چکیده مبسوط مقالات
هفتمین سمینار
جبرخطی و کاربردهای آن
۷ و ۸ اسفند ۱۳۹۲

$$\begin{bmatrix} f(C^*AC) \le C^*f(A)C & \mathcal{N}(A)^\perp = \mathcal{R}(A^t) \\ |\langle x, y\rangle| \le \|x\|\,\|y\| & s(A+B) \prec_w s(A)+s(B) \end{bmatrix}$$

Extended Abstracts
The 7 Seminar on Linear Algebra and its Applications
26-27 February 2014


آرامگاه نادرشاه افشار – مشهد

Ferdowsi University of Mashhad
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Extended Abstracts

The 7th Seminar on Linear Algebra and its Applications

Ferdowsi University of Mashhad, Iran
26-27th February 2014

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Welcome

We are pleased to organize the 7$^{\text{th}}$ Seminar on Linear Algebra and its Applications during 26-27 February 2014 in Iran. The seminar provides a forum for mathematicians worldwide and scholar students to present their latest results about all aspects of linear algebra and its applications and a means to discuss their recent researches with each other. The organizing committee of the seminar warmly welcomes the participants to Mashhad, hoping that their stay in Mashhad will be happy and fruitful. About 200 participants have taken part in this seminar. We have made every effort to make the seminar as worthwhile as possible. We wish to express our thanks to all whose help has made this gathering possible. In particular, we would like to express our gratitude to the administration of Ferdowsi University of Mashhad and the Iranian Mathematical Society.

Chair
Mohammad Sal Moslehian

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Short Presentations

The Extended Abstracts of Talks
The 7$^{\text{th}}$ Seminar on Linear Algebra and its Applications
26-27$^{\text{th}}$ February 2014, Ferdowsi University of Mashhad, Iran

A GENERALIZATION OF ORTHOGONAL BASIS

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Abstract. Let H be a Hilbert space and $(h_i)_{i=1}^m$ be a finite sequence of vectors in H. The sequence $(h_i)_{i=1}^m$ is called to benon-orthogonal neighbors (NON) sequence provided that, $\langle v_i, v_j \rangle \neq 0$ if and only if $|i - j| \leq 1$ for all $i, j \in \{1, 2, ..., m\}$. In this paper we will prove that a necessary and sufficient condition for existing a non-orthogonal neighbors sequence with m vectors in a Hilbert space H is that $m - 1 \leq dim(H)$. Furthermore, we extend the notion of non orthogonal neighbors such that orthogonal basis is an especial case of it and will offer a conjecture regarding this notion.

1. Introduction

The concept of orthogonality goes a long way back in time. Usually this notion is associated with Hilbert spaces or, more generally, inner product spaces. Various extensions have been introduced through the decades. G. Birkhoff [2], B. D. Roberts [4], R. C. James [3] and I. Singer [5] offered various definitions of this notion even in normed spaces.
In this short paper we introduce another kind of orthogonality (with a taste of combinatorics) in Hilbert spaces which includes traditional definition of orthogonality as an especial case. Throughout this paper

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* Speaker.

H will denote a Hilbert space over a real or complex field. The vectors u and v of H are called to be orthogonal provided that $\langle u, v\rangle = 0$ and a set or sequence is called to be orthogonal if all pair of distinct vectors in the set or sequence are orthogonal. To find more details on Hilbert spaces see [1].

Definition 1.1. Let H be an inner product space and $(h_i)_{i=1}^m$ be a finite sequence of vectors in H. For a non negative integer k, the sequence $(h_i)_{i=1}^m$ is called to be **k-non orthogonal neighbors** sequence provided that, $\langle v_i, v_j\rangle \neq 0$ if and only if $|i-j| \leqslant k$ for all $i, j \in \{1, 2, ..., m\}$.

Obviously a 0-non orthogonal neighbors sequence is an orthogonal sequence with non zero vectors. Therefore if there exists a 0-non orthogonal neighbors sequence with m vectors in H the dimension of H is at least m.

In a 1-non orthogonal neighbors sequence the vectors which are not neighbors, i.e. are not consecutive, are orthogonal together. For given real number x, let $\lceil x\rceil$ denote the smallest integer more than or equal to x. Let $(h_i)_{i=1}^m$ be a 1-non orthogonal neighbors sequence in H. Since the subsequence $(h_{2i-1})_{i=1}^{\lceil m/2\rceil}$ is an orthogonal sequence in H, the dimension of H is at least $\lceil m/2\rceil$ but we will prove that the dimension of H is much more than $\lceil m/2\rceil$, in fact, it is at least $m-1$.

Our aim is to find some fantastic facts regarding 1-non orthogonal neighbors sequences. More precisely we will prove that a necessary and sufficient condition for existing a 1-non orthogonal neighbors sequence with m vectors in a Hilbert space H is that $m-1 \leq dim(H)$. Our conjecture is that a necessary and sufficient condition in order to exist a k-non orthogonal neighbors sequence with m vectors in a Hilbert space H is that $m-k \leq dim(H)$. To simplify notation, the expression "non orthogonal neighbors" or (NON) is used instead of 1-non orthogonal neighbors.

2. Main results

The following lemma is a vital tool to prove the first theorem.

Lemma 2.1. *Let H be a Hilbert space of dimension n. There is no non orthogonal neighbors sequence of length more than $n+1$ in H which spans H.*

Proof. Towards a contradiction, suppose there exists a non orthogonal sequence $(h_i)_{i=1}^m$ such that $m \geq n+2$ and $H = span\{h_i\}_{i=1}^m$. By the definition of (NON) we have $h_i \neq 0 (i = 1, 2, ..., m)$ and $\langle h_i, h_j\rangle = 0$ if

and only if $|i-j|>1$. The set $\{h_i\}_{i=1}^m$ spans H, so we can find a subset β of $\{h_i\}_{i=1}^m$ of n vectors which is a basis for H. There are three cases to consider. In each of these cases we obtain a contradiction.
(a) If h_1 and h_m are not the elements of β, then h_1 can be written as $h_1 = c_2h_2 + \cdots + c_{m-1}h_{m-1}$ where c_i's are scalars. Taking the inner product of both sides of this equation with h_m to conclude $c_{m-1} = 0$. Now, h_1 can be written as $h_1 = c_2h_2 + \cdots + c_{m-2}h_{m-2}$. Similarly, take the inner product of both sides of the reduced linear combination of h_1 with $h_{m-1}, h_{m-2}, \cdots, h_3$, consecutively, to get $c_{m-2} = c_{m-3} = \cdots = c_2 = 0$. This implies that $h_1 = 0$, which is a contradiction.
(b) If just one of the h_1 or h_m isn't in β, without loss of generality we may assume that $h_1 \in \beta$. Since β has n elements, there exists a $j \in \{2, 3, \cdots, m-1\}$ such that h_j isn't in β. If $j = m-1$, we can write $h_{m-1} = c_1h_1 + \cdots + c_{m-2}h_{m-2}$. Taking the inner product of both sides of this equation with h_m to get $\langle h_{m-1}, h_m\rangle = 0$, which is not true due to the definition of (NON), so $j \neq m-1$. Now, h_j has a representation of the form

$$h_j = c_1h_1 + \cdots + c_{j-1}h_{j-1} + c_{j+1}h_{j+1} + \cdots + c_{m-1}h_{m-1}.$$

Taking the inner product of both sides of this equation with h_m and conclude that $c_{m-1} = 0$. Now, h_j can be written as $h_j = c_1h_1 + \cdots + c_{j-1}h_{j-1} + c_{j+1}h_{j+1} + \cdots + c_{m-2}h_{m-2}$. Similarly, taking the inner product of both sides of the reduced linear combination of $_j$ with $h_{m-1}, h_{m-2}, \cdots, h_{j+2}$, consecutively, to get $c_{m-2} = \cdots = c_{j+1} = 0$. This implies that $h_j = c_1h_1 + \cdots + c_{j-1}h_{j-1}$. Taking the inner product of both sides of this equation with h_{j+1} to achieve $\langle h_j, h_{j+1}\rangle = 0$, which contradicts the definition of (NON).
(c) Finally, if both h_1 and h_m belong to β, then $\beta = \{h_{m_1}, h_{m_2}, \cdots, h_{m_n}\}$ where $1 = m_1 < m_2 < \cdots < m_{n-1} < m_n = m$. Since $m_i \in \{1, 2, \cdots, m\}$ $(1 \le i \le n)$ and $m > n+1$ there exists an integer j such that $m_j - m_{j-1} > 1$. Choose an integer k such that $m_{j-1} < k < m_j$. We claim that $\beta' = \beta \bigcup \{h_k\} - \{h_{m_n}\}$ is a basis for H. If we prove this claim the proof is complete by part (b). To prove the claim, h_k has a representation of the form

$$h_k = c_{m_1}h_{m_1} + \cdots + c_{m_{j-1}}h_{m_{j-1}} + c_{m_j}h_{m_j} + \cdots + c_{m_n}h_{m_n}.$$

If $c_{m_n} \neq 0$, then h_{m_n} has a linear combination by the elements of $\beta \bigcup \{h_k\}$, and so β' is a basis. Therefore we can assume that $c_{m_n} = 0$. This implies that h_k has a representation of the form

$$h_k = c_{m_1}h_{m_1} + \cdots + c_{m_{j-1}}h_{m_{j-1}} + c_{m_j}h_{m_j} + \cdots + c_{m_{n-1}}h_{m_{n-1}}.$$

Taking the inner product of both sides of this equation with $h_{m_{n-1}+1}$ to get $c_{m_{n-1}} = 0$. Similarly, taking the inner product of both sides of the reduced linear combination of h_k with $h_{m_{n-2}+1}, \cdots, h_{m_j+1}$, consecutively, to get $c_{m_{n-2}} = \cdots = c_{m_j} = 0$. This implies that h_k has a representation of the form

$$h_k = c_{m_1} h_{m_1} + \cdots + c_{m_{j-1}} h_{m_{j-1}}.$$

Now, taking the inner product of both sides of this equation with h_{k+1} to get $\langle h_k, h_{k+1} \rangle = 0$, which is an obvious contradiction. □

Theorem 2.2. *If a non orthogonal neighbors sequence $(h_i)_{i=1}^m$ in a Hilbert space H spans H then the dimension of H is either $m-1$ or m.*

Theorem 2.3. *There exists a non orthogonal neighbors sequence of length m in Hilbert space H if and only if $m - 1 \leq dim(H)$.*

This leads us to state the following conjecture which is an extension of Theorem 2.3.

Conjecture 2.4. *Let m be a fixed positive integer. For each integer $k \in \{1, 2, ..., m-1\}$ there exists a k-non orthogonal neighbors sequence of length m in Hilbert space H if and only if $m - k \leq dim(H)$.*

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ON THE GRAPHS ASSOCIATED TO MATRIX RINGS

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Abstract. In this talk, we study the zero-divisor graph associated to some special subrings of upper triangular matrix rings.

1. Introduction

Let R be a commutative ring with identity $1 \neq 0$. Let $Z(R)$ denote the set of zero-divisors of R, and $Z(R)^* = Z(R) \setminus \{0\}$ be the nonzero zero-divisors of R. The *zero-divisor graph* of R, denoted by $\Gamma(R)$, is the undirected graph whose vertices are the elements of $Z(R)^*$, and two distinct vertices r and s are adjacent if and only if $rs = 0$.

Zero-divisor graphs were first defined for commutative rings by Beck in [2]. However, he let all elements of a ring R be vertices of the graph and was mainly interested in colorings. In [1], Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are the

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* Speaker.

non-zero zero-divisors of R. They studied the interplay between the ring-theoretic properties of a commutative ring and the graph-theory properties of its zero-divisor graph. In 2002, Redmond [5] introduced the concept of the zero-divisor graph for a noncommutative ring R. Also, in [4], the zero-divisor graph of triangular matrix rings is studied.

The zero-divisor graph of a noncommutative ring R is a directed graph, which is denoted by $\overrightarrow{\Gamma}(R)$. An element $r \in R$ is a left zero-divisor if there exists $0 \neq s \in R$ such that $rs = 0$, and likewise $r \in R$ is a right zero-divisor if there exists $0 \neq s \in R$ such that $sr = 0$. In R the sets of nonzero left zero-divisors and nonzero right zero-divisors are denoted by $Z_l(R)^*$ and $Z_r(R)^*$, respectively. The vertex set of $\overrightarrow{\Gamma}(R)$ is $V(\overrightarrow{\Gamma}(R)) = Z_l(R)^* \cup Z_r(R)^*$, and there is an arc from r to s, denoted by $r \to s$, if and only if $rs = 0$. For general background on graph theory, please see [3].

In this talk, we study the zero-divisor graphs of some subrings of upper triangular matrix rings over commutative rings with identity.

2. Main results

Let R be a commutative ring with nonzero identity. Let $T_n(R)$ denote the $n \times n$ upper triangular matrix ring over R and let $S_n(R)$ be a subring of $T_n(R)$ defined as follows, where $n \geq 2$.

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} : a, a_{ij} \in R \right\}.$$

If there is no confusion, we write S instead of $S_n(R)$. Also we denote the underlying graph of $\overrightarrow{\Gamma}(S)$ by $\Gamma(S)$.

In a graph G, the *distance* between two distinct vertices a and b, denoted by $d(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, we set $d(a, b) := \infty$. The *diameter* of a graph G is $diam(G) = sup\{d(a, b) : a$ and b are distinct vertices of $G\}$. Also the *girth* of a graph G is the length of the shortest cycle in G.

Proposition 2.1. *Let $n \geq 3$. Then the following statements hold.*
(a) The girth of $\Gamma(S)$ is 3.
(b) $diam\Gamma(R) \leq diam\Gamma(S) \in \{2, 3\}$.

Recall that a graph is said to be *planar* if it can be drown in the plane, so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

We denote the underlying graph of $\overrightarrow{\Gamma}(S)$ by $\Gamma(S)$.

Theorem 2.2. *$\Gamma(S)$ is planar if and only if $n = 2$ and R is isomorphic to one of the following rings:*

$$\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{F}_4, \mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_5.$$

A graph is *outerplanar* if it can be drawn in the plane without crossing in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.

The following corollary follows from Theorem 2.2.

Corollary 2.3. *$\Gamma(S)$ is outerplanar if and only if $n = 2$ and R is isomorphic to one of the following rings:*

$$\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{F}_4, \mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2).$$

Let x be a vertex of a graph G. The neighborhood of x, denoted by $N(x)$, is the set of vertices which are adjacent to x.

Let G and H be graphs. A *homomorphism* f from G to H is a map from $V(G)$ to $V(H)$ such that for any $a, b \in V(G)$, a is adjacent to b implies that $f(a)$ is adjacent to $f(b)$. Moreover, if f is bijective and its inverse mapping is also a homomorphism, then we call f an *isomorphism* from G to H, and in this case we say G is isomorphic to H, denoted by $G \cong H$. A homomorphism (resp, an isomorphism) from G to itself is called an *endomorphism* (resp, *automorphism*) of G. An endomorphism f is said to be *half-strong* if $f(a)$ is adjacent to $f(b)$ implies that there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that c is adjacent to d. By $End(G)$, we denote the set of all the endomorphisms of G. It is well-known that $End(G)$ is a monoid with respect to the composition of mappings. Let S be a semigroup. An element a in S is called *regular* if $a = aba$ for some $b \in S$ and S is called regular if every element in S is regular. Also, a graph G is called *end-regular* if $End(G)$ is regular.

Proposition 2.4. *Let $n \geq 3$. Then $\Gamma(S)$ is not end-regular.*

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ON ADDITIVE PRESERVERS OF THE HYPER-RANGE AND HYPER-KERNEL

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Abstract. In this paper we study the bijective additive maps $\phi : B(H) \longrightarrow B(H)$ which preserve upper(lower) semi-Fredholm and finite rank nilpotent operators and we show that if for every finite dimension or codimension subspace A of H, $T(A) \subseteq A \Longrightarrow \phi(T)(A) \subseteq A$ and $\phi(T_A)(A) = \phi(T)(A)$, then there exists an non-zero scaler $\mu \in \mathbb{C}$ such that $\phi(T) = \mu T$ for all $T \in B(H)$. We also determine the form of surjective additive maps $\phi : B(H) \longrightarrow B(H)$ which weakly preserve in both directions the hyper-range's codimension (i.e. $codimR^{\infty}(T) < \infty \Longleftrightarrow codimR^{\infty}(\phi(T)) < \infty$) and the hyper-kernel's dimension (i.e. $dimN^{\infty}(T) < \infty \Longleftrightarrow dimN^{\infty}-(\phi(T)) < \infty$). We show that if $\phi : B(H) \longrightarrow B(H)$ weakly preserves in both directions the hyper-range's codimension and the hyper-kernel's dimension, then there exists an invertible bounded linear or conjugate linear operator $A : H \longrightarrow H$ and non-zero complex number $\mu \in \mathbb{C}$ such that $\phi(T) = \mu ATA^{-1}$ for all $T \in B(H)$.

1. Introduction

Let X be a complex Banach space and let H be a seperable complex infinite dimension Hilbert space. The algebra of all bounded linear operators acting on X is denoted by $B(X)$.
For all operator $T \in B(X)$, write $N(T)$ for its kernel and $R(T)$ for its

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* Speaker.

range. The hyper-kernel and the hyper-range of T are defined respectively by $N^{\infty}(T) := \cup_{n\in N} N(T^n)$ and $R^{\infty}(T) := \cap_{n\in N} R(T^n)$.
A surjective additive map $\phi : B(X) \longrightarrow B(X)$ preserves property P of operators in both directions if $T \in P \Longleftrightarrow \phi(T) \in P$.

A semi-Fredholm operator is a bounded linear operator between Banach spaces that has a finite dimension kernel or finite codimensional range and closed range.
Upper(resp.lower) semi-Fredholm operators are those which have finite dimensional kernels (resp. codimensional ranges). The set of such operators is denoted by $F_{+}(X)$ $(F_{-}(X))$.
The ascent $a(T)$ and descent $d(T)$ of $T \in B(X)$ are defined by

$$min\{\ n \geq 0 : N(T^n) = N(T^{n+1})\};$$

$$min\{\ n \geq 0 : R(T^n) = R(T^{n+1})\}.$$

As for notation, if A is a closed subspace of Hilbert space H and $T \in B(H)$, we consider

$$\begin{aligned} T_A : A \oplus B &\longrightarrow A \oplus B \\ (x_1, x_2) &\longrightarrow T(x_1). \end{aligned}$$

Let us introduce the following subsets:
(i) $A(X) := \{T \in B(X) : a(T) < \infty\}$ the set of finite ascent operators;
(ii) $D(X) := \{T \in B(X) : d(T) < \infty\}$ the set of finite descent opera tors;
(iii) $B_{+}(X) := F_{+}(X) \cap A(X)$ the set of upper semi-Browder operators;
(iv) $B_{-}(X) := F_{-}(X) \cap D(X)$ the set of lower semi-Browder operators;
(V) $B(X) := B_{+}(X) \cap B_{-}(X)$ the set of Browder operators.
In [3], the form of all surjective linear map $\phi : B(H) \longrightarrow B(H)$ preserving nilpotent operators is determined and it is shown also that if $\phi : B(H) \longrightarrow B(H)$ preserves nilpotent operators then there exists a non-zero complex number μ and a bounded bijective linear operator $A : H \longrightarrow H$ such that either
(i) $\phi(T) = cATA^{-1}$ for every $T \in B(H)$ or
(ii) $\phi(T) = cAT^{tr}A^{-1}$ for every $T \in B(H)$, where T^{tr} denotes the transpose of T relative to a fixed but arbitrary orthonormal basis.
In this paper, we obtain the following resulte:

Theorem 1.1. *Let $\phi : B(H) \longrightarrow B(H)$ be a bijective additive map. If ϕ preserves upper(lower) semi-Fredholm and finite rank nilpotent operators and if for every finite dimension or codimension subspace A of H we have $T(A) \subseteq A \Longrightarrow \phi(T)(A) \subseteq A$ and $\phi(T_A)(A) = \phi(T)(A)$,* *then there exists an non-zero scaler* $\mu \in \mathbb{C}$ *such that* $\phi(T) = \mu T$ *for all* $T \in B(H)$.

In [2] we saw that a surjective additive map $\phi : B(X) \longrightarrow B(X)$ preserves hyper-kernel's dimension if $dimN^{\infty}(\phi(T)) = dimN^{\infty}(T)$ for all $T \in B(X)$ and also preserves hyper-range's codimension if $codimR^{\infty}(\phi(T)) = codimR^{\infty}(T)$ for all $T \in B(X)$.
In the same article the forms of all additive map $\phi : B(X) \longrightarrow B(X)$ preserving the hyper-kernel's dimension or the hyper range's codimension are determined, and it is established also that $\phi : B(X) \longrightarrow B(X)$ preserves the hyper-range or the hyper-kernel, then there exists a nonzero scaler $\mu \in \mathbb{C}$ such that $\phi(T) = \mu T$ for all $T \in B(X)$.
In the present paper, we define surjective additive map $\phi : B(X) \longrightarrow B(X)$ weakly preserves in both directions hyper-kernel's dimension if $dimN^{\infty}(T) < \infty \Longleftrightarrow dimN^{\infty}(\phi(T)) < \infty$ and also weakly preserves in both directions hyper-range's codimension if $codimR^{\infty}(T) < \infty \Longleftrightarrow codimR^{\infty}(\phi(T)) < \infty$ and we determine the forms of all surjective additive maps $\phi : B(H) \longrightarrow B(H)$ that weakly preserve the hyper-kernel's dimension and the hyper -range's codimension in both directions, and consequently we obtain;

Theorem 1.2. *Let* H *be a infinite-dimensional Banach space, let* $\phi : B(H) \longrightarrow B(H)$ *be a surjective additive map. Then the following assertions are equivalent:*
(i) ϕ *preserves closed range operators and weakly preserves hyper-kernel of* T *in both directions.*
(ii) ϕ *weakly preserves hyper-range of* T *in both directions.*
(iii)There exists an invertible bounded linear or conjugate linear operator $A : H \longrightarrow H$ *and nonzero complex number* μ *such that* $\phi(T) = \mu ATA^{-1}$ *for all* $T \in B(H)$.

Theorem 1.3. *Let* $\phi : B(H) \longrightarrow B(H)$ *be a surjective additive map. Then the following assertions are equivalent:*
(i) ϕ *weakly preserves hyper-kernel and hyper-range of* T *in both directions.*
(ii) there exists an invertible bounded linear, or conjugate linear, operator $A : H \longrightarrow H$ *and a non-zero complex number* μ *such that either* $\phi(T) = \mu ATA^{-1}$ *for all* $T \in B(H)$, *or* $\phi(T) = \mu AT^{*}A^{-1}$ *for all* $T \in B(H)$.

2. Main Results

We prove the Theorems 1.1, 1.2 and 1.3.

Proof. (1.1). Let $T \in B_+(H)$. By [4, Theorem10] there exists a decomposition $H = H_1 \oplus H_2$ such that $dimH_1 < \infty$, $TH_i \subseteq H_i$ $(i = 1, 2)$, $T\mid_{H_1}$ is nilpotent and $T\mid_{H_2}$ is bounded below and the space H_1 is uniquely determined by $H_1 = N^\infty(T)$. Since $T\mid_{H_2}$ is bounded below thus $inf\{\|T_2(x)\|, x \in \ H_2, \|x\| = \ 1\} > 0$, since $codimH_2 < \infty$ by [4, Theorem 8] T_{H_2} is upper semi-Fredholm. Now assumption implies that $\phi(T_{H_2})$ is upper semi-Fredholm. By [4, Theorem 8] there exists a closed subspace $H_3 \subseteq H$ of finite codimension such that $inf\{\|\phi(T_{H_2})(x)\|, x \in H_3, \|x\| = 1\} > 0$. We claim that $H_3 = H_2$. Since $T_{H_2}\mid_{H_1} = 0$ and $\phi(T_{H_2})(H_1) = \phi(T_{H_{2H_1}})(H_1) = 0$ thus $H_3 \subseteq H_2$. If $H_3 \subsetneq H_2$ then there is a $x_0 \in H_3$ such that $\|x_0\| = 1$. Put $H_0 := \overline{Spanx_0}$, $\phi(T_{H_{2H_0}}) = \phi(T_{H_2})(H_0) = 0$. Since ϕ is injective therefore $T_{H_{2H_0}} = 0$ but $T_{H_2}(x_0) \neq 0$. Thus $H_2 = H_3$. Now since $T\mid_{H_1}$ is finite rank nilpotent, $dim(H_1) < \infty$ and $T_{H_1}(H_1) \subseteq H_1$ therefore $\phi(T_{H_1})H_1 \subseteq H_1$. Now we have $\phi(T)\mid_{H_1} = \phi(T_{H_1})\mid_{H_1}$, $\phi(T)\mid_{H_2} = \phi(T_{H_2})\mid_{H_2}$ thus by [4, Theorem 10] $\phi(T) \in B_+(H)$ and $N^\infty(\phi(T)) = H_1 = N^\infty(T)$. According to [2, Theorem 3.6] there exists a non-zero scaler $\mu \in \mathbb{C}$ such that $\phi(T) = \mu T$ for all $T \in B(H)$. $\square$

Proof. (1.2). It is clear that by the first part of [4, Proposition 8] ϕ preserves the upper semi-Browder operators in both directions and also by the second part of [4, Proposition 8] ϕ preserves the lower semi-Browder operators in both directions. Now by [1, Theorem A] we conclude that (i), (ii) and (iii) are equivalent. $\square$

Proof. (1.3). It is easy to see that by the third part of [4, proposition 8] ϕ preserves the Browder operators in both directions. then by [1, Theorem B], (i) and (ii) are equivalent. $\square$

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EXTENDING THE CENTROSYMMETRIC PROPERTY FOR MULTIDIMENSIONAL MATRICES

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Abstract. In this paper, the centrosymmetric property of ordinary matrices is extended for multidimensional matrices. Then, an application of such matrices is presented after a brief study of some of their basic properties and features.

1. Introduction

Centrosymmetric matrices (also known as centrally-symmetric matrices) have been introduced since 1960 [5]. The centrosymmetric property has been used in [1] to capture the symmetric behavior of a dynamic system by a fuzzy linguistic modeling structure. A first-order fuzzy relational model structure is considered in this regard, in which a square fuzzy relational matrix represents the truth values of the associated fuzzy rule-base. Thus, in [1], some results are obtained about the stability of the first-order fuzzy relational dynamic systems based on the special features of the centrosymmetric fuzzy relational matrices.

Wishing to deal with more complex systems, fuzzy relational models with higher orders are required in which multidimensional fuzzy relational matrices are involved. This is a good motivation for defining and

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* Speaker.

using multidimensional centrosymmetric matrices which is done in this paper.

Therefore, multidimensional centrosymmetric matrices and some of their basic properties are introduced in the next section. Also, the application of the proposed matrices in fuzzy linguistic modeling of complex dynamic systems with symmetric behavior is briefly discussed.

Throughout this paper, $\mathbf{M}(i_1, i_2, \ldots, i_n)$ denotes the $(i_1, i_2, \ldots, i_n)$-th element of a multidimensional matrix $\mathbf{M}$.

2. Main results

In this section, the centrosymmetric property is extended for multidimensional matrices and is applied to fuzzy relational modeling of complex dynamic systems.

2.1. Introducing multidimensional centrosymmetric matrices.

A center-wise matrix transposition is the essence of centrosymmetric matrices.

Definition 2.1. Let $\mathbf{M}$ be a $p_1 \times p_2 \times \ldots \times p_n$ matrix. The *central transposition* of the n-dimensional matrix $\mathbf{M}$ (denoted by $\mathbf{M}^{T\cdot}$) is defined as:

$$\mathbf{M}^{T\cdot}(i_1, i_2, \ldots, i_n) = \mathbf{M}(p_1 + 1 - i_1, p_2 + 1 - i_2, \ldots, p_n + 1 - i_n).$$

It is followed immediately from Definition 2.1 that $\left(\mathbf{M}^{T\cdot}\right)^{T\cdot} = \mathbf{M}$, since $p + 1 - (p + 1 - i) = i$.

Definition 2.2. Let $\mathbf{R}$ and $\mathbf{S}$ be two multidimensional matrices.

(1) $\mathbf{R}$ is *centrosymmetric (or centrally-symmetric)* if $\mathbf{R}^{T\cdot} = \mathbf{R}$.
(2) $\mathbf{S}$ is *skew centrosymmetric* if $\mathbf{S}^{T\cdot} = -\mathbf{S}$.

Every multidimensional matrix can be written as the addition of a centrosymmetric matrix and a skew centrosymmetric matrix of the same size.

Lemma 2.3. *Let $\boldsymbol{M}$ be a multidimensional matrix. Then, $\boldsymbol{M}$ can be written as $\boldsymbol{M} = \boldsymbol{R} + \boldsymbol{S}$, where:*

$$\begin{cases} \boldsymbol{R} = \frac{1}{2}(\boldsymbol{M} + \boldsymbol{M}^{T\cdot}) \\ \boldsymbol{S} = \frac{1}{2}(\boldsymbol{M} - \boldsymbol{M}^{T\cdot}) \end{cases}$$

Remark 2.4. The multiplication of two compatible multidimensional matrices is performed as a special case of the fuzzy relational inner composition introduced in [2]. Accordingly, a $p_1 \times p_2 \times \ldots \times p_n$ matrix $\mathbf{M}_1$ and a $q_1 \times q_2 \times \ldots \times q_n$ matrix $\mathbf{M}_2$ are compatible (for performing $\mathbf{M}_1\mathbf{M}_2$, when $p_n = q_1$. Obviously, for the summation of two multidimensional matrices, they should be of the same size.

The (skew) centrosymmetric property of multidimensional matrices is preserved under most operations.

Lemma 2.5. *Let $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ be two multidimensional centrosymmetric matrices and $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ be two multidimensional skew centrosymmetric matrices. Then:*

(1) $-\boldsymbol{R}_1$ *is centrosymmetric.*
(2) $\boldsymbol{R}_1 + \boldsymbol{R}_2$ *is centrosymmetric (when $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are of the same size).*
(3) $\boldsymbol{R}_1\boldsymbol{R}_2$ *is centrosymmetric (when $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are compatible).*
(4) $\boldsymbol{S}_1\boldsymbol{S}_2$ *is centrosymmetric (when $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are compatible).*
(5) $-\boldsymbol{S}_1$ *is skew centrosymmetric.*
(6) $\boldsymbol{S}_1 + \boldsymbol{S}_2$ *is skew centrosymmetric (when $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are of the same size).*
(7) $\boldsymbol{R}_1\boldsymbol{S}_2$ *(as well as $\boldsymbol{S}_1\boldsymbol{R}_2$) is skew centrosymmetric (when $\boldsymbol{R}_1$ and $\boldsymbol{S}_2$ (or $\boldsymbol{S}_1$ and $\boldsymbol{R}_2$) are compatible).*

2.2. **Motivation and Application.** Centrosymmetric matrices have been used in [1] for modeling dynamic systems with symmetric behavior around the operating point via fuzzy relational models. Fuzzy relational models are used for modeling various types of systems, as can be seen for example in [4]. Investigating the stability of a model is a very important issue for dynamic systems. This issue is very challenging in the fuzzy linguistic modeling area. Reference [3] initiates a matrix-based approach to this problem and starts the way toward the goal with first-order fuzzy relational models. On the basis of [3], a symmetrical-behavior analysis has been performed in [1]. A stability analysis method has been made possible therein by using some features of centrosymmetric matrices, i.e., the special properties of the eigenvalues and eigenvectors of a centrosymmetric matrix. An important limitation of [1] is that only first-order models can be handled. Higher-order models involve multidimensional fuzzy relational matrices.

It is interesting that for higher-order models (which may include external inputs as well as internal inputs), the symmetric behavior of the dynamic system around its equilibrium point is represented as a centrosymmetric multidimensional matrix as proposed in this paper. The following examples clarify this issue.

Example 2.6. Suppose that we want to model a first-order dynamic system with a symmetric behavior around the origin. Let x, y be two linguistic variables such that $x \in \{N, Z, P\}$ and $y \in \{N, Z, P\}$, where x, y correspond respectively to the input and output of the system. Two dual fuzzy rules of the fuzzy model are as follows:

Rule 1**:** if x:P, then y:N.
Rule 1′**:** if x:N, then y:P.

Therefore, the elements of the fuzzy relational matrix associated with "Rule 1" and "Rule 1′" are equal.

Note that the fuzzy modeling in general is a local modeling procedure and there are many systems that show locally symmetric behavior around their equilibrium points exactly or at least approximately. In such cases, for any given fuzzy rule in a model, a dual fuzzy rule is associated with the same truth degree.

Example 2.7. Suppose that we want to model a higher-order dynamic system with a symmetric behavior around the origin. Let x_1, x_2, y be three linguistic variables such that $x_1 \in \{NN, N, Z, P, PP\}$, $x_2 \in \{N, Z, P\}$, and $y \in \{N, Z, P\}$, where x_1, x_2 correspond to the inputs (one internal input and one external input) and y corresponds to the output of the system. Consider the following dual rules from among the rules of the fuzzy rule-base.

Rule 1**:** if x_1:NN & x_2:P, then y:N.
Rule 1′**:** if x_1:PP & x_2:N, then y:P.

Therefore, $\mathbf{R}$ is a $5 \times 3 \times 3$ matrix and $\mathbf{R}(1, 3, 1) = \mathbf{R}(5, 1, 3)$, since "Rule 1" and "Rule 1′" are both valuated equally. Likewise, $\mathbf{R}(i, j, k) = \mathbf{R}(6 - i, 4 - j, 4 - k)$ for every valid i, j, k.

More advanced topics about the proposed multidimensional centrosymmetric matrices are needed to be studied in future works in order to have powerful analytical tools for symmetric fuzzy relational modeling structures.

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A PRECONDITIONED GAOR ITERATIVE METHOD FOR SOLVING LINEAR SYSTEM OF EQUATIONS

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Abstract. Recently, Wang et al. have propounded some preconditioned GAOR methods. It has been proved that applying one of the preconditioners leads to the superior convergence rate than other mentioned preconditioners. In the present paper we examine a new type of preconditioner which outperforms those proposed by the authors in the above referred work.

1. Introduction

Consider the following linear system

$$Hy = f, \tag{1.1}$$

where

$$H = \begin{pmatrix} I-B & U \\ L & I-C \end{pmatrix}, \tag{1.2}$$

is a nonsingular matrix with $B = [b_{ij}]_{p\times p}$, $C = [c_{ij}]_{q\times q}$, $L = [l_{ij}]_{q\times p}$, $U = [u_{ij}]_{p\times q}$ and $p + q = n$.

Throughout this paper, we consider the decomposition $H = \widehat{D} - \widehat{L} - \widehat{U}$ in which

$$\widehat{D} = I, \qquad \widehat{L} = \begin{pmatrix} 0 & 0 \\ -L & 0 \end{pmatrix}, \qquad \widehat{U} = \begin{pmatrix} B & -U \\ 0 & C \end{pmatrix}. \tag{1.3}$$

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* Speaker.

Before presenting the main contribution of this paper, we recollect some useful definitions and principles. For an arbitrary matrix $A \in \mathbb{R}^{n\times n}$, the decomposition $A = M - N$ is called a splitting if $M, N \in \mathbb{R}^{n\times n}$ and M is nonsingular. Consider the linear system of equations $Ax = b$ where A is nonsingular and $b \in \mathbb{R}^n$. Let $A = M - N$ be an arbitrary splitting for the matrix A. Basically, a stationary iterative method for solving $Ax = b$ has the following form:

$$x^{(k+1)} = Tx^{(k)} + M^{-1}b, \quad k = 0, 1, 2, \dots, \tag{1.4}$$

where the initial vector $x^{(0)}$ is given and $T = M^{-1}N$ is called the iteration matrix. Suppose that $\rho(T)$ stands for the spectral radius of the iteration matrix based on the splitting $A = M - N$. It is well-known that the convergence analysis of the iterative method (1.4) relies on the spectral radius of the iteration matrix T, i.e., $\rho(T)$. For large values of k, at each step, the corresponding error reduces in magnitude approximately by a factor of $\rho(T)$. That is, the smaller $\rho(T)$ is the quicker the convergence is. The GAOR method is defined by the following splitting

$$M_{r,\omega} = \widehat{D} - r\widehat{L}, \quad N_{r,\omega} = (1-\omega)\widehat{D} + (\omega - r)\widehat{L} + \omega\widehat{U}.$$

The iteration matrix of the GAOR method is represented by $T_{r,\omega}$ and defined as follows (see [3] and the references therein):

$$T_{r,\omega} = (\widehat{D} - r\widehat{L})^{-1}[(1-\omega)\widehat{D} + (\omega - r)\widehat{L} + \omega\widehat{U}].$$

Definition 1.1. (Woznicki [5]). The splitting $A = M - N$ is called

(1) a regular splitting of A if $M^{-1} \geq 0$ and $N \geq 0$,
(2) a nonnegative splitting of A if $M^{-1} \geq 0$, $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$,
(3) a weak nonnegative splitting of A if $M^{-1} \geq 0$ and either $M^{-1}N \geq 0$ (the first type) or $NM^{-1} \geq 0$ (the second type),
(4) a convergent splitting of A if $\rho(M^{-1}N) < 1$.

Theorem 1.2. *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weak nonnegative splittings of A where $A^{-1} \geq 0$, if $M_1^{-1} \geq M_2^{-1}$ then $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$.*

Theorem 1.3. *Let A be a Z-matrix. Moreover, suppose that $A = M - N$ is a weak nonnegative splitting of the first type. Then $\rho(M^{-1}N) < 1$ if and only if A is an M-matrix.*

By the theoretical results proved in [5], we can instantly conclude Theorem 1.2. Moreover, the proof of Theorem 1.3 is elaborated in [1].

2. Main results

The preconditioned GAOR methods are defined by applying GAOR method on a preconditioned linear system; for further details see [2, 3, 4]. For instance, Wang et al. have considered the preconditioned linear system $\tilde{H}y = \tilde{f}$ where $\tilde{H} = (I + \tilde{S})H$ and $\tilde{f} = (I + \tilde{S})f$. The authors have assumed that

$$\tilde{S} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix},$$

and the application of the following preconditioners have been investigated

$$S_1 = \begin{pmatrix} 0 & b_{12} & \cdots & 0 & 0 \\ b_{21} & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & b_{p-1,p} \\ 0 & 0 & \cdots & b_{p,p-1} & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ b_{21} & 0 & \cdots & 0 & 0 \\ \vdots & b_{32} & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{p,p-1} & 0 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 0 & b_{12} & \cdots & 0 & 0 \\ 0 & 0 & b_{23} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & b_{p-1,p} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

It has been proved that the preconditioner which is obtained based on S_1 works better. We focuss on the following preconditioned linear system

$$\tilde{H}^* y = \tilde{f}^*, \tag{2.1}$$

where $\tilde{H}^* = (I + \tilde{S}^*)H$ and $\tilde{f}^* = (I + \tilde{S}^*)f$ such that

$$\tilde{S}^* = \begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix} \quad \text{and} \quad V_1 = \begin{pmatrix} 0 & c_{12} & \cdots & 0 & 0 \\ c_{21} & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & c_{q-1,q} \\ 0 & 0 & \cdots & c_{q,q-1} & 0 \end{pmatrix}.$$

We point out here that the preconditioner which is mentioned in the above relation has an analogues structure with the preconditioners employed in [4]. It is not difficult to verify that

$$\tilde{H}^* = \begin{pmatrix} I - B^* & U^* \\ L^* & I - C^* \end{pmatrix},$$

where $B^* = B - S_1(I - B)$, $U^* = (I + S_1)U$, $L = (I + V_1)L$ and $C = C - V_1(I - C)$. In the sequel, we assume that the matrices B, C, L and

U gratify the following conditions

$$B \geq 0, \quad C \geq 0, \quad L \leq 0 \quad \text{and} \quad U \leq 0.$$

which show that the matrix H is a Z-matrix. More precisely, we may demonstrate that under the above conditions the GAOR splitting is a regular splitting; see [1]. If the GAOR method is convergent for solving linear system $Hy = f$, then Theorem 1.3 indicates that H is in fact an M-matrix. Now, we may establish the following theorems by using Theorem 1.2 which reveals that the preconditioned GAOR method with the preconditioner $P^* = I + \tilde{S}^*$ has faster convergence rate than the GAOR method.

Theorem 2.1. *Let $T_{r,\omega}$ and $\tilde{T}^*_{r,\omega}$ be the iteration matrices corresponding to GAOR method and preconditioned GAOR method with the preconditioner $P^* = I + \tilde{S}^*$. Suppose that the matrix H is irreducible with $b_{i,i+1} > 0$, $b_{i+1,i} > 0$ for some $i \in \{1, 2, \ldots, p-1\}$, $b_{ii} > 0$ whenever $b_{i,i+1} > 0$, $b_{i+1,i} > 0$ for $i \in \{1, 2, \ldots, p-1\}$, $c_{j,j+1} > 0$, $c_{j+1,j} > 0$ for some $j \in \{1, 2, \ldots, q-1\}$, $c_{jj} > 0$ whenever $c_{j,j+1} > 0$, $c_{j+1,j} > 0$ for $j \in \{1, 2, \ldots, q-1\}$. If $\rho(T_{r,\omega}) < 1$ then $\rho(T_{r,\omega}) < \rho(\tilde{T}^*_{r,\omega})$ in which $0 < \omega \leq 1$ and $0 \leq r < 1$.*

The subsequent theorem turns out that our proposed preconditioned GAOR method surpasses those examined in [3].

Theorem 2.2. *Presume that $\tilde{T}_{r,\omega}$ and $\tilde{T}^*_{r,\omega}$ stand for the iteration matrices associated with the preconditioners $P = I + \tilde{S}$ and $P^* = I + \tilde{S}^*$, respectively. Under the assumptions of Theorem 2.1, we deduce that*

$$\rho(\tilde{T}^*_{r,\omega}) < \rho(\tilde{T}_{r,\omega}).$$

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SOME REFINEMENTS OF THE OPERATOR HERMITE–HADAMARD INEQUALITY

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Abstract. In this paper we obtain some classes of refinements of the operator Hermite–Hadamard inequality by partitioning $[0,1]$ into suitable equal parts and using some monotonicity properties of operator convex functions. These refinements become sharper by increasing the points of divisions.

1. Introduction

Let $B(H)$ denote the C^*-algebra of all bounded linear operators acting on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$, $B_h(H)$ be the set of all self–adjoint operators in $B(H)$ and I be the identity operator on H. In the case when $\dim H = n$, we identify $B(H)$ with the full matrix algebra $M_n(\mathbb{C})$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. An operator $A \in B(H)$ is called positive if $\langle Ax, x \rangle \geq 0$ holds for every $x \in H$ and then we write $A \geq 0$. If A is positive and invertible, we write $A > 0$. For self–adjoint operators $A, B \in B(H)$, we say $A \leq B$ if $B - A \geq 0$.
A continuous real valued function f defined on an interval J of $\mathbb{R}$ is

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* Speaker.

said to be operator convex if

$$f(tA+(1-t)B) \leq tf(A)+(1-t)f(B)$$

for all $A, B \in B(H)$ with spectra in J and all $t \in [0,1]$. A function f is called operator concave if $-f$ is operator convex [2].

Recently we extended some important inequalities for operator convex functions [5].

The Löwner–Heinz Inequality [4] asserts that $f(x) = x^r$ $(0 \leq r \leq 1)$ is operator monotone on $[0, \infty)$. For more information on operator inequalities see [2].

Let f be an operator convex function on an interval J of the real line and A, B be self-adjoint operators with spectra in J. The operator Hermite–Hadamard inequality reads as follows [3]

$$f\Big(\frac{A+B}{2}\Big) \leq \int_0^1 f((1-t)A+tB)dt \leq \frac{f(A)+f(B)}{2}. \tag{1.1}$$

Recently, Dragomir [1] presented the following generalization of the above operator Hermite–Hadamard inequality:

$$\begin{aligned} f\Big(\frac{A+B}{2}\Big) &\leq \frac{1}{2}\Big[f\Big(\frac{3A+B}{4}\Big)+f\Big(\frac{A+3B}{4}\Big)\Big] \\ &\leq \int_0^1 f((1-t)A+tB)dt \\ &\leq \frac{1}{2}\Big[f\Big(\frac{A+B}{2}\Big)+\frac{f(A)+f(B)}{2}\Big] \\ &\leq \frac{f(A)+f(B)}{2}. \end{aligned} \tag{1.2}$$

2. Main results

In this section we present some new refinements of the operator Hermite–Hadamard inequality by dividing the interval $[0,1]$ into some equal parts and using some monotonicity properties of operator convex functions. In the next theorem we divide the interval $[0,1]$ into n equal parts.

Theorem 2.1. *Let $f : J \to \mathbb{R}$ be an operator convex function and A, B be self–adjoint operators with spectra in J. Then*

$$\begin{aligned} f\Big(\frac{A+B}{2}\Big) &\leq \frac{1}{2n}\Big[2\sum_{i=1}^{n-1} f(X_i^{(n)})+f\Big(\frac{X_0^{(n)}+X_1^{(n)}}{2}\Big)+f\Big(\frac{X_{n-1}^{(n)}+X_n^{(n)}}{2}\Big)\Big] \\ &\leq \int_0^1 f((1-t)A+tB)dt \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2n}\left[2\sum_{i=1}^{n-1}\frac{f(X_{i-1}^{(n)})+f(X_{i+1}^{(n)})}{2}+\frac{f(X_0^{(n)})+f(X_1^{(n)})}{2}\right.\\
&\qquad\left.+\frac{f(X_{n-1}^{(n)})+f(X_n^{(n)})}{2}\right]\\
&\leq \frac{f(A)+f(B)}{2}, \qquad (2.1)
\end{aligned}$$

where $X_i^{(n)}=\big(1-\frac{i}{n}\big)A+\frac{i}{n}B$ *for* $i=0,\cdots,n$.

The next theorem is another refinement of the operator Hermite–Hadamard inequality by dividing the interval $[0,1]$ to 2^n equal parts.

Theorem 2.2. *Let* $f:J\to\mathbb{R}$ *be an operator convex function and* A,B *be self–adjoint operators with spectra in* J*. Then*

$$\begin{aligned}
f\Big(\frac{A+B}{2}\Big) &\leq \frac{1}{2^{n-1}}\sum_{i=1}^{2^{n-1}}f(X_{2i-1}^{(2^n)})\\
&\leq \int_0^1 f((1-t)A+tB)dt\\
&\leq \frac{1}{2^{n-1}}\sum_{i=1}^{2^{n-1}}\frac{f(X_{2i-2}^{(2^n)})+f(X_{2i}^{(2^n)})}{2}\\
&\leq \frac{f(A)+f(B)}{2} \qquad (2.2)
\end{aligned}$$

which $X_i^{(2^n)}=\big(1-\frac{i}{2^n}\big)A+\frac{i}{2^n}B$ *for* $i=0,\cdots,2^n$*. Moreover the lower bound of integral is an increasing sequence while the upper bound is decreasing, both converging to the integral.*

Finally, we present some monotonicity properties of operator convex functions which lead to another refinement of the operator Hermite–Hadamard inequality [5].

Theorem 2.3. *Let* $f:J\to\mathbb{R}$ *be an operator convex function and* A,B *be self-adjoint operators with spectra in* J*. Then for each* $n=1,2,\cdots,$

$$\frac{1}{n+2}\sum_{i=0}^{n+1}f\left(A+i\frac{B-A}{n+1}\right)\leq\frac{1}{n+1}\sum_{i=0}^{n}f\left(A+i\frac{B-A}{n}\right) \qquad (2.3)$$

$$\frac{1}{n}\sum_{i=1}^{n}f\left(A+i\frac{B-A}{n+1}\right)\leq\frac{1}{n+1}\sum_{i=1}^{n+1}f\left(A+i\frac{B-A}{n+2}\right). \qquad (2.4)$$

If f *is operator concave, all inequalities are reversed.*

Theorem 2.4. *Let $f : J \to \mathbb{R}$ be an operator convex function on J and let A, B be self-adjoint operators with spectra in J. Then*

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq \frac{1}{m}\sum_{i=1}^{m} f\left(A + i\frac{B-A}{m+1}\right) \\ &\leq \int_0^1 f\big((1-t)A + tB\big)dt \\ &\leq \frac{1}{n+1}\sum_{i=0}^{n} f\left(A + i\frac{B-A}{n}\right) \\ &\leq \frac{f(A)+f(B)}{2} \quad (m, n = 1, 2, \cdots). \end{aligned} \tag{2.5}$$

If f is operator concave, all inequalities are reversed.

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LINEAR ISOMETRIES BETWEEN REAL BANACH ALGEBRAS OF CONTINUOUS COMPLEX-VALUED FUNCTIONS

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Abstract. Let X and Y be compact Hausdorff spaces and let τ and η be topological involutions on X and Y, respectively. In 1991, Kulkarni and Arundhathi characterized linear isometries from a real uniform function algebra A on (X,τ) onto a real uniform function algebra B on (Y,η) applying their Choquet boundaries and showed that these mappings are weighted composition operators.

In this paper we characterize all onto linear isometries and certain into linear isometries between $C(X,\tau)$ and $C(Y,\eta)$ applying the extreme points in the unit balls of $C(X,\tau)^*$ and $C(Y,\eta)^*$.

1. Introduction

Let $\mathfrak{X}$ be a normed space over $\mathbb{F}$. We denote by $\mathfrak{X}^*$ and $B_{\mathfrak{X}}$ the dual space of $\mathfrak{X}$ and the closed unit ball of $\mathfrak{X}$, respectively. For a subset E of $\mathfrak{X}$, let $Ext(E)$ denote the set of all extreme points of E. Kulkarni and Limaye showed [5, Theorem 2] that if A is a nonzero linear subspace of $\mathfrak{X}$ and $\varphi \in Ext(B_{A^*})$, then φ has an extension to some $\psi \in Ext(B_{\mathfrak{X}^*})$.

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* Speaker.

We know that if $\mathfrak{X}$ and $\mathcal{Y}$ are normed spaces over $\mathbb{F}$ and $T : \mathfrak{X} \to \mathcal{Y}$ is a linear isometry from $\mathfrak{X}$ onto $\mathcal{Y}$ over $\mathbb{F}$, then T is a bijection mapping between $Ext(B_{\mathfrak{X}})$ and $Ext(B_{\mathcal{Y}})$.

Let X be a compact Hausdorff space. We denote by $C_{\mathbb{F}}(X)$ the unital commutative Banach algebra of all continuous functions from X into $\mathbb{F}$, with the uniform norm $||f||_X = \sup\{|f(x)| : x \in X\}$, $f \in C_{\mathbb{F}}(X)$. We write $C(X)$ instead as $C_{\mathbb{C}}(X)$. For $x \in X$, we consider the linear functional $e_{C_{\mathbb{F}}(X),x}$ on $C_{\mathbb{F}}(X)$ defined by $e_{C_{\mathbb{F}}(X),x}(f) = f(x)$ ($f \in C_{\mathbb{F}}(X)$), which is called the evaluation functional on $C_{\mathbb{F}}(X)$ at x. Clearly, $\lambda e_{C_{\mathbb{F}}(X),x} \in B_{C_{\mathbb{F}}(X)^*}$ for all $(x, \lambda) \in X \times \mathbb{F}$. It is known that

$$Ext(B_{C_{\mathbb{F}}(X)^*}) = \{\lambda e_{C_{\mathbb{F}}(X),x} : (x, \lambda) \in X \times \mathbb{S}_{\mathbb{F}}\}.$$

Let A be a real linear subspace of $C(X)$ and let S be nonempty subset of X. We say that A is *extremely regular* at S if for every open neighborhood U of S and for each $\varepsilon > 0$, there is a function $f \in A$ with $||f||_X = 1$ such that $f(x) = 1$ for all $x \in S$ and $|f(y)| < 1$ for all $y \in X \backslash U$.

Let X be a topological space. An involution τ on X is called a *topological involution* on X, if τ is continuous.

Let X be a compact Hausdorff space and let τ be a topological involution on X. We define

$$C(X, \tau) = \{f \in C(X) : f \circ \tau = \overline{f}\}.$$

Then $C(X, \tau)$ is a unital uniformly closed self-adjoint real subalgebra of $C(X)$ which separates the points of X and does not contain i_X, the constant function with value i on X. Note that $Re\ f \in C(X, \tau)$ for all $f \in C(X, \tau)$. Moreover, $C(X) = C(X, \tau) \oplus iC(X, \tau)$ and

$$\max\{||f||_X, ||g||_X\} \leq ||f + ig||_X \leq 2 \max\{||f||_X, ||g||_X\},$$

for all $f, g \in C(X, \tau)$. In fact, the complex Banach algebra $(C(X), ||.||_X)$ can be regarded as the complexification of the real Banach algebra $(C(X, \tau), ||.||_X)$. Note that $C(X, \tau) = C_{\mathbb{R}}(X)$ if and only if τ is the identity self-map on X. Hence, the class of real Banach algebras $C(X, \tau)$ is, in fact, larger than the class of real Banach algebras $C_{\mathbb{R}}(X)$.

The real Banach algebra $C(X, \tau)$ and its real linear subspaces were first considered by Kulkarni and Limaye in [3]. For a detailed account of several properties of $C(X, \tau)$, we refer to [4].

Throughout the rest of this paper, X and Y are compact Hausdorff spaces, τ and η are topological involutions on X and Y, respectively.

In this paper we characterize all linear isometries from $C(X, \tau)$ onto $C(Y, \eta)$ and certain linear isometries from $C(X, \tau)$ into $C(Y, \eta)$ applying the extreme points in $B_{C(X,\tau)^*}$ and $B_{C(Y,\eta)^*}$. Our results in Section

2 are some of the given results by Kulkarni and Arundhathi in [2] that we obtain by applying the extreme points in $B_{C(X,\tau)^*}$ and $B_{C(Y,\eta)^*}$. The main result in Section 3 is a generalization of the given result by Holsztyński in [1] for certain into isometries.

2. Onto linear isometries

We first determine unit-preserving linear isometries from $C(X,\tau)$ onto $C(Y,\eta)$.

Theorem 2.1. *Let $T : C(X,\tau) \to C(Y,\eta)$ be a linear isometry from $C(X,\tau)$ onto $C(Y,\eta)$ with $T1_X = 1_Y$. Then there exists a homeomorphism h from Y onto X with $h \circ \eta = \tau \circ h$ on Y such that*

$$(Tf)(y) = f(h(y)), \quad \forall f \in C(X,\tau),\ \forall y \in Y.$$

Corollary 2.2. *Let $T : C(X,\tau) \to C(Y,\eta)$ be an isometry mapping from $C(X,\tau)$ onto $C(Y,\eta)$ with $T(0) = 0$ and $T1_X = 1_Y$. Then T is an isomorphism.*

We now study the onto case (not necessarily unit-preserving).

Lemma 2.3. *Let $f \in C(X,\tau)$ with $||f||_X = 1$. Then $|f(x)| = 1$ for all $x \in X$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $g \in C(x,\tau)$ with $||g||_X \geq \varepsilon$ implies that*

$$\max\{||f+g||_X, ||f-g||_X\} \geq 1+\delta.$$

Lemma 2.4. *Let $T : C(X,\tau) \to C(Y,\eta)$ be a linear isometry from $C(X,\tau)$ onto $C(Y,\eta)$ and let $a = T1_X$. Then $|a(y)| = 1$ for all $y \in Y$.*

In the following result, we show that every linear isometry from $C(X,\tau)$ onto $C(Y,\eta)$ is a weighted composition operator.

Theorem 2.5. *Let $T : C(X,\tau) \to C(Y,\eta)$ be a linear isometry from $C(X,\tau)$ onto $C(Y,\eta)$. Then, there exist a function $a \in C(Y,\eta)$ with $|a(y)| = 1$ for all $y \in Y$ and a homeomorphism h from Y onto X with $ho\eta = \tau oh$ on Y such that*

$$(Tf)(y) = a(y)f(h(y)), \quad \forall f \in C(X,\tau),\ \forall y \in Y.$$

3. Into linear isometries

We formulate our main result in this section which is a version for into linear isometries of $C(X,\tau)$-spaces of a known Holsztyńskis's theorem for into linear isometries of $C_{\mathbb{F}}(X)$-spaces . We first study unit-preserving linear isometries from $C(X,\tau)$ into $C(Y,\eta)$

Theorem 3.1. *Let $T : C(X,\tau) \to C(Y,\eta)$ be a linear isometry from $C(X,\tau)$ into $C(Y,\eta)$ satisfying the following conditions:*

(i) $T1_X = 1_Y$.

(ii) *If* $y \in Y$ *and there is a point* $x \in X$ *such that* $Re((Tf)(y)) = Re(f(x))$ *for all* $f \in C(X,\tau)$, *then* $T(C(X,\tau))$ *is extremely regular at* $\{y, \tau(y)\}$.

Then, there exist a η*-invariant closed boundary* Y_0 *for* $T(C(X,\tau))$ *and a continuous map* h *from* Y_0 *onto* X *with* $h \circ \eta = \tau \circ h$ *on* Y_0 *such that*

$$(Tf)(y) = f(h(y)), \quad \forall f \in C(X,\tau), \quad \forall y \in Y_0.$$

We now study the into case (not necessarily unit-preserving).

Theorem 3.2. *Let* $T : C(X,\tau) \to C(Y,\eta)$ *be a linear isometry from* $C(X,\tau)$ *into* $C(Y,\eta)$ *satisfying the following conditions:*

(i) $|T1_X(y)| = 1$ *for all* $y \in Y$.

(ii) *If* $y \in Y$ *and there is a point* $x \in X$ *such that* $Re(\overline{T1_X(y)}(Tf)(y)) = Re(f(x))$ *for all* f *in* $C(X,\tau)$, *then* $(\overline{T1_X})T(C(X,\tau))$ *is extremely regular at* $\{y, \tau(y)\}$.

Then, there exist a η*-invariant closed boundary* Y_0 *for* $T(C(X,\tau))$, *a function* $a \in C(Y,\eta)$ *with* $|a(y)| = 1$ *for all* $y \in Y$ *and a continuous map* h *from* Y_0 *onto* X *with* $h \circ \eta = \tau \circ h$ *on* Y_0 *such that*

$$(Tf)(y) = a(y)f(h(y)), \quad \forall f \in C(X,\tau), \quad \forall y \in Y_0.$$

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A q-NUMERICAL RADIUS INEQUALITY AND CHARACTERIZATION OF SOME TYPES OF MATRICES

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Abstract. We discuss a characterization of square matrices $A \in M_n$ that satisfy the inequality $r_q(A) \leq \frac{r(A)(1+q)}{2}$, for every $0 \leq q \leq 1$. Also we show that this inequality is equivalent to the inequality $\rho(AB) \leq r(A)r(B)$, for every $B \in M_n$, with $\text{rank}(B) = 1$. Here $r_q(A)$, $r(A)$ and $\rho(A)$, denote the q-numerical radius, ordinary numerical radius and spectral radius of A respectively.

1. Introduction

The study of numerical range of operators on Banach or Hilbert spaces represents one of the active research areas in operator theory [2, 4]. Also inequalities about numerical radius is an important topic in the field of numerical range. A new good reference which has gathered various types of these inequalities is [3]. One of the interesting questions is characterizing all matrices $A \in M_n$ that satisfy the inequity $\rho(AB) \leq r(A)r(B)$, for every $B \in M_n$. It is proved that this inequality implies that A is a radial matrix which is unitarily similar to a matrix of the form $\|A\|(I_k \oplus B)$ and the numerical range of B is a subset of $\{z : |z - \frac{1}{2}| \leq \frac{1}{2}\}$ [1]. The converse is not true and it is a conjecture that A is unitarily similar to a matrix of the form $\|A\|(I_k \oplus 0_r \oplus B)$ with $\text{Re}(W(B^{-1})) \geq 1$ (B is omitted if $k + r = n$). Another question that arises from this inequality is about equivalency of it with the inequality $\rho(AB) \leq r(A)r(B)$, for every $B \in M_n$, with $\text{rank}(B) = 1$. In this paper

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we get an equivalent inequality in the form of q-numerical range for the last inequality and discuss about the matrices which satisfy in it.

