Preface

The 8th seminar on Linear Algebra is the latest in a successful series of meetings that began in more than 15 years ago. The meeting is being organized in cooperation with the Iranian Mathematical Society (IMS) and covers a wide and inclusive range of topics in applied and core linear algebra, as well as applications, both emerging and established.

The organizing committee of the seminar warmly welcomes the participants to Sanandaj. We wish you a very happy and fruitful stay in Sanandaj. About 150 participants have taken part in this seminar. We have made endeavor to make the seminar as worthwhile as possible. We wish to express our thanks to all whose help has made this gathering possible. In particular, we would like to express our gratitude to the administration of University of Kurdistan, the Iranian Mathematical Society, and Tejarat Bank.

Meeting Themes

Meeting themes include, but are not limited to:

- All issues related to linear algebra
- Operator Theory
- Matrix Analysis
- Matrix computation
- Direct and iterative methods for solving linear systems
- Numerical methods for solving eigenvalue problems and least squares problems

Invited Plenary Speakers

IMS and the Seminar Organizing Committee are proud to announce that the following mathematicians and scientists have accepted their invitations to speak at the seminar.

- Faezeh Toutounian, Ferdowsi University of Mashhad
- Siegfried M. Rump, Hamburg University of Technology
- Takeshi Ogita, Tokyo Woman's Christian University

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The Extended Abstracts of
The 8th Seminar on Linear Algebra and its Applications
13-14th May 2015, University of Kurdistan, Iran

GENERALIZATIONS OF THE LSMR AND NSCG METHODS FOR SOLVING MATRIX EQUATIONS

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Abstract. In this paper, We present the generalizations of the LSMR and NSCG methods for solving matrix equations. First, based on the LSMR algorithm, the Bl-LSMR and Gl-LSMR algorithms are derived by minimizing the Frobenius norm of residual matrix of normal equation. In addition, by extending the idea of LSMR algorithm, we also present the LSMR-M algorithm for solving the general coupled matrix equations. Next, based on NSCG and NS-CGNR methods, we establish the iterative methods which are inner/outer iterations for solving the sylvester equation and matrix equation $AXB = C$. Convergence conditions of each method are studied in dept and by using the numerical experiments the efficiency of the methods versus some well-known iterative method are shown. We also show that the Hermitian splitting and quasi-Hermitian splitting can induce accurate, robust, and effective preconditioned Krylov subspace methods.

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STRUCTURED PERTURBATIONS

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Abstract. For a symmetric matrix it was shown by Bunch, Demmel and van Loan that there is no difference between the general and the structured condition number. We show that this may change for other structures.

1. Introduction

The sensitivity of a problem determines, in general, the maximally achievable accuracy when solved by a numerical algorithm. For example, for a linear system $Ax = b$ the sensitivity is characterized by the condition number $\|A^{-1}\|\|A\|$. This is true if there are no perturbations in the right hand side; otherwise the condition number increases at most by a factor 2.

Assume a linear system with condition number of about 10^{10} is solved using IEEE754 double precision with a relative rounding error unit $u = 2^{-53} \approx 10^{-16}$. Then that means that at most 6 decimal figures of the solution components can be expected to be correct. An algorithm is *stable* if the computed solution achieves approximately that maximally achievable accuracy. The concept extends to general numerical problems.

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Key words and phrases. Structered perturbation, stability, condition number, structured condition number, normwise perturbation, componentwise perturbation.

When solving special linear systems, for example with symmetric, Toeplitz or circulant matrix, special solvers are available. Such solvers are often much faster than a general solver such as Gaussian elimination. Input parameters are only the necessary parameters to identify the matrix.

For example, it suffices to know the first row of a symmetric positive definite Toeplitz matrix to identify the matrix, and a fast algorithm needs only $\mathcal{O}(n^2)$ operations compared to $\mathcal{O}(n^3)$ operations for Gaussian elimination. However, as only the first row is input to the Toeplitz solver, perturbations are necessarily restricted to Toeplitz perturbations.

Therefore it is necessary to adapt the definition of the sensitivity, i.e. the condition number, to structured perturbations. This opens the mathematical question how big is the difference between the sensitivity with respect to general perturbations compared to structured perturbations.

For a symmetric matrix it was shown by Bunch, Demmel and van Loan that there is no difference between the general and the structured condition number. We show that this may change for other structures. For example, the condition number may square for linear systems with circulant matrix. For that structure particularly fast algorithms, based on Fast Fourier Transformations, requiring only $\mathcal{O}(n \log(n))$ operations are available.

If, as in the example above, the general condition number is of the order 10^{10}, the structured condition number may be of the order 10^5. Thus a solution computed by Gaussian elimination cannot be more accurate than 6 decimal figures, whereas a circulant solver might achieve 11 correct digits. To my knowledge, such an algorithm is not available, a challenge for numerical analysts.

For general Toeplitz matrices the situation is even worse. The factor for a linear system with n unknowns between the general and the structured condition number may be up to about a factor 2^n. This result was obtained together with Sekigawa, and it has intimate relations to the ratio $\|PQ\|$ and $\|P\|\|Q\|$ for polynomials P and Q. The infimum of that ratio is explicitly known for a number of norms. For the spectral norm, which is used here, it is only known up to some factor.

The situation changes for the matrix condition number (instead of the condition number for linear systems). Here I can show that for quite a number of structures there is no difference at all between the general and the structured condition number. Such structures include symmetric, persymmetric, skewsymmetric, symmetric Toeplitz, general Toeplitz, persymmetric Hankel, general Hankel and circulant matrices.

The result can be interpreted in the way that amongst the worst perturbations there is always a structured one.

The famous Eckart-Young theorem states that the condition number is equal to the reciprocal of the distance to the nearest singular matrix. That 80-year old theorem is quite intuitive: Approaching a singular problem should necessarily increase the condition number, i.e. the sensitivity. It is shown that the theorem extends to all the structures mentioned above.

For simplicity and convenience, perturbations are often normwise. However, that bears a significant disadvantage. All components of a matrix may be perturbed, in particular zero components. However, zero components are often determined by the structure of the underlying problem, for example the discretization of some continuous equations. Therefore, a more versatile and general point of view are componentwise perturbations. The most common case are entrywise relative perturbations. In that case, zero components are not changed.

The natural question arises whether there is a difference between general and structured componentwise perturbations. In that case the picture changes completely. We could construct parametrized examples where the general structured condition number tends to infinity for the parameter approaching zero. However, the structured componentwise condition number is always less than 8. That means the solution of the problems are insensitive to structured perturbations, but arbitrarily sensitive to general perturbations.

Thus there is no relation between the general and the componentwise structured condition number. Similarly, for structured perturbations, there is no relation between the componentwise condition number and the distance to the nearest singular matrix. More precisely, there are parametrized examples of structured matrices with structured componentwise condition number tending to infinity, whereas the structured componentwise distance to the nearest singular matrix is always equal to 1. In other words, no entrywise change of less than 100 per cent can produce a singular matrix, that is the distance to the nearest singular matrix is equal to the distance to the zero matrix.

Attempting to rescue a version of the Eckart-Young theorem, it remains the questions whether there is a relation between the general (unstructured) componentwise condition number and the general componentwise distance to the singular matrix. That was an open problem for some time. We solved that question by developing an extension the 100-year old Perron-Frobenius theory from positive or non-negative matrices to general real or complex matrices. The result characterizes the relation up to a constant factor less than 6.

The condition number characterizes the maximally achievable accuracy of a numerical solution. However, in general the condition number is not known. That lead Wilkinson to his famous definition of the textitbackward error: What is the smallest perturbation of a given problem such that a computed solution is the exact solution of the perturbed problem. If such a perturbation is of the size of the relative rounding error unit, an algorithm is stable.

Many algorithms have been analyzed to be stable, and that is possible without knowing the condition number. Again, the question about structures arises. It is well-known that for normwise perturbations there is no difference between the general and the structured backward error. For other structures that problem has been analyzed as well.

At the 19th Householder meeting 2014 in Spa, Belgium, Jim Demmel posed the question about the relation between the general and symmetric backward error under componentwise perturbations. That question arose in practice: Algorithms are known to be backward stable in the general componentwise sense, but it was not clear whether that also implies stability with respect to symmetric componentwise perturbations.

We solved that problem not only with respect to symmetric, but with respect to all the structures mentioned above. The result will appear this year in SIMAX. It says that for all those structures there are problems being arbitrarily sensitive with respect to normwise structured perturbations, but not sensitive at all with respect to componentwise structured perturbations.

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ITERATIVE REFINEMENT FOR THE SYMMETRIC EIGENVALUE DECOMPOSITION

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Abstract. An efficient refinement algorithm is proposed for the symmetric eigenvalue problems. The algorithm is simple, and it mainly consists of matrix multiplications. It constructs an arbitrarily accurate eigenvalue decomposition, up to the limit of computational precision. Since the proposed algorithm is based on Newton's method, it converges quadratically for simple eigenvalues. Numerical results demonstrate the excellent performance of the proposed algorithm in terms of convergence rate and overall computational cost.

1. Introduction

Let A be a real symmetric $n \times n$ matrix. We are concerned with the symmetric eigenvalue problem $Ax = \lambda x$, where $\lambda \in \mathbb{R}$ is an eigenvalue of A and $x \in \mathbb{R}^n$ is an eigenvector of A associated to λ. To solve the

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Key words and phrases. accurate numerical algorithm, iterative refinement, symmetric eigenvalue decomposition.

* Speaker.

problem is important, since it is one of the significant tasks in scientific computing. Excellent overviews can be found in [4].

The purpose of this paper is to develop an algorithm for calculating an arbitrarily accurate result of the eigenvalue decomposition

$$A = XDX^{\mathrm{T}}, \tag{1.1}$$

where $X \in \mathbb{R}^{n\times n}$ is the orthogonal matrix whose ith columns are the eigenvectors $x^{(i)}$ of A and $D \in \mathbb{R}^{n\times n}$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues λ_i, i.e., $D_{ii} = \lambda_i$ for $i = 1, \dots, n$. For this purpose we discuss iterative refinement methods for (1.1) together with the convergence analysis.

Several efficient numerical algorithms for (1.1) have been developed such as the bisection method with inverse iteration, the QR algorithm, the divide-and-conquer algorithm or the MRRR algorithm via Householder's tridiagonalization, the Jacobi algorithm and so forth. Since they have actively been studied in numerical linear algebra for decades, there are highly reliable implementations of them such as LAPACK routines.

We stress that we do not intend to compete with such existing algorithms but develop an iterative refinement algorithm for improving the results obtained by any of them, i.e., the proposed algorithm can be regarded as a supplement to the existing ones for constructing (1.1). In fact, we assume that an approximation $\widehat{X}$ of X in (1.1) is given, where $\widehat{X}$ should be accurate to some extent. A sufficient condition for the convergence of the iterations is given by our analysis.

Another possible approach to achieving an accurate eigenvalue decomposition is to use some multiple precision arithmetic library such as MPFR with GMP in the Householder's tridiagonalization and the subsequent algorithm. In this case, however, the accuracy of the results depends on the arithmetic precision in use. Let $\mathbf{u}$ denote the relative rounding error unit according to working precision (For example, $\mathbf{u} = 2^{-53}$ for IEEE 754 binary64). Define $\lambda_{\max} := \max_i |\lambda_i|$. Then it is known (cf. e.g., [4]) from the standard error analysis for the existing algorithms that the absolute error bound of each computed eigenvalue $\widehat{\lambda}_i$ for λ_i is given as

$$|\widehat{\lambda}_i - \lambda_i| \le p(n)\mathbf{u} \cdot \lambda_{\max},$$

where $p(n)$ is a modestly growing function of n. This means that small eigenvalues may not achieve high relative accuracy. If $\lambda_{\min} :=$

$\min_i |\lambda_i| \neq 0$, then the relative error bound of $\widehat{\lambda}_i$ becomes

$$\frac{|\widehat{\lambda}_i - \lambda_i|}{|\lambda_i|} \leq \frac{|\widehat{\lambda}_i - \lambda_i|}{\lambda_{\min}} \leq p(n)\mathbf{u} \cdot \frac{\lambda_{\max}}{\lambda_{\min}} = p(n)\mathbf{u} \cdot \kappa(A),$$

where $\kappa(A)$ is the spectral condition number of A. In general, we do not know beforehand how much arithmetic precision suffices to achieve the desired result accuracy, especially for such small eigenvalues. Moreover, the use of such multiple precision arithmetic for entire computations is often much more time-consuming than pure floating-point arithmetic due to the difficulty of the optimization for today's computer architecture. Therefore, we prefer the approach by the iterative refinement rather than that by simply using the multiple precision arithmetic. More precisely, the use of higher precision arithmetic in our proposed algorithms is basically restricted to matrix multiplication, which accounts for the majority of the computational cost. For example, XBLAS [1], the extended and mixed precision BLAS, and other accurate and efficient algorithms for dot products [2, 5] and matrix products [3] based on error-free transformations are available for practical implementation.

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Lectures

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AN ITERATIVE ALGORITHM FOR HERMITIAN TRIDIAGONAL LEAST-SQUARES SOLUTIONS OF QUATERNION MATRIX EQUATIONS

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Abstract. This paper is concerned with examining the applicability of the conjugate gradient least-squares (CGLS) method for determining the Hermitian tridiagonal least-squares solutions of the quaternion matrix equation $AX_1B + CX_2D = E$. All computations in the presented algorithm are quaternionic.

1. Introduction

The quaternions arise in various fields such as computer graphics, control theory, signal processing, and etc. For an overview on the properties of quaternions, one may refer to [1, 2] and references therein. Throughout the current work, the set of all $m \times n$ matrices over the quaternion ring is denoted by $\mathbb{Q}^{m\times n}$, i.e.,

$$\mathbb{Q} = \{a_1+a_2i+a_3j+a_4k \mid i^2 = j^2 = k^2 = ijk = -1, \quad a_1, a_2, a_3, a_4 \in \mathbb{R}\}.$$

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* Speaker.

For a given $n \times n$ matrix A, the symbols A^H and $\mathrm{tr}(A)$ are respectively exploited to represent the conjugate transpose and the trace of A. The real part of a quaternion a is denoted by $\mathrm{Re}(a)$, i.e., if $a = a_1 + a_2 i + a_3 j + a_4 k$ then $\mathrm{Re}(a) = a_1$. The set of all $n \times n$ Hermitian, anti-Hermitian, Hermitian tridiagonal and anti-Hermitian tridiagonal quaternion matrices are respectively indicated by $\mathbb{HQ}^{n\times n}$, $\mathbb{AHQ}^{n\times n}$, $\mathbb{THQ}^{n\times n}$, and $\mathbb{TAHQ}^{n\times n}$. Moreover, the notation $\mathbb{KQ}^{n\times n}$ stands for the set of all $n \times n$ quaternion matrices so that their main diagonal, the first diagonal below this, and the first diagonal above the main diagonal are zero. The inner product over $\mathbb{Q}^{m\times n}$ is defined by $\langle A, B\rangle = \mathrm{Re}(\mathrm{tr}(B^H A))$ for $A, B \in \mathbb{Q}^{m\times n}$ and the induced matrix norm is specified by

$$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{\mathrm{Re}(\mathrm{tr}(A^H A))}.$$

As a natural extension for the matrix groups $X = (X_1, X_2)$ and $Z = (Z_1, Z_2)$, the inner product between X and Z is elucidated by

$$\langle (X_1, X_2), (Z_1, Z_2)\rangle = \langle X_1, Z_1\rangle + \langle X_2, Z_2\rangle .$$

Thence $\|X\|^2 = \|X_1\|^2 + \|X_2\|^2$ for $X = (X_1, X_2)$.

In this paper, we focus on the solutions of the following problem.

Problem 1. Presume that the $n \times n$ quaternion matrices A, B, C, D and E are given. Find the matrices X_1 and X_2 such that

$$\|AX_1B + CX_2D - E\| = \min, \tag{1.1}$$

and $X = (X_1, X_2) \in \mathbb{THQ}^{n\times n} \times \mathbb{THQ}^{n\times n}$.

Here we would like to point out that Ling et al. [2] have extended the LSQR algorithm [3] to determine the Hermitian tridiagonal least-squares solutions of the matrix equation $AXB = E$.

For simplicity, we consider the linear operator $\mathcal{M} : \mathbb{Q}^{n\times n} \times \mathbb{Q}^{n\times n} \to \mathbb{Q}^{n\times n}$ such that for $X = (X_1, X_2)$

$$\mathcal{M}(X) = AX_1B + CX_2D. \tag{1.2}$$

Using (1.2), the matrix equation $AX_1B + CX_2D = E$ can be reformulated to $\mathcal{M}(X) = E$.

Definition 1.1. Let $\mathcal{G}$ be a linear operator from $\mathbb{Q}^{m\times n}$ to $\mathbb{Q}^{m\times n}$, then the adjoint of $\mathcal{G}$ is denoted by $\mathcal{G}^*$ and satisfies $\langle \mathcal{G}(X), Z\rangle = \langle X, \mathcal{G}^*(Z)\rangle$.

2. Main results

Lately, Peng [4, 5] has successfully extended the CGLS method to determine the least-squares solutions of two kinds of matrix equations over two special classes of matrices. In this section we develop the CGLS method for solving problem (1) which is propounded in Algorithm 1. The following lemma has a fundamental role for developing the CGLS algorithm to solve Problem 1.

Lemma 2.1. *Presume that* $\mathbb{THQ}^{n\times n}$, $\mathbb{TAHQ}^{n\times n}$ *and* $\mathbb{KQ}^{n\times n}$ *are defined as before. Then,*

$$\mathbb{Q}^{n\times n} = \mathbb{THQ}^{n\times n} \oplus \mathbb{TAHQ}^{n\times n} \oplus \mathbb{KQ}^{n\times n},$$

where $\oplus$ *stands for the orthogonal direct sum with respect to the inner product* $\langle .,. \rangle$.

Remark 2.2. For simplicity, we mention the linear operator $\mathcal{L}$ from $\mathbb{Q}^{n\times n}$ to $\mathbb{THQ}^{n\times n}$ as follows:

$$\begin{array}{rcl} \mathcal{L} : \mathbb{Q}^{n\times n} & \to & \mathbb{THQ}^{n\times n} \\ A & \mapsto & A_1 \end{array}$$

where $A_1 \in \mathbb{THQ}^{n\times n}$, $A_2 \in \mathbb{TAHQ}^{n\times n}$ and $A_3 \in \mathbb{KQ}^{n\times n}$ is the unique representation of matrix A, i.e., $A = A_1 + A_2 + A_3$.

For an arbitrary $A \in \mathbb{Q}^{n\times n}$, suppose that the operators $\mathcal{H}$ gives the Hermitian part of matrix A, i.e., $\mathcal{H}(A) = \frac{1}{2}(A + A^H)$. In addition, assume that $\mathcal{K}(A)$ reset to zero all elements of the matrix A except the main diagonal, the first diagonal below this and above the main diagonal where A is a given arbitrary matrix. It is not difficult to see that $\mathcal{L}(A) = \mathcal{K}(\mathcal{H}(A))$.

Algorithm 1. `The CGLS method for solving problem` (1).
Data: `Input` A, B, C, D, E; `the initial guess` $X(0) = (0, 0)$ `and choose tolerance` ϵ.
Initialization:

- $k = 0$
- $R(0) = E - \mathcal{M}(X(0))$
- $P(0) = \mathcal{L}(\mathcal{M}^*(R(0)))$
- $Q(0) = P(0)$

`While` $\|P(k)\| > \epsilon$ `Do:`

- $X(k+1) = X(k) + \frac{\|P(k)\|^2}{\|\mathcal{M}(Q(k))\|^2} Q(k)$
- $R(k+1) = R(k) - \frac{\|P(k)\|^2}{\|\mathcal{M}(Q(k))\|^2} \mathcal{M}(Q(k))$
- $P(k+1) = \mathcal{L}\left(\mathcal{M}^*(R(k+1))\right)$

- $Q(k+1) = P(k+1) + \frac{\|P(k+1)\|^2}{\|P(k)\|^2} Q(k)$
- $k := k + 1$.

`EnDDo`

Now we present the following useful theorem which represents properties of the sequences produced by the proposed algorithm. By Theorem 2.3, it turns out that the solution of problem (1) can be computed by Algorithm 1 within finite number of steps in exact arithmetic.

Theorem 2.3. *Suppose that k steps of Algorithm 1 have been performed, i.e., $P(l) \neq 0$ and $\mathcal{M}(Q(l)) \neq 0$ for $l = 0, 1, \ldots, k$. The sequences $P(l)$ and $Q(l)$ $(l = 0, 1, \ldots, k)$ produced by Algorithm 1 satisfy*

(1) $\langle P(i), P(j) \rangle = 0$,
(2) $\langle \mathcal{M}(Q(i)), \mathcal{M}(Q(j)) \rangle = 0$,
(3) $\langle Q(i), P(j) \rangle = 0$,

for $i, j = 0, 1, 2, \ldots, k$ $(i \neq j)$.

The following theorem shows that the residual matrices associated with the approximate solutions produced by Algorithm 1 satisfy an optimality property.

Theorem 2.4. *Suppose that m steps of Algorithm 1 have been performed, i.e., $P(l) \neq 0$ and $\mathcal{M}(Q(l)) \neq 0$ for $l = 0, 1, \ldots, m-1$. Presume that the subspaces $\mathcal{K}_m = span\{Q(0), \ldots, Q(m-1)\}$ and $\mathcal{MK}_m = span\{\mathcal{M}(Q(0)), \ldots, \mathcal{M}(Q(m-1))\}$. Assume that $P(m) \neq 0$, then the mth approximate solution obtained by Algorithm 1 satisfies*

$$\|E - \mathcal{M}(X(m))\| = \min_{X \in X(0) + \mathcal{K}_m} \|E - \mathcal{M}(X)\|.$$

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ON GENERALIZED EVEN MAJORIZATION

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Abstract. In this paper we deal with gs-majorization, introduce generalized even majorization(ge-majorization) and prove that gs-majorization and ge-majorization coincide on $\mathbb{R}^n$ when $n \geq 3$.

1. Introduction

Let $x = (x_1, \ldots, x_n)^t$ and $y = (y_1, \ldots, y_n)^t$ be two vectors in $\mathbb{R}^n$ and let $x^{\downarrow} = \left(x_1^{\downarrow}, \ldots, x_n^{\downarrow}\right)$ be the vector obtained by rearranging the coordinates of x in decreasing order. i.e. $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \ldots \geq x_n^{\downarrow}$. Then x is said to be majorized by y (written $x \prec y$) if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \qquad k = 1, 2, \ldots, n \tag{1.1}$$

and $\sum_{i=1}^{n} x_i^{\downarrow} = \sum_{i=1}^{n} y_i^{\downarrow}$. A matrix D is called doubly stochastic if it is nonnegative and summation of all entries in each row and each column

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Key words and phrases. Doubly stochastic matrix, Even permutation, ge-Majorization.

* Speaker.

equals to 1. Let $x \prec y$, then there exists a doubly stochastic matrix D such that $x = Dy$. If $x = Dy$ for some generalized doubly stochastic matrix(a matrix with all row and column summations equal to one, but not necessarily nonnegative), we say that x is gs-majorized by y. We know that every generalized doubly stochastic matrix is an affine combination of permutations. If we consider only even permutations, we say that x is ge-majorized by y. In this paper we prove that gs-majorization and ge-majorization coincide on $\mathbb{R}^n$ when $n \geq 3$.

2. GE-MAJORIZATION

First we state some Known facts about majorization. Let $e = (1, 1, ..., 1)^t$. We have the following theorem for doubly stochastic matrices.

Theorem 2.1. *(Birkhoff's theorem) The set of $n \times n$ doubly stochastic matrices is a convex set whose extreme points are the permutation matrices.*

Definition 2.2. A matrix D is called even doubly stochastic if it is a convex combination of even permutation matrices.

Definition 2.3. A matrix D is called g-doubly stochastic(ge-doubly stochastic) if it is an affine combination of permutation(respectively even permutation) matrices.

We have some definitions about majorizations corresponding to variety of doubly stochastic matrices.

Definition 2.4. We say that x is even majorized by y, if $x = Dy$ for some even doubly stochastic matrix D and write $x \prec_e y$.

Definition 2.5. We say that x is gs-majorized(ge-majorized) by y if $x = Dy$ for some g-doubly stochastic(respectively ge-doubly stochastic) matrix D and write $x \prec_{gs} y$(respectively $x \prec_{ge} y$).

Our aim is to show that gs-majorization and ge-majorization coincide on $\mathbb{R}^n$ when $n \geq 3$.

Theorem 2.6. *The concepts of gs-majorization and ge-majorization coincide on $\mathbb{R}^n$ when $n \geq 3$, i.e. $x \prec_{gs} y$ if and only if $x \prec_{ge} y$.*

Proof. We prove this theorem in two parts:

Part 1: $n = 3$.

Let $y_0 = (y_1, y_2, y_3)^t \in \mathbb{R}^3$. We consider two cases:

Case 1: If $y = \alpha e$ for some $\alpha \in \mathbb{R}$. Then

$$\{x; x \prec_{gs} y_0\} = \{x; x \prec_{ge} y_0\} = \{y_0\},$$

and we are done.
Case 2: If $y_0 \neq \alpha e$ for all $\alpha \in \mathbb{R}$. Without loss of generality assume $y_1 \neq y_2$. All permutations of y_0 lie in the hyper plane $x+y+z = y_1+y_2+y_3$. We show even permutations of y_0 are affinely independent. Suppose $y_0' = (y_3, y_1, y_2)^t$ and $y_0'' = (y_2, y_3, y_1)^t$ be two even permutaions of y_0. We show that $y_0' - y_0$ and $y_0'' - y_0$ are linearly independent. If they are linearly dependent then $y_0' - y_0 = \lambda(y_0'' - y_0)$ for some $\lambda \in \mathbb{R}$ i.e.

$$\begin{cases} y_1 - y_3 = \lambda\,(y_1 - y_2) \\ y_2 - y_1 = \lambda\,(y_2 - y_3) \\ y_3 - y_2 = \lambda\,(y_3 - y_1). \end{cases}$$

Simplifying these relations shows that

$$(y_1 - y_2)\left(\lambda^2 - \lambda + 1\right) = 0.$$

Since $y_1 \neq y_2$ and $\lambda^2 - \lambda + 1 = 0$ has no real solution we get to contradiction. So $y_0' - y_0$ and $y_0'' - y_0$ are linearly independent.

Part 2: $n > 3$.

First we show that every transposition can be written as an affine combination of even permutations. Without loss of generality assume $y = (y_2, y_1, y_3, ..., y_n)^t$, $y_0 = (y_1, y_2, y_3, ..., y_n)^t$, $z = (y_1, y_2, y_3)^t$, $z' = (y_3, y_1, y_2)^t$, $z'' = (y_2, y_3, y_1)^t$ and $z_0 = (y_2, y_1, y_3)^t$. We see in part 1 that

$$z_0 = \alpha_1 z + \alpha_2 z' + \alpha_3 z'',$$

for some α_1, α_2 and α_3 such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Now

$$y = \left(\alpha_1\left(z^t, y_4, ..., y_n\right) + \alpha_2\left(z'^t, y_4, ..., y_n\right) + \alpha_3\left(z''^t, y_4, ..., y_n\right)\right)^t$$

and we conclude that every transposition can be written as an affine combination of even permutations. Now suppose $\Pi = \Pi_k \Pi_{k-1} ... \Pi_1$ where Π_i $(1 \leq i \leq k)$ are transpositions. We use induction on k. If $k = 1$, it is obvious by part 1. Now, suppose that the theorem holds for $k-1$. We prove that the relation holds for k.

$$\Pi_{k-1}\Pi_{k-2}...\Pi_1\left(y_0\right) = \sum_{i=1}^{m} \alpha_i P_i\left(y_0\right),$$

where $\sum \alpha_i = 1$, P_is are even permutations and m is an integer. Now

$$\begin{aligned} \Pi_k\left(\Pi_{k-1}, ..., \Pi_1\left(y_0\right)\right) &= \Pi_k\left(\sum_{i=1}^{m} \alpha_i P_i\left(y_0\right)\right) \\ &= \sum_{i=1}^{m} \alpha_i \Pi_k P_i\left(y_0\right). \end{aligned}$$

$$\Pi_k P_i(y_0) = \sum_{j=1}^{m'} \beta_j Q_j(P_i(y_0)),$$

where $\sum \beta_i = 1$ and Q_js are even permutations. Hence

$$\begin{aligned}\Pi_k(\Pi_{k-1}...\Pi_1)(y_0) &= \sum_{i=1}^{m} \alpha_i \left(\sum_{j=1}^{m'} \beta_j Q_j P_i(y_0)\right) \\ &= \sum_{i=1}^{mm'} \alpha'_i P'_i(y_0),\end{aligned}$$

where $\sum_{i=1}^{mm'} \alpha'_i = 1$ and P'_is are even permutations. The proof is complete. □

If $n = 1$ or $n = 2$, then the only even permutation is identity and hence ge-majorization and equality coincide.

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SELECTION CRITERIA FOR CHOOSING WINDOW LENGTH IN SINGULAR SPECTRUM ANALYSIS

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Abstract. Singular Spectrum Analysis (SSA) is a powerful non-parametric time series analysis method which mainly is based on matrix analysis and classical time series. In order to apply this method, at the first stage, window length must be determined. An improper value of window length yields misleading results in SSA. In this paper, we review the criteria that should be consider to find a suitable value for window length.

1. Introduction

Singular Spectrum Analysis (SSA) is a powerful non-parametric technique in time series analysis that has been developed and applied to many practical problems. Most recent developments in the theory and methodology of SSA can be found in [1].
The aim of SSA is to decompose the original series into the sum of a small number of independent and interpretable components such as a slowly varying trend, oscillatory components and a structureless noise. The whole procedure of SSA depends upon two basic, but very important, parameters:

- the window length, L,

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* Speaker.

- the way of grouping.

Choosing improper values of these parameters yields incomplete reconstruction and misleading results in forecasting. In spite of the importance of the choosing parameters, no theoretical solution has been yet proposed to this problem. Of course, there are worthwhile efforts and various techniques for selecting the appropriate value of L (see, for example, [2] and references therein). In this paper, we consider choosing window length based on several criteria.

The structure of this paper is as follows: Section 2 presents a brief description of the SSA. Section 3 shows the main results. The final section discusses the conclusions of the study.

2. Short Description of SSA

The basic SSA method consists of four steps: In the first step, we transfer a one-dimensional time series $Y_N = (y_1, \ldots, y_N)$ into the trajectory matrix $\mathbf{X} = [X_1, \cdots, X_K]$ with $X_j = [y_j, \cdots, y_{L+j-1}]^T$ where $L (2 \leq L \leq N-1)$, is the window length and $K = N - L + 1$. Then, we use singular value decomposition (SVD) of $\mathbf{X}$ and decompose it to the sum of L elementary matrices: $\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_L$. In the next step, we are splitting the elementary matrices into several groups and summing the matrices within each group. Finally, by diagonal averaging, we convert each matrix, resulted from the grouping step, to the form of a Hankel matrix which can be subsequently converted to a time series. The resulting filtered time series can be used for further analysis such as forecasting.

3. Main Results

3.1. **Lag-Covariance.** Let us first consider the behavior of matrix $\mathbf{X}\mathbf{X}^T/K$, that is lag-covariance matrix. Denote the trace of the matrix $\mathbf{X}\mathbf{X}^T$ as $T_{\mathbf{X}}^{L,N} = tr(\mathbf{X}\mathbf{X}^T)$ we have the following theoretical results:

Lemma 3.1. *Consider the trajectory matrix* $\mathbf{X}$ *as defined before. Then*

$$T_{\mathbf{X}}^{L,N} = T_{\mathbf{X}}^{K,N} = \sum_{j=1}^{N} w_j^{L,N} x_j^2 \tag{3.1}$$

where, $w_j^{L,N} = \min\{\min\{L, K\}, j, N-j+1\} = w_j^{K,N}$ *and* $K = N-L+1$.

Proof. See [3] □

Lemma 3.2. *$T_{\mathbf{X}}^{L,N}$ is an increasing function of L on $\{2, ..., \left[\frac{N+1}{2}\right]\}$, a decreasing function on $\{\left[\frac{N+1}{2}\right] + 1, ..., N-1\}$, and*

$$\max_{L\in\{2,...,N-1\}} T_{\mathbf{X}}^{L,N} = T_{\mathbf{X}}^{L_{\max},N} \tag{3.2}$$

where $L_{\max}$ is the median of $\{1, \ldots, N\}$.

Proof. See [3] □

Theorem 3.3. *$T_{\mathbf{X}}^{L,N}/K$ is an increasing function of L on $\{2, ..., N-1\}$.*

Proof. See [1] □

3.2. **Rank of trajectory matrix.**

Theorem 3.4. *Let d denotes the rank of the trajectory matrix $\mathbf{X}$ in SSA. Thus, the maximum rank of the trajectory matrix $\mathbf{X}$ is achieved at $L = L_{\max}$.*

Proof. See [1] □

3.3. **Separability.** SSA decomposition of the series Y_N can only be successful if the resulting additive components of the series are approximately separable from each other. We have two types of separability, strong and weak, that are explained in the following.

3.3.1. *Strong separability.* Assume that $Y_N = Y_N^{(1)} + Y_N^{(2)}$. Moreover let $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are corresponding trajectory matrices respectively. In addition, assume that we have $\mathbf{X} = \mathbf{X}_{\text{SVD}}^{(1)} + \mathbf{X}_{\text{SVD}}^{(2)}$ by using SVD and grouping. Then strong separability of the series $Y_N^{(1)}$ and $Y_N^{(2)}$, happens if $\mathbf{X}_{\text{SVD}}^{(1)} = \mathbf{X}^{(1)}$ & $\mathbf{X}_{\text{SVD}}^{(2)} = \mathbf{X}^{(2)}$. Although, strong separability does happens in practice rarely; theoretically it happens if the columns (rows) of $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are orthogonal and singular values are isolated. Considering the coefficient of variation for the differentiation of singular values, we checked and found that the maximum separation for a wide class of series will be obtained when $L = L_{\max}$.

3.3.2. *Weak separability.* Quantity *w-correlation* is a natural measure of similarity between two series $Y_N^{(i)} = [y_1^{(i)}, \cdots, y_N^{(i)}]^T$ $(i = 1, 2)$:

$$\rho_{12}^{(w)} = \frac{\left(Y_N^{(1)}, Y_N^{(2)}\right)_w}{\sqrt{\left(Y_N^{(1)}, Y_N^{(1)}\right)_w}\sqrt{\left(Y_N^{(2)}, Y_N^{(2)}\right)_w}}$$

where, $\left(Y_N^{(i)}, Y_N^{(j)}\right)_w = \sum_{p=1}^{N} w_p^{L,N} y_p^{(i)} y_p^{(j)}$, $(i, j = 1, 2)$ (See, [4]). If the absolute value of the w-correlations is small, then the corresponding

series are almost w-orthogonal, but, if it is large, then the two series are far from being w-orthogonal and are therefore weakly separable. Hassani et al ([5]) showed the minimum value of w-correlation attains at $L = L_{\max}$, for a wide class of time series.

3.4. **Number of entities of trajectory matrix.** There are $L \times K$ observations in the trajectory matrix. Now, it is easy to see that $L = L_{\max}$ gives maximum value of the number of entities of the trajectory matrix.

3.5. **Accuracy Measure.** Mahmoudvand et al ([2]) have considered this issue by means of simulation and showed that there are a meaningful differences among the optimal values of window length that used for reconstruction and forecasting. They showed that, optimal value for reconstruction is again close to $L_{\max}$, but it is depend on the several conditions for the forecasting procedure. Moreover, Golyandina ([4]) have considered the basic model "signal + residual" and showed that the error behaviour depends on the type of residuals, deterministic or stochastic, and whether the noise is white or red. In her paper, the choice of L close to one-half of the time series length was approved to be appropriate in most cases.

4. Conclusion

The considered theoretical criteria in this paper show that choosing $L = L_{\max}$ is the best value for reconstruction in a wide class of time series. However, there is not any general recommendation for the forecasting procedure.

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TRUNCATED GENERALIZED GLOBAL ARNOLDI FACTORIZATION

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Abstract. In this paper, we show the dependence of the residual matrix on the starting matrix, and give necessary and sufficient conditions for a starting matrix to produce a zero residual for solving the matrix equation $AXB = C$ where A and B are nonsingular matrices.

1. Introduction

In this paper, we will consider the following matrix equation

$$AXB = C, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{s\times s}$ are nonsingular matrices and $X, C \in \mathbb{R}^{n\times s}$. Different methods are devoted to find the special solution structures of the matrix equation (1.1) such as symmetric, skew-symmetric or symmetric positive definite solution X; see [2, 3, 4] .

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* Speaker.

In this paper, we will develop nessesary and sufficient conditions for a starting matrix to produce a zero residual matrix.
This paper is organized as follows. In section 2 we show the functional dependence of the residual matrix on the starting matrix. In section 3 we give nessesary and sufficient conditions for a starting matrix to produce a zero residual matrix.
Thoroughout this paper, we use the following notations. Let $\mathbb{R}^{m\times n}$ be the set of $m\times n$ real matrices. The symbols A^T, $\|A\|_2$ and $trace(A)$ will denote the transpose, 2-Norm and trace, respectively, of a matrix $A\in\mathbb{R}^{m\times n}$. For any matrices A and B in $\mathbb{R}^{n\times s}$, $< A, B >_F= trace(A^TB)$ denotes the inner product. The associated norm is the Frobenius norm obtained by $\|.\|_F$.
Further, $vec(.)$ will stand for the vec operator, i.e.
$vec(A) = (a_1^T, a_2^T, ..., a_s^T)^T$ for the matrix $A = (a_1, a_2, ..., a_s) \in \mathbb{R}^{n\times s}$, where $a_j, j = 1, 2, ..., s$ is the j-th column of A and $A\otimes B = (a_{ij}B)$ denotes the Kronecker product of the matrices A and B. Let $A = [A_1, ..., A_p] \in \mathbb{R}^{n\times ps}$ and $B = [B_1, ..., B_l] \in \mathbb{R}^{n\times ls}$, where A_j and B_j are $n\times s$ matrices. The matrix $A^T\diamond B$ are defined by
$(A^T\diamond B)_{ij} =< A_i, B_j >_F$.

2. The generalized Global Arnoldi Factorization

We can easily see that the matrix equation (1.1) is equivalent to the following linear system of equations

$$(B^T\otimes A)vec(X) = vec(C).$$

Now , we need to define the matrix Krylov subspace associated with triplet (A, V, B) as follows.

Definition 2.1. Let $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{s\times s}$, and $V\in\mathbb{R}^{n\times s}$. The matrix Krylov subspace is defined by

$$\mathcal{G}K_k = \mathcal{G}K_k(A, V, B) = span\{V, AVB, ..., A^{k-1}VB^{k-1}\}. \tag{2.1}$$

The generalized global Arnoldi algorithm [4] leads to

$$A\mathcal{V}_k(I_k\otimes B) = \mathcal{V}_k(H_k\otimes I_s) + R_k(e_k^T\otimes I_s), \tag{2.2}$$

where $R_k = h_{k+1,k}V_{k+1}$, $\mathcal{V}_k\in\mathbb{R}^{n\times ks}$, $\mathcal{V}_k^T\diamond\mathcal{V}_k = I_k$, $H_k\in\mathbb{R}^{k\times k}$ is upper Hessenberg. The matrix R_k is the residual matrix and is F-orthogonal to the columns of $\mathcal{V}_k$. If the rsidual R_k is the zero matrix then equation (2.2) is called a truncated generalized global Arnoldi factorization when $k < ns$.
We expose the functional dependence of the residual matrix on the starting matrix, and give necessary and sufficient for a starting matrix

to produce a zero residual.
The following result is an extension of the implicit Q- Theorem proved in [1].

Theorem 2.2. *Suppose*

$$A\mathcal{V}_k(I_k \otimes B) = \mathcal{V}_k(H_k \otimes I_s) + R_k(e_k^T \otimes I_s),$$
$$A\mathcal{U}_k(I_k \otimes B) = \mathcal{U}_k(G_k \otimes I_s) + F_k(e_k^T \otimes I_s),$$

where columns $\mathcal{V}_k$ and $\mathcal{U}_k$ are F-orthogonal and H_k, G_k are both upper Hessenberg with positive subdiagonal entries.
If $\mathcal{V}_k(e_1 \otimes I_s) = \mathcal{U}_k(e_1 \otimes I_s)$ and $\mathcal{V}_k^T \diamond R_k = \mathcal{U}_k^T \diamond F_k = 0$, then $\mathcal{V}_k = \mathcal{U}_k$, $H_k = G_k$ and $R_k = F_k$.

Proof. The result can be easily proved by induction on k. □

In the following, we show relations between a generalized global Arnoldi factorization and block Krylov matrices.
Consider the block Krylov matrix and the companion matrix, respectively,

$$K = (V_1, AV_1B, ..., A^{k-1}V_1B^{k-1}),$$
$$C = [e_2, ..., e_k, c] \in \mathbb{R}^{k\times k},$$

where $c = (c_1, ..., c_k)^T$.
Observe that

$$AK(I_k \otimes B) - K(C \otimes I_s) = \hat{R}(e_k^T \otimes I_s), \tag{2.3}$$

where $\hat{R} = A^kV_1B^k - \sum_{i=0}^{k-1} c_{i+1}A^iV_1B^i = A^kV_1B^k - K(c \otimes I_s)$. Also, we can easily show that $\hat{r} = vec(\hat{R}) = (((B^k)^T \otimes A^k) - \sum_{i=0}^{k-1} c_{i+1}(B^i)^T \otimes A^i))vec(V_1)$. Note that $\hat{r} = p(B^T; A)vec(V_1)$, where $p(x, y) = x^ky^k - \sum_{i=0}^{k-1} c_{i+1}x^iy^i$. The vector c must be chosen such that solves the least squares problem

$$min_{c\in\mathbb{R}^k}\|A^kV_1B^k - K(c \otimes I_s)\|_F = min_{c\in\mathbb{R}^k}\|p(B^T; A)vec(V_1)\|_2. \tag{2.4}$$

To solve the minimization problem (2.4), consider the global QR factorization $K = \mathcal{Q}(R \otimes I_s)$ where $\mathcal{Q}$ is F-orthonormal, R is upper triangular and $\rho_{ii} = e_i^TRe_i > 0$.
Multiplying (2.3) on the right $(R^{-1} \otimes I_s)$, we obtain

$$AK(I_k \otimes B)(R^{-1} \otimes I_s) - K(C \otimes I_s)(R^{-1} \otimes I_s) = \hat{R}_k(e_k^T \otimes I_s)(R^{-1} \otimes I_s).$$

By Kronecker product propertices, we can rewrite the above equation as follows:

$$A\mathcal{Q}(I_k \otimes B) - \mathcal{Q}(G \otimes I_s) = F(e_k^T \otimes I_s),$$

where $G = RCR^{-1}$, $F = \frac{1}{\rho_{kk}}\hat{R}_k$. It can be esaily shown that $\mathcal{Q}(e_1 \otimes I_s) = \mathcal{V}_k(e_1 \otimes I_s)$ and $\mathcal{Q}^T \diamond \hat{R} = 0$ so $\mathcal{Q}^T \diamond \hat{F} = 0$. Thus extension

Q-implicit theorem will imply $\mathcal{Q} = \mathcal{V}_k$, $G = H_k$ and $F = R_k$. Putting $H_k = G$ yields

$$\beta_k = e_{k+1}^T H_k e_k = \frac{\rho_{k+1,k+1}}{\rho_{k,k}}.$$

Moreover, Let $\hat{p}_k$ solves minimization problem in (2.4). Also, $R_k = \frac{\hat{R}_k}{\rho_{k,k}}$. Thus, we have

$$r_k = vec(R_k) = \frac{vec(\hat{R}_k)}{\rho_{k,k}} = \frac{\hat{r_k}}{\rho_{k,k}}.$$

It follows that

$$\rho_{k+1,k+1} = \|p_k(\hat{B^T}; A)vec(V_1)\|_2.$$

The following Theorem summarizes the preceding discussion.

Theorem 2.3. *Suppose* $A\mathcal{V}_k(I_k \otimes B) = \mathcal{V}_k(H_k \otimes I_s) + R_k(e_k^T \otimes I_s)$ *be a k-step generalized global Arnoldi factorization, and suppose that* $dim(\mathcal{G}K_k(A, V, B)) = ks$. *Then*

$$r_k = \frac{p_k(\hat{B^T}; A)vec(V_1)}{\|p_k(\hat{B^T}; A)vec(V_1)\|_2}.$$

Moreover, $\hat{p}_k$ *solves* $min\|p_k(\hat{B^T}; A)vec(V_1)\|_2$.
In the following, we will develop nessesary and sufficient conditions for a starting matrix to produce a zero residual matrix.

Theorem 2.4. *suppose* $A\mathcal{V}_k(I_k \otimes B) = \mathcal{V}_k(H_k \otimes I_s) + R_k(e_k^T \otimes I_s)$ *be a k-step generalized global Arnoldi factorization with* H_k *unreduced. Then* $R_k = 0$ *if and only if* $V_1 = \mathcal{X}(y \otimes I_s)$ *where* $A\mathcal{X}(I_k \otimes B) = \mathcal{X}(J \otimes I_s)$ *with* $rank(\mathcal{X}) = ks$ *and and* J *a Jordan matrix of order* k .

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THE INVERSE OF HIGHER ORDER TENSORS IN THE MAX-ALGEBRA SENSE

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Abstract. Recently we generalize the max algebra system to the class of nonnegative tensors. In this paper we give some basic properties for the left (right) inverse, under the new system. The existence of order 2 left (right) inverses of tensors is characterized.

1. Introduction

Definition 1.1. [1] The max algebraic addition ($\oplus$) and multiplication ($\otimes$) are defined as follows:
(i) Suppose that $\mathbb{A}, \mathbb{B} \in \Re_+^{[m,n]}$ then we have $\mathbb{A} \oplus \mathbb{B} \in \Re_+^{[m,n]}$ and

$$(\mathbb{A} \oplus \mathbb{B})_{i_1 \dots i_m} = a_{i_1 \dots i_m} \oplus b_{i_1 \dots i_m} = \max\left(a_{i_1 \dots i_m}, b_{i_1 \dots i_m}\right). \tag{1.1}$$

(ii) Suppose that $\mathbb{A} \in \Re_+^{[m,n]}$ and $\mathbb{B} \in \Re_+^{[k,n]}$ where $m \geq 2\,, k \geq 1$ then we have $\mathbb{A} \otimes \mathbb{B} \in \Re_+^{[(m-1)(k-1)+1,n]}$ and

$$\begin{aligned}(\mathbb{A} \otimes \mathbb{B})_{i\alpha_1 \dots \alpha_{m-1}} &= \bigoplus_{i_2 \dots i_m=1}^{n} a_{ii_2 \dots i_m} b_{i_2\alpha_1} \dots b_{i_m\alpha_{m-1}} \\ &= \max_{1 \leq i_2 \dots i_m \leq n} \left\{a_{ii_2 \dots i_m} b_{i_2\alpha_1} \dots b_{i_m\alpha_{m-1}}\right\},\end{aligned} \tag{1.2}$$

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* Speaker.

where $i \in \{1, ..., n\}$, $\alpha_1, ..., \alpha_{m-1} \in [n]^{k-1}$. In particular for $x \in \Re_+^n$ we have

$$(\mathbb{A} \otimes x)_i = \max_{1 \le i_2 ... i_m \le n} \{a_{ii_2...i_m} x_{i_2} ... x_{i_m}\}.$$

Theorem 1.2. [1] *Let $\mathbb{A}$ (and $\mathbb{B}$, $\mathbb{C}$) be an order $m+1$ (and order $k+1$, order $r+1$), dimension n tensor, respectively. Then we have*

$$\mathbb{A} \otimes (\mathbb{B} \otimes \mathbb{C}) = (\mathbb{A} \otimes \mathbb{B}) \otimes \mathbb{C}.$$

Definition 1.3. A diagonal tensor is a tensor that only the entries of which all the indices are equal can be different from zero. A diagonal tensor with all diagonal entries equal to 1 is called the unit tensor and denoted by $\mathbb{I}$.

Theorem 1.4. [1] *Let $\mathbb{A}, \mathbb{I} \in \Re_+^{[m,n]}$, then*
(i). *If $\mathbb{A} \otimes \mathbb{I} = 0$, then $\mathbb{A} = 0$.*
(ii). *If $\mathbb{I} \otimes \mathbb{A} = 0$, then $\mathbb{A} = 0$.*

Lemma 1.5. *Let $\mathbb{A} \in \Re_+^{[m,n]}$ and P, Q are both matrices, then*

$$(P \otimes \mathbb{A} \otimes Q)_{i_1 ... i_m} = \max_{1 \le j_1, ..., j_m \le n} \{a_{j_1 ... j_m} p_{i_1 j_1} q_{j_2 i_2} ... q_{j_m i_m}\}.$$

2. Main sections and results

Since the operation $\oplus$ in max algebra is not invertible, inverse matrices are almost non existent. It is known that in max algebra, generalized permutation matrices are the only type of invertible matrices

Theorem 2.1. [3] *The inverse of a nonnegative matrix A is nonnegative if and only if A is a generalized permutation matrix.*

Recently in [2] the left and right inverse of tensors under the general product, are defined. In conventional multilinear algebra we know that, not all tensors have inverses. We will see that in max algebra the invertible tensors are even more limited.

Definition 2.2. Let $\mathbb{A}$ be a tensor of order m and dimension n, and let $\mathbb{B}$ be a tensor of order k and dimension n. If $\mathbb{A} \otimes \mathbb{B} = \mathbb{I}$, then $\mathbb{A}$ is called an order m left inverse of $\mathbb{B}$ in the max algebra sense, and $\mathbb{B}$ is called an order k right inverse of $\mathbb{A}$ in the max algebra sense.

Theorem 2.3. *Let $\mathbb{A} \in \Re_+^{[m,n]}$ be a diagonal tensor. Then*
$\mathbb{A}$ has an order k left inverse if and only if $a_{ii...i} \neq 0$, $i = 1, 2, ..., n$. Moreover, an order k diagonal tensor $\mathbb{L}$ with diagonal entry $l_{ii...i} = a_{ii...i}^{-(k-1)}$ is the unique order k left inverse of $\mathbb{A}$.

Theorem 2.4. *Let $\mathbb{A} \in \Re_+^{[m,n]}$ be a diagonal tensor. Then $\mathbb{A}$ has an order k right inverse if and only if $a_{ii\ldots i} \neq 0$, $i = 1, 2, ..., n$. In this case, an order k diagonal tensor $\mathbb{R}$ with diagonal entry $r_{i\ldots i} = \sqrt[(m-1)]{a_{ii\ldots i}^{-1}}$ is the unique order k right inverse of $\mathbb{A}$.*

We will now characterize the left (right) inverse of order 2 for a tensor $\mathbb{A} \in \Re_+^{[m,n]}$, for this purpose we require the following lemma.

Lemma 2.5. *If $\mathbb{A} \in \Re_+^{[m,n]}$ and $\mathbb{A}$ has an order 2 left (right) inverse, then*
(i). *This left (right) inverse does not have a row such that all entries are nonzero (zero).* **(ii).** *This left (right) inverse does not have a column such that all entries are nonzero (zero).* **(iii).** *$\mathbb{A}$ does not have a face such that all entries are nonzero (zero).*

Proof. This follows from Definition 1.1 and Definition 2.2. □

Theorem 2.6. *If $\mathbb{A} \in \Re_+^{[m,n]}$, and $\mathbb{A}$ has an order 2 left (right) inverse G, then G must be a generalised permutation matrix.*

Theorem 2.7. *If $\mathbb{A} \in \Re_+^{[m,n]}$, then $\mathbb{A}$ has an order 2 left inverse if and only if there exists a generalised permutation matrix G such that $\mathbb{A} = G \otimes \mathbb{I}$. Moreover, G^{-1} is the unique order 2 left inverse of $\mathbb{A}$.*

Proof. If $\mathbb{A} = G \otimes \mathbb{I}$, for a generalised permutation matrix G, then $\mathbb{A}$ has an order 2 left inverse G^{-1}. Assume C is an order 2 left inverse of $\mathbb{A}$, then $C \otimes \mathbb{A} = \mathbb{I}$, this equation conclude that C must be a generalised permutation matrix (by Theorem 2.6), thus $\mathbb{A} = C^{-1} \otimes \mathbb{I}$. Suppose that B is also an order 2 left inverse of $\mathbb{A}$, we can also get $\mathbb{A} = B^{-1} \otimes \mathbb{I}$. Hence $(C^{-1} - B^{-1}) \otimes \mathbb{I} = 0$, By Theorem 1.4, we have $C^{-1} = B^{-1}$. By the fact that a nonsingular matrix has a unique inverse matrix, it follows that $B = C$ and the desired results hold. □

Theorem 2.8. *If $\mathbb{A} \in \Re_+^{[m,n]}$, then $\mathbb{A}$ has an order 2 right inverse if and only if there exists a generalised permutation matrix Q such that $\mathbb{A} = \mathbb{I} \otimes Q$. In this case, Q^{-1} is the unique order 2 right inverse of $\mathbb{A}$.*

Proof. If $\mathbb{A} = \mathbb{I} \otimes Q$ for a generalised permutation matrix Q, then $\mathbb{A}$ has an order 2 left inverse Q^{-1}. If T is an order 2 right inverse of $\mathbb{A}$, then $\mathbb{A} \otimes T = \mathbb{I}$, imply that T is a generalised permutation matrix (by Theorem 2.6). So $\mathbb{A} = \mathbb{I} \otimes T^{-1}$. Hence if $\mathbb{A}$ has an order 2 right inverse, then there exists a generalised permutation matrix T such that $\mathbb{A} = \mathbb{I} \otimes T$.
If $\mathbb{R}$ is any order 2 right inverse of $\mathbb{A}$, then $\mathbb{A} \otimes R = \mathbb{I} \otimes Q \otimes R = \mathbb{I}$. Set $D = Q \otimes R$, then $\mathbb{I} = \mathbb{I} \otimes D$. By Definition 1.1, D must be the identity matrix of dimension n. Hence the proof is complete. □

Notice that for $m = 2$ (when $\mathbb{A}$ is a matrix), we have the right inverse is equal to left inverse, (refer to max algebra theory).

Theorem 2.9. *Let $\mathbb{A}$ and $\mathbb{B}$ be tensors such that $\mathbb{A} \otimes \mathbb{B} = 0$. Then the following hold:* **(i).** *If the order 2 left inverse of a tensor $\mathbb{A}$ (resp. $\mathbb{B}$) exists, then $\mathbb{B} = 0$ (resp. $\mathbb{A} = 0$).* **(ii).** *If the order 2 right inverse of a tensor $\mathbb{A}$ (resp. $\mathbb{B}$) exists, then $\mathbb{B} = 0$ (resp. $\mathbb{A} = 0$).*

Definition 2.10. We define a new class for tensors as follows:

$$\Gamma = \left\{ \begin{array}{l} \mathbb{A} \in \Re_+^{[m,n]} : \mathbb{A} = G \otimes \mathbb{I} = \mathbb{I} \otimes G, \\ \textit{where } G \textit{ is a generalized permutation matrix} \end{array} \right\}.$$

For example the unit tensor is belong to this class.

The following theorem is an interesting and fundamental extension of Theorem 2.1 for tensors, in which we charactrize the invertible tensors completely.

Theorem 2.11. *Let $\mathbb{A} \in \Re_+^{[m,n]}$. Then a matrix B such that*

$$\mathbb{A} \otimes B = \mathbb{I} = B \otimes \mathbb{A},$$

exists if and only if $\mathbb{A}$ is belong to Γ.

Proof. Let $\mathbb{A} \in \Gamma$, thus there exists a generalised permutation matrix G such that $\mathbb{A} = G \otimes \mathbb{I} = \mathbb{I} \otimes G$. By puting $B = G^T$, we will have $\mathbb{A} \otimes B = \mathbb{I} = B \otimes \mathbb{A}$. On the other hand, if $\mathbb{A} \otimes B = \mathbb{I} = B \otimes \mathbb{A}$, Theorems 2.7 and 2.8 conclude that $\mathbb{A}$ is belong to Γ. □

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SOLVING SYSTEM OF LINEAR FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS ON UNBOUNDED DOMAIN

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Abstract. The purpose of this study is to present a matrix method for solving system of linear Fredholm integro-differential equations of the second kind on unbounded domain with degenerate kernels in terms of generalized Laguerre polynomials. The method is based on the approximation by the truncated generalized Laguerre series.

1. Introduction

The main object of this paper is to approximate the solution system of Fredholm integro-differential equations of the second kind on a semi-infinite domain of the following form [1, 2]:

$$U'(x) = F(x) + \rho \int_0^\infty w(t)K(x,t)U(t)dt, \quad x \in \mathbb{R}_+, \tag{1.1}$$

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* Speaker.

along with initial condition $U(0) = A$, where $\rho \in R$, and

$$\begin{aligned} U(x) &= [u_1(x), u_2(x), ..., u_m(x)]^T, \\ F(x) &= [f_1(x), f_2(x), ..., f_m(x)]^T, \\ K(x,t) &= [k_{ij}],\ i,j = 1,2,...,m, \\ A &= [a_1, a_2, ..., a_m]^T. \end{aligned}$$

In system (1.1), $w(t) = t^\alpha e^{-t} (\alpha > -1)$ and the known kernel $K(x,t)$ might has singularity in the region $D = \{(x,t) : 0 \leq x,t < \infty\}$ and $F(x)$ is continuous function and A is fixed constant vector, and $U(x)$ is the unknown vector function of the solution that will be determined. Let $\Lambda = [0,\infty)$ and $w^{(\alpha)}(x) = x^\alpha e^{-x}$ be a weight function on Λ in the usual sense. We define

$$L^2_{w^{(\alpha)}}(\Lambda) = \{v : v \textit{ is measurable on } \Lambda \textit{ and } \|v\|_{w^{(\alpha)}} < \infty\},$$

equipped with the following inner product and norm:

$$(u,v)_{w^{(\alpha)}} = \int_\Lambda u(x)v(x)w^{(\alpha)}(x)dx,\ \ \|v\|_{w^{(\alpha)}} = (v,v)^{\frac{1}{2}}_{w^{(\alpha)}}.$$

Next, suppose $L_n^{(\alpha)}(x)$ be the generalized Laguerre polynomials of degree n, defined by the following:

$$L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^x \partial_x^n \left(e^{-x} x^{n+\alpha}\right),\ n = 0,1,....$$

Generalized Laguerre polynomials are orthogonal in $L^2_{w^{(\alpha)}}(\Lambda)$ Hilbert space with the weight function $w^{(\alpha)}(x) = x^\alpha e^{-x}$ satisfy in the following relation

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)dx = \gamma_n^\alpha \delta_{n,m},\ \forall n,m \geq 0,$$

where $\delta_{n,m}$ is the Kronecher delta function and $\gamma_n^\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}$.

2. Approximation of functions

A function $f(x) \in L^2_{w^{(\alpha)}}(\Lambda)$ may be expressed in terms of generalized Laguerre polynomials as:

$$f(x) = \sum_{i=0}^\infty f_i^{(\alpha)} L_i^{(\alpha)}(x),$$

where the generalized Laguerre coefficients $f_i^{(\alpha)}$ are given by

$$f_i^{(\alpha)} = \int_0^\infty \frac{L_i^{(\alpha)}(x)}{\binom{i+\alpha}{i}} \cdot \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)} \cdot f(x)dx,\ i = 0,1,....$$

In practice, only the first $(n+1)$ terms of generalized Laguerre polynomials are considered. Then we have

$$f(x) \simeq \sum_{i=0}^{n} f_i^{(\alpha)} L_i^{(\alpha)}(x) = F^T L_x,$$

$$F = [f_0^{(\alpha)}, f_1^{(\alpha)}, \ldots, f_n^{(\alpha)}]^T, \; L_x = [L_0^{(\alpha)}(x), L_1^{(\alpha)}(x), \ldots, L_n^{(\alpha)}(x)]^T. \quad (2.1)$$

We can also approximate the function of two variables, $k(x,t) \in L^2_{w^{(\alpha)}}(\Lambda^2)$ as follows:

$$k(x,t) \simeq \sum_{i=0}^{n}\sum_{j=0}^{n} L_i^{(\alpha)}(x) k_{ij}^{(\alpha)} L_j^{(\alpha)}(t) = L_x^T K L_t.$$

Here the entries of matrix $K = [k_{ij}^{(\alpha)}]_{(n+1)\times(n+1)}$ will be obtained by

$$k_{ij}^{(\alpha)} = \frac{(L_i^{(\alpha)}(x), (k(x,t), L_j^{(\alpha)}(t)))}{(L_i^{(\alpha)}(x), L_i^{(\alpha)}(x))(L_j^{(\alpha)}(t), L_j^{(\alpha)}(t))}, \quad for \; i,j = 0,1,...,n.$$

3. Main sections and results

Theorem 3.1. *Suppose L_x be the generalized Laguerre vector defined in (2.1) then*

$$\int_0^x L_t dt \simeq P L_x,$$

where P is the $(n+1)\times(n+1)$ operational matrix for integration as entries follows:

$$\Omega(i,j,\alpha) = \sum_{k=0}^{i}\sum_{r=0}^{j} \frac{(-1)^{k+r} j! \Gamma(i+\alpha+1)\Gamma(k+\alpha+r+2)}{(i-k)!(j-r)!(k+1)! r! \Gamma(k+\alpha+1)\Gamma(r+\alpha+1)}.$$

we consider the *ith* equation of (1.1) as follows:

$$u_i'(x) = f_i(x) + \rho \int_0^\infty t^\alpha e^{-t} \sum_{j=1}^{m} k_{ij}(x,t) u_j(t) dt, \; u_i(0) = a_i. \quad (3.1)$$

In order to approximate the solution of equation (3.1), we approximate functions $f_i(x)$, $u_i(x)$ and $k_{ij}(x,t)$ with respect to generalized Laguerre polynomials (basis) as mentioned in the previous section as follows:

$$f_i(x) \simeq F_i^T L_x, u_i'(x) \simeq C_i'^T L_x, u_i(0) \simeq C_{i0}^T L_x, k_{ij}(x,t) \simeq L_x^T K_{ij} L_t. \quad (3.2)$$

By substituting the approximations (3.2) into equation (3.1), we obtain:

$$C_i' = F_i + \rho \sum_{j=1}^{m} K_{ij} Q (P^T C_j' + C_{j0}), \quad i = 1,...,m, \quad (3.3)$$

where

$$Q = \int_0^\infty t^\alpha e^{-t} L_t L_t^T dt = [q_{ij}^{(\alpha)}], \quad i, j = 0, 1, ..., n.$$

By solving linear system of algebraic equations (3.3) by direct or iterative methods, we can achieve the vector C_i' for $i = 1, ..., m$, then we will have

$$C_i^T = C_i'^T P + C_{i0}^T \Longrightarrow u_i(x) \simeq C_i^T L_x, \ i = 1, ..., m.$$

That are the approximate solution for our system of (1.1).

Example 3.2. For the first example, consider the following system of linear Fredholm integro-differential equations on the half line (constructed):

$$\begin{aligned} u_1'(x) &= f_1(x) + \int_0^\infty t^{\frac{1}{2}} e^{-t}(2x + t^2)(u_1(t) + u_2(t))dt, \\ u_2'(x) &= f_2(x) + \int_0^\infty t^{\frac{1}{2}} e^{-t}(t - x^2)(u_1(t) - u_2(t))dt, \end{aligned} \tag{3.4}$$

where $f_1(x) = 3x^2 - \frac{87307746120759955}{2251799813685248}x - \frac{6631788499575074881}{18014398509481984}$ and $f_2(x) = \frac{9878247846805983 7}{9007199254740992}x^2 + 2x - \frac{853121404951425865}{18014398509481984}$. Subject to initial conditions $u_1(0) = 1$ and $u_2(0) = 1$. The exact solutions of this problem are $u_1(x) = x^3 + 2x + 1$ and $u_2(x) = x^2 + 1$. If we apply the presented method in this paper and solve equation (3.4) with $n = 3$. For this system we get:

$$\begin{aligned} u_1(x) &= \frac{137}{8}L_0(x) - \frac{113}{4}L_1(x) + 21L_2(x) - 6L_3(x) = x^3 + 2x + 1, \\ u_2(x) &= \frac{19}{4}L_0(x) - 5L_1(x) + 2L_2(x) + (0)L_3(x) = x^2 + 1, \end{aligned}$$

which is the exact solution.

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MATRIX POLYNOMIALS WITH k PRESCRIBED DISTINCT EIGENVALUES

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Abstract. Consider an $n \times n$ matrix polynomial $P(\lambda)$ and a set Σ consisting of $k \leq n$ distinct complex numbers. In this paper, a (weighted) perturbation of $P(\lambda)$ is constructed such that the perturbed matrix polynomial has the specified set Σ in its spectrum.

1. Introduction

In the last decades, the study of matrix polynomials, especially with regard to their spectral analysis, has received the attention of several researchers and has met many applications. Some basic references for the theory and applications of matrix polynomials are [2, 3] and references therein. In 2012, Psarrakos [4] introduced and studied a spectral norm distance from an $n \times n$ matrix polynomial $P(\lambda)$ to the set of $n \times n$ matrix polynomials that have a prescribed scalar $\mu \in \mathbb{C}$ as a eigenvalue of a desired algebraic multiplicity. In particular, he computed lower and upper bounds for this distance, constructing an associated perturbation of $P(\lambda)$ for the upper bound. In this paper, spectrum updating problems, we construct a perturbation of $P(\lambda)$ for which the perturbed matrix polynomial has $k \leq n$ prescribed distinct eigenvalues.

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* Speaker.

For $A_j \in \mathbb{C}^{n\times n}$ $(j = 0, 1, \ldots, m)$ and a complex variable λ, we define the *matrix polynomial*

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0 = \sum_{j=0}^{m} A_j\lambda^j. \qquad (1.1)$$

If for a scalar $\mu \in \mathbb{C}$ and some nonzero vector $v \in \mathbb{C}^n$, it holds that $P(\mu)v = 0$, then the scalar μ is called an *eigenvalue* of $P(\lambda)$ and the vector v is known as a *(right) eigenvector* of $P(\lambda)$ corresponding to μ. Similarly, a nonzero vector $\nu \in \mathbb{C}^n$ is known as a *(left) eigenvector* of $P(\lambda)$ corresponding to μ when $\nu^*P(\mu) = 0$. The *spectrum* of $P(\lambda)$, denoted by $\sigma(P)$, is the set of its eigenvalues.

The multiplicity of an eigenvalue $\lambda_0 \in \sigma(P)$ as a root of the scalar polynomial $\det P(\lambda)$ is called the *algebraic multiplicity* of λ_0, and the dimension of the null space of the (constant) matrix $P(\lambda_0)$ is known as the *geometric multiplicity* of λ_0. The singular values of $P(\lambda)$ are the nonnegative roots of the eigenvalue functions of $P(\lambda)^*P(\lambda)$, and they are denoted by $s_1(P(\lambda)) \geq s_2(P(\lambda)) \geq \cdots \geq s_n(P(\lambda))$ (i.e., they are considered in a nondecreasing order).

Definition 1.1. Let $P(\lambda)$ be a matrix polynomial as in (1.1) and let $\Delta_j \in \mathbb{C}^{n\times n}$ $(j = 0, 1, \ldots, m)$ be arbitrary matrices. Consider perturbations of the matrix polynomial $P(\lambda)$ of the form

$$Q(\lambda) = P(\lambda) + \Delta(\lambda) = \sum_{j=0}^{m} (A_j + \Delta_j)\lambda^j. \qquad (1.2)$$

For $\varepsilon > 0$ and a set of given nonnegative weights $w = \{\omega_0, \ldots, \omega_m\}$, with $\omega_0 > 0$, define the class of admissible perturbed matrix polynomials

$$\mathcal{B}(P, \varepsilon, w) = \{Q(\lambda) \text{ as in (1.2)} : \|\Delta_j\| \leq \varepsilon\omega_j,\ j = 0, 1, \ldots, m\},$$

and the scalar polynomial $w(\lambda) = \omega_m\lambda^m + \omega_{m-1}\lambda^{m-1} + \cdots + \omega_1\lambda + \omega_0$.

Definition 1.2. Consider a complex function f and k distinct scalars $\mu_1, \mu_2, \ldots, \mu_k \in \mathbb{C}$. The *divided difference relative to μ_i and μ_{i+t}* $(1 \leq i \leq k-1,\ 1 \leq t \leq k-i)$ is denoted by $f[\mu_i, \mu_{i+1}, \ldots, \mu_{i+t}]$ and is defined by the following recursive formula [1]:

$$f[\mu_i, \ldots, \mu_{i+k}] = \frac{f[\mu_i, \mu_{i+1}, \ldots, \mu_{i+k-1}] - f[\mu_{i+1}, \mu_{i+2}, \ldots, \mu_{i+k}]}{\mu_i - \mu_{i+k}}.$$

Definition 1.3. Suppose that $P(\lambda)$ is a matrix polynomial as in (1.1) and a set of distinct complex numbers $\Sigma = \{\mu_1, \mu_2, \ldots, \mu_k\}$ $(k \leq n)$ is

given. For any scalar $\gamma \in \mathbb{C}$, define the $nk \times nk$ matrix

$$F_\gamma\left[P,\Sigma\right] = \begin{bmatrix} P(\mu_1) & 0 & \cdots & 0 \\ \gamma P[\mu_1,\mu_2] & P(\mu_2) & \cdots & 0 \\ \gamma^2 P[\mu_1,\mu_2,\mu_3] & \gamma P[\mu_2,\mu_3] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^{k-1}P[\mu_1,\ldots,\mu_k] & \gamma^{k-2}P[\mu_2,\ldots,\mu_k] & \cdots & P(\mu_k) \end{bmatrix}.$$

2. Construction of a perturbation

In this section, we construct an $n \times n$ matrix polynomial $\Delta_\gamma(\lambda)$ such that the given set $\Sigma = \{\mu_1, \mu_2, \ldots, \mu_k\}$ $(k \leq n)$ is included in the spectrum of the perturbed matrix polynomial $Q_\gamma(\lambda) = P(\lambda) + \Delta(\lambda)$. Without loss of generality, hereafter we can assume that the parameter γ is real nonnegative [4]. Also, for convenience, we set $\rho = nk - k + 1$.

Definition 2.1. Suppose that

$$u(\gamma) = \begin{bmatrix} u_1(\gamma) & \cdots & u_k(\gamma) \end{bmatrix}^T, v(\gamma) = \begin{bmatrix} v_1(\gamma) & \cdots & v_k(\gamma) \end{bmatrix}^T \in \mathbb{C}^{nk},$$

is a pair of left and right singular vectors of $s_\rho\left(F_\gamma\left[P,\Sigma\right]\right)$, respectively, such that $(u_j(\gamma), v_j(\gamma) \in \mathbb{C}^n; j = 1,\ldots,k)$. Define the $n \times k$ matrices

$$U(\gamma) = \left[\, u_1(\gamma) \ \cdots \ u_k(\gamma) \,\right] \quad \text{and} \quad V(\gamma) = \left[\, v_1(\gamma) \ \cdots \ v_k(\gamma) \,\right].$$

Suppose that $\gamma > 0$ and $\operatorname{rank}(V(\gamma)) = k$. Consider the quantities

$$\theta_{ij} = \frac{\gamma}{\mu_i - \mu_j}, \quad 1 \leq i < j \leq k,$$

and for $p = 2, 3, \ldots, k$ define the following vectors

$$\hat{v}_1(\gamma) = v_1(\gamma), \quad \hat{v}_p(\gamma) = v_p(\gamma) + \sum_{i=1}^{p-1} \left[(-1)^i \left(\prod_{j=p-i}^{p-1} \theta_{jp} \right) v_{p-i}(\gamma) \right],$$

the vectors $\hat{u}_p(\gamma), p = 1,\ldots,p$ are defined similarly. Analogously to Definition 2.1, we define the $n \times k$ matrices

$$\hat{U}(\gamma) = \left[\, \hat{u}_1(\gamma) \ \cdots \ \hat{u}_k(\gamma) \,\right] \quad \text{and} \quad \hat{V}(\gamma) = \left[\, \hat{v}_1(\gamma) \ \cdots \ \hat{v}_k(\gamma) \,\right].$$

We also consider the quantities

$$\alpha_{i,s} = \frac{1}{w\left(|\mu_i|\right)} \sum_{j=0}^{m} \left(\left(\frac{\bar{\mu}_i}{|\mu_i|} \right)^j \mu_s^j \omega_j \right) \text{ and } \beta_s = \frac{1}{k} \sum_{i=1}^{k} \alpha_{i,s}, \ i,s = 1,\ldots,k,$$

where we set $\alpha_{i,s} = 0$ whenever $\mu_i = 0$. Then, define the $n \times n$ matrix

$$\Delta_\gamma = -s_\rho(F_\gamma[P,\Sigma])\hat{U}(\gamma)\,\operatorname{diag}\left\{ \frac{1}{\beta_1}, \frac{1}{\beta_2}, \ldots, \frac{1}{\beta_k} \right\} \hat{V}(\gamma)^\dagger,$$

where $\hat{V}(\gamma)^\dagger$ denotes the *Moore-Penrose pseudoinverse* of $\hat{V}(\gamma)$, and the $n \times n$ matrix polynomial $\Delta_\gamma(\lambda) = \sum_{j=0}^{m} \Delta_{\gamma,j}\lambda^j$, where

$$\Delta_{\gamma,j} = \frac{1}{k}\sum_{i=1}^{k}\left(\frac{1}{w(|\mu_i|)}\left(\frac{\bar{\mu}_i}{|\mu_i|}\right)^j \omega_j \Delta_\gamma\right), \quad j = 1, 2, \ldots, k. \tag{2.1}$$

By straightforward computations, we verify that the matrix polynomial $\Delta_\gamma(\lambda)$ satisfies $\Delta_\gamma(\mu_i) = \beta_i \Delta_\gamma$, for $i = 1, \ldots, k$. Notice that $\text{rank}(V(\gamma)) = k$ implies $\hat{v}_i(\gamma) \neq 0$, $(i = 1, \ldots, k)$ and $\hat{V}(\gamma)^\dagger \hat{V}(\gamma) = I_k$, where I_k denotes the $k \times k$ identity matrix.

