
- [10] M. S. Tallini, Sottospazi, spazi quozienti ed omomorfismi tra spazi ipervettoriali. Rivista di Mat. Pure e applicata, Univ. di Udine, 18 (1996) 71-84.

VALI-E-ASR UNIVERSITY OF RAFSANJAN, DEPARTMENT OF MATHEMATICS, P. O. BOX 7713936417, RAFSANJAN, IRAN.

E-mail address: z.nazari@vru.ac.ir

Vali-e-Asr University of Rafsanjan, Department of Mathematics, P. O. Box 7713936417, Rafsanjan, Iran.

E-mail address: e.zangiabadi@vru.ac.ir

VALI-E-ASR UNIVERSITY OF RAFSANJAN, DEPARTMENT OF MATHEMATICS, P. O. BOX 7713936417, RAFSANJAN, IRAN.



#### A NOTE ON INVERTIBLE WEAK HYPERVECTOR SPACES

ELHAM ZANGIABADI, ZOHREH NAZARI, AND AFSANEH RANJBAR

ABSTRACT. In this paper, the concept of invertible weak hypervector space is introduced and some of properties are investigated.

Keywords: Weak hypervector space; Invertible weak hypervector space; Normal weak hypervector space.

#### 1. INTRODUCTION

Hyperstructures theory was born in 1934, when Marty [3] defined hypergroups, began to analysis their properties and applied them to groups, rational algebra functions. Now they are widely studies from theoretical point of view and their applications to many subjects of pure and applied properties. Since then many researchers have work on hyperalgebraic structures and developed this theory ([1],[2] and [7]). Tallini introduced the notation of hypervector spaces ([5], [6]) and studied basic properties of them. Note that the hypervector spaces used in this paper are the special case where there is only one hyperoperation, the external one, all the others are ordinary operations.

In this paper we We need some preliminary definitions for to state our results. In this section we state them.

**Definition 1.1.** [6] A weak hypervector space over a field K is a quadruplet  $(V, +, \circ, K)$  such that (V, +) is an abelian group and

$$\circ: K \times V \to p^*(V)$$

is a mapping of  $K \times V$  into the power set of V (deprived of the empty set), such that the following properties holds:

 $(1.1) a \circ (x+y) \cap (a \circ x) + (a \circ y) \neq \emptyset, \ \forall a \in K, \ \forall x, y \in V,$ 

$$(1.2) (a+b) \circ x \cap [(a \circ x) + (b \circ x)] \neq \emptyset, \ \forall a, b \in K, \ \forall x \in V,$$

- $(1.3) a \circ (b \circ x) = (ab) \circ x, \ \forall a, b \in K, \ \forall \ x \in V,$
- $(1.4) x \in 1 \circ x, \ \forall x \in V,$
- (1.5)  $a \circ (-x) = -a \circ x, \forall a \in K, \forall x \in V.$

**Example 1.2.** The set  $\mathbb{R}^2$  with usual sum and the following scalar product is a weak hypervector space on  $\mathbb{R}$ .

$$a \circ x = \begin{cases} \overline{ox} & x \neq 0\\ \{0\} & x = 0 \end{cases},$$

Where  $\overline{ox}$  is the line passing through origion and the point x.

<sup>2010</sup> Mathematics Subject Classification. 20N20, 22A30.

Speaker: Afsaneh Ranjbar.

**Lemma 1.3.** [4] If V is a weak hypervector space over K,  $0 \neq a \in K$  and  $x \in V$ , then there exist an element z in  $a \circ x$  such that  $x \in a^{-1} \circ z$ .

By the above lemma we have the following definition.

**Definition 1.4.** Let V is a weak hypervector space over K,  $a \in K$  and  $x \in V$ . Essential point of  $a \circ x$ , that denoted by  $e_{a \circ x}$ , for  $a \neq 0$  is the element of  $a \circ x$  such that  $x \in a^{-1} \circ e_{a \circ x}$ . For a = 0, we define  $e_{a \circ x} = 0$ .

Note that  $e_{a \circ x}$  is not unique necessarily. Hence, the set of all these elements is denoted by  $E_{a \circ x}$ . In example 1.2  $E_{a \circ x}$  is equal to the set

$$\begin{cases} \overline{ox} & x \neq 0\\ \{0\} & x = 0 \end{cases}$$

**Definition 1.5.** Let  $(X, +, \circ)$  and  $(Y, +', \circ')$  be two weak hypervector spaces over K. A map

$$T: X \to Y$$

 $is \ called$ 

- i) a weak linear map if and only if T(x+y) = T(x) + T(y),  $T(E_{a \circ x}) \subseteq a \circ T(x)$ ,
- ii) a linear map if and only if T(x+y) = T(x) + T(y),  $T(a \circ x) \subseteq a \circ T(x)$ ,

for all  $x, y \in X$  and  $a \in K$ .

#### 2. Main results

In this section we define invertible weak hypervector space and we study some results of them.

**Definition 2.1.** A weak hypervector space V over K is said to be invertible if and only if  $u \in a \circ v$  implies that  $v \in a^{-1} \circ u$ , for all  $u, v \in V$  and  $a \in K$ .

**Lemma 2.2.** Let V be an invertible weak hypervector space. Then  $E_{a \circ x} = a \circ x$ . *Proof.* The proof is straightforward.

**Theorem 2.3.** Let  $(X, +, \circ)$  and  $(Y, +', \circ')$  be two weak hypervector spaces over K and  $T: X \to Y$  be a weak linear map between them. If X be invertible then T is a linear map.

*Proof.* By the use of Lemma 2.2, we have  $E_{a \circ x} = a \circ x$ . So  $T(a \circ x) = T(E_{a \circ x}) \subseteq a \circ' T(x)$ . It completes the proof.

**Definition 2.4.** Let X be a weak hypervector space over K. V is called a normal weak hypervector space, if it satisfies in the following conditions:

- i)  $(E_{a \circ x} + E_{b \circ x}) \cap E_{(a+b) \circ x} \neq \emptyset$ ,
- ii)  $(E_{a \circ x} + E_{a \circ y}) \cap E_{a \circ (x+y)} \neq \emptyset$ , for all  $x, y \in V$  and  $a, b \in K$ .

**Theorem 2.5.** Every invertible weak hypervector space is normal.

*Proof.* According to Lemma 2.2, we have  $E_{a \circ x} = a \circ x$ . Since X be a weak hypervector space, we have

$$E_{(a+b)\circ x} \cap (E_{a\circ x} + E_{b\circ x}) = a \circ (x+y) \cap (a \circ x) + (a \circ y) \neq \emptyset$$

$$E_{(a+b)\circ x} \cap (E_{a\circ x} + E_{b\circ x}) = (a+b) \circ x \cap [(a \circ x) + (b \circ x)] \neq \emptyset.$$

It completes the proof.

#### References

- [1] P. Corsini, Prolegomena of Hypergroup Theory, Second edition, Aviani Editore, 1993.
- [2] P. Corsini, V. Leoreanu, Applications of Hyperstructure Theory, Kluwer Academic Publishers, Advances in Mathematics, 2003.
- [3] F. Marty. Sur nue generalization de la notion do group, 8<sup>th</sup> congress of the Scandinavic Mathematics, Stockholm, (1934), 45-49.
- [4] A. Taghavi, R. Hosseinzadeh, A note on dimension of weak hypervector space. Ita. J. Pure Appl. Mat. 33 (2014), 7-14.
- M.S. Tallini, Hypervector spaces, Proceedings of the forth international congress of algebraic hyperstructures and applications, Xanthi, Greece, (1990) 167-174.
- [6] M. S. Tallini, Weak hypervector space and norms in such spaces, Algebraaic Hyper Structurs and Applications, Jast, Rumania, Hadronic Press, (1994), 199-206.
- [7] T. Vougiuklis, Hyperstructures and their representations, Hadronic Press, Inc, 1994.

Vali-e-Asr University of Rafsanjan, Department of Mathematics, P. O. Box 7713936417, Rafsanjan, Iran.

E-mail address: e.zangiabadi@vru.ac.ir

Vali-e-Asr University of Rafsanjan, Department of Mathematics, P. O. Box 7713936417, Rafsanjan, Iran.

E-mail address: z.nazari@vru.ac.ir

VALI-E-ASR UNIVERSITY OF RAFSANJAN, DEPARTMENT OF MATHEMATICS, P. O. BOX 7713936417, RAFSANJAN, IRAN.



## A COMPARISON OF TWO FAMILIES OF GLOBAL KRYLOV TYPE METHODS FOR MATRIX EQUATION

GHODRAT EBADI AND SOMAIYEH RASHEDI

ABSTRACT. Global Krylov subspace methods have been proposed for solving nonsymmetric linear systems with multiple right-hand sides. In this paper, we consider the well-known global Krylov subspace methods based on short recurrences, i.e. Gl-BiCG and Gl-BiCR. Then, we present new variants of the Gl-BiCG and Gl-BiCR algorithms. Numerical experiments demonstrate that BiCR variants often converge faster than their BiCG counterpart.

Keywords: Sparse linear systems; Multiple right-hand sides; Global methods; Krylov subspace.

#### 1. INTRODUCTION

Global Krylov subspace methods (with preconditioning techniques) are widely used for iterative solution of simultaneous linear systems with the same coefficient matrix A and s right-hand sides (r.h.s.)  $b_i$ ,

which we can write in the block form as the matrix equation

$$AX = B$$
,

where the matrix  $A \in \mathbb{R}^{n \times n}$  is a real sparse nonsymmetric matrix,  $X = [x_1, \ldots, x_s]$ and  $B = [b_1, \ldots, b_s] \in \mathbb{R}^{n \times s}$  with usually  $s \ll n$ . Such linear systems with multiple right-hand sides arise frequently in computational science and engineering.

Recently, Jbilou et al. [2] proposed the global bi-conjugate gradient method (Gl-BiCG) method for solving the nonsymmetric matrix equation (1.1). Also, The global bi-conjugate residual method (Gl-BiCR) was proposed by J. Zhang et al. [1].

In order to further stabilize the convergence behavior and hopefully to accelerate the convergence speed of the Gl-BiCG and Gl-BiCR algorithms, new variants of the Gl-BiCG and Gl-BiCR algorithms will be developed. Then, we compare the Gl-BiCG variants with their Gl-BiCR counterpart in terms of computational time and the number of iterations. Finally, The efficiency of these modifications are demonstrated by numerical experimental results arising in PDEs.

<sup>2010</sup> Mathematics Subject Classification. 65F10, 65F30, 65F50, 65H10. Speaker: Somaiyeh Rashedi.

#### 2. An observation of deriving GL-BiCG and GL-BiCR algorithms

In Global methods we take the s systems (1.1) into one big, tensorized  $sn \times sn$  system

(2.1) 
$$(I \otimes A)x = b$$
, where  $b = \begin{bmatrix} b_1 \\ \vdots \\ b_s \end{bmatrix}$ ,  $x = \begin{bmatrix} x_1 \\ \vdots \\ x_s \end{bmatrix}$ ,

and then apply a standard Krylov subspace method to the system (2.1). Although the recurrence formulas for updating an approximation and a residual matrix are the same in Gl-BiCG and Gl-BiCR approaches, the iteration coefficients  $\alpha_j$  and  $\beta_j$  are different. These coefficients are determined by the following orthogonality conditions:

$$R_j \perp_F W, \quad AP_j \perp_F W.$$

Choosing  $W = Kj(A^T, \tilde{R}_0)$  leads to Gl-BiCG given in the following Algorithm. The algorithm breaks down if the nominator in  $\alpha_j$  or  $\beta_j$  is zero. In a similar spirit,

Gl-BiCR algorithm can be derived by choosing  $W = A^T K j(A^T, \tilde{R}_0)$  and replacing the coefficients  $\alpha_j$  and  $\beta_j$  of Gl-BiCG with the following coefficients

$$\alpha_j = \frac{\langle \hat{R}_j, AR_j \rangle_F}{\langle A^T \tilde{P}_j, AP_j \rangle_F}, \ \beta_j = \frac{\langle \hat{R}_{j+1}, AR_{j+1} \rangle_F}{\langle \tilde{R}_j, AR_j \rangle_F}.$$

Note that one can use  $AP_{j+1} = AR_{j+1} + \beta_j AP_j$  in the Gl-BiCR algorithm to reduce the number of matrix-matrix multiplications per iteration step. The residual  $R_j$  of the Gl-BiCG and Gl-BiCR algorithms is defined as

$$(2.2) R_j = \phi_j(A)R_0,$$

where  $\phi_j$  is a polynomial of degree j with scalar coefficients satisfying  $\phi_j(0) = 1$ and the direction  $P_j$  is defined as

$$(2.3) P_j = \psi_j(A)R_0,$$

where  $\psi_j$  is a polynomial with scalar coefficients. Note that  $\widetilde{R}_j$  and  $\widetilde{P}_j$  can also be expressed as  $\widetilde{R}_j = \phi_j(A^T)\widetilde{R}_0$  and  $\widetilde{P}_j = \psi_j(A^T)\widetilde{R}_0$ . These scalar polynomials are related by the following recurrence formulas

(2.4) 
$$\phi_{j+1}(t) = \phi_j(t) - \alpha_j t \psi_j(t),$$

and

$$\psi_{j+1}(t) = \phi_{j+1}(t) + \beta_j \psi_j(t),$$

with  $\phi_0(t) = \psi_0(t) = 1$ , for  $t \in \mathbb{R}$ .

## 3. GL-BICR STABILIZED ALGORITHM (GL-BICRSTAB)

In this section, we derive a new variant of Gl-BiCR, called Gl-BiCRStab, which speeds up the convergence of the Gl-BiCR. The residual of Gl-BiCRStab is defined by

$$R_j = \widetilde{\phi}_j(A)\phi_j(A)R_0,$$

where  $\phi_j$  is a polynomial defined by a coupled two-term recurrence and  $\phi_j$  is defined in (2.4). The polynomial recurrence for the polynomial  $\phi_{j+1}$  is defined as

(3.1) 
$$\begin{aligned} \widetilde{\phi}_0(A) &= I, \\ \widetilde{\phi}_{j+1}(A) &= \widetilde{\phi}_j(A) - w_j A \widetilde{\phi}_j(A), \end{aligned}$$

for scalar sequence  $w_i$ . One can write (3.1) as

$$\widetilde{\phi}_{j+1}(A) = (I - w_0 A)(I - w_1 A) \dots (I - w_j A).$$

We have

$$R_{j+1} = \widetilde{\phi}_{j+1}(A)R_{j+1}^{Gl-BiCR}$$

where  $R_j^{Gl-BiCR}$  and  $P_j^{Gl-BiCR}$  are defined in (2.2) and (2.3), respectively. Thus, we have

$$\begin{aligned} R_{j+1}^{Gl-BiCR} &= R_j^{Gl-BiCR} - \alpha_j A P_j^{Gl-BiCR}, \\ P_{j+1}^{Gl-BiCR} &= R_{j+1}^{Gl-BiCR} + \beta_j P_j^{Gl-BiCR}. \end{aligned}$$

The appropriate recurrence relations for Gl-BiCRStab can be derived by updating

$$\widetilde{\phi}_{j+1}R_{j+1}^{Gl-BiCR}, \quad \widetilde{\phi}_{j}R_{j+1}^{Gl-BiCR}, \quad \widetilde{\phi}_{j+1}P_{j+1}^{Gl-BiCR}, \quad \widetilde{\phi}_{j+1}P_{j}^{Gl-BiCR}.$$

The above recurrences are respectively converted into

$$R_{j+1} = (I - w_j A)S_j,$$
  

$$R_{j+1} = R_j - \alpha_j AP_j - w_j AS_j,$$
  

$$P_{j+1} = R_{j+1} + \beta_j (I - w_j A)P_j.$$
  

$$S_i = R_j - \alpha_j AP_j.$$

The iteration coefficients  $\alpha_j$  and  $\beta_j$  can be computed in the way that  $R_j$  and  $AP_j$  are orthogonal to the Krylov subspace  $A^T K_j(A^T, \tilde{R}_0)$ . Thus, these coefficients cab be determined as:

$$\begin{aligned} \alpha_j &= \frac{\langle R_j, A^T R_0 \rangle_F}{\langle A P_j, A^T \widetilde{R}_0 \rangle_F}, \\ \beta_j &= -\frac{\langle A S_j, A^T \widetilde{R}_0 \rangle_F}{\langle A P_j, A^T \widetilde{R}_0 \rangle_F} \end{aligned}$$

Also the parameter  $w_j$  is selected to minimize the F-norm of  $R_j$ , so we have

$$w_j = \frac{\langle S_j, AS_j \rangle_F}{\langle AS_j, AS_j \rangle_F}.$$

#### 4. Numerical experiments

In all runs, the iteration was started with  $X_0 = 0$ , B = rand(n, 10),  $\tilde{R}_0 = R_0$ , the stopping criterion was  $\frac{\|R^{(k)}\|_F}{\|R^{(0)}\|_F} \leq 10^{-10}$  or when the number of iterations exceeded 800. For all examples we will also report results using (right) no-fill ILU preconditioner. Note that Its, CPU and  $\|R\|_F$  stand for the number of iterations, consuming time in seconds and final residual norm, respectively.

**Example 4.1.** We consider the elliptic PDE [3]

$$-u_{xx} - u_{yy} + 2\alpha_1 u_x + 2\alpha_2 u_y - 2\alpha_3 u = 0,$$

on the unit square with Dirichlet boundary conditions, where  $\alpha_1 = \alpha_3 = 10$ ,  $\alpha_2 = 20$ . Applying the central finite differences on  $[0,1] \times [0,1]$ , with the mesh size  $h = \frac{1}{201}$  yields the five-diagonal linear systems.

	no	precond	itioning	ILU	J precond	litioning
Methods	Its	CPU(s)	$\ R\ _F$	Its	CPU(s)	$\ R\ _F$
Gl-BiCG	700	19.21	2.34e-010	203	22.56	2.49e-010
Gl-BiCGStab	483	14.51	2.65e-010	127	12.13	1.57e-011
Gl-BiCR	655	15.57	2.65e-010	202	17.11	2.82e-010
Gl-BiCRStab	393	11.37	2.16e-010	120	11.23	1.63e-010

Table 1: Numerical results of Example 4.1.

We can see from Table 1 that Gl-BiCGStab and Gl-BiCRStab require less iteration steps and computational time and give much smoother convergence behavior than the Gl-BiCG and Gl-BiCR, respectively. Moreover, Gl-BiCR variants converge faster than their Gl-BiCG counterparts in the all cases. In the presence of preconditioning, where the preconditioned matrix is better conditioned and convergence is reached after a relatively small number of iterations, Gl-BiCGStab and Gl-BiCRStab yield superior performance in terms of iteration steps and CPU-time.

#### References

- J. Zhang, H. Dai, J. Zhao, A new family of global methods for linear systems with multiple right-hand sides, J. Comput. Appl. Math. 236 (2011), 1562-1575.
- [2] K. Jbilou, H. Sadok, A. Tinzefte, Oblique projection methods for linear systems with multiple right-hand sides, Electron. Trans. Numer. Anal. 20 (2005), 119-138.
- [3] D. Y. Hu, L. Reichel, Krylov Subspace Methods for the Sylvester Equations, Linear Algebra Appl. 174 (1992), 283-314.

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF TABRIZ, TABRIZ, IRAN *E-mail address*: ghodrat\_ebadi@yahoo.com

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF TABRIZ, TABRIZ, IRAN *E-mail address*: s\_rashedi\_t@yahoo.com



# A FAST MULTIPOLE ACCELERATED MESHLESS TECHNIQUE WITHIN A MULTILEVEL NEWTON FRAMEWORK FOR TWO-DIMENSIONAL NONLINEAR CUBIC SCHRÖDINGER EQUATIONS

#### HADI ROOHANI GHEHSAREH

ABSTRACT. In this paper, an efficient numerical technique is performed to approximate the solution of two-dimensional cubic nonlinear Schrödinger equations. The method is based on the coupling of fast multipole strategy and non-symmetric radial basis function collocation method within a multilevel Newton framework. In the proposed process, 3-dimensional radial basis functions are used as the basis function. For solving the resulting nonlinear system, multilevel algorithm based on the Newton approach is constructed and applied. In the multilevel algorithm, to deal with the large scale and ill-conditioned system a fast technique based on the preconditioned GMRES method is used for solving the ill-conditioned system. Finally, the presented method is used for solving an example of the governing problem.

Keywords: Meshless methods; Nonlinear Schrödinger equations; Newton algorithm; Fast multipole method.

#### 1. INTRODUCTION

Here we will consider the following generalized two-dimensional cubic nonlinear Schrödinger equation with wave function  $\Psi(x, y, t)$ :

$$(1.1) \quad i\frac{\partial\Psi}{\partial t} + a\frac{\partial^2\Psi}{\partial x^2} + a\frac{\partial^2\Psi}{\partial y^2} + q \mid \Psi \mid^2 \Psi + \omega(x,y)\Psi = 0, \quad (x,y) \in \Omega, \ t \ge 0,$$

with the following initial and Dirichlet boundary conditions:

(1.2) 
$$\begin{aligned} \Psi(x,y,0) &= \Psi_0(x,y), \quad (x,y) \in \Omega, \\ \Psi(x,y,t) &= g(x,y,t), \quad (x,y) \in \partial\Omega, \ t > 0, \end{aligned}$$

where,  $\Psi(x, y, t)$  is an unknown complex function,  $i = \sqrt{-1}$ ,  $\Omega = [a, b] \times [a, b]$  is a rectangular domain,  $\partial \Omega$  is the boundary of  $\Omega$ ,  $\Psi_0(x, y)$  and g(x, y, t) are given sufficiently smooth functions and  $\omega(x, y)$  is an arbitrary potential function.

In our numerical process we first decompose the unknown complex function  $\Psi$  in the (1.1) into its real and imaginary parts by considering:

(1.3) 
$$\Psi(x, y, t) = u(x, y, t) + iv(x, y, t),$$

where u(x, y, t) and v(x, y, t) are unknown real functions. By substituting Eq.(1.3) in the governing problem (1.1)-(1.2), the following coupled system of nonlinear

<sup>2010</sup> Mathematics Subject Classification. 65M70, 65N55.

Speaker: Hadi Roohani Ghehsareh.

equations will be obtained in a matrix-vector form as

(1.4) 
$$\begin{aligned} \frac{\partial \mathbf{U}}{\partial t} + \mathbf{A_1} \Delta \mathbf{U} + \mathbf{F}(\mathbf{U}) \mathbf{U} + \mathbf{A_2} \mathbf{U} &= 0, \\ \mathbf{U}(x, y, 0) &= \mathbf{C_1}(x, y), \quad (x, y) \in \partial\Omega, \ t > 0, \\ \mathbf{U}(x, y, t) &= \mathbf{C_2}(x, y, t), \quad (x, y) \in \partial\Omega, \ t > 0. \end{aligned}$$

Where

$$\mathbf{U} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathbf{A}_1 = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} 0 & \mathbf{z} \\ -\mathbf{z} & 0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix},$$
$$\Delta \mathbf{U} = \mathbf{U}_{\text{ex}} + \mathbf{U}_{\text{ex}} \quad \mathbf{z} = a(u^2 + v^2) \quad \mathbf{C}_1(x, y) = \begin{pmatrix} Re(\Psi_0) \\ -\omega \end{pmatrix} \text{ and } \mathbf{C}_2(x, y, t) = \mathbf{C}_2(x, y, t)$$

also,  $\Delta \mathbf{U} = \mathbf{U}_{xx} + \mathbf{U}_{yy}, \mathbf{z} = q(u^2 + v^2), \mathbf{C}_1(x, y) = \begin{pmatrix} I \mathbf{U}(\mathbf{x} 0) \\ I m(\Psi_0) \end{pmatrix}$  and  $\mathbf{C}_2(x, y, t) = \langle Re(g) \rangle$ 

$$\binom{Im(g)}{Im(g)}$$

In the next section, an efficient computational technique based on the combination of a fast multipole strategy and meshfree radial basis functions collocation method will be formulated and implemented for the resulting nonlinear system (1.4).