2. Main results

In this section e_i denotes the i-th vector of the standard basis in $\mathbb{C}^n$. For a matrix $A \in M_n$, the diagonal matrix whose diagonal is equal to diagonal of A is displayed by $\mathrm{diag}(A)$. Also the Euclidean norm on $\mathbb{C}^n$ is displayed by $|.|$. We remember that the numerical range of $A \in M_n$ is defined by

$$W(A) = \{x^*Ax : x^*x = 1\},$$

and for $0 \leq q \leq 1$, the q-numerical range of A is as follow [5]:

$$W_q(A) = \{y^*Ax : y^*y = x^*x = 1,\ y^*x = q\}.$$

It is clear that $W_1 = W$. The numerical radius $r(.)$ and q-numerical radius $r_q(.)$ are defined as follow:

$$r(A) = \sup\{|\lambda| : \lambda \in W(A)\},$$

and

$$r_q(A) = \sup\{|\lambda| : \lambda \in W_q(A)\}.$$

Also Crawford number of A is defied by

$$c(A) = \inf\{|\lambda| : \lambda \in W(A)\},$$

Lemma 2.1. *Let A and $\tilde{A}$ be two unitarily similar matrices in M_n. Then $\rho(AB) \leq r(A)r(B)$, for every $B \in M_n$ with rank$(B) = 1$, if and only if $\rho(\tilde{A}B) \leq r(\tilde{A})r(B)$, for every $B \in M_n$ with rank$(B) = 1$,*

Proof. Suppose that U is a unitary matrix in M_n such that $\tilde{A} = U^*AU$ and $\rho(AB) \leq r(A)r(B)$, for every $B \in M_n$ with $\mathrm{rank}(B) = 1$. Now for every $B \in M_n$ with $\mathrm{rank}(B) = 1$, we have

$$\rho(\tilde{A}B) = \rho(U^*AUBU^*U) = \rho(AUBU^*) \leq r(A)r(UBU^*) = r(\tilde{A})r(B).$$

□

Theorem 2.2. *For $A \in M_n$, two following conditions are equivalent:*
(i) $\rho(AB) \leq r(A)r(B)$, for every matrix B with rank$(B) = 1$. (2.1)
(ii) $r_q(A) \leq \frac{r(A)(1+q)}{2}$, for every $0 \leq q \leq 1$.

Proof. Let B be a rank one matrix in M_n with $\|B\| = 1$. Then there exist unit vectors $x, y \in \mathbb{C}^n$ such that $B = xy^*$. Now choose a unitary matrix U in such a way that its first row is equal to x^* and its second row is $\frac{y^*-(y^*x)x^*}{\sqrt{1-|y^*x|^2}}$. Then $Ux = e_1$ and $Uy = x^*ye_1 + \sqrt{1-|y^*x|^2}e_2$. Therefore

$$xy^* = U^*(e_1(y^*xe_1^* + \sqrt{1-|y^*x|^2}e_2^*))U = U^*(y^*xE_{11} + \sqrt{1-|y^*x|^2}E_{22})U.$$

Hence
$$r(B) = \frac{1+|y^*x|}{2}.$$
On the other hand we have
$$\rho(AB) = \rho(Axy^*) = \rho(y^*Ax) = |y^*Ax|.$$
Therefore (i) holds if and only if (ii) holds.

□

Corollary 2.3. *Let $A = (a_{ij})$ be a matrix in M_n that satisfies the condition (2.1). Then*
$$max\{|a_{ij}| \ : \ i \neq j\} \leq \frac{r(A)}{2}.$$

Proof. For $i \neq j$, setting $y = e_i$ and $x = e_j$, by Theorem 2.2 we have,
$$|a_{ij}| = |e_i^* A e_j| \leq r_0(A) \leq \frac{r(A)}{2}.$$

□

Corollary 2.4. *Let A be a matrix in M_n that satisfies the condition (2.1). Then*
$\|A - diag(A)\|_1, \|A - diag(A)\|_\infty \leq \frac{r(A)\sqrt{n-1}}{2}$.

Proof. Let $a_{rs} = |a_{rs}|\exp(i\theta_{rs})$, $y = e_r$ and $x = \frac{\sum_{r\neq s}^n \exp(-i\theta_{rs})e_r}{\sqrt{n-1}}$ and using Theorem 2.2, for every $1 \leq r \leq n$, we have,
$$\frac{\sum_{s\neq r}^n |a_{rs}|}{\sqrt{n-1}} = |y^*Ax| \leq r_0(A) \leq \frac{r(A)}{2}.$$
Hence $\|A - \text{diag}(A)\|_1 \leq \frac{\|A\|\sqrt{n-1}}{2}$. Similarly we can show that $\|A - \text{diag}(A)\|_\infty \leq \frac{r(A)\sqrt{n-1}}{2}$.

□

Corollary 2.5. *Let A be a matrix in M_n that satisfies the condition (2.1). Then*
$$W(A) \subseteq D(tr(\frac{A}{n}), \frac{r(A)\sqrt{n-1}}{2}).$$

Proof. By [5, Theorem 1.3.4], A is unitarily similar to a matrix B whose diagonal entries are equal to $\text{tr}(\frac{A}{n})$. Now considering [5, Theorem 1.5.2], a discussion same as Corollary 2.3, gets the desired result.

□

Lemma 2.6. *Let A be a matrix in M_n that satisfies the condition (2.1). Then A is a radial matrix.*

Proof. We can take unit vectors $x, y \in \mathbb{C}^n$ in such a way that $\|A\| = |y^*Ax|$ and $y^*x \geq 0$. Since xy^* is a rank one matrix, by theorem 2.2, we have:

$$\|A\| = |y^*Ax| \leq \frac{r(A)(1 + y^*x)}{2} \leq r(A).$$

Therefore A is a radial matrix.

□

Lemma 2.7. *Let A be an invertible matrix in M_n that satisfies the condition (2.1). Then*

(i) $\frac{1}{\|A^{-1}\|} \leq [\frac{1+c(A^{-1})}{2}]r(A)$.

(ii) *inf* $\{|Ax| \ : \ |x| = 1\} \geq \frac{2}{r(A)[1+r(A^{-1})]}$.

Proof. For every unit vector x in $\mathbb{C}^n$ we have

$$\begin{aligned} \frac{1}{|A^{-1}x|} &= \frac{|x^*A(A^{-1}x)|}{|A^{-1}x|} = \rho(\frac{A(A^{-1}x)x^*)}{|A^{-1}x|}) \\ &\leq r(A)\frac{r(A^{-1}xx^*)}{|A^{-1}x|} = r(A)\frac{1 + x^*A^{-1}x}{2}. \end{aligned}$$

Now taking the infimum (supremum) over x gets (i)((ii)).

□

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HILBERT C^*-MODULES WHICH ARE MADE INTO HILBERT SPACES

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Abstract. We know that every Hilber space is a left Hilbert $\mathbb{C}$-module, but the converse is not true in general. In this talk we show that under which conditions a Hilbert C^*-module is a Hilbert space.

1. Introduction

Hilbert C^*-modules are essentially objects like Hilbert spaces, except that the inner product, instead of being complex-valued, take its values in a C^*-algebra. Throughout the paper $\mathcal{A}$ is a C*-algebra (not necessarily unital). A (right) pre-Hilbert module over a C^*-algebra $\mathcal{A}$ is a complex linear space $\mathcal{X}$, which is an algebraic right $\mathcal{A}$-module and $\lambda(xa) = (\lambda x)a = x(\lambda a)$ equipped with an $\mathcal{A}$-valued inner product $\langle ., . \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying,
(i) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ iff $x = 0$,
(ii)$\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle$,
(iii)$\langle x, ya\rangle = \langle x, y\rangle a$,
(iv) $\langle y, x\rangle = \langle x, y\rangle^*$.

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* Speaker.

for each $x, y, z \in \mathcal{X},\ \lambda \in \mathbb{C},\ a \in \mathcal{A}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{X}$ is called a Hilbert $\mathcal{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. Left Hilbert $\mathcal{A}$-modules are defined in a similar way. For example every C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with respect to inner product $\langle x, y\rangle = x^*y$, and every inner product space is a left Hilbert $\mathbb{C}$-module.

Suppose that $\mathcal{A}$ is a C^*-algebra and $\mathscr{S}(\mathcal{A})$ is the state space of $\mathcal{A}$. Then $\mathscr{S}(\mathcal{A})$ is convex and weak*-compact, and so it has extreme points. The set of all extreme points of $\mathscr{S}(\mathcal{A})$ is denoted by $\mathscr{P}(\mathcal{A})$ and its members are called pure states of $\mathcal{A}$. By Krein-Milman theorem $\mathscr{S}(\mathcal{A})$ is the weak*-closed convex hull of $\mathscr{P}(\mathcal{A})$, or $\mathscr{S}(\mathcal{A}) = \overline{co}(\mathscr{P}(\mathcal{A}))$. For more details one can see [1, 2, 4]

Theorem 1.1. (see [3, Theorem 4.3.8]) Suppose $\mathcal{A}$ is a C^*-algebra and $\mathscr{P}(A)$ is the set of all its pure states and $a \in A$, if $\rho(a) = 0$ for all $\rho \in \mathscr{P}(\mathcal{A})$, then $a = 0$.

2. Hilbert C^*-modules over FPS C^*-algebras

Definition 2.1. A C^*-algebra $\mathcal{A}$ is called FPS (finitely pure state) if $card(\mathscr{P}(\mathcal{A})) < \infty$.

Theorem 2.2. (see [3, Theorem 3.4.7]) A non-zero functional ρ on $C(X)$ is a pure state of $C(X)$ if and only if it is multiplicative.

Theorem 2.3. (see [3, Corollary 3.4.2]) Each pure state ρ of $C(X)$ corresponds to a point $x_0 \in X$ such that $\rho(f) = f(x_0)$ for each $f \in C(X)$.

Corollary 2.4. *If X is a finite set, then $C(X)$ is a FPS C^*-algebra.*

Theorem 2.5. *If $\mathcal{X}$ is a Hilbert C^*-module over a FPS C^*-algebra $\mathcal{A}$, then $\mathcal{X}$ is a Hilbert space.*

Note there is a relation between the norm induced by $\langle ., .\rangle_\varphi$ and the norm induced by $\langle ., .\rangle$. In fact

$$||x||_\varphi^2 = \langle x, y\rangle_\varphi = \varphi(\langle x, x\rangle) \leq ||\varphi||||\langle x, x\rangle|| = ||x||^2.$$

Since $(\mathcal{X}, \langle ., .\rangle_\varphi)$ is a Hilbert space by Riesz representation theorem, if $\zeta : \mathcal{X} \to \mathbb{C}$ is a bounded linear functional, then there exists $y \in \mathcal{X}$ such that $||\zeta|| = ||y||$ and $\zeta(x) = \langle x, y\rangle_\varphi$ for every $x \in \mathcal{X}$.

Theorem 2.6. *If $\mathcal{X}$ is a Hilbert C^*-module over a FPS C^*-algebra $\mathcal{A}$ and $f \in \mathcal{X}'$ and ψ is a bounded linear functional on $\mathcal{A}$ (for example a state) and $\varphi = \frac{\sum_{\psi \in P(\mathcal{A})} \psi}{card(P(\mathcal{A}))}$, then $\zeta : \mathcal{X} \to \mathbb{C}$, which maps x to $\psi(f(x))$*

is a bounded linear functional on $\mathcal{X}$ and there exists $y \in \mathcal{X}$ such that $||\zeta|| = ||y||$ and

$$\zeta(x) = \varphi(\langle x, y \rangle),$$

for each $x \in \mathcal{X}$.

Theorem 2.7. *If $\mathcal{X}$ is a Hilbert C^*-module over FPS C^*-algebra $\mathcal{A}$ and $\varphi = \frac{\sum_{\psi \in P(\mathcal{A})} \psi}{card(P(\mathcal{A}))}$ and ζ is a bounded linear functional such that $\varphi \leq \zeta$, then $\langle x, y \rangle_\zeta = \zeta(\langle x, y \rangle)$ defines an inner product on $\mathcal{X}$, which makes it to a Hilbert space.*

We will denote the Hilbert space $(\mathcal{X}, \langle x, x \rangle_\zeta)$ by $\mathcal{X}^\zeta$.

Theorem 2.8. *Suppose that $\mathcal{X}$ is a Hilbert C^*-module over FPS C^*-algebra $\mathcal{A}$. If $\varphi = \frac{\sum_{\psi \in P(\mathcal{A})} \psi}{card(P(\mathcal{A}))}$ and $\varphi \leq \zeta$, then $\mathcal{X}^\zeta \subseteq \mathcal{X}^\varphi$.*

Theorem 2.9. *If $\mathcal{A}$ is a FPS C^*-algebra, then $\mathcal{A}$ is a Hilbert space.*

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AN EXTENSION OF THE GELFAND-MAZUR THEOREM

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Abstract. It is proved that if A is a complex unital fundamental division F-algebra with bounded elements and the dual space A^* of A separates the points on A, then A is isomorphic to the complex numbers $\mathbb{C}$.

1. Introduction

Let A be a unital complex algebra with unit 1. The Gelfand-Mazur theorem states that if A is a normed division algebra, then A is isomorphic to the complex numbers $\mathbb{C}$ (see [4]). The problem whether the Gelfand-Mazur theorem is valid for the F-algebras (i.e. completely metrizable topological algebras), is an open question (see [5]). Żelazko proved this theorem for the locally bounded algebras and also for the locally convex F-algebras. The similar result was proved by Arens on continuous inverse algebras (i.e. locally convex topological algebras with open unit group and continuous inversion), every element has non-empty compact spectrum. In 1965, Allan showed if A is a locally convex division algebra with bounded elements, then A is isomorphic to $\mathbb{C}$ [1, Corollary 3.9].

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* Speaker.

Recall that a topological algebra A is called a topological algebra with bounded elements if all elements of A are bounded, that is, for each $a \in A$, there is non-zero complex number z such that sequence $(\frac{1}{z^n}a^n)$ converges to zero. The radius of boundedness $\beta(a)$ of a bounded element a is defined by

$$\beta(a) = \inf\{|z| : (\frac{1}{z}a)^n \text{ converges to zero } \}.$$

Here, we prove the Gelfand-Mazur theorem in absence of local convexity and local boundedness.
In 1990, generalizing the Cohen factorization theorem, Ansari [3] introduced the notion of fundamental topological algebras (see Definition 2.1). It is easy to see that every locally bounded algebra and every locally convex algebra is fundamental. We prove that if A is a fundamental topological F-algebra with bounded elements and dual space A^* of A separates the points on A, then the spectrum $\sigma(a)$ of $a \in A$ is compact and nonempty. Therefore, the Gelfand-Mazur theorem is valid for fundamental division F-algebras with bounded elements on which dual space separates the points.

Recall that the spectrum $\sigma(x)$ of an element x in topological algebra A is given by the formula $\sigma(x) = \{\lambda \in \mathbb{C} : \lambda.1 - x \notin G(A)\}$, where $G(A)$ is the set of all unit elements in A.

2. Main results

Definition 2.1. A topological vector space A is said to be fundamental if there exists $b > 1$ such that for every sequence (x_n) in A, if the sequence $b^n(x_n - x_{n-1})$ tends to zero as $n \to \infty$, then (x_n) is Cauchy. A fundamental topological algebra is a topological algebra whose underlying topological vector space is fundamental.

Here we give an example of (neither locally convex nor locally bounded) fundamental F-algebras with bounded elements on which dual space separates the points.

Example 2.2. Let A be a commutative locally bounded and non-locally convex topological algebra such that A^* separates the points on A and X be a locally convex and non-locally bounded topological vector space. Suppose $(a, x) \mapsto xa$ is a bilinear and separately continuous mapping of $A \times X$ into X, satisfying $x(a_1a_2) = (xa_1)a_2$ and $x.1 = x$ for all $a_1, a_2 \in A$ and $x \in X$. Then X is a topological unit linked right $A-$module with module multiplication defined by $(a, x) \mapsto xa$, and $Z = X \times A$ is a non-locally bounded, non-locally convex, fundamental topological vector space. Define the multiplication on Z by

$$(x_1, a_1)(x_2, a_2) = (x_1a_2 + x_2a_1, a_1a_2)$$

for all $x_1, x_2 \in X$ and $a_1, a_2 \in A$. Now, Z is an algebra and since the module multiplication is separately continuous, Z is a topological algebra. Every element in Z is bounded. It is clear that $(0, 1)$ is the unity element of Z. Therefore, Z is a unital fundamental topological algebra with bounded elements. Also Z^* separates the points on Z since A^* and X^* separate the points on A and X, respectively. If, moreover, A and X are completely metrizable, then Z is completely metrizable.

Gelfand's proof of Gelfand-Mazur theorem is based upon an abstract theory of analytic functions (see [4]). We extend this theorem on fundamental topological algebras by a similar process.

Theorem 2.3. *Let A be a fundamental F-algebra with bounded elements such that A^* separates the points on A. If A is a division algebra, then A is isomorphic to $\mathbb{C}$.*

Proof. Our proof breaks into four steps. For details of the proof of these steps we refer the reader to [2].

Step 1: Let $a \in A$ with $\beta(a) < 1$. Then we prove that $1 - a \in G(A)$ and

$$(1 - a)^{-1} = 1 + \sum_{n=1}^{\infty} a^n.$$

Step 2: Let $a \in A$. We prove that the spectrum $\sigma(a)$ of a is a closed subset of $\mathbb{C}$. Moreover, the mapping $F(z) = (z - a)^{-1}$ is a holomorphic mapping of $\mathbb{C} \setminus \sigma(a)$ into A, that is, $\phi o F$ is a holomorphic mapping of $\mathbb{C} \setminus \sigma(a)$ into $\mathbb{C}$ for all $\phi \in A^*$.

Step 3: By result obtained in Step 2 and applying Liouville's theorem, we prove that for every $a \in A$, the spectrum $\sigma(a)$ is nonempty.

Step 4: Let $a \in A$. Since by Step 3, $\sigma(a) \neq \emptyset$, we can choose $\lambda \in \sigma(a)$. Then $\lambda.1 - a \notin G(A)$ and so $\lambda.1 - a = 0$. Hence $a = \lambda.1$. $\square$

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MORE PROPERTIES OF THE COMPLEX BIMATRIX VARIATE BETA DISTRIBUTION

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Abstract. In this short note, We determined exact form of characteristic function and several propertise of the complex bimatrix beta distribution with the assumption of X_1 and X_2 being random Hermitian positive definite matrixs.

1. Introduction

We give a brief review of some definitions and notations. We adhere to standard notations. For a given $A \in C^{p\times p}$, A^H denotes the conjugate transpose of A, $tr(A) = a_{11}+\cdots+a_{pp}$; $etr(A) = exp(tr(A))$; $|A| =$ determinant of A; $A = A^H > 0$ means that A is a Hermitian positive definite, and $A^{\frac{1}{2}}$ denotes the unique Hermitian positive definite square

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* Speaker.

root of $A = A^H > 0$.

Definition 1.1. A $p \times p$ random Hermitian positive definite matrix U is said to have a complex matrix variate wishart distribution with parameters n, p and $\Sigma = \Sigma^H > 0$, denoted as $U \sim CW_p(n, \Sigma)$, if its p.d.f is given by

$$(\widetilde{\Gamma}_p(n)|\Sigma|^n)^{-1}|U|^{n-p} etr(-\Sigma^{-1}U), U = U^H > 0,\ n \geq p.$$

Definition 1.2. Let $S_1 \sim CW_p(n_1, \Sigma)$, $S_2 \sim CW_p(n_2, \Sigma)$ and $B \sim CW_p(m, \Sigma)$ be independently distribution. Define

$$X_i = (I_p + B^{-\frac{1}{2}} S_i B^{-\frac{1}{2}})^{-\frac{1}{2}} B^{-\frac{1}{2}} S_i B^{-\frac{1}{2}} (I_p + B^{-\frac{1}{2}} S_i B^{-\frac{1}{2}})^{-\frac{1}{2}}, \quad i = 1, 2,$$

where $B^{\frac{1}{2}} B^{\frac{1}{2}} = B$ and $0 < X_i < I_p, i = 12$. Furthermore, $n_i > (p-1)$, $i = 1, 2$ and $m > (p-1)$. The joint density of (X_1, X_2) is

$$\left\{\widetilde{B}_p(n_1, n_2, m)\right\}^{-1} \left\{\prod_{i=1}^{2} |X_i|^{n_i - p}\right\} |I_p - X_1|^{(n_2+m)-p} |I_p - X_2|^{(n_1+m)-p}$$
$$\times |I_p - X_1 X_2|^{-(n_1+n_2+m)}$$

where $\tilde{B}_p(.,.,.)$ is the complex multivariate beta function and $\widetilde{B}_p(n_1, n_2, m) = \frac{\widetilde{\Gamma}_p(n_1)\widetilde{\Gamma}_p(n_2)\widetilde{\Gamma}_p(m)}{\widetilde{\Gamma}_p(n_1+n_2+m)}$, and $\widetilde{\Gamma}_p(.)$ is the complex multivariate gamma function. The above density is the complex bimatrix variate beta distribution and is denoted as $(X_1, X_2) \sim CBB_p^{IV}(n_1, n_2, m)$.

2. Characteristic Function

In this section we obtain the characteristic function and several properties of the complex bimatrix variate beta distribution [1] and [2].

Theorem 2.1. *Suppose that $X = [X_1 : X_2]$ and $T = [T_1 : T_2]$, then the characteristic function of the complex bimatrix variate beta distribution is*

$$\begin{aligned}
&\varphi_X(T) \\
&= \left\{\widetilde{B}_p(n_1, n_2, m)\right\}^{-1} \sum_{k=0}^{\infty} \sum_{t=0}^{\infty} \sum_{z=0}^{\infty} \sum_{\kappa} \sum_{\tau} \sum_{\zeta} \sum_{\delta \in \kappa, \tau} \sum_{\gamma \in \kappa, \zeta} \frac{(n_1 + n_2 + m)_\kappa}{k! t! z!} \\
&\quad \times \frac{(n_2)_\delta \widetilde{B}_p(n_2, n_1 + m) \widetilde{g}^{\delta}_{\kappa,\tau}}{(n_1 + n_2 + m)_\delta} \frac{(n_1)_\gamma \widetilde{B}_p(n_1, n_2 + m) \widetilde{f}^{\gamma}_{\kappa,\zeta}}{(n_1 + n_2 + m)_\gamma} \\
&\quad \times \frac{\widetilde{C}_\tau(iT_2) \widetilde{C}_\delta(I_p) \widetilde{C}_\zeta(iT_1) \widetilde{C}_\gamma(I_p)}{\widetilde{C}_\kappa(I_p) \widetilde{C}_\tau(I_p) \widetilde{C}_\zeta(I_p)}
\end{aligned}$$

where $\widetilde{C}_\tau(.)$ and $[.]_\kappa$ are respectively the complex zonal polynomial with weight τ and the complex multivariate hypergeometric coefficient. For more details see [3].
Furthermore Corresponding to the partitions κ and τ of k and t, respectively, the product $\widetilde{C}_\kappa(X)\widetilde{C}_\tau(X)$ is defined as Chikuse[4]

$$\widetilde{C}_\kappa(X)\widetilde{C}_\tau(X) = \sum_{\delta\in\kappa,\tau} \widetilde{g}^{\delta}_{\kappa,\tau}\widetilde{C}_\delta(X),$$

where δ is the partition of the integer $k+t$ and $\widetilde{g}^{\delta}_{\kappa,\tau}$ is the coefficient of $\widetilde{C}_\delta(X)$ in $\widetilde{C}_\kappa(X)\widetilde{C}_\tau(X)$.

Proof. The proof is similar to the proof of Theorem 2 of [1]. □

Theorem 2.2. *Let* $(X_1, X_2) \sim CBB^{IV}_p(n_1, n_2, m)$*, then the density function of* $U = X_2^{\frac{1}{2}} X_1 X_2^{\frac{1}{2}}$ *is*

$$\begin{aligned}&\frac{\widetilde{B}_p(n_1+m, n_2+m)}{\widetilde{B}_p(n_1, n_2, m)}|U|^{n_1-p}|I_p-U|^{m-p}\\ &\times {}_2\widetilde{F}_1(n_1+m, n_1+m, n_1+n_2+2m; I_p-U)\end{aligned}$$

Where ${}_p\widetilde{F}_q(\underbrace{.,\dots,.}_{p-times},\underbrace{.,\dots,.}_{q-times};.)$ the generalized hypergeometric function of Hermitian matrix argument.

Proof. Consider the transformation $U = X_2^{\frac{1}{2}} X_1 X_2^{\frac{1}{2}}$, $X_2 = X_2$, with $J(X_1, X_2 \to U, X_2) = |X_2|^{-p}$, then the joint density of U and X_2 is

$$\begin{aligned}&\left\{\widetilde{B}_p(n_1, n_2, m)\right\}^{-1}|U|^{n_1-p}|X_2|^{-(n_1+m)}|I_p-U|^{-(n_1+n_2+m)}\\ &\times |I_p-X_2|^{(n_1+m)-p}|X_2-U|^{(n_2+m)-p}, 0<U<X_2<I_p\end{aligned}$$

Now let us make the change of variable
$T = (I_p-U)^{-\frac{1}{2}}(I_p-X_2)(I_p-U)^{-\frac{1}{2}}$ and $U = U$. Noting that $J(U, X_2 \longrightarrow U, T) = |I_p-U|^p$, then the joint density of U, and T is

$$\begin{aligned}&\{\widetilde{B}_p(n_1, n_2, m)\}^{-1}|U|^{n_1-p}|T|^{(n_1+m)-p}|I_p-U|^{m-p}|I_p-T|^{(n_2+m)-p}\\ &\times |I_p-(I_p-U)T|^{-(n_1+m)} \quad 0<T<I_p, 0<U<I_p\end{aligned}$$

Integrating with respect to T, so obtain the stated marginal density function for U. □

Theorem 2.3. *Let* $(X_1, X_2) \sim CBB_p^{IV}(n_1, n_2, m)$*, then*

$$P(U > I_m) = 1 - \frac{\widetilde{B}_p(n_1+m, n_2+m)\widetilde{B}_p(m, n_1)}{\widetilde{B}_p(n_1, n_2, m)} \times {}_3\widetilde{F}_2(m, n_1+m, n_1+m, n_1+m, n_1+n_2+2m, I_m)$$

Theorem 2.4. *Let* $(X_1, X_2) \sim CBB_p^{IV}(n_1, n_2, m)$*, then*

$$E[tr(X_1X_2)]^t = \sum_{k=0}^{\infty}\sum_{\kappa\in k}\sum_{\tau}\sum_{\delta\in\kappa,\tau} \frac{[n_1+n_2+m]_\kappa [n_1]_\delta [n_2]_\delta \widetilde{g}_{\kappa,\tau}^{\delta}}{k![n_1+n_2+m]_\delta [n_1+n_2+m]_\delta} \times \frac{\widetilde{\Gamma}_p(n_1+m)\widetilde{\Gamma}_p(n_2+m)}{\widetilde{\Gamma}_p(n_1+n_2+m)\widetilde{\Gamma}_p(m)}\widetilde{C}_\delta(I_P)$$

Proof.

$$\begin{aligned}
&E[tr(X_1X_2)]^t \\
&= \left\{\widetilde{B}_p(n_1, n_2, m)\right\}^{-1} \int_{0<X_1=X_1^H<I_p}\int_{0<X_2=X_2^H<I_p} (tr(X_1X_2))^t \left\{\prod_{\imath=1}^{2}|X_\imath|^{n_\imath-p}\right\} \\
&\quad \times |I_p - X_1|^{(n_2+m)-p}|I_p - X_2|^{(n_1+m)-p}|I_p - X_1X_2|^{-(n_1+n_2+m)}dX_2dX_1 \\
&= \sum_{k=0}^{\infty}\sum_{\kappa\in k}\sum_{\tau}\sum_{\delta\in\kappa,\tau} \frac{[n_1+n+2+m]_\kappa \widetilde{g}_{\kappa,\tau}}{k!\widetilde{B}_p(n_1, n_2, m)} \frac{\widetilde{B}_p(n_2, n_1+m)(n_2)_\delta}{(n_1+n_2+m)_\delta} \\
&\quad \times \int_{0<X_1=X_1^H<I_p} |X_1|^{n_1-p}|I_p - X_1|^{(n_2+m)-p}\widetilde{C}_\delta(X_1)dX_1.
\end{aligned}$$

Now,by integration respect to X_1 we get the desired result. □

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S-NUMBERS AND NON-SEPARABLE HILBERT SPACES

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ABSTRACT. We give a survey of some recent results on s-numbers and unitarily invariant norms in infinite dimensional Hilbert spaces.

1. Introduction

Suppose that H is a Hilbert space, $T \in B(H)$ and $i \in \mathbb{N}$. The ith s-number of T is denoted by $s_i(T)$ and is defined as follows

$$s_i(T) = \inf\{\|T - F\| : F \in B(H) \text{ has rank } < i\}. \qquad (1.1)$$

It is clear that $s_1(\cdot) \geq s_2(\cdot) \geq \cdots \geq 0$.

If H is finite dimensional, that is, $H = \mathbb{C}^n$ and $B(H) = M_n(\mathbb{C})$, then the set of s-numbers of T is exactly the set of singular valued of the positive matrix $|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$.

The singular value decomposition theorem shows that the s-numbers have an important role in matrix analysis. This theorem asserts that for each matrix $T \in M_n(C)$, there are unitary matrices U, V and diagonal matrix $D = diag(s_1(T), \cdots, s_n(T))$ such that $T = UDV$. Consequently, s-numbers have a major role in description of unitarily invariant norms.

Note that a norm $\|\cdot\|$ on a non-zero ideal J of $B(H)$ is called *unitarily invariant* if $\|UTV\| = \|T\|$ for every $T \in J$ and arbitrary unitary

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operators $U, V \in B(H)$. Important examples of unitarily invariant norms are "operator norm", "trace norm", "Schatten p-norms", and also Ky-Fan k-norms which defined by $N_k(\cdot) = s_1(\cdot) + \cdots + s_k(\cdot)$.

For an ideal J of $B(H)$, Ky Fan's dominance theorem is said to hold for a unitarily invariant norm $\|\cdot\|$ if for every $T, R \in J$, where $N_k(T) \leq N_k(R)$ for all $k \in \mathbb{N}$, the inequality $\|T\| \leq \|R\|$ holds. Also, Ky Fan's dominance theorem holds for J if Ky Fan's dominance theorem holds for all unitarily invariant norms on J.

In the finite dimensional case, by the singular value decomposition (SVD), there is a nice representation of unitarily invariant norms as symmetric gauge functions, which plays a major role in solving problems and proving theorems in the finite dimensional case.

In the infinite dimensional case, as the SVD is absent, there does not exist such a representation. In [2], authors propose an technique that can be considered as the key to solving problems in the infinite dimensional case. In fact, they show that if H is a separable Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$, then the equality $\|\cdot\| = \|\sum_{i=1}^{\infty} s_i(\cdot) e_i \otimes e_i\|$ holds for all unitarily invariant norms on $B(H)$. Consequently, Ky Fans dominance theorem is valid on $B(H)$, when H is separable.

In this note, we show that if H is a non-separable infinite dimensional Hilbert space, then Ky Fan's dominance theorem dose not hold for $B(H)$.

Lemma 1.1. *Let H be an arbitrary Hilbert space and $T, R \in B(H)$. If $\overline{Ran(T)}$ and $\overline{Ran(R)}$ are separable and $N_k(T) \leq N_k(R)$, for all $k \in \mathbb{N}$, then $\|T\| \leq \|R\|$.*

Theorem 1.2. *Let H be a Hilbert space. Ky Fan's dominance theorem holds for $B(H)$ if and only if H is separable or finite dimensional. Moreover, the Continuum Hypothesis implies that, for every non-zero ideal J in $B(H)$, Ky Fan's theorem holds for J if and only if $J \subseteq I_c$, where $c = Card(\mathbb{R})$ and*

$$I_c = \{T \in B(H) : \; dim M < c, \text{ for every closed subspace } M \text{ of } Ran(T)\}$$

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G-FRAMES ARE OPERATORS

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Abstract. In this note, we aim to show that several known generalizations of frames are special kind of continuous frame defined by Ali et al. in 1993. Indeed, it is shown that these generalizations can be considered as an operator between two Hilbert spaces.

1. Introduction

Frames have a myriad of applications in mathematics and engineering including sampling theory, wavelet theory, signal and image processing, quantum computing and more. Various kinds of frames have been proposed recently. In [5] Sun introduced a g-frame which generalized the mentioned frames above but not the continuous frame defined by Ali et al. in [2]. In recent years, some researchers have generalized Sun's g-frame and continuous frame. Our aim is to show that all of these g-frames are equivalent to the continuous frame defined by Ali et al. in [2]. In fact, the main idea in this note is that these g-frames can be considered as operators of some Hilbert space H to some Hilbert space K.

At this point, we suppose that the reader is familiar with discrete frame.

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Key words and phrases. G-frame, continuous frame, Sun's g-frame.

* Speaker.

This paper is organized as follows. In Section (2) we give some basic definitions of several kinds of frames from [5, 1, 3, 4]. In Section (3) we introduce our g-frame and show that it is equivalent to the continuous frames. Then, we show that the g-frames mentioned in Section (2) are equivalent to the continuous frame defined by Ali et al. in [2].

2. Some basic definitions and preliminaries

In this section we recall definitions of some g-frames from [5, 1, 3, 4] which will be required in the sequel.

Definition 2.1. Let (Ω, μ) be a measure space and H be a Hilbert space. The mapping $F : (\Omega, \mu) \to H$ is called a continuous frame with bounds A, B if $\omega \to \langle f, F(\omega)\rangle$ is a measurable function on Ω for every $f \in H$ and

$$A\|f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leq B\|f\|^2, \quad (f \in H).$$

The mapping F is called a Bessel map if the right hand inequality holds.

The following definition is [5, Definition 1.1.]

Definition 2.2. Let H be a Hilbert space, $(K_j)_{j\in J}$, (J is at most countable) be a family of Hilbert spaces and $\Lambda_j : H \to K_j$, $j \in J$ be bounded linear operators. The set $\{\Lambda_j : j \in J\}$ is called a Sun g-frame if there exist $A, B > 0$ such that $A\|x\|^2 \leq \sum_{j\in J} \|\Lambda_j x\|^2 \leq B\|x\|^2$ for all $x \in H$.

Definition 2.3. ([1]) Let H be a Hilbert space, (Ω, μ) be a measure space and $(K_\omega)_{\omega\in\Omega}$ be a family of Hilbert spaces. A family $\{\Lambda_\omega \in B(H, K_\omega) : \omega \in \Omega\}$ is called a continuous g-frame for H with respect to $(K_\omega)_{\omega\in\Omega}$ if the mapping $\Omega \to \mathbb{C}$, defined by $\omega \to \|\Lambda_\omega f\|$ is a measurable function on Ω for any $f \in H$ and there are also two constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2, \quad (f \in H).$$

Definition 2.4. ([4]) Let $v : (\Omega, \mu) \to (0, \infty)$ be a measurable function, H_C be the collection of all nonzero closed subspaces of H. A mapping $F : \Omega \to H_C$ is called a continuous frame of subspaces with respect to v for H if $\omega \to \pi_{F(\omega)}$ is measurable function from Ω to $B(H)$ (π_M is the projection to M) and there exist $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \int_\Omega v^2(\omega)\|\pi_{F(\omega)}(f)\|^2 d\mu(\omega) \leq B\|f\|^2, \quad (f \in H).$$

Definition 2.5. ([3]) Let H be a Hilbert space and (Ω, μ) be a measure space with positive measure μ. Also, suppose that B_H is the collection of all Bessel sequences in H and I is an at most countable index set. A mapping $F : \Omega \to B_H$ defined by $\omega \to (f_i(\omega))_{i\in I}$ is called a generalized continuous frame with respect to (Ω, μ) if F is weakly measurable, i.e. for all $f \in H, i \in I$, the function $\omega \to \langle f, f_i(\omega)\rangle$ is measurable on Ω and there exist positive constants A, B such that

$$A\|f\|^2 \leq \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 d\mu(\omega) \leq B\|f\|^2, \quad (f \in H).$$

3. Main results

In this section we first introduce our g-frame and show that it is equivalent to the continuous frames. Then, we show that the g-frames mentioned in Section 2 are equivalent to our g-frame and so are equivalent to the continuous frame defined by Ali et al. in 1993,[2].

Now we give our new generalization of frame.

Definition 3.1. Let H and K be two Hilbert spaces. A linear operator $\Lambda : H \longrightarrow K$ is called a generalized frame or simply a g-frame for H with respect to K if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \|\Lambda f\|^2 \leq B\|f\|^2$$

for all $f \in H$.

We can consider a discrete frame as a bounded linear operator from some Hilbert space H to some Hilbert space K.

Remark 3.2. From now on throughout this paper we always mean by a g-frame the one defined as in Definition 3.1.

We intend to show that the Sun g-frame defined in Definition 2.2 is a special case of our g-frame.

Proposition 3.3. *The Sun's g-frame as in Definition 2.2, is a special kind of our g-frame.*

Ali et al. continuous frame can be consider as a special kind of our g-frame.

Proposition 3.4. *The continuous frames in the sense of Ali et al. as in Definitin (2.1), are a special kind of our g-frame.*

Theorem 3.5. *Our g-frame is a special kind of the continuous frame defined by Ali et al. as in Definition (2.1).*

Corollary 3.6. *Our g-frame is equivalent to the continuous frames.*

Corollary 3.7. *The Sun's g-frame as in Definition 2.2, is equivalent to the continuous frames.*

We show that some of other g-frames also are equivalent to the continuous frames.

Theorem 3.8. *The g-frame of Dao-Xin Ding as in Definition 2.5, is equivalent to the continuous frames.*

Proposition 3.9. *Continuous g-frame as in Definition 2.3, is a special kind of our g-frame.*

Corollary 3.10. *Continuous g-frame as in Definition 2.3, is equivalent to the continuous frames.*

Proposition 3.11. *The continuous frame of subspaces defined as in Definition 2.4, is equivalent to the continuous frames.*

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REVERSES OF CALLEBAUT INEQUALITY FOR OPERATORS

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Abstract. In this talk, we show some reverses of Callebaut inequality under some mild conditions. In particular, we present

$$\Big(\sum_{j=1}^{n} A_j\sigma B_j\Big)\sharp\Big(\sum_{j=1}^{n} A_j\sigma^{\perp} B_j\Big) \geq \Big(\frac{\sum_{j=1}^{n} A_j}{2}\Big)\sharp\Big(\frac{\sum_{j=1}^{n} B_j}{2}\Big),$$

where A_j, B_j $(1 \leq j \leq n)$ are positive invertible operators such that $\lambda A_j \leq B_j \leq 2\lambda A_j$ $(1 \leq j \leq n)$ for some $\lambda \in (0, +\infty)$ and σ is a connection with a certain representing function.

1. Introduction

Let $\mathbb{B}(\mathbb{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathbb{H}$. An operator $A \in \mathbb{B}(\mathbb{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{H}$. The set of all positive invertible operators is denoted by $\mathbb{B}(\mathbb{H})_+$
The axiomatic theory for operator means of positive invertible operators have been developed by Kubo and Ando [2]. A binary operation σ on $\mathbb{B}(\mathbb{H})_+$ is called a connection, if the following conditions are satisfied:
(i) $A \leq C, B \leq D$ imply $A\sigma B \leq C\sigma D$;
(ii) $A_n \downarrow A, B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$, where $A_n \downarrow A$ means that

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$A_1 \geq A_2 \geq \cdots$ and $A_n \to A$ as $n \to \infty$ in the strong operator topology;
(iii) $T^*(A\sigma B)T \leq (T^*AT)\sigma(T^*BT) \quad (T \in \mathbb{B}(\mathbb{H}))$;
There exists an affine order isomorphism between the class of connections and the class of positive operator monotone functions f defined on $(0,\infty)$ via $f(t)I = I\sigma(tI)\,(t > 0)$. In addition, $A\sigma B = A^{\frac{1}{2}}f(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}$ for all $A, B \in \mathbb{B}(\mathbb{H})_+$. The operator monotone function f is called the representing function of σ. The dual $\sigma^\perp$ of a connection σ with the representing function f is the connection with the representing function $t/f(t)$. A connection σ is a mean if it is normalized, i.e. $I\sigma I = I$.
the Callebaut inequality

$$\Big(\sum_{j=1}^{n} x_j y_j\Big)^2 \leq \sum_{j=1}^{n} x_j^{1+s} y_j^{1-s} \sum_{j=1}^{n} x_j^{1-s} y_j^{1+s} \leq \Big(\sum_{j=1}^{n} x_j^2\Big)\Big(\sum_{j=1}^{n} y_j^2\Big)$$

where x_j, y_j $(j = 1, \cdots, n)$ and $s \in [0,1]$.
Wada [4] gave an operator version of the Callebaut inequality by showing that if $A, B \in \mathbb{B}(\mathbb{H})_+$ and if σ is an operator mean, then

$$\begin{aligned}(A\sharp B) \otimes (A\sharp B) &\leq \frac{1}{2}\left\{(A\sigma B) \otimes (A\sigma^\perp B) + (A\sigma^\perp B) \otimes (A\sigma B)\right\} \\ &\leq \frac{1}{2}\left\{(A \otimes B) + (B \otimes A)\right\}.\end{aligned}$$

In [3] the authors showed that

$$\sum_{j=1}^{n}\Big(A_j\sharp B_j\Big) \leq \Big(\sum_{j=1}^{n} A_j\sigma B_j\Big)\sharp\Big(\sum_{j=1}^{n} A_j\sigma^\perp B_j\Big) \leq \Big(\sum_{j=1}^{n} A_j\Big)\sharp\Big(\sum_{j=1}^{n} B_j\Big), \tag{1.1}$$

where $A_j, B_j \in \mathcal{P}$ $(1 \leq j \leq n)$ and σ is an operator mean.

In this paper, we present a reverse of inequality (1.1) under some mild conditions and some related results.

2. Main results

In the this section, we provide some reverses of Callebaut inequality under some mild conditions. First we need following lemma.

Lemma 2.1. [1] *Let $A_j, B_j \in \mathbb{B}(\mathbb{H})_+$ $(1 \leq j \leq n)$ such that $\lambda A_j \leq B_j \leq 2\lambda A_j$ $(1 \leq j \leq n)$ for some $\lambda \in (0,+\infty)$ and let σ be a connection with the representing function f such that $f(0) = 0$. Then*

$$2\sum_{j=1}^{n}(A_j\sigma B_j) \geq (\sum_{j=1}^{n} A_j)\sigma(\sum_{j=1}^{n} B_j).$$

Our first result reads as follows.

Theorem 2.2. *Let $A_j, B_j \in \mathbb{B}(\mathbb{H})_+$ $(1 \le j \le n)$ such that $\lambda A_j \le B_j \le 2\lambda A_j$ $(1 \le j \le n)$ for some $\lambda \in (0, +\infty)$ and let σ be a connection with the representing function f such that $f(0) = 0$ and $\lim_{t\to 0^+} tf(t)^{-1} = 0$. Then*

$$\Big(\sum_{j=1}^{n} A_j\sigma B_j\Big)\sharp\Big(\sum_{j=1}^{n} A_j\sigma^{\perp} B_j\Big) \ge \Big(\frac{\sum_{j=1}^{n} A_j}{2}\Big)\sharp\big(\frac{\sum_{j=1}^{n} B_j}{2}\big).$$

In the case $\sigma = \sharp_\alpha$, where $\alpha \in [0,1]$, due to $\sharp_\alpha^{\perp} = \sharp_{1-\alpha}$, we infer that the Callebaut inequality.

Corollary 2.3. *Let $A_j, B_j \in \mathbb{B}(\mathbb{H})_+$ $(1 \le j \le n)$ such that $\lambda A_j \le B_j \le 2\lambda A_j$ $(1 \le j \le n)$ for some $\lambda \in (0, +\infty)$. Then*

$$\sqrt{2}\Big(\big(\sum_{i=1}^{n} A_i\sharp_\alpha B_i\big)\sharp\big(\sum_{i=1}^{n} A_i\sharp_{1-\alpha} B_i\big)\Big) \ge \Big(\sum_{i=1}^{n} A_i\Big)\sharp\Big(\sum_{i=1}^{n} B_i\Big).$$

Theorem 2.4. *Let $A_j, B_j \in \mathbb{B}(\mathbb{H})_+$ $(1 \le j \le n)$ and let σ be a connection with the representing function f such that $\sqrt{\lambda t} \le f(t) \le \sqrt{2\lambda t}$ $(t \in (0,\infty))$ for some $\lambda \in (0, +\infty)$. Then*

$$2\sum_{j=1}^{n}(A_j\sharp B_j) \ge \Big(\sum_{j=1}^{n} A_j\sigma B_j\Big)\sharp\Big(\sum_{j=1}^{n} A_j\sigma^{\perp} B_j\Big) \ge \Big(\frac{\sum_{j=1}^{n} A_j}{2}\Big)\sharp\big(\frac{\sum_{j=1}^{n} B_j}{2}\big).$$

Corollary 2.5. *Let $A_j, B_j \in \mathbb{B}(\mathbb{H})_+$ $(1 \le j \le n)$ such that $A_j \le B_j \le 2A_j$ $(1 \le j \le n)$ and σ_f be a connection with the representing function $f(t) = \frac{1+\sqrt{2}}{2}\sqrt{t}$. Then*

$$2\sum_{j=1}^{n}(A_j\sharp B_j) \ge \Big(\sum_{j=1}^{n} A_j\sigma_f B_j\Big)\sharp\Big(\sum_{j=1}^{n} A_j\sigma_f^{\perp} B_j\Big) \ge \Big(\frac{\sum_{j=1}^{n} A_j}{2^{\frac{1}{4}}}\Big)\sharp\big(\frac{\sum_{j=1}^{n} B_j}{2^{\frac{1}{4}}}\big).$$

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FUNCTIONS OF BOUNDED p-VARIATION OF SECOND ORDER

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Abstract. The generalized functionals of Merentes type generate a scale of spaces connecting the class of functions of bounded second p-variation with the Sobolev space of functions with p-integrable second derivative. We prove some limiting relations for these functionals as well as sharp estimates in terms of the fractional modulus of order $2-1/p$. These results extend some results for functions of bounded variation but are not consequence of the last.

Let f be a 1-periodic function on the real line. A set $\Pi = \{x_0, \dots x_n\}$ of points such that $x_0 < x_1 < \dots < x_n$, where $x_n = x_0 + 1$ will be called a *partition*. For a partition Π we denote by $\|\Pi\| = \max_j (x_{j+1} - x_j)$ its norm. Let $p > 1$, $p' := \frac{p}{p-1}$ and $0 \le \alpha \le 1/p'$. We denote

$$v(f;\Pi) = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|,$$

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and call

$$v(f) = \sup_{\Pi} \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \, .$$

the variation of f. A function f is called of **bounded variation** iff $v(f) < \infty$. (C. Jordan (1880)) We have that

(a) (C. Jordan): f bounded variation if and only if f is a difference of two increasing functions

(b) if f is derivable and f' continuous then $v(f) = \int_0^1 |f'| = \|f'\|_1$.

Similarly, $v_p(f) = \sup\limits_{\Pi} \left(\sum\limits_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|^p \right)^{1/p}$ is called the p-variation of f and a function f is called of p-**bounded variation** iff $v_p(f) < \infty$. (Wiener, N. (1924))

F. Riesz (1910) introduced

$$v_{p,1/p'}(f) = \sup_{\Pi} \left(\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|^p \, \frac{1}{(x_{k+1} - x_k)^{p/p'}} \right)^{1/p}$$

the $p, 1/p'$-variation of f. and showed that$V_{p,1/p'} = W_p^1$ (a variational characterization of the space of abs. cont. functions with p-integrable first derivative) and $\|f'\|_p = v_{p,1/p'}(f)$. Clearly $v_p(f) \leq v_{p,1/p'}(f)$, i.e. $W_p^1 \subseteq V_p$.

The space W_p^1 was also characterized by A. P. Terehin (1965, 1967) in terms of the fractional continuity modulus of order $1 - 1/p$

$$\omega_{1-1/p}(f;t) = \sup_{\|\Pi\| \leq t} v_p(f;\Pi), \quad 0 < t \leq 1,$$

namely, if $f \in W_p^1$, then $\omega_{1-1/p}(f;t) \leq t^{1/p'} \|f'\|_p$ and conversely, if $\omega_{1-1/p}(f;t) = O(t^{1/p'})$ then $f \in W_p^1$.

C. J de la Valée Poussin (1908) introduced the class of **functions of bounded second variation** as follows. Let Π be a partition of the form

$$x_0 < y_1 \leq z_1 < x_1 < y_2 \leq z_2 < x_2 < \ldots < y_n \leq z_n < x_n = x_0 + 1,$$

and

$$v^{(2)}(f, \Pi) = \sum_{k=1}^{n-1} \left| \frac{f(x_{k+1}) - f(z_{k+1})}{x_{k+1} - z_{k+1}} - \frac{f(y_{k+1}) - f(x_k)}{y_{k+1} - x_k} \right| .$$

A function is called of *bounded second variation* and we write $f \in V^{(2)}$ if $v^{(2)}(f) = \sup\limits_{\Pi} v^{(2)}(f, \Pi) < \infty$, where the supremum is taken over all refined partitions Π. The function f is of bdd. second variation iff f

can be represented as a difference of two convex functions. The notion is used by HLP, Robert and Varberg (bdd. convexity), Huggins and Web ("bdd. slope variation"), A.M. Russel- bdd. second variation (see e.g Robert and Varberg, Convex Functions (1973)) The following were proved

(a) $v^{(2)}(f) < \infty$ **iff** $f(x) = \int_0^x g$ with g of bdd. variation
(b) if $v^{(2)}(f) < \infty$ then f' exists a.e.
(c) If f is derivable on $[0,1]$ then $v^{(2)}(f) = v(f')$
(d) (Riesz-Nagy, 1955) If f is derivable on $[0,1]$ and the derivative absolutely continuous then $v^{(2)}(f) = \|f''\|_1$

N. Merentes (1992) introduced the class $V^{(2)}_{p,1/p'}$ of functions of bounded $(p, 1/p')$ second variation as the class of those functions for which

$$v^{(2)}_{p,1/p'}(f)^p =$$

$$\sup_{\Pi} \sum_{k=0}^{n-1} \left| \frac{f(x_{k+1}) - f(z_{k+1})}{x_{k+1} - z_{k+1}} - \frac{f(y_{k+1}) - f(x_k)}{y_{k+1} - x_k} \right|^p \frac{1}{(x_{k+1} - x_k)^{p/p'}} < \infty.$$

If $v^{(2)}_{p,1/p'}(f) < \infty$ than f' exists everywhere. A variational characterization of the Sobolev space W^2_p in terms of functions of bounded second $(p, 1/p')$-variation was given; namely, a function belongs to W^2_p (f' abs. cont. and f'' p integrable) if and only if $f \in V^{(2)}_{p,1/p'}$ and, moreover $v^{(2)}_{p,1/p'}(f) = \|f''\|_p$. Observe that from the characterization of F. Riesz and N. Merentes it follows that $v_{p,1/p'}(f') = v^{(2)}_{p,1/p'}(f)$.

A. P. Terehin (1965), Volosivets (1993, 1996) showed that $\omega_{2-1/p}(f; t) = \sup_{0<h\leq t} \omega_{1-1/p}(\Delta_h f; h) \quad (0 < t \leq 1)$. where $\Delta_h f(x) = f(x+h) - f(x)$ (*fractional modulus of order* $2 - 1/p$) is $O(\delta^{2-1/p})$ if and only if $f \in W^2_p$.

Martin Lind (2013) studied the scale of spaces $V_{p,\alpha}$ between the class V_p of functions of bounded p-variation and the Sobolev space W^1_p. He gave sharp estimates of $v_{p,\alpha}(f)$ in terms of fractional continuity modulus of order $1 - 1/p$.

$$v_{p,\alpha}(f; \Pi) = \left(\sum_{k=0}^{n-1} \frac{|f(x_{k+1}) - f(x_k)|^p}{(x_{k+1} - x_k)^{\alpha p}} \right)^{1/p}.$$

and

$$v_{p,\alpha}(f) = \sup_{\Pi} v_{p,\alpha}(f; \Pi)$$

M. Lind (2013) has proved that if $f \in V_p$ and $I_{p,\alpha} = \int_0^1 \left(t^{-\alpha}\omega_{1-1/p}(f;t)\right)^p \frac{dt}{t} < \infty$ then

$$v_{p,\alpha}(f) \leq A\left(v_p(f) + p'\alpha^{1/p}(1/p'-\alpha)^{1/p} I_{p,\alpha}(f)\right)$$

The constant $\alpha^{1/p}(1/p'-\alpha)^{1/p}$ has the optimal asymptotic behavior, namely if $\alpha \to 0+$ and if $I_{p,\alpha_0} < \infty$ for some $0 < \alpha_0 < 1/p'$, the second term of the right hand side goes to 0 and the left hand side to $v_p(f)$. If $\alpha \to 1/p'-$ then $\lim(1/p'-\alpha)^{1/p} I_{p,\alpha}(f) = p^{-1/p}\|f'\|_p$, $v_p(f) \leq \|f'\|_p$. The hypothesis cannot be weakened.

Analogously, we can introduce a scale of spaces $V_{p,\alpha}^{(2)}$ between the class $V_p^{(2)}$ of functions of bounded p-second variation and the Sobolev space W_p^2. We would prove sharp estimates of $v_{p,\alpha}^{(2)}$ via integrals defined in terms of fractional continuity modulus of order $2-1/p$.
The difficulty here is that we have just the inequality

$$\omega_{2-1/p}(f;t) \leq t\omega_{1-1/p}(f';t)$$

and a simple substitution in the old result will lead to a weaker inequality than that needed.

$$v_{p,\alpha}^{(2)}(f) = \sup_{\Pi} v_{p,\alpha}^{(2)}(f;\Pi)$$

where

$$v_{p,\alpha}^{(2)}(f;\Pi)^p =$$

$$\sum_{k=0}^{n-1} \left| \frac{f(x_{k+1}) - f(z_{k+1})}{x_{k+1} - z_{k+1}} - \frac{f(y_{k+1}) - f(x_k)}{y_{k+1} - x_k} \right|^p \frac{1}{(x_{k+1} - x_k)^{\alpha p}}.$$

The spaces $V_{p,\alpha}^{(2)}$ of all functions such that $v_{p,\alpha}^{(2)}(f) < \infty$ form a scale between the space of functions of bounded second p-variation and the Sobolev space W_p^2. If $\alpha > 1/p'$, the space $V_{p,\alpha}^{(2)}$ contains only linear functions.

A Marchau- Timmar inequality would imply the desired inequality but not with the right asymptotic behavior of the constant.

We proved that if $f \in V_p^{(2)}$ and $I_{p,\alpha}^{(2)} = \int_0^1 \left(t^{-\alpha-1}\omega_{2-1/p}(f;t)\right)^p \frac{dt}{t} < \infty$ then $f \in V_{p,\alpha}^{(2)}$ and

$$v_{p,\alpha}^{(2)}(f) \leq A\left(v_p^{(2)}(f) + p'\alpha^{1/p}(1/p'-\alpha)^{1/p} I_{p,\alpha}^{(2)}(f)\right)$$

where A is an absolut constant.

The constant has the right behavior and implies the result of Merentes.

M. Lind (2013) described also explicitly Peetre's K-functional:

$$K(f;t;V_p,W_p^1) \approx \omega_{1-1/p}(f;t^{p'})$$

and S. Barza, P. Silvestre gave an explicit expression for $K(f;t;V_p^2,W_p^2)$.

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NEW APPROACH FOR SOLVING SOME RELATED PROBLEMS OF M- MATRICES ON THE BASIS OF DIFFERENTIAL EQUATIONS

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Abstract. This paper develops a novel linear system based approach for computing $\|A^{-1}\|_\infty$, the Skeel condition number of an M–Matrix A, and the positive diagonal matrices D guaranteeing that AD be a strictly diagonally dominant matrix.

1. Introduction

A matrix $A \in Z^{n,n} = \{(a_{ij}) \in R^{n,n} : a_{ij} \leqslant 0, i \neq j; i, j = 1, 2, \cdots, n\}$ is called an M–matrix if A^{-1} is a nonnegative matrix (denoted by $A^{-1} \geqslant 0$). The kind of M-matrices is a very important special class of real matrices, and has been widely applied to many areas. In this paper,we will deal with the following three problems of an arbitrary M-matrix A:

(a) Computing $\|A^{-1}\|_\infty$;
(b) Computing the Skeel condition number of A;

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* Speaker.

(c) Finding a suitable positive diagonal matrix D such that AD is a strictly diagonally dominant matrix.

We will develop a novel linear system based approach to solve the problems (a),(b),(c). The linear system model can be described by the following differential equation

$$\frac{dx(t)}{dt} = -Ax(t) + u, \tag{1.1}$$

where $A = (a_{ij})_{n\times n}$ is an M–matrix, and $u = (u_1, u_2, \cdots, u_n)^T > 0$ is a constant external input vector.

2. Preliminaries

For the sake of convenience and completeness, we introduce some notations,and definitions in this section.
For a vector $x = (x_1, x_2, \cdots, x_n)^T \in R^n$, $x > 0$ denotes each element x_i is positive, and $\|x\|_1$ denotes a vector norm defined by $\|x\|_1 = \sum_{i=1}^n |x_i|$. $e = (1, 1, \cdots, 1)^T$ denotes a column vector each element of which equals 1. A^{-1} denotes the inverse of A; $|A|$ denotes the absolute value matrix given by $|A| = (a_{ij})_{n\times n}$; $A \geqslant 0 (A > 0)$ denotes A is a nonnegative(positive) matrix, the infinity norm of A is defined as $\|A\|_\infty = max_{1\leqslant i\leqslant n} \sum_{j=1}^n |a_{ij}|$.

Definition 2.1. For $A = (a_{ij}) \in C^{n,n}$, if

$$v_i = |a_{ii}| - \sum_{j=1, j\neq i}^{n} |a_{ij}| > 0, \qquad i = 1, 2, \cdots, n,$$

A is said to be strictly row diagonally dominant and $v = (v_1, v_2, \cdots, v_n)^T$ is called the diagonal dominance vector.

Definition 2.2. For $A = (a_{ij}) \in C^{n,n}$, the condition number is defined as $k(A) = \|A\|_\infty \|A^{-1}\|_\infty$, and the Skeel condition number is defined as $Cond(A) = \||A^{-1}||A|\|_\infty$.

Definition 2.3. The equilibrium point δ for system (1.1) is said to be globally exponentially stable if there exist constant k and $\alpha > 1$ such that

$$\|x(t) - \delta\| \leqslant \alpha \|x_0 - \delta\| e^{-kt}, \qquad \forall x_0 \in R^n, \forall t > 0,$$

where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ is the solution of system (1.1) with the initial value x_0 and k is called the degree of exponential stability.

3. Theoretic analysis

Theorem 3.1. *Let the input vector $u = e$ in linear system (1.1). If $\delta = (\delta_1, \delta_2, \cdots, \delta_n)^T$ is an equilibrium point for system (1.1),then $\|A^{-1}\|_\infty = max_{1\leqslant i\leqslant n}(\delta_i)$.*

Theorem 3.2. *Let $u = |A|e$ in linear system (1.1) If $\delta = (\delta_1, \delta_2, \cdots, \delta_n)^T$ is an equilibrium point for system (1.1),then $Cond(A) = \||A^{-1}||A|\|_\infty = max_{1\leqslant i\leqslant n}(\delta_i)$.*

Theorem 3.3. *Assume that $\delta = (\delta_1, \delta_2, \cdots, \delta_n)^T$ is an equilibrium point for system (1.1), and $D = diag(\delta_1, \delta_2, \cdots, \delta_n)$. Then AD is a strictly row diagonally dominant matrix, and u is the diagonal dominance vector, that is,*

$$u_i = a_{ii}\delta_i - \sum_{j=1, i\neq j}^{n} |a_{ij}|\delta_j > 0, \qquad i = 1, 2, \cdots, n.$$

4. example

Example 4.1. For a given M–matrix

$$A = \begin{bmatrix} 1 & 0 & -0.2 \\ -0.8 & 1 & -0.1 \\ -0.9 & 0 & 1 \end{bmatrix}$$

Let $u = e$ and the desired precision be 0.0001. We can obtain at $t = 14$ that

$$\delta = (1.4634, 2.4024, 2.3171)^T \text{ and } \|A^{-1}\|_\infty = max_{1\leqslant i\leqslant n}(\delta_i) = 2.4024.$$

To show the advantage of our approach in terms of the accuracy of computation, we compare the value of $\|A^{-1}\|_\infty$ computed by our approach with the bounds derived in $[1-3]$ in the following table.