Moreover, since $u(\gamma), v(\gamma)$ is a pair of left and right singular vectors of $s_\rho(F_\gamma[P, \Sigma])$, we have $F_\gamma[P, \Sigma]v(\gamma) = s_\rho(F_\gamma[P, \Sigma])\, u(\gamma)$. Substituting $\hat{u}_1(\gamma), \ldots, \hat{u}_k(\gamma)$ and $\hat{v}_1(\gamma), \ldots, \hat{v}_k(\gamma)$ into these equations yields

$$s_\rho(F_\gamma[P, \Sigma])\, \hat{u}_i(\gamma) = P(\mu_i)\, \hat{v}_i(\gamma), \quad i = 1, 2, \ldots, k.$$

Therefore, for the matrix polynomial

$$Q_\gamma(\lambda) = P(\lambda) + \Delta_\gamma(\lambda) = \sum_{j=0}^{m}(A_j + \Delta_{\gamma,j})\lambda^j \tag{2.2}$$

and for every $i = 1, 2, \ldots, k$, it follows

$$Q_\gamma(\mu_i)\, \hat{v}_i(\gamma) = P(\mu_i)\, \hat{v}_i(\gamma) + \Delta_\gamma(\mu_i)\, \hat{v}_i(\gamma) = 0.$$

As a consequence, if $\text{rank}(V(\gamma)) = k$, then $\mu_1, \mu_2, \ldots, \mu_k$ are eigenvalues of the matrix polynomial $Q_\gamma(\lambda)$ in (2.2) with $\hat{v}_1(\gamma), \hat{v}_2(\gamma), \ldots, \hat{v}_k(\gamma)$ as their associated eigenvectors, respectively.

Theorem 2.2. *Consider a matrix polynomial $P(\lambda)$ as in (1.1) and a given set of $k \leq n$ distinct complex numbers $\Sigma = \{\mu_1, \mu_2, \ldots, \mu_k\}$. For every $\gamma > 0$ such that rank$(V(\gamma)) = k$, the scalars $\mu_1, \mu_2, \ldots, \mu_k$ are eigenvalues of the matrix polynomial $Q_\gamma(\lambda)$ in (2.2), with corresponding eigenvectors $\hat{v}_1(\gamma), \hat{v}_2(\gamma), \ldots, \hat{v}_k(\gamma)$, respectively.*

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A NEW METHOD FOR SOLVING GENERALIZED SADDLE POINT PROBLEMS

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Abstract. A new method is presented for solving generalized saddle point problems. In this method, the problem is split into two smaller subsystems. One of them is solved directly by the Cholesky factorization and the other by an iterative method. Convergence of the method is investigated. Some numerical are given to show the effectiveness of the method.

1. Introduction

We consider large and sparse saddle point problems of the form

$$\mathcal{A}u \equiv \begin{pmatrix} A & B^T \\ -B & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} \equiv b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is non-symmetric positive definite ($x^TAx > 0$ for all $0 \neq x \in \mathbb{R}^n$), $C \in \mathbb{R}^{m\times m}$ is symmetric positive semidefinite and $B \in \mathbb{R}^{m\times n}$, $m \leq n$, is of full rank. Moreover, $x, f \in \mathbb{R}^n$ and $y, g \in \mathbb{R}^m$. Such systems arise in a variety of scientific computing such as mixed finite element discretization of the Navier-Stokes equations [3]. When $C = 0$ and the matrix A is symmetric positive definite, Li et al. in [4] proposed a preconditioner of the form

$$\mathcal{P}_\alpha = \begin{pmatrix} I_n & -B^T(BB^T)^{-1} \\ 0 & I_m \end{pmatrix} \begin{pmatrix} I_n & 0 \\ B & -\alpha BB^T \end{pmatrix}, \tag{1.2}$$

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for the system (1.1) and investigated its properties. Here, I_m and I_n are the identity matrices of dimension m and n, respectively, and $\alpha > 0$. Recently, Zhu et al. have used the preconditioner $\mathcal{P}_\alpha$ for the system (1.1) when $C = 0$ and A is positive definite and analyzed its properties. In this paper, we generalized the preconditioner $\mathcal{P}_\alpha$ to the system (1.1).

2. Generalization of the preconditioner

Let

$$\mathcal{P}_{\alpha,\gamma} = \begin{pmatrix} I_n & -B^T(\gamma I_m + \alpha C)^{-1}(BB^T)^{-1} \\ 0 & I_m \end{pmatrix} \begin{pmatrix} I_n & 0 \\ \gamma B & \alpha BB^T \end{pmatrix}, \quad (2.1)$$

where $\alpha, \gamma > 0$. Premultiplying both sides of the system (1.1) by $\mathcal{P}_{\alpha,\gamma}$ yields

$$\begin{pmatrix} \tilde{A} & 0 \\ \tilde{B} & \tilde{C} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tilde{f} \\ \tilde{g} \end{pmatrix}, \quad (2.2)$$

where

$$\begin{aligned} \tilde{A} &= A + \alpha B^T(\gamma I_m + \alpha C)^{-1}B - \gamma B^T(\gamma I_m + \alpha C)^{-1}(BB^T)^{-1}BA, \\ \tilde{B} &= B(\gamma A - \alpha B^T B), \quad \tilde{C} = BB^T(\gamma I_m + \alpha C), \\ \tilde{f} &= f - B^T(\gamma I_m + \alpha C)^{-1}\left[\gamma(BB^T)^{-1}Bf - \alpha g\right], \quad \tilde{g} = B(\gamma f - \alpha B^T g). \end{aligned}$$

System (2.2) is equivalent to

$$\begin{cases} (M_{\alpha,\gamma} - N_{\alpha,\gamma})\, x = \tilde{f}, \\ BB^T(\gamma I_m + \alpha C)y = \tilde{g} - B(\gamma A - \alpha B^T B)x, \end{cases} \quad (2.3)$$

where $M_{\alpha,\gamma} = A + \alpha B^T(\gamma I_m + \alpha C)^{-1}B$ and $N_{\alpha,\gamma} = \gamma B^T(\gamma I_m + \alpha C)^{-1}(BB^T)^{-1}BA$. From Eq. (2.3) we can compute the vector x from the first equation and then from the second equation we can obtain the vector y. Both of the matrices BB^T and $\gamma I_m + \alpha C$ are symmetric positive definite and of small size. Hence, we can use the Cholesky factorization of these matrices to compute y. Therefore, all we need is to present a method to compute the vector x from the first equation in (2.3). To do so, we propose the following stationary iterative method

$$M_{\alpha,\gamma}x^{(k+1)} = N_{\alpha,\gamma}x^{(k)} + \tilde{f}, \quad (2.4)$$

where $x^{(0)}$ is an initial guess. Since A is positive definite and the matrix $\alpha B^T(\gamma I_m + \alpha C)^{-1}B$ is symmetric positive definite, it follows that the matrix $M_{\alpha,\gamma}$ is positive definite. Hence, one may use a direct method such as the LU factorization or an iterative method such as the GMRES(m) algorithm (see [3]) to solve the system with the coefficient matrix $M_{\alpha,\gamma}$. Hence all it remains is to investigate the convergence of the stationary iterative method (2.4).

Theorem 2.1. *For a fixed value $\alpha > 0$, $\rho(M_{\alpha,\gamma}^{-1}N_{\alpha,\gamma}) \to 0$ as $\gamma \to 0$.*

Proof. Let (λ, x) be an eigenpair of $G_{\alpha,\gamma} = M_{\alpha,\gamma}^{-1}N_{\alpha,\gamma}$ with $\|x\|_2 = 1$. Then

$$| \lambda | = \frac{| x^H N_{\alpha,\gamma} x |}{| x^H M_{\alpha,\gamma} x |} \leq \frac{| x^H B^T (I_m + rC)^{-1} (BB^T)^{-1} BAx |}{\Re(x^H Ax) + r x^H B^T (I_m + rC)^{-1} Bx},$$

where $r = \alpha/\gamma$. Let $\mu_1 = \cdots = \mu_k = 0 < \mu_{k+1} \leq \cdots \leq \mu_m$ be the eigenvalues of C. Since C is symmetric, there is an orthogonal matrix U such that $C = U^T D U$, where $D = \mathrm{diag}(\mu_1, \ldots, \mu_m)$. Now, we have

$$\begin{aligned}
&| x^H B^T (I_m + rC)^{-1} (BB^T)^{-1} BAx |^2 \\
&\quad \leq \|Ax\|_2^2 \|B^T (BB^T)^{-1} (I_m + rC)^{-1} Bx\|_2^2 \\
&\quad \leq \|A\|_2^2 \left(x^H B^T (I_m + rC)^{-1} (BB^T)^{-1} (I_m + rC)^{-1} Bx \right) \\
&\quad \leq \frac{\|A\|_2^2}{\sigma_{\min}^2(B)} \left(x^H B^T (I_m + rC)^{-2} Bx \right) \\
&\quad = \frac{\|A\|_2^2}{\sigma_{\min}^2(B)} \left(w^H (I_m + rD)^{-2} w \right),
\end{aligned}$$

where $w = UBx \neq 0$ and $\sigma_{\min}(B)$ is the smallest singular value of B. On the other hand, we see that

$$r x^H B^T (I_m + rC)^{-1} Bx = r w^H (I_m + rD)^{-1} w.$$

We have

$$\lim_{r\to\infty} w^H (I_m + rD)^{-1} w = \lim_{r\to\infty} w^H (I_m + rD)^{-2} w = \|\tilde{w}\|_2^2, \tag{2.5}$$

where $\tilde{w} = (w_1, \ldots, w_k, 0, \ldots, 0)^T \in \mathbb{C}^m$. Hence

$$| \lambda | \leq \frac{\|A\|_2}{\sigma_{\min}(B)} \frac{\sqrt{w^H (I_m + rD)^{-2} w}}{\Re(x^H Ax) + r w^H (I_m + rD)^{-1} w}.$$

From Eq. (2.5) we deduce that the right-hand side of the latter inequality tends to zero as $r \to \infty$ (for α fixed and $\gamma \to 0$) and this completes the proof. □

From Theorem 2.1, it follows that for fixed α and a sufficiently small value of γ, we have $\rho(M_{\alpha,\gamma}^{-1}N_{\alpha,\gamma}) < 1$ which guarantees the convergence of the method.

3. Numerical experiments

We consider the steady-state Navier-Stokes equation

$$\begin{cases} -\nu \triangle \mathbf{u} + (\mathbf{u}.\nabla)\mathbf{u} + \nabla p &= \mathbf{f}, \\ \nabla.\mathbf{u} &= 0, \end{cases} \quad \text{in} \quad \Omega = [0,1] \times [0,1],$$

TABLE 1. Numerical results for the test problem.

		Proposed method $(\alpha,\gamma)=(1,10^{-6})$		IHSS method		
Grid	(n,m)	Iters	CPU	α	Iters	CPU
8×8	(162,62)	3	0.02	0.07	233	0.83
16×16	(578,256)	8	0.30	0.03	281	3.55
32×32	(2178,1024)	5	6.36	0.02	557	22.23
64×64	(8450,4096)	7	40.11	-	-	-

where $\nu > 0$. By the IFISS package [2], this problem is linearized by the Picard iteration and then discretized by using stabilized Q1-P0 finite elements (See [3]). This yields a generalized saddle point problem of the form (1.1). In the implementation of the proposed method the linear system with the coefficient matrix $M_{\alpha,\gamma}$ is solved by the GMRES(50) algorithm with the stoping criterion $\|r_k\|_2 < 10^{-7}\|r_0\|_2$ (r_k is the residual vector) and the remaining systems are solved directly by the Cholesky factorization. We use the stopping criterion $\delta_k = \|x^{(k)} - x^{(k-1)}\|_\infty < 10^{-7}$. The right-hand side vectors f and g are taken such that x and y are two vectors of all ones. Matrix properties of the test problem for different sizes together with the numerical results are given in Table 1. In this table "Iters" and "CPU" stand for the number of iterations and CPU time (second). To show the effectiveness of the proposed method the obtained results are compared with that of the inexact Hermitian and skew-Hermitian (IHSS) method proposed by Benzi and Golub in [1]. As seen, the proposed method is superior to the IHSS method in terms of both CPU time and iterations. For the grid of dimension 64×64 and for best value α the IHSS method does not converge in 1000 iterations and we have $\delta_{1000} = 0.009$.

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CONGRUENCE NUMERICAL RANGE OF MATRICES

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Abstract. In this paper, some algebraic and geometrical properties of the $C-$congruence numerical range of matrices are studied. The emphasis is on the study of some geometrical properties for the special case $C = diag(1, 0, \ldots, 0)$.

1. Introduction

Let M_n be the algebra of all $n \times n$ complex matrices, and let $\mathcal{U}_n$ be the group of unitary matrices in M_n. Suppose that $A, C \in M_n$. The notion of $C-$congruence numerical range of A was first introduced by R.C. Thompson in 1980, e.g., see [5], as follows:

$$R_C(A) = \{tr(CU^TAU) \ : \ U \in \mathcal{U}_n\}. \tag{1.1}$$

In fact, Thompson replaced the conjugate transpose of the underlying matrix in the definition of $C-$numerical range, which has important application in quantum control and quantum information, e.g., see [1] and its references, by the transpose in order to give the definition of the corresponding $C-$congruence numerical range.
In this note, we are going to study some algebraic and geometrical properties of the $C-$congruence numerical range of matrices. For this,

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* Speaker.

in Section 2, Theorem 2.1 characterizes the shape of this set for the case $n \geq 3$. Also, we introduce the notions of congruence zero set and the congruence level sets related to the $E_{11}-$congruence numerical range of matrices, where $\{E_{ij} : 1 \leq i, j \leq n\}$ is the standard basis of M_n. The emphasis is on the study the connectivity of the congruence zero set.

2. Main results

The following theorem can be found in [2].

Theorem 2.1. *Let $A, C \in M_n$. If $n \geq 3$, then $R_C(A)$ is a circular disk centered at the origin in the complex plane.*

For the case that $n = 2$ and $C \in M_2$ is arbitrary, we have no characterization of the shape of $R_C(A)$. But if we consider $C = E_{11} \in M_n$, then by the result from [4], we have:

$$R(A) := R_{E_{11}}(A) = \begin{cases} \{z \in \mathbb{C} : |z| \leqq \sigma\} & if \quad n \geq 2 \\ \{z \in \mathbb{C} : |z| = \sigma\} & if \quad n = 1, \end{cases}$$

where σ is the largest singular values of $\frac{1}{2}(A + A^T)$.

Now, we introduce and study the notions of the congruence zero set and the congruence level sets of matrices.

Definition 2.2. For $A \in M_n$,
(i) the congruence zero set of A is defined and denoted by

$$Z(A) = \{x \in \mathbb{C}^n \ : \ x \neq 0, \ x^T A x = 0\};$$

(ii) Let $\lambda \in R(A)$. The congruence level set of A corresponding to λ is defined as

$$LS(A, \lambda) = \{x \in \mathbb{C}^n \ : \ x^T A x = \lambda\}.$$

Example 2.3. let A be the Jordan block of order 2 with eigenvalue 0, that is, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. In view of Definition 2.2, we have

$$Z(A) = \{(x, 0)^T \ : \ x \in \mathbb{C}, \ x \neq 0\} \ \cup \ \{(0, x)^T \ : \ x \in \mathbb{C}, \ x \neq 0\}.$$

Also, we have

$$LS(A, 0) = \{(x, 0)^T \ : \ x \in \mathbb{C}, |x| = 1\} \ \cup \ \{(0, x)^T \ : \ x \in , |x| = 1\}.$$

By the fact that $0 \in R(A)$, we have $LS(A, 0) \neq \emptyset$. Also, since $LS(A, 0) \subseteq Z(A)$, $Z(A) \neq \emptyset$. Furthermore, we have the following important properties of $LS(A, \lambda)$ and $Z(A)$.

Theorem 2.4. *Let $A \in M_n$, and $\lambda \in R(A)$. Then the following assertions are true:*
(i) $LS(A,\lambda) = LS(\frac{A+A^T}{2},\lambda)$;
(ii) $LS(e^{it}A,\lambda) = e^{-i\frac{t}{2}}LS(A,\lambda) = LS(A,e^{-it}\lambda)$ *for all* $t \in \mathbb{R}$;
(iii) $Z(A) = Z(\frac{A+A^T}{2})$.

Proof. By Definition 2.2, and a simple calculation, the results can be easily verified. □

Remark 2.5. Let $A \in M_n$. We know that $\frac{A+A^T}{2}$ is a symmetric matrix. So, by Theorem 2.4((i), (iii)), to study the algebraic and topological properties of $LS(A,\lambda)$ and $Z(A)$, it is enough to consider the case that A is a symmetric matrix.

Next, we study the relationship between the connectivity of $Z(A)$ and $LS(A,0)$. For this, we need the following lemma, which can be found in [3].

Lemma 2.6. *Let X be a topological space, and $C \subseteq X$ be a connected set. Suppose that for every $x \in C$ there exists a connected subset Y_x of X such that $x \in Y_x$ and $X = \bigcup_{x\in C} Y_x$. Then X is connected.*

Theorem 2.7. *Let $A \in M_n$. Then the following assertions are true:*
(i) Let $U \in \mathcal{U}_n$. Then $Z(A)$ is connected if and only if $Z(U^TAU)$ is connected;
(ii) $Z(A)$ is connected if and only if $LS(A,0)$ is connected.

Proof. (i); We know that $UU^* = I_n$. So, if $x \in Z(A)$, then

$$\begin{aligned}(U^*x)^T(U^TAU)(U^*x) &= x^T(UU^*)^TA(UU^*)x\\ &= x^TAx\\ &= 0,\end{aligned}$$

and hence $U^*x \in Z(U^TAU)$. Therefore, the mapping

$$f : Z(A) \longrightarrow Z(U^TAU)$$
$$f(x) = U^*x$$

is well defined. Since U is unitary, f is one by one, onto, and $\|f(x) - f(y)\| = \|x - y\|$ for all $x, y \in Z(A)$. So, f is homeomorphism, and hence the result holds.
(ii); The normalization $x \longrightarrow \frac{x}{\|x\|}$ is a continuous mapping from $Z(A)$ onto $LS(A,0)$. So, $LS(A,0)$ is connected if $Z(A)$ is connected.
Conversely, assume that $LS(A,0)$ is connected. It is easy to see that

$$Z(A) = \bigcup_{x\in LS(A,0)} \{\alpha x \ : \ \alpha > 0\}.$$

So, the result follows from Lemma 2.6, by setting $X = Z(A)$ and $C = LS(A, 0)$. □

At the end of this section, we study the connectivity of the congruence zero set for 2×2 complex matrices.

Theorem 2.8. *Let $A \in M_2$, and $\sigma_1 \geq \sigma_2 \geq 0$ be the singular values of the matrix $\frac{A+A^T}{2}$. Then $Z(A)$ is connected if and only if $\sigma_2 = 0$.*

Proof. By Theorem 2.4(iii), we have $Z(A) = Z(B)$, where $B = \frac{A+A^T}{2}$. Since B is a symmetric matrix, Takagi's factorization of B implies that there exists a unitary matrix $U \in M_2$ such that $B = U^T \Sigma U$, where $\Sigma = diag(\sigma_1, \sigma_2)$. So, by Theorem 2.7(i), $Z(A) = Z(B)$ is connected if and only if $Z(\Sigma)$ is connected. Assume that $\sigma_2 > 0$. By Definition 2.2, we have:

$$\begin{aligned} Z(\Sigma) &= \{(x,y) \in \mathbb{C}^2 \ : \ (x,y) \neq (0,0),\ \sigma_1 x^2 + \sigma_2 y^2 = 0\} \\ &= \{(x,y) \in \mathbb{C}^2 \ : \ (x,y) \neq (0,0) \\ &, \quad (\sqrt{\sigma_1}x - i\sqrt{\sigma_2}y)(\sqrt{\sigma_1}x + i\sqrt{\sigma_2}y) = 0\} \\ &= \{(x, \frac{\sqrt{\sigma_1}}{\sqrt{\sigma_2}}\ ix) \ : \ x \in \mathbb{C}, x \neq 0\} \\ &\cup \quad \{(x, -\frac{\sqrt{\sigma_1}}{\sqrt{\sigma_2}}\ ix) \ : \ x \in \mathbb{C}, x \neq 0\}. \end{aligned}$$

Since two sets in the last equality are nonempty and separated, $Z(\Sigma)$ is disconnected. Hence, if $Z(A)$ is connected, then $\sigma_2 = 0$.
Conversely, let $\sigma_2 = 0$. Then by Definition 2.2, we have either $Z(\Sigma) = \mathbb{C}^2 \setminus \{0\}$ or $Z(\Sigma) \cong \mathbb{C} \setminus \{0\}$. So, $Z(A) = Z(\Sigma)$ is connected. □

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BAND MATRIX AND NON-POLYNOMIAL SPLINE

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Abstract. We investigate the monotonicity and inverse of band matrices which are used in the study of convergence analysis for finite difference, spline and non-polynomial spline methods. The application of the results is discussed in two-point boundary value problem and some numerical results are given to illustrate the efficiency of method.

1. Introduction

In this section we recall some details concerning band matrices. First we define a band matrix [1, 2, 3].

Definition 1: The matrix $M_{n\times n}$ is called a band matrix if integers r and s, $1 < r, s < n$ exist with the property that $m_{ij} = 0$, whenever $i - j \geq s, j - i \geq r$. The bandwidth w for such a matrix is defined to be $w = r + s - 1$.

Definition 2: Let $A = (a_{ij}) \in R_{n\times n}$ The matrix A is said to be *monotone* if $AX \geq 0$ implies $X \geq 0$, where $X = [x_1, x_2, ..., x_n]^t$. The matrix A is monotone if and only if the elements of the inverse matrix A^{-1} are nonnegative (see [2] pages 202-203.)

Band matrices have been used in the error analysis of numerical solutions of two-point boundary value problems in ordinary differential equations by the finite difference and spline methods [4, 5]. Whenever

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finally we faced the linear system of equations we usually require determinants, inverses, and bounds for inverses of these matrices.
Let us consider a mesh with nodal points x_i on $[a, b]$ such that:

$$\Delta : a = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = b, x_i = a + ih, \tag{1.1}$$

where $h = \frac{b-a}{n+1}$ for $i = 0(1)n + 1$.

Let $P_n(x, z, y)$, be a tridiagonal matrix given by

$$P_n(x, z, y) = \begin{pmatrix} z & -y & & & \\ -x & z & -y & & \\ & \ddots & \ddots & \ddots & \\ & & -x & z & -y \\ & & & -x & z \end{pmatrix}, \tag{1.2}$$

Denoting $D_n = \det[P_n(x, z, y)]$, then (1.2) the expansion of the determinant in terms of elements of the last row leads to the following recurrence relation

$$D_n = zD_{n-1} - xyD_{n-2}, n = 1, 2, ..., \quad D_{-1} = 0, D_0 = 1, (z > 0). \tag{1.3}$$

Several new results can now be derived directly from the difference equation (1.3)

Lemma 1. If $z - xy \neq 1$ then $\sum_{n=0}^{i-1} D_n = \frac{xyD_{i-1} - D_i + 1}{1 - z + xy}$.

Lemma 2. $D_n = \det[P_n(x, z, y)] = l_1 \times l_2 \times ... \times l_n$, $D_i = l_i D_{i-1}, i = 1, 2, ..., n$ where $l_i = z - \frac{xy}{l_{i-1}}, i \geq 2, l_1 = z$.

Theorem 1. If $R = P^n + \lambda P^{n-1}$, where P is given by (1.2) and $\lambda \in \mathbb{R}^+, n \in \mathbb{N}$, then R is a monotone matrix and

$$R^{-1} = \frac{1}{\lambda}[P^{-(n-1)} - \frac{P^{-(n-2)}}{\lambda}(I + \frac{P}{\lambda})^{-1}], \quad \|R^{-1}\| \leq \frac{1}{\lambda}(\frac{(b-a)^2}{8h^2})^{(n-1)},$$

provided $\rho(\frac{P}{\lambda}) < 1$.

By considering $F(z) = P_n(-1, z, -1)$ and $G(z') = P_n(1, z', 1)$ then we have the following lemmas.

Lemma 3. If we consider the matrices $F(z)$ and $G(z')$ then:

$$(F^n(z)G^n(z'))^{-1} = \frac{1}{(z + z')^n} \sum_{i=0}^{n} \binom{n}{i} F^{-(n-i)}(z)G^{-i}(z'), z \neq -z'. \tag{1.4}$$

Lemma 4. By consider the matrices $F(z)$ and $G(z')$ then we have:

$$(F^n(z)F^n(z'))^{-1} = \frac{1}{(z - z')^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} F^{-(n-i)}(z')F^{-i}(z), z \neq z'.$$

2. Aplication

We consider the nonlinear two-point boundary value problem

$$u^{(10)} = f(x,u), u(a) = \lambda, u(b) = \mu, a \leq x \leq b. \tag{2.1}$$

with boundary conditions:

$$\begin{cases} u(a) = \lambda_1, u^{(2)}(a) = \lambda_2, u^{(4)}(a) = \lambda_3, u^{(6)}(a) = \lambda_4, u^{(8)}(a) = \lambda_5, \\ u(b) = \mu_1, u^{(2)}(b) = \mu_2, u^{(4)}(b) = \mu_3, u^{(6)}(b) = \mu_4, u^{(8)}(b) = \mu_6, \end{cases} \tag{2.2}$$

where $f(x,u)$ is sufficiently differentiable on $[a,b]$ and $a, b, \lambda_i, (i = 1,2,3,4,5)$ and $\mu_i, (i = 1,2,3,4,5)$, are arbitrary real finite constants. For each segment $[x_i, x_{i+1}], i = 0,1,2,...,n-1$ by using the non-polynomial spline relation derived in [5] we have

$$\begin{aligned}[(u_{i+5} + u_{i-5}) - 10(u_{i+4} + u_{i-4}) + 45(u_{i+3} + u_{i-3}) - 120(u_{i+2} + u_{i-2}) + \\ 210(u_{i+1}+u_{i-1}) - 252u_i] = h^{10}[\alpha(l_{i+5}+l_{i-5}) + \beta(l_{i+4}+l_{i-4}) + \gamma(l_{i+3}+l_{i-3}) \\ + \delta(l_{i+2} + l_{i-2}) + \eta(l_{i+1} + l_{i-1}) + \eta l_i], i = 5,6,...,n-5. \end{aligned} \tag{2.3}$$

At the mesh point x_i the proposed differential equation (2.1) may be discretized by:

$$l_i = f(x_i, u_i), \tag{2.4}$$

To obtain unique solution for the nonlinear system (2.3) we need eight more equations to be associate, so that we use the boundary conditions. To obtain the eight-order boundary formula we use equations (12)-(19) in [5].

3. Numerical results

We Consider the following boundary-value problem

$$u^{(10)}(x) + u(x) = -10(2x\sin(x) - 9\cos(x)), -1 \leq x \leq 1.$$

The exact solution of this problem is $(x^2-1)\cos(x)$, and the boundary conditions (2.2) obtain by the exact solution and [5]. We applied our presented method to solve this problem with different value of parameters $\alpha, \beta, \gamma, \delta, \eta, \tau$ in [5].
Method (I) for $\alpha = 0, \beta = 0, \gamma = 0, \delta = \frac{7}{144}, \eta = \frac{2}{9}$, and $\tau = \frac{11}{24}$.
Method (II) for $\alpha = 0, \beta = 0, \gamma = \frac{17}{12096}, \delta = \frac{9}{224}, \eta = \frac{109}{448}$, and $\tau = \frac{1301}{3024}$.
Method (III) for $\alpha = 0, \beta = \frac{1}{362880}, \gamma = \frac{251}{181440}, \delta = \frac{913}{22680}, \eta = \frac{44117}{181440}$, and $\tau = \frac{15619}{36288}$.
Method (IV) for $\alpha = \frac{1}{47900160}, \beta = \frac{61}{23950080}, \gamma = \frac{22103}{15966720}, \delta = \frac{11477}{285120}, \eta = \frac{215687}{887040}$, and $\tau = \frac{1718069}{3991680}$.

TABLE 1. Observed maximum absolute errors for $n = 16, 32$

n	Method (I)	Method (II)	Method (III)	Method (IV)
16	1.66×10^{-6}	8.67×10^{-7}	2.82×10^{-6}	4.24×10^{-6}
32	5.01×10^{-9}	3.01×10^{-9}	7.51×10^{-9}	9.52×10^{-9}
Order	8.3	8.7	8.5	8.8

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LINEAR ALGEBRAIC SYSTEMS ARISING FROM THE KURAMOTO–SIVASHINSKY EQUATION

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Abstract. In this paper, we obtain algebraic systems from solving Kuramoto−Sivashinsky equation. Then the algebraic system is solved by the gaussian elimination method. Also the solution of Kuramoto−Sivashinsky equation is approximated as a linear combination of quartic B-spline functions.

1. Introduction

In this paper, we obtain algebraic systems for the Kuramoto−Sivashinsky (KS) equation

$$u_t + uu_x + \alpha u_{xx} + \upsilon u_{xxxx} = 0, \quad x \in [a,b], \ t \in [0,T], \tag{1.1}$$

with the initial condition

$$u(x,0) = f_0(x), \ a \leq x \leq b, \tag{1.2}$$

and boundary conditions

$$u(a,t) = g_0(t), \ u(b,t) = g_1(t), \tag{1.3}$$

$$u_x(a,t) = u_x(b,t) = 0, \tag{1.4}$$

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* Speaker.

$$u_{xx}(a,t) = u_{xx}(b,t) = 0, \tag{1.5}$$

where υ is a positive constant and α is a real constant. The KS equation is commonly used in various fields of science and engineering such as reaction diffusion systems, flame propagation and viscous flow problems, for example see[2]. The layout of the paper is as follows: In Section 2, we obtain algebraic systems for KS. In Section 3, we report our numerical results.

2. Solution of KS equations via quartic B-spline

We define quartic B-splines by the following relationships

$$B_i(z) = \frac{1}{h^4}\begin{cases} (z-z_{i-2})^4, & z \in [z_{i-2}, z_{i-1}), \\ (z-z_{i-2})^4 - 5(z-z_{i-1})^4, & z \in [z_{i-1}, z_i), \\ (z-z_{i-2})^4 - 5(z-z_{i-1})^4 & \\ +10(z-z_i)^4, & z \in [z_i, z_{i+1}), \\ (z_{i+3}-z)^4 - 5(z_{i+2}-z)^4, & z \in [z_{i+1}, z_{i+2}), \\ (z_{i+3}-z)^4, & z \in [z_{i+2}, z_{i+3}), \\ 0, & otherwise. \end{cases}$$

To solve the KS equation by collocation method with quartic B-splines as basis functions, we define the approximation for $u(x,t)$ as

$$U(x,t) = \sum_{i=-2}^{N+1} c_i(t) B_i(x), \tag{2.1}$$

where $c_i(t)$ are time-dependent quantities to be determined from the boundary and collocation conditions and $B_i(x)$ are the quartic B-spline basis functions. The interval $[a,b]$ is partitioned into a mesh of uniform length $h = (b-a)/2$, by the knots z_j where $j = 0, 1, 2, \ldots, N$ such that $a = z_0 < z_1 ... z_{N-1} < z_N$ and $z_j = z_0 + jh$. Also the numerical solution $u(x,t)$ is given at mid knots $x_i = \frac{(z_{i+1}+z_i)}{2}$. We discretize the time derivative of the KS equation using a finite-difference formula. Using the finite difference method, we can write

$$\frac{u^{n+1}-u^n}{\Delta t} + \frac{1}{2}\Big((uu_x)^{n+1} + \alpha(u_{xx})^{n+1} + \upsilon(u_{xxxx})^{n+1}\Big) + (\frac{1}{2})\Big((uu_x)^n + \alpha(u_{xx})^n + \upsilon(u_{xxxx})^n\Big) = 0, \tag{2.2}$$

where Δt is a time step size and $u_i^n = u(x_i, t^{n-1} + \Delta t)$. To linearized the non-linear term $(uu_x)^{n+1}$ in (2.2) we can use the Taylor expansions.

$$(uu_x)^{n+1} = u^n u_x^{n+1} + u^{n+1} u_x^n - (uu_x)^n.$$

With substituting the approximate solution (2.1) for u and putting the values of the mid values U, its derivatives and using boundary conditions (1.3) and (1.5), we obtain the following algebraic systems for KS equation.

$$AC = Q,$$

where

$$A = \begin{pmatrix} \acute{a}_3 - \acute{a}_4 & \acute{a}_2 - \acute{a}_5 & \acute{a}_1 & 0 & 0 & \dots & 0 \\ \acute{a}_4 - \acute{a}_5 & \acute{a}_3 & \acute{a}_2 & \acute{a}_1 & 0 & \dots & 0 \\ \acute{a}_5 & \acute{a}_4 & \acute{a}_3 & \acute{a}_2 & \acute{a}_1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots & & & \\ 0 & \dots & \acute{a}_5 & \acute{a}_4 & \acute{a}_3 & \acute{a}_2 & \acute{a}_1 \\ 0 & \dots & 0 & \acute{a}_5 & \acute{a}_4 & \acute{a}_3 & \acute{a}_2 - \acute{a}_1 \\ 0 & \dots & 0 & 0 & \acute{a}_5 & \acute{a}_4 - \acute{a}_1 & \acute{a}_3 - \acute{a}_2 \end{pmatrix},$$

$$C = (c_0^{n+1}, c_1^{n+1}, \dots, c_{N-2}^{n+1}, c_{N-1}^{n+1})^T,$$

$$Q = \Big(\Psi^n(x_0) - (\acute{a}_4 + \acute{a}_5)\frac{g_0(t^{n+1})}{12}, \Psi^n(x_1) - \acute{a}_5\frac{g_0(t^{n+1})}{12}, \Psi^n(x_2),$$
$$\dots, \Psi^n(x_{N-3}), \Psi^n(x_{N-2}) - \acute{a}_1\frac{g_1(t^{n+1})}{12}, \Psi^n(x_{N-1}) - (\acute{a}_1 + \acute{a}_2)\frac{g_1(t^{n+1})}{12}\Big)^T,$$

with
$\acute{a}_1 = \frac{1}{16} + \frac{\Delta t u^n}{4h} + \frac{\Delta t u_x^n}{32} + \frac{3\alpha\Delta t}{2h^2} + \frac{12\upsilon\Delta t}{h^4}, \acute{a}_2 = \frac{76}{16} + \frac{11\Delta t u^n}{2h} + \frac{76\Delta t u_x^n}{32} + \frac{12\alpha\Delta t}{2h^2} - \frac{96\upsilon\Delta t}{2h^4}, \acute{a}_3 = \frac{230}{16} + \frac{230\Delta t u_x^n}{32} - \frac{30\alpha\Delta t}{2h^2} + \frac{144\upsilon\Delta t}{2h^4}, \acute{a}_4 = \frac{76}{16} - \frac{11\Delta t u^n}{2h} + \frac{76\Delta t u_x^n}{32} + \frac{12\alpha\Delta t}{2h^2} - \frac{96\upsilon\Delta t}{2h^4}, \acute{a}_5 = \frac{1}{16} - \frac{\Delta t u^n}{4h} + \frac{\Delta t u_x^n}{32} + \frac{3\alpha\Delta t}{2h^2} + \frac{12\upsilon\Delta t}{h^4},$
and
$\Psi^n = u^n - \frac{\Delta t}{2}\big(\alpha(uu_x)^n + \upsilon(u_{xxxx})^n\big).$

Then we obtain the linear system consists of N equations in N unknowns. This system solved by gaussian elimination method. Also the above algebraic systems can be solved by iterative method, for example jacobi method. In the jacobi method, we use the following formula

$$x^{k+1} = D^{-1}(E + F)x^k + D^{-1}Q,$$

where E is an upper triangular matrix with the diagonal elements zero and F is a lower triangular matrix with the diagonal elements zero, also D is diagonal matrix such that$A = D - E - F$.

3. Numerical examples

We now obtain the numerical solutions of the KS equation. In order to illustrate the efficiency of the present method for our problem in comparison with the exact solution, we report the global relative error

(GRE) using formula

$$GRE = \frac{\sum_i |U(x_i,t)-u(x_i,t)|}{\sum_i |u(x_i,t)|}.$$

where U is numerical solution and u denotes exact solution.

Consider the KS equation with $\alpha = 1$ and $\upsilon = 1$ in the interval $[-30, 30]$, with the exact solution
$u(x,t) = b + \frac{15}{19}(\frac{11}{19})^{\frac{1}{2}}(-9\tanh(k(x-bt-x_0)) + 11\tanh^3(k(x-bt-x_0)))$.
The boundary conditions and the initial conditions is taken from the exact solution. We have taken $b = 5, k = \frac{1}{2}(\frac{11}{19})^{\frac{1}{2}}$ and $x_0 = -12$. In Table 1 we give a comparison between the global relative error found by our method and method in [1]. Note that we have computed the numerical results by Mathematica (9) programming.

Table 1: Global relative error for example.

partitions $Time$	$\Delta t = 0.01, N = 100$	$\Delta t = 0.01, N = 200$	$\Delta t = 0.0001, N = 1000$ $Method\ in$[1]
1	4.02778e-4	1.03495e-4	6.7923e-04
2	6.03004e-4	1.50962e-4	1.1503e-03
3	7.6671e-4	1.94541e-4	1.5941e-03
4	9.50268e-4	2.39339e-4	2.0075e-03

4. Conclusion

In this paper the B-spline collocation method is used to solve the Kuramoto-Sivashinsky equation with initial and boundary conditions. The numerical results given in the previous section demonstrate the good accuracy of the scheme proposed in this research.

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ON EXCESS OF FUSION FRAMES

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Abstract. In this manuscript, we study the notion of excess of fusion frames. We state some results in alternate dual fusion frames.

1. Introduction

Fusion frames or frame of subspaces, were introduced in 2003 by Casazza and Kutyniok [1]. Fusion frames arise in more applications, both of practical as well as theoretical nature. The notion of excess of fusion frames were introduced in 2008 [3]. In this paper, we represent some results in excess of fusion frames in an easer way. In 2007, Gavruta introduced alternate dual of fusion frames [2]. We state some results about alternate dual of fusion frames.

1.1. Fusion Frames.

First, we review the basic definitions related to the fusion frames. Throughout this paper, $\mathcal{H}$ is a real or complex Hilbert space and I is a countable index set.

Definition 1.1. Let $\{W_i\}_{i\in I}$ be a sequence of closed subspaces in $\mathcal{H}$ and $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. We say

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* Speaker.

that the family $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ a fusion frame or frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i\in I} v_i^2 \|P_{W_i}(x)\|^2 \leq B\|x\|^2 \quad \forall x \in \mathcal{H},$$

where P_{W_i} denotes the orthogonal projection onto W_i. The fusion frame $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ is called a *tight* fusion frame, if $A = B$ and *Parseval*, if $A = B = 1$. If all v_i's take the same value v, then $\{W_i\}_{i\in I}$ is called v-*uniform*. Moreover, $\{W_i\}_{i\in I}$ is called an orthonormal fusion basis for $\mathcal{H}$ if $\mathcal{H} = \bigoplus_i W_i$. If $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ possesses an upper fusion frame bound but not necessarily a lower bound, we call it a Bessel fusion sequence with Bessel fusion bound B.

Notation For any family $\{\mathcal{H}_i\}_{i\in I}$ of Hilbert spaces, we use

$$(\sum_{i\in I} \oplus \mathcal{H}_i)_{\ell_2} = \left\{ \{f_j\} : f_j \in \mathcal{H}_j, \sum_{j\in J} \|f_j\|^2 < \infty \right\}$$

with the inner product

$$\langle \{f_j\}, \{g_j\} \rangle = \sum_{j\in J} \langle f_j, g_j \rangle, \quad \{f_j\}, \{g_j\} \in (\sum_{i\in I} \oplus \mathcal{H}_i)_{\ell_2}.$$

It is easy to show that $(\sum_{i\in I} \oplus \mathcal{H}_i)_{\ell_2}$ is a Hilbert space.

Definition 1.2. Let $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$. The synthesis operator $T_{\mathcal{W}} : (\sum_{i\in I} \oplus W_i)_{\ell_2} \to \mathcal{H}$ is defined by

$$T_{\mathcal{W}}(\{f_i\}_{i\in I}) = \sum_{i\in I} v_i f_i, \quad \{f_i\}_{i\in I} \in (\sum_{i\in I} \oplus W_i)_{\ell_2}.$$

In order to map a signal to the representation space, i.e., to analyze it, the analysis operator $T^*_{\mathcal{W}}$ is employed, which is defined by

$$T^*_{\mathcal{W}} : \mathcal{H} \to (\sum_{i\in I} \oplus W_i)_{\ell_2} \quad with \quad T^*_{\mathcal{W}}(f) = \{v_i P_{W_i}(f)\}_{i\in I},$$

for any $f \in \mathcal{H}$. The fusion frame operator $S_{\mathcal{W}}$ for $\mathcal{W}$ is defined by

$$S_{\mathcal{W}}(f) = T_{\mathcal{W}} T^*_{\mathcal{W}}(f) = \sum_{i\in I} v_i^2 P_{W_i}(f).$$

It follows from [1] that for any fusion frame the operator $S_{\mathcal{W}}$ is invertible and positive and $AI \leq S_{\mathcal{W}} \leq B$. Any $f \in \mathcal{H}$, has the representation $f = \sum_{i\in I} v_i^2 S^{-1}_{\mathcal{W}} P_{W_i}(f)$.

Definition 1.3. We call a fusion frame $\{W_i\}_{i\in I}$ for $\mathcal{H}$, a *Riesz decomposition* of $\mathcal{H}$, if for every $f \in \mathcal{H}$, there exists a unique $f_i \in W_i$ such that $f = \sum_{i\in I} f_i$.

Definition 1.4. If $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$, then ${S_{\mathcal{W}}}^{-1}\mathcal{W} = \{({S_{\mathcal{W}}}^{-1}W_i, v_i)\}_{i\in I}$ is called the canonical dual fusion frame.

The alternate dual of a fusion frame were introduced in [2].

Definition 1.5. Let $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ with the frame operator $S_{\mathcal{W}}$. The fusion Bessel sequence $\mathcal{W}' = \{(W_i', w_i)\}_{i\in I}$ is called an alternate dual fusion frame if

$$x = \sum_{i\in I} v_i w_i P_{W_i'} S_{\mathcal{W}}^{-1} P_{W_i}(x), \quad for \quad all \quad x \in \mathcal{H}.$$

The pair $(\mathcal{W}, \mathcal{W}')$ is called dual pair.

Like discrete frames, it is easy to show that if $\mathcal{W}$ and $\mathcal{W}'$ are Bessel fusion and $(\mathcal{W}, \mathcal{W}')$ is a dual pair, then both $\mathcal{W}$ and $\mathcal{W}'$ are fusion frames.
In an analogous way as in frame theory, the concept of excess were introduced for fusion frames in [3].

Definition 1.6. Let $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ with synthesis operator $T_{\mathcal{W}}$. The excess of $\mathcal{W}$ is defined as

$$e(\mathcal{W}) = dim N(T_{\mathcal{W}}),$$

where $N(T_{\mathcal{W}}) = ker(T_{\mathcal{W}})$.

Definition 1.7. Let $\mathcal{W}$ and $\mathcal{W}'$ be fusion frames with respect to the same family of weights. We say that they are unitary equivalent, if there exist an unitary operator on $\mathcal{H}$, such that $W_i = U(W_i')$.

2. Main results

For unitary equivalent fusion frames, we have the following result.

Proposition 2.1. *Let $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ and $\{(W_i', v_i)\}_{i\in I}$ be unitary equivalent fusion frames . Then $e(\mathcal{W}) = e(\mathcal{W}')$.*

Similarly, the following statement holds for equivalent weights.

Proposition 2.2. *Let $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ be a fusion frame and $\{w_i\}_{i\in I} = \{\alpha v_i\}_{i\in I}$, for some $\alpha > 0$. Then $\mathcal{V} = \{(W_i, w_i)\}_{i\in I}$ is a fusion frame and $e(\mathcal{W}) = e(\mathcal{V})$.*

As in the case of ordinary frames, we have the following result for Riesz decomposition for $\mathcal{H}$.

Lemma 2.3. *Let $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ be a Riesz decomposition of $\mathcal{H}$. Then the excess of $\mathcal{W}$ is equal to zero.*

For alternate dual fusion frames, we have the following proposition.

Proposition 2.4. *Let $\mathcal{W}$ be a fusion frame for $\mathcal{H}$ and $\mathcal{V}$ be its alternate dual. The following statements hold.*

(1) *If $\mathcal{W}$ is a Parseval fusion frame for $\mathcal{H}$, then $\mathcal{W}$ is alternate dual for itself. Hence, in general each fusion frame is not necessary an alternate dual for itself;*
(2) *There is no need of $\mathcal{W}$ be the alternate dual of $\mathcal{V}$;*
(3) *$e(\mathcal{W})$ is not equal to $e(\mathcal{V})$, in general. (Notice that, the excess of a frame is equal to the excess of its alternate dual.)*

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NON-POLYNOMIAL SPLINE METHOD FOR COMPUTING EIGENVALUES OF SPECIAL TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract. This brief note describes some new non-polynomial splines methods of order 2 and 4 for approximating eigenvalues of a two point boundary-value problem involving the differential equation $u''(x) + (\lambda q(x) - p(x))u(x) = 0$. Convergence analysis of these methods is discussed. Numerical results are given to illustrate the efficiency of methods.

1. Introduction

We shall consider the second-order homogeneous linear differential equation

$$u''(x) + (\lambda q(x) - p(x))u(x) = 0, \tag{1.1}$$

associated with one of the following pairs of homogeneous boundary conditions where $q(x) > 0$ and $p(x) \geq 0$, on $[a, b]$:

$$\begin{cases} (I) & u(a) = u(b) = 0, \\ (II) & u'(a) = u'(b) = 0, \\ (III) & u'(a) - \delta u(a) = 0, \quad u'(b) + \sigma u(b) = 0. \end{cases} \tag{1.2}$$

Such boundary-value problems occur frequently in modem physics and engineering.

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2. Non-polynomial cubic spline functions

We consider a uniform mesh Δ with nodal points x_l on $[a,b]$ such that

$$\Delta : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

where $x_l = a + lh$, for $l = 0, 1, 2, \cdots, n$, where by $h = \frac{b-a}{n}$. Also we denote a function value $u(x_l)$ by u_l.

A function $S_\Delta(x)$ of class $C^\infty[a,b]$ which interpolates $u(x)$ at the mesh points $x_l, l = 1, 2, ..., n$, depends on a parameter τ, is termed as parametric cubic spline function and reduces to ordinary cubic spline $S_\Delta(x)$ in [a,b] as $\tau \to 0$ see [1].

For each segment $[x_l, x_{l+1}], l = 0, 1, 2, ..., n-1$ the polynomial, $S_\Delta(x)$, has the form

$$S_\Delta(x) = a_l + b_l(x - x_l) + c_l \sin\tau(x - x_l) + d_l \cos\tau(x - x_l), \quad l = 0, 1, 2, ..., n \tag{2.1}$$

where a_l, b_l, c_l, and d_l are constants and τ is free parameter.

Let u_l be an approximation to $u(x_l)$, obtained by the segment $S_\Delta(x)$ of the mixed splines function passing through the points (x_l, u_l) and (x_{l+1}, u_{l+1}), to obtain the necessary conditions for the coefficients introduced in (2.1), we do not only require that $S_\Delta(x)$ satisfies (3.3) at x_l and x_{l+1} and that the boundary conditions of (3.3) are fulfilled, but also the continuity of first derivative at the common nodes (x_l, u_l).

To derive expression for the coefficients of (2.1), in terms of u_l, u_{l+1}, M_l and M_{l+1}, we first denote:

$$S_\Delta(x_l) = u_l, S_\Delta(x_{l+1}) = u_{l+1},$$

$$S''_\Delta(x_l) = M_l, S''_\Delta(x_{l+1}) = M_{l+1}. \tag{2.2}$$

From algebraic manipulation we get the following expression:

$$a_l = u_l + \frac{M_l}{\tau^2}, \qquad b_l = \frac{u_{l+1} - u_l}{h} + \frac{M_{l+1} - M_l}{\tau\theta},$$

$$c_l = \frac{M_l \cos\theta - M_{l+1}}{\tau^2 \sin\theta}, \qquad d_l = \frac{-M_l}{\tau^2}, \tag{2.3}$$

where $\theta = \tau h$.

Following [1] using the continuity of the first derivative at (x_l, u_l), that is $S'_{\Delta_{l-1}}(x_l) = S'_{\Delta_l}(x_l)$, we get the following consistency relations for $l = 1, ..., n$.

$$\alpha M_{l+1} + 2\beta M_l + \alpha M_{l-1} = (\frac{1}{h^2})(u_{l+1} - 2u_l + u_{l-1}), \tag{2.4}$$

where

$$\alpha = (\frac{1}{\theta^2})(\theta \csc\theta - 1), \beta = (\frac{1}{\theta^2})(1 - \theta\cot\theta),$$

and $\theta = \tau h$.

When $\tau \to 0$, that $\theta \to 0$, then $(\alpha, \beta) \to (\frac{1}{6}, \frac{1}{3})$, and the relations defined by (2.4), reduce into ordinary cubic spline relation:

$$\frac{h^2}{6}(M_{l+1} + 4M_l + M_{l-1}) = (u_{l+1} - 2u_l + u_{l-1}). \tag{2.5}$$

3. Convergence analysis

The problem (3.3) has an infinite sequence of real and positive eigenvalues $0 < \lambda_1 < \lambda_2 < ...$ Applying the (2.4) with $\alpha = \frac{1}{12}, \beta = \frac{5}{12}$ we get the $n-1$ linear system of equations

$$(J + h^2 BP)Y - h^2 B\Lambda QY = 0, \tag{3.1}$$

where Λ is an approximation to λ and $J = (J_{ij}), B = (b_{ij}), R = \text{diag}(r_i), Q = \text{diag}(q_i), Y = [y_1, ..., y_{n-1}]^t$ has the following form

$$J_{ij} = \begin{cases} 2, & i = j = 1, 2, ..., n-1, \\ -1, & |i-j| = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{3.2}$$

$$b_{ij} = \begin{cases} \frac{10}{12}, & i = j = 1, 2, ..., n, \\ \frac{1}{12}, & |i-j| = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{3.3}$$

The matrices $B^{-1}J + h^2 P$ and Q are symmetric [3]. The eigenvalues $h^2\Lambda$ will be real and positive if the matrices Q and $B^{-1}J + h^2 P$ are symmetric and positive definite. we have

$$(B^{-1}J + h^2 P)Y - h^2 \Lambda QY = 0, \tag{3.4}$$

$$(B^{-1}J + h^2 P)\mathbf{Y} - h^2 \lambda Q\mathbf{Y} = B^{-1}T, \tag{3.5}$$

$$\mathbf{Y} = [y(x_1), ..., y(x_{n-1})],$$

we get

$$\begin{cases} |\lambda - \Lambda| \equiv O(h^4), & (\alpha = \frac{1}{12}, \beta = \frac{5}{12}), \\ |\lambda - \Lambda| \equiv O(h^2), , & (\alpha = \frac{1}{6}, \beta = \frac{1}{3}), \end{cases} \tag{3.6}$$

4. Numerical results

In this section, two numerical examples are solved by the method outlined in Section 2.

Example 1. $u''(x) + \lambda u(x) = 0$, and $u(0) = u(1) = 0$. the eigenvalues of Example 1 are $\lambda_m = (m\pi)^2, m = 0, 1, 2, ...$

Example 2. $u''(x) + \frac{\lambda}{x^2}u(x) = 0$, and $u(1) = u(e) = 0$, and also the eigenvalues of Example 2 are $\lambda_m = (m\pi)^2 + \frac{1}{4}, m = 0, 1, 2, ...$

Table 1. Observed maximum absolute errors for $n = 8, 16, 32, 64$

	Example 1		Example 2	
n	$\alpha = \frac{1}{6}, \beta = \frac{1}{3}$	$\alpha = \frac{1}{12}, \beta = \frac{5}{12}$	$\alpha = \frac{1}{6}, \beta = \frac{1}{3}$	$\alpha = \frac{1}{12}, \beta = \frac{5}{12}$
8	1.27×10^{-1}	9.84×10^{-4}	1.56×10^{-2}	4.82×10^{-3}
16	3.17×10^{-2}	6.12×10^{-5}	3.84×10^{-3}	2.96×10^{-4}
32	7.92×10^{-3}	3.82×10^{-6}	9.56×10^{-4}	1.84×10^{-5}
64	1.98×10^{-3}	2.39×10^{-7}	2.38×10^{-5}	1.15×10^{-6}

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CHARACTERIZATIONS OF SOME SUB-CLASSES OF MOORE-PENROSE INVERTIBLE OPERATORS

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Abstract. In this paper, we find some characterizations of the sub-classes of normal operators, self-adjoint operator, partial isometries and normal partial isometries in the class of closed range operators (Moore-Penrose invertible operators) using some operator inequalities.

1. Introduction

Let H be a complex Hilbert space and $B(H)$ be a space of all bounded linear operators on H. For an operator $A \in B(H)$, we write A^* for its adjoint, $R(A)$ for its range, and $N(A)$ for its kernel. An operator $A \in B(H)$ is said to be self-adjoint or Hermitian if $A = A^*$ and normal if $AA^* = A^*A$.

Recall that an operator $A \in B(H)$ is said a partial isometry provided that $\|Ax\| = \|x\|$ for every $x \in N(A)$.

Seddik, studied some inequalities on elementary operators and could find some characterizations for special subclasses of invertible operators such as invertible self-adjoint operators, invertible normal operators and ..., using these inequalities.

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In [1, 2], the author focus on closed range operators i.e. Moore-Penrose invertible operators instead of invertible operators, but the results are restricted to EP-operators. In this paper we discuss very recent results about this topics.

For this purpose we need the following preliminaries.

Definition 1.1. Let $\mathcal{A}$ be an algebra with involution and $a \in \mathcal{A}$. If there exists an element $x \in \mathcal{A}$ satisfied the following four equations

$$\begin{array}{ll} axa = a & xax = x \\ (ax)^* = ax & (xa)^* = xa, \end{array} \qquad (*)$$

then x is called a Moore-Penrose inverse of a and denoted by $a^\dagger$.

It is easy to show that the Moore-Penrose inverse of an element a is unique.

A well-known result about this type of inverse states that if $\mathcal{H}$ is a Hilbert space and $T \in \mathbb{B}(\mathcal{H})$ then T has a Moore-Penrose inverse if and only if T has a closed range.

In this case $TT^\dagger$ is the projection on $\mathcal{R}(T)$ and $T^\dagger T$ is the projection on $\mathcal{R}(T^*)$.

A closed range operator T is called EP, if the ranges of S and S^* are the same or equivalently $S^\dagger S = SS^\dagger$.

The ascent and descent of $T \in B(H)$ are respectively defined by

$$asc(T) = \min\{p \in \mathbb{N} \cup \{0\} : \ ker(T^p) = ker(T^{p+1})\},$$

and

$$dsc(T) = \min\{p \in \mathbb{N} \cup \{0\} : \ R(T^p) = R(T^{p+1})\}.$$

If they are finite, they are equal, and their common value is called the index of T and it is denoted by $ind(T)$.

2. Main results

The main results of this paper are as follows.

Theorem 2.1. [3] *Assume that $ind(T) < \infty$. Then the following properties are equivalent:*

(i) *T is normal,*
(ii) $\forall X \in B(H); \|TXT^\dagger\| + \|T^\dagger XT\| \geq 2\|TT^\dagger XT^\dagger T\|$;
(iii) $\forall X \in B(H); \|T^2X\| + \|XT^2\| \geq 2\|TXT\|$.

Theorem 2.2. [3] *Assume that $ind(T) < \infty$. Then the following properties are equivalent:*

(i) *T is a complex coefficient of a self-adjoint operator,*
(ii) $\forall X \in B(H); \|TXT^\dagger + T^\dagger XT\| \geq 2\|TT^\dagger XT^\dagger T\|$;
(iii) $\forall X \in B(H); \|T^2X + XT^2\| \geq 2\|TXT\|$.

The following theorem, first was proved in [1] with the additional assumption that A is an EP-operator and recently improved for all closed range operators by Menkad[4].

Theorem 2.3. *Let $T \in B(H)$ be a non-zero operator with closed range. Then the following statements are equivalent:*

(i) $\frac{T}{\|T\|}$ *is a partial isometry,*
(ii) $\forall X \in B(H), \|T^*XT^\dagger + T^\dagger XT^*\| = 2\|TT^\dagger XT^\dagger T\|$,
(iii) $\|T^* \otimes T^\dagger + T^\dagger \otimes T^*\|_\lambda = 2$.

Theorem 2.4. *Let $T \in B(H)$ be a non-zero operator with closed range. Then the following statements are equivalent:*

(i) $\frac{T}{\|T\|}$ *is a normal partial isometry,*
(ii) $\forall X \in B(H), \|TXT^\dagger\| = \|T^\dagger TXTT^\dagger\|$,
(iii) $\forall X \in B(H), \|TXT^\dagger\| + \|T^\dagger XT\| = 2\|TT^\dagger XT^\dagger T\|$.

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BIRKHOFF TYPE ORTHOGONALITY IN C*-SEMI-INNER PRODUCT SPACES

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Abstract. In this note, we investigate on some Birkhoff-james type orthogonality in C*-semi-inner product spaces which are a generalization of Hilbert C*-modules.

1. Introduction

The notion of orthogonality in an arbitrary normed space, with the norm not necessarily coming from an inner product, may be introduced in various ways. Among these, the one which is frequently met with in the literature is the Birkhoff-James orthogonality. It is defined as follows.

Definition 1.1. Let X be a normed space. For $x, y \in X$, we say that x is Birkhoff-James orthogonal to y, and denote it by $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{C}$.

It is well-known that Birkhoff-James orthogonality is equivalent to semi-inner product orthogonality for some suitable semi-inner product on the normed space that generates the norm of the space [3].

First, we recall the definition of semi-inner product in the sense of Lumer and Giles.

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Key words and phrases. Birkhoff-James orgthogonality, C*-semi-inner product space, Hilbert modules.

Definition 1.2. A semi-inner product on a complex vector space X is a complex valued function $[x, y]$ on $X \times X$ with the following properties:

i) $[\lambda y + z, x] = \lambda[y, x] + [z, x]$ and $[x, \lambda y] = \bar{\lambda}[x, y]$, for all complex λ,
ii) $[x, x] \geq 0$, for all $x \in X$ and $[x, x] = 0$ implies $x = 0$;
iii) $|[x, y]|^2 \leq [x, x][y, y]$.

In [4], authors defined a generalization of semi-inner product space as follows.

Definition 1.3. Let A be a C*- algebra and X be a right A-module. A mapping $[.,.] : X \times X \to A$ is called a C*-semi-inner product or C*-s.i.p., in brief, if the following properties are satisfied:

(i) $[x, x] \geq 0$, for all $x \in X$ and $[x, x] = 0$ implies $x = 0$;
(ii) $[x, \alpha y_1 + \beta y_2] = \alpha[x, y_1] + \beta[x, y_2]$, for all $x, y_1, y_2 \in X$ and $\alpha, \beta \in \mathbb{C}$;
(iii) $[x, ya] = [x, y]a$ and $[xa, y] = a^*[x, y]$, for all $x, y \in X$ and $a \in A$;
(iv) $|[y, x]|^2 \leq \|[y, y]\|[x, x]$.

The triple $(X, A, [.,.])$ is called a C*-semi-inner product space or we say X is a semi-inner product A-module.

Let $(X, A, [.,.])$ be a C*-semi-inner product space. For any $x \in X$, the function $\|x\| := \|[x, x]\|^{\frac{1}{2}}$, defines a norm on X. Also we can define $|x| := [x, x]^{\frac{1}{2}}$.

Our purpose in this work, is to investigate some Birkhoff type orthogonality in C*-semi-inner product spaces.

2. Main results

In fact, on semi-inner product A-module X, we can define three similar type of orthogonality:

$$\begin{array}{lll} x\perp_1 y & \Leftrightarrow & \|x + ya\| \geq \|x\| \qquad \forall a \in A; \\ x\perp_2 y & \Leftrightarrow & |x + ya| \geq |x| \qquad \forall a \in A; \\ x\perp_3 y & \Leftrightarrow & |x + ya|^2 \geq |x|^2 \qquad \forall a \in A. \end{array}$$

It can be easily seen that

$$x\perp_3 y \Rightarrow x\perp_2 y \Rightarrow x\perp_1 y \Rightarrow x\perp_B y.$$

In [1], Arambasic and Rajic studied these types of orthogonality in Hilbert C^*-modules.

Proposition 2.1. *Let X be a Hilbert module over a C^*-algebra A. Then for each $x, y \in X$,*

$$\langle x, y\rangle = 0 \Rightarrow x\perp_1 y \Rightarrow x\perp_B y.$$

The converse of both relation is not true.

Theorem 2.2. *For a Hilbert module X over a C^*-algebra A, the following relations are equivalent:*

i) $\langle x, y\rangle = 0$;
ii) $x\perp_2 y$;
iii) $x\perp_3 y$.

Also in [2], they found some conditions under which the first orthogonality coincide with each one of Birkhoff-James orthogonality or inner product orthogonality.

Theorem 2.3. *Let A be a C^*-algebra and X be a semi-inner product A-module. For $x, y \in X$, if $[x, y] = 0$ or $[y, x] = 0$ then $x\perp_1 y$ and therefore $x\perp_B y$.*

Theorem 2.4. *Let A be a C^*-algebra and X be a semi-inner product A-module. For $x, y \in X$, if $[x, y] = [y, x] = 0$ then $x\perp_3 y$.*

For the converse, we have the following theorem.

Theorem 2.5. *Let A be a C^*-algebra and X be a semi-inner product A-module and $x, y \in X$ such that $x\perp_2 y$ and $[x, y][y, x]$ is positive. Then we have $[x, y][y, x] = 0$.*

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A NEW REFINEMENT OF THE HERMITE-HADAMARD INEQUALITY AND IT'S APPLICATIONS

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Abstract. In this paper by a new refinement of the Hermite-Hadamard integral inequality, as:

$$f(\frac{a+b}{2}) \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{1}{16}(7f(a)+2f(\frac{a+b}{2})+7f(b)) \leqslant \frac{f(a)+f(b)}{2},$$

where, f is a real-valued convex function on the interval $[a,b]$, we obtain a different refinement of Heinz inequality.

1. Introduction

Let $\mathcal{B}(H)$ be the space of all bounded operators on a Hilbert space H. A norm $|||.|||$ is called unitarily invariant norm if $|||UAV||| = |||A|||$ for all $A \in \mathcal{B}(H)$ and for all unitary operators $U, V \in \mathcal{B}(H)$.
If A, B, X are operators on a complex separable Hilbert space such that A and B are positive, then for every unitarily invariant norm $|||.|||$, the function $f(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||$ is convex on the interval $[0,1]$, attains its minimum at $\nu = \frac{1}{2}$, and attains its maximum at $\nu = 1$ and $\nu = 0$. Moreover, $f(\nu) = f(1-\nu)$ for $0 \leqslant \nu \leqslant 1$. From [1] we know

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that for every unitarily invariant norm, we have the Heinz inequalities

$$2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| \leqslant \left|\left|\left|A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}\right|\right|\right| \leqslant |||AX+XB|||\,. \quad (1.1)$$

In this paper, we use a similar method to [2, 3] and get different refinement of (1.1). Our results are better than those in [4] and different from [2, 3].

2. Main results

From [3], we know the following Hermite-Hadamard integral inequality for convex functions.

Lemma 2.1. *Let f be a real-valued function which is convex on the interval $[a,b]$. Then*

$$f(\frac{a+b}{2}) \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{f(a)+f(b)}{2}.$$

We will use the following lemma.

Lemma 2.2. *Let f be a real-valued function which is convex on the interval $[a,b]$. Then*

$$\begin{aligned} f(\frac{a+b}{2}) \leqslant \frac{1}{b-a}\int_a^b f(t)dt &\leqslant \frac{1}{16}\left(7f(a)+2f(\frac{a+b}{2})+7f(b)\right) \\ &\leqslant \frac{f(a)+f(b)}{2}. \end{aligned}$$

Proof. Using the previous lemma, we can easily verify the inequality

$$\frac{1}{16}\left(7f(a)+2f(\frac{a+b}{2})+7f(b)\right) \leqslant \frac{f(a)+f(b)}{2}.$$

Next we will prove the following inequality:

$$\frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{1}{16}(7f(a)+2f(\frac{a+b}{2})+7f(b)).$$

We have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(t)dt &= \frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}} f(t)dt + \int_{\frac{a+b}{2}}^b f(t)dt\right)\\ &\geqslant \frac{1}{b-a}\left(\frac{f(a)+f(\frac{a+b}{2})}{2}.\frac{b-a}{2} + \frac{f(b)+f(\frac{a+b}{2})}{2}.\frac{b-a}{2}\right)\\ &= \frac{1}{4}\left(f(a)+2f(\frac{a+b}{2})+f(b)\right)\\ &= \frac{1}{16}\left(4f(a)+8f(\frac{a+b}{2})+4f(b)\right)\\ &\leqslant \frac{1}{16}\left(4f(a)+2f(\frac{a+b}{2})+3(f(a)+f(b))+4f(b)\right)\\ &= \frac{1}{16}\left(7f(a)+2f(\frac{a+b}{2})+7f(b)\right).\end{aligned}$$

□

Applying the previous lemma to the function

$$f(\nu) = \left|\left|\left|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\right|\right|\right|$$

on the interval $[\mu, 1-\mu]$ when $0 \leqslant \mu \leqslant \frac{1}{2}$ and on the interval $[1-\mu, \mu]$ when $\frac{1}{2} \leqslant \mu \leqslant 1$, we obtain a refinement of the first inequality in (1.1).

Theorem 2.3. *Let $A, B, X \in \mathcal{B}(H)$ such that A, B are poitive. Then for $0 \leqslant \mu \leqslant 1$ and for every unitarily invariant norm, we have*

$$\begin{aligned}2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| &\leqslant \frac{1}{|1-2\mu|}\left|\int_\mu^{1-\mu}\left|\left|\left|A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu\right|\right|\right|d\nu\right|\\ &\leqslant \frac{1}{8}(7\left|\left|\left|A^\mu XB^{1-\mu}+A^{1-\mu}XB^\mu\right|\right|\right| + 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|)\\ &\leqslant \left|\left|\left|A^\mu XB^{1-\mu}+A^{1-\mu}XB^\mu\right|\right|\right|.\end{aligned}$$

Proof. First assume that $0 \leqslant \mu \leqslant \frac{1}{2}$. Then it is follows by previous lemma that

$$\begin{aligned}f(\frac{1-\mu+\mu}{2}) &\leqslant \frac{1}{1-2\mu}\int_\mu^{1-\mu} f(t)dt\\ &\leqslant \frac{1}{16}\left(7f(\mu)+2f(\frac{1-\mu+\mu}{2})+7f(1-\mu)\right)\\ &\leqslant \frac{f(\mu)+f(1-\mu)}{2},\end{aligned}$$

and hence

$$\begin{aligned} f(\frac{1}{2}) &\leqslant \frac{1}{1-2\mu}\int_{\mu}^{1-\mu} f(t)dt \\ &\leqslant \frac{1}{8}\left(7f(\mu)+2f(\frac{1}{2})\right) \\ &\leqslant f(\mu). \end{aligned}$$

Thus,

$$\begin{aligned} 2\left|\!\left|\!\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\!\right|\!\right| &\leqslant \frac{1}{1-2\mu}\int_{\mu}^{1-\mu}\left|\!\left|\!\left|A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}\right|\!\right|\!\right| d\nu \\ &\leqslant \frac{1}{8}(7\left|\!\left|\!\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\!\right|\!\right|+2\left|\!\left|\!\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\!\right|\!\right|) \\ &\leqslant \left|\!\left|\!\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\!\right|\!\right|. \end{aligned}$$

Now, assume that $\frac{1}{2}\leqslant\mu\leqslant 1$. Then by applying above way to $1-\mu$, it follows that

$$\begin{aligned} 2\left|\!\left|\!\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\!\right|\!\right| &\leqslant \frac{1}{2\mu-1}\int_{\mu}^{1-\mu}\left|\!\left|\!\left|A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}\right|\!\right|\!\right| d\nu \\ &\leqslant \frac{1}{8}(7\left|\!\left|\!\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\!\right|\!\right|+2\left|\!\left|\!\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\!\right|\!\right|) \\ &\leqslant \left|\!\left|\!\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\!\right|\!\right|. \end{aligned}$$

Since

$$\begin{aligned} &\lim_{\mu\to\frac{1}{2}}\frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu}\left|\!\left|\!\left|A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}\right|\!\right|\!\right| d\nu\right| \\ &= \lim_{\mu\to\frac{1}{2}}\frac{1}{8}(7\left|\!\left|\!\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\!\right|\!\right|+2\left|\!\left|\!\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\!\right|\!\right|) \\ &= \left|\!\left|\!\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\!\right|\!\right|, \end{aligned}$$

the result follows by combining previous inequalities. □

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POSITIVITY PRESERVING SCHEME FOR REACTION-DIFFUSION-CONVECTION PROBLEMS

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Abstract. Nonstandard finite differences (NSFD) schemes can improve the accuracy and reduce computational costs of traditional finite-difference schemes. In addition, NSFDs produce numerical solutions which also exhibit essential properties of solution. In this paper, we derive a NSFD scheme based on Mickens' rules for reaction-diffusion-convection (RDC) systems. The new scheme is a modification of the fully implicit method by using a nonlocal approximation for reaction term of RDC. The proposed scheme improves the accuracy and guarantees the positivity requirement, as is demanded for the solution of RDC. Numerical simulations on a *catalytic particle* are used for illustrating the accuracy and performance of the proposed scheme as a compared to standard finite difference scheme.

1. Introduction

In order to solve reaction-diffusion-convection equation, analytical and numerical solution techniques are used. Analytical solution of this equation may be carried out when simple and idealized conditions are satisfied. However, if the solution parameters change in time, use of the numerical solution techniques is necessary for the solution of advection-diffusion reaction equation.

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Key words and phrases. Finite differences schemes, Catalytic particle model, Positivity preserving, M-matrix.

It is desirable that the numerical solutions satisfy the same properties as the exact solution such as positivity, total variation dimensioning or monotonicity [3, 4]. In this paper, we propose a modification of fully implicit scheme for the RDC problem in *catalytic particle* [1]. The proposed scheme enables us to solve accurately the examined PDE and it is the *positivity preservation* of the solution which exhibit essential property of solution.