## 2. Main results

2.1. **RBFs collocation method.** In this section a meshfree method based on the unsymmetric RBF collocation method (Kansas method) [1] is employed to approximate the solution of the problem (1.4). For this purpose, the unknown function **U** shall be approximated by 3-dimensional RBFs functions as:

(2.1) 
$$\mathbf{U}(x,y,t) \simeq \sum_{i=1}^{N} \lambda_{\mathbf{i}} \varphi(r_i) = \begin{pmatrix} \sum_{i=1}^{N} \lambda_i^{(1)} \varphi^{(1)}(r_i) \\ \sum_{i=1}^{N} \lambda_i^{(2)} \varphi^{(2)}(r_i) \end{pmatrix},$$

where  $\varphi^{(1)}(.)$  and  $\varphi^{(2)}(.)$  are 3-dimensional globally supported radial basis function and  $r_k$  denotes the Euclidean distance between (x, y, t) and  $(x_k, y_k, t_k)$ . Nodes  $\{\mathbf{x}_k = (x_k, y_k, t_k), k = 0, ..., N\}$  are collocation points in  $(\Omega \times [0, T])$ , usually called centers nodes. Also  $\{\lambda_i\}_{i=1}^N$  is a set of 2N unknown coefficients to be determined. In the fast multipole method, instead of directly evaluating (2.1), we approximate it by using the following functional approximation

(2.2) 
$$\varphi(\|\mathbf{x} - \mathbf{x}\|) = \sum_{l} \sum_{m} \varphi(\|\mathbf{x}_{l} - \mathbf{x}_{m}\|) Q_{M}(\eta_{l}, \eta) Q_{M}(\xi_{m}, \xi),$$

where  $\{\eta_l\}_{l=1}^M$  and  $\{\xi_l\}_{l=1}^M$  are two identical sets of Chebyshev points in  $[-1, 1]^3$ ,  $\eta$  and  $\xi$  are the linear transformations of **x** and **y**, respectively. Moreover, we have

$$Q_M(\eta,\xi) = P_M(\eta_1,\xi_1)P_M(\eta_2,\xi_2)P_M(\eta_3,\xi_3), \quad \eta = (\eta_1,\eta_2,\eta_3), \ \xi = (\xi_1,\xi_2,\xi_3),$$

with  $P_M(\eta,\xi) = \frac{1}{M} + \frac{2}{M} \sum_{i=1}^{M-1} T_i(\eta) T_i(\xi)$ , where  $T_i(.)$  is the first kind Chebyshev polynomial of order *i*. Now the unknown solution U(x, y, t) at the collocation points  $\mathbf{x}_k$  could be approximated as follows:

(2.3) 
$$U(\mathbf{x}_k) = \sum_{l=1}^{M} U(\mathbf{x}_l) Q_M(\eta_l, \eta_k), \quad k = 1, 2, \dots, N.$$

at the computational cost of O(MN). Now based on the Kansa's collocation method, by substituting Eq (2.3) in relations (1.4) and also by choosing N collocation nodes in the computational domain and on the boundary a nonlinear system of algebraic equations is concluded.

		Real part	Imaginary part	
N	с	€∞	Erel	CPU Time
$5^{3}$	0.256	$2.091 \times 10^{-4}$	$3.653 \times 10^{-4}$	1.8
$10^{3}$	0.348	$8.070 \times 10^{-5}$	$8.121 \times 10^{-5}$	15.6
$15^{3}$	0.443	$4.613 \times 10^{-5}$	$4.537 \times 10^{-5}$	48.2
$20^{3}$	0.545	$1.986 \times 10^{-5}$	$1.931 \times 10^{-5}$	115.1
30 <sup>3</sup>	0.619	$7.749 \times 10^{-6}$	$7.899 \times 10^{-6}$	158.8
$40^{3}$	0.785	$2.974 \times 10^{-6}$	$2.035 \times 10^{-6}$	210.2

TABLE 1. The absolute  $\varepsilon_{\infty}$  errors and CPU times for various values of N.

2.2. Multilevel Newton Iteration for nonlinear system. In this section a multilevel Newton algorithm would be used for solving a nonlinear PDE  $\mathcal{L}u = f$  as follow:

Algorithm1: Smoothed Newton iteration (A multigrid like algorithm).

Create nested point sets X<sub>1</sub> ⊂ X<sub>2</sub> ⊂ ... X<sub>K</sub> ⊂ Ω, and an initial guess u<sub>0</sub>.
 For k = 1, 2, ...

 a) Solve the linearized problem on the coarse grid X<sub>k</sub>
 L<sub>uk-1</sub> v = f − Lu<sub>k-1</sub>, on X<sub>k</sub>.
 b) Perform the smoothing of the Newton update
 v̂ = S<sub>tk</sub> v.
 c) Update the previous approximation
 u<sub>k</sub> = u<sub>k-1</sub> + v̂.

Where in this algorithm  $L_{u_{k-1}}$  is the linearization operator of the  $\mathcal{L}$  at  $u_{k-1}$  and  $S_{t_k}$  is the smoothing operator. Here we will use the algorithm.1 for solving the resultant nonlinear system of equations, for this purpose in the first step we will pick  $\mathbf{U}_0 = \{C_1(\mathbf{x}_j)\}_1^N$  as initial guess and the linearization operator  $L_{\mathbf{U}_{k-1}}$  of the nonlinear differential operator  $\mathcal{L}\mathbf{U}$  will be given by

$$L_{\mathbf{U}_{k-1}}\mathbf{V} := \frac{\partial \mathbf{V}}{\partial t} + \mathbf{A}_{1}\Delta\mathbf{V} + \mathbf{J}_{\mathbf{U}_{k-1}}\mathbf{V} + \mathbf{A}_{2}\mathbf{V},$$

where  $\mathbf{J}_{\mathbf{U}_{\mathbf{k}-1}}$  is the Jacobian matrix contains the partial derivatives of  $\mathbf{F}(\mathbf{U})\mathbf{U}$ . So for this model problem, in line 2a of Algorithm.1 at each iteration step the following linear system of equation should be solved

37.7

=

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A_1} \Delta \mathbf{V} + \mathbf{J_{U_{k-1}}} \mathbf{V} + \mathbf{A_2} \mathbf{V}$$
  
=  $\frac{\partial \mathbf{U}_{k-1}}{\partial t} + \mathbf{A_1} \Delta \mathbf{U}_{k-1} + \mathbf{F}(\mathbf{U_{k-1}}) \mathbf{U_{k-1}} + \mathbf{A_2} \mathbf{U}_{k-1}$ 

It would be mentioned here, a preconditioned GMRES method is used to solve the linearized problem. The proposed fast multipole strategy reduces the computational cost of the matrix vector multiplication significantly.

2.3. Numerical Results. Consider a nonlinear model of problem (1.1)-(1.2), in the square domain  $(x, y) \in [0, \pi]^2$ , as [2]:

(2.4) 
$$i\frac{\partial\Psi}{\partial t} + \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \beta \mid \Psi \mid^2 \Psi = 0, \quad (x,y) \in [0,\pi]^2, \ t \ge 0,$$

The equation describes a progressive plane wave. The numerical results are achieved by letting  $A = \frac{1}{2}$ ,  $k_1 = k_2 = \beta = 1$  and 3-dimensional inverse multiquadrics(IMQ) functions as trial functions are reported in Table 1. From the obtained results it is evident that the method approximated the solution with high accuracy.

## References

- E.J. Kansa, Multiquadrics scattered data approximation scheme with applications to computational fluid-dynamics I, surface approximations and partial derivative estimates, Comput. Math. Appl, 19 (1990)127-145.
- [2] S. Abbasbandy, H. Roohani Ghehsare, I. Hashim, A meshfree method based on the radial basis function for the solution of two-dimensional cubic nonlinear Schrödinger equation, Engineering Analysis with Boundary Elements, 37(2013) 885-898.

DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICS, MALEK ASHTAR UNIVERSITY, SHAHIN SHAHR, ISFAHAN, IRAN.

 $E\text{-}mail\ address:\ \texttt{hadiroohani61@gmail.com}$ 



## RELATION BETWEEN FUZZY CONTINUITY AND FUZZY BOUNDEDNESS

MORTAZA SAHELI

ABSTRACT. In this paper, we consider three fuzzy topology on fuzzy normed linear space defined by Felbin. We show that every fuzzy bounded linear operator is fuzzy continuous. Moreover, by example we show that fuzzy continuity is not equivalent to fuzzy boundedness.

Keywords: Fuzzy norm; Fuzzy topology; Locally convex topological vector space.

#### 1. INTRODUCTION

The notion of fuzzy norm on a linear space was first introduced by Katrasas [9]. Feblin [6] gave an idea of a fuzzy norm on a linear space whose associated metric is Kalva type [7]. Cheng and Menderson [3] considered a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [8]. Felbins definition of a fuzzy norm of a linear operator between two fuzzy normed spaces was generalized by Xiao and Zhu [11]. Bag and Samanta [2] introduced a notion of boundedness of a linear operator between fuzzy normed spaces, and studied the relation between fuzzy continuity and fuzzy boundedness. They also considered fuzzy bounded linear functionals, the concept of fuzzy dual spaces, and established some fundamental theorems in the area of fuzzy functional analysis.

In [4], Das and Das define a fuzzy topology on the fuzzy normed linear space defined by Felbin and study some basic properties of this fuzzy topology. After that, Fang [5] showed that X with this topology is not topological vector space and modifies the fuzzy topology and proved some results. Also, Xu and Fang defined another fuzzy topological space and studied these spaces [12]. Recently, we defined two fuzzy topology on the fuzzy normed linear space defined by Bag and Samanta and study some properties of these fuzzy topologies [10].

In this paper, first we define fuzzy norm defined by Felbin and fuzzy bounded linear operator from fuzzy normed linear operator X to fuzzy normed linear space Y. Then we define three fuzzy topology on fuzzy normed linear spaces. Finally, we study relation between fuzzy continuity and fuzzy boundedness.

## 2. Fuzzy norm

**Definition 2.1.** ([6]) Let X be a vector space over  $\mathbb{R}$ . Assume the mappings  $L, R : [0,1] \times [0,1] \longrightarrow [0,1]$  are symmetric and non-decreasing in both arguments, and that L(0,0) = 0 and R(1,1) = 1. Let  $\|.\| : X \longrightarrow F^+(\mathbb{R})$ . The quadruple  $(X, \|.\|, L, R)$  is called a fuzzy normed linear space (briefly, FNS) with the fuzzy norm  $\|.\|$ , if the following conditions are satisfied:  $(F_1)$  if  $x \neq 0$  then  $\inf_{0 \le \alpha \le 1} \|x\|_{\alpha}^{-} > 0$ ,

<sup>2010</sup> Mathematics Subject Classification. 26E50, 54A40, 46S40. Speaker: Mortaza Saheli.

 $\begin{array}{l} (F_2) \ \|x\| = 0 \ if \ and \ only \ if \ x = 0, \\ (F_3) \ \|rx\| = |r| \|x\| \ for \ x \in X \ and \ r \in \mathbb{R}, \\ (F_4) \ for \ all \ x, y \in X, \\ (F_4L) \|x + y\| (s+t) \ge L(\|x\|(s), \|y\|(t)) \ whenever \ s \le \|x\|_1^-, \ t \le \|y\|_1^- \ and \ s+t \le \|x + y\|_1^-, \\ (F_4R) \|x + y\| (s+t) \le R(\|x\|(s), \|y\|(t)) \ whenever \ s \ge \|x\|_1^-, \ t \ge \|y\|_1^- \ and \ s+t \ge \|x + y\|_1^-. \end{array}$ 

In the sequel we fix  $L(s,t) = \min(s,t)$  and  $R(s,t) = \max(s,t)$  for all  $s,t \in [0,1]$ . And we write  $(X, \|.\|)$  or simply X when L and R are as indicated just above.

**Definition 2.2.** Let  $(X, \|.\|)$  and  $(Y, \|.\|)$  be fuzzy normed linear spaces. Furthermore, let  $T : X \longrightarrow Y$  be a linear operator. The operator T is said to be fuzzy bounded, if there is a fuzzy real number  $\eta$  such that

 $||Tx|| \le \eta ||x||, \quad for \quad all \quad x \in X.$ 

The set of all fuzzy bounded linear operators,  $T: X \longrightarrow Y$ , is denoted by B(X, Y).

**Theorem 2.3.** Let  $(X, \|.\|)$  be a fuzzy normed linear space. Then there exists a topology  $\tau^{\dagger}$  on X such that  $(X, \tau^{\dagger})$  is a metrizable Hausdorff topological vector space having  $U = \{U_{\epsilon,\alpha} : \epsilon > 0, \alpha \in (0,1]\}$  as a neighborhood base of 0, where  $U_{\epsilon,\alpha} = \{x \in X : \|x\|_{\alpha}^{+} < \epsilon\}.$ 

## 3. Fuzzy Topology

**Definition 3.1.** ([4]) A fuzzy topology on a set X is a family  $\tau$  of fuzzy subsets of X satisfying the following:

(i) The fuzzy subsets 1 and 0 are in  $\tau$ ,

(ii)  $\tau$  is closed under finite intersection of fuzzy subsets,

(iii)  $\tau$  is closed under arbitrary union of fuzzy subsets.

The pair  $(X, \tau)$  is called a fuzzy topological space.

**Definition 3.2.** ([5]) A stratified fuzzy topology  $\tau$  on a vector space X is said to be an fuzzy vector topology, if the following two mappings

 $f: X \times X \longrightarrow X, \ (x,y) \longrightarrow x+y \ and \ g: \mathbb{K} \times X \longrightarrow X, \ (t,x) \longrightarrow tx,$ 

are continuous, where  $\mathbb{K}$  is equipped with the fuzzy topology induced by the usual topology and  $X \times X$  and  $\mathbb{K} \times X$  are equipped with the corresponding product fuzzy topologies. A vector space X with an fuzzy vector topology  $\tau$ , denoted by  $(X, \tau)$  is called an fuzzy topological vector space (FTVS).

**Definition 3.3.** A mapping  $f : (X, \tau_1) \longrightarrow (Y, \tau_2)$  is called fuzzy continuous at some point  $x \in X$  if  $f^{-1}(\mu)$  is a nbhd of x for each nbhd  $\mu$  of f(x). f is called fuzzy continuous if f is fuzzy continuous at every  $x \in X$ . This is equivalent to inverse of every fuzzy open subset of Y is fuzzy open in X.

**Definition 3.4.** Let  $(X, \|.\|)$  be a fuzzy normed linear space.  $\omega(\tau^{\dagger})$  is defined an induced fuzzy topology by the crisp vector topology  $\tau^{\dagger}$  determined by fuzzy norm  $\|.\|$  on X.

**Definition 3.5.** Let  $(X, \|.\|)$  be a fuzzy normed linear space and let  $x \in X$ ,  $\alpha \in (0,1)$  and  $\epsilon > 0$  the fuzzy set  $\mu_{\alpha}(x, \epsilon)$  defined in X by

$$\mu_{\alpha}(x,\epsilon)(y) = \begin{cases} \alpha & , & \|x-y\|_{\alpha}^{+} < \epsilon \\ 0 & , & o.w. \end{cases}$$

is said to be an  $\alpha$ -open sphere in X.

**Definition 3.6.** Let  $(X, \|.\|)$  be fuzzy normed linear space. A fuzzy set  $\mu$  on X is said to be  $\|.\|$ -strongly open if for every  $x \in \text{supp}\mu$  and  $\alpha \in (0, \mu(x))$ , there exists  $\epsilon > 0$  such that  $\mu_{\alpha}(x, \epsilon) \subseteq \mu$ .

**Theorem 3.7.** Let  $(X, \|.\|)$  be a fuzzy normed linear space. Then a family

$$\tau_{\parallel,\parallel}^* = \{\mu \in I^X : \mu \text{ is } \parallel . \parallel - \text{strongly open}\}$$

is a fuzzy topology on X and  $(X, \tau^*_{\parallel,\parallel})$  is a locally convex Hausdorff topological vector space.

**Definition 3.8.** Let  $(X, \|.\|)$  be a fuzzy normed linear space and  $\epsilon > 0$ . The fuzzy set  $B_{\epsilon}$  on X by

$$B_{\epsilon}(x) = \sup\{1 - \alpha \in (0, 1] : \|x\|_{\alpha}^{+} \le \epsilon\}$$

is said to be a fuzzy sphere with center 0 and radius  $\epsilon$  in X.

**Theorem 3.9.** Let  $(X, \|.\|)$  be a fuzzy normed linear space. Then a family

 $\tau_{\parallel . \parallel} =$ 

 $\{\mu \in I^X : \forall \ x \in supp \mu \ and \ 0 < r < \mu(x) \ \exists \epsilon > 0 \ s.t. \ x + B_\epsilon \cap \underline{r} \subseteq \mu\}$ 

is a fuzzy topology on X.

4. Relation between Fuzzy continuity and Fuzzy Boundedness

**Theorem 4.1.** Let  $(X, \|.\|_1)$  and  $(Y, \|.\|_2)$  be fuzzy normed linear spaces and  $T : X \longrightarrow Y$  be a fuzzy bounded linear operator. Then  $T : (X, \tau^*_{\|.\|_1}) \longrightarrow (Y, \tau^*_{\|.\|_2})$  is fuzzy continuous.

*Proof.* Let T be a fuzzy bounded. Then there exists  $\eta \in F^+(\mathbb{R})$  such that

 $||T(x)||_2 \le \eta ||x||_1$ , for all  $x \in X$ .

Assume that  $\mu \in \tau_{\parallel,\parallel_2}^*$ ,  $T^{-1}(\mu)(x) > 0$  and  $0 < \alpha < T^{-1}(\mu)(x)$ . Hence  $0 < \alpha < \mu(T(x))$ . Since  $\mu \in \tau_{\parallel,\parallel_2}^*$ , there exists  $\epsilon > 0$  such that  $\mu_{\alpha}(T(x), \epsilon) \subseteq \mu$ . Suppose that  $\mu_{\alpha}(x, \epsilon/\eta_{\alpha}^+)(y) = \alpha$ . Thus  $\|x - y\|_{1\alpha}^+ < \epsilon/\eta_{\alpha}^+$ . So  $\|T(x) - T(y)\|_{2\alpha}^+ = \|T(x - y)\|_{2\alpha}^+ \le \eta_{\alpha}^+ \|x - y\|_{1\alpha}^+ < \epsilon$ . Therefore  $\mu_{\alpha}(T(x), \epsilon)(T(y)) = \alpha \le \mu(T(y)) = T^{-1}(\mu)(y)$ . This implies that  $\mu_{\alpha}(x, \epsilon/\eta_{\alpha}^+) \subseteq T^{-1}(\mu)$ . Hence  $T^{-1}(\mu) \in \tau_{\|.\|_1}^*$ . Thus T is a fuzzy continuous.

**Theorem 4.2.** Let (X, |||.||) be a fuzzy normed linear space. Then  $\tau^*_{||.||} = \omega(\tau^{\dagger})$  if and only if for any  $\alpha, \beta \in (0, 1), ||.||^+_{\alpha}$  is equivalent to  $||.||^+_{\beta}$ .

**Corollary 4.3.** Let  $(X, \|.\|_1)$  and  $(Y, \|.\|_2)$  be fuzzy normed linear spaces and  $T: X \longrightarrow Y$  be a fuzzy bounded linear operator. Moreover, assume that for any  $\alpha, \beta \in (0, 1), \|.\|_{1\alpha}^+$  is equivalent to  $\|.\|_{1\beta}^+$  and  $\|.\|_{2\alpha}^+$  is equivalent to  $\|.\|_{2\beta}^+$ . Then  $T: (X, \omega(\tau_1^{\dagger})) \longrightarrow (Y, \omega(\tau_2^{\dagger}))$  is fuzzy continuous.

**Theorem 4.4.** Let (X, ||||.||) be a fuzzy normed linear space. Then  $\tau^*_{||.||} = \tau_{||.||}$  if and only if for each  $\alpha \in (0, 1)$ ,  $||.||^+_{\alpha}$  is equivalent to  $||.||^+_1$ .

**Corollary 4.5.** Let  $(X, \|.\|_1)$  and  $(Y, \|.\|_2)$  be fuzzy normed linear spaces and  $T: X \longrightarrow Y$  be a fuzzy bounded linear operator. Moreover, assume that for any  $\alpha \in (0,1), \|.\|_{1\alpha}^+$  is equivalent to  $\|.\|_{11}^+$  and  $\|.\|_{2\alpha}^+$  is equivalent to  $\|.\|_{21}^+$ . Then  $T: (X, \tau_{\|.\|_1}) \longrightarrow (Y, \tau_{\|.\|_2})$  is fuzzy continuous.

**Example 4.6.** Let X be a vector space and  $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$  a basis for X (dim  $X = \infty$ ). Define fuzzy norms  $\|.\|_1$  and  $\|.\|_2$  on X by

$$[\|x\|_1]_{\alpha} = [\sum_{i=1}^n |a_i|, \sum_{i=1}^n (1/\alpha)^i |a_i|] \text{ and } [\|x\|_2]_{\alpha} = [\sum_{i=1}^n |a_i|, \sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|, \sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|, \sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n a_i e_i, \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n |a_i| = [\sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n |a_i|] \text{ where } x = \sum_{i=1}^n |a_i|$$

for all  $\alpha \in (0,1]$ . It is clear that  $\|.\|_1$  and  $\|.\|_2$  are fuzzy norms on X.

We define  $T: (X, \|.\|_1) \to (X, \|.\|_2)$  by  $Te_n = ne_n$ , for all  $e_n \in \mathcal{B}$ . Assume that  $\mu \in \tau^*_{\|.\|_2}$ ,  $T^{-1}(\mu)(x) > 0$  and  $0 < \alpha < T^{-1}(\mu)(x)$ . Hence there exists  $\epsilon > 0$  such that  $\mu_{\alpha}(T(x), \epsilon) \subseteq \mu$ . Suppose that  $\mu_{\alpha}(x, \epsilon)(y) = \alpha$ . Thus  $\|x - y\|^+_{\alpha} < \epsilon$ . So

$$||T(x) - T(y)||_{2\alpha}^{+} = ||T(x - y)||_{\alpha}^{+} = \sum_{i=1}^{k} |ia_{i}| = \sum_{i=1}^{k} i|a_{i}| \le \sum_{i=1}^{k} (1/\alpha)^{i}|a_{i}| < \epsilon.$$

Therefore  $\mu_{\alpha}(x, \epsilon) \subseteq T^{-1}(\mu)$ . Hence  $T^{-1}(\mu) \in \tau_{\|.\|_1}^*$ . Thus T is a fuzzy continuous. Now we show that T is not fuzzy bounded. If T is fuzzy bounded, then there exists fuzzy real number  $\eta$  such that  $\|Tx\|_0 \leq \eta \|x\|$ , for all  $x \in X$ , and hence  $\|Tx\|_0^+ \leq \eta_1^+ \|x\|_1^+$ , for all  $x \in X$ . However,  $\|Te_n\|_{21}^+ = \|ne_n\|_{21}^+ = |n| = n \leq \eta_1^+ \|e_n\|_1^+ = \eta_1^+$ , hence  $n \leq \eta_1^+$ , for all  $n \in \mathbf{N}$ , and thus  $\eta_1^+ = +\infty$ , which is a contradiction.

#### References

- T. Bag, S. K. Samanta, Finite dimensional fuzzy normed linear spaces, J. Fuzzy Math., 11 (3) (2003), 687-705.
- [2] T. Bag, S. K. Samanta, Fuzzy bounded linear operators, Fuzzy Sets and Systems, 151 (2005), 513-547.
- [3] S. C. Cheng, J. N. Mordeson, Fuzzy linear operators and fuzzy normed linear spaces, Bull. Cal. Math. Soc., 86 (1994), 429-436.
- [4] N. F. Das, P. Das, Fuzzy topology generated by fuzzy norm, Fuzzy Sets and Systems, 107 (1999), 349-354.
- [5] J. -X. Fang, On I-topology generated by fuzzy norm, Fuzzy Sets and Systems, 157 (2006), 2739-2750.
- [6] C. Felbin, Finite dimensional fuzzy normed linear space, Fuzzy Sets and Systems, 48 (1992), 239-248.
- [7] O. Kaleva, S. Seikkala, On fuzzy metric spaces, Fuzzy Sets and Systems, 12 (1984), 215-229.
- [8] I. Karmosil, J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica, 11(1975), 326-334.
- [9] A. K. Katsaras, Fuzzy topological vector spaces II, Fuzzy Sets and Systems, 12 (1984), 143-154.
- [10] M. Saheli, On fuzzy topology and fuzzy norm, Annals of Fuzzy Mathematics and Informatics, 10 (4) (2015), 639647.
- [11] J. Xiao, X. Zhu, Fuzzy normed space of operators and its completeness, Fuzzy Sets and Systems, 133 (2003) 389-399.
- [12] G. -H. Xu, J. -X. Fang, A new I-vector topology generated by a fuzzy norm, Fuzzy Sets and Systems, 158 (2007), 2375-2385.

DEPARTMENT OF MATHEMATICS, VALI-E-ASR UNIVERSITY OF RAFSANJAN, RAFSANJAN, IRAN *E-mail address*: saheli@vru.ac.ir



### EIGENVALUES FOR TRIDIAGONAL 2-TOEPLITZ MATRICES

MARYAM SHAMS SOLARY

ABSTRACT. In this paper, we try to introduce an algorithm for finding eigenvalue of tridiagonal 2-Toeplitz matrices of even order. When the order of a tridiagonal 2-Toeplitz matrix is even, a problem occurs that causes the eigenvalues are found implicitly. We try to solve and decompose eigenproblem for these matrices.

Keywords: 2-Toeplitz matrix; Eigenvalue; Eigenvector.

#### 1. INTRODUCTION

In this paper we consider the properties of the characteristic equation of a tridiagonal 2-Toeplitz matrix  $B_n = [a_{ij}]$  of even order n = 2m where  $a_{ij} = 0$  if  $|i - j| > 1, a_{ij} = a_{kl}$  if  $(i, j) \equiv (k, l) \mod 2$ , i.e.

(1.1) 
$$B_n = tridiag_n(c_1, a_1, b_1; c_2, a_2, b_2) = \begin{pmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & & \\ & c_2 & a_1 & b_1 & & \\ & & c_1 & a_2 & b_2 & & \\ & & & c_2 & a_1 & \ddots & \\ & & & & \ddots & \ddots & \end{pmatrix}.$$

In [3] Rimas has given the explicit expression for powers of a tridiagonal 2-Toeplitz matrix of odd order.

We try to generalize this method for a tridiagonal 2-Toeplitz matrix of even order In Theorem 2.4 and Theorem 3.1 of [2], Gover proved the following results.