Table 1. The infinity norm bound

	Shivakumar [3]	*Cheng* [1]	*Li* [2]	*Our approach*	*True value*
$\|A^{-1}\|_\infty$	$\leqslant 4.0278$	$\leqslant 3.8455$	$\leqslant 3.9670$	$= 2.4024$	$= 2.4024$

5. Conclusion

In this paper, based on linear system theories, we propose a novel approach to compute $\|A^{-1}\|_\infty$ the Skeel condition number of an M-matrix A, and the positive diagonal matrices D guaranteeing that AD be a strictly diagonally dominant matrix.

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LINEAR ALGEBRA METHOD FOR COUNTING THE NUMBER OF PERFECT MATCHINGS IN GRAPHS

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Abstract. We apply linear algebra method to give upper bounds on perfect matchings in pfaffian graphs. These upper bounds which are sharper then Bregman's upper bounds for the number of perfect matchings. We show that some of our upper bounds are sharp for 3 and 4-regular pfaffian graphs. We apply our results to fullerene graphs.

1. Introduction

Let $G = (V, E)$ be a simple undirected graph. Denote $n := |V|$ the number of vertices; $m := |E|$ the number of edges; $d(v)$ the degree of $v \in V$; perfmat G the number of perfect matches in G; $A(G)$ the adjacency matrix of G; $S(G)$ is the skew symmetric adjacency matrix of the digraph $DG = (V, Arc)$, where Arc is an orientation of edges E; $G \times H$ the cartesian product of the graphs G and H; K_n a complete graph on n vertices, $K_{r,r}$ a complete bipartite r-regular graph on $2r$ vertices, C_n a cycle on n vertices.

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A variation of Bregman's inequality yields [1]

$$\text{perfmat } G \leq \prod_{v \in V} (d(v)!)^{\frac{1}{2d(v)}}. \tag{1.1}$$

Equality holds if and only if G is a union of complete regular bipartite graphs. Since $\det S(G)$ is the square of the pfaffian corresponding to $S(G)$ we deduce the well known inequality

$$\det S(G) \leq (\text{perfmat } G)^2. \tag{1.2}$$

A graph is called *pfaffian* if there is an orientation Arc of E such that equality holds in (1.2). It was shown by Kasteleyn [5] that every planar graph is pfaffian. An example of a pfaffian nonplanar graph is $K_4 \times K_2$.

Let $B = [b_{uv}]_{u,v \in V}$ a real symmetric nonnegative definite matrix of order $|V|$, which is denoted by $B \succeq 0$. The generalized Hadamard-Fischer inequality, abbreviated here as H-F inequality, states

$$\det B[U \cup W] \det B[U \cap W] \leq \det B[U] \det B[W] \text{ where } U, W \subset V. \tag{1.3}$$

Here $\det B[\emptyset] = 1$. (See for example [3] and references therein.) Assume that $U \cap W = \emptyset$. Then the above inequality is equivalent to the Hadamard-Fischer inequality. Furthermore if the left hand-side of (1.3) is positive then equality holds if and only if $B[U \cup W]$ is a block diagonal matrix $\operatorname{diag}(B[U], B[W])$. In particular Hadamard's determinat inequality for $B \succeq 0$ is

$$\det B \leq \prod_{v \in V} b_{vv}. \tag{1.4}$$

If all diagonal entries of B are positive then equality holds if and only if B is a diagonal matrix.

We apply Hadamard's determinat inequality to give upper bounds for the number of perfect matchings in Pfaffian graphs. We also, derive a number of improvements of our upper bounds using the Hadamard-Fischer inequality mainly for graphs with no 4-cycles. These inequalities apply to fullerene graphs.Fullerene graphs are 3-regular planar graph,all whose faces are pentagon or hexagons.by Euler formula it is easily shown that every fullerene have 12 pentagons.

2. Main results

The main results of this paper are upper bounds for the prefect matching in pfaffian graphs using Hadamard's determinant inequality and its generalizations. Apply first the Hadamard's determinant inequality to $S(G)$ to deduce that for any pfaffian G one has

$$\operatorname{perfmat} G \leq \prod_{v \in V} d(v)^{\frac{1}{4}}. \tag{2.1}$$

First we show that a complete bipartite graph $K_{r,r}$ does not satisfy the above inequality unless $r = 1, 2$. Next we show that the above upper bound is sharp for the following d-regular connected pfaffian graphs. For $d = 3$ two planar graphs: K_4 and $C_4 \times K_2$. For $d = 4$: a unique planar graph is octahedron (a complement of a perfect match in K_6 and hence planar). Furthermore, if G is a planar 3-regular graph having $3^{\frac{n}{4}}$ perfect matchings then G is a disjoint union of K_4 and $C_4 \times K_2$. Also, we prove the following improvements of (2.1).

Theorem 2.1. *Let $G = (V, E)$ be a pfaffian connected graph. Denote by $G' = (V, E')$ be the induced graph by G, where $(u, v) \in E'$ if and only if the distance between u and v in G is 2. Let M' be a match in G'. Assume that G does not have 4-cycles. Then*

$$\operatorname{perfmat} G \leq \Big(\prod_{(u,v) \in M'} (d(u)d(v) - 1) \Big)^{\frac{1}{4}} \Big(\prod_{v \in V \setminus V(M')} d(v) \Big)^{\frac{1}{4}}. \tag{2.2}$$

One can use the following proposition to estimate $|M'|$.

Proposition 2.2. *Assume that $G = (V, E), n = |V|$ is a connected graph with a path of length $l \geq 3$. Then the induced graph G' given in Theorem 2.1 has a match M' of cardinality $2\lfloor \frac{l}{4} \rfloor$ and $\lfloor \frac{l}{2} \rfloor$ if l is even or odd respectively.*

Theorem 2.3. *Every cubic pfaffian graph with n vertices, which has a perfect match and no 4-cycles, has at most $8^{\frac{n}{12}} 3^{\frac{n}{12}}$ perfect matching.*

Theorem 2.4. *Let $G = (V, E), n = |E|$ be a semi-circular cubic graph where $n_1 = |V_1|$ and $n_0 = |V_0|$ is the number of vertices of G in the the closed disk bounded by the first circle O_1 and its interior respectively. If G does not have 3 and 4-cycles then*

$$\operatorname{perfmat} G \leq 20^{\frac{n-n_1}{12}} 3^{\frac{n_1}{4}} \quad \text{if } n_0 > 0, \tag{2.3}$$

$$\operatorname{perfmat} G \leq 20^{\frac{n}{12}} \quad \text{if } G \text{ is circular, i.e. } n_0 = 0. \tag{2.4}$$

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A NEW PRECONDITIONED AOR ITERATIVE METHOD AND COMPARISON THEOREMS

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Abstract. In this paper, we introduce a new preconditioners for solving linear systems $Ax = b$. We propose preconditioned associated accelerated overrelaxation (AOR) iterative methods with this new preconditioners, and give the corresponding convergence results.

1. Introduction

Consider the following linear system

$$Ax = b \tag{1.1}$$

where $A \in R^{n\times n}$, $b \in R^n$ are given and $x \in R^n$ is unknown. For simplicity, without loss of generality, we assume throughout this paper that $A = I - L - U$, where I is the identity matrix, and L and U are strictly lower and upper triangular matrices obtained from A, respectively. The AOR iteration matrix associated with A is

$$L(r, w) = (I - rL)^{-1}[(1 - w)I + (w - r)L + wU] \tag{1.2}$$

where w and r are real parameters with $w \neq 0$. We now transform the original system in equation (1.1) into the preconditioned form $PAx = Pb$. Then, we apply the AOR iterative method for solving preconditioned form. In [2]-[4] some different preconditioners have been proposed by several authors. In this paper, we propose a new preconditioned AOR iterative method with a preconditioner $P_{\alpha\beta} = I + S_{\alpha\beta}$, where

$$S_{\alpha\beta} = (s_{ij})_{n\times n},\ s_{ij} = \begin{cases} -\alpha_{j-1}(a_{ij} + \beta_{j-1}) & i = 1, j = 2, \cdots, n \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

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* Speaker.

and α_i, β_i are real parameters for $i = 1, 2, \cdots, n-1$. Let $S_{\alpha\beta}A = E_{\alpha\beta} - F_{\alpha\beta}$, where $E_{\alpha\beta}$ is diagonal matrix and $F_{\alpha\beta}$ is upper triangular matrix, respectively. Assume that $A_{\alpha\beta} = P_{\alpha\beta}A = D_{\alpha\beta} - L_{\alpha\beta} - U_{\alpha\beta}$, where $D_{\alpha\beta} = I + E_{\alpha\beta}$, $L_{\alpha\beta} = L$, $U_{\alpha\beta} = U + F_{\alpha\beta}$. Then the AOR iteration matrices associated with $A_{\alpha\beta}$ are

$$L_{\alpha\beta}(r, w) = (D_{\alpha\beta} - rL_{\alpha\beta})^{-1}[(1-w)D_{\alpha\beta} + (w-r)L_{\alpha\beta} + wU_{\alpha\beta}]. \quad (1.4)$$

We first give some terminologies which will be used in the next sections. A matrix A is a Z-matrix if $a_{ij} \leq 0$ for all $i, j = 1, ..., n$ such that $i \neq j$. Also if $a_{ii} > 0$, for $i = 1, 2, ..., n$, the matrix is called an L-matrix. Furthermore, a Z-matrix is a nonsingular M-matrix, if A is nonsingular and $A^{-1} \geq 0$. An $n \times n$ matrix $A = (a_{ij})$ is reducible if we may partition $\{1, \cdots, n\}$ into two nonempty subsets E, F such that $a_{ij} = 0$ if $i \in E$ and $j \in F$. If A is not a reducible matrix, we call A is an irreducible matrix.

Lemma 1.1. *(Varga [1]) Let $A \geq 0$ be an irreducible matrix. Then*

(1) *A has a positive real eigenvalue equal to its spectral radius.*
(2) *$\rho(A)$ is a simple eigenvalue of A corresponding to an eigenvector $x > 0$.*
(3) *$\rho(A)$ increases when any entry of A increases.*

Lemma 1.2. *(Varga [1]) Let A be a nonnegative matrix. Then*

(1) *If $\alpha x \leq Ax$ for some nonnegative and nonzero vector x, then $\alpha \leq \rho(A)$.*
(2) *If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$, for some nonnegative vector x, then $\alpha \leq \rho(A) \leq \beta$ and x is a positive vector.*

Lemma 1.3. *(Varga [1]) Let $A = M - N$ be an M-spliting of A. Then $\rho(M^{-1}N) < 1$ if and if A a nonsingular M-matrix.*

2. Main results

The spectral radius of the iterative matrix is conclusive for the convergence and stability of the method, and the smaller it is, the faster the method converges when the spectral radius is smaller than 1. In this section, some results for the AOR iterative method with preconditioner $P_{\alpha\beta}$ is given.

Theorem 2.1. *Let $L(r, w)$ and $L_{\alpha\beta}(r, w)$ be the iteration matrices of the AOR method given by equations (1.2) and (1.4). If A is an irreducible L-matrix with $a_{v+1,1}a_{1,v+1} > 0$, $\beta_v \in (\frac{1-a_{v+1,1}a_{1,v+1}}{a_{v+1,1}}, -a_{1,v+1}) \cap (0, -a_{1,v+1})$, $\alpha_v \in (0, 1]$, $(v = 1, \cdots, n-1)$, $\sum_{v=1}^{n-1} \alpha_v \leq 1$ and $0 \leq r < w \leq 1$, then $L(r, w)$ and $L_{\alpha\beta}(r, w)$ are nonnegative irreducible matrices.*

Proof. Because A is an irreducible L-matrix, L is a nonnegative strictly lower triangular matrix and U is a nonnegative strictly upper triangular matrix.

By equation (1.2), we have

$$\begin{aligned} L(r,w) &= [I + rL + r^2L^2 + \cdots + r^{n-1}L^{n-1}] \\ &\quad \times[(1-w)I + (w-r)L + wU] \\ &= (1-w)I + (w-r)L + wU + \text{nonnegative terms} \end{aligned}$$

Since $0 \le r < w \le 1$, it follows that $L(r,w)$ is nonnegative. We can also get that $(1-w)I + (w-r)L + wU$ is irreducible for irreducible A, and hence $L(r,w)$ is also irreducible. Now, we show that $D_{\alpha\beta} > 0$, $L_{\alpha\beta} \ge 0$, $U_{\alpha\beta} \ge 0$, and $E_{\alpha\beta} \le 0$. We obtain $D_{\alpha\beta} = diag(\zeta, 1, \cdots, 1)$, where

$$\zeta = 1 - \alpha_1 a_{21}(a_{12} + \beta_1) - \alpha_2 a_{31}(a_{13} + \beta_2) - \cdots - \alpha_{n-1}a_{n,1}(a_{1,n} + \beta_{n-1})$$

$$L_{\alpha\beta} = (l_{ij})_{n\times n},\ l_{ij} = \begin{cases} -a_{ij} & \text{if } i > j \\ 0 & \text{if } i \le j \end{cases}$$

$$U_{\alpha\beta} = (u_{ij})_{n\times n},\ u_{ij} = \begin{cases} 0 & \text{if } i \ge j, \\ -a_{ij} & \text{if } i > j, i \ne 1, \end{cases},\ \textit{and}$$

$$\begin{aligned} u_{1j} &= -a_{1j} + \alpha_1 a_{2,j}(a_{12} + \beta_1) + \alpha_2 a_{3,j}(a_{13} + \beta_2) + \cdots \\ &\quad + \alpha_{n-1}a_{n,j}(a_{1n} + \beta_{n-1}). \end{aligned}$$

For $\beta_v \in (\frac{1-a_{v+1,1}a_{1,v+1}}{a_{v+1,1}}, -a_{1,v+1}) \cap (0, -a_{1,v+1})$, $\alpha_v \in (0,1]$ $(v = 1, 2, \cdots, n-1\ )$, $\sum_{v=1}^{n-1} a_v \le 1$, we can write

$$\begin{aligned} 1 - \sum_{v=1}^{n-1} \alpha_v a_{v+1,1}(a_{1,v+1} + \beta_v) &> 1 - \sum_{v=1}^{n-1} \alpha_v a_{v+1,1}(a_{1,v+1} + \frac{1 - a_{v+1,1}a_{1,v+1}}{a_{1,v+1}}) \\ &= 1 - \sum_{v=1}^{n-1} \alpha_v \ge 0 \end{aligned} \tag{2.1}$$

By considering equations (2.1), we get $D_{\alpha\beta} > 0$ and $E_{\alpha\beta} \le 0$. We can also write

$$\begin{aligned} -a_{1j} + \sum_{v=1}^{n-1} \alpha_v a_{v,j}(a_{1,v+1} + \beta_v) &\ge -a_{1j} + \sum_{v=1}^{n-1} \alpha_v a_{v,j}(a_{1,v+1} - a_{1,v+1}) \\ &= -a_{1j} \ge 0 \end{aligned} \tag{2.2}$$

Therefore $L_{\alpha\beta} \ge 0$ and $U_{\alpha\beta} \ge 0$. Now from equation (1.4), we have

$$\begin{aligned} L_{\alpha\beta}(r,w) &= (I - rD_{\alpha\beta}^{-1}L_{\alpha\beta})^{-1}[(1-w)I + (w-r)D_{\alpha\beta}^{-1}L_{\alpha\beta} + wD_{\alpha\beta}^{-1}U_{\alpha\beta}] \\ &= [I + rD_{\alpha\beta}^{-1}L_{\alpha\beta} + r^2(D_{\alpha\beta}^{-1}L_{\alpha\beta})^2 + \cdots + r^{n-1}(D_{\alpha\beta}^{-1}L_{\alpha\beta})^{n-1}] \\ &\quad \times [(1-w)I + (w-r)D_{\alpha\beta}^{-1}L_{\alpha\beta} + wD_{\alpha\beta}^{-1}U_{\alpha\beta}] \\ &= (1-w)I + (w-r)D_{\alpha\beta}^{-1}L_{\alpha\beta} + wD_{\alpha\beta}^{-1}U_{\alpha\beta} \\ &\quad + \text{nonnegative terms} \end{aligned} \tag{2.3}$$

By the above results, we can see $L_{\alpha\beta}(r,w)$ are nonnegative irreducible matrices, and the proof is complete. $\square$

Theorem 2.2. *Let $L(r,w)$, $L_{\alpha\beta}(r,w)$ be the iteration matrices of the AOR method given by equations (1.2) and (1.4). If A is an irreducible L-matrix with $a_{v+1,1}a_{1,v+1} > 0$, $\beta_v \in (\frac{1-a_{v+1,1}a_{1,v+1}}{a_{v+1,1}}, -a_{1,v+1}) \cap (0, -a_{1,v+1})$, $\alpha_v \in (0,1]$ $(v = 1, \cdots, n-1)$, $\sum_{v=1}^{n-1} \alpha_v \leq 1$ and $0 \leq r < w \leq 1$, then we have*

(1) $\rho\left(L_{\alpha\beta}(r,w)\right) < \rho\left(L(r,w)\right)$, *if* $\rho\left(L(r,w)\right) < 1$
(2) $\rho\left(L_{\alpha\beta}(r,w)\right) = \rho\left(L(r,w)\right)$, *if* $\rho\left(L(r,w)\right) = 1$
(3) $\rho\left(L_{\alpha\beta}(r,w)\right) > \rho\left(L(r,w)\right)$, *if* $\rho\left(L(r,w)\right) > 1$.

Proof. Theorem 2.1 implies that $L(r,w)$ is a nonnegative irreducible matrix. Hence there exists a positive vector x, such that $L(r,w)x = \lambda x$, where $\rho(L(r,w)) = \lambda$ or equivalently

$$[(1-w)I + (w-r)L + wU]x = \lambda(I - rL)x. \tag{2.4}$$

By equation (2.4) we can obtain

$$L_{\alpha\beta}(r,w) - \lambda x = (\frac{\lambda - 1}{\lambda})(D_{\alpha\beta} - rL_{\alpha\beta})^{-1}[-E_{\alpha\beta} + (1-w)S_{\alpha\beta} + (w-r)S_{\alpha\beta}L + wS_{\alpha\beta}U]x$$

Now let

$$Q = (D_{\alpha\beta} - rL_{\alpha\beta})^{-1}[-E_{\alpha\beta} + (1-w)S_{\alpha\beta} + (w-r)S_{\alpha\beta}L + wS_{\alpha\beta}U] \tag{2.5}$$

From $S_{\alpha\beta} \geq 0$, $S_{\alpha\beta}L \geq 0$, $E_{\alpha\beta} \leq 0$, $S_{\alpha\beta}U \geq 0$ we have

$$[-E_{\alpha\beta} + (1-w)S_{\alpha\beta} + (w-r)S_{\alpha\beta}L + wS_{\alpha\beta}U] \geq 0,$$

Since $rD_{\alpha\beta}^{-1}L_{\alpha\beta}$ is a strictly lower triangular matrix, so $\rho(rD_{\alpha\beta}^{-1}L_{\alpha\beta}) = 0 < 1$. By considering Lemma 1.3, we have R is a nonsingular M-matrix. Therefore $(D_{\alpha\beta} - rL_{\alpha\beta})^{-1} \geq 0$, and so $Q \geq 0$. $If\ \lambda < 1$, then $L_{\alpha\beta}(r,w)x - \lambda x \leq 0$. Therefore $L_{\alpha\beta}(r,w)x \leq \lambda x$. By using Lemma 1.2, we get $\rho(L_{\alpha\beta}(r,w)) < \lambda = \rho(L(r,w))$; $If\ \lambda = 1$, then $L_{\alpha\beta}(r,w)x - \lambda x = 0$. Therefore $L_{\alpha\beta}(r,w)x = \lambda x$. By using Lemma 1.2, we get $\rho(L_{\alpha\beta}(r,w)) = \lambda = \rho(L(r,w))$; $If\ \lambda > 1$, then $L_{\alpha\beta}(r,w)x - \lambda x \geq 0$. Therefore $L_{\alpha\beta}(r,w)x \geq \lambda x$. By using Lemma 1.2, we get $\rho(L_{\alpha\beta}(r,w)) \geq \lambda = \rho(L(r,w))$.

□

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POLAR DECOMPOSITION OF THE ALUTHGE TRANSFOMATION IN HILBERT C^*-MODULES

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Abstract. Let $T = U|T|$ and $S = V|S|$ be the polar decomposition of adjointable operators T and S, respectively on a Hilbert C^*-module. In this talk, we determine these pairs of operators for which their products TS accepts the polar decomposition as $TS = UV|TS|$. Specially, we provide sufficient conditions for a certain operator T such that its Aluthge transform $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ admits the polar decomposition.

1. Introduction

A pre-Hilbert C^*-module $\mathcal{X}$ over a C^*-algebra $\mathcal{A}$, is a right $\mathcal{A}$-module together with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying the conditions:

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ for $x, y, z \in \mathcal{X}, \alpha, \beta \in \mathbb{C}$,
(ii) $\langle x, ya\rangle = \langle x, y\rangle a$ for $x, y \in \mathcal{X}, a \in \mathcal{A}$,
(iii) $\langle x, y\rangle^* = \langle y, x\rangle$ for $x, y \in \mathcal{X}$,
(iv) $\langle x, x\rangle \geq 0$ for $x \in \mathcal{X}$,
(v) $\langle x, x\rangle = 0$ if and only if $x = 0$.

We can define a norm on $\mathcal{X}$ by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{X}$ is called a right Hilbert C^*-module over $\mathcal{A}$ if it is complate with respect to its norm. Each Hilbert space is a Hilbert $\mathbb{C}$-module

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* Speaker.

and each C^*-algebra $\mathcal{A}$ can be regarded as a Hilbert $\mathcal{A}$-modules via $\langle a, b\rangle = a^*b(a, b \in \mathcal{A})$. Note that, some properties of Hilbert spaces do not hold in Hilbert C^*-modules. For example, a bounded operator T on Hilbert C^*-module $\mathcal{X}$ might not admit a bounded operator T^* as its adjoint operator, that satisfy in the condition $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for any $x, y \in \mathcal{X}$. But, it is easy to see that every adjointable operator T is a bounded linear $\mathcal{A}$-module mapping. Also, in the general, a closed submodule F of $\mathcal{X}$ might not be orthogonal complemented and $F^{\perp\perp}$ is usually larger than F. Recall that a closed submodule F of $\mathcal{X}$ is said to be orthogonally complemented if $\mathcal{X} = F \oplus F^{\perp}$, where $F^{\perp} = \{x \in \mathcal{X} : \langle x, x'\rangle = 0 \text{ for all x}' \in \text{F}\}$. Lance proved that if an adjointable operator $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ has closed range, then

(i) $N(T)$ is a complemented of $\mathcal{X}$, with complement $R(T^*)$,
(ii) $R(T)$ is a complemented of $\mathcal{Y}$, with complement $N(T^*)$.

The basic theory of Hilbert C^*-modules can be found in [5]. In this paper, $\mathcal{A}$ denotes a C^*-algebra and $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules and $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is set of all adjointable operators between $\mathcal{X}$ and $\mathcal{Y}$ and write $\mathcal{B}(\mathcal{X})$ for $\mathcal{B}(\mathcal{X}, \mathcal{X})$. We used $D(.)$, $N(.)$ and $R(.)$ for domain, kernel and range of operators, respectively.

The following theorem gives some conditions under which the product of two operators acting on Hilbert C^*-modules again admits polar decomposition.

Theorem 1.1. *Let $T, S \in \mathcal{B}(\mathcal{X})$ admit the polar decomposition with $N(T) = N(S^*)$. Then the operator TS admits the polar decomposition.*

The following lemma gives a necessary and sufficient condition for operators that satisfy in statement (i) of lemma 1.2. The proof of this lemma is based on the polar decomposition property for operators that acting between Hilbert C^*-modules.

Lemma 1.2. *Let $T = U|T|$ and $S = V|S|$ be the polar decomposition of $T, S \in \mathcal{B}(\mathcal{X})$, respectively. Then the following assertions are equivalent:*

(i) $[T, S] = 0$ *and* $[T, S^*] = 0$.
(ii) *The following equetions are satisfy:*

$$\begin{array}{ccccc} [S, U] = 0 & , & [S^*, U] = 0 & , & [U, |S|] = 0, \\ [T, V] = 0 & , & [T^*, V] = 0 & , & [V, |T|] = 0, \\ [U, V] = 0 & , & [U^*, V] = 0 & , & [|T|, |S|] = 0. \end{array}$$

2. Main results

The following theorem presents the polar decomposition of the product of operators.

Theorem 2.1. *Let $T = U|T|$ and $S = V|S|$ be the polar decomposition of $T, S \in \mathcal{B}(\mathcal{X})$, respectively. If $[T, S] = 0, [T, S^*] = 0$ and $N(T) = N(S^*)$, then $TS = UV|TS|$ is the polar decomposition of TS. That is TS is a partial isometry with $N(UV) = N(|T||S|)$ and $|T||S| = |TS|$.*

Lemma 2.2. *Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{X})$. Then for any $q \geq 0$,*

(i) $|T|^q = U^*U|T|^q$ *is the polar decomposition of* $|T|^q$,
(ii) $|T^*|^q = UU^*|T^*|^q$ *is the polar decomposition of* $|T^*|^q$.

Now we present a relationship between polar decomposition of a binormal operator and its Aluthge transform.
An operator $T \in \mathcal{B}(\mathcal{X})$ is said to be binormal if $[|T|, |T^*|] = 0$, where $[A, B] = AB - BA$. Binormality of operators was defined by Campbell. He was obtained some properties of this operators. As a consequent, if T is a binormal operator, then we have $|T|^{1/2}|T^*|^{1/2} = \left||T|^{1/2}|T^*|^{1/2}\right|$.

Lemma 2.3. *Let $T = U|T|$ be the polar decomposition of a binormal operator $T \in \mathcal{B}(\mathcal{X})$ with $N(T) = N(T^*)$. Then*

$$|T|^{1/2}|T^*|^{1/2} = U^*UUU^*||T|^{1/2}|T^*|^{1/2}|$$

is the polar decomposition of $|T|^{1/2}|T^*|^{1/2}$.

Theorem 2.4. *Let $T = U|T|$ be the polar decomposition of a binormal operator $T \in \mathcal{B}(\mathcal{X})$ with $N(T) = N(T^*)$. Then $\widetilde{T} = U^*UU|\widetilde{T}|$ is the polar decomposition of $\widetilde{T}$.*

As an extension of $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$, we consider $\widetilde{T} = |T|^qU|T|^q$ for a positive number q which is not necessarily $\frac{1}{2}$ and call it the generalized Aluthge transformation. In the following theorem we obtain the polar decomposition of the generalized Aluthge transform.

Theorem 2.5. *Let $T = U|T|$ be the polar decomposition of a binormal operator $T \in \mathcal{B}(\mathcal{X})$ with $N(T) = N(T^*)$. Then the generalized Aluthge transformation $\widetilde{T} = |T|^qU|T|^q$ accepts the polar decomposition.*

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STABILITY ANALYSIS OF A N-COUPLED CELL SYSTEM WITH DELAY

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ABSTRACT. In this paper, we consider a system of coupled cells in a ring with n cells with delays, considering the time delay between the signal transmission of the cells. The $n-$coupled cell system is investigated from the viewpoint of distribution of zeros of the associated characteristic equation. In fact, we obtain some sufficient conditions by using the results of circle block matrices and their eigenvalues for the zero solution of the model to be asymptotically stable.

1. Introduction

In the last decades interest in studying of coupled systems has arisen. Since systems of coupled differential equations are used as models in a wide range of applications [1, 4]. Golubitsky and coworkers in their papers such as [2, 3] have shown that many phase relations observed in animal gaits can be modeled by coupled cell systems. If we regard the individual unit in these models as a 'cell', with deterministic dynamics, then a coupled cell system may be regarded as a set of individual but interacting dynamical systems. They formalized these ideas into a general theory of networks of coupled cells (a general review of this work appears in [2]). In this paper, we consider the system of delay differential equations (DDE) representing the models containing n

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* Speaker.

cells in a ring with time-delayed connections such that every cell is a two-dimensional ordinary differential equation. We would like to know about distribution of zeros of the associated characteristic equation. Its characteristic equation is very complicated because of existence time-delay and the number of cells. We applied some results in eigenvalue problems of circle block matrices and then obtain some important results about the stability of the n-coupled cell system. To the best of our knowledge, it is the first time to deal with the stability of n-coupled cell systems with time-delays in a ring.

2. Main results

We consider the n-coupled cell system in a ring with the following equations

$$\begin{aligned} \dot{x}_i(t) &= y_i(t) + f(x_i(t)) + ax_{i-1}(t-\tau) + ax_{i+1}(t-\tau) + cx_i(t-\tau) \\ \dot{y}_i(t) &= -x_i(t) + g(y_i(t)), \end{aligned}$$

where $i = 2, ..., n-1$. Moreover, the dynamics of cells for $i = 1$ and n is as follows:

$$\begin{aligned} \dot{x}_1(t) &= y_1(t) + f(x_1(t)) + ax_n(t-\tau) + ax_2(t-\tau) + cx_1(t-\tau) \\ \dot{y}_1(t) &= -x_1(t) + g(y_1(t)), \end{aligned}$$

and

$$\begin{aligned} \dot{x}_n(t) &= y_n(t) + f(x_n(t)) + ax_{n-1}(t-\tau) + ax_1(t-\tau) + cx_n(t-\tau) \\ \dot{y}_n(t) &= -x_n(t) + g(y_n(t)). \end{aligned}$$

Let

$$f(0) = 0, \quad g(0) = 0, \quad f'(0) = d, \quad g'(0) = b.$$

Therefore, the associated eigenvalue problem of the n-coupled cell system takes the form

$$\det \triangle(\lambda, \tau) = 0$$

when

$$\triangle(\lambda, \tau) = \lambda I_{2n} - \begin{bmatrix} A_0 & A_1 & ... & A_{n-1} \\ A_{n-1} & A_0 & ... & A_{n-2} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ A_1 & A_{n-1} & ... & A_0 \end{bmatrix}$$

and

$$A_0 = \begin{bmatrix} d + ce^{-\lambda\tau} & 1 \\ -1 & b \end{bmatrix}, \quad A_1 = A_{n-1} = \begin{bmatrix} ae^{-\lambda\tau} & 0 \\ o & 0 \end{bmatrix}, A_i = 0$$

if $i = 2, ..., n-2$.

Theorem 2.1. *If the following assumption is satisfied, then the zero solution of the n-coupled cell system is asymptotically stable for all* $\tau \geq 0$*,*

$$\theta = b + d + c + 2a\cos(\frac{2\pi l}{n}) < 0, \quad \delta = 1 + b(d + c + 2a\cos(\frac{2\pi l}{n})) > 0.$$

Proof. First note that by [4] the associated eigenvalue problem of the n-coupled cell system with delays is

$$\det(\triangle(\lambda, \tau)) = \Pi_{l=0}^{n-1} \det(\lambda I_2 - h(u_l)) \tag{2.1}$$

where

$$h(x) = A_0 + xA_1 + x^2A_2 + ... + x^{n-1}A_{n-1}$$

and u_l is the l-th root of equation $x^n = 1$. Then, Eq(2.1) is equivalent to

$$\begin{aligned} \det(\triangle(\lambda, \tau)) &= \Pi_{l=0}^{n-1} \det(\begin{bmatrix} \lambda - d - ce^{-\lambda\tau} - 2a\cos(\frac{2\pi l}{n})e^{-\lambda\tau} & -1 \\ 1 & \lambda - b \end{bmatrix}) \\ &= \Pi_{l=0}^{n-1}(\lambda^2 - \theta\lambda + \delta). \end{aligned}$$

Therefore, for $\tau = 0$, $\theta < 0$ and $\delta > 0$ all roots of the characteristic equation have the negative real parts. $\square$

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ON CHARACTERIZATION OF INNER PRODUCT SPACES WITH RESPECT TO ANGULAR DISTANCE

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Abstract. In this talk we present a new characterization of inner product spaces which is similar to the well-known Ficken characterization. More precisely, we discuss that for which $t \in \mathbb{R}$, the angular distance equality $\alpha[x + ty, y] = \alpha[y + tx, x]$ characterizes an inner product space.

1. Introduction and Preliminaries

There are a lot of significant natural geometric properties, which fail in general normed linear spaces, such as non Euclidean ones. Some of these interesting properties hold just when the space is an inner product one. This is the most important motivation for studying characterizations of inner product spaces. The first norm characterization of inner product spaces was given by Frechet in 1935. In 1936, Jordan and von Neumann showed that a normed linear space $(X, \|.\|)$ is an inner product one if and only if the parallelogram law $\|x-y\|^2+\|x+y\|^2 = 2\|x\|^2+2\|y\|^2$ holds for all $x, y \in X$. Since then, the problem of finding necessary and sufficient conditions for a normed linear space to be an inner product one has been investigated by many mathematicians who considered some geometric aspects of underlying

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spaces.
In 1943, Ficken [4] showed that a normed linear space is an inner product space if and only if a reflection about a line in any two-dimensional subspace is an isometric mapping (see Theorem 1.3 below).

In 1936, Clarkson introduced the concept of angular distance between nonzero elements x and y in a normed linear space X as $\alpha[x,y] = \|\frac{x}{\|x\|} - \frac{y}{\|y\|}\|$. There are interesting characterizations of inner product spaces connected with the concept of angular distance; see [2, 3] and refrences therein.

In this talk we present a new characterization of inner product spaces which is similar to the well-known Ficken characterization. More precisely, we discuss that for which $t \in \mathbb{R}$, the angular distance equality $\alpha[x+ty,y] = \alpha[y+tx,x]$ characterizes an inner product space.

Our basic tools are norm derivations and some characterizations of inner product spaces along with them. In a normed linear space $(X, \|.\|)$, the various norm derivatives are given for fixed $x, y \in X$ by the following expressions

$$G(x,y) := \lim_{\lambda \to \pm 0} \frac{\|x+\lambda y\| - \|x\|}{\lambda} \quad \text{(Gateaux derivative)}$$

and

$$\rho'_{\pm}(x,y) := \lim_{\lambda \to \pm 0} \frac{\|x+\lambda y\|^2 - \|x\|^2}{2\lambda} \quad \text{(norm derivatives)} \tag{1.1}$$

Some characterizations of inner product spaces given in terms of norm derivatives were reported in [1, 2]. From $\rho'_{\pm}(x,y) = \|x\|G(x,y)$ we conclude that the norm has Gateaux derivative if and only if $\rho'_+ = \rho'_-$. Although instead of considering Gateaux derivative it is more convenient to apply the norm derivatives $\rho'_{\pm}(x,y)$ because when the norm comes from an inner product $\langle .,. \rangle$, we obtain $\rho'_{\pm}(x,y) = \langle .,. \rangle$ i.e. functionals $\rho'_{\pm}$ are perfect generalizations of inner products.

The next two theorems describe several properties of ρ'_+ and ρ'_-, see [1, 2].

Theorem 1.1. [1] *Let $(X, \|.\|)$ be a normed linear space and ρ'_+ and ρ'_- be given by (1.1). Then*
(i) *ρ'_+ and ρ'_- are well defined;*
(ii) *$\rho'_{\pm}(0,y) = \rho'_{\pm}(x,0) = 0$ for all $x, y \in X$;*
(iii) *$\rho'_{\pm}(x,x) = \|x\|^2$ for all $x \in X$;*
(iv) *$\rho'_{\pm}(\alpha x,y) = \rho'_{\pm}(x,\alpha y) = \alpha\rho'_{\pm}(x,y)$ for all $x, y \in X$ and $\alpha \geq 0$;*
(v) *$\rho'_{\pm}(\alpha x,y) = \rho'_{\pm}(x,\alpha y) = \alpha\rho'_{\mp}(x,y)$ for all $x, y \in X$ and $\alpha \leq 0$;*
(vi) *$|\rho'_{\pm}(x,y)| \leq \|x\|\|y\|$ for all $x, y \in X$.*

Theorem 1.2. [1, 2] *Let* $(X, \|.\|)$ *be a normed linear space. Then the following statements are mutually equivalent:*
(i) $\rho'_+(x,y) = \rho'_+(y,x)$ *for all* $x, y \in X$*;*
(ii) $\rho'_-(x,y) = \rho'_-(y,x)$ *for all* $x, y \in X$*;*
(iii) $(X, \|.\|)$ *is an inner product space.*

In the following theorem we state the well-known Ficken characterization [4].

Theorem 1.3. [4] *Let* $(X, \|.\|)$ *be a normed linear space. Then the norm comes from an inner product if and only if* $\|x + ty\| = \|y + tx\|$ *for all* $x, y \in X$ *and* $t \in \mathbb{R}$*.*

2. Main results

We start this section with the following theorem, which provides the necessary condition for our characterization.

Theorem 2.1. *Let* $(X, \langle \cdot, \cdot \rangle)$ *be an inner product space and* x, y *be two independent vectors. Then*
$\|x\| = \|y\| \Rightarrow \alpha[x + ty, y] = \alpha[y + tx, x], for\ all\ t \in \mathbb{R}.$

Now we can state the next theorem as a characterization of inner product spaces.

Theorem 2.2. *Let* $(X, \|.\|)$ *be a normed linear space. Then the following statements are equivalent:*
(i) *For independent vectors x,y with the same norm there exists* $h \in \mathbb{R}$ *such that* $\|x + hy\| = \|y + hx\|$ *and*

$$\alpha[x + ty, y] = \alpha[y + tx, x] \qquad (for\ all\ t \leq h);$$

(ii) $(X, \|.\|)$ *is an inner product space.*

Here we can reformulate our result for unit vectors as the following corollary.

Corollary 2.3. *Let* $(X, \|.\|)$ *be a normed linear space. Then the following statements are equivalent:*
(i) *For independent vectors* $u, v \in S_X$ *there exists* $h \in \mathbb{R}$ *such that*

$$\|u + hv\| = \|v + hu\| \text{ and } \alpha_p[u + tv, v] = \alpha_p[v + tu, u] \quad (for\ all\ t \leq h);$$

(ii) $(X, \|.\|)$ *is an inner product space.*

3. THE CASES $t = 1$ AND $t = -1$

The results in the previous section has been based on satisfying the angular distance equality

$$\alpha[x + ty, y] = \alpha[y + tx, x] \tag{3.1}$$

for infinite values of t. In 2009, Senlin Wu [5] proved that in a normed linear space $(X, \|.\|)$ the norm comes from an inner product if and only if $\alpha[u + v, u] = \alpha[u + v, v]$ (for all $u, v \in S_X, u \neq -v$). we also show that in order to obtain the characterization of inner product spaces it is sufficient that (3.1) holds only for $t = -1$. First we need the following lemma which provides an equivalent condition for the characterization of inner product spaces due to Isoscelis orthogonality.

Lemma 3.1. *Let $(X, \|.\|)$ be a normed linear space. If for any independent vectors x, y with the same norm $\alpha[x - y, y] = \alpha[y - x, x]$, then*

$$a \perp_I b \Rightarrow a \perp_I \frac{\|a - b\| + \|b\|}{\|b\|}\, b \qquad (a, b \neq 0).$$

Theorem 3.2. *A normed linear space $(X, \|.\|)$ is an inner product space if and only if for any $u, v \in S_X$ with $u \neq v$*

$$\alpha[u - v, v] = \alpha[v - u, u].$$

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E-FRAMES FOR SEPARABLE HILBERT SPACES

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Abstract. The purpose of this paper is to introduce the concept of E−frames for a separable Hilbert space $\mathcal{H}$, where E is an invertible infinite matrix mapping on the Hilbert space $\overset{\infty}{\underset{n=1}{\oplus}} \mathcal{H}$. We investigate and study some properties of E−frames and characterize all E−frames for $\mathcal{H}$. Further more, we characterize all dual E−frames associated with a given E−frame. A similar characterization is also established for E−orthonormal bases, E−Reisz bases and dual E−Riesz bases. Our results generalize the concept of frames because the ordinary frames are the special case of E−frames in which the matrix E be replaced by the identity matrix operator I on $\overset{\infty}{\underset{n=1}{\oplus}} \mathcal{H}$.

1. Introduction

Suppose that X and Y be two sequence spaces and $\mathrm{A} = (a_{n,k})_{n,k\geq 1}$ be an infinite matrix of real or complex numbers. Then, we say that A defines a matrix mapping from X into Y, and we denote it by writing $\mathrm{A} : X \to Y$, if for every sequence $x = \{x_n\}_{n=1}^{\infty} \in X$, the sequence $Ax = \{(Ax)_n\}_{n=1}^{\infty}$, the A-transform of x, is in Y, where

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[*] Speaker.

$$(\mathrm{A}x)_n = \sum_{k=1}^{\infty} a_{n,k} x_k, \;\; n = 1, 2, \dots \,.$$

Let $(\mathcal{H}_n)_{n=1}^{\infty}$ be a sequence of Hilbert spaces, and let

$$\bigoplus_{n=1}^{\infty} \mathcal{H}_n = \left\{ (h_n)_{n=1}^{\infty} : \;\; h_n \in \mathcal{H}_n, \;\; \sum_{n=1}^{\infty} \|h_n\|^2_{\mathcal{H}_n} < \infty \right\}.$$

Define an inner product $\langle ., . \rangle$ on $\bigoplus_{n=1}^{\infty} \mathcal{H}_n$ by:

$$\langle (g_n)_{n=1}^{\infty}, (h_n)_{n=1}^{\infty} \rangle := \sum_{n=1}^{\infty} \langle g_n, h_n \rangle_{\mathcal{H}_n}.$$

With respect to this inner product, $\bigoplus_{n=1}^{\infty} \mathcal{H}_n$ is a Hilbert space, called the Hilbert space direct sum of the $(\mathcal{H}_n)_{n=1}^{\infty}$ [1].

Next, let $(\mathcal{H}, \langle ., . \rangle)$ be a separable Hilbert space. A countable family $\{f_k\}$ in $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist positive real numbers A and B such that

$$A\|f\|^2 \leq \|\{\langle f, f_k \rangle\}_{k=1}^{\infty}\|^2_{\ell^2} \leq B\|f\|^2, \;\; \forall f \in \mathcal{H}.$$

The sequence $\{f_k\} \subset \mathcal{H}$ is called a Bessel sequence for $\mathcal{H}$, when it satisfies the upper frame inequality [2].

In this paper, we are going to introduce and study the concept of E-frames for a Hilbert space $\mathcal{H}$. Our results generalize the concept of frames because the ordinary frames are the special case of E$-$frames in which the matrix E be replaced by the identity matrix operator I on $\bigoplus_{n=1}^{\infty} \mathcal{H}$. Thought this paper we suppose that $\mathcal{H}$ is a separable Hilbert space and E is an invertible infinite matrix mapping on the Hilbert space direct sum $\bigoplus_{n=1}^{\infty} \mathcal{H}$.

2. Main results

We begin with definition of an E$-$Bessel sequence.

Definition 2.1. A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is called an E$-$Bessel sequence if there exists a constant $B > 0$ such that

$$\left\| \left\{ \left\langle f, \left(E \{f_j\}_{j=1}^{\infty} \right)_k \right\rangle \right\}_{k=1}^{\infty} \right\|^2_{\ell^2} \leq B\|f\|^2, \;\; \forall f \in \mathcal{H}. \tag{2.1}$$

Any number B satisfying (2.1) is called a E$-$Bessel bound for $\{f_k\}_{k=1}^{\infty}$. The optimal bound for a given E$-$Bessel sequence $\{f_k\}_{k=1}^{\infty}$ is the smallest possible value of $B > 0$ satisfying (2.1).

Definition 2.2. Consider a sequence $\{g_k\}_{k=1}^{\infty}$ of vectors in $\mathcal{H}$.

(1) The sequence $\{g_k\}_{k=1}^{\infty}$ is an E−complete if

$$\overline{span}\left\{\left(E\{g_j\}_{j=1}^{\infty}\right)_k\right\}_{k=1}^{\infty}=\mathcal{H}.$$

(2) The sequence $\{g_k\}_{k=1}^{\infty}$ is an E−basis(E− Schauder basis) for $\mathcal{H}$, if for each $f\in\mathcal{H}$ there exist unique scalar coefficients $\{c_k(f)\}_{k=1}^{\infty}$ such that

$$f=\sum_{k=0}^{\infty}c_k(f)\left(E\{g_j\}_{j=1}^{\infty}\right)_k. \tag{2.2}$$

(3) An E−basis $\{g_k\}_{k=1}^{\infty}$ is unconditional, if the series (2.2) converges unconditionally for each $f\in\mathcal{H}$.

(4) An E−basis $\{g_k\}_{k=1}^{\infty}$ is E−orthonormal basis, if $\{g_k\}_{k=1}^{\infty}$ is an E−orthonormal system, i.e., if

$$\left\langle\left(E\{g_j\}_{j=1}^{\infty}\right)_n,\left(E\{g_j\}_{j=1}^{\infty}\right)_k\right\rangle=\delta_{n,k}=\begin{cases}1 & n=k,\\ 0 & n\neq k.\end{cases}$$

Theorem 2.3. *Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Then the* E−*orthonormal bases for $\mathcal{H}$ are precisely the sets $\left\{U\left(E^{-1}\{e_j\}_{j=1}^{\infty}\right)_k\right\}_{k=1}^{\infty}$, where U is an unitary operator on $\mathcal{H}$.*

Definition 2.4. An E−Riesz basis for $\mathcal{H}$ is a family of the form $\left\{U\left(E^{-1}\{e_j\}_{j=1}^{\infty}\right)_k\right\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and U is a bounded bijective operator on $\mathcal{H}$.

Theorem 2.5. *Let $\{f_k\}_{k=1}^{\infty}$ be an* E−*Riesz basis for $\mathcal{H}$, then $\{f_k\}_{k=1}^{\infty}$ is an* E−*Bessel sequence. Furthermore, there exists a unique sequence $\{g_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ such that*

$$f=\sum_{k=1}^{\infty}\left\langle f,\left(E\{g_j\}_{j=1}^{\infty}\right)_k\right\rangle\left(E\{f_j\}_{j=1}^{\infty}\right)_k,\ \forall f\in\mathcal{H}. \tag{2.3}$$

The sequence $\{g_k\}_{k=1}^{\infty}$ is also an E−*Riesz basis, and the series (2.3) converges unconditionally for all $f\in\mathcal{H}$.*

Definition 2.6. A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in $\mathcal{H}$ is an E−frame for $\mathcal{H}$ if there exist positive real numbers A and B such that

$$A\|f\|^2\leq\left\|\left\{\left\langle f,\left(E\{f_j\}_{j=1}^{\infty}\right)_k\right\rangle\right\}_{k=1}^{\infty}\right\|_{\ell^2}^2\leq B\|f\|^2,\qquad\forall f\in\mathcal{H}.$$

More precisely, the sequence $\{f_k\}_{k=1}^{\infty}$ is an E−frame for $\mathcal{H}$ if its E−transform is a frame for $\mathcal{H}$.

It is clear that every E$-$frame is E$-$complete. Also, since an E$-$frame $\{f_k\}_{k=1}^{\infty}$ is an E$-$Bessel sequence, the operator

$$\mathrm{T}:\ \ell^2 \to \mathcal{H},\ \ \mathrm{T}\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k \Big(E\{f_j\}_{j=1}^{\infty}\Big)_k,$$

is bounded. T is called the pre E$-$frame operator. Its adjoint, the analysis operator, is given by

$$\mathrm{T}^*:\ \mathcal{H} \to \ell^2,\ \ \mathrm{T}^* f = \left\{\left\langle f, \Big(E\{f_j\}_{j=1}^{\infty}\Big)_k \right\rangle\right\}_{k=1}^{\infty}.$$

Composing T and T*, the E$-$frame operator

$$\mathrm{S} = \mathrm{T}\mathrm{T}^* : \mathcal{H} \to \mathcal{H},\ \ \mathrm{S}f = \sum_{k=1}^{\infty} \left\langle f, \Big(E\{f_j\}_{j=1}^{\infty}\Big)_k \right\rangle \Big(E\{f_j\}_{j=1}^{\infty}\Big)_k.$$

is obtained. Since $\{f_k\}_{k=1}^{\infty}$ is an E$-$Bessel sequence, the series defining S converges unconditionally for all $f \in \mathcal{H}$.

Definition 2.7. Let $\{f_k\}_{k=1}^{\infty}$ be an E$-$frame for $\mathcal{H}$. An E$-$frame $\{g_k\}_{k=1}^{\infty}$ is called a dual E$-$frame of $\{f_k\}_{k=1}^{\infty}$ for $\mathcal{H}$, if

$$f = \sum_{k=1}^{\infty} \left\langle f, \Big(E\{g_j\}_{j=1}^{\infty}\Big)_k \right\rangle \Big(E\{f_j\}_{j=1}^{\infty}\Big)_k, \quad \forall f \in \mathcal{H}.$$

Theorem 2.8. *Let $\{e_k\}_{k=1}^{\infty}$ be an arbitrary orthonormal basis for $\mathcal{H}$. The* E$-$*frames for $\mathcal{H}$ are precisely the families* $\big\{\mathrm{U}(E^{-1}\{e_i\}_{i=1}^{\infty})_k\big\}_{k=1}^{\infty}$, *where* U *is bounded surjective operator on $\mathcal{H}$.*

Theorem 2.9. *Let $\{f_k\}_{k=1}^{\infty}$ be an* E$-$*frame for $\mathcal{H}$. The dual* E$-$*frames of $\{f_k\}_{k=1}^{\infty}$ are precisely the families*

$$\{g_k\}_{k=1}^{\infty} = \left\{\mathrm{S}^{-1}f_k + h_k - \sum_{n=1}^{\infty} \left\langle \Big(E\{\mathrm{S}^{-1}f_j\}_{j=1}^{\infty}\Big)_k, \Big(E\{f_j\}_{j=1}^{\infty}\Big)_n \right\rangle h_n \right\}_{k=1}^{\infty},$$

where $\{h_k\}_{k=1}^{\infty}$ is an E$-$*Bessel sequence in $\mathcal{H}$.*

Corollary 2.10. *An* E$-$*Riesz basis $\{f_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ is an* E$-$*frame for $\mathcal{H}$ and the dual* E$-$*Riesz basis equals the canonical dual* E$-$*frame* $\{\mathrm{S}^{-1}f_k\}_{k=1}^{\infty}$.

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POSITIVE BLOCK MATRICES ON HILBERT AND KREIN C^*-MODULES

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Abstract. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert C^*-modules. In this paper we give some necessary and sufficient conditions for the positivity of a block matrix on the Hilbert C^*-module $\mathscr{H}_1 \oplus \mathscr{H}_2$. If $(\mathscr{H}_1, J_1)$ and $(\mathscr{H}_2, J_2)$ are two Krein C^*-modules, we study the $\tilde{J}$-positivity of 2×2 block matrix

$$\begin{pmatrix} A & X \\ X^\sharp & B \end{pmatrix}$$

on the Krein C^*-module $(\mathscr{H}_1 \oplus \mathscr{H}_2, \tilde{J} = J_1 \oplus J_2)$, where $X^\sharp = J_2 X^* J_1$ is the (J_2, J_1)-adjoint of the operator X.

1. Introduction

Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. Actually Hilbert C^*-modules can be considered as a 'quantization' of the Hilbert space theory; see e.g. [4].

Let $\mathcal{A}$ be a C^*-algebra. A complex linear space $\mathscr{H}$ is said to be an inner product $\mathcal{A}$-module if $\mathscr{H}$ is a right $\mathcal{A}$-module together with a C^*-valued map $(x, y) \mapsto \langle x, y\rangle : \mathscr{H} \times \mathscr{H} \to \mathcal{A}$ such that

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* Speaker.

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ $(x, y, z \in \mathscr{H}, \alpha, \beta \in \mathbb{C})$;
(ii) $\langle x, ya\rangle = \langle x, y\rangle a$ $(x, y \in \mathscr{H}, a \in \mathcal{A})$;
(iii) $\langle y, x\rangle = \langle x, y\rangle^*$ $(x, y \in \mathscr{H})$;
(iv) $\langle x, x\rangle \geq 0$ and if $\langle x, x\rangle = 0$, then $x = 0$ $(x \in \mathscr{H})$.
An inner product $\mathcal{A}$-module $\mathscr{H}$ which is complete with respect to the induced norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ $(x \in \mathscr{H})$ is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$.
Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert C^*-modules. We denote by $\mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ the Banach algebra of all adjointable operators $T : \mathscr{H}_1 \to \mathscr{H}_2$.
Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert C^*-module. Then $\mathcal{L}(\mathscr{H}) := \mathcal{L}(\mathscr{H}, \mathscr{H})$ is a C^*-algebra with the identity operator $I_{\mathscr{H}}$. An operator $T \in \mathcal{L}(\mathscr{H})$ is called selfadjoint if $T^* = T$ and is positive if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathscr{H}$. We denote by $T^{\frac{1}{2}}$ the unique positive square root of T. If T is a positive invertible operator we write $T > 0$. For selfadjoint operators T and S on $\mathscr{H}$, we say $T \leq S$ if $S - T \geq 0$. The operator $T \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ is called a contraction if $T^*T \leq I_{\mathscr{H}_1}$. We write $\mathscr{R}(T)$ and $\mathscr{N}(T)$ for the range and null space of the operator T, respectively.
Every operator $\mathbf{A} \in \mathcal{L}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ is uniquely determined by operators $A_{ij} \in \mathcal{L}(\mathscr{H}_j, \mathscr{H}_i)$ $(1 \leq i, j \leq 2)$ defined by $A_{ij} = \pi_i \mathbf{A} \tau_j$, where τ_j is the canonical embedding of $\mathscr{H}_j$ in $\mathscr{H}_1 \oplus \mathscr{H}_2$ and π_i is the natural projection from $\mathscr{H}_1 \oplus \mathscr{H}_2$ onto $\mathscr{H}_i$. Let us represent $\mathbf{A}$ by the block matrix

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{1.1}$$

Clearly the operator $\mathbf{A}$ is selfadjoint if and only if $\mathbf{A}$ is of the form $\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$, where A_{11} and A_{22} are selfadjoint operators on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. The diagonal block matrix $\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$ is denoted by $A_{11} \oplus A_{22}$.
Krein spaces as an indefinite generalization of Hilbert spaces were formally defined by Ginzburg; see e.g. [1]. The notion of a Krein C^*-modules is a natural generalization of a Krein space. In sequel we present the standard terminology and some basic results on Krein C^*-modules.
Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$. A nontrivial selfadjoint involution J on $\mathscr{H}$ produce an $\mathcal{A}$-valued indefinite inner product on $\mathscr{H}$. In this case, $(\mathscr{H}, J)$ is called a Krein C^*-module. Trivially a Krein space is a Krein C^*-module over the C^*-algebra of complex number.
Let $(\mathscr{H}_1, J_1)$ and $(\mathscr{H}_2, J_2)$ be Krein C^*-modules. The (J_1, J_2)-adjoint

operator of $A \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ is defined by

$$[Ax, y]_{J_2} = [x, A^\sharp y]_{J_1} \qquad (x \in \mathscr{H}_1\,,\, y \in \mathscr{H}_2),$$

Let $(\mathscr{H}, J)$ be a Krein C^*-module. An operator $A \in \mathcal{L}(\mathscr{H})$ is said to be J-selfadjoint if $A^\sharp = A$. For J-selfadjoint operators A and B, the J-order, denoted as $A \leq^J B$, is defined by

$$[Ax, x]_J \leq [Bx, x]_J \qquad (x \in \mathscr{H}).$$

It is easy to see that $A \leq^J B$ if and only if $JA \leq JB$. The J-selfadjoint operator $A \in \mathcal{L}(\mathscr{H})$ is said to be J-positive if $A \geq^J 0$.
Positivity of 2×2 block matrices of operators on Hilbert spaces have been studied by many authors; see e.g. [2, 3] and references therein. In this paper, we study the positivity of 2×2 block matrices of adjointable operators on Hilbert C^*-modules. We also investigate the positivity of 2×2 block matrices on Krein C^*-modules.

2. Main result

The following lemma characterize the relation between contractions and the positivity of a block matrix of operators on Hilbert C^*-modules.

Lemma 2.1. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert $\mathcal{A}$-modules. An operator $C \in \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1)$ is a contraction if and only if the block matrix $\begin{pmatrix} I_{\mathscr{H}_1} & C \\ C^* & I_{\mathscr{H}_2} \end{pmatrix}$ $\in \mathcal{L}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ is positive.*

A characterization of positive 2×2 block matrices can be obtained by using Lemma 2.1 as follows:

Theorem 2.2. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert C^*-modules and let $A \in \mathcal{L}(\mathscr{H}_1)$ and $B \in \mathcal{L}(\mathscr{H}_2)$ such that $\mathscr{R}(A)$ and $\mathscr{R}(B)$ be closed submodules of $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Then the block matrix $\begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \in \mathcal{L}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ is positive if and only if $A \geq 0$, $B \geq 0$ and there exists a contraction G such that $C = A^{\frac{1}{2}} G B^{\frac{1}{2}}$.*

Let $(\mathscr{H}_1, J_1)$ and $(\mathscr{H}_2, J_2)$ be Krein C^*-modules. It is easy to see that $\tilde{\mathbf{J}} = J_1 \oplus J_2$ is a selfadjoint involution on $\mathscr{H}_1 \oplus \mathscr{H}_2$. Let $\mathbf{A}$ be the block matrix introduced in (1.1). Then we have

$$\mathbf{A}^\sharp = \tilde{\mathbf{J}} \mathbf{A}^* \tilde{\mathbf{J}} = \begin{pmatrix} J_1 A_{11}^* J_1 & J_1 A_{21}^* J_2 \\ J_2 A_{12}^* J_1 & J_2 A_{22}^* J_2 \end{pmatrix}.$$

Therefore $\mathbf{A}$ is $\tilde{\mathbf{J}}$-selfadjoint if and only if $\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^\sharp & A_{22} \end{pmatrix}$ in which A_{11} is J_1-selfadjoint and A_{22} is J_2-selfadjoint.

Theorem 2.3. *Let $(\mathscr{H}_1, J_1)$ and $(\mathscr{H}_2, J_2)$ be Krein C^*modules. Suppose that A is J_1-selfadjoint and B is J_2-selfadjoint. If A is invertible, then the operator $\begin{pmatrix} A & X \\ X^\sharp & B \end{pmatrix}$ is $\tilde{\mathbf{J}}$-positive if and only if $A \geq^{J_1} 0$, $B \geq^{J_2} 0$ and $X^\sharp A^{-1} X \leq^{J_2} B$.*

Let $(\mathscr{H}, J)$ be a Krein C^*-module. An operator $X \in \mathcal{L}(\mathscr{H})$ is called a J-contraction if $X^\sharp X \leq^J I_{\mathscr{H}}$.

Remark 2.4. The $\tilde{\mathbf{J}}$-positivity of block matrix $\begin{pmatrix} I_{\mathscr{H}_1} & X \\ X^\sharp & I_{\mathscr{H}_2} \end{pmatrix} \in \mathcal{L}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ implies that $J_1 \geq 0$ and $J_2 \geq 0$ which is impossible. Therefore in contrast to operators on Hilbert C^*-modules Lemma 2.1 is not valid in the setting of Krein C^*-modules.

