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IMPROVING QUALITY OF ANSWERS IN CONVEX FEASIBILITY PROBLEM

MOKHTAR ABBASI AND TOURAJ NIKAZAD

Abstract. Iterative methods for solving Convex Feasibility Problems(CFP) often find a solution which may not be desired enough. In this paper we try to steer solutions from iterative methods toward desired solution by using Superiorization algorithm. The advantages of our method are demonstrated by applying it on an example taken from the field of computerized tomography.

Keywords: Convex Feasibility Problem (CFP); Superiorization; Computerized tomography.

1. Introduction

Suppose $C_1, \cdots, C_n$ be closed and convex subsets of Hilbert space X such that $C = \bigcap_{i=1}^{n} C_i \neq \emptyset$, finding a point $x^* \in C$ is called Convex Feasibility Problem (CFP).The CFP has many applications in diverse areas of mathematics and physical science, most notably, computerized tomography.During three past decades, a lot of ierative methods have been introduced by researchers for solving CFP. In most cases a CFP don't have a unique solution and the iterative methods used for it find a solution may not be desire enough from application point of view.In some problems there are prori information about the solution of CFP which can be used to decide about utility of CFP solutions. In recent years, a new algorithmic structure called Superiorization has been developed and successfully applied for solving CFP ,specially CFP problems in the field of image reconstruction from projections [1]. Suppose $f(x)$ be a convex function and according to priori information we know for the solution of a CFP,namely x^*, the value of $f(x)$ is not so large. In summery,The Superiorization algorithm try to steer solutions of iterative methods, for solving CFP, toward those which have smaller $f(x)$ in compared to solutions obtained from original iterative methods (without using Superiorization).

2. preleminaries and definitions

the $\Im$-class operators family is defined as follow [5]:

Definition 2.1. *The $\Im$-class consist operators as $T : D \subseteq \mathbb{R}^n \to D$ which have two following properties:*

(I) *For every $z \in \operatorname{Fix} T$ there exist one nonnegative real function ϕ_z such that*

$$\|z - Tx\| \leq \|z - x\| - \phi_z(x) \quad \forall x \in D. \tag{2.1}$$

(II) *If the sequnce $\{x_m\}$ converges to α and $\lim_{m \to \infty} \phi_z(x_m) = 0$ then $\alpha \in \operatorname{Fix} T$*

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Many of well known operators for solving CFP are belong to -class such as metric projections, subgradient projections, useful part of cutter opertors(closed at zero cutter operators). Most of iterative methods for solving CFP problems are made by using composition and convex combination of operators T_i where $T_i \in$ with $\operatorname{Fix} T_i = C_i$. The -class has an important property, closedness under composition and convex combination of its members. the follwing lemma is a straight result from [5, Proposition 13].

Lemma 2.2. *Suppose* $T_1, \cdots, T_n \in$ *and* $\bigcap_{i=1}^{n} \operatorname{Fix} T_i \neq \emptyset$ *then*

(1) $T = T_n \cdots T_1 \in$ *and* $\operatorname{Fix} T = \bigcap_{i=1}^{n} \operatorname{Fix} T_i$

(2) $U(x) = \sum_{i=1}^{n} w_i T_i(x) \in$ *and* $\operatorname{Fix} U = \bigcap_{i=1}^{n} \operatorname{Fix} T_i$ *where* $\sum_{i=1}^{n} w_i = 1$ *and* $w_i > 0, \forall i = 1, \cdots, n$

3. Main results

In this section we consider the consistent linear system $Ax = b$ as a special case of CFP which called Linear Feasibility Problem (LFP). For solving a LFP, we use a method called block EMR. Block EMR is belong to Landweber Type iterations. Suppose the linear system $Ax = b$ partitioned into p blocks such that any equation in linear system should be belong to at least a block. So we have p smaller linear systems as follow:

$$A_i x = b_i, \qquad i = 1, \cdots, p. \tag{3.1}$$

for any linear system $A_i x = b_i$ in 3.1, an operator T_i is defined as follow:

$$T_i(x) = x + \lambda(x) A_i^T M_i (b_i - A_i x), \qquad i = 1, \cdots, p. \tag{3.2}$$

where

$$\lambda(x) = \frac{\|A_i^T M_t (b_i - A_i x)\|_2^2}{\|M_i^{\frac{1}{2}} A_i A_i^T M_i (b_i - A_i x)\|_2^2} \tag{3.3}$$

and $\{M_i\}_{i=1}^{p}$ are symmetric positive definite(SPD) matrices with appropriate size. It's not difficult to see that $\operatorname{Fix} T_i = \{x | A_i x = b_i\}$ and according to [4, Lemma 5.2] we have:

Lemma 3.1. *The operators* $\{T_i\}_{i=1}^{p}$ *defined by 3.2 and 3.3 belong to* .