**Theorem 1.1.** [2]. The eigenvalues of the tridiagonal 2-Toeplitz matrix of order n = 2m given in (1.1) are the solutions of the quadratic equations

(1.2) 
$$(a_1 - \lambda)(a_2 - \lambda) - [b_1c_1 + \sqrt{b_1b_2c_1c_2}Q_k + b_2c_2] = 0, \quad k = 1, 2, \dots, m$$

where  $Q_k$ , k = 1, 2, ..., m, are the zeros of  $q_m(y) = U_m(\frac{y}{2}) + \beta U_{m-1}(\frac{y}{2})$ ,  $U_m(\frac{y}{2})$ is the Chebyshev polynomial of degree m of the second kind with  $m \in N \cup \{-1, 0\}$ and

(1.3) 
$$\beta = \sqrt{\frac{b_2 c_2}{b_1 c_1}}.$$

All Chebyshev polynomials, among  $U_m(y)$  satisfies the three-term recurrence relations:

$$U_{m+1}(y) = 2yU_m(y) - U_{m-1}(y)$$
  $(U_{-1}(y) = 0, U_0(y) = 1, U_1(y) = 2y).$ 

<sup>2010</sup> Mathematics Subject Classification. 15A18, 15B05. Speaker: Maryam Shams Solary.

**Theorem 1.2.** [2]. The eigenvector of  $B_{2m}$ , in (1.1) associated with the eigenvector  $\lambda_k$ , which is a solution of (1.2) is given by

(1.4) 
$$x = [q_0(Q_k), -\frac{a_1 - \lambda_k}{b_1} p_0(Q_k), sq_1(Q_k), -s\frac{a_1 - \lambda_k}{b_1} p_1(Q_k), \\ \dots, s^{m-1} q_{m-1}(Q_k), -s^{m-1} \frac{a_1 - \lambda_k}{b_1} p_{m-1}(Q_k)]^T,$$

that

(1.5) 
$$s = \sqrt{\frac{c_1 c_2}{b_1 b_2}},$$

$$(1.6) p_m(y) = U_m(\frac{y}{2}),$$

and  $Q_k$ ,  $k = 1, 2, \ldots, m$ , are the zeros of

(1.7) 
$$q_m(y) = U_m(\frac{y}{2}) + \beta U_{m-1}(\frac{y}{2}).$$

#### 2. Main results

In this Section, we try to prove the following Theorem:

**Theorem 2.1.** Let  $a_i \in R$ ,  $b_i > 0$ ,  $c_i > 0$  (i = 1, 2) and  $\beta$  is given from (1.3). Then roots of

(2.1) 
$$q_m(x) = U_m(x) + \beta U_{m-1}(x)$$

that n = 2m of matrix  $B_n$  in (1.1) associated with: i. We claim, it is sufficient to study  $\beta > 0$  since results for  $\beta \leq 0$  gives trivially from the following proof. ii. When  $\beta = 1$ , there are m roots on [-1, 1]. iii. For some  $\beta \leq \frac{m+1}{m}$  exist m roots on the interval [-1,1] and m-1 roots with |x| < 1 for  $|\beta| > \frac{m+1}{m}$ . iv. All roots are real-valued and antisymmetric for all  $\beta$ .

Proof. Theorem is proved by some details abouts continuatioun and bifurcation discussed in [1].  $\square$ 

Here we try to compute roots  $\alpha$  for fixed  $\beta$  in (2.1) by companion matrix. For fixed  $\beta$  we calculate roots of equation (2.1) by computing the eigenvalues of a companion matrix  $C_U$  for Chebyshev polynomials of the second kind,

(2.2) 
$$C_U = \begin{pmatrix} 0 & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ & \frac{1}{2} & 0 & \frac{1}{2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & & \dots & 0 & \frac{1}{2} & -\frac{\beta}{2} \end{pmatrix}, \quad \alpha = eig(C_U).$$

**Theorem 2.2.** Let  $a_i \in R$ ,  $b_i > 0$ ,  $c_i > 0$  (i = 1, 2) and  $\beta$  is given from (1.3). Consider  $B_n = tridiag_n(c_1, a_1, b_1; c_2, a_2, b_2)$  be tridiagonal 2-Toeplitz matrix (1.1)

of order n = 2m  $(m \in N)$  and  $\alpha_k$ ,  $1 \le k \le m$  are the zeros of (2.1). Then all eigenvalues of  $B_n$  are real, simple and are given by: (2.3)

$$\lambda_k = \begin{cases} \frac{a_1 + a_2}{2} - \sqrt{\left(\frac{a_1 - a_2}{2}\right)^2 + b_1 c_1 + \sqrt{b_1 b_2 c_1 c_2} \alpha_k + b_2 c_2}, & \text{if } 1 \le k \le m, \\ \frac{a_1 + a_2}{2} + \sqrt{\left(\frac{a_1 - a_2}{2}\right)^2 + b_1 c_1 + \sqrt{b_1 b_2 c_1 c_2} \alpha_k + b_2 c_2}, & \text{if } m + 1 \le k \le n. \end{cases}$$

*Proof.* Theorem is proved by  $\lambda_k$  in (2.3) and the relations in Theorem 1.2.

Now we try to find the inverse matrix  $T^{-1}$ , because the powers of even order matrix  $B_n$   $(n = 2m, m \in N)$  given in (1.1), can be found applying

(2.4) 
$$(B_n)^l = TJ^lT^{-1}, \ l \in N,$$

that J is the Jordan's form of the matrix  $B_n$ ,  $T_k$ ,  $k = \overline{1, n}$  is the eigenvector of  $B_n$  corresponding to eigenvalue  $\lambda_k$ ,  $k = \overline{1, n}$  in (2.6),

(2.5) 
$$T = [T_1, T_2, \dots, T_n] \text{ and } J = diag(\lambda_1, \lambda_2, \dots, \lambda_n).$$

**Theorem 2.3.** Consider  $a_i \in R$ ,  $b_i > 0$ ,  $c_i > 0$  (i = 1, 2). Then the eigenvector  $T_k$  of  $B_{2m} = tridiag_{2m}(c_1, a_1, b_1; c_2, a_2, b_2)$  in (1.1) associated with the eigenvalues  $\lambda_k$  in (2.3) and  $\beta$  is given from (1.3),

(2.6) 
$$T_{k} = \begin{pmatrix} s^{0} \left( U_{0} \left( \frac{\alpha_{k}}{2} \right) + \beta U_{-1} \left( \frac{\alpha_{k}}{2} \right) \right) \\ s^{0} \frac{\lambda_{k} - a_{1}}{b_{1}} U_{0} \left( \frac{\alpha_{k}}{2} \right) \\ s^{1} \left( U_{1} \left( \frac{\alpha_{k}}{2} \right) + \beta U_{0} \left( \frac{\alpha_{k}}{2} \right) \right) \\ s^{1} \frac{\lambda_{k} - a_{1}}{b_{1}} U_{1} \left( \frac{\alpha_{k}}{2} \right) \\ \vdots \\ s^{m-1} \left( U_{m-1} \left( \frac{\alpha_{k}}{2} \right) + \beta U_{m-2} \left( \frac{\alpha_{k}}{2} \right) \right) \\ s^{m-1} \frac{\lambda_{k} - a_{1}}{b_{1}} U_{m-1} \left( \frac{\alpha_{k}}{2} \right) \end{pmatrix}, \quad k = 1, 2, \dots, n.$$

**Theorem 2.4.** Consider  $a_i \in R$ ,  $b_i > 0$ ,  $c_i > 0$  (i = 1, 2). Then the inverse of the matrix T in (2.4) for n = 2m  $(m \in N)$  is shown by

(2.7) 
$$T^{-1} = [H_1, H_2, \dots, H_n]^T,$$

$$H_{k} = \begin{pmatrix} \frac{t_{k}}{s^{0}} \left( U_{0}(\frac{\alpha_{k}}{2}) + \beta U_{-1}(\frac{\alpha_{k}}{2}) \right) \\ \frac{t_{k}}{s^{0}} \frac{\lambda_{k} - a_{1}}{c_{1}} U_{0}\left(\frac{\alpha_{k}}{2}\right) \\ \frac{t_{k}}{s^{1}} \left( U_{1}(\frac{\alpha_{k}}{2}) + \beta U_{0}(\frac{\alpha_{k}}{2}) \right) \\ \frac{t_{k}}{s^{1}} \frac{\lambda_{k} - a_{1}}{c_{1}} U_{1}\left(\frac{\alpha_{k}}{2}\right) \\ \vdots \\ \frac{t_{k}}{s^{m-1}} \left( U_{m-1}(\frac{\alpha_{k}}{2}) + \beta U_{m-2}(\frac{\alpha_{k}}{2}) \right) \\ \frac{t_{k}}{s^{m-1}} \frac{\lambda_{k} - a_{1}}{c_{1}} U_{m-1}\left(\frac{\alpha_{k}}{2}\right) \end{pmatrix},$$

(2.8) 
$$t_k = \frac{1}{S_1 + S_2}, \quad k = 1, 2, \dots, n,$$

that

(2.9) 
$$S_1 = \sum_{i=1}^m \left( U_{i-1}\left(\frac{\alpha_k}{2}\right) + \beta U_{i-2}\left(\frac{\alpha_k}{2}\right) \right)^2, \quad S_2 = \frac{(\lambda_k - a_1)^2}{b_1 c_1} \sum_{i=1}^m U_{i-1}^2\left(\frac{\alpha_k}{2}\right),$$

and s,  $\alpha_k$ ,  $\lambda_k$  are defined by (1.5), (2.2) and (2.3), respectively.

**Theorem 2.5.** Let  $a_i \in R$ ,  $b_i > 0$ ,  $c_i > 0$  (i = 1, 2) and  $B_n = tridiag_n(c_1, a_1, b_1; c_2, a_2, b_2)$ be tridiagonal 2-Toeplitz matrix in (1.1) of order n = 2m associated with the eigenvalues  $\lambda_k$   $(k = \overline{1, n})$  in (2.3). Then (2.10)

$$\begin{bmatrix} B_n^l \end{bmatrix}_{ij} = \begin{cases} \sum_{k=1}^n \lambda_k^l t_k s^{\frac{i-j}{2}} \left( U_{\frac{i-1}{2}}(\frac{\alpha_k}{2}) + \beta U_{\frac{i-3}{2}}(\frac{\alpha_k}{2}) \right) \left( U_{\frac{j-1}{2}}(\frac{\alpha_k}{2}) + \beta U_{\frac{j-3}{2}}(\frac{\alpha_k}{2}) \right) \\ i, j = 1, 3, 5, \dots, n-1, \\ \sum_{k=1}^n \lambda_k^l t_k s^{\frac{i-j}{2}} \frac{(\lambda_k - a_1)^2}{b_1 c_1} U_{\frac{i-2}{2}}(\frac{\alpha_k}{2}) U_{\frac{j-2}{2}}(\frac{\alpha_k}{2}) \\ i, j = 2, 4, 6, \dots, n, \\ \sum_{k=1}^n \lambda_k^l t_k s^{\frac{i-j+1}{2}} \left( U_{\frac{i-1}{2}}(\frac{\alpha_k}{2}) + \beta U_{\frac{i-3}{2}}(\frac{\alpha_k}{2}) \right) \frac{(\lambda_k - a_1)}{c_1} U_{\frac{j-2}{2}}(\frac{\alpha_k}{2}) \\ i = 1, 3, 5, \dots, n-1; j = 2, 4, 6, \dots n, \\ \sum_{k=1}^n \lambda_k^l t_k s^{\frac{i-j-1}{2}} \frac{(\lambda_k - a_1)}{b_1} U_{\frac{i-2}{2}}(\frac{\alpha_k}{2}) \left( U_{\frac{j-1}{2}}(\frac{\alpha_k}{2}) + \beta U_{\frac{j-3}{2}}(\frac{\alpha_k}{2}) \right) \\ i = 2, 4, 6, \dots n; j = 1, 3, 5, \dots, n-1; \end{cases}$$

for  $\beta$ , s,  $\alpha_k$ ,  $\lambda_k$  and  $t_k$  are defined by (1.3), (1.5), (2.2), (2.3) and (2.8), respectively and for all  $l \in N$  and  $1 \leq i, j \leq n$ .

**Example 2.6.** Let  $B_6 = tridiag_6(16, 36, 25; 1, 9, 4)$ . Finding eigenvalues and  $B_6^3$  by the following algorithm in this paper. In the first step, we should try to find zeros for  $U_3(\frac{\alpha}{2}) + 0.1U_2(\frac{\alpha}{2}) = 0$ .

This work is done by finding eigenvalues of companion matrix, thus

 $\alpha_1 = -1.440349746683088, \ \alpha_2 = 1.390287324807722, \ \alpha_3 = -0.049937578124635.$ 

Now we try to find eigenvalues of  $B_6$  by (2.3):

 $\lambda = \begin{pmatrix} -0.492085815181635\\ -2.834985553426094\\ -1.671315580146121\\ 45.492085815181639\\ 47.834985553426094\\ 46.671315580146121 \end{pmatrix}.$ 

Now we try to find  $B_6^3$  by (2.10):

	/ 7.905599999	5.262500000	0.810000000	0.250000000	0.000000000	0.000000000 `	\
$B_6^3 = 10^4$	3.368000000	2.254500000	1.002000000	0.540000000	0.040000000	0.000000000	1
	0.129600000	0.250500000	7.938000000	5.272500000	0.809999999	0.249999999	
	0.025600000	0.086400000	3.374400000	2.254500000	1.001999999	0.539999999	·
	0.000000000	0.001600000	0.129599999	0.250499999	7.937999999	5.262499999	1
	0.000000000	0.000000000	0.025600000	0.086399999	3.368000000	2.232900000	/

#### Acknowledgement

The author wishes to thank Professor John P. Boyd in University of Michigan for exactly constructive comments.

#### References

- J. P. Boyd, Solving Transcendental Equations: The Chebyshev Polynomial Proxy and Other Numerical Rootfinders, SIAM, 2014.
- [2] M.J.C. Gover, The eigenproblem of a tridiagonal 2-Toeplitz matrix, Linear Algebra Appl., 197(1994), 63-78.
- [3] J. Rimas, Explicit expression for powers of tridiagonal 2-Toeplitz matrix of odd order, Linear Algebra Appl., 436(2012), 3493-3506.

Department of Mathematics, Payame Noor University, Po Box 19395-3697 Tehran, IRAN

 $E\text{-}mail\ address:\ \texttt{shamssolary} \texttt{Qpnu.ac.ir} \quad \texttt{or} \quad \texttt{shamssolary} \texttt{Qgmail.com}$ 



### ON THE MUTATIVE METHOD IN MAX ALGEBRA

SEYYED MAHMOUD MANJEGANI AND HOJR SHOKOOH SALJOOGHI

ABSTRACT. In the present paper, we introduce a new concept named mutation defined by using determinant. We calculate the max-eigenvalue and max-eigenvector for a nonnegative irreducible  $n \times n$  matrix with the aid of the mutation. By analogy with former methods, this one will either minimize the long process of finding the answer or cut down the expenditure by means of simple and clear-cut method. Besides, this method of mutation has some ways of recognizing the reducibility and irreducibility of matrices.

Keywords: Matrix mutation; Max-eigenvalue; Max-eigenvector.

#### 1. INTRODUCTION

One of the offshoots of linear algebra that has drawn a lot of attentions recently is the max algebra system. The max algebra system consists of the nonnegative real numbers  $\mathbb{R}_+$  equipped with the operation of multiplication  $a \otimes b = ab$ , and maximization  $a \oplus b = \max\{a, b\}$ . Furthermore, the max algebra is isomorphic to the max-plus algebra, which consists of the set  $\mathbb{R} \cup \{-\infty\}$  with operations of maximization and addition [2, 3, 4, 5]. This algebra system and its isomorphic version raise the possibility of changing the non-linear phenomena in different areas such as parallel computation, transportation networks, timetabled programs, IT, dynamic systems, combinatorial optimization, and mathematical physics to linearalgebra. Furthermore, this algebra system has been used directly in areas such as algorithm. Vetrbi, analyzing DNA and in AHP for ranking matrices In this algebra system, there is no deduction but many of appeared problems in linear algebra like equation systems, eigenvalue, projections, subspaces, singular value decomposition, duality theory have developed and have reached other areas like functional analysis, algebra topology and combinatorial optimization. One can see more about the max algebra in [1].

Let  $A = [a_{ij}] \in M_n(\mathbb{R}_+)$  be an  $n \times n$  nonnegative irreducible matrix. The weighted directed graph associated with A denoted by D(A) = (V, E), where A has vertex set  $V = \{1, 2, ..., n\}$  and edges (i, j) from i to j with weight  $a_{ij}$  if and only if  $a_{ij} > 0$ . A circuit of length k is a sequence of k edges  $(i_1, i_2), ..., (i_k, i_1)$ , where  $i_1, i_2, ..., i_k$  are distinct. This has *circuit product*  $a_{i_1, i_2}, ..., a_{i_k, i_1}$  with positive  $K^{th}$  root as the *circuit geometric mean*. The maximum circuit geometric mean in D(A) is denoted by  $\mu(A)$ . A circuit with circuit geometric mean equal to  $\mu(A)$  is called a *critical circuit*, and vertices on critical circuits are called *critical vertices*.

For an  $n \times n$  matrix A, the conventional eigenequation for eigenvalue  $\lambda$  and corresponding eigenvector x is  $Ax = \lambda x$ . In the max algebra system the eigenequation for nonnegative matrix matrix  $A = [a_{ij}]$  is  $A \otimes x = \lambda x$ , where  $(A \otimes x)_i = \max(a_{ij}x_j), x = (x_1, x_2, \dots, x_n)^t$  and  $\lambda$  is the maximum circuit geometric mean

<sup>2010</sup> Mathematics Subject Classification. 15A80, 46L05.

Speaker: Hojr Shokooh Saljooghi.

and is denoted by  $\mu(A)$ . In the max algebra  $\mu(A)$  is called *max-eigenvalue* of A and x is a *max-eigenvector*. This max-eigenvector is unique (up to scalar multiples) for an  $n \times n$  nonnegative irreducible matrix [3].

#### 2. MUTATIONS OF A MATRIX

One of the fundamental features of a matrix is its determinant. First we try to introduce the concept of mutation with the terms of determinant. Suppose that  $S = \{1, 2, ..., n\}$  is nonempty and  $S_n = \{\varphi_1, ..., \varphi_n\}$  is all permutations in the set S. Determinant of a matrix  $A = [a_{ij}]$  can be defined by means of permutation as:

$$\det A = |A| = \sum_{\sigma \in s_n} (\sigma) a_{1\sigma 1}, a_{2\sigma 2}, \dots, a_{n\sigma n}.$$

But it is essential to calculate the roots of matrix determinant  $(A - \lambda I)$ , that is to say, calculating polynomial expression of matrix A before working out the eigenvalue and eigenvector of a matrix. To calculate the max-eigenvalue and max-eigenvector of a matrix in the max algebra system, there is no need to calculate the polynomial expression of a matrix, because the max algebra system is slightly different from linear algebra. The concept of mutation will be used in working out these values that has been introduced by matrix determinant and in this case the relationship between usual linear algebra and max algebra system has been determined much well.

**Definition 2.1.** Suppose that  $A = [a_{ij}]$  is an  $n \times n$  matrix. Each factor of  $a_{1\sigma 1}, a_{2\sigma 2}, \ldots, a_{n\sigma n}$  that has been used in the determinant of matrix A is a mutation. If all factors in a mutation are of elements other than main diagonal, the mutation is major and otherwise it is minor.

**Example 2.2.** Consider the following  $n \times n$  matrix

$$A = \begin{pmatrix} 1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9 \end{pmatrix},$$

then we have

$$\det A = 1 \cdot 5 \cdot 9 - 1 \cdot 6 \cdot 8 - 2 \cdot 4 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 3 \cdot 5 \cdot 7,$$

where  $3 \cdot 4 \cdot 8$  and  $2 \cdot 6 \cdot 7$  are major mutations and other terms are minor mutations.

Recall that, a matrix  $A \in M_n(\mathbb{R}_+)$  is *reducible* if either n = 1 and A = 0 or if  $n \ge 2$ , there is a permutation matrix  $p \in M_n$ , and there is an integer number r with  $1 \le r \le n-1$  such that

$$P^T A P = \begin{pmatrix} B & C \\ o & D \end{pmatrix},$$

where  $B \in M_r(\mathbb{R}_+)$ ,  $D \in M_{n-r}(\mathbb{R}_+)$ ,  $C \in M_{n-r}(\mathbb{R}_+)$ , and  $0 \in M_{n-r,r}(\mathbb{R})$  is a zero matrix. A matrix  $A \in M_n$  is said to be irreducible if it is not reducible. The following theorem is our first result.

**Theorem 2.3.** Let  $A \in M_{n \times n}$   $(n \ge 2)$  be a nonnegative matrix, then A is reducible if only if at least n - 1 elements of  $a_{ij}a_{ji}$  in major mutations are zero.

Example 2.4. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 6 \\ 5 & 0 & 3 \end{pmatrix}.$$
  
det  $A = 1 \cdot 2 \cdot 3 - 1 \cdot 6 \cdot 0 + 0 \cdot 4 \cdot 3 - 0 \cdot 6 \cdot 5 + 0 \cdot 4 \cdot 0 - 2 \cdot 2 \cdot 5$ ,

since the product of three elements  $a_{12}a_{21}, a_{13}a_{31}, a_{32}a_{23}$ , are zero the matrix is reducible.

Now, we are in position to introduce our mutative algorithm for calculating  $\mu(A)$  and x. In [3], introduced the power method algorithm to compute  $\mu(A)$  and max-eigenvector x.

**Lemma 2.5.** Let A be an  $n \times n$  nonnegative matrix then a portion of mutation that has no elements of principal diagonal contain a circuit.

*Proof.* Let  $\alpha$  be such portion of a mutation of matrix A. Then it is contain elements  $a_{i1i2}, a_{i2i3}, \ldots, a_{iki1}$ . Thus by definition of circuit,  $\alpha$  has one circle.

**Corollary 2.6.** Let A be an  $n \times n$  nonnegative matrix. Then major mutations are the circuit.

**Theorem 2.7 (Main Theorem).** Let A be an  $n \times n$  nonnegative irreducible matrix, then  $\mu(A)$  is equal the maximum  $K^{th}$ -root of a portion in mutation that formed a circuit, where k is the number of elements in circuit of mutation, that is

$$\mu(A) = \max\left\{\sqrt[k]{\Pi a_{ij}}; 1 \le i, j \le n\right\}.$$

For matrix A in the following example, max-eigenvalue and max-eigenvector are computed with power method algorithm in [3]. We try to calculate them by our method.

Example 2.8. Let

$$A = \begin{pmatrix} 0 & e^3 & 0 \\ e^2 & 0 & e \\ e & e^2 & e^2 \end{pmatrix}.$$
  
det  $A = 0 \cdot 0 \cdot e^2 - 0 \cdot e \cdot e^2 - e^3 \cdot e^2 \cdot e^2 + e^3 \cdot e \cdot e + 0 \cdot e^2 \cdot e^2 + 0 \cdot 0 \cdot e$ .

2 ~ `

Since the product of two elements  $a_{12}a_{21}, a_{23}a_{32}$ , are non zero then the matrix is irreducible. By main theorem,

$$\mu(A) = \max\{\sqrt[3]{e^3 \cdot e^2 \cdot e^2}, \sqrt[2]{e^3 \cdot e^2}\} = \sqrt[2]{e^5}.$$

To calculate max-eigenvector x for an irreducible nonnegative matrix  $A = [a_{ij}]$ in the max algebra system, by properties of max algebra and setting

$$A^* = [a_{ij}^*],$$

where

 $a_{ij}^* = \begin{cases} a_{ij}, & a_{ij} \text{ is an element in the row which appear in the mutation that make } \mu(A), \\ 0, & \text{otherwise}, \end{cases}$ 

we have  $A^*x = \mu(A)x$ . Thus, we can simply obtain max-eigenvalue and maxeigenvector of A. For matrix A in the above example, we have

$$A^* = \begin{pmatrix} 0 & e^3 & 0\\ e^2 & 0 & 0\\ 0 & e^2 & 0 \end{pmatrix}.$$

Let  $x = (x_1, x_2, x_3)^T$  be the max-eigenvector for A corresponding to  $\mu(A)$ . Then

$$\begin{pmatrix} 0 & e^3 & 0 \\ e^2 & 0 & 0 \\ 0 & e^2 & 0 \end{pmatrix} (x_1, x_2, x_3)^T = \sqrt[2]{e^5} (x_1, x_2, x_3)^T.$$

Therefore,

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = (e^{\frac{1}{2}}, e^0, e^{\frac{-1}{2}}).$$

#### Acknowledgment

This work was supported by the Department of Mathematical Sciences at Isfahan University of Technology, Iran.

#### References

- R.B. Bapat, A max version of the Perron-Frobenius theorem, Linear Algebra Appl. 275-276 (1998) 3-18.
- [2] R.B. Bapat, D.P. Stanford, P. van den Driessche, Pattern properties and spectral inequalities in max algebra, SIAM J. of Matrix Analysis and Applications 16 (1995) 964-976.
- [3] L. Esner, P. Van Driessche, On the power method in max algebra, Linear Algebra Appl. 302-303 (1999) 17-32.
- [4] L. Esner, P. Van Driessche, Modifying the method in max algebra, Linear Algebra Appl. 332-385 (2001) 3-13.
- [5] H. Shokoh Saljooghi, Spectral Properties of Matrix Polynomias in the Max Algebra [master thesis], Shahid Bahonar University of Kerman, 2012.