Example 2.5. Consider the Minkowski space $(\mathbb{C}^2, J_0)$ with $J_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $X = \begin{pmatrix} i & i \\ i & 2i \end{pmatrix}$. Then

$$X^\sharp = J_0 X^* J_0 = \begin{pmatrix} -i & i \\ i & -2i \end{pmatrix}.$$

Therefore $X^\sharp X \leq^{J_0} I$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It means that X is a J_0-contraction. Now let $\tilde{\mathbf{J_0}} = J_0 \oplus J_0$ and $\mathbf{T} = \begin{pmatrix} I & X \\ X^\sharp & I \end{pmatrix}$. Then

$$\tilde{\mathbf{J_0}}\mathbf{T} = \begin{pmatrix} 1 & 0 & i & i \\ 0 & -1 & -i & -2i \\ -i & i & 1 & 0 \\ -i & 2i & 0 & -1 \end{pmatrix}.$$

The matrix $\tilde{\mathbf{J_0}}\mathbf{T}$ is not positive. It follows that $\mathbf{T}$ is not $\tilde{\mathbf{J_0}}$-positive, while X is a J_0-contraction.

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JORDAN GENERALIZED DERIVATIONS ON TRIVIAL EXTENSION ALGEBRAS

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Abstract. We investigate those conditions under which a Jordan generalized derivation on a trivial extension algebra is a generalized derivation. Some applications to triangular algebras are also presented.

1. Introduction

Let A be a unital algebra over a unital commutative ring and let X be a unital A-module. A linear map $f : A \to X$ is called a generalized derivation if there exists a linear map $d : A \to X$ such that

$$f(ab) = f(a)b + ad(b) \qquad (a, b \in A).$$

We will call a linear map $f : A \to X$ a Jordan generalized derivation if there exists a linear map $d : A \to X$ such that

$$f(a \circ b) = f(a) \circ b + a \circ d(b) \qquad (a, b \in A);$$

where $\circ$ stands for the Jordan product $a \circ x = x \circ a = ax + xa$ of the elements $a \in A, x \in X$.

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[*] Speaker.

It is obvious that every generalized derivation is a Jordan generalized derivation. It should be also remarked that every (Jordan) derivation f is a (Jordan) generalized derivation whose associated linear map id f itself. Jordan derivations on a wide variety of algebras are studied by many authors; see for example [1, 2, 3, 4, 6] and references therein.

We define the triangular algebra

$$\mathrm{Tri}(A, X, B) = \left\{ \begin{pmatrix} a & x \\ & b \end{pmatrix} : \ a \in A, x \in X, b \in B \right\};$$

where, A and B are unital algebras and X is a unital $(A, B)-$module, which is faithful as a left $A-$module as well as a right $B-$module.

Jordan generalized derivations on triangular algebras have already studied by Li and Benkovič [5]. They showed that:

Theorem 1.1 ([5, Theorem 2.5])**.** *Let f be a Jordan generalized derivation on triangular algebra* $\mathrm{Tri}(A, X, B)$ *with an associated linear map d. Then f is a generalized derivation of the form $f(x) = f(1)x + d(x)$ for all $x \in A$.*

Let A be an algebra and let X be an $A-$module. Then the direct product $A \times X$ under its usual pairwise addition, scalar product and the multiplication

$$(a, x)(b, y) = (ab, ay + xb), \qquad (a, b \in A, x, y \in X),$$

is an algebra called the *trivial extension* of A by X and denoted by $A \ltimes X$. The class of trivial extensions contains a wide variety of algebras includes a triangular algebra $\mathrm{Tri}(A, X, B)$. Every triangular algebra $\mathrm{Tri}(A, X, B)$ can be identified with the trivial extension algebra $(A \ltimes B) \ltimes X$. In this talk we present the structure of Jordan generalized derivations on a trivial extension algebra $A \times X$.

2. The main results

In this section we consider Jordan generalized derivations on a the trivial extension algebra $A \ltimes X$. It is obvious that every linear mapping $f : A \ltimes X \to A \ltimes X$ is in the form

$$f(a, x) = (f_1(a) + h_1(x), f_2(a) + h_2(x)) \quad (a \in A, x \in X);$$

for some linear maps $f_1 : A \to A, f_2 : A \to X, h_1 : X \to A$ and $h_2 : X \to X$. A direct verification also reveals that f is a generalized lie derivation with an associated linear map d (with the presentation

$$d(a, x) = (J_A(a) + T(x), J_X(a) + S(x)) \quad (a \in A, x \in X))$$

if and only if the components of f and d satisfy the following assertions; $(a, b \in A, x, y \in X)$.

(a) $f_1(a \circ b) = f_1(a) \circ b + a \circ J_A(b)$.
(b) $f_2(a \circ b) = f_2(a) \circ b + a \circ J_X(b)$.
(c) $h_1(x) \circ y + x \circ T(y) = 0$.
(d) $h_1(a \circ x) = a \circ T(x)$.
(e) $h_2(a \circ x) = f_1(a) o x + a \circ S(x) = h_2(x) \circ a + x \circ J_A(a)$.

This characterization leads us to the next result.

Theorem 2.1. *Let A be a unital algebra (over a 2-torsion free commutative unital ring) and let X be a faithful A-module. If $\pi_X(Z(A \ltimes X)) = X$ then a Jordan generalized derivation $f : A \ltimes X \to A \ltimes X$ with an associated linear map d is a generalized derivation if and only if $J_X([a,b]) = 0$ and $T(x) \in Z(A)$ for all $a, b \in A$ and $x \in X$.*

We also present some results on certain trivial extension extending the main result of Li and Benkovič [5].

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A NEW METHOD TO SOLVE ILL-CONDITIONED LINEAR SYSTEMS

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Abstract. Finding accurate solution of ill-conditioned systems is very difficult.There are some methods to solve this kind of systems. The pin-pointing method propose an alghorithm that changes the system in two stages.This method uses the truncated singular value decomposition of the initial coefficient matrix at the first stage and the Gaussian elimination procedure for reduced system of linear equations at the second stage. Using Gaussian elimination procedure make a considerable error in the solution. Tikhonove method is another way to solve ill-conditioned systems. In this paper we proposed a new idea to overcome this kind of problems and some numerical results are given to show the efficiency of the proposed method.

1. Introduction

Consider an ill-conditioned linear system

$$Ax = b \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ and $x, b \in \mathbb{R}^n$. The system (1.1) is ill-conditioned if $cond(A) =\| A \|\| A^{-1} \|$ is dagerousely high. Some methods have been proposed to overcome this kind of problems. One approach is to use

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* Speaker.

the scaling strategy. Another method is to consider the problem as a least squares (LS) problem . The best known method is using singular value decomposition (SVD) to solve it. Let

$$A = U\Sigma V^T = \sum_{i=1}^{n} \delta_i u_i v_i^T \tag{1.2}$$

where the columns of V and U are the right and left singular vectors, respectively and $\Sigma = diag(\delta_1, \delta_2, ..., \delta_n)$ where $\delta_1 \geq \delta_2 \geq, ..., \geq \delta_n \geq 0$ are singular values of A. We suppose that all of singular values are greater than zero and the solution of Equation (1.1) takes the form

$$x = A^{+}b = \sum_{i=1}^{n} \frac{u_i^T b}{\delta_i} v_i \tag{1.3}$$

The SVD solution (1.3) may be inaccurate because of the appearance of small singular values for badly conditioned matrices.The most famous method is TSVD to solve this kind of problems that omit the small singular values. Given $\epsilon > 0$ is a small number, let $\epsilon > \delta_{r+1} \geq \delta_{r+2} \geq, ..., \geq \delta_n > 0$ be the dangerous small singular values of A. By neglecting these singular values and let them be zero, the SVD changes to TSVD and the solution (1.3) takes the form

$$x = \sum_{i=1}^{r} \frac{u_i^T b}{\delta_i} v_i \tag{1.4}$$

The TSVD is widely used for regularization of ill-conditioned linear systems.

Volokh and Vilnay in [1] proposed a method for pin-pointing this solution. Let $U_1 = [u_1, u_2, ..., u_r]$ and $V_1 = [v_1, v_2, ..., v_r]$ be the left and right singular matrices of TSVD and $U_2 = [u_{r+1}, ..., u_n] \equiv U_1^{\perp}$ and $V_2 = [v_{r+1}, ..., v_n] \equiv V_1^{\perp}$ be their orthogonal complements.Equation (1.1) is written as follows:

$$U^T AVV^T x = U^T b \tag{1.5}$$

by separating U_1 , U_2 , V_1 , V_2 and using $U_1^T AV_1 = diag(\delta_1, ..., \delta_r)$ equation (1.5) takes the form

$$\tilde{A}\, z = \begin{pmatrix} diag(\delta_1, ..., \delta_r) & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{1.6}$$

where $\tilde{A} = U^T AV$,$C = U_2^T AV_2$, $b_1 = U_1 b$, $b_2 = U_2 b$,$z = V^T x$. The system (1.6) is equal to two systems

$$diag(\delta_1, ..., \delta_r) z_1 = b1 \tag{1.7}$$

$$Cz_2 = b_2 \tag{1.8}$$

that $x = Vz = V_1 z_1 + V_2 z_2$. The first system of equations is diagonal and easily solve. It can be seen that the singular values of C are $\delta_{r+1} \geq \delta_{r+2} \geq, ..., \geq \delta_n$ and the system $Cz_2 = b_2$ is better than $AX = b$ because

$$cond(A) = \| A \|_2 \| A^{-1} \|_2 = \frac{\delta_1}{\delta_n}$$

$$cond(C) = \| C \|_2 \| C^{-1} \|_2 = \frac{\delta_{r+1}}{\delta_n}$$

Volokh solved $Cz_2 = b_2$ by Gaussian elimination procedure. Although $cond(C) \ll cond(A)$ but the system (1.8) remain ill-conditioned. It seems that its better we change the system to more stages. Unfortunatly in practical examples its not true because the bad effects of very small singular values is remaind.

The Tikhonove regularization method [4] proposed replacing the least square problem by the penalized least squares problem

$$min \| Ax - b \|^2 + \mu^2 \| x \|^2 \tag{1.9}$$

Choosing $\mu > 0$ guarantees that $A^T A + \mu^2 I$ is invertible. The explicit solution to (1.9) is given by

$$x_\mu = (A^T A + \mu^2 I)^{-1} A^T b \tag{1.10}$$

the solution vector x_μ can be expressed as

$$x_\mu = V(\Sigma^T \Sigma + \mu^2 I)^{-1} \Sigma^T U^T b = \sum_{i=1}^{n} u_i^T b \frac{\delta_i}{\delta_i^2 + \mu^2} v_i \tag{1.11}$$

that is referred to as smoothing. The most profoundly affected vectors in V are the ones with highest index .This is because of the relative size of the small singular values to the regularization constant μ . When δ_i is much larger than μ, adding μ^2 to δ_i^2 will not have a very profound effect on the magnitude of the sum. On the other hand, very small singular values will be greatly affected by the addition.

2. PROPOSED REGULARIZATION METHOD

Our proposed regularization method will use properties of both Volokh and Vilnay method and Tikhonov regularization. In [1] Volukh and Vilnay suggested solving $Cz_2 = b_2$ by Gaussian elimination that is not an accurate method for ill-conditioned systems(see table1). We proposed solving system (1.8) by Tikhonove method that implies a good solution for system(1.8) (see table1). Consider $C = U' diag(\delta_{r+1}, \cdots, \delta_n) V'^T$

that all of singular values are dangerousely small ,the solusion of system $Cz_2 = b_2$ by (1.8) is approximated by

$$z_{2_\mu} = \sum_{i=r+1}^{n} u_i'^T b \frac{1}{\mu^2 + \sigma_i^2} v_i' \tag{2.1}$$

where $\sigma_r < \mu \leq \sigma_{r+1}$ we have

$$x = V1z_1 + V2z_{2_\mu}$$

Example In this example we have tested our proposed method by badly ill-posed matrix $A = Hilbert(14)$ that $cond(A) = 2.37 \times 10^{17}$. where $b = A * (1, 1, ..., 1)^T$ .As we can see in Table1 , the results of proposed method is very better than other methods.

Table 1. Numerical results for example 1

exact	proposed method	Volokh-Vilnay	multistage Volokh-Vilnay	Tikhonove
1	1.00000000000	1.0000001	1.00000005	1.0000000002
1	0.99999999715	0.9999850	0.99999976	0.999999984
1	1.00000004146	1.0005064	0.99995743	1.00000024
1	0.99999975336	0.9927372	1.00054006	0.9999984
1	1.00000069767	1.0537086	0.99752948	1.00000424
1	0.99999910063	0.7808433	1.00513709	0.9999954
1	1.00000025583	1.4583887	0.99495015	0.99999874
1	1.00000038287	0.8472320	1.00323021	1.00000406
1	0.99999995767	-0.692613	0.99694855	1.0000025
1	0.99999979201	5.6870251	0.99817069	0.9999972
1	0.99999985983	-5.264741	1.00979225	0.9999956
1	1.00000003714	5.766978	0.99334689	1.00000053
1	1.00000026126	0.979787	0.99864940	1.00000590
1	0.99999986302	1.3497372	1.00174824	0.99999695

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TILING SETS AND AFFINE SYSTEM WITH COMPOSITE DILATIONS

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Abstract. Consider an AB-affine system $\mathcal{A}_{AB}(\Psi)$, in which $A, B \subseteq GL_n(\mathbb{R})$ and $\Psi \in L^2(\mathbb{R}^n)$. By choosing A, B and Ψ appropriately, it can be made $\mathcal{A}_{AB}(\Psi)$ an orthonormal basisor parseval frame for $L^2(\mathbb{R}^n)$. In this paper, we show that, there exist a relationship between an orthonormal basis and tiling set and also, between a parsval frame and packing set. Moreover, we construct an orthonormal AB-multiwavelet that arises from AB-multiresolution analysis.

1. Introduction

The construction and the study of orthonormal bases and parsval frames is of major importance in several areas of mathematics and applications, recently. The motivation for this study comes partly from

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signal processing, where such bases are useful in image compression and feature extraction.
The subspace $\mathcal{L}$ in $\mathbb{R}^n$ is a lattice if $\mathcal{L} = A\mathbb{Z}^n$, where $A \in GL_n(\mathbb{R})$. Given a measurable set $\Omega \subseteq \mathbb{R}^n$ and a lattice $\mathcal{L}$ in $\mathbb{R}^n$, it to be said Ω *tiles* $\mathbb{R}^n$ by $\mathcal{L}$ translation, or Ω is a *fundamental domain* of $\mathcal{L}$ if the following properties hold :

(i) $\cup_{l\in\mathcal{L}}(\Omega + l) = \mathbb{R}^n \quad a.e.$,
(ii) $\mu((\Omega + l) \cap (\Omega + l')) = 0$ for any $l \neq l' \in \mathcal{L}$.

It is called Ω *packs* $\mathbb{R}^n$ by $\mathcal{L}$ translation if only (ii) holds. In general, Blanchard in [1] considers the deffinition of tiling sets, for an arbitrary group G. Let G be a group acting from right on a measurable set $S \subseteq \mathbb{R}^n$. Then Ω is a *G-tiling set* for S, if

(i) $\cup_{g\in G}\Omega g = S \;\; a.e.$
(ii) $\mu(\Omega g_1 \cap \Omega g_2) = 0$ for $g_1 \neq g_2 \in G$.

Let A and B be a countable subset of $GL_n(\mathbb{R})$. A collection of the form

$$\mathcal{A}_{AB}(\Psi) = \{D_a D_b T_k \Psi : k \in \mathbb{Z}^n, a \in A, b \in B\},$$

is called **Affine systems with composite dilation**, or AB-**Affine system**, where $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$, and the operators T_k and D are called the translations and dilations, respectively. If $\mathcal{A}_{AB}(\Psi)$ is an orthonormal basis (ON) or, more generally, a parsval frame (PF) for $L^2(\mathbb{R}^n)$, then Ψ is called an ON *AB-multiwavelet* or a *PF AB-multiwavelet*, respectively.

Associated with the Affine system with composite dilation, is the following generalization of the classical Multiresolution Analysis, that will be useful to construct more examples of AB multiwavelets, as well as examples with properties that are of great potentional in applications.
Let $B = \{b^j : j \in \mathbb{Z}\}$ in which $b \in GL_2(\mathbb{Z})$, with $|detb^j| = 1$, and $A \in GL_2(\mathbb{Z})$. A sequence $\{V_i\}_{i\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^2)$ is called an AB-Multiresolution Analysis (AB-MRA) if the following holds :

(i) $D_{b^j} T_k V_o = V_0$, for any $j \in \mathbb{Z}, k \in \mathbb{Z}^2$,
(ii) $V_i \subset V_{i+1}$, for each $i \in \mathbb{Z}$, where $V_i = D_a^{-i} V_o$,
(iii) $\cap_{i\in\mathbb{Z}} V_i = \{0\}$ and $\overline{\cup_{i\in\mathbb{Z}} V_i} = L^2(\mathbb{R}^2)$,
(iv) there exists $\phi \in L^2(\mathbb{R}^2)$ such that $\Phi_B = \{D_{b^j} T_k \phi : j \in \mathbb{Z}, k \in \mathbb{Z}^2\}$ is a semi- orthogonal Parsval frame for V_0 ; that is, Φ_B is a parsval frame for V_o and in addition, $D_{b^j} T_k \phi \perp D_{b^{j'}} T_{k'} \phi$ for any $j \neq j', j, j' \in \mathbb{Z}, \; k \neq k', k, k' \in \mathbb{Z}^2$.

The space V_0 is called an AB scaling space and the function ϕ is an AB scalling function for V_0 (see [3]).

2. Main results

We show that, there exists a relationship between an orthonormal basis and a fundamental domain. Also, there exists a relationship between a parsval frame and packing set. Therefore, we have the following:

Proposition 2.1. *Let $\Omega \subseteq \widehat{\mathbb{R}}^n$, be a measurable set and $\widehat{\psi} = \chi_\Omega$, in $L^2(\Omega)$. Then, the collection $\{(T_k\psi)^\wedge = e^{2\pi i \xi k}\chi_\Omega : k \in \mathbb{Z}^n\}$ is an orthonormal basis for $L^2(\Omega)$ if and only if Ω is a fundamental domain.*

Proposition 2.2. *Let $\Omega \subseteq \widehat{\mathbb{R}}^n$, be a measurable set and $\widehat{\psi} = \chi_\Omega$, in $L^2(\Omega)$. Then, the collection $\{(T_k\psi)^\wedge = e^{2\pi i \xi k}\chi_\Omega : k \in \mathbb{Z}^n\}$ is a parsval frame for $L^2(\Omega)$ if and only if Ω is a packing set by translation of $\mathbb{Z}^n$, for $\widehat{\mathbb{R}}^n$. i.e. $\mu((\Omega + k) \cap (\Omega + k')) = 0$ for $k \neq k' \in \mathbb{Z}^n$.*

In the sequal we costruct an orthonormal AB-multiwavelet that arises from AB-multiresolution analysis.

Example 2.3. Let $a = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and $b = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$. Suppose

$$S_0 = \{\xi = (\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2 : |\xi_2 - \xi_1| \leq 1\},$$

and define

$$V_0 = L^2(S_0)^\vee = \{f \in L^2(\mathbb{R}^2) : supp\hat{f} \subset S_0\}.$$

Consider

$$S_i = S_0 a^i = \{\xi = (\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2 : \ |\xi_2 - \xi_1| \leq 2^i\},$$

and

$$V_i = \{f \in L^2(\mathbb{R}^2) : \ supp\hat{f} \subset S_i\}.$$

It is easy to check that the space $\{V_i\}_{i\in\mathbb{Z}}$ satisfy the following properties :
(1)$D_{b^j}T_kV_0 = V_0$; (2)$V_i \subset V_{i+1}, i \in \mathbb{Z}$; (3)$D_a^{-i}V_o = V_i$; (4)$\cap_{i\in\mathbb{Z}} V_i = \{0\}$; (5)$\overline{\cup_{i\in\mathbb{Z}} V_i} = L^2(\mathbb{R}^2)$.
Let $A = \{a^i : i \in \mathbb{Z}\}$, $B = \{b^j : j \in \mathbb{Z}\}$, and $U = U_1 \cup U_2$, where U_1 is a triangle with vertices at $(0, 0), (-1, 0), (0, 1)$, and $U_2 = \{\xi \in \widehat{\mathbb{R}}^2 : -\xi \in U_1\}$. Define φ by $\widehat{\varphi}(\xi) = \chi_U(\xi)$. Thus $\{V_i\}_{i\in\mathbb{Z}}$ is an AB-MRA with scaling function φ.
Assume that W_0 be an orthogonal complement V_0 in V_1, that is, $V_1 = V_0 \oplus W_0$. We define $W_0 = L^2(S_1 \backslash S_0)^\vee$, and set : $R_0 := S_1 \backslash S_0$. Consider the following subset of R_0:

$$E_1 = E_1^+ \cup E_1^-, E_2 = E_2^+ \cup E_2^-, E_3 = E_3^+ \cup E_3^-,$$

where

$$E_1^+ = \{\xi = (\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2 : -2 \leq \xi_1 \leq -1, 0 \leq \xi_2 \leq \xi_1 + 2\},$$
$$E_2^+ = \{\xi = (\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2 : -1 \leq \xi_1 \leq 0, \xi_1 + 1 \leq \xi_2 \leq 1\},$$
$$E_3^+ = \{\xi = (\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2 : -1 \leq \xi_1 \leq 0, 1 \leq \xi_2 \leq \xi_1 + 2\},$$

and $E_l^- = \{\xi \in \widehat{\mathbb{R}}^2 : -\xi \in E_l^+\}, l = 1, 2, 3$.
We then define $\psi^l, l = 1, 2, 3$, by setting $\hat{\psi}^l = \chi_{E_l}, l = 1, 2, 3$. It's easy to see that $\cup_{l=1}^3 \cup_{j\in\mathbb{Z}} E_l b^{-j} = R_0$. The dilations operators D_a^i, for each $i \in \mathbb{Z}$, maps R_0 into R_i, in which $R_i = R_0 a^i$. Consider $L^2(R_i)^\vee = W_i$. Then $\{D_a^i D_b^j T_k \psi^l : k \in \mathbb{Z}^2, i, j \in \mathbb{Z}, l = 1, 2, 3\}$, is an orthonormal basis of $L^2(\mathbb{R}^2) = \oplus_{i\in\mathbb{Z}} W_i$, that is, $\Psi = \{\psi^1, \psi^2, \psi^3\}$ is an ON AB-multiwavelet.

The following result establishes the number of generators needed to obtain an orthonormal MRA AB-wavelet.

Theorem 2.4. [3] *Let $\Psi = \{\psi^1, \ldots, \psi^L\}$ be an orthonormal MRA AB-multiwavelet for $L^2(\mathbb{R}^n)$, and let $N = |B/aBa^{-1}|$ (= the order of quotient group B/aBa^{-1}). Assume that $|deta| \in \mathbb{N}$. Then $L = N|deta| - 1$.*

By using this theorem, we can calculate the number of AB-multiwavelet.

Remark 2.5. In example 2.3, the set B is considred as $B = \{b^j : j \in \mathbb{Z}\}$ in which, $b^j = \begin{bmatrix} j+1 & j \\ -j & -j+1 \end{bmatrix}$. By a simple calculation, we get $ab^j a^{-1} = b^j$, thus, $aBa^{-1} =< b >$ and it is clearly $B =< b >$. Then $B/aBa^{-1} \simeq I_{2\times 2}$. So $N = |B/aBa^{-1}| = 1$. Therefore, $L = N|deta| - 1 = 1.4 - 1 = 3$.

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EXTENDED EIGENVALUE OF DUHAMEL MULTIPLICATION OPERATORS

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Abstract. In this paper under certain conditions, we calculate extended eigenvalues and extended eigenvectors of some weighted shift operator on $\ell^p(\beta)$. Also we show that $\ell^p(\beta)$ with generalized Duhamel product $\circledast$ is a Banach algebra.

1. Introduction

Let $\{\beta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with $\beta(0) = 1$ and $1 \leq p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}$ such that

$$\|f\|_\beta^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

We shall use the formal notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ whether or not the series converges for any complex values of z. These are called formal power series. Sources on formal power series include [4, 5]. Throughout this paper, we consider the space $\ell^p(\beta)$ to be defined by

$$\ell^p(\beta) = \{f : f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n, \ \|f\|_\beta < \infty\}.$$

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Let $\{\delta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with $\delta_0 = 1$. Given arbitrary two functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ and $g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n$ of the space $\ell^p(\beta)$, define the following generalized Duhamel product series (see [2])

$$f \circledast g = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\delta_{m+n}}{\delta_n \delta_m} \hat{f}(n)\hat{g}(m) z^{m+n}. \tag{1.1}$$

Some results of this article is a generalization of the work done in [2, 3]. In this paper under certain conditions, we calculate extended eigenvalues and extended eigenvectors of some weighted shift operator on $\ell^p(\beta)$. Also we show that $\ell^p(\beta)$ with generalized Duhamel product $\circledast$ is a Banach algebra.

Let X be a separable Banach space. We denote by $B(X)$ the set of bounded linear operators on X. An operator $T \in B(X)$ is called a well splitting operator in X if for every $x \in X$ there exists a bounded linear operator B_x such that $T^n x = B_x y_n$ for every $n \in \mathbb{N} \cup \{0\}$ and for some complete system $\{y_n\}_{n\geq 0}$ of the space X. A complex number λ is called an extended eigenvalue of $T \in B(X)$ if there exists non-zero operator $A \in B(X)$ such that $\lambda AT = TA$ (see [1]) and references therein.

2. Main results

Let $1 < p < \infty$ and let q be the conjugate exponent to p. For each $n, k, M \in \mathbb{N} \cup \{0\}$, define

$$C_o := \sup_{n\geq 0} \sum_{k=0}^{n} \left(\frac{\delta_n \beta(n)}{\delta_k \delta_{n-k} \beta(k) \beta(n-k)} \right)^q, \tag{2.1}$$

and

$$b_{M,k} := \sup_{n\geq M+1} \frac{\delta_{n+k}\beta(n+k)}{\delta_n \delta_k \beta(n)\beta(k)} \ . \tag{2.2}$$

Throughout this section we assume that $1 < p < \infty$, $C_o < \infty$ and $\lim b_{M,k} = 0$ when $M \to \infty$.

Lemma 2.1. *The weighted shift operator T defined on $\ell^p(\beta)$ as $T(f) = \sum_{n=0}^{\infty} \frac{\delta_{n+1}}{\delta_n} \hat{f}(n) z^{n+1}$ is bounded and*

$$\| T^N \| = \sup_{n\geq 0} \frac{\beta(n+N)\delta_{n+N}}{\beta(n)\delta_n}, \qquad N \in \mathbb{N} \cup \{0\}.$$

Let $\ell^0(\beta)$ be the set of all formal power series. For each $f \in \ell^p(\beta)$, let $D_f : \ell^p(\beta) \to \ell^0(\beta)$ defined by $D_f(g) = f \circledast g$ be its corresponding $\circledast$-multiplication linear operator. When $f(z) = z$, we take $D_z = M_{\circledast,z}$. It is easy to see that $M_{\circledast,z}(f) = \sum_{n=0}^{\infty} \frac{\delta_{n+1}}{\delta_n \delta_1} \hat{f}(n) z^{n+1}$ and $M_{\circledast,z}^N(f) = \frac{\delta_N}{\delta_1^N} z^N \circledast f$, for all $N \in \mathbb{N} \cup \{0\}$ and $f \in \ell^p(\beta)$. So $M_{\circledast,z}$ is bounded and

$$\|M_{\circledast,z}^N\| = \sup_{n \in \mathbb{N} \cup \{0\}} \frac{\beta(n+N)\delta_{n+N}}{\delta_1^N \delta_n \beta(n)}, \qquad N \in \mathbb{N} \cup \{0\}.$$

Theorem 2.2. *Let T and D_f be the above mentioned operators. Then we have the following statements.*

(a) For each $f \in \ell^p(\beta)$, D_f is bounded on $\ell^p(\beta)$; that is $D_f(\ell^p(\beta)) \subseteq \ell^p(\beta)$.

(b) The weighted shift operator T is a well splitting operator on $\ell^p(\beta)$.

(c) Let λ be a nonzero complex number with $|\lambda| \leq 1$, and let $A \in B(\ell^p(\beta))$ be a nonzero operator. Then $\lambda AT = TA$; i.e., λ is an extended eigenvalue of T, if and only if $A\Lambda_{\{\lambda\}} = D_{A(1)}$, where $\Lambda_{\{\lambda\}}(z^n) := \lambda^n z^n$.

Proof. (a) Let $f, g \in \ell^p(\beta)$. Using (1.1), it is easy to see that

$$\widehat{(f \circledast g)}(n) = \sum_{k=0}^{n} \frac{\delta_n}{\delta_k \delta_{n-k}} \hat{f}(k)\hat{g}(n-k).$$

Let q be the conjugate exponent to p. By using Hölder inequality and (2.1) we have

$$\begin{aligned}
\|D_f(g)\|_\beta^p &= \sum_{n=0}^{\infty} |\widehat{(f \circledast g)}(n)|^p \beta(n)^p \\
&\leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{\delta_n \beta(n)}{\delta_k \delta_{n-k} \beta(k)\beta(n-k)} |\hat{f}(k)||\hat{g}(n-k)|\beta(k)\beta(n-k) \right)^p \\
&\leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left(|\hat{f}(k)|\beta(k)|\hat{g}(n-k)|\beta(n-k) \right)^p \right)^{\frac{p}{p}} \\
&\quad \times \left(\sum_{k=0}^{n} \left(\frac{\delta_n \beta(n)}{\delta_k \delta_{n-k} \beta(k)\beta(n-k)} \right)^q \right)^{\frac{p}{q}} \\
&\leq C_o^{\frac{p}{q}} \sum_{n=0}^{\infty} \sum_{k=0}^{n} |\hat{f}(k)|^p \beta(k)^p |\hat{g}(n-k)|^p \beta(n-k)^p
\end{aligned}$$

$$= C_o^{\frac{p}{q}} \left(\sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p \right) \left(\sum_{n=0}^{\infty} |\hat{g}(n)|^p \beta(n)^p \right) = C_o^{\frac{p}{q}} \|f\|_\beta^p \ \|g\|_\beta^p.$$

Consequently, we get that

$$\|D_f(g)\|_\beta = \|f \circledast g\|_\beta \leq C_o{}^{\frac{1}{q}} \|f\|_\beta \|g\|_\beta,$$

and so $\|D_f\| \leq C_o{}^{\frac{1}{q}} \|f\|_\beta$.

(b) For $f \in \ell^p(\beta)$, since $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$, then we have

$$T^N(f) = f \circledast \delta_N z^N = D_f(\delta_N z^N).$$

Using (a), since D_f is bounded and defined on the whole of $\ell^p(\beta)$, so T is well splitting operator on $\ell^p(\beta)$.

(c) Let $\lambda AT = TA$. Then $\lambda^n AT^n = T^n A$, for all $n \in \mathbb{N} \cup \{0\}$. In particular, $A\lambda^n T^n(1) = T^n A(1)$. By using (2.1), we obtain that

$$T^n(1) = 1 \circledast \delta_n z^{^n} = \delta_n z^n \Rightarrow \lambda^n A(\delta_n z^n) = A(1) \circledast \delta_n z^n = \delta_n z^n \circledast A(1).$$

Thus $\delta_n A(\lambda^n z^n) = \delta_n(z^n \circledast A(1))$ and so $A\Lambda_{\{\lambda\}}(z^n) = z^n \circledast A(1)$. This implies that $A\Lambda_{\{\lambda\}}(P) = P \circledast A(1)$, for all polynomials P. Since polynomials are dense in $\ell^p(\beta)$, $A\Lambda_{\{\lambda\}}(f) = D_{A(1)}(f)$ for all $f \in \ell^p(\beta)$, which yields $A\Lambda_{\{\lambda\}} = D_{A(1)}$.

Conversely, suppose $A\Lambda_{\{\lambda\}} = D_{A(1)}$. Then we have

$$\begin{aligned} TA(z^n) &= TA\Lambda_{\{\lambda\}}\Lambda_{\{\frac{1}{\lambda}\}}(z^n) = TD_{A(1)}\Lambda_{\{\frac{1}{\lambda}\}}(z^n) \\ &= TD_{A(1)}(\frac{z^n}{\lambda^n}) = \delta_1 z \circledast D_{A(1)}(\frac{z^n}{\lambda^n}) = D_{\delta_1 z} D_{A(1)}(\frac{z^n}{\lambda^n}) \\ &= \lambda A\Lambda_{\{\lambda\}}(\lambda_n \frac{z^{n+1}}{\lambda^{n+1}}) = \lambda A\Lambda_{\{\lambda\}}\Lambda_{\{\frac{1}{\lambda}\}}(\lambda_n z^{n+1}) = \lambda A(T(z^n)). \end{aligned}$$

Thus for any polynomial P, $TA(P) = \lambda AT(P)$ and so $TA(f) = \lambda AT(f)$ for all $f \in \ell^p(\beta)$. $\square$

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DOUGLAS RANGE FACTORIZATION THEOREM FOR ADJOINTABLE OPERATORS ON HILBERT C^*-MODULES

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Abstract. We discuss relationships among the concepts majorization, range inclusion and factorization for adjointable operators acting on a Hilbert C^*-module. Indeed, we extend Douglas range factorization theorem for bounded operators and closed densely defined operators on a Hilbert space to the context of bounded adjointable operators and regular operators acting on a Hilbert C^*-module. We also give some applications of this extension.

1. Introduction

Douglas in [1] introduced two conditions for bounded operators T and S on a Hilbert space H which are equivalent to the existence of a bounded operator C such that $S = TC$. Theorem 1 of [1] states that the following statements are equivalent:
(i) $ran(S) \subseteq ran(T)$.
(ii) $\|S^*(x)\| \leq \lambda^2 \|T^*(x)\|$ for some $\lambda \geq 0$ and all $x \in X$.
(iii) There exists a bounded operator C on H so that $S = TC$.
Furthermore, Douglas extended this result to the case of unbounded operators on Hilbert spaces in Theorem 2 of [1] as follows: suppose t

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* Speaker.

and s are closed densely defined operators on H then
(i) If $ss^* \leq tt^*$, then there is a contraction V so that $s \subseteq tV$.
(ii) If $ran(s) \subseteq ran(t)$, then there exists a densely defined operator V so that $s = tV$ and a number $M \geq 0$ so that $\|V(x)\|^2 \leq M(\|x\|^2 + \|s(x)\|^2)$ for all $x \in Dom(V)$. Moreover, if s is bounded, then V is bounded and if t is bounded, then V is closed.
We will refer to these results as Douglas range factorization theorem. More recently, Douglas range factorization theorem have been extended to the context of bounded adjointable operators and regular operators on Hilbert C^*-modules in [5] and [2], respectively.
We recall that a Hilbert C^*-module E over a C^*-algebra A is a right A-module equipped with an A-valued inner product $< .,. >$ and such that E is complete with respect to the norm $\|x\| := \| < x, x > \|^{\frac{1}{2}}$. Although Hilbert C^*-modules are very similar to Hilbert spaces, with C^*-algebra elements playing the role of scalars, lack of analogue of projection theorem in Hilbert C^*-modules causes many difficulties to obtain properties of them parallel to Hilbert spaces. In particular, one need to employ the notion of Moore-Penrose inverse to generalize Douglas range factorization Theorem to the context of operators on Hilbert C^*-modules because of this obstruction.
In this talk, we first recall our results on Douglas range factorization theorem for adjointable operators on Hilbert C^*-module. Then we employ these results to investigate the problem concerning the existence of solutions for the equation $t = s_1X + s_2Y$ whenever t, s_1 and s_2 are regular operators on a Hilbert C^*-module E. For this, we first consider this problem for bounded adjointable operators on a Hilbert C^*-module and then we use the concept of bounded transform of a regualr operators to study the case of regular operators.
We conclude this section with recalling some basic definitions concerning adjointable operators on Hilbert C^*-modules.
We denote the set of all A-linear maps $T : E \to F$ for which there is a map $T^* : F \to E$ such that $< T(x), y >=< x, T^*(y) >$ for all $x \in E$ and $y \in F$ by $B(E, F)$. The existence of adjoint operator for T implies that T is bounded. So we call T a bounded adjointable operator. As for Hilbert spaces, one need to study unbounded adjointable operators, which are known as regular operators. We call a densely defined closed operator $t : D(t) \subseteq E \to E$ on a Hilbert C^*-module E regular if its adjoint is dense and $I + t^*t$ has dense range.
Recall that a regular operator $t^\dagger$ acting on a Hilbert C^*-module E is called the Moore-Penrose inverse of regular operator t on E if $tt^\dagger t = t$,

$t^\dagger t t^\dagger = t^\dagger$, $(tt^\dagger)^* = \overline{tt^\dagger}$ and $(t^\dagger t)^* = \overline{t^\dagger t}$.

2. Main results

We begin this section with the following theorem which can be considered as an improvement of Theorem 1 of [5] by replacing the assumption of $ran(S)$ being closed with Moore-Penrose inevitability of S.

Theorem 2.1. *Let T and S be bounded adjointable operators on Hilbert C^*-module E. Suppose S has the Moore-Penrose inverse then following statements are equivalent:*
(i) $ran(T) \subseteq ran(S)$;
(ii) $TT^* \leq \lambda SS^*$ for some $\lambda \geq 0$;
(iii) $T = SQ$ for some bounded adjointable operator Q on E.

The above theorem leads to the following propositions.

Proposition 2.2. *Let T be a bounded adjointable operator on a Hilbert C^*-module E admitting the Moore-Penrose inverse. Then $ran(T) = ran((TT^*)^{\frac{1}{2}})$.*

Proposition 2.3. *Let T and S be positive bounded adjointable operators with closed range. Then $T + S$ has closed range if and only if $ran(T) + ran(S)$ is closed.*

In the next two theorems, we extend Douglas range factorization to the context of regular operators on a Hilbert C^*-module.

Theorem 2.4. *Let t and s be regular operators on E. If s has a bounded adjointable generalized inverse and $tt^* \leq \lambda ss^*$ for some $\lambda \geq 0$, then there exists a bounded adjointable Q on E such that $Qs^* \subseteq t^*$.*

Theorem 2.5. *Suppose t and s are regular operators, and s has a generalized inverse. If $ran(t) \subseteq ran(s)$, then there exists a densely defined operator r of E such that $t = sr$. Moreover, if s is bounded adjointable, then r is closed densely defined and its graph is orthogonally complemented in $E \oplus E$, and if t is bounded adjointable, then r is bounded adjointable.*

We remark here that our generalization of Douglas range factorization theorem to the context of regular operators on a Hilbert C^*-module motivated us to extend this well-known theorem of Douglas to the setting of unbounded operators on a Banach space in [3].
The rest of this section is devoted to study the problem concerning the

existence of solution for the equation $t = s_1X + s_2Y$ whenever t, s_1 and s_2 are regular operators on Hilbert C^*-module E (see, [4]). Lets start with the case in which all operators are bounded adjointable.

Theorem 2.6. *Let T, S_1 and S_2 be bounded adjointable operators on a Hilbert C^*-module E. Suppose $ran(S_1) + ran(S_2)$ is closed then following statements are equivalent:*
(i) $ran(T) \subseteq ran(S_1) + ran(S_2)$;
(ii) $TT^* \leq \lambda(S_1S_1^* + S_2S_2^*)$ for some $\lambda \geq 0$;
(iii) $T = S_1X + S_2Y$ for some bounded adjointable operators X and Y on E.

We remark that Theorem (2.1) enable us to replace the condition of $ran(S_1) + ran(S_2)$ being closed in Theorem (2.6) by orthogonal complementability of $\overline{ran(S_1) + ran(S_2)}$ in E.

Theorem 2.7. *Let t, s_1 and s_2 be regular operators on a Hilbert C^*-module E with $ran(s_1) + ran(s_2)$ being closed. Suppose that $ran(t) \subseteq ran(s_1) + ran(s_2)$ then there exists densely defined operators r_1 and r_2 on E such that $t = s_1r_1 + s_2r_2$.*

Theorem 2.8. *Let T be a bounded adjointable operator on Hilbert C^*-module E and s_1, s_2 be regular operators on E with $ran(s_1) + ran(s_2)$ being closed. Suppose that $ran(T) \subseteq ran(s_1) + ran(s_2)$ then there exist bounded adjointable operators R_1 and R_2 on E such that $t = s_1R_1 + s_2R_2$.*

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ADDITIVE MAPS OF SOME OPERATOR ALGEBRAS BEHAVING LIKE DERIVATIONS AT IDEMPOTENT-PRODUCT ELEMENTS

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Abstract. Let $\mathcal{N}$ be a nest on a Banach space $\mathcal{X}$ and, suppose that $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{X})$ contains all rank one operators in $Alg\mathcal{N}$ and identity operator I, which is a Banach algebra with respect to some norm. Let there exists a non-trivial idempotent $P \in \mathcal{A}$ with $P(\mathcal{X}) \in \mathcal{N}$. We show that if $d : \mathcal{A} \to \mathcal{B}(\mathcal{X})$ is an additive mapping derivable at P (i.e. $d(AB) = Ad(B) + d(A)B$ for any $A, B \in \mathcal{A}$ with $AB = P$), then d is a derivation. As applications of the above result, we characterize the additive mappings derivable at P on Banach space nest algebras and standard operator algebras.

1. Introduction

Throughout this article, all linear spaces and algebras are over the complex field $\mathbb{C}$. Let $\mathcal{A}$ be an algebra, $\mathcal{M}$ be an $\mathcal{A}$-bimodule and $d : \mathcal{A} \to \mathcal{M}$ be an additive mapping. d is said to be a *derivation* if $d(AB) = Ad(B) + d(A)B$ for all $A, B \in \mathcal{A}$. Each linear mapping of the form $A \mapsto AM - MA$, where $M \in \mathcal{M}$, is a derivation which will be called an *inner derivation.* We say that d is *derivable* at a given point $Z \in \mathcal{A}$ if $d(AB) = Ad(B) + d(A)B$ for any $A, B \in \mathcal{A}$ with $AB = Z$. As well known, derivations are very important mappings both in theory and applications, and have been studied intensively (see [3] and the

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references therein). Clearly, a derivation from $\mathcal{A}$ into $\mathcal{M}$ is derivable at any $Z \in \mathcal{A}$. The converse is, in general, not true.

There have been a number of papers concerning the characterize of derivable mappings of operator algebras. In [1, 2, 4] the authors study the derivable mappings at zero for some algebras. In [5] Zhu and Xiong show that every strongly operator topology continuous linear derivable mapping at any orthogonal projection operator P_N $(0 \neq N \in \mathcal{N})$ from a nest algebra $Alg\mathcal{N}$ into itself is a derivation, where $\mathcal{N}$ is a continuous nest on a complex and separeble Hilbert space $\mathcal{H}$. Also, derivable mappings at a non-trivial idempotent are studied in [2, 3]. In this note we study the derivable mappings at a non-trivial idempotent on special operator algebras in Banach spaces. Indeed, if $\mathcal{A}$ is a subalgebra of $Alg\mathcal{N}$, contains the identity operator I and all rank one operators in $Alg\mathcal{N}$, such that $\mathcal{A}$ is a Banach algebra with respect to some norm and there exists a non-trivial idempotent $P \in \mathcal{A}$ with $P(\mathcal{X}) \in \mathcal{N}$ and $\delta : \mathcal{A} \to \mathcal{B}(\mathcal{X})$ is an additive mapping derivable at P or $I - P$, we show that δ is a derivation. Also we give several applications of this result for Banach space nest algebras and standard operator algebras.

Let us embark some notations. For a Banach space $\mathcal{X}$ with topological dual $\mathcal{X}^*$, denote by $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear operators on $\mathcal{X}$. Note that the rank one operator $x \otimes f$ on $\mathcal{X}$ is defined by $(x \otimes f)(z) = f(z)x$ for $z \in \mathcal{X}$ and $f \in \mathcal{X}^*$. A *nest* $\mathcal{N}$ on $\mathcal{X}$ is a chain of closed (under norm topology) subspaces of $\mathcal{X}$ which is closed under the formation of arbitrary intersection and closed linear span, and which includes $\{0\}$ and $\mathcal{X}$. The *nest algebra* associated to the nest $\mathcal{N}$, denote by $Alg\mathcal{N}$, is the weak closed operator algebra of the form

$$Alg\mathcal{N} := \big\{T \in \mathcal{B}(\mathcal{X}) \mid T(N) \subseteq N \ for\ all\ N \in \mathcal{N}\big\}.$$

When $\mathcal{N} \neq \{\{0\}, \mathcal{X}\}$, we say that $\mathcal{N}$ is non-trivial. It is clear that if $\mathcal{N}$ is trivial, then $Alg\mathcal{N} = \mathcal{B}(\mathcal{X})$.

For a Banach space $\mathcal{X}$ denote by $\mathcal{F}(\mathcal{X})$ the algebra of all finite rank operators in $\mathcal{B}(\mathcal{X})$. Recall that an *standard operator algebra* $\mathcal{U}$ on $\mathcal{X}$ is a subalgebra of $\mathcal{B}(\mathcal{X})$ containing $\mathcal{F}(\mathcal{X})$ which is a Banach algebra with respect to some norm.

2. Main results

From this point up to the last section $\mathcal{X}$ (resp. $\mathcal{H}$) is a Banach (resp. Hilbert) space, $\mathcal{N}$ is a nest on $\mathcal{X}$ (resp. $\mathcal{H}$) and $Alg\mathcal{N}$ is the nest algebra associated to the nest $\mathcal{N}$. Also $\mathcal{A}$ denote a subalgebra of $\mathcal{B}(\mathcal{X})$ (resp. $\mathcal{B}(\mathcal{H})$) containing all rank one operators in $Alg\mathcal{N}$ and identity operator I, which is a Banach algebra with respect to some norm.

The following is our main result.

Theorem 2.1. *Let there exists a non-trivial idempotent $P \in \mathcal{A}$ with $P(\mathcal{X}) \in \mathcal{N}$. If $d : \mathcal{A} \to \mathcal{B}(\mathcal{X})$ is an additive mapping derivable at P or $I - P$, then d is a derivation.*

Note that if $\mathcal{A} = Alg\mathcal{N}$ in above proposition, then any additive mapping $d : Alg\mathcal{N} \to \mathcal{B}(\mathcal{X})$ derivable at P or $I - P$ is a derivation. Indeed, each additive mapping $d : Alg\mathcal{N} \to Alg\mathcal{N}$ derivable at P or $I - P$ is a derivation.

Every additive derivation from $Alg\mathcal{N}$ into $\mathcal{B}(\mathcal{X})$ is inner. So by Theorem 2.1 we obtain the following corollary.

Corollary 2.2. *Suppose that there exists a non-trivial idempotent $P \in Alg\mathcal{N}$ with $P(\mathcal{X}) \in \mathcal{N}$. If $d : Alg\mathcal{N} \to \mathcal{B}(\mathcal{X})$ is an additive mapping derivable at P or $I - P$, then d is inner derivation.*

If $\mathcal{X}$ is an infinite dimensional Banach space, then every additive derivation of $Alg\mathcal{N}$ is linear. Also every linear derivation of a nest algebra on a Banach space is continuous and all continuous linear derivations of a nest algebra on a Banach space are inner derivations. So we have the following corollary.

Corollary 2.3. *Suppose that there is a non-trivial idempotent $P \in Alg\mathcal{N}$ with $P(\mathcal{X}) \in \mathcal{N}$. If $d : Alg\mathcal{N} \to Alg\mathcal{N}$ is an additive mapping derivable at P or $I - P$, then:*

(i) *if d is linear, then d is an inner derivation.*
(ii) *if $\mathcal{X}$ is an infinite dimensional Banach space, then d is an inner derivation.*

Let there exists a non-trivial idempotent $P \in \mathcal{A}$ with $P(\mathcal{H}) \in \mathcal{N}$. If $\delta : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a linear mapping, then δ is an inner derivation. Hence by Theorem 2.1, we have the following corollary.

Corollary 2.4. *Let there exists a non-trivial idempotent $P \in \mathcal{A}$ with range $P(\mathcal{H}) \in \mathcal{N}$. If $d : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a linear mapping derivable at P or $I - P$, then d is an inner derivation.*

Any linear derivation from a nest algebra into an ideal $\mathcal{I}$ of $\mathcal{B}(\mathcal{H})$ is an inner derivation. So we conclude the next corollary.

Corollary 2.5. *Suppose that $\mathcal{I}$ is an ideal of $\mathcal{B}(\mathcal{H})$. Let there exists a non-trivial idempotent $P \in Alg\mathcal{N}$ with range $P(\mathcal{H}) \in \mathcal{N}$. Then each linear mapping $d : Alg\mathcal{N} \to \mathcal{I}$ derivable at P or $I - P$ is an inner derivation.*

From Theorem 2.1 we have the next theorem.

Theorem 2.6. *Let $\mathcal{U}$ be an standard operator algebra on $\mathcal{X}$ containing the identity operator I and P be a non-trivial idempotent operator in $\mathcal{U}$. Let $d : \mathcal{U} \to \mathcal{B}(\mathcal{X})$ be an additive mapping which is derivable at P. Then d is a derivation.*

For proof of Theorem 2.1 and Theorem 2.6 we refer the reader to [3].

By above theorem for any non-trivial idempotent $P \in \mathcal{B}(\mathcal{X})$, every additive mapping $d : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ derivable at P is a derivation.

Let $\mathcal{U}$ be an standard operator algebra on $\mathcal{X}$. Any linear derivation $d : \mathcal{U} \to \mathcal{B}(\mathcal{X})$ is an inner derivation. So we have the following.

Corollary 2.7. *Suppose that $\mathcal{U}$ is a standard operator algebra on $\mathcal{X}$ containing the identity operator I and P is a non-trivial idempotent operator in $\mathcal{U}$. Let $d : \mathcal{U} \to \mathcal{B}(\mathcal{X})$ be a linear mapping which is derivable at P. Then d is an inner derivation.*

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THE INVARIANT SUBSPACE PROBLEM FOR QUASINILPOTENT OPERATOR

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Abstract. In this paper we study the relation ship between the decomposition of operators on the basis of the decomposition of space into invariant subspaces. Furthermore, subspaces will be examined by considering the bounded and unbounded operators. At end, some equations of the non-trivial invariant subspaces will be shown.

1. Introduction

The matter of invariant subspaces have more than one hundred years antiquity; from when the issue of decomposition of matrixes into simpler matrixes has been propounded. since each bounded operator an be analogize with its matrix, Jordan analysis is viewed as a method of matrix simplification. This method works on the basis of decomposition of spaces into decreasing subspaces of that matrix which itself is a special invariant of a invariant subspaces and decomposes the operators

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* Speaker.

in the Banach space.
Definition1 : if T is a linear operator on X in Banach spaces and W is a subspace of X, then W under T is invariant when $TW \subseteq W$. In addition, if W under any continuous operator is invariant, W is called hyper invariant subspaces of T.
Definition2 : if $T : X \rightarrow X$ is a bounded linear operator, $X = X_1 \bigoplus X_2$, that X_1, X_2 are subspaces of X then $X_1(X_2)$ are called reducing subspace of T when X_1, X_2 are invariant under T.
It can be shown that the existence of invariant subspaces is a necessary condition for decomposition of it.In other word, The existence of reducing subspace of a linear operator will lead to the decomposition of that operator.[2, 3, 4]
we know that if each T operator has some closed subspace such as X_1 on the Banach space, then X_1 will be invariant under T if and only if $PTP = TP$ [2]. Also if $X = X_1 \bigoplus X_2$, and P is projection operator on X_1 then X_1 will be a reducing subspace of T if and only $TP = PT$.
The issue of invariant subspaces has been started with the following question. Does any continuous linear operator $T : X \rightarrow X$ on the Banach space, has non-trivial invariant subspaces?
Enflo in 1987 by presenting a example [3] of a continuous linear operator on separable Banach space (which has not non-trivial invariant subspace)indicated that the invariant subspace issue had negative answer in its general from,IN 1985 ,Read, defined on operator L_1 which had not non-trivial stable subspace [4] while the existence of any invariant subspace for a continuous linear operator on the separable Hilbert space has not been proved or rejected until now. In other words, this mater is considered as unsolved problem on mathematics . Although after that many examples of linear operators (which have not invariant subspace) has been presented, the received negative answers do not damage its generality. In fact these negative answers present some special forms of the issue in a more limited condition. For some special operators this problem have answers.
It has been shown that if $T : X \rightarrow X$ is not a one-to-one operator or has not density , T will have non-trivial invariant subspace like ker$T$ and Rang$T$.
The matter of subspace of operators can be examine from space dimension. for example, does any continuous and linear operator $T : X \rightarrow X$ has non-trivial stable subspace on the bounded space .
operator in the real numbers space ,the operator will have non-trivial invariant subspace because :
Example : if X is a separable infinite Banach space ,the pivot's produced space $x \neq 0, x \in X$ or $Y = span < x, Tx, Tx^2, ... >$is non-trivial

invariant subspace. so it is clear that Y is invariant.
Question: if $T : X \to X$is a continuous linear operator in the space with separable unbounded dimension. an we say that it has non-trivial stable subspace?
answer : if λ is a characteristic value of T operator, the produced subspace by λ it means $N_\lambda < x \in X : Tx = \lambda x >$ is a non-trivial invariant subspace T operator . for every $x \in N_\lambda$ we have $Tx \in N_\lambda$.
if $\sigma_p(T)$ is non-empty, T operator has non-trivial invariant subspace . Furthermore, if $\sigma_r(T)$ non-empty. T has non-trivial invariant subspace on X because if $0 \neq \lambda \in c$ then $T(\overline{(\lambda I - T)x}) \subseteq \overline{(\lambda I - T)x}$
if $\sigma(T)$ is disconnected and A is its connected component, then we will have separated and open sets of V and W.when $A \subseteq V$, $\sigma(T)A \subseteq W$ the following function is defined as
so $f^2 = f$ and f in neighborhood of $V \cup W$ from $\sigma(T)$ is analytic and

$$f(t) = \frac{1}{2\pi i} \int_p f(\lambda)(\lambda I - T)^{-1} d\lambda$$

Here P is a Jordan curve and closed piece of $V \bigcup W$. if $P_A = f(t)$ so $P_A^2 = P_A$ there fore P_A is a project operator and rang P_A , kerP_A are decreasing subspaces of T it can be said that they are non-trivial invariant subspace of T . if H is a separated unbounded Hilbert space and K is direct addition of H so we will have

$$K = \{(x_1, x_2, ...); x_i \in H, \sum_{i=1}^{\infty} \|x_i\|^2 < \infty\}$$

left Transitional T operator $T : K \to K$ is an universal operator [14] one of the equations of invariant subspaces is related to the geometry of Hilbert space .
It means if two decomposition are given to the direct sum of a separable unbounded Hilbert space ; $H = K + L = M + N$, then this question is raised : is there necessarily any equal non-trivial subset decomposition? words are there any subspaces of K_0 from K , L_0 from L,M_0 from M and N_0 from N,

$$\{0\} \neq K_0 + L_0 \neq H \ , \ K_0 + L_0 = M_0 + N_0$$

Nordgern, Radjari and Rosenthal [1,2,4], indicated that this issue is equal with the general issue of stable subspace.
There fore, each operator has a non-trivial invariant subspace, if and only if each pair of operator idempotent has a common non-trivial invariant subspaces. As the last example of this part the following issue is presented here.
Halmos in 1968, when studied this issues defined the meaning quasi

triangular operator.
Conway in 1977 claimed that the main issue of stable subspace is that the biqusitriangular operators have non-trivial stable subspace.
n this part, non-trivial,invariant subspaces of empty like Dower operator will be disccussed . In 1996, enflo and Ansari [3]. used vectors for solving the problem of stable subspace of empty like dower operators on Hilbert space. They proved the existance of vectors for linear operators on Hilbert space in their common article.
Anyhow, this method was not extended to the ideal Banach space so in 2003 Vladimir G.Troitsky , by considering equality of weak topology and weak-topology on the reflex Banach space , proved the existsne of such vectors for bounded linear operators. In addition, he presented a general and comptrehen sive definition of these vectors on the Banach space. According to this definition, the minimal vectors were used on the other spaces.
The VladimirG.Troitsky definition of minimal vector:
let X is a Banach space ,$T : X \to X$ is a linear operator, $\epsilon > 0$ as $\parallel x \parallel > \epsilon$ if we suppose that $n \in N$ is stable then $c = T^{-n}B(x, \epsilon)$. In this case C is a convex nonempty and closed set.
if $d = \inf\{\parallel z \parallel : z \in \underset{\smile}{C}\}$, then $d > 0$ therefore there is $\{z_n\}$ subsequence in G as $\parallel T^n z_n - x \parallel \leq \epsilon, \|z_n\| \to d$ so there is bounded sequence in X; so the following sequence $\{z_{n_k}\}$ is in X as $z_{n_k} \to z \in C$ and because X is a reflex space so weak topology and weak* topology are coinded on X, There fore $\{z_n\}$ sequence in C is convex to Z so $\parallel z_n \parallel \to \parallel z \parallel$
These two vector are called minimal vectors dependent on (T, ζ, x_0) and are shown.
corollary 5 : suppose X is a reflex space and $T \varepsilon L(X)$ which is dense in X also let $x_0 \varepsilon X, \epsilon > 0$, exists as $(T^n x_n)_{n \geq 1}$, $\|X_0\| > \epsilon$ is convergent in norm then T is a non- trivial stable cloud subspace.

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SMITH'S DETERMINANT OVER MULTIPLICATIVE FUNCTIONS

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ABSTRACT. For a given arithmetical function f, the square matrix $[f(i,j)]_{n\times n}$ given by $f(i,j) = f(\gcd(i,j))$ where $\gcd(i,j)$ denotes the greatest common divisor of the numbers i and j, is called the greatest common divisor matrix related by f (namely f-GCD matrix), and its determinant is known as Smith's determinant. In this paper, we evaluate Smith's determinant of f-GCD matrices for some multiplicative functions, including the generalized divisor function, Möbius function, and Euler function. Finally, we develop a method to approximate the value of $\det[f(i,j)]_{n\times n}$ for strongly multiplicative functions f.

1. INTRODUCTION

In 1876 Smith [5] studied the square matrix having the greatest common divisor $\gcd(i,j)$ of the integers i and j as its i,j-entry, and he proved that $\det[\gcd(i,j)]_{n\times n} = \varphi(1)\varphi(2)\cdots\varphi(n)$, where $\varphi(n)$ is the Euler function. His argument allows us to evaluate more general determinants.

For given arithmetical function f, the square matrix $[f(i,j)]_{n\times n}$ defined by $f(i,j) = f(\gcd(i,j))$ as its i,j-entry, is called the greatest common divisor matrix related by f (namely f-GCD matrix), and its

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determinant is known as Smith's determinant. Smith's argument implies that

$$\det[f(i,j)]_{n\times n} = \prod_{k=1}^{n} g(k),$$

where $f(n) = \sum_{d|n} g(d)$. By using the Möbius inversion formula, we get $g(n) = \sum_{d|n} \mu(d)f(n/d)$, where μ is the Möbius function. Since Smith's paper [5] this field has been studied extensively. For a classical and a recent account of the theory of GCD matrices we refer the reader to [2] and [4], respectively, and the references given there.

A remarkable class of the arithmetical functions are multiplicative function. We recall that f is multiplicative if it satisfies $f(mn) = f(m)f(n)$ for $\gcd(m,n) = 1$. If f is multiplicative, then $f(1) = 1$, and the above function g is multiplicative, too. Thus, for primes p and integers $\alpha \geqslant 1$ we have $g(k) = \prod_{p^\alpha \| k} g(p^\alpha)$ and $g(p^\alpha) = f(p^\alpha) - f(p^{\alpha-1})$. Hence, for any multiplicative function f we obtain

$$\det[f(i,j)]_{n\times n} = \prod_{k=2}^{n} \prod_{p^\alpha \| k} \left(f(p^\alpha) - f(p^{\alpha-1}) \right). \tag{1.1}$$

In this paper, we evaluate Smith's determinant of f-GCD matrices for some multiplicative functions, including the generalized divisor function, Möbius function, and Euler function. Finally, we develop a method to approximate the value of $\det[f(i,j)]_{n\times n}$ for strongly multiplicative functions f.