2. Modified fully implicit (MFI) scheme

Consider underlying equation, which can describe the reaction-diffusion-convection problem in a catalytic particle:

$$\frac{\partial C(x,t)}{\partial t} = \frac{\partial^2 C(x,t)}{\partial x^2} - Pe\frac{\partial C(x,t)}{\partial x} - \phi^2 C(x,t), \tag{2.1}$$

with initial and boundary conditions:

$$C(x,0) = 0, \quad C(0,t) = 1, \quad C(1,t) = 1.$$

The function $C(x,t)$ corresponds to the normalized concentration and endowed, Pe is the Peclet number, which denotes the relationship between the convective and diffusive transport and ϕ is Thiele modulus, which relates chemical reaction rate and the diffusive transport; the dimensionless parameters $x \in [0,1]$ and $t > 0$ denote the spatial coordinate and time, respectively.

Making use of the nonstandard discretization [2] of the reaction term $\phi^2 C$ in (2.1), it is now desired to find an accurate modified fully implicit scheme which is positivity preserving and can be written as:

$$C(S,t) = a(C_{j+1}^n + C_{j-1}^n) + b(C_{j+1}^{n+1} + C_{j-1}^{n+1}) + (\frac{1}{2} - a - b)(C_j^{n+1} + C_j^n), \tag{2.2}$$

here a and b are arbitrary parameters to be determined below. The stencil of the involved nodes of the scheme is displayed in Figure 1.

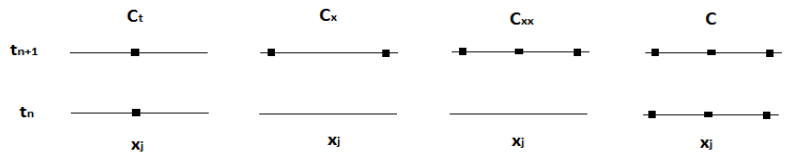

Figure 1. Involved nodes in the modified fully implicit scheme.

The corresponding finite difference approximation provides the equation difference

$$PC^{n+1} = NC^n, \tag{2.3}$$

where P and N are the following tridiagonal matrices:

$$P = tridiag\left\{-\frac{Pe}{2\Delta x} - \frac{1}{\Delta x^2} + b\phi^2; \frac{1}{\Delta t} + \frac{2}{\Delta x^2} + \phi^2(\frac{1}{2} - a - b); \frac{Pe}{2\Delta x} - \frac{1}{\Delta x^2} + b\phi^2\right\}, \quad (2.4)$$

$$N = tridiag\left\{-a\phi^2; \frac{1}{\Delta t} - \phi^2(\frac{1}{2} - a - b); -a\phi^2\right\}. \quad (2.5)$$

The parameters a, b and Δt are chosen according to the following theorem:

Theorem 1. Sufficient for scheme (2.3) to be positive is,

$$a \leq 0, \quad b \leq -\frac{Pe^2}{16\phi^2}, \quad \Delta t \leq \frac{1}{\phi^2(\frac{1}{2} - a - b)}. \quad (2.6)$$

Theorem 2. Under conditions of Theorem 1, the fully implicit variant scheme is stable and convergent with local truncation error $O(\Delta t, \Delta x^2)$.

3. Numerical results

To obtain a reference solution of (2.1) the Laplace transform was applied and for the analytical solution we found

$$F[C(x,s)] = \frac{exp(m_2 x)[exp(m_1) - 1] + exp(m_1 x)[1 - exp(m_2)]}{exp(m_1) - exp(m_2)} \quad (3.1)$$

with

$$m_1 = \frac{Pe - \sqrt{Pe^2 + 4(s + \phi^2)}}{2} \quad , \quad m_2 = \frac{Pe + \sqrt{Pe^2 + 4(s + \phi^2)}}{2},$$

where $F[C(x,s)]$ is the Laplace transform of $C(x,t)$. Unfortunately, the inverse Laplace transform for $F[C(x,s)]$ is not available. In order to determinate the solution in the time-domain, we have used the numerical inversion by Zakians algorithm [5].
We apply new scheme to (2.1) with different values of Pe and ϕ, see Figure 2-3. If one of the conditions in (2.6) is violated, then the numerical solution may exhibit spurious oscillations, see Figure 4.

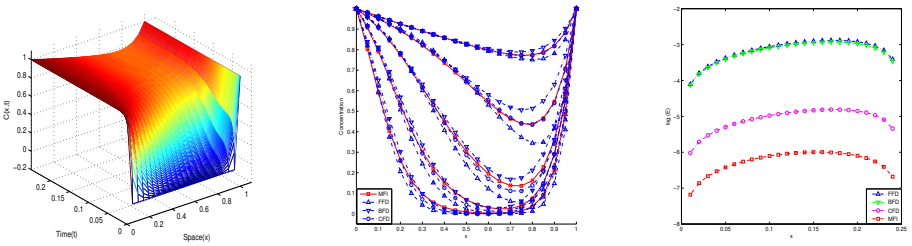

Figure 2. Concentration profiles at different times and their error logarithms when $Pe = 10$ and $\phi = 2$.

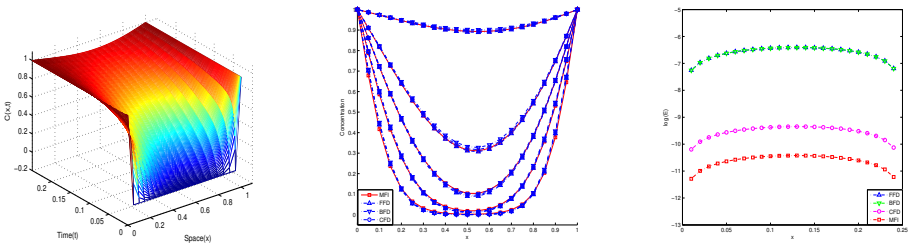

FIGURE 3. Concentration profiles at different times and their error logarithms when $Pe = 1$ and $\phi = 0.1$.

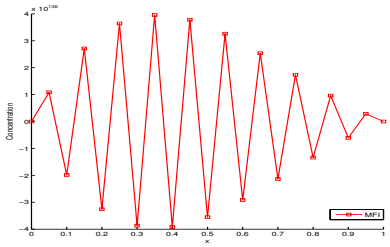

FIGURE 4. Numerical results for the modified fully implicit scheme with $b = -\frac{Pe^2}{16\phi^2} + 300$.

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A NUMERICAL METHOD FOR DISCRETE FRACTIONAL–ORDER PREDATOR–PREY MODEL DERIVED FROM NSFD SCHEME

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Abstract. In this paper, we introduce fractional–order into a predator–prey model. The nonstandard finite difference (NSFD) scheme is implemented to study the dynamic behaviors in the fractional–order predator–prey model. Numerical results show that the NSFD approach is easy and accurate for implementing when applied to fractional–order predator–prey model.

1. Introduction

Study of the fractional differential equations (FDEs) as a dynamical system is a novel and appealing subject which has motivated the leading research literatures in recent years. FDEs have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry, engineering and biological sciences [2]. Recently, most of the dynamical systems based on the integer–order calculus have been modified into the fractional order domain due to the extra degrees of freedom and the flexibility which can be used to precisely fit the experimental data much better than the integer–order modeling.

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Key words and phrases. Fractional differential equations, Nonstandard finite difference scheme, Predator–prey model, Stability.

* Speaker.

This paper is devoted to the construction of a nonstandard discretization scheme given by Mickens to the Grünwald–Letnikov (GL) discretization process for solving the fractional–order predator–prey model.

2. Preliminaries

Derivatives of fractional–order have been introduced in several ways. In this paper we consider GL approach. The GL method of approximation for the one–dimensional fractional derivative takes the following form [2]

$$D^{\alpha}x(t) = f(t, x(t)), \qquad x(0) = x_0, \qquad t \in [0, t_f] \tag{2.1}$$

$$D^{\alpha}x(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{[t/h]} (-1)^j \binom{\alpha}{j} x(t - jh),$$

where $0 < \alpha \leq 1$, D^{α} denotes the fractional derivative, h is the step size and $[\frac{t}{h}]$ represents the integer part of $\frac{t}{h}$. Therefore, Eq. (2.1) is discretized in the next form

$$\sum_{j=0}^{n} c_j^{\alpha} x(t_{n-j}) = f(t_n, x(t_n)), \qquad n = 1, 2, 3, ...$$

where $t_n = nh$ and c_j^{α} are the GL coefficients defined as

$$c_j^{\alpha} = (1 - \frac{1+\alpha}{j}) c_{j-1}^{\alpha}, \qquad c_0^{\alpha} = h^{-\alpha}, \qquad j = 1, 2, 3, ...$$

The nonstandard discretization technique is a general scheme where we replace the step size h by a function $\phi(h)$ [3]. By applying this technique and using the GL discretization method, it yields the following relations

$$x(t_{n+1}) = \frac{-\sum_{j=1}^{n+1} c_j^{\alpha} x(t_{n+1-j}) + f(t_{n+1}, x(t_{n+1}))}{c_0^{\alpha}},$$

where $c_0^{\alpha} = \phi(h)^{-\alpha}$.

3. NSFD scheme for fractional–order predator–prey model

Consider the fractional–order predator–prey system [1],

$$D^{\alpha_1}x(t) = x(t)(r - ax(t) - by(t)),$$
$$D^{\alpha_2}y(t) = y(t)(-d + cx(t)),$$

where $0 < \alpha_i \leq 1$, for $i = 1, 2$ and $x \geq 0$, $y \geq 0$ are prey and predator densities, respectively, and all constants r, a, b, c and d are positive.

Applying Mickens scheme by replacing the step size h by a function $\phi(h)$ and using the GL discretization method, yields the following equations:

$$\sum_{j=0}^{n+1} c_j^{\alpha_1} x(t_{n+1-j}) = rx(t_n) - ax(t_{n+1})x(t_n) - bx(t_{n+1})y(t_n),$$

$$\sum_{j=0}^{n+1} c_j^{\alpha_2} y(t_{n+1-j}) = -dy(t_{n+1}) + cx(t_{n+1})y(t_n).$$

Invoking some algebraic manipulations to Eqs. (3.1), the following relations are obtained

$$x(t_{n+1}) = \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_1} x(t_{n+1-j}) + rx(t_n)}{c_0^{\alpha_1} + ax(t_n) + by(t_n)},$$

$$y(t_{n+1}) = \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_2} y(t_{n+1-j}) + cx(t_{n+1})y(t_n)}{c_0^{\alpha_2} + d},$$

where

$$c_0^{\alpha_1} = \phi_1(h)^{-\alpha_1}, \qquad c_0^{\alpha_2} = \phi_2(h)^{-\alpha_2},$$

with

$$\phi_1(h) = \frac{e^{rh} - 1}{r}, \qquad \phi_2(h) = \frac{e^{dh} - 1}{d}.$$

The equilibrium points of system are $E_0 = (0, 0)$, $E_1 = (\frac{r}{a}, 0)$ and $E_2 = (\frac{d}{c}, \frac{cr-ad}{cb})$. The stability analysis of such kind of system have been studied [1]. In the special case $a = 0$ it is known that the equilibrium point E_2 is a centre for the integer order system $\alpha_1 = \alpha_2 = 1$. In the fractional case $0 < \alpha_i \leq 1$, for $i = 1, 2$ the equilibrium point E_2 is locally asymptotically stable. The numerical simulations in the next section will support this result.

4. Numerical results

In this section, numerical results from the implementation of NSFD scheme for the fractional–order predator–prey system are presented.

In Fig. 1 is depicted phase trajectory of the fractional–order predator–prey system for commensurate order $\alpha_1 = \alpha_2 = 1$ with parameters $a = 0, b = 1, c = 1, r = 2, d = 3$ with the initial conditions $x(0) = 1$, $y(0) = 2$, for simulation time 300s and step size $h = 0.1$.

In Fig. 2 is depicted phase trajectory of the fractional–order predator–prey system for incommensurate order $\alpha_1 = 0.90, \alpha_2 = 0.80$ and parameters $a = 0, b = 1, c = 1, r = 2, d = 3$ with the initial conditions $x(0) = 1$ and $y(0) = 2$, for simulation time 300s and step size $h = 0.1$. Therefore Fig. 2 illustrate that the equilibrium point $E_2 = (3, 2)$ is locally asymptotically stable for all $\alpha \in (0, 1)$.

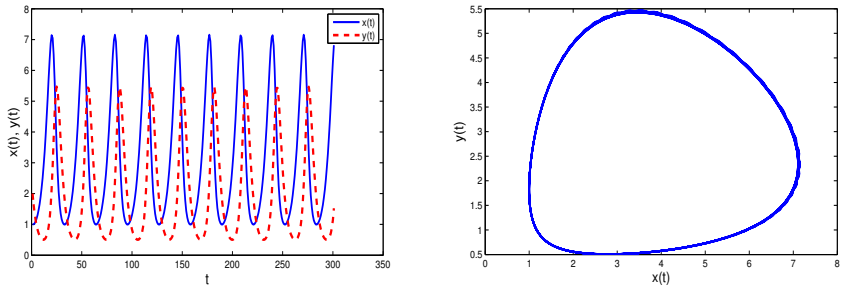

Fig. 1. Plot of populations x, y over time for the case $\alpha_1 = \alpha_2 = 1$ and $h = 0.1$.

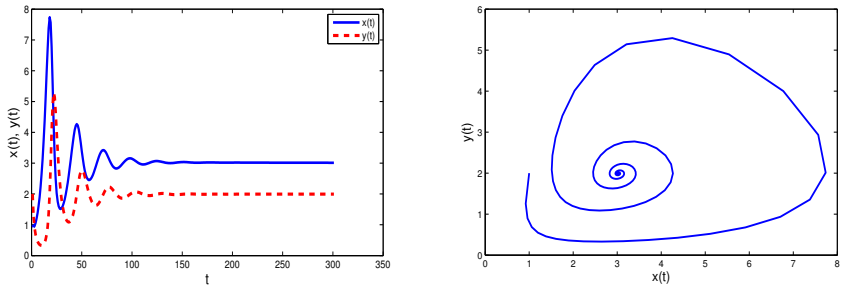

Fig. 2. Plot of populations x, y over time for the case $\alpha_1 = 0.90, \alpha_2 = 0.80$ and $h = 0.1$.

5. CONCLUSION

In this paper we study the fractional–order predator–prey model. Numerical solutions of this model are given. The reason for considering a fractional–order system instead of its integer–order counterpart is that fractional–order differential equations are generalizations of integer order differential equations. Also using fractional–order differential equations can help us to reduce the errors arising from the neglected parameters in modeling real life phenomena.

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2-ADJOINTABLE OPERATORS ON 2-PRE HILBERT C^*-MODULE SPACES

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Abstract. In this paper, first, we introduce the new concept of 2-pre Hilbert C^*-module. Next, we present the concept of C^*-2 linear operators which coincides with Lewandowska's definition [Lewandowska, Z.: On 2-normed sets, Glasnik Mat., **38 (58)**, 99–110(2003)]. Finally , we define the notion of 2-adjointable mappings between 2-pre Hilbert C^*-modules.

1. Introduction

The concept of 2-inner product has been intensively studied by many outhors in the last three decades.The basic definitions and elementery properties of 2-inner product spaces can be found in [1] and [2].

Recently, M.Frank and e.t. defined the notion ϕ-perturbation of an adjointable mapping and proved the supperstability of an adjointable mapping on Hilbert C^*-modules(see [3]).

In this paper, first, we introduce the definition 2-pre Hibert C^*-module spaces and give several important properties. Next, we present the concept of A-2 linear operators which coincides with Lewandowska's definition (see [4, 5]). Also, we define 2-adjoinable mappings between

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Key words and phrases. 2-adjointable mapping, C^*-algebra, 2-pre Hilbert C^*-module, C^*- 2 linear operator, supperstability.

* Speaker.

2-pre Hilbert C^*-modules and prove an analogue of ϕ-perturbation of adjoitable mappings in paper([3]).

2. Main sections and results

Let X be a left module over a C^*-algebra A. An action of $a \in A$ on X is denoted by $a.x \in X$, $x \in X$.

Definition 2.1. A 2-pre Hilbert A-module is a left A-module X equipped with A-valued function defined on $X \times X \times X$ satisfing the following conditions:

I_1) $(x, x|z)$ is a positive element in A for any $x, z \in X$ and $(x, x|z) = 0$ if and only if x and z are linearly dependent;

I_2) $(x, x|z) = (z, z|x)$ for any $x, z \in X$;

I_3) $(y, x|z) = (x, y|z)^*$ for any $x, y, z \in X$;

I_4) $(\alpha x + x', y|z) = \alpha(x, y|z) + (x', y|z)$ for any $\alpha \in \mathbb{C}$ and $x, x', y, z \in X$;

I_5) $(ax, y|z) = a(x, y|z)$ for any $x, y, z \in X$ and any $a \in A$.

The map $(.,.|.)$ is called A-valued 2-inner product and $(X, (.,.|.))$ is called 2-pre Hilbert C^*-module space.

Example 2.2. Every 2-inner product space is a 2-pre Hilbert $\mathbb{C}$-module.

Example 2.3. Let A be a C^*-algebra and $J \subseteq A$ be a left ideal. Then J can be equipped with the structure of 2-pre Hilbert A-module with A-valued inner product $(x, y|z) := xy^*zz^* - xz^*zy^*$ for any $x, y, z \in A$.

Definition 2.4. Let X be a 2-pre Hilbert A-module. we can define a function $||.|.||_X$ on $X \times X$ by $||x|z||_X = ||(x, x|z)||^{\frac{1}{2}}$ for all $x, z \in X$.

Proposition 2.5. *$||.|.||_X$ satisfies the following conditions:*

$N1$) $||ax|z||_X \leq ||a||\ ||x|z||_X$ for any $x, z \in X$ and $a \in A$;

$N2$) $(x, y|z)\ (y, x|z) \leq \|y|z\|_X^2\ (x, x|z)$ for any $x, y, z \in X$;

$N3$) $||(x, y|z)||^2 \leq ||(x, x|z)||\ ||(y, y|z)||$

Proof. $N1$ is obvious; $N3$ follows from $N2$, so let us prove $N2$.
Let ϕ be a positive linear functional on A. Then $\phi((.,.|.))$ is usual 2-inner product on X. Applying the Schwartz inequality for 2-inner product (see [2], page 3) we obtain for all $x, y, z \in X$,

$$\begin{aligned}
\phi((x, y|z)\ (y, x|z)) &= \phi((x, y|z)y, x\ |z)) \\
&\leq \phi((x, x|z))^{\frac{1}{2}}\ \phi(((x, y|z)y\ , (x, y|z)y\ |z))^{\frac{1}{2}} \\
&\leq \phi((x, x|z))^{\frac{1}{2}}\ \phi((x, y|z)\ (y, y|z)\ (x, y|z)^*)^{\frac{1}{2}} \\
&\leq \phi((x, x|z))^{\frac{1}{2}}\ ||(y, y|z)||^{\frac{1}{2}}\ \phi((x, y|z)\ (y, x|z))^{\frac{1}{2}}.
\end{aligned}$$

Thus, for any positive linear functional ϕ, we have

$$\phi((x,y|z)\ (y,x|z)) \leq ||y|z||_X^2\ \phi((x,x|z))$$

hence

$$(x,y|z)\ (y,x|z) \leq ||y|z||_X^2\ (x,x|z).$$

□

Theorem 2.6. *The function* $||.|.||_X$ *is a* 2*-norm on* X*.*

Proof. Now, we verify that $||.|.||_X$ satisfies the following properties of 2-norms:

1) I_3 and I_4 show that $||\alpha x|y||_X = ||(\alpha x, \alpha x|y)||^{\frac{1}{2}} = |\alpha|\ ||x|y||_X$ for all $x, y \in X$ and $\alpha \in \mathbb{C}$.

2) I_1 follows that $||x|y||_X = 0$ if and only if x and y are linearly dependent for all $x, y \in X$.

3) it follows from I_2 that $||x|y||_X = ||(x,x|y)||^{\frac{1}{2}} = ||y|x||_X$ for all $x, y \in X$.

4) By proposition 2.5 $(N3)$, we have

$$\begin{aligned} ||x+x'|y||_X^2 &= ||(x+x', x+x'|y)|| \\ &= ||(x,x|y) + (x',x|y + (x,x'|y) + (x',x'|y))|| \\ &\leq ||(x,x|y)|| + 2||(x,x'|y)|| + ||(x',x'|y)|| \\ &\leq (||(x,x|y)||^{\frac{1}{2}} + ||(x',x'|y)||^{\frac{1}{2}})^2 = (||x|y||_X + ||x'|y||_X)^2 \end{aligned}$$

for all $x, x', y \in X$. This show that $(X, ||.|.||_X)$ is a 2-normed space. □

2.1. **2-adjointable mappings.** In continue, we let A be a C^*-algebra. Now, we start with following definition.

Definition 2.7. Let X and Y be two 2-pre Hilbert A-modules. An operator $f : X \times X \to Y$ is said to be A-2 linear if it satisfies the following conditions:

1) $f(x+y, z+w) = f(x,z) + f(x,w) + f(y,z) + f(y,w)$ for all $x, y, z, w \in X$;

2) $f(\alpha x, \beta y) = \alpha\ \overline{\beta}\ f(x,y)$ for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in X$;

3) $f(ax, by) = a.\ b^*.f(x,y)$ for all $x, y \in X$ and $a, b \in A$.

Example 2.8. Let X be a 2-pre Hilbert A-module and $z \in X$. Define $f : X \times X \to A$ by $f(x,y) = (x,y|z)$. Then f is a A- 2 linear operator.

Definition 2.9. Let X and Y be two 2-pre Hilbert A-modules. A mapping $f : X \times X \to Y$ is called 2-adjointable if there exists a mapping $g : Y \times Y \to X$ such that

$$(f(x,y), s \mid t) = (x, y \mid g(s,t)) \tag{2.1}$$

for all $x, y \in X$ and $s, t \in Y$. The mapping g is denoted by f^* and is called the 2-adjointable of f.

Proposition 2.10. *Let X be a 2-pre Hilbert A-module and $\dim(X) > 1$. If $(x, y|z) = 0$ for all $y, z \in X$, then $x = 0$.*

Proof. Suppose $x \neq 0$. Let x and y be linearly independent. Then by hyphosits $(x, x, |y) = 0$ and this is contradiction. □

Lemma 2.11. *Every 2-adjonable mapping is A- 2 linear.*

Proof. Let $f : X \times X \to Y$ be a 2-adjoinable mapping. Then there exists a mapping $g : Y \times Y \to X$ such that (2.1) holds. For every $x, y, z, w \in X$, every $s, t \in Y$, every $\alpha, \beta \in \mathbb{C}$, every $a, b \in A$, we have

$$\begin{aligned}
&(f(\alpha ax + y, \beta bz + w), s \mid t) = (\alpha ax + y, \beta bz + w \mid g(s,t)) \\
&= \alpha\overline{\beta}ab^* \ (x, z \mid g(s,t)) + \alpha a \ (x, w \mid g(s,t)) + \overline{\beta}b^* \ (y, z \mid g(s,t)) \\
&+ (y, w \mid g(s,t)) \\
&= \alpha\overline{\beta}ab^* \ (f(x,z), s \mid t) + \alpha a \ (f(x,w), s \mid t) \\
&+ \overline{\beta}b^* \ (f(y,z), s \mid t) + (f(y,w), s \mid t) \\
&= (\alpha\overline{\beta}ab^* \ f(x,z) + \alpha a \ f(x,w) + \overline{\beta}b^* \ f(y,z) + f(y,w), s \mid t).
\end{aligned}$$

It follows from proposition 2.10 that f is A-2 linear. □

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SOLVING THE MIXED FREDHOLM - VOLTERRA INTEGRO - DIFFERENTIAL EQUATION VIA HAAR WAVELET BASES

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Abstract. In this work, we present a method for numerical approximation of fixed point operator, particularly for the mixed Volterra - Fredholm integro-differential equations, The main tool for error analysis is the Banach fixed point theorem. The advantage of this method is that it does not use numerical integration, we use the properties of rationalized Haar wavelets for approximate of integral. The cost of our algorithm increases accuracy and reduces the calculation, considerably. Some examples are provided to illustrate its high accuracy and numerical results are compared with other methods in the other papers.

1. Introduction

2. Introduction and preliminaries

Several applications for integro-differential equations exist, for example natural sciences and engineering and mathematical modeling of spatiotemporal developments, epidemic modeling, whereas analytical solutions of integro-differential equations, either do not exist or are hard to find, therefore we use several numerical methods that have been developed for finding approximate solutions of these equations.

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The orthogonal set of Haar functions is a group of square waves with magnitude of $+2^{i/2}$, $-2^{i/2}$ and 0, for any $i = 0, 1, \ldots$. The aim of this work is to present a numerical method for approximating the solution of nonlinear mixed Volterra - Fredholm integro-differential equations of the second kind as follows:

$$\lambda u'(x) = f(x) + \alpha \int_0^1 K_1(x, t, u(t))dt + \beta \int_0^x K_2(x, t, u(t))dt \quad (2.1)$$

where $x, t \in [0, 1], u \in C([0, 1], \mathbb{R})$, and $|\alpha| + |\beta| \neq 0$ and $u(0) = u_0$ and $\lambda \in \mathbb{R} - \{0\}$ and $f : [0, 1] \to \mathbb{R}$ and $K_1, K_2 : [0, 1]^2 \times \mathbb{R} \to \mathbb{R}$ are assumed to be known continuous functions satisfying the Lipschitz condition that is, there exist $M_1, M_2 \geqslant 0$ such that:

$$|K_i(x, t, y) - K_i(x, t, z)| \leqslant M_i|y - z| \ i = 1, 2$$

where $y, z \in \mathbb{R}$, and the unknown function to be determined is $u : [0, 1] \to \mathbb{R}$. We first define operator T within the space Banach space $C([0, 1])$ that are continuous and real valued functions defined on $[0, 1]$ (usual sup-norm), the integral operator $T : (X, \|.\|_\infty) \to (X, \|.\|_\infty)$ is defined as

$$T(u)(x) = \frac{1}{\lambda}u_0 + \frac{1}{\lambda}\int_0^x f(t)dt + \frac{\alpha}{\lambda}\int_0^x (\int_0^1 K_1(s, t, u(t))ds)dt \quad (2.2)$$

$$+\frac{\beta}{\lambda}\int_0^x (\int_0^s K_2(s, t, u(t))ds)dt \quad (2.3)$$

where $t \in [0, 1],\ u \in C([0, 1])$. For all $y_1, y_2 \in C([0, 1])$ we have $\|Ty_1 - Ty_2\| \leq \tilde{M}\|y_1 - y_2\|$.

3. Numerical approximation of the solution

We now describe the idea of our proposed numerical method. We using an initial function $u_0 \in C[0, 1]$, and since in general we cannot calculate $T(u_0)$ explicitly, we approximate this function as follows. For each $t, s \in [0, 1],\ u_i \in C([0, 1])$ and $m \in \mathbb{N}$, we define recursively:

$$u_i(x) = \frac{1}{\lambda}u_0 + \frac{1}{\lambda}\int_0^x f(t)dt + \frac{\alpha}{\lambda}\int_0^x (\int_0^1 K_1(s, t, u_{i-1}(t))ds)dt \quad (3.1)$$

$$+\frac{\beta}{\lambda}\int_0^x (\int_0^s K_2(s, t, u_{i-1}(t))ds)dt \quad (3.2)$$

also we define:

$$\psi_{i-1}(t, s) := K_1(t, s, u_{i-1}(s)),\ \varphi_{i-1}(t, s) := K_2(t, s, u_{i-1}(s)) \quad (3.3)$$

If Q_m is an orthogonal projection defined by:

$$Q_m(\psi)(t,s)=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}K_{ij}h_i(t)h_j(s),\ Q_m(\varphi)(t,s)=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}L_{ij}h_i(t)h_j(s). \tag{3.4}$$

The matrix form of K and L are:

$$Q_m(\psi)(t,s)=\mathbf{h^T}(t)\ \mathbf{K}\ \mathbf{h}(s),\ Q_m(\varphi)(t,s)=\mathbf{h^T}(t)\ \mathbf{L}\ \mathbf{h}(s), \tag{3.5}$$

Note that wherein:

$$\mathbf{h}(t)=[h_0(t),h_1(t),\ldots,h_{m-1}(t)]^T \tag{3.6}$$

And $\mathbf{K}=[K_{lq}]_{m\times m}$, $\mathbf{L}=[L_{lq}]_{m\times m}$ that:

$K_{lq}=2^{(i+j)/2}\langle h_l(t),\langle\psi(t,s),h_q(s)\rangle\rangle$ $L_{lq}=2^{(i+j)/2}\langle h_l(t),\langle\varphi(t,s),h_q(s)\rangle\rangle$

$l=2^j+k$ and $q=2^i+k'$ with $i,j=0,1,\ldots,$ and $k=0,1,\ldots,2^j-1$ and $k'=0,1,\ldots,2^i-1$, and $0\leq l,q\leq m-1$ Now, by using the RH function vector $\mathbf{h}(t)$, the matrix $\hat{\mathbf{h}}_{m\times m}$ is defined as:

$$\hat{\mathbf{h}}_{m\times m}=[\mathbf{h}(\frac{1}{2m}),\mathbf{h}(\frac{3}{2m}),\ldots,\mathbf{h}(\frac{2m-1}{2m})]. \tag{3.7}$$

Notice that using the equation 3, we have

$$\mathbf{K}=(\hat{\mathbf{h}}_{m\times m}^{-1})^{\mathbf{T}}\ \hat{\mathbf{K}}\ \hat{\mathbf{h}}_{m\times m}^{-1}\ ,\ \mathbf{L}=(\hat{\mathbf{h}}_{m\times m}^{-1})^{\mathbf{T}}\ \hat{\mathbf{L}}\ \hat{\mathbf{h}}_{m\times m}^{-1} \tag{3.8}$$

where $\hat{\mathbf{K}}=[\hat{\mathbf{K}}_{ij}]_{m\times m}$, $\hat{\mathbf{L}}=[\hat{\mathbf{L}}_{ij}]_{m\times m}$ and for $i,j=1,2,\ldots,m$, we have:

$$\hat{\mathbf{K}}_{ij}=\psi(\frac{2i-1}{2m},\frac{2j-1}{2m})\ ,\ \hat{\mathbf{L}}_{ij}=\varphi(\frac{2i-1}{2m},\frac{2j-1}{2m}) \tag{3.9}$$

4. Error analysis

In this section, by using the Banach fixed point theorem, we get an upper bound for the error of the our method.

Lemma 4.1. *Let $\psi\in C([0,1]^2)$, $L=\max\{\|\partial\psi/\partial t\|_\infty,\|\partial\psi/\partial s\|_\infty\}$ and $\{Q_n\}_{n\geq0}$ be the sequence of projections associated with the basis $\{h_n\}_{n\geq0}$, then*

$$\|\psi-Q_n(\psi)\|_\infty\leq\frac{4L}{2^l}, \tag{4.1}$$

holds where $n=2^l+k$ with $l=0,1,\ldots,$ and $k=0,1,\ldots,2^l-1$.

Lemma 4.2. *Let $K_1,K_2:[0,1]^2\times\mathbb{R}\to\mathbb{R}$ be continiouse and Lipschitzian with Lipschitz constants M_1 and M_2, then T has an unique fixed point and for all $u_0\in C([0,1])$*

$$\|u-T^m(u_0)\|_\infty\leq\|T(u_0)-u_0\|_\infty\times\sum_{j=m}^{\infty}q^j \tag{4.2}$$

where u is the fixed point of T and $q = \frac{1}{|\lambda|}(|\alpha|M_1 + |\beta|M_2) < 1$

Theorem 4.3. *Assume that $\{u_m\}_{m\geq 1}$ is a subset of $C([0,1])$, and $K_1, K_2 \in C([0,1]^2 \times \mathbb{R})$ are lipschitzian functions at their third variable with Lipschitz constants M_1, M_2 and u is the unique fixed point of the integral operator $T : C([0,1]) \to C([0,1])$ defined by (2.2). Then,*

$$\|u - u_m\|_\infty \leq \|T(u_0) - u_0\|_\infty \sum_{j=m}^{\infty} q^j + \sum_{j=1}^{m} q^{m-j}\varepsilon_j \tag{4.3}$$

holds, and $\varepsilon_1, \ldots, \varepsilon_m > 0$ we have $(|\frac{\alpha}{\lambda}| + |\frac{\beta}{\lambda}|)\frac{4L_{m-1}}{2^j} < \varepsilon_m$

5. Numerical examples

In this section by using the method presented in (3) are solved some examples from different references.

Example 5.1. Let us consider the nonlinear integral equation

$$u'(x) = 2x - \frac{1}{192}x^3\pi + \frac{1}{96}x^3 Ln(2) - \frac{1}{105}x^{10} + \frac{1}{24}\int_0^1 arctan(u)tx^3 dt \tag{5.1}$$

$$+\frac{1}{15}\int_0^x t^2x^3u^2(t)dt, \tag{5.2}$$

with $u(0) = 0$ with exact solution $u(x) = x^2$. Numerical results for Example 5.1 after 5 and 6 iterations are indicated in Table1, (see Figure 1).

Table 1. Absolute errors for Example 5.1

t_i	our method for $m = 32$	our method for $m = 64$
0.1	$2.03E{-}8$	$1.93E{-}8$
0.2	$3.10E{-}7$	$5.19E{-}8$
0.3	$1.22E{-}6$	$2.01E{-}7$
0.4	$8.66E{-}7$	$1.36E{-}6$
0.6	$7.54E{-}7$	$7.13E{-}6$
0.7	$2.77E{-}5$	$1.55E{-}5$
0.9	$1.99E{-}4$	$1.69E{-}4$

Example 5.2. Let us consider the nonlinear integral equation in [4]

$$\acute{u}(x) = xe^x + e^x - x + \int_0^1 xu(t)dt,\ u(0) = 0, \tag{5.3}$$

with exact solution $u(x) = xe^x$.

The results in Table 4 show that, our method are the same or even better than results obtained in [1], [2], [3] and [4] by CAS wavelet approximating, differential transformation methods, homotopy perturbation (HMP) methods and Schreuder bases with $m_h = j^2$ Schreuder bases respectively.

Table 2. Absolute errors for Example 5.2

t_i	*CAS wavelet* [1]	*DT* [2]	*HPM* [3]	*Schreuder bases* with $j = 17$ in [4]	*our* method
0.1	$1.3492E{-}3$	$1.0012E{-}2$	$0.2315E{-}5$	$1.0179E{-}7$	$1.8512E{-}6$
0.3	$5.6715E{-}3$	$5.0873E{-}2$	$0.2083E{-}4$	$1.0178E{-}6$	$1.4019E{-}6$
0.5	$1.3233E{-}2$	$9.7189E{-}2$	$0.5787E{-}4$	$2.3089E{-}6$	$5.0330E{-}7$
0.7	$1.4120E{-}2$	$1.0413E{-}1$	$0.1134E{-}3$	$3.9780E{-}6$	$2.2070E{-}6$
0.9	$1.3205E{-}2$	$1.0013E{-}2$	$0.1875E{-}3$	$6.1354E{-}6$	$3.4610E{-}6$

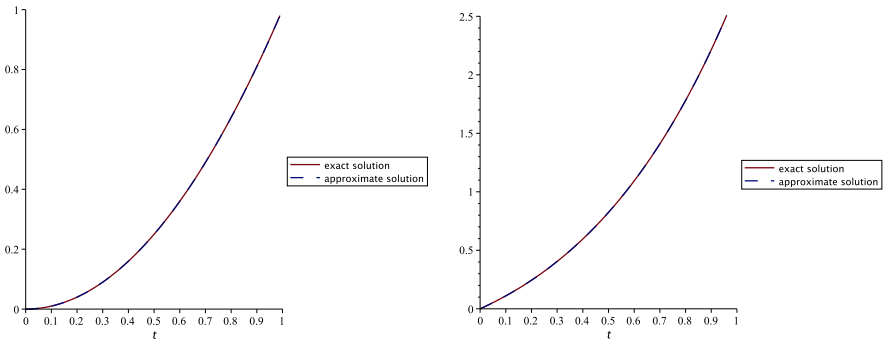

FIGURE 1. Comparison of exact and approximate solutions for Example 5.1,5.2.

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ITERATIVE METHOD FOR APPROXIMATING THE MOORE-PENROSE AND DRAZIN INVERSES

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Abstract. In the present paper an iterative method is proposed to approximate Moore-Ponrose and Drazin inverses. This method is based on Global Least Square algorithm (GL-LSQR). To do so, we define some linear matrix equations systems. By using the approximate solutions of these systems, obtained by GL-LSQR algorithm, Moore-Ponrose and Drazin inverses are approximated. Finally, some numerical examples are given to show the efficiency of the new method.

1. Introduction

Throughout this paper, for two matrices X and Y in $\mathbb{C}^{n\times m}$, we denote $\langle X, Y\rangle_F$ as the Frobenius inner product, i.e., $\langle X, Y\rangle_F = tr(Y^H X)$, and associated Frobenius norm is denoted by $||.||_F$, where A^H is conjugate transpose of A. $ind(A)$ denotes the index of A, i.e., it is the size of the largest Jordan block corresponding to the zero eigenvalue of A.

Moore-Penrose inverse of $A \in \mathbb{C}^{n\times m}$, denoted by $A^\dagger$, is the unique matrix satisfying in the following Penrose equations
(a):$AXA = A$ (b):$XAX = X$ (c):$(AX)^H = AX$ (d):$(XA)^H = XA$.
Drazin inverse of $A \in \mathbb{C}^{n\times n}$, denoted by A^D, is the unique matrix satisfying the following equations

$$(a'): \quad A^{k+1}A^D = A^k \quad (b'): \quad A^D A A^D = A^D \quad (c'): \quad AA^D = A^D A,$$

where k is the index of A [1].

Moore-Penrose and Drazin inverses have important spectral properties that make them extremely useful in many applications [4]. Iterative methods are very useful for solving matrix equations with sparse matrices. Although they can also apply to solve low dimension matrix equations with dense matrices [2].

In this paper, we define some linear matrix equations systems associated to each of Penrose and Drazin conditions. By using the approximate

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* Speaker.

solutions of these linear matrix equations systems, we approximate Moore-Penrose and Drazin inverses. Finally, some numerical experiments are presented.

2. Main results

In this section, we give some theorems and lemmas which are clear to prove.

Theorem 2.1. *Let Z and $W \in \mathbb{C}^{n\times m}$ be the solutions of the following systems, respectively*

$$(\dot{a}): \begin{cases} AZA &= A \\ (ZA)^H &= ZA, \end{cases} \quad (\dot{b}): \begin{cases} AWA &= A \\ (AW)^H &= AW, \end{cases}$$

then ZAW satisfies in the Penrose equations $(a), (b), (c)$ and (d).

Remark 2.2. Since Moore-Penrose inverse is unique, so ZAW is the Moore-Penrose inverse of A.

We can apply the GL-LSQR method [3] for solving the systems $(\dot{a})$ and $(\dot{b})$ but, it is better to implement this iterative method for solving the equivalent systems. Because in each iteration of the algorithm, the number of matrix-vector multiplications is lower, so less time is required to run. Therefore, we have the following lemma.

Lemma 2.3. *Let*

$$(\ddot{a}): \begin{cases} A\tilde{Z}A &= A \\ AA^H\tilde{Z}^H &= A, \end{cases} \quad \text{and } (\ddot{b}): \begin{cases} A\tilde{W}A &= A \\ \tilde{W}^HA^HA &= A. \end{cases}$$

The systems $(\ddot{a})$ and $(\ddot{b})$ are equivalent with the systems $(\dot{a})$ and $(\dot{b})$, respectively, means that any solution of system $(\dot{a})$ is a solution of system $(\ddot{a})$ and vice versa. Similarly, any solution of system $(\dot{b})$ is a solution of system $(\ddot{b})$ and vice versa

Now, we apply the GL-LSQR algorithm for solving systems $(\ddot{a})$ and $(\ddot{b})$. For this purpose,we define linear operator $\mathcal{A}$ as follows. Let $A \in \mathbb{C}^{m\times n}$

$$\mathcal{A}: \mathbb{C}^{n\times m} \to \mathbb{C}^{2m\times n},\ \mathcal{A}(\tilde{Z}) = \begin{bmatrix} A\tilde{Z}A \\ AA^H\tilde{Z}^H \end{bmatrix}.$$

So, the system $(\ddot{a})$ can be reduced to the operator form $\mathcal{A}(\tilde{Z}) = \begin{bmatrix} A \\ A \end{bmatrix} = \mathcal{B}$.

Definition 2.4. Let $\mathcal{A}$ be the operator mentioned above. Then $\mathcal{A}^* : \mathbb{C}^{2m\times n} \to \mathbb{C}^{n\times m}$ is said the adjoint of $\mathcal{A}$ if
$\langle \mathcal{A}(X), \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \rangle_F = \langle X, \mathcal{A}^*\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \rangle_F$, where $Y_i \in \mathbb{C}^{m\times n}$ $i = 1,\ 2$.

$$\text{Let } \begin{cases} \mathcal{A}^* : \mathbb{C}^{2m\times n} \to \mathbb{C}^{n\times m}, \\ \mathcal{A}^*\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = A^HY_1A^H + Y_2^HAA^H. \end{cases}$$

It is easy to show that $\mathcal{A}^*$ is the adjoint of operator $\mathcal{A}$ in the Frobenius inner product.

A linear operator and its adjoint can be similarly defined for the linear system $(\ddot{b})$. The following theorem is similar to Theorem (2.1).

Theorem 2.5. *Let W be a solution of the following linear matrix equations system*

$$(\hat{d}): \begin{cases} A^{k+1}W = A^k \\ AW - WA = 0 \\ AWA = A, \end{cases}$$

where k is $ind(A)$. Then $A^D = WAW$ is the Drazin inverse of A.

Remark 2.6. To approximate the group inverse, i.e., $ind(A) = 0$ or 1, it is enough to solve linear matrix equations $\begin{cases} A^{k+1}W = A^k, \ k = 0, 1 \\ WA - AW = 0. \end{cases}$

Then it is easy to prove that $A^g = WAW$ is the group inverse of matrix A. For applying Gl-LSQR algorithm to approximate the solutions of the systems $(\ddot{a}), (\ddot{b})$ and $(\hat{d})$, we need the the following bidiagonal process.
Bidiag-Process (starting matrix $\mathcal{B}$; reduction to lower bidiagonal matrix form):

$$\beta_1 = ||\mathcal{B}||_F, \ \mathcal{U}_1 = \frac{\mathcal{B}}{\beta_1}, \ \alpha_1 = ||\mathcal{A}^*(\mathcal{U}_1)||_F, \mathcal{V}_1 = \frac{\mathcal{A}^*(\mathcal{U}_1)}{\alpha_1}$$

$$\left.\begin{array}{c} \beta_{i+1} = ||\mathcal{A}(\mathcal{V}_i) - \alpha_i\mathcal{U}_i||_F, \ \mathcal{U}_{i+1} = \frac{\mathcal{A}(\mathcal{V}_i) - \alpha_i\mathcal{U}_i}{\beta_{i+1}} \\ \alpha_{i+1} = ||\mathcal{A}^*(\mathcal{U}_{i+1}) - \beta_{i+1}\mathcal{V}_i||_F, \ \mathcal{V}_{i+1} = \frac{\mathcal{A}^*(\mathcal{U}_{i+1}) - \beta_{i+1}\mathcal{V}_i}{\alpha_{i+1}} \end{array}\right\}, \ i = 1, 2, ..., m.$$

Now, the main step of the GL-LSQR algorithm can be summarized as follows.
Algorithm: GL-LSQR algorithm

(1) Set $\mathcal{X}_0 = 0$
(2) $\beta_1 = ||\mathcal{B}||_F$, $\mathcal{U}_1 = \mathcal{B}/\beta_1$, $\alpha_1 = ||\mathcal{A}^*(\mathcal{U}_1)||_F$, $\mathcal{V}_1 = \mathcal{A}^*(\mathcal{U}_1)/\alpha_1$
(3) Set $\mathcal{W}_1 = \mathcal{V}_1$, $\bar{\phi}_1 = \beta_1$, $\bar{\rho}_1 = \alpha_1$
(4) For $i = 1, 2, ...$ until convergence, Do:
(5) $\quad \hat{W}_i = \mathcal{A}(\mathcal{V}_i) - \alpha_i\mathcal{U}_i$, $\beta_{i+1} = ||\hat{W}_i||_F$, $\mathcal{U}_{i+1} = \hat{W}_i/\beta_{i+1}$
(6) $\quad \hat{S}_i = \mathcal{A}^*(\mathcal{U}_{i+1}) - \beta_{i+1}\mathcal{V}_i$, $\alpha_{i+1} = ||\hat{S}_i||_F$, $\mathcal{V}_{i+1} = \hat{S}_i/\alpha_{i+1}$
(7) $\quad \rho_i = (\bar{\rho}_i^2 + \beta_{i+1}^2)^{1/2}$, $c_i = \bar{\rho}_i/\rho_i$, $s_i = \beta_{i+1}/\rho_i$, $\theta_{i+1} = s_i\alpha_{i+1}$
(8) $\quad \bar{\rho}_{i+1} = c_i\alpha_{i+1}$, $\phi_i = c_i\bar{\phi}_i$, $\bar{\phi}_{i+1} = -s_i\phi_i$
(9) $\quad \mathcal{X}_i = \mathcal{X}_{i-1} + (\phi_i/\rho_i)\mathcal{W}_i$
(10) $\quad \mathcal{W}_{i+1} = \mathcal{V}_{i+1} - (\theta_{i+1}/\rho_i)\mathcal{W}_i$
(11) If $|\bar{\phi}_{i+1}|$ is small enough then stop
(12) EndDo.

3. Numerical results

In this section, two examples are given for computation of the Moore-Penrose and Drazin inverses, respectively. Codes were written in Matlab with double precision.

Example 3.1. Let $A = rand(n, m)$, where *rand* is MATLAB function which generates random number between 0 and 1. We applied the GL-LSQR algorithm to approximate the Moore-Penrose inverse and the numerical results are given in Table (1).

Table 1. Numerical results for Example 3.1.

$n \times m$	70×20	70×30	70×40
Iteration	95	253	513
Time(in seconds)	0.0502	0.1479	0.3558
$\|\|AXA - A\|\|_F$	$2.9214e-08$	$8.9649e-08$	$8.1573e-08$
$\|\|XAX - X\|\|_F$	$7.8579e-10$	$4.9744e-09$	$1.0772e-08$
$\|\|(AX)^H - AX\|\|_F$	$1.6490e-08$	$4.2249e-08$	$4.1780e-08$
$\|\|(XA)^H - XA\|\|_F$	$4.4330e-09$	$5.9464e-08$	$4.3161e-08$

Example 3.2. Let $A = tridiag(-1, 2, 1)$, $A(1, n) = -1$ and $A(n, 1) = -1$ with $ind(A) = 1$. We applied GL-LSQR algorithm for computing the Drazin inverse of A and the numerical results are given in Table (2).

Table 2. Numerical results for Example 3.2.

n	50	100	150
Iteration	1926	14282	> 20000
Time(in seconds)	0.5707	11.4414	58.3015
$\|\|A^2X - A\|\|_F$	$3.5379e-12$	$8.4626e-12$	$1.2775e-07$
$\|\|XA - AX\|\|_F$	$1.5356e-12$	$3.6057e-11$	$3.8981e-09$
$\|\|XAX - X\|\|_F$	$1.6408e-10$	$7.2547e-09$	$1.6286e-04$

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A DIFFERENCE TYPE REVERSE OF THE OPERATOR BELLMAN INEQUALITY

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Abstract. The operator Bellman inequality says that if A, B are positive contraction operators, Φ is a unital positive linear map on $\mathbb{B}(\mathcal{H})$ and $p \geq 1$, then

$$\Phi\left((I-A)^{1/p}\nabla_\lambda(I-B)^{1/p}\right) \leq (\Phi(I-A\nabla_\lambda B))^{1/p} .$$

In this paper, we use the Mond–Pečarić method to establish a reverse of the operator Bellman inequality.

1. Introduction and preliminaries

Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with the identity I. In the case when $\dim\mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the matrix algebra $\mathcal{M}_n(\mathbb{C})$ of all $n\times n$ matrices with entries in the complex field. An operator $A \in \mathbb{B}(\mathcal{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$ and in a case we write

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* Speaker.

$A \geq 0$. We write $A > 0$ if A is a positive invertible operator. The set of all positive invertible operators is denoted by $\mathbb{B}(\mathcal{H})_+$. For self-adjoint operators $A, B \in \mathbb{B}(\mathcal{H})$, we say $B \geq A$ if $B - A \geq 0$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$-isomorphism between the C^*-algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of a self-adjoint operator A and the C^*-algebra generated by A and the identity operator I. If $f, g \in C(\mathrm{sp}(A))$, then $f(t) \geq g(t)$ $(t \in \mathrm{sp}(A))$ implies that $f(A) \geq g(A)$.

Let f be a continuous real valued function defined on an interval J. It is called operator monotone if $A \leq B$ implies $f(A) \leq f(B)$ for all self-adjoint operators $A, B \in \mathbb{B}(\mathcal{H})$ with spectra in J. It said to be operator concave if $\lambda f(A) + (1-\lambda) f(B) \leq f(\lambda A + (1-\lambda)B)$ for all self-adjoint operators $A, B \in \mathbb{B}(\mathcal{H})$ with spectra in J and all $\lambda \in [0,1]$. Every nonnegative continuous function f is operator monotone on $[0, +\infty)$ if and only if f is operator concave on $[0, +\infty)$; see [2, Theorem 8.1]. A map $\Phi : \mathbb{B}(\mathcal{H}) \longrightarrow \mathbb{B}(\mathcal{K})$ is called positive if $\Phi(A) \geq 0$ whenever $A \geq 0$, where $\mathcal{K}$ is a complex Hilbert space and is said to be unital if $\Phi(I_{\mathcal{H}}) = I_{\mathcal{K}}$. We denote by $\mathbf{P}[\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{K})]$ the set of all positive linear maps $\Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K})$ and by $\mathbf{P}_N[\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{K})]$ the set of all unital positive linear maps.

Bellman [1] proved that if p is a positive integer and A, B, a_j, b_j $(1 \leq j \leq n)$ are positive real numbers such that $A^p \geq \sum_{j=1}^n a_j^p$ and $B^p \geq \sum_{j=1}^n b_j^p$, then

$$\left(A^p - \sum_{j=1}^n a_j^p\right)^{1/p} + \left(B^p - \sum_{j=1}^n b_j^p\right)^{1/p} \leq \left((A+B)^p - \sum_{j=1}^n (a_j + b_j)^p\right)^{1/p}.$$

In [3] the authors showed an operator Bellman inequality as follows:

$$\Phi\left((I-A)^{1/p} \nabla_\lambda (I-B)^{1/p}\right) \leq \left(\Phi(I - A \nabla_\lambda B)\right)^{1/p},$$

whenever A, B are positive contraction operators, Φ is a unital positive linear map on $\mathbb{B}(\mathcal{H})$ and $p \geq 1$.

In this paper, we use the Mond–Pečarić method to present some reverses of the operator Bellman inequality under some mild conditions.

2. A difference type reverse of the operator Bellman inequality

Let $A_j \in \mathbb{B}(\mathcal{H})$ be self-adjoint operators with $\mathrm{sp}(A_j) \subseteq [m, M]$ for some scalars $m < M$, Φ_j be unital positive linear maps on $\mathbb{B}(\mathcal{H})$ $(j = 1, \cdots, n)$, $\omega_1, \cdots, \omega_n \in \mathbb{R}_+$ be any finite number of positive real

numbers such that $\sum_{j=1}^{n}\omega_j = 1$ and f be a strictly concave differentiable function. Then

$$\beta_f I + \sum_{j=1}^{n}\omega_j\Phi_j\big(f(A_j)\big) \geq f\left(\sum_{j=1}^{n}\omega_j\Phi_j(A_j)\right) \tag{2.1}$$

where $\beta_f = \max_{m\leq t\leq M}\{f(t) - \mu_f t - \nu_f\}$; see also [2, Corollary 2.16].

Proposition 2.1 (A difference type reverse of the operator Bellman inequality)**.** *Let $A_j, \Phi_j, \omega_j, j = 1, \cdots, n$ be as above, A_j are contractions such that $0 < mI_{\mathcal{H}} \leq A_j \leq M_{\mathcal{H}} < I_{\mathcal{H}}$ and $0 < p < 1$. Then*

$$\delta I_{\mathcal{K}} + \sum_{j=1}^{n}\omega_j\Phi_j\left((I_{\mathcal{H}} - A_j)^p\right) \geq \left(\sum_{j=1}^{n}\omega_j\Phi_j(I_{\mathcal{H}} - A_j)\right)^p, \tag{2.2}$$

where $\delta = (1-p)\left(\frac{1}{p}\frac{(1-m)^p-(1-M)^p}{M-m}\right)^{\frac{p}{p-1}} + \frac{(1-M)(1-m)^p-(1-m)(1-M)^p}{M-m}$.

In particular,

$$\delta I_{\mathcal{K}} + \Phi\left(\sum_{j=1}^{n}\omega_j(I_{\mathcal{H}} - A_j)^p\right) \geq \left(\Phi\left(\sum_{j=1}^{n}(I_{\mathcal{H}} - A_j)\right)\right)^p, \tag{2.3}$$

Corollary 2.2 (Generalized of the difference type classical Bellman inequality)**.** *If m, n, p are positive integers, $\omega_1, \cdots, \omega_n \in \mathbb{R}_+$ are any finite number of positive real numbers such that $\sum_{j=1}^{n}\omega_j = 1$ and a_{ij} $(j = 1, \cdots n,\ i = 1, \cdots, m)$ are positive real numbers such that $1 \geq \sum_{i=1}^{m} a_{ij}^p \quad (j = 1, \cdots n)$, then*

$$\left(1 - \frac{1}{p}\right)p^{\frac{1}{1-p}} + \sum_{j=1}^{n}\omega_j\left(1 - \sum_{i=1}^{m}a_{ij}^p\right)^{\frac{1}{p}} \geq \left(1 - \sum_{j=1}^{n}\sum_{i=1}^{m}\omega_j a_{ij}^p\right)^{\frac{1}{p}}. \tag{2.4}$$

Corollary 2.3. *Let $\Phi \in \mathbf{P}_N[\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{K})]$, $0 < mI_{\mathcal{H}} \leq A_j \leq MI_{\mathcal{H}}$ be positive operators and $0 \leq \omega_j \leq 1$ $(j = 1, \cdots, n)$ such that $\sum_{j=1}^{n}\omega_j = 1$. Then*

$$\log\left[\frac{1}{e}\left(\frac{M^m}{m^M}\right)^{\frac{1}{M-m}}L(m, M)\right] + \Phi\left(\sum_{j=1}^{n}\omega_j\log A_j\right) \geq \log\left(\sum_{j=1}^{n}\omega_j\Phi(A_j)\right)$$

where $L(a, b) = \begin{cases} \frac{b-a}{\log b - \log a} & ; a \neq b \\ a & ; a = b \end{cases}$ *is the Logarithmic mean of positive real numbers a and b.*

Corollary 2.4. *Let* $\Phi \in \mathbf{P}_N[\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{K})]$, $0 < mI_{\mathcal{H}} \leq A_j \leq MI_{\mathcal{H}}$ *be positive operators and* $0 \leq \omega_j \leq 1$ $(j = 1, \cdots, n)$ *such that* $\sum_{j=1}^{n} \omega_j = 1$. *Then*

$$I(m, M) - L(1/M, 1/m)^{-1} + \Phi\left(\sum_{j=1}^{n} \omega_j A_j\right) \log\left(\Phi\Big(\sum_{j=1}^{n} \omega_j A_j\Big)\right) \geq \Phi\left(\sum_{j=1}^{n} \omega_j A_j \log A_j\right).$$

where $I(a, b) = \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & ; a \neq b \\ a & ; a = b \end{cases}$ *is the Identric mean of positive real numbers* a *and* b.

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AN EXACT FINITE DIFFERENCE SCHEME FOR TELEGRAPH EQUATION

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Abstract. In this paper, we present a finite difference scheme for telegraph equation. A new version of exact nonstandard finite difference (NSFD) scheme for telegraph equation is proposed using the analytical solution.

1. Introduction

During the last few decades, the telegraph equation has played an important role in electrical engineering. The standard form of the telegraph equation is given by [2]

$$u_{xx} = au_{tt} + bu_t + cu,$$

where $u = u(x,t)$ is the resistance, and a, b and c are constants related to the inductance, capacitance and conductance of the cable respectively. The telegraph equation arises in the propagation of electrical signals along a telegraph line. Consider the following telegraph equation

$$u_{xx} = u_{tt} + u_t - u, \tag{1.1}$$

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* Speaker.

with initial conditions

$$u(x,0) = e^x, \qquad u_t(x,0) = -2e^x, \qquad x \in [0, X],$$

and boundary conditions

$$u(0,t) = e^{-2t}, \qquad u_x(0,t) = e^{-2t}, \qquad t \in [0, T].$$

The nonstandard discretization technique is a general scheme where we replace the stepsize h by a function $\phi(h)$ [3]. NSFD schemes were firstly proposed by Mickens [1] for either ordinary differential equations or partial differential equations and successively, their use has been investigated in several fields.
This paper is devoted to the construction of an exact difference scheme based on Mickens method for solving the telegraph equation.

2. Analysis of NSFD scheme for telegraph equation

In this section, we illustrate an exact finite difference scheme for telegraph equation (1.1). Analytical solution to this equation is given by [2]

$$u(x,t) = e^{x-2t}. \tag{2.1}$$

If we choose $h = 2\Delta t$, then one can easily obtain $u(x-h,t) = u(x,t+\Delta t)$ and the following equations:

$$\begin{aligned}
\frac{1}{u(x,t)} &= \frac{1}{e^{x-2t}},\\
\frac{1}{u(x+h,t)} &= \frac{1}{e^{x+h-2t}},\\
\frac{1}{u(x-h,t)} &= \frac{1}{e^{x-h-2t}},\\
\frac{1}{u(x,t+\Delta t)} &= \frac{1}{e^{x-2\Delta t-2t}},\\
\frac{1}{u(x,t-\Delta t)} &= \frac{1}{e^{x+2\Delta t-2t}}.
\end{aligned} \tag{2.2}$$

According to (2.2) we can write

$$\begin{aligned}
\frac{1}{u(x,t)} - \frac{1}{u(x+h,t)} &= \frac{1}{u(x+h,t)}(e^h - 1),\\
\frac{1}{u(x,t)} - \frac{1}{u(x-h,t)} &= \frac{1}{u(x-h,t)}(e^{-h} - 1),\\
\frac{1}{u(x,t)} - \frac{1}{u(x,t+\Delta t)} &= \frac{1}{u(x,t+\Delta t)}(e^{-2\Delta t} - 1),\\
\frac{1}{u(x,t)} - \frac{1}{(x,t-\Delta t)} &= \frac{1}{u(x,t-\Delta t)}(e^{2\Delta t} - 1).
\end{aligned}$$

Let the step functions are $\psi_1 = (e^h - 1)$, $\psi_2 = (1 - e^{-h})$, $\varphi_1 = \dfrac{e^{2\Delta t} - 1}{2}$ and $\varphi_2 = \dfrac{1 - e^{-2\Delta t}}{2}$, so $2\varphi_1 = \psi_1$ and $2\varphi_2 = \psi_2$. Thus, we can have the forward and backward difference quotients with the special stepsize functions:

$$\begin{aligned} \partial_x u &= \frac{u(x+h,t) - u(x,t)}{\psi_1} = u(x,t), \\ \bar{\partial}_x u &= \frac{u(x,t) - u(x-h,t)}{\psi_2} = u(x,t), \\ \partial_t u &= \frac{u(x,t+\triangle t) - u(x,t)}{\varphi_2} = -2u(x,t), \\ \bar{\partial}_t u &= \frac{u(x,t) - u(x,t-\triangle t)}{\varphi_1} = -2u(x,t). \end{aligned} \tag{2.3}$$

2.1. **An explicit finite difference scheme.** If we select $u_{xx} = \partial_x \bar{\partial}_x u$ and $u_{tt} = \partial_t \bar{\partial}_t u$ then using (2.3) we can get

$$\begin{aligned} \partial_x \bar{\partial}_x u - \partial_t \bar{\partial}_t u &= \frac{u(x,t) - u(x-h,t)}{\psi_2} - \frac{-2u(x,t) + 2u(x,t-\Delta t)}{\varphi_1} \\ &= \frac{6u(x,t) - 3u(x+h,t) - 3u(x-h,t)}{(e^h - 1)(1 - e^{-h})} \\ &= 2\frac{u(x,t) - u(x+h,t) + u(x,t) - u(x-h,t)}{(e^h - 1)(1 - e^{-h})} \\ &+ \frac{2u(x,t) - u(x+h,t) - u(x-h,t)}{(e^h - 1)(1 - e^{-h})} \\ &= 2\frac{u(x,t)(1 - e^h) + u(x-h,t)(e^h - 1)}{(e^h - 1)(1 - e^{-h})} \\ &+ u(x,t)\frac{2 - e^h - e^{-h}}{(e^h - 1)(1 - e^{-h})} \\ &= 2\frac{u(x-h,t) - u(x,t)}{(1 - e^{-h})} - u(x,t) \\ &= \frac{u(x,t+\Delta t) - u(x,t)}{\varphi_2} - u(x,t). \end{aligned} \tag{2.4}$$

Based upon the analytical solution (2.1) we write U_j^n as

$$U_j^n = u(x_j, t_n) = e^{x_j - 2t_n}. \tag{2.5}$$

Then we can write an explicit NSFD scheme according to (2.4) and (2.5) in the following form

$$\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\psi_1\psi_2} = \frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{\varphi_1\varphi_2} + \frac{U_j^{n+1} - U_j^n}{\varphi_2} - U_j^n. \quad (2.6)$$

Hence the following theorem is hold.

Theorem 2.1. *An explicit exact NSFD scheme for telegraph equation (1.1) is given by (2.6). The stepsize satisfies $h = 2\Delta t$, and the stepsize functions satisfy*

$$\psi_1 = (e^h - 1), \qquad \varphi_1 = \frac{e^{2\Delta t} - 1}{2},$$
$$\psi_2 = (1 - e^{-h}), \qquad \varphi_2 = \frac{1 - e^{-2\Delta t}}{2}.$$

3. Conclusion

In this paper, we peresent an exact NSFD scheme for telegraph equation based on the analytical solution. The proposed step functions can be constructed using the method in the Mickens papers.

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PRECONDITIONED GSOR ITERATION METHOD FOR A CLASS OF COMPLEX SYMMETRIC LINEAR SYSTEMS

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Abstract. For solving a broad class of complex symmetric system of linear equations, a preconditioned variant of the generalized successive overrelaxation (GSOR) iterative method is presented. The iteration matrix of the method involves two parameters α and ω. We study the convergence properties of the method and determine the optimal values of iteration parameters. Numerical examples are given to illustrate the performance of the proposed method.

1. Introduction

Consider the system of complex linear equations of the form

$$Ax \equiv (W + iT)x = b, \qquad x, b \in \mathbb{C}^n, \tag{1.1}$$

where $i = \sqrt{-1}$ and $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semidefinite matrices. Note that A is nonsingular if and only if $null(W) \cap null(T) = \{0\}$ and i is not a generalized eigenvalue of the matrix pair (W, T) (i.e., $Tx \neq iWx$ for any $x \neq 0$), where $null(\cdot)$ denotes the nullspace of the corresponding matrix. For more details about the practical backgrounds of this class of problems, the reader is referred to [3].

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* Speaker.

Bai et al. in [2] described the Hermitian and skew-Hermitian splitting (HSS) method to solve non-Hermitian positive definite system of linear equations. Based on the particular form of the coefficient matrix (1.1), Bai et al. in [1] proposed the modified HSS (MHSS) iteration method for solving the complex system (1.1).

To avoid the complex arithmetic, Salkuyeh et al. in [4], by letting $x = u + iv$ and $b = p + iq$ where $u, v, p, q \in \mathbb{R}^n$, rewrote the complex system (1.1) in the following real equivalent form

$$\begin{bmatrix} W & -T \\ T & W \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}, \tag{1.2}$$

and, based on the successive overrelaxation (SOR) method, presented the generalized SOR (GSOR) method for solving the real equivalent system (1.2) when at least one of the matrices W and T is positive definite.

In this paper, we present a preconditioned variant of the GSOR iterative method, called preconditioned GSOR (PGSOR) iteration method, and study the convergence properties of the method and determine the optimal values of involved parameters.

2. THE PGSOR ITERATIVE METHOD

Let

$$\mathcal{P}_\omega = \begin{bmatrix} \omega I & I \\ -I & \omega I \end{bmatrix}.$$

We apply $\mathcal{P}_\omega$ to the system (1.2) to obtain the following preconditioned linear system

$$\mathcal{R}u \equiv \begin{bmatrix} \tilde{W}_\omega & -\tilde{T}_\omega \\ \tilde{T}_\omega & \tilde{W}_\omega \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix} \equiv d, \tag{2.1}$$

where $\tilde{W}_\omega = \omega W + T$, $\tilde{T}_\omega = \omega T - W$, $\tilde{p} = \omega p + q$ and $\tilde{q} = \omega q - p$. By splitting the coefficient matrix $\mathcal{R}$ as

$$\mathcal{R} = \begin{bmatrix} \tilde{W}_\omega & 0 \\ 0 & \tilde{W}_\omega \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -\tilde{T}_\omega & 0 \end{bmatrix} - \begin{bmatrix} 0 & \tilde{T}_\omega \\ 0 & 0 \end{bmatrix},$$

the PGSOR method for solving (2.1) is given by

$$\begin{cases} \tilde{W}_\omega x^{(k+1)} = (1-\alpha)\tilde{W}_\omega x^{(k)} + \alpha\tilde{T}_\omega y^{(k)} + \alpha\tilde{p}, \\ \tilde{W}_\omega y^{(k+1)} = -\alpha\tilde{T}_\omega x^{(k+1)} + (1-\alpha)\tilde{W}_\omega y^{(k)} + \alpha\tilde{q}. \end{cases} \tag{2.2}$$

It is straightforward to see that

$$\mathcal{L}(\omega, \alpha) = \begin{bmatrix} I & 0 \\ \alpha\tilde{S}_\omega & I \end{bmatrix}^{-1} \begin{bmatrix} (1-\alpha)I & \alpha\tilde{S}_\omega \\ 0 & (1-\alpha)I \end{bmatrix},$$

is the iteration matrix of the PGSOR method, where $\tilde{S}_\omega = \tilde{W}_\omega^{-1}\tilde{T}_\omega$.

We can easily find that $\omega W+T$ is symmetric positive definite. Hence, the PGSOR method is well defined and the linear system with coefficient matrix $\omega W + T$ can be exactly solved by the Cholesky factorization or inexactly by the CG algorithm. We now state the following theorem.

Theorem 2.1. *Let $W, T \in \mathbb{R}^{n\times n}$ be symmetric positive semidefinite satisfying $null(W) \cap null(T) = \{0\}$. Also, let $\mu_{\min}$ and $\mu_{\max}$ be the smallest and largest generalized eigenvalues of the matrix pair (W, T), respectively. Then, for every positive parameter ω, the PGSOR iteration method is convergent if and only if*

$$0 < \alpha < \frac{2}{1+\rho(\tilde{S}(\omega))},$$

where $\tilde{S}(\omega) = (\omega W + T)^{-1}(\omega T - W)$ and

$$\rho(\tilde{S}(\omega)) = \max\left\{\frac{1-\omega\mu_{\min}}{\omega+\mu_{\min}}, \frac{\omega\mu_{\max}-1}{\omega+\mu_{\max}}\right\}.$$

In addition, the optimal values of the parameters α and ω are given by

$$\alpha^* = \frac{2}{1+\sqrt{1+\xi^2}} \quad \text{and} \quad \omega^* = \frac{1-\eta_{\min}\eta_{\max}+\sqrt{(1+\eta_{\min}^2)(1+\eta_{\max}^2)}}{\eta_{\min}+\eta_{\max}},$$

where $\xi = \frac{1-\omega^*\eta_{\min}}{\omega^*+\eta_{\min}} \left(= \frac{\omega^*\eta_{\max}-1}{\omega^*+\eta_{\max}}\right)$ and the corresponding optimal convergence factor is given by

$$\rho(\mathcal{L}(\omega^*, \alpha^*)) = 1 - \alpha^* = 1 - \frac{2}{1+\sqrt{1+\xi^2}}.$$

Moreover, we have $\rho(\mathcal{L}(\omega^*, \alpha^*)) < \frac{\sqrt{2}-1}{\sqrt{2}+1} \approx 0.172$, and also if $\hat{\omega} = 1$ and $\hat{\alpha} = \frac{2}{\sqrt{2}+1}$ then $\rho(\mathcal{L}(\hat{\omega}, \hat{\alpha})) = \frac{\sqrt{2}-1}{\sqrt{2}+1} \approx 0.172$.

In the sequel, we present a numerical example that is computed in double precision using some MATLAB codes on a Laptop, Intel Core i7, 1.8 GHz with 6GB RAM. We use a null vector as an initial guess and the stopping criterion $\|b - Ax^{(k)}\|_2 < 10^{-6}\|b\|_2$.

Example 2.2. Let

$$W = K + \frac{3-\sqrt{3}}{h}I \quad \text{and} \quad T = K + \frac{3+\sqrt{3}}{h}I,$$

where $K = I \otimes V_m + V_m \otimes I$, with $V_m = h^{-2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$, $h = 1/(m+1)$ and m is a natural number. Therefore $A = W + iT \in$

TABLE 1. Numerical results for Example 2.2.

	m	16	32	64	128	256
MHSS	IT(CPU)	40(0.08)	54(0.30)	73(1.53)	98(8.50)	133(52.57)
	α^*	1.06	0.75	0.54	0.40	0.30
GSOR	IT(CPU)	19(0.02)	22(0.02)	24(0.07)	26(0.31)	27(1.40)
	α^*	0.550	0.495	0.457	0.432	0.418
PGSOR	IT(CPU)	4(0.01)	4(0.01)	5(0.03)	5(0.12)	5(0.52)
	α^*	0.990	0.987	0.986	0.984	0.983
	ω^*	0.657	0.624	0.602	0.590	0.583
PGSOR	IT(CPU)	9(0.01)	9(0.02)	9(0.04)	8(0.13)	8(0.61)
	$\hat{\alpha} = \frac{2}{\sqrt{2}+1}$					
	$\hat{\omega} = 1$					

$\mathbb{R}^{n\times n}$ where $n = m^2$. Moreover, the right-hand side vector is given by

$$b_j = \frac{(1-i)j}{\tau(j+1)^2}, \quad j = 1, 2, \ldots, n.$$

Numerical results for Example 2.2 are listed in TABLE 1. This table presents optimal value of the parameters and the number of iteration steps and the CPU times, where the CPU times are shown in parentheses. This table shows ITs and CPUs for the MHSS and GSOR method when the optimal values of the parameters are used and for PGSOR method when the optimal values (α^*, ω^*) and approximation values $\hat{\alpha} = \frac{2}{\sqrt{2}+1}$ and $\hat{\omega} = 1$ used. As seen the PGSOR method is more effective than the MHSS and GSOR method in terms of both iterations and CPU times and the performance of the PGSOR method with the approximation parameters is also close that of the optimal parameters.

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ON SOME SPECTRAL PROBLEMS INVOLVING A SHIFT OPERATOR

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Abstract. For an arbitrary shift operator S on a separable Hilbert space $\mathcal{H}$, we study the eigenoperators of the linear transformations $\Gamma_i : B(\mathcal{H}) \longrightarrow B(\mathcal{H})$ that are defined by $\Gamma_1(X) = S^*XS$, $\Gamma_2(X) = SX - XS^*$, $\Gamma_3(X) = SX - XS$ and $\Gamma_4(X) = S^*X - XS$, where $B(\mathcal{H})$ is the space of all bounded linear operators on $\mathcal{H}$.