DEPARTMENT OF MATHEMATICAL SCIENCES, ISFAHAN UNIVERSITY OF TECHNOLOGY ISFAHAN, IRAN, 84156-83111

E-mail address: manjgani@cc.iut.ac.ir

DEPARTMENT OF MATHEMATICAL SCIENCES, ISFAHAN UNIVERSITY OF TECHNOLOGY ISFAHAN, IRAN, 84156-83111

E-mail address: h.shokooh@math.iut.ac.ir



## PARAMETERIZATION OF FEEDBACK MATRIX IN TIME-DELAY SYSTEMS

NARGES TAHMASBI AND HOJJAT AHSANI TEHRANI

ABSTRACT. In this paper, we obtain a state feedback matrix by elementary similarity operations such that the eigenvalues of the closed-loop system lie in the self-conjugate eigenvalue spectrum  $\Lambda$  by a parametric matrix. The main idea is a convert the discrete-time delay linear system into the standard system without delay. For stabilization discrete-time delay systems, it is necessary to place all the eigenvalues of the closed-loop systems in unit circle.

Keywords: Parameterization; State feedback matrix; Time-delay.

#### 1. INTRODUCTION

Parameterization using state feedback controls the assignment of eigenvalues in recent decades has been the subject of many investigations. Different methods of parametric eigenvalue assignment for systems have been proposed in [1]-[4]. Karbassi and Bell in [4] introduced a new method for feedback controller parameterization. They have shown that by using a simple algorithm based vectors derived from primary functions and properties, constants Kronecker, a group of parametric controllers, linear parameters can be produced. Location parameter can be achieved using Graph model [3]. In this research, with expansion method Karbassi and Bell, a general framework to obtain the controller parameters using a nonlinear parameters build [1].

#### 2. Problem statement

Consider a linear time-delay system defined by the state equation

(2.1) 
$$x_{i+1} = \sum_{k=0}^{p} A_k x_{i-k} + \sum_{k=0}^{q} B_k u_{i-k},$$

where  $x \in \mathbb{R}^n$  is the state vector,  $u \in \mathbb{R}^m$  is the input vector and the matrices  $A_i \in \mathbb{R}^{n \times n}$  and  $B_i \in \mathbb{R}^{n \times m}$ . respectively, with  $rank(B_i) = m$ . by definition state vector such as

(2.2) 
$$X_{i+1} = \begin{bmatrix} x_{i+1} \ x_i \ \cdots \ x_{i-(p-1)} \ u_i \ u_{i-1} \ \cdots \ u_{k-(i-1)} \end{bmatrix}^T$$

The system (2.1) with p delays in state and q delays in input vector can be rewriten as a standard system

where  $X \in \mathbb{R}^{\tilde{n}}$ ,  $\tilde{n} = n(p+1) + mq$ .

We define control low as  $u_i = KX_i$ , where K is a feedback gain. Therefore, the system (2.1) changes to a standard closed-loop system  $X_{i+1} = \Gamma X_i$ .

Speaker: Narges Tahmasbi.

The aim of eigenvalue assignment is to design a state feedback controller K, producing a closed-loop system with a satisfactory response by shifting controllable poles from undesirable to desirable locations( $\Lambda$ ).

## 3. Main results

Consider the state transformation  $X_i = T\tilde{X}_i$ , where T can be obtained by elementary similarity operations as described in [4]. By replace that in equation (2.3) we have

(3.1) 
$$\tilde{X}_{i+1} = T^{-1}AT\tilde{X}_i + T^{-1}B\tilde{u}_i$$

In this way,  $\tilde{A} = T^{-1}AT$  and  $\tilde{B} = T^{-1}B$  are in a compact canonical form known as vector companion form:

(3.2) 
$$\tilde{A} = \begin{bmatrix} G_0 \\ I_{\tilde{n}-m} \\ , & O_{\tilde{n}-m,m} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_0 \\ O_{\tilde{n}-m,m} \end{bmatrix}$$

Here  $G_0$  is a  $m \times \tilde{n}$  matrix and  $B_0$  is an  $m \times m$  upper triangular matrix. Note that if the Kronecker invariants of the pair (A, B) are regular, then  $\tilde{A}$  and  $\tilde{B}$  are always in the above form [4].

The state feedback matrix which assigns all the eigenvalues to zero, for the transformed pair  $(\tilde{A}, \tilde{B})$ , is then chosen as

(3.3) 
$$u = -B_0^{-1} G_0 \tilde{X} = \tilde{F}_p \tilde{X}, \ F_p = \tilde{F}_p T^{-1}$$

The transformed closed-loop matrix  $\tilde{\Gamma_0}=\tilde{A}+\tilde{B}\tilde{F}_p$  assumes a compact Jordan form with zero eigenvalues

(3.4) 
$$\tilde{\Gamma_0} = \begin{bmatrix} O_{m,\tilde{n}} \\ I_{\tilde{n}-m} & , & O_{\tilde{n}-m,m} \end{bmatrix}$$

**Theorem 3.1.** [4] Let  $\tilde{A}_{\lambda}$  be any matrix in companion form, i.e.,

(3.5) 
$$\tilde{A}_{\lambda} = \begin{bmatrix} G_{\lambda} \\ I_{\tilde{n}-m} & , & O_{\tilde{n}-m,m} \end{bmatrix}.$$

with the eigenvalue spectrum  $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_{\tilde{n}}\}$ , a set of self conjugate eigenvalues. Then  $\tilde{K} = B_0^{-1}(-G_0 + G_{\lambda})$ .

The above theorem leads to a general framework for obtaining the parametric controllers in general. Thus, let

(3.6) 
$$det(\tilde{A}_{\lambda} - \lambda I) = P_{\tilde{n}}(\lambda) = 0,$$

where

$$(3.7) P_n(\lambda) = (-1)^{\tilde{n}} (\lambda^{\tilde{n}} + C_1 \lambda^{\tilde{n}-1} + \dots + C_{\tilde{n}-1} \lambda + C_{\tilde{n}}),$$

is the characteristic polynomial of the closed-loop system. Since it is required that the zeros of this polynomial lie in the set  $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{\tilde{n}}\}$ , it is clear that

(3.8) 
$$P_{\tilde{n}}(\lambda) = (-1)^{\tilde{n}} (\lambda - \lambda_1) (\lambda - \lambda_2) \cdots (\lambda - \lambda_{\tilde{n}})$$

By equating these two equations the coefficients  $C_i$ ,  $(i = 1, 2, \dots, \tilde{n})$  can be obtained as follows:

(3.9) 
$$\begin{cases} C_1 = -\sum_{i=1}^n \lambda_i \\ C_2 = \sum_{i=1}^{\tilde{n}} \sum_{j=0}^{\tilde{n}} \lambda_i \lambda_j \\ \vdots \\ C_{\tilde{n}} = (-1)^{\tilde{n}} \prod_{i=1}^{\tilde{n}} \lambda_i. \end{cases}$$

Now by direct computation of  $det(\tilde{A}_{\lambda} - \lambda I)$  in parametric form and equating the coefficients of the characteristic polynomial with (3.9), the following nonlinear system of equations is obtained:

$$(3.10) \qquad \begin{cases} f_1(g_{11}, g_{12}, \dots, g_{1\tilde{n}}, g_{21}, \dots, g_{2\tilde{n}}, \dots, g_{m1}, g_{m2}, \dots, g_{m\tilde{n}}) = C_1 \\ f_2(g_{11}, g_{12}, \dots, g_{1\tilde{n}}, g_{21}, \dots, g_{2\tilde{n}}, \dots, g_{m1}, g_{m2}, \dots, g_{m\tilde{n}}) = C_2 \\ \vdots \\ f_n(g_{11}, g_{12}, \dots, g_{1\tilde{n}}, g_{21}, \dots, g_{2\tilde{n}}, \dots, g_{m1}, g_{m2}, \dots, g_{m\tilde{n}}) = C_{\tilde{n}}, \end{cases}$$

where  $g_{ij}$ ,  $(i = 1, ..., m, j = 1, ..., \tilde{n})$ , are the elements of  $G_{\lambda}$ :

(3.11) 
$$G_{\lambda} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1\tilde{n}} \\ \vdots & \vdots & \vdots & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{m\tilde{n}} \end{bmatrix}$$

In this way, a nonlinear system of  $\tilde{n}$  equations with  $\tilde{n} \times m$  unknowns is obtained. By choosing  $N = \tilde{n}(m-1)$  unknowns arbitrarily it is then possible to solve the system.

#### 4. NUMERICAL EXAMPLE

#### **Example 4.1.** Consider a discrete-time linear system with delay in input vector

$$x_{k+1} = \begin{bmatrix} 0.9512 & 0\\ 0 & 0.9048 \end{bmatrix} x_k + \begin{bmatrix} 4.8770 & 4.8770\\ 0 & 0 \end{bmatrix} u_k + \begin{bmatrix} 0 & 0\\ -1.1895 & 3.5890 \end{bmatrix} u_{k-1}$$

It is desired to obtion parametric state feedback controllers which assign the eigenvalue  $\Lambda = \{0.1, -0.1, 0.5, -0.5\}$  to the closed-loop system. first we convert this system to (2.3), then the transformed pair  $(\tilde{A}, \tilde{B})$  in vector companion from and the corresponding transformation matrix are

à =	$ \begin{array}{c} 0.9055 \\ 0.0002 \\ 1 \\ 0 \end{array} $	$0.1345 \\ 0.9505 \\ 0 \\ 1$	0 0 0 0	0 - 0 0	$, \tilde{B} =$	$\begin{bmatrix} 1\\ 0\\ 0\\ 0 \end{bmatrix}$	0 - 1 0 0	
		1	0	υ.	]	L U	0_	J

and

$$K = \left[ \begin{array}{ccc} -0.2470 & 0.2954 & -0.3351 & 1.1697 \\ -0.0255 & -0.0851 & 0.1126 & -0.3922 \end{array} \right],$$

with norm 1.3428. Now, let us consider

$$\tilde{A}_{\lambda} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

therefore, of equation (3.7) we have:

$$det(\tilde{A}_{\lambda} - \lambda I) = P_4(\lambda) = (\lambda^4 - 0.26\lambda^2 + 0.0025)$$

By equating the coefficients of equations (3.9) and (3.10) we obtain the nonlinear system of 4 equations with 8 unknowns. If we choose  $g_{13} = -g_{24}$  and  $g_{23} = -g_{21} = 1$ , then we will obtion

(4.1) 
$$G_{\lambda} = \begin{bmatrix} g_{11} & 2g_{11}g_{24} - g_{24}^2 - 0.0025 & -g_{24} & -0.0025 - g_{24}^2 \\ -1 & -g_{11} & 1 & g_{24} \end{bmatrix}$$

The nonlinear parametric controller matrix  $K = B_0^{-1}(-G_0 + G_\lambda)T^{-1}$ . For example, with  $g_{24} = 0$ , have

$$K_1 = \begin{bmatrix} -0.0014 & 0.2977 & -0.3914 & -0.1303 \\ -0.0014 & -0.0843 & -0.9936 & -1.4512 \end{bmatrix},$$

with norm 1.3694, the following can be the state feedback matrix obtained so that has minimum norms. Therefore, we assume:

$$K = B_0^{-1} (-G_0 + G_\lambda) T^{-1} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} =$$

(4.2)

 $\begin{bmatrix} 0.154g_{11} + 0.051g_{12} + 0.162g_{13} + 0.0537g_{14} - 0.15 & 0.154g_{11} + 0.051g_{12} + 0.162g_{13} + 0.0537g_{14} - 0.05 \\ 0.189g_{21} - 0.189g_{21} - 0.209g_{13} + 0.209g_{13} + 0.209g_{14} + 0.17 & 0.189g_{21} - 0.209g_{13} + 0.209g_{14} - 0.17 \\ 0.249g_{11} - 0.249g_{12} - 0.79g_{13} - 0.262g_{14} - 0.23 & 0.249g_{21} - 0.249g_{22} - 0.79g_{23} - 0.262g_{24} + 0.26 \\ 0.751g_{12} - 0.751g_{11} - 0.79g_{13} - 0.262g_{14} + 0.68 & 0.751g_{22} - 0.751g_{21} - 0.79g_{23} - 0.262g_{24} - 0.68 \end{bmatrix}^{T}$ 

With the help of equations (3.10) and (4.2) we are defining the optimization problem as follows:

Minimize 
$$\sum_{i=1}^{2} \sum_{J=1}^{4} K_{ij}$$
  
s.t. 
$$\begin{cases} -(g_{11} + g_{22}) = 0\\ g_{11}g_{24} - g_{12}g_{23} + g_{13}g_{22} - g_{14}g_{21} = 0\\ -g_{13} - g_{24} + g_{11}g_{22} - g_{12}g_{21} = -0.26\\ g_{13}g_{24} - g_{14}g_{23} = 0.0025 \end{cases}$$

The purpose of solving this problem, find the unknown parameters so that in addition assign the eigenvalue  $\Lambda$  to the closed-loop system, norm feedback matrix is also minimized.

With the help of iterative methods for nonlinear programming problems such as Levenberg-Marquart Method with initial point  $x_0 = \{0, 0, 0, 0, 0, 0, 0, 0, 0\}^T$ , we have

$$K_3 = \left[ \begin{array}{ccc} -0.1287 & 0.1055 & -0.3704 & 0.4442 \\ -0.0787 & -0.1144 & 0.2181 & -0.4741 \end{array} \right],$$

with norm 0.8086.

### 5. Conclusion

In this paper, we compute a parametric matrix state feedback of linear systems with delay. We first by defining an augmented vector, time-delay systems changed to standard systems, then by elementary similarity opearations obtain nonlinear parametric feedback matrix minimum norm, therefore the energy or cost input for u = Kx the substantially reduced.

## References

- M. H. Amin, A. M. Elabadall, parametrization of a class of dead-beat controllers via the theory of decoupling, IEEE Transactions on Automatic control. 33(1988), 1185–1188.
- [2] H. Katayama, A. Ichikawa, Pole assignment by state transition garph, IEEE transactions on Automatic Control. 37(1992), 1196–1201.
- [3] J. Katuski, N. K. Nichols, P. Van Dooren, Robust pole assignments in linear state feedback, International journal of control. 41(1985), 1129–1155.
- [4] S. M. Karbassi, D. J. Bell, New method of parametric eigenvalue assignment in state feedback control, IEE Processing. D. 141(1994), 223–226.

PHD STUDENT, SHAHROOD UNIVERSITY OF TECHNOLOGY, SHAHROOD, IRAN. *E-mail address*: n.tahmasbi@shahroodut.ac.ir

Associate Professor, Department of Mathematics Sciences, Shahrood University of Technology, Shahrood, Iran.

E-mail address: hahsani@shahroodut.ac.ir



## NORM MINIMIZATION CONTROLLER IN FRACTIONAL SYSTEMS WITH DELAY

#### NARGES TAHMASBI AND HOJJAT AHSANI TEHRANI

ABSTRACT. This paper presents a method for control fractional-order system with time-delay. The main idea is a convert the fractional system with timedelay to system without delay with define an augmented vector. Then by similarity transformations and using the closed-loop matrix graph, we find the parametric state feedback matrix. Finally we get the the controlled optimal matrix with minimum norm.

Keywords: Fractional; State feedback; Norm; Eigenvalue Assignment.

#### 1. INTRODUCTION

Kouvaritakis and Cameron [3] considered controllers with dyadic structure which could be used to relocate one real pole or a complex conjugate pair of poles. All available degrees of freedom were utilised in minimizing the Frobenius norm of the controller matrix. In the case of real eigenvalue assignment, an explicit expression was given for the feedback law; the control in the case of complex pair assignment was calculated following the numerical solution of a quartic equation.

Minimization of the norm of feedback controllers has received considerable attention in recent years by many authors. It is well established that the norm of the feedback matrix should be kept as small as possible, i.e., minimum, in order to improve the transient response of the closed-loop system. However, these two objectives are conflicting and each require different numerical treatment. Indeed, a parameterized state feedback controller is needed for a constructive investigation into these aspects of the work. An important advantage of this latter method is its linearity in its parameters which makes it attractive for mathematical operations.

#### 2. Problem statement

Consider the fractional discrete-time linear system described by

(2.1) 
$$\Delta^{\alpha} x_{k+1} = A x_k + B u_k,$$

where  $\alpha$  is the fractional-order which  $0 < \alpha \leq 1$ ,  $x_k \in \mathbb{R}^n$  is the state vector,  $u_k \in \mathbb{R}^m$  is the input vector. It is assumed that  $A \in \mathbb{R}^{n \times n}$  is open-loop matrix and  $B \in \mathbb{R}^{n \times m}$  is input matrix with rank(B) = m and  $1 \leq m \leq n$ . Also  $x_0$  is a nonzero definite vector. Fractional order derivative  $\alpha$  in (2.1) is defined by [4]

(2.2) 
$$\Delta^{\alpha} x_{k} = \sum_{i=0}^{k+1} (-1)^{i} {\alpha \choose i} x_{k-i},$$

<sup>2010</sup> Mathematics Subject Classification. 34A08, 93B52, 15A60. Speaker: Narges Tahmasbi.

using the definition (2.2) we can write the equation (2.1) in the form

(2.3) 
$$x_{k+1} + \sum_{i=1}^{k+1} (-1)^i \binom{\alpha}{i} x_{k-i+1} = Ax_k + Bu_k.$$

With difine  $c_i = (-1)^i {\alpha \choose i+1}$  for  $i \in \mathbb{N}$  we have:

(2.4) 
$$x_{k+1} = A_{\alpha} x_k + \sum_{i=1}^{k+1} c_i x_{k-i} + B u_k, \ A_{\alpha} = A + \alpha I.$$

And, it is assumed that i is bounded by natural number k + 1. this system is system with unlimited delays. By definition the sequence  $c_i$  converges to zero. Getting  $c_i = 0$  for i > h (greater h is better) the system (2.4) will be a time delay system with h delays

(2.5) 
$$x_{k+1} = A_{\alpha} x_k + \sum_{i=1}^h c_i x_{k-i} + B u_k.$$

Note that the system (2.6) describes an discrete-time linear system with h delays and can be rewriten as a standard system

where

(2.7) 
$$X_k = \begin{bmatrix} x_k & x_{k-1} & x_{k-2} & \cdots & x_{k-h} \end{bmatrix}^T \in \mathbb{R}^{\tilde{n}}, \ U_k = u_k, \ \tilde{n} = n(h+1)$$

Now, consider the fractional discrete-time linear system with time-delay described by

(2.8) 
$$\Delta^{\alpha} x_{k+1} = \sum_{i=0}^{p} A_i x_{k-i} + \sum_{i=0}^{q} B_i u_{k-i},$$

with using transformations (2.7), can be system (2.8) with p + h delays in state and q delays in input vector can be rewriten as a standard system

$$(2.9) X_{k+1} = AX_k + Bu_k$$

We define control low as  $u_k = FX_k$ , Where F is a feedback gain. Therefor, the system (2.9) changes to a standard closed-loop system  $X_{k+1} = (\Gamma = A + BF)X_k$ . In this paper we determined the state feedback matrix F, In addition to being the at the eigenvalues of the closed-loop system  $\Gamma$  lie in the self-conjugate eigenvalue spectrum  $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{\tilde{n}}\}$  also, have norm minimum. Consider the state transformation

where T can be obtained by elementary similarity operations as described in [1]. By replace (2.10) in equation (2.9) we have

(2.11) 
$$\tilde{X}_{k+1} = T^{-1}AT\tilde{X}_k + T^{-1}B\tilde{u}_k,$$

in this way,  $\tilde{A} = T^{-1}AT$  and  $\tilde{B} = T^{-1}B$  are in a compact canonical form known as vector companion form:

(2.12) 
$$\tilde{A} = \begin{bmatrix} G_0 \\ I_{\tilde{n}-m} & , & O_{\tilde{n}-m,m} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_0 \\ O_{\tilde{n}-m,m} \end{bmatrix}$$

The transformed closed-loop matrix  $\tilde{\Gamma} = \tilde{A} + \tilde{B}\tilde{F}_p$  assumes a compact Jordan form with zero eigenvalues. It is from this form that the state feedback matrix, which assigns a set of arbitrary eigenvalues to the system and also the location of parameters, can be obtained. The controller matrix  $\tilde{F}_p$  is then modified by adding a diagonal matrix  $D = \{\lambda_1, \lambda_2, \ldots, \lambda_{\tilde{n}}\}$  for an arbitrary set of self-conjugate

(2.13) 
$$\tilde{A}_{\lambda} = \begin{bmatrix} G_{\lambda} \\ I_{\tilde{n}-m} & , & O_{\tilde{n}-m,m} \end{bmatrix},$$

where  $G_{\lambda}$  is the first  $m \times \tilde{n}$  sub-matrix of  $\tilde{A}_{\lambda}$ . Obviously,  $\tilde{A}_{\lambda}$  possesses the desired set of eigenvalues and is in the same canonical form as  $\tilde{A}$ .

Thus, the primary feedback matrix which gives rise to the assignment of eigenvalues becomes  $\tilde{F}_1 = \tilde{F}_p + B_0^{-1}G_\lambda = B_0^{-1}(-G_0 + G_\lambda)$ .

#### 3. Main result

The parametric feedback matrix is defined as

(3.1) 
$$F = F_1 + F_\alpha, \ F_\alpha = B_0^{-1} G_\alpha T^{-1},$$

where  $G_{\alpha}$  is an  $m \times n$  matrix containing free parameters only. The simplest way to locate the parameters is the method of state transition graph[2] applied to the compact Jordan form. Therefore, the parametric feedback matrix F is then

$$(3.2) F = F_1 + VHW.$$

The Frobenius norm of F is then

(3.3) 
$$||F||^2 = \sum_{i=1}^m \sum_{j=1}^n (F_{ij})^2,$$

in minimizing the norm, we must have

(3.4) 
$$\frac{\partial \parallel F \parallel^2}{\partial h_{sr}} = 0 \ \forall \ s \le m, \ r \le n$$

differentiating  $||F||^2$  with respect to each  $h_{sr}$  yields

$$(3.5) V^T F_1 W^T + V^T V H W W^T = 0$$

Thus, by defining

(3.6) 
$$\begin{cases} P = V^T V, \ Q = W W^T \\ C = V^T F_1 W^T, \ H = -P^{-1} C Q^{-1}. \end{cases}$$

## 4. Numerical Example

**Example 4.1.** Consider a fractional-order linear system with delay in state and input vector

$$\Delta^{0.5} x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} x_k + \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} x_{k-1} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} u_k + \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} u_{k-1} + \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} u_{k-2},$$

where h = 3. then  $\tilde{n} = n(\max(h, p) + 1) + mq = 12$ . We find feedback matrix F which the eigenvalues of the closed-loop system assign in spectrum

$$\{0, 0.1, 0.2, 0.3, 0.4, 0.5, -0.1, -0.2, -0.3, -0.4, -0.5, -0.6\}$$

This system unstable, for stability system, state feedback matrix which assign all the eigenvalues of the closed-loop system is found to be:

<b>F</b> _	-1.0530	-0.0154	-1.3826	1.5363	-0.0324	0.0131	-0.0189	0.0131	1.4928	1.1209	0.4830	0.2316	1
r = [	0.6325	-0.4103	0.5098	-0.4917	0.0180	-0.0094	0.0074	-0.0034	-0.5007	-0.6449	-0.1920	0.1282	] '
With norm 3.3083. Either  $[g_{12}, g_{14}, g_{16}, g_{18}, g_{110}, g_{112}]$  is the effective parameters which produce parametric feedback matrix:

C	0	-1.8006	0	-3.1885	0	-3.5617	0	-1.3316	0	-0.8717	0	-0.1631	1	
$G_{\alpha} =$	0	0	0	0	0	0	0	0	0	0	0	0	] '	
there	efore													

$F_1 = \begin{bmatrix} & & \\ & & \\ & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & & \\ & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & $	-0.2813	-0.6146	-0.1826	-0.0685	-0.0612	-0.1079	0.0433	-0.0940	0.1219	-0.0804	0.0203	-0.0380 ]
	0.6325	-0.4103	0.5098	-0.4917	0.0180	-0.0094	0.0074	-0.0034	-0.5007	-0.6449	-0.1920	0.1282



FIGURE 1. State response in example 2

with the Frobenius norm 1.5279. The system response and compares the state variables with minimum norm shown in Fig. 1. As you can see, system of this example from initial value  $x_0 = [-3, 3, 5, -3, 2, 4, -2, -4, 2, 4, 6, -3]^T$  is practically stable.

## 5. Conclusion

We presented the method [?], matrix feedback parameter in linear fractional systems with time delays such that all eigenvalues of matrix closed-loop found in the area of sustainability are in unit circle as well as the parameters of matrix norms have been able to significantly reduce the state feedback, Therefore, the energy or cost input  $u_k = Fx_k$  the substantially reduced.