All concepts and topics used from analytic number theory in this paper could be find in any standard book in this filed. The book [1] covers all required matters.

2. Smith's determinant of some multiplicative functions

1. The generalized divisor function is defined by $\sigma_s(n) = \sum_{d|n} d^s$ for any $s \in \mathbb{C}$. It is multiplicative, and $\sigma_s(p^\alpha) = (p^{s(\alpha+1)} - 1)/(p^s - 1)$ for $s \neq 0$. Also, $\sigma_0(p^\alpha) = \alpha + 1$. Thus, by using (1.1), for $s \neq 0$ we obtain

$$\det[\sigma_s(i,j)]_{n\times n} = \prod_{k=2}^{n} \prod_{p^\alpha \| k} p^{s\alpha} = \prod_{k=2}^{n} k^s = (n!)^s.$$

Also, $\det[\sigma_0(i,j)]_{n\times n} = 1$. Thus, the later is valid for $s = 0$, too.

2. For $s \in \mathbb{C}$ we define $I_s(n) = n^s$. We write

$$\prod_{p^\alpha \| k} \left(I_s(p^\alpha) - I_s(p^{\alpha-1}) \right) = \prod_{p^\alpha \| k} p^{s\alpha} \left(1 - \frac{1}{p^s} \right) = J_s(k),$$

where J_s is the Jordan's generalization of the Euler function, and it is defined by $J_s(k) = k^s \prod_{p|k}(1 - p^{-s})$. Thus, for any $s \in \mathbb{C}$ we obtain

$$\det[I_s(i,j)]_{n\times n} = \prod_{k=2}^{n} J_s(k).$$

In [3] we proved that $\ln\det[I_1(i,j)]_{n\times n} = n\ln n + \beta n + \frac{1}{2}\ln n + O(\ln\ln n)$, where β is an absolute constant, and it is defined by the following summation over all prime numbers $\beta = -1 + \sum_p (\ln(1-1/p))/p$.

3. We have $\varphi(p^\alpha) - \varphi(p^{\alpha-1}) = p^\alpha(1-1/p^2)$. Hence

$$\prod_{p^\alpha\|k}\left(\varphi(p^\alpha) - \varphi(p^{\alpha-1})\right) = \prod_{p^\alpha\|k} p^\alpha\left(1-\frac{1}{p^2}\right) = k\prod_{p|k}\left(1-\frac{1}{p^2}\right) = \frac{J_2(k)}{k}.$$

Thus, we imply that

$$\det[\varphi(i,j)]_{n\times n} = \prod_{k=2}^{n}\frac{J_2(k)}{k} = \frac{\det[I_2(i,j)]_{n\times n}}{n!}.$$

4. The difference $\mu(p^\alpha) - \mu(p^{\alpha-1})$ is -2 if $\alpha = 1$, is 1 if $\alpha = 2$, and is 0 for $\alpha \geqslant 3$. Since $2^3|n$ for $n \geqslant 8$, thus

$$\det[\mu(i,j)]_{n\times n} = 0, \qquad \text{for } n \geqslant 8.$$

For $1 \leqslant n \leqslant 7$, a simple computation gives the following values.

n	1	2	3	4	5	6	7
$\det[\mu(i,j)]_{n\times n}$	1	-2	4	4	-8	-32	64

3. Smith's determinant of strongly multiplicative functions

The multiplicative function f is said to be strongly multiplicative, if $f(p^\alpha) = f(p)^\alpha$ for primes p and integers $\alpha \geqslant 1$. If f is strongly multiplicative, then $f(p^\alpha) - f(p^{\alpha-1}) = f(p^\alpha)(1 - 1/f(p))$, and (1.1) becomes

$$\det[f(i,j)]_{n\times n} = n!\prod_{k=2}^{n}\prod_{p|k}\left(1-\frac{1}{f(p)}\right) = n!\prod_{p\leqslant n}\left(1-\frac{1}{f(p)}\right)^{\left[\frac{n}{p}\right]}.$$

Let us denote the last product by $D(n)$. Assuming that $f(p) \notin [0,1]$ for all primes p, we write

$$\ln D(n) = \sum_{p\leqslant n}\left[\frac{n}{p}\right]\ln\left(1-\frac{1}{f(p)}\right) = n\sum_{p\leqslant n}h_2(p) + O\Big(\sum_{p\leqslant n}h_1(p)\Big),$$

where

$$h_1(p) = \ln\left(1 - \frac{1}{f(p)}\right), \qquad \text{and} \qquad h_2(p) = \frac{h_1(p)}{p}.$$

Hence, estimating $D(n)$ and consequently $\det[f(i,j)]_{n\times n}$ for strongly multiplicative functions f reduces to estimating sums over primes. To evaluate $S_h(n) := \sum_{p\leqslant n} h(p)$ for $h \in C^1(1,\infty)$, we use the Stieltjes integral to write $S_h(n) = \int_{2^-}^{n} h(t)\mathrm{d}\pi(t)$, where $\pi(t)$ counts the numbers of primes not exceeding t. Then, integration by parts gives

$$S_h(n) = \pi(n)h(n) - \int_2^n \pi(t)h'(t)\mathrm{d}t.$$

Therefore, we obtain

$$\det[f(i,j)]_{n\times n} = n!\mathrm{e}^{nS_{h_2}(n)+O(S_{h_1}(n))}.$$

Some very good approximations for the prime counting function are known in literature, and one may use them to obtain good approximations for $\det[f(i,j)]_{n\times n}$, when f is strongly multiplicative.

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ON M.V-BANACH ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be a Banach algebra. An element a of $\mathcal{A}$ has the mean value property if we have

$$\sum_{n=1}^{\infty}\frac{(\beta a)^n}{n!}-\sum_{n=1}^{\infty}\frac{(\alpha a)^n}{n!}=(\beta-\alpha)\sum_{n=1}^{\infty}\frac{c_{\alpha,\beta}^{n-1}a^n}{(n-1)!}$$

A Banach algebra $\mathcal{A}$ is called *M.V* or *M.V- Banach algebra* if every element of $\mathcal{A}$ has the mean value property. In this study the following result is proved:
Let $\mathcal{A}$ be a unital domain and $\delta : D(\delta) \subseteq \mathcal{A} \to \mathcal{A}$ be a closed derivation. Furthermore, assume that a is an element of $D(\delta)$ with the mean value property. If $e^a\delta(a) = \delta(a)e^a$ then $\delta(a) = 0$.

1. Introduction and preliminaries

Derivations are defined by the familiar Leibnitz formula. As operators they may be bounded or unbounded. Now, we are going to recall the definition of a derivation. Let $\mathcal{A}$ be a Banach algebra, δ a linear mapping in $\mathcal{A}$. Then δ is said to be a derivation in $\mathcal{A}$ if it satisfies the following conditions:
1) The domain $D(\delta)$ of δ is a dense subalgebra in $\mathcal{A}$;
2) $\delta(ab) = \delta(a)b + a\delta(b)$ $(a, b \in D(\delta))$.
If $D(\delta) = \mathcal{A}$, then δ is said to be a derivation on $\mathcal{A}$. If $\mathcal{A}$ contains a unit element $\mathbf{1}$, we will always assume $\mathbf{1} \in D(\delta)$ and then $\mathbf{1}^2 = \mathbf{1}$.

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Clearly, $\delta(\mathbf{1}) = 0$. In 1955, Singer and Wermer proved the classical theorem: Every continuous derivation on a commutative Banach algebra maps into Jacobson radical. They conjectured that the assumption of continuity is unnecessary and Thomas proved this conjecture. Indeed, he proved that every derivation on a commutative Banach algebra maps into its Jacobson radical. Clearly, if the algebra $\mathcal{A}$ is also semi-simple, then every derivation on $\mathcal{A}$ is identically zero. Next, some attempts have been also made to eliminate the commutativity assumption of Singer-Wermer's Theorem. As a generalization of a derivation, a linear mapping $d : \mathcal{A} \to \mathcal{A}$ is called a σ-derivation if $d(ab) = d(a)\sigma(b)+\sigma(a)d(b)$ for all $a, b \in \mathcal{A}$, where $\sigma : \mathcal{A} \to \mathcal{A}$ is a linear mapping. Hosseini et al ([2], [3]) have presented some results about the range of a σ-derivation and derivation. The following assertions are of the most famous conjectures about derivations on Banach algebras:
(1) Every derivation on a Banach algebra has a nilpotent separating ideal.
(2) Every derivation on a semi-prime Banach algebra is continuous.
(3) Every derivation on a prime Banach algebra is continuous.
(4) Every derivation on a Banach algebra leaves each primitive ideal invariant.
Obviously, if (1) is true, then the same for (2) and (3). Mathieu and Runde in [4] proved that (1), (2) and (3) are equivalent. The conjecture (4) is known as the non commutative Singer-Wermer conjecture, and it has been proved in [1] that if each of the conjectures (1), (2), or (3) holds, then (4) is also true. Moreover Runde [5] proved that the following four statements are equivalent:
(R1) Every derivation on a commutative Banach algebra has a nilpotent separating space.
(R2) Every derivation on a semi-prime Banach algebra is continuous.
(R3) Every derivation on a Banach algebra which ia an integral domain is continuous.
(R4) Every derivation on a topologically simple, commutative Banach algebra other than $\mathbb{C}$ is continuous.
In this paper, we prove the following theorem:
Let $C^* - algebra$ $\mathcal{A}$ be a unital domain and let $\delta : D(\delta) \subseteq \mathcal{A} \to \mathcal{A}$ be a closed derivation. Suppose that $a \in D(\delta)$ is a self-adjoint element such that $a\delta(a) = \delta(a)a$ and $C^*(a) \subseteq D(\delta)$. Then there is a continuous function $h : Sp(a) \to (0, 1)$ such that $e^x - 1 = xe^{xh(x)}$ for all $x \in Sp(a)$. If $\delta(G^{-1}(h)) = 0$, where G is the Gelfand transform, then $\delta(a) = 0$. Moreover, we define the mean value property for an element a of a Banach algebra $\mathcal{A}$ as follows:

An element a of $\mathcal{A}$ has the mean value property if for every closed interval $[\alpha, \beta] \subseteq \mathbb{R}$ there exists an element $c_{\alpha,\beta} \in (\alpha, \beta)$ such that

$$\sum_{n=1}^{\infty} \frac{(\beta a)^n}{n!} - \sum_{n=1}^{\infty} \frac{(\alpha a)^n}{n!} = (\beta - \alpha) \sum_{n=1}^{\infty} \frac{c_{\alpha,\beta}^{n-1} a^n}{(n-1)!}$$

In the case $\mathcal{A}$ is unital, the above condition is translated into $e^{\beta a} - e^{\alpha a} = (\beta - \alpha) a e^{a c_{\alpha,\beta}}$. A Banach algebra $\mathcal{A}$ is called *M.V* or *M.V- Banach algebra* if every element of $\mathcal{A}$ has the mean value property. In this article, we show that every closed derivation on an M.V-Banach algebra which is an integral domain is identically zero. By using the above-mentioned theorem, we prove that if $\mathcal{A}$ is an M.V-Banach algebra which is an integral domain and $\delta : \mathcal{A} \to \mathcal{A}$ is a derivation, then the following statements are equivalent:
(H1) $\delta(e^a) = \sum_{n=0}^{\infty} \frac{\delta(a^n)}{n!}$ for all $a \in \mathcal{A}$;
(H2) δ is identically zero;
(H3) δ is continuous.

2. Main results

Throughout this paper, $\mathcal{A}$ denotes a complex Banach algebra. If the algebra $\mathcal{A}$ is unital then $\mathbf{1}$ will show its unit element. Let $\mathcal{G}$ be an open subset of $\mathbb{C}$. A map $f : \mathcal{G} \subseteq \mathbb{C} \to \mathcal{A}$ is said to be differentiable at point z_0 of $\mathcal{G}$ if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists. This limit is called the derivative of f at point z_0 and is denoted by $f^{'}(z_0)$. If a is an element of $\mathcal{A}$ then the *spectrum* of a is denoted by $Sp(a)$ and $Sp(a) = \{\lambda \in \mathbb{C} \mid \lambda \mathbf{1} - a \textit{ is not invertible}\}$.

Definition 2.1. An algebra $\mathcal{A}$ is a domain if $\mathcal{A} \neq \{0\}$ and either $a = 0$ or $b = 0$ whenever $ab = 0$ in $\mathcal{A}$. A commutative algebra which is a domain is called an integral domain.

Theorem 2.2. *Let $C^* -$ algebra $\mathcal{A}$ be a unital domain and $\delta : D(\delta) \subseteq \mathcal{A} \to \mathcal{A}$ be a closed derivation. Suppose that $a \in D(\delta)$ is a self-adjoint element of $\mathcal{A}$ such that $a\delta(a) = \delta(a)a$ and $C^*(a) \subseteq D(\delta)$. Then, there is a continuous function $h : Sp(a) \to (0, 1)$ such that $e^x - 1 = xe^{xh(x)}$ for all $x \in Sp(a)$. If $\delta(G^{-1}(h)) = 0$, where G is the Gelfand transform, then $\delta(a) = 0$.*

Definition 2.3. An element a of $\mathcal{A}$ has the mean value property if for every closed interval $[\alpha, \beta] \subseteq \mathbb{R}$ there exists an element $c_{\alpha,\beta} \in (\alpha, \beta)$ such that

$$\sum_{n=1}^{\infty} \frac{(\beta a)^n}{n!} - \sum_{n=1}^{\infty} \frac{(\alpha a)^n}{n!} = (\beta - \alpha) \sum_{n=1}^{\infty} \frac{c_{\alpha,\beta}^{n-1} a^n}{(n-1)!}$$

In the case $\mathcal{A}$ is unital, the above-mentioned condition is translated into $e^{\beta a} - e^{\alpha a} = (\beta - \alpha)ae^{ac_{\alpha,\beta}}$. An algebra $\mathcal{A}$ is called *M.V* or *M.V-algebra* if every element of $\mathcal{A}$ has the mean value property.

For example, let a be an idempotent element of a unital Banach algebra $\mathcal{A}$, has the mean value property.

Proposition 2.4. *Let $\mathcal{A}$ be a unital Banach algebra and a an element of $\mathcal{A}$. Suppose that the resolvent function of a has the following property: $f_a(\beta) - f_a(\alpha) = (\beta - \alpha)f_a^{'}(c)$ for some $c \in (\alpha, \beta)$. Then there exists a real number t_0 such that $a = t_0\boldsymbol{1}$*

Theorem 2.5. *Let $\mathcal{A}$ be a unital domain and $\delta : D(\delta) \subseteq \mathcal{A} \to \mathcal{A}$ be a closed derivation. Furthermore, assume that a is an element of $D(\delta)$ with the mean value property. If $e^a\delta(a) = \delta(a)e^a$ then $\delta(a) = 0$. In particular, if $\mathcal{A}$ is an M.V-Banach algebra and $e^a\delta(a) = \delta(a)e^a$ for all $a \in D(\delta)$ then δ is identically zero.*

Theorem 2.6. *Suppose that the M.V-Banach algebra $\mathcal{A}$ is a unital integral domain and let $\delta : \mathcal{A} \to \mathcal{A}$ be a derivation. Then the following assertions are equivalent:*
(1) $\delta(e^a) = \sum_{n=0}^{\infty} \frac{\delta(a^n)}{n!}$ for all $a \in \mathcal{A}$;
(2) δ is identically zero;
(3) δ is continuous.

Theorem 2.7. *Suppose that $\mathcal{A}$ is a unital domain and a is an element in $\mathcal{A}$ with the mean value property. If b is an element in $\mathcal{A}$ such that $(ab - ba)e^a = e^a(ab - ba)$ then $ab = ba$. In particular, if $\mathcal{A}$ is an M.V-Banach algebra and $(ab - ba)e^a = (ab - ba)e^a$ for all $a \in \mathcal{A}$, then $b \in Z(\mathcal{A})$, where $Z(\mathcal{A})$ is the center of $\mathcal{A}$.*

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ADDITIVE MAPS PRESERVING THE FIXED POINTS OF OPERATORS

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Abstract. Let $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on a complex Banach space $\mathcal{X}$ with $\dim \mathcal{X} \geq 3$ and let $F(A)$ be the space of all fixed points of an operator $A \in \mathcal{B}(\mathcal{X})$. We characterize the forms of additive maps $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ satisfying $F(A) = F(\phi(A))$.

1. Introduction

The study of maps on operator algebras preserving certain properties is a topic which attracts much attention of many authors. Some of these problems are concerned with preserving a certain property of products of operators (see [3] and [4]). Let $\mathcal{B}(\mathcal{X})$ denotes the algebra of all bounded linear operators on a complex Banach space $\mathcal{X}$. For $A \in \mathcal{B}(\mathcal{X})$, denote by Lat$A$ the lattice of A, that is, the set of all invariant subspaces of A. $\mathcal{X}^*$ denotes the dual space of $\mathcal{X}$. The main motivation for our paper is the following result from Jafarian and Sourour [1]. They proved that a linear map $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ satisfying $\mathrm{Lat}(A) = \mathrm{Lat}(\phi(A))$, for every $A \in \mathcal{B}(\mathcal{X})$ if and only if it is of the form $\phi(A) = \alpha A + \rho(A)I$ for every $A \in \mathcal{B}(\mathcal{X})$, where α is a complex number and $\rho \in \mathcal{X}^*$.

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* Speaker.

Recall that $x \in \mathcal{X}$ is a fixed point of an operator $A \in \mathcal{B}(\mathcal{X})$, whenever we have $Ax = x$. Denote by $F(A)$, the set of all fixed points of A. It is clear that $F(A) \in \mathrm{Lat}A$. We denote by $\dim F(T)$, the dimension of $F(T)$. Let $\dim \mathcal{X} \geq 3$. In [2], we characterized the forms of surjective maps $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ such that $\dim F(AB) = \dim F(\phi(A)\phi(B))$, for every $A, B \in \mathcal{B}(\mathcal{X})$.

In this lecture, we characterize the forms of additive maps $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ which satisfy $F(A) = F(\phi(A))$, for every $A \in \mathcal{B}(\mathcal{X})$.

2. Main results

In order to prove the main result of this section, first we prove some auxiliary lemmas.

Lemma 2.1. *Let $x \in \mathcal{X}$ and $A \in \mathcal{B}(\mathcal{X})$. If x and Ax are linear independent vectors, then there exists a rank one idempotent P such that $x \in F(A + P)$.*

Suppose $\phi : \mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X})$ is a surjective additive map which satisfies the following condition:

$$F(A) = F(\phi(A)) \qquad (A \in \mathcal{B}(\mathcal{X})).$$

Lemma 2.2. *$\phi(P) = P$ for every rank one idempotent operator P.*

Lemma 2.3. *For any rank one idempotent P, there exists an $\eta(P) \in \mathbb{C}$ such that $\phi(A) + P = \eta(P)(A + P)$, for every $A \in \mathcal{B}(\mathcal{X})$.*

Theorem 2.4. *Let $\mathcal{X}$ be a complex Banach space with $\dim \mathcal{X} \geq 3$. Suppose $\phi : \mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X})$ is a surjective additive map which satisfies the following condition:*

$$F(A) = F(\phi(A)) \qquad (A \in \mathcal{B}(\mathcal{X})).$$

Then $\phi(A) = A$, for every $A \in \mathcal{B}(\mathcal{X})$.

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P-MAJORIZATION (RP-MAJORIZATION) AND ITS LINEAR PRESERVERS ON $\mathbb{R}^2$ ($\mathbb{R}_2$)

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Abstract. A matrix is said to be doubly substochastic if it has nonnegative components and each row and each column sum is at most 1. A square nonnegative and symmetric matrix P with zero diagonal elements is called a predistance matrix. For $x, y \in \mathbb{R}^n$ ($x, y \in \mathbb{R}_n$), it is said that x p-majorized (rp-majorized) by y and denoted by $x \prec_p y$ ($x \prec_{rp} y$) if there exists a predistance doubly substochastic matrix P such that $x = Py$ ($x = yP$). In the present work, we characterize the structure of all (strong) linear preservers of $\prec_p$ ($\prec_{rp}$) on $\mathbb{R}^2$ ($\mathbb{R}_2$).

1. Introduction

Majorization is a topic of much interest in various areas of mathematics and statistics that has studied more in linear algebra and its applications. Although, this notion is rather old, it is also an active field in researches. Some new kinds of majorization have been introduced in [1, 2, 3].

Some of our notations and symbols are explained the following.
$\mathbf{M}_{m,n}$: the set of all $m \times n$ real matrices.
$\mathbb{R}^n$: the set of all $n \times 1$ real column vectors.

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* Speaker.

$\mathbb{R}_n$: the set of all $1 \times n$ real row vectors.
$\mathcal{D}_n^{ps}$: the collection of all $n \times n$ predistance doubly substochastic matrices.
$\{e_1, \ldots, e_n\}$: the standard basis of $\mathbb{R}^n$.
$\{\varepsilon_1, \ldots, \varepsilon_n\}$: the standard basis of $\mathbb{R}_n$.
$\mathbb{N}_k$: the set $\{1, \ldots, k\} \subset \mathbb{N}$.
A^t: the transpose of a given matrix A.
$[T]$: the matrix representation of a linear function $T : \mathbb{R}^n \to \mathbb{R}^n$ ($T : \mathbb{R}_n \to \mathbb{R}_n$) with respect to the standard basis.
Co(A): the set $\{\sum_{i=1}^m \lambda_i a_i \mid m \in \mathbb{N}, \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1, a_i \in A, i \in \mathbb{N}_m\}$ where $A \subseteq \mathbb{R}^n$ ($\mathbb{R}_n$).
Let $\sim$ be a relation on $\mathbf{M}_{m,n}$. A linear function $T : \mathbf{M}_{m,n} \longrightarrow \mathbf{M}_{m,n}$ is said to be a linear preserver (or strong linear preserver) of $\sim$ if $x \sim y$ implies that $T(x) \sim T(y)$ (or $x \sim y$ if and only if $T(x) \sim T(y)$).

1.1. Doubly substochastic and p-majorization(rp-majorization).

Here, we introduce p-majorization (rp-majorization) on $\mathbb{R}^n$ ($\mathbb{R}_n$) and we investigate some properties of this relation on $\mathbb{R}^2$ ($\mathbb{R}_2$).

Definition 1.1. A matrix is said to be doubly substochastic if it has nonnegative components and each row and each column sum is at most 1.

Definition 1.2. An $n \times n$ nonnegative and symmetric matrix P with zero diagonal elements is called a predistance matrix.

We consider the predistance doubly substochastic matrices and introduce the following new type of majorization.

Definition 1.3. For $x, y \in \mathbb{R}^n$ ($x, y \in \mathbb{R}_n$), we say x is p-majorized (rp-majorized) by y and we write $x \prec_p y$ ($x \prec_{rp} y$), if there exists $P \in \mathcal{D}_n^{ps}$ such that $x = Py$ ($x = yP$).

The following proposition provides a criterion for p-majorization (rp-majorization) on $\mathbb{R}^2$ ($\mathbb{R}_2$).

Proposition 1.4. *Let $x = (x_1, x_2)^t$, $y = (y_1, y_2)^t \in \mathbb{R}^2$ ($x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}_2$). Then $x \prec_p y$ ($x \prec_{rp} y$) if and only if $x_1 \in Co\{0, y_2\}$, $x_2 \in Co\{0, y_1\}$, and $x_1 y_1 = x_2 y_2$.*

Proof. If $x \prec_p y$, then there exists $P \in \mathcal{D}_2^{ps}$ such that $x = Py$. We observe that $P = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$, for some $0 \leq \alpha \leq 1$. So $x_1 \in Co\{0, y_2\}$, $x_2 \in Co\{0, y_1\}$, and $x_1 y_1 = x_2 y_2$.

Next, assume that $x_1 \in Co\{0, y_2\}$, $x_2 \in Co\{0, y_1\}$, and $x_1 y_1 = x_2 y_2$. Then there exist α, β ($0 \leq \alpha, \beta \leq 1$) such that $x_1 = \alpha y_2$ and $x_2 = \beta y_1$.

If $y_1y_2 \neq 0$, as $x_1y_1 = x_2y_2$, then $\alpha = \beta$. Put $P = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$. If $y_1 = 0$, then set $P = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$, and if $y_2 = 0$, then put $P = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}$. We see in each case $p \in \mathcal{D}_2^{ps}$ and $x = Py$. So $x \prec_p y$.
In a similar process, we can prove the statements for $\prec_{rp}$. ☐

Now, we bring some properties of $\prec_p$ on $\mathbb{R}^2$. Also, these statements hold about $\prec_{rp}$ on $\mathbb{R}_2$.

Proposition 1.5. *Let $x = (x_1, x_2)^t$, $y = (y_1, y_2)^t \in \mathbb{R}^2$. Then*

(a) $x \prec_p y \nRightarrow y \prec_p x$.
(b) $x \prec_p y$ *and* $y \prec_p x \nRightarrow x = y$.
(c) $x \prec_p y$ *and* $y \prec_p z \nRightarrow x \prec_p z$.

Proof. The proof is easy. ☐

2. Main results

In this section, the structure of all (strong) linear preservers of p-majorization (rp-majorization) on $\mathbb{R}^2$ ($\mathbb{R}_2$) will be characterized. The following theorem characterizes all the linear preservers of $\prec_p$ ($\prec_{rp}$) on $\mathbb{R}^2$ ($\mathbb{R}_2$).

Theorem 2.1. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ ($T : \mathbb{R}_2 \to \mathbb{R}_2$) be a linear function, and let $[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then T preserves $\prec_p$ ($\prec_{rp}$) if and only if $a = d$ and $b = c$. That is, $[T] = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$.*

Proof. First, assume that T preserves $\prec_p$. Since $e_1 \prec_p e_2$ and $e_2 \prec_p e_1$, so $Te_1 \prec_p Te_2$ and $Te_2 \prec_p Te_1$, and hence $a \in Co\{0, d\}$, $c \in Co\{0, b\}$, and $d \in Co\{0, a\}$, $b \in Co\{0, c\}$. Then $a = d$ and $b = c$.

Next, suppose that $a = d$ and $b = c$. Let $x = (x_1, x_2)^t$, $y = (y_1, y_2)^t \in \mathbb{R}^2$, and let $x \prec_p y$. Then $Tx = (ax_1 + bx_2, bx_1 + ax_2)^t$ and $Ty = (ay_1+by_2, by_1+ay_2)^t$. Since $x_1y_1 = x_2y_2$, we observe that $(Tx)_1(Ty)_1 = (Tx)_2(Ty)_2$. By the hypothesis, $x_1 = \alpha y_2$ and $x_2 = \beta y_1$ for some $0 \leq \alpha, \beta \leq 1$. If $y_1y_2 \neq 0$, then $\alpha = \beta$. So $(Tx)_1 \in Co\{0, (Ty)_2\}$ and $(Tx)_2 \in Co\{0, (Ty)_1\}$. Thus $Tx \prec_p Ty$. If $y_1y_2 = 0$, then clearly $Tx \prec_p Ty$.
As a similar fashion, the case $\prec_{rp}$ can proved. ☐

Lemma 2.2. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ ($T : \mathbb{R}_2 \to \mathbb{R}_2$) be a linear function. If T strongly preserves $\prec_p$ ($\prec_{rp}$), then T is invertible.*

Proof. Let $x \in \mathbb{R}^2$ ($\mathbb{R}_2$) and let $Tx = 0$. Since $Tx = T0$ and T strongly preserves $\prec_p$ ($\prec_{rp}$), it shows that $x \prec_p 0$ ($x \prec_{rp} 0$). So $x = 0$ and hence T is invertible. ☐

The following theorem finds the structure of strong linear preserves of p-majorization (rp-majorization) on $\mathbb{R}^2$ ($\mathbb{R}_2$).

Theorem 2.3. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ ($T : \mathbb{R}_2 \to \mathbb{R}_2$) be a linear function. Then T strongly preserves $\prec_p$ ($\prec_{rp}$) if and only if $[T] = \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$, where $a^2 - b^2 \neq 0$.*

Proof. First, suppose that T strongly preserves $\prec_p$. So T is invertible and T preserves $\prec_p$, and hence by Theorem 2.1, $[T] = \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$, where $a^2 - b^2 \neq 0$.

For the converse, it is enough to prove that $x \prec_p y$ whenever $T(x) \prec_p T(y)$. Let $x,y \in \mathbb{R}^2$ such that $T(x) \prec_p T(y)$. Thus there exists $P \in \mathcal{D}_2^{ps}$ such that $T(x) = PT(y)$. Hence $x = (T^{-1}PT)y$. Since $T^{-1}PT = P$, we have $x = Py$. Therefore, $x \prec_p y$.
One can prove the theorem for $\prec_{rp}$ in a similar fashion. $\square$

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The Extended Abstracts of Talks
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AN EXTENSION OF H^*-ALGEBRAS

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Abstract. In this talk we introduce an extension of H^*-algebras and show that they have a C^*-algebraic structure.

1. Introduction

An H^*-algebra E is a Banach algebra which satisfies the following properties:

i) The underlying Banach space of E is a Hilbert space.
ii) For each $x \in E$ there is an element of E, denoted by x^* and called the adjoint of x, such that $\langle xy, z\rangle = \langle y, x^*z\rangle$ and $\langle yx, z\rangle = \langle y, zx^*\rangle$ for all $y, z \in E$.

An H^*-algebra E is called proper if it satisfies the following two equivalent conditions

i) The only x in E such that $xE = 0$ is $x = 0$
ii) The only x in E such that $Ex = 0$ is $x = 0$

where $xE = \{xy : y \in E\}$ and $Ex = \{yx : y \in E\}$. In a proper H^*-algebra every element has a unique adjoint, so the map $* : E \longrightarrow E$ which maps x to its adjoint (i.e. x^*) is an involution on E. If E is a

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proper H^*-algebra and $D = \{\sum_{i=1}^{n} x_i y_i : n \in \mathbb{N}, x_i, y_i \in E\}$, then D is dense in E and for every $x \in E$ and $y \in D$ we have $\langle x, y\rangle = \langle y^*, x^*\rangle = \overline{\langle y, x\rangle}$. Since D is dense in E we conclude that $\langle x, y\rangle = \langle y^*, x^*\rangle = \overline{\langle y, x\rangle}$ for every $x, y \in E$. For more details about H^*-algebras see [1].

Hilbert C^*-modules are extensions of Hilbert spaces, where the scalar field $\mathbb{C}$ is replaced by a C^*-algebra.

Definition 1.1. Suppose A is a C^*-algebra and M is a right A-module. M is called pre-Hilbert A-module or pre-Hilbert C^*-module over C^*-algebra A, if there exists a sesquilinear form $\langle\ ,\ \rangle : M \times M \longrightarrow A$ with the following properties

i) $\langle x, x\rangle \geq 0$ (i.e. $\langle x, x\rangle$ is a positive element of C^*-algebra A);
ii) $\langle x, x\rangle = 0$ implies $x = 0$;
iii) $\langle y, x\rangle = \langle x, y\rangle^*$;
iv) $\langle x, ya\rangle = \langle x, y\rangle a$,
for each $x, y, \in M$ and $a \in A$.

The map $\langle\ ,\ \rangle$ is called A-valued inner product. Let M be a pre-Hilbert A-module and $x \in M$. Put $||x||_M = ||\langle x, x\rangle||^{\frac{1}{2}}$. The function $||.||_M$ is a norm on M. A pre-Hilbert A-module M is called a Hilbert A-module or a Hilbert C^*-module over C^*-algebra A, if it is complete with respect to the norm $||.||_M$. For details see [3].

It is natural to ask what happens if one replace the $\mathbb{C}$-valued inner product of an H^*-algebra, with an A-valued inner product, where A is a C^*-algebra. In fact if we do this, the underlying Banach space of E (which is a Hilbert space) will be a Hilbert A-module.

Definition 1.2. An extended H^*-algebra E over C^*-algebra A is a Banach algebra which satisfies the following properties:

i) The underlying Banach space of E is a Hilbert A-module.
ii) For each $x \in E$ there is an element of E, denoted by x^* and called the adjoint of x, such that $\langle xy, z\rangle = \langle y, x^*z\rangle$ and $\langle yx, z\rangle = \langle y, zx^*\rangle$ for all $y, z \in E$.
iii) $\langle xa, y\rangle = \langle x, ya^*\rangle$ for every $x, y \in E$ and $a \in A$.

A proper extended H^*-algebra over C^*-algebra A is an extended H^*-algebra over C^*-algebra A which satisfies the following two equivalent properties:

i) The only x in E such that $xE = 0$ is $x = 0$
ii) The only x in E such that $Ex = 0$ is $x = 0$

In a proper extended H^*-algebra, every element has a unique adjoint, so the map $* : E \longrightarrow E$ which maps x to its adjoint (i.e. x^*) is an involution on E. If E is a proper extended H^*-algebra over C^*-algebra

A and $D = \{\sum_{i=1}^{n} x_i y_i : n \in \mathbb{N}, x_i, y_i \in E\}$, then D is dense in E and $\langle x, y\rangle = \langle y^*, x^*\rangle = \langle y, x\rangle^*$ for every $x, y \in E$.

Definition 1.3. An element x in a proper extended H^*-algebra over C^*-algebra A is called self-adoint if $x^* = x$.

Remark 1.4. If E is a proper extended H^*-algebra over C^*-algebra A and $x, y \in E$ then if for every $z \in E$ we have $\langle x, z\rangle = \langle y, z\rangle$ or $\langle z, x\rangle = \langle z, y\rangle$ then $x = y$. Also if $y, z \in E$ and for every $x \in E$, $xy = xz$ then $y = z$

Lemma 1.5. *If E is a proper extended H^*-algebra over C^*-algebra A then for every $x, y \in E$ and $a \in A$ we have $(xy)a = x(ya) = (xa)y$.*

Proof. For every z in E we have

$$\langle (xy)a, z\rangle = a^*\langle xy, z\rangle = a^*\langle x, zy^*\rangle = \langle xa, zy^*\rangle = \langle (xa)y, z\rangle,$$

now by above remark $(xy)a = (xa)y$. On the other hand

$$\langle (xy)a, z\rangle = a^*\langle xy, z\rangle = a^*\langle y, x^*z\rangle = \langle ya, x^*z\rangle = \langle x(ya), z\rangle.$$

So $(xy)a = x(ya)$. □

Lemma 1.6. *If E is a proper extended H^*-algebra over C^*-algebra A then for every $x \in E$ and $a \in A$ we have $(xa)^* = x^*a^*$.*

Proof. We have

$$\begin{aligned}\langle y, z(xa)^*\rangle = \langle y(xa), z\rangle = \langle (yx)a, z\rangle = \langle yx, za^*\rangle &= \langle y, (za^*)x^*\rangle \\ &= \langle y, z(x^*a^*)\rangle.\end{aligned}$$

□

2. Main results

If M is a proper extended H^*-algebra over C^*-algebra A, then M has a self adjoint element e such that $||e|| = 1$, since if x is a non-zero element of M, then xx^* is self adjoint and if one put $e = \frac{xx^*}{||xx^*||}$ then e is self adjoint. Moreover for every such element the map $\varphi_e : M \longrightarrow A$ which maps x to $\langle e, x\rangle$ is a bounded A-linear map since

$$||\varphi_e(x)|| = ||\langle e, x\rangle|| \leq ||e||||x|| = ||x||,$$

so we have $||\varphi_e|| \leq 1$. A-linearity of φ_e is straightforward.

Lemma 2.1. *If M is a proper extended H^*-algebra over C^*-algebra A and e is a unital self adjoint element of M and one define $. : M \times M \longrightarrow M; (x, y) \mapsto x.y = x\varphi_e(y) = x\langle e, y\rangle$, then with the above multiplication M is a normed algebra.*

Lemma 2.2. *If M is a proper extended H^*-algebra over C^*-algebra A and $. : M \times M \longrightarrow M; (x, y) \mapsto x.y = x\langle e, y\rangle$, then $(x.y)^* = x^*.y^*$*

Theorem 2.3. *Suppose M is a proper extended H^*-algebra over C^*-algebra A, then M or a quotient of M has a C^*-algebraic structure.*

Remark 2.4. If M is a C^*-algebra, then it is an extended H^*-algebra.

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FUNCTIONAL INTERVAL MATRICES

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Abstract. Interval matrix is one of the new branches of applied mathematics that in recent years are studied many in papers. We extend this theory to matrices with entries belong special C*-algebras.

1. Introduction

By defining a relation on a C*-algebra we tried that extend an interval matrix to the interval matrix whose entries belong to an arbitrary C*-algebra. In general this relation is not a totally ordered set, but with regarding an special C*-algebra we obtain an extended interval matix , Such that we can extend the results of the interval matrix to this extended theory. T. Nimala and et.al in 2011, presented a paper on interval matrix [5] . In this paper for instance we extend some results of Nirmala's paper to this theory.

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2. Main results

Definition 2.1. Let A is an algebra, We say that $a \in A$ is *invertible* if there is an element b in A such that $ab = ba = 1$. In this case b is unique and written a^{-1} .

Definition 2.2. An **involution** on an algebra A is a conjugate-linear map $a \mapsto a^*$ on A, such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. The pair $(A, *)$ is called an *involutive* algebra, or a *-algebra. A Banach *-algebra is a *-algebra A together with a complete submultiplicative norm such that $\| a^* \|=\| a \|$ $(a \in A)$. If in addition, A has a unit 1 such that $\| 1 \|= 1$, we call A a **unital Banach *-algebra**.
A, C^*-algebra is a Banach *-algebra such that $\| a^*a \|=\| a \|^2$ $(a \in A)$.An element a of a C*-algebra A is called hermitian if $a = a^*$ and is called **positive** if is hermitian and $\sigma(a) \subseteq R^+$. We write $a \geq 0$ to mean that a is positive, and denote by A_{Sa} and A^+ respectively, the set of hermitian elements and positive elements of A, if A is a C^*-algebra, we make A_{Sa} a poset by defining $a \leq b$ to be mean $b - a \in A^+$.

Lemma 2.3. *The sum of two positive elements in a C^*-algebra is a positive element.*

Proof. See [4] □

Note that if Ω is any compact Hausdorff space and $C(\Omega)$ the algebra of all continuous real function on Ω with componentwise operation and conjulate as involution and supremum norm, then $f(\Omega)$ is positive if and only if $f(\omega) \geq 0$ for all $\omega \in \Omega$ and f is invertible if and only if f never vanishes on Ω and in this case we denote the inverse of such $f \in C(\Omega)$ by $f^{-1} = \frac{1}{f}$. from lemma 2.3, the relation $\leq$ on a C^*-algebra is transitive and obviously is reflexive and furtheremore since $A^+ \cap -A^+ = \{0\}$ this relation is antisymmetric and therefore is a partially ordered set .Note that if $f, g \in C(\Omega)$, then the function h_1, h_2 defined by

$$h_1(x) = min\{f(x), g(x)\} = \frac{f(x) + g(x) - |f(x) - g(x)|}{2}$$

and

$$h_2(x) = max\{f(x), g(x)\} = \frac{f(x) + g(x) + |f(x) - g(x)|}{2}$$

are in $C(\Omega)$.
In this paper, $M_{m\times n}$ denotes the set of all $m \times n$ matrices with entry belong $C(\Omega)^+$ and never vanishes on Ω ,in which $C(\Omega)^+$ is the set of all positive element in $C(\Omega)$.

Definition 2.4. Let $\underline{A}$, $\overline{A}$ are two matrices in $M_{m\times n}$ such that $\underline{A} \leq \overline{A}$, then the set of matrices $\mathbb{A} = \lceil \underline{A}, \overline{A} \rceil = \{A : \underline{A} \leq A \leq \overline{A}\}$, is called an extended interval matrix, and the matrices $\underline{A}$, $\overline{A}$ are called its bounds. Comment : If $\underline{A} = \underline{(f_{i,j})}$ and $\overline{A} = \overline{(f_{i,j})}$, then $\mathbb{A}$ is the set of all matrices $A = (f_{i,j})$ satisfying

$$\underline{f_{i,j}} \leq f_{i,j} \leq \overline{f_{i,j}} \qquad (i = 1, \ldots, m, j = 1, \ldots, n) \qquad (*)$$

It is worth noting that each coefficient may attain any value in its interval (*) independently of the values taken on by other coefficients. Notice that interval matrices are typeset in boldface letters.

Notation: In many cases it is more advantageous to express the data in terms of the center matrix and radius matrix $A_c = \frac{1}{2}(\underline{A} + \overline{A})$ (1)
and $\Delta = \frac{1}{2}(\overline{A} - \underline{A})$ (2)
which is always nonnegative.
Comment : From (1), (2) we easily obtain that $\underline{A} = A_c + \Delta$ and $\overline{A} = A_c - \Delta$.
So that $\mathbb{A}$ can be given either as $\lceil \underline{A}, \overline{A} \rceil$, or as $\mathbb{A} = \lceil A_c - \Delta, A_c + \Delta \rceil$. In the sequel we employ both forms and we switch freely between them according to which one is more useful in the current context. If both X and Y are arbitrary intervals in $C(\Omega)$, the most important bases arithmetic operators are summarized as follows:
$(a) X + Y = \lceil \underline{f}, \overline{f} \rceil + \lceil \underline{g}, \overline{g} \rceil = \lceil \underline{f} + \underline{g} \rceil + \lceil \overline{f} + \overline{g} \rceil$
$(b) X - Y = \lceil \underline{f}, \overline{f} \rceil - \lceil \underline{g}, \overline{g} \rceil = \lceil \underline{f}, \overline{f} \rceil + \lceil -\overline{g}, \underline{g} \rceil = \lceil \underline{f} - \overline{g}, \overline{f} - \underline{g} \rceil$
$(c) S(x) = \{\underline{f(x)}\overline{g(x)}, \overline{f(x)}\underline{g(x)}, \underline{f(x)}\ \underline{g(x)}, \overline{f(x)}\ \overline{g(x)}\}$
$h_1(x) = min\{S(x) : x \in \Omega\}$
$h_2(x) = max\{S(x) : x \in \Omega\}$
$(d) X.Y = \lceil h_1, h_2 \rceil$
$(e) \frac{1}{X} = \lceil \frac{1}{\overline{f}}, \frac{1}{\underline{f}} \rceil$ in which for each $x \in \Omega$, $\overline{f}(x) < 0, \underline{f}(x) > 0$ i.e an interval X is invertible if and only if for each $x \in \Omega$, 0 is not element of $\lceil \underline{f}(x), \overline{f}(x) \rceil$.
$(f) \frac{X}{Y} = X.(\frac{1}{Y})$.

Definition 2.5. Let $\mathbb{A}$ be a square interval matrix. The adjoint matrix $\mathbb{A}^*$ of $\mathbb{A}$ is the transpose of the matrix of cofactors of the elements of $\mathbb{A}$. That is, $\mathbb{A}^* = adj(\mathbb{A}) = (g_{ij})$, where $g_{ij} = (-1)^{i+j}|\mathbb{A}_{ij}|$ for all $i, j = 1, 2, \ldots, n$. We set $det\mathbb{A} = |\mathbb{A}| = \sum f_{ij}\mathbb{A}_{ij}$where $\mathbb{A}_{ij}$ is the $(i, j)-$ cofactor of with usual meaning. It is easy to see that most of the properties of determinants of classical matrices are hold good (up to equivalent) for the determinants of extended interval matrices under the modified interval arithmetic.

Definition 2.6. A square interval matrix $\mathbb{A}$ is said to be non singular or regular if $|\mathbb{A}|$ is invertible. Alternatively, a square matrix $\mathbb{A}$ is said to be invertible if $|\mathbb{A}|$ is invertible .

Example 2.7. Let $\mathbb{A} = \lceil \underline{A}, \overline{A} \rceil = \begin{pmatrix} \lceil 1, 2 \rceil & \lceil 3, 4 \rceil \\ \lceil -9, 1 \rceil & \lceil 8, 10 \rceil \end{pmatrix}$ We extended interval matrix with constant function as entries .Then $|\mathbb{A}| = \lceil 1, 2 \rceil \lceil 8, 10 \rceil - \lceil 3, 4 \rceil \lceil -9, 1 \rceil = \lceil 8, 20 \rceil - \lceil 36, 4 \rceil = \lceil 8, 20 \rceil + \lceil -4, 36 \rceil = \lceil 4, 54 \rceil$. We see 0 is not element of $|\mathbb{A}| = \lceil 4, 54 \rceil$ and hence $|\mathbb{A}|$ is invertible. So that $\mathbb{A}$ is regular.

Definition 2.8. For any $\mathbb{A} \in M_{m \times n}$, if $|\mathbb{A}|$ is invertible, then the common solution of equations $\mathbb{A}X = I$ and $X\mathbb{A} = I$ is called the inverse of $|\mathbb{A}|$ and is denoted by $\mathbb{A}^{-1} = \frac{adj(\mathbb{A})}{|\mathbb{A}|} = \frac{\mathbb{A}^*}{|\mathbb{A}|} = \frac{\mathbb{A}^*}{dual(|\mathbb{A}|)}$

Theorem 2.9. *Let $\mathbb{A}^*$ be the adjoint matrix of $\mathbb{A} \in M_{m \times n}$. Then $\mathbb{A}\mathbb{A}^* = \mathbb{A}^*\mathbb{A} = |\mathbb{A}|I$.*

Proof. Let $\mathbb{A} = (f_{ij}), \mathbb{A}^* = (g_{ij})$ so that $g_{ij} = \mathbb{A}_{ji}$. Then for $i, j = 1, \ldots, n$, we have

$$(\mathbb{A}\mathbb{A}^*)_{ij} = \sum_{k=1}^{n} f_{ik} g_{kj} = \sum_{k=1}^{n} f_{ik} \mathbb{A}_{kj} = |\mathbb{A}|\delta_{ij} = |\mathbb{A}|I \qquad (*)$$

and

$$(\mathbb{A}\mathbb{A}^*)_{ij} = \sum_{k=1}^{n} g_{ik} f_{kj} = \sum_{k=1}^{n} f_{kj} \mathbb{A}_{ki} = |\mathbb{A}|\delta_{ij} = |\mathbb{A}|I \qquad (**)$$

From equations (*) and (**), we see that $\mathbb{A}\mathbb{A}^* = \mathbb{A}^*\mathbb{A} = |\mathbb{A}|I$. □

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COMPLEX MATRICES WITH THE PERRON-FROBENIUS PROPERTY

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Abstract. By real matrices which have the Perron-Frobenius (resp. strong Perron-Frobenius) property, we have two collections of real matrices WPFn and PFn. We extend here the collections WPFn and PFn to complex matrices. Also, we state here some sufficient conditions and some necessary and sufficient conditions for a complex matrix to be in WPFn or PFn.

1. Introduction

In 1907, Perron [5] proved that, if A is an $n \times n$ entrywise positive matrix, then $\rho(A) > 0$ is an eigenvalue of A and it has an entrywise positive eigenvector with respect to $\rho(A)$. Later in 1912, Frobenius [2] extend this result to irreducible $n \times n$ matrices with all nonnegative entries, and the associated eigenvector$x$ is now called the Perron-Frobenius eigenvector. Recently, in 2012, Noutsos and Varga [4] stated two extensions of this property to complex matrices.

As in [4], there are two types (Type I and Type II) of extensions of the Perron-Frobenius property to complex matrices. By λ_i, $i = 1, \ldots n$, we mean n eigenvalues of an $n \times n$ complex matrix. For extension of Type I, we have

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Definition 1.1. We say that λ_1 is a dominant eigenvalue of $A \in \mathbb{C}^{n\times n}$ if it is a largest in modulus eigenvalue, i.e., $|\lambda_1| \geq |\lambda_i|$, $i = 1, \ldots, n$ so that $|\lambda_1| = \rho(A)$.

Definition 1.2. A matrix $A \in \mathbb{C}^{n\times n}$ has the *Perron-Frobenius property* if it has an eigenvalue $\lambda_1 = \rho(A) > 0$, and an associated nonzero column eigenvector x, which has all nonnegative components. This vector is called right Perron-Frobenius eigenvector of A.

Definition 1.3. A matrix $A \in \mathbb{C}^{n\times n}$ has the *strong Perron-Frobenius property* if it has a simple eigenvalue λ_1, with $|\lambda_1| > |\lambda_i|$, for all remaining eigenvalues λ_i, $i = 2, \ldots, n$, of A, so that $\lambda_1 = \rho(A) > 0$, and to λ_1, there corresponds a column eigenvector x, which has all positive components. this vector is called a strong right Perron-Frobenius eigenvector.

For extension of Type II, we have the following two definitions from [4],

Definition 1.4. A matrix $A \in \mathbb{C}^{n\times n}$ has the *complex Perron-Frobenius property* if it has a dominant eigenvalue λ_1, which is positive and its associated (nonzero) eigenvector x can be chosen so that $Rex \geq 0$, i.e., if $x = [x_1 \; x_2 \; \ldots \; x_n]$, then $Rex_j \geq 0$ for all $j = 1, \ldots, n$. The vector x is called the complex right Perron-Frobenius eigenvector.

Definition 1.5. A matrix $A \in \mathbb{C}^{n\times n}$ has the *strong complex Perron-Frobenius property* if it has a dominant eigenvalue λ_1 which is positive, simple, with $\lambda_1 > |\lambda_i|$, $i = 2, \ldots, n$, and for the associated eigenvector x, there holds: $Rex > 0$, i.e., $Rex_j > 0$ for all $j = 1, \ldots, n$. This vector x is called a strong complex right Perron-Frobenius eigenvector.

2. MAIN RESULTS

We define two collections of complex matrices which have the Perron-Frobenius or complex Perron-Frobenius property. Actually, these are extensions of WPFn and PFn of real matrices. For more information about WPFn and PFn of real matrices, we refer the reader to [1] and [3]. We denote by WPFn the set of all matrices $A \in \mathbb{C}^{n\times n}$ such that $\rho(A)$ is an eigenvalue of A and A has left and right nonnegative eigenvectors with respect to $\rho(A)$. Also, we define PFn, the collection of all matrices $A \in \mathbb{C}^{n\times n}$ such that $\rho(A)$ is a positive, simple and strictly dominant eigenvalue of A, also, A has left and right positive eigenvectors with respect to $\rho(A)$.

The superscript* stands for the conjugate transpose of a matrix or a vector.

Theorem 2.1. *The following assertions are true*

(i) *$A \in WPFn$ if and only if A and A^* posses the Perron-Frobenius property.*
(ii) *$A \in PFn$ if and only if A and A^* posses the strong Perron-Frobenius property.*

The results of this paper depends on Theorem 2.2 from [4], which follows.

Theorem 2.2. [4] *Assume that both the matrices $A \in \mathbb{C}^{n\times n}$ and A^* have the Perron-Frobenius property, with the dominant eigenvalue $\lambda_1 = \rho(A) = \rho(A^*)$ being simple with $\lambda_1 > |\lambda_i(A)| = |\lambda_i(A^*)|$, for all $i = 2, 3, \ldots, n$, and that X and Y are, respectively, the normalized right and left Perron-Frobenius. Then,*

$$\lim_{k\to\infty} \frac{1}{\lambda_1^k} A^k = xy^T \geq 0.$$

Moreover, if A and A^* both have the strong Perron-frobenius property, then there exists a positive integer k_0 such that $Re(A^k) > 0$ for all $k \geq K_0$.

Following, we introduce two types (Type I and Type II) of extensions of eventually nonnegative (or positive) to complex matrices.

Definition 2.3. (Type I) A complex $n\times n$ matrix A is called eventually nonnegative (resp. positive) if there exists a nonnegative integer k_0 such that $A^k \geq 0$ (resp. $A^k > 0$) for all $k \geq k_0$.

Definition 2.4. (Type II) A complex $n \times n$ matrix A is called complex eventually nonnegative (resp. positive) if there exists a nonnegative integer k_0 such that $Re(A)^k \geq 0$ (resp. $Re(A)^k > 0$) for all $k \geq k_0$.

Following, we extend Theorem 1 from [3].

Theorem 2.5. *A complex matrix A is in PFn if and only if A is eventually positive.*

Theorem 2.6. *Assume that $A \in \mathbb{C}^{n\times n}$ and $\rho(A)$ is simple and strictly dominant then $A \in WPFn$ if and only if A is eventually nonnegative.*

Theorem 2.7. *Let $A \in \mathbb{C}^{n\times n}$ be nonnilpotent, and let A be complex eventually nonnegative. Then, A and A^* have the complex Perron-Frobenius property.*

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b-NUMERICAL RANGE OF b-HILBERT SPACES

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Abstract. In this paper we introduce the concept of b-bounded linear operator on a b-Hilbert space and investigate adjointability of it. Then we define b-numerical range of these operators and state some properties of them.

1. Introduction

The concept of 2-inner product spaces have been extensively studied by many authors ([2, 4, 5]). Let X be a linear space of dimension greater than 1 over the field $\Bbbk (= \mathbb{R}$ or $\mathbb{C})$. Suppose $\|.,.\|$ is a real-valued function on $X \times X$ satisfying the following conditions:

$a)$ $\|x, y\| \geq 0$ and $\|x, y\| = 0$ if and only if x and y are linearly dependent.
$b)$ $\|x, y\| = \|y, x\|$ for all $x, y \in X$.
$c)$ $\|\lambda x, y\| = |\lambda| \|x, y\|$ for all $\lambda \in \Bbbk$ and all $x, y \in X$.
$d)$ $\|x + y, z\| \leq \|x, z\| + \|x, z\|$ for all $x, y, z \in X$.

Then $\|.,.\|$ is called a 2-norm on X and $(X, \|.,.\|)$ is called a linear 2-normed space. For a fixed $b \in X$, $p_b(x) = \|x, b\|,\ x \in X$ is a seminorm on X. If $A \subseteq X$, is closed in the seminormed space $(X, \|., b\|)$, then we say A is b-closed.

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Let $b, x, y \in X$ and A be a subspace of X. Then x is said to be b-orthogonal to y if $\|x, b\| \neq 0$ and for all scalars α, $\|x, b\| \leq \|x + \alpha y, b\|$ and $y \neq b$. In this case we write $x \perp^b y$. If M_1 and M_2 are subsets of X, then we say M_1 is b-orthogonal to M_2 if $x \perp^b y$ for each $x \in M_1$ and $y \in M_2$. If $b \in X \backslash x - A$, then we say y_0 is b-best approximation for $x \in X$, if $x - y_0 \perp^b A$. The set A is called b-Chebyshev if for every $x \in X \backslash A + L_b$ (the subspace generated by b is denoted by L_b),there exists a unique b-best approximation for x.
Let X be a linear space with $dim(X) > 1$. Suppose that $\langle ., .|.\rangle$ is a $\mathbb{k}$-valued function defined on $X \times X \times X$ satisfying the following conditions:
$a)$ $\langle x, x|z\rangle \geq 0$ and $\langle x, x|z\rangle = 0$ if and only if x and z are linearly dependent,
$b)$ $\langle x, x|z\rangle = \langle z, z|x\rangle$,
$c)$ $\langle x, y|z\rangle = \overline{\langle y, x|z\rangle}$,
$d)$ $\langle \alpha x, y|z\rangle = \alpha\langle x, y|z\rangle$,
$e)$ $\langle x + x', y|z\rangle = \langle x, y|z\rangle + \langle x', y|z\rangle$,
where $\alpha \in \mathbb{k}$ and $x, y, z \in X$. Then $\langle ., .|.\rangle$ is called a 2-inner product on X and $(X, \langle ., .|.\rangle)$ is called a 2-inner product space. We can define a 2-norm on $X \times X$ by $\|x, y\| = \langle x, x|y\rangle^{\frac{1}{2}}$, where $x, y \in X$. Let $b \in X$, $x, y \in X - L_b$ and $A \subseteq X$. Then $x \perp^b y$ if and only if $\langle x, y|b\rangle = 0$ and we mean by b-orthogonal complement of A which is denoted by $A_b^{\perp}$, the set of elements $x \in X$ in which $\langle x, y|b\rangle = 0$, for all $y \in A$. Also we have the Cauchy-Schwarz inequality $\langle x, y|b\rangle^2 \leq \|x, b\|^2\|y, b\|^2$ for every $x, y \in X$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is b-Cauchy (b-convergent) sequence if it is Cauchy (convergent) in the seminormed space $(X, \|., b\|)$. We call X is b-Hilbert if every b-Cauchy sequence is b-convergent. Recall that every non empty b-closed, convex A in a b-Hilbert space X with $A \cap \langle b\rangle = \emptyset$ is b-Chebyshev and if A is a b-Chebyshev subspace of X with $A \cap \langle b\rangle = \{0\}$, then $X = A_b^{\perp} \oplus A$ ([5]).

Example 1.1. The set $\mathbb{R}^2$ of all ordered pairs on $\mathbb{R}$ with the pointwise addition and scalar multiplication and $\langle (a_1, a_2), (b_1, b_2)|(c_1, c_2)\rangle = a_1b_1c_2^2 + a_2b_2c_1$, is a $(1, 0)$-Hilbert space, where $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$. In fact we have
(i) $\langle (a_1, a_2), (a_1, a_2)|(1, 0)\rangle = a_2^2 \geq 0$ and it is equal to zero if and only if $a_2 = 0$. This implies that $(a_1, a_2), (1, 0)$ are linearly dependent.
(ii) $\langle \lambda(a_1, a_2) + (a_3, a_4), (b_1, b_2)|(1, 0)\rangle = \lambda\langle (a_1, a_2), (b_1, b_2)|(1, 0)\rangle +$
$\langle (a_3, a_4), (b_1, b_2)|(1, 0)\rangle = \lambda a_2b_2 + a_4b_2$.
(iii) $\langle (a_1, a_2), (b_1, b_2)|(1, 0)\rangle = \overline{\langle (b_1, b_2), (a_1, a_2)|(1, 0)\rangle} = a_2b_2$.
(iv) $\langle (a_1, a_2), (a_1, a_2)|(1, 0)\rangle = \langle (1, 0), (1, 0)|(a_1, a_2)\rangle = a_2^2$.
Also $\|(a_1, a_2), (1, 0)\| = \langle (a_1, a_2), (a_1, a_2)|(1, 0)\rangle^{\frac{1}{2}} = |a_2|$.

Now if $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ is a $(1,0)$-Cauchy sequence, then $\{b_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and so $\lim_{n\to\infty} b_n = b_0$ for some b_0 in $\mathbb{R}$. An easy verification shows that $\|(a_n, b_n) - (0, b_0), (1, 0)\| = |b_n - b_0| \to 0$ as n tends to infinity.

Let X be a vector space and $b, y_1, y_2 \in X$. Then y_1 is said to be b-congruent to y_2, if $y_1 - y_2 \in L_b$. In this paper we introduce the concept of b-bounded linear operator on a b-Hilbert space and investigate adjointability of it. Then we define b-numerical range of these operators and state some properties of them.

2. Main results

Definition 2.1. Let X be a b-Hilbert space and $T : X \to X$ be a linear operator. We say that T is b-bounded if T invariants the subspace generated by b, i.e. $T(L_b) \subseteq L_b$ and there exists $M > 0$, in which $\|T(x), b\| \le \|x, b\|$ for each $x \in X$. In this case we define $\|T\|$, infimum of such M. One can see that $\|T\| = \sup\{\|T(x), b\| : \ \|x, b\| \le 1\}$. Also the linear functional $f : X \to \mathbb{C}$ is b-bounded if $T(L_b) = \{0\}$ and there exists $M > 0$, in which $|f(x)| \le \|x, b\|$ for each $x \in X$.

Example 2.2. Let X be $(1,0)$-Hilbert space in the Example 1.1 and $T : X \to X$ be defined by $T((a_1, a_2)) = \lambda(a_1, a_2)$ for some $\lambda \in \mathbb{R}$ and $S : X \to X$ be defined by $S((a_1, a_2)) = (a_1 + a_2, 0)$. Obviously T, S are linear operators and invariants $L_{(1,0)}$. On the other hand $\|T((a_1, a_2), (1, 0))\| = \|\lambda(a_1, a_2), (1, 0)\| = |\lambda a_2| = |\lambda|\|(a_1, a_2), (1, 0)\|$ and $\|S((a_1, a_2)), (1, 0)\| = \|(a_1 + a_2, 0), (1, 0)\| = 0$. So T, S are $(1,0)$-bounded.