1. Introduction

Throughout this paper the Hilbert space $\mathcal{H}$ is separable infinite dimensional and the space of all bounded linear operators on $\mathcal{H}$ is denoted by $B(\mathcal{H})$. One can identify $\mathcal{H}$ with the set of all one-side square summable sequence that denoted by l^2. The canonical basis of l^2 is $\{e_n\}_{n=0}^{\infty}$ where e_n is the sequence consisting of zero, except for the n-th place which is one. The Hardy space $\mathcal{H}^2 = \mathcal{H}^2(\mathbb{D})$ is the collection of all analytic functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$ on open unit disk $\mathbb{D}$ satisfying the norm condition $\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty$.

The unilateral (forward) shift operator on l^2 is defined by $U(e_n) = e_{n+1}$ for $n \geq 0$. A Toeplitz operator T on a Hilbert space $\mathcal{H}$ is one whose matrix has constant diagonals and it can be characterized by the

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* Speaker.

equation $U^*TU = T$. The matrix representation of a Hankel operator has constant diagonals orthogonal to the main diagonal. One can find H is an Hankel operator if and only if it satisfies the equation $U^*H = HU$.

Barria and Halmos [2] introduce generalized Toeplitz operator T which is satisfied in the equation $U^*TU = \lambda T$ for a complex number λ. Indeed, as a spectral problem, they asked what are the eigenoperators of the linear transformation $\Gamma : B(\mathcal{H}) \longrightarrow B(\mathcal{H})$ defined by $\Gamma(X) = U^*XU$. For example, all of Toeplitz operators are eigenoperators corresponding to eigenvalue $\lambda = 1$. S. Sun [5] characterises all eigenoperators of Γ. An isometry operator S on a Hilbert space $\mathcal{H}$ is called a shift operator if the sequence $\{S^{*n}\}$ strongly converges to zero. A more general spectral problem was introduced here by replacing U with a shift operator S in the definition of Γ. The operator $T \in B(\mathcal{H})$ is called S-Toeplitz if $S^*TS = T$. Therefore S-Toeplitz operators are the eigenoperators of Γ corresponding to $\lambda = 1$. This spectral problem was thoroughly solved in section 2 using the techniques of S. Sun [5] and R. Martínez-Avendaño [3].

A λ-Hankel operator is introduced in [3] as one satisfying the operator equation $U^*X - XU = \lambda X$, where λ is a complex number. Some other operator equations involving U such as $UX - XU = \lambda X$ is introduced as well. In the case $\lambda = 0$, every Hankel operator is a 0-Hankel operator and all analytic Toeplitz operators are those that satisfy in the last equation. In Section 2, we replace the unilateral shift operator by a general shift operator and investigate these operator equations.

2. Some equations involving forward and backward shift operators

Let S be a shift operator on a Hilbert space $\mathcal{H}$, $\mathcal{K}$ is the kernel of S^* and $P_0 = I - S^*S$ is the projection $\mathcal{H}$ onto $\mathcal{K}$. Every operator $A \in B(\mathcal{H})$ is associated with a matrix of operators on the Hilbert space $\mathcal{K}$ which is $A \sim [A_{ij}]_{i,j=0}^{\infty}$, where $A_{ij} = P_0 S^{*i} A S^j P_0$ for $i, j \geq 0$. Therefore, the matrix representation of a S-Toeplitz operator is of the form

$$[T_{ij}]_{i,j=0}^{\infty} = \begin{bmatrix} T_0 & T_{-1} & T_{-2} & \cdots \\ T_1 & T_0 & T_{-1} & \cdots \\ T_2 & T_1 & T_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.1}$$

(see [4] for more details).

Now we recall the linear transformation $\Gamma_1 : B(H) \longrightarrow B(H)$ which is defined by $\Gamma_1(X) = S^*XS$. For $|\lambda| > 1$, this mapping has no eigenoperator. In the following we find the eigenoperators of Γ_1 for the cases $|\lambda| = 1$ and $|\lambda| < 1$ disjointedly.

Theorem 2.1. *Let $S \in B(\mathcal{H})$ be a shift operator. If λ is a complex number with modules one then all solution of the equation $S^*XS = \lambda X$ are of the form WT, for some unitary operator W and all S-Toeplitz operators T.*

Theorem 2.2. *Let $|\lambda| < 1$ and S be shift operator. Then all solutions of equation $S^*XS = \lambda X$ are of the form*

$$X = \sum_{n=0}^{\infty} \sum_{i \in \Lambda} \lambda^n ((S^n f_i) \otimes (S^n \zeta_i) + (S^n \zeta_i) \otimes (S^n g_i)) \qquad (2.2)$$

where f_i, g_i are some elements of $\mathcal{H}$ for all $i \in \Lambda$ and $\{\zeta_i\}_{i \in \Lambda}$ is an orthonormal basis of $ker K^*$.

According to hypothesis of a shift operator S, we can show that only zero operator satisfies $SX = XS^*$. As a general case if $\lambda \in \mathbb{C}$, we show that this result is true for the equation $SX - XS^* = \lambda X$. This is equivalent to the linear transformation $\Gamma_2 : B(H) \longrightarrow B(H)$ defined by $\Gamma_2(X) = SX - XS^*$ which has no eigenoperator.

Theorem 2.3. *Let S be a shift operator and A be a bounded operator on a Hilbert space $\mathcal{H}$ with $ker A = \{0\}$. If B is an arbitrarily bounded operator then $X = 0$ is the only solution of the operator equation $AX = BXS^*$.*

Corollary 2.4. *The linear transformation $\Gamma_2 : B(H) \longrightarrow B(H)$ defined by $\Gamma_2(X) = SX - XS^*$ has no eigenoperator.*

A is called analytic S-Toeplitz operator if $SA = AS$. One can show that A is a S-Toeplitz operator with lower triangular matrix representation. A generalized analytic S-Toeplitz is one that satisfies in the operator equation $SX - XS = \lambda X$. But in the case that S is a shift operator with finite multiplicity (dimension of $\ker S^*$) we show that the last equation has no non-zero solution for $\lambda \neq 0$. This is equivalent to $\lambda = 0$ is the only eigenvalue of the mapping $\Gamma_3 : B(H) \longrightarrow B(H)$ defined by $\Gamma_3(X) = SX - XS$.

Theorem 2.5. *Let $\lambda \neq 0$ and S be a shift operator with finite multiplicity on a Hilbert space $\mathcal{H}$. Then $X = 0$ is the only bounded solution of the operator equation $SX - XS = \lambda X$.*

An operator H is called S-Hankel if $S^*H = HS$. Therefore, the S-Hankel operators are eigenoperators of the mapping $\Gamma_4 : B(\mathcal{H}) \longrightarrow B(\mathcal{H})$ defined by $\Gamma_4(X) = S^*X - XS$ corresponding to eigenvalue $\lambda = 1$. The existence of eigenoperators for all cases of λ is studied in the following.

Theorem 2.6. *Let S be a shift operator on a Hilbert space $\mathcal{H}$. If $|\lambda| < 2$ then the operator equation $S^*X - XS = \lambda X$ has a non-zero solution.*

According to the above theorem $\sigma(\Gamma_4) = \{z \in \mathbb{C} : |z| \leq 2\}$. This implies that if $|\lambda| > 2$ then $X = 0$ is the only solution of the equation operator $S^*X - XS = \lambda X$. For the case $|\lambda| = 2$ we use the next Theorem to show that Γ_4 has no eigenoperator.

Theorem 2.7. *Let the operators A and B be contractions (with norm less than or equal to one) and act on a Hilbert space $\mathcal{H}$ and $\alpha \in \mathbb{C}$ with modules one. If $AX - XB = 2\alpha X$ for some operator X, then $AX = \alpha X$ and $XB = -\alpha X$.*

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ON RANK ONE SOLUTIONS FOR NONLINEAR OPERATOR EQUATIONS

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Abstract. Let S be a Banach space and A, B are bounded linear operator on Banach Space S. This note concerns the operator equations $AX - XB = 0$, $XAX = AX$ and , where X is unknown operator. We shall introduce rank one operators as a solution for mentioned operator equations in Banach space by using eigenvalue of A (or B).

1. Introduction

Let S be a Banach space and A be a bounded linear operator on S. Suppose that A^* is adjoint operator on dual space S' which is defined by $(A^*\phi)(x) = \phi(Ax)$ for any $x \in S$ and $\phi \in S'$. A bounded linear operator A on S is said to be of rank one if the dimension of the range of A is one. An operator A is of rank one if and only if there exist $\phi \in S'$ and $u \in S$ such that $A = \phi \otimes u$, where $(\phi \otimes u)x := \phi(x)u$, for any $x \in S$.
It is easy to show that for every $u,\ v \in S$, $\phi,\ \psi \in S'$ and operator A on S, we have

(1) $A(\phi \otimes u) = \phi \otimes Au$,

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* Speaker.

(2) $(\phi \otimes u)A = A^*\phi \otimes u$,
(3) $(\phi \otimes u)(\psi \otimes v) = \psi \otimes \phi(v)u$.

This note is motivated by [1].

2. Main results

First, we consider the equation $AX - XB = 0$. It is well-known in matrix theory that $AX - XB = 0$ has a nonzero solution if A and B have a common eigenvalue (see [4]). Next proposition introduce a rank one explicit solution by using the common eigenvalue of A and B^* in Banach space.

Proposition 2.1. *Let A and B be two nonzero bounded linear operators and X be an unknown operator on Banach space S. If A and B^* have a common eigenvalue, then $AX - XB = 0$ has a nonzero rank one solution.*

Proof. Suppose that λ is a common eigenvalue of A and B^*. So there exists a nonzero $u \in S$ and $\phi \in S'$ such that $Au = \lambda u$ and $B^*\phi = \lambda\phi$, respectively. Then $X = \phi \otimes u$ satisfies the equation $AX - XB = 0$, because

$$\begin{aligned} AX - XB = &= A(\phi \otimes u) - (\phi \otimes u)B \\ &= (\phi \otimes Au) - (B^*\phi \otimes u) \\ &= \lambda(\phi \otimes u) - \lambda(\phi \otimes u) = 0, \end{aligned}$$

thus $X = \phi \otimes u$ is the rank one solution. □

In the following, we consider the equation $XAX = AX$ in Banach space. Holbrook et al. [2] verified that every solution of the equation $XAX = AX$ in Hilbert space is an idempotent precisely when every restriction of A to an invariant subspace has a dense range. Next proposition illustrate a rank one idempotent solution for this equation in Banach space.

Proposition 2.2. *Let A be a nonzero bounded linear operators and X be an unknown operator on Banach space S. If A has an eigenvalue, then $XAX = AX$ has a nonzero idempotent rank one solution.*

Proof. Suppose that λ is an eigenvalue of A, so there exists a unit eigenvector $u \in S$ such that $Au = \lambda u$. By the Hahn-Banach Theorem there exists a bounded linear functional ϕ on S such that $\phi(u) = \|u\| = 1$. Then $X = \phi \otimes u$ satisfies the $XAX = AX$, because

$$\begin{aligned} XAX &= (\phi \otimes u)A(\phi \otimes u) \\ &= (A^*\phi \otimes u)(\phi \otimes u) \\ &= \phi \otimes A^*\phi(u).u \\ &= \phi \otimes A^*u \\ &= A(\phi \otimes u) = AX, \end{aligned}$$

thus $X = \phi \otimes u$ is the rank one solution. It is sufficient to show that X is idempotent.

$$\begin{aligned} X^2 &= (\phi \otimes u)(\phi \otimes u) \\ &= \phi \otimes \phi(u).u \\ &= \phi \otimes u = X. \end{aligned}$$

□

Proposition 2.3. *Let A be a nonzero bounded linear operators and X be an unknown operator on Banach space S. If A and A^* have eigenvalues such that $\lambda_A\overline{\lambda_A^*} = 1$, then $AXA = X$ has a nonzero rank one solution.*

Proof. Suppose that λ_A and λ_A^* are eigenvalues of A and A^*, respectively. So there exists a nonzero $u \in S$ such that $Au = \lambda_A u$ and a nonzero $\phi \in S'$ such that $A^*\phi = \lambda_A^*\phi$. Then $X = \phi \otimes u$ satisfies the $AXA = X$, because

$$\begin{aligned} AXA &= A(\phi \otimes u)A \\ &= (A^*\phi \otimes Au) \\ &= \lambda_A\overline{\lambda_A^*}(\phi \otimes u) = X. \end{aligned}$$

□

Proposition 2.4. *Let A be a nonzero bounded linear operators and X be an unknown operator on Banach space S. If A has an eigenvalue, then there exist nilpotent rank one operators X and Y such that $AX + XA = Y$.*

Proof. It is straight result of Theorem 2.1 in [1]. □

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THE WEIGHTED HARMONIC–GEOMETRIC–ARITHMETIC MEAN INEQUALITY OF SEVERAL POSITIVE OPERATORS

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Abstract. In this paper we define weighted harmonic, geometric and arithmetic mean of several positive operators in $\mathbb{B}(\mathcal{H})$ and we show that the inequality between of them. Also we show that the generalized geometric – arithmetic mean inequality.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathbb{B}(\mathcal{H})$ be all bounded linear operators on $\mathcal{H}$, we write $A \leq B$ if A and B are self-adjoint and $B-A$ is positive. If $\mathbf{b} = (b_1, \ldots, b_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ are two sequences of positive numbers, such that $\lambda_1 + \cdots + \lambda_n = 1$. We have

$$\mathbf{a_n}(\mathbf{b};\lambda) = \sum_{j=1}^{n} \lambda_j b_j \,, \quad \mathbf{g_n}(\mathbf{b};\lambda) = \prod_{j=1}^{n} b_j^{\lambda_j} \,, \quad \mathbf{h_n}(\mathbf{b};\lambda) = \left(\sum_{j=1}^{n} \lambda_j b_j^{-1} \right)^{-1} ,$$

are weighted arithmetic, geometric and harmonic means respectively. Let A and B are two positive operators and $p \in [0,1]$, the weighted

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* Speaker.

arithmetic, geometric and harmonic means of A and B as follows

$$A\nabla_p B = (1-p)A + pB\,,$$
$$A\sharp_p B = A^{1/2}(A^{-1/2}BA^{-1/2})^p A^{1/2}\,,$$
$$A!_p B = ((1-p)A^{-1} + pB^{-1})^{-1}\,.$$

2. The weighted arithmetic, geometric and harmonic mean

We take $\mathbf{a_2}(A,B) = A\nabla_p B$, $\mathbf{g_2}(A,B) = A\sharp_p B$ and $\mathbf{h_2}(A,B) = A!_p B$.

Definition 2.1. With the above notations, if $\mathbf{B} = (B_1,\ldots,B_n)$ be a sequence of positive operators and $\lambda = (\lambda_1,\ldots,\lambda_n)$ be a sequence of positive numbers such that $\lambda_1+\cdots+\lambda_n = 1$. We define weighted arithmetic, geometric and harmonic mean of several operators as follows

$$\mathbf{a_n}(\mathbf{B},\lambda) = \mathbf{a_2}(B_1, \mathbf{a_{n-1}}(\mathbf{B}',\lambda'),\lambda_1),$$
$$\mathbf{g_n}(\mathbf{B},\lambda) = \mathbf{g_2}(B_1, \mathbf{g_{n-1}}(\mathbf{B}',\lambda'),\lambda_1),$$
$$\mathbf{h_n}(\mathbf{B},\lambda) = \mathbf{h_2}(B_1, \mathbf{h_{n-1}}(\mathbf{B}',\lambda'),\lambda_1),$$

where $\mathbf{B}' = (B_2,\ldots,B_n)$ and $\lambda' = (\lambda'_2,\ldots,\lambda'_n)$, that $\lambda'_j = \frac{\lambda_j}{1-\lambda_1}$, $j = 2,\ldots,n$.

Theorem 2.2. $\mathbf{h_n}(\mathbf{B},\lambda) \le \mathbf{g_n}(\mathbf{B},\lambda) \le \mathbf{a_n}(\mathbf{B},\lambda)$.

We need a lemma before we prove the theorem.

Lemma 2.3. *Let X be a positive operator, then for all $0 \le \lambda \le 1$, $X^\lambda \le (1-\lambda)I + \lambda X$.*

Proof. If we take $f(\lambda) = 1-\lambda+\lambda t - t^\lambda$, for $t > 0$, f is concave on $[0,1]$, and $f(0) = f(1) = 0$ for all $t \ne 1$, thus f is nonnegative. If $t = 1$, it is clear.

Proof of theorem. If we choose $X = B_1^{-1/2}B_2B_1^{-1/2}$ in the Lemma 2.3, then

$$B_1\sharp_p B_2 \le B_1\nabla_p B_2 \tag{2.1}$$

for all positive operators B_1, B_2. Since

$$\begin{aligned}
\mathbf{g_n}(\mathbf{B},\lambda) &= \mathbf{g_2}(B_1, \mathbf{g_{n-1}}(\mathbf{B}',\lambda'),\lambda_1)\\
&\le \mathbf{a_2}(B_1, \mathbf{g_{n-1}}(\mathbf{B}',\lambda'),\lambda_1) \text{ (by (2.1))}\\
&\le \mathbf{a_2}(B_1, \mathbf{a_{n-1}}(\mathbf{B}',\lambda'),\lambda_1) \text{ (by monotonicity)}\\
&= \mathbf{a_n}(\mathbf{B},\lambda)
\end{aligned}$$

we get the result by induction on n. It is clear that $\mathbf{g_n}(\mathbf{B},\lambda)^{-1} = \mathbf{g_n}(\mathbf{B}^{-1},\lambda)$ where $\mathbf{B}^{-1} = (B_1^{-1},\dots,B_n^{-1})$. Therefore

$$\mathbf{h_n}(\mathbf{B},\lambda) = \mathbf{a_n}(\mathbf{B}^{-1},\lambda)^{-1} \leq \mathbf{g_n}(\mathbf{B}^{-1},\lambda)^{-1} = \mathbf{g_n}(\mathbf{B},\lambda).$$

This complete the proof.

3. The generalized Geometric - Arithmetic inequality

Let $\mathbf{x} = (x_1,\dots,x_n)$, $\lambda = (\lambda_1,\dots,\lambda_n)$ where $x_i > 0$, $\lambda_i > 0$ and $\sum \lambda_i = 1$. We define a mean of order t by

$$M_t(\mathbf{x},\lambda) = \left(\sum_{i=1}^{n} \lambda_i x_i^t\right)^{1/t} \quad t \neq 0,$$

and

$$M_0(\mathbf{x},\lambda) = \lim_{t\to 0} M_t(\mathbf{x},\lambda) = \prod_{i=1}^{n} x_i^{\lambda_i}.$$

Specifically, the values $t = -1, 0, 1$ give expressions known, respectively, as the harmonic, geometric and arithmetic mean of $x_1,\dots,x_n$. If A and B are two positive operators, $t \in [0,1]$ and $r \in \mathbb{R}$, we define

$$A\ m_{r,t}\ B = A^{1/2}[1 - t + t(A^{-1/2}BA^{-1/2})^r]^{1/r}A^{1/2} = F_2[(A,B),r,t]\,.$$

By using induction on n, we define

$$F_n(\mathbf{B},r,t) = F_2[B_1, F_{n-1}(\mathbf{B}',r,t'),r,t_1],$$

where $\mathbf{B} = (B_1,\dots,B_n)$, $\mathbf{B}' = (B_2,\dots,B_n)$, $t = (t_1,\dots,t_n) \in [0,1]^n$ such that $t_1 + \cdots + t_n = 1$ and $t' = (t_2',\dots,t_{n-1}')$ that $t_i' = \frac{t_i}{1-t_1}$. We show that by induction, if $B_i,\ i = 1,\dots,n$ be commuting then

$$F_n(\mathbf{B},r,t) = (\alpha_1 B_1^r + \cdots + \alpha_n B_n^r)^{1/r},$$

where $0 \leq \alpha_i \leq 1$, $\alpha_1 + \cdots + \alpha_n = 1$ and each α_i depending to $t_1,\dots,t_n$.

Remark 3.1. The $F_n(\mathbf{B},1,t)$ is arithmetic mean, i.e. $F_n(\mathbf{B},1,t) = t_1B_1 + \cdots + t_nB_1$ where $t_1 + \cdots + t_n = 1$; also it is easy to show that $F_n(\mathbf{B},0,t) = \mathbf{g_n}(\mathbf{B},t)$ and $F_n(\mathbf{B},-1,t) = \mathbf{h_n}(\mathbf{B},t)$.

Theorem 3.2. *If $s \leq r$, then $F_n(\mathbf{B},s,t) \leq F_n(\mathbf{B},r,t)$.*

Corollary 3.3. *By the Theorem 3.2, we have*

$$\mathbf{h_n}(\mathbf{B},t) = F_n(\mathbf{B},-1,t) \leq F_n(\mathbf{B},0,t) = \mathbf{g_n}(\mathbf{B},t) \leq F_n(\mathbf{B},1,t) = \mathbf{a_n}(\mathbf{B},t).$$

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ON AUTOMORPHISMS OF QUATERNION RINGS

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Abstract. Let S be a ring and $R = S \oplus Si \oplus Sj \oplus Sk$ be the quaternion ring over S. In this work we characterize the automorphisms of R when S is a formally real commutative ring.

1. Introduction

The structure of automorphisms of matrix rings has been studied in many papers (see, for example [1],[2]). For upper triangular matrix rings we know that an automorphism is a composition of an inner automorphism with a permutation similarity (see [2]). In [3], Ghosseiri has determined the structure of automorphisms of generalized traingular matrix rings.

The ring of real quaternions was discovered by William Rowan Hamilton (1805-1865) in 1843. Hamilton's quaternions have many applications other than in physics. They are extesnively used in computer graphics to describe motion in 3-space, and more recently, they have been used in multiple antennae communications systems. In some ways

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we can think of the quaternions as an extension of the complex numbers. Throughout this paper, we assume that S a formally real commutative ring. A commutative ring S is called formally real, if $\sum_i \alpha_i^2 = 0$ ($\alpha_i \in S$) implies that $\alpha_i = 0$ for all i. Any formally real ring is an orderable domain and its characteristice is zero (see [4]). For example, $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ are formally real, but $\mathbb{C}$ and $\mathbb{Z}_n$ are not.

The ring defined by $R = \{q_0 + q_1 i + q_2 j + q_3 k : q_i \in S\} = S \oplus Si \oplus Sj \oplus Sk$, where $i^2 = j^2 = k^2 = ijk = -1$ is called a quaternion ring over S. Note that $ij = -ji, ik = -ki, jk = -kj$, so multiplication in R is not commutative in general. If $S = \mathbb{R}$, then R is called Hamilton's ring of real quaternions. Set $S(i) = S + Si$. We can show that

$$R \cong \Big\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \quad \alpha, \beta \in S(i) \Big\},$$

where

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; i \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; j \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; k \leftrightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

It is clear that under this isomorphism R is a subring of $M_2(S(i))$(see [4]).

2. Main results

The main aim of this work is to prove the following theorem.

Theorem 2.1. *Let S be a formally real commutative ring and let $R = S \oplus Si \oplus Sj \oplus Sk$. If $\varphi : R \longrightarrow R$ is an automorphism, then there exist an automorphism ψ of R fixing i, j, k and invertible element $u \in R$ such that for each $p \in R$*

$$\varphi(p) = Inn_u \circ \psi(p).$$

Proof. (Sketch) First we show that $\varphi(i) = ai+bj+ck$, $\varphi(j) = a'i+b'j+c'k$, and $\varphi(k) = a''i+b''j+c''k$, where $a^2+b^2+c^2 = 1$, $a'^2+b'^2+c'^2 = 1$, $a''^2 + b''^2 + c''^2 = 1$, $a'' = bc' - cb'$, $b'' = ca' - ac'$, $c'' = ab' - ba'$, and $aa'+bb'+cc' = 0$. Then we show that there exists an invertible element $u \in R$ such that

$$\varphi(i) = uiu^{-1}, \quad \varphi(j) = uju^{-1}, \quad \varphi(k) = uku^{-1}.$$

Now, setting, $\psi = Inn_{u^{-1}} \circ \varphi$. We have $\psi(i) = i$, $\psi(j) = j$, $\psi(k) = k$. Then one can show that $\psi \mid_S$ is an automorphism of S and

$$\psi(x + yi + wj + zk) = \psi(x) + \psi(y)i + \psi(w)j + \psi(z)k,$$

for all $x, y, w, z \in S$. Thus we have $\varphi = Inn_u \circ \psi$. □

Corollary 2.2. *Let $S = \mathbb{R}$. Then any automrphism of the Hamilton's real quaternion ring $\mathbb{H}$ is inner.*

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SIMILARITY OF A MATRIX AND ITS H-TRANSPOSE

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Abstract. Let $M_n(\mathbb{C})$ as the linear space of complex n-by-n matrices and $H \in M_n(\mathbb{C})$ as a involutory matrix. For $A \in M_n(\mathbb{C})$, its H-transpose is defined as $A^{[T]_H} = HA^TH$. Our aim is to show that every matrix in $M_n(\mathbb{C})$ is similar to its H-transpose. But we first prove this assertion for Jordan blocks and Jordan matrices.

1. Introduction

We recall, that an involutory matrix is a matrix that is its own inverse. That is, matrix A is an involution if and only if $A^2 = I$. Involutory matrices are all square roots of the identity matrix. An involutory matrix which is also symmetric is an orthogonal matrix, and thus represents an isometry. Clearly, any block-diagonal matrices constructed from involutory matrices will also be involutory, as a consequence of the linear independence of the blocks. Let $M_n(\mathbb{C})$ as the linear space of complex n-by-n matrices and $H \in M_n(\mathbb{C})$ be an involutory matrix. For $A \in M_n(\mathbb{C})$, define its H-transpose as $A^{[T]_H} = HA^TH$. We show

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that every matrix in $M_n(\mathbb{C})$ is similar to its H-transpose. But, this assertion is first proved for an arbitrary Jordan block. Throughout this paper, Jordan blocks are shown by J or J_t.

2. Main sections and results

2.1. **H-transpose of Jordan matrices.** In this section, we show that every Jordan matrix is similar to its H-traspose, for some involutory matrix. We first prove this assertion for an arbitrary Jordan block.

Proposition 2.1. *Every Jordan block $J_m(\lambda) \in M_m(\mathbb{C})$, is similar to its H-transpose, for every involutory matrix $H \in M_m(\mathbb{C})$.*

Proof. Let B as the m-by-m backward matrix, i.e., $B = [\delta_{i,n+1-j}] = B^{-1}$, and define $S = HB$, in which H is an arbitrary involutory matrix. Because of nonsingularity of H and B, S is also nonsingular. H and B are involutory matrices. Thus $H^{-1} = H$ and $B^{-1} = B$ and we have

$$S^{-1}(J_m(\lambda))^{[T]_H}S = BH(H(J_m(\lambda))^T H)HB = B(J_m(\lambda))^T B = J_m(\lambda).$$

Note that the last equality is because of nature of left and right multiplication by B. Thus $(J_m(\lambda))^{[T]_H}$ and $J_m(\lambda)$ are similar, via $S = HB$. □

Proposition 2.2. *Every Jordan matrix $J = J_{n_1}(\lambda_1) \oplus ... \oplus J_{n_k}(\lambda_k)$, where $0 < k \leq n$, $n_1 + ... + n_k = n$, is similar to its H-transpose, for some involutory matrix H.*

Proof. Let $H = H_1 \oplus ... \oplus H_k$, in which $H_i \in M_{n_i}(\mathbb{C})$ be an involutory matrix for $i = 1, ..., k$, and for each Jordan block $J_{n_i}(\lambda_i)$, consider its corresponding H_i and define $(J_{n_i}(\lambda_i))^{[T]_{H_i}} = H_i(J_{n_i}(\lambda_i))^T H_i$. According to proposition (2.1), each $(J_{n_i}(\lambda_i))^{[T]_{H_i}}$ is similar to $J_{n_i}(\lambda_i)$, via nonsingular matrix S_i. Define $S = S_1 \oplus ... \oplus S_k$. S as a direct sum of nonsingular matrices is nonsingular. Thus $S^{-1}J^{[T]_H}S = S^{-1}HJ^THS = S_1^{-1}(J_{n_1}(\lambda_1))^{[T]_{H_1}}S_1 \oplus ... \oplus S_k^{-1}(J_{n_k}(\lambda_k))^{[T]_{H_k}}S_k = J_{n_1}(\lambda_1) \oplus ... \oplus J_{n_k}(\lambda_k) = J$. and each Jordan matrix is similar to its H-transpose, for some involutory matrix H. □

Example 2.3. Let $H = I_r \oplus (-I_{n-r})$, $0 \leq r \leq n$. Then every Jordan matrix is similar to its H-transpose.

Example 2.4. Let Jordan matrix J and involutary matrix H as $J = J_3(2) \oplus J_2(4)$ and $H = H_1 \oplus H_2$, where $H_1 = I_1 \oplus (-I_2)$ and $H_2 = -I_2$. By considering B_1 and B_2 as 3-by-3 and 2-by-2 backward matrices, respectively, define

$$S_1 = H_1B_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad S_2 = H_2B_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Thus

$$S_1^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad S_2^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

and

$$(J_3(2))^{[T]_{H_1}} = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}, (J_2(4))^{[T]_{H_2}} = \begin{pmatrix} 4 & 0 \\ 1 & 4 \end{pmatrix},$$

and we have, $S_1^{-1}(J_3(2))^{[T]_{H_1}} S_1 = J_3(2)$ and $S_2^{-1}(J_2(4))^{[T]_{H_2}} S_2 = J_2(4)$. Set $S = S_1 \oplus S_2$. We see, $J = S^{-1} J^{[T]_H} S$. Thus J and $J^{[T]_H}$ are similar.

2.2. **H-transpose and H-conjugate transpose of square matrices.** In this section, we show that every square matrix is similar to its H-traspose, for some involutory matrix, also present several equivalent statements to H-conjugate transpose concept.

Theorem 2.5. *Every matrix $A \in M_n(\mathbb{C})$, is similar to its H-transpose, for some involutory matrix H.*

Proof. Let J as the Jordan canonical form of A. Thus there exists nonsingular matrix S_1 such that

$$A = S_1^{-1} J S_1 \Rightarrow J = S_1 A S_1^{-1}. \tag{2.1}$$

On the other hand, every Jordan matrix is similar to its H-transpose, for some involutory matrix H. Therefore, there exists nonsingular matrix S such that

$$J = S^{-1} J^{[T]_H} S = S^{-1} H J^T H S = (HS)^{-1} J^T (HS) \Rightarrow J^T = (HS) J (HS)^{-1}. \tag{2.2}$$

Thus $A^{[T]_H} = H A^T H = H(S_1^{-1} J S_1)^T H = H S_1^T J^T S_1^{-T} H$, and by (2.2), this yields $A^{[T]_H} = H S_1^T (HS) J (HS)^{-1} S_1^{-T} H$, and by (2.1), this yields $A^{[T]_H} = H S_1^T H S S_1 A S_1^{-1} S^{-1} H S_1^{-T} H$. Let $P = S_1^{-1} S^{-1} H S_1^{-T} H$. As the product of nonsingular matrices, P is nonsingular and $A^{[T]_H} = P^{-1} A P$. Thus A and its H-transpose are similar. □

Example 2.6. Let $H = I_r \oplus (-I_{n-r})$, $0 \leq r \leq n$. Then each matrix is similar to its H-transpose.

Definition 2.7. If the H-conjugate transpose of $A \in M_n(\mathbb{C})$, is denoted by $A^{[*]_H} (= HA^*H)$, for every involutory matrix $H \in M_n(\mathbb{C})$, $A \in M_n(\mathbb{C})$ is said to be H-Hermitian if $A = A^{[*]_H}$.

Theorem 2.8. *Let $A \in M_n(\mathbb{C})$ be given. Then for some involutory matrix H, each of the following statements yields the next statement:*
a: A is similar to a matrix $B \in M_n(\mathbb{R})$.

b: A is similar to $A^{[*]_H}$.
c: A is similar to $A^{[*]_H}$ via a H-Hermitian similarity transformation.
d: $A = LK$, in which $L, K \in M_n(\mathbb{C})$ are H-Hermitian with at least one nonsingular.
e: $A = LK$, in which $L, K \in M_n(\mathbb{C})$ are H-Hermitian.
In addition, (d) yields (b).

Proof. If (a) holds, then there exists nonsingular $S \in M_n(\mathbb{C})$ such that $S^{-1}AS = B$. But according to theorem (2.5), B is similar to $B^{[T]_H}$. Thus there is nonsingular $T \in M_n(\mathbb{C})$ where $S^{-1}AS = B = T^{-1}B^{[T]_H}T = T^{-1}B^{[*]_H}T = T^{-1}(S^{-1}AS)^{[*]_H}T = T^{-1}S^{[*]_H}A^{[*]_H}(S^{-1})^{[*]_H}T = P^{-1}A^{[*]_H}P$, in which $P = (S^{-1})^{[*]_H}T$ is nonsingular. Thus, $A = SP^{-1}A^{[*]_H}PS^{-1}$. If (b) holds, then there exist nonsingular matrix $S \in M_n(\mathbb{C})$ where $S^{-1}AS = A^{[*]_H}$ and observe that $T^{-1}AT = A^{[*]_H}$ if $T = \alpha S$ for any nonzero $\alpha = re^{i\theta} \in \mathbb{C}$. Thus, $AT = TA^{[*]_H}$ or, equivalently, $AT^{[*]_H} = T^{[*]_H}A^{[*]_H}$. Adding these two identities, $A(T+T^{[*]_H}) = (T+T^{[*]_H})A^{[*]_H}$, and if $(T+T^{[*]_H})$ were nonsingular, this would mean that A is similar to $A^{[*]_H}$, via the H-Hermitian matrix $(T+T^{[*]_H})$. But α may be chosen so as to make $(T+T^{[*]_H})$ nonsingular, since $(T+T^{[*]_H})$ is nonsingular if and only if $T^{-1}(T+T^{[*]_H}) = I+T^{-1}T^{[*]_H}$ is, if and only if -1 is not in $\sigma(T^{-1}T^{[*]_H})$. However, $T^{-1}T^{[*]_H} = e^{-2i\theta}S^{-1}S^{[*]H}$ and since α may be chosen to produce any $\theta \in [0, 2\pi)$, we need only pick θ so that $-e^{2i\theta}$ not be in $\sigma(S^{-1}S^{[*]_H})$. Thus (b) implies (c). Next suppose that (c) holds and write $R^{-1}AR = A^{[*]_H}$ with $R \in M_n(\mathbb{C})$ nonsingular and H-Hermitian. Then $R^{-1}A = A^{[*]_H}R^{-1}$ and $A = R(A^{[*]_H}R^{-1})$. But $(A^{[*]_H}R^{-1})^{[*]_H} = R^{-1}A = A^{[*]_H}R^{-1}$, so that A is the product of the two H-Hermitian matrices R and $A^{[*]_H}R^{-1}$, of which R is nonsingular, and (d) holds. Obviously (d) implies (e). If (d) holds and $A = LK$ with L nonsingular, then $L^{-1}AL = KL = (LK)^{[*]_H} = A^{[*]_H}$, and (b) holds. □

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ORTHONORMAL BASIS OF A NON-DEGENERATE SUBSPACE

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Abstract. In this paper, a special indefinite inner product on $\mathbb{C}^n$ and its corresponding nonsingular Hermitian matrix shall be introduced. Then it shall be proven that under this defined indefinite inner product, the product of the elements of each basis on a nondegenerated subspace, has a fixed form.

1. Introduction

Let $\mathbb{C}^n$ be the n-dimensional Hilbert space consisting of all column vectors x with complex coordinates x_i, $i = 1, 2, ..., n$. The typical column vector x will be written in the form $x = \langle x_1, ..., x_n \rangle$. We know that any indefinite scalar product satisfies all the properties of a standard inner product with the possible exception that $[x, x]$ may be nonpositive for $x \neq 0$. That is, linearly in the first argument and it has antisymmetriy and nondegeneracy properties.

Example 1.1. Let $x = \langle x_1, ..., x_n \rangle$, $\langle y_1, ..., y_n \rangle \in \mathbb{C}^n$ and define

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* Speaker.

$$[x,y]=\sum_{i=1}^{r}x_i\overline{y_i}-\sum_{i=r+1}^{n}x_i\overline{y_i}.$$

It is easily checked that $[.,.]$ is an indefinite inner product. Actually, this product is linearly in the first argument and it has antisymmetriy property. But for nondegeneracy property, if $[x,y]=0$ for all $y\in\mathbb{C}^n$, then in special case, $[x,y]=0$ for $y=\langle x_1,...,x_r,-x_{r+1},...,-x_n\rangle$. Thus,

$$\sum_{i=1}^{r}x_i\overline{x_i}-(-\sum_{i=r+1}^{n}x_i\overline{x_i})=\sum_{i=1}^{r}|x_i|^2+\sum_{i=r+1}^{n}|x_i|^2=\sum_{i=1}^{n}|x_i|^2=0,$$

and this yields that $x_i=0$, for $i=1,...,n$. So, $x=0$.

By [1], we know that for every $n\times n$ invertible Hermitian matrix H, the formula

$$[x,y]=(Hx,y),\qquad x,y\in\mathbb{C}^n \tag{1.1}$$

determines an indefinite inner product on $\mathbb{C}^n$ and conversely, for every indefinite inner product $[.,.]$ on $\mathbb{C}^n$ there exists an $n\times n$ invertible and Hermitian matrix H such that (1.1) holds and the established correspondence $[.,.]\leftrightarrow H$ is a bijection between the set of all indefinite inner products on $\mathbb{C}^n$ and the set of all $n\times n$ invertible Hermitian matrices.

Example 1.2. Remind Example (1.1). It is proved that the corresponding nonsingular Hermitian matrix to that indefinite inner product is written in the form $J=I_r\oplus(-I_{n-r})$.

Definition 1.3. A subspace $\mathbb{M}$ is said to be nondegenerate, with respect to the indefinite inner product $[.,.]$ if $x\in\mathbb{M}$ and $[x,y]=0$ for all $y\in\mathbb{M}$ imply that $x=0$. Otherwise $\mathbb{M}$ is degenerate.

Definition 1.4. If $\mathbb{M}$ is any nondegenerate nonzero subspace of $\mathbb{C}^n$, then the basis $x_1,...,x_k$ for $\mathbb{M}$ is said to be a orthogonal basis with respect to the indefinite inner product $[.,.]$ if $[x_i,x_j]=0,\forall i\neq j$, and is said to be an orthonormal basis if in addition to orthogonality, $[x_i,x_i]=\pm1,\forall i=1,...,k$. If the indefinite inner product $[.,.]$ in this definition, be the special indefinite inner product where presented in Example (1.1), then the above definitions of orthogonal basis and orthonormal basis are said J-orthogonal and J-othonormal basis, respectively.

We will prove that if $\mathbb{M}$ is any k-dimensional nondegenerate nonzero subspace with respect to the indefinite inner product $[.,.]=(J.,.)$ where $J=I_r\oplus(-I_{n-r})$ and $0<r<n$ and $k\geq r$, then a J-orthonormal basis $x_1,...,x_k$ in $\mathbb{M}$ with respect to this indefinite inner product and by re-election of i indexes, if it is required, satisfying

$$[x_i,x_j]=\begin{cases}+1 & i=j=1,...,r;\\ -1 & i=j=r+1,...,k,\\ 0 & i\neq j.\end{cases}\quad.$$

Definition 1.5. Let $\mathbb{M}$ any nondgenerate nonzero subspace and let Q as an invertible Hermitian linear transformation from $\mathbb{M}$ to $\mathbb{M}$. We define the signature of Q as the difference between the number of positive eigenvalues of Q and the number of negative eigenvalues in Q (in both cases counting with multiplicities) and denotes by $sigQ$.

For example, let $J = I_r \oplus (-I_{n-r})$ as the matrix representation of an invertible Hermitian linear transformation Q. Then $sigQ = sigJ = r - (n-r) = 2r - n$.

2. Main sections and results

Proposition 2.1. *Let $[.,.] = (H.,.)$ be an indefinite inner product with corresponding Hermitian invertible matrix H, and let $x_1, ..., x_k$ be an orthonormal (with respect to $[.,.]$) basis in a nondegenerate subspace $\mathbb{M} \subseteq \mathbb{C}^n$. Then the sum $\sum_{i=1}^{k}[x_i, x_i]$ coincides with the signature of the Hermitian linear transformation $PH|_{\mathbb{M}} : \mathbb{M} \to \mathbb{M}$, where P is the orthogonal projection (with respect to $(.,.)$) of $\mathbb{C}^n$ onto $\mathbb{M}$.*

Proof: See [1].

Now, if $H = J = I_r \oplus (-I_{n-r})$ and $x_1, ..., x_k$, $k \geq r$, be a J-orthonormal basis in a nondegenerate subspace $\mathbb{M}$ then the difference between the number of positive and negative eigenvalues of $PJ|_{\mathbb{M}}$ is $2r - k$. For this, by $\mathbb{C}^n = M \oplus M^{[\perp]}$, it's clear that the orthogonal projection P has the matrix representation as $I_k \oplus 0_{n-k}$. On the other hand, $J = I_r \oplus (-I_{n-r})$. Thus, the structure of $PJ|_M$ is as $I_r \oplus (-I_{k-r}) \oplus 0_{n-k}$ and we have $sigPJ|_{\mathbb{M}} = r - (k-r) = 2r - k$. Hence,

$$\sum_{i=1}^{k}[x_i, x_i] = sigPJ|_{\mathbb{M}} = 2r - k. \tag{2.1}$$

Moreover, according to the definition of J-orthonormal basis, $[x_i, x_i] = +1$ or -1. If the number of x_is where $[x_i, x_i] = +1$ is m and the number of x_is where $[x_i, x_i] = -1$ is t, then, $m + t = k$ and according to (2.1), $m(1) + t(-1) = 2r - k$. Thus, by the following system

$$\begin{cases} m + t = k \\ m(1) + t(-1) = 2r - k \end{cases},$$

we earn $m = r$ and $t = k - r$. Thus the number of x_is where $[x_i, x_i] = +1$ is r and the number of x_is where $[x_i, x_i] = -1$ is $k - r$. Therefore, by re-election of i indexes, if it is required, we have

$$[x_i, x_i] = \begin{cases} +1, & \text{i=1,...,r;} \\ -1, & \text{i=r+1,...,k .} \end{cases}$$

According to the above description, it is clear that $[x_i, x_i] = +1$, for $k > r$ and $i = 1, ..., k$. Thus, the following proposition is satisfied:

Proposition 2.2. *If $J = I_r \oplus (-I_{n-r})$ and $x_1, ..., x_k$, $k \geq r$, be an J-orthonormal basis in a nondegenerate subspace $\mathbb{M}$ then by changing the indexes, if necessary, $x_1, ..., x_k$ satisfying*

$$[x_i, x_j] = \begin{cases} +1 & i = j = 1, ..., r; \\ -1 & i = j = r+1, ..., k; \\ 0 & i \neq j. \end{cases}$$

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ONE TEST FOR SOLVABILITY OF A CLASS OF INTEGRAL EQUATION OF CONVOLUTION TYPE WITH TWO KERNELS

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Abstract. This paper is devoted to introduce a simple test. With my test, the solvability of an integral equation of convolution type where has two kernels without solving can be showed. It is exact for both case of kernel, when it is conservative and semi conservative.

1. Introduction

Integral equation on the half line with a kernel that depend on the difference of arguments, called Wiener-Hopf integral equations(WHIE), have important and multiple application in different areas of mathematical natural sciences. Numerous investigations have been devoted to them.(See [2] ,[4]) We consider the following integral equation

$$f(x) = g(x) + \int_0^\infty K(x-t)f(t)dt. \tag{1.1}$$

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Let

$$K(x) \geq 0, \quad x \in (-\infty, \infty), \quad \nu_0 = \int_{-\infty}^{\infty} K(x)dx = 1.$$

Denote

$$\nu_m = \int_{-\infty}^{\infty} x^m K(x)dx, \qquad m = 0, 1, 2, 3, ...$$

Consider the following symbol function

$$A(\lambda) = 1 - \int_{-\infty}^{\infty} e^{i\lambda t} K(t)dt, \qquad -\infty \leq \lambda \leq \infty.$$

We know that $ind_\gamma a = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz$ denotes the index of the closed curve γ with respect to a.
With last notations we have two following theorems :

Theorem 1.1. *If $\nu_1 \neq 0$, then*

$$A(\lambda) = \frac{\lambda}{\lambda + i}.B(\lambda)$$

where

$$B(\lambda) \neq 0, \qquad -\infty \leq \lambda \leq \infty, \qquad indB(\lambda) = \begin{cases} 0 & ;\nu_1 > 0 \\ -1 & ;\nu_1 < 0 \end{cases}$$

Proof: See [1].

Theorem 1.2. *If $\nu_1 = 0$ and $\nu_2 < \infty$, then*

$$A(\lambda) = \left(\frac{\lambda}{\lambda + i}\right)^2 B(\lambda)$$

where

$$B(\lambda) \neq 0, \qquad -\infty \leq \lambda \leq \infty, \qquad indB(\lambda) = -1.$$

Proof: See [1].

Now we consider following dual integral equation

$$f(x) = g(x) + \int_0^{\infty} K_1(x-t)f(t)dt + \int_{-\infty}^{0} K_2(x-t)f(t)dt$$

here we suppose that

$$K_j \geq 0, \qquad x \in (-\infty, \infty), \qquad j = 1, 2$$

$$\nu(j, m) = \int_{-\infty}^{\infty} x^m K_j(x) dx, \qquad j = 1, 2, \qquad m = 0, 1, 2, ...$$

$$\nu(j, 0) = \int_{-\infty}^{\infty} K_j(x) dx \leq 1, \qquad j = 1, 2$$

Now we consider the symbol functions

$$A_j(\lambda) = 1 - \int_{-\infty}^{\infty} e^{i\lambda x} K_j(x) dx \quad , \quad j = 1, 2; \quad -\infty \leq \lambda \leq \infty$$

Let

$$A_1(\lambda) = \left(\frac{\lambda}{\lambda + i}\right)^m B_1(\lambda), B_1(\lambda) \neq, -\infty \leq \lambda \leq \infty$$

$$A_2(\lambda) = \left(\frac{\lambda}{\lambda - i}\right)^n B_2(\lambda), B_2(\lambda) \neq, -\infty \leq \lambda \leq \infty$$

our main goal is to find

$$\chi = ind \frac{B_1(\lambda)}{B_2(\lambda)}$$

See [3].
I have tested all cases and I get the following results.

2. semi conservative and cosevative cases

For semi conservative types we can consider the following cases:

- $\nu(1, 0) = 1$ and $\nu(2, 0) = 1$,
  (1) $\chi = \begin{cases} 0 & ; \quad \nu(1, 1) > 0, \\ -1 & ; \quad \nu(1, 1) < 0. \end{cases}$
  (2) $\chi = -1$.
- $\nu(1, 0) = 1$ and $\nu(2, 0) < 1$,
  (1) $\chi = \begin{cases} -1 & ; \quad \nu(2, 1) > 0, \\ 0 & ; \quad \nu(2, 1) < 0. \end{cases}$
  (2) $\chi = -1$.

For coservative types only one condition $\nu(1,0)=\nu(2,0)=1$, should be hold, and in this case we have,

$$(1)\ \chi=\begin{cases}-1 & ; \quad \nu(1,1)>0 \quad , \quad \nu(2,1)>0,\\ 0 & ; \quad \nu(1,1)>0 \quad , \quad \nu(2,1)<0,\\ -2 & ; \quad \nu(1,1)<0 \quad , \quad \nu(2,1)>0,\\ -1 & ; \quad \nu(1,1)>0 \quad , \quad \nu(2,1)<0.\end{cases}$$

$$(2)\ \chi=\begin{cases}-2 & ; \quad \nu(2,1)>0,\\ -1 & ; \quad \nu(2,1)<0.\end{cases}$$

$$(3)\ \chi=\begin{cases}-1 & ; \quad \nu(1,1)>0,\\ -2 & ; \quad \nu(1,1)<0.\end{cases}$$

$$(4)\ \chi=-2.$$

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MATRIX INVERSE YOUNG INEQUALITIES

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Abstract. We use operator monotone and operator convex functions to prove an inverse to the Young inequality for eigenvalues of positive definite matrices and then apply it to obtain a matrix inverse Young inequality.

1. Introduction

Some of the most important inequalities as well as some equalities in complex numbers admit generalisations in matrix context. The equality $|z\overline{w}| = |z|\,|w|$, the triangle inequality $|z+w| \leq |z| + |w|$ and the arithmetic-geometric mean inequality $|z\overline{w}| \leq \frac{1}{2}(|z|^2 + |w|^2)$, are all in evidence. (see [4] and [2] for generalisations of the second and the third one in complex matrices.)
Another such inequality is the Young inequality:

$$|z\overline{w}| \leq \frac{1}{p}|z|^p + \frac{1}{q}|w|^q \tag{1.1}$$

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* Speaker.

in which $p, q \in (1, \infty)$ are conjugate exponents, that is $1/p + 1/q = 1$. Note that (1.1), can also be written in the form

$$|z\overline{w}| \leq \nu\, |z|^{\frac{1}{\nu}} + (1-\nu)|w|^{\frac{1}{1-\nu}} \quad \text{for} \quad \nu \in (0,1). \tag{1.2}$$

An analytic investigation of the function $f(t) = \nu t - t^{\nu}$ on the ray $(0, \infty)$ shows that for $\nu > 1$, the function f is strictly increasing on $(0, 1)$ and strictly decreasing on $(1, \infty)$ also f attains its maximum $(= \nu - 1)$ at $t = 1$. Hence for $\nu > 1$ the inequality $a^{\nu} b^{1-\nu} \geq \nu a + (1-\nu)b$ holds for positive a, b with equality if and only if $a = b$. Therefore the inequality

$$|z\overline{w}| \geq \nu\, |z|^{\frac{1}{\nu}} + (1-\nu)|w|^{\frac{1}{1-\nu}} \tag{1.3}$$

holds for non zero complex numbers z, w with equality if and only if $|z|^{\frac{1}{\nu}} = |w|^{\frac{1}{1-\nu}}$, which can be called the inverse Young inequality. The main result established in this paper is an extension of the inverse Young inequality to complex matrices.

In what follows $M_n(C)$ shall denotes the set (the C^*-algebra) of all n by n complex matrices. A hermitian matrix $A \in M_n(C)$ is called positive semidefinite (resp. positive definite) if $\langle Ax, x\rangle \geq 0$ (resp.$\langle Ax, x\rangle > 0$) for each $x \in C^n$. The set $M_n^+(C)$ of all positive semidefinite matrices then is a cone in $M_n(C)$ and makes the set of all hermitian matrices into a partially ordered set, for hermitian matrices A and B, $A \leq B$ if and only if $B - A \in M_n^+(C)$. If $X \in M_n(C)$, $|X|$ stands for the unique positive square root of X^*X. For a hermitian matrix A we arrange eigenvalues of A in nonincreasing order as $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ (see [2] for details).

The Young inequality was extended to complex matrices in [1] by T. Ando in the following theorem.

Theorem 1.1. *For each complex matrices A and B there exists a unitary matrix U such that for each conjugate exponents p and q,*

$$U^*|AB^*|U \leq \frac{1}{p}|A|^p + \frac{1}{q}|B|^q. \tag{1.4}$$

Equality holds in (1.4) if and only if $|A|^p = |B|^q$ [3].

2. Main results

An almost immediate consequence of the Gelfand theory is that (1.3) holds for a commuting pair A, B of positive matrices.

Lemma 2.1. *Suppose a, b be invertible positive elements in a unital commutative C^*-algebra $\mathcal{A}$ and $\nu \in (1, \infty)$. Then*

$$ab \geq \nu a^{\frac{1}{\nu}} + (1-\nu)b^{\frac{1}{1-\nu}}\,.$$

Proof. Let $\phi : \mathcal{A} \to C(\Omega)$ be the Gelfand *-isomorphism. Given invertible positive elements $a, b \in \mathcal{A}$, let $f = \phi(a)$ and $g = \phi(b)$. f and g are then invertible positive elements of $C(\Omega)$ and for each $\omega \in \Omega$ by (1.3), $f(\omega)g(\omega) \geq \nu f(\omega)^{\frac{1}{\nu}} + (1-\nu)g(\omega)^{\frac{1}{1-\nu}}$ hence $fg \geq \nu f^{\frac{1}{\nu}} + (1-\nu)g^{\frac{1}{1-\nu}}$ which implies via ϕ that $ab \geq \nu a^{\frac{1}{\nu}} + (1-\nu)b^{\frac{1}{1-\nu}}$. □

Remark 2.2. The fact that the inequality $AB \geq \nu A^{\frac{1}{\nu}} + (1-\nu)B^{\frac{1}{1-\nu}}$ holds for a commuting pair A, B of positive matrices can also be proved by a simultaneous diagonalisation from (1.3) and does not requires theories of C^*-algebras.

The following example shows that (1.3) does not hold for arbitrary matrices A and B that $AB \neq BA$.

Example 2.3. Let $X = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Then for $\nu = 2$,

$$|XB| \not\geq 2|X|^{\frac{1}{2}} + (1-2)B^{-1}.$$

Two special cases of operator functions we will use to prove our main results are the following lemmas:

Lemma 2.4. *The function $t \to t^r$ is operator monotone and operator concave on $[0, \infty)$ for $0 \leq r \leq 1$. That is*

(a) *If $0 \leq X \leq Y$ then $X^r \leq Y^r$.*
(b) *If Q is a projection then $QA^rQ \leq (QAQ)^r$ for each positive semi-definite matrix A.*

Lemma 2.5. *The function $t \to t^r$ is inversely operator monotone and operator convex on $(0, \infty)$ for $-1 \leq r \leq 0$. That is*

(a) *If $0 < X \leq Y$ then $Y^r \leq X^r$.*
(b) *If Q is a projection then $(QAQ)^r \leq QA^rQ$ for each positive definite matrix A.*

See [2, Ch. V] for details of both lemmas and more on operator monotone, operator convex and operator concave functions.

By using Ando's methods [1] and above lemmas, we prove the following theorem to show that the inverse Young inequality holds for eigenvalues of positive definite matrices.

Theorem 2.6. *Assume that $\nu > 1$. Then for each complex matrix A and invertible complex matrix B,*

$$\lambda_j(|AB^*|) \geq \lambda_j\Big(\nu|A|^{\frac{1}{\nu}} + (1-\nu)|B|^{\frac{1}{1-\nu}}\Big), \quad j = 1, 2, \cdots, n.$$

Corollary 2.7. *IF A and B are two complex matrices in $M_n(C)$ such that B is invertible and $\nu \in (1, \infty)$, then there exists a unitary matrix U such that*

$$U^*|AB^*|\,U \geq \nu\,|A|^{\frac{1}{\nu}} + (1-\nu)|B|^{\frac{1}{1-\nu}} \tag{2.1}$$

Moreover, equality holds if and only if $|A|^{\frac{1}{\nu}} = |B|^{\frac{1}{1-\nu}}$.

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THE IMPORTANCE OF THE PERRON-FROBENIUS THEOREM

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Abstract. The Perron-Frobenius theorem provides a simple characterization of the eigenvectors and eigenvalues of certain types of matrices with nonnegative entries. The importance of the Perron-Frobenius theorem stems from the fact that eigenvalue problems on these types of matrices frequently arise in many different fields of science and engineering. In this paper, we will examine the statement of the Perron-Frobenius theorem and discuss several examples of its use, such as in graph theory, stochastic matrices, Markov chains, power-control problems, and finding PageRank scores.

1. Introduction

The Perron-Frobenius theorem is valuable as it describes several eigenvalue/eigenvector properties that hold true for specific matrices that are actually somewhat common in the real world. These properties help us solve more complex eigenvalue/eigenvector problems. Many real world problems can be written in vector notation, to which we can then apply the Perron-Frobenius theorem to arrive at a solution.

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* Speaker.

In 1907, Oskar Perron first discovered that these properties held true for strictly positive matrices [5]. Later, in 1912, Ferdinand Frobenius discovered that these properties extended to certain nonnegative matrices [1].

To understand the theorem, we recall several key linear algebra topics in section 2. The individual statements of the Perron-Frobenius theorem are provided in section 3. The Perron-Frobenius theorem helps determine what types of vectors are special for many types of matrices encountered in the real world, such as stochastic matrices. A few examples showing the usefulness of the theorem are given in section 4.

This paper is derived from a survey that the author is writing about the application of non-negative operators in modelling real world phenomena. The relevant papers and Book are [2], [3] and [4].

2. Prerequisites

We say a matrix A or a vector x is *positive* if all its entries are positive, *nonnegative* if all its entries are nonnegative. We write $A > B$ ($A \geq B$) to mean $A - B$ is positive (nonnegative). Similarly, we use the notations $x > y$ ($x \geq y$) to mean $x - y$ is positive (nonnegative)

For an $n \times n$ matrix A, we assume the reader is familiar with the *eigenvalue* and *eigenvector* of A. The set of all distinct eigenvalues of A, denoted $\sigma(A)$, is referred to as the *spectrum* of A. Furthermore, the *spectral radius* of A, represented as $\rho(A)$, is the maximum magnitude over all eigenvalues of A.

A *permutation matrix* is a matrix where each row and column has only one nonzero entry, which is 1. Thus, if you have a permutation matrix P and a matrix A, taking P^TAP is called conjugating A. It reorders the rows and columns of A the same way.

Using the idea of conjugation, we define an *irreducible matrix* to be a matrix A where, for any permutation matrix P, $P^TAP \neq \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$, where X and Z are both square matrices. In other words, no conjugation of A results in a block upper triangular matrix.

The Perron-Frobenius theorem focuses on nonnegative, irreducible matrices. To check the irreducibility of a matrix, the following theorem is quite direct and helpful [6].

Theorem 2.1. *A nonnegative $n \times n$ matrix A is irreducible if and only if $(I_n + A)^{n-1} > 0$.*

We note that the power in the above expression contains the same n as in the size of the matrix. Since computer software is readily available to compute matrix powers, the above may be readily checked.

3. The Perron-Frobenius Theorem

Here we provide the statement of the Perron-Frobenius theorem:

Theorem 3.1. The Perron-Frobenius Theorem
For any irreducible, nonnegative, $n \times n$ matrix A, the following hold:

- *Matrix A has a real eigenvalue r, called* **Perron root** *of A, which is positive and*

$$r = \rho(A) \in \sigma(A)$$

- *There exists a strictly positive vector x for r.*
- *The algebraic multiplicity of r is 1.*

Such an eigenvalue is called the "dominant eigenvalue" of A. We should point out that other eigenvalues may be positive, negative or complex. Complex eigenvalues are a real possibility as only symmetric matrices are guaranteed to not have them, and very few of the matrices we have been discussing, in application, will be symmetric with the notable exception of undirected graphs.

4. Some Applications of the Perron-Frobenius Theorem

4.1. **Markov Chains.** Suppose A is the transition matrix of a Markov Chain. Then it is nonnegative and its entries, A_{ij} is the probability of going from state j to state i in one transition. If A is reducible, then we can see that it is possible to go from states above the zeo block to any states, but only from states adjacent to the zero block to themselves.

4.2. **Graphs.** If we consider directed graphs, then each has associated with it a nonnegative matrix with all entries 0 or 1, with $A_{ij} = 1$ if there is an arc from vertex i to vertex j. If the associated matrix is irreducible then one can get from any vertex to any other vertex (perhaps in several steps) whereas if it is reducible then (like the Markov Chain case), there are vertices from which one cannot travel to all other vertices.

4.3. **Economic growth.** We consider an economy, with activity level $x_i \geq 0$ in sector i, $i = 1, \cdots, n$. given activity level x in period t, in period $t+1$ we have $x_{t+1} = Ax_t$, with $A \geq 0$. $A_{ij} \geq 0$ means activity in sector j does not decrease activity in sector i.

4.4. **Dynamical Systems.** Suppose $(x_t)_i$ denotes number of individuals in group i at period t. For example, in a population model, groups could be by age, location, health, marital status, etc. Its dynamics is given by $x_{t+1} = Ax_t$, with $A \geq 0$. A_{ij} gives the fraction of members of group j that move to group i, or the number of members in group i created by members of group j (e.g., in births). $A_{ij} \geq 0$ means the

more we have in group j in a period, the more we have in group i in the next period.

There are many other applications of Perron-Frobinius theorem. This is because Non-negative matrices arises in classification of players (in different field of sports), linear optimization, queueing theory, modeling of an uplink problem in networks, power control, linear Lyapunov functions and more.

5. Conclusion

The Perron-Frobenius Theorem has proven to be a consistently powerful result for examining certain nonnegative matrices arising in discrete models. It has been shown that careful consideration need be given to what hypothesis is used; depending on whether one has an irreducible or primitive matrix. In applications, knowledge of the dominant eigenvalue and eigenvector is very helpful and also attainable while knowledge of the rest of the "spectrum" is both unnecessary and computationally extensive.

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Relevant papers and book [2], [3] and [4]

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HIGH ORDER MATRIX TRANSFORM METHOD FOR THE RIESZ SPACE FRACTIONAL DIFFUSION EQUATION

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Abstract. In this paper, a matrix transform method coupled with fractional centered difference scheme for solving fractional diffusion equation with Riesz fractional derivative is considered. The stability and convergence of proposed method has been discussed by using the matrix analysis method. Finally, numerical example is presented to confirm the analytical results.

1. Introduction

Fractional calculus is a useful mathematical tool for applied sciences and recently fractional differential equations have found new applications in engineering, physics, chemistry, hydrology and other sciences [1, 2]. In this paper, we consider the following Riesz space fractional diffusion equations derived from the aforementioned model as follows

$$\frac{\partial u(x,t)}{\partial t} = K_\alpha \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha}, \quad x \in [0,L],\, t \in [0,T], \tag{1.1}$$

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* Speaker.

subject to the initial value and boundary conditions given by

$$u(x,0) = f(x) \tag{1.2}$$

$$u(0,t) = u(L,t) = 0 \tag{1.3}$$

where K_α denotes the anomalous diffusion coefficient and $f(x)$ is assumed to be smooth enough. We suppose here that the exact solution of the problem is smooth enough to satisfy the requirements of discretization. $\frac{\partial^\alpha \varphi}{\partial |x|^\alpha}$ denotes the Riesz space fractional derivative of order $\alpha,\ 1 < \alpha \leq 2$ defined by [3]

$$\frac{\partial^\alpha \varphi(x,t)}{\partial |x|^\alpha} = -c_\alpha ({}_0D_x^\alpha + {}_xD_L^\alpha)\varphi(x,t), \tag{1.4}$$

where $c_\alpha = \frac{1}{2cos(\frac{\pi\alpha}{2})}$, ${}_0D_x^\alpha$ and ${}_xD_L^\alpha$ are the left and right Riemann-Liouville space fractional derivatives of order α given by

$${}_0D_x^\alpha \varphi(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_0^x (x-\xi)^{1-\alpha} \varphi(\xi,t) d\xi,$$

$${}_xD_L^\alpha \varphi(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_x^L (\xi-x)^{1-\alpha} \varphi(\xi,t) d\xi.$$

The outline of the paper is organized as follows: In section 2 we derive a numerical method by [3,1] Padé approximation coupled with fractional centered difference scheme for the Riesz space fractional diffusion equation. In section 3 we prove the convergence of the numerical method by showing consistency and stability. In Section 4 we conduct numerical experiments to examine the performance of the proposed numerical method. Finally, the conclusions to this work are drawn in Section 5.

2. Discretisation of the fractional diffusion equation

We shall solve (1.1) with initial value (1.2) and homogeneous Dirichlet boundary conditions (1.3). Let M and N be the number of grid points in the x- and t-direction, respectively. Let $h = L/M$, $k = T/N$ and let $x_i = ih,\ i = 0,1,...,M,\ t_j = jk,\ j = 0,1,...,N$. Numerical approximations of $u(x,t)$ at the grid are denoted by

$$u_{i,j} = u(x_i, t_j) \tag{2.1}$$

Assume that $u(x,t)$ is sufficiently smooth function and replace the Riesz fractional partial derivatives in (1.1) with respect to x by the fractional centered difference estimate:

$$\frac{\partial^\alpha u(x_i,t_j)}{\partial |x|^\alpha} = -\frac{1}{h^\alpha} \sum_{s=-M+i}^{i} \omega_s^{(\alpha)} u(x_{i-s}, t_j) + O(h^2), \tag{2.2}$$

where $\omega_{s,\alpha} = (-1)^s \frac{\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-s+1)\Gamma(\frac{\alpha}{2}+s+1)}$, $\alpha > -1$ for $s = 0, 1, 2, ...$ are the fractional centered difference coefficients.
This discretization results in an initial-value problem of the form:

$$\frac{dU(t)}{dt} = -AU(t), \quad U(0) = U_0, \tag{2.3}$$

where the matrix A is as follows:

$$A_{i,j} = \frac{\omega_{|i-j|,\alpha}}{h^\alpha}, \quad \text{for} \quad i, j = 1, 2, ..., M-1.$$

the exact solution of (2.2) can be written as

$$U(t) = exp(-tA)U_0, \tag{2.4}$$

Replacing t by $t + k$, we can write the exact solution (2.3) as

$$U(t+k) = exp(-(t+k)A)U_0 = exp(-kA)U(t), \tag{2.5}$$

which satisfies the recurrence formula:

$$U(t_{n+1}) = exp(-kA)U(t_n), \quad \text{for} \quad n = 0, 1, ... \tag{2.6}$$

We shall use [3, 1] Padé approximations of the matrix exponential functions $exp(-kA)$ to construct a family of time stepping schemes. then we obtain the following numerical scheme for solving (1.1)

$$V^{n+1} = (24I - 6(kA))(24I + 18(kA) + 6(kA)^2 + (kA)^3)^{-1}V^n, \tag{2.7}$$

V is the approximate value of U.

3. Stability Analysis

The following theorems proved via matrix analysis method [4]

Theorem 3.1. *The truncation error of the difference scheme (2.7) is* $O(h^2 + k^5)$.

Theorem 3.2. *Numerical algorithm (2.7) is unconditionally stable.*

4. Numerical Results

Example 4.1. Consider the following Riesz fractional diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = K_\alpha \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha}, \quad 0 < x < 1, \quad t > 0, \quad 1 < \alpha \leq 2, \tag{4.1}$$

associated with the initial value and zero Dirichlet boundary conditions

$$u(x, 0) = sin(\pi x), \quad 0 \leq x \leq 1, \tag{4.2}$$

$$u(0, t) = u(1, t) = 0, \quad 0 \leq t \leq T, \tag{4.3}$$

This Riesz fractional diffusion equation has the exact solution as follow [5]

$$u(x,t) = sin(\pi x)\exp(-[K_\alpha(\pi^2)^{\frac{\alpha}{2}}]t)$$

To examine the performance of this recurrence method for this example problem, In Table 1 the maximum numerical error at time $t = 1.0$ was computed for different value of Δx and Δt. It can be seen that the Padé approximation-fractional centered difference scheme is in excellent agreement with the exact solution.

TABLE 1
Maximum error behavior for the example problem at $T_{end} = 1$

$h = \Delta x$	$k = \Delta t$	Maximum error for $\alpha = 1.8$
0.05	0.05	1.2297e-007
0.05	0.01	2.0267e-010
0.05	0.001	1.2474e-013
0.01	0.05	1.7752e-005
0.01	0.01	4.5882e-010
0.01	0.001	1.9479e-013

5. CONCLUSIONS

In this work, we proposed a method to find the solution of the Riesz fractional diffusion equations via a matrix transform method coupled with fractional centered difference scheme. The method is twice order in space and fifth order in time direction, respectively. It is shown through analysis that the difference scheme is unconditionally stable and convergence, and numerical experiments are conducted to test its high accuracy and to show its reasonableness.

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INTERPOLATION OF TRIPLE QUASI-BANACH SPACES ON $\mathbb{R}^3$

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Abstract. Let $\bar{A} = (A_0, A_1, A_2)$ be a compatible triple of quasi-Banach spaces. We may define the interpolation method in $\mathbb{R}^3$, and prove some related lemma and theorem.

1. Introduction

Let A_0 and A_1 be two topological vector spaces. Then we shall say that A_0 and A_1 are compatible if there is a hausdorff topological vector spaces $\mathcal{U}$ such that A_0 and A_1 are sub-spaces of $\mathcal{U}$.

In this paper φ denotes any sub-category of the category $\mathcal{N}$ of all normed vector spaces. We let φ_1 stand for the category of all compatible couples $\bar{A}$ of spaces in φ.

Let $\bar{A} = (A_0, A_1)$ be a given in φ_1. Then a space A in φ will be called an intermediate space between A_0 and A_1 (or with respect to $\bar{A}$) if $\triangle(\bar{A}) \subset A \subset \sum(\bar{A})$ $(\triangle(\bar{A}) = A_0 \cap A_1, \sum(\bar{A}) = A_0 + A_1)$ With continuous inclusions. The space A is called an interpolation space between A_0 and A_1 (or with to $\bar{A}$) if in addition

$$T : \bar{A} \longrightarrow \bar{A} \quad implies \quad T : A \longrightarrow A$$

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Key words and phrases. interpolation space, Banach spaces, quasi-Banach spaces.

* Speaker.

(linear operator $T : \bar{A} \longrightarrow \bar{A}$ which means, as usual, that $T : A_0 + A_1 \longrightarrow A_0 + A_1$ is linear and the restrictions $T|_{A_j}$ are bounded operators from A_j to A_j for $j = 0, 1$).

2. Main section and results

Our main reference to the theory of interpolation space is [1]. Let $\bar{A} = (A_0, A_1, A_2)$ be a compatible triple of quasi-Banach spaces, we define the space $F(\bar{A})$ of all function f with values in $\sum(\bar{A})$, which are bounded and continuous on the $\overline{S} = \{x = (x_0, x_1, x_2) \in \mathbb{R}^3 \ : \ 0 \leq x_0 \leq 1\}$ and analytic on the $S = \{x = (x_0, x_1, x_2) \in \mathbb{R}^3 \ : \ 0 < x_0 < 1\}$. And moreover, the functions $t \longrightarrow f(0, j, t)(j = 0, 1)$ and $t \longrightarrow f(1, 1, t)$ are continuous function from the real line into A_0, A_1 and A_2,which tend to zero as $|t| \longrightarrow \infty$. Clearly, $F(\bar{A})$ is a vector space. We provided F with the quasi-norm

$$\|f\|_F = max(sup\|f(0, 0, t)\|_{A_0}, sup\|f(0, 1, t)\|_{A_1}, sup\|f(1, 1, t)\|_{A_2}).$$

Theorem 2.1. *The space F is a quasi-Banach space.*

We define the interpolation functor $\bar{A}_{[\theta]}$. The space $\bar{A}_{[\theta]}$ consists of all $a \in \sum(\bar{A})$ such that $a = f(\theta)((\theta, 0, 0) = \theta)$ for some $f \in F(\bar{A})$. The quasi-norm on $\bar{A}_{[\theta]}$ is

$$\|a\|_{[\theta]} = inf\{\|f\|_F : \ f(\theta) = a, \ f \in F\}.$$

Now we may introduce a second interpolation method in $\mathbb{R}^3$. This is based on a space $\tilde{F}(\bar{A})$ of analytic functions, defined as follows. The functions g in $\tilde{F}(\bar{A})$ are defined on S with values in $\sum(\bar{a})$. Moreover they have the following properties:
(i) $\|g\|_{\sum(\bar{A})} \leq c(1 + |x|), \quad (|x| = (\sum_{i=0}^{i=2} x_i^2)^{1/2})$
(ii) g is continuous on $\overline{S}$ and analytic on S,
(iii) $g(0, j, t_1) - g(0, j, t_2)$ has values in A_j and $g(1, 1, t_1) - g(1, 1, t_2) \in A_2$ for all real values of t_1 and t_2 and for $j = 0, 1$, and

$$\|g\|_{\tilde{F}} = max(sup_{t_1,t_2}\|\frac{g(0, 0, t_1) - g(0, 0, t_2)}{t_1 - t_2}\|_{A_0}$$

$$, sup_{t_1,t_2}\|\frac{g(0, 1, t_1) - g(0, 1, t_2)}{t_1 - t_2}\|_{A_1}, sup_{t_1,t_2} \ |\frac{g(1, 1, t_1) - g(1, 1, t_2)}{t_1 - t_2}\|_{A_2}).$$

is finite.

Lemma 2.2. *We have*

$$(A, A, A)_{[\theta]} = A \ \ if \ 0 < \theta < 1.$$

Theorem 2.3. *Let $F_0(\bar{A})$ be the space of all linear combinations of functions of the form*

$$exp(\delta z^2)\sum_{n=1}^{N} a_n exp(\lambda_n z)$$

Where an $a_n \in \triangle(\bar{A})$, λ_n real and $\delta > 0$. Then $F_0(\bar{A})$ is dense in $\tilde{F}(\bar{A})$.