## References

- S. M. Karbassi, D. J. Bell, Parametric time-optimal control of linear discrete-time systems by state feedback-Part 1:Regular Kronecker invariants, International journal of control, 57(1993), 817–830.
- [2] H. Katayama, A. Ichikawa, Pole assignment by state transition garph, IEEE transactions on Automatic Control. 37(1992), 1196–1201.
- [3] B. Kouvaritakis1, R.G Cameron, Pole placement with minimised norm controllers Control Theory and Applications, IEE Proceedings. 127(1)(1980), 32–36.
- M. Rivero, S. V. Rogosin, J. A. Tenreiro, Stability of fractional-order systems, Mathematical Problems in Engineering. 4(2013), 114–120.

PHD STUDENT, SHAHROOD UNIVERSITY OF TECHNOLOGY, SHAHROOD, IRAN. E-mail address: n.tahmasbi@shahroodut.ac.ir

Associate Professor, Department of Mathematics Sciences, Shahrood University of Technology, Shahrood, Iran.

E-mail address: hahsani@shahroodut.ac.ir



# EFRON-STEIN INEQUALITY FOR OPERATOR-VALUED RANDOM MATRICES

#### ALI TALEBI AND MOHAMMAD SAL MOSLEHIAN

ABSTRACT. The main purpose of the paper is to state a version of the Efron–Stein inequality for operator-valued random matrices.

Keywords: Efron-Stein inequality; Random matrix; Non-commutative probability.

## 1. INTRODUCTION

Assume that  $Z = f(X_1, X_2, ..., X_n)$  is a symmetric function of independent random variables  $X_1, X_2, ..., X_n$  in a probability space  $(\Omega, F, P)$ . Efron and Stein proved that

$$\operatorname{var}(Z) \leq \frac{1}{2} \sum_{j=1}^{n} \mathbb{E}\left[ \left( Z - Z'_{j} \right)^{2} \right],$$

where  $Y_1, Y_2, \ldots, Y_n$  are independent copies of  $X_1, X_2, \ldots, X_n$ , respectively, and  $Z'_i = f(X_1, \ldots, Y_j, \ldots, X_n)$ .

Recently, Paulin et al. [2] established new Efron–Stein inequalities that describe the concentration properties of a matrix-valued function of independent random variables. We present a operator-value matrix version of Efron–Stein inequality in the main theorem.

In the sequel,  $(\mathcal{M}, \tau)$  denotes a non-commutative probability space, that is, a von Neumann algebra  $\mathcal{M}$  equipped with a normal faithful finite trace  $\tau$  with  $\tau(1) = 1$ , where 1 stands for the identity of  $\mathcal{M}$ . We denote the self-adjoint elements of  $\mathcal{M}$  by  $\mathcal{M}_{sa}$ . For  $1 \leq p < \infty$  the Banach space  $L_p(\mathcal{M})$  is the completion of  $\mathcal{M}$ with respect to the *p*-norm  $||x||_p := \tau(|x|^p)^{1/p}$ . The elements of  $L_1(\mathcal{M})$  are called (noncommutative) random variables. [1, 4, 5].

Let x be a normal random variable and  $e^x$  be the unique spectral measure on the Borel subsets  $\mathfrak{B}(\mathbb{C})$  of  $\mathbb{C}$ . We use the notation  $W^*(x)$  for the von Neumann subalgebra  $W^*(\{e^x(B): B \in \mathfrak{B}(\mathbb{C})\})$  of  $\mathcal{M}$ .

If  $\mathcal{N}$  is a von Neumann subalgebra of  $\mathcal{M}$ , then there exists a map  $\mathcal{E}_{\mathcal{N}} : \mathcal{M} \longrightarrow \mathcal{N}$ , which satisfies the following properties:

- (i)  $\mathcal{E}_{\mathcal{N}}$  is normal positive contractive projection from  $\mathcal{M}$  onto  $\mathcal{N}$ ;
- (ii)  $\mathcal{E}_{\mathcal{N}}(axb) = a\mathcal{E}_{\mathcal{N}}(x)b$  for every  $x \in \mathcal{M}$  and  $a, b \in \mathcal{N}$ ;
- (iii)  $\tau \circ \mathcal{E}_{\mathcal{N}} = \tau$ .

It is known that  $\mathcal{E}_{\mathcal{N}}$  can be extended to a contractive positive projection, denoted by the same  $\mathcal{E}_{\mathcal{N}}$ , from  $L_p(\mathcal{M})$  onto  $L_p(\mathcal{N})$ , named the conditional expectation with respect to  $\mathcal{N}$ . The reader is referred to [1] for more information on non-commutative probability spaces.

<sup>2010</sup> Mathematics Subject Classification. Primary 15A60; Secondary 46L53, 47A30, 60F99. Speaker: Ali Talebi.

Let  $d \in \mathbb{N}$  and  $\mathbb{M}_d(\mathbb{C}) \otimes \mathcal{M} \cong \mathbb{M}_d(\mathcal{M})$  be the  $d \times d$  matrices

$$\mathbf{X} = \begin{bmatrix} X_{11} & X_{12} & \dots & x_{1d} \\ X_{21} & X_{22} & \dots & X_{2d} \\ \vdots \\ X_{d1} & X_{d2} & \dots & X_{dd} \end{bmatrix}$$

with entires in  $\mathcal{M}$ . Then the trace  $\overline{\tau}$  is defined by

$$\overline{\tau}(\mathbf{X}) := \tau\left(\overline{\operatorname{tr}}(\mathbf{X})\right) := \frac{1}{d} \sum_{i=1}^{d} \tau(X_{ii}),$$

is a normalized normal faithful finite trace on  $\mathbb{M}_d(\mathcal{M})$ , where  $\overline{\mathrm{tr}} = \frac{1}{d}\mathrm{tr}$ . Similarly, let  $\mathbb{M}_d(\mathcal{N})$  denote the subalgebra of  $d \times d$  matrices with entires in  $\mathcal{N}$ . Then  $\mathbb{M}_d(\mathcal{N})$  is a von Neumann algebra of  $\mathbb{M}_d(\mathcal{M})$  and the corresponding conditional expectation  $\mathbb{E} = I_{\mathbb{M}_d} \otimes \mathcal{E}_{\mathcal{N}}$  is given by  $\mathbb{E}(\mathbf{X}) = (\mathcal{E}_{\mathcal{N}}(X_{ij}))_{d \times d}$ . Note that  $\tau(\mathbf{X})$  is defined by  $(\tau(X_{ij}))_{d \times d}$ .

### 2. Main results

The notion of independence is one of the main notions in the classical probability and noncommutative probability setting. However, there are more noncommutative notions of independence [3] such as the tensor independence and the free independence. We said von Neumann subalgebras  $\mathcal{A}_j$   $(1 \leq j \leq n)$  to be tensor independent if

$$\tau\left(\prod_{i=1}^{m}\left(\prod_{k=1}^{n}a_{ki}\right)\right) = \prod_{k=1}^{n}\tau\left(\prod_{i=1}^{m}a_{ki}\right),$$

whenever  $a_{kj} \in \mathcal{A}_k$   $(1 \leq j \leq m; 1 \leq k \leq n; m \in \mathbb{N});$ and the  $\mathcal{A}_j$   $(1 \leq j \leq n)$  to be free independent if

$$\tau \left( a_1 a_2 \dots a_m \right) = 0,$$

whenever  $a_k \in \mathcal{A}_{i_k}$ ,  $i_1 \neq i_2 \neq \ldots \neq i_m$  and  $\tau(a_k) = 0$  for all k.

**Theorem 2.1** (Operator-valued matrix Efron–Stein inequality). Let  $\mathcal{A}_j$  be a sequence of either freely independent or tensor independent von Neumann subalgebras of  $(\mathbb{M}_d(\mathcal{M}), \overline{\tau})$  and  $A_j \in L_1(\mathcal{A}_j)$   $(1 \leq j \leq n)$  be self-adjoint matrices and let  $F : L_1(\mathbb{M}_d(\mathcal{M}))_{sa} \times \ldots \times L_1(\mathbb{M}_d(\mathcal{M}))_{sa} \longrightarrow L_1(\mathbb{M}_d(\mathcal{M}))_{sa}$  be an arbitrary function.

Let  $B = F(A_1, \ldots, A_n)$  and  $B^{(j)} = F(A_1, \ldots, A_{j-1}, A'_j, A_{j+1}, \ldots, A_n)$  be such that  $\overline{\tau}(BB^{(j)}) \leq \|\mathcal{E}_j(B)\|_2^2$  and  $\overline{\tau}((B^{(j)})^2) \geq \overline{\tau}(B^2)$  for all  $j = 1, 2, \ldots, n$ , where  $A'_1, A'_2, \ldots, A'_n \in L_1(\mathbb{M}_d(\mathcal{M}))_{sa}$  and  $\mathcal{E}_j$  denotes the conditional expectation of  $\mathbb{M}_d(\mathcal{M})$  with respect to  $\mathcal{A}_j$ . Then the inequality

$$\tau\left(\operatorname{tr}\left(\left(B-\tau\left(B\right)\right)^{2}\right)\right) \leq \frac{1}{2}\tau\left(\operatorname{tr}\left(\sum_{j=1}^{n}\left(B-B^{(j)}\right)^{2}\right)\right),$$

holds.

Applying Theorem 2.1, we now deduce a version of the Efron–Stein inequality for random matrices. Assume that  $(X_1, X_2, \ldots, X_n)$  is a random vector of mutually independent random variables on a probability space  $(\Omega, F, P)$ .

Let  $F : \mathbb{R}^n \longrightarrow \mathbb{H}_d$  be a bounded measurable function, where  $\mathbb{H}_d$  denotes the space of the  $d \times d$  Hermitian matrices in  $\mathbb{M}_d(\mathbb{C})$ . Set the random matrix

$$\mathbf{Y} := \mathbf{Y} \left( X_1, X_2, \dots, X_n \right) := F \left( X_1, X_2, \dots, X_n \right) - \mathbb{E} F \left( X_1, X_2, \dots, X_n \right).$$

Now suppose that  $X'_1, X'_2, \ldots, X'_n$  are independent copies of  $X_1, X_2, \ldots, X_n$ . Consider the random vectors

$$(X_1,\ldots,X'_j,\ldots,X_n).$$

Construct the random matrices

$$\mathbf{Y}^{(j)} := \mathbf{Y}^{(j)} \left( X_1, \dots, X'_j, \dots, X_n \right) := F \left( X_1, \dots, X'_j, \dots, X_n \right) - \mathbb{E}F \left( X_1, X_2, \dots, X_n \right)$$

With the above notations, the matrix Efron–Stein inequality reads as follows.

Corollary 2.2 (Matrix Efron–Stein inequality). The inequality

$$\mathbb{E} \left\| oldsymbol{Y} 
ight\|_2^2 \leq rac{1}{2} \mathbb{E} \left\| \sum_{j=1}^n \left( oldsymbol{Y} - oldsymbol{Y}^{(j)} 
ight)^2 
ight\|_1,$$

holds, where  $\|.\|_p$  denote the Schatten p-norm.

#### References

- M. Junge and Q. Xu, Non-commutative Borkholder/Rosenthal inequalities, Ann. Probab. 31 (2003), no. 2, 948–995.
- [2] D. Paulin, L. Mackey and J. A. Tropp, *Efron-Stein Inequalities for Random Matrices*, Ann. Probab. (2015), In press.
- [3] D.V. Voiculescu, K.J. Dykema and A. Nica, *Free random variables*, volume 1 of CRM Monograph Series, American Mathematical Society, Providence, RI, 1992.
- [4] A. Talebi and M. S. Moslehian, A variance bound for a general function of independent noncommutative random variables and random matrices, submitted.
- [5] A. Talebi, M. S. Moslehian and Gh. Sadeghi, Noncommutative Blackwell-Ross martingale inequality, Infin. Dimens. Anal. Quantum Probab. Relat. Top., to appear.

DEPARTMENT OF PURE MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, IRAN, *E-mail address*: alitalebimath@yahoo.com;

DEPARTMENT OF PURE MATHEMATICS, CENTER OF EXCELLENCE IN ANALYSIS ON ALGEBRAIC STRUCTURES (CEAAS), FERDOWSI UNIVERSITY OF MASHHAD, P.O. BOX 1159, MASHHAD 91775, IRAN,

E-mail address: moslehian@um.ac.ir



## NONCOMMUTATIVE ETEMADI INEQUALITY

#### ALI TALEBI

ABSTRACT. In this paper, based on a maximal inequality type result from a paper of Cuculescu we establish a noncommutative maximal inequality, named Etemadi inequality.

Keywords: Noncommutative probability space; Trace; Noncommutative Etemadi inequality; Weakly fully independent.

#### 1. INTRODUCTION

One of important types of probability inequalities relates tail probabilities for the maximal partial sum of independent random variables such as Etemadi inequalities.

In [2], Etemadi established that if  $X_1, X_2, \ldots, X_n$  are independent random variables defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , then

$$\mathbb{P}\left(\max_{1 \le k \le n} \left| \sum_{i=1}^{k} X_i \right| \ge 3t \right) \le 3 \max_{1 \le k \le n} \mathbb{P}\left( \left| \sum_{i=1}^{k} X_i \right| \ge t \right)$$

for any positive real number t > 0.

One of the major problems occurring in the noncommutative probability theory concerns with the extensions of classical inequalities to the noncommutative setup.

A von Neumann algebra  $\mathfrak{M}$  on a Hilbert space  $\mathcal{H}$  with the unit element 1 equipped with a normal faithful tracial state  $\tau : \mathfrak{M} \to \mathbb{C}$  is called a noncommutative probability space. The elements of  $\mathfrak{M}$  are called (non-commutative) random variables. We denote by  $\leq$  the usual order on the self-adjoint part  $\mathfrak{M}^{sa}$  of  $\mathfrak{M}$ . For each self-adjoint operator  $a \in \mathfrak{M}$ , there exists a unique spectral measure E as a  $\sigma$ -additive mapping with respect to the strong operator topology from the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$  of  $\mathbb{R}$  into the set of all orthogonal projections such that for every Borel function  $f : \sigma(a) \to \mathbb{C}$  the operator f(x) is defined by  $f(a) = \int f(\lambda) dE(\lambda)$ , in particular,  $\mathbf{1}_B(a) = \int_B dE(\lambda) = E(B)$ . For  $p \geq 1$ , the noncommutative  $L_p$ -space  $L_p(\mathfrak{M})$  is defined as the completion of

For  $p \ge 1$ , the noncommutative  $L_p$ -space  $L_p(\mathfrak{M})$  is defined as the completion of  $\mathfrak{M}$  with respect to the  $L_p$ -norm  $||a||_p := (\tau(|a|^p))^{1/p}$ . Further, for a positive element  $a \in \mathfrak{M}$ , it holds that

(1.1) 
$$\|a\|_p^p = \int_0^\infty p t^{p-1} \tau(\mathbf{1}_{[t,\infty)}(a)) dt.$$

For further information we refer the reader to [4, 3] and references therein.

Let  $\mathcal{P}$  be the lattice of projections of  $\mathfrak{M}$ . Set  $p^{\perp} = 1 - p$  for  $p \in \mathcal{P}$ . For a collection of projections  $(p_{\lambda})_{\lambda \in \Lambda} \subseteq \mathcal{P}$ , we denote by  $\vee_{\lambda \in \Lambda} p_{\lambda}$  (resp.,  $\wedge_{\lambda \in \Lambda} p_{\lambda}$ ) the projection from  $\mathfrak{H}$  onto the closed subspace generated by  $p_{\lambda}(\mathfrak{H})$  (resp. onto the subspace  $\cap_{\lambda \in \Lambda} p(\mathfrak{H})$ ). Consequently,  $(\vee_{\lambda \in \Lambda} p_{\lambda})^{\perp} = \wedge_{\lambda \in \Lambda} p_{\lambda}^{\perp}$ . Two projections p and

<sup>2010</sup> Mathematics Subject Classification. Primary 15A60; Secondary 46L53, 47A30, 60F99. Speaker: Ali Talebi.

q are called equivalent if there exists a partial isometry  $v \in \mathfrak{M}$  such that  $v^*v = p$ and  $vv^* = q$ . In this case, we write  $p \sim q$ . If p is equivalent to a projection  $q_1 \leq q$ , we write  $p \prec q$ . We need the following elementary properties of projections (see [5]).

**Lemma 1.1.** Let p and q be two projection of  $\mathfrak{M}$ . Then (i)  $p \lor q - q \sim p - p \land q$ .

(ii) If  $p \wedge q = 0$  then  $p \prec q^{\perp}$ 

(iii) If p and q are equivalent projections in  $\mathfrak{M}$  then  $\tau(p) = \tau(q)$ .

(iv) If  $(p_{\lambda})_{\lambda \in \Lambda}$  is a family of projections in  $\mathfrak{M}$  then  $\tau (\forall_{\lambda \in \Lambda} p_{\lambda}) \leq \sum_{\lambda \in \Lambda} \tau(p_{\lambda})$ .

Our appraoches need several essential operator algebra techniques. Employing Cuculescu's approach [1] we establish some noncommutative maximal inequalities such as Hajek–Renyi and Kolmogorov inequality.

## 2. Main results

We say a sequence  $(x_k)_{k=1}^n$  is weakly fully independent if for any 1 < j < n the subalgebras  $W^*(x_1, \ldots, x_{j-1})$  and  $W^*(x_j, \ldots, x_n)$  are independent, where  $W^*(A)$  denotes the von Neumann algebra generated by the spectral projections of the real and imaginary parts of any member of  $A \subset \mathfrak{M}$ .

To establish a noncommutative version of the Etemadi inequality we need the following simple lemma.

**Lemma 2.1.** Let p, q and r be projections in  $\mathfrak{M}$  satisfying  $p \leq q$  and  $p \leq r$ . Then  $\tau(p) \leq \tau(qr)$ .

In particular, for arbitrary projections p, q it holds that  $\tau (p \land q) \leq \tau (pq)$ .

Proof.

$$\tau(p) = \tau\left(p^{\frac{1}{2}}pp^{\frac{1}{2}}\right) \le \tau\left(p^{\frac{1}{2}}rp^{\frac{1}{2}}\right) = \tau\left(r^{\frac{1}{2}}pr^{\frac{1}{2}}\right) \le \tau\left(qr\right).$$

**Theorem 2.2** (Noncommutative Etemadi inequality). Let  $a_1, a_2, \ldots, a_n$  be weakly fully independent self-adjoint random variables in  $\mathfrak{M}$  such that  $s_k s_n = s_n s_k$ , where  $s_k = \sum_{j=1}^k a_j$  for all  $1 \le k \le n$ . Then for every  $\alpha > 0$  there exists a projection q in  $\mathfrak{M}$  such that

$$\tau(q) \le 2\tau \left( \mathbf{1}_{[\alpha,\infty)} \left( |s_n| \right) \right) + \max_{1 \le k \le n} \tau \left( \mathbf{1}_{[\alpha,\infty)} \left( |s_k| \right) \right) \le 3 \max_{1 \le k \le n} \tau \left( \mathbf{1}_{[\alpha,\infty)} \left( |s_k| \right) \right).$$
$$\tau(1-p) \le \tau \left( \mathbf{1}_{[0,3\alpha)} \left( |a_1| \right) \right).$$

**Remark 2.3.** Notice that the projection q in the previous theorem is nonzero provided that  $\tau(\mathbf{1}_{[3\alpha,\infty)}(|s_k|)) \neq 0$  for some  $1 \leq k \leq n$ .

Remark 2.4. From the Etemadi's inequality, we may conclude that

$$\left(1 - \max_{1 \le k \le n} \tau\left(\mathbf{1}_{(2\alpha,\infty)}\left(|s_n - s_k|\right)\right)\right) \tau(q) \le \tau\left(\mathbf{1}_{(\alpha,\infty)}\left(|s_n|\right)\right)$$

and hence

$$\tau(q) \le \frac{\tau\left(\mathbf{1}_{(\alpha,\infty)}\left(|s_n|\right)\right)}{1 - \max_{1 \le k \le n} \tau\left(\mathbf{1}_{(2\alpha,\infty)}\left(|s_n - s_k|\right)\right)}$$

provided that  $\max_{1 \le k \le n} \tau \left( \mathbf{1}_{(2\alpha,\infty)} \left( |s_n - s_k| \right) \right) \ne 1.$ 

**Corollary 2.5.** Let  $X_1, X_2, \ldots, X_n$  be independent random variables. If t > 0 is an arbitrary positive real number, then

$$\mathbb{P}\left(\max_{1\leq k\leq n} \left|\sum_{i=1}^{k} X_{i}\right| \geq 3t\right) \leq 3\max_{1\leq k\leq n} \mathbb{P}\left(\left|\sum_{i=1}^{k} X_{i}\right| \geq t\right).$$

## References

- [1] I. Cuculescu, Martingales on von Neumann algebras, J. Multivariate Anal. 1 (1971), 17–27.
- [2] N. Etemadi, On some classical results in probability theory, Sankhy Ser A 47 (1985), no. 2.
- [3] A. Talebi, M. S. Moslehian and Gh. Sadeghi, *Etemadi and Kolmogorov inequalities in non*commutative probability spaces, submitted.
- [4] A. Talebi, M. S. Moslehian and Gh. Sadeghi, Noncommutative Blackwell-Ross martingale inequality, Infin. Dimens. Anal. Quantum Probab. Relat. Top., to appear.
- [5] Q. Xu, Operator spaces and noncommutative L<sub>p</sub>, Lectures in the Summer School on Banach spaces and Operator spaces, Nankai University China, 2007.

DEPARTMENT OF PURE MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, IRAN, *E-mail address*: alitalebimath@yahoo.com;



# ON THE SOLUTIONS OF OPERATOR EQUATIONS BXA = B = AXB VIA \*-ORDER

MAHDI VOSOUGH AND MOHAMMAD SAL MOSLEHIAN

ABSTRACT. In this paper, we establish some necessary and sufficient conditions for the existence of solutions to the system of operator equations BXA = B = AXB in the setting of bounded linear operators on a Hilbert space, where the unknown operator X is called the inverse of A along B. After that, under some mild conditions we prove that an operator X is a solution of

BXA = B = AXB if and only if  $B \leq AXA$ , where the \*-order  $C \leq D$  means  $CC^* = DC^*, C^*C = C^*D$ . Moreover we present the general solution of the above equation.

Keywords: Moore-Penrose inverse; Operator equations.

### 1. INTRODUCTION

Throughout the paper,  $\mathscr{H}$  and  $\mathscr{K}$  are complex Hilbert spaces. An operator  $A \in \mathbb{B}(\mathscr{H})$  is a generalized projection if  $A^2 = A^*$ . Let  $\mathscr{S}(\mathscr{H}), \mathscr{Q}(\mathscr{H}), \mathscr{OP}(\mathscr{H}), \mathscr{GP}(\mathscr{H})$  be set of all self-adjoint operators on  $\mathscr{H}$ , the set of all idempotents, the set of orthogonal projections and the set of all generalized projections on  $\mathscr{H}$ , respectively. For  $A \in \mathbb{B}(\mathscr{H}, \mathscr{H})$ , let  $\mathscr{R}(A)$  and  $\mathscr{N}(A)$  be the range and the null space of A, respectively. The projection corresponding to a closed subspace  $\mathscr{M}$  of  $\mathscr{H}$  is denoted by  $P_{\mathscr{M}}$ . The symbol  $A^-$  stands for an arbitrary generalized inner inverse of A, that is, an operator  $A^-$  satisfying  $AA^-A = A$ . The Moore–Penrose inverse of a closed range operator A is the unique operator  $A^{\dagger} \in \mathbb{B}(\mathscr{H})$  satisfying the following equations

$$AA^{\dagger}A = A, \qquad A^{\dagger}AA^{\dagger} = A^{\dagger}, \qquad (AA^{\dagger})^* = AA^{\dagger}, \qquad (A^{\dagger}A)^* = A^{\dagger}A.$$

Then  $A^*AA^{\dagger} = A^* = A^{\dagger}AA^*$  see [2]. For  $A, B \in \mathscr{S}(\mathscr{H}), A \leq B$  means  $B - A \geq 0$ . The order  $\leq$  is said to be the Löwner order on  $\mathscr{S}(\mathscr{H})$ . If there exists  $C \in \mathscr{S}(\mathscr{H})$  such that AC = 0 and A + C = B, then we write  $A \leq B$ . The order  $\leq$  is said to be the logic order on  $\mathscr{S}(\mathscr{H})$  see [5, 4]. For  $A, B \in \mathbb{B}(\mathscr{H})$ , let  $A \leq B$  mean

It is known that if  $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$  has closed range, then by considering

$$\mathscr{H} = \mathscr{R}(A^*) \oplus \mathscr{N}(A) \text{ and } \mathscr{K} = \mathscr{R}(A) \oplus \mathscr{N}(A^*)$$

we can write

(1.2) 
$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix},$$

<sup>2010</sup> Mathematics Subject Classification. pirmary 15A60, secondary 47A05, 15A09. Speaker: Mahdi Vosough.

where  $A_1 : \mathscr{R}(A^*) \to \mathscr{R}(A)$  is invertible, see [3, 1]. Therefore, the Moore–Penrose generalized inverse of A can be represented as

(1.3) 
$$A^{\dagger} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{bmatrix}.$$

## 2. Main results

We need the following essential lemma.