Slightly modification in the proof of Theorem 3.5. of [4] gives the following theorems.

Theorem 2.3. *Let A be a b-closed subspace of a b-Hilbert space X, in which $A \cap L_b = \{0\}$, and T be a b-bounded linear functional on X with $f(A_b^{\perp}) = \{0\}$. Then there exists a unique $y \in A$ up to b-congruent such that $f(x) = \langle x, y|b\rangle$ and $\|f\| = \|y, b\|$.*

Theorem 2.4. *Let A be a b-closed subspace of a b-Hilbert space X, in which $A \cap L_b = \{0\}$, and $T : X \to X$ be a b-bounded linear operator with $T(A_b^{\perp}) = \{0\}$. Then there exists a b-bounded linear operator $T^* : X \to X$ in which $\langle T(x), y|b\rangle = \langle x, T^*(y)|b\rangle$ for every $x, y \in X$.*

The operator T^* in the above theorem is called b-adjoint of T and b-adjointable operator U is called b-unitary, if $UU^* = U^*U = I$. Let X be a b-Hilbert space and M_b the algebriac complement of L_b in X. So $L_b \oplus M_b = X$. We define semi definite inner product $\langle x, y\rangle_b = \langle x, y|b\rangle$

on X for each $x, y \in X$. It is well known that this semi inner product induced an inner product on the quotient space $\frac{X}{L_b}$ as $\langle x+L_b, y+L_b\rangle = \langle x, y\rangle_b$ for each $x, y \in X$. By identifying $\frac{X}{L_b}$ by M_b in obvious way, we obtain an inner product on M_b. Define $\|x\| = \langle x, x\rangle_b^{\frac{1}{2}}$ for each $x \in X$. Then $(M_b, \|.\|)$ is a normed space. Let X_b be the completion of the inner product space M_b ([1]). Let $T : X \to X$ be a b-bounded linear operator, then $T_b : X_b \to X_b$ defined by $T_b(\lim_{n\to\infty} x_n + L_b) = \lim_{n\to\infty} T(x_n) + L_b$ is well defined. Since $\{x_n + L_b\}$ is a Cauchy sequence in X_b. We show that $\{T_b(x_n + L_b)\}$ is a Cauchy sequence in X_b and so convergent. For, $\|T(x_n - x_m) + L_b\| = \|T(x_n - x_m)\|_b = \|T(x_n - x_m), b\| \leq \|T\|\|x_n - x_m, b\| \to 0$ as $n \to \infty$.

Definition 2.5. Let X be a b-Hilbert space and $T : X \to X$ be a b-bounded linear operator. We define b-numerical range of T and denote it by $W_b(T)$, the set $\{\langle T(x), x|b\rangle : \|x, b\| = 1\}$. It is obvious that $W_b(T) = W_b(T_b)$.

In the Example 2.2, $W_{(1,0)}(T) = \{\lambda^2\}$ and $W_{(1,0)}(S) = \{0\}$. Note that $T_{(1,0)} = ((a_1, a_2) + L_{(1,0)}) = \lambda(0, a_2) + L_{(1,0)}$. So by relating every b-bounded linear operator T to a bounded linear operator T_b, one can see that the results of numerical ranges hold in the b-Hilbert spaces. For instance we have the following theorem.

Theorem 2.6. *Let $T : X \to X$ be a b-bounded linear operator a b-Hilbert space X. Then we have*
(a) $W_b(\alpha + \beta T) = \alpha + \beta W_b(T)$.
(b) if T has a b-adjoint, then $W_b(T^*) = \{\overline{\lambda} : \lambda \in W_b(T)\}$.
(c) if U is a b-unitary operator, then $W_b(U^*TU) = W_b(T)$.

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USING MATCHING THEORY TO IMPROVE THE EFFICIENCY OF THE QR BASED TURNBACK METHOD

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Abstract. Here, we show how we can improve the efficiency of the QR based Turnback method, for computing a sparse null space basis of a large and sparse matrix, by using the concepts of the matching theory of bipartite graphs. Then, we present numerical results to justify the efficiency of the resulting algorithm.

1. Introduction

Let $A \in R^{m\times n}$ be a sparse and large matrix that has full rank, where $m < n$. If $N \in R^{(n-m)\times n}$ has full rank and satisfies $AN = 0$, then N is called a null space basis of A. In the following we will call the individual columns of N, a null vector. One efficient algorithm for computing a sparse null space basis is the Turnback method [4]. Here, we examine the effect of the Dulmage-Mendelsohn decomposition on the efficiency of the QR-based Turnback method. The Dulmage-Mendelsohn decomposition is a canonical decomposition of bipartie graphs, which gives us some sparse submatrices of A, named the Head part and the Body part. On the other hand, if we compute a null space basis for the Head part of A, then, we can use it to derive a null space of A. Therefore, one

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idea for improving the efficiency of the Turnback method is to first apply the Dulamge-Mendelsohn decomposition of A to compute its Head part and apply the Turnback method to compute a sparse null space basis for the Head part of A. In section 1, we describe the QR-based Turnback method. In section 2, we explain the Dulmage-Mendelsohn decomposition and consider its relation to the null space basis algorithms. In section 3, we present some Tables of numerical results to justify the efficiency of our proposed algorithm.

2. QR-based Turnback method

In this section, we describe the QR-based Turnback method. Let $A = (a_1, a_2, \cdots, a_n)$, where a_j denotes the jth column of A. The column a_s is called a start column, if it is linearly dependent on the lower-indexed columns .i.e if the ranks of $(a_1, a_2, \cdots, a_s)$ and $(a_1, a_2, \cdots, a_{s-1})$ are equal. Note that the coeficients of this linear dependency give a null vector whose highest-indexed nonzero is in position s. It is easy to see that the number of start columns is $n - m$ .i.e. the dimension of the null space of A. For more explanation, let the scalars $\lambda_1, \cdots, \lambda_{s-1}$ be so that $\lambda_1 a_1 + \cdots + \lambda_{s-1} a_{s-1} = a_s$. Set $x = (x_1, ..., x_n)^T$, where, if $j = 1, ..., s-1$ then $x_j = \lambda_j, x_s = 1$ and for $j = s+1, \cdots, n$ we have $x_j = 0$. Clearly, the vector x is a null vector of A. We can use the QR decomposition algorithm, to obtain the start columns of A. Let $A = QR$, then, as is a start column if the higest nonzero position in column s of R is no larger than the highest nonzero position in earlier columns of R. After the determination of the start columns, the QR-based Turnback method finds one null vector for each start column. Let a_s be the start column, then, the Turnback method starts with the column s and finds thee smallest k for which columns as, $a_{s-1}, ..., a_{s-k}$ are linearly dependent. The nonzero entries of the null vector are among positions $s - k$ through s. If the value of k is small, for most of the start columns, then, the resulting null basis have small number of nonzero entries. To guarantee the linear independency of generated null vectors we do not let the start column as to participate in the dependency, relating to the sequent null vectors. Based on the above considerations, in the QR-based Turnback method, we apply the QR decomposition algorithm, to find the start columns and for each start column, we compute a null vector so that the computed null vectors are linearly independent. To find a null vector, using the start column as, we form a set of active columns A_c. At first A_c just contains the start column as. Then, we choose an inactive column a_c, $c < s$. If a_c is linearly independent of the columns in $A_c - \{a_s\}$, then, we add a_c to A_c. Otherwise, we stop, A_c is the desired set of indices of independent columns. The linear dependence or independence is determined by using the QR factorization algorithm. Note that small cardinality of the active sets A_c, will lead to sparse null vectors.

3. Dulmage-Mendelsohn decomposition

In this section we explain the Dulmage-Mendelsohn decomposition and its relation to sparse null space problem. Given a large and sparse matrix $A \in R^{m\times n}$ $(m < n)$ with $rank(A) = m$, assume that we can partition A so that the resulting algorithm has the form $A = \begin{bmatrix} B & 0 \\ 0 & H \end{bmatrix}$ where, $B \in R^{m_1\times n_1}$, $H \in R^{m_2\times n_2}$, $m_2 < n_2$, $m_1+m_2 = m$ and $n_1+n_2 = n$. The Dulmage-Mendelsohn decompsition, provide us with such a partition [1, 2]. B is called he Body part and H is called the head part of A. To explain the Dulmage- Mendelsohn decomposition let M be an arbitrary matching of A and $G = (C, R, E)$ be the nipartite graph of A, where the vertex set C, corresponds to the columns of A and the vertex set R corresponds to rows of A. The edge $e = (i, j)$ belongs to E iff the entry of A in row i and column j is nonzero. let C_U be the set of unmatched column indices, R_U be the set of unmatched row indices. $R_M(C_U)$ be the set of all row indices that are reachable from some unmatched column, $C_M(R_U)$ be the set of all column indices that are reachable from some unmatched row, $C_M(C_U)$ be the set of all column indices that are reachable from some unmatched column, $R_M(R_U)$ be the set of all row indices that are reachable from some unmatched row. Let

$$C_{MM} = C(C{\cup}C_M(C_U){\cap}C_M(R_U)), R_{MM} = R{-}(R{\cap}R_M(R_U){\cap}R_M(R_U)).$$

Moreover, for simplicity in notation let $C_1 = C_M(R_U)$, $C_2 = C_{MM}$, $C_3 = C_U \cap C_M(R_U)$, $R_1 = R_U \cap R_M(R_U)$, $R_2 = R_{MM}$, $R_3 = R_M(C_U)$. It can be shawn that the submatrix of A corresponding to R_2 and C_2 is the Body of A and the submatrix corresponding to R_3 and C_3 is the Head part of A. If we compute a null space for the Head part of A i.e. H, then we can use it to obtain a null space of A. Moreover, assuem that $v \in R^{n_2}$ is a null vector of H i.e. $H_v = 0$.Then, if we let $v = [0, v]T$ then, $Av = 0$ and v is a null vector for A.

4. Numerical results

In this section we present numerical results to investigate the efficiency of the proposed algorithm of this manuscript. We implemented the algorithms in Matlab envronment, using an 2.8 GHz Intell Corei7 processor with 4 GB memory. In Table 1, we provided 19 coefficient matrices, arising from real world applications, for which our proposed algorithm generates a null space basis with less number of nonzero elements than the QR-based Turnback method as applied to the original coefficient matrix. These 19 instances, are coefficient matrices from the NETLIB set of test problems [3]. The NETLIB is a set of standard test problems, arising from real world applications. In this table, m and n denote the number of rows and number of columns of the coeffi- cient matrix. nnz1 and t1 denote the number of nonzero elements and the computing time (in seconds) for the null space generated by the QR-bsed Turnback method when applied to the original coefficient matrix. Similarly, nnz2 and t2 denote the number of nonzero elements and the computing time for the null space generated by the the proposed algorithm of this manuscript. For example for problem sc205, the QR-based turnback method computes a nul space basis with 4724 number of nonzeros while our proposed algorithm computes a nul space basis with 1102 number of nonze- rosin approximately the same computing time. For all test problems, the computing time os both algortihms are aproximately, the same. This justifies the efficiency of our proposed algorithm.

Table 1. Comparison between two algorithms

problem	m	n	nnz1	t1	nnz2	t2
25fv47	821	1876	14005	1.84	11032	1.83
adlittle	56	118	861	0.03	763	0.04
sc205	205	317	4724	0.09	1102	0.10
sc105	105	163	1643	0.06	436	0.05
greenbeb	50	78	243	0.01	155	0.02
share1b	117	253	2780	0.27	1902	0.23

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NEW ITERATIVE SOLUTION FOR LINEAR SYSTEMS WITH DOMINANT SKEW-SYMMETRIC PART

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Abstract. We present an efficient iterative method for solving linear systems which the skew-symmetric part of their coefficient matrix is dominant. This method is actually inner/outer iterations, which employs the CGNR method as inner iteration to approximate each outer iterate. Convergence conditions of this method are studied and numerical experiments show the efficiency of this method and it's preconditioned version.

1. Introduction

Consider system of linear equations

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a large sparse matrix. The iterative solution of this system requires some forms of splitting. Among these forms of splitting which named Hermitian and skew-Hermitian splitting is

$$A = H + S, \tag{1.2}$$

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* Speaker.

where $H = (A + A^T)/2$ and $S = (A - A^T)/2$. When H dominates S, we have $||H^{-1}S||_2 < 1$ and may use nested splitting conjugate gradient (NSCG) method [1]. But if S dominates H, it means that $||H^{-1}S||_2$ is large and we cannot guarantee convergence of the NSCG method. When the skew-symmetric part S is dominant, it is typically harder to solve the linear system (1.1) and it is challenging from a numerical point of view.

In this paper, we present an efficient iterative method, named NS-CGNR, for case which the skew-symmetric component S dominates the symmetric component H.

2. Main results

Consider the Hermitian and skew-Hermitian splitting (1.2). Since the matrix S may be singular, we introduce a shift ($\alpha > 0$) and define

$$A = (H - \alpha I) + (S + \alpha I) = H_\alpha + S_\alpha. \tag{2.1}$$

Then the system of linear equations (1.1) is equivalent to

$$S_\alpha x = b - H_\alpha x.$$

Given an initial guess $x^{(0)} \in \mathbb{R}^n$, suppose that we have computed approximations $x^{(1)}, x^{(2)}, \cdots, x^{(l)}$ to the solution $x^* \in \mathbb{R}^n$ of the system (1.1). Then the next approximation $x^{(l+1)}$ may be defined as either an exact or an inexact solution of the system of linear equations

$$S_\alpha x = b - H_\alpha x^{(l)}. \tag{2.2}$$

Now we solve the system of linear equations (2.2) by the CGNR method [3]. The parameter α should be chosen as the splitting (2.1) satisfies the condition $\rho(S_\alpha^{-1} H_\alpha) < 1$. The following lemma states an inequality for the spectral radius of product of matrices.

Lemma 2.1. [2] *Let $H \in \mathbb{R}^{n\times n}$ (respectively $S \in \mathbb{R}^{n\times n}$) be a symmetric (respectively skew-symmetric) matrix. If $H_\alpha = H - \alpha I$ and $S_\alpha = S + \alpha I$ with $\alpha \in \mathbb{R}_0^+$, then*

$$\rho(S_\alpha^{-1} H_\alpha) \leq \rho(S_\alpha^{-1})\rho(H_\alpha).$$

Now, consider the function $f(\alpha) = \rho(S_\alpha^{-1})\rho(H_\alpha)$. Assume that the largest and the smallest eigenvalues of H are denoted by $\lambda_{\max}$ and $\lambda_{\min}$, respectively and let $i\nu$ be the eigenvalue of S with smallest modulus. Therefore, the spectral radii of H_α and S_α^{-1} are

$$\rho(H_\alpha) = \max\{|\lambda_{max} - \alpha|, |\lambda_{min} - \alpha|\} \quad \text{and} \quad \rho(S_\alpha^{-1}) = |\nu^2 + \alpha^2|^{-1}.$$

Without loss of generality we can assume that $\lambda_{\max} > 0$. Then,

$$f(\alpha) = \frac{1}{2|\alpha + i\nu|}\left(|2\alpha - (\lambda_{\max} + \lambda_{\min})| + (\lambda_{\max} - \lambda_{\min})\right),$$

and f has a single critical point at $\alpha_c = \frac{\lambda_{\max}+\lambda_{\min}}{2}$ and

$$f(\alpha_c) = \frac{\lambda_{\max} - \lambda_{\min}}{\sqrt{(\lambda_{\max} + \lambda_{\min})^2 + 4\nu^2}}.$$

For a symmetric positive definite matrix $\mathcal{B}$, we denote by $\kappa(\mathcal{B}) = ||\mathcal{B}||_2||\mathcal{B}^{-1}||_2$ its Euclidean condition number, and we define the $||\cdot||_{\mathcal{B}}$ norm of a vector $x \in \mathbb{R}^n$ as $||x||_{\mathcal{B}} = \sqrt{x^T\mathcal{B}x}$. Then the induced $||\cdot||_{\mathcal{B}}$ norm of a matrix $H \in \mathbb{R}^{n\times n}$ is define as $||H||_{\mathcal{B}} = ||\mathcal{B}^{\frac{1}{2}}H\mathcal{B}^{-\frac{1}{2}}||_2$. $A = B - C$ is called a contractive splitting if $||B^{-1}C|| < 1$ for some matrix norm.

We can prove the following theorem for convergence of the NS-CGNR method.

Theorem 2.2. *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular and non-symmetric matrix, and $A = H_\alpha + S_\alpha$ a contractive (with respect to the $||\cdot||_{S_\alpha^T S_\alpha}$-norm). Suppose that the NS-CGNR method is started from an initial guess $x^{(0)} \in \mathbb{R}^n$, and produces an iterative sequence $\{x^{(l)}\}_{l=0}^{\infty}$, where $x^{(l)} \in \mathbb{R}^n$ is the lth approximation to the solution $x^* \in \mathbb{R}^n$ of the system of linear equations (1.1), obtained by solving the linear system (2.2) with k_l steps of CGNR iterations. Then*
(a) $||x^{(l)} - x^*||_{S_\alpha^T S_\alpha} \le \gamma^{(l)}||x^{(l-1)} - x^*||_{S_\alpha^T S_\alpha}$, $l = 1, 2, 3, \cdots$,
(b) $||b - Ax^{(l)}||_{S_\alpha^T S_\alpha} \le \widetilde{\gamma}^{(l)}||b - Ax^{(l-1)}||_{S_\alpha^T S_\alpha}$, $l = 1, 2, 3, \cdots$,
where

$$\gamma^{(l)} = 2\left(\frac{\kappa(S_\alpha) - 1}{\kappa(S_\alpha) + 1}\right)^{k_l}(1+\varrho)+\varrho,\ \widetilde{\gamma}^{(l)} = \gamma^{(l)}\kappa(S_\alpha)\frac{1+\varrho}{1-\varrho},\ l = 1, 2, 3, \cdots$$

and $\varrho = ||S_\alpha^{-1}H_\alpha||_{S_\alpha^T S_\alpha} = ||H_\alpha S_\alpha^{-1}||_2$.

Moreover, for some $\gamma \in (\varrho, \varrho_1)$ with $\varrho_1 = \min\{1, 2 + 3\varrho\}$, and

$$k_l \ge \frac{\ln((\gamma - \varrho)/(2(1 + \varrho)))}{\ln((\kappa(S_\alpha) - 1)/(\kappa(S_\alpha) + 1))},\ l = 1, 2, 3, \cdots,$$

we have $\gamma^{(l)} \le \gamma$ $(l = 1, 2, 3, \cdots)$, and the sequence $\{x^{(l)}\}_{l=0}^{\infty}$ converges to the solution x^* of the system of linear equations (1.1). For $\varrho \in (0, r)$, which r is the positive root of quadratic equation $\kappa(S_\alpha)\varrho^2 + (\kappa(S_\alpha) + 1)\varrho - 1 = 0$, and some $\widetilde{\gamma} \in ((1 + \varrho)\varrho\kappa(S_\alpha)/(1 - \varrho), 1)$, and

$$k_l \ge \frac{\ln(((1 - \varrho)\widetilde{\gamma} - \varrho(1 + \varrho)\kappa(S_\alpha))/(2(1 + \varrho)^2\kappa(S_\alpha)))}{\ln((\kappa(S_\alpha) - 1)/(\kappa(S_\alpha) + 1))},\ l = 1, 2, 3, \cdots,$$

we have $\widetilde{\gamma}^{(l)} \leq \widetilde{\gamma}$ $(l = 1, 2, 3, \cdots)$, *and the residual sequence* $\{b - Ax^{(l)}\}_{l=0}^{\infty}$ *converges to zero.*

To improve the performance of the NS-CGNR method, we use the IC(0) preconditioner [3] and denote this preconditioned method with PNS-CGNR. In the following example, we show the efficiency of the NS-CGNR and PNS-CGNR methods versa the GMRES(20) method.

Example 2.3. The constant coefficient advection-diffusion equation is

$$-\triangle u + \beta \frac{\partial u}{\partial x} = f, \tag{2.3}$$

with $\beta \geq 0$ on the unit square $(0, 1) \times (0, 1)$ and Dirichlet conditions prescribed on the boundary. The domain is discredited with mesh size h and solved by the finite difference method. In the following table, we report the number of total outer iterations and CPU time (in parentheses), and compare the NS-CGNR method and its preconditioned variant with the GMRES(20) method. All iterations started from the zero vector for initial $x^{(0)}$ and terminated when the current itrate satisfies $||r^{(k)}||_2 \leq 10^{-10}||r^{(0)}||_2$.

Table 1. Results for problem (2.3) with $\beta = 10^5$

h	$\|\|H\|\|_2$	$\|\|S\|\|_2$	NS-CGNR	PNS-CGNR	GMRES(20)
1/32	6.3706e+3	3.1863e+6	5(0.018)	8(0.006)	21(0.047)
1/64	2.5566e+4	6.3927e+6	6(0.034)	9(0.010)	92(0.418)
1/128	1.0236e+5	1.2796e+7	27(0.109)	13(0.024)	391(1.953)
1/256	4.0956e+5	2.5598e+7	66(0.624)	17(0.063)	227(1.447)
1/512	1.6383e+6	5.1199e+7	67(1.652)	37(0.514)	140(1.738)

By focusing on the results presented in the Table 1, one can observe that the NS-CGNR method is superior to the GMRES(20) method in both term iteration numbers and CPU time. Moreover, the use of the IC(0) precondition can induce accurate and effective.

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GENERALIZATION OF THE LIE SYMMETRY THEORY TO THE FRACTIONAL-PARTIAL EQUATIONS

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Abstract. In this paper we introduce the concept of fractional derivatives and prolongation formula for this kind of derivatives. Method of Lie group analysis are applied to investigate symmetry properties of the fractional diffusion equations $D_t^{\alpha} u = (k(u)u_x)_x$ for different orders of $\alpha (0 < \alpha \leq 1)$ and different types of derivatives(integer, Riemann-Liouville and Caputo).

1. Introduction

Lie symmetry group theory is one of the most important methods in analysis of differential equations in order to constructing the exact solutions, mapping them to another solutions by linear transformations and symmetry reductions. Such Lie groups are invertible point transformations that causing movement dependent and independent

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* Speaker.

variables of the differential equations. Many PDEs in the applied sciences are continuity equations which appear mathematical modeling. In recent years, the fractional calculus is increasingly used as effective tool to describe physical, chemical and biological processes. Most of numerical method allows one to find solutions only for limited classes of linear equations, but modern group analysis can be effectively used to find exact solutions of this type of equations. Recently Gazizov et al.(2007) adapted methods of Lie group for symmetry analysis to fractional differential equations(FDE) and proposed prolongation formulas for fractional derivatives. In this paper nonlinear fractional diffusion equations $D_t^\alpha u = (k(u)u_x)_x$, are considerd whit $\alpha (0 < \alpha \leq 1)$, that describe the processes of subdiffusion for $\alpha \in (0,1)$ and normal diffusion for $\alpha = 1$, It is noteworthy, we utilize of Riemann-Liouville type for fractional derivative .

2. Main results

Suppose $\Delta(x, u^n, u^\alpha)$ be a fractional differential equation defined over the total space $M = X \times U$, whose coordinates represent the independent and dependent variables and the derivatives of dependent variables up to integer order n and faractional value α, that is called the $n+\alpha$-th order jet space on the underlying space $X \times U$. In nonlinear fractional diffusion equations

$$D_t^\alpha u = (k(u)u_x)_x, \tag{2.1}$$

u is a function of independent variables t and x, $u_x = \frac{\partial u}{\partial x}$ and $D_t^\alpha u$ is fractional derivatives of u with respect to t which can be of Riemann-Liouville type

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial t^m} \int_0^t \frac{u(x,z)}{(t-z)^{\alpha+1-m}} dz,$$

that $\Gamma(z) = \int_0^\infty \exp(-t) t^{z-1} dt$, Specially for $n \in \mathbb{Z}$ $\Gamma(n+1) = n!$ also $0 < m-1 < \alpha \leq m$ for $m \in \mathbb{N}$. one of important property is the generalized Leibnitz rule in fractional derivatives [4]

$$D_t^\alpha (f(t)g(t)) = \Sigma_{n=0}^\infty C(\alpha, n) D_t^{\alpha-n} f(t) D_t^n g(t). \tag{2.2}$$

A local invertible group transformation acting on M is given by one parameter Lie group

$$\begin{aligned} &G : I \times M \longrightarrow M \qquad I \subseteq R, \\ &(a, (x,u)) \mapsto \varphi(a) = (x^1 + a\xi^1(x,u) + O(a^2), ...). \end{aligned}$$

Let $C-$curve is the graph of G on M, that at each it's point the tangent vector

$$\mathbf{v} = \dot{\varphi}(a) = \frac{d\varphi}{da}\Big|_{a=0} = \sum_{i=1}^{p} \xi^i(x,u)\partial_{x^i} + \sum_{\alpha=1}^{q} \eta^\alpha(x,u)\partial_{u^\alpha},$$

be infinitesimal transformation of G in $\underline{g}$ that act on independent and dependent variables and the derivatives of dependent variables, namely $M \times U^{(1)}$. G is a symmetry group which transforms solutions of the system to other solutions. According to Lie theory, the construction of the symmetry group G is equivalent to determination of it's infinitesimal transformations

$$\overline{x} = t + a\xi^1(x,t,u), \quad \overline{t} = t + a\xi^2(x,t,u), \quad \overline{u} = t + a\eta(x,t,u), \quad (2.3)$$

so $\mathbf{v} = \xi^1(x,t,u)\frac{\partial}{\partial x} + \xi^2(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}$ is in $\underline{g}$. For determining symmetry group through the classical infinitesimal symmetry condition, we must check the following system

$$\mathrm{Pr}^{(n+\alpha)}\mathbf{v}\big[\Delta(x,u^n,u^\alpha)\big] = 0 \qquad \text{whenever } \Delta(x,u^n,u^\alpha) = 0, \quad (2.4)$$

where $\mathrm{Pr}^{(n+\alpha)}\mathbf{v}$ that called $(n+\alpha) - th$ prolongation of $\mathbf{v}$ is

$$\mathrm{Pr}^{(n+\alpha)}\mathbf{v} = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \xi^3\frac{\partial}{\partial u_x} + \xi^4\frac{\partial}{\partial u_{xx}} + \xi^\alpha\frac{\partial}{\partial D_t^\alpha u}, \quad (2.5)$$

To construct prolongation V we must apply generalization of (2.3) to usual partial derivatives u_x, u_{xx} and $D_t^\alpha u$, according to [4] we have

$$\begin{array}{l} \overline{u}_{\overline{x}}(\overline{x},\overline{t}) = u_x(x,t) + a(\xi^3) + o(a), \\ \overline{u}_{\overline{x}\overline{x}}(\overline{x},\overline{t}) = u_{xx}(x,t) + a(\xi^4) + o(a), \end{array}$$

where ξ^3 and ξ^4 are defined by prolongation formula

$$\begin{array}{l} \xi^3 = D_x\eta - u_t D_x\xi^2 - u_x D_x \xi^1, \\ \xi^4 = D_x\xi^3 - u_{xt}D_x\xi^2 - u_{xx}D_x\xi^1, \end{array}$$

now according to [5] we can determine infinitesimal transformations of fractional derivatives, that depend on the type of derivatives. In Riemann-Liouville type we have

$$D_{\overline{t}}^\alpha \overline{u}(\overline{x},\overline{t}) = D_t^\alpha u(x,t) + a\xi^\alpha + o(a),$$

where

$$\begin{array}{l} \xi^\alpha = D_t^\alpha(\eta) + \xi^1 D_t^\alpha(u_x) - D_t^\alpha(\xi^1 u_x) \\ +D_t^\alpha(D_t(\xi^2)u) - D_t^{\alpha+1}(\xi^2 u) + \xi^2 D_t^{\alpha+1}(u), \end{array}$$

The general case ξ^α depends on x, t, u, derivatives u_t, u_{tt},... and on an infinite set of the fractional derivatives $D_t^{\alpha-n}u$, $D_t^{\alpha-n}u_x(n =$

$0, 1, 2, ...)$.Therefore

$$\begin{aligned}&\mathrm{Pr}^{n+\alpha}\mathbf{v}\big[\Delta(x,u^n,u^\alpha)\big] = \mathrm{Pr}^{n+\alpha}\mathbf{v}(-D_t^\alpha u + (k(u)u_x)_x\\ &\quad= \mathrm{Pr}^{n+\alpha}\mathbf{v}(-D_t^\alpha u + k'(u)u_x^2 + k(u)u_{xx})\\ &\quad= \big[-\xi^\alpha + \eta(k''(u)u_x^2 + k'(u)u_{xx}) + 2u_x k'(u)\xi^3 + k(u)\xi^4\big]\big|_\Delta = 0\end{aligned}$$

this equation defines all infinitesimal symmetries of equation (2.1), and it is called *determining equation.* It is a linear FDE for unknown functions ξ^1, ξ^2 and η. Coefficients of this equations depend on variables u_x, u_{xx}, u_{xt}, u_t, u_{tt}, ... and $D_t^{\alpha-n}u$, $D_t^{\alpha-n}u_x$ for n=0, 1, 2,..., which are considered independent. Now by properties of fractional derivatives, the determining equation are as below

$$\begin{aligned}&-D_t^\alpha(D_t(\xi^2)u) + D_t^{\alpha+1}(\xi^2 u) - \xi^2 D_t^{\alpha+1}(u) = 0,\\ &-\xi^1 D_t^\alpha(u_x) + D_t^\alpha(\xi^1 u_x) = 0,\\ &-D_t^\alpha(\eta) + (k''(u)u_x^2 + k'(u)u_{xx})\eta = 0,\\ &\quad 2k'(u)u_x\xi^3 = 0,\\ &\quad k(u)\xi^4 = 0,\end{aligned}$$

so

$$D_t^1(\xi^1) = 0, \quad D_t^2(\xi^2) = 0, \quad \xi^3 = 0, \quad \xi^4 = 0,$$

and $\eta = 0$ but if $k(u) = exp(u)$ then $\eta = f(x)$.

Theorem 2.1. *The Lie algebras of infinitesimal projectable symmetries of fractional diffusion equations are spanned by the vector fields*

$$\mathbf{X}_1 = \partial_x, \qquad \mathbf{X}_2 = x\partial_x + t\partial_t, \qquad \mathbf{X}_3 = x\partial_x, \tag{2.6}$$

where C_1 is arbitrary constant, and if $\alpha = 1$ we have $\mathbf{X}_4 = \partial_t$.

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SUBDIRECT SUM OF INVERSE-POSITIVE MATRICES

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Abstract. In this note, we investigate two problems about k-subdirect sum of matrices. The first one is whether the k subdirect sum of inverse-positive matrices forms an inverse positive matrix? and the second one is, whether an inverse-positive matrix can be written of the form of k-subdirect sum of two inverse-positive matrices.

1. Introduction

The notation of subdirect sum was introduced by Fallat and Johnson [1] as follows:

Let $0 \leq k \leq m, n$ and suppose that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{M}_n(\mathbb{C}) \quad \& \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in \mathbb{M}_m(\mathbb{C})$$

in which $A_{22}, B_{11} \in \mathbb{M}_k(\mathbb{C})$. Then we call the matrix

$$C = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} + B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \end{bmatrix} \in \mathbb{M}_{m+n-k}(\mathbb{C}),$$

the k-subdirect sum of A and B, which we denote by $A \oplus_k B$.

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* Speaker.

This definition is the generalization of direct sum and has some applications in various problems such as matrix completion problems.

An interesting problem about subdirect sum of matrices is that if A, B belong to a special class of matrices, Does $A \oplus_k B$? Furthermore the converse of this problem is discussed. That is, if C belongs to a special class of matrices and k is a positive number, can we write C as the k-subdirect sum of two matrices in that class? For giving answer to the above questions, one should note that there is some difference between the case $k = 1$ and $k > 1$.

In [1], Fallat and Johnson consider some different class of matrices such as:

- positive (semi)-definite matrices (PD, (PSD));
- P-matrices (positive principal minors) (P);
- P_0-matrices (nonnegative principal minors) (P_0);
- M-matrices (nonpositive off-diagonal entries and positive principal minors) (M);
- totally nonnegative matrices (all minors nonnegative) (TN);
- completely positive matrices (matrices of the form BB^T with B entry-wise nonnegative) (CP);
- doubly nonnegative matrices (positive semidefinite and entry-wise nonnegative) (DN);
- symmetric M-matrices (SM).

It is well-known that the answers of the above questions for these classes of matrices are affirmative for direct sum of two matrices. Also for the case that $k = 1$, they are true. But if $k > 1$, then some of the questions have negative answers.

In this note, we focuss on the class of inverse-positive matrices and the answer of the above questions.

2. MAIN RESULTS

Definition 2.1. Let A be a non-singular real matrix. The matrix A is called an inverse-positive matrix if all the entries of A^{-1} are nonnegative.

The inverse-positivity is preserved by multiplication.

Berman and Plemmons [3] state some equivalent conditions for inverse-positivity. As an example, a matrix A is inverse positive if and only if for all vector x, the vector Ax has positive entries only if x has positive entries (i.e. A is monotone).

If A, B are two inverse-positive matrices, the subdirect sum $A \oplus_k B$ is not necessarily inverse-positive.

Example 2.2. Let

$$A = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ -1 & -1 & 6 & -4 \\ -1 & -1 & -1 & 1 \end{bmatrix} \qquad \& \qquad B = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 8 & -1 & 0 & 0 \\ -1 & -1 & -1 & 2 \\ -1 & -1 & 3 & -1 \end{bmatrix}$$

are inverse positive, but $A \oplus_2 B$ is not.

In 2011, Abad et.al. [2] proved that the 1-subdirect sum of inverse-positive matrices is inverse-positive and conversely, every inverse-positive matrix can be written as a 1-subdirect sum of two inverse-positive matrices.

Every non-singular M-matrix is an inverse-positive matrix. So by the results mentioned in introduction the k-subdirect sum of two non-singular M-matrix is M-matrix which is inverse-positive.

Now we summarize some other conditions under which this relation holds. Here we state the results for $k > 1$. See [2].

Theorem 2.3. *Let*

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{M}_n(\mathbb{C}) \quad \& \quad B = \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \in \mathbb{M}_m(\mathbb{C})$$

such that $A_{22}, B_{11} \in \mathbb{M}_k(\mathbb{C})$.

- *If* $H = A_{22}^{-1} + B_{11}^{-1}$ *is inverse-positive, then* $C = A \oplus_k B$ *is inverse positive.*
- *If* A_{21}, B_{21} *have non-positive entries and* $H' = A_{22} + B_{11}$ *is inverse-positive, then* $C = A \oplus_k B$ *is inverse-positive.*

Theorem 2.4. *Let*

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{M}_n(\mathbb{C}) \quad \& \quad B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} \in \mathbb{M}_m(\mathbb{C})$$

such that $A_{22}, B_{11} \in \mathbb{M}_k(\mathbb{C})$. *If* A_{21}, B_{12} *have non-positive entries and* $H' = A_{22} + B_{11}$ *is inverse-positive, then* $C = A \oplus_k B$ *is inverse-positive.*

For the converse, we have this theorem.

Theorem 2.5. *The inverse-positive matrices of the form*

$$C = \begin{bmatrix} C_{11} & 0 & 0 \\ C_{21} & C_{22} & 0 \\ 0 & C_{32} & C_{33} \end{bmatrix} \qquad \text{or} \qquad C = \begin{bmatrix} C_{11} & 0 & 0 \\ C_{21} & C_{22} & C_{23} \\ 0 & 0 & C_{33} \end{bmatrix},$$

where $C_{22} \in \mathbb{M}_k$ *can be expressed as a* k*-subdirect sum of two inverse-positive matrices.*

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SUBADDITIVITY INEQUALITIES FOR CONVEX FUNCTIONS OF OPERATORS

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Abstract. It is shown that for a continuous convex function f, the inequality $f(A) + f(B) \leq f(A + B)$ holds true for self-adjoint operators A and B under proper conditions. Some convexity inequalities for Hilbert space operators are also presented.

1. Introduction

Assume that $\mathbb{B}(\mathcal{H})$ is the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ and I denote the identity operator. If $\dim \mathcal{H} = n$, then we identify $\mathbb{B}(\mathcal{H})$ with the C^*-algebra $\mathcal{M}_n$ of all $n \times n$ matrices with complex entries.

If $f : [0, \infty) \to \mathbb{R}$ is a convex function with $f(0) = 0$, then

$$f(a) + f(b) \leq f(a + b) \tag{1.1}$$

holds for all positive scalars a, b. But in the case when a, b are replaced by positive matrices (operators), the inequality (1.1) may not hold, although f be operator convex. For example if $f(t) = t^2$ and A, B are the following two positive matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix},$$

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then there is no relationship between A^2+B^2 and $(A+B)^2$ under the operator order. Many authors tried to establish an operator version of inequality (1.1), see e.g. [4, 5].

A continuous real function $f: J \to \mathbb{R}$ is said to be operator convex if $f(tA+(1-t)B) \le tf(A)+(1-t)f(B)$ for all self-adjoint operators A, B with spectra in J and every $t \in [0,1]$. If $f: J \to \mathbb{R}$ is operator convex, then the so-called Choi–Davis–Jensen inequality $f(\Phi(A)) \le \Phi(f(A))$ holds for every unital positive linear map Φ on $\mathbb{B}(\mathcal{H})$. However, it is easy to see that this inequality may not hold if f is convex in the usual sense, see [?].

We give some conditions under which inequality (1.1) holds true for positive operators. Also we introduce a subset Ω of $\mathbb{B}(\mathcal{H}) \times \mathbb{B}(\mathcal{H})$ such that if f is convex, then $f(tA+(1-t)B) \le tf(A)+(1-t)f(B)$ for every $(A,B) \in \Omega$.

2. Main results

We start this section by the following operator extension of (1.1).

Theorem 2.1. *If $f:[0,\infty) \to \mathbb{R}$ is a continuous convex function with $f(0) \le 0$, then*

$$\sum_{i=1}^{n} f(A_i) \le f\left(\sum_{i=1}^{n} A_i\right) \tag{2.1}$$

for all positive operators A_i such that $A_i \le M \le \sum_{i=1}^{n} A_i \ \ (i=1,\dots,n)$ for some scalar $M \ge 0$. If f is concave, then a reverse inequality holds. In particular, if f is convex, then

$$f(A)+f(B) \le f(A+B) \tag{2.2}$$

for all positive operators A, B such that $A \le MI \le A+B$ and $B \le MI \le A+B$ for some scalar $M \ge 0$.

For every positive operator A, assume that $\widetilde{A} = \frac{1}{2} - \frac{1}{M-m}\left|A - \frac{m+M}{2}\right|$. If $f:[0,M] \to \mathbb{R}$ is convex, then the scalar $\delta_f = f(0)+f(M)-2f\left(\frac{M}{2}\right)$ is positive. With this notation, we give a refinement of (2.2).

Theorem 2.2. *if $f:[0,\infty) \to \mathbb{R}$ is convex and $f(0) \le 0$, then*

$$f(A)+f(B) \le f(A+B) - \delta_f \widetilde{X} \le f(A+B)$$

for all positive operators A, B such that $A \le MI \le A+B$ and $B \le MI \le A+B$ for some scalar $M \ge 0$, where

$$\widetilde{X} = 1 - \left|\frac{A}{M} - \frac{1}{2}\right| - \left|\frac{B}{M} - \frac{1}{2}\right|.$$

Next, we give some conditions under which the Choi–Davis–Jensen inequality holds true for continuous convex functions. Let the subset Ω of $\mathbb{B}(\mathcal{H}) \times \mathbb{B}(\mathcal{H})$ be defined [2] by

$$\Omega = \left\{ (A, B) \mid A \leq m \leq \frac{A+B}{2} \leq M \leq B, \quad \text{for some} \quad m, M \in \mathbb{R} \right\}.$$

Theorem 2.3. [3] *Let f be a continuous convex function on an interval J containing m, M. Let Φ_i, $i = 1, \ldots, n$, be positive linear mappings on $\mathbb{B}(\mathcal{H})$ with $\sum_{i=1}^n \Phi_i(I) = I$. If $(A_i, B_i) \in \Omega$, $i = 1, \ldots, n$, then*

$$f\left(\sum_{i=1}^n \Phi_i \left(\frac{A_i + B_i}{2} \right) \right) \leq \sum_{i=1}^n \Phi_i \left(\frac{f(A_i) + f(B_i)}{2} \right). \tag{2.3}$$

If f is concave, then (2.3) is reversed.

As a corollary, we give a converse of (2.3).

Corollary 2.4. *under the same assumptions as in the Theorem 2.3,*

$$\sum_{i=1}^n \Phi_i \left(f\left(\frac{A_i + D_i}{2} \right) \right) \leq \frac{1}{2} \left[f\left(\sum_{i=1}^n \Phi_i(A_i) \right) + f\left(\sum_{i=1}^n \Phi_i(D_i) \right) \right].$$

Corollary 2.5. *If f is a convex function on an interval J containing m, M, then*

$$f(tA + (1-t)B) \leq tf(A) + (1-t)f(B) \tag{2.4}$$

for all $(A, B) \in \Omega$ and all $t \in [0, 1]$. If f is concave, then inequality (2.4) is reversed.

Note that in Theorem 2.3, Corollary 2.4 and Corollary 2.5, the function f need not to be operator convex.

Corollary 2.6. *Let $A, B, C, D \in \mathbb{B}(\mathcal{H})$ be such that $I \leq A \leq m \leq B, C \leq M \leq D$ for two real numbers $m < M$. If one of the following conditions*

$$\begin{array}{ll} \text{(i)} & B + C \leq A + D \quad \textit{and} \quad p \geq 1 \\ \text{(ii)} & A + D \leq B + C \quad \textit{and} \quad p \leq 0 \end{array}$$

is satisfied, then

$$B^p + C^p \leq A^q + D^q$$

for each $q \geq p$.

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NEAREST MATRIX WITH k PRESCRIBED DISTINCT EIGENVALUES

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Abstract. For an $n \times n$ matrix A and a set of given complex numbers $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ such that $k \leq n$, we compute a spectral norm distance from A to the set of matrices that Λ included in their spectrum. Also construction of optimal perturbation, Δ, to matrix A such that spectrum of $A + \Delta$ consist of Λ is considered.

1. Introduction

Let A be a complex $n \times n$ matrix and let L be the set of complex $n \times n$ matrices with a multiple zero eigenvalue. In 1999, Malyshev has computed the spectral norm distance from A to L as following [4]

$$\rho_2(A, L) = \min_{B \in L} \|A - B\|_2 = \max_{\gamma \geq 0} s_{2n-1}\left(\begin{bmatrix} A & \gamma I_n \\ 0 & A \end{bmatrix}\right),$$

where s_i is the ith singular value of the corresponding matrix that ordered in nonincreasing order. Also he obtain a matrix Δ, such that $A + \Delta \in L$ and Δ is the optimal perturbation of A. The spectral norm distance from an $n \times n$ matrix A to the set of matrices that have two prescribed eigenvalues and nearest matrix with two prescribed eigenvalues were studied by Gracia [2]. In 2003, Ikramov and Nazari [3] obtain a spectral norm distance from A to the set of matrices that have

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triple zero eigenvalue. Mengi [5] extends the Malyshev formula and finds nearest matrix to A with an eigenvalue of prespecified algebraic multiplicity. In this paper we study the norm distance from an $n \times n$ matrix A to the set $\mathcal{M}_k$ of matrices that have $k \leq n$ perspecified distinct eigenvalues. Also the optimal perturbation of A with minimum norm is constructed.

Suppose that a set of complex numbers $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ such that $k \leq n$ and an $n \times n$ matrix A, are given. For

$$\gamma = [\gamma_{1,1}, \gamma_{1,2}, \cdots, \gamma_{1,k-1}, \gamma_{2,1}, \gamma_{2,2}, \cdots \gamma_{2,k-2}, \cdots, \gamma_{k-1,1}] \in \mathbb{C}^{\frac{k(k-1)}{2}}$$

define the $nk \times nk$ upper triangular matrix $Q(\gamma)$ as

$$Q(\gamma) = \begin{bmatrix} B_1 & \gamma_{1,1} I_n & \cdots & \gamma_{1,k-1} I_n \\ 0 & B_2 & & \gamma_{2,k-2} I_n \\ \vdots & 0 & \ddots & \gamma_{k-1,1} I_n \\ 0 & \cdots & 0 & B_k \end{bmatrix},$$

where $B_i = A - \lambda_i I_n,\ (i = 1 \ldots k)$, and $\gamma_{i,1}, (i = 1, \ldots, k-1)$ are pure real variables.

It is clear that $Q(\gamma)$ can be assume as a matrix function of variables $\gamma_{i,j}$ such that $i = 1 \ldots k-1,\ j = 1 \ldots k-i$. Also hereafter, for simplicity the positive integer $nk - (k-1)$ will be denoted by κ.

If we denote the norm distance from an $n \times n$ matrix A to the set $\mathcal{M}_k$ by $\rho_2(A, \mathcal{M}_k)$, then is proved that

$$\rho_2(A, \mathcal{M}_k) = \min_{M \in \mathcal{M}_k} \|A - M\|_2 = s_\kappa \left(Q(\gamma^*)\right),$$

where γ^* is a point that the singular value $s_\kappa\left(Q(\gamma)\right)$ attains its maximum value, such that $\prod_{i=1}^{k-1} \gamma^*_{i,1} \neq 0$.

2. Properties of $s_\kappa\left(Q(\gamma)\right)$ and its corresponding singular vectors

In this section some properties of $s_\kappa\left(Q(\gamma)\right)$ and its corresponding singular vectors are studied. This properties are applied in next section to obtain an optimal perturbation and the minimum norm distance from A to $\mathcal{M}_k$. In this section, some definitions and lemmas of [1, 2, 3] are applied and rewritten but for our purpose.

Lemma 2.1. *If $A \in \mathbb{C}^{n\times n}$ has $\lambda_1, \lambda_2, \cdots, \lambda_k$ as some of its eigenvalues, then $s_\kappa\left(Q(\gamma)\right) = 0$.*

Definition 2.2. Let γ_* be a point where the singular value $s_\kappa\left(Q(\gamma)\right)$ attains its maximum value (if any). If $\gamma = \gamma_*$, then we set

$$s_* = \max_{\gamma} s_\kappa\left(Q(\gamma)\right) = s_\kappa\left(Q(\gamma_*)\right),$$

If $s_* = 0$, then we can see that A has $\lambda_1, \lambda_2, \cdots, \lambda_k$ as some of its eigenvalues. So, in remainder of paper we assume that $s_* > 0$.

Definition 2.3. Let vectors $u(\gamma) = \begin{bmatrix} u_1(\gamma) \\ \vdots \\ u_k(\gamma) \end{bmatrix}, v(\gamma) = \begin{bmatrix} v_1(\gamma) \\ \vdots \\ v_k(\gamma) \end{bmatrix} \in \mathbb{C}^{nk}(u_j(\gamma), v_j(\gamma) \in \mathbb{C}^n, j = 1, \cdots, k)$ be a pair of left and right singular vectors of $s_\kappa\left(Q(\gamma)\right)$ respectively, (for some γ). We define the two $n \times k$ matrices

$$U(\gamma) = [u_1(\gamma), \cdots, u_k(\gamma)], \qquad \text{and} \qquad V(\gamma) = [v_1(\gamma), \cdots, v_k(\gamma)].$$

Without less than generality we can assume that $u(\gamma)$ and $v(\gamma)$ are unit vectors. Also note that $\gamma_{i,1}, (i = 1, \ldots, k-1)$ are real variables, while other components of the vector γ are complex variables. Thus, if for $j > 1$, we set $\gamma_{i,j} = \gamma_{i,j,R} + i\gamma_{i,j,I}$ then we can assume that γ has $(k-1)^2$ real components.

Next lemma obtained by applying the Lemma 2.10 of [5] for $Q(\gamma)$ (see also [4], Lemma 5).

Lemma 2.4. *Let γ_* and s_* are such as defined in Definition 2.2 and let $\gamma_{*i,1}, (i = 1, \ldots, k-1)$ are real nonzero variables. Then there exist a pair $u(\gamma_*) = \begin{bmatrix} u_1(\gamma_*) \\ \vdots \\ u_k(\gamma_*) \end{bmatrix}, v(\gamma_*) = \begin{bmatrix} v_1(\gamma_*) \\ \vdots \\ v_k(\gamma_*) \end{bmatrix} \in \mathbb{C}^{nk}(u_j(\gamma_*), v_j(\gamma_*) \in \mathbb{C}^n, j = 1, \cdots, k)$ of left and right singular vectors of s_* respectively, that verifies following relations*

1. $u_i^*(\gamma_*)v_{i+j}(\gamma_*) = 0, \qquad i = 1, \cdots k-1, \; j = 1, \cdots, k-i,$ *and*
2. $u_i^*(\gamma_*)u_j(\gamma_*) = v_i^*(\gamma_*)v_j(\gamma_*), \qquad i, j = 1, \ldots, k.$

Corollary 2.5. *For matrices $U(\gamma_*)$ and $V(\gamma_*)$, we have*

$$U^*(\gamma_*)U(\gamma_*) = V^*(\gamma_*)V(\gamma_*).$$

This equation implies that $U(\gamma_*)$ and $V(\gamma_*)$ have the same nonzero singular values and the same associated right singular vectors. Thus, there exists an $n \times n$ unitary matrix W such that $U(\gamma_*) = WV(\gamma_*)$. Consequently if $V(\gamma_*)$ is full rank matrix, then

$$\left\|U(\gamma_*)V^\dagger(\gamma_*)\right\|_2 = \left\|WV(\gamma_*)V^\dagger(\gamma_*)\right\|_2 = \|W\|_2 = 1.$$

Lemma 2.6. *The two matrices $U(\gamma_*)$ and $V(\gamma_*)$ are full rank.*

3. Finding $\rho_2(A, \mathcal{M}_k)$ and construction of perturbation

According to the Lemma 2.1 and considering the Weyl inequalities for singular values (for example, see [1, Corollary 5.1]), a lower bound for minimum norm of perturbation is obtained in next lemma.

Lemma 3.1. *Let an $n \times n$ matrix A and a set of complex numbers $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ such that $k \leq n$, are given. If Δ is a matrix such that $A + \Delta \in \mathcal{M}_k$, then*

$$\|\Delta\|_2 \geq s_\kappa \left(Q(\gamma) \right).$$

Finally in the next lemma, optimal perturbation Δ under the condition $A + \Delta \in \mathcal{M}_k$ is constructed.

Lemma 3.2. *Let γ_* be a point where the singular value $s_\kappa \left(Q(\gamma) \right)$ attains its maximum value, such that $\prod_{i=1}^{k-1} \gamma_{i,1}^* \neq 0$. Suppose that $U(\gamma_*)$ and $V(\gamma_*)$ be as in Corollary 2.5 and define*

$$\Delta = -s_* U(\gamma_*) V^\dagger(\gamma_*),$$

where $V(\gamma_*)^\dagger$ is the Moore-Penrose pseudoinverse of $V(\gamma_*)$. Then for the distance $\rho_2(A, \mathcal{M}_k)$ we have $\|\Delta\|_2 = s_* \left(= \rho_2(A, \mathcal{M}_k)\right)$, and also $A + \Delta \in \mathcal{M}_k$.

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STAGNATION OF THE ITERATIVE METHOD GMRES

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Abstract. In this note we characterize all $n \times n$ nonsingular normal matrices A such that GMRES(A,b) stagnates completely for some vector b. A necessary and sufficient condition for the nonexistence of real stagnation vector for real normal matrices was studied. Moreover, we characterize all the eigenvalues of nonsingular normal matrices $A \in M_3(\mathbb{C})$ such that GMRES(A,b) stagnates completely for some $b \in \mathbb{C}^3$.

1. Main Results

The iterative method GMRES is a Krylov method based on the Arnoldi orthogonalization process for solving the system of linear equations $Ax = b$. The initial residual is $r_0 = b - Ax_0$, where x_0 is the starting vector and the Krylov subspace of order m based on A and r_0 is $\mathbf{K}_m(A, r_0) = \text{span}\{r_0, Ar_0, ..., A^{m-1}r_0\}$. The approximate solution x_m at iteration m is of the form $x_m \in x_0 + \mathbf{K}_m(A, r_0)$, and minimizes the norm of the residual vector $r_m = b - Ax_m$. It is readily seen that the residual norms are monotonically nonincreasing and the GMRES method terminates with the exact solution in at most n iterations. In this paper we study the problem of stagnation of the GMRES algorithm. For a given matrix A and right hand side vector b, we say there is partial stagnation if, for some $1 \leq m \leq n-1$,

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$\|b\| = \|r_0\| = \cdots = \|r_m\|$. In other words, the norms of the first m residuals are equal. In this case we say that GMRES(A,b) has partial stagnation of order m. Complete stagnation corresponds to $m = n-1$, so we have $\|b\| = \|r_0\| = \cdots = \|r_{n-1}\| > \|r_n\| = 0$. In this case we say that GMRES(A,b) stagnates completely and the degree of the minimal polynomial of A equals n. A vector $b \in \mathbb{R}^n$(or $\mathbb{C}^n$) is called a real (or complex) stagnation vector for $A \in M_n(\mathbb{C})$, if $\|b\| = 1$ and GMRES(A,b) stagnates completely. Note that, for normal matrix A, the GMRES completely stagnates only if A has distinct eigenvalues since otherwise the minimal polynomial has degree less than n. Complete stagnation of GMRES has been studied in [2, 3, 5]. Motivated by the study of convergence of iterative methods for solving linear systems, O. Nevanlinna in [4] introduced the notion of the polynomial numerical hull of order m of a matrix $A \in M_n(\mathbb{C})$, which is defined and denoted by $V^m(A) = \{\xi \in \mathbb{C} : |p(\xi)| \leq \|p(A)\| \text{ for all } p(z) \in \Pi_m[\mathbb{C}]\}$, where $\Pi_m[\mathbb{C}]$ is the set of complex polynomials with degree at most m (for details see [1]).

2. Main results

Let $A \in M_n(\mathbb{C})$ be a nonsingular normal matrix. It is clear that the origin is not in the numerical range of A if and only if there exists a unit vector b such that $GMRES(A, b)$ has a partial stagnation of order 1. The joint numerical range for $(A_1, A_2, \dots, A_m) \in M_n(\mathbb{C}) \times \cdots \times M_n(\mathbb{C})$ is defined and denoted by $W(A_1, \dots, A_m) := \{(x^*A_1x, \dots, x^*A_mx) : x \in \mathbb{C}^n, x^*x = 1\}$. Now, we state a characterization of partial stagnation of GMRES.

Lemma 2.1. *Suppose that $A \in M_n(\mathbb{C})$ is a nonsingular matrix. Then there is a partial stagnation of GMRES of order m if and only if $(0, \dots, 0) \in W(A, A^2, \dots, A^m)$.*

By the above lemma we obtain the following.

Proposition 2.2. *Suppose that $A \in M_n(\mathbb{C})$ is a nonsingular normal matrix. Then there exists a vector $b \in \mathbb{C}^n$ such that GMRES(A,b) has a partial stagnation of order m if and only if $0 \in V^m(A)$.*

There exists a vector $b \in \mathbb{C}^n$ such that GMRES(A,b) has a partial stagnation of order m if and only if the vector b is orthogonal to the subspace $A\mathbf{K}_m(A, b)$, or equivalently, $b^*A^jb = 0,\ j = 1, \dots, m$. Then, $0 \in V^m(A)$. □

Theorem 2.3. *Let $A \in M_n(\mathbb{C})$ be a nonsingular normal matrix and $n \geq 3$. Then the following assertions are equivalent:*

(i) $V^{n-1}(A) = \sigma(A) \cup \{0\}$.
(ii) There is a vector $b \in \mathbb{C}^n$ *such that* $GMRES(A, b)$ *stagnates completely.*

In the following example we show that the condition $n \geq 3$ is necessary in Theorem 2.3.

Example 2.4. Let $A = \mathrm{diag}(-1, 1) \in M_2(\mathbb{R})$. It is clear that $V^1(A) = [-1, 1] \neq \{-1, 0, 1\} = \sigma(A) \cup \{0\}$. But $(1/\sqrt{2}, 1/\sqrt{2})A(1/\sqrt{2}, 1/\sqrt{2})^T = 0$. This means that there exists a vector $b \in \mathbb{C}^n$ such that $GMRES(A, b)$ stagnates completely.

Theorem 2.5. *Let* $A \in M_n(\mathbb{R})$ *be a normal matrix. Then the matrix* A *has a complex stagnation vector if and only if* A *has a real stagnation vector.*

Now, by using the results of G. Meurant in [3, Theorem 2.2], we state a necessary and sufficient condition for real stagnation vector of normal matrices $A \in M_n(\mathbb{R})$.

Theorem 2.6. *Let* $A \in M_n(\mathbb{R})$ *be a normal matrix and* $n \geq 2$. *Then the following assertions are equivalent:*
(i) There is no real vector $b \in \mathbb{R}^n$ *such that GMRES(A,b) stagnates completely.*
(ii) There exist real numbers $a_1, \ldots, a_{n-1}$ *such that* $a_1A_1 + \cdots + a_{n-1}A_{n-1}$ *is definite, where* $A_i = A^i + (A^i)^T,\ i = 1, \ldots, n-1$.

In the case $n \leq 4$, the above Theorem is true for any matrix $A \in M_n(\mathbb{R})$. But an open and interesting question is to prove or disprove $(i) \to (ii)$ for any matrix $A \in M_n(\mathbb{R}), n > 4$, see [3]. In the following example, we will show that there exists a nonsingular normal matrix $A \in M_n(\mathbb{R})$ with distinct eigenvalues such that A has infinitely many real stagnation vectors.

Example 2.7. Let $A = A_{11} \oplus A_{22} \in M_4(\mathbb{R})$, where $A_{11} = \left[\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right], A_{22} = \left[\begin{smallmatrix} -1 & 1 \\ -1 & -1 \end{smallmatrix}\right]$ and let $S = \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1/2\}$. Easy computation shows that $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T A^2\mathbf{x} = \mathbf{x}^T A^3\mathbf{x} = 0, \forall \mathbf{x} \in S$. Hence A has infinitely many real stagnation vectors.

We now characterize the eigenvalues of all nonsingular normal matrices $A \in M_3(\mathbb{C})$ such that GMRES(A,b) stagnates completely for some $b \in \mathbb{C}^3$.

Theorem 2.8. *Suppose that* $A \in M_3(\mathbb{C})$ *is a nonsingular normal matrix. Then the following assertions are equivalent:*

(i) *There exists a vector* $b \in \mathbb{C}^3$ *such that GMRES(A,b) stagnates completely.*
(ii) $\sigma(A) = \{e^{i\theta}(\alpha - i\eta), e^{i(\theta+\pi)}(\beta + i\eta), \gamma e^{i(\theta+\pi/2)}\}$*, where* $\alpha, \beta, \gamma, \theta$ *are positive numbers and* $\eta = \frac{-\gamma+\sqrt{\gamma^2+4\alpha\beta}}{2}$.

Now, we state a characterization for all nonsingular normal matrices $A \in M_3(\mathbb{R})$ such that GMRES(A,b) stagnates completely for some $b \in \mathbb{R}^3$.

Proposition 2.9. *Suppose that* $A \in M_3(\mathbb{R})$ *is a nonsingular, normal matrix with distinct eigenvalues. Then there exists a real stagnation vector* b *if and only if there exists a real orthogonal matrix* W *such that* $We_n = b$ *and* $A = WBW^T$*, where* B *has one of the following forms* $B_1 = \begin{bmatrix} s & s & r \\ \delta & s & 0 \\ 0 & r & 0 \end{bmatrix}$ *or* $B_2 = \begin{bmatrix} s & -s & r \\ \delta & s & 0 \\ 0 & -r & 0 \end{bmatrix}$, $r, s \in \mathbb{R}$, $r \neq 0$ *and* $\delta = \pm\sqrt{s^2 + r^2}$.