Theorem 2.4. *For any $\bar{A} = (A_0, A_1, A_2)$, we have*

$$\bar{A}_{[\theta]} \subset \bar{A}^{[\theta]} \quad and \quad \|a\|^{[\theta]} \leq \|a\|_{[\theta]}.$$

We prove two theorems concerning multilinear interpolation.

Theorem 2.5. *Let $\bar{A}^{(\nu)}(\nu = 1, 2, ..., n)$ and $\bar{B}$ be compatible triple quasi-Banach spaces. assume that $T : \sum_{1\leq\nu\leq n}^{\oplus} \triangle(\bar{A}^{(\nu)}) \longrightarrow \triangle(\bar{B})$ is multilinear and*

$$\|T(a_1, ..., a_n)\|_{B_j} \leq M_j \prod_{\nu=1}^{n} \|a_\nu\|_{A_j^\nu}. \quad (a_\nu \in \triangle(\bar{A}^{(\nu)}),\ j = 0, 1, 2)$$

Then T may be uniquely extended to a multilinear mapping from $\sum_{1\leq\nu\leq n}^{\oplus} \triangle(\bar{A}^{(\nu)})$ to $\bar{B}_{[\theta]}$ with quasi-norm at most $M_0^{1-(\eta+\theta)} M_1^\theta M_2^\eta$ $(0 \leq \theta + \eta \leq 1)$.

Theorem 2.6. *Let $\bar{A}^{(\nu)}(\nu = 1, 2, ..., n)$ and $\bar{B}$ be compatible triple quasi-Banach spaces. assume that $T : \sum \bar{A}^{(1)} \oplus \sum_{2\leq\nu\leq n}^{\oplus} \triangle(\bar{A}^{(\nu)}) \longrightarrow \triangle(\bar{B})$ is multilinear and, the restriction to $\bar{A}_i^{(1)}$ having values in $B_i, i = 0, 1, 2$, and*

$$\|T(a^{(1)}, ..., a^{(n)}\|_{B_j} \leq M_j \prod_{\nu=1}^{n} \|a^{(\nu)}\|_{A_j^\nu}. \quad (a^{(1)} \in \triangle(\bar{A}_j^{(1)}),\ j = 0, 1, 2)$$

Then T may be uniquely extended to a multilinear mapping from $\sum \bar{A}^{(1)} \oplus \sum_{2\leq\nu\leq n}^{\oplus} \triangle(\bar{A}^{(\nu)})$ to $\bar{B}^{[\theta]}$ with quasi-norm at most $M_0^{1-(\eta+\theta)} M_1^\theta M_2^\eta$ $(0 < \theta + \eta < 1)$.

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SOME REMARKS ON A DETERMINANT RELATED TO THE PRIME COUNTING FUNCTION

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Abstract. In this note we consider the numbers $k_1, k_2, \ldots, k_n$ appearing in the expansion $\log x - \frac{x}{\pi(x)} = 1 + \sum_{j=1}^{n} \frac{k_j}{\log^j x} + O_n(\frac{1}{\log^{n+1} x})$ as $x \to \infty$, where as usual $\pi(x)$ denotes the number of prime numbers not exceeding x. Indeed, k_n can be determined in terms of a determinant. We study asymptotic behaviour of this determinant, and some other related numbers.

1. Introduction

We let $\pi(x)$ denotes the number of primes not exceeding x. The prime number theorem with error term implies

$$\pi(x) = x\left(\frac{1}{\log x} + \frac{1!}{\log^2 x} + \cdots + \frac{n!}{\log^{n+1} x}\right) + O\left(\frac{x}{\log^{n+2} x}\right), \quad (1.1)$$

for each integer $n \geqslant 1$. We define the function $A : [2, \infty) \to \mathbb{R}$ by

$$A(x) = \log x - \frac{x}{\pi(x)}.$$

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* Speaker.

This function appears in approximations of $\pi(x)$ in the form

$$\pi(x) = \frac{x}{\log x - A(x)}.$$

Although A.-M. Legendre [3] suggested the approximate value $A(x) \approx 1.08336$, nowadays it is known that $A(x) \to 1$ as $x \to \infty$. Moreover, Panaitopol [4] showed that (1.1) implies the expansion

$$A(x) = 1 + \frac{k_1}{\log x} + \frac{k_2}{\log^2 x} + \cdots + \frac{k_n}{\log^n x} + O_n\Big(\frac{1}{\log^{n+1} x}\Big), \qquad (1.2)$$

where $k_1, k_2, \ldots, k_n$ are constants given by the recurrence relation

$$k_n + 1!k_{n-1} + 2!k_{n-2} + \cdots + (n-1)!k_1 = n \cdot n!, \quad (n = 1, 2, 3, \ldots). \quad (1.3)$$

and some initial values are $k_1 = 1, k_2 = 3, k_3 = 13, k_4 = 71$. Recently we have studied the function $A(x)$ in [2]. Our main results are as follows.

Theorem 1.1. *Let $x_0 = 32299$. We have*

$$1 + \frac{0.56}{\log x} \leqslant A(x) \leqslant 1 + \frac{1.51}{\log x},$$

where the left hand side inequality is valid for each $x \geqslant x_0$ and fails for $x < x_0$, and the right hand side inequality is valid for each $x \geqslant 2$.

Corollary 1.2. *The function $A(x)$ takes the maximum value only at the prime $x = 24137$, and we have*

$$\max_{x \geqslant 2} A(x) = A(24137) = \log(24137) - \frac{24137}{2688} \approxeq 1.11196252139.$$

The Riemann zeta function is defined for $\Re(s) > 1$ by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, and extended by analytic continuation to the complex plan with one simple pole at $s = 1$ with residue 1. The Riemann hypothesis states that non-trivial zeros of the Riemann zeta function all lie on the line $\Re(s) = \frac{1}{2}$. Among various equivalences for the Riemann hypothesis, the following one is about some certain matrices.

Let R_n be the $0-1$ matrix with size $n \times n$ defined by $R_{ij} = 1$ if $j = 1$ or if i divides j, and $R_{ij} = 0$ otherwise. Then, the Riemann hypothesis holds if and only if for every $\varepsilon > 0$ there is a $C(\varepsilon) > 0$ such that

$$|\det(R_n)| < C(\varepsilon) n^{\frac{1}{2}+\varepsilon}.$$

The Riemann hypothesis has a remarkable effect on the error term in the prime number theorem. Considering such effects, we obtained some sharper results concerning $A(x)$, as follows.

Theorem 1.3. *Assume that the Riemann Hypothesis is true. Then one has*

$$-\frac{29}{\log^2 x} \leqslant A(x) - \left(1 + \frac{1}{\log x}\right) \leqslant \frac{23}{\log^2 x} \qquad \text{for } x \geqslant 5 \times 10^9,$$

and

$$-\frac{136}{\log^3 x} \leqslant A(x) - \left(1 + \frac{1}{\log x} + \frac{3}{\log^2 x}\right) \leqslant \frac{114}{\log^3 x} \qquad \text{for } x \geqslant 9 \times 10^{12}.$$

The sequence k_n defined in (1.2) is addressed on the on-line encyclopedia of integer sequences with the sequence number A233824 [5]. A Maple code to compute any arbitrary value, which is based on the recurrence relation (1.3) is

```
k:=proc(n) option remember;
if n=1 then 1 else n*n!-add((n-j)!*k(j), j=1..n-1); fi;
end;
```

Running the above program on Maple, we get some more initial values of the sequence $(k_n)_{n\geqslant 1}$ as follows

$$1, 3, 13, 71, 461, 3447, 29093, 273343, 2829325, 31998903, 392743957.$$

Also, k_n can be determined in terms of a determinant as follows

$$k_n = \det \begin{bmatrix} n \cdot n! & 1! & 2! & \dots & (n-1)! \\ (n-1) \cdot (n-1)! & 0! & 1! & \dots & (n-2)! \\ \vdots & \vdots & \vdots & & \vdots \\ 2 \cdot 2! & 0 & 0 & \dots & 1! \\ 1 \cdot 1! & 0 & 0 & \dots & 0! \end{bmatrix}_{n\times n}.$$

Computations led us to some observations about the values of k_n and some other related sequences. We list a number of them in the following conjectures.

(1) The sequence $(k_n)_{n\geqslant 1}$ is strictly increasing and positive. Moreover, as $n \to \infty$ one has

$$k_n \sim n \cdot n!.$$

(2) The sequence $(a_n)_{n\geqslant 1}$ with general term $a_n = \frac{n\cdot n! - k_n}{n!}$ is strictly increasing for $n \geqslant 7$. Also, $a_n \to 1$ as $n \to \infty$, and so

$$k_n = (n + O(1))\, n!,$$

and one has

$$\max_{n\geqslant 1} a_n = a_7 = \frac{6187}{5040}, \quad \text{and} \quad \min_{n\geqslant 1} a_n = a_1 = 0.$$

(3) For any $n \geqslant 3$ the ratio $b_n := \frac{k_n}{n \cdot n!}$ increases strictly. Hence, for any $n \geqslant 2$ one gets $k_n < n \cdot n!$. Also, one has

$$\max_{n \geqslant 1} b_n = b_1 = 1, \quad \text{and} \quad \min_{n \geqslant 1} b_n = b_3 = \frac{13}{18}.$$

(4) For any $n \geqslant 6$ the difference $d_n := b_n - b_{n-1}$ decreases strictly, and $d_n \to 0$ as $n \to \infty$. Hence one obtains $d_n > 0$ for any $n \geqslant 4$. Moreover, one imply that

$$\max_{n \geqslant 2} d_n = d_6 = \frac{71}{2400}, \quad \text{and} \quad \min_{n \geqslant 2} d_n = d_2 = -\frac{1}{4}.$$

In this presentation we introduce evidences and computational results leading to the above conjectures, and we suggest sketch of proofs, for some of them.

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JOINT $C-$NUMERICAL RANGES AND RADII OF MATRICES

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Abstract. In this paper, using the notion of joint $C-$numerical range, some algebraic properties of the $C-$numerical range of matrix polynomials are investigated. Moreover, the notion of joint $C-$numerical radius of matrices are introduced and studied.

1. Introduction

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices and $\mathcal{U}_n$ be the group of unitary matrices in $\mathbb{M}_n$. For $A, C \in M_n$, the *$C-$numerical range* and the *$C-$numerical radius* of A are defined and denoted, respectively, by:

$$W_C(A) = \{tr(CU^*AU) \ : \ U \in \mathcal{U}_n\} \ and \ \ r_C(A) = \max_{z \in W_C(A)} |z|,$$

where $tr(.)$ denotes the *trace*. The $C-$numerical range and $C-$numerical radius of matrices are related to optimization problems, and have important applications in quantum control and quantum information;

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e.g., see [3] and its references. Let C and A have eigenvalues $\gamma_1, \ldots, \gamma_n$, and $\alpha_1, \ldots, \alpha_n$, respectively. The $C-$spectrum of A is defined as:

$$\sigma_C(A) = \{ \sum_{j=1}^{n} \gamma_j \alpha_{i_j} \ : \ (i_1, \ldots, i_n) \ is \ a \ permutation \ of \ \{1, 2, \ldots, n\} \ \}.$$

It is known that $\sigma_C(A) \subseteq W_C(A)$. The concept of $C-$spectrum of A is very useful in the study of $W_C(A)$. For a comprehensive survey of $W_C(A)$, $r_C(A)$ and $\sigma_C(A)$, see [5].
One of generalizations of the $C-$numerical ranges is the notion of joint $C-$numerical ranges. Suppose that $C \in \mathbb{M}_n$ and $\mathbf{A} = (A_0, \ldots, A_m) \in \mathbb{M}_n^{m+1}$. The joint $C-$numerical range of $\mathbf{A}$ is defined and denoted, e.g., see [1, 2] and their references, by:

$$W_C(\mathbf{A}) = \{(tr(CU^*A_0U), \ldots, tr(CU^*A_mU)) \ : \ U \in \mathcal{U}_n\}.$$

Since $W_C(A_0, \ldots, A_m)$ can be viewed as the range of the continuous function

$$U \longmapsto (tr(CU^*A_0U), \ldots, tr(CU^*A_mU))$$

from the compact connected set $\mathcal{U}_n$ to $\mathbb{C}^{m+1}$, we have that:
$W_C(A_0, A_1, \ldots, A_m)$ is also compact and connected set in $\mathbb{C}^{m+1}$.
At the end of this section, we give some information about matrix polynomials. Notice that matrix polynomials arise in many applications and their spectral analysis is very important when studying linear systems of ordinary differential equations with constant coefficients. Suppose that

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0 \qquad (1.1)$$

is a matrix polynomial, where $A_i \in \mathbb{M}_n$ $(i = 0, 1, \ldots, m)$, $A_m \neq 0$ and λ is a complex variable. A scalar $\lambda_0 \in \mathbb{C}$ is an *eigenvalue* of $P(\lambda)$ if the system $P(\lambda_0)x = 0$ has a nonzero solution $x_0 \in \mathbb{C}^n$. This solution x_0 is known as an *eigenvector* of $P(\lambda)$ corresponding to λ_0, and the set of all eigenvalues of $P(\lambda)$ is said to be the *spectrum* of $P(\lambda)$; namely,

$$\sigma[P(\lambda)] = \{\mu \in \mathbb{C} \ : \ det(P(\mu)) = 0\}.$$

The *(classical) numerical range* of $P(\lambda)$ is defined and denoted by

$$W[P(\lambda)] := \{\mu \in \mathbb{C} \ : \ x^*P(\mu)x = 0 \ for \ some \ nonzero \ x \in \mathbb{C}^n\},$$

which is closed and contains $\sigma[P(\lambda)]$. The numerical range of matrix polynomials plays an important role in the study of overdamped vibration systems with finite number of degrees of freedom, and it is also related to the stability theory; e.g., see [4] and its references. In Section 2 of this paper, using the notion of joint $C-$numerical range, we investigate some algebraic properties of the $C-$numerical range of

matrix polynomials. Moreover, we introduce and study the notion of joint $C-$numerical radius of matrices.

2. Main results

We begin this section by introducing the notion of the $C-$numerical range of matrix polynomials. Let $P(\lambda)$ be a matrix polynomial as in (1.1). For a given matrix $C \in M_n$, the $C-$numerical range of $P(\lambda)$ is defined and denoted by

$$W_C[P(\lambda)] = \{\mu \in \mathbb{C} \ : \ tr(CU^*P(\mu)U) = 0 \ \ for\ some\ U \in \mathcal{U}_n\}.$$

Now, in the following theorem, we state the relationship between the $C-$numerical range of $P(\lambda)$ and the joint $C-$numerical range of its coefficients. Also, using the $C-$numerical range of diagonal matrix polynomials, we can approximate the shape of the $C-$numerical range of any matrix polynomial.

Theorem 2.1. *Let $C \in M_n$, and $P(\lambda)$ be a matrix polynomial as in (1.1). Then the following assertions are true:*
(i) $W_C[P(\lambda)] = \{\mu \in \mathbb{C} \ : \ a_m\mu^m+\cdots+a_1\mu+a_0 = 0, \ (a_0, a_1, \ldots, a_m) \in W_C(A_0, A_1, \ldots, A_m)\}$;
(ii) $W_C[P(\lambda)] = \bigcup W_C[D(\lambda)]$, *where the union is taken over all diagonal matrix polynomials $D(\lambda)$ of degree m and order n such that $JW_C[D(\lambda)] \subseteq JW_C[P(\lambda)]$, and note that $JW_C[P(\lambda)]$ is considered as the joint $C-$numerical range of its coefficients.*

Proof. The result in (i) follows easily from the definitions of $W_C[P(\lambda)]$ and $W_C(A_0, \ldots, A_m)$.
To prove (ii), by (i), $\supseteq$ is clear. Let now $\mu \in W_C[P(\lambda)]$ be given. By (i), there exists a $(a_0, a_1, \ldots, a_m) \in JW_C[P(\lambda)]$ such that $a_m\mu^m+\cdots+a_1\mu+a_0 = 0$. By setting $D(\lambda) = \frac{a_m}{tr(C)}I\lambda^m+\cdots+\frac{a_1}{tr(C)}I\lambda+\frac{a_0}{tr(C)}I$, we have $JW_C[D(\lambda)] = \{ \ (a_0, a_1, \ldots, a_m) \ \} \subseteq JW_C[P(\lambda)]$, and $\mu \in W_C[D(\lambda)]$. Hence, the proof of $\subseteq$ is complete. □

Corollary 2.2. *Let $C \in M_n$, and $P(\lambda)$ be a matrix polynomial as in (1.1). If $(0, 0, \ldots, 0) \in JW_C[P(\lambda)]$, then $W_C[P(\lambda)] = \mathbb{C}$.*

At the end of this section, we introduce and study some properties of the notion of joint $C-$numerical radius of matrices.

Definition 2.3. Let $\mathbf{A} = (A_0, A_1, \ldots, A_m) \in \mathbb{M}_n^{m+1}$ and $C \in \mathbb{M}_n$. The joint $C-$numerical radius of $\mathbf{A}$ is defined and denoted by

$$r_C(\mathbf{A}) = r_C(A_0, \ldots, A_m) := \sup_{(a_0,\ldots,a_m)\in W_C(A_0,\ldots,A_m)} l_2(a_0, \ldots, a_m),$$

where $l_2(a_0, \ldots, a_m) = (\sum_{j=0}^m |a_j|^2)^{\frac{1}{2}}$.

Now, in the following theorem, we state some algebraic properties of the joint $C-$numerical radius of matrices.

Theorem 2.4. *Let $\boldsymbol{A} = (A_0, A_1, \ldots, A_m) \in \mathbb{M}_n^{m+1}$ and $C \in \mathbb{M}_n$. Then*
(i) $r_C(.)$ is a seminorm on $\mathbb{M}_n^{m+1}$;
(ii) If $tr(C) \neq 0$ and C is not a scalar matrix, then $r_C(.)$ is a norm on $\mathbb{M}_n^{m+1}$;
(iii) $r_C(\boldsymbol{A}) \leq \|\boldsymbol{A}\|_C := \sup_{(\lambda_0,\lambda_1,\ldots,\lambda_m)\in S} \|(\lambda_0 A_0 + \cdots + \lambda_m A_m)\|_C$, where $S = \{(z_0, \ldots, z_m) \in \mathbb{C}^{m+1} \ : \ |z_0|^2 + \cdots + |z_m|^2 = 1\}$;

(iv) Let $C \in \mathbb{M}_n$ be a non-scalar matrix. Suppose that $\tau := |tr(C)| > 0$ and $\delta := diam(W(C))$, where $W(C)$ is the numerical range of C. Then

$$\alpha\|\boldsymbol{A}\| \leqslant r_C(\boldsymbol{A}) \leqslant \beta\|\boldsymbol{A}\|,$$

where $\|\boldsymbol{A}\| := \|\boldsymbol{A}\|_{E_{11}}$, and
(a) $\alpha = \tau/2$ if $tr(C) \in W(C)$;
(b) $\alpha = \delta\tau/|4tr(C) - 2(\gamma_1 - \gamma_n)|$ if $tr(C) \notin W(C)$ and μC is Hermitian with eigenvalues $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_n$ for some non-zero $\mu \in \mathbb{C}$;
(c) $\alpha = \delta\tau/(4(1-1/n)\tau + \delta)$ if $tr(C) \notin W(C)$ and μC is not Hermitian for any non-zero $\mu \in \mathbb{C}$;
(d) $\beta = \sum_{i=1}^{n} s_i(C)$.

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A FAST COMPUTATIONAL METHOD FOR CALCULATE THE MOORE–PENROSE INVERSE FULL AND SPARSE MATRICES

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Abstract. In This paper, we provide a fast computational method in order to calculate the Moore-Penrose inverse full and sparse of matrices. Our method proves to be much faster and has significantly better accuracy than the already proposed methods.

1. Introduction

Let $\mathbb{R}^{m\times n}$ denote the set of all $m \times n$ matrices over the field of real numbers, $\mathbb{R}$. The symbols A^T, $\text{rank}(A^T)$ will stand for the transpose and rank of $A \in \mathbb{R}^{m\times n}$, respectively. For a matrix $A \in \mathbb{R}^{m\times n}$, the Moore–Penrose inverse of A, denoted by $A^\dagger$, is the unique matrix $X \in \mathbb{R}^{n\times m}$ satisfying the following equations

$$AXA = A, \quad XAX = X, \quad (AX)^T = AX, \quad (XA)^T = XA. \tag{1.1}$$

There are several methods for computing the Moore–Penrose inverse matrix [1, 2, 5]. In [3], a method (qrginv) for computing the Moore–Penrose inverse of an arbitrary matrix was presented. They made use of the QR–factorization, as well as an algorithm based on a known reverse

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* Speaker.

order law for generalized inverse matrices, and also they apply a method (ginv), presented in [2]. In the current article, we improved **qrginv** algorithm using the QR–factorization by Gram–Schmidt orthonormalization (GSO) and Theorem 2.1 for faster computing Moore–Penrose inverse of arbitrary matrices (including singular and rectangular). We should note that we invoke **ginv** algorithm.

2. Main Results

In order to support and state our achievement we need to remind Gram-Schmidt orthonormalization (GSO) and the QR–factorization.

2.1. **The Gram–Schmidt procedure.** Let us remind a generalization of the Gram–Schmidt orthonormalization process (shortly GSO) which is applied for singular matrices. Let $\mathcal{A} = \{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{R}^m$ be a set of vectors spanning a subspace V. This process generates a set of mutually orthonormal vectors such as $\mathcal{Q} = \{q_1, q_2, \ldots, q_r\} \subseteq \mathbb{R}^m$ having the property that $\mathcal{Q}$ is an orthonormalization basis for V. $\mathcal{Q}$ is obtained using the Gram–Schmidt orthonormalization process (shortly GSO) as follows.

- $q_1 = \frac{a_{c_1}}{\|a_{c_1}\|}, \qquad\qquad if\ a_{c_1} \neq 0 = a_j\ for\ 1 \leq j < c_1,$
- $\hat{q}_j = a_j - \sum_{i=1}^{k-1}(a_j,\ q_i)q_i, \qquad j = c_{k-1}+1,\ c_{k-2}+2, \ldots,\ c_k,$
- $q_k = \frac{\hat{q}_j}{\|\hat{q}_j\|}, \quad if\ \hat{q}_{c_k} \neq 0 = q_j,\ for\ c_{k-1}+1 \leq j < c_k,\ k = 2, \ldots,\ r.$

The integer r found by the GSO process is the dimension of the subspace V. The integers $\{c_1, \ldots, c_r\}$ are the indices of a maximal linearly independent subset $\{a_{c_1}, \ldots, a_{c_r}\}$ of $\mathcal{A}$.

2.2. **The QR–factorization.** Let us remind the QR–factorization for arbitrary matrices (including singular and rectangular). Let the orthonormal set $\{q_1, \ldots, q_r\}$ be obtained from the set of vectors $\{a_1, \ldots, a_n\}$ by the GSO process described in above and let

$$\tilde{Q} = [q_1, \ldots, q_r] \in \mathbb{R}^{m\times r}, \qquad A = [a_1, \ldots, a_n] \in \mathbb{R}^{m\times n},$$

with $\mathrm{rank}(A) = r > 0$, be the corresponding matrices. Then there exist matrices $\tilde{A}$, Q and R such that

$$\tilde{A} = AP = QR, \tag{2.1}$$

where

- P is a permutation matrix and $Q = [\tilde{Q}\ Z] \in \mathbb{R}^{m\times m}$, where $\tilde{Q}$ and Z denote matrices whose columns are an orthonormal basis of $R(A)$ and $N(A^T)$, resectively;

- $R = [\tilde{R} \quad 0]^T \in \mathbb{R}^{m\times n}$, where $\tilde{R} \in \mathbb{R}^{r\times n}$ is upper triangular matrix with $\text{rank}(\tilde{R}) = r$.

From (2.1) we can obtain $\tilde{A} = [\tilde{Q}\ Z][\tilde{R} \quad 0]^T = \tilde{Q}\tilde{R}$. It follows that $\tilde{A}$ has a $\tilde{Q}\tilde{R}$–factorization. A nonzero matrix can be expressed as the product of a matrix of full column rank and a matrix of full row rank. In fact, for given $A \in \mathbb{R}^{m\times n}$ ($\text{rank}(A) = r > 0$) there exist matrices $F \in \mathbb{R}^{m\times r}$ and $G \in \mathbb{R}^{r\times n}$ such that $A = FG$ [1]. Such factorization, which is so-called a full rank factorization, turn out to be a powerful tool in the study of generalized inverses. The following theorem is due to C.C. MacDuffe [1] who apparently was the first one to point out, that a full rank factorization of a matrix A leads to an explicit formula for its Moore–Penrose inverse, $A^\dagger$.

Theorem 2.1. *If $A \in \mathbb{R}^{m\times n}$ matrix, with $\text{rank}(A) = r > 0$, has a full rank factorization $A = FG$, then $A^\dagger = G^T(F^TAG^T)^{-1}F^T$.*

As a direct consequence of Theorem 2.1, we have:

Corollary 2.2. *Let $A \in \mathbb{R}^{m\times n}$, $\text{rank}(A) = r > 0$ and $\tilde{A} = \tilde{Q}\tilde{R}$ be the $\tilde{Q}\tilde{R}$–factorization of $\tilde{A}$, then $A^\dagger = P^T(\tilde{R}^T)(\tilde{R}\tilde{R}^T)^{-1}\tilde{Q}^T$.*

Proof. With $A = \tilde{A}$, $F = \tilde{Q}$ and $G = \tilde{R}$ in Theorem 2.1 we have $(\tilde{A})^\dagger = (AP)^\dagger = \tilde{R}^T(\tilde{R}\tilde{R}^T)^{-1}\tilde{Q}^T$, and consequently $A^\dagger = P\tilde{R}^T(\tilde{R}\tilde{R}^T)^{-1}\tilde{Q}^T$. □

To calculate the rank of $\tilde{R}$, one needs only the number of its columns having at least one value above a tolerance level in absolute terms. This tolerance is set to be equal to 10^{-5}, which is also used by Katsikis *et al.* [3], and turns out to provide accurate results. The proposed method is called **Frginv**.

2.3. **Numerical examples.** We compare the performance of the proposed method **Frginv** to that of Katsikis et al. [3] for the computation of Moore–Penrose inverse matrices. Testing **qrginv** and **Frginv** was performed separately for random singular and for some sparse matrices. Both algorithms were carefully implemented and tested in Matlab (R2010b) on a common personal computer.

2.4. **Random singular matrices.** We are computing the performance of the proposed method **Frginv** to that of **qrginv** function [3] . In the same way of [3] we tested on a series of random sigular matrices of size $m\times n$, with $n = 2^k, k = 7, \ldots, 11$ and $m = 2n$, which are rank deficient, with rank $r = \frac{7n}{8}$. In addition, the accuracy of the results is examined with the matrix $2-$norm in error matrices corresponding to the four properties characterizing the Moore–Penrose inverses shown in Table 1. The computation error is less than 10^{-12} per coefficient in the error matrices, in all cases. The computation time (in seconds) are reported

in Table 1. We observe that the computation time of **Frginv** method is substantially less than those of the **qrginv** method.

Table 1. Error and Computational Time Results for Random Singular Matrices

n	Method	Time	$\| AA^\dagger A - A \|_2$	$\| A^\dagger AA^\dagger - A^\dagger \|_2$	$\| (AA^\dagger)^T - (AA^\dagger) \|_2$	$\| (A^\dagger A)^T - (A^\dagger A) \|_2$
2^7	qrginv	0.03	3.41e–14	0.26e–16	5.76e–15	3.26e–15
	Frginv	0.01	1.61e–14	3.72e–16	3.03e–15	2.52e–15
2^8	qrginv	0.11	1.00e–13	9.92e–16	8.02e–15	4.04e–15
	Frginv	0.07	2.98e–14	3.40e–16	3.85e–15	3.11e–15
2^9	qrginv	1.05	1.36e–13	1.01e–15	1.10e–14	4.94e–15
	Frginv	0.82	5.34e–14	5.01e–15	5.01e–15	3.83e–15
2^{10}	qrginv	10.83	2.62e–13	1.06e–15	1.49e–14	5.90e–15
	Frginv	8.29	8.85e–14	2.91e–16	6.38e–15	4.88e–15
2^{11}	qrginv	81.14	4.99e–13	1.18e–15	2.18e–14	7.25e–15
	Frginv	55.07	1.66e–13	2.59e–16	8.37e–15	6.55e–15

2.5. **Matrix–Market Sparse matrices.** For sparse matrices, we have chosen some matrices from Matrix–Market collection [4]. We follow the same method as in [5], and we have the rank deficient matrices as $AZ \equiv [A\ Z]$, where A is one of the chosen matrices and Z is a zero matrix of order $m \times 100$. The results of the methods are presented in Table 5. We observe that, the Moore–Penrose inverses obtained by **Frginv** are reasonably accurate in all cases, the computation time required by the **Frginv** method is significantly less than the time required by the **Frginv** methods. On the other hand, we can see that the accuracy computation of the **Frginv** method is less than the **qrginv** method, however in some cases, the accuracy of the results of both methods is low. We can conclude that **Frginv** method is a robust and efficient tool for obtaining the Moore–Penrose inverse of large sparse and rank deficient matrices.

Table 5. Error and Computational Time Results for sparse matrices

Matrix	Method	Time	$\| AA^\dagger A - A \|_2$	$\| A^\dagger AA^\dagger - A^\dagger \|_2$	$\| (AA^\dagger)^T - (AA^\dagger) \|_2$	$\| (A^\dagger A)^T - (A^\dagger A) \|_2$
$Well1850Z$	qrginv	3.83	1.02e–12	5.22e–11	5.67e–11	3.59e–13
	Frginv	1.76	4.01e–14	6.37e–12	1.91e–12	7.76e–14
$Watt1Z$	qrginv	10.51	7.13e–07	1.88e–05	6.11	7.49e–09
	Frginv	3.10	7.13e–07	6.41e–06	6.12	7.49e–09
$GR-30-30Z$	qrginv	3.11	2.31e–11	5.35e–11	3.52e–10	3.73e–13
	Frginv	1.68	3.27e–13	1.27e–11	4.97e–12	2.33e–13
$NOS3Z$	qrginv	4.38	1.66e–09	2.63e–09	5.46e–08	2.19e–11
	Frginv	3.02	8.69e–11	8.42e–11	9.94e–10	2.17e–11

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NESTED SPLITTING CONJUGATE GRADIENT METHOD FOR BTTB SYSTEMS

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Abstract. We present a class of nested iteration schemes for solving BTTB system $T_{m,n}(f)x = b$, where $T_{m,n}(f)$ denotes the $m \times m$ block Toeplitz matrix with $n \times n$ Toeplitz blocks (BTTB) generated by a $(2\pi, 2\pi)$-periodic continous function $f(x, y)$. These schemes are actually inner/outer iteration, which employ the conjugate gradient (CG) method as inner iteration to approximate each outer iterate, while each outer iteration is induced by a convergent splitting of the coefficient matrix. Convergence conditions of this method are studied in depth and numerical experiments show the efficiency of this method.

1. Introduction

Consider the solution of large linear equations

$$T_{m,n}(f)x = b, \tag{1.1}$$

where $T_{m,n}(f)$ is an $m \times m$ block Toeplitz matrix with $n \times n$ Toeplitz blocks (BTTB) of the form

$$T_{m,n} = [T_{(i-j)}]_{i,j=1}^{m}, \tag{1.2}$$

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where each $T_{(j)}$ is a Toeplitz matrix of order n

$$T_{(j)} = [t_{j,k-l}]_{k,l=1}^{n}, \qquad j = 0, \pm 1, \pm 2, \ldots \tag{1.3}$$

We assume that the BTTB matrix $T_{m,n}(f)$ is generated by a two-variate function $f(x, y)$ in C_Ω where $\Omega = (-\pi, \pi)^2$ and C_Ω is the set of all $(2\pi, 2\pi)$-periodic continuous functions. That is, the kth diagonal of the block T_j is given by

$$t_{j,k} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y) e^{-i(jx+ky)} dx dy, \quad j, k = 0, \pm 1, \pm 2, \ldots. \tag{1.4}$$

In the sequel, we assume that $f(x, y) = h_1(x, y) + ih_2(x, y)$ where $h_i(x, y)$ are real-valued and $h_i(x, y) > 0$ for $i = 1, 2$. Furtheremore we assume that

$$\sup_{(x,y)\in\Omega} \frac{h_2(x, y)}{h_1(x, y)} \leq 1. \tag{1.5}$$

With the above assumptions, for a BTTB matrix $T_{m,n}(f)$, we can choose the splitting $T_{m,n} = H - S$ as

$$H = \frac{T_{m,n} + T_{m,n}^H}{2}, \qquad S = \frac{T_{m,n}^H - T_{m,n}}{2}, \tag{1.6}$$

and

$$H = T_{m,n}(h_1), \quad S = iT_{m,n}(h_2), \tag{1.7}$$

where $T_{m,n}(.)$ is defined as in Eqs. (1.2), (1.3), and (1.4).

2. The NSCG method

We let $T_{m,n}(f) = B - C$ be a splitting of the matrix $T_{m,n}(f)$, namely B is nonsingular, and the spectral radius of the matrix $B^{-1}C$ is less than one, i.e., $\rho(B^{-1}C) < 1$. Furthermore we assume that B is a symmetric positive definte matrix and define the $\|.\|_B$ norm of a vector x as $\|x\|_B = \sqrt{x^H Bx}$. Then the induced $\|.\|_B$ norm of a matrix X is defined as $\|H\|_B = \|B^{\frac{1}{2}} H B^{-\frac{1}{2}}\|_2$.

Given a starting vector $x^{(0)}$, suppose that we have computed approximations $x^{(0)}, \ldots, x^{(l)}$ to the solution x^* of the system of linear equations (1.1). Then next approximation $x^{(l+1)}$ may be defined as either an exact or an inexact solution of the system of linear equations

$$Bx = Cx^{(l)} + b. \tag{2.1}$$

In [1], the authors proposed to solve the linear system of Eq.(2.1) by the CG method and they established the following theorem about the convergence properties of this method which called NSCG method.

Theorem 2.1. [1] *Let $T_{m,n}(f)$ be a nonsingular and non-symmetric matrix, and $T_{m,n} = B - C$ a contractive (with respect to the $\|.\|_B$-norm) and symmetric positive definite splitting. Suppose that the NSCG method is started from an initial guess $x^{(0)}$, and produces an iterative sequence $\{x^{(l)}\}_{l=0}^{\infty}$, where $x^{(l)}$ is the lth approximation to the solution x^* of the system of linear Eq. (1.1), obtained by solving the linear system $Bx = Cx^{(l)} + b$ with k_l steps of CG iterations. Then*

(a) $\|x^{(l)} - x^*\|_B \leq \gamma^{(l)}\|x^{(l-1)} - x^*\|_B, \qquad l = 1, 2, 3, \ldots,$
(b) $\|b - T_{m,n}x^{(l)}\|_B \leq \tilde{\gamma}^{(l)}\|b - T_{m,n}x^{(l-1)}\|_B, \qquad l = 1, 2, 3, \ldots,$

where

$$\gamma^{(l)} = 2(\frac{\sqrt{\kappa(B)} - 1}{\sqrt{\kappa(B)} + 1})^{k_l}(1 + \rho) + \rho, \quad \tilde{\gamma}^{(l)} = \gamma^{(l)}\frac{1 + \rho}{1 - \rho}, \quad l = 1, 2, 3, \ldots$$

and $\rho = \|B^{-1}C\|_B$. Moreover, for some $\gamma \in (\rho, 1)$, and

$$k_l \geq \frac{ln((\gamma - \rho)/(2(1 + \rho)))}{ln(\sqrt{\kappa((B)} - 1)/(\sqrt{\kappa(B)} + 1))}, \quad l = 1, 2, 3, \ldots,$$

we have $\gamma^{(l)} \leq \gamma$ $(l = 1, 2, 3, \ldots)$, and the sequence $\{x^{(l)}\}_{l=0}^{\infty}$ converges to the solution x^* of the system of linear Eq.(1.1). For $\rho \in (0, \sqrt{2} - 1)$ and some $\tilde{\gamma} \in ((1 + \rho)\rho/(1 - \rho), 1)$, and

$$k_l \geq \frac{ln(((1 - \rho)\tilde{\gamma} - \rho(1 + \rho))/(2(1 + \rho)^2))}{ln(\sqrt{\kappa((B_{m,n})} - 1)/(\sqrt{\kappa(B_{m,n})} + 1))}, \quad l = 1, 2, 3,$$

we have $\tilde{\gamma}^{(l)} \leq \tilde{\gamma}$ $(l = 1, 2, 3, \ldots)$, and the residual sequence $\{b - T_{m,n}x^{(l)}\}_{l=0}^{\infty}$ converges to zero.

By choosing $B = H$ and $C = S$ in Eq.(1.6), we obtain the following positive definite linear system of equations

$$Hx = Sx^{(l)} + b \tag{2.2}$$

For obtaining $x^{(l+1)}$, we can solve the matrix Eq. (2.2) iteratively by the CG method.

2.1. **Convergence analysis.** In the sequel, we need the following Theorem

Theorem 2.2. [2]*Let f and g be two-variate nonnegative (g not identically zero) Lebesgue integrable functions on Ω. Then the following facts are true:*

$$r := \inf_{(x,y)\in Q}\frac{g(x, y)}{f(x, y)} < \lambda(T_{m,n}^{-1}(f)T_{m,n}(g)) < R := \sup_{(x,y)\in Q}\frac{g(x, y)}{f(x, y)}, \tag{2.3}$$

where by $\lambda(X)$ we denote a generic eigenvalue of X.

The following lemma shows that the $\|H^{-1}S\|_H$ is less than one.

Lemma 2.3. *Suppose that H and S are in Eq. (1.6). Then $T_{m,n} = H - S$ is a contractive splitting (with respect to $\|.\|_H$-norm), i.e., $\|H^{-1}S\|_H < 1$.*

Proof.

$$\|H^{-1}S\|_H = \|H^{-\frac{1}{2}}SH^{-\frac{1}{2}}\|_2.$$

Since $H^{-1/2}SH^{-1/2}$ is skew-hermitian we deduce

$$\|H^{-1}S\|_H = \max_i |\lambda_i(H^{-1/2}SH^{-1/2})| = \max_i |\lambda_i(H^{-1}S)|. \tag{2.4}$$

In conclusion, by Eq. (1.7) we infer that

$$\|H^{-1}S\|_H = \lambda_{max}(T_{m,n}(h_1)^{-1}T_{m,n}(h_2)). \tag{2.5}$$

The use of Theorem 2.2 and Eq. (1.5) implies that

$$\|H^{-1}S\|_H < \sup_{(x,y)\in Q} \frac{h_2(x,y)}{h_1(x,y)} <= 1$$

and the proof is complete. □

By Theorem 2.1 and Lemma 2.3, we conclude that the NSCG method for BTTB matrices is convergent.

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A GENERALIZED UPPER BOUND FOR NOISE ERROR IN PROJECTED SIMULTANEOUS ITERATIVE RECONSTRUCTION TECHNIQUES

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Abstract. We provide an upper bound for noise error that used in analyzing of semi-convergence phenomena of a class of non-stationary iteration methods, often referred to as simultaneous iterative reconstruction technique, including Landweber, Cimmino, CAV, DROP and symmetric Kaczmarz iterations, with a focus on their constrained companion versions that include projection in each iteration.

1. Introduction

Computed tomography and many other large-scale imaging problems lead to large linear systems of equations (often inconsistent) with noisy data, of the form:

$$Ax \backsimeq b,\ b = \bar{b} + \delta b,\ A \in \mathbb{R}^{m\times n} \tag{1.1}$$

where $\bar{b} = A\bar{x}$ denotes the exact data and δb is the perturbation consisting of additive noise. The matrix A comes from the discretization of an ill-posed linear problem, such as the Radon transform, and $\bar{x}$ denotes the exact image. The numerical solution of these systems calls

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* Speaker.

for the use of iterative regularization methods, and it is often necessary to incorporate additional constraints to the reconstruction, e.g. non-negativity constraints.

The main goal of our paper is providing an upper bound for noise error to analyze semi-convergence for a class of non-stationary iteration methods in $\mathbb{R}^n$, often referred to as simultaneous iterative reconstruction technique (SIRT), including Landweber and Cimmino iterations, with a focus on their constrained companion versions that include projection in each iteration.

2. Providing an upper bound for noise error

Algorithm P-SIRT. Let $\mathcal{C}$ denote a closed convex set, and let $P_{\mathcal{C}}$ be the metric projection onto the set $\mathcal{C}$. Choose an arbitrary initial vector $x^0 \in \mathbb{R}^n$ and a sequence of positive relaxation parameters $\{\lambda_k\}$, and update the iteration vector x^k by means of

$$\begin{aligned} x^{k+1} &= P_{\mathcal{C}}\left(x^k + \lambda_k A^T M(b - Ax^k)\right) \\ &= P_{\mathcal{C}}\left(x^k + \lambda_k u^k\right) \end{aligned} \tag{2.1}$$

This is an instance of the gradient projection algorithm, for $M = I$ often called the projected Landweber iteration. The convergence result of P-SIRT is given in [1, Corollary 3.7.3].

Let x^k and $\bar{x}^k$ denote the iterative Algorithm using the noisy and the noise-free right-hand side, respectively. Then the error in the kth iterate clearly satisfies $x^k - \bar{x} = x^k - \bar{x}^k + \bar{x}^k - \bar{x}$, and therefore

$$\|x^k - \bar{x}\| \leq \|x^k - \bar{x}^k\| + \|\bar{x}^k - \bar{x}\| \tag{2.2}$$

Hence the error decomposes into two components- the *noise error* $x^k - \bar{x}^k$ and the *iterative error* $\bar{x}^k - \bar{x}$. We consider following equalities:

$$B = A^T M A,\ \bar{Q}_k = I - \bar{\lambda}_k B,\ \delta = \|A^T M \delta b\|,\ e_N^k = \|x^k - \bar{x}^k\| \tag{2.3}$$

Also we assumption that λ_k and $\bar{\lambda}_k$ for any k are nonnegative and decreasing. So if we set $\bar{q}_k = \|I - \bar{\lambda}_k B\|$ then we have $\bar{q}_k \leq 1 - \bar{\lambda}_k \sigma_n^2$ for $k \geq i$. Consider the two following iteration.
Iterative method without noise:

$$\begin{aligned} x^{k+1} = \ & P_{\mathcal{C}}\left(x^k + \lambda_k A^T M(b - Ax^k)\right) \\ & = P_{\mathcal{C}}\left(x^k + \lambda_k u^k\right)\ \ k = 1, 2, \ldots. \end{aligned} \tag{2.4}$$

Iterative method with noise in right hand side of (1.1):

$$\begin{aligned} \bar{x}^{k+1} = \ & P_{\mathcal{C}}\left(\bar{x}^k + \bar{\lambda}_k A^T M(\bar{b} - A\bar{x}^k)\right) \\ & = P_{\mathcal{C}}\left(\bar{x}^k + \bar{\lambda}_k \bar{u}^k\right)\ \ k = 1, 2, \ldots. \end{aligned} \tag{2.5}$$

We want give an bound for noise error that is generalized version of error bounds presented in [2, 3, 4]. Actually we give noise error analysis with diffrent value of relaxation parameters in (2.4) and (2.5).

Theorem 2.1. *The noise error in Algorithm P-SIRT is bounded above by*

$$\|x^k - \bar{x}^k\| \leq \left(\frac{(\lambda_0 - \bar{\lambda}_{k-1})\sigma_1 \sqrt{\langle r^0, Mr^0 \rangle}}{\sigma_n \bar{\lambda}_{k-1}} + \frac{\bar{\lambda}_0 \sigma_1 \|M^{\frac{1}{2}} \delta b\|}{\sigma_n \bar{\lambda}_{k-1}} \right) \Psi^k(\sigma_n, \bar{\lambda}_{k-1}), \tag{2.6}$$

with

$$\Psi^k(\sigma, \lambda) \equiv \frac{1 - \left(1 - \lambda\sigma^2\right)^k}{\sigma} \tag{2.7}$$

Proof. We give a summary of main prove here. For more details see [5]. Let $r^k = b - Ax^k$. By some calculation we have:

$$\begin{aligned} x^k - \bar{x}^k &= \bar{Q}_{k-1}(x^{k-1} - \bar{x}^{k-1}) + (\lambda_{k-1} - \bar{\lambda}_{k-1})(A^T M(b - Ax^{k-1})) \\ &\quad + \bar{\lambda}_{k-1} A^T M \delta b \\ &= \sum_{j=1}^{k-2} (\lambda_j - \bar{\lambda}_j) \prod_{i=j+1}^{k-1} \bar{Q}_i u^j + (\lambda_{k-1} - \bar{\lambda}_{k-1}) u^{k-1} \qquad (2.8) \\ &\quad + \sum_{j=1}^{k-2} (\bar{\lambda}_j) \prod_{i=j}^{k-1} \bar{Q}_i A^T M \delta b + \bar{\lambda}_{k-1} A^T M \delta b \qquad (2.9) \end{aligned}$$

Let $\lambda_0 = \bar{\lambda}_0$ or $\lambda_0 \geq \bar{\lambda}_0$, we have $|\lambda_j - \bar{\lambda}_j| \leq (\lambda_0 - \bar{\lambda}_j)$ for $j = 1, 2, \cdots, k-1$, so by affect norm in both sides of previous inequality and using nonexpansivity of projection operator $P_{\mathcal{C}}$, see [1, Corollary 2.2.23], we have:

$$\|x^k - \bar{x}^k\| \leq \left(\frac{(\lambda_0 - \bar{\lambda}_{k-1})\sigma_1 \sqrt{\langle r^0, Mr^0 \rangle}}{\sigma_n \bar{\lambda}_{k-1}} + \frac{\bar{\lambda}_0 \sigma_1 \|M^{\frac{1}{2}} \delta b\|}{\sigma_n \bar{\lambda}_{k-1}} \right) \Psi^k(\sigma_n, \bar{\lambda}_{k-1}) \tag{2.10}$$

□

Now we assume that M is SPD and $\bar{\lambda}_{k-1} \in (0, \frac{1}{\sigma^2})$, according to the [3, Theorem 3.7] we have

$$\Psi^k(\sigma_n, \bar{\lambda}_{k-1}) \leq \sqrt{\bar{\lambda}_{k-1}} \frac{1 - \zeta_k^k}{\sqrt{1 - \zeta_k}} \tag{2.11}$$

here ζ_k are roots of a certain polynomial such that $0 < \zeta_k < \zeta_{k+1}$ and $\lim_{k\to\infty} \zeta_k = 1$. So we have find that:

$$\|x^k - \bar{x}^k\| \leq \frac{\sigma_1}{\sigma_n}\left(\frac{(\lambda_0 - \bar{\lambda}_{k-1})\sqrt{\langle r^0, Mr^0\rangle}}{\sqrt{\bar{\lambda}_{k-1}}} + \frac{\bar{\lambda}_0\|M^{\frac{1}{2}}\delta b\|}{\sqrt{\bar{\lambda}_{k-1}}}\right)\frac{1-\zeta_k^k}{\sqrt{1-\zeta_k}}. \tag{2.12}$$

Using

$$\frac{(1-\zeta_k^k)^2}{1-\zeta_k} < \frac{1-\zeta_k^k}{1-\zeta_k} = 1 + \zeta_k + \zeta_k^2 + \cdots + \zeta_k^{k-1} < k$$

We can write:

$$\|x^k - \bar{x}^k\| \leq \left(\frac{(\lambda_0 - \bar{\lambda}_{k-1})\sigma_1\sqrt{\langle r^0, Mr^0\rangle}}{\sigma_n\sqrt{\bar{\lambda}_{k-1}}} + \frac{\bar{\lambda}_0\sigma_1\|M^{\frac{1}{2}}\delta b\|}{\sigma_n\sqrt{\bar{\lambda}_{k-1}}}\right)\sqrt{k}$$

Remark 2.2. If we set $x^0 = 0$ the upper bound can be written as:

$$\|x^k - \bar{x}^k\| \leq \left(\frac{(\lambda_0 - \bar{\lambda}_{k-1})\sigma_1}{\sigma_n\sqrt{\bar{\lambda}_{k-1}}}\|M^{\frac{1}{2}}b\| + \frac{\sigma_1\bar{\lambda}_0}{\sigma_n\sqrt{\bar{\lambda}_{k-1}}}\|M^{\frac{1}{2}}\delta b\|\right)\sqrt{k}.$$

This error bound is generalized version of [2, 3, 4]. If we set $\lambda_k = \bar{\lambda}_k$ for $k = 0, 1, \ldots$ then we have same error bound with those presented in [2, 3, 4].

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ACCELERATION OF BLOCK ITERATIVE METHODS FOR SOLVING LINEAR SYSTEM OF EQUATIONS

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Abstract. We present a generalized relaxation of a composition operator which is based on class of strictly relaxed cutter operators on a general Hilbert space for solving linear system of equation. The goal of our work is to study the implementation of the block algebraic iterative methods on state-of- the-art operators.

1. Introduction

Algebraic iterative methods are routinely used for solving the ill-posed sparse linear systems arising in tomographic image reconstruction. Here we consider the algebraic reconstruction technique (ART) and the simultaneous iterative reconstruction techniques (SIRT). First, we begin with block iterative methods which are used for solving linear systems of equations (inequalities). Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ be given. We assume the consistent linear system of equations

$$Ax = b.$$

Let A and b be partitioned into p row-blocks $\{A_t\}_{t=1}^p$ and $\{b^t\}_{t=1}^p$ respectively. Consider the following algorithm.

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* Speaker.

Algorthm: Sequential Block Iteration
Initialization: $x^0 \in \mathbb{R}^n$ is arbitrary.
Iterative Step: Given x^k, compute for all $k \geq 0$

$$\begin{aligned} x^{k,0} &= x^k \\ x^{k,t} &= x^{k,t-1} + \alpha_t A_t^T M_t (b^t - A_t x^{k,t-1}) \\ &= T_t(x^{k,t-1}),\ t = 1, \ldots, p, \\ x^{k+1} &= x^{k,p} \end{aligned}$$

where

$$T_t(x) = x + \alpha_t A_t^T M_t (b^t - A_t x). \tag{1.1}$$

Here $\{\alpha_t\}_{t=1}^p$ and $\{M_t\}_{t=1}^p$ are relaxation parameters and symmetric positive definite weight matrices respectively.

In this paper we present an accelerated scheme of sequential block iteration method based on operators that are defined in (1.1).

2. Preliminaries and Notations

Throughout this paper, we consider $T : H \to H$ with nonempty fixed point set, i.e., $FixT \neq \emptyset$ where H is a Hilbert space and Id denotes the identity operator on H. Followings are some definitions from [3] which will be useful in our future analysis.

Definition 2.1. Let $T : H \to H$ and $\alpha \in [0, 2]$. The operator T_α defined by

$$T_\alpha := (1 - \alpha)Id + \alpha T$$

is called an α-relaxation or, shortly, relaxation of the operator T.

Based on [3, Remark 2.1.31], an $\alpha-$relaxed cutter operator is defined as follows.

Definition 2.2. Let $T : H \to H$ has a fixed point. Then the operator T is an α-relaxed cutter, or, shortly, relaxed cutter where $\alpha \in [0, 2]$, if

$$\langle T_\alpha(x) - x, z - x \rangle = \alpha \langle T(x) - x, z - x \rangle \geq \|T(x) - x\|^2$$

for all $x \in H$ and $z \in FixT$. If $\alpha \in (0, 2)$, then T_α is called a strictly relaxed cutter operator of T.

We next reintroduce, see also [2, 3, 4, 5], the generalized relaxation of an operator, which allows to accelerate locally a fixed point iteration method.

Definition 2.3. Let $T : H \to H$ and $\sigma : H \to (0, \infty)$ be a *step size function.* The generalized relaxation of T is defined by

$$T_{\sigma,\lambda}(x) = x + \lambda\sigma(x)(T(x) - x) \tag{2.1}$$

where λ is a relaxation parameter in $[0, 2]$.

3. Main Results

In this section we consider a fixed point iteration method based on the generalized relaxation of a relaxed cutter operator and present its convergence analysis. We deal in this paper with a finite family of cutter operators $T_i : H \to H, i = 1, \ldots, p$, with $\bigcap_{i=1}^{p} FixT_i \neq \emptyset$, and with compositions of $\{T_i\}_{i=1}^{p}$. We propose local acceleration techniques for algorithms which apply this operation. Consider $x^0 \in H$ is arbitrary. For an operator $T_i : H \to H$, we define the operator $T_{\sigma,\lambda} : H \to H$ by

$$x^{k+1} = T_{\sigma,\lambda}(x^k) := x^k + \lambda_k \sigma(x^k)(T(x^k) - x^k) \; k = 0, 1, \ldots \; . \tag{3.1}$$

Let $T_i : H \to H, i = 1, \ldots, p$ be strictly relaxed cutter operators with $\bigcap_{i=1}^{p} FixT_i \neq \emptyset$. Define the operator $T : H \to H$ as the composition

$$T := T_p T_{p-1} \cdots T_1. \tag{3.2}$$

Now denote $S_0 = Id$ and $S_i := T_i T_{i-1} \cdots T_1$ for $i = 1, \ldots, p$. We define the step size function $\sigma_{max} : H \to (0, \infty)$ in the following form:

$$\sigma_{max}(x) = \begin{cases} \dfrac{\sum_{i=1}^{p} \left\langle T(x) - S_i(x) + \frac{1}{\alpha_i}\zeta_i(x), \zeta_i(x) \right\rangle}{\|T(x) - x\|^2}, & x \in H \backslash FixT \\ 1, & x \in FixT \end{cases} \tag{3.3}$$

where $x \in H \backslash FixT$ and $\zeta_i(x) = S_i(x) - S_{i-1}(x)$. Next, we give convergence result of $T_{\sigma,\lambda}$, defined by (3.1). We call an operator T of the form (3.2) a cyclic relaxed cutter.

Theorem 3.1. *Let $0 < \sigma \leq \sigma_{max}$ be step-size function defined in (3.3) and $\lambda_k \in [\varepsilon, 2-\varepsilon]$ for an arbitrary constant $\varepsilon \in (0, 1)$. Then generalized relaxation operator defined by (3.1) converges to a point in $FixT$.*

Proof. see [5, Theorem 2] □

If $0 < \varepsilon \leq \alpha_t \leq 2 - \varepsilon$ for $t = 1, \ldots, p$, then the operator T_t of (1.1) is strictly relaxed cutter.

Therefore the step size function (3.3) for the operator (1.1)can be written as

$$\sigma_{max} = \frac{\left(\sum_{i=1}^{p} \dfrac{\alpha_i(2 - \alpha_i)}{\rho(A_i^T M_i A_i)^2} \|A_i^T M_i (b_i - A_i S_{i-1})\|^2 + \|U(x) - x\|^2 \right)}{2\|U(x) - x\|^2}. \tag{3.4}$$

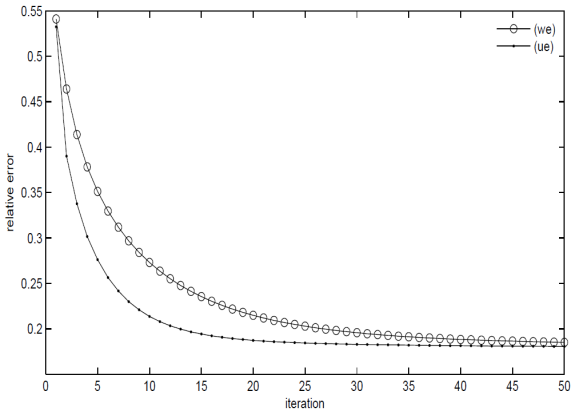

FIGURE 1. The history of relative error within 50 iterations.

4. Numerical Results

We give some numerical results where the operators (1.1) are used in (3.2).

The test is taken from the field of image reconstruction from projections using the SNARK93 software package [1]. The phantom is discretized into 63×63 pixels which satisfies the linear system of equations $Ax = b$. We are using 16 projections with 99 rays per projection. Let A and b be partitioned into 16 row blocks $\{A_t\}_{t=1}^{16}$ and $\{b_t\}_{t=1}^{16}$ respectively. We use Cimmino's $M-$matrix in (1.1). Figure 1 demonstrates iteration history for the relative error using extrapolation (ue) and without extrapolation (we) within 50 iterations.

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CHARACTRIZATION OF LEFT DERIVABLE MAPS AT NON-TRIVIAL IDEMPOTENTS ON NEST ALGEBRAS

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Abstract. Let $Alg\mathcal{N}$ be a nest algebra associated with the nest $\mathcal{N}$ on a (real or complex) Banach space X. Suppose that there exists a non-trivial idempotent $P \in Alg\mathcal{N}$ with range $P(\mathcal{X}) \in \mathcal{N}$ and $\delta : Alg\mathcal{N} \longrightarrow Alg\mathcal{N}$ is a continuos linear mapping (generalized) left derivable at P, i.e. $\delta(ab) = a\delta(b) + b\delta(a)(\delta(ab) = a\delta(b) + b\delta(a) - ba\delta(I))$ for any $a, b \in Alg\mathcal{N}$ with $ab = P$. we show that δ is a (generalized) Jordan left derivation.

1. Introduction

Throughout this paper all algebras and vector spaces will be over $\mathbb{F}$, where $\mathbb{F}$ is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let $\mathcal{A}$ be analgebra with unity 1, $\mathcal{M}$ be a left $\mathcal{A}$-module and $\delta : \mathcal{A} \to \mathcal{M}$ be a linear mapping. δ is said to be a left derivation (or a generalized left derivation) if $\delta(ab) = 2a\delta(b)+b\delta(a)$ (or $\delta(ab) = 2a\delta(b)+b\delta(a)-ba\delta(1)$) for any $a, b \in \mathcal{A}$. It is called a Jordan left derivation (or a generalized Jordan left derivation) if $\delta(a^2) = 2a\delta(a)$ (or $\delta(a^2) = 2a\delta(a) - a^2\delta(1)$) for any $a \in \mathcal{A}$. Obviously, any (generalized) left derivation is a (generalized) Jordan left derivation, but in general the converse is not true (see [6], Example 1.1). The concepts of left derivation and Jordan

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left derivation was introduced by Brešar and Vukman in [1]. For results concerning left derivation and Jordan left derivations we refer the readers to [4].

In recent years, several authors studied the linear (additive) maps that behave like homomorphisms, derivations or left derivations when acting on special products (for instance, see [3, 5, 6] and the references therein). In this article we study the continuous linear maps on nest algebras behaving like left derivations at idempotent-product.

Let $\mathcal{A}$ be analgebra with unity 1, $\mathcal{M}$ be a left $\mathcal{A}$-module and $\delta : \mathcal{A} \to \mathcal{M}$ be a linear mapping. We say that δ is left derivable (or a generalized left derivable) at a given point $z \in \mathcal{M}$if $\delta(ab) = 2a\delta(b) + b\delta(a)$ (or $\delta(ab) = 2a\delta(b) + b\delta(a) - ba\delta(1)$) for any $a, b \in \mathcal{A}$ with $ab = z$. In this paper we characterize the continuos linear maps on nest algebras which are (generalized) left derivable at a non-trivial idempotent operator P.

The following are the notations and terminologies which are used throughout this article.

Let $\mathcal{X}$ be a Banach algebra. Let $\mathcal{X}$ be a Hilbert space. We denote by $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear operators on $\mathcal{X}$. A subspace lattice $\mathcal{L}$ on a Hilbert space $\mathcal{X}$ is a collection of closed (under norm topology) subspaces of $\mathcal{X}$ which is closed under the formation of arbitrary intersection (denoted by $\wedge$) and closed linear span (denoted by $\vee$), and which includes $\{0\}$ and $\mathcal{X}$. For a subspace lattice $\mathcal{L}$, we define the *associated subspace lattice* $Alg\mathcal{L}$ by

$$Alg\mathcal{L} = \{T \in \mathcal{B}(\mathcal{X}) \,|\, T(N \subseteq N \; for \; all \; N \in \mathcal{L}\}.$$

A totally ordered subspace lattice $\mathcal{N}$ on $\mathcal{X}$ is called a *nest* and $Alg\mathcal{N}$ is called a nest algebra. When $\mathcal{N} \neq \{\{0\}, \mathcal{X}\}$, we say $\mathcal{N}$ is non-trivial. It is clear that if $\mathcal{N}$ is trivial, then $Alg\mathcal{N} = \mathcal{B}(\mathcal{X})$. The identity element of nest algebras denote by I and an element P in a nest algebra is called a non-trivial idempotent if $P \neq 0, I$ and $P^2 = P$.

Let $\mathcal{N}$ be a non-trivial nest on a Banach space $\mathcal{X}$. If there exists a non-trivial idempotent $P \in Alg\mathcal{N}$ with range $P(\mathcal{X}) \in \mathcal{N}$, then we have $(I - P)(Alg\mathcal{N})P = \{0\}$ and hence

$$Alg\mathcal{N} = P(Alg\mathcal{N})P \dotplus P(Alg\mathcal{N})(I - P) \dotplus (I - P)(Alg\mathcal{N})(I - P)$$

as sum if linear spaces. This is so-called the Peirce decompositon of $Alg\mathcal{N}$. The sets $P(Alg\mathcal{N})P, P(Alg\mathcal{N})(I - P)$ and $(I - P)(Alg\mathcal{N})(I - P)$ are closed in $Alg\mathcal{N}$. In fact $P(Alg\mathcal{N})P$ and $(I - P)(Alg\mathcal{N})(I - P)$ are Banach $(P(Alg\mathcal{N})(I - P), P(Alg\mathcal{N})(I - P))$-bimodule. Also $P(Alg\mathcal{N})(I - P)$ is fithful as a left $P(Alg\mathcal{N})P$-mofule as well as right $(I-P)(Alg\mathcal{N})(I-P)$-module. For more information on nest algebras, we refer to [2].

A subspace lattice $\mathcal{L}$ on a Hilbert space $\mathbb{H}$ is called a *commutative subspace lattice*, or a *CSL*, if the projections of $\mathbb{H}$ onto the subspaces of $\mathcal{L}$ commute with each other. if $\mathcal{L}$ is a CSL, then $Alg\mathcal{L}$ is called a CSL-algebra. Eech nest algebra on a Hilbert space is a CSL-algebra.

2. Main sections and results

In order to prove our results we need the following result.

Theorem 2.1. [3] *Let $\mathcal{X}$ be a linear space and let $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}$ be a bilinear map satisfying*

$$a, b \in \mathcal{A}, \quad ab = 1 \Rightarrow \phi(a, b) = \phi(1, 1)$$

Then

$$\phi(a, a) = \phi(a^2, 1)$$

for all $a \in \mathcal{A}$.

Proposition 2.2. *Let $\mathcal{X}$ be a Banach algebra with unity 1 and $\mathcal{M}$ be a unital Banach left $\mathcal{A}$-module. Let $\delta : \mathcal{A} \longrightarrow \mathcal{M}$ be a continuos linear map. if δ is left derivable at 1, then δ is a Jordan left derivation.*

the following is our main result.

Theorem 2.3. *Let $\mathcal{N}$ be a nest on Banach space $\mathcal{X}$, and exists a non-trivial idempotent $P \in Alg\mathcal{N}$ with range $P(\mathcal{X}) \in \mathcal{N}$. If $\delta : Alg\mathcal{N} \longrightarrow Alg\mathcal{N}$ is continous left derivable map at P, then δ is a Jordan left derivation.*

Proof. As a notational convenience, we denote $\mathbb{A} = Alg\mathcal{N}$, $\mathbb{A}_{11} = P\mathbb{A}P$, $\mathbb{A}_{12} = P\mathbb{A}(I - P)$ and $\mathbb{A}_{22} = (I - P)\mathbb{A}(I - P)$. As mentioned in the introduction $\mathbb{A} = \mathbb{A}_{11} \dot{+} \mathbb{A}_{12} \dot{+} \mathbb{A}_{22}$. Throughout the proof, a_{ij} and b_{ij} will denote arbitrary elements in $\mathbb{A}_{ij}$ for $1 \leq i, j \leq 2$.

First we show that $\delta(P) = 0$. Since $P^2 = P$, we have $2P\delta(P) = \delta(P)$. So $2P\delta(P) = P\delta(P)$ and $(I - P)\delta(P) = 0$. Thus $P\delta(P) = 0$ and hence $\delta(P) = 0$.

we complete the proof by checking some steps.

Step 1. $P\delta(a_{11}^2)P = 2a_{11}P\delta(a_{11})P$ and $P\delta(a_{11}^2)(I-P) = 2a_{11}P\delta(a_{11})(I-P)$.

Step 2. $P\delta(a_{22}) = 0$.

Step 3. $P\delta(a_{12}) = 0$.

Step 4. $(I - P)\delta(a_{11})(I - P) = 0$.

Step 5. $(I - P)\delta(a_{12})(I - P) = 0$.

Step 6. $(I - P)\delta(a_{22})(I - P) = 0$.

Since $ab = PaPbP + PaPb(I-P) + Pa(I-P)b(I-P) + (I-P)a(I-P)b(I-P)$, for any $a, b \in \mathbb{A}$, by steps 1-6, it follows that δ is a Jordan left derivation. □

Our next result characterizes the linear mappings on $Alg\mathcal{N}$ which are generalized left derivable at P.

Theorem 2.4. *Let $\mathcal{N}$ be a nest on Banach space $\mathcal{X}$, and exists a non-trivial idempotent $P \in Alg\mathcal{N}$ with range $P(\mathcal{X}) \in \mathcal{N}$. If $\delta : Alg\mathcal{N} \longrightarrow Alg\mathcal{N}$ is continous generalized left derivable map at P, then δ is a generalized Jordan left derivation.*

Corollary 2.5. *Let $\mathcal{N}$ be a non-trivial nest on Hilbert $\mathbb{H}$. Let P be a non-trivial idempotent in $Alg\mathcal{N}$ with range $P(\mathbb{H}) \in \mathcal{N}$ and $\delta : Alg\mathcal{N} \longrightarrow Alg\mathcal{N}$ is continous linear map.*

(1) *If δ is left derivable at P, then δ is zero.*
(2) *if δ generalized left derivable map at P, then $\delta(a) = a\delta(1)$ for all $a \in Alg\mathcal{N}$*

is a generalized Jordan left derivation.

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ON SOME PROPERTIES OF PSEUDOSPECTRUM

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Abstract. In this paper we will consider some properties of pseudospectrum of matrices.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all $n \times n$ complex equipped with the operator norm $\|.\|$ induced by the usual vector norm $\|x\| = (x^*x)^{1/2}$ on $\mathbb{C}^n$, i.e.,

$$\|A\| = max\{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\}.$$

Let $\epsilon > 0$. If we convent that $\|(A - zI)^{-1}\| = \infty$ for $\lambda \in \sigma(A)$, the pseudospectrum of a matrix $A \in \mathbb{M}_n(\mathbb{C})$ is defined and denoted by

$$\sigma_\epsilon(A) = \{z \in \mathbb{C} : \|(A - zI)^{-1}\| > 1/\epsilon\}. \tag{1.1}$$

Some authors use $\|(A - zI)^{-1}\| \geqslant 1/\epsilon$ instead of $\|(A - zI)^{-1}\| > 1/\epsilon$ in the definition of $\sigma_\epsilon(A)$. One can easily adapt our results and proofs using this different definition. The pseudospectrum of a matrix A for a given ϵ consists of all eigenvalues of matrices which are $\epsilon-$close to A.

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* Speaker.

From either of (1.1) it follows that the pseudospectrum associated with various ϵ are nested sets,

$$\sigma_{\epsilon_1}(A) \subseteq \sigma_{\epsilon_2}(A), \qquad 0 < \epsilon_1 \leqslant \epsilon_2$$

and that the intersection of all the $\epsilon-$pseudospectra is the spectrum,

$$\bigcap_{\epsilon>0} \sigma_\epsilon(A) = \sigma(A). \tag{1.2}$$

Also, note that if U is a unitary matrix (i.e., $U^* = U^{-1}$), then

$$\sigma_\epsilon(A) = \sigma_\epsilon(U^*AU) \qquad \forall \epsilon > 0. \tag{1.3}$$

Consequently, for a normal matrix, the pseudospectrum is just the union of the open $\epsilon-$disks about the points of the spectrum.

Theorem 1.1. [3] *Let $\epsilon > 0$ and $A \in \mathbb{M}_n(\mathbb{C})$. Then the following assertions are equivalent:*
(i) $z \in \sigma_\epsilon(A)$;
(ii) There is unit vector $x \in \mathbb{C}^n$ such that $\|(A - zI)x\| < \epsilon$;
(iii) There are $x \in \mathbb{C}^n$ and $E \in \mathbb{M}_n(\mathbb{C})$ such that $(A + E)x = zx$;
(iv) If $s_1(A) \geqslant \cdots \geqslant s_n(A)$ are the singular values of $A \in \mathbb{M}_n(\mathbb{C})$, then $s_n(zI - A) < \epsilon$.

In our discussion we assume that $D(a, r) = \{\mu \in \mathbb{C} : |\mu - a| < r\}$, where $a \in \mathbb{C}$ and $r \geq 0$.
We recall the following properties of pseudospectrum that are useful; see [5, 2].

Proposition 1.2. *Let $\epsilon > 0$ and $A \in \mathbb{M}_n(\mathbb{C})$.*
(i) $\sigma_\epsilon(A^*) = \overline{\sigma_\epsilon(A)}$.
(ii) If $A = A_1 \oplus A_2$, then $\sigma_\epsilon(A) = \sigma_\epsilon(A_1) \cup \sigma_\epsilon(A_2)$.
(iii) If $A = \begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix}$ with matrix blocks, then $\sigma_\epsilon(A_1) \subseteq \sigma_\epsilon(A)$.
(iv) For any $\lambda \in \mathbb{C}, \sigma_\epsilon(A + \lambda I) = \lambda + \sigma_\epsilon(A)$.
(v) For any nonzero $\lambda \in \mathbb{C}, \sigma_{|\lambda|\epsilon}(\lambda A) = \lambda\sigma_\epsilon(A)$.
(vi) $\sigma_\epsilon(A)$ is nonempty, open and bounded, with at most n connected components, each containing one or more eigenvalues of A.

In [2], the authors characterized the pseudospectrum for every upper triangle $A \in \mathbb{M}_2(\mathbb{C})$, in the following form:

Proposition 1.3. *Suppose $\epsilon > 0$ and $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathbb{M}_2(\mathbb{C})$. Then*

$$\sigma_\epsilon(A) = \{\mu \in \mathbb{C} : \sqrt{(|\mu - a| + |\mu - c|)^2 + |b|^2} - \sqrt{(|\mu - a| - |\mu - c|)^2 + |b|^2} < 2\epsilon\}.$$

Consequently, $b = 0$ if and only if $\sigma_\epsilon(A) = D(a, \epsilon) \cup D(c, \epsilon)$. If $a = c = 0$, then $\sigma_\epsilon(A) = \{\mu : |\mu| < \sqrt{\epsilon(\epsilon + |b|)}\}$.

using (1.3), Proposition 1.3 and Schur's theorem, the pseudospectrum has been characterized for every $A \in \mathbb{M}_2(\mathbb{C})$.

2. Main results

In view of equality (1.2) and Theorem 1.1, we obtain the following theorem.

Theorem 2.1. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then*

$$\sigma(A) = \limsup_{\epsilon>0,\, \|E\|<\epsilon} \sigma(A+E) = \bigcap_{\epsilon>0} \bigcup_{\|E\|<\epsilon} \sigma(A+E).$$

Theorem 2.2. [5, Theorem 2.2] *Let $\epsilon > 0$ and $A \in M_n(\mathbb{C})$. Then*

$$\sigma(A) + D(0,\epsilon) \subseteq \sigma_\epsilon(A).$$

The set equality holds if A is normal.