Lemma 2.1. Let  $A, B \in \mathbb{B}(\mathscr{H})$  and  $\overline{\mathscr{M}}$  denote the closure of a space  $\mathscr{M}$ . (a)  $AA^* = BA^* \iff A = BP_{\overline{\mathscr{M}(A^*)}} \iff A = BQ$  for some  $Q \in \mathscr{OP}(\mathscr{H})$ ; (b)  $A^*A = A^*B \iff A = P_{\overline{\mathscr{M}(A)}}B \iff A = PB$  for some  $P \in \mathscr{OP}(\mathscr{H})$ ; (c)  $A \stackrel{*}{\leq} B \iff B = A + P_{\mathscr{N}(A^*)}BP_{\mathscr{N}(A)}$ ; (d)  $A \stackrel{*}{\leq} B \iff A = P_{\overline{\mathscr{M}(A)}}B = BP_{\overline{\mathscr{M}(A^*)}} = P_{\overline{\mathscr{M}(A)}}BP_{\overline{\mathscr{M}(A^*)}}$ ; (e)  $A \stackrel{*}{\leq} B \iff A = A_1 \bigoplus 0, B = A_1 \bigoplus B_1$ ; where  $A_1 \in \mathbb{B}(\overline{\mathscr{M}(A^*)}, \overline{\mathscr{M}(A)}), B_1 \in \mathbb{B}(\mathscr{N}(A), \mathscr{N}(A^*))$  and  $A \bigoplus B$  means the block matrix  $\begin{bmatrix} A & 0\\ 0 & B \end{bmatrix}$ . Proposition 2.2. Let  $A, B \in \mathbb{B}(\mathscr{H})$ . Then

\*

- (a) If A has closed range and  $B \stackrel{*}{\leq} A$ , then  $X = A^{\dagger}$  is a solution of the system BXA = B = AXB.
- (b) If B has closed range and  $B \stackrel{*}{\leq} A$ , then  $X = B^{\dagger}$  is a solution of the system BXA = B = AXB.

**Theorem 2.3.** Let  $A, B \in \mathbb{B}(\mathscr{H})$  where A has closed range. If the system BXA = B = AXB is solvable, then the system  $XB = A^{\dagger}B, BX = BA^{\dagger}$  is solvable. Conversely, If  $B \stackrel{*}{\leq} A$  and the system  $XB = A^{\dagger}B, BX = BA^{\dagger}$  is solvable, then the system BXA = B = AXB is solvable.

**Corollary 2.4.** Let  $A, B \in \mathbb{B}(\mathscr{H})$  and  $B \stackrel{*}{\leq} A$ . Then the following statements are equivalent:

- (a) There exists a solution  $X \in \mathbb{B}(\mathscr{H})$  of the system BXA = B = AXB;
- (b)  $B \stackrel{*}{\leq} AXA$

#### References

- L. Long and S. Gudder. On the supremum and infimum of bounded quantum observables. Journal of Mathematical physices, 52:122101, 2011.
- [2] Z. Mousavi, F. Mirzapour and M.S. Moslehian. Positive definite solutions of certain nonlinear matrix equations. Operators and Matrices, 10:113–126, 2016.

- [3] M. Vosough and M. S. Moslehian, Solutions of the system of operator equations BXA=B=AXB via \*-order, Electron. J. Linear Algebra (to appear)
- [4] Q.-W. Wang and C.-Z. Dong. Positive solutions to a system of adjointable operator equations over Hilbert C<sup>\*</sup>-modules. *Linear Algebra and its Applications*, 433:1481–1489, 2010.
- [5] X.M. Xu, H.K. Du, X.C. Fang and Y. Li. The supremum of linear operators for the \*-order. *Linear Algebra and its Applications*, 433:2198–2207, 2010.

Department of Pure Mathematics, Ferdowsi University of Mashhad.  $E\text{-}mail\ address: \texttt{vosough.mehdi@yahoo.com}$ 

DEPARTMENT OF PURE MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, P.O. Box 1159, MASHHAD 91775, IRAN.

 $E\text{-}mail\ address: \texttt{moslehian}\texttt{Gum.ac.ir}$ 



# ON THE SOLVABILITY OF THE MATRIX INEQUALITY $AXA^* + BX^*B^* \ge C$

#### MAHDI VOSOUGH AND MOHAMMAD SAL MOSLEHIAN

ABSTRACT. In this talk, we state some necessary and sufficient conditions for the existence of a solution of  $AXA^* + BX^*B^* \ge C$ , where A, B are arbitrary matrices and C is a Hermitian matrix. In the special case when B = A, we determine the general solution of  $A(X + X^*)A^* \ge C$ , where A is an arbitrary matrix and C is a Hermitian matrix.

Keywords: Matrix equation; Matrix inequality; Generalized inverse.

#### 1. INTRODUCTION

Throughout this paper,  $\mathbb{C}^{m \times n}$  and  $\mathbb{C}_{H}^{m}$  stand for the sets of  $m \times n$  complex matrices and  $m \times m$  complex Hermitian matrices, respectively. We denote by  $I_{n}$  the identity matrix of  $\mathbb{C}^{n \times n}$ . We write A > 0  $(A \ge 0)$  if A is positive definite (positive semid efinite). Two Hermitian matrices  $A, B \in \mathbb{C}^{n \times n}$  are said to satisfy the inequality  $A \ge B$  in sense of the Löwner partial ordering if A - B is positive semidefinite. For  $A \in \mathbb{C}^{m \times n}$ , let  $\mathcal{R}(A), r(A), A^{T}$ , and  $A^{*}$  be the range space, the rank, the transpose, and the conjugate transpose of A, respectively. The notation  $\begin{bmatrix} A & B \end{bmatrix}$  denotes a matrix with columns  $A \in \mathbb{C}^{m \times n}$  and  $B \in \mathbb{C}^{m \times k}$ . Similarly  $\begin{bmatrix} A \\ C \end{bmatrix}$  is a matrix with rows  $A \in \mathbb{C}^{m \times n}$  and  $C \in \mathbb{C}^{\ell \times n}$ . The Moore–Penrose inverse of a matrix  $A \in \mathbb{C}^{m \times n}$  is the matrix  $A^{\dagger} \in \mathbb{C}^{n \times m}$  satisfying the following equations

(1.1)  $AA^{\dagger}A = A, \qquad A^{\dagger}AA^{\dagger} = A^{\dagger}, \qquad (AA^{\dagger})^* = AA^{\dagger}, \qquad (A^{\dagger}A)^* = A^{\dagger}A.$ 

Furthermore,

(1.2)  $A^*AA^{\dagger} = A^* = A^{\dagger}AA^*$  and  $(A^{\dagger})^* = (A^*)^{\dagger}$ .

A Hermitian matrix A is said to be a *projection* if  $A^2 = A$ . Further, for  $A \in \mathbb{C}^{m \times n}$ , we set

$$E_A := I_m - AA^{\dagger}$$
 and  $F_A := I_n - A^{\dagger}A$ .

It is known that  $E_A$  and  $F_A$  are projections such that  $r(E_A) = m - r(A)$  and  $r(F_A) = n - r(A)$ .

The inertia of a Hermitian matrix A is defined by the triplet  $In(A) = (i_+(A), i_-(A), i_0(A))$ , where  $i_+(A), i_-(A)$  and  $i_0(A)$  are the numbers of the positive, negative and zero eigenvalues of A counted with multiplicities, respectively. The numbers  $i_+(A)$  and  $i_-(A)$  are called the *positive* and the *negative index of inertia*, respectively. For  $A \in \mathbb{C}_H^m$ , it is known that  $r(A) = i_+(A) + i_-(A)$  and  $i_0(A) = m - r(A)$ .

<sup>2010</sup> Mathematics Subject Classification. pirmary 15A60, secondary 15A24; 15A09; 15B57. Speaker: Mahdi Vosough.

$$(1.3) AXB + CYD = G$$

is a more generalization of the generalized Sylvester's equation. In 2016, Wang [5] investigated matrix equation (1.3) and gave the general solution of this equation. In 2015, Liu [2] investigated the matrix inequality

$$(1.4) AXB + (AXB)^* \ge C,$$

determined the general solution, and obtained the Hermitian solutions of the equation  $AXA^* = B$  subject to  $CXC^* \ge D$  see [3, 4]

## 2. Main results

In this section, we first determine the general solution of the matrix equation

$$AXA^* + BX^*B^* = C,$$

where  $A, B \in \mathbb{C}^{m \times n}$  and  $C \in \mathbb{C}^{m \times m}$ . To achieve our aim, we need the next proposition.

**Proposition 2.1.** Let  $A, B \in \mathbb{C}^{m \times n}$  and  $C \in \mathbb{C}^{m \times m}$ . Set  $M := E_A B$ . Then the following statements are equivalent:

(i) Eq. (2.1) is solvable;  
(ii) 
$$\begin{bmatrix} E_M E_A \\ E_A \end{bmatrix} C \begin{bmatrix} I_m & E_B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ E_A C & 0 \end{bmatrix}$$
;  
(iii)  $r \begin{bmatrix} C & A & B \end{bmatrix} = r \begin{bmatrix} A & B \end{bmatrix} = r \begin{bmatrix} C^* & A & B \end{bmatrix}$ .  
Moreover, the general solution of (2.1) can be written as  
 $\begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix}^{\dagger} C \left( \begin{bmatrix} A & B \end{bmatrix}^{\dagger} \right)^* + F_{\begin{bmatrix} A & B \end{bmatrix}} V + WF_{\begin{bmatrix} A & B \end{bmatrix}}$ ,  
where  $V = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}$  and  $W = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}$  are arbitrary. And  
 $X = (A^{\dagger} - A^{\dagger} B M^{\dagger} E_A) C (A^{\dagger} - A^{\dagger} B M^{\dagger} E_A)^* + F_A V_1 + W_1 F_A - A^{\dagger} B F_M V_3$ ;  
 $X^* = M^{\dagger} E_A C (M^{\dagger} E_A)^* + F_M V_4 + W_4 F_M - W_3 A^{\dagger} B F_M$ ;  
 $(A^{\dagger} - A^{\dagger} B M^{\dagger} E_A) C (M^{\dagger} E_A)^* + F_A V_2 + W_2 F_M - A^{\dagger} B F_M V_4 = 0$ ;  
 $M^{\dagger} E_A C (A^{\dagger} - A^{\dagger} B M^{\dagger} E_A)^* + F_M V_3 + W_3 F_M = 0$ .

In particular, if V = W = 0, then

$$X + (A^{\dagger}B)X^{*}(A^{\dagger}B)^{*} = A^{\dagger}C(A^{\dagger})^{*}.$$

Now we ready to state main result.

**Theorem 2.2.** If  $A, B \in \mathbb{C}^{m \times n}$  and  $C \in \mathbb{C}^m_H$ , then the following statements are equivalent:

(i) There exists a matrix  $X \in \mathbb{C}^{n \times n}$  such that

$$AXA^* + BX^*B^* \ge C;$$

(ii)

$$E \begin{bmatrix} A & B \end{bmatrix}^{CE} \begin{bmatrix} A & B \end{bmatrix} \le 0$$

and

(2.3) 
$$r\left(E_{\begin{bmatrix}A & B\end{bmatrix}}CE_{\begin{bmatrix}A & B\end{bmatrix}}\right) = r\left(E_{\begin{bmatrix}A & B\end{bmatrix}}C\right);$$

(iii) 
$$i_{+}(N) = r \begin{bmatrix} A & B \end{bmatrix}$$
 and  $i_{-}(N) = r \begin{bmatrix} C & A & B \end{bmatrix}$ , where  $N = \begin{bmatrix} C & A & B \\ A^{*} & 0 & 0 \\ B^{*} & 0 & 0 \end{bmatrix}$ 

In this case, the general Hermitian solution of (2.2) can be written in the following parametric forms

$$\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix}^{\dagger} W \left( \begin{bmatrix} A & B \end{bmatrix}^{\dagger} \right)^{*} + F_{\begin{bmatrix} A & B \end{bmatrix}} V + V^{*} F_{\begin{bmatrix} A & B \end{bmatrix}},$$

where

$$W = C - CE_{\begin{bmatrix} A & B \end{bmatrix}} \begin{pmatrix} E_{\begin{bmatrix} A & B \end{bmatrix}} CE_{\begin{bmatrix} A & B \end{bmatrix}} \end{pmatrix}^{\dagger} E_{\begin{bmatrix} A & B \end{bmatrix}} C + UU^*,$$

and  $V, U \in \mathbb{C}^{n \times n}$  are arbitrary matrices.

**Corollary 2.3.** Let  $A \in \mathbb{C}^{m \times n}$  and  $C \in \mathbb{C}^m_H$ . Then the following statements are equivalent:

(i) There exists a matrix  $X \in \mathbb{C}^{n \times n}$  such that

(2.4) 
$$A(X + X^*)A^* \ge C;$$
  
(ii)  $E_A C E_A \le 0$  and  $r(E_A C E_A) = r(E_A C);$   
(iii)  $i_+(N) = r(A)$  and  $i_-(N) = r \begin{bmatrix} C & A & A \\ A^* & 0 & 0 \\ A^* & 0 & 0 \end{bmatrix}$ 

In this case, the general Hermitian solution of (2.4) can be written in the following parametric forms

$$\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} = \begin{bmatrix} A & A \end{bmatrix}^{\dagger} W \left( \begin{bmatrix} A & A \end{bmatrix}^{\dagger} \right)^* + F \begin{bmatrix} A & A \end{bmatrix}^V + V^* F \begin{bmatrix} A & A \end{bmatrix},$$

where

$$W = C - CE_A \left( E_A CE_A \right)^{\dagger} E_A C + UU^*,$$

and  $V, U \in \mathbb{C}^{n \times n}$  are arbitrary matrices.

## References

- F. O. Farid, M. S. Moslehian, Q.-W. Wang and Z.-C. Wu, On the Hermitian solutions to a system of adjointable operator equations, Linear Algebra Appl. 437 (2012), no. 7, 1854-1891.
- [2] X. Liu, The Hermitian solution of AXA<sup>\*</sup> = B subject to CXC<sup>\*</sup> ≥ D, Appl. Math. Comput. 270 (2015), 890–898.
- [3] M. Vosough and M. S. Moslehian, Solutions of the system of operator equations BXA=B=AXB via \*-order, Electron. J. Linear Algebra (to appear)
- [4] M. Vosough and M. S. Moslehian, Solvability of the matrix inequality  $AXA^* + BX^*B^* \ge C$ , submitted.
- [5] Q. W. Wang and Z.-H. He, The Common Solution of Some Matrix Equations, Algebra Colloquium, 23 (2016), no. 1, 71–81.

Department of Pure Mathematics, Ferdowsi University of Mashhad. E-mail address: vosough.mehdi@yahoo.com

 $2\_$  Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran.

 $E\text{-}mail\ address:\ \texttt{moslehian}\texttt{Cum.ac.ir}$ 



## NEW ADDITIVE RESULT ON DRAZIN INVERSE

MANSOUR DANA AND RAMESH YOUSEFI

ABSTRACT. We present a new computational formula for the Drazin inverse of the sum P + Q, under conditions weaker than those used in some literature on this subject. Finally, numerical example is given to illustrate our results.

Keywords: Drazin inverse; Index.

### 1. INTRODUCTION

The Drazin inverse plays an important role in diverse fields like Markov chains, singular differential and difference equations, and iterative methods [1].

Let A be a square complex matrix. As we know, the Drazin inverse [1] of A, denoted by  $A^d$ , is the unique matrix satisfying the following three equations

$$A^{k+1}A^d = A^k, \qquad A^d A A^d = A^d, \qquad A A^d = A^d A,$$

where k is the smallest non-negative integer such that  $rank(A^{k+1}) = rank(A^k)$ , i.e., k = ind(A), the index of A. In the case that ind(A) = 1, the Drazin inverse is called the group inverse of A and it is denoted by  $A^{\sharp}$ . Clearly, ind(A) = 0 if and only if A is nonsingular, and in that case  $A^d = A^{-1}$ . We denote by  $A^{\pi}$  the eigenprojection of A corresponding to the eigenvalue 0 that is given by  $A^{\pi} = I - AA^d$ .

Suppose  $P, Q \in \mathbb{C}^{n \times n}$ . In 1958, Drazin (see [2]) studied the problem of finding the formula for  $(P+Q)^d$  and he offered the formula  $(P+Q)^d = P^d + Q^d$ , which is valid when PQ = QP = 0. In the present, there is no formula for  $(P+Q)^d$  without any side condition for matrices P and Q, so this problem remains open. However, many authors have considered this problem and provided a formula for  $(P+Q)^d$ with some specific conditions for matrices P and Q. For example, Hartwig et al. [3] extended Drazin's result to the situation when only PQ = 0, but with a more complicated formula for  $(P+Q)^d$ . In 2011, Yang and Liu [4] gave the result of  $(P+Q)^d$  when  $Q^2P = 0$  and PQP = 0.

In this paper, we give the formula of  $(P+Q)^d$  under the conditions  $P^2QP = 0$ ,  $PQP^2 = 0$  and  $Q^2P = 0$ . These result extend the formulas in [2, 4].

Before giving the main results, we first introduce several lemmas as follows.

**Lemma 1.1** ([1]). Let  $P \in \mathbb{C}^{m \times n}$  and  $Q \in \mathbb{C}^{n \times m}$ . Then  $(PQ)^d = P((QP)^d)^2 Q$ . **Lemma 1.2** ([3]) Let  $P \cap \subset \mathbb{C}^{n \times n}$  be such that ind(P) = s and ind(Q) = t.

**Lemma 1.2** ([3]). Let  $P, Q \in \mathbb{C}^{n \times n}$  be such that ind(P) = s and ind(Q) = t. If PQ = 0, then

$$(P+Q)^{d} = Q^{\pi} \sum_{i=0}^{t-1} Q^{i} (P^{d})^{i+1} + \sum_{i=0}^{s-1} (Q^{d})^{i+1} P^{i} P^{\pi}.$$

Recall that N is nilpotent of index k when  $N^k = 0$  but  $N^{k-1} \neq 0$ .

<sup>2010</sup> Mathematics Subject Classification. 15A09.

Speaker: Ramesh Yousefi.

Lemma 1.3. Let  $P, Q \in \mathbb{C}^{n \times n}$ .

- (1) PQ is nilpotent if and only if QP is nilpotent.
- (2) If PQ be nilpotent of index k, then QP is nilpotent of index l where  $l \in \{k-1, k, k+1\}$ .

## 2. Main results

In this section, we first give the formula of the Drazin inverse of P + Q under the conditions that  $P^2QP = 0$ ,  $PQP^2 = 0$  and  $Q^2P = 0$ , which will be the main tool in our following development.

**Theorem 2.1.** Let  $P, Q \in \mathbb{C}^{n \times n}$  be such that  $P^2QP = 0$ ,  $PQP^2 = 0$  and  $Q^2P = 0$ , then

$$(P+Q)^{d} = (P+Q) \left( \sum_{i=0}^{l-1} (P^{d})^{i+2} Q^{i} Q^{\pi} + \sum_{i=0}^{r-1} P^{\pi} P^{i} (Q^{d})^{i+2} \right) + \sum_{i=0}^{\lfloor (\frac{l-1}{2}) \rfloor} \left( P((QP)^{d})^{i+1} + ((QP)^{d})^{i+1} Q \right) Q^{2i} Q^{\pi} + \sum_{i=0}^{s-2} \left( P(QP)^{\pi} (QP)^{i+1} + (QP)^{\pi} (QP)^{i+1} Q \right) (Q^{d})^{2(i+2)} - QP^{d} Q^{d} - PP^{d} Q^{d} - QP (Q^{d})^{3} - (QP) (QP)^{d} Q^{d} - (PQ)^{2} (PQ)^{d} (Q^{d})^{3},$$

where r = ind(P), l = ind(Q) and s = ind(QP).

Proof. From the definition of the Drazin inverse, we have that

$$(P+Q)^d = (P+Q)((P+Q)^d)^2 = (P+Q)(P^2 + QP + Q^2 + PQ)^d.$$

Denote by  $F = Q^2 + PQ$  and  $G = P^2 + QP$ . From  $PQP^2 = 0$  and  $Q^2P = 0$ , we get FG = 0. Then applying Lemma (1.2), we obtain

(2.1) 
$$(P+Q)^d = (P+Q) \left( \sum_{i=0}^{ind(G)-1} G^{\pi} G^i (F^d)^{i+1} + \sum_{i=0}^{ind(F)-1} (G^d)^{i+1} F^i F^{\pi} \right).$$

Now, we calculate  $F^d$ . Since  $Q^2 P = 0$ , then we can apply Lemma (1.2) to get

(2.2) 
$$(F^d)^n = \sum_{i=0}^s (PQ)^\pi (PQ)^i (Q^d)^{2(i+n)} + \sum_{i=0}^{\lfloor (\frac{l-1}{2}) \rfloor} ((PQ)^d)^{i+n} Q^{2i} Q^\pi,$$

for every  $n \in \mathbb{N}$ . Similarly, from  $P^2QP = 0$  and Lemma (1.2), for every  $n \in \mathbb{N}$  we have

(2.3) 
$$(G^d)^n = ((QP)^d)^n + (P^d)^{2n} + Q(P^d)^{2n+1}.$$

After computation we get

$$\left\{ \begin{array}{ll} G^n = P^{2n} + QP^{2n-1} + (QP)^n, & \text{if } n \geq 2, \\ F^n = \sum_{i=0}^n (PQ)^{n-i}Q^{2i}, & \text{if } n \geq 1. \end{array} \right.$$

After substituting this expressions, (2.3) and (2.2) into (2.1) we complete the proof.  $\hfill\square$ 

The next theorem is a symmetrical formulation of Theorem (2.1).

**Theorem 2.2.** Let  $P, Q \in \mathbb{C}^{n \times n}$  be such that  $PQP^2 = 0$ ,  $P^2QP = 0$  and  $PQ^2 = 0$ , then

$$(P+Q)^{d} = \left(\sum_{i=0}^{l-1} Q^{\pi} Q^{i} (P^{d})^{i+2} + \sum_{i=0}^{r-1} (Q^{d})^{i+2} P^{i} P^{\pi}\right) (P+Q) + \sum_{i=0}^{\lfloor \left(\frac{l-1}{2}\right) \rfloor} Q^{\pi} Q^{2i} \left( ((PQ)^{d})^{i+1} P + Q((PQ)^{d})^{i+1} \right) + \sum_{i=0}^{s-2} (Q^{d})^{2(i+2)} \left( (PQ)^{\pi} (PQ)^{i+1} P + Q(PQ)^{\pi} (PQ)^{i+1} \right) - Q^{d} P^{d} Q - Q^{d} P P^{d} - (Q^{d})^{3} P Q - Q^{d} (PQ) (PQ)^{d} - (Q^{d})^{3} (QP)^{2} (QP)^{d},$$

where r = ind(P), l = ind(Q) and s = ind(PQ).

Notice that one special case of Theorem (2.1) is when matrix QP is nilpotent. Similarly, a special case of Theorem (2.2) is when matrix PQ is nilpotent. The following additive formulas are corollaries of these cases, respectively.

**Corollary 2.3.** Let  $P, Q \in \mathbb{C}^{n \times n}$ , let QP be nilpotent of index k. If  $P^2QP = 0$ ,  $PQP^2 = 0$  and  $Q^2P = 0$ , then

$$(P+Q)^{d} = (P+Q) \left( \sum_{i=0}^{l-1} (P^{d})^{i+2} Q^{i} Q^{\pi} + \sum_{i=0}^{r-1} P^{\pi} P^{i} (Q^{d})^{i+2} \right) + \sum_{i=0}^{k-2} \left( P(QP)^{i+1} + (QP)^{i+1} Q \right) (Q^{d})^{2(i+2)} - QP^{d} Q^{d} - PP^{d} Q^{d} - QP (Q^{d})^{3},$$

where l = ind(Q) and r = ind(P).

**Corollary 2.4.** Let  $P, Q \in \mathbb{C}^{n \times n}$ , let PQ be nilpotent of index k. If  $P^2QP = 0$ ,  $PQP^2 = 0$  and  $PQ^2 = 0$ , then

$$(P+Q)^{d} = \left(\sum_{i=0}^{l-1} Q^{\pi} Q^{i} (P^{d})^{i+2} + \sum_{i=0}^{r-1} (Q^{d})^{i+2} P^{i} P^{\pi}\right) (P+Q) + \sum_{i=0}^{k-2} (Q^{d})^{2(i+2)} \left( (PQ)^{i+1} P + Q(PQ)^{i+1} \right) - Q^{d} P^{d} Q - Q^{d} P P^{d} - (Q^{d})^{3} P Q,$$

where l = ind(Q) and r = ind(P).

In the special case when k = 2, we get the next results.

**Corollary 2.5.** Let  $P, Q \in \mathbb{C}^{n \times n}$ , let QP be 2-nilpotent. If  $P^2QP = 0$ ,  $PQP^2 = 0$ and  $Q^2P = 0$ , then

$$(P+Q)^{d} = (P+Q) \left( \sum_{i=0}^{l-1} (P^{d})^{i+2} Q^{i} Q^{\pi} + \sum_{i=0}^{r-1} P^{\pi} P^{i} (Q^{d})^{i+2} \right) + PQP(Q^{d})^{4} - QP^{d} Q^{d} - PP^{d} Q^{d},$$

where l = ind(Q) and r = ind(P).

**Corollary 2.6.** Let  $P, Q \in \mathbb{C}^{n \times n}$ , let PQ be 2-nilpotent. If  $P^2QP = 0$ ,  $PQP^2 = 0$ and  $PQ^2 = 0$ , then

$$(P+Q)^{d} = (Q^{d})^{4} P Q P - Q^{d} P^{d} Q - Q^{d} P P^{d} + \left(\sum_{i=0}^{l-1} Q^{\pi} Q^{i} (P^{d})^{i+2} + \sum_{i=0}^{r-1} (Q^{d})^{i+2} P^{i} P^{\pi}\right) (P+Q),$$

where l = ind(Q) and r = ind(P).