Remark 2.10. It would be nice to characterize all the eigenvalues of those (normal) matrices $A \in M_n(\mathbb{C})$, $n \geq 4$, such that GMRES(A,b) stagnates completely for some vector $b \in \mathbb{C}^n$.

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TOTALLY POSITIVE MATRICES AND TOTALLY POSITIVE PRESERVING FUNCTIONS

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Abstract. An $m \times n$ matrix A is called totally positive (TP) if all its minors of all sizes are positive. Totally positive matrices have been studies in great detail and occur in many applications. For example there is an interesting connection between totally positivity and canonical basis for quantum groups. A a real-valued function f is a tatally positive preserving function if $f(A)$ is totally positive whenever A is an $m \times n$ totally positive matrix . We will introduce totally positive matrices and will show that under some conditions the function $f(t) = t^{\alpha}$ is a totally positive preserving function.

1. Introduction

Remarkable properties of square matrices with positive enteries were discovered by Perron in 1907. Frobenius later generalized this work by extending Perron's results to non-negative matrices [5]. The class of totally positive matrices arises in a variety of important applications. For example, in statistics, mathematica biology, combiatories, dynamics, approximation theory, operator theory, and geometry [3]. Since there has been a considerable amount of work accomplished on totally positivity, some of which is contained in the exaceptional survey paper

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by T. Ando [1]. One more recent paper [2] take the point of view of bidiagonal factorization of totally nonegative matrices.

2. Totally positive matrices

The set of all m-by-n matrices over field $\mathbb{F}$ will be denoted by $M_{m,n}(\mathbb{F})$. If $\mathbb{F}=\mathbb{R}$ the set of real numbers, then we will use the notation $M_{m,n}$. For matrix $A \in M_{m,n}(\mathbb{F})$, $\alpha \subseteq \{1,2,3,\dots,m\}$ and $\beta \subseteq \{1,2,3,\dots,n\}$, $A[\alpha\ ,\ \beta]$ will denote the submatrix of A lying in rows indexed by α and columns indexed by β. $A(\alpha\ ,\ \beta)$ will denote the submatrix of A delating the rows indexed by α and columns indexed by β. That is

$$A(\alpha\ ,\ \beta) = A[\alpha^c\ ,\ \beta^c].$$

If $\alpha=\beta$, then $A[\alpha]$ is called the **principal** submatrix of A. $A(\alpha)$ is cllaed **complementary** submatrix of A. $\det A[\alpha\ ,\ \beta]$ where $|\alpha|=|\beta|$ is called the **minor** of A.

Definition 2.1. A matrix $A \in M_{m,n}$ is called Totally Positive(TP) or Totally Nonnegative(TNN) if all minors are positive or nonnegative.

Example 2.2. Matrices $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 8 \end{pmatrix}$ are TP. Vandermond matrix, Cauchy matrix, and Pascal matrix are totally positive.

Theorem 2.3. *Let $A \in M_n$ ba a TP matrix. Then A has n positive distinct eigenvalues.*

Theorem 2.4. *Let $A \in M_n$ ba a TP matrix. Suppose that the eigenvalues of A are given by $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$ and let eigenvalues of $A(k)$ where $1 \le k \le n$ be $\mu_1 > \mu_2 > \cdots > \mu_{n-1} > 0$ then*

$$\lambda_{j-1} > \mu_j > \lambda_{j+1} \qquad , \qquad \lambda_0 = \lambda_1.$$

Theorem 2.5. *If $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$, then there is an n-by-n TP matrix A whose eigenvalues are $\lambda_1, \lambda_2, \cdots, \lambda_n > 0$.*

One can see the proof of above theorems in [3].

3. Totally positive preserving functions

Definition 3.1. The *Hadamard(Shur) product* of two $m \times n$ matrices $A=[a_{ij}]$ and $B=[b_{ij}]$ is defined and denoted by $A \circ B = [a_{ij}b_{ij}]$. *The Hadamard power* of A is defined by $A^{(t)} = [a_{ij}^t]$. Let f be a real function. Then the Hadamard matrix fuction $f(A)$ is defined $f(A) = [f(a_{ij})]$.

We are interested in characterizing positive functions that preserve totally positivity that is $f(A)$ is totally positive whenever A is totally positive. We try to test the function $f(t) = t^\alpha$. It is clear that if A is an $m \times n$ totally positive matrix where $m = 2$ or $n = 2$, then the Hadamard power $A^{(\alpha)}$ is totally positive for every $\alpha \geq 0$.

It was shown that if $A \in M_n(\mathbb{R})$ is a positive semi-definite matrix with nonnegative entries, then $A^{(\alpha)}$ is positive semi-definite for every $\alpha \geq n-2$. Moreover, the lower bound is sharp[4]. We will show that the same result does not hold for a totally positive $m \times n$ matrix A when m and n are greater than 3.

Theorem 3.2. *Let A be a totally positive $m \times n$ matrix where $m = 3$ or $n = 3$. Then the Hadamard power $A^{(t)}$ is totally positive for $t \geq 1$.*

The following examples show that $n-2$ is not the lower bound in general,

Example 3.3. Let

$$A = \begin{pmatrix} 1 & 11 & 22 & 20 \\ 6 & 67 & 139 & 140 \\ 16 & 182 & 395 & 445 \\ 12 & 138 & 309 & 375 \end{pmatrix}.$$

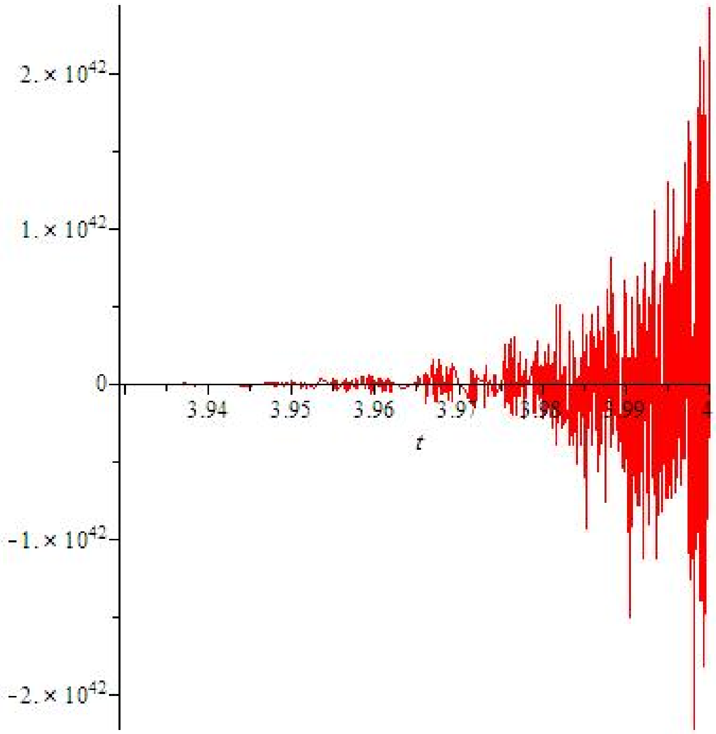
Then A is TP but the graph for the function $f(t) = \det A^{(t)}$ shows that $A^{(t)}$ is not totally positive for every $t \geq 2$.

Proposition 3.4. *Let A be a totally positive and symmetric 4×4 matrix. Then $A^{(t)}$ is TP for $t \geq 2$.*

Example 3.5. Let

$$A = \begin{pmatrix} 31626 & 453445 & 3758022 & 16959141 & 12065898 \\ 453445 & 6502215 & 53893656 & 243222036 & 173052633 \\ 3758022 & 53893656 & 446729566 & 2016161339 & 1434547607 \\ 16959141 & 243222036 & 2016161339 & 9099415347 & 6474561371 \\ 12065898 & 173052633 & 1434547607 & 6474561371 & 4606954729 \end{pmatrix}.$$

A is a totally positive symmetric matrix. We draw the following graph for the function $f(t) = \det A^{(t)}$. As we see $f(t)$ is negative from some $t \geq 3$ that means A is not totally positive for every $t \geq 3$.



From Theorem 6.4 in [1] we understand $A^{(\alpha)}$ is totally positive for sufficiently large α. It will be interesting if we could find a sharp lower bound for α.

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JOINT HIGHER RANK NUMERICAL RANGE OF THREE PAULI MATRICES

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Abstract. Pauli channel is a central class of quantum operations in quantum computing. For a noisy quantum channel, a quantum error correcting code of dimension k exists if and only if the joint rank-k numerical range associated with the error operators of the channel is non-empty. In the present paper, joint higher rank numerical range of three Pauli matrices are completely characterized.

1. Introduction

Quantum computation is a fast-growing and a multi-disciplinary research field. If large-scale quantum computers can be manufactured, they will be able to solve certain problems much faster than any current classical ones. For example, Shor's algorithm can be used to break public-key cryptography schemes such as the widely used RSA scheme. One of the biggest hurdles faced by quantum computing researchers is called "decoherence" (i.e. the tendency of quantum systems to be disturbed). There are several proposals to resolve this. Quantum error correction codes (QECC) along with the celebrated Knill-Laflamme (KL) conditions [3] totally form one of the most promising candidates to

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* Speaker.

suppress environmental noise, which leads to decoherence [2]. KL conditions assert that given a quantum channel $\Phi : M_n \to M_n$ with error operators $\{E_i\}_{1\leq i,j\leq r}$, subspace $\mathcal{V} \subset \mathbb{C}^n$ is a QECC of Φ if and only if there exist scalars λ_{ij} and orghogonal projection operator $P \in M_n$with the range space $\mathcal{V}$ such that $PE_i^*E_jP = \lambda_{ij}P$.

KL conditions prompted the Choi et.al. [1] to introduce the notion of (joint) higher rank numerical range of an ($m-$tuple of) operator(s). For $1 \leq k < n$ joint rank-k numerical range of $A_1, \ldots, A_m \in M_n$ is defined by

$$\begin{array}{l}\Lambda_k\left(A_1,\ldots,A_m\right) = \\ \left\{(\lambda_1,\ldots,\lambda_m) : \exists U \in M_{k,n}, UU^* = I_k, \forall_j, UA_jU^* = \lambda_j I_k\right\}.\end{array}$$

In fact, there is a QECC of dimension k for quantum channel Φ described above if and only if $\Lambda_k\left(A_1,\ldots,A_m\right)$ is non-empty for

$$\left(A_1,\ldots,A_m\right) = \left(E_1^*E_1, E_1^*E_2, \ldots, E_r^*E_r\right).$$

There are many open problems for joint higher rank numerical range. In the 9th Workshop on "Numerical Ranges and Numerical Radii", researchers presented some open problems [4] about this set, showing their interest in characterizing it for Pauli matrices, which are matrices in the form $P_1 \otimes P_2 \otimes \ldots \otimes P_m$ where

$$m \in \mathbb{N}, \{P_1,\ldots,P_m\} \subset \{I, X, Y, Z\},$$

and

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Li et.al. [5] recently studied emptiness of

$$\Lambda_k\left(\overset{n}{\underset{i=1}{\otimes}} X, \overset{n}{\underset{i=1}{\otimes}} Y, \overset{n}{\underset{i=1}{\otimes}} Z\right).$$

But that problem is still open. The main purpose of the present paper is to determine joint higher rank numerical range of three Pauli matrices.

Throughout our paper, the following notations will be fixed:
Let $n > 2$ be a power of 2, and I_n be Identity matrix of size $n-by-n$. Define

$$\begin{array}{l} I_1 = [1], I = I_2, \\ \mathcal{P}_1 = \{I_1\}, \\ \mathcal{P}_n = \left\{\begin{array}{c} H \in M_n : H^* = H, \sigma(H) = \{-1,1\}, algebraic\ multiplicity \\ of\ each\ eigenvalue\ of H \text{ is } \frac{n}{2} \end{array}\right\}. \end{array}$$

Also consider that $P, Q \in \mathcal{P}_n, P \neq \pm Q$ and define

$$\begin{aligned} S_I(P,Q) &= \{R \in \mathcal{P}_n : RP = PR, RQ = QR, \pm R \notin \{P,Q\}\}, \\ S_Z(P,Q) &= \{R \in \mathcal{P}_n : RP = PR, RQ = -QR, R \neq \pm P\}, \\ S_Y(P,Q) &= \{R \in \mathcal{P}_n : RP = -PR, RQ = -QR\}. \end{aligned}$$

2. Main results

Theorem 2.1. *Let $n > 2$ and $k \leq \frac{n}{2}$ be positive integers and $P, Q \in \mathcal{P}_n$ be Pauli matrices such that $P \neq \pm Q$.*

(a) *If $PQ = -QP$. Then*

$$\Lambda_k(P,Q) = \left\{(x,z) : x, z \in \mathbb{R}, x^2 + z^2 \leq 1\right\}.$$

(b) *Let $PQ = -QP, R \in S_I(P,Q)$. If $k \leq \frac{n}{4}$, then*

$$\Lambda_k(P,Q,R) = \left\{(x,y,z) : x,y,z \in \mathbb{R}, x^2 + y^2 \leq 1, z \in [-1,1]\right\}.$$

Otherwise

$$\Lambda_k(P,Q,R) = \{(0,0,0)\}.$$

(c) *Let $PQ = -QP, R \in S_Z(P,Q)$. If $k \leq \frac{n}{4}$, then*

$$\Lambda_k(P,Q,R) = \left\{(x,y,z) : x,y,z \in \mathbb{R}, y^2 + \max\left\{x^2, z^2\right\} \leq 1\right\}.$$

Otherwise

$$\Lambda_k(P,Q,R) = \{(0,y,0) : y \in [-1,1]\}.$$

(d) *Let $PQ = -QP, R \in S_Y(P,Q)$. Then*

$$\Lambda_k(P,Q,R) = \left\{(x,y,z) : x,y,z \in \mathbb{R}, x^2 + y^2 + z^2 \leq 1\right\}.$$

(e) *Let $P, Q \in \mathcal{P}_n$ be Pauli matrices and $PQ = QP$. If $k \leq \frac{n}{4}$, then*

$$\Lambda_k(P,Q) = \{(x,y) : x,y \in \mathbb{R}, max(|x|,|y|) \leq 1\}.$$

Otherwise

$$\Lambda_k(P,Q) = \{(0,0)\}.$$

(f) *Let $P_1, P_2, P_3 \in \mathcal{P}_n$ be Pauli matrices such that $P_1P_2P_3 = \pm I_n$ and for any $i, j \in \{1,2,3\}$,*

$$P_iP_j = P_jP_i, \pm P_i \notin \{P_1,P_2,P_3\} \setminus \{P_i\}.$$

If $k \leq \frac{n}{4}$, then

$$\Lambda_k(P_1,P_2,P_3) =$$
$$\{(x,y,\pm z) : x,y,z \in \mathbb{R}, x+y+z \in [-1, 1 + 2\min\{x,y,z\}]\}.$$

Otherwise

$$\Lambda_k(P_1,P_2,P_3) = \emptyset.$$

(g) *Let $P_1, P_2, P_3 \in \mathcal{P}_n$ be Pauli matrices such that $P_1P_2P_3 \neq \pm I_n$ and for any $i, j \in \{1, 2, 3\}$,*

$$P_iP_j = P_jP_i, \pm P_i \notin \{P_1, P_2, P_3\} \setminus \{P_i\}.$$

If $k \leq \frac{n}{8}$, then

$$\Lambda_k (P_1, P_2, P_3) = \{(x, y, z) : x, y, z \in \mathbb{R}, max (|x|, |y|, |z|) \leq 1\}.$$

If $k \in \left[\frac{n}{8} + 1, \frac{n}{4}\right]$, then

$$\Lambda_k (P_1, P_2, P_3) = \{(x, y, z) : x, y, z \in \mathbb{R}, |x| + |y| + |z| \leq 1\}.$$

Otherwise

$$\Lambda_k (P_1, P_2, P_3) = \{(0, 0, 0)\}.$$

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ROOT-APPROXIMABILITY ON AUTOMORPHISMS OF THE UNIT BALL OF $\mathfrak{B}(\mathcal{H})^n$

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Abstract. In this paper we study the root-approximable of free holomorphic function on

$[\mathfrak{B}(\mathcal{H})^n_1] = \{(X_1, \cdots, X_n) \in \mathfrak{B}(\mathcal{H})^n \ : \ \|X_1X_1^* + \cdots + X_nX_n^*\| < 1\}$,

where $\mathfrak{B}(\mathcal{H})$ is the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$.

1. Introduction

In [1] a generalization of Jensen and Bernstein-Doetsch theorem has been proved for the nonlinear structure of a topological group. For this end, the authors introduced the notion of a root-approximable group.

Definition 1.1. Let G be a group endowed with a topology τ. An element $x \in G$ is said to be root-approximable if there exists a sequence (y_n) in G such that

$$\lim_{n\to\infty} y_n = e \ , \ y_n^{2^n} = x.$$

For all $n \in \mathbb{N}$, where e is the identity element of G. The topological group G is called root-approximable if every element of G is root-approximable.

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Let $\Omega \subset G$ be an open set, a function $f : \Omega \longrightarrow \mathbb{R}$ is said to be midconvex on Ω, if each $a \in \Omega$ and ay, ay^{-1} in Ω for a y in G,

$$2f(a) \leq f(ay) + f(ay^{-1}).$$

The authors in [1] presented a generalization of Bernstein-Doetsch theorem a midcovex real-valued function in an open subset of a root-approximable topological group which is bounded from above in some neighborhood of a point is everywhere continuous. Many classical group, but not all are root-approximable. For instance $\mathrm{Gl_n}(\mathbb{C})$ has this property while $\mathrm{Gl_n}(\mathbb{R})$ has not. In [2] we showed that the $\mathrm{Aut}(\mathbb{B}_n)$ is root-approximable group.
If $L = [a_{ij}]_{n\times n}$ is a bounded linear operator on $\mathbb{C}^n$, it generates a free holomorphic function on $[\mathfrak{B}(\mathcal{H})_1^n]$ by setting

$$\Phi_L(X_1, \ldots, X_n) = [X_1, \ldots, X_n]L = \left[\sum_{i=1}^{n} a_{i1}X_i, \ldots, \sum_{i=1}^{n} a_{in}X_i\right]$$

where $L = [a_{ij}I_{\mathcal{H}}]_{n\times n}$ by abuse of notation we also write $\Phi_L(X) = XL$. The automorphism group of $[\mathfrak{B}(\mathcal{H})_1^n]$ denoted by $\mathrm{Aut}(\mathfrak{B}(\mathcal{H})_1^n)$, consists of all free biholomorphic functions of $[\mathfrak{B}(\mathcal{H})_1^n]$. It is clear that $\mathrm{Aut}(\mathfrak{B}(\mathcal{H})_1^n)$ is a group with respect to the composition of free holomorphic functions.
Now, we consider an important case: Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{B}_n$, we define $\Delta_\lambda = (1 - \|\lambda\|^2)I_{\mathbb{C}}$ and $\Delta_{\lambda^*} = (I_{\mathcal{H}} - \lambda^*\lambda)^{1/2}$. Suppose $\Theta_\lambda(X_1, \ldots, X_n) = -\lambda + \Delta_\lambda(I_{\mathcal{H}} - \sum \bar{\lambda}_i X_i)^{-1}[X_1, \ldots, X_n]\Delta_{\lambda^*}$, Θ_λ is an automorphism on $[\mathfrak{B}(\mathcal{H})_1^n]$, also $\Phi_U(X_1, \ldots, X_n) = [X_1, \ldots, X_n]U \in \mathrm{Aut}(\mathfrak{B}(\mathcal{H})_1^n)$, for any unitary operator U on $\mathbb{C}^n$.

Theorem 1.2. [4] *If $\Psi \in \mathrm{Aut}(\mathfrak{B}(\mathcal{H})_1^n)$ and $\lambda = \Psi^{-1}(0)$, then there is a unique unitary operator U on $\mathbb{C}^n$ such that $\Psi = \Phi_U \circ \Psi_\lambda$ where $\Psi_\lambda = -\Theta_\lambda$.*

In [4] Popescu showed that $\mathrm{Aut}(\mathfrak{B}(\mathcal{H})_1^n)$ and $\mathrm{Aut}(\mathbb{B}_n)$ are isomorphism by $\Gamma : \mathrm{Aut}(\mathfrak{B}(\mathcal{H})_1^n) \longrightarrow \mathrm{Aut}(\mathbb{B}_n)$, $[\Gamma(\Psi)](z) = \phi_\lambda(z)U$ where U is an unitary operator on $\mathbb{C}^n$ and $\Psi = \Phi_U \circ \Psi_\lambda$, $\lambda = \Psi^{-1}(0) \in \mathbb{B}_n$.

Corollary 1.3. $\Gamma(\Theta_\lambda) = \psi_\lambda$ *and* $\Gamma(\Phi_U) = U$.

2. Main results

In this section we show that the $\mathrm{Aut}(\mathfrak{B}(\mathcal{H})_1^n)$ is root-approximable. Since Γ is a group isomorphism, then $\Gamma(\Theta_\lambda) = \psi_\lambda$ therefore $\Gamma(\Theta_{U\lambda}) = \psi_{U\lambda} = U\psi_\lambda U^{-1}$ and hence $\Theta_{U\lambda} = \Phi_U\Theta_\lambda\Phi_U^{-1}$. Suppose that $0 < r < 1$,

$\lambda = re_1 = (r, 0')$ then $\Delta_\lambda = \sqrt{1-r^2} I_{\mathbb{C}}$,

$$\Delta_{\lambda^*} = \left(I_{\mathbb{C}^n} - \begin{pmatrix} r^2, 0 \\ 0, 0 \end{pmatrix} \right)^{1/2} I_{\mathcal{H}} = \begin{pmatrix} \sqrt{1-r^2} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} I_{\mathcal{H}}$$

since $(I_{\mathcal{H}} - \sum \bar{\lambda}_i X_i)^{-1} = (I_{\mathcal{H}} - rX_1)^{-1}$ we have $\Theta_{re_1}(X_1, \cdots, X_n) = (-r, 0') + ((1-r^2)(I_{\mathcal{H}} - rX_1)^{-1}(\sqrt{1-r^2}X_1, X')$, therefore $\Theta_{re_1}(e_1) = (-r, 0') + \sqrt{1-r^2}(1-r)^{-1}(\sqrt{1-r^2}, 0')I_{\mathcal{H}} = e_1$, in a similar way $\Theta_{re_1}(-e_1) = -e_1$. Conversely, suppose that $0 < r < 1$ and $\Theta_{re_1}(X_1, \cdots, X_n) = (X_1, \cdots, X_n)$ then we have

$$(-r, 0') + ((1-r^2)(I_{\mathcal{H}} - rX_1)^{-1}X_1, \sqrt{1-r^2}(I_{\mathcal{H}} - rX_1)^{-1}X') = (X_1, X'),$$

$$(1-r^2)(I_{\mathcal{H}} - rX_1)^{-1}X_1 = r + X_1 \Rightarrow X_1 = \pm I_{\mathcal{H}},$$

$$\sqrt{1-r^2}(I_{\mathcal{H}} - rX_1)^{-1}X' = X' \Rightarrow X_j = 0, \ j = 2, 3, \ldots, n.$$

Hence $X = (X_1, \cdots, X_n) = \pm e_1$.
Since $\Theta_\lambda = \Phi_U \Theta_{re_1} \Phi_U^{-1}$ where $\lambda = U(re_1)$, $r = \|\lambda\|_2$, therefore $\Theta_\lambda(X) = X \Rightarrow \Theta_{re_1}(\Phi_U^{-1}(X)) = \Phi_U^{-1}(X)$ thus $\Phi_U^{-1}(X) = \pm e_1$ and hence $X = \pm\Phi_U^{-1}(e_1) \in \partial\mathbb{B}_n$. Therefore we show that:

Theorem 2.1. *Θ_λ has only two fixed points $\pm\frac{\lambda}{\|\lambda\|_2}$.*

As a consequence of the Brouwer fixed point theorem every $\Psi \in \mathrm{Aut}(\mathfrak{B}(\mathcal{H})_1^n)$ fixes at least one point of $\overline{\mathfrak{B}(\mathcal{H})_1^n}$. By Heyden-Suffridge theorem (see [3, 5]), if $\Psi \in \mathrm{Aut}(\mathfrak{B}(\mathcal{H})_1^n)$ and Ψ fixes three points of $\mathbb{S}$, then Ψ fixes at least one fix point of $\mathfrak{B}(\mathcal{H})_1^n$. Therefore if Ψ has not any fixed point in $\mathfrak{B}(\mathcal{H})_1^n$, it has at most two fixed point in $\mathbb{S}$, hence we can propound the following definition:

Definition 2.2. Assume that $\Psi \in \mathrm{Aut}(\mathfrak{B}(\mathcal{H})_1^n)$. We say Ψ is

(a) elliptic if it has some fixed points in $\mathfrak{B}(\mathcal{H})_1^n$.
(b) hyperbolic if it has no fixed point in $\mathfrak{B}(\mathcal{H})_1^n$ and has two distinct fixed points in $\mathbb{S}$
(c) parabolic if it has no fixed point in $\mathfrak{B}(\mathcal{H})_1^n$ and has only one fixed point in $\mathbb{S}$.

Lemma 2.3. *For each $\zeta \in \mathbb{S}$, $\mathbb{T}_\zeta = \{\Theta_{r\zeta};\ -1 < r < 1\}$ is root-approximable abelian group.*

The following Theorem is main result of this paper.

Theorem 2.4. *The group $\mathrm{Aut}(\mathfrak{B}(\mathcal{H})_1^n)$ is root-approximable.*

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MOORE-PENROSE INVERSE AND RANGE PROPERTY

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ABSTRACT. Suppose T is a bounded adjointable operator in a Hilbert C*-module. It is well known that an operator T has closed range, if and only if its Moore-Penrose inverse $(T)^\dagger$ exists. In this article we find a relation between T with $(T)^\dagger$.

1. INTRODUCTION

Hilbert C^*-modules are essentially objects like Hilbert spaces, except that the inner product, instead of being complex-valued, take its values in a C^*-algebra. Throughout the paper $\mathcal{A}$ is a C*-algebra (not necessarily unital). A (right) pre-Hilbert module over a C^*-algebra $\mathcal{A}$ is a complex linear space $\mathcal{X}$, which is an algebraic right $\mathcal{A}$-module and $\lambda(xa) = (\lambda x)a = x(\lambda a)$ equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying,
(i) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ iff $x = 0$,
(ii)$\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle$,
(iii)$\langle x, ya\rangle = \langle x, y\rangle a$,
(iv) $\langle y, x\rangle = \langle x, y\rangle^*$.
for each $x, y, z \in \mathcal{X},\ \lambda \in \mathbb{C},\ a \in \mathcal{A}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{X}$ is called a Hilbert $\mathcal{A}$-module if it is complete with respect to the norm

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* Speaker.

$\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. Left Hilbert $\mathcal{A}$-modules are defined in a similar way. For example every C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with respect to inner product $\langle x, y\rangle = x^*y$, and every inner product space is a left Hilbert $\mathbb{C}$-module.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. Then $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the set of all maps $T : \mathcal{X} \to \mathcal{Y}$ for which there is a map $T^* : \mathcal{Y} \to \mathcal{X}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for each $x \in \mathcal{X}$, $y \in \mathcal{Y}$. It is known that any element T of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(xa) = (Tx)a$ for $x \in \mathcal{X}$ and $a \in \mathcal{A}$ [1, Page 8]. We use the notations $\mathcal{L}(\mathcal{X})$ in place of $\mathcal{L}(\mathcal{X}, \mathcal{X})$, and $\ker(\cdot)$ and $\operatorname{ran}(\cdot)$ for the kernel and the range of operators, respectively.

Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $\mathcal{Y}$ is a closed submodule of $\mathcal{X}$. We say that $\mathcal{Y}$ is orthogonality complemented if $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Y}^\perp$, where $\mathcal{Y}^\perp := \{y \in \mathcal{X} : \langle x, y\rangle = 0 \text{ for all } x \in \mathcal{Y}\}$ denotes the orthogonal complement of $\mathcal{Y}$ in $\mathcal{X}$. The reader is referred to [1, 4] and the references cited therein for more details.

Recall that a closed submodule in a Hilbert module is not necessarily orthogonally complemented, however Lance proved that certain submodules are orthogonally complemented as fallows.

Theorem 1.1. (see [1, Theorem 3.2]) Let $\mathcal{X}$, $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then

- $\ker(T)$ is orthogonally complemented in $\mathcal{X}$, with complement $\operatorname{ran}(T^*)$.
- $\operatorname{ran}(T)$ is orthogonally complemented in $\mathcal{Y}$, with complement $\ker(T^*)$.
- The map $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ has closed range.

If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, then an operator $S \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ is called an *inner inverse* of T if $TST = T$. If $T \in$ has an inner inverse S, then the operator $T^\times = STS$ in $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ is called a *generalized inverse* of T if

$$TT^\times T = T \quad \text{and} \quad T^\times T T^\times = T^\times. \tag{1.1}$$

It is known that a bounded adjointable operator T has generalized inverse if and only if $\operatorname{ran}(T)$ is closed.

Definition 1.2. Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The Moore-Penrose inverse $T^\dagger$ of T (if it exists) is an element of $\mathcal{L}(\mathcal{Y}, \mathcal{X})$, which satisfies

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T. \tag{1.2}$$

Motivated by these conditions $T^\dagger$ is unique and $T^\dagger T$ and $TT^\dagger$ are orthogonal projections. (Recall that an orthogonal projection is a self-adjoint idempotent operator, that its range is closed.) Clearly, T is

Moore-Penrose invertible if and only if T^* is Moore-Penrose invertible, and in this case $(T^*)^\dagger = (T^\dagger)^*$.

Theorem 1.3. (see [5, Theorem 2.2]) Let $\mathcal{X}$, $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then the Moore-Penrose inverse $T^\dagger$ of T exists if and only if T has closed range.

By (1.2), we have

$$\begin{aligned} \mathrm{ran(T)} &= \mathrm{ran(T\,T^\dagger)} & \qquad \mathrm{ran(T^\dagger)} &= \mathrm{ran(T^\dagger T)} \\ \ker(T) &= \ker(T^\dagger T) & \qquad \ker(T^\dagger) &= \ker(T\,T^\dagger) \end{aligned}$$

and by Theorem 1.1, we know that

$$\begin{aligned} \mathcal{X} &= \ker(T) \oplus \mathrm{ran(T^\dagger)} = \ker(\mathrm{T^\dagger T}) \oplus \mathrm{ran(T^\dagger T)} \\ \mathcal{Y} &= \ker(T^\dagger) \oplus \mathrm{ran(T)} = \ker(\mathrm{T\,T^\dagger}) \oplus \mathrm{ran(T\,T^\dagger)}. \end{aligned}$$

In this article we find relation between T with $(T)^\dagger$, in this cases that ran(T) make with other orthogonal submodules. If T has closed range, By theorem 1.3 $T^\dagger$ exists, but for composition T, $T^\dagger$, T^* and $(T^\dagger)^*$ and more property, similarly invertible, closed range, injective in Hillbert C^*-module remained. we like to admit some results with let conditions on lemma and theorems.

Since every C^*-algebra is a Hilbert C^*-module over itself, our results are also remarkable in the case of bounded adjointable operators on C^*-algebras.

Definition 1.4. Let $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules, just as for Hilbert spaces (see [3, Page 50]) $V \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is called a partial isometry if for each $x \in (\ker V)^\perp$, we have $\|Vx\| = \|x\|$.

Similarly [3, Theorem 2.3.4], each $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has a polar decomposition $T = V|T|$, where $V \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a partial isometry and $|T| = (T^*T)^{\frac{1}{2}}$ and $\ker(V) = \ker(T)$, $\quad \mathrm{ran(V)} = \overline{\mathrm{ran(T)}}$, $\quad \mathrm{ker(V^*)} = \mathrm{ker(T^*)}$, $\quad \mathrm{ran(V^*)} = \overline{\mathrm{ran(|T|)}}$ and $V^*T = |T|$.

2. MAIN RESULT

In this section we like to admit some results with Moore-Penrose inverse property and orthogonal submodules.

Lemma 2.1. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, $T \in L(\mathcal{X})$ has a closed range the following assertions are equivalent:*

(1) $T = P_{\mathrm{ran(T)}} P_{(\ker(T))^\perp} = P_{\mathrm{ran(T)}} P_{\mathrm{ran(T^*)}} = P_{(\ker(T^*))^\perp} P_{(\ker(T))^\perp}$.
(2) $T^2 = TT^*T$;
(3) $|Tx|^2 = \langle Tx, x\rangle$, *for all* $x \in (\ker(T))^\perp$.

Theorem 2.2. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, $T \in L(\mathcal{X})$ has a closed range, then $T^2 = TT^*T$ if and only if $T^* = T^\dagger T^2 T^\dagger$.*

Lemma 2.3. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, $T \in L(\mathcal{X})$ and M a closed submodule of $\mathcal{X}$ and P_M the orthogonal projection onto M. Then $M^\perp \oplus (M \cap \ker T) \subset \ker(P_{(\ker T)^\perp} P_M)$.*

Corollary 2.4. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, $T \in L(\mathcal{X})$ has a closed range and $T^2 = TT^*T$. Then $(\ker(T^*) \oplus (\mathrm{ran(T)} \cap \mathrm{ker\,T})) \subset \mathrm{ker(T^*)}$.*

Lemma 2.5. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, $T \in L(\mathcal{X})$ has a closed range, then the following assertions are equivalent:*

(1) $\mathrm{ran(T)} \oplus \mathrm{ran(T^*)} = \mathcal{X}$
(2) $T^\dagger T + TT^\dagger = 1$
(3) $(T^\dagger T - TT^\dagger)^2 = 1$

Theorem 2.6. *Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $T \in \mathcal{L}(\mathcal{X})$ is a self adjoint operator with closed range. Then $1 - T^\dagger T + T$ is invertible.*

Theorem 2.7. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, $T \in L(\mathcal{X})$ has a closed range. If $\mathrm{ran(T)} \oplus \mathrm{ran(T^*)} = \mathcal{X}$, then $TT^\dagger - TT^*$ is invertible.*

Theorem 2.8. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, $T \in L(\mathcal{X})$ has a closed range and $\ker(T^*) = \ker(T)$. Then $T^\dagger T^2 = T^2 T^\dagger$.*

Theorem 2.9. *Let $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be an operator with closed range and $T = V|T|$, be the polar decomposition of T. Then $V = T|T|^\dagger$.*

Proposition 2.10. *Suppose $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are Hilbert $\mathcal{A}$-modules, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$, TS has closed ranges and T is an isometric with complemented range then S has closed range.*

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REAL INTERPOLATION OF LORENTZ SPACES WITH RESPECT TO VECTOR MEASURES

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Abstract. In this paper we describe the real interpolated spaces between two quasi-normed Lorentz spaces $\Lambda^{P}_{\|m\|}(\varphi)$ with respect to the vector measure m.

1. Introduction

Let $m : \Sigma \to X$ be a vector measure defined on a σ-algebra of subsets of a nonempty set Ω, this will always means that m is countably additive on Σ with range in real Banach space X. We denote by X^* its dual space and by X^{**} its bidual. Also $B(X)$ denotes the unit ball of X. The semivariation of m is the set function $\|m\|(A) = \sup\{|\langle m, x^*\rangle|(A) : x^* \in B(X^*)\}$, for each $A \in \Sigma$, where $|\langle m, x^*\rangle|$ is the variation of the scalar measure $\langle m, x^*\rangle$. A measurable function $f : \Omega \to \mathbb{R}$ is called weakly integrable (with respect to m) if $f \in L^1(|\langle m, x^*\rangle|)$ for all $x^* \in X^*$ and for each $A \in \Sigma$ there exists an element $\int_A f dm \in X^{**}$ such that $\langle \int_A f dm, x^*\rangle = \int_A f d\langle m, x^*\rangle$ for all $x^* \in X^*$. The space $L^1_w(m)$ of all weakly integrable functions becomes a Banach lattice when it is

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endowed with the norm

$$\|f\|_1 = \sup\left\{\int_\Omega |f|d|\langle m, x^*\rangle| : x^* \in B(X^*)\right\}.$$

We say that a weakly integrable function f is integrable (with respect to m) if the vector $\int_A f dm \in X$ for all $A \in \Sigma$.
Let $1 < p < \infty$. We say that a measurable function f is p–integrable with respect to m, if $|f|^p \in L^1(m)$. We denote by $L^p(m)$ the corresponding spaces of p–integrable function with respect to m, which is a Banach space when equipped with the norm

$$\|f\|_p = \sup\left\{\left(\int_\Omega |f|^p d|\langle m, x^*\rangle|\right)^{\frac{1}{p}} : x^* \in B(X^*)\right\}.$$

see [4] for a detailed study of vector measures.

2. Lorentz spaces with respect to a vector measure

For the measurable function f on a measure space (Ω, m) where m is a vector measure, we define its distribution function by $\|m\|_f(t) := \|m\|(\{w \in \Omega : |f(w)| > t\})$, where $\|m\|$ is the semivariation of the measure m. Also, the decreasing rearrangement of f, defined for all $s > 0$, as the function $f_*(s) := inf\{t > 0 : \|m\|_f(t) \leq s\}$.
Now we define the Lorentz spaces with respect to a vector measure m and derive some of their elementry properties. The Lorentz spaces $\Lambda^p_{\|m\|}(\varphi)$ with respect to a vector measure m, $0 < p \leq \infty$, $\varphi(t) \geq 0$, is defined to be the collection of all functions f for which the quantity

$$\|f\|_{\Lambda^p_{\|m\|}(\varphi)} = \left(\int_0^\infty (f_*(t)\varphi(t))^p \frac{dt}{t}\right)^{\frac{1}{p}} \qquad 0 < p < \infty,$$

is finite. Moreover (see [1]), integration by parts yields

$$\int_0^\infty (f_*(t)\varphi(t))^p \frac{dt}{t}) = p\int_0^\infty y^{p-1}\left\{\int_0^{\|m\|_f(y)} \varphi^p(t)\frac{dt}{t}\right\} dy \qquad 0 < p < \infty,$$

and hence

$$\int_0^\infty (f_*(t)\varphi(t))^p \frac{dt}{t}) = \int_0^\infty y^{p-1} w^p(\|m\|_f(y))dy.$$

Where $w(t) = \{\int_0^t \varphi^p(s)\frac{ds}{s}\}^{\frac{1}{p}}$ is a positive, non-decreasing. From now on, we delete the subscript $\|m\|$.
for $p = \infty$ we define

$$\|f\|_{\Lambda^\infty(\varphi)} = \|f\|_{\Lambda^\infty(w)} = sup_s f_*(s)w(s) < \infty$$

Theorem 2.1. *$\Lambda^p(\varphi)$ is a quasi-Banach space if its "fundamental function" $w(t) = \{\int_0^t \varphi^p(s)\frac{ds}{s}\}^{1/p}$ satisfied the Δ_2-condition $w(2t) \leq cw(t)$ for some $c > 0$.*

Theorem 2.2. *$\Lambda^p(\varphi_0) \subset \Lambda^p(\varphi_1)$ if $w_1(t) < cw_0(t)$, for all $t > 0$*

3. INTERPOLATION WITH A PARAMETER FUNCTION WITH RESPECT TO A VECTOR MEASURE

Let (A_0, A_1) be a couple of quasi-Banach spaces. For every $f \in A_0 + A_1$, we define the k-functional

$$k(t, f, A_0, A_1) = k(t, f) = inf_{f_0+f_1=f}(\|f_0\|_{A_0} + t\|f_1\|_{A_1}),$$

where $f_i \in A_i$, $i = 0, 1$ and $0 < t < \infty$. For each f, $0 < q \leq \infty$ and each measurable function ϱ, the space

$$(A_0, A_1)_{\varrho,q} = \left\{f; f \in A_0 + A_1, \int_0^\infty \left(\frac{k(t,f)}{\varrho(t)}\right)^q \frac{dt}{t} < \infty\right\}$$

is an interpolation space. Now we collect properties that we will need in what follows. The following lines $A \preceq B$ means that $A \leq cB$ for some positive constant c.

Proposition 3.1. *Let f be a measurable function and $1 \leq p, q < \infty$. Then*

$$\int_0^\infty t^{p-1}(\|m\|_f(t))^{\frac{p}{q}} dt = \frac{1}{q}\int_0^\infty (s^{\frac{1}{q}} f_*(s))^p \frac{ds}{s}.$$

Proposition 3.2. *Let f be a function in $L^1(m)$. Then*

1. $tf_*(t) \preceq k(t, f, L^1(m), L^\infty(m)), \quad t > 0,$
2. $k(t, f, L^1(m), L^\infty(m)) \preceq \int_0^t f_*(s)ds, \quad t > 0.$

For the proof of above propositions you can see [2].

Proposition 3.3. *Let $0 < q \leq \infty, 0 < p < \infty$ and $h(t)$ be a positive and nonincreasing function. Then*

$$\left(\int_0^\infty (\varphi(t))^q \left(\int_0^t (h(u)\psi(u))^p \frac{du}{u}\right)^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{1}{q}}$$
$$\leq \quad c\left(\int_0^\infty (\varphi(t)h(t)\psi(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

For the proof see[3]

Lemma 3.4. *Let $0 < q \leq \infty$. Then $(L^1(m), L^\infty(m))_{\varrho,q} = \Lambda^q(\frac{t}{\varrho(t)})$.*

Now we have the following fundamental interpolation theorem associated with vector measures.

Theorem 3.5. *Let φ_0, φ_1 be two weights and ϱ be a parameter function. If $0 < p_0, p_1 < \infty, 0 < q < \infty$, then*

$$(\Lambda^{p_0}(\varphi_0), \Lambda^{p_1}(\varphi_1))_{\varrho,q} = \Lambda^q(\varphi),$$

where $\varphi(t) = \frac{t}{\varrho(t)}$.

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GENERALIZED LIE DERIVATIONS ON RINGS WITH A NON-TRIVIAL IDEMPOTENT

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Abstract. We investigate Lie derivations and generalized Lie derivations on unital rings with a non-trivial idempotent. It is shown under some mild conditions that each Lie derivation and generalized Lie derivation on this kind of rings is proper. Some applications to triangular algebras are also presented.

1. Introduction

Throughout the paper, R is a unital ring with a non-trivial idempotent e. Recall that such a ring has the following presentation

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix},$$

where $R_{11} = eRe, R_{12} = eR(1-e), R_{21} = (1-e)Re$ and $R_{22} = (1-e)R(1-e)$.

Set $[x, y] = xy - yx$ for $x, y \in R$. An additive map d on R is called a derivation if

$$d(xy) = d(x)y + xd(y), \quad (x, y \in R).$$

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For a fixed element $b \in R$, the map $\delta_b : x \longrightarrow [x, b]$ is a derivation of R, which is called an inner derivation implemented by b.

An additive map L of R is said to be a Lie derivation if

$$L([x, y]) = [L(x), y] + [x, L(y)] \qquad (x, y \in R).$$

A Lie derivation L is called proper, if there exist a derivation d and a mapping $\tau : R \longrightarrow \mathcal{Z}(R)$ such that $L = d + \tau$.

Definition 1.1. Let $f : R \longrightarrow R$ be an additive map.

If there is a derivation $d : R \longrightarrow R$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$, then f is called a generalized derivation associated to d.

If there is an additive Lie derivation $L : R \longrightarrow R$ such that $f([x, y]) = f(x)y - f(y)x + xL(y) - yL(x)$ for all $x, y \in R$, then f is called a generalized Lie derivation associated to L.

Hvala [4] proved that every additive generalized Lie derivation on a prime ring R with characteristic not 2 and $deg(R) > 3$ is in the form of $\tau + \xi$, where $\tau : R \longrightarrow R_{\mathcal{C}}$ is a generalized derivation and $\xi : R \longrightarrow R$ is an additive map sending commutators to zero, where R is the extended centroid and $R_{\mathcal{C}}$ is the central closure of R.

In this talk we present Lie derivations and generalized Lie derivations on a general ring with a non-trivial idempotent.

2. Main results

For a unital ring R with a non-trivial idempotent e, if R_{12} is faithful as a (R_{11}, R_{22})-module. Then a direct verification reveals that the center $\mathcal{Z}(R)$of R is

$$\mathcal{Z}(R) = \left\{ \begin{pmatrix} a & 0 \\ o & b \end{pmatrix}; am = mb, bn = na \ for\ each\ m \in R_{12}, n \in R_{21} \right\}.$$

Furthermore , $\pi_{R_{11}}(\mathcal{Z}(R)) \subseteq \mathcal{Z}(R_{11})$ and $\pi_{R_{22}}(\mathcal{Z}(R)) \subseteq \mathcal{Z}(R_{22})$ and there exists a unique algebraic isomorphism φ from $\pi_{R_{11}}(\mathcal{Z}(R))$ to $\pi_{R_{22}}(\mathcal{Z}(R))$ such that $am = m\varphi(a)$ and $\varphi(a)n = na$ for every $m \in R_{12}, n \in R_{21}$.

The next result shows the structure of Lie derivations on such a ring.

Proposition 2.1. *Let R be a unital ring with nontrivial idempotent e. A Lie derivation L on R is of the form*

$$L\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} S_1(a) - mn_0 - m_0n + W_1(b) & am_0 - m_0b + T_3(m) \\ n_0a - bn_0 + U_2(n) & S_4(a) + n_0m + nm_0 + W_4(b) \end{pmatrix}$$

where $m_0 \in R_{12}$, $n_0 \in R_{21}$ *and* $S_1 : R_{11} \longrightarrow R_{11}$, $W_1 : R_{22} \longrightarrow \mathcal{Z}(R_{11})$, $S_4 : R_{11} \longrightarrow \mathcal{Z}(R_{22})$, $W_4 : R_{22} \longrightarrow R_{22}$, $T_3 : R_{12} \longrightarrow R_{12}$ *and* $U_2 : R_{21} \longrightarrow R_{21}$ *are additive maps satisfy the following conditions,*

(1) S_1 *and* W_4 *are Lie derivations.*
(2) $S_4[a, a'] = 0$ *and* $W_1[b, b'] = 0$.
(3) $T_3(am) = S_1(a)m - B_m S_4(a) + aT_3(m)$,
$T_3(mb) = mW_4(b) - W_1(b)m + T_3(m)b$.
(4) $U_2(na) = nS_1(a) - S_4(a)n + U_2(n)a$,
$U_2(bn) = W_4(b)n - nW_1(b) + bU_2(n)$.
(5) $S_1(mn) - W_1(nm) = mU_2(n) + T_3(m)n$,
$W_4(nm) - S_4(mn) = U_2(n)m + nT_3(m)$.

Proposition 2.2. *Let R be a unital ring with a non-trivial idempotent such that R_{12} is faithful as a (R_{11}, R_{22})-module. A Lie derivation L on R of the form*

$$L\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} S_1(a) - mn_0 - m_0n + W_1(b) & am_0 - m_0b + T_3(m) \\ n_0a - bn_0 + U_2(n) & S_4(a) + n_0m + nm_0 + W_4(b) \end{pmatrix}$$

can be written as $d+\tau$, where d is a derivation and τ maps into $\mathcal{Z}(R)$ if and only if

$$S_4(A) \subseteq \pi_{R_{22}}(\mathcal{Z}(R)), \quad W_1(B) \subseteq \pi_{R_{11}}(\mathcal{Z}(R)) \text{ and }$$
$$W_1(n'm) + S_4(mn') \in \mathcal{Z}(R)$$

We give some sufficient conditions under which every Lie derivation on a unital ring with a non-trivial idempotent is proper.

Theorem 2.3. *Let R be a unital ring with a non-trivial idempotent such that R_{12} is faithful as a (R_{11}, R_{22})-module. A Lie derivation L on R of the form*

$$L\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} S_1(a) - mn_0 - m_0n + W_1(b) & am_0 - m_0b + T_3(m) \\ n_0a - bn_0 + U_2(n) & S_4(a) + n_0m + nm_0 + W_4(b) \end{pmatrix}$$

can be written as $d+\tau$, where d is a derivation and τ maps into $\mathcal{Z}(R)$ if and only if

(1) *there exist $x, y \in R_{11}$ such that $[x, y] \in \mathcal{Z}(R_{11})$ and is invertible in R_{11} or $S_4(R_{11}) \subseteq \pi_{R_{22}}(\mathcal{Z}(R))$ or $\pi_{R_{22}}(\mathcal{Z}(R)) = \mathcal{Z}(R_{22})$ or R_{11} is generated as an algebra by commutators and idempotents.*
(2) *there exist $s, t \in R_{22}$ such that $[s, t] \in \mathcal{Z}(R_{22})$ and is invertible in R_{22} or $W_1(R_{22}) \subseteq \pi_{R_{11}}(\mathcal{Z}(R)$ or $\pi_{R_{11}}(\mathcal{Z}(R_{11})) = \mathcal{Z}(R_{11})$ or R_{22} is generated as an algebra by commutators and idempotents.*
(3) $W_1(n'm) + S_4(mn') \in \mathcal{Z}(R)$ *or either* $\mathcal{Z}(R_{11})$ *or* $\mathcal{Z}(R_{22})$ *is a field.*

Now, we are ready to characterize generalized Lie derivations on rings with a non-trivial idempotent.

Theorem 2.4. *Consider a unital 2-torsion free ring R with a non-trivial idempotent e where R_{12} is faithful (R_{11}, R_{22})-module and $\pi_{ii}(\mathcal{Z}(R)) = \mathcal{Z}(R_{ii})$, $(i = 1, 2)$. If $L : R \longrightarrow R$ be a generalized Lie derivation with an associated linear map d, then $L = \Delta + \tau$, where $\Delta : A \longrightarrow A$ is a generalized derivation and $\tau : R \longrightarrow \mathcal{Z}(R)$ is a linear map that vanishes on all commutators of R.*

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A NEW FAMILY OF LINEAR DISPERSION SPACE-TIME CODES OF HIGH PERFORMANCE

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Abstract. Space-time coding is a coding way introduced for multiple input multiple output (MIMO) communication systems. After orthogonal space-time codes, linear dispersion codes were introduced to increase rate of transmission and decrease loss of information. In this talk we introduce a new approach to design square linear dispersion space-time codes. The introduced codes are full-rate, full-diversity and information lossless.

1. Introduction

The use of multiple antennas in wireless communication systems is a powerful method in order to increase rate of transmission and performance of the system. A space-time code provides a scheme for data transmission from a multiple antennas transmitter by transmitting copies of information symbols in different time slots from different

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* Speaker.

antennas. In the receiver side, each antenna receives faded replicas of transmitted symbols. In [4] orthogonal space-time codes were introduced. These codes are full-diversity and easily decodable but their rate of transmission does not exceed 3/4 for more than two antennas. Furthermore, orthogonal space-time codes are not information lossless in general. Hassibi and Hochwald [2] have proposed high date rate codes named linear dispersion space time block codes or simply LD codes for arbitrary number of transmit and receive antennas with high rate of trnsmission. Here, we introduce a family of LD codes which satisfy three important criteria. That is, the proposed family of codes are full-diversity, full rate and information lossless.

Definition 1.1. A linear dispersion (LD) space-time code with variables $s_1, s_2, \cdots, s_k$ is an $n \times t$ matrix whose entries are complex linear combinations of $s_1, s_2, \cdots, s_k$ and $\bar{s_1}, \bar{s_2}, \cdots, \bar{s_k}$.

An LD space-time code can be written in the form

$$\mathbf{X}(\mathbf{s}) = \sum_{i=1}^{k} \mathbf{A}_i s_i + \sum_{i=1}^{k} \mathbf{B}_i \bar{s}_i. \tag{1.1}$$

Here, k denotes the total number of information symbols, $\mathbf{s}$ denotes a $k \times 1$ transmission symbol vector $\mathbf{s} = (s_1, s_2, \cdots, s_k)^T$ and each of $\mathbf{A}_i$ and $\mathbf{B}_i$ denotes an $n \times t$ complex matrix. the rate of $\mathbf{X}(\mathbf{s})$ is defined as $R = k/t$. $\mathbf{X}(\mathbf{s})$ is called full-rate if $R = n$.

For a subset S of complex numbers (constellation set), each matrix obtained by $\mathbf{X}(\mathbf{s})$ when taking values from S is called a codeword. The set of all codewords is called a codebook. The LD-code $\mathbf{X}(\mathbf{s})$ is said to be full-diversity over S if for any two distinct codewords $\mathbf{C}_1$, $\mathbf{C}_2$ the matrix $(\mathbf{C}_1 - \mathbf{C}_2)^*(\mathbf{C}_1 - \mathbf{C}_2)$ is full rank.

For an $n \times m$ matrix $\mathbf{A}$ by $\mathrm{Vec}(\mathbf{A})$ we mean the $nm \times 1$ column vector obtained by columns of $\mathbf{A}$. An LD code is called information lossless or simply lossless if it uses the capacity of the channel completely. In matrix language for the code $\mathbf{X}(\mathbf{s})$ defined in (1.1) we let

$$\mathcal{A} = [\mathrm{Vec}(\mathbf{A}_1), \mathrm{Vec}(\mathbf{A}_2), \cdots, \mathrm{Vec}(\mathbf{A}_k)]$$

$$\mathcal{B} = [\mathrm{Vec}(\mathbf{B}_1), \mathrm{Vec}(\mathbf{B}_2), \cdots, \mathrm{Vec}(\mathbf{B}_k)]$$

$$\mathcal{F} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^* & \mathcal{A}^* \end{pmatrix}.$$

It was shown in[1] that in case $k = nt$ if $\mathcal{F}$ is a unitary matrix then the LD code $\mathbf{X}(\mathbf{s})$ is lossless.

2. Main results

As mentioned before we construct an LD code for a MIMO system with n transmit antennas and m receive antennas. The designed code transmits the information in $t = n$ time slots.
Let S be an arbitrary finite subset of the complex field which will play the role of constellation set for our designed code. Consider the field generated by S over the field of rational numbers, $F = \mathbb{Q}(S)$. In order to design a code it suffices to construct the matrices $\mathcal{A}$ and $\mathcal{B}$ introduced in the previous section. Here we set $\mathcal{B} = 0$ and to design $\mathcal{A}$, we start with a unitary matrix $\mathbf{U} \in M_{n^2}(\mathbb{Q})$ with rows $\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_{n^2}$ where $\mathbf{u}_{(i-1)n+i}$ has nonzero entries for $1 \le i \le n$. For instance one may let $\mathbf{U}$ be the matrix

$$\frac{1}{n^2}\begin{bmatrix} n^2-2 & 2 & 2 & \cdots & 2 \\ -2 & -(n^2-2) & 2 & \cdots & 2 \\ -2 & 2 & -(n^2-2) & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & 2 & 2 & \cdots & -(n^2-2) \end{bmatrix}.$$

Now, we choose the complex number ω with $|\omega| = 1$ such that for any nonzero polynomial P of degree less than n^2 with coefficients in F, $P(\omega) \neq 0$. This is possible since F is a finitely generated extension of $\mathbb{Q}$. Define

$$\mathbf{U}_1 = \mathbf{U}.\mathrm{diag}(1, \omega, \omega^2, \ldots, \omega^{n^2-1}).$$

Notice that $\mathbf{U}_1$ is a unitary matrix with coefficients in the field $K = F(\omega)$. The next step is to choose the complex number α with $|\alpha| = 1$ such that for any nonzero polynomial P of degree less than $n+1$ with coefficients in K, $P(\alpha) \neq 0$. As mentioned before, it is possible to choose such an α. For instance, one may choose ω and α such that the set $\{\omega, \alpha\}$ be algebraically independent over $\mathbb{Q}$. Note that each of the numbers ω and α may be an algebraic or a transcendental number. Now define

$$\mathcal{A} = \mathrm{diag}(d_1, d_2, \cdots, d_{n^2})\mathbf{U}_1 \tag{2.1}$$

where $d_r = \alpha$ for all r of the form $r = (i-1)n + i$, $i = 1, 2, \cdots, n$ and $d_r = 1$ otherwise.

Theorem 2.1. *For any finite subset S of complex numbers, the proposed code is an information lossless, full-rate square LD-code which is full diversity over S.*

Example 2.2. In this section we consider a system with two antennas in the transmitter and one or two antennas in the receiver. The information symbols are selected from a constellation set S which is a finite subset of $\mathbb{Z}[i]$ and two transmission time slots is considered. In this case $F = \mathbb{Q}(S) = \mathbb{Q}[i]$. Following our construction algorithm if we let

$$U = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

then the related space-time code is the matrix

$$A = \begin{bmatrix} \alpha(s_1 + \omega s_2 + \omega^2 s_3 + \omega^3 s_4) & -s_1 + \omega s_2 - \omega^2 s_3 + \omega^3 s_4 \\ -s_1 - \omega s_2 + \omega^2 s_3 + \omega^3 s_4 & \alpha(-s_1 + \omega s_2 + \omega^2 s_3 - \omega^3 s_4) \end{bmatrix}.$$

In this code ω is a complex number of absolute value 1 which does not satisfy any polynomial of degree less than 4 over the field $\mathbb{Q}[i]$ and α is also a complex number of absolute value one which does not satisfy any polynomial of degree less than 3 over the field $K = \mathbb{Q}(i, \omega)$.

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SOME LEWENT TYPE DETERMINANTAL INEQUALITY

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ABSTRACT. In this paper, by using of the convexity of the function $f(A) = \log\det(I + A) - \log\det(I - A)$ over the set of positive semi definite contractions, we present two new version of Lewent type determinantaly inequality by a Jensen-Mercer type inequality.

1. Introduction and Preliminaries

A. McD. Mercer in [4], proved that if $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ and $w_k (1 \leq k \leq n)$ are positive weights associated with these x_k such that $\sum_{k=1}^{n} w_k = 1$, and f is a convex function on an interval containing the x_k then

$$f\left(x_1 + x_n - \sum_{k=1}^{n} w_k x_k\right) \leq f(x_1) + f(x_n) - \sum_{k=1}^{n} w_k f(x_k)\,. \qquad (1.1)$$

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* Speaker.

The following numerical inequality is due to Lewent [1]

$$\frac{1+\sum_{j=1}^{n}\lambda_j x_j}{1-\sum_{j=1}^{n}\lambda_j x_j} \leq \prod_{j=1}^{n}\left(\frac{1+x_j}{1-x_j}\right)^{\lambda_j}, \tag{1.2}$$

where $x_j \in [0,1)$ and $\sum_{j=1}^{n}\lambda_j = 1,\ \lambda_j \geq 0\ (j = 1,\cdots,n)$ are scaler weights.

In what follows we assume that $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, $\mathbb{B}(\mathcal{H})$ and $\mathbb{B}(\mathcal{K})$ are C^*-algebras of all bounded linear operators on the appropriate Hilbert space and $\mathbf{P}[\mathbb{B}(\mathcal{H}),\mathbb{B}(\mathcal{K})]$ is the set of all positive linear maps from $\mathbb{B}(\mathcal{H})$ to $\mathbb{B}(\mathcal{K})$. When $\dim\mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the full matrix algebra $\mathcal{M}_n(\mathbb{C})$ of all $n\times n$ matrices with entries in the complex field $\mathbb{C}$ and denote its identity by I_n.

A positive operator A (written $A \geq 0$) has a unique positive square root B with $B^2 = A$; we write $B = A^{1/2}$. For two selfadjoint operators A and B, we say $A \geq B$ whenever $A - B \geq 0$. For any operator A, one defines its *absolute value* $|A| = (A^*A)^{1/2}$. We say that A is *strictly contractive* if $\|A\| < 1$, such an A is called a *contraction*. An operator A is said to be *trace class operator* if

$$\|A\|_1 := \sum_{x\in E}\langle |A|(x), x\rangle < +\infty\,,$$

where E is an orthonormal basis of $\mathcal{H}$.