Remark 2.3. It is clear that, if $\epsilon > 0$ and $A \in M_n(\mathbb{C})$, then

$$\sigma(A) + D(0,\epsilon) = \bigcup_{\lambda_i \in \sigma(A)} D(\lambda_i, \epsilon).$$

Theorem 2.4. [4, 2.4.8.1] *Suppose that $A, B \in M_n(\mathbb{C})$ commute. Then $\sigma(A+B) \subseteq \sigma(A) + \sigma(B)$.*

Theorem 2.5. *Let $\epsilon > 0$ and $A, B \in M_n(\mathbb{C})$. If A and B are normal and commute, then $\sigma_\epsilon(A+B) \subseteq \sigma_\epsilon(A) + \sigma_\epsilon(B)$.*

Proof. Since A and B commute, by [4, Theorem 2.3.3] there is a unitary matrix $U \in M_n(\mathbb{C})$ and there are diagonal matrices $\Lambda_1, \Lambda_2 \in M_n(\mathbb{C})$ such that $U^*AU = \Lambda_1 = [\lambda_i]$ and $U^*BU = \Lambda_2 = [\mu_j]$. Since A and B are normal, then $A+B$ is normal. Also, according to Theorem 2.2 and Remark 2.3, we have

$$\sigma_\epsilon(A) = \{\lambda_1, ..., \lambda_n\} + D(0,\epsilon) = \bigcup_{\lambda_i \in \sigma(A)} D(\lambda_i, \epsilon),$$

$$\sigma_\epsilon(B) = \{\mu_1, ..., \mu_n\} + D(0,\epsilon) = \bigcup_{\mu_j \in \sigma(B)} D(\mu_j, \epsilon)$$

and

$$\sigma_\epsilon(A+B) = \{\lambda_1 + \mu_1, ..., \lambda_n + \mu_n\} + D(0,\epsilon) = \bigcup_{i=1}^{n} D(\lambda_i + \mu_i, \epsilon).$$

On the other hand, by Minkowski sum [1], it is known that $D(a,r) + D(b,r) = D(a+b, 2r)$. Hence

$$\begin{aligned}\sigma_\epsilon(A) + \sigma_\epsilon(B) &= \{\lambda_i + \mu_j : 1 \leqslant i, j \leqslant n\} + D(0, 2\epsilon) \\ &= \bigcup_{\substack{\lambda_i \in \sigma(A) \\ \mu_j \in \sigma(B)}} D(\lambda_i + \mu_j, 2\epsilon).\end{aligned}$$

Therefore

$$\sigma_\epsilon(A+B) \subseteq \sigma_\epsilon(A) + \sigma_\epsilon(B).$$

□

The numerical range, also known as the field of values, of $A \in M_n(\mathbb{C})$ is defined by

$$W(A) = \{x^* A x : x \in \mathbb{C}^n, \|x\| = 1\}.$$

This is a closed, convex subset of $\mathbb{C}$ that contains the spectrum $\sigma(A)$. Following theorem shows generalized state:

Theorem 2.6. [5] *Let $\epsilon > 0$ and $A \in M_n(\mathbb{C})$. Then*

$$\sigma_\epsilon(A) \subseteq W(A) + D(0, \epsilon).$$

This fact implies that pseudospectra cannot be much larger than the numerical range.

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BIRKHOFF-JAMES-ORTHOGONALITY AND APPROXIMATE BIRKHOFF-JAMES-ORTHOGONALITY FOR OPERATORS

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Abstract. Here we speak about well-known Birkhoff-James-orthogonality for a spacial classes of normed spaces named the space of square matrices and the space of operators on an arbitrary Hilbert space. Also the definition of approximate BJ-orthogonality are introduced and it has been considered in such spaces.

1. Introduction

An element x of the normed space $\mathcal{X}$ is Birkhoff-James-orthogonal to element y of $\mathcal{X}$, denoted by $x \perp_{BJ} y$, if

$$\|x + \lambda y\| \geq \|x\| \quad (\lambda \in \mathbb{C}).$$

This type of orthogonality was first introduced by Birkhoff for real normed spaces in [2] and then by James for complex normed space in [5]. Since then this type of orthogonality was investigated by many authors who paid their attention to a certain normed space such as spaces of square complex matrices or more generally bounded operators on a

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* Speaker.

Hilbert space, C^*-algebra and Hilbert C^*-modules. For example in [1] the authors present some characterizations of Birkhoff-James orthogonality in such spaces. Also some types of approximate Birkhoff-James orthogonality are considered. In an inner product space $(\mathcal{X}, \langle ., . \rangle)$, approximate orthogonality is defined by

$$x \perp^{\varepsilon} y \quad \text{if and only if} \quad |\langle x, y \rangle| \leq \varepsilon \|x\| \|y\| \quad (x, y \in \mathcal{X}).$$

The definition of approximate BJ-orthogonality in general normed spaces have been introduced by Dragomir in [4] as follows:

$$x \perp_{BJD}^{\varepsilon} y \Leftrightarrow \forall \lambda \in \mathbb{C};\ \|x + \lambda y\| \geq (1 - \varepsilon)\|x\|.$$

The interesting fact about this type of approximate orthogonality is that it coincides with the approximate orthogonality in inner product spaces. In addition, an alternative definition of approximate orthogonality was given in [3]. Utilizing the results of [1] for Birkhoff-James-orthogonality, some similar results for approximate Birkhoff-James-Orthogonality in square matrix spaces $\mathbb{M}(n)$ and the space of bonded linear operators on Hilbert space $\mathcal{H}$, denoted by $\mathbb{B}(\mathcal{H})$ in this paper, can be given.

In addition to the definition of Birkhoff-James-orthogonality several other definitions for orthogonality have been offered. For example isosceles orthogonality in a real normed space $\mathcal{X}$ is as follows,

$$x \perp_I y \Leftrightarrow \|x - y\| = \|x + y\|.$$

In complex normed space this definition may be generalized by

$$x \perp_I' y \Leftrightarrow \forall \theta_1, \theta_2 \in \mathbb{R}; \left\| \frac{x}{\|x\|} + e^{i\theta_1} \frac{y}{\|y\|} \right\| = \left\| \frac{x}{\|x\|} + e^{i\theta_2} \frac{y}{\|y\|} \right\|.$$

In this paper we mainly speak about the BJ-orthogonality and approximate BJ-orthogonality and their characterizations in the $\mathbb{M}(n)$ and $\mathbb{B}(\mathcal{H})$. Very recently we have investigated this issue in another approach in [6].

2. Main sections and results

First we present the following Theorem from [1].

Theorem 2.1. *Let $A, B \in \mathbb{M}(n)$. Then A is BJ-orthogonal to B if and only if there exists a unit vector x such that $\|Ax\| = \|A\|$ and $\langle Ax, Bx \rangle = 0$.*

This theorem is generalized to $\mathbb{B}(\mathcal{H})$ as follows,

Theorem 2.2. *Let $A, B \in \mathbb{B}(\mathcal{H})$. Then $A \perp_{BJ} B$ if and only if there exists a sequence of unit vectors $\{\xi_n\}$ in $\mathcal{H}$ such that $\lim_{n\to\infty} \|A\xi_n\| = \|A\|$ and $\lim_{n\to\infty}\langle A\xi_n, B\xi_n\rangle = 0$.*

Using the method of these theorems we reach the following similar results for approximate BJ-orthogonality as well.

Theorem 2.3. *Let $A, B \in \mathbb{M}(n)$ and $\varepsilon \in [0,1)$ such that $A \perp^{\varepsilon}_{BJD} B$. Then there exists a unit vector x such that*

$$|\langle Ax, Bx\rangle| \leq \varepsilon\|A\|\|B\|.$$

This theorem is generalized to $\mathbb{B}(\mathcal{H})$ as follows,

Theorem 2.4. *Let $A, B \in \mathbb{B}(\mathcal{H})$ and $\varepsilon \in [0,1)$ such that $A \perp^{\varepsilon}_{BJD} B$. Then there exists a sequence of unit vectors $\{\xi_n\}$ in $\mathcal{H}$ such that*

$$\lim_{n\to\infty} |\langle A\xi_n, B\xi_n\rangle| \leq \varepsilon\|A\|\|B\|.$$

The following theorem tells a relationship between isosceles orthogonality and approximate BJ-orthogonality in general normed spaces.

Theorem 2.5. *Let x and y be two vectors in a normed space $\mathcal{X}$ such that $x \perp'_I y$. Then $x \perp^{\varepsilon}_{BJD} y$ for some $\varepsilon \in [0,1)$.*

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NOTS ON DERIVATIONS OF THRIANGULAR BANACH ALGEBRAS

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Abstract. Let $\mathcal{A}, \mathcal{B}$ are two Banach algebras and $\mathcal{M}$ be a Banach A, B-module. Let $\Gamma = \{\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a \in A, b \in B, m \in M\}$ with ususal operations on matrices and $\| \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \| = \|a\|_A + \|b\|_B + \|m\|_M$ is a Banach algebra.This Banach algebras is called triangular Banach algebra. In this paper we varify some properties of derivations on Γ and their duals.

1. Introduction

Triangular Banach algebras $\begin{bmatrix} \mathcal{A} & M \\ 0 & \mathcal{B} \end{bmatrix}$, introduced by Forrest and Marcoux motivated by Gilfeather and Smith. Forrest and Marcoux also studied derivations on triangular Banach algebra. In this paper we varify some propoerties of derivation on tri angular Banach algebras.

Assume that $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras and $\mathcal{M}$ is a Banach $\mathcal{A}, \mathcal{B}$-module. Let $\Gamma = \{\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M}\}$. Then Γ equipped with the usual 2×2 matrix operations, obvious internal module actions,

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* Speaker.

is a complex algebra and the Banach space norm $\|\begin{bmatrix}\mathcal{A} & \mathcal{M}\\ 0 & \mathcal{B}\end{bmatrix}\| = \|a\|_{\mathcal{A}} + \|m\|_{\mathcal{M}} + \|b\|_{\mathcal{B}}$, Γ is a Banach algbera, which is called triangular Banach algebra.

Let A be a Banach algebra and X be a Banach A-bimodule, a continuouse derivation on A is a bounded linear operator $D : A \longrightarrow X$ such that $D(ab) = a \cdot D(b) + D(a) \cdot b$. The derivation D is called inner if there exists some $x \in X$ such that $D(a) = a \cdot x - x \cdot a$ for all $a \in A$. The inner derivation D is called approximately inner if there exists a net $\{x_\alpha\} \subseteq X$ such that $D(a) = \lim_{\alpha}(a \cdot x_\alpha - x_\alpha \cdot a)$.

The module actions of Γ on Γ^* behaves in natural way by $\tau(h) = \alpha(a)+\mu(m)+\beta(b)$ For $h = \begin{bmatrix}a & m\\ 0 & b\end{bmatrix} \in \Gamma$ and $\tau = \begin{bmatrix}\alpha & \mu\\ 0 & \beta\end{bmatrix} \in \Gamma^*$. Suppose that $\omega = \begin{bmatrix}x & z\\ 0 & y\end{bmatrix} \in \Gamma$. Then we have

$$\langle \omega \cdot \tau, h\rangle = \begin{bmatrix}x \cdot \alpha + z \cdot \mu & y \cdot \mu\\ 0 & y \cdot \beta\end{bmatrix} \cdot \begin{bmatrix}a & m\\ 0 & b\end{bmatrix}$$

The reader can see this in [4].

2. Main sections and results

Proposition 2.1. *Suppose that $m \geq 0$ and $D : A \longrightarrow A^{(2m+1)}$ is a bounded derivation. Then $\bar{D} : \Gamma \longrightarrow \Gamma^{(2m+1)}$ defined by*

$$\bar{D}(\begin{bmatrix}a & m\\ 0 & b\end{bmatrix}) = \begin{bmatrix}D_A(a) & 0\\ 0 & 0\end{bmatrix}$$

is a bounded derivation. D is inner derivation if and only if $\bar{D}$ be inner. Similarly, let $D : B \longrightarrow B^{(2m+1)}$ be a bounded derivation. Then $\bar{D} : \Gamma \longrightarrow \Gamma^{(2m+1)}$ defined by

$$\begin{bmatrix}a & m\\ 0 & b\end{bmatrix}) \longmapsto \begin{bmatrix}0 & 0\\ 0 & D_B(b)\end{bmatrix}$$

is also a bounded derivation if and only if $\bar{D}$ is inner.

Proposition 2.2. *Let $n \geq 0$, and $D : \Gamma \longrightarrow \Gamma^{(n)}$ be a continuouse derivation. Then $D^* : \Gamma^{(n+1)} \longrightarrow \Gamma^*$, the dual operator of D, satisfies*

$$d^*(\begin{bmatrix}a & m\\ 0 & b\end{bmatrix} \cdot \begin{bmatrix}\alpha & \mu\\ 0 & \beta\end{bmatrix}) = \begin{bmatrix}a & m\\ 0 & b\end{bmatrix} \cdot D^*(\begin{bmatrix}\alpha & \mu\\ 0 & \beta\end{bmatrix}) - (D(\begin{bmatrix}a & m\\ 0 & b\end{bmatrix}) \cdot \begin{bmatrix}\alpha & \mu\\ 0 & \beta\end{bmatrix}) \mid_\Gamma$$

for $\begin{bmatrix}a & m\\ 0 & b\end{bmatrix} \in \Gamma$ and $\begin{bmatrix}\alpha & \mu\\ 0 & \beta\end{bmatrix} \in \Gamma^{(n+1)}$.

Proposition 2.3. *Let $k \geq 0$ is an integer. Suppose that $D : \Gamma \longrightarrow \Gamma^{(k)}$ is a continuouse derivation. So for $n \geq o$, $D^{(2n+1)} : \Gamma^{(k+2n+1)} \longrightarrow \Gamma(2n+1)$. The $(2n+1)$-th dual operator of D where satisfies*

$$D^{(2n+1)}(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} \alpha & \mu \\ 0 & \beta \end{bmatrix}) = \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \cdot D^{(2n+1)}(\begin{bmatrix} \alpha & \mu \\ 0 & \beta \end{bmatrix}) - (D(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} \alpha & \mu \\ 0 & \beta \end{bmatrix})) \mid_{\Gamma^{(2m)}}$$

$$D^{(2n+1)}\begin{bmatrix} \alpha & \mu \\ 0 & \beta \end{bmatrix} \cdot \begin{bmatrix} a & m \\ 0 & b \end{bmatrix})) = D^{(2n+1)}(\begin{bmatrix} \alpha & \mu \\ 0 & \beta \end{bmatrix} \cdot \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} - \begin{bmatrix} \alpha & \mu \\ 0 & \beta \end{bmatrix} \cdot D^{(2n)}(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix})) \mid_{\Gamma^{(2n)}}$$

for each $\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in \Gamma^{(2n)}, \begin{bmatrix} \alpha & \mu \\ 0 & \beta \end{bmatrix} \in \Gamma^{(k+2n+1)}$.

Proposition 2.4. *For each integer $(2n+1) \geq 0$, let $D_1 : A \longrightarrow A^{(2n+1)}$ and $D_2 : B \longrightarrow B^{(2n+1)}$ be a continuouse derivation. So $\bar{D} : \Gamma \longrightarrow \Gamma^{(2n+1)}$ defined by*

$$\bar{D}(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}) = \begin{bmatrix} D(a) & -D^{(2n+1)}(m) \\ 0 & D(b) \end{bmatrix}$$

is a continuouse derivation. Forthermore

(1) *If $\bar{D}$ is inner, then D is inner.*

(2) *If $D_1 : A \longrightarrow A^{(2n+1)}$ and $D_2 : B \longrightarrow B^{(2n+1)}$ are inner, then a contenuouse derivation $\tilde{D} : \Gamma \longrightarrow \Gamma^{(2n+1)}$ can be set so that $\tilde{D}(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for $a \in A, b \in B$, and $\bar{D} - \tilde{D}$ is inner.*

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Please cite your relevant papers but at most total 5 papers/books.

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ON CHAINS OF PRIME SUBMODULES OF A FINITELY GENERATED MODULE

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Abstract. Let R be a commutative ring with identity. In this paper, we study the chain of prime submodules of a finitely generated R-module M and the supremum of the lengths of these chains which is called the dimension of M and denoted by $\dim M$. For this purpose, the interplay between this dimension and the vector space dimension of some related vector spaces with M are considered. We show that if M is a non-torsion finitely generated module over a domain R, then $\dim R \leq \dim M$. Moreover if $\dim R$ is finite, then so is $\dim M$.

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unital. Let R be a ring. For a submodule N of an R-module M, $(N : M)$ is the ideal $\{r \in R \mid rM \subseteq N\}$ of R. As usual, $(0 : M)$ is called the annihilator of M and denoted by $\mathrm{Ann}(M)$. A proper submodule N of M is called a prime submodule if for $r \in R$, $m \in M$, $rm \in N$ implies that $r \in (N : M)$ or $m \in N$. This class of submodules has been extensively studied by several authors (see, for

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* Speaker.

example, [3, 4]). It is well known that every maximal submodule of a module is a prime submodule and so every finitely generated module has a prime submodule. It is easily seen that if N is a prime (resp. maximal) submodule of M, then $(N : M)$ is a prime (resp. maximal) ideal of R. Also, if N is a submodule of M such that $(N : M)$ is a maximal ideal of R, then N is a prime submodule of M. Thus every proper subspace N of a vector space M over a field R is a prime submodule of M.
The (Krull) dimension of a ring R, denoted by $\dim(R)$, was originally defined to be the supremum of the lengths of all chains of prime ideals in R. This dimension has been generalized from rings to modules in several ways (see, for example, [1, 2]). Here, the dimension of a module is considred as follows;

$$\dim M = \sup\{n \mid N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_n\}.$$

If this supremum dose not exist, then $\dim(M)$ is defined to be ∞. In the rest of paper, the vector space dimension of a vector space M will be denoted by $\mathrm{vdim}(M)$. It is easily seen that if M is a vector space with $\mathrm{vdim}(M) = n$, then $\dim(M) = n-1$, since every prime submodule of a vector space M is a prime submodule of M.
We know that if $\mathrm{Ann}(M) = m$ is a maximal ideal of R, then M/mM is an R/m vector space. Using the structure of this vector space, we obtain several properties for the chain of prime submodules of a finitely generated module M and in particular the dimension of M.

2. Main results

It is well-known that any two basis of a vector space have the same cardinal number [5, Theorem 1.12]. It may be compared with the following result for finitely generated modules:

Lemma 2.1. *If M is a finitely generated R-module with $\mathrm{Ann}(M) = m$ as a maximal ideal of R, then every minimal generating set has the same cardinal number. In particular $\dim M = n - 1$, where n is the number of elements of a minimal generating set of M.*

Of course the maximality of $\mathrm{Ann}(M)$ in the above lemma is essential. For example, if $R = \mathbb{Z}_{12}$ and $M = \mathbb{Z}_3 \oplus \mathbb{Z}_4$, then M is a free R-module with the basis $\{(\bar{1}, \bar{1})\}$. Moreover $\{(\bar{1}, \bar{0}), (\bar{0}, \bar{1})\}$ is a minimal generating set of M. Thus Lemma 2.1 is not true because $Ann(M) = \{\bar{0}\}$ which is not maximal in R.
We recall that if N is a prime submodule of an R-module M, then $(N : M) = p$ is a prime ideal of R. In this case, N is also called a

p-prime submodule.
If N is a prim submodule of an R-module M and S be a multiplicatively closed subset of R, then $S^{-1}N$ is a prime submodule of $S^{-1}R$-module of fractions $S^{-1}M$ [3, Proposition 1]. In particular, there is a one-to-one corresponding between the set of all prime submodules of M disjoint from S and the set of all prime submodules of $S^{-1}R$-module $S^{-1}M$. Let R be a domain and $S^{-1}R$ be the field of fractions of R. Let M be an R-module and $S^{-1}M$ be the $S^{-1}R$-vector space such that $\text{vdim}(S^{-1}M) = n$. It is clear that $S^{-1}R$-vector space $S^{-1}M$ has a chain of the length n of 0-prime submodules. Thus M has a chain of the length n of prime submodules. On the other hand, let $N_0 \subset N_1 \subset \cdots \subset N_n$ be a chain of 0-prime submodules of M. Since $S = R - \{0\}$ and therefore $S \cap (N_i : M) = \emptyset$ for all $0 \leq i \leq n$, by [3, Proposition 1], $S^{-1}N_0 \subset S^{-1}N_1 \subset \cdots \subset S^{-1}N_n$ is a chain of prime submodule of $S^{-1}R$-vector space $S^{-1}M$. Thus $\dim(S^{-1}(M)) \geq n$ which is a contradiction to the given fact in the end of Introduction. It follows that there is not a chain of the length $n+1$ of 0-prime submodules in M. Thus we have the following theorem:

Theorem 2.2. *Let R be a domain with $S^{-1}R$ as the field of fractions and let M be an R-module. If $S^{-1}M$ is an n-dimensional vector space over $S^{-1}R$, then M has a chain of the length n of 0-prime submodules and has no any chain of 0-prime submodule of the length $n+1$.*

Corollary 2.3. *Let R be a domain, p a prime ideal of R and M an R-module. If $S^{-1}(R/p)$ is the field of fractions of R/p and $S^{-1}(M/pM)$ is an n-dimensional vector space over $S^{-1}(R/p)$, then M has a chain of the length n of p-prime submodules and has no any chain of p-prime submodule of the length $n+1$.*

Let R be a local ring with only maximal ideal m and M/mM be an n-dimensional R/m-vector space. If $N_1, \cdots, N_k$ are distinct maximal submodules of M, then for each $1 \leq i \leq n$ the ideal $(N_i : M)$ is maximal in R and so $(N_i : M) = m$. It implies $(\cap_{i=1}^{j} N_i : M) = \cap_{i=1}^{j}(N_i : M) = m$ for all $2 \leq j \leq k$. Hence we have the chain $\cap_{i=1}^{k} N_i \subset \cap_{i=1}^{k-1} N_i \subset \cdots \subset N_1$ of distinct prime submodules of M. Now since M/mM is an n-dimensional R/m-vector space, we have $k \leq n$ by Corollary 2.3. Therefore we have the following.

Theorem 2.4. *Let (R, m) be a local ring and M/mM be an n-dimensional R/m-vector space. Then M has at most n distinct maximal submodules.*

Corollary 2.5. *Let (R, m) be a local ring and M an R-module generated by n elements. Then M has at most n distinct maximal submodules.*

Let M be an R-module. The torsion submodule of M, denoted $T(M)$, is the set $\{m \in M \mid rm = 0 \text{ for some } 0 \neq r \in R\}$. M is called a torsion module if $T(M) = M$. In otherwise, we say that M is non-torsion. If M is a non-torsion module over an integral domain R, then it is easy to verify that $T(M)$ is a 0-prime submodule of M. Now if moreover we suppose that M is finitely generated, then $\dim R \leq \dim M$. Indeed, if $0 \subset p_1 \subset p_2 \subset \cdots \subset p_n$ is a chain of prime ideals of R, then by [4, Theorem 3.8] there exist prime submodules N_i containing $T(M)$ such that $(N_i : M) = p_i$ for all $1 \leq i \leq n$. Hence we have the the chain $T(M) \subset N_0 \subset N_1 \subset \cdots \subset N_n$ of prime submodules. Therefore $\dim R \leq \dim M$. We show that if $\dim R$ is finite, then so is $\dim M$. For this purpose, first assume $\dim R = 0$, i. e., R is a field. Thus M is a finite dimensional vector space over R and so $\dim M = \text{vdim}(M) - 1$, as desired. Now suppose $\dim R = n > 0$ and t is the cardinal number of a minimal generating set of M. Thus by[5, Theorem 1.9 (3)], for each prime ideal p of R, the vector space dimension of the vector space $S^{-1}M/pM$ over R/p is at most t. Hence by Corollary 2.3, there is not any chain of p-prime submodules of the length $t + 1$. Let $N_0 \subset N_1 \subset \cdots \subset N_k$ be a chain of prime submodules of M. Since $\dim R = n > 0$. Without loss of generality, assume that $(N_0 : M) = (N_1 : M) = \cdots = (N_{s_0-1} : M) = p_0$. Thus by Corollary 2.3, $s_0 < t$. By repeating this process, there are at most $n + 1$ positive integers $s_0, \cdots, s_n$ such that $s_i < t$ for all $0 \leq i \leq n$. Thus $k = s_0 + \cdots + s_n - 1 < (n + 1)t - 1$ and so $\dim M < (n + 1)m - 1$. We summarize these facts in the following theorem:

Theorem 2.6. *Let M be a finitely generated non-torsion module. Then $\dim R \leq \dim M$. Moreover, if $\dim R$ is finite, then so is $\dim M$.*

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NONLINEAR PRESERVERS OF MAJORIZATION

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Abstract. By a matrix majorization relation $X \prec_{\ell s} Y$ on the collection M_{nm} of all $n \times n$ real matrices, we mean $X = RY$ for some $n \times n$ row substochastic matrix R. An operator $T\colon M_{nm} \to M_{nm}$ is said to be a preserver (resp. strong preserver) of the relation $\prec_{\ell s}$ on M_{nm} if $X \prec_{\ell s} Y$ implies that $TX \prec_{\ell s} TY$ (resp. if $X \prec_{\ell s} Y$ if and only if $TX \prec_{\ell s} TY$). It is shown that if $T\colon M_{nm} \to M_{nm}$ is surjective (not necessarily linear) strong preserver of $\prec_{\ell s}$, then $T(0) = 0$ and T is invertible.

1. Introduction

Throughout the paper, the notation M_{nm} is fixed for the space of all $n \times m$ real matrices; this is further abbreviated by M_n when $m = n$. the space M_{n1} of all $n \times 1$ real vectors is denoted by the usual notation $\mathbb{R}^n$. An $n \times m$ matrix $R = [r_{ij}]$ is called *row stochastic* (resp. *row substochastic*) if $r_{ij} \geq 0$ and $\Sigma_{k=1}^{m} r_{ik}$ is equal (resp. at most equal) to 1 for all i, j. By a matrix majorization relation $X \prec_{\ell} Y$ (resp. $X \prec_{\ell s} Y$) on the collection M_{nm} of all $n \times n$ real matrices, we mean $X = RY$ for some $n \times n$ row stochastic (resp. row substochastic) matrix R. Also,

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* Speaker.

we define $C(A) := \{X \in M_{nm} : X \prec_{\ell s} A\}$. For information on this subject, we refer the reader to[1], [3], [4] and [5].

Next Section deals with nonlinear operators which strongly preserves $\prec_{\ell s}$ and, in particular, we will show that, invertibility of T and the condition $T(0) = 0$ are necessary for surjective operators (not necessarily linear) T to be a preserver of $\prec_{\ell s}$. It is shown that [5] if $m \geq 2$ and if $T\colon M_{nm} \to M_{nm}$ is surjective strong preserver of $\prec_{\ell}$, then $T - T(0)$ is equivalent to a linear strong preserver of $\prec_{\ell}$. In the following , we state some results from [5].

Definition 1.1. [5, Definition 1.2] Two strong preservers T and τ of $\prec_{\ell}$ on M_{nm} are said to be equivalent, if $TX \sim \tau X$ for all $X \in M_{nm}$.

Theorem 1.2. [5, Theorem 2.9] *Let $T\colon M_{nm} \to M_{nm}$ be a strong preserver of $\prec \ell$. Then $T-T(0)$ is equivalent to a linear strong preserver of $\prec_{\ell}$. In fact, there exists an invertible $K \in M_m$ such that $TX - T(0) \sim XK^{-1}$ for all $X \in M_{nm}$.*

Corollary 1.3. [5, Corollary 2.10] *For a linear strong preserver $T\colon M_{nm} \to M_{nm}$ there exists a permutation matrix P and an invertible matrix $L \in M_m$ such that $TX = PXL$ for all $X \in M_{nm}$.*

2. Main sections and results

In this section we study the structure of strong preserver $T\colon M_{nm} \to M_{nm}$ of matrix majorization $\prec_{\ell s}$ which are not necessarily linear. We will show that such mapping must be invertible and $T(0) = 0$. Let throughout the rest of the paper, $C(A) := \{X \in M_{nm} : X \prec_{\ell s} A\}$.

Theorem 2.1. [2, Theorem 3.4] *A linear operator $T : M_{nm} \to M_{nm}$ strongly preserves the majorization relation $\prec_{\ell s}$ if and only if there exists P and an invertible matrix L in M_m such that $T(X) = PXL$ for all $X \in M_{nm}$.*

Theorem 2.2. *Let $T\colon M_{nm} \to M_{nm}$ be a (not necessarily linear) strong preserver of $\prec_{\ell s}$. Then the following assersions are true:*

(1) *Assume T is surjective. Then $C(T(A)) = T(C(A))$ for all $A \in M_{nm}$.*

(2) *Assume $C(T(A)) = T(C(A))$ for all $A \in M_{nm}$. Then $T(0) = 0$.*

Corollary 2.3. *If $T\colon M_{nm} \to M_{nm}$ be a surjective strong preserver of $\prec_{\ell s}$, then $T(0) = 0$.*

Theorem 2.4. *Let $T\colon M_{nm} \to M_{nm}$ be an operator (not necessarily linear) that is surjective and strongly preserves $\prec_{\ell s}$. Then T is invertible.*

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ON SOME PROPERTIES OF FIXED POINTS OF OPERATORS

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Abstract. Let $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on a Banach space $\mathcal{X}$. In this lecture, we state some properties of fixed points of sum and product of elements belonging to $\mathcal{B}(\mathcal{X})$. Moreover, we state some properties of the dimension of fixed points of sum and product of such elements.

1. Introduction

Let $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on a Banach space $\mathcal{X}$. Recall that $x \in X$ is a fixed point of an operator $T \in B(X)$, whenever we have $Tx = x$. Denote by $F(T)$, the set of all fixed points of T. It is clear that for an linear operator T, $F(T)$ is a vector space. Denote by $\dim F(T)$, the dimension of $F(T)$. The ordered pair (S, T) is called a Banach operator pair if the set $F(T)$ is S-invariant, namely $S(F(T) \subseteq F(T)$.

In this lecture, we state some properties of fixed points of sum and product of operators. Also, we state some properties of the dimension of fixed points of such elements.

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* Speaker.

1.1. **Main Results.**

Lemma 1.1. *Let $A, B \in \mathcal{B}(\mathcal{X})$. Then $A = B$ if and only if $\dim F(AT) = \dim F(BT)$, for every $T \in \mathcal{F}_1(\mathcal{X})$.*

Lemma 1.2. *Let $A \in \mathcal{B}(\mathcal{X})$. $A \in \mathbb{C}^*I$ if and only if $\dim F(AP) = 0$, for every $P \in \mathcal{P}_1(\mathcal{X})$.*

Lemma 1.3. *Let A and B be non-scalar operators. If $\dim F(AP) = \dim F(BP)$, for every $P \in \mathcal{P}_1(\mathcal{X})$, then there exists an $\lambda \in \mathbb{C} \setminus \{1\}$ such that $B = \lambda I + (1 - \lambda)A$.*

Lemma 1.4. *Let A and B be non-scalar operators. If $\dim F(AR) = \dim F(BR)$, for every $R \in \mathcal{P}_1(\mathcal{X}) \cup \mathcal{N}_1(\mathcal{X})$, then $A = B$.*

Lemma 1.5. *Let $A, B \in \mathcal{B}(\mathcal{X})$ be non-scalar operators. If $F(A+P) = F(B+P)$, for every $P \in \mathcal{P}_1(\mathcal{X})$, then there exists an $\lambda \in \mathbb{C} \setminus \{1\}$ such that $B = \lambda I + (1 - \lambda)A$.*

Lemma 1.6. *Let $A, B \in \mathcal{B}(\mathcal{X})$. If $F(A + R) = F(B + R)$, for every $R \in \mathcal{F}_1(\mathcal{X})$, then $A = B$.*

Lemma 1.7. *Let $A \in \mathcal{B}(\mathcal{X})$. $A \in \mathbb{C}^*I$ if and only if $F(PAP) = 0$, for every $P \in \mathcal{P}_1(\mathcal{X})$.*

Lemma 1.8. *Let A and B be non-scalar operators. If $F(PAP) = F(PBP)$, for every $P \in \mathcal{P}_1(\mathcal{X})$, then there exists an $\lambda \in \mathbb{C} \setminus \{1\}$ such that $B = \lambda I + (1 - \lambda)A$.*

Lemma 1.9. *Let $A, B \in \mathcal{B}(\mathcal{X}) \setminus \{0, I\}$. If $F(RAR) = F(RBR)$, for every $R \in \mathcal{F}_1(\mathcal{X})$, then $A = B$.*

Theorem 1.10. *Let $A \in \mathcal{B}(\mathcal{X})$. If for any $T \in \mathcal{B}(\mathcal{X})$, (A, T) is a Banach operator pair, then A is a scalar operator.*

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ON THE RESIDUAL OF A SUBMODULE BY A FREE MODULE OF FINITE RANK

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Abstract. Let R be a unique factorization domain with identity and F a free R-module of finite rank with the basis $\{f_1, \cdots, f_m\}$. Let G be a submodule of F generated by the set $\{g_1, \cdots, g_k\}$ and $A = (a_{ij})_{m\times k}$ be the matrix of multipliers which describes the elements g_i $(1 \leq i \leq k)$ as a linear combination of $\{f_1, \cdots, f_m\}$. We show that if $\det A \neq 0$, then $[G : F] = Rt$ in which t divides $\det A$.

1. Introduction

Let R be a commutative ring and M an R-module. Moreover, let N be a submodule of M. The residual submodule of N by M is the ideal $[N : M] = \{r \in R \mid rM \subseteq N\}$. A submodule P of M is called a prime submodule of M if whenever $rm \in P$, then $r \in [P : M]$ or $m \in P$ (see, for example, [2, 4]). It is easily seen that if P is a prime submodule of M, then $[P : M]$ is a prime ideal of R. The radical of a submodule N of M, denoted $\operatorname{rad} N$, is the intersection of all prime submodules of N (see, for example, [2, 3, 5]). A challenging problem in recent years has been to find a good description of the radical of a submodule over

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* Speaker.

a commutative ring. A number of authors have tried to describe this radical, either in terms of its elements, or as some sort of decomposition. As for the former, most of the efforts were directed towards finding a description similar to the well-known formula for the elements of the radical of an ideal; namely, for an ideal I of a ring (commutative, with identity)R, then $\sqrt{I} = \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{Z}\}$. A method for computing the radical of a submodule G of a free module F was given in [3], using the symmetric algebra of F. Moreover, the radical and the residual of submodules by a free module of rank at most 2 over a unique factorization domain have been investigated in [4]. It has been shown that if I ia an ideal of R and F is a free R-module F, then $[IF : F] = I$ and $\operatorname{rad} IF = \sqrt{I}F$ [4, Proposition 2.2]. In general, if G is a finitely generated submodule of a free R-module of finite rank, then $[\operatorname{rad} G : F] = \sqrt{[G : F]}$ [2, Theorem 3.8 and Proposition 5.3]. Thus the computing of the radical of G depends on that of $[G : F]$, which was the motivation of this work.

2. Main results

We start with a lemma which is used in the rest of paper frequently. Some well-known elementary results from matrix theory over commutative rings may be found in [1].

Lemma 2.1. *Let R be a commutative ring and F be a free R-module of finite rank with the basis $\{f_1, \cdots, f_m\}$. Let $G = Rg_1 + \cdots + Rg_k$ and $A = (a_{ij})_{k\times m}$ in which a_{ij} $(1 \le i \le k, 1 \le j \le m)$ are the multipliers appeared in the linear combinations $g_i = a_{i1}f_1 + \cdots + a_{im}f_m$. Then $r \in [N : M]$ if and only if there exists a matrix $B = (b_{ij})_{m\times k}$ with entries in R such that $BA = rI_{m\times m}$.*

Example 2.2. Let $R = K[X, Y]$ be a ring of polynomial rings in variables X and Y over a field K and let $F = R \oplus R$.

(1) If $G = R(X, Y)+R(Y, X)$, then by Lemma 2.1, $A = \begin{pmatrix} X & Y \\ Y & X \end{pmatrix}$. If $r \in [G : F]$, then there exist the elements $b_{11}, b_{12}, b_{21}, b_{22} \in R$ such that

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} X & Y \\ Y & X \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}.$$

Therfore

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} rX/(X^2 - Y^2) & -rY/(X^2 - Y^2) \\ -rY/(X^2 - Y^2) & rX/(X^2 - Y^2) \end{pmatrix}$$

Now $b_{ij} \in R$ $(1 \le i, j \le 2)$ if and only if $r \in R(X^2 - Y^2)$. Hence $[G : F] = R(X^2 - Y^2)$. Moreover, we have $(X - Y)(X +$

$Y) \in [G : F]$, but $(X - Y) \notin [G : F]$ and $(X + Y) \notin [G : F]$. Thus $[G : F]$ is not a prime ideal of R and so G is not a prime submodule of F. Furthermore, we have $[G : F] = R \det A$.

(2) If $G = R(X, 0) + R(0, X)$, then by Lemma 2.1, $A = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$ If $r \in [G : F]$, then there exist the elements $b_{11}, b_{12}, b_{21}, b_{22} \in R$ such that

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}.$$

Therfore

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} r/X & 0 \\ 0 & r/X \end{pmatrix}$$

Now $b_{ij} \in R$ $(1 \leq i, j \leq 2)$ if and only if $r \in RX$. Hence $[G : F] = RX$. In this case, the generator X of $[G : F]$ divides $\det A = X^2$.

Theorem 2.3. *Let R, F, G and A be as in* Lemma 2.1. *Then* $\det A \in [G : F]$.

Proof. let $c_{ik} = (-1)^{i+k} \det A_{ki}$ in which A_{ki} is the $(m-1) \times (m-1)$ matrix obtained by deleting the row k and the column i of A. Thus $\det A = \sum_{i=1}^{m} a_{ik} c_{ik}$. In other words, $AC = (\det A)I$ where $C = (c_{ik})_{m \times m}$ and I is the identity matrix. Now by Lemma 2.1, $\det A \in [G : F]$. □

Theorem 2.4. *Let R be a unique factorization domain. Let F, G and A be as in* Lemma 2.1 *such that* $\det A \neq 0$. *Then there exists $q \in R$ such that q divides* $\det A$ *and* $[G : F] = Rq$.

Proof. By Theorem 2.3, there exists a matrix $C = (c_{ij})_{m \times m}$ with entries in R such that $AC = (\det A) I_{m \times m}$. Since $\det A \neq 0$, A is invertible and in particular $A^{-1} = (1/\det A)C$. Since R is a unique factorization domain, every entry $c_{ij}/\det A$ can be uniquely factorized into irreducible elements c'_{ij}/p_{ij} of R such that $\gcd(c'_{ij}, p_{ij}) = 1$. Now let $q = \operatorname{lcm}\{p_{ij}\}_{i,j=1}^{m}$. Then q divides $\det A$. We claim that $[G : F] = Rq$. First assume that $r \in [G : F]$. Hence by Lemma 2.1, there exists a matrix $D_{m \times m}$ such that $DA = rI_{m \times m}$. Thus $D = rA^{-1} = (r/\det A)C$. It follows that $rc'_{ij}/p_{ij} \in R$ for all $i, j = 1, \cdots, m$. Now since $\gcd(c'_{ij}, p_{ij}) = 1$, $p_{ij} \mid r$ for all $i, j = 1, \cdots, m$ and so $q \mid r$, i. e., $r \in Rq$. Hence we have, $[G : F] \subseteq Rq$. For the reverse containment, let $r \in Rq$. Then there is $s \in R$ such that $r = sq$ and for all i, j, $p_{ij} \mid q$. Therefore, $rc_{ij}/\det A = sqc'_{ij}/p_{ij} \in R$. Now by considering $D = (r/\det A)C$, we have

$$DA = (r/\det A)CA = (r/\det A)(\det A)I_{m \times m} = rI_{m \times m}.$$

Thus by Lemma 2.1, $r \in [G : F]$. Hence $[G : F] = Rq$. □

Theorem 2.5. *Let R be a domain, F be a free module of finite rank m and $G = R_{g_1} + \cdots + R_{g_k}$ be a submodule of F such that $k \leq m$. Then $[G : F] = 0$.*

Proof. Let R_S and G_S be the ring of fractions of R and the module of fractions of G respectively. The generating set $\{g_1/1, \cdots, g_k/1\}$ for the R_S-vector space G_S is contained in a basis for G_S. Thus $\dim_{R_S} G_S \leq k$ and so $\dim_{R_S} G_S < \dim_{R_S} F_S$. Hence we have $G_S \neq F_S$. Now assume that $\{f_1, \cdots, f_m\}$ is a basis for the R-module F. It is easily seen that $(a_i)_S \cap G_S = 0$ for each $1 \leq i \leq n$. Hence we have

$$[G : F] = \cap_{i=1}^{m} [G : a_i] = 0.$$

□

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PERTURBATION ANALYSIS FOR LINEAR SYSTEM OF EQUATION CORRESPONDING TO SOME NUMERICAL METHODS BASED ON WEBFEM

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Abstract. Discretizing a linear parabolic differential equation leads to a linear system of equations. The behavior of this matrix plays an important role to analysis of stability and convergence. Due to this, in this paper, using the WEB-spline Finite Element Method (WEBFEM), a two dimensional linear parabolic differential equation is discretized. Then, the perturbation analysis of the obtained matrix from the linear system of equations is provided. These results show that, the condition number of this matrix leads to infinite, as discretization is refined. Finally, three theorems about the right, left and general perturbation are given.

1. Introduction

Let us consider the parabolic problem of the form

$$\begin{array}{cc} u_t = \nabla.\,(D(x,y)\nabla u) - \lambda(x,y)u + F(x,y) & ((x,y),t) \in \Omega \times (t_0,T) \\ u(x,y,t) = 0 & (x,y) \in \partial\Omega,\ t \in (t_0,T) \\ u(x,y,t_0) = \phi(x,y) & (x,y) \in \Omega \subset \mathbb{R}^2, \end{array} \tag{1.1}$$

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* Speaker.

where Ω is a bounded domain in $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$, the diffusion coefficient D is known continuous function, λ and F belong to $H^0(\Omega) = \left\{ v = v(x,y,t) : \int\limits_{\Omega} v^2 d\Omega < +\infty \right\}$ and satisfy the following conditions

$$0 < d_0 \leqslant D \leqslant d_1, \quad 0 < \lambda_1 \leqslant \lambda \leqslant \lambda_2 \ , \tag{1.2}$$

for fixed real numbers $\lambda_1, \lambda_2, d_0$ and d_1. According to these conditions the parabolic problem (1.1) has a unique solution [1]. The main purpose of this paper is perturbation analysis of a linear system of equation obtained from discretizing the problem (1.1) using WEBFEM. This method that developed by Klaus Hollig [2, 3], uses regular grids and R-functions to establish basis functions, then build a system through Galerkin's method [2]. In next section, this process, is given in brief. After that in section 3, the perturbation analysis of obtained linear system is provided.

2. Mathematical formulation

The WEBFEM is based on applying the weighted extended bivariate B-spline (WEB-spline) basis in the finite element method as basis functions. B-spline tensor product is an extension of B-spline in higher dimensions. So, in order to make a bivariate B-spline of degree n, $b^n_{\mathbf{k},h}$, one dimensional B-spline tensor product is used as follows

$$b^n_{\mathbf{k},h}(x,y) = b^n_{k_1,h}(x) \otimes b^n_{k_2,h}(y), \qquad \mathbf{k} = (k_1,k_2) \in K,$$

where $b^n_{\ell,h}(x) = b^n(x/h - \ell)$ is one dimensional B-spline, the set K includes all $\mathbf{k}$ indices in such a way that for some $\mathbf{x} \in \Omega$, $b^n_{\mathbf{k},h} \neq 0$ and h is increment of variables x and y. In addition, the support of bivariate B-spline basis is given by $[k_1, k_1+n+1]h \times [k_2, k_2+n+1]h$. The WEB-spline, $B_{\mathbf{i}}$, after some computations is obtained as

$$B_{\mathbf{i}} = \frac{\omega}{\omega(x_{\mathbf{i}})}(b_{\mathbf{i}} + \sum_{\mathbf{j}\in J} e_{\mathbf{ij}} b_{\mathbf{j}}), \quad (b_{\mathbf{k}} = b^n_{\mathbf{k},h}),$$

where $\omega(\mathbf{x})$ is a weight function which is determined by using the Rvachev method (or R-function). For more details, see [2, 3].
To obtain an approximate solution of problem (1.1), using the forward finite difference method in a discretizing process with step size $\Delta t = \frac{T-t_0}{n}$ (for arbitrarily $n \in \mathbb{N}$), the problem (1.1) is reduced to a system of linear elliptic problems of the form

$$\begin{array}{ll} -\nabla.\left(D(x,y)\nabla u_i\right) - \left(\lambda(x,y) + \frac{1}{\Delta t}\right) u_i = F(x,y) + \frac{1}{\Delta t} u_{i-1} & (x,y) \in \Omega \\ u_i(x,y) = 0 & (x,y) \in \partial\Omega, \end{array} \tag{2.1}$$

where u_i is the approximated value of $u(x, y, t_i)$ for $t_i = t_0 + i\Delta t$, $i = 1, 2, \ldots, n$ and $u_0 = \phi(x, y)$. To approximate the solution u_i, $i = 1, 2, \ldots, n$, the WEBFEM is used for each equation in (2.1). For this purpose, put $u_i \simeq \sum_{j=1}^{N} c_j^{(i)} B_j(x, y)$ where $c_j^{(i)}$, are unknown parameters that should be derived. Solving the following linear system of equation, these unknown parameters are determined

$$\begin{gathered} AC^{(i)} = B^{(i)}, \\ \left(B^{(i)}\right)_j = \int_{\Omega} (F(x,y) + \tfrac{1}{\Delta t} u_{i-1}(x,y)) B_j(x,y) d\Omega, \quad \left(C^{(i)}\right)_j = c_j^{(i)} \\ (A)_{j,k} = \int_{\Omega} (D(x,y) \nabla B_j(x,y) . \nabla B_k(x,y))\, d\Omega - \int_{\Omega} (\lambda(x,y) + \tfrac{1}{\Delta t}) B_j(x,y) B_k(x,y) d\Omega. \end{gathered} \tag{2.2}$$

3. Perturbation analysis

Theorem 3.1. *Consider the coefficient matrix A and suppose that the provided assumptions in (1.2) are satisfied. Then*

$$Cond(A) \asymp \frac{(d_1 + \lambda_2)\Delta t + 1}{(d_0 + \lambda_1)\Delta t + 1} h^{-2}.$$

Proof. The coefficient matrix A is symmetric. Thus, the condition number A is obtained from the quotient of the largest and smallest eigenvalue. To determine these two eigenvalues, the Rayleigh quotient

$$r(C^{(i)}) = \frac{{C^{(i)}}^t A C^{(i)}}{{C^{(i)}}^t C^{(i)}},$$

should be maximized and minimized [2]. Since, $u_i \simeq \sum\limits_{j=1}^{N} c_j^{(i)} B_j$, we obtain

$$r(C^{(i)}) = \frac{\int_{\Omega} D(x,y) \left\|\nabla u_i\right\|^2 d\Omega + \int_{\Omega} \left(\lambda(x,y) + \frac{1}{\Delta t}\right) |u_i|^2 d\Omega}{\left\|C^{(i)}\right\|^2}.$$

The upper bound is determind as

$$\int_{\Omega} D(x,y) \left\|\nabla u_i\right\|^2 d\Omega + \int_{\Omega} \left(\lambda(x,y) + \frac{1}{\Delta t}\right) |u_i|^2 d\Omega \leqslant d_1 \left\|u_i\right\|_1^2 + \left(\lambda_2 + \frac{1}{\Delta t}\right) \left\|u_i\right\|_1^2,$$

where $\|\,.\,\|_1$ is the norm on $H^{1,0}(\Omega)$. Then by the Bernstein inequlity (see [2]), we obtain

$$\begin{aligned} &\int_{\Omega} D(x,y) \left\|\nabla u_i\right\|^2 d\Omega + \int_{\Omega} \left(\lambda(x,y) + \frac{1}{\Delta t}\right) |u_i|^2 d\Omega \\ &\qquad \leqslant \left(d_1 h^{-2} h^m + \left(\lambda_2 + \frac{1}{\Delta t}\right) h^{-2} h^m\right) \left\|C^{(i)}\right\|^2, \end{aligned}$$

where m is the degree of WEB-splines. Thus

$$r(C^{(i)}) \leqslant \left(d_1 h^{-2} h^m + \left(\lambda_2 + \frac{1}{\Delta t} \right) h^{-2} h^m \right).$$

On the other hand, by the Poincare-Friedrichs inequality (see [2]), the lower bound is determined as

$$\int_\Omega D(x,y) \left\| \nabla u_i \right\|^2 d\Omega + \int_\Omega \left(\lambda(x,y) + \frac{1}{\Delta t} \right) |u_i|^2 d\Omega \geqslant d_0 \left\| u_i \right\|_0^2 + \left(\lambda_1 + \frac{1}{\Delta t} \right) \left\| u_i \right\|_0^2,$$

where $\| \, . \, \|_0$ is the norm on $H^0(\Omega)$. Then by [2, 3], we obtain

$$\int_\Omega D(x,y) \left\| \nabla u_i \right\|^2 + \int_\Omega \left(\lambda(x,y) + \frac{1}{\Delta t} \right) |u_i|^2 d\Omega \geqslant \left(d_0 h^m + \left(\lambda_1 + \frac{1}{\Delta t} \right) h^m \right) \left\| C^{(i)} \right\|^2.$$

Thus

$$r(C^{(i)}) \geqslant \left(d_0 h^m + \left(\lambda_1 + \frac{1}{\Delta t} \right) h^m \right).$$

This, with the above estimate, implies the proof. □

So we have three following theorems.

Theorem 3.2. ***Right Perturbation Theorem*** *Let $\delta C^{(i)}$ and $\delta B^{(i)}$ be the perturbation of $C^{(i)}$ and $B^{(i)}$ in the (2.2), respectively. Also, A is assumed to be nonsingular and $B^{(i)} \neq 0$. Then*

$$\frac{\left((d_0 + \lambda_1) \Delta t + 1 \right) \left\| \delta B^{(i)} \right\|}{\left((d_1 + \lambda_2) \Delta t + 1 \right) \left\| B^{(i)} \right\|} h^2 \leqslant \frac{\left\| \delta C^{(i)} \right\|}{\left\| C^{(i)} \right\|} \leqslant \frac{\left((d_1 + \lambda_2) \Delta t + 1 \right) \left\| \delta B^{(i)} \right\|}{\left((d_0 + \lambda_1) \Delta t + 1 \right) \left\| B^{(i)} \right\|} h^{-2}.$$

Theorem 3.3. ***Left Perturbation Theorem*** *Let A be nonsingular and $B^{(i)} \neq 0$. Suppose that δA and $\delta C^{(i)}$ are the perturbation of A and $C^{(i)}$ in the (2.2), respectively. Then*

$$\frac{\left\| \delta C^{(i)} \right\|}{\left\| C^{(i)} \right\|} \leqslant \left(\frac{\left((d_1 + \lambda_2) \Delta t + 1 \right) \left\| \delta A \right\|}{\left((d_0 + \lambda_1) \Delta t + 1 \right) \left\| A \right\|} h^{-2} \right) / \left(1 - \frac{\left((d_1 + \lambda_2) \Delta t + 1 \right) \left\| \delta A \right\|}{\left((d_0 + \lambda_1) \Delta t + 1 \right) \left\| A \right\|} h^{-2} \right).$$

Theorem 3.4. ***General Perturbation Theorem*** *Let A be nonsingular, $B^{(i)} \neq 0$ and $\left\| \delta A \right\| < \frac{1}{\left\| A^{-1} \right\|}$. Then*

$$\frac{\left\| \delta C^{(i)} \right\|}{\left\| C^{(i)} \right\|} \leqslant \left(\frac{\left((d_1 + \lambda_2) \Delta t + 1 \right)}{h^2 \left((d_0 + \lambda_1) \Delta t + 1 \right) - \left((d_1 + \lambda_2) \Delta t + 1 \right) \frac{\left\| \delta A \right\|}{\left\| A \right\|}} \right) \left(\frac{\left\| \delta A \right\|}{\left\| A \right\|} + \frac{\left\| \delta B^{(i)} \right\|}{\left\| B^{(i)} \right\|} \right).$$

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AN ITERATIVE METHOD BASED ON DIAGONAL DOMINANCE FOR SOLVING SYSTEMS OF LINEAR EQUATIONS

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Abstract. A system of linear equations, $Ax = b$, is rewritten in an equivalent form $x = Gx + c$, where G is a diagonal dominant matrix thanks to a control parameter α. Then a homotopy based method is applied for solving this new system where G plays the role of an iterative matrix. We discuss the properties of G and present a sufficient condition for the general method to be convergent.

1. Introduction

The system of linear equations $Ax = b$, is probably the most important problem which has led matrix analysis and its variants to its current status. Most phenomena in various fields culminate in the solution of a system of linear equations. It is the spring board for most of modern concepts in advanced matrix analysis.

The solution strategies for the system $Ax = b$, could be classified mainly as direct methods and iterative ones. Direct methods are mostly classic techniques which are only applicable when relatively small size problems are under consideration. But for large systems which are

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* Speaker.

most appealing ones in applied sciences, an iterative method would be of more interest [1]. In this work we aim to present a new technique to handle a nonsingular system of linear equations. A "control parameter" is used to rewrite the system $Ax = b$ in an equivalent form which can be solved by an iterative technique. One can choose the control parameter in a way that the iterative matrix gets a diagonally dominant structure. Under some conditions the approach will lead us to the desired solution.

2. The basic idea

Consider a system of linear equations, $Ax = b$, where A is a nonsingular matrix of size n and $x, b \in \mathbb{R}^n$. Without loss of generality we can assume that A has nonzero diagonal elements. If D_A denotes the diagonal matrix consisting of diagonal elements of A, then D_A^{-1} is well-defined. We use a control parameter, real and nonzero number α, to define $A_\alpha = A - \alpha D_A$. Employing this notation we can rewrite the system $Ax = b$ in an equivalent form $x = Gx + c$, where $G = \dfrac{-1}{\alpha} D_A^{-1} A_\alpha$ and $c = \dfrac{1}{\alpha} D_A^{-1} b$. According to column elements of A we define the scalar M to be

$$M := \max_{1 \le j \le n} \sum_{i \neq j}^{n} |\frac{a_{ij}}{a_{ii}}|.$$

Lemma 2.1. *If α is chosen to satisfy $|\alpha - 1| > M$ then matrix G would be a strictly column diagonally dominant matrix.*

Remark 2.2. A similar argument to the preceding one could be stated for the row elements which leads to a row diagonally dominant matrix.

2.1. **Solving $x = Gx + c$.** We solve the system $x = Gx + c$ using a homotopy based method. We define the following homotopy equation

$$Gx - qx + c = 0, \tag{2.1}$$

where $q \in [0, 1]$ is a homotopy parameter. Thus for every $q \in [0, 1]$, we have a system of linear equations whose solution is dependent upon q. When $q = 0$ the system is the trivial case $Gx = -c$, in the case $q = 1$ system (2.1) will be equivalent to the original system under study. Now, if we assume that the solution to (2.1) could be represented as an infinite series in the form

$$x = x^{(\circ)} + x^{(1)}q + x^{(2)}q^2 + x^{(3)}q^3 + \cdots, \tag{2.2}$$

then substituting series (2.2) in (2.1) we would have

$$G(x^{(\circ)} + x^{(1)}q + x^{(2)}q^2 + \cdots) - q(x^{(\circ)} + x^{(1)}q + \cdots) + c = 0. \tag{2.3}$$

This holds for all $q \in [0,1]$, so we have

$$\begin{aligned} Gx^{(\circ)} &= -c, \\ Gx^{(1)} &= x^{(\circ)}, \\ Gx^{(k)} &= x^{(k-1)},\ k \geq 2. \end{aligned} \tag{2.4}$$

Each of the above systems could be solved by a classic iterative method like Gauss-Seidel or Jacobi or it can be treated by LU decomposition method (a direct method), and it would be an easy task because G is a diagonally dominant matrix.

Each system has a unique solution, which, from a *theoretical* point of view, are:

$$\begin{aligned} x^{(\circ)} &= -G^{-1}c, \\ x^{(1)} &= G^{-1}x^{(\circ)} &&= -(G^{-1})^2 c, \\ x^{(k)} &= G^{-1}x^{(k-1)} &&= -(G^{-1})^{k+1}c,\ k \geq 2. \end{aligned} \tag{2.5}$$

Now the series solution, according to $q = 1$, would be

$$\begin{aligned} x &= x^{(\circ)} + x^{(1)} + x^{(2)} + x^{(3)} + \cdots, \\ &= -(G^{-1} + G^{-2} + \cdots)c, \end{aligned} \tag{2.6}$$

and under the assumption $\rho(G^{-1}) < 1$ the above series converges to

$$= c - (I - G^{-1})^{-1}c. \tag{2.7}$$

Theorem 2.3. *The solution (2.7) is a solution to the original system* $Ax = b$.

2.2. **Algorithm.** Although, yet, we have not any factual values for the auxiliary parameter α and only have a coarse limitation ($|\alpha - 1| > M$), we can put the technique in an algorithm.

Algorithm 1

Input: Matrix A, Vector b, N
Output: Vector x

1: {Step 1} **Define** $M := \max\limits_{1 \leq j \leq n} \sum\limits_{i \neq j}^{n} |\frac{a_{ij}}{a_{ii}}|$, **Choose** α
2: {Step 2} **Define** $A_\alpha := A - \alpha D_A$, $G := \frac{-1}{\alpha} D_A^{-1} A_\alpha$, $c := \frac{1}{\alpha} D_A^{-1} b$
3: {Step 3} **Solve** $Gx^{(\circ)} = -c$
4: **for** $k = 1$ to N **do**
5: **Solve** $Gx^{(k)} = x^{(k-1)}$
6: **end for**
7: {Step 4} **Put** $x := x^{(\circ)} + x^{(1)} + x^{(2)} + \cdots + x^{(N)}$

3. Properties of the iterative matrix

For the following properties we accept that $|\alpha - 1| > M$.

Lemma 3.1. $\|G\|_1 \leq 2|\frac{\alpha-1}{\alpha}|$.

Lemma 3.2. *If* $\alpha < -M + 1$ *then* $\|G^{-1}\|_1 > \frac{M-1}{2M}$.

Lemma 3.3. *If* $\alpha > M + 1$ *then* $\rho(G) < 2$.

Lemma 3.4. *The eigenvalues of* G *have positive real parts.*

Lemma 3.5. *If* A *happens to be a symmetric matrix with equal diagonal elements then* G *would be a positive definite matrix.*

4. Convergence discussion

For the method to be convergent one has to find suitable values of α. According to Section 2 these values belong to the following set:

$$\Lambda := \{\alpha : \ |\alpha - 1| > M \text{ and } \rho(G^{-1}) < 1\}. \tag{4.1}$$

If the set Λ is empty then the approach will fail. So the question of convergence can be rephrased as follows:
Is the set Λ *nonempty? if so how could one specify a member of it?*
We use the concept of minimal Gerschgorin disks [2] to provide a sufficient condition for convergence. According to the weighted Gerschgorin disk, we have the following inequality for any eigenvalue of G:

$$|\lambda_G - \frac{\alpha - 1}{\alpha}| \leq \frac{1}{|\alpha|} \max_{1\leq j\leq n} \sum_{i\neq j}^{n} \frac{|a_{ij}|}{|a_{ii}|} \frac{x_j}{x_i}, \tag{4.2}$$

where $x_i > 0$ for $i = 1, \ldots, n$. Now we define L to be the minimal radii for such discs, which would be the solution to the *minmax* problem:

$$L := \min_{x_1,\ldots,x_n>0} \{ \max_{1\leq j\leq n} \sum_{i\neq j}^{n} \frac{|a_{ij}|}{|a_{ii}|} \frac{x_j}{x_i} \}. \tag{4.3}$$

Theorem 4.1. *For any eigenvalue of* G *we have* $|\lambda_G| \geq \dfrac{|\alpha - 1| - L}{|\alpha|}$.

Corollary 4.2. *If* $L < 1$ *and* $\alpha < 0$ *then* $\rho(G^{-1}) < 1$, *which means that the set* Λ *is not empty.*

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NONCONVEX CASES OF JOINT HIGHER RANK NUMERICAL RANGE OF TRIPLE OF PAULI GROUP ELEMENTS

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Abstract. In [2015, Joint higher rank numerical range of Pauli group, Linear and Multilinear Algebra, 63(3), 439-454], authors showed that joint higher rank numerical range of triple of Pauli group elements is often convex, and characterized it exactly in the case that it is convex. In this talk, I completely characterize joint higher rank numerical range of triple of Pauli group elements when it is a nonconvex set.

1. Introduction

For $1 \leq k < n$ joint rank$-k$ numerical range of $A_1, \ldots, A_m \in M_n$ is defined by

$$\Lambda_k(A_1, \ldots, A_m) = \{(\lambda_1, \ldots, \lambda_m) : \exists U \in M_{k,n}, UU^* = I_k, \forall_j, UA_jU^* = \lambda_j I_k\}.$$

In the 9th workshop on "Numerical Ranges and Numerical Radii", scholars presented some open problems [2] of this set, showing their

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* Speaker.

interest in characterizing it for $m-$tuple of elements of $N-$qubit Pauli group. This group is defined by $\mathcal{G}_N = \bigcup_{s\in\{\pm 1,\pm i\}} s\mathbb{P}_N$, where

$$\mathbb{P}_N = \{P_1 \otimes \cdots \otimes P_N : N \in \mathbb{N}, P_1, \ldots, P_N \in \{I, X, Y, Z\}\}$$

and $I \in M_2$ is identity and X, Y, Z are famous Pauli matrices:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Throughout our paper, the following notations will be fixed:
Let $N \in \mathbb{N}, n = 2^N$, and I_n be Identity matrix of size $n-by-n$. Define

$$I_1 = [1],$$
$$\mathcal{P}_n = \left\{ \{P_1, \ldots, P_m\} : \forall i \exists \underset{unitary}{U} \in M_n, U^* P_i U \in \mathbb{P}_N; \forall i \neq j, P_i P_j = \pm P_j P_i \right\},$$

and consider $\{P, Q, R\} \in \mathcal{P}_n$.
Mousavi and Salemi [3] recently showed that $\Lambda_k(P, Q, R)$,where $k \leq \frac{n}{2}$, often is convex except probably in the case with the following assumptions:

$$(*) \qquad PQR = \pm iI_n, PQ = -QP, RP = -PR, RQ = -QR,$$

and derived an approximation in this case:

Theorem 1.1. [3, Theorem 3.4(i1), Corollary 3.5(I)] *Let P, Q, R satisfies in (*) and $k \leq \frac{n}{2}$. Then*

$$\begin{aligned} \Lambda_{\frac{n}{2}}(P, Q, R) &= \{(x, y, z) : x, y, z \in \mathbb{R}, x^2 + y^2 + z^2 = 1\}, \\ \operatorname{conv}(\Lambda_k(P, Q, R)) &= \{(x, y, z) : x, y, z \in \mathbb{R}, x^2 + y^2 + z^2 \leq 1\}, \end{aligned}$$

where $\operatorname{conv}(S)$ *is the convex hull of the set S.*

From other point of view, we [1] tried to improve this approximation:

Theorem 1.2. [1, Theorem 2.1(e),Theorem 3.7(e)] *Let P, Q, R satisfies in (*) and $k \leq \frac{n}{2}$. Then*

(a) *(P, Q, R) is simultaneously unitarily similar to*

$$\left(Z \otimes I_{\frac{n}{2}}, X \otimes I_{\frac{n}{2}}, \pm Y \otimes I_{\frac{n}{2}}\right).$$

(b) *If $k \leq \frac{n}{4}$, then*

$$\Lambda_k(P, Q, R) = \left\{(x, y, z) : x, y, z \in \mathbb{R}, x^2 + y^2 + z^2 \leq 1\right\}.$$

Otherwise

$$\begin{aligned} &\{(x, y, z) : x, y, z \in \mathbb{R}, x^2 + y^2 + z^2 = 1\} \subset \\ &\Lambda_k(P, Q, R) \subset \\ &\{(x, y, z) : x, y, z \in \mathbb{R}, x^2 + y^2 + z^2 \leq 1\}. \end{aligned}$$

Now, in this paper, I complete previous statements by proving the following theorem:

Theorem 1.3. *Let P, Q, R satisfies in (*) and $k \in \left[\frac{n}{4}+1, \frac{n}{2}\right]$. Then*

$$\Lambda_k(P,Q,R) = \left\{(x,y,z) : x,y,z \in \mathbb{R}, x^2+y^2+z^2 = 1\right\}.$$

2. Main sections and results

Proof of Theorem 1.3.
(ad absurdum) Let $(a,b,c) \in \Lambda_k(P,Q,R)$ and $a^2+b^2+c^2 < 1$. Using Theorem 1.2(a), there exists matrix $W \in M_{k,n}$ such that

$$\begin{aligned} W\left(I \otimes I_{\frac{n}{2}}\right) W^* &= I_k, \\ W\left(Z \otimes I_{\frac{n}{2}}\right) W^* &= aI_k, \\ W\left(X \otimes I_{\frac{n}{2}}\right) W^* &= bI_k, \\ W\left(Y \otimes I_{\frac{n}{2}}\right) W^* &= cI_k. \end{aligned}$$

Now consider $W = [A, B]$ that $A, B \in M_{k,\frac{n}{2}}$. Then

$$(**): \begin{cases} AA^* = \frac{1+a}{2} I_k, \\ BB^* = \frac{1-a}{2} I_k, \\ BA^* = \frac{b-ic}{2} I_k, \\ AB^* = \frac{b+ic}{2} I_k. \end{cases}$$

By singular value decomposition, we see that there exist unitary matrices $V_A, V_B \in M_k, W_A, W_B \in M_{\frac{n}{2}}$ such that

$$\begin{aligned} A &= V_A \left[\begin{array}{cc} \sqrt{\frac{1+a}{2}} I_k & 0_{k,\frac{n}{2}-k} \end{array}\right] W_A, \\ B &= V_B \left[\begin{array}{cc} \sqrt{\frac{1-a}{2}} I_k & 0_{k,\frac{n}{2}-k} \end{array}\right] W_B. \end{aligned}$$

Therefore (**) implies that

$$V_A \left[\begin{array}{cc} \sqrt{\frac{1+a}{2}} I_k & 0_{k,\frac{n}{2}-k} \end{array}\right] W_A W_B^* \left[\begin{array}{c} \sqrt{\frac{1-a}{2}} I_k \\ 0_{\frac{n}{2}-k,k} \end{array}\right] V_B^* = AB^* = \frac{b+ic}{2} I_k$$

i.e.

$$\left[\begin{array}{cc} \sqrt{\frac{1+a}{2}} I_k & 0_{k,\frac{n}{2}-k} \end{array}\right] W \left[\begin{array}{c} \sqrt{\frac{1-a}{2}} I_k \\ 0_{\frac{n}{2}-k,k} \end{array}\right] = \frac{b+ic}{2} I_k$$

where $W = W_A W_B^*$.

Now assume that $W = \left[\begin{array}{cc} C & D \\ E & F \end{array}\right]$, hence

$$\sqrt{\frac{1+a}{2}} \left[\begin{array}{cc} C & D \end{array}\right] \left[\begin{array}{c} \sqrt{\frac{1-a}{2}} I_k \\ 0_{\frac{n}{2}-k,k} \end{array}\right] = \frac{b+ic}{2} I_k.$$

Which implies that $C = \frac{b+ic}{\sqrt{1-a^2}} I_k$.
Furthermore $WW^* = I_{\frac{n}{2}}$ implies that

$$\begin{bmatrix} \frac{b^2+c^2}{1-a^2} I_k + DD^* & DF^* \\ FD^* & EE^* + FF^* \end{bmatrix} = \begin{bmatrix} C & D \\ E & F \end{bmatrix} \begin{bmatrix} C & E^* \\ D^* & F^* \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ 0 & I_{\frac{n}{2}-k} \end{bmatrix}.$$

Therefore $DD^* = \frac{1-a^2-b^2-c^2}{1-a^2} I_k$, which is contradiction because

$$rank\,(DD^*) \leq \min\left\{rank\,(D)\,, rank\,(D^*)\right\} \leq \frac{n}{2} - k < k.$$

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ON THE POWERS OF A SQUARE MATRIX OF ORDER TWO AND PELL'S EQUATION

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Abstract. In this note we consider an application of the powers of a square matrix of order two to solve Pell's equation $x^2-Dy^2=1$ $(x,y\in\mathbb{Z})$, where $D>1$ is an integer which is not perfect square.

1. Introduction

Pell's equation

$$x^2-Dy^2=1, \tag{1.1}$$

where $D>1$ is an integer which is not perfect square, is one of the most important quadratic Diophantine equations. While the pair $(1,0)$ is an obvious solution of (1.1), it is known (see for example [2]) that it has always a non-trivial solution, which is different from $(1,0)$. Because solutions can be arranged in sets of four by combinations of signs $\pm x$ and $\pm y$, all solutions will be known once all positive solutions have been. In this note we explain a method to generate all solutions of the Pell's equation when we have just the smallest non-trivial solution. As the working engine of the method we explain here, the following result from Matrix theory (see for example [1]) is required.

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* Speaker.

Proposition 1.1. *Assume that M is a square matrix of order two with complex elements, and let λ_1 and λ_2 be its eigenvalues.*

(1) *If $\lambda_1 \neq \lambda_2$, then for each integer $n \geqslant 1$ one has*

$$M^n = \lambda_1^n \Lambda_2 + \lambda_2^n \Lambda_1,$$

with $\Lambda_1 = \lambda'(M - \lambda_1 I_2)$, $\Lambda_2 = \lambda'(M - \lambda_2 I_2)$ and $\lambda' = \frac{1}{\lambda_1 - \lambda_2}$.

(2) *If $\lambda_1 = \lambda_2 = \lambda$, then for each integer $n \geqslant 1$ one has*

$$M^n = \lambda^n I_2 + n\lambda^{n-1}(M - \lambda I_2).$$

2. Pell's equation

D	(x_1, y_1)	D	(x_1, y_1)	D	(x_1, y_1)	D	(x_1, y_1)
2	(3,2)	6	(5,2)	10	(19,6)	13	(649, 180)
3	(2,1)	7	(8,3)	11	(10,3)	14	(15,4)
5	(9,4)	8	(3,1)	12	(7,2)	15	(4,1)

We let $\mathrm{Sol}(D)$ be the set of all positive integers to the equation (1.1), and for given positive integers x and y we define

$$M_{(x,y)} = \begin{bmatrix} x & Dy \\ y & x \end{bmatrix}.$$

Hence $(x, y) \in \mathrm{Sol}(D)$ if and only if $\det M_{(x,y)} = 1$. By simple computation we get $M_{(x,y)} M_{(u,v)} = M_{(xu+Dyv, xv+yu)}$, and this implies the following important property.

$$\text{If } (x, y), (u, v) \in \mathrm{Sol}(D) \text{ then } (xu + Dyv, xv + yu) \in \mathrm{Sol}(D). \quad (2.1)$$

By repeated utilization of (2.1) we deduce that for each integer $n \geqslant 1$ we have $(x_n, y_n) \in \mathrm{Sol}(D)$ where the sequences x_n and y_n are defined by

$$\begin{bmatrix} x_n & Dy_n \\ y_n & x_n \end{bmatrix} = M_{(x_1,y_1)}^n,$$

and (x_1, y_1) is the fundamental solution to (1.1). While by considering the identity $M_{(x_1,y_1)}^{n+1} = M_{(x_1,y_1)} M_{(x_1,y_1)}^n$ one may describe x_n and y_n by a recursive system, Proposition 1.1 allows us to obtain explicit formulae for x_n and y_n in terms of x_1, y_1 and n.

The characteristic equation of $M_{(x_1,y_1)}$ is $\lambda^2 - 2x_1\lambda + 1 = 0$, and it has the solutions $\lambda_{1,2} = x_1 \pm \sqrt{x_1^2 - 1} = x_1 \pm y_1\sqrt{D}$, since we have $x_1^2 - Dy_1^2 = 1$. Now, we apply Proposition 1.1 to compute $M_{(x_1,y_1)}^n$ and

consequently x_n and y_n. We obtain

$$x_n = \frac{1}{2}\left(\left(x_1 + y_1\sqrt{D}\right)^n + \left(x_1 - y_1\sqrt{D}\right)^n\right)$$
$$= \sum_{0\leqslant k\leqslant \frac{n}{2}} \binom{n}{2k} x_1^{n-2k} y_1^{2k} D^k, \tag{2.2}$$

and

$$y_n = \frac{1}{2\sqrt{D}}\left(\left(x_1 + y_1\sqrt{D}\right)^n - \left(x_1 - y_1\sqrt{D}\right)^n\right)$$
$$= \sum_{1\leqslant k\leqslant \frac{n+1}{2}} \binom{n}{2k-1} x_1^{n-2k+1} y_1^{2k-1} D^{k-1}, \tag{2.3}$$

for $n \geqslant 1$. This is valid for $n = 0$ admitting the trivial solution $(x_0, y_0) = (1, 0)$. The following result guarantees that the above mentioned solutions are all solutions to (1.1).