Next, we give a numerical example of Theorem (2.1) which does not satisfy the conditions from [4], but it satisfies the conditions of our Corollary (2.3).

**Example 2.7.** Consider the two matrices  $P, Q \in \mathbb{C}^{6 \times 6}$ , where

for every nonzero  $a, b \in \mathbb{C}$ . We have

From  $PQP \neq 0$ ,  $QPQ \neq 0$  and  $P^2 \neq 0$ , formula for  $(P+Q)^d$  from [4, Theorem (2.2)] fail to apply. But it satisfies  $(PQ)^2 = 0$ ,  $PQP^2 = 0$ ,  $P^2QP = 0$  and  $Q^2P = 0$ , also we have

$$ind(P) = 2$$
,  $ind(Q) = 3$ .

Applying Corollary (2.3), we get

## References

- A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, second ed., Springer Verlag, New York, 2003.
- [2] M.P. Drazin, Pseudoinverse in associative rings and semigroups, Amer. Math. Monthly. 65 (1958) 506-514.
- [3] R.E. Hartwig, G. Wang, Y. Wei, Some additive results on Drazin inverse, Linear Algebra Appl. 322 (2001) 207-217.
- [4] H. Yang and X. Liu, The Drazin inverse of the sum of two matrices and its applications, J. Comput. Appl. Math. 235 (2011) 1412-1417.

University of Kurdistan, Sanandage, 66177 Islamic Republic of Iran<br/>  $E\text{-}mail\ address:\ mdana@uok.ac.ir$ 

UNIVERSITY OF KURDISTAN, SANANDAGE, 66177 ISLAMIC REPUBLIC OF IRAN *E-mail address*: Ramesh.Yousefi@Yahoo.com



## ON DRAZIN INVERSE OF SUM OF TWO OPERATOR MATRICES

MANSOUR DANA AND RAMESH YOUSEFI

ABSTRACT. In this short paper, We give formula for the Drazin inverse of the sum of two operators under special conditions. Finally, numerical example is given to illustrate our result.

Keywords: Drazin inverse; Index.

## 1. INTRODUCTION

Let  $\mathcal{B}(\mathcal{X})$  denote the Banach algebra of all bounded operators on the complex Banach space  $\mathcal{X}$ . If  $T \in \mathcal{B}(\mathcal{X})$  we write  $\sigma(T)$ ,  $\rho(T)$  and r(T) for the spectrum, the resolvent set and the spectral radius of T, respectively. For  $\lambda \in \rho(T)$  we denote the resolvent  $(\lambda I - T)^{-1}$  by  $R(\lambda, T)$ . If 0 is an isolated point of  $\sigma(T)$ , then the spectral projection of T associated with  $\{0\}$  is defined by

$$T^{\pi} = \frac{1}{2\pi i} \int_{\gamma} R(\lambda, T) d\lambda$$

where  $\gamma$  is a small circle surrounding 0 and separating 0 from  $\sigma(T) \setminus \{0\}$ . The Drazin inverse for bounded linear operators on complex Banach spaces was investigated by Caradus [2].

An operator  $T \in \mathcal{B}(\mathcal{X})$  is said to be Drazin invertible if there exists an operator  $T^d \in \mathcal{B}(\mathcal{X})$  such that

 $TT^d = T^dT, \qquad T^dTT^d = T^d, \qquad T^{k+1}T^d = T^k$ 

for some integer  $k \ge 0$ . The smallest integer  $k \ge 0$ , in the latter identity is called the index ind(T) of T. If we define  $T^0 = I$ , then the previous conditions hold with k = 0 if and only if T is invertible. We note that if T is nilpotent, then it is Drazin invertible,  $T^d = 0$ , and ind(T) = r, where r is the power of nilpotency of T. If Tis Drazin invertible then  $T^{\pi} = I - TT^d$ . In this case  $R(\lambda, T)$  has a pole of order rat  $\lambda = 0$  and it can be expressed by [2]

$$R(\lambda,T) = \sum_{n=1}^{r} \frac{T^{n-1}T^{\pi}}{\lambda^n} - \sum_{n=0}^{\infty} \lambda^n (T^d)^{n+1}$$

in the region  $0 < |\lambda| < (r(T^d))^{-1}$ .

If P and Q are two Drazin invertible operators such that PQ = QP = 0 then  $(P+Q)^d = P^d + Q^d$ . This result was originally proved by Drazin [3] in the contexts of associative rings and semigroups. In the present, there is no formula for  $(P+Q)^d$  without any side condition for matrices P and Q, so this problem remains open. However, many authors have considered this problem and provided a formula for

<sup>2010</sup> Mathematics Subject Classification. 15A09.

Speaker: Ramesh Yousefi.

 $(P+Q)^d$  with some specific conditions for matrices P and Q. In 2009, Martínez-Serrano and Castro- González [4] gave a result of  $(P+Q)^d$  when  $P^2Q = 0$  and  $Q^2 = 0$ . In 2012, Bu et al. [1] gave the representation of  $(P+Q)^d$  when  $P^2Q = 0$  and  $Q^2P = 0$ .

In this paper, we give the formula of  $(P+Q)^d$  under the conditions  $PQP^2 = 0$ ,  $PQ^2 = 0$  and  $QP^3 = 0$ . These result extend the formulas in [3, 4, 1].

Before giving the main results, we first introduce several lemmas as follows.

Lemma 1.1 ([4]). if  $A \in \mathcal{B}(\mathcal{X})$  and  $B \in \mathcal{B}(\mathcal{Y})$  are Drazin invertible with ind(A) = m and ind(B) = n. Then  $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$  is also Drazin invertible and  $M^d = \begin{pmatrix} A^d & S \\ 0 & B^d \end{pmatrix}$ , where  $S = \sum_{i=0}^{n-1} (A^d)^{i+2} CB^i B^{\pi} + A^{\pi} \sum_{i=0}^{m-1} A^i C(B^d)^{i+2} - A^d CB^d$ 

**Lemma 1.2** ([4]). Let  $P, Q \in \mathcal{B}(\mathcal{X})$  be Drazin invertibles such that PQ = 0, then P + Q is Drazin invertible and

$$(P+Q)^{d} = Q^{\pi} \sum_{i=0}^{t-1} Q^{i} (P^{d})^{i+1} + \sum_{i=0}^{s-1} (Q^{d})^{i+1} P^{i} P^{\pi},$$

where s = ind(P) and t = ind(Q).

## 2. Main results

**Theorem 2.1.** Assume P, Q and PQ are Drazin invertible. If  $PQP^2 = 0$ ,  $PQ^2 = 0$  and  $QP^3 = 0$  then

$$(P+Q)^{d} = \sum_{i=0}^{k+1} \left( (P^{d})^{2i+1} + (Q^{d})^{2i+1} \right) \left( (QP)^{i} (QP)^{\pi} + (PQ)^{i} (PQ)^{\pi} \right)$$
$$+ \sum_{i=0}^{k} \left( P^{2i+1} P^{\pi} + Q^{2i+1} Q^{\pi} \right) \left( ((QP)^{d})^{i+1} + ((PQ)^{d})^{i+1} \right)$$
$$+ \sum_{i=0}^{k} Q^{\pi} Q^{2i} \left( P^{2} ((QP)^{d})^{i+2} Q + QP^{2} ((QP)^{d})^{i+2} \right)$$
$$+ \sum_{i=0}^{k} (Q^{d})^{2(i+2)} \left( P^{2} (QP)^{i} (QP)^{\pi} Q + QP^{2} (QP)^{i} (QP)^{\pi} \right)$$
$$- P^{d} - Q^{d} - Q^{d} P^{2} (QP)^{d} - P (PQ)^{d} - (Q^{d})^{2} P^{2} Q (PQ)^{d},$$

where  $k = max\{ind(P^2), ind(Q^2), ind(QP)\}$ .

**Example 2.2.** Consider the two matrices  $P, Q \in \mathbb{C}^{6 \times 6}$ , where

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & a & b \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & c & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for every nonzero  $a, b, c \in \mathbb{C}$ . We have

	0	1	0	0	1	0	١		0	0	0	-1	0	0	\
$P^d =$	0	1	0	0	1	0		$Q^d =$	0	0	0	0	0	0	
	0	-1	0	0	-1	0	,		0	0	0	3	0	0	
	0	0	0	0	0	0			0	0	0	-1	0	0	·
	0	0	0	0	0	0			0	0	0	0	0	0	
	0	0	0	0	0	0 /	/		0	0	0	0	0	0	/

Since  $QP^2 \neq 0$  and  $P^2 \neq 0$  representations for $(P+Q)^d$  from [4, 1] fail to apply. After calculating, we get that  $PQP^2 = 0$ ,  $QP^3 = 0$  and  $PQ^2 = 0$ . Hence, the conditions of Theorem (2.1) are satisfied, also we have

$$ind(P) = 3$$
,  $ind(Q) = 2$ .

Applying Theorem (2.1), we get

## References

- C. Bu, C. Feng and S. Bai, Representations for the Drazin inverses of the sum of two matrices and some block matrices, J. Appl. Math. Comput. 218 (2012) 10226-10237.
- [2] S.R. Caradus, Operator Theory of the Generalized Inverse, Science Press, New York, 2004.
- [3] M.P. Drazin, Pseudoinverse in associative rings and semigroups, Amer. Math. Monthly. 65 (1958) 506-514.
- [4] M.F. Martínez-Serrano and N. Castro-González, On the Drazin inverse of block matrices and generalized Schur complement, Appl. Math. Comput. 215 (2009) 2733-2740.

UNIVERSITY OF KURDISTAN, SANANDAGE, 66177 ISLAMIC REPUBLIC OF IRAN *E-mail address*: mdana@uok.ac.ir

UNIVERSITY OF KURDISTAN, SANANDAGE, 66177 ISLAMIC REPUBLIC OF IRAN *E-mail address*: Ramesh.Yousefi@Yahoo.com



# GENERALIZED JOINT RANK-k NUMERICAL RANGES AND QUANTUM ERROR CORRECTION

MOHSEN ZAHRAEI AND ESMAEIL FOOLADI H.

ABSTRACT. For a noisy quantum channel, a quantum error correcting code of dimension k exists if and only if the joint rank-k numerical range associated with the error operators of the channel is non empty. In this paper, the notion of joint numerical range and joint rank-k numerical range of rectangular matrices are introduced. Some algebraic and geometrical properties are investigated. The proposed definitions yield a natural generalization of the standard joint numerical ranges and joint higher rank numerical ranges.

Keywords: Quantum error correction; Joint numerical range; Joint rank-k numerical range.

#### 1. INTRODUCTION

In quantum computing, information is stored in quantum bits, abbreviated as qubits. Mathematically, a qubit is represented by a  $2 \times 2$  rank one Hermitian matrix  $Q = vv^*$ , where  $v \in \mathbb{C}^2$  is a unit vector. A state of N-qubits  $Q_1, \ldots, Q_N$  is represented by their tensor products is  $M_n$  with  $n = 2^N$ . A quantum channel for states of N-qubits corresponds to a trace preserving completely positive linear map, there are  $T_1, \ldots, T_r \in M_n$  with  $\sum_{j=1}^r T_j^* T_j = I_n$  such that  $\phi(X) = \sum_{j=1}^r T_j X T_j^*$ , where,  $T_1, \cdots, T_r$  are known as the error operators.

Let  $\mathcal{P}_k$  be the set of rank-k orthogonal projections in  $M_n$ . Define the joint rank-k numerical range of an m-tuple of matrices  $A = (A_1, \ldots, A_m) \in M_n^m$  by

$$\Lambda_k(A) = \{(a_1, \dots, a_m) \in \mathbb{C}^m : \text{ there is } P \in \mathcal{P}_k \text{ such that } PA_jP = a_jP \text{ for } j = 1, \dots, m\}.$$

In this connection, for any  $A, B \in M_{n \times m}$  with  $B \neq 0$ , and any vector norm  $\|\cdot\|$  on  $M_{n \times m}$ , the numerical range of A with respect to B is defined and denoted by  $W_{\|\cdot\|}(A; B) = \{\mu \in \mathbb{C} : \|A - \lambda B\| \ge |\mu - \lambda|, \quad \forall \lambda \in \mathbb{C}\}$ . Also, the joint numerical range of  $A = (A_1, A_2, \ldots, A_s)$  is defined as teh following:

(1.1) 
$$W(A) = \{ (x^*A_1x, \dots, x^*A_sx) : x \in \mathbb{C}^s, \ x^*x = 1 \}.$$

## 2. Main results

It is natural to use a formula analogous to (1.1) to propose a definition of the joint numerical range of rectangular matrices.

**Definition 2.1.** Let  $A = (A_1, \ldots, A_s) \in M^s_{n \times m}$ ,  $B \in M_{n \times m}$  and  $\|\cdot\|$  be a vector norm on  $M_{n \times m}$ . The numerical range of A with respect to B is defined and denoted by

 $W_{\|\cdot\|}(A;B) = \{(a_1, \dots, a_s) \in \mathbb{C}^s : a_i \in W_{\|\cdot\|}(A_i;B) \text{ for } i = 1, \dots, s\},\$ where  $W_{\|\cdot\|}(A_i;B) = \{\mu \in \mathbb{C} : \|A_i - \lambda B\| \ge |\mu - \lambda|, \ \forall \lambda \in \mathbb{C}\}.$  [1]

Speaker: Mohsen Zahraei.

The following proposition follows from Definition 2.1 and Proposition

 $\begin{array}{l} \textbf{Proposition 2.2. Let } A = (A_1, \dots, A_s) \in M_{n \times m}^s, B \in M_{n \times m} \ and \|\cdot\| \ be \ a \ vector \ norm \ on \ M_{n \times m}. \ Then \ the \ following \ assertions \ are \ true: \\ (i) \ If \|B_i\| > 1 \ for \ i = 1, \dots, s, \ then \ \{(\mu_1, \dots, \mu_s) \in \mathbb{C}^s : B_i \bot_{BJ}(A_i - \mu_i B_i), \ for \ i = 1, \dots, s\} \subseteq W_{\|\cdot\|}(A; B). \ The \ equality \ holds \ if \ \|B_i\| = 1, \ for \ i = 1, \dots, s; \\ (ii) \ If \ the \ norm \ is \ unitarily \ invariant, \ then \ W_{\|\cdot\|}(UAV; UBV) = W_{\|\cdot\|}(A; B), \ where \\ U \in \mathcal{U}_n, V \in \mathcal{U}_m, UAV = (UA_1V, \dots, UA_sV) \ and \ UBV = (UB_1V, \dots, UB_sV); \\ (iii) \ W_{\|\cdot\|}(aA + bB; B) = aW_{\|\cdot\|}(A; B) + b, \ where \ a, b \in \mathbb{C} \ and \ aA + bB = (aA_1 + bB_1, \dots, aA_s + bB_s); \\ (iv) \ \{(\mu_1^{-1}, \dots, \mu_s^{-1}) \in \mathbb{C}^s : \mu_i \in W_{\|\cdot\|}(A_i; B), |\mu_i| \ge 1 \ for \ i = 1, \dots, s\} \subseteq W_{\|\cdot\|}(B; A); \\ (v) \ For \ any \ nonzero \ b \in \mathbb{C}, \\ \left\{ \begin{array}{c} if \ |b| = 1, \ then \ W_{\|\cdot\|}(A; bB) = b^{-1}W_{\|\cdot\|}(A; B); \\ if \ |b| < 1, \ then \ W_{\|\cdot\|}(A; bB) \subseteq b^{-1}W_{\|\cdot\|}(A; B); \\ if \ |b| < 1, \ then \ W_{\|\cdot\|}(A; bB) \supseteq b^{-1}W_{\|\cdot\|}(A; B); \\ if \ |b| > 1, \ then \ W_{\|\cdot\|}(A; bB) \supseteq b^{-1}W_{\|\cdot\|}(A; B). \\ where, \ bB = (bB_1, \dots, bB_s). \end{array} \right.$ 

Now, we use a formula analogous to to propose a definition of the joint higher rank numerical range of an *s*-tuple of rectangular matrices  $A = (A_1, \ldots, A_s) \in M^s_{n \times m}$ . For this mind, we introduce the following set: [3] (2.1)

$$\chi = \{ (X,Y) \in \chi_{n,n-k+1} \times \chi_{m,m-k+1} : \begin{cases} Y = \left( \begin{array}{c|c} X & 0 \\ \hline 0 & U_{m-n} \end{array} \right) & \text{if } m \ge n, \\ \\ X = \left( \begin{array}{c|c} Y & 0 \\ \hline 0 & U_{n-m} \end{array} \right) & \text{if } m \le n \end{cases}$$

**Definition 2.3.** Let  $A = (A_1, \ldots, A_s) \in M_{n \times m}^s, B \in M_{n \times m}, 1 \le k \le \min\{n, m\}$ be a positive integer, and  $\chi$  be the set as in (2.1). Moreover, let  $\|\cdot\|$  be a vector norm on  $M_{(n-k+1)\times(m-k+1)}$ . The joint rank-k numerical range of A with respect to B is defined and denoted by

$$\Lambda_{k,\|\cdot\|}(A;B) = \{(a_1, \dots, a_s) \in \mathbb{C}^s : a_i \in \Lambda_{k,\|\cdot\|}(A_i;B) \text{ for } i = 1, \dots, s\},\$$

where

$$\Lambda_{k,\|\cdot\|}(A_i;B) = \{\mu \in \mathbb{C} : \|X^*(A_i - \lambda B_i)Y\| \ge |\mu - \lambda|, \ \forall \ \lambda \in \mathbb{C}, \ \forall \ (X,Y) \in \chi\},$$
  
for  $i = 1, \dots, s$ .

In the following theorem, we give a formula, analogous to , for  $\Lambda_{k,\|\cdot\|}(\cdot,\cdot)$ .

**Theorem 2.4.** Let  $A = (A_1, \ldots, A_s) \in M^s_{n \times m}, B \in M_{n \times m}, 1 \le k \le \min\{n, m\}$  be a positive integer, and  $\chi$  be the set as in (2.1). Moreover, let  $\|\cdot\|$  be a vector norm on  $M_{(n-k+1)\times(m-k+1)}$ . Then

$$\Lambda_{k,\|\cdot\|}(A;B) = \bigcap_{(X,Y)\in\chi} W_{\|\cdot\|}(X^*AY;X^*BY),$$

where  $X^*AY = (X^*A_1Y, \ldots, X^*A_sY)$  and  $\chi$  is the set as in (2.1). For the case k = 1, if the vector norm  $\|\cdot\|$  is Unitarily invariant, then

$$\Lambda_{1,\|\cdot\|}(A;B) = W_{\|\cdot\|}(A;B)$$

**Corollary 2.5.** Let  $A = (A_1, \ldots, A_s) \in M^s_{n \times m}, B \in M_{n \times m} \in M^s_n$  and  $1 \le k \le n$ be a positive integer, and  $\chi$  be the set as in (2.1). Moreover, let  $\|\cdot\|$  be a vector norm on  $M_{(n-k+1)}$ . Then

$$\Lambda_{k,\|\cdot\|}((A_1,\ldots,A_s);B) = \bigcap_{X \in \chi_{n,n-k+1}} W_{\|\cdot\|}((X^*A_1X,X^*A_1X,\ldots,X^*A_sX);X^*BX)$$

Consequently, for the case  $\|\cdot\| = \|\cdot\|_2$  and  $B = I_n$ ,  $\Lambda_{k,\|\cdot\|}((A_1,\ldots,A_s);I_n) =$  $\Lambda_k(A_1,\ldots,A_s).$ 

**Remark 2.6.** By Corollary 2.5, the notion of joint rank-k numerical range of rectangular matrices can be considered as a generalization of the joint rank-k numerical range of square matrices.

In the following proposition, we investigate some of algebraic and geometrical properties of  $\Lambda_{k,\parallel\cdot\parallel}((A_1,\ldots,A_s);B)$ .

**Proposition 2.7.** Let  $A = (A_1, \ldots, A_s) \in M^s_{n \times m}, B \in M_{n \times m}, 1 \le k \le \min\{n, m\}$ be a positive integer, and  $\chi$  be the set as in (2.1). Moreover, let  $\|\cdot\|$  be a vector norm on  $M_{(n-k+1)\times(m-k+1)}$ . Then

(*i*)  $\Lambda_{k,\parallel \mid \parallel}((UA_1V, \dots, UA_sV); B) = \Lambda_{k,\parallel \mid \parallel}((A_1, \dots, A_s); B)$ 

(ii) Let  $1 \le k_2 \le k_1 \le \min\{n, m\}$  be two positive integers. Moreover, let  $\|\cdot\|$  be a unitarily invariant norm on  $M_{(n-k_2+1)\times(m-k_2+1)}$  and  $||| \cdot |||$  be the vector norm on  $M_{(n-k_1+1)\times(m-k_1+1)}$  as in (2.1). Then

$$_{k_1,|||\cdot|||}(A;B) \subseteq \Lambda_{k_2,||\cdot||}(A;B);$$

 $\Lambda_{k_1,|||\cdot|||}(A;B) \subseteq \Lambda_{k_2,||\cdot||}(A;B);$ (*iii*)  $\Lambda_{k,||\cdot||}((A_1+b_1B,\ldots,A_s+b_sB);B) = \Lambda_{k,||\cdot||}(A;B)+b;$ (iv)  $\Lambda_{k+1,\parallel\cdot\parallel}(A;B) \subseteq \Lambda_{k,\parallel\cdot\parallel}(A;B).$ 

*Proof.* The above results follow from Definition (2.3), Theorem 2.6, Theorem 2.7 and Proposition 2.10 in [3]. 

**Proposition 2.8.** Let  $A = (A_1, \ldots, A_s) \in M^s_{n \times m}, B \in M_{n \times m}, 0 \neq b \in \mathbb{C}$ , and  $1 \leq c$  $k \leq \min\{n,m\}$  be a positive integer. Moreover, let  $\|\cdot\|$  be a vector norm on  $M_{(n-k+1)\times(m-k+1)}$ . Then the following assertions are true: (i) If |b| = 1, then  $\Lambda_{k,\|\cdot\|}((A_1, \dots, A_s); bB) = b^{-1}\Lambda_{k,\|\cdot\|}((A_1, \dots, A_s); B);$ (*ii*) If |b| < 1, then  $\Lambda_{k,\|\cdot\|}((A_1, \ldots, A_s); bB) \subseteq b^{-1}\Lambda_{k,\|\cdot\|}((A_1, \ldots, A_s); B);$ (*iii*) If |b| > 1, then  $\Lambda_{k, \|\cdot\|}((A_1, \dots, A_s); bB) \supseteq b^{-1} \Lambda_{k, \|\cdot\|}((A_1, \dots, A_s); B)$ .

*Proof.* Using Definition 2.3, we have  $(\mu_1, \ldots, \mu_s) \in \Lambda_{k, \|\cdot\|}(A_1, \ldots, A_s; bB) \Leftrightarrow \mu_i \in A_{k, \|\cdot\|}(A_1, \ldots, A_s; bB)$  $\Lambda_{k,\|\cdot\|}(A_i; bB) \forall i = 1, 2, \dots, s.$  Now, the results follow from [3, Proposition 2.9].  $\Box$ 

#### References

- [1] C. Chorianopoulos, S. Karanasios and P. Psarrakos, A definition of numerical range of rectangular matrices, Linear Multilinear Algebra 51(2009), 459-475.
- [2] C. K Li and Y. T. Poon, Generalized numerical ranges and quantum error correction, J. Operator Theory 66(2011), 335-351.
- [3] M. Zahraei and Gh. Aghamollaei, Higher rank numerical ranges of rectangular matrices, Ann. Func. Anal. 6(2015), 133-142.

DEPARTMENT OF MATHEMATICS, AHVAZ BRANCH, ISLAMIC AZAD UNIVERSITY, AHVAZ, IRAN E-mail address: m.zahraei@iauahvaz.ac.ir and mzahraei56@yahoo.com

DEPARTMENT OF MATHEMATICS, BUSHEHR BRANCH, TEACHER EDUCATION UNIVERSITY, BUSHEHR, IRAN

E-mail address: fooladi6@gmail.com



## ON THE GEOMETRY OF BOUNDED LINEAR OPERATORS

## ALI ZAMANI

ABSTRACT. In this talk, we study the Birkhoff–James orthogonality for operators on a complex Hilbert space. In addition, we investigate the problem of characterization of norm–parallelism for bounded linear operators.

Keywords: Bounded linear operators; Orthogonality; Norm-parallelism.

## 1. INTRODUCTION

Let  $\mathbb{B}(H)$  denote the algebra of all bounded linear operators acting on a complex separable Hilbert space  $(H, \langle \cdot, \cdot \rangle)$ . By  $\mathbb{K}(H)$  we denote the algebra of all compact operators on H, and by  $\mathbb{T}(H)$  the algebra of all trace-class operators on H.

Let  $(X, \|\cdot\|)$  be a complex normed space. We recall that  $x \in X$  is orthogonal to  $y \in X$  in the Birkhoff–James sense, denoted by  $x \perp_B y$ , if

$$||x + \gamma y|| \ge ||x||$$
 for all  $\gamma \in \mathbb{C}$ .