Corollary 3.2. *Lemma 2.2 and 3.1 lead to that the operator* $T = T_1 T_2 \cdots T_p \in$ *and* $\operatorname{Fix} T = \bigcap_{i=1}^{n} \operatorname{Fix} T_i = \{x | Ax = b\}$.

Theorem 3.3. *Suppose* $\{v_k\}$ *be a bounded sequence in* $\mathbb{R}^n$ *and* $\{\beta_k\} \subset \mathbb{R}^+$ *be a summable sequence (i.e.* $\sum_{k=1} \beta_k < \infty$ *) and* $T \in$ *then the sequence*

$$x^{k+1} = T(x^k + \beta_k v_k) \tag{3.4}$$

converges to a point in $\operatorname{Fix} T$.

Proof. see [5, Remark 23]. ∎

4. Numerical results

Discrete image reconstruction from projections resulted in large, sparse and often undetermined linear systems. When according to priori information, it is known that the reconstructed image is an $s \times t$ smooth image (the image in which adjacent pixels have similar values), the function $TV : \mathbb{R}^n \to \mathbb{R}^+$ is defined as follow:

$$TV(x) = \sum_{k=1}^{s-1} \sum_{l=1}^{t-1} \sqrt{(p_{k+1,l} - p_{k,l})^2 + (p_{k,l+1} - p_{k,l})^2} \tag{4.1}$$

where $p_{k,l}$ denotes the value of (i,j)-th pixel for $k = 1, \cdots, s$ and $l = 1, \cdots, t$.Somehow the TV function measures smoothness of image, the less TV the more smoothness image. From calculus we know any multi-variable function at each point most decreases in the opposite direction of its gradient at that point, we use this fact and by using operator T in 3.2 for solving linear system define a sequence

$$x^{k+1} = T(x^k + \beta_k v_k)$$

in which $v_k = -\frac{\nabla TV(x^k)}{\|\nabla TV(x^k)\|}$ and $\beta_k = \beta^k$ where $0 < \beta < 1$. According to theorem 3.3 and 3.2 the above iterative method converges to the solution of linear system $Ax = b$, in addition the term $\beta_k v_k$ in each step try to steer solution toward a solution with less TV function values. In fact we use a special case of Superiorization algorithm see [3] for general definition of Superiorization algorithm and [6] for utility of Superiorization algorithm in discrete tomography.

In the following, we solve an example of image reconstruction from projection field. To producing data for this example we use Matlab Air-tools package [2].The image which should be reconstructed is a 450×450 pixel image and coefficient matrix A is a 80550×202500 which partitioned into 16 equal size blocks. Figure 1 demonstrate relative error $(\frac{\|x^* - x^k\|}{\|x^*\|})$ for EMR method and its superiorization version, as it is seen, after 10 iteration EMR with superiorization hav about 70% better relative error than EMR.

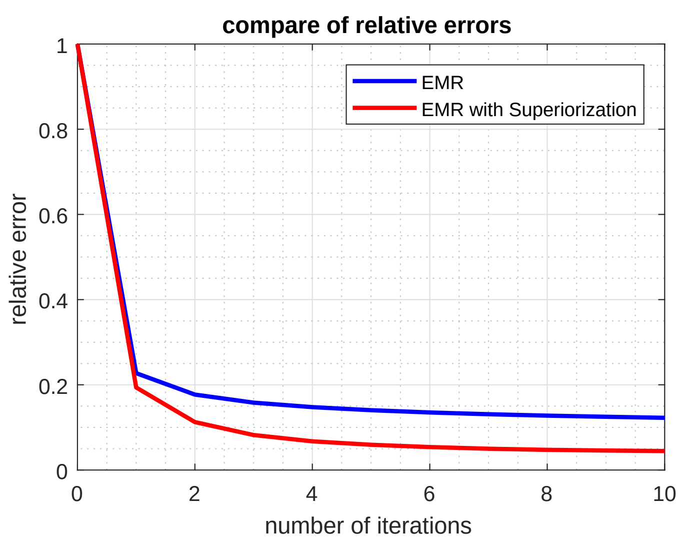

Table 1. Summery of results

Images	TV	Relative error	CPU time (in seconds)
Original image	2968	-	-
EMR image	18288	0.1204	83.4
EMR with Superiorization	4023	0.0436	83.8

Figure 1. Relative error for EMR method (blue) and EMR with Superiorization (red)

Figure 2 shows original image (middle) and reconstructed image with EMR (left) and reconstructed image with Superiorization (right).Table 1 summarizes results of our example.

Figure 2. Recunsructed image with EMR (left),original image (middle), Recunsructed image with EMR and Superiorization (right)

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