Theorem 2.1. *We have* $\mathrm{Sol}(D) = \{(x_n, y_n) \mid n \geqslant 0\}$.

Example 2.2. We find all solutions to the equation $x^2 - 7y^2 = 1$. We have $(x_1, y_1) = (8, 3)$, and hence, by using (2.2) and (2.3) we get

$$x_n = \sum_{0\leqslant k\leqslant \frac{n}{2}} \binom{n}{2k} 8^{n-2k} 3^{2k} 7^k,$$

and

$$y_n = \sum_{1\leqslant k\leqslant \frac{n+1}{2}} \binom{n}{2k-1} 8^{n-2k+1} 3^{2k-1} 7^{k-1}.$$

By computation, some first values of (x_n, y_n) are as follows.

$x_0 = 1$	$y_0 = 0$	$x_5 = 514088$	$y_5 = 194307$
$x_1 = 8$	$y_1 = 3$	$x_6 = 8193151$	$y_6 = 3096720$
$x_2 = 127$	$y_2 = 48$	$x_7 = 130576328$	$y_7 = 49353213$
$x_3 = 2024$	$y_3 = 765$	$x_8 = 2081028097$	$y_8 = 786554688$
$x_4 = 32257$	$y_4 = 12192$	$x_9 = 33165873224$	$y_9 = 12535521795$

3. Some remarks on the the fundamental solutions

$$g := x_1 + y_1\sqrt{D},$$

then $g = \lambda_1$ and $g^{-1} = x_1 - y_1\sqrt{D} = \lambda_2$, and we can rewrite (2.2) and (2.3) as $x_n = \frac{1}{2}(g^n + g^{-n})$ and $y_n = \frac{1}{2\sqrt{D}}(g^n - g^{-n})$ for each $n \geqslant 0$.

Since $x_n^2 - Dy_n^2 = 1$ we obtain $x_n > \sqrt{D}y_n > y_n$ and $g > g^{-1} > 0$. Hence, as $n \to \infty$ we imply that

$$x_n \sim \frac{1}{2}g^n, \quad \text{and} \quad y_n \sim \frac{1}{2\sqrt{D}}g^n.$$

Consequently $\frac{x_n}{y_n} \sim \sqrt{D}$. This observation leads us to propose the question of finding the asymptotic behaviour of the square-mean value

$$\frac{1}{X}\sum_{1\leqslant n\leqslant X}\left(\frac{x_n}{y_n} - \sqrt{D}\right)^2 \quad \text{as } X \to \infty.$$

We end this note by proposing another question, asking the asymptotic size of the average of the fundamental solutions x_1 and y_1. Regarding to this question, it is known [4] that

$$\frac{g}{2} < x_1 < \frac{g}{2} + \frac{1}{4\sqrt{D}}, \quad \text{and} \quad \frac{g}{2\sqrt{D}} - \frac{1}{4D} < y_1 < \frac{g}{2\sqrt{D}}.$$

Moreover, it is known [3, 4] that $\log(2\sqrt{D}) < \log g < \sqrt{D}\,(2 + \log(4D))$. Therefore, we obtain

$$\sqrt{D} < x_1 < \frac{(4D)^{\sqrt{D}}\mathrm{e}^{2\sqrt{D}}}{2} + \frac{1}{4\sqrt{D}}, \quad \text{and} \quad 1 \leqslant y_1 < \frac{(4D)^{\sqrt{D}}\mathrm{e}^{2\sqrt{D}}}{2\sqrt{D}}.$$

Now, we let $x_1 = x_1(D)$ and $y_1 = y_1(D)$ be the fundamental solution to (1.1) and we ask about the asymptotic behaviour of the averages

$$\frac{1}{X}\sum_{\substack{D\leqslant X\\ \sqrt{D}\notin\mathbb{N}}} x_1(D) \quad \text{and} \quad \frac{1}{X}\sum_{\substack{D\leqslant X\\ \sqrt{D}\notin\mathbb{N}}} y_1(D) \quad \text{as } X \to \infty.$$

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PRESERVERS OF 2-MAJORIZATION ON $\mathbb{R}^2$

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Abstract. Majorization is a basic concept in matrix theory that has found applications in numerous settings over the past century. In this article we introduce the definition of 2-majorization. We find all linear transformations from, $\mathbb{R}^2$ to $\mathbb{R}^2$ that preserve 2-majorization.

1. Introduction

Majorization is one of the interesting concepts in linear algebra and there are important researches on linear transformations that preserve majorization. Majorization is a powerful, easy-to-use and flexible mathematical tool which can be applied to a wide variety of problems in quantum mechanics. An important problem in quantum information theory is the mathematical characterization of the phenomenon of quantum catalysis: when can the surrounding entanglement be used to perform transformations of a jointly held quantum state under LOCC (local operations and classical communication)? Mathematically, the answers to this question is given by trumping and specially power majorization [1, 2].

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Here we consider $\mathbb{R}^n$ as the vector space of all real $1 \times n$ vectors. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and let $x^{\downarrow} = (x_1^{\downarrow}, x_2^{\downarrow}, \ldots, x_n^{\downarrow}) \in \mathbb{R}^n$ be the vector consisting elements of x reordered such that

$$x_1^{\downarrow} \geq x_2^{\downarrow} \geq ... \geq x_n^{\downarrow}.$$

Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. We say x majorized by y, $x \prec y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}, \quad 1 \leq k \leq n$$

with the equality when $k = d$.[**?**]

Let x and y be two quantum state with Schmidt decomposition $x = \sum_{i=1}^{n} \sqrt{x_i} a_i \otimes b_i$ and $y = \sum_{i=1}^{n} \sqrt{y_i} c_i \otimes d_i$. Nielsen in [3] proved that we can LOCC-transform x into y if and only if $x \prec y$. There are states x can not be LOCC-transformed into y (i.e. $x \not\prec y$), but with the help of a catalyst state $z \in \mathbb{R}^n$, the transformation $x \otimes z \to y \otimes z$ becomes possible (i.e. $x \otimes z \prec y \otimes z$). Mathematically, if $x \not\prec y$, but $x \otimes z \prec y \otimes z$, for some $z \in \mathbb{R}^n$, then we say that x is trumped by y and write $x \prec_T y$. In [2, Theorem 2.1.], the authors explore the connections between trumping and power majorization.

Definition 1.1. Let x and y be vectors of non-negative components. We say x is power majorized by y, $x \prec_p y$, if $x_1^p + x_2^p + ... + x_n^p \leq y_1^p + y_2^p + ... + y_n^p$ for all $p \geq 1$, $p \leq 0$ and the inequality switches when $0 \leq p \leq 1$.

2. 2-majorizations and their linear preservers on $\mathbb{R}^2$

In this section we introduce the concept of r-majorization for a constant $r \in \mathbb{R}$. Actually we define a similar definition to power majorization and instead having all $p \in \mathbb{R}$ just using a constant real number r and adding another condition, precisely as the following:

Definition 2.1. Let x and y be vectors of non-negative components and $r \in \mathbb{R}$. We say x is r-majorized by y, denoted by $x \prec_r y$ in the following way:

a) If $r \geq 1$, $r \leq 0$, $\sum_{i=1}^{k} x_i^r \leq \sum_{i=1}^{k} y_i^r$, for $1 \leq k \leq n$

b) If $0 \leq r \leq 1$, $\sum_{i=1}^{k} x_i^r \leq \sum_{i=1}^{k} y_i^r$, for $1 \leq k \leq n$.

We imidiately have the following lemma.

Lemma 2.2. *If $x \prec_r y$ for every $r \in \mathbb{R}$, then $x \prec_P y$.*

We may extend the definition of 2-majorization to real vectors.

Definition 2.3. Let x and y be real vectors. We say x is 2-majorized by y, denoted by $x \prec_2 y$ if $\sum_{i=1}^{k} x_i^r \leq \sum_{i=1}^{k} y_i^r$, for $1 \leq k \leq n$.

Now we have the following Lemma.

Lemma 2.4. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear preserver of $\prec_2$, with the matrix presentation $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $(b, d) \prec_2 (a, c)$.*

Proof. We know $(0, \sqrt{x^2 + y^2}) \prec_2 (x, y)$, for every $x, y \in \mathbb{R}$ that emplies $(b\sqrt{x^2 + y^2}, d\sqrt{x^2 + y^2}) \prec_2 (ax + by, cx + dy)$. Hence we have

$$b^2(x^2 + y^2) \leq (ax + by)^2$$

and

$$b^2(x^2 + y^2) + d^2(x^2 + y^2) \leq (ax + by)^2 + (cx + dy)^2$$

which are equivalent to

$$b^2x^2 \leq a^2x^2 + 2abxy, \tag{2.1}$$

and

$$(b^2 + d^2)x^2 \leq (a^2 + c^2)x^2 + (2ab + 2cd)xy. \tag{2.2}$$

Since x and y are arbitrary we may choose $y = 0$, hence we have the relations $b^2x^2 \leq a^2x^2$ and $(b^2 + d^2)x^2 \leq (a^2 + c^2)x^2$ which emply $b^2 \leq a^2$ and $b^2 + d^2 \leq a^2 + c^2$. □

Theorem 2.5. *If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear preserver of $\prec_2$, then $T = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $d, a \in \mathbb{R}$ and $d^2 \leq a^2$ or $T = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$, $a, c \in \mathbb{R}$.*

Proof. Let the matrix presentation of T be

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Hence in (2.1) and (2.2), we may choose $x = ky$, where $k \in \mathbb{R}$, consquently we have

$$b^2(ky)^2 \leq a^2(ky)^2 + 2ab(ky)y$$

and

$$(b^2 + d^2)(ky)^2 \leq (a^2 + c^2)(ky)^2 + (2ab + 2cd)(ky)y$$

that emply

$$b^2k^2 \leq a^2k^2 + 2abk, \tag{2.3}$$

and

$$(b^2 + d^2)k^2 \leq (a^2 + c^2)k^2 + (2ab + 2cd)k. \tag{2.4}$$

Also we may choose $y = kx$, with $k \in \mathbb{R}$, hence in the similar way we have

$$b^2 \leq a^2 + 2abk, \tag{2.5}$$

and

$$(b^2 + d^2)k^2 \leq a^2 + c^2 + (2ab + 2cd)k. \tag{2.6}$$

Now in (2.3) and (2.4) suppose $k \longrightarrow +\infty$, we have $ab \geq 0$ and $ab+cd \geq 0$. Similarly if in (2.5) and (2.6) $k \longrightarrow -\infty$, then $ab \leq 0$ and $ab+cd \leq 0$. Hence $ab = 0$ and $ab + cd = 0$ which emplies $cd = 0$

By the previous Lemma we have $b^2 \leq a^2$, hence $ab = 0$ empies $b = 0$. Now we have two following cases:

Case1: If $c = 0$, then $T = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, and by the privious Lemma $d^2 \leq a^2$.

Case2: If $d = 0$, then $T = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$, $a, c \in \mathbb{R}$. □

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SOME RESULTS ON THE WEIGHTED DRAZIN INVERSE OF A MODIFIED MATRIX WITH NEW CONDITIONS

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Abstract. In this article, we present conditions under which the weighted Drazin inverse of a modified matrix $A - CWD_{d,w}WB$ can be expressed in terms of the weighted Drazin inverse of A and the generalized Schur complement $D - BWA_{d,w}WC$. The results extend the earlier works about the Drazin inverse.

1. Introduction

The Drazin inverse and the weighted Drazin inverse are very useful because of their various applications which can be found in [1,5]. Let $\mathbb{C}^{n\times n}$ denote the set of $n \times n$ complex matrices. For $A \in \mathbb{C}^{n\times n}$, the smallest nonnegative integer k such that $rank(A^{k+1}) = rank(A^k)$ is called the index of A, and denoted by $k = ind(A)$.

Definition 1.1. Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$, and $X \in \mathbb{C}^{n\times n}$ be a matrix such that

$$A^{k+1}X = A^k, \qquad XAX = X, \qquad AX = XA,$$

then X is called the Drazin inverse of A, and is denoted $X = A^d$.

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Definition 1.2. Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ with $ind(AW) = k$, and $X \in \mathbb{C}^{m\times n}$ be a matrix such that

$$(AW)^{k+1}XW = (AW)^k, \qquad XWAWX = X, \qquad AWX = XWA,$$

then X is called the W-weighted Drazin inverse of A, and is denoted by $X = A_{d,w}$. In particular, when A is an square matrix and $W = I$, where I is the identity matrix with proper size, $A_{d,w} = A^d$.

Mosic [4] and Shakoor, Yang, Ali [2,3] give representations for the Drazin inverses of a modified matrix $A - CD^dB$ under new conditions to generalize some results in the literature. These results extends the formula of Sherman-Morrison-Woodbury type

$$(A - CD^{-1}B)^{-1} = A^{-1} + A^{-1}C(D - BA^{-1}C)^{-1}BA^{-1},$$

where the matrices A, D and the Schur complement $D - BA^{-1}C$ are invertible.

Throughout this paper, let $A, B, C, D \in \mathbb{C}^{n\times n}$ and $W \in \mathbb{C}^{n\times n}$, $(WA)^\pi = I - WAWA_{d,w}$ and $(AW)^\pi = I - AWA_{d,w}W$. The generalized Schur complements will be denoted by $M = A - CWD_{d,w}WB$, and $Z = D - BWA_{d,w}WC$.

2. The weighted Drazin inverse of a modified matrix

Some conclusions in [2] are obtained directly from the results. Let $A, B, C, D \in \mathbb{C}^{n\times n,}$ and $W \in \mathbb{C}^{n\times n,}$. Throughout this section , we use the following notations:

$$M = A - CWD_{d,w}WB, \qquad Z = D - BWA_{d,w}WC$$

and

$$K = A_{d,w}WC, \qquad H = BWA_{d,w}W,$$

First, we present the following theorem.

Theorem 2.1. *Let A, B, C, D and W be complex matrices, where $ind(AW) = k$. If $(AW)^\pi C = C(WD)^\pi$, $CWD_{d,w}(WZ)^\pi WB = 0$, $CWZ_{d,w}W(DW)^\pi BW = 0$, and $C(WZ)^\pi WD_{d,w}WB = 0$, then*

$$(MW)^d = X - \sum_{i=0}^{k-1} X^{i+1}KWZ_{d,w}WBW(AW)^i(AW)^\pi$$

where $X = A_{d,w}W + KWZ_{d,w}WH$, and $ind(MW) \le ind(AW)$.

Corollary 2.2. *Let A, B, C, and D be complex matrices, where $ind(AW) = k$. If $(AW)^{\pi}C = C(WD)^{\pi}$, $(WZ)^{\pi}WB = 0$, $(DW)^{\pi}BW = 0$, and $C(WZ)^{\pi} = 0$, then*

$$(MW)^{d} = X - \sum_{i=0}^{k-1} X^{i+1} KWZ_{d,w} WBW(AW)^{i}(AW)^{\pi}$$

where $X = A_{d,w}W + KWZ_{d,w}WH$, and $ind(MW) \leq ind(AW)$.

In the same way, we give a new theorem.

Theorem 2.3. *Let A, B, C, and D be complex matrices, where $ind(AW) = k$. If $B(WA)^{\pi} = (DW)^{\pi}B$, $CWD_{d,w}(WZ)^{\pi}WB = 0$, $C(WZ)^{\pi}WD_{d,w}WB = 0$ and $C(WD)^{\pi}WZ_{d,w}WBW = 0$, then*

$$\begin{aligned}(MW)^{d} &= A_{d,w}W + KWZ_{d,w}WH \\ &\quad - \sum_{i=0}^{k-1}(AW)^{\pi}(AW)^{i}CWZ_{d,w}WH(A_{d,w}W + KWZ_{d,w}WH)^{i+1}\end{aligned}$$

and $ind(MW) \leq ind(AW)$.

Corollary 2.4. *Let A, B, C, and D be complex matrices, where $ind(AW) = k$. If $B(WA)^{\pi} = (DW)^{\pi}B$, $C(WZ)^{\pi} = 0$, $(WZ)^{\pi}WB = 0$, $C(WD)^{\pi} = 0$ then*

$$\begin{aligned}(MW)^{d} &= A_{d,w}W + KWZ_{d,w}WH \\ &\quad - \sum_{i=0}^{k-1}(AW)^{\pi}(AW)^{i}CWZ_{d,w}WH(A_{d,w}W + KWZ_{d,w}WH)^{i+1}\end{aligned}$$

and $ind(MW) \leq ind(AW)$.

Theorem 2.5. *Let A, B, C, and D be complex matrices, where $ind(AW) = k$. If $Z = 0$, $(AW)^{\pi}C = C(WD)^{\pi}$, $(DW)^{\pi}BW = 0$, and $WDW(\Gamma W)^{\pi} = 0$ then*

$$\begin{aligned}(MW)^{d} &= (I - KW\Gamma_{d,w}WH)(AW)^{d}(I - KW\Gamma_{d,w}WH) + \\ &\quad \sum_{i=0}^{k-1}[(I - KW\Gamma_{d,w}WH)(AW)^{d}]^{i+2}KW\Gamma_{d,w}WBW(AW)^{i}(AW)^{\pi}\end{aligned}$$

and $ind(MW) \leq ind(AW)$.

Theorem 2.6. *Let A, B, C, and D be complex matrices, where $ind(AW) = k$. If $Z = 0$, $B(WA)^{\pi} = (DW)^{\pi}B$, $C(WD)^{\pi} = 0$, and $(\Gamma W)^{\pi}WDW = 0$ then*

$$(MW)^d = (I - KW\Gamma_{d,w}WH)(AW)^d(I - KW\Gamma_{d,w}WH)+$$
$$\sum_{i=0}^{k-1}(AW)^{\pi}(AW)^i CW\Gamma_{d,w}WH[(AW)^d(I - KW\Gamma_{d,w}WH)]^{i+2}$$

and $ind(MW) \leq ind(AW)$.

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DEFLATED AND AUGMENTED CONJUGATE GRADIENT METHOD FOR THE MATRIX EQUATION $AX = B$

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Abstract. Deflation and augmentation techniques are used to accelerate the convergence of Krylov subspace methods. In this paper, we consider deflated and augmented conjugate gradient (CG) algorithm for solving the matrix equation $AX = B$ in the case where A is symmetric positive definite.

1. Introduction

We consider the matrix equation

$$AX = B, \tag{1.1}$$

where A is an $m \times m$ symmetric positive definite matrix, B and X are $m \times s$ rectangular matrices with $s \ll m$.

For solving the matrix equation (1.1), the block CG (Bl-CG) algorithm [1] and the global CG (Gl-CG) algorithm [2] were presented. The global methods are effective as compared to block methods.

Deflation and augmentation are two techniques for quickening the convergence of Krylov subspace methods. In augmentation approaches,

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* Speaker.

the search space of the method is made larger by a appropriately selected subspace. Since eigenvalues of the operator close to zero incline to reduce speed of convergence of the Krylov subspace methods, these eigenvalues are essentially deflated from the spectrum of the matrix, with adding the corresponding eigenvectors to the search space. In deflation, for removing components that decelerate convergence, the linear system $Ax = b$ is multiplied with a properly selected projection.

Saad et al. [3] described a deflated version of the CG algorithm. Also, Viuk et al. [4] applied deflated CG with Incomplete Cholesky preconditioning for the solution of a class of layered problems with extreme contrasts in the coefficients.

In this paper, we consider augmented and deflated CG method for solving the matrix equation (1.1).

2. Deflated and augmented CG method

In this section, we consider deflation and augmentation approaches to the CG method for the solution of matrix equation (1.1). Given an initial guess $X_0 \in \mathbb{R}^{m\times s}$ and an n-dimensional subspace S_n of $\mathbb{R}^{m\times s}$. We consider an approximation X_n to the solution X of the form

$$X_n \in X_0 + S_n,$$

so that the corresponding residual satisfies

$$R_n := B - AX_n \perp S_n.$$

Now, we suppose that the search space S_n is augmented with the space $\mathcal{U}$:

$$S_n = \widetilde{K}_n + \mathcal{U},$$

here, $\mathcal{U}$ is called the augmentation space and the subspace $\widetilde{K}_n$ has dimension n. Let matrix $U \in \mathbb{R}^{m\times k}$ such that

$$E := U^T AU \in \mathbb{R}^{k\times k}$$

is nonsingular. We define the $m \times m$-matrices

$$\begin{aligned} Q &:= UE^{-1}U^T, \\ P &:= I - AQ, \\ \widetilde{P} &:= I - QA. \end{aligned}$$

In deflation and augmentation approaches, we first solve deflated matrix equation $\widehat{A}X = \widehat{B}$ where $\widehat{A} = PA$ and $\widehat{B} = PB$. Then substituting the solution $\widehat{X}_n$ of this matrix equation in the relation $X_n = \widetilde{P}\widehat{X}_n + QB$, we obtain the solution X_n of the matrix equation (1.1).

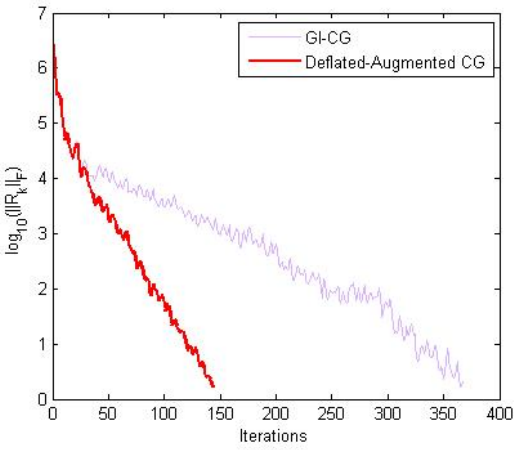

FIGURE 1. The convergence history for NOS5.

3. Numerical example

In this section, we present a numerical example to illustrate the effectiveness of deflated and augmented CG method to solve large and sparse matrix equation (1.1). We consider matrix NOS5 from Matrix Market and the right-hand side B is chosen such that the exact solution X is a matrix of order $m \times s$ whose ith column has all entries equal to one except the ith entry which is zero. The stopping criterion is

$$\frac{\|B - AX\|}{\|B\|} < 10^{-7}.$$

Figure 1 reports on results obtained from the Gl-CG and the deflated and augmented CG method for the matrix NOS5. As seen from this figure, the deflated and augmented CG method converges faster than the Gl-CG method.

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APPLICATION OF GENERALIZED ARNOLDI METHOD IN COMPUTING PAGERANK

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Abstract. In this paper, we study the role of weight matrix G in the G-norm of a generalized and refined Arnoldi method, from the literature, for computing PageRank. Two new ways for defining matrix G is proposed and the propoerties of the two new variants, made by them, are studied. Numerical results show the performance improvement of the new variants.

1. Introduction

PageRank is related to a link analysis algorithm used by the Google internet search engine that assigns a numerical weighting to each element of a hyperlinked set of documents in the World Wide Web [3]. Here the weighting obtained by PageRank provides the relative importance of each document. The PageRank weighting are the entries of the dominant eigenvector of the modified adjacency matrix

$$A = \alpha P + (1 - \alpha)E,$$

where P is a column-stochastic matrix (which means that the danging nodes are already replaced by columns with $1/n$), α is a damping factor, and E is a rank-one matrix [2]. Usually, the matrix A involved is extremely large, so iterative methods based on matrix-vector products have attracted the most attentions during the last ten years. The

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largest eigenvalue of PageRank matrix A is known to be 1. This fact have been used in [1] to propose an Arnoldi-type method combined with SVD by considering the known largest eigenvalue as a shift so that the computation of the largest Ritz value is avoided. Recently, a generalized and refined Arnoldi method based on the general inner product was introduced for computing the PageRank [4]. This method enjoys weighted Arnoldi process combined with the advantage of the Arnoldi-type algorithm [4].

In this paper, we investigate the role of weight matrix in the method introduced in [4]. We propose two different variants of the method and compare them with the one introduced in [4].

2. Generalized Arnoldi method

Let $G \in \mathbb{R}^{n\times n}$ be a symmetric positive definite (SPD) matrix and $x, y \in \mathbb{R}^n$ be two vectors, then G-inner product is defined as $(x, y) = x^T G y = \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}x_i y_j$, where g_{ij} is the ith row and jth column element of G. This inner product is well defined if and only if the matrix G is SPD. For a diagonal matrix $G = \text{diag}\{g_1, g_2, \ldots, d_n\}$ with $g_i > 0,\ i = 1, 2, \ldots, n$, the norm associated with this inner product (called G-norm) is defined by $\|x\|_G = \sqrt{(x, x)_G} = \sqrt{x^T G x} = \sqrt{\sum_{i=1}^{n} g_i x_i^2}$.

Given a general non-Hermitian matrices $A \in \mathbb{R}^{n\times n}$ and an initial vector $q_0 \in \mathbb{R}^n$, the generalized Arnoldi method (GArnoldi) uses a G-norm and constructs a G-orthogonal basis of

$$\mathcal{K}_m(A, q_0) = \{q_1, Aq_1, \ldots, A^{m-1}q_1\}$$

starting with a vector q_0. In fact, for $Q_m = [q_1, q_2, \ldots, q_m] \in \mathbb{R}^{n\times m}$, $Q_m^T G Q_m = I$ [4]. Let $H_m \in \mathbb{R}^{m\times m}$ be Hessenberg matrix produced using GArnoldi method. Then, $Q_m^T G A Q_m = H_m$, i. e. the eigenvalues of H are the approximation to the eigenvalues of GA. A detailed presentation of GArnoldi can be found in [4].

3. Adaptively Accelerated Arnoldi method for computing PageRank

Considering the fact that the largest eigenvalue of PageRank matrix A is 1, special modifications can be made on GArnoldi algorithm for computing PageRank. One idea is a direct application of the Arnoldi-type method, see [1]. A recent extension of this idea is shortly presented in Algorithm 1 which, in each iteration, performs a practical choice of G from the viewpoint of solving weighted least squares problems. It adaptively assigns, in every outer iteration, the suitable weights in

the residual of the approximate PageRank vector for improving the convergence performance [4].

Algorithm 1 **Adaptively Accelerated Arnoldi method for computing PageRank**

(1) Choose q_0
(2) Set $G = I$ and $\|q_0\|_G = 1$
(3) For l=1,2, ..., until convergence, Do
(4) Compute $[Q_m, H_{m+1,m}] = \text{GArnoldi}(A, G, q_0, m)$
(5) Compute SVD $H_{m+1,m} - [I; 0]^T = U\Sigma V^T$
(6) Compute $q_0 = Q_m v_m$
(7) Compute $r = Aq_0 - q_0$
(8) If $\|r\|_1 < TOL$ then stop; EndIF
(9) Set G =diag $\{\frac{|r|}{\|r\|_1}\}$
(10) EndDo

In each iteration, for the q_0 computed in step 6 of Algorithm 1, a new weight matrix G (here a diagonal matrix G =diag$(g_1, g_2, \ldots, g_n)$ is considered) such that it produces the minimum possible value for $\|Aq_0 - q_0\|_G = \|r\|_G = \sqrt{\sum_{i=1}^{n} g_i r_i^2}$.

The suggested and used G in Algorithm 1 is

$$G_0 = \text{diag}\{|r|/\|r\|_1\}. \tag{3.1}$$

Here, we propose and apply the following two weight matrices

$$G_1 = \text{diag}\{|r|/\|r\|_G\}, \tag{3.2}$$

and

$$G_2 = \text{diag}\{|r|/\sqrt{n}\|r\|_G\}, \tag{3.3}$$

as weight matrix G in step 9 of Algorithm 1. We denote the variants of Algorithm 1, made by the matrices in (3.1), (3.2), and (3.3) by G0Arnoldi, G1Arnoldi, and G2Arnoldi respectivley.

4. Numerical experiments

We compare the variant G0Arnoldi proposed in [4] with G1Arnoldi, and G2Arnoldi. The three variants have the same intermediate memory requirement and computation cost at each iteration except for step 9 of the algorithm.

The test matrices are produced using `surfer` function in MATLAB. We made several test matrices from different Websites, but we only report on an example about University of Tehran. We use the statement $[U, G]$ = `surfer('http://www.ut.ac.ir',1000)` to access the home page of University of Tehran and generate a 1000-by-1000 test matrix. We use Tol = 1e-12 as tolerance of stopping criterion.

Table 1 shows CPU time versus dampimg factor α for three different values of restart number m. Figure 1, illustrates the norm of residual versus the number of iteratons, when $\alpha = 0.95$, and $m = 7$.

TABLE 1. CPU time versus dampimg factor. α

No. of restarts	α	G0Arnoldi	G1Arnoldi	G2Arnoldi
5	0.85	0.28	0.14	0.15
	0.90	0.22	0.17	0.18
	0.95	0.25	0.22	0.23
6	0.85	0.15	0.14	0.15
	0.90	0.21	0.15	0.18
	0.95	0.34	0.23	0.24
7	0.85	1.58	0.18	0.16
	0.90	0.84	0.18	0.19
	0.95	0.31	0.23	0.22

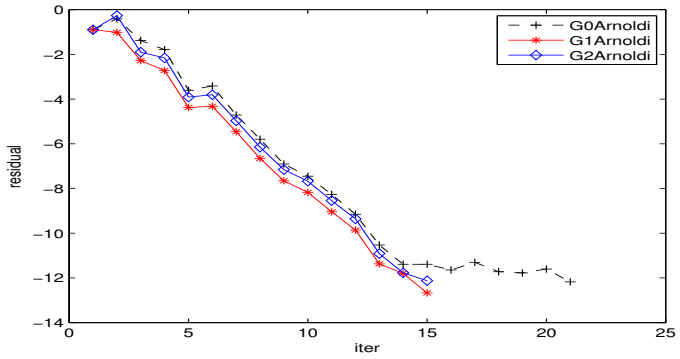

FIGURE 1
Logarithmic scale of residual norm versus the number of iteration, when $\alpha = 0.95$, and $m = 7$.

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THE RELATIONS BETWEEN THE COMPACT OPERATORS AND SOME TOPOLOGICAL CONCEPTS

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Abstract. Compact operators have many connections with other mathematical concepts. In this paper, the relationship between these operators and some topological properties such as compactness and ameanability is discussed.

1. Introduction

Let G be a locally compact group with a fixed left Haar measure λ and ω be a weight on G (which is a continuous function $\omega : G \to (0, +\infty)$ with $\omega(xy) \leq \omega(x) \leq \omega(y)$ for all $x, y \in G$). There is an interesting relationship between the group G and the group algebra $L^1(G)$, weighted group algebra $L^1(G, \omega)$ and the measure algebra $M(G)$. For example, G is ameanable if and only if $L^1(G)$ is ameanable. In this paper, the relationship between operators on these algebras and their dual spaces and topological properties of G is considered.

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* Speaker.

Recall that for a Banach algebra A, a bounded linear operator $T : A \to A$ is called a left (right) multiplier,if $T(ab) = T(a)b$ $(T(ab) = aT(b))$, for all $a, b \in A$. An element $a \in A$ is called (right) compact if the mapping $b \to ba$ is a compact operator on A.

2. Main results

In the first we describe some known results. In 1964 S.Sakai proved that for a locally compact non-compact group G the constant $g = 0$ is the only compact element of $L^1(G)$. In the following in 1967 D. Aceman proved that for a compact group G each $g \in L^1(G)$ is a compact operator (i.e. the operator $f \to f * g$ is a compact operator on $L^1(G)$). In the next compact element in weighted group algebra $L^1(G, \omega)$ characterized by F. Ghahramani ([1]):

Theorem 1 [1]: An elemet $g \in L^1(G, \omega)$ is compact if and only if the function F_g defined on G by

$$F_g = \int (\omega(st)/\omega(s))|g(t)|d\lambda(t),$$

vanishes at infinity.

Now we focuse on compact operators on second dual algebras.

Theorem 2 [2]: Suppose that G is a locally compact group. Then the following are equivalent:
(a) G is ameanable;
(b) There is a compact right multiplier $T : L^1(G)^{**} \to L^1(G)^{**}$;
(c) There is a weakly compact right multiplier $T : L^1(G)^{**} \to L^1(G)^{**}$.

Theorem 3 [5]: Suppose that G is a locally compact group. Then the following statements are equivalent:
(a) G is compact;
(b) $M(G)^{**}$ has an element m that the map $n \to mn : M(G)^{**} \to M(G)^{**}$ is a compact operator;
(c) $M(G)^{**}$ has an element m that the map $n \to mn : M(G)^{**} \to M(G)^{**}$ is a weakly compact operator.

Theorem 4 [3]: Suppose that G is a locally compact group. Then the following are equivalent:
(a) G is compact;
(b) $L^1(G)^{**}$ has a compact left multiplier T with $< T(n), 1 > \neq 0$ for

some $n \in L^1(G)^{**}$;
(c) $L^1(G)^{**}$ has a weakly compact left multiplier T with $< T(n), 1 > \neq 0$ for some $n \in L^1(G)^{**}$.

However, the natural question is wether these theorems hold for the weighted case. We now answer this question for theorems 2, 3 and 4. For a weight ω on G, the weight ω^* defined on G by,

$$\omega^*(x) = \omega(x)\omega(x^{-1}), (x \in G).$$

Note that by [4] $L^1(G, \omega)$ is ameanable if and only if G is ameanable and ω^* is bounded.
In the case $G = (Z, +)$ let $\omega_\alpha(n) = (1 + |n|)^\alpha, (n \in Z, \alpha > 0)$. Then for each $\alpha > 0$, ω_α is a weight on Z with disceret topology.
Next proposition express that theorem 3 is not valid by replace $L^1(G)$ by $L^1(G, \omega)$.

Proposition 4: There is a compact multiplier $T : l^1(Z, \omega_\alpha)^{**} \to L^1(Z, \omega_\alpha)^{**}$, such that for some $n \in L^1(Z, \omega_\alpha)^{**}, < T(n), 1 > \neq 0$, (but Z is not compact).
Proof: Let . be the first Arens Product on $l^1(Z, \omega_\alpha)^{**}$ and $*$ be the be the first Arens Product on $l^1(Z, \omega_\alpha)^{***}$ where $l^1(Z, \omega_\alpha)^\infty$ is considered with pointwise multiplication. Define the maps $\phi : l^1(Z, \omega_\alpha) \to l^1(Z, \omega_\alpha)^*$ and and $\psi : l^1(Z, \omega_\alpha)^{***} \to l^1(Z, \omega_\alpha)^{**}$ by $\phi(f) = 1.f$ and $\psi(F) = F * m$ where m is a topologically left invariant mean on $l^\infty(Z, \omega_\alpha)$. Now let $T = \psi o \phi^{**}$ then for $q, p \in l^1(Z, \omega_\alpha)^{**}$ and $f \in l^\infty(Z, \omega_\alpha)$ we have,

$$\begin{aligned}
< T(q.p), f > &= < \psi o \phi^{**}(q.p), f > = < \phi^{**}(q.p) * m, f > \\
&= < \phi^{**}(q.p), m * f > = < 1, q.p >< 1, m * f > \\
&= < 1, q >< 1, p >< m, f > = < 1, q >< p.m, f > \\
&= < 1, q >< m.p, f > = < 1, q >< m, p.f > \\
&= < 1, q >< m, 1 * p.f > = < 1, q >< 1, m * p.f > \\
&= < \phi^{**}(q), m * p.f > = < \phi^{**}(q) * m, p.f > \\
&= < \psi o \phi^{**}(q), p.f > = < T(q), p.f > \\
&= < T(q).p, f > .
\end{aligned}$$

Hence T is a left multiplier on $l^1(Z, \omega_\alpha)^{**}$. In addition T is weakly compact because by Arens regularity of $l^1(Z, \omega_\alpha)$ ϕ is weakly compact. As well we have

$$\begin{aligned} < T(m), 1 > &= < \psi o \phi^{**}(m), 1 > \\ &= < \phi^{**}(m) * m, 1 > \\ &= < \phi^{**}(m), m * 1 > \\ &= < m, 1 >^2 \neq 0. \end{aligned}$$

Proposition 5: Suppose that G is a locally compact group. Then the following statement are hold.
(a) If $L^1(G,\omega)$ is amenable, then there is a compact right multiplier $T : L^1(G,\omega)^{**} \to L^1(G,\omega)^{**}$;
(b) The converse of (a) not hold in general.
Proof: (a) Let the weighted group $L^1(G,\omega)$ be amenable. Then by [4] G is amenable and therefore by theorem 2 there is a compact right multiplier $T : L^1(G)^{**} \to L^1(G)^{**}$. On the other hand by [6], there is a (algebra) isomorphism $S : L^1(G) \to L^1(G,\omega)$. Now the operator $S^{**-1} o T o S^{**}$, is a compact right multiplier on $L^1(G,\omega)^{**}$.
(b) By proposition 4 there is a compact right multiplier on $L^1(Z,\omega_\alpha)^{**}$. But $L^1(Z,\omega_\alpha)$ is not amenable since ω_α^* is not bounded.

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Please cite your relevant papers but at most total 5 papers/books.

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Posters

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THE MAX PLUS ALGEBRA ON HIGHER ORDER TENSOR

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Abstract. In this paper we generalize the max plus algebra system of real matrices to the class of real tensors and derive its fundamental properties.

1. Introduction

The main aim of this paper is to study the max plus algebra system for a class of higher order tensors. A tensor can be regarded as a higher order generalization of a matrix, which takes the form

$$\mathbb{A} = (a_{i_1,\dots,i_m}), \quad a_{i_1,\dots,i_m} \in \Re, \quad 1 \leq i_1, \dots, i_m \leq n,$$

where $\Re$ is a real field. Such a multi-array $\mathbb{A}$ is said to be an mth order n-dimensional square real tensor with n^m entries $a_{i_1,\dots,i_m}$. In this regard, a vector is a first order tensor and a matrix is a second order tensor. Tensors of order more than two are called higher order tensors. The operations $\oplus$ and $\otimes$ are defined by $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$. $\oplus$ and $\otimes$ are extended to matrices and vectors over $\Re_{\max} = \Re \cup \{-\infty\}$ as in the conventional linear algebra: If A, B, C are matrices of compatible sizes with entries from $\Re_{max}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} + b_{ij}$ for

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all $i, j \in [n]$. Also we write $C = (c_{il}) = A \otimes B$, if $c_{il} = \max_k (a_{ik} + b_{kl})$, for all $i, l \in [n]$ and $c \otimes A = A \otimes c = (c \otimes a_{ij})$ for $c \in \bar{\Re}_{\max}$. Refer [2]. Now the question arises Is it possible to extend the max plus algebra to higher orders tensors? In this paper we show the answer is affirmative.

2. Main sections and results

In this section we define the max plus algebra system on tensors.

Definition 2.1. The max plus algebraic addition ($\oplus$) and multiplication ($\otimes$) are defined as follows:
(i). Suppose that $\mathbb{A}, \mathbb{B}$ are mth order n dimensional tensors with entries from $\Re_{\max}$ then we have $\mathbb{A} \oplus \mathbb{B}$ is mth order n dimensional tensor and

$$(\mathbb{A} \oplus \mathbb{B})_{i_1 \ldots i_m} = a_{i_1 \ldots i_m} \oplus b_{i_1 \ldots i_m} = \max (a_{i_1 \ldots i_m}, b_{i_1 \ldots i_m}) . \tag{2.1}$$

(ii). Suppose that $\mathbb{A} \in \Re_{max}^{[m,n]}$ and $\mathbb{B} \in \Re_{max}^{[k,n]}$ where $m \geq 2 , k \geq 1$ then we have $\mathbb{A} \otimes \mathbb{B} \in \Re_{max}^{[(m-1)(k-1)+1,n]}$ and

$$\begin{aligned} (\mathbb{A} \otimes \mathbb{B})_{i\alpha_1 \ldots \alpha_{m-1}} &= \bigoplus_{i_2, \ldots, i_m = 1}^{n} a_{i i_2 \ldots i_m} \otimes b_{i_2 \alpha_1} \otimes \ldots \otimes b_{i_m \otimes \alpha_{m-1}} \\ &= \max_{1 \leq i_2, \ldots, i_m \leq n} \left\{ a_{i i_2 \ldots i_m} + b_{i_2 \alpha_1} + \ldots + b_{i_m \alpha_{m-1}} \right\} , \end{aligned} \tag{2.2}$$

where $i \in \{1, \ldots, n\}$, $\alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1}$. In particular for $x \in \Re_{max}^{n}$ we have

$$(\mathbb{A} \otimes x)_i = \max_{1 \leq i_2 \ldots i_m \leq n} \left\{ a_{i i_2 \ldots i_m} + x_{i_2} + \ldots + x_{i_m} \right\} .$$

Example 2.2. Let $\mathbb{A}$ and $\mathbb{B}$ be third-order two-dimensional tensors of the following form:

$$\begin{array}{llll} a_{111} = 1 & a_{121} = 2 & a_{112} = 1 & a_{122} = 2 \\ a_{211} = 2 & a_{221} = 1 & a_{212} = -2 & a_{222} = -8, \end{array}$$

$$\begin{array}{llll} b_{111} = 2 & b_{121} = 0 & b_{112} = 4 & b_{122} = -1 \\ b_{211} = 10 & b_{221} = -3 & b_{212} = 1 & b_{222} = 0, \end{array}$$

if $\mathbb{C} = \mathbb{A} \otimes \mathbb{B}$, then for example $c_{12112} = 5$.
If $x = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ then $(\mathbb{A} \otimes x) = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

The max plus algebraic addition ($\oplus$) and multiplication ($\otimes$) have the following properties:

Theorem 2.3. *Let $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \Re_{max}^{[m,n]}$, then*
(i). $\mathbb{A} \oplus \mathbb{B} = \mathbb{B} \oplus \mathbb{A}$. **(ii).** $\mathbb{A} \oplus (\mathbb{B} \oplus \mathbb{C}) = (\mathbb{A} \oplus \mathbb{B}) \oplus \mathbb{C}$.
(iii). $\mathbb{A} \oplus \mathbb{E} = \mathbb{A} = \mathbb{E} \oplus \mathbb{A}$ *where $\mathbb{E}$ is an mth order n dimensional*

tensor whose all entries are ε.
(iv). $\mathbb{A} \oplus \mathbb{B} \geq \mathbb{A}$. **(v).** $\mathbb{A} \oplus \mathbb{B} = \mathbb{A}$ *if and only if* $\mathbb{A} \geq \mathbb{B}$.

Theorem 2.4. *Let* $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \Re_{max}^{[m,n]}$ *and* $\alpha \in \bar{\Re}_{\max}$, *then*
(i). $(\alpha \otimes \mathbb{A})_{i_1 i_2 \ldots i_m} = (\mathbb{A} \otimes \alpha)_{i_1 i_2 \ldots i_m} = \alpha + a_{i_1 i_2 \ldots i_m}$.
(ii). $\mathbb{A} \otimes E = \mathbb{E} = E \otimes \mathbb{A}$, *where* E *is an* $n \times n$ *matrix whose all entries are* ε.
(iii). $\alpha \otimes (\mathbb{B} \oplus \mathbb{C}) = (\alpha \otimes \mathbb{B}) \oplus (\alpha \otimes \mathbb{C})$.
(iv). $\mathbb{B} \otimes (\alpha \otimes \mathbb{C}) = ((m-1)\,\alpha) \otimes (\mathbb{B} \otimes \mathbb{C})$ *and* $(\alpha \otimes \mathbb{B}) \otimes \mathbb{C} = \alpha \otimes (\mathbb{B} \otimes \mathbb{C})$.
(v). *Let* $\mathbb{A}_1, \mathbb{A}_2 \in \Re_{max}^{[m,n]}$ *and* $\mathbb{B} \in \Re_{max}^{[k,n]}$ *then* $(\mathbb{A}_1 \oplus \mathbb{A}_2) \otimes \mathbb{B} = (\mathbb{A}_1 \otimes \mathbb{B}) \oplus (\mathbb{A}_2 \otimes \mathbb{B})$.
(vi). *Let* A *be an* $n \times n$ *matrix and* $\mathbb{B}_1, \mathbb{B}_2 \in \Re_{max}^{[k,n]}$ *then* $A \otimes (\mathbb{B}_1 \oplus \mathbb{B}_2) = (A \otimes \mathbb{B}_1) \oplus (A \otimes \mathbb{B}_2)$. *(Note that in general when* A *is not a matrix, then the right distributivity doesn't hold.)*

Now we use a method similar with the proof of Theorem (3.4) in [1] and Theorem (1.1) in [3] to show the associative law.

Theorem 2.5. *Let* $\mathbb{A}$ *(and* $\mathbb{B}$, $\mathbb{C}$*) be an order* $m+1$ *(and order* $k+1$, *order* $r+1$*), dimension* n *tensor, respectively. Then we have*

$$\mathbb{A} \otimes (\mathbb{B} \otimes \mathbb{C}) = (\mathbb{A} \otimes \mathbb{B}) \otimes \mathbb{C}.$$

Theorem 2.6. *Let* $\mathbb{A}, \mathbb{B}$ *and* $\mathbb{C}$ *are tensors over* $\Re_{max}$ *of compatible sizes and* $\alpha \in \bar{\Re}_{\max}$, *then*
(i). $\mathbb{A} \geq \mathbb{B} \Rightarrow (\mathbb{A} \oplus \mathbb{C}) \geq (\mathbb{B} \oplus \mathbb{C})$.
(ii). $\mathbb{A} \geq \mathbb{B} \Rightarrow (\mathbb{A} \otimes \mathbb{C}) \geq (\mathbb{B} \otimes \mathbb{C})$.
(iii). $\mathbb{A} \geq \mathbb{B} \Rightarrow (\mathbb{C} \otimes \mathbb{A}) \geq (\mathbb{C} \otimes \mathbb{B})$.
(iv). $\mathbb{A} \geq \mathbb{B} \Rightarrow (\alpha \otimes \mathbb{A}) \geq (\alpha \otimes \mathbb{B})$.

Definition 2.7. A square matrix is called diagonal if all its diagonal entries are real numbers and off-diagonal entries are ε. A diagonal with all diagonal entries equal to 0 is called the unit matrix and denoted I.

Theorem 2.8. *Let* $\mathbb{A} \in \Re_{max}^{[m,n]}$. *Then*

$$\mathbb{A} \otimes I = \mathbb{A} = I \otimes \mathbb{A},$$

whenever I *is of a suitable dimension.*

Definition 2.9. A permutation matrix is a matrix in which each row and each column contains exactly one entry equal to 0 and all other entries are equal to ε.

Definition 2.10. A matrix that has the same ε-pattern as a permutation matrix is called a generalized permutation matrix.

Definition 2.11. An mth order n dimensional tensor $\mathbb{A}$ is called diagonal if all its diagonal entries are real numbers and off-diagonal entries are ε. A diagonal tensor with all diagonal entries equal to 0 is called the unit tensor and denoted by $\mathbb{I}$.

Theorem 2.12. *Let $\mathbb{A}, \mathbb{I} \in \Re_{max}^{[m,n]}$, and $\mathbb{E}$ be a tensor whose all entries are ε.*
(i). *If $\mathbb{A} \otimes I = \mathbb{E}$, then $\mathbb{A} = \mathbb{E}$.*
(ii). *If $I \otimes \mathbb{A} = \mathbb{E}$, then $\mathbb{A} = \mathbb{E}$.*
(iii). *If $\mathbb{A} \otimes \mathbb{I} = 0$, then $\mathbb{A} = 0$.*
(iv). *If $\mathbb{I} \otimes \mathbb{A} = 0$, then $\mathbb{A} = 0$.*

Definition 2.13. A tensor $\mathbb{A} \in \Re_{max}^{[m,n]}$ is called reducible, if there exists a nonempty proper index subset $I \subset \{1, ..., n\}$ such that

$$a_{i_1,...,i_m} = \varepsilon, \ \ \forall i_1 \in I, \ \ \forall i_2, ..., i_m \notin I,$$

If $\mathbb{A}$ is not reducible, then we call $\mathbb{A}$ irreducible.

Lemma 2.14. *Let $\mathbb{A} \in \Re_{max}^{[m,n]}$ and P, Q are both matrices, then*

$$(P \otimes \mathbb{A} \otimes Q)_{i_1 \ldots i_m} = \max_{1 \le j_1, ..., j_m \le n} \{a_{j_1 \ldots j_m} + p_{i_1 j_1} + q_{j_2 i_2} + ... + q_{j_m i_m}\}.$$

Theorem 2.15. *Let $\sigma \in S_n$ be a permutation on the set $\{1, ..., n\}$, $P = P_\sigma = (p_{ij})$ be the corresponding permutation matrix of σ (where $p_{ij} = 0 \Leftrightarrow j = \sigma(i)$). Let $\mathbb{A}, \mathbb{B} \in \Re_{max}^{[m,n]}$ such that $\mathbb{B} = P \otimes \mathbb{A} \otimes P^T$, Then we have:*
(i). $b_{i_1 \ldots i_m} = a_{\sigma(i_1) \ldots \sigma(i_m)}$.
(ii). $P \otimes \mathbb{I} \otimes P^T = \mathbb{I}$.
(iii). *Let $D = diag(d_{11}, ..., d_{nn})$ be an invertible diagonal matrix. Then*

$$D^{-(m-1)} \otimes \mathbb{I} \otimes D = \mathbb{I}.$$

Theorem 2.16. *Let $\mathbb{A} \in \Re_{max}^{[m,n]}$ be a reducible tensor and P be an $n \times n$ permutation matrix. Then $P \otimes \mathbb{A} \otimes P^T$ is a reducible tensor.*

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USING OF NORMAL MATRICES FOR DIGITAL IMAGE WATERMARKING

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Abstract. This paper present a technique for digital image watermarking based on eigenvalue decomposition of normal matrices. The proposed method transforms host image and watermark to the space of normal matrices and deals with properties of its spectral decompositions to obtain a watermarked image. Watermark extraction has also a similar procedure to embedding. Numerical experiments are provided to illustrate the validity of the method.

1. Introduction

Nowadays, everyone can access easily to digital images via the Internet and the other communication means. In addition, powerful software and hardware tools are also available to edit any kind of digital multimedia. So, digital images may become a problematic issue if it have significant authentications and information. To cope with this problem, *digital image watermarking* can be employed. Digital image watermarking is procedure of embedding additional information called *watermark* into an image, called *host image*, which preserves perceptual quality of the original host image. During normal use, the watermark that carries important information, is hidden and it can be visible only by pursuing some particular processes. The watermark should be

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detectable or extractable from host image by owner identification or integrity verification of the watermarked image.

Watermarking has a long list of applications in areas such as protecting copyright ownership of digital media, verifying the authenticity of a bill, making the information imperceptible, keeping the existence of information secret and so on. The study of digital image watermarking has received a great deal of attention. Some references for theory and application of it are [3, 4] and references therein.

Theory of linear algebra provides helpful instruments for digital image watermarking and also in a variety of other fields of digital image processing. One of the most useful tool for digital image watermarking is employing the Singular Value Decomposition (SVD) [2].

Motivated by influences and traits of singular values of A in digital image watermarking, and regarded to this point that singular values of A are positive square roots of the eigenvalues of matrices AA^* and A^*A, this paper concerns eigenvalue of normal matrices $A + A^*$ and $A - A^*$ in order to obtain a technique for digital image watermarking which is more effective and yield better results, as well as need less computations.

A matrix $M \in \mathbb{C}^{n\times n}$ is called *normal* if $M^*M = MM^*$. Assume that M is an n-square normal matrix, then there exists an orthonormal basis of $\mathbb{C}^{n\times n}$ consisting of eigenvectors of M and M is unitarily diagonalizable [1]. A general square matrix M satisfying $M = B + C$, where the symmetric matrix $B = \dfrac{M + M^*}{2}$ is called *symmetric part* of M and analogously, the skew-symmetric matrix $C = \dfrac{M - M^*}{2}$ is called *skew-symmetric* part of M. Consequently, every square matrix can be written as sum of two normal matrices: a symmetric matrix and a skew-symmetric one. This point is specially used in the proposed watermarking scheme.

2. Digital image watermarking method and results

Let X and W be square matrices representing the host and watermark image, respectively, and let α be a given scaling factor. The parameter α regulates strength of embedding watermark into the host image. As was mentioned, every matrix is equal to summation of its symmetric and skew-symmetric parts. In the proposed technique, therefore, a given scaling factor is shared equally between symmetric and skew-symmetric parts of the host image. Denote by B_X and B_W the symmetric parts of X and W and denote by C_X and C_W the skew-symmetric parts of it, respectively. We can factor these matrices as

follows

$$B_X = U_{B_X}\Lambda_{B_X}U_{B_X}^*, \qquad \Lambda_{B_X} = \mathrm{diag}(\lambda_{B_X,1},\ldots,\lambda_{B_X,n}),$$
$$B_W = U_{B_W}\Lambda_{B_W}U_{B_W}^*, \qquad \Lambda_{B_W} = \mathrm{diag}(\lambda_{B_W,1},\ldots,\lambda_{B_W,n}),$$
$$C_X = U_{C_X}\Lambda_{C_X}U_{C_X}^*, \qquad \Lambda_{C_X} = \mathrm{diag}(\lambda_{C_X,1},\ldots,\lambda_{C_X,n}),$$
$$C_W = U_{C_W}\Lambda_{C_W}U_{C_W}^*, \qquad \Lambda_{C_W} = \mathrm{diag}(\lambda_{C_W,1},\ldots,\lambda_{C_W,n}).$$

Now, for scaling factor α embed the watermark into the host image by modifying the eigenvalues of X as detailed below

$$\hat{C}_X = U_{C_X}\hat{\Lambda}U_{C_X}^*, \qquad \hat{\Lambda} = \Lambda_{C_X} + \frac{\alpha}{2}\Lambda_{C_W},$$
$$\tilde{B}_X = U_{B_X}\tilde{\Lambda}U_{B_X}^*, \qquad \tilde{\Lambda} = \Lambda_{B_X} + \frac{\alpha}{2}\Lambda_{B_W},$$

therefore, the watermarked image $\mathcal{X}$ will be $\mathcal{X} = \tilde{B}_X + \hat{C}_X$.

Henceforth, extracting the watermark W from watermarked image $\mathcal{X}$ is considered. The extraction process needs two diagonal matrices Λ_{B_X} and Λ_{C_X} and also, two unitary matrices U_{B_W} and U_{C_W}. In the presented watermark extraction scheme, at first, two matrices $\bar{\Lambda}_{B_W}$ and $\bar{\Lambda}_{C_W}$ are computed as estimations of Λ_{B_W} and Λ_{C_W}, respectively. Next, extracted watermark $\mathcal{W}$ will be obtained by approximating the two matrices B_W and C_W. Assume now, $\mathcal{X}$ is an acceptable approximation of the original image X. Form the symmetric and skew-symmetric parts of $\mathcal{X}$, namely $B_{\mathcal{X}}$ and $C_{\mathcal{X}}$, in the order already mentioned, then compute its eigenvalue decompositions as shown below

$$B_{\mathcal{X}} = \frac{\mathcal{X} + \mathcal{X}^T}{2}, \qquad B_{\mathcal{X}} = U_{B_{\mathcal{X}}}\Lambda_{B_{\mathcal{X}}}U_{B_{\mathcal{X}}}^*,$$
$$C_{\mathcal{X}} = \frac{\mathcal{X} - \mathcal{X}^T}{2}, \qquad C_{\mathcal{X}} = U_{C_{\mathcal{X}}}\Lambda_{C_{\mathcal{X}}}U_{C_{\mathcal{X}}}^*.$$

Now, two diagonal matrices $\bar{\Lambda}_{B_W}$ and $\bar{\Lambda}_{C_W}$ are obtained from the following relations

$$\bar{\Lambda}_{B_W} = \frac{\Lambda_{B_{\mathcal{X}}} - \Lambda_{B_X}}{\alpha}, \qquad \bar{\Lambda}_{C_W} = \frac{\Lambda_{C_{\mathcal{X}}} - \Lambda_{C_X}}{\alpha},$$

also if we denote by $\bar{B}_W$ and $\bar{C}_W$ estimations of B_W and C_W, respectively, then

$$\bar{B}_W = U_{B_W}\bar{\Lambda}_{B_W}U_{B_W}^*, \qquad \text{and} \qquad \bar{C}_W = U_{C_W}\bar{\Lambda}_{C_W}U_{C_W}^*.$$

Finally, the extracted watermark $\mathcal{W}$ will be $\mathcal{W} = \bar{B}_W + \bar{C}_W$.

In remainder of this paper, the validity and influence of the presented watermarking method is examined by experiments.The Peak Signal to Noise Ratio (PSNR) is calculated to measure the quality of the

watermarked image. PSNR for gray scale images of size $M \times N$, where its pixels are represented with 8 bits, is computed by

$$PSNR = 10 \log_{10}^{\frac{255^2}{MSE}}; \qquad MSE = \frac{1}{MN} \sum_{i,j} |X_{i,j} - \mathcal{X}_{i,j}|^2,$$

where $X_{i,j}$ and $\mathcal{X}_{i,j}$ denote elements of the original and watermarked images, respectively. In the above relation, MSE denotes the Mean Square Error between the host image and the watermarked image pixels. We considered the host 512×512 gray scale images Lena the watermark image Yazd University emblem of size 128×128, in performed experiments. Results are shown for some values of the scaling factor α in Table 1. The results obtained by the presented method techniques are compared to the results achieved by digital image watermarking method using SVD as a prevalent watermarking technique.

TABLE 1. PSNR results for the host and the watermark image

α	Proposed Method	SVD Method
0.2000	74.8323	74.8500
0.4000	68.8117	68.8294
0.6000	65.2899	65.3076
0.8000	62.7911	62.8088
1.0000	60.8529	60.8706
1.2000	59.2693	59.2870
1.4000	57.9304	57.9481
1.6000	56.7705	56.7882
1.8000	55.7475	55.7652
2.0000	54.8323	54.8500

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FUZZY SEMI-INNER PRODUCT SPACES

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Abstract. In the present paper, we investigate a connection between two fuzzy semiinner product which one arise form Felbin's fuzzy norm and another one is based on Bag and Samanta's fuzzy norm. Also we show that, considering a fuzzy inner product space, how one can construct another kind of fuzzy inner product on this space.

1. Introduction

We know that in classical mechanics the algebra of observables is the commutative algebra of functions on some space. Moreover, in quantum mechanics or quantum field theory, the observables are operators on a Hilbert space, and the algebra of operators on a Hilbert space is a non-commutative algebra. On the one hand, there are no solutions to equations like $rs - sr = 1$ in commutative algebra. In fact, it has no solution in operators on a finite dimensional Hilbert space. So in the dynamics of quantum theory, one must study operators on infinite dimensional Hilbert spaces. On the other hand, the usual uncertainty principle of Heisenberg ultimates generalized uncertainty principle, this has been motivated by string theory and non-commutative geometry. In strong quantum gravity regime spacetime points are determined in a fuzzy manner. For this reason, fuzzy structure, fuzzy Hilbert space

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and operators on a fuzzy Hilbert space are required. Thus, one needs to discuss a new family of fuzzy inner product space.

2. Preliminaries

Definition 2.1. ([3]) A mapping $\eta : R \longrightarrow [0,1]$ is called a fuzzy real number with α-level set $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies the following conditions:
(N1) there exists $t_0 \in R$ such that $\eta(t_0) = 1$.
(N2) for each $\alpha \in (0,1]$, there exist real numbers $\eta_\alpha^- \leq \eta_\alpha^+$ such that the $\alpha-$level set $[\eta]_\alpha$ is equal to the closed interval $[\eta_\alpha^-, \eta_\alpha^+]$.
The set of all fuzzy real numbers is denoted by $F(R)$. Since each $r \in R$ can be considered as the fuzzy real number $r \in F(R)$ defined by

$$r(t) = \begin{cases} 1 & , \quad t = r \\ 0 & , \quad t \neq r, \end{cases}$$

it follows that R can be embedded in $F(R)$.

Definition 2.2. ([1]) A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a t-norm if it satisfies the following for every $r, s, t, u \in [0,1]$:
(i) $1 * t = t$,
(ii) $t * s = s * t$,
(iii) $t * (r * s) = (t * r) * s$,
(iv) If $t \leq s$ and $r \leq u$ then $t * r \leq s * u$.

Definition 2.3. ([4]) A fuzzy inner product space is a triplet $(X, F, *)$, where X is a real vector space, $*$ is a t-norm and F is a fuzzy set on $X^2 \times R$ satisfying the following conditions for every $x, y, z \in X$ and $t, s \in R$:
(FI1) $F(x, y, 0) = 0$,
(FI2) $F(x, y, t) = F(y, x, t)$,
(FI3) $F(x, x, t) = 1$, for all $t > 0$, if and only if $x = 0$,
(FI4) For any real number $c \in R$ and $t \neq 0$,

$$F(cx, y, t) = \begin{cases} F(x, y, t/c) & , \quad c > 0 \\ H(t) & , \quad c = 0 \\ 1 - F(x, y, t/c) & , \quad c < 0, \end{cases}$$

where

$$H(t) = \begin{cases} 1 & , \quad t > 0 \\ 0 & , \quad t \leq 0, \end{cases}$$

(FI5) $F(x, z, t) * F(y, z, s) \leq F(x + y, z, t + s)$, for all $t, s > 0$,
(FI6) $\lim_{t \to \infty} F(x, y, t) = 1$.

Example 2.4. ([4]) Let $(X, < ., . >)$ be an ordinary inner product space. We define a mapping $F : X^2 \times R \longrightarrow [0, 1]$ as follows

$$F(x, y, t) = \begin{cases} t^{1/2}/(t^{1/2} + | < x, y > |^{1/2}) & , \quad t > 0 \\ 0 & , \quad t = 0 \\ | < x, y > |^{1/2}/((-t)^{1/2} + | < x, y > |^{1/2}) & , \quad t < 0, \end{cases}$$

let $*$ be arbitrary t-norm. Then $(X, F, *)$ is a fuzzy inner product space.

We assume that
(FI7) $F(x + y, z, r) = \sup\{F(x, z, t) * F(y, z, s) : t + s = r, t, s > 0\}$.
(FI8) $F(x, x, t) = 0$, for all $t < 0$.
(FI9) There exists $0 < t_x$ such that $F(x, x, t_x) = 0$.
(FI10) $F(x, y, .)$ is a strictly increasing on the subset $\{t > 0 : 0 < F(x, , y, t) < 1\}$ of R, for all $x, y \in X$.
(FI11) $F(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ $(F(x, y, .) : (-\infty, 0) \longrightarrow [0, 1])$ is continuous, for all $x, y \in X$.

Definition 2.5. ([3]) A fuzzy inner product space is a pair $(X, < ., . >)$, where X is a real vector space and $< ., . >$ is a function $\langle ., . \rangle : X \times X \longrightarrow F(R)$ such that for all vectors $x, y, z \in X$ and all $c \in R$, we have:
(IP1) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$,
(IP2) $\langle cx, y \rangle = c\langle x, y \rangle$,
(IP3) $\langle x, y \rangle = \langle y, x \rangle$,
(IP4) $\langle x, x \rangle \geq 0$,
(IP5) $0 < \inf_{\alpha \in (0,1]} \langle x, x \rangle_\alpha^-$ if $x \neq 0$,
(IP6) $\langle x, x \rangle = 0$ if and only if $x = 0$.

Lemma 2.6. *([3]) A fuzzy inner product space $(X, < ., . >)$ satisfy the Schwarz inequality*

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle}\sqrt{\langle y, y \rangle}, \text{ for all } x, y \in X.$$

We assume that
(IP7) for any decreasing sequence $\{\alpha_k\}$ in $(0, 1]$ such that $\alpha_k \longrightarrow \alpha \in (0, 1]$ implies that $< x, y >_{\alpha_k}^+ \longrightarrow < x, y >_\alpha^+$, for all $x, y \in X$.

3. Fuzzy Semi-Inner Product Spaces

Definition 3.1. A fuzzy semi inner product space is a pair $(X, < ., . >)$, where X is a real vector space and $< ., . >$ is a function $\langle ., . \rangle : X \times X \longrightarrow F(R)$ such that for all vectors $x, y, z \in X$ and all $c \in R$, we have:
(IP1) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$,

(IP2) $\langle cx, y\rangle = c\langle x, y\rangle$,
(IP3) $\langle x, y\rangle = \langle y, x\rangle$,
(IP4) $\langle x, x\rangle \geq 0$,
(IP5) $\langle x, x\rangle = 0$ if and only if $x = 0$.

Theorem 3.2. *Let (X, F, min) be a fuzzy inner product space satisfying (FI7), (FI8) and (FI9). Suppose that*

$$< x, y >_{\alpha}^{-} = \inf\{t < 0 : \ \alpha \leq F(x, y, t)\},$$

and

$$< x, y >_{\alpha}^{+} = \sup\{t > 0 : \ F(x, y, t) \leq 1 - \alpha\},$$

for all $\alpha \in (0, 1]$. Then there is a fuzzy semiinner product $< ., . >$ on X such that

$$[< x, y >]_{\alpha} = [< x, y >_{\alpha}^{-}, < x, y >_{\alpha}^{+}] \text{ for all } \alpha \in (0, 1] \text{ and } x, y \in X.$$

Theorem 3.3. *Let $(X, < ., . >)$ be a fuzzy semiinner product space satisfying (IP7) and $[< x, y >]_{\alpha} = [< x, y >_{\alpha}^{-}, < x, y >_{\alpha}^{+}]$, for all $\alpha \in (0, 1]$. Moreover, let F be a functions on $X \times X \times R$ defined by*

$$F(x, y, t) = \begin{cases} 1 - \inf\{\alpha \in (0, 1] :< x, y >_{\alpha}^{+} \leq t\} & , \quad t > 0 \\ 0 & , \quad t = 0 \\ \inf\{\alpha \in (0, 1] :< x, y >_{\alpha}^{-} \geq t\} & , \quad t < 0, \end{cases}$$

Then $(X, F, *)$ is a fuzzy inner product space for any t-norm $*$.

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EFFICIENT METHODS WITH POSITIVITY PROPERTY FOR BLACK-SCHOLES EQUATION

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Abstract. When one solves the Black-Scholes partial differential equation (PDE) it is of great important that numerical scheme be free of spurious oscillations and satisfies positivity requirement. In this paper, first we show that the positivity is not ensured with a class of standard finite difference method (we refer to it θ-method) when applied to the Black-Scholes equation for very small time steps. Next, by reforming the discretization of the reaction term of equation a class of method is derived that is free of spurious oscillations around discontinuities and preserving positivity.

1. Introduction

Mathematical finance is a field of applied mathematics, concerned with financial markets. In the market of financial derivatives the most important problem is the so called *option valuation problem*, i.e. to compute a fair value for the option. The solution of the Black-Scholes equation determines the option price, respectively according to the used initial conditions. Numerical methods based on standard finite difference approach e.g. Crank-Nicolson and fully implicit schemes (substituting $\theta = 1/2, 1$ respectivly in θ-method) are consistent with the original differential equation and guarantee convergence of the discrete solution to the exact one, but in the presence of discontinuous payoff and low volatility, essential qualitative properties of the solution are

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not transferred to the numerical solution. Spurious oscillations and negative values might be occurred in the solution. Therefor, we need to construct positivity preserving schemes [**?**, **?**, **?**, **?**], that avoid unrealistic negative values for the solution. One possibility is modification of θ-method. In this paper we are interested in the following Black-Scholes pricing PDE:

$$-\frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} - rV = 0, \tag{1.1}$$

where $V(s,t)$ is the price of the option and endowed with initial and boundary conditions:

$$V(s,0) = max(s-K,0)1_{[L,U]}(s), \qquad V(s,t) \to 0 \quad as \quad s \to 0 \quad or \quad s \to \infty,$$

with updating of the initial condition at the monitoring dates t_i, $i = 1, ..., F$:

$$V(s,t_i) = V(s,t_i^-)1_{[L,U]}(s), \qquad 0 = t_0 < t_1 < ... < t_F = T$$

where $1_{[L,U]}(s)$ is the indicator function, i.e.,

$$1_{[L,U]}(s) = \begin{cases} 1 & if \quad s \in [L,U] \\ 0 & if \quad s \notin [L,U] \end{cases},$$

here the parameter $r > 0$ is the interest rate and the reference volatility is $\sigma > 0$.
With θ-method we mean discretization $\frac{\partial V}{\partial s}$ and $\frac{\partial^2 V}{\partial s^2}$ as

$$\frac{\partial V}{\partial s} = \frac{V_{j+1}^{n+1} - V_{j-1}^{n+1}}{2\Delta s},$$

$$\frac{\partial^2 V}{\partial s^2} = (1-\theta)\frac{V_{j-1}^{n} - 2V_{j}^{n} + V_{j+1}^{n}}{\Delta s^2} + \theta\frac{V_{j-1}^{n+1} - 2V_{j}^{n+1} + V_{j+1}^{n+1}}{\Delta s^2}.$$

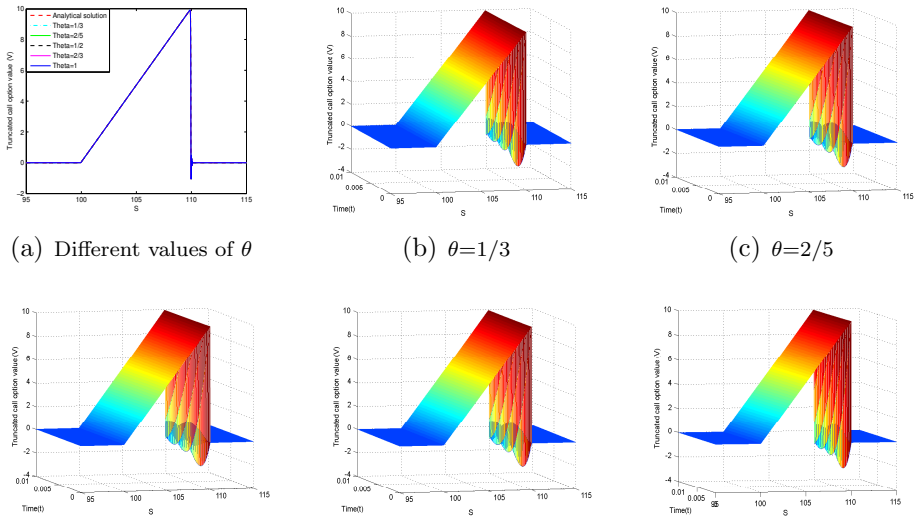
The standard scheme lead to a difference equation

$$AV^{n+1} = BV^n$$

with A and B be the following tridiagonal matrices:

$$A = tridiag\left\{\frac{r}{2}\frac{s_j}{\Delta s} - \frac{\theta}{2}(\frac{\sigma s_j}{\Delta s})^2; \frac{1}{\Delta t} + \theta(\frac{\sigma s_j}{\Delta s})^2 + r; -\frac{r}{2}\frac{s_j}{\Delta s} - \frac{\theta}{2}(\frac{\sigma s_j}{\Delta s})^2\right\},$$
$$B = tridiag\left\{\frac{1-\theta}{2}(\frac{\sigma s_j}{\Delta s})^2; \frac{1}{\Delta t} - (1-\theta)(\frac{\sigma s_j}{\Delta s})^2; \frac{1-\theta}{2}(\frac{\sigma s_j}{\Delta s})^2\right\}.$$

As we can see in Figure 1 the θ-method for different values of θ give negative values and spurious oscillations.



(a) Different values of θ (b) θ=1/3 (c) θ=2/5

(d) θ=1/2 (e) θ=2/3 (f) θ=1

FIGURE 1. Truncated call option value for different values of θ with $\Delta s = 0.01$, $\Delta t = 10^{-4}$. parameters: $L = 90$, $K = 100$, $U = 110$, $r = 0.05$, $\sigma = 0.001$, $T = 0.01$, $S_{max} = 120$.