It is easy to see that the Birkhoff–James orthogonality is equivalent to the usual orthogonality in case X is an inner product space.

Further properties of the Birkhoff–James orthogonality in Hilbert  $C^*$ -modules can be found in [1, 2, 3].

Furthermore, we say that  $x \in X$  is norm–parallel to  $y \in X$ , in short  $x \parallel y$ , if

 $||x + \lambda y|| = ||x|| + ||y|| \quad \text{for some} \quad \lambda \in \mathbb{T} = \{\alpha \in \mathbb{C} : |\alpha| = 1\}.$ 

In the case of inner product spaces the norm–parallel relation is exactly the usual vectorial parallel relation, that is,  $x \parallel y$  if and only if x and y are linearly dependent, or equivalently

$$|\langle x, y \rangle| = ||x|| ||y||.$$

In the case of normed linear spaces two linearly dependent vectors are norm– parallel, but the converse is false in general.

Notice that the norm–parallelism is symmetric (i.e.,  $x \parallel y \Leftrightarrow y \parallel x$ ) and homogenous (i.e.,  $x \parallel y \Leftrightarrow \alpha x \parallel \beta y$  for all  $\alpha, \beta \in \mathbb{R}$ ), but not transitive (i.e.,  $x \parallel y$  and  $y \parallel z \neq x \parallel z$ ) (see[4]).

Some characterizations of the norm-parallelism for elements of an arbitrary Hilbert  $C^*$ -module were given in [4, 5].

In the next section, we study the Birkhoff–James orthogonality and norm– parallelism of bounded linear operators on a complex Hilbert space.

<sup>2010</sup> Mathematics Subject Classification. 47L05, 47A30, 46B20.

Speaker: Ali Zamani.

## 2. Main results

In the following result we establish a characterization of the Birkhoff–James orthogonality for bounded linear operators.

**Theorem 2.1.** Let  $T, S \in \mathbb{B}(H)$ . Then the following statements are equivalent:

(i)  $T \perp_B S$ .

(ii)  $||T + \gamma S||^2 \ge ||T||^2 + |\gamma|^2 m^2(S)$   $(\gamma \in \mathbb{C}),$ 

where  $m(S) := \inf\{\|S\xi\| : \xi \in H, \|\xi\| = 1\}$  is the minimum modulus of S.

It is well known that Pythagoras' equality does not hold in  $\mathbb{B}(H)$ . The following result is a kind of Pythagorean inequality for bounded linear operators.

**Corollary 2.2.** Let  $T, S \in \mathbb{B}(H)$  with m(S) > 0. Then there exists a unique  $\xi \in \mathbb{C}$ , such that

$$\left\| (T+\xi S)+\gamma S \right\|^2 \ge \left\| T+\xi S \right\|^2 + |\gamma|^2 m^2(S) \qquad (\gamma \in \mathbb{C}).$$

One of the typical linear preserver problems on  $\mathbb{B}(H)$  is the study of those linear operators  $\Phi : \mathbb{B}(H) \to \mathbb{B}(H)$  which preserve a certain relation  $\sim$  in both directions on  $\mathbb{B}(H)$ , that is,

$$T \sim S$$
 if and only if  $\Phi(T) \sim \Phi(S)$ .

**Theorem 2.3.** Let  $\Phi : \mathbb{B}(H) \to \mathbb{B}(H)$  be a linear surjective mapping preserving the Birkhoff-James orthogonality in both directions, that is,

$$T \perp_B S$$
 if and only if  $\Phi(T) \perp_B \Phi(S)$ .

Then there exist a nonzero complex number  $\alpha$  and unitaries U and V such that for all  $X \in \mathbb{B}(H)$  the mapping  $\Phi$  is of the form

$$\Phi(X) = \alpha U X V \quad or \quad \Phi(X) = \alpha U X^t V,$$

where  $X^t$  denotes the transpose of X relative to a fixed but arbitrary orthonormal basis of H.

In the following result we characterize the norm–parallelism for bounded linear operators.

**Theorem 2.4.** Let  $T, S \in \mathbb{B}(H)$ . Then the following statements are equivalent:

(i)  $T \parallel S$ .

(ii) There exists a sequence of unit vectors  $\{\xi_n\}$  in H such that

$$\lim_{n \to \infty} \left| \langle T\xi_n, S\xi_n \rangle \right| = \|T\| \, \|S\|.$$

**Corollary 2.5.** Let  $T, S \in \mathbb{K}(H)$ . Then the following statements are equivalent: (i)  $T \parallel S$ .

(ii) There exists a unit vector  $\xi$  in H such that

$$\left| \langle T\xi, S\xi \rangle \right| = \|T\| \, \|S\|.$$

**Corollary 2.6.** Let  $T, S \in \mathbb{T}(H)$ . Then the following statements are equivalent:

(i)  $T \parallel S$ .

(ii) There exist isometries V, W and  $\lambda \in \mathbb{T}$  such that

$$|T + \lambda S| = V|T|V^* + W|S|W^*.$$

#### Acknowledgement

This work is a part of a postdoctoral research under supervision of Prof. M.S.Moslehian. I would like to thank the Research Office of Ferdowsi University of Mashhad.

#### References

- [1] Lj. Arambašić and R. Rajić, A strong version of the Birkhoff–James orthogonality in Hilbert  $C^*$ -modules, Ann. Funct. Anal. 5 (2014), no. 1, 109–120.
- M. S. Moslehian and A. Zamani, Characterizations of operator Birkhoff–James orthogonality, Canad. Math. Bull. (to appear) doi:10.4153/CMB-2017.004-5.
- [3] K. Paul, D. Sain and P. Ghosh, Birkhoff–James orthogonality and smoothness of bounded linear operators, Linear Algebra Appl. 506 (2016), 551–563.
- [4] A. Zamani and M. S. Moslehian, Norm-parallelism in the geometry of Hilbert C\*-modules, Indag. Math. 27 (2016), no. 1, 266–281.
- [5] A. Zamani, The operator-valued parallelism, Linear Algebra Appl. 505 (2016), 282-295.

FARHANGIAN UNIVERSITY, SEMNAN, IRAN *E-mail address*: Zamani.ali85@yahoo.com



# ON THE CONSTRUCTION OF HNN-EXTENSIONS FOR DIALGEBRAS

CHIA ZARGEH

ABSTRACT. In this talk we construct HNN extensions for dialgebras. Given a dialgebra D with a subalgebra  $D_0$  and a derivation  $d: D_0 \to D$  the HNN extension contains D and d extends to a derivation equations.

Keywords: Dialgebra; Groebner-Shirshov basis; HNN extension.

## 1. INTRODUCTION

The Higman-Neumann-Neumann extensions (HNN-extensions) of groups were constructed in [3], and have been used since then for the proof of many embedding theorems, such that every countable group is embeddable into a group with two generators. Wasserman [6] introduced HNN- extension for Lie algebras and used it to obtain some analogous results to group theory. As an application of HNNextension of Lie algebras, Wasserman proved that Markov properties of finitely presented Lie algebras are undecidable.

In 1965, Bloh [1] introduced the notion of Leibniz algebras as non-antisymmetric generalization of Lie algebras. Afterwards, Loday investigated the structure of Leibniz algebras with more details in [4]. He also gave the definition of a new class of algebras, dialgebras [5], which is closely connected to the notion of Leibniz algebras in the same way as associative algebras connected to Lie algebras. Our pupose in this talk is to construct HNN extension for dialgebras and we use the Composition-Diamond lemma as a strong tool to define HNN extension.

## 2. Main results

**Definition 2.1.** A diassociative algebra is a k-linear space, equipped with two klinear map  $\dashv$ , $\vdash$  :  $D \otimes D \mapsto D$  called respectively the left product and the right product such that the products  $\dashv$  and  $\vdash$  are associative and satisfy the following properties:

- (1)  $x \dashv (y \vdash z) = x \dashv (y \dashv z),$
- $(2) \ (x\dashv y)\vdash z=x\vdash (y\vdash z),$
- (3)  $x \vdash (y \dashv z) = (x \vdash y) \dashv z$ .

For diassociative algebra D and D', a morphism of diassociative algebra  $f: D \to D'$ is a k-linear map such that:

$$f(x \dashv y) = f(x) \dashv f(y) \text{ and } f(x \vdash y) = f(x) \vdash f(y)$$

for all  $x, y \in D$ .

<sup>2010</sup> Mathematics Subject Classification. 16S15, 17A99. Speaker: Chia Zargeh.

**Definition 2.2.** For a diassociative algebra D, a derivation map  $d : D \to D$  is defined as follows:

$$d(x \dashv y) = d(x) \dashv y + x \dashv d(y) \text{ and } d(x \vdash y) = d(x) \vdash y + x \vdash d(y)$$

for all  $x, y \in D$ .

**Definition 2.3.** A subspace  $D_0$  of a diassociative algebra D is called a subalgebra if  $x \dashv y$  and  $x \vdash y$  are in  $D_0$  for  $x, y \in D_0$ .

**Remark 2.4.** Denote by [u] a normal diword in free dialgebra D(X) which is of the form

$$[u] = x_{-m} \vdash \ldots \vdash x_0 \dashv \ldots \dashv x_n$$

where  $x_i \in X, n \ge 0, m \ge 0$  and  $x_0$  is called the center element. Any diword can be written in the normal form uniquely.

## 3. Composition-Diamond Lemma for Dialgebra

We use the composition of polynomials to show that the new presentation for HNN extension of dialgebra is trivial under composition. A composition of elements of  $S \subset D(X)$  is called trivial modulo S if it is a linear combination of polynomials insluding  $s \in S$  as subwords. If any composition of polynomials in S is trivial modulo S then S is called a Groebner-Shirshov basis. All possible compositions of polynomials in the free dialgebra D(X) are composition of left (right) multiplication, composition of inclusion and composition of intersection. The next theorem states Compositon-Diamond lemma for dialgebras and it is proved in[2].

**Definition 3.1.** The monomial ordering < on dialgebra polynomials [u] and [v] is defined as follows:

 $[u] < [v] \Leftrightarrow wt([u]) < wt([v])$  lexicographically

where  $wt([u]) = (n + m + 1, m, x_{-m}, ..., x_0, ..., x_n)$ 

**Theorem 3.2.** (CompositionDiamond lemma). Let  $S \subset D(X)$  be a monic set and the ordering  $\langle$  as before, Id(S) is the ideal generated by S. Then  $(i) \Rightarrow (ii) \Leftrightarrow$  $(ii)' \Leftrightarrow (iii)$ , where

- (i) S is a Groebner-Shirshov basis in D(X).
- (ii)  $f \in Id(S) \Rightarrow [\overline{f}] = [a[\overline{s}]b]$  for some  $s \in S$ , a, b in free monoid on X and [asb] a normal S-diword.
- (ii)'  $f \in Id(S) \Rightarrow f = \alpha_1[a_1s_1b_1] + ... \alpha_n[a_ns_nb_n]$  with  $[a_1[\bar{s_1}]b_1] > ... > [a_n[\bar{s_n}]b_n]$ , where  $[a_i[s_i]b_i]$  is normal  $s_i$ -diword.
- (iii)  $Irr(S) = \{[u] | u \neq [a[\bar{s}]b], s \in S, a, b \in X^*, [asb] is normals diword\}.$

Proof. [2].

#### 4. The HNN Construction

Let D be a dialgebra and  $D_0$  be a subalgebra of D and consider a derivation  $d: D_0 \to D$ . We define HNN extension by the presentation

$$H = \langle D, t | t \vdash a = d(a) \text{ for all } a \in D_0 \rangle.$$

We consider a new presentation for H through linear basis of D. Let denote the bases of D and  $D_0$  by X and B respectively. It is obvious that B is a subset of X such that the elements of B are smaller than  $X \setminus B$ . The arbitrary elements of X

are denoted by x, y, z, u, v and the elements of B are a, b. We write the multiplication table of D for basis elements as follows:

•  $x \dashv y = \sum_{v} \alpha_{xy}^{v} v$ •  $x \vdash y = \sum_{v} \beta_{xy}^{v} v$ •  $x \dashv x = \sum_{v} \gamma_{xx}^{v} v$ •  $x \vdash x = \sum_{v} \delta_{xx}^{v} v$ 

where the scalars  $\{\alpha_{xy}^{v}, \beta_{xy}^{v}, \gamma_{xx}^{v}, \delta_{xx}^{v}\}$  satisfies the conditions of axioms (1), (2) and (3) of definition of dialgebras. Since  $D_0$  is a subalgebra of D then

$$\alpha^v_{ab}, \beta^v_{ab}, \gamma^v_{aa}, \delta^v_{aa} = 0$$

for  $a, b \in B$  and  $v \notin B$ . For derivation  $d: D_0 \to D$  we have

$$d(a) = \sum_{v} \lambda_a^v v$$

for  $a \in B$ . Let x > y then we consider the following equivalent presentation for H:

$$\begin{split} H &= \langle X, t | x \dashv y = \sum_{v} \alpha_{xy}^{v} v, x \vdash y = \sum_{v} \beta_{xy}^{v} v \\ x \dashv x = \sum_{v} \gamma_{xx}^{v} v, x \vdash x = \sum_{v} \delta_{xx}^{v} v, \\ t \vdash a = \sum_{v} \lambda_{a}^{v} v \ a \in B \rangle. \end{split}$$

## References

- A. Bloh, A generalization of the concept of a Lie algebra. Sov. Math. Dokl. 6 (1965), 1450– 1452.
- [2] L. A. Bokut, Y. Chen, C. Liu, Grobner-Shirshov bases for dialgebras, Internat. J.Algebra Comput. 20(3) (2010) 391415.
- [3] G. Higman, B.H. Neumann and H. Neumann, Embedding theorems for groups, J. London Math. Soc. 24 (1949), 247-254. MR 11:322d.
- [4] J.-L. Loday, Cyclic homology. Grundl. Math. Wiss. Bd. vol. 301. Springer-Verlag, Berlin, 1992.
- [5] J.-L. Loday, Algebras with two associative operations (dialgebras), C. R. Acad. Sci. Paris 321 (1995) 141146.
- [6] A. Wasserman, A derivation HNN construction for Lie algebras, Israil J. Math., 106, (1998), 76-92.

Deparetment of Mathematics, Santiago de Compostela University, Spain E-mail address: c.zargeh@rai.usc.es



# ROTA THEOREM FOR FINITE DIMENSIONAL BANACH SPACES

## RAHIM ALIZADEH

ABSTRACT. Rota theorem states that for a Hilbert space H and  $T \in B(H)$ , the equality

 $\rho(T) = \inf\{\|P^{-1}TP\| : P \text{ is invertible in } B(H)\},\$ 

holds, where  $\rho$  denotes the spectral radius. In this paper, using an elementary method, prove this result for finite dimensional Banach spaces endowed by an absolute norm. Also we show that for every arbitrary induced norm  $\|.\|$  on  $M_n$ , the algebra of  $n \times n$  matrices with complex entries, the following equality holds

 $\rho(T) = \inf\{r(P^{-1}TP) : P \text{ is invertible in } M_n\},\$ 

where r denotes the numerical radius related to  $\|.\|$ .

Keywords: Spectral radius; Numerical radius; Absolute norm.

## 1. INTRODUCTION

A direct result of Rota theorem [3] is that for every bounded linear operator T on a Hilbert space, the spectral radius of T is equal to the infimum of the norms of all operators which are similar to T. In this paper, using an elementary method, we refine this result for finite dimensional Banach spaces that which are endowed with absolute norms. Before it, we need some terminologies.

Let  $M_n$  be the algebra of all n by n matrices with complex entries. We denote a diagonal matrix with the entries  $\lambda_1, \dots, \lambda_n$  on its diagonal by  $diag(\lambda_1, \dots, \lambda_1)$ . The spectral radius of a matrix in  $A \in M_n$  is displayed by  $\rho(A)$ . For every two vectors  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  in  $\mathbb{C}^n$ , the product partial order is defined on  $\mathbb{C}^n$  by  $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$ , if  $|x_i| \leq |y_i|$ , for all  $i = 1, \dots, n$ . A norm  $\|.\|$  on  $\mathbb{C}^n$  is said to be *absolute* if

$$||(x_1,\ldots,x_n)|| = ||(|x_1|,\ldots,|x_n|)||, \quad x = (x_1,\ldots,x_n) \in \mathbb{C}^n,$$

and it is *monotone* if

$$||(x_1,\ldots,x_n)|| \le ||(y_1,\ldots,y_n)||,$$

for every  $x, y \in \mathbb{C}^n$  with  $x \leq y$ . It is well known that a norm on  $\mathbb{C}^n$  is absolute if and only if it is monotone [2, Theorem 5.5.10] and [1, Proposition IV.1.1]. The numerical range and numerical radius related to a norm  $\|.\|$  on  $\mathbb{C}^n$  is defined as follows respectively

$$W(A) = \{y^*Ax : \|y\|^D = \|x\| = y^*x = 1\},$$
$$r(A) = \sup\{|\lambda| : \ \lambda \in W(A)\}.$$

<sup>2010</sup> Mathematics Subject Classification. 15A60, 15A18. Speaker: Rahim Alizadeh.

206

Here  $\|.\|^D$  is the dual norm of  $\|.\|$  [2, Section 5.4]. A pair  $(x, y) \in \mathbb{C}^n \times \mathbb{C}^n$  of unit vectors is called a dual pair if  $\|x\| = \|y\|^D = y^*x = 1$ . Clearly the inequalities  $\rho(A) \leq r(A) \leq \|A\|$  hold.

## 2. Main results

In this section, we show that for every arbitrary induced norm  $\|.\|$  on  $M_n$  the following equality holds

$$\rho(T) = \inf\{r(P^{-1}TP) : P \text{ is invertible in } M_n\},\$$

In addition, we prove that for a finite dimensional complex normed space X that is endowed by an absolute norm, the equality

$$\rho(T) = \inf\{\|P^{-1}TP\| : P \text{ is invertible}\},\$$

holds on B(X). Note that since X is finite dimensional, we can substitute X with  $\mathbb{C}^n$  and B(X) with  $M_n$  (with respect to some basis). We say that the norm on X is absolute if, its related norm on  $\mathbb{C}^n$  is absolute.

**Theorem 2.1** Let  $\|.\|$  be an arbitrary induced norm on  $M_n$ .

Then for every  $A \in M_n$ , we have

(2.1) 
$$\rho(A) = \inf\{r(P^{-1}AP) : P \text{ is invertible}\}$$

**Proof.** At first note that for every induced norm  $\|.\|$  on  $M_n$  we have

$$r(P^{-1}AP) \ge \rho(P^{-1}AP) = \rho(A).$$

Hence

$$\rho(A) \leq \inf\{r(P^{-1}AP) : P \text{ is invertible}\}.$$

Now let  $\|.\|$  be an induced norm that satisfies the equality (2.2). Suppose that  $\epsilon > 0$  and A is an arbitrary element in  $M_n$  with not necessarily distinct eigenvalues  $\lambda_1, \ldots, \lambda_n$ . By [2, Theorem 2.4.7] there exists an invertible matrix  $P_{\epsilon}$  such that  $P_{\epsilon}^{-1}AP_{\epsilon} = T_{\epsilon} = (t_{ij}(\epsilon))$  is an upper triangular matrix with  $|t_{ij}(\epsilon)| < \epsilon$ ,  $1 \le i < j \le n$ . Then we have

$$W(P_{\epsilon}AP_{\epsilon}^{-1}) \subseteq W(\operatorname{diag}(\lambda_{1},\ldots,\lambda_{n})) + \epsilon W(\sum_{1 \leq i < j \leq n} E_{ij})$$
$$= Co(\sigma(A)) + \epsilon W(\sum_{1 \leq i < j \leq n} E_{ij}).$$

Hence

$$\inf\{r(P^{-1}AP): P \text{ is invertible}\} \le r(P_{\epsilon}AP_{\epsilon}^{-1}) \le \rho(A) + \|\sum_{1 \le i < j \le n} E_{ij}\|\epsilon.$$

Therefore the equality (2.1) holds.

**Theorem 2.2** Let  $\|.\|$  be an induced norm on  $M_n$  such that for every diagonal matrix diag $(a_1, \ldots, a_n)$  in  $M_n$  the following equality holds

$$\|\operatorname{diag}(a_1,\ldots,a_n)\| = \max\{|a_1|,\ldots,|a_n|\}.$$

Then for every  $A \in M_n$ , we have

(2.2) 
$$\rho(A) = \inf\{\|P^{-1}AP\| : P \text{ is invertible}\}$$

**Proof.** For every induced norm  $\|.\|$  on  $M_n$  we have

$$||P^{-1}AP|| \ge \rho(P^{-1}AP) = \rho(A).$$

Hence

$$\rho(A) \le \inf\{\|P^{-1}AP\| : P \text{ is invertible}\}.$$

Suppose that  $\epsilon > 0$  and A is an arbitrary element in  $M_n$  with not necessarily distinct eigenvalues  $\lambda_1, \ldots, \lambda_n$  and let  $P_{\epsilon}$  and  $T_{\epsilon} = (t_{ij}(\epsilon))$  be same as the proof of Theorem 2.1. then we have

$$\begin{split} \rho(A) &= \rho(T_{\epsilon}) = \|(diag(\lambda_{1}, \dots, \lambda_{n})\| \\ &\geq \|T_{\epsilon}\| - \epsilon \sum_{1 \leq i < j \leq n} \|E_{ij}\| \\ &\geq \inf\{\|P^{-1}AP\| : P \text{ is invertible}\} - \epsilon \sum_{1 \leq i < j \leq n} \|E_{ij}\|. \end{split}$$

Hence

$$\rho(A) \ge \inf\{\|P^{-1}AP\| : P \text{ is invertible}\}.$$

Therefore the equality (2.2) holds.

**Lemma 2.3** Let  $\|.\|$  be a norm on  $\mathbb{C}^n$  that induces the induced norm  $\|.\|'$  on  $M_n$ . Then the equality

$$\|\operatorname{diag}(a_1,\ldots,a_n)\|' = \max\{|a_1|,\ldots,|a_n|\}.$$

holds for every diagonal matrix  $\operatorname{diag}(a_1, \ldots, a_n)$  in  $M_n$ , if and only if  $\|.\|$  is an absolute norm.

**Proof.** Let  $\|.\|$  be an absolute norm on  $\mathbb{C}^n$  and consider a diagonal matrix  $\operatorname{diag}(a_1, \ldots, a_n)$  in  $M_n$ . Since absolute norms are monotone, we have

$$\begin{aligned} \|\text{diag}(a_1, \dots, a_n)\|' &= \sup\{\| (|\lambda_1 a_1|, \dots, |\lambda_n a_n|) \| : \|(\lambda_1, \dots, \lambda_n)\| \le 1\} \\ &\le \sup\{\max\{|a_1|, \dots, |a_n|\}\|(\lambda_1, \dots, \lambda_n)\| : \|(\lambda_1, \dots, \lambda_n)\| \le 1\} \\ &= \max\{|a_1|, \dots, |a_n|\}. \end{aligned}$$

On the other hand for every diagonal matrix  $\operatorname{diag}(a_1, \ldots, a_n)$  in  $M_n$  and for every  $i = 1, 2, \ldots, n$ , we have

$$\|\text{diag}(a_1, \dots, a_n)\|' \ge \|\frac{|a_i|e_i|}{\|e_i\|} = |a_i|.$$

Hence the desired equality holds. Conversely, suppose that the equality

$$\|\text{diag}(a_1,\ldots,a_n)\| = \max\{|a_1|,\ldots,|a_n|\}.$$

holds for every diagonal matrix diag $(a_1, \ldots, a_n)$  in  $M_n$  and consider  $(x_1, \ldots, x_n)$ and  $(y_1, \ldots, y_n)$  in  $\mathbb{C}^n$ , with  $|x_i| \leq |y_i|$ ,  $i = 1, 2, \ldots, n$ . Let  $x_i = \lambda_i y_i$ , with

207

 $|\lambda_1|, \ldots, |\lambda_n| \leq 1$ . Then we have

$$\begin{aligned} \|(x_1,\ldots,x_n)\| &= \|(\lambda_1y_1,\ldots,\lambda_ny_n)\| \\ &\leq \|\operatorname{diag}(\lambda_1,\ldots,\lambda_n)\|'\|(y_1,\ldots,y_n)\| \\ &= \max\{|\lambda_1|,\ldots,|\lambda_n|\}\|(y_1,\ldots,y_n)\| \\ &\leq \|(y_1,\ldots,y_n)\|. \end{aligned}$$

Hence  $\|.\|$  is monotone and it is an absolute norm.

The following corollary is an immediate consequence of Theorem 2.2 and Lemma 2.3.

**Corollary 2.4** Let  $\|.\|$  be an absolute norm on a finite dimensional normed space X. Then the equality (2.2) holds on B(X).

Finally, we remark that characterizing those norms for which the equality (2.2) holds, seems to be an interesting problem.

### References

[1] R. Bhatia, Matrix Analysis, Graduate texts in mathematics, 169. Springer-Verlag, 1997.

[2] R.A. Horn, C.R. Johnson, Matrix analysis, Cambridge university press, Cambridge, 1990.

[3] G.C. Rota, On models for linear operators, Comm Pure Apll Math, (13) 1960, 469-472.

Department of Mathematics, Shahed university P. O. Box:18151-159, Tehran, Iran E-mail address: alizadeh@shahed.ac.ir