2. Construction of new scheme

The proposed scheme discretized the reaction term $-rV$ in equation (**??**) as follows:

$$V(S, t + \Delta t) = a(V_{j+1}^{n+1} + V_{j-1}^{n+1}) + (1 - 2a)V_j^n,$$

here a is arbitrary constant to be determined following Theorem 1. The finite difference approximation provides the equation difference

$$PV^{n+1} = NV^n \tag{2.1}$$

where P and N be the following tridiagonal matrices:

$$P = tridiag\left\{ra + \frac{rS_j}{2\Delta S} - \frac{\sigma^2 S_j^2\theta}{2\Delta S^2}; \frac{1}{\Delta t} + \frac{\sigma^2 S_j^2\theta}{\Delta S^2}; ra - \frac{rS_j}{2\Delta S} - \frac{\sigma^2 S_j^2\theta}{2\Delta S^2}\right\},$$
$$N = tridiag\left\{\frac{\sigma^2 S_j^2(1-\theta)}{2\Delta S^2}; \frac{1}{\Delta t} - \frac{\sigma^2 S_j^2(1-\theta)}{\Delta S^2} - r(1-2a); \frac{\sigma^2 S_j^2(1-\theta)}{2\Delta S^2}\right\}.$$

Using the strategy discussed in [**?**, **?**] for positivity and stability, we conclude the following theorems:

Theorem 1. Sufficient for scheme (**??**) to be positive is,

$$a \le -\frac{r}{8\sigma^2\theta} \quad , \quad \Delta t < \frac{1}{r(1-2a) + (1-\theta)(\sigma M)^2}. \tag{2.2}$$

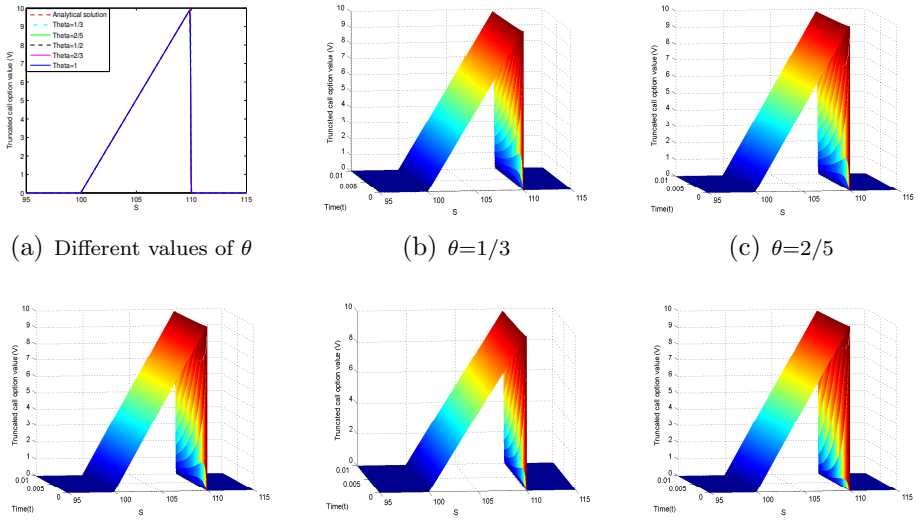
Theorem 2. Under conditions (**??**), the new scheme is stable and

convergent with local truncation error $O(\Delta t, \Delta S^2)$.

Theorem 3. Under the condition $\Delta t < \frac{1}{r+(1-\theta)(\sigma M)^2}$, then $P^{-1}N$ admits M real positive and distinct eigenvalues $\lambda_i(P^{-1}N)$ and $\lambda_i(P^{-1}N) \in (0, 1)$, then spurious oscillations are avoided.

3. Numerical results

To illustrate the advantage of the designed positive NSFD, we presents the numerical results for different values of θ:



(a) Different values of θ (b) θ=1/3 (c) θ=2/5

(d) θ=1/2 (e) θ=2/3 (f) θ=1

Figure 2. Truncated call option value for different values of θ with $\Delta s = 0.01$, $\Delta t = 10^{-4}$. parameters: $L = 90$, $K = 100$, $U = 110$, $r = 0.05$, $\sigma = 0.001$, $T = 0.01$, $S_{max} = 120$.

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AN ITERATIVE METHOD FOR SOLVING THE GENERAL AND CONSTRAINED LINEAR MATRIX EQUATIONS SYSTEM

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Abstract. In this paper, an iterative approach based on the GL-LSQR method is proposed which is more general and includes all of matrix equations and some constrained linear matrix equations system. When the matrix equations are consistent, a least norm solution can be obtained. Moreover, the optimal approximate solution to a given group of matrices by new method, can be derived. Finally, some numerical experiments are given to compare the new iterative method with some existent methods.

1. Introduction

Throughout this paper, the notations $tr(A)$ and A^H are used to denote the trace and the transpose conjugate of a matrix A, respectively. For two matrices X and Y in $\mathbb{C}^{m\times n}$, $\langle X, Y\rangle_F$ denotes the Frobenius inner product, i.e., $\langle X, Y\rangle_F = tr(Y^H X)$, and associated Frobenius norm is denoted by $||.||_F$. Let $\mathfrak{M} = \mathbb{C}^{m_1\times n_1} \times \mathbb{C}^{m_2\times n_2} \times \ldots \times \mathbb{C}^{m_l\times n_l}$ and $\mathfrak{N} = \mathbb{C}^{p_1\times q_1} \times \mathbb{C}^{p_2\times q_2} \times \ldots \times \mathbb{C}^{p_s\times q_s}$ be two matrix spaces.

Matrix equations appear frequently in many areas of applied mathematics and play vital roles in many applications [1]. Due to their wide applications, many authors have presented several methods to solve some special kinds of matrix equations and constrained linear matrix

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* Speaker.

equations [2], in recent years. Some of the common constraints on the solution of matrix equations are Symmetric, Centrosymmetric, Reflxive constraint and so on that can be found in [1]. Here we consider a general form of these constraints as follows:

$$X = \tilde{L}(X), \tag{1.1}$$

where $\tilde{L}$ is a self-conjugate involution operator on $\mathfrak{M}$, i.e., $\tilde{L}^* = \tilde{L}$ and $\tilde{L}^2 = \mathfrak{I}$, where $\mathfrak{I}$ is the identity operator. Let $\mathfrak{Cm}$ denote the set of constraint matrices defined by (1.1).

In this paper, we show that all of these problems can be solved by the new proposed method. So, we consider the constrained linear matrix equations system that is more general than unconstrained linear matrix equations system. Moreover, when the matrix equations are consistent, the optimal approximate solution to a given group of matrices, can be derived.

2. Main result

In this section, we consider the generalized linear matrix equations of the form

$$\begin{cases} \sum_{j=1}^{l} A_{ij}X_jB_{ij} + \sum_{j=1}^{l} C_{ij}X_j^HD_{ij} = E_i, & i = 1, 2, \ldots, s, \\ X_j \in \mathfrak{Cm}, & j = 1, \ldots, l, \end{cases} \tag{2.1}$$

where A_{ij} and $C_{ij} \in \mathbb{C}^{p_i \times m_j}$, B_{ij} and $D_{ij} \in \mathbb{C}^{n_j \times q_i}$ and $E_i \in \mathbb{C}^{p_i \times q_i}$, $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, l$, are given matrices and $X_i \in \mathbb{C}^{m_i \times n_i}$, $i = 1, 2, \ldots, l$, are unknown matrices to be determined. By setting $\tilde{L} = \mathfrak{I}$ in (1.1), the constrained generalized linear system (2.1) is reduced to the unconstrained linear matrix equations system. Now, we define the following linear matrix operator $L : \mathfrak{M} \to \mathfrak{N}$, $L(\mathcal{X}) = (L_1(\mathcal{X}), \ldots, L_s(\mathcal{X}))$, where $\mathcal{X} = (X_1, \ldots, X_l)$ and

$$L_i : \mathfrak{M} \to \mathbb{C}^{p_i \times q_i}, \quad L_i(\mathcal{X}) = \sum_{j=1}^{l} A_{ij}X_jB_{ij} + \sum_{j=1}^{l} C_{ij}X_j^HD_{ij}, \quad 1 \le i \le s.$$

Therefore, the constrained system (2.1) can be written as follows

$$L(\mathcal{X}) = E \ \ and \ \ \tilde{L}(\mathcal{X}) = \mathcal{X}, \tag{2.2}$$

where $E = (E_1, E_2, \ldots, E_s)$ and $\tilde{L}$ is the operator (1.1).

Remark 2.1. Any solution of the (2.2) is a solution of the following pair of operator equations

$$L(\mathcal{X}) = E \ \ and \ \ L\tilde{L}(\mathcal{X}) = E, \tag{2.3}$$

So, the pair of operator equation (2.3) can be rewritten as the following new operator form

$$\bar{L} : \mathfrak{M} \to \mathfrak{N} \times \mathfrak{N}, \quad \bar{L}(\mathcal{X}) = (L_1(\mathcal{X}), \ldots, L_s(\mathcal{X}), \bar{L}_1(\mathcal{X}), \ldots, \bar{L}_s(\mathcal{X})),$$

where $\bar{L}_i(\mathcal{X}) = \sum_{j=1}^{l} A_{ij}\tilde{L}(X_j)B_{ij} + \sum_{j=1}^{l} C_{ij}\tilde{L}^H(X_j)D_{ij}$, $1 \leq i \leq s$, with $\tilde{L}^H(X_j) = \tilde{L}(X_j^H)$, $1 \leq j \leq l$, and X_j satisfies in (1.1). Therefore, (2.3) can be rewritten as follows

$$\bar{L}(\mathcal{X}) = \bar{E}, \tag{2.4}$$

where $\bar{E} = (E_1, \ldots, E_s, E_1, \ldots, E_s)$. Now we need to define a proper inner product and its associated norm which are utilized throughout this work.

Definition 2.2. Let $\mathcal{X} = (X_1, X_2, \ldots, X_l)$, $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_l) \in \mathfrak{M}$, then $\langle\langle \mathcal{X}, \mathcal{Y} \rangle\rangle = Re(tr(\sum_{j=1}^{l} Y_j^H X_j))$, and its associated norm is denoted by $\sharp\mathcal{X}\sharp = \sqrt{\langle\langle \mathcal{X}, \mathcal{X} \rangle\rangle}$, where $Re(a)$ is the real part of the complex number a.

Remark 2.3. Let $\bar{L}$ be the linear operator (2.4). Then the adjoint of operator $\bar{L}$ is

$$\bar{L}^* : \mathfrak{N} \times \mathfrak{N} \to \mathfrak{M}$$
$$\bar{L}^*(\mathcal{Y}) = (\bar{T}_1(\mathcal{Y}), \bar{T}_2(\mathcal{Y}), \ldots, \bar{T}_l(\mathcal{Y})),$$

where $\mathcal{Y} = ((Y_1, \ldots, Y_s), (Z_1, \ldots, Z_s))$ and for $1 \leq j \leq l$

$$\bar{T}_j(\mathcal{Y}) = \sum_{i=1}^{s} A_{ij}^H Y_i B_{ij}^H + \sum_{i=1}^{s} D_{ij} Y_i^H C_{ij} + \sum_{i=1}^{s} \tilde{L}(A_{ij}^H Z_i B_{ij}^H) + \sum_{i=1}^{s} \tilde{L}(D_{ij} Z_i^H C_{ij}).$$

Now we can apply the $\mathcal{L} - GL - LSQR$ algorithm [3] for solving

$$\bar{L}(\mathcal{X}) = \bar{E}. \tag{2.5}$$

Hence, we have the following theorem that its proof is easy by induction.

Theorem 2.4. *Let $\mathcal{X}_i$, $i = 1, 2, \ldots, k$ be the approximate solution of (2.5) which is obtained by the $\mathcal{L}-GL-LSQR$ algorithm then $\mathcal{X}_i$ is an approximate solution of the system (2.2).*

Theorem 2.5. [4] *The $\mathcal{L} - GL - LSQR$ algorithm returns the minimum norm solution.*

Let the system (2.2) is solvable and CS_X denote its solution set. For a given element $\breve{\mathcal{X}} \in \mathfrak{Cm}$, we can find $\hat{\mathcal{X}} \in CS_X$ by the $\mathcal{L} - GL - LSQR$ such that $\sharp\hat{\mathcal{X}} - \breve{\mathcal{X}}\sharp = \min_{\mathcal{X} \in CS_X} \sharp\mathcal{X} - \breve{\mathcal{X}}\sharp$,

3. Numerical results

In this section, two example are given to compar new method with some existent method.

Example 3.1. In this example, we consider the Example 3 of [5]. For various matrix size m, we apply our algorithm, Algorithm 1, the CGNE and the CGNR methods [5] and the sequences of numerical results with respect to the CPU times in seconds and the number of iterations, are listed in Table 1. Note that the CPU time is shown in parenthesis.

TABLE 1. Numerical results for Example 3.1.

m	$\mathcal{L}-GL-LSQR$	Algorithm 1	CGNE	CGNR
40	935(0.1858)	965(0.1047)	945(0.1969)	939(0.3476)
50	1385(0.3664)	1416(0.1925)	1396(0.3730)	1388(0.6698)
60	1915(0.6094)	1951(0.3842)	1929(0.6942)	1919(1.2183)

Example 3.2. Consider the Example 1 of [2]. We applied the $\mathcal{L}-GL-LSQR$ algorithm to solve this example and after 2 iterations we obtained $||R_2||_F = 6.1455e-015$ and $\delta_2 = 3.1402e-16$, where R_n is the residual after n iteration and $\delta_k = \frac{||X_k - X^*||_F}{||X^*||_F}$. Also, we applied Algorithm 1 of [2] to solve this example with different values of w. The numerical results are depicted in Figure 1. As is seen from Figure 1, the $\mathcal{L}-GL-LSQR$ algorithm converges faster than Algorithm 1.

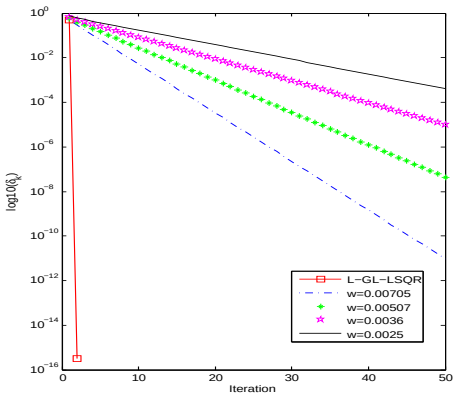

FIGURE 1. The numerical results for Example 3.2 using $\mathcal{L}-GL-LSQR$ and Algorithm 1.

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A NEW METHOD FOR COMPUTING THE GROUP-INVERSE SOLUTION OF SINGULAR LINEAR EQUATIONS

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Abstract. DGMRES is an algorithm to solve the Drazin–inverse solution of large scale and sparse consistent or inconsistent singular linear systems with an arbitary index. In this paper, we present an algorithm based on the DGMRES algorithm is proposed for computing the group–inverse solution of singular linear equations.

1. Introduction

Consider the linear system

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{C}^{N\times N}$ is a singular matrix and ind(A) is arbitrary. Here ind(A), the index of A is the smallest nonnegative integer a such that rank$(A^{a+1}) =$ rank(A^a) . We recall that the Drazin-inverse solution of (1.1) is the vector $A^D b$, where A^D is the Drazin-inverse of the singular matrix A, see [3]. In the special case $a = 1$, this matrix is called the group inverse of A and denoted by A_g. The group inverse has various applications in the theory of finite Markov chains and others[1]. DGMRES [3] is a Krylov subspace method which finds Drazin-inverse solution by minimizing 2-norm of residual $A^a r_m$.

In this paper, we develop the GrGMRES algorithm which is another implementation of DGMRES, for solving the singular linear system (1.1) with index one.

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Key words and phrases. Singular linear systems; DGMRES method; Group-inverse solution; Drazin-inverse solution; Krylov subspace methods.

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2. Main results

DGMRES method starts with an initial vectors x_0 and generates a sequence of vectors $x_0, x_1, \ldots$, as $x_m = x_0 + \sum_{i=1}^{m-a} c_i A^{a+i-1} r_0$, $r_0 = b - Ax_0$. Then $r_m = b - Ax_m = b - \sum_{i=1}^{m-a} c_i A^{a+i} r_0$. The Krylov subspace we will use is $\mathcal{K}_{m-a}\{A; A^a r_0\} = \text{span}\{A^a r_0, A^{a+1} r_0, \ldots, A^{m-1} r_0\}$. The vector x_m produced by DGMRES satisfies

$$\|A^a r_m\|_2 = \min_{x \in x_0 + \mathcal{K}_{m-a}\{A; A^a r_0\}} \|A^a (b - Ax)\|_2. \tag{2.1}$$

We starts by orthogonalizing the krylov vectors $A^a r_0, A^{a+1} r_0, \ldots, ,$ by the modified Gram–Schmidt process. Consequently, we have a set of orthonormal vectors $v_1, v_2, \ldots,$ that satisfies

$$Av_i = \sum_{j=1}^{i+1} v_j h_{ji}, \quad i = 1, 2, \ldots, \tag{2.2}$$

as long as $i \leq q - 1$, where q is the degree of the minimal polynomial of A with respect to $A^a r_0$. Furthermore, for each k, $\mathcal{K}_k(A; A^a r_0) = \text{span}\{\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_k\}$. If we now define the $N \times k$ matrix $\hat{V}_k$ by $\hat{V}_k = [v_1, v_2, \ldots, v_k]$, $k = 1, 2, \ldots,$ we can write $x_m = x_0 + \hat{V}_{m-a} \xi_m$, for some $\xi_\mathrm{m} \in \mathbb{C}^{\mathrm{m-a}}$, we should find ξ_m. Since $A^a r_m = A^a r_0 + A^{a+1} \hat{V}_{m-a} \xi_m = \beta v_1 - A^{a+1} \hat{V}_{m-a} \xi_m$. By (2.2), we have $A\hat{V}_k = \hat{V}_{k+1} \bar{H}_k; \quad k \leq q - 1$, where $\bar{H}_k \in \mathbb{C}^{(k+1)\times k}$ is an uper Hessenberge matrix. Therefore, provided $m \leq q - 1$, we have [3], $\|A^a r_m\|_2 = \min_{\xi \in \mathbb{C}^{m-a}} \|\beta e_1 - \hat{H}_m \xi\|_2$, where $\hat{H}_m = \bar{H}_m \bar{H}_{m-1} \ldots \bar{H}_{m-a}$. We can reduce the $n \times (m - a)$ least squares problem of (2.1) to the $(m + 1) \times (m - a)$ least squares problem and $\hat{H}_m \in \mathbb{C}^{(m+1)\times(m-a)}$. For more details we refer the reader to [3].
Now, let $\bar{H}_{i\bullet}^{(m)}$ and $\bar{H}_{\bullet j}^{(m)}$ represent the row i and the column j of $\bar{H}_m$, respectively. By partitioning $\bar{H}_m$ and $\hat{H}_m = \bar{H}_m \bar{H}_{m-1}$ as

$$\bar{H}_m = \begin{bmatrix} \bar{H}_{1\bullet}^{(m)} & \bar{R}_m \end{bmatrix}^T \text{ and } \widehat{H}_m = \begin{bmatrix} d_m^T & F_m \end{bmatrix}^T, \tag{2.3}$$

respectively, where $\bar{R}_m$ is an $m \times m$ upper triangular matrix, F_m is an $m \times (m-1)$ upper Hessenberg matrix, and $d_m^T \in \mathbb{R}^{1\times(m-1)}$, we see that

$$d_2^T = \bar{H}_{1\bullet}^{(2)} \bar{H}_{\bullet 1}^{(1)}, \; d_{m+1}^T = (d_m^T | \bar{H}_{1\bullet}^{(m+1)} \bar{H}_{\bullet m}^{(m)}), \; m = 2, 3, \ldots \tag{2.4}$$

and

$$F_2 = \bar{R}_2 \bar{H}_{\bullet 1}^{(1)}, \; F_{m+1} = (\tilde{F}_m | \bar{R}_{m+1} \bar{H}_{\bullet m}^{(m)}), \; m = 2, 3, \ldots, \tag{2.5}$$

where $\tilde{F}_m = \begin{bmatrix} F_m & 0 \end{bmatrix}^T$. If m steps of the Arnoldi process have been taken and $\hat{V}_{m+1}$, the Arnoldi basis associated with DGMRES is of full rank, then F_m is of full rank. This result follows from the fact that

for the elements $\hat{h}_{i+2,i}$, $i = 1, 3, \ldots, m-1$, of $\hat{H}$, we have $\hat{h}_{i+2,i} = \bar{h}_{i+2,i+1}\bar{h}_{i+1,i} \neq 0$. In what follows we suppose that F_m is of full rank. In order to get the solution ξ_m of the least square problem, we can consider the normal equation $\hat{H}_m^T\hat{H}_m\xi_m = \beta\hat{H}_m^T e_1$. The use of (2.3) implies that $\left(d_m d_m^T + F_m^T F_m\right)\xi_m = \beta\widehat{H}_m^T e_1 = \beta d_m$. Let $\lambda_m = \beta - d_m^T\xi_m$, then, we have $F_m^T F_m\xi_m = \lambda_m d_m$. By defining $u_m = \frac{\xi_m}{\lambda_m}$, we can obtain $F_m^T F_m u_m = d_m$. For solving this positive definite system, we form the QR factorization $Q_m F_m = \begin{bmatrix} R_m & o \end{bmatrix}^T$, where $R_m \in \mathbb{C}^{(m-1)\times(m-1)}$ is an nonsingular upper triangular matrix and $Q \in \mathbb{R}^{m\times m}$ is an unitary matrix. This gives $R_m^T R_m u_m = d_m$. With setting $z_m = R_m u_m$, the vector u_m can be computed by solving the lower and upper triangular systems $R_m^T z_m = d_m$ and $R_m u_m = z_m$, respectively. From (2.13), we have $u_m^T d_m = \|R_m u_m\|^2 \geq 0$. So, from $\lambda_m = \beta - d_m^T\xi_m$, imply that $\lambda_m = \frac{\beta}{1+d_m^T u} > 0$ and $\xi_m = \lambda_m u_m$, which can be used for computing λ_m and ξ_m. We note that F_{m+1} can be obtained as a simple update of F_m by first appending a row of zeros at the bottom of F_m and following that by appending the $(m+1)$-vector $\bar{R}_{m+1}\bar{H}_{\bullet m}^{(m)}$ as the mth column. After forming F_m, for obtaining R_m, we can factorize F_m by appling a series of Givens rotations to the columns of F_m. We can show that 2-norm of Ar_m is $\|Ar_m\|_2 = \lambda_m\sqrt{1 + \|z_m\|_2^2}$. Now, we summarize the steps of the new method, called GrGMRES method, as follows:

Algorithm 1 GrGMRES algorithm

1. Pick x_0 and compute $r_0 = b - Ax_0$ and Ar_0.
2. Compute $\beta = \|A^a r_0\|_2$ and set $v_1 = \beta^{-1}(A^a r_0)$.
3. Orthogonalize the vectors: $A^a r_0, A^{a+1} r_0, \ldots,$ by the modified Gram-Schmidt process,
   For $i = 1, 2, \ldots,$ do
   Compute $h_{ji} = (v_j, Av_i), j = 1, 2, \ldots, i.$ and $\hat{v}_i = Av_i - \sum_{j=1}^{i} v_j h_{ij}$.
   Let $h_{i+1,i} = \|\hat{v}_i\|$ and set $v_{i+1} = \hat{v}_i / h_{i+1,i}$.
4. For $k = 1, 2, \ldots,$ form the matrices $\hat{V}_k \in \mathbb{C}^{N\times k}$ and $\bar{H}_k \in \mathbb{C}^{(k+1)\times k}$.
5. Form the vector d_m and the matrix F_m by using the recursive formula
$$d_2^T = \bar{H}_{1\bullet}^{(2)}\bar{H}_{\bullet 1}^{(1)},\ d_{k+1}^T = (d_k^T | \bar{H}_{1\bullet}^{(k+1)}\bar{H}_{\bullet k}^{(k)}),\ k = 2, 3, \ldots, m-1$$
   and
$$F_2 = \bar{R}_2\bar{H}_{\bullet 1}^{(1)},\ F_{k+1} = (\tilde{F}_k | \bar{R}_{k+1}\bar{H}_{\bullet k}^{(k)}),\ k = 2, 3, \ldots, m-1,$$
   where$\tilde{F}_k = \begin{bmatrix} F_k & 0 \end{bmatrix}^T$ and $\bar{R}_k$ is defined in (2.3).
6. Compute the QR factorization of F_m.
7. Solve $R_m^T z_m = d_m$ and $R_m u_m = z_m$.
8. Compute $\lambda_m = \frac{\beta}{1+d_m^T u_m}$ and $\xi_m = \lambda_m u_m$.
9. Compute $x_m = x_0 + \hat{V}_{m-1}\xi_m$, and $\|Ar_m\|_2 = \lambda_m\sqrt{1 + \|z_m\|_2^2}$.

2.1. **Numerical examples.** we tested th GrGMRES method for example taken from the paper dy A. Sidi [3]. Our example, which have a singular coefficient matrix with index one, is derived by the finite difference method for elliptic partial differential equations. The numerical experiments presented in this subsection were computed in double precision using some MATLAB codes on a Pentium 4 PC, with a 3.40 GHz CPU and 8GB of RAM. All iterations started from the zero vector for initial x_0 and terminated when the curren titeration satisfes $\|Ar_m\|_2 \leq 10^{-12}$.

Example We compute the linear system $Ax = b$ by discretizing Poisson equation with Neumann boundary conditions:

$$\begin{cases} (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})u(x,y) = f(x,y), & (x,y) \in \Omega = [0,1] \times [0,1] \\ \frac{\partial}{\partial n}u(x,y) = \varphi(x,y) & x,y \in \partial\Omega. \end{cases}$$

Let M be an odd integer, we discretize the Poisson equation on a uniform grid of mesh size $h = 1/M$ via central differences, and then by taking the unknowns in the red-black order we obtain the system $Ax = b$. The numerical experiment was done for $M = 31, 63, 127$.

Table 1

Application of GrGMRES implementation to the consistent singular system for Example

Size of A	1024×1024			4096×4096			16384×16384		
Method	Its	Time	Error	Its	Time	Error	Its	Time	Error
DGMRES	164	0.33	$8.87e-013$	310	4.20	$9.92e-13$	471	29.46	$9.00e-13$
GrGMRES	164	0.24	$8.67e-013$	310	3.23	$9.88e-13$	471	24.34	$9.83e-13$

Table 2

Application of GrGMRES implementation to the inconsistent singular system for Example

Size of A	1024×1024			4096×4096			16384×16384		
Method	Its	Time	Error	Its	Time	Error	Its	Time	Error
DGMRES	164	0.38	$8.74e-013$	310	4.04	$9.90e-13$	471	29.48	$9.96e-13$
GrGMRES	164	0.31	$8.67e-013$	310	3.35	$9.88e-13$	471	22.99	$9.83e-13$

Numerical experiment show GrGMRES algorithm is a robust and efficient tool for computing the group-inverse solution of singular linear equations.

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THE GENERALIZED GAUSS-SEIDEL ITERATION METHOD FOR SOLVING ABSOLUTE VALUE EQUATIONS

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Abstract. Based on the Gauss-Seidel splitting, we present a new matrix splitting iteration method, called generalized Gauss-Seidel (GGS) iteration method, for solving the large sparse absolute value equation (AVE) $A\mathbf{x} - |\mathbf{x}| = \mathbf{b}$ where $A \in \mathbb{R}^{n\times n}$ and $\mathbf{b} \in \mathbb{R}^n$ and investigate its convergence properties. Moreover, by preconditioning AVE, a preconditioned variant of the GGS (PGGS) method is presented. Numerical experiments illustrate the efficiency of both GGS and PGGS iterations.

1. Introduction

Consider the absolute value equation (AVE)

$$A\mathbf{x} - |\mathbf{x}| = \mathbf{b}, \tag{1.1}$$

where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $\mathbf{b} \in \mathbb{R}^n$ and $|\cdot|$ denotes the absolute value. The general NP-hard linear complementarity problem (LCP) [1], which subsumes many mathematical programming problems, can be formulated as an AVE (1.1) [3].

In this paper, based on the the Gauss-Seidel splitting, we present a new matrix splitting iteration method, called generalized Gauss-Seidel

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* Speaker.

(GGS) iteration method to solve the AVE (1.1). Moreover, we consider the Gauss-Seidel iteration method to solve preconditioned AVE (1.1) and suggest an efficient preconditioner to expedite the convergence rate of the method when the coefficient matrix A is a Z-matrix. For this reason, we transform the AVE (1.1) into the preconditioned form

$$P_\beta A\mathbf{x} - P_\beta|\mathbf{x}| = P_\beta\mathbf{b}, \tag{1.2}$$

where $P_\beta = D + \beta F$. The GGS iteration method corresponding to the preconditioned system will be referred to as the preconditioned GGS (PGGS) iteration method. It is noteworthy that the motivation of choosing this preconditioner stems from [2] in which Kotakemori et al. applied the preconditioner P_β for solving the linear system $A\mathbf{x} = \mathbf{b}$ with the Gauss-Seidel method when A is a strictly diagonally dominant Z-matrix.

2. The Generalized Gauss-Seidel iterative method

Let

$$A = D - E - F, \tag{2.1}$$

where D, E and F are diagonal, strictly lower triangular and strictly upper triangular matrices, respectively. We assume that the diagonal entries of A are nonzero. By rewriting the AVE (1.1) as the fixed-point equation

$$(D - E)\mathbf{x} - |\mathbf{x}| = F\mathbf{x} + \mathbf{b}, \tag{2.2}$$

we can define the generalized Gauss-Seidel (GGS) iteration for solving (1.1) as follows

$$(D - E)\mathbf{x}^{(k+1)} - |\mathbf{x}^{(k+1)}| = F\mathbf{x}^{(k)} + \mathbf{b}, \quad k = 0, 1, 2, \ldots, \tag{2.3}$$

where $\mathbf{x}^{(0)}$ is a given initial guess.

Let $D(\mathbf{x}) = \mathrm{diag}(\mathrm{sign}(\mathbf{x}))$. Without loss of generality we can assume that the diagonal entries of A are positive (otherwise we set $\mathbf{x} = D(\mathrm{diag}(A))\mathbf{y}$ and consider the equation $AD(\mathrm{diag}(A))\mathbf{y} - |\mathbf{y}| = \mathbf{b}$).

Algorithm 1. Generalized Gauss-Seidel Iteration Method

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(1) For $k = 0, 1, \ldots,$ until convergence, Do
(2) Set $s = b_1$
(3) For $i = 1, 2, \ldots, n$, Do
(4) If $s \geq 0$, then
(5) $x_i^{(k+1)} := s/(a_{ii} - 1)$
(6) Else

(7) $x_i^{(k+1)} := s/(a_{ii}+1)$

(8) EndIf

(9) set $s = b_{i+1} - \sum_{j=1}^{i} a_{ij}x_j^{(k+1)} + \sum_{j=i+2}^{n} a_{ji}x_j^{(k)}$

(10) EndDo

(11) EndDo

Now, we are ready to describe the convergence properties of the proposed method.

Theorem 2.1. *Suppose that the AVE (1.1) is solvable. Let the diagonal entries of A are greater than one and the matrix $D - E - I$ is strictly row diagonally dominant. If*

$$\|(D-E)^{-1}F\|_\infty < 1 - \|(D-E)^{-1}\|_\infty, \tag{2.4}$$

then the equation (1.1) has a unique solution $\boldsymbol{x}^*$ and the sequence $\{\boldsymbol{x}^{(k)}\}$ obtained from (2.3) converges to the solution $\boldsymbol{x}^*$.

Theorem 2.2. *Let the AVE (1.1) be solvable and $A - I$ be an H_+-matrix. Then, the sequence $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty}$ generated by the iteration method (2.3) converges to the unique solution $\boldsymbol{x}^*$ of the AVE (1.1) for any initial vector $\boldsymbol{x}^{(0)}$.*

3. Preconditioned generalized Gauss-Seidel

In this section, we express the PGGS iteration method for solving the AVE (1.1) while the matrix A is a Z-matrix. Consider the preconditioned AVE (1.2) and let

$$P_\beta A = \tilde{D} - \tilde{E} - \tilde{F}, \tag{3.1}$$

in which $\tilde{D}$ is a diagonal matrix, $\tilde{E}$ is a strictly lower triangular matrix and $\tilde{F}$ is a strictly upper triangular matrix. Hence, the PGGS iteration method for solving (1.1) is defined as follows

$$(\tilde{D} - \tilde{E})\mathbf{x}^{(k+1)} - D|\mathbf{x}^{(k+1)}| = -\beta F|\mathbf{x}^{(k)}| + \tilde{F}\mathbf{x}^{(k)} + P_\beta \mathbf{b}, \tag{3.2}$$

or equivalently

$$D^{-1}(\tilde{D} - \tilde{E})\mathbf{x}^{(k+1)} - |\mathbf{x}^{(k+1)}| = -\beta D^{-1}F|\mathbf{x}^{(k)}| + D^{-1}\tilde{F}\mathbf{x}^{(k)} + D^{-1}P_\beta \mathbf{b}, \tag{3.3}$$

where $\mathbf{x}^{(0)}$ is a given initial guess.

Let

$$FE = \bar{D} - \bar{E} - \bar{F}, \tag{3.4}$$

where $\bar{D}$, $-\bar{E}$ and $-\bar{F}$ are the diagonal, strictly lower triangular and strictly upper triangular matrices. Then, from $P_\beta = D + \beta F$ and Eq. (3.1), we have

$$\tilde{D} = (D^2 - \beta\bar{D}), \quad \tilde{E} = (DE - \beta\bar{E}) \quad \text{and} \quad \tilde{F} = (DF - \beta FD - \beta\bar{F} + \beta F^2). \tag{3.5}$$

Note that, when the diagonal entries of $D^{-1}(\tilde{D} - \tilde{E})$ exceed 1, $\mathbf{x}^{(k+1)}$ can be obtained by Algorithm 1 by a simple modification. In the next lemma, we show that, if A is a Z-matrix with diagonal entries greater than 1, then the diagonal entries of $D^{-1}(\tilde{D} - \tilde{E})$ exceed 1.

Lemma 3.1. *Let $A = (a_{ij})$ be a Z-matrix having diagonal entries greater than 1. Assume that*

$$0 < \beta < \frac{a(a-1)}{\max_i d_{ii}}, \tag{3.6}$$

where $a = \min_{1 \le i \le n} a_{ii}$ and $\bar{D} = diag(d_{11}, d_{22}, \ldots, d_{nn})$. Then the diagonal entries of $D^{-1}(\tilde{D} - \tilde{E})$ are greater than 1.

The next theorem presents sufficient conditions for the convergence of the PGGS iteration method.

Theorem 3.2. *Let the conditions of Lemma 3.1 hold and $P_\beta = D + \beta F$, where $\beta \in \mathbb{R}$. Let also $D - E - I$ be strictly row diagonally dominant. There exists an $r > 0$ such that if $0 < |\beta| < r$, then $\|(\tilde{D} - \tilde{E})^{-1} D\|_\infty < 1$. Also, If the AVE (1.1) is solvable and*

$$\beta\|(\tilde{D} - \tilde{E})^{-1} F\|_\infty + \|(\tilde{D} - \tilde{E})^{-1}\tilde{F}\|_\infty \le 1 - \|(\tilde{D} - \tilde{E})^{-1} D\|_\infty, \tag{3.7}$$

then the solution of the equation AVE (1.1) is unique and the PGGS iteration method (3.3) converges to the solution.

The numerical results showed that the proposed schemes are applicable to large and sparse AVE and the PGGS iteration method is significantly superior to other methods.

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MULTIPLIERS WITH CLOSED RANGE ON CHARACTER INNER AMENABLE BANACH ALGEBRAS

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Abstract. In this paper we study of character inner amenable Banach algebra A. We prove that for a character inner amenable Banach algebra A, if there exists $b_0 \in A$ such that $R(b_0^2)a = aR(b_0^2)$ for every $a \in A$ and R with closed range, then for each $\varphi \in \Phi_{R(A)}$ the Banach algebra $R(A)$ is φ-inner amenable.

1. Introduction

Let A be a Banach algebra. We recall from [2] that a left multiplier on A is an element L in $L(A)$ (linear maps on A) such that $L(ab) = L(a)b,\ (a, b \in A)$ and a right multiplier on A is an element R in $L(A)$ such that $R(ab) = aR(b)$. A multiplier is a pair (L, R) where L and R are left and right multipliers on A respectively and $aL(b) = R(a)b\ (a, b \in A)$.
Recently Jabbari et al. [4] have introduced the φ-version ($\varphi \in \Phi_A$) of inner amenability. A Banach algebra A is said to be φ-inner amenable if there exists a $m \in A^{**}$ satisfying $m(\varphi) = 1$ and $m(fa) = m(af)(f \in A^*,\ a \in A)$. Such a m will sometimes be referred to as a φ-inner mean and A is said to be character inner amenable if and only if A is φ-inner amenable for every $\varphi \in \Phi_A$. They also gave several characterizations

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of φ-inner amenability. For instance, as in the case of φ-amenability in [4, Theorem 1.4], they have shown that a φ-inner mean is in fact some w^*-cluster point of a bounded net $(a_\alpha) \in A$ satisfying $\|a_\alpha a - a a_\alpha\| \to 0$, for all $a \in A$ and $\varphi(a_\alpha) = 1$ for all α; [4, Theorem 2.1].

2. Main section and results

Proposition 2.1. *Let A be a character inner amenable Banach algebra. If there exists $b_0 \in A$ such that $R(b_0^2)a = aR(b_0^2)$ for every $a \in A$ and R with closed range, then for each $\varphi \in \Phi_{R(A)}$ the Banach algebra $R(A)$ is φ-inner amenable.*

Proof. For arbitrary $\varphi \in \Phi_{R(A)}$ we can choose $\varphi(R(b_0^2)) = 1$. If now define the linear functional $\widetilde{\varphi}$ on A by $\widetilde{\varphi}(a) := \varphi(aR(b_0^2))$ for $a \in A$, then $\widetilde{\varphi}$ is multiplicative and non-zero, and the definition of $\widetilde{\varphi}$ is independent of the choice of b_0. Therefore $\widetilde{\varphi} \in \Phi_A$. As we mentioned in preliminaries, by $\widetilde{\varphi}$ amenability of A, there exist a net $(u_\alpha)_{\alpha \in I}$ in A such that $\widetilde{\varphi}(u_\alpha) = 1$ for all $\alpha \in I$, and $\|u_\alpha a - a u_\alpha\| \to 0$ for each $a \in A$. Now for each $\alpha \in I$, set $v_\alpha := u_\alpha R(b_0^2)$. So we have $\varphi(v_\alpha) = 1$ and for each $a \in A$

$$\|v_\alpha R(a) - R(a)v_\alpha\| \leq \|R(b_0^2)\| \|u_\alpha R(a) - R(a)u_\alpha\| \to 0$$

and this complete the proof.

□

A similar argument is also valid the left multiplier L on A.

Corollary 2.2. *Let A be a character amenable Banach algebra. If there exists $b_0 \in A$ such that $R(b_0^2)a = aR(b_0^2)$ for every $a \in A$ and R with closed range, then for each $\varphi \in \Phi_{R(A)}$ the Banach algebra $R(A)$ is φ-inner amenable.*

Proof. Suppose that A is character amenable. Then A has a bounded approximate identity and so by [4, Corollary 2.2] A is a character inner amenable. Then by proposition 2.1, $R(A)$ is φ-inner amenable for each $\varphi \in \Phi_{R(A)}$. □

The following corollary is a special case of proposition 2.1.

Corollary 2.3. *Let A be a commutative character inner amenable Banach algebra and $R : A \to A$ with closed range. Then for each $\varphi \in \Phi_{R(A)}$ the Banach algebra $R(A)$ is φ-inner amenable.*

We recall that a *character* on A is a non-zero homomorphism from A into the scalar field. The set of all characters on A is the character space of A, denoted by Φ_A.

Proposition 2.4. *Let A be a Banach algebra and $\varphi \in \Phi_A$. If the ideal* $\ker \varphi$ *has a central approximate identity, Then A is φ-inner amenable.*

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ITERATIVE METHODS FOR LINEAR EQUATIONS WITH MULTIPLE RIGHT-HAND SIDES AND APPLICATIONS IN QCD

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Abstract. In this paper, we focus on iterative solution to multiple linear systems with several right-hand sides. Such situations arise, for example, in lattice quantum chromodynam- ics (QCD) simulations. For solving such systems efficiently, we present new block and global Krylov subspace methods which, based on a block and global Lanczos process. Numerical examples demonstrate that the new methods can be more competitive than some other block and global solvers.

1. Introduction

In this paper we consider the linear systems with multiple right-hand sides of the form

$$Ax_i = b_i, \ i = 1, \ldots, p, \tag{1.1}$$

with the same coefficient matrix and different right-hand sides. When all the b_i exist simultaneously, we can rewrite the systems (1.1) as a

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* Speaker.

matrix equation

$$AX = B, \tag{1.2}$$

where A is an $N \times N$ sparse non-Hermitian matrix, $X = [x_1, x_2, \ldots, x_p] \in C^{N\times p}$ and $B = [b_1, b_2, \ldots, b_p] \in C^{N\times p}$ with usually $p << N$.

Systems like these arise naturally in quantum chromodynamics (QCD), the physical theory of the strong interaction between the quarks as constituents of matter. The Wilson-Dirac matrix D_W in lattice QCD arises as a discretization of the Dirac operator from QCD. It represents a periodic nearest neighbor coupling on a four-dimensional lattice, and one has 12 variables per lattice point, one for each possible combination of four spin and three color indices. The couplings are defined via a gluon background field which is drawn from a statistical distribution and therefore varies irregularily from one lattice point to the next. In lattice QCD, the task is to solve the block system (1.1) with $A = D_W$ and $b_i = e_i,\ i = 1, \ldots, 12$. where $e_1, \ldots, e_{12}$ are the first twelve unit vectors (corresponding to the twelve variables at one lattice point). We refer to [3] for details on how the Dirac-Wilson matrix is obtained from the Dirac equation

Throughout this paper, we use the follow notations. Let $E = C^{m\times n}$ for X and Y in E, we define the inner product $\langle X, Y\rangle_F = tr(Y^H X)$, where $tr(Z)$ and Z^H denote the trace and the conjugate transpose, respectively. The associated norm is the well-known Frobenius norm denoted by $\|.\|_F$. A set of members of E is said to be F -orthogonal($\perp_F$) if it is orthogonal with respect to the scalar product $\langle ., .\rangle_F$.

2. Disscusion

One can solve the systems (1.2) separately by the Krylov subspace methods but this can take considerably more computation time. Therefore, it is more acceptable to solve the block systems (1.2)simultaneously.

Methods capable of solving all right-hand sides at once in systems like (1.2) have been around for more than thirty years and are called block and global methods. In the block methods, approximate solution X_k satisfies $X_k - X_0 \in K_k(A, R_0)$ where $K_k(A, R_0)$ is the block Krylov subspace defined as:

$$K_k(A, R_0) = colspan\Big\{R_0, AR_0, \ldots, A^{k-1}R_0\Big\} \subset C^{n\times p},$$

where "colspan" is defined such that

$$K_k(A, R_0) = \Big\{ \sum_{i=0}^{i=k-1} A^i R_0 \gamma_i,\ \gamma_i \in C^{p\times p},\ i = 0, 1, \ldots, k-1\Big\},$$

which implies that

$$X_k(:,j) - X_0(:,j) \in \Big\{ \sum_{j=1}^{p} \sum_{i=0}^{k-1} A^i R_0(:,j)\gamma_{i,j},\ \gamma_{i,j} \in C, \forall(i,j),\ i = 0,1,\ldots,k-1 \Big\}$$
$$= \sum_{j=1}^{p} K_k(A, R(:,j)),$$

with $K_k(A, R(:,j))$ is a Krylov subspace generated by each initial residual $R(:,j)$. The first block solver is block biconjugate gradient (Bl-BiCG) proposed by OLeary [1] for solving the problem (1.2). Bl-BiCG computes two sets of direction matrices $\{P_0, \ldots, P_k\}$ and $\{\widetilde{P}_0, \ldots, \widetilde{P}_k\}$ that span the block Krylov subspaces $K_{k+1}(A, R_0)$ and $K_{k+1}(A^H, \widetilde{R}_0)$, where $R_0 = B - AX_0$ and $\widetilde{R}_0$ is an arbitary matrix. Letting $P_0 = R_0,\ \widetilde{P}_0 = \widetilde{R}_0$, the iterates satisfies the following recursions:

$$P_{k+1} = R_{k+1} - P_k\beta_k, \quad \widetilde{P}_{k+1} = \widetilde{R}_{k+1} - \widetilde{P}_k\widetilde{\beta}_k,$$
$$R_{k+1} = R_k - AP_k\alpha_k, \quad \widetilde{R}_{k+1} = \widetilde{R}_k - A^H\widetilde{P}_k\widetilde{\alpha}_k,$$

By the biorthogonality conditions

$$\widetilde{R}_j^H R_k = 0, \quad \widetilde{P}_j^H A P_k = 0, \quad k > j,$$

coefficients can be obtained as:

$$\beta_k = -(\widetilde{R}_k^H R_k)^{-1}\widetilde{R}_{k+1}^H R_{k+1}, \quad \widetilde{\beta}_k = -(R_k^H\widetilde{R}_k)^{-1}R_{k+1}^H\widetilde{R}_{k+1},$$
$$\alpha_k = (\widetilde{P}_k^H A P_k)^{-1}\widetilde{R}_k^T R_k, \quad \widetilde{\alpha}_k = (P_k^H A^H\widetilde{P}_k)^{-1}R_k^H\widetilde{R}_k.$$

For global methods, matrix Krylov subspace $\mathbb{K}_k(A, R_0)$ can be defined as:

$$\mathbb{K}_k(A, R_0) = \{R_0, AR_0, \ldots, A^{k-1}R_0\} \subset C^{n\times p}$$
$$= \Big\{ \sum_{i=0}^{i=k-1} \alpha_i A^i R_0,\ \alpha_i \in C,\ i = 0,1,\ldots,k-1 \Big\}.$$

Recently, Jbilou et al. [2] have presented the global FOM (Gl-FOM) and global GMRES (Gl-GMRES) algorithms for solving the nonsymmetric matrix equation (1.2).
Similarly, the global biconjugate gradient (Gl-BiCG) algorithm can be deriven as:

$$P_{k+1} = R_{k+1} - \beta_k P_k, \quad \widetilde{P}_{k+1} = \widetilde{R}_{k+1} - \overline{\beta}_k\widetilde{P}_k,$$
$$R_{k+1} = R_k - \alpha_k AP_k, \quad \widetilde{R}_{k+1} = \widetilde{R}_k - \overline{\alpha}_k A^H\widetilde{P}_k,$$
$$\beta_k = -\frac{(R_{k+1}, \widetilde{R}_{k+1})_F}{(R_k, \widetilde{R}_k)_F}, \quad \alpha_k = \frac{(R_k, \widetilde{R}_k)_F}{(AP_k, \widetilde{P}_k)_F}.$$

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USING FRAMES OF SUBSPACES IN ITERATIVE METHODS FOR SOLVING OPERATOR EQUATIONS

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Abstract. We apply frames of subspaces in order to give some approximated solutions to an operator equation. We design some iterative methods based on knowledge of the bounds of frame of subspaces and then investigate the convergence and optimality of them.

1. Introduction and preliminaries

Let that H be a separable Hilbert space and Λ be a countable indexing set. For a family of weights $\{v_\lambda\}_{\lambda\in\Lambda}$, i.e, $v_\lambda > 0$ for all $\lambda \in \Lambda$, a family of subspaces $\{H_\lambda\}_{\lambda\in\Lambda}$ of a Hilbert space H is called a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ for H, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{\lambda\in\Lambda} v_\lambda^2 \|\pi_{H_\lambda}(f)\|^2 \leq B\|f\|^2 \quad \forall f \in H, \tag{1.1}$$

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where π_{H_λ} denotes the orthogonal projection onto the subspace H_λ. As in the well known frame situation, the frame operator $S_{H,v}$ for $\{H_\lambda\}_{\lambda\in\Lambda}$ and $\{v_\lambda\}_{\lambda\in\Lambda}$ is defined by

$$S_{H,v}(f) = \sum_{\lambda\in\Lambda} v^2 \pi_{H_\lambda}(f).$$

The frame operator $S_{H,v}$ for $\{H_\lambda\}_{\lambda\in\Lambda}$ and $\{v_\lambda\}_{\lambda\in\Lambda}$ is a bounded, self-adjoint and invertible operator on H with $AI \leq S_{H,v} \leq BI$, where A and B are the bounds of the frame of subspaces. Further, the following reconstruction formula satisfies:

$$f = \sum_{\lambda\in\Lambda} v_\lambda^2 S_{H,v}^{-1} \pi_{H_\lambda}(f) \quad \forall f \in H.$$

It is proved that $\{S_{H,v}^{-1} H_\lambda\}_{\lambda\in\Lambda}$ is a frame with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$. For more details see [2].
We use frames of subspaces in order to give some approximated solutions to the operator equation

$$Lu = f, \tag{1.2}$$

where $L : H \to H$ is a bounded, invertible and self adjoint on a separable Hilbert space H.

Proposition 1.1. *Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$, and let $L : H \to H$ be a bounded invertible operator on H. Then $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ is a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$.*

In this case if u is the solution of equation (1.2) and $S^{'}$ is the frame operator of the frame of subspaces $\{LH_\lambda\}$ then

$$u = \sum_{\lambda\in\Lambda} v_\lambda^2 S_{H,v}'^{-1} \pi_{LH_\lambda} u.$$

In [4, 1] you can see the development of numerical methods for solving the problem by using frames.

2. An iterative method based on the knowledge of the bounds of frame of subspaces

Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ for a separable Hilbert space H with frame operator $S_{H,v}$. By Proposition 1.1, $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ also is a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$. We denote the frame operator for $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ and $\{v_\lambda\}_{\lambda\in\Lambda}$, by $S'_{H,v}$. Also since L is bounded invertible then there exist two positive constants c_1 and c_2 such that

$$c_1\|u\|_H \leq \|Lu\|_H \leq c_2\|u\|_H, \quad \forall u \in H. \tag{2.1}$$

Theorem 2.1. *Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ for H with frame operator $S_{H,v}$ and let L be as in (1.2). Let $u_0 = 0$ and for $k \geq 1$,*

$$u_k = u_{k-1} + \frac{2}{c_1^2A + c_2^2B} LS'_{H,v}(f - Lu_{k-1}),$$

where $S'_{H,v}$ is the frame operator for the frame of subspaces $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ with bounds A, B, and c_1, c_2 as in (2.1). Then

$$\|u - u_k\|_H \leq (\frac{c_2^2B - c_1^2A}{c_1^2A + c_2^2B})^k \|u\|_H.$$

In particular the vectors u_k converges to u as $k \to \infty$.

3. Chebyshev method based on the upper and lower bounds of a frame of subspaces

Now let $h_n = \sum_{k=1}^n d_{n_k} u_k$, where u_k as in Theorem 2.1, such that $\sum_{k=1}^n d_{n_k} = 1$, that is guaranteed if $u_k = u$ for all $1 \leq k \leq n$ then $h_n = u$. Defining $R = I - \frac{2}{c_1^2A+c_2^2B} LS^{'}L$ and $Q_n(x) = \sum_{k=1}^n d_{n_k} x^k$, we obtain

$$u - h_n = Q_n(R)(u - u_0), \tag{3.1}$$

therefore

$$\begin{aligned} \|u - h_n\| = \|Q_n(R)(u - u_0)\| &\leq \|Q_n(R)\|\|u - u_0\| \\ &\leq \max_{|x|\leq\alpha_0} |Q_n(x)|\|u - u_0\|, \end{aligned} \tag{3.2}$$

where $\alpha_0 = \frac{c_2^2B - c_1^2A}{c_2^2B + c_1^2A}$.
The aim is to minimize this error. This is done by Chebyshev polynomials, satisfying the recurrence relation

$$C_0(x) = 1,\ C_1(x) = x,\ C_n(x) = 2xC_{n-1}(x) - C_{n-2}(x),\quad \forall n \geq 2.$$

For more details see [3].
Now we can design the following algorithm base on the Chebyshev method and using frames of subspaces.

Algorithm$[L, \epsilon, A, B, c_1, c_2] \to u_\epsilon$
(i) Let $\alpha_0 = \frac{c_2^2B - c_1^2A}{c_2^2B + c_1^2A}$, $\sigma = \frac{c_2\sqrt{B} - c_1\sqrt{A}}{c_2\sqrt{B} + c_1\sqrt{A}}$
(ii) $h_0 := 0$, $h_1 := \frac{2}{c_1^2A+c_2^2B} LS^{'}f$, $\beta_1 = 2$, $n = 1$
(iii) While $\frac{2\sigma^n}{1+\sigma^{2n}} \frac{\|f\|}{c_1} > \epsilon$
(1) $n := n + 1$
(2) $\beta_n = (1 - \frac{\alpha_0^2}{4}\beta_{n-1})^{-1}$

(3) $h_n = \beta_n(h_{n-1} - h_{n-2} + \frac{2}{c_1^2 A + c_2^2 B} LS'(f - Lh_{n-1})) + h_{n-2}, \quad n \geq 2$
(iv) $u_\epsilon := h_n$.

The following theorem holds.

Theorem 3.1. *The approximated solution h_n in the* ***Algorithm*** $[L, \epsilon, A, B, c_1, c_2]$ *satisfies*

$$\|u - h_n\| \leq \frac{2\sigma^n}{1 + \sigma^{2n}} \frac{\|f\|}{c_1}.$$

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CHARACTRIZATION OF STRONG OPERATOR TOPOLOGY CONTINUOS LINEAR MAPS ON NEST ALGEBRAS

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Abstract. Let $Alg\mathcal{N}$ be a nest algebra associated with the nest $\mathcal{N}$ on a (real or complex) Banach space X. Suppose that there with each $N \in \mathcal{N}$ complemented in $\mathcal{X}$ whenever $N_- = N$ and $\delta : Alg\mathcal{N} \longrightarrow Alg\mathcal{N}$ is a linear map. we characteraize the strongly operator topology continuos linear map δ on some nest algebra $Alg\mathcal{N}$ with property that $\delta(P) = 2P\delta(P)$ or $\delta(P) = 2P\delta(P) - P\delta(I)$ every idempotent P in $Alg\mathcal{N}$.

1. Introduction

Throughout this paper all algebras and vector spaces will be over $\mathbb{F}$, where $\mathbb{F}$ is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let $\mathcal{A}$ be analgebra with unity 1, $\mathcal{M}$ be a left $\mathcal{A}$-module and $\delta : \mathcal{A} \to \mathcal{M}$ be a linear mapping.

In recent years, several authors studied the linear (additive) maps that behave like homomorphisms, derivations or left derivations when acting on special products (for instance, see [2, 3, 6] and the references

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therein). In this article we study the continuous linear maps on nest algebras behaving like left derivations at idempotent-product.

Let $\mathcal{N}$ be a nest on a Banach space $\mathcal{X}$ and $\delta : Alg\mathcal{N} \to Alg\mathcal{N}$ be a strong operator topology continuos linear map. In this paper we describe the strongy continuos linear maps δ on some nest algebra $Alg\mathcal{N}$ with property that $\delta(P) = 2P\delta(P)$ or $\delta(P) = 2P\delta(P) - P\delta(I)$ every idempotent P in $Alg\mathcal{N}$.

The following are the notations and terminologies which are used throughout this article.

Let $\mathcal{X}$ be a Banach algebra. Let $\mathcal{X}$ be a Hilbert space. We denote by $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear operators on $\mathcal{X}$, and $\mathcal{F}(\mathcal{X})$ denotes the algebra of all finite rank operators in $\mathcal{B}(\mathcal{X})$. A subspace lattice $\mathcal{L}$ on a Hilbert space $\mathcal{X}$ is a collection of closed (under norm topology) subspaces of $\mathcal{X}$ which is closed under the formation of arbitrary intersection (denoted by $\wedge$) and closed linear span (denoted by $\vee$), and which includes $\{0\}$ and $\mathcal{X}$. For a subspace lattice $\mathcal{L}$, we define the *associated subspace lattice* $Alg\mathcal{L}$ by

$$Alg\mathcal{L} = \{T \in \mathcal{B}(\mathcal{X}) \,|\, T(N \subseteq N \; for \; all \; N \in \mathcal{L}\}.$$

A totally ordered subspace lattice $\mathcal{N}$ on $\mathcal{X}$ is called a *nest* and $Alg\mathcal{N}$ is called a nest algebra. When $\mathcal{N} \neq \{\{0\}, \mathcal{X}\}$, we say $\mathcal{N}$ is non-trivial. It is clear that if $\mathcal{N}$ is trivial, then $Alg\mathcal{N} = \mathcal{B}(\mathcal{X})$. Denote $Alg\mathcal{N} := Alg_{\mathcal{F}}\mathcal{N} \cap \mathcal{F}(\mathcal{X})$, the set of all finite rank operators in $Alg\mathcal{N}$ and for $N \in \mathcal{N}$, let $N_- = \vee\{M \in \mathcal{N} | M \subset N\}$. The identity element of nest algebras denote by I and an element P in a nest algebra is called a non-trivial idempotent if $P \neq 0, I$ and $P^2 = P$.

Let $\mathcal{N}$ be a non-trivial nest on a Banach space $\mathcal{X}$. If there exists a non-trivial idempotent $P \in Alg\mathcal{N}$ with range $P(\mathcal{X}) \in \mathcal{N}$, then we have $(I - P)(Alg\mathcal{N})P = \{0\}$ and hence

$$Alg\mathcal{N} = P(Alg\mathcal{N})P \dotplus P(Alg\mathcal{N})(I - P) \dotplus (I - P)(Alg\mathcal{N})(I - P)$$

as sum if linear spaces. This is so-called the Peirce decompositon of $Alg\mathcal{N}$. The sets $P(Alg\mathcal{N})P, P(Alg\mathcal{N})(I - P)$ and $(I - P)(Alg\mathcal{N})(I - P)$ are closed in $Alg\mathcal{N}$. In fact $P(Alg\mathcal{N})P$ and $(I - P)(Alg\mathcal{N})(I - P)$ are Banach $(P(Alg\mathcal{N})(I - P), P(Alg\mathcal{N})(I - P))$-bimodule. Also $P(Alg\mathcal{N})(I - P)$ is fithful as a left $P(Alg\mathcal{N})P$-mofule as well as right $(I-P)(Alg\mathcal{N})(I-P)$-module. For more information on nest algebras, we refer to [1].

A subspace lattice $\mathcal{L}$ on a Hilbert space $\mathbb{H}$ is called a *commutative subspace lattice*, or a *CSL*, if the projections of $\mathbb{H}$ onto the subspaces of $\mathcal{L}$ commute with each other. if $\mathcal{L}$ is a CSL, then $Alg\mathcal{L}$ is called a CSL-algebra. Eech nest algebra on a Hilbert space is a CSL-algebra.

2. Main sections and results

Theorem 2.1. [2] *Let $\mathcal{X}$ be a linear space and let $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}$ be a bilinear map satisfying*

$$a, b \in \mathcal{A}, \quad ab = 1 \Rightarrow \phi(a, b) = \phi(1, 1)$$

Then

$$\phi(a, a) = \phi(a^2, 1)$$

for all $a \in \mathcal{A}$.

Proposition 2.2. *Let $\mathcal{X}$ be a Banach algebra with unity 1 and $\mathcal{M}$ be a unital Banach left $\mathcal{A}$-module. Let $\delta : \mathcal{A} \longrightarrow \mathcal{M}$ be a continuos linear map. if δ is left derivable at 1, then δ is a Jordan left derivation.*

Corollary 2.3. *Let $\mathcal{X}$ be a Banach algebra with unity 1 and $\mathcal{M}$ be a unital Banach left $\mathcal{A}$-module. Let $x, y \in \mathcal{A}$ with $x + y = 1$ and let $\delta : \mathcal{A} \longrightarrow \mathcal{M}$ be a continuos linear map. if δ is left derivable at x and y then δ is a Jordan left derivation.*

Remark 2.4. If $\mathcal{A}$ is a CSL-algebra or a unital semisinple Banach algebra, then by [4] and [5] every continous Jordan left derivation on $\mathcal{A}$ is zero. Hence from 2.2 every continous linear map $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ which is left derivable at 1 is zero.

the following is our main result. Now, we characterize the strongly operator topology continous (generalized) left derivations on some nest algebras.

Theorem 2.5. *Let $\mathcal{N}$ be a nest on a Banach space $\mathcal{X}$, with each $N \in \mathcal{N}$ complemented in $\mathcal{X}$ whenever $N_- = N$. Let $\delta : Alg\mathcal{N} \longrightarrow Alg\mathcal{N}$ be a strong operator topology continuos linear map. Then:*

(i) *If $\delta(P) = 2P\delta(P)$ for every idempotent P in $Alg\mathcal{N}$, then $\delta = 0$.*
(ii) *If $\delta(P) = 2P\delta(P) - P\delta(I)$ for every idempotent P in $Alg\mathcal{N}$, then $\delta(a) = a\delta(I)$ for all $a \in Alg\mathcal{N}$.*

Proof. (i) For arbitrary idempotent operator $P \in Alg\mathcal{N}$, by hypothesis we have $\delta(P) = 0$. Notice that $Alg_{\mathcal{F}}\mathcal{N}$ is contained in the linear span of the idempotent in $Alg\mathcal{N}$. So we see that $\delta(F) = 0$ for all $F \in Alg_{\mathcal{F}}\mathcal{N}$. since δ is continous under SOT and $\overline{Alg_{\mathcal{F}}\mathcal{N}}^{SOT} = Alg\mathcal{N}$. we find that $\delta(a) = 0$ for all $a \in Alg\mathcal{N}$.

(ii) Difine $\Delta : Alg\mathcal{N} \longrightarrow Alg\mathcal{N}$ by $\Delta(a) = \delta(a) - a\delta(I)$. So by (i) we have $\Delta = 0$ and $\delta(a) = a\delta(I)$ for all $a \in Alg\mathcal{N}$. □

It is obvious that the nests on Hilbert spaces, finite nests and the nests having order-type $w + 1$ or $1 + w^*$, where w is the order-type of the nutural numbers, satisfy the condition in 2.5 automatically.

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SOME NORM INEQUALITIES ON POSITIVE BLOCK MATRICES

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Abstract. Let H be a positive definite matrix partitioned into four $n \times n$ blocks. We have a sharp inequality between $||H||$ and the norm of partial trace of H, if the off-diagonal blocks of H are Hermitian. In addition there is a norm inequality for H in the case the right upper block of H is accretive. In this paper is given more general norm inequalities for H with accretive off-diagonal blocks.

1. Introduction

Let M_n denote the space of $n \times n$ complex matrices, and let M_n^+ be the subspace of positive semi-definite matrices. We refer the reader to [1] for reminding basic concepts like symmetric norm and Ky Fan norm. A positive 2×2-block matrix $H \in M_{n+m}^+$ can be written in the form

$$H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

with $A \in M_n^+$, $B \in M_m^+$.

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* Speaker.

If the off-diagonal blocks of H are Hermitian, $(X = X^*)$ then H has the form

$$H = \begin{pmatrix} A & X \\ X & B \end{pmatrix} \in M_{2n}^+.$$

In this case we have a remarkable norm inequality:

Theorem 1.1. [4] *Given any matrix $H \in M_{2n}^+$ written in Hermitian blocks in M_n, we have*

$$||H|| = || \begin{pmatrix} A & X \\ X & B \end{pmatrix} || \leq ||A + B||$$

for all symmetric norms.

In general case which $H \in M_{n+m}^+$ we have a decomposition for H.

Lemma 1.2. [2] *For every matrix H in M_{n+m}^+ there exist some unitaries $U, V \in M_{n+m}$ such that*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} V^*. \tag{1.1}$$

The above decomposition yields a norm inequality for all symmetric (or unitarily invariant) norms:

$$|| \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} || \leq ||A|| + ||B||. \tag{1.2}$$

In general, the sum of the norms $||A|| + ||B||$ can not be replaced by the norm of the sum $||A + B||$ in (1.2), even if X is normal [3].

We are searching for better estimations for special cases. If X is accretive we have

Theorem 1.3. [4] *If the matrix $H \in M_{2n}^+$ is written in blocks of same size such that the right upper block X is accretive, then*

$$|| \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} || \leq ||A + B|| + ||ReX||.$$

Remember $ReX = \frac{X+X^*}{2}$ and "X is accretive" means $ReX \geq 0$. In the next section we improve Theorem 1.3 and obtain more general estimations for $||H||$.

2. A generalized inequality

It is simple to see the matrix

$$W = \begin{pmatrix} \sqrt{\alpha}I & -\sqrt{1-\alpha}I \\ \sqrt{1-\alpha}I & \sqrt{\alpha}I \end{pmatrix}$$

is a unitary matrix in M_{2n} where I is the identity of M_n and $0 < \alpha < 1$. Now consider

$$K = W^* \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} W$$

which equals

$$\begin{pmatrix} \alpha A + (1-\alpha)B + \sqrt{\alpha - \alpha^2}(X + X^*) & Y \\ Y^* & (1-\alpha)A + \alpha B - \sqrt{\alpha - \alpha^2}(X + X^*) \end{pmatrix}$$

or

$$\begin{pmatrix} lphaA + (1-\alpha)B + 2\sqrt{\alpha - \alpha^2}ReX & Y \\ Y^* & (1-\alpha)A + \alpha B - 2\sqrt{\alpha - \alpha^2}ReX \end{pmatrix} \tag{2.1}$$

The matrix K is unitarily equivalent to $H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ and so has the same norm, for every symmetric norm.

Theorem 2.1. *For any matrix $H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ in M_{2n}^+ written in blocks of same size such that the right upper block X is accretive, we have*

$$||H|| \leq ||\alpha A + (1-\alpha)B|| + ||(1-\alpha)A + \alpha B|| + 2\sqrt{\alpha - \alpha^2}||ReX|| \tag{2.2}$$

for all symmetric norms $||.||$, where $0 \leq \alpha \leq 1$.

Proof. By ky Fan dominance principle it suffices to show (2.2) for Ky Fan k-norms $||.||_k$, $k = 1, 2, \cdots, 2n$.

Applying (1.1) and (2.1) yields

$$\begin{aligned} ||H||_k &= ||W^*HW||_k \\ &\leq ||\alpha A + (1-\alpha)B + 2\sqrt{\alpha - \alpha^2}ReX||_k \\ &+ ||(1-\alpha)A + \alpha B - 2\sqrt{\alpha - \alpha^2}ReX||_k \end{aligned}$$

for $0 < \alpha < 1$. Since $ReX \geq 0$, we obtain

$$\begin{aligned} ||H||_k &\leq ||\alpha A + (1-\alpha)B + 2\sqrt{\alpha - \alpha^2}ReX||_k + ||(1-\alpha)A + \alpha B||_k \\ &\leq ||\alpha A + (1-\alpha)B||_k + 2\sqrt{\alpha - \alpha^2}||ReX||_k \\ &+ ||(1-\alpha)A + \alpha B||_k. \end{aligned}$$

for all $0 < \alpha < 1$. Tending $\alpha \to 0$ and $\alpha \to 1$ and applying Ky Fan dominance principle show the inequality (2.2) for $0 \leq \alpha \leq 1$.

□

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ON THE OPERATOR JENSEN-MERCER INEQUALITY

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Abstract. We prove the operator Jensen-Mercer inequality for non-unital n-tuples of positive linear mappings and convex functions and superquadratic functions.

1. Introduction

The well-known Jensen-Mercer inequality for the convex functions states that if f is a convex function on an interval $[m, M]$, then

$$f(m + M - \sum_{j=1}^{k} w_j x_j) \leq f(m) + f(M) - \sum_{j=1}^{k} w_j f(x_j) \tag{1.1}$$

for every $x_1, x_2, ..., x_k \in [m, M]$ and nonnegative real numbers $w_1, w_2, ...,$ w_k with $\sum w_j = 1$.

The following operator variant of (1.1) was given in [3].

Theorem 1.1. *Let $A_1, A_2, ..., A_k \in B(H)$ be selfadjoint operators with spectra in $[m, M]$ and let $\Phi_1, \Phi_2, ..., \Phi_k$ be positive linear maps from $B(H)$ into $B(K)$ with $\sum_{j-1}^{k} \Phi_j(1_H) = 1_K$. If $f \in C([m, M])$ is a*

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convex function, then

$$f\left(m1_K + M1_K - \sum_{j=1}^{k}\Phi_j(A_j)\right) \leq f(m)1_K + f(M)1_K - \sum_{j=1}^{k}\Phi_j(f(A_j)). \tag{1.2}$$

In [1], a refinement of (1.2) was obtained for superquadratic functions. Here, we use ideas from [2, 4] to show that the mentioned results hold true for positive linear maps $\Phi_1, ..., \Phi_k$ with $\sum_{j=1}^{k}\Phi_j(1_H) \leq 1_k$.

2. Main results

Theorem 2.1. *Let f be a continuous convex function on interval $[m, M]$ containing 0 with $f(0) \geq 0$ and $A_1, A_2, ..., A_k \in B(H)$ be self-adjoint operators with spectra in $[m, M]$. Suppose $\Phi_1, \Phi_2, ..., \Phi_k$ are positive linear maps from $B(H)$ into $B(K)$ with $\sum_{j=1}^{k}\Phi_j(1_H) \leq 1_K$. Then*

$$f\left(m1_K + M1_K - \sum_{j=1}^{k}\Phi_j(A_j)\right) \leq f(m)1_K + f(M)1_K - \sum_{j=1}^{k}\Phi_j(f(A_j)).$$

Proof. Put $C = 1_K - \sum_{j=1}^{k}\Phi_j(1_H) \geq 0$ and $D = \sqrt{C}$. Define the positive linear map ϕ from B(H) to B(K) by

$$\Phi(A) = DAD.$$

Now we have

$$f\left(m.1_K + M.1_K - \sum_{j=1}^{k}\Phi_j(A_j)\right) =$$

$$f\left(m.1_K + M.1_K - \sum_{j=1}^{k}\Phi_j(A_j) - \Phi(0)\right)$$

$$\leq f(m).1_K + f(M).1_K - \sum_{j=1}^{k}\Phi_j(f(A_j)) - \Phi(f(0))$$

$$\leq f(m).1_K + f(M).1_K - \sum_{j=1}^{k}\Phi_j(f(A_j)).$$

□

Corollary 2.2. *Let $[m, M]$ be an interval containing $-1, 1$ and $\Phi_1, \Phi_2, ..., \Phi_k$ be positive linear maps from $B(H)$ into $B(K)$ such that $\sum_{j=1}^{k} \Phi_j(1_H) \leq 1_K$. If $f \in C([m, M])$ is a convex function, then*

$$f\left(m.1_K + M.1_K + \sum_{j=1}^{n} \Phi_j(1_H) - \sum_{j=n+1}^{k} \Phi_j(1_H)\right) \leq$$

$$f(m).1_K + f(M).1_K - f(-1)\sum_{j=1}^{n} \Phi_j(1_H) - f(1)\sum_{j=n+1}^{k} \Phi_j(1_H).$$

Corollary 2.3. *Let $A_1, A_2, ..., A_k \in B(H)$ be selfadjiont operators with spectra contained in $[m, M]$ containing 0 and $w_1, w_2, ..., w_k$ be nonnegative real numbers such that $\sum_{j=1}^{k} w_j \leq W_k$. If $f \in C([m, M])$ is a convex function such that $f(0) \geq 0$, then*

$$f\left(m + M - \frac{1}{W_k}\sum_{j=1}^{k} w_j A_j\right) \leq f(m) + f(M) - \frac{1}{W_k}\sum_{j=1}^{k} w_j f(A_j).$$

Theorem 2.4. *Let $A_1, A_2, ..., A_k \in B(H)$ be selfadjiont operators with spectra contained in $[0, M]$ and $\phi_1, \phi_2, ..., \phi_k \in P[B(H), B(K)]$ positive linear maps such that $\sum_{j=1}^{k} \phi_j(I_H) \leq I_K$.If $f \in C([0, \infty])$ is a superquadratic function such that $f(0) = 0$, then*

$$f\left(M - \sum_{j=1}^{k} \phi_j(A_j)\right) \leq f(M) - \sum_{j=1}^{k} \phi_j(f(A_j))$$

$$-\frac{1}{M}\sum_{j=1}^{k} \phi_j(A_j f(M - A_j)) - \frac{1}{M}\sum_{j=1}^{k} \phi_j((M - A_j)f(A_j))$$

$$-\frac{1}{M}\left(\sum_{j=1}^{k} \phi_j(A_j)\right) f\left(M - \sum_{j=1}^{k} \phi_j(A_j)\right)$$

$$-\frac{1}{M}\left(M - \sum_{j=1}^{k} \phi_j(A_j)\right) f\left(\sum_{j=1}^{k} \phi_j(A_j)\right).$$

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