In [3], Matković, Pečarić and Perić extended the (1.1) as follows: Let $A_1,\cdots,A_n \in \mathbb{B}(\mathcal{H})$ be selfadjoint operators with spectra in $[m,M]$ for some scalers $m < M$ and $\phi_1,\cdots,\phi_n \in \mathbf{P}[\mathbb{B}(\mathcal{H}),\mathbb{B}(\mathcal{K})]$ positive linear maps with $\sum_{j=1}^{n}\phi_j(I_{\mathcal{H}}) = I_{\mathcal{K}}$. If f is continuous convex function on $[m,M]$, then

$$f\left(mI_{\mathcal{K}} + MI_{\mathcal{K}} - \sum_{j=1}^{n}\phi_j(A_j)\right) \leq f(m)I_{\mathcal{K}} + f(M)I_{\mathcal{K}} - \sum_{j=1}^{n}\phi_j(f(A_j))\,. \tag{1.3}$$

In [2], Lin proved that if $A_j\ (j = 1,\cdots,n)$ are contractive trace class operators, then

$$\left|\det\left(\frac{I+\sum_{j=1}^{n}\lambda_j A_j}{I-\sum_{j=1}^{n}\lambda_j A_j}\right)\right| \leq \prod_{j=1}^{n}\det\left(\frac{I+|A_j|}{I-|A_j|}\right)^{\lambda_j}, \tag{1.4}$$

where $\sum_{j=1}^{n}\lambda_j = 1,\ \lambda_j \geq 0\ (j = 1,\cdots,n)$.

2. Main results

All the operators considered in this paper are trace class operators over a separable Hilbert space $\mathcal{H}$ and I is the identity operator. First, we state an important proposition for main theorem.

Proposition 2.1. [2, Proposition 5] *Let A, B be positive trace class operators with $A \geq B$ and C be any selfadjoint trace class operator. Then* $\operatorname{tr}(AC)^2 \geq \operatorname{tr}(BC)^2$.

Lemma 2.2. [2, Lemma 6] *The function* $f(A) = \log\det(I + A) - \log\det(I - A)$ *is convex over the set of positive semi definite contractions.*

In the next theorems, by using an idea of [2, Lemma 6], we present two new version of Lewent determinantal inequality as follows:

Theorem 2.3. *Let $0 < A_1 \leq A_2 \leq \cdots \leq A_n$ be contractive trace class operators. Then*

$$\det\left(\frac{I + A_1 + A_n - \sum_{j=1}^n \lambda_j A_j}{I - A_1 - A_n + \sum_{j=1}^n \lambda_j A_j}\right) \tag{2.1}$$

$$\leq \det\left[\left(\frac{I + A_1}{I - A_1}\right)\left(\frac{I + A_n}{I - A_n}\right)\prod_{j=1}^n \left(\frac{I + A_j}{I - A_j}\right)^{-\lambda_j}\right],$$

where $\sum_{j=1}^n \lambda_j = 1,\ \lambda_j \geq 0\ (j = 1, \cdots, n)$.

Theorem 2.4. *Let $A_1, \cdots, A_n \in \mathcal{M}_\ell(\mathbb{C})$ be Hermitian matrices with spectra in $[m, M]$ for some scalers $m < M$ and $\phi_1, \cdots, \phi_n \in \mathbf{P}[\mathcal{M}_\ell(\mathbb{C}), \mathcal{M}_k(\mathbb{C})]$ positive linear maps with $\sum_{j=1}^n \phi_j(I_\ell) = I_k$. Then*

$$\det\left(\frac{(1 + m + M)I_k - \sum_{j=1}^n \phi_j(A_j)}{(1 - m - M)I_k + \sum_{j=1}^n \phi_j(A_j)}\right) \tag{2.2}$$

$$\leq \frac{(1 + m)(1 + M)}{(1 - m)(1 - M)} \prod_{j=1}^n \det\left(\phi_j\left(\frac{I_\ell - A_j}{I_\ell + A_j}\right)\right).$$

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QUANTUM ERROR CORRECTION AND COMMUTATIVE PAULI CHANNELS

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Abstract. For a noisy quantum channel, a quantum error correction code of dimension k exists if and only if the joint rank-k numerical range associated with the error operators of the channel is non-empty. In this paper, we obtained the largest k, for which the joint rank-2^k numerical range associated with error operators of some given commutative Pauli channel is nonempty.

1. Introduction

Four extremely useful matrices in the study of quantum computation and quantum information are known as the Pauli matrices, represented as follows

$$\sigma_0 := I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 := \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \ \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $\imath = \sqrt{-1}$. These four matrices form an orthogonal basis for the algebra of 2×2 complex matrices with the Hilbert–Schmidt inner product $\langle A, B \rangle = \text{trace}(B^*A)$.

Let N be a natural number and $n = 2^N$. The *N-qubit Pauli group* $\mathcal{P}_N$ is defined to consist of all N-fold tensor products of Pauli matrices(say

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* Speaker.

N-qubit Pauli matrices), with multiplicative factors ± 1 and $\pm \imath$, as follows

$$\mathcal{P}_N := \left\{ \alpha \sigma_{(j_1, j_2, \ldots, j_N)} \ : \ j_l \in \{0,1,2,3\}, \alpha \in \{\pm 1, \pm \imath\} \right\},$$

where $\sigma_{(j_1, j_2, \ldots, j_N)} := \sigma_{j_1} \otimes \sigma_{j_2} \cdots \otimes \sigma_{j_N}$.

Let M_n be the algebra of $n \times n$ complex matrices. A *quantum channel* is a *trace preserving completely positive linear map* $\Phi : M_n \to M_n$. By the structure of completely positive linear maps, there are $E_1, \ldots, E_m \in M_n$ with $\sum_{j=1}^m E_j^* E_j = I_n$ such that

$$\Phi(\rho) = \sum_{j=1}^{m} E_j \rho E_j^*.$$

The matrices $E_1, \ldots, E_m$ are interpreted as the *error operators* of the quantum channel $\Phi := \{E_1, \ldots, E_m\}$. A k-dimensional subspace $\mathbb{V}$ of $\mathbb{C}^n$ is called a *quantum error correction code* for the quantum channel Φ, if there exists another quantum channel $\Psi : M_n \to M_n$ such that $\Psi \circ \Phi(\rho) = \rho$, for all $\rho \in P M_n P$, where P is the orthogonal projection of $\mathbb{C}^n$ onto $\mathbb{V}$.

By the result in [1], this happens if and only if there are scalars γ_{ij} with $1 \leq i, j \leq r$ such that

$$P E_i^* E_j P = \gamma_{ij} P.$$

Motivated by the above discussion, researchers study the (*joint*) *higher rank numerical range* defined as follows;

Definition 1.1. Given $\mathbf{A} = (A_1, \ldots, A_m) \in M_n^m$. The joint rank-$k$ numerical range $\Lambda_k(\mathbf{A})$ is defined as the collection of vectors $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{C}^{1 \times m}$ such that $P\mathbf{A}P = \mathbf{a}P$ for some rank-k orthogonal projection $P \in M_n$, where $P\mathbf{A}P = (PA_1P, \ldots, PA_mP)$ and $\mathbf{a}P = (a_1 P, \ldots, a_m P)$.

Let $\Phi = \{E_1, \ldots, E_m\}$ be a quantum channel. Then Φ has a 2^k-dimensional quantum error correction code if and only if

$$\Lambda_{2^k}(E_1^* E_1, E_1^* E_2, \ldots, E_1^* E_m, E_2^* E_1, \ldots, E_m^* E_m) \neq \emptyset.$$

In the context of quantum error correction, this means that there exists an N-qubit encoding which accommodates a k-qubit data states.

A quantum channel is called a *Pauli channel* if each of its error operators are scalar multiple of elements in Pauli group $\mathcal{P}_N$. The Pauli channels are a central class of quantum channels in quantum computing, see [2, Problem 1].

We say that $A_1, \ldots, A_m \in \mathcal{P}_N$ are *completely independent*, if the set $\left\{A_{i_1}A_{i_2}\cdots A_{i_j} : 1 \le i_1 < i_2 < \cdots < i_j \le m, 1 \le j \le m\right\}$ is independent.

Let $\mathcal{H}_n$ be the real linear space of Hermitian matrices in M_n. The joint spectrum of Hermitian m-tuple $\mathbf{A} \in \mathcal{H}_n^m$ is defined as

$$\operatorname{spec}(\mathbf{A}) := \left\{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^{1\times m} : \exists\ 0 \neq x \in \mathbb{C}^n \text{ s.t. } \mathbf{A}x = \boldsymbol{\lambda}x\right\}.$$

2. Main results

Let $\Phi = \{I_{2^N}, E_1, E_2, \ldots, E_m\}$ be a commutative Pauli channel. In this section, we are looking to find the largest k such that joint rank-2^k associated with Φ is nonempty.

Proposition 2.1 ([4]). *Let $\{A_1, \ldots, A_m\}$ be a commuting family in $\mathcal{H}_n$, and let $\mathbf{A} = (A_1, \ldots, A_m)$. If $\operatorname{spec}(\mathbf{A}) = \{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n\}$, then*

$$\Lambda_k(\mathbf{A}) \subseteq \Omega_k(\mathbf{A}) = \bigcap_{\substack{\Gamma\subseteq\{1,\ldots,n\}\\ |\Gamma|=n-k+1}} \operatorname{conv}\{\boldsymbol{\lambda}_j \ : \ j \in \Gamma\}.$$

Let $\{A_1, \ldots, A_m\}$ be a commuting family in $\mathcal{H}_n$, and let

$$\mathbf{A} = (A_1, \ldots, A_m), \quad \text{with} \quad \operatorname{spec}(\mathbf{A}) = \{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n\}.$$

We define $\Delta_k(\mathbf{A})$ as the set of those $\mathbf{a} \in \mathbb{R}^m$ such that for k disjoint subsets $S_1, S_2, \ldots, S_k$ of $\{1, 2, \ldots, n\}$ we have $\mathbf{a} \in \operatorname{conv}(\{\boldsymbol{\lambda}_j \ : \ j \in S_i\})$ for each i. By this definition we have the following proposition.

Proposition 2.2 ([4]). *Let $\{A_1, \ldots, A_m\}$ be a commuting family in $\mathcal{H}_n$, and let $\mathbf{A} = (A_1, \ldots, A_m)$. Then $\Delta_k(\mathbf{A}) \subseteq \Lambda_k(\mathbf{A})$, for every $1 \le k \le n$.*

Now, we state a key lemma in this paper.

Lemma 2.3. *Let $A_1, \ldots, A_m \in \mathcal{P}_N$ be commutative, Hermitian and completely independent matrices, where $N \ge m$. Then the m-tuple $\mathbf{A} = (A_1, \ldots, A_m)$ is unitarily equivalent to the following*

$$\left(\sigma_{(3,0,0,\ldots,0)}, \sigma_{(0,3,0,\ldots,0)}, \ldots, \sigma_{(0,0,\ldots,0,3)}\right) \otimes I_{2^{N-m}}.$$

Theorem 2.4. *Let $\mathbf{A} = (A_1, A_2, A_3) \in \mathcal{P}_N^3$ be a triple of commutative Hermitian matrices. Then $\Lambda_k(\mathbf{A}) = \Omega_k(\mathbf{A})$, for every $1 \le k \le 2^N$.*

Theorem 2.5. *Let $\mathbf{A} = \left(\sigma_{(i_1,i_2,\ldots,i_m)} \otimes I_{2^{N-m}}\right)_{(i_1,i_2,\ldots,i_m)\in\{0,3\}^m}$, where $N \ge m$. Then*

$$\Lambda_{2^{N-m+1}}(\mathbf{A}) = \emptyset, \quad \textit{and} \quad \Lambda_{2^{N-m}}(\mathbf{A}) = \operatorname{conv}(\operatorname{spec}(\mathbf{A})).$$

Remark 2.6. Suppose $E_1, E_2, \ldots, E_m$ are completely independent matrices in $\mathcal{P}_N$, and

$$\Phi = \left\{E_{i_1}E_{i_2}\cdots E_{i_j} \ : \ 1 \le i_1 < i_2 < \cdots < i_j \le m, 1 \le j \le m\right\} \cup \{I_{2^N}\}.$$

By using Lemma 2.3, such a channel can be rewritten (up to a constant unitary correction) in the form

$$\Phi = \left\{\sigma_{(i_1,i_2,\ldots,i_m)} \otimes I_{2^{N-m}} : \ i_j \in \{0,3\}, 1 \le j \le m\right\}.$$

So, if an $(N-m)$-qubit state ρ is encoded as $\hat{\rho} = (e_1e_1^*) \otimes \rho$, where $e_1^* = \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix} \in \mathbb{C}^{1\times m}$, then $\Phi(\hat{\rho}) = \hat{\rho}$ and our recovery channel is $\Psi = \{I_{2^N}\}$; See [3].

Theorem 2.7. *Let $N \ge 3$ and let*

$$\mathbf{A} = \left(\sigma_{(3,0,0)}, \sigma_{(0,3,0)}, \sigma_{(0,0,3)}, \sigma_{(3,3,0)}, \sigma_{(3,0,3)}, \sigma_{(3,0,3)}\right) \otimes I_{2^{N-3}}.$$

Then $\Lambda_{2^{N-1}}(\mathbf{A}) = \emptyset$ and $\mathbf{0} \in \Lambda_{2^{N-2}}(\mathbf{A})$,

Remark 2.8. Suppose E_1, E_2, E_3 are Hermitian and completely independent matrices in $\mathcal{P}_N$ and $\Phi = \{I_{2^N}, E_1, E_2, E_3\}$. In the context of quantum error correction, Φ is written as $\Phi(\rho) = \sum_{i=0}^{3} p_i E_i \rho E_i$, where the error operator E_i is happened by the given probability $p_i \ge 0$, $\sum_{i=0}^{3} p_i = 1$ and $E_0 = I_{2^N}$.

By using Lemma 2.3 and Theorem 2.7, there exist $U \in M_{2^N \times 2^{N-2}}$, such that $U^*U = I_{2^{N-2}}$ and $U^*\mathbf{A}U = \mathbf{0}$. Define

$$R = [U \quad E_1U \quad E_2U \quad E_3U] \in M_{2^N}.$$

Therefore R is a unitary matrix and so a quantum gate. If an $(N-2)$-qubit state ρ is encoded as $\hat{\rho} = R\left((e_1e_1^*) \otimes \rho\right)R^*$, where $e_1^* = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}$, then $R^*\Phi(\hat{\rho})R = \mathrm{diag}\{p_0, p_1, p_2, p_3\} \otimes \rho$.

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INEQUALITIES OF LÖWNER TYPE

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ABSTRACT. In this talk, we investigate several norm inequalities corresponding to the Löwner–Heinz inequality as well as some operator inequalities involving the strict positivity, operator convex functions and operator monotone functions.

1. Introduction and preliminaries

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. In the case when $\dim\mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field.

There are three types of ordering on the real space of all self-adjoint operators as follows. Let $A, B \in \mathbb{B}(\mathscr{H})$ be self-adjoint. Then
(1) $A \geq B$ if $\langle Ax, x\rangle \geq \langle Bx, x\rangle$.
(2) $A \succ B$ if $\langle Ax, x\rangle > \langle Bx, x\rangle$ holds for all non-zero elements $x \in \mathscr{H}$.
(3) $A > B$ if $A \geq B$ and $A - B$ is invertible.
Clearly (3)$\Rightarrow$ (2)$\Rightarrow$ (1) but the reverse implications are not valid in general. For instance, if A is the diagonal operator $(1, 1/2, 1/3, \cdots)$ on ℓ^2, then $A \succ 0$ but $A \not> 0$. Of course, in the case where H is of finite dimension, (2) and (3) are equivalent; see [3, 5].

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2. Results

The Löwner–Heinz inequality says that if $0 \le A \le B$, then $A^r \le B^r$ for any $0 < r \le 1$.

Löwner (MZ- 1934) proved the inequality for $r = 1/2$. Heinz (MA-1951) proved it for positive operators acting on a Hilbert space of arbitrary dimension.

Chan and Kwong (AMM, 1985) showed that if $A \ge B \ge 0$, $C \ge D \ge 0$, $AC = CA$ and $BD = DB$, then

$$A^{1/2}C^{1/2} = B^{1/2}D^{1/2}\,.$$

Yoshino (PJA, 1988) proved that this is equivalent to the Löwner–Heinz inequality.

There exist several norm inequalities each of which is equivalent to the Löwner–Heinz inequality. One of them is the so-called the Araki–Cördes inequality

$$\|A^rB^r\| \le \|AB\|^r,$$

in which A and B are positive operators and $0 < r \le 1$.

The other inequality equivalent to the the Löwner–Heinz inequality is the Heinz–Kato inequality stating that if $Q \in \mathbb{B}(\mathscr{H})$, $A \ge 0, B \ge 0$, $\|Qx\| \le \|Ax\|, \|Q^*y\| \le \|By\|$ for all $x, y \in \mathscr{H}$, then

$$|\langle Qx, y\rangle| \le \|A^\nu x\|\|B^{1-\nu}y\|$$

for all $\nu \in [0,1]$; cf. Yoshino's paper (PJA, 1988).

An extension of the Löwner–Heinz inequality:
The Furuta inequality (PAMS, 1987), which says that if $A \ge B \ge 0$, then

$$(A^{r/2}A^pA^{r/2})^{1/q} \ge (A^{r/2}B^pA^{r/2})^{1/q}$$

holds for $r \ge 0$, $p \ge 0$ and $q \ge 1$ with $(1+r)q \ge p + r$.

Recently the behavior of operator monotone functions on unbounded intervals with respect to the relation of strictly positivity has been investigated. In fact, an extension of the Löwner–Heinz inequality reads as follows.

Theorem 2.1 (A)**.** [5] *Let $A, B \in \mathbb{B}(\mathscr{H})$ be positive operators such that $A - B \ge m > 0$ and $0 < r \le 1$. Then*

$$A^r - B^r \ge ||A||^r - (||A|| - m)^r$$

and

$$\log A - \log B \ge \log\|A\| - \log(\|A\| - m) > 0.$$

Here $m = \|(A-B)^{-1}\|^{-1}$.

Let f be a real-valued function defined on an interval J. If for each self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$ with spectra in J,

- $A \leq B$ implies $f(A) \leq f(B)$, then f is called operator monotone;
- $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$ for all $\lambda \in [0,1]$, then f is said to be operator convex.

If f is an operator monotone function on $[0,\infty)$, then f can be represented as

$$f(t) = f(0) + \beta t + \int_0^\infty \frac{\lambda t}{\lambda + t} d\mu(\lambda)\,,$$

where $\beta \geq 0$ and μ is a positive measure on $[0,\infty)$ and if f is an operator convex function on $[0,\infty)$, then f can be represented as

$$f(t) = f(0) + \beta t + \gamma t^2 + \int_0^\infty \frac{\lambda t^2}{\lambda + t}\, d\mu(\lambda)\,,$$

where $\gamma \geq 0$, $\beta = f'_+(0) = \lim_{t\to 0^+} \frac{f(t)-f(0)}{t}$ and μ is a positive measure on $[0,\infty)$. The integral representation of operator convex and operator monotone on bounded intervals is different; see [3].

If the relation $\leq$ is replaced by $<$ throughout the above definitions, then we reach the notions of strictly operator monotone and strictly operator convex functions.
Kwong (PAMS, 1975) showed that if $A > B$ ($A \succ B$, resp.), then $A^r > B^r$ ($A^r \succ B^r$, resp.) for $0 < r \leq 1$. Uchiyama (IEOT, 2000) showed that for every non-constant operator monotone function f on an interval J, $A \succ B$ implies $f(A) \succ f(B)$ for all self-adjoint operators A, B with spectra in J.

Very recently, the following generalization of Theorem A is given in [2].

Theorem 2.2. *If $A > B \geq 0$ and f is a non-constant operator monotone function on $[0,\infty)$, then $f(A) - f(B) \geq f(\|B\| + m) - f(\|B\|) > 0$, where $m = \|(A-B)^{-1}\|^{-1}$.*

As a consequence, we have the following improvement of Theorem A.

Theorem 2.3. *If $A > B \geq 0$ and $0 < r \leq 1$, then*

$$A^r - B^r \geq (\|B\| + m)^r - (\|B\|)^r > 0$$

and $\log A - \log B \geq \log(\|B\| + m) - \log\|B\| > 0$, where $m = \|(A - B)^{-1}\|^{-1}$.

Now we treat the behavior of operator convex functions on bounded intervals with respect to the relation of strictly positivity.

Theorem 2.4. [3] *Let f be a non linear operator convex function on (a, b). Then f is strictly operator convex on (a, b).*

A one-variable version of some unitarily invariant norms are presented below.

Theorem 2.5. [4] *The following mutually equivalent inequalities hold*:

$|||AXA^{-1} + A^{-1}XA||| \geq 2|||X|||$ *for any* $A > 0$ *and* $X = X^*$;

$|||AX + X^*A^*||| \geq 2||||XA|||$

for any invertible operator A *and* $X \in \mathcal{I}$ *such that* XA *is self-adjoint*;

$|||A^2X + XA^2||| \geq 2|||AXA|||$ *for any* $A > 0$ *and* $X = X^*$.

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GENERALIZED ARITHMETIC-GEOMETRIC MEAN TYPE INEQUALITY

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Abstract. We show that for two continuous functions f, g on $(0, \infty)$ such that $\frac{f(t)}{g(t)}$ is an Kwong function and $f(t)g(t) \leq t$, any positive matrices A, B and any matrix X, it holds that

$$|||f(A)Xg(B) + g(A)Xf(B)||| \leq |||AX + XB|||$$

for each unitarily invariant norm $|||.|||$.

1. Introduction

A continuous real valued function g defined on the interval $(0, \infty)$ is called Kwong or Anti-Löwner if the matrix

$$K_g = \left(\frac{g(\lambda_i) + g(\lambda_j)}{\lambda_i + \lambda_j} \right)_{i,j=1}^{n} \tag{1.1}$$

is positive semidefinite for any distinct real numbers $\lambda_1, \ldots \lambda_n$ in $(0, \infty)$. In this note we assume that all matrices are in M_n and the symbol $|||.|||$ denotes a unitarily invariant norm on M_n, that is $|||UAV||| = |||A|||$ for all matrices A, U, V with U, V are unitary. The arithmetic-geometric mean inequality says that if A, B and X are arbitrary operators with

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A, B positive definite, then

$$||A^p X B^{1-p} + A^{1-p} X B^p|| \leq ||AX + XB||, \tag{1.2}$$

where $||.||$ is the operator norm. Bhatia and Davis [2] extended 1.2 for any unitarily invariant norm $|||.|||$ and n by n matrices A, B and X with A, B positive semidefinite.

2. Main results

We start Main result by the following Theorems:

Theorem 2.1. *Let f be a continuous function on $(0, \infty)$. Then f is an Kwong function if and only if $AB + BA \geq 0$ implies that $f(A)B + Bf(A) \geq 0$ for each invertible positive operator A and positive operator B.*

Theorem 2.2. *Let f, g be non-negative operator monotone functions on $[0, \infty)$. Then $h(x) = \frac{f(x)g(x)}{x}$ is Kwong.*

Theorem 2.3. *Let f be an operator monotone function on $(0, \infty)$. Then $h(x) = xf^{'}(x)$ is Kwong.*

Theorem 2.4. *Let f be an Kwong function. Then*

(i) *The function $g(x) = x^{1-p} f(x^p)$ is Kwong for each $0 \leq p \leq 1$.*

(ii) *The function $h(x) = f(x^p)^{\frac{1}{p}}$ is Kwong for each $0 < p \leq 1$. Moreover, if f is non-negative operator monotone then $h(x) = f(x^p)^{\frac{1}{p}}$ is also Kwong for each $1 \leq p \leq 2$.*

Example 2.5. The function $f(x) = \text{arcsinh}\,(x)$ is Kwong. Indeed

$$f(x) = \text{arcsinh}\,(x) = \ln(x + (x^2 + 1)^{\frac{1}{2}})$$

By Theorem 2.4 $g(x) = (x^2 + 1)^{\frac{1}{2}}$ is Kwong and $g(x) \geq 1$, therefore $\ln(x + g(x))$ is Kwong.

Our next theorem is the main result of this section.

Theorem 2.6. *Suppose that A, B are any positive matrices and f and g are two continuous functions on $(0, \infty)$ such that $h(x) = \frac{f(x)}{g(x)}$ is Kwong and $f(x)g(x) \leq x$. Then for any matrix X we have*

$$|||f(A)Xg(B) + g(A)Xf(B)||| \leq |||AX + XB||| \tag{2.1}$$

Example 2.7. By taking $f(x) = x^p$ and $g(x) = x^{1-p}$ in Theorem 2.6 for each $0 \leq p \leq 1$, we get well-known Heinz inequality stating that for complex matrices A, B, X with A, B positive semidefinite,

$$|||A^p X B^{1-p} + A^{1-p} X B^p||| \leq |||AX + XB|||,$$

where $|||.|||$ is any unitarily invariant norm.

An immediate consequence of Theorem 2.6 and Corollary **??** is the following corollary.

Corollary 2.8. *Let A, B be positive matrices and let f be a non-negative operator monotone function on $(0, \infty)$. If $g(x) = \frac{x}{f(x)}$, then*

$$|||f(A)Xg(B) + g(A)Xf(B)||| \leq |||AX + XB|||$$

for any matrix X and any unitarily invariant norm $|||.|||$.

Corollary 2.9. *Let A, B be positive matrices and let f be a non-negative operator monotone function on $[0, \infty)$ such that $f(0) = 0$ and $f'(0) = \lim_{x \to 0^+} f'(x) < \infty$. Then*

$$|||f(A)X + Xf(B)||| \leq f'(0)|||AX + XB|||$$

for every unitarily invariant norm.

Example 2.10. Note that the function $f(x) = \log(x + 1)$ is operator monotone on $[0, \infty)$. We easily see that $f'(0) = 1$. Hence for positive matrices A, B and arbitrary matrix X we have

$$|||\log(A + I)X + X\log(B + I)||| \leq |||AX + XB|||$$

for each unitarily invariant norm $|||.|||$.

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ON THE FERMI ISOMORPHISM

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Abstract. Let S be an $n-$surface and $\alpha : I \to S$ be a unit speed curve. Let the smooth vector field X tangent to S along α, is everywhere orthogonal to α. We define the concept of Fermi parallelism of X and show that if $a \in I$ and $v \in T_{\alpha(a)S}$ is orthogonal to the derivative of α, then there exists a unique vector field V along α that V is Fermi parallel and $V(a) = v$. Then we show that this vector field defines a vector space isomorphism between the orthogonal complements of any two vector space $< \dot{\alpha}(a) >$ and $< \dot{\alpha}(b) >$.

1. Introduction

Let S be an $n-$surface, $\alpha : I \to S$ be a smooth parameterized curve and X tangent to S along α. According to [3], X is said to be parallel, if the covariant derivative of X is equal to zero. If X and Y are parallel vector fields along α then $\|X\|$, $X.Y$ and the angle between X and Y are constant along α. Moreover $X + Y$ and λX for $\lambda \in R$, are parallel [3] if $t_0 \in I$ and $v \in T_{\alpha(t_0)}S$, then there exists a unique vector field

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* Speaker.

V tangent to S along α which is parallel and $V(t_0) = v$ [1, 3]. In this study a different kind of parallelism, named Fermi parallelism, is defined for the vector fields which are everywhere orthogonal to α. It is shown that if X and Y are parallel vector fields orthogonal to α then $\|X\|$, $X.Y$ and the angle between X and Y are constant along α. Also it is shown that if $t_0 \in I$ and $v \in R^{n+1}$, then there exists a unique vector field V defined on S along α which is parallel and $V(t_0) = v$. At last, a linear, one to one, onto and dot product preserving map $F_\alpha : \dot{\alpha}(a)^\perp \to \dot{\alpha}(b)^\perp$ is constructed[2].

2. Main results

Definition 2.1. Let $\alpha : I \to S$ be a parameterized curve in an $n-$surface S and let X be a smooth vector field defined on S along α which is everywhere orthogonal to α. i.e., $X(t).\dot{\alpha}(t) = 0$ for all $t \in I$. Then the Fermi derivative of X is the vector field $\hat{X}$ defined by

$$\hat{X}(t) = \dot{X}(t) - [\dot{X}(t).\dot{\alpha}(t)]\dot{\alpha}(t)$$

Theorem 2.2. *If X and Y are smooth vector fields defined on S along α which are everywhere orthogonal to α and f a smooth function along α. Then*

(1) $(X \hat{+} Y) = \hat{X} + \hat{Y}$,
(2) $(f\hat{X}) = \hat{f}X + f\hat{Y}$,
(3) $(X\hat{.}Y) = \hat{X}Y + X\hat{Y}$.

Proof. Let X and Y are smooth vector fields defined on S along α everywhere orthogonal to α and f be a smooth function along α, then

$$\begin{aligned} (X \hat{+} Y)(t) &= (X \dot{+} Y)(t) - [(X \dot{+} Y)(t).\dot{\alpha}(t)]\dot{\alpha}(t) \\ &= \dot{X}(t) - [\dot{X}(t).\dot{\alpha}(t)]\dot{\alpha}(t) + \dot{Y}(t) - [\dot{Y}(t).\dot{\alpha}(t)]\dot{\alpha}(t) \\ &= \hat{X}(t) + \hat{Y}(t). \end{aligned}$$

$$\begin{aligned} (f\hat{X})(t) &= (f\dot{X})(t) - [(f\dot{X})(t).\dot{\alpha}(t)]\dot{\alpha}(t) \\ &= \dot{f}(t)X(t) + f(t)\dot{X}(t) - [(\dot{f}(t)X(t))\dot{\alpha}(t) + (f(t)\dot{X}(t))\dot{\alpha}(t)] \\ &= \hat{f}(t)X(t) + f(t)[\dot{X}(t) - (\dot{X}(t)\dot{\alpha}(t))\dot{\alpha}(t)] \\ &= \hat{f}(t)X(t) + f(t)\hat{X}(t). \end{aligned}$$

$$\begin{aligned} (X\hat{Y})(t) &= [\hat{X}(t) + (\dot{X}(t)\dot{\alpha}(t))\dot{\alpha}(t)]Y(t) \\ &\quad +X(t)[\hat{Y}(t) + (\dot{Y}(t)\dot{\alpha}(t))\dot{\alpha}(t)] = \hat{X}(t)Y(t) + X(t)\hat{Y}(t). \end{aligned}$$

□

Definition 2.3. For an $n-$surface S in U and a parameterized curve $\alpha : I \to S$, a smooth vector field X defined on S along α which is everywhere orthogonal to α is said to be Fermi parallel or simply parallel, if $\hat{X} = 0$.

Fermi parallelism has the following properties:

Theorem 2.4. *If X is parallel along α and $c \in R$, then*
i) X has constant length along α,
ii) XY is constant along α,
iii) $X + Y$ and cX are parallel along α,
iv) The angle between X and Y is constant along α.

Proof. If X and Y are parallel vector fields along α, then the Theorem 2.2 implies that,

$$\frac{d}{dt}\|X\|^2 = 0, \frac{d}{dt}XY = 0, (X \hat{+} Y) = 0, (c\hat{X}) = c\hat{X} = 0$$

□

The following Theorem is a reformulation of the fundamental existence and uniqueness theorem for solutions of systems of first order differential equations and is the basic theorem of this paper[1].

Theorem 2.5. *Let S be an $n-$surface in U, let α be a unit speed parameterized curve in S, let $t_0 \in I$ and $v \in R^{n+1}$ is orthogonal to $\dot{\alpha}(t_0)$. There exists a unique vector field V defined on S along α orthogonal to α, which is parallel and $V(t_0) = v$.*

Proof. We require a vector field V defined on S along α which is everywhere orthogonal to α satisfying $\hat{V} = 0$. But

$$\begin{aligned}
\hat{V}(t) &= \dot{V}(t) - [\dot{V}(t)\dot{\alpha}(t)]\dot{\alpha}(t) \\
&= \dot{V}(t) - [(V(t)\dot{\alpha}(t))^{\cdot} - V(t)\ddot{\alpha}(t)]\dot{\alpha}(t) \\
&= \dot{V}(t) + [V(t)\ddot{\alpha}(t)]\dot{\alpha}(t).
\end{aligned}$$

So $\hat{V} = 0$ if and only if

$$\dot{V}(t) + [V(t)\ddot{\alpha}(t)]\dot{\alpha}(t) = 0 \tag{2.1}$$

This is a first order differential equation in V, so there exists a unique vector field V along α satisfying (2.1) together with the initial condition $V(t_0) = v$. To see that V is indeed everywhere orthogonal to α, note that

$$\begin{aligned}
(V(t)\dot{\alpha}(t))^{\cdot} &= \dot{V}(t)\dot{\alpha}(t) + V(t)\ddot{\alpha}(t) \\
&= [(-V(t)\ddot{\alpha}(t))\dot{\alpha}(t)]\dot{\alpha}(t) + V(t)\ddot{\alpha}(t) = 0.
\end{aligned}$$

Therefore $V(t)\dot{\alpha}(t)$ is constant along α. But since $V(t_0)\dot{\alpha}(t_0) = v\dot{\alpha}(t_0) = 0$, this constant must be zero. □

Now let p and q be two given points in an $n-$surface S, a parameterized curve α in S from p to q is a smooth map $\alpha : [a, b] \to S$ with $\alpha(a) = p$ and $\alpha(b) = q$. Such a parameterized curve α, determines a map $F_\alpha : \dot{\alpha}(a)^\perp \to \dot{\alpha}(b)^\perp$ by $F_\alpha(v) = V(b)$, where for $v \in T_p(S)$ with $v \perp \dot{\alpha}(a)$ and V is the unique parallel vector field along α with $V(a) = v$. Then F_α is called the Fermi transport of v along α from p to q.

Theorem 2.6. *Let S be an $n-$surface in U, let $p, q \in S$ and α be a smooth parameterized curve in S from p to q. Then for $v, w \in \dot{\alpha}(a)^\perp \cap T_pS$ and $c \in R$ we have*

(1) $F_\alpha(v) \cdot F_\alpha(w) = V(b) \cdot W(b) = v \cdot w$,
(2) $F_\alpha(v + w) = F_\alpha(v) + F_\alpha(w)$,
(3) $F_\alpha(cv) = cF_\alpha(v)$,
(4) F_α *is one to one and onto.*

Proof. Let $v, w \in \dot{\alpha}(a)^\perp \cap T_pS$, then there exist vector fields V and W along α orthogonal to α, such that $\hat{V} = \hat{W} = 0$ and $V(a) = v, W(a) = w$. Then, according to the Theorem 2.4 we have,

$$F_\alpha(v) \cdot F_\alpha(w) = V(b) \cdot W(b) = V(a) \cdot W(a) = v \cdot w$$

$V + W$ and cV are parallel vector fields and

$$(V + W)(a) = V(a) + W(a) = v + w, (cV)(a) = cV(a) = cv$$

So

$$\begin{aligned} F_\alpha(v + w) &= (V + W)(b) = V(b) + W(b) = F_\alpha(v) + F_\alpha(w) \\ F_\alpha(cv) &= (cV)(b) = cV(b) = cF_\alpha(v) \end{aligned}$$

Let $F_\alpha(v) = 0$, then there exists a unique vector field V which is orthogonal to α along α such that $\hat{V} = 0$ and $V(b) = 0$. But since such a vector field is unique by Theorem 2.5 and zero vector field has this properties, then the Theorem 2.4 implies that $v = V(a) = 0$ and F_α is one to one. Now since $dim\dot{\alpha}(a)^\perp = dim\dot{\alpha}(b)^\perp$, F_α is onto. □

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FUNCTIONAL CALCULUS. PART I: LINEAR ALGEBRA

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Abstract. Functional calculus is a strong tool for constructing, deconstructing and reconstructing linear operators and it all begins with the simple but very important theory of polynomials of operators. The present survey article was motivated by extending the well-known primary and cyclic decomposition theorems together with their byproducts, namely the rational canonical and Jordan canonical forms, to infinite dimensional spaces which succeeded with finding simpler and shorter proofs for the finite dimensional cases. Also, it extends the functional calculus in a novel way to unbounded symmetric operators.

1. Linear Algebra

Introduction. In general, an algebra homomorphism Φ from an algebra of $\mathbb{F}$-valued functions to the algebra $L(V)$ of linear operators on a linear space V over the field $\mathbb{F}$ is called a **functional calculus of** T if $\Phi(\mathbf{1}) = I$ and $\Phi(\mathbf{id}) = T$, where $\mathbf{1}$ and $\mathbf{id}$ stand for the constant function $\mathbf{1}(x) \equiv 1$ and the identity function $\mathbf{id}(x) \equiv x$. The most elementary algebraic manipulation of an operator T is to find the linear combinations of its powers T^n including $T^0 = I$; i.e., polynomials in T. Polynomials to operators are like prism to the visible light; when an operator is tested by various polynomials, its spectrum shows up. The

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most important polynomials for an operator are those which magnify some part of the operator and diminish the rest of the operator. An invariant subspace of $T \in L(V)$ is a linear subspace W of V such that $Tw \in W$ for all $w \in W$. The zero subspace and the whole space are called the trivial invariant subspaces of T. It is easy to see that W is an invariant subspace of T if and only if $f(T)W \subset W$ for all $f \in \mathbb{F}[x]$. One way to construct invariant subspaces is to choose any subset Ω of V and define $Z(\Omega, T) := \vee\{g(T)\omega : g \in \mathbb{F}[x]; \omega \in \Omega\}$, where $\vee$ stands for all (finite) linear combinations. This is called the minimal invariant subspace of T containing Ω or, in another words, the **invariant subspace of T generated by** Ω. In case $\Omega = \{\omega\}$, it is denoted by $Z(\omega, T)$ and is called the **cyclic subspace of T generated by** ω. In this case, $Z(\omega, T) = \{p(T)\omega : \forall p \in \mathbb{F}[x]\} = \vee\{\omega, T\omega, T^2\omega, T^3\omega, \cdots\}$.

Let W be an invariant subspace of T. We may or may not be able to refine T by finding a **complementary invariant subspace** U such that $V = W \oplus U$ and, hence, $T = T|_W \oplus T|_U$. If such a complementary invariant subspace U exists, it has the property that if $0 \neq u \in U$, then $f(T)u \notin U$ for all $f \in \mathbb{F}[x]$. To find such a subspace U is not easy and sometimes it is impossible. For example, if V has a countable basis $\{e_1, e_2, e_3, \cdots\}$ and T is the backwardshift $Te_n = e_{n-1}$ with $e_0 = 0$, then the totality of all invariant subspaces of T form a chain $\{0\} \subset \vee\{e_1\} \subset \vee\{e_1, e_2\} \subset \vee\{e_1, e_2, e_3\} \subset \cdots \subset V$; hence, there are no complementary invariant subspaces except, of course, the trivial $V = \{0\} \oplus V$. We say $f \in \mathbb{F}[x]$ is a TW**-absorber** of $y \in V$ if $f(T)y \in W$. The set of all TW-absorbers of y form an ideal whose generator is a monomial called the **minimal** TW**-absorber** of y.

If $W = \{0\}$, any TW-absorber of y is called a local annihilator of T at y and the corresponding minimal TW-absorber is called the **local minimal polynomial** of T at y. If f is the local minimal polynomial of T at some point and a local annihilator at all other points, then f is called the **minimal polynomial** of T. Annihilators exist in the finite dimension but may or may not exist in general. The global minimal polynomial of an operator is its local annihilator at every point of its domain. The backward shift just described has no global annihilator but every vector $\omega \in V$ has a local annihilator $f_\omega(x) = x^n$, $n = 1, 2, 3, \cdots$.

The subspace $Z(\omega, T)$ is finite dimensional if and only if T has a local minimal polynomial f_ω at ω; in this case, $\mathbf{dim}(Z(\omega, T)) = \mathbf{deg}(f_\omega)$ and f_ω is the (global) minimal polynomial of $T|_{Z(\omega,T)}$. If Ω_α ($\alpha \in \Lambda$) is a family of subsets of V, then $Z(\cup_{\alpha\in\Lambda}\Omega_\alpha, T) = \vee_{\alpha\in\Lambda}Z(\Omega_\alpha, T)$; moreover, if Ω_α ($\alpha \in \Lambda$) is a monotone chain, then $Z(\cup_{\alpha\in\Lambda}\Omega_\alpha, T) = \cup_{\alpha\in\Lambda}Z(\Omega_\alpha, T)$.

Spectral subspaces are another type of invariant subspaces defined in terms of polynomials. If $h \in \mathbb{F}[x]$ is a given polynomial, the linear subspace $\mathcal{S}(h,T) = \cup_{n=0}^{\infty}\mathbf{ker}(h^n(T))$ is called the **spectral subspace** of T corresponding to h.

The proof of the following lemma is left to the interested reader.

Lemma 1.1. *Let W be an invariant subspace of $T \in L(V)$. The following assertions are true.*

(1) *If p is a prime factor of the minimal TW-absorber of some $y \in V \backslash W$, then there exists $z \in V \backslash W$ such that $p(T)z \in W$.*
(2) *If f is the minimal polynomial of $T|_W$, then f is the local minimal polynomial of T at some $w \in W$.*
(3) *$\mathcal{S}(h, T|_W) = \mathcal{S}(h,T) \cap W$ for any $h \in \mathbb{F}[x]$.*
(4) *If $W = \mathcal{S}(f,T)$ and if f and h are relatively prime in $\mathbb{F}[x]$, then $h(T)|_W$ is bijective.*

The notation of this section is fixed throughout the remainder of the paper. The vacuous operations $\oplus_\emptyset$, $\Sigma_\emptyset$ and $\vee_\emptyset$ over certain objects are defined to be the corresponding zero objects. The operator 0 has the minimal polynomial $\mathbf{1}$ or $\mathbf{id}$ according to whether $V = \{0\}$ or $V \neq \{0\}$; in either case, T has the local minimal polynomial $\mathbf{1}$ at $0 \in V$.

The primary decomposition theorem. The primary decomposition theorem is about the decomposition of a finite dimensional space into the spectral subspaces corresponding to the prime factors of the minimal polynomial of an operator. Note that if f is the minimal polynomial of $T \in L(V)$, then $V = \mathcal{S}(f,T)$. The proof of the finite dimensional case can be used to show that if V and T are arbitrary, then $\vee_{p\in\mathcal{P}}\mathcal{S}(p,T) = \oplus_{p\in\mathcal{P}}\mathcal{S}(p,T)$, where $\mathcal{P} \subset \mathbb{F}[x]$ is the collection of all prime polynomials.

Additive spectral measures. Let $f \in \mathbb{F}[x]$ and $W = \mathbf{ker}(f(T))$. Assume further that f is the minimal polynomial of $T|_W$. Consider the prime factorization $f = p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$. For $i = 1, 2, \cdots, k$, let $f_i = f/p_i^{m_i}$ and choose $\psi_i \in \mathbb{F}[x]$ such that $\Sigma_{i=1}^k \psi_i f_i = \mathbf{1}$. Letting $S = T|_W$ and $q_i = \psi_i f_i$, it follows that $\Sigma_{i=1}^k q_i(S) = I|_W$ and $q_i(S)q_j(S) = \delta_{ij}I|_W$; $(i, j = 1, 2, \cdots, k)$. Now, let p/q be any rational function such that $\mathbf{gcd}(f,q) = 1$, and choose $h \in \mathbb{F}[x]$ such that $hq \equiv \mathbf{1}(\mathbf{mod}\ f)$. Then,

$$(p/q)(S) := \Sigma_{i=1}^k (ph)(q_i(S)S)q_i(S)y = \int_{\sigma(S)} p(t)h(t)dE(t)y, \qquad (1.1)$$

where $\sigma(S) := \{q_1(S)S, q_2(S)S, \cdots, q_k(S)S\}$ is called the **spectrum** of S, and $E : 2^{\sigma(S)} \to L(V)$ defined by $E(\{q_i(S)S\}) = q_i(S)$ is called the **spectral measure** of S. (The relation (1.1) is independent of the choice of h.) If f is the minimal polynomial of T, the values of

E are idempotent-valued and E is called the **spectral measure** of T; in this case, $q_i(T) \neq 0$ for all $i = 1, 2, \cdots, k$ and the set $\sigma(T) := \{q_1(T)T, q_2(T)T, \cdots, q_k(T)T\}$ is called the **spectrum** of T.

Splitting annihilators. With the notation of the previous section, assume f splits in $\mathbb{F}$; that is $f(x) = (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\cdots(x-\lambda_k)^{m_k}$ for some $\lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbb{F}$. Then the spectrum can be identified as $\sigma(T) = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$, where λ_i, for a while, is just a shortened notation for $Tq_i(T) = (T|_{\mathcal{S}(\mathbf{id}-\lambda_i,T)}) \oplus 0$. Let g be a rational function with poles off $\{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ and $m = \max\{m_1, m_2, \cdots, m_k\}$. Then $g(x) = g(\lambda_i)+g'(\lambda_i)(x-\lambda_i)/1!+g''(\lambda_i)(x-\lambda_i)^2/2!+g'''(\lambda_i)(x-\lambda_i)^3/3!+\cdots$. (Here and in the next integral, λ_i's act as scalars.) Hence,

$$g(T)y = \Sigma_{i=1}^k g(q_i(T)T)q_i(T)y = \Sigma_{i=1}^k \Sigma_{j=0}^m \frac{g^{(j)}(\lambda_i)}{j!} q_i(T)y. \tag{1.2}$$

Rational canonical forms. The **cyclic decomposition theorem** says that if V is finite dimensional, then $V = \oplus_{j=1}^r \oplus_{\alpha\in\Lambda_j} Z(\alpha, T)$, where $V \supset \Lambda_j \neq \emptyset$, and T has a local minimal polynomial f_j at all $\alpha \in \Lambda_j$ such that $f_1 = f$, $f_{j+1}|f_j$ and $f_{j+1} \neq f_j$, $j = 1, 2, \cdots, r$. (Set $f_{r+1} = 1$.) Let $e_0 = 0$ and $\{e_1, e_2, e_3, \cdots\}$ be a Hamel basis in a vector space V. The backward shift $Te_n = e_{n-1}$ $(n = 1, 2, 3, \cdots)$ reveals that any invariant subspace of T is of the form $Z(e_n, T) \subset \mathcal{S}(\mathbf{id}, T) = V$ for some positive integer n and, hence, the cyclic decomposition theorem does not hold. Note that although T has a local minimal polynomial at each point of V, the operator T has no minimal polynomial.

Let $h \in \mathbb{F}[x]$ and assume f is the minimal polynomial of $T|_{\mathbf{ker}(h(T))}$. The following is an extension of the cyclic decomposition theorem to infinite dimensional spaces:

$$\mathbf{ker}(h(T)) = \mathbf{ker}(f(T)) = \oplus_{j=1}^r \oplus_{\alpha\in\Lambda_j} Z(\alpha, T) \tag{1.3}$$

with f_j's and Λ_j's as in the cyclic decomposition theorem just stated for finite dimensional spaces.

The **rational canonical form** is the matrix representation of T with respect to the basis $\{T^i\alpha : i = 0, 1, 2, \cdots, \mathbf{deg}(f_j)-1; \alpha \in \Lambda_j; j = 1, 2, \cdots, r\}$, where f_j is the local minimal polynomial of T at $\alpha \in \Lambda_j$.

The uniqueness of the cyclic decomposition cannot be formulated as such in the infinite dimension; it is unique in the finite dimension and its proof follows the standard methods given in classical textbooks [2, 3].

Jordan canonical forms. Here, again, assuming T has a minimal polynomial f splitting as $f = (x-\lambda_1)^{m_1}\cdots(x-\lambda_n)^{m_n}$, the primary decomposition theorem implies that $V = \oplus_{i=1}^n \mathcal{S}_i$ which yields a decomposition $T = \oplus_{i=1}^n T|_{\mathcal{S}_i}$, where $\mathcal{S}_i = \mathcal{S}(\mathbf{id} - \lambda_i\mathbf{id}, T)$. Letting

$N_i = (T - \lambda_i I)|_{\mathcal{S}_i}$, it follows that each N_i is a nilpotent operator with a minimal polynomial $\mathbf{id}^{m_i}$ for $i = 1, 2, \cdots, n$. By the cyclic decomposition theorem applied to each N_i,

$$V = \oplus_{i=1}^n \oplus_{\gamma \in \Gamma_i} Z(\gamma, N_i) \text{ and } T = \oplus_{i=1}^n \oplus_{\gamma \in \Gamma_i} (\lambda_i I + N_{i\gamma}), \qquad (1.4)$$

where $N_i = \oplus_{\gamma \in \Gamma_i} N_{i\gamma}$ and each $N_{i\gamma}$ can be represented as a matrix of finite size with 1's on the upper diagonal and 0's elsewhere. The matrix representation of $\lambda_i I + N_{i\gamma}$ is called a **Jordan block** of T and the totality of them is called the **Jordan canonical form** of T.

Complexification. When $\mathbb{R}$ is imbedded in $\mathbb{C}$, any vector space $V = \mathbb{R}^n$ is also imbedded in $V_{\mathbb{C}} = \mathbb{C}^n$; therefore, any operator $T \in L(\mathbb{R}^n)$ can be extended in a canonical way to an operator $T_{\mathbb{C}} \in L(\mathbb{C}^n)$ such that $[T_{\mathbb{C}}]_{\mathcal{E}} := [T_{\mathbb{R}}]_{\mathcal{E}}$, where $\mathcal{E} \subset \mathbb{R}^n \cap \mathbb{C}^n$ is the standard $\{0,1\}$-basis. In general, if $\mathbb{G}$ is an extension of the general field $\mathbb{F}$, and $\mathcal{A}$ is an abstract set, then $\mathbb{F}^{\mathcal{A}} \subset \mathbb{G}^{\mathcal{A}}$ are linear spaces over $\mathbb{F}$ and $\mathbb{G}$, respectively. Also, $V_{\mathbb{F}} \subset V_{\mathbb{G}}$, where $V_{\mathbb{F}}$ (resp. $V_{\mathbb{G}}$) is the linear space of all functions $(x_a)_{a \in \mathcal{A}}$ with values in $\mathbb{F}$ (resp. $\mathbb{G}$) such that $x_a = 0$ for all but finitely many $a \in \mathcal{A}$. Also, the set $\mathcal{A}$, identified as the totality of all $\{(\delta_{ab})_{a \in \mathcal{A}} : b \in \mathcal{A}\}$, can be viewed as the common (standard) $\{0,1\}$-basis of $V_{\mathbb{F}}$ and $V_{\mathbb{G}}$. Let $[t_{ab}]_{(a,b) \in \mathcal{A} \times \mathcal{A}}$ be a matrix with entries in $\mathbb{F}$ such that, for each $b \in \mathcal{A}$, $t_{ab} = 0$ for all but finitely many $a \in \mathcal{A}$. Define $T_{\mathbb{F}} \in L(V_{\mathbb{F}})$ and $T_{\mathbb{G}} \in L(V_{\mathbb{G}})$ by the matrix representations $[T_{\mathbb{F}}]_{\mathcal{A}} = [T_{\mathbb{G}}]_{\mathcal{A}} = [t_{ab}]_{(a,b) \in \mathcal{A} \times \mathcal{A}}$. Now, if $\mathcal{A}$ is a Hamel basis of V and $T \in L(V)$, then V and T can be naturally identified as $V_{\mathbb{F}}$ and $T_{\mathbb{F}}$; The space $V_{\mathbb{G}}$ and the operator $T_{\mathbb{G}}$ are called the **generalized complexifications** of $V \equiv V_{\mathbb{F}}$ and $T \equiv T_{\mathbb{F}}$, respectively.

Symmetrization. Fix $V \equiv V_{\mathbb{F}}$, $T = T_{\mathbb{F}}$, $V_{\mathbb{G}}$ and $T_{\mathbb{G}}$ as in the previous section. Let h, f be as in (1.3) and assume $\mathbb{G}$ is the splitting field of the minimal polynomial f of $T|_{\mathbf{ker}(h(T))}$. We may and shall assume without loss of generality that $V = \mathbf{ker}(h(T))$. The purpose of this section is to find a relation between the rational canonical forms of $T_{\mathbb{F}}$ and the Jordan canonical forms of $T_{\mathbb{G}}$.

Assume $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the distinct eigenvalues of T which are the roots of the minimal polynomial f of T. If $f = q_1^{k_1} q_2^{k_2} \cdots q_s^{k_s}$ is the prime factorization of f in $\mathbb{F}[x]$, then each λ_i is the root of exactly one prime factor and, hence, in view of the primary decomposition theorem, we can further assume without loss of generality that $f = q^k$ for some irreducible polynomial $q \in \mathbb{F}[x]$. Define $\pi_i : \mathbb{G} \to \mathbb{G}$ to be any field automorphism leaving the elements of $\mathbb{F}$ fixed and sending λ_1 to λ_i. Accordingly, extend π_i componentwise (resp. entrywise) to the vectors (resp. matrices) with components (resp. entries) in $\mathbb{F}$. Note that $\pi_i([T]_{\mathcal{A}}) = [T]_{\mathcal{A}}$. Also, note that $\pi_i(\mathcal{S}(\mathbf{id} - \lambda_1, T_{\mathbb{G}})) = \mathcal{S}(\mathbf{id} - \lambda_i, T_{\mathbb{G}})$

for all $i = 1, 2, \cdots, n$. With Λ_j as in (1.3), $Z(\alpha, T_{\mathbb{F}}) = \oplus_{i=1}^{n} Z(\alpha_i, T_{\mathbb{F}})$ for all $\alpha \in \Lambda_j$, where $\alpha_i = \pi_i(\alpha_1)$ is the projection of α into the summand $\mathcal{S}(\mathbf{id} - \lambda_i, T_{\mathbb{G}})$, $i = 1, 2, \cdots, n$. Therefore, (1.3) can be further refined to yield the following reformulation of (1.4):

$$\begin{aligned} V_{\mathbb{G}} &= \oplus_{j=1}^{r} \oplus_{\alpha \in \Lambda_j} \oplus_{i=1}^{n} Z(\alpha_i, T_{\mathbb{G}}), \\ T_{\mathbb{G}} &= \oplus_{j=1}^{r} \oplus_{\alpha \in \Lambda_j} \oplus_{i=1}^{n} (\lambda_i I + N_{i\alpha}), \end{aligned} \tag{1.5}$$

where α_i is the projection of α into $\mathcal{S}(\mathbf{id} - \lambda_i \mathbf{1}, T_{\mathbb{G}})$ and $N_{i\alpha}$ is nilpotent. The formulas (1.5) are **symmetric** in the sense that each Λ_j is the image of Λ_1 under a field automorphism of $\mathbb{G}$ leaving the elements of $\mathbb{F}$ fixed; in particular, $\alpha_i = \pi_i(\alpha_1)$.) In the following paragraphs, a (not necessarily symmetric) Jordan canonical form of $T_{\mathbb{G}}$ is given and the aim is to symmetrize it to obtain a rational canonical form for $T_{\mathbb{F}}$.

By the symmetric core of $\mathbb{G}$ with respect to $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, we mean the smallest subfield $\mathbb{H}$ of $\mathbb{G}$ containing $\mathbb{F}$ and all sums of the form $\Sigma_{i=1}^{n} \lambda_i^t$, $t = 0, 1, 2, 3, \cdots$. Let $\mathbb{H}(\lambda_1) \subset \mathbb{G}$ be the smallest field containing $\mathbb{H}$ and λ_1. Then $\mathbb{H}(\lambda_1) = \{\Sigma_{\ell=0}^{N} b_\ell \lambda_1^\ell : \ b_0, b_1, \cdots, b_N \in \mathbb{H}\}$, where $N = \mathbf{deg}(f) - 1$. Hence, $\mathcal{S}(\mathbf{id} - \lambda_1, T_{\mathbb{G}}) = \mathbf{ker}(T_{\mathbb{G}} - \lambda_1 I)^N \subset \mathbb{G} V_{\mathbb{H}(\lambda_1)}$, where $V_{\mathbb{H}(\lambda_1)} = \{\Sigma_{\ell=0}^{N} \beta_\ell \lambda_1^\ell : \beta_0, \beta_1, \cdots, \beta_N \in V_{\mathbb{H}}\}$. This implies that the vector α_1 in (1.5) can be rescaled to lie in $V_{\mathbb{H}(\lambda_1)}$. Thus, assuming α_1 is rescaled, it follows that for each $\alpha \in \Lambda_j$, there exist $\beta_0, \beta_1, \cdots, \beta_N \in V_{\mathbb{H}}$ such that $\xi_{j\alpha} := \Sigma_{i=1}^{n} \pi_i(\alpha_1) = \Sigma_{i=1}^{n} \Sigma_{\ell=0}^{N} \beta_\ell \lambda_i^\ell = \Sigma_{\ell=0}^{N} (\sigma_{i=1}^{n} \lambda_i^\ell) \beta_\ell \in V_{\mathbb{H}}$ and, hence,

$$\begin{aligned} V_{\mathbb{G}} &= \oplus_{i=1}^{n} \mathcal{S}(\mathbf{id} - \lambda_i, T_{\mathbb{G}}) = \oplus_{i=1}^{n} \pi_i(\mathcal{S}(\mathbf{id} - \lambda_1, T_{\mathbb{G}})) \\ &= \oplus_i \pi_i(\oplus_{j=1}^{r} \oplus_{\alpha \in \Lambda_j} Z(\alpha_1, T_{\mathbb{G}})) = \oplus_{i,j,\alpha} Z(\pi_i(\alpha_1), T_{\mathbb{G}}) \\ &= \oplus_{j,\alpha} Z(\Sigma_{i=1}^{n} \pi_i(\alpha_1), T_{\mathbb{G}}) = \oplus_{j,\alpha} Z(\xi_{j\alpha}, T_{\mathbb{G}}). \end{aligned}$$

Thus, $V_{\mathbb{H}} = \oplus_{j,\alpha} Z(\xi_{j\alpha}, T_{\mathbb{G}}) \cap V_{\mathbb{H}} = \oplus_{j,\alpha} Z(\xi_{j\alpha}, T_{\mathbb{H}})$.

By symmetry, the minimal polynomial of $T_{\mathbb{F}}$ is of the form

$$q^k(x) = [\Pi_{i=1}^{n} (x - \lambda_i)]^{kh} = [\Pi_{j=1}^{m} (x^r - \mu_j)]^s,$$

where $\mu_1, \mu_2, \cdots, \mu_m \in \mathbb{G}$ are distinct values of $\lambda_1^r, \lambda_2^r, \cdots, \lambda_n^r$ and h, r, s are positive integers such that $\mathbf{char}(\mathbb{F}) \nmid s$ and r is a power of $\mathbf{char}(\mathbb{F})$. Since $\mathbf{char}(\mathbb{F}) \nmid s$, it follows that $\mathbb{F}$ is the symmetric core of $\mathbb{G}$ with respect to $\{\mu_1, \mu_2, \cdots, \mu_m\}$. Assume without loss of generality that $\pi_i(\mu_1) = \mu_i$ and that $\xi_{j\alpha}$ is rescaled to guarantee its projection $\xi_{j\alpha 1}$ into $\mathcal{S}(\mathbf{id}^r - \mu_1 \mathbf{1}, T_{\mathbb{G}})$ lies in $V_{\mathbb{F}(\mu_1)}$ for all possible values of i, j, α. Again, here, $V_{\mathbb{F}} = \oplus_{j,\alpha} Z(\zeta_{j\alpha}, T_{\mathbb{F}})$, where $\zeta_{j\alpha} := \Sigma_{i=1}^{m} \pi_i(\xi_{j\alpha 1}) \in V_{\mathbb{F}}$.

Complexification of a real Hilbert Space. As in the section of complexification of a general linear space, let $\mathcal{A}$ be an abstract set and consider the real linear space $\mathbb{R}^{\mathcal{A}}$ naturally imbedded in the complex linear space $\mathbb{C}^{\mathcal{A}}$. Define the real Hilbert space $H_{\mathbb{R}} := \{(x_a)_{a \in \mathcal{A}} \in \mathbb{R}^{\mathcal{A}} :$

$\Sigma_{a\in\mathcal{A}}|x_a|^2 < \infty\}$ which is naturally imbedded in the complex Hilbert space $H_{\mathbb{C}} := \{(x_a)_{a\in\mathcal{A}} \in \mathbb{C}^{\mathcal{A}} : \Sigma_{a\in\mathcal{A}}|x_a|^2 < \infty\}$. Now, if H is a Hilbert space with an orthonormal basis $\mathcal{A}$, then depending on whether H be real or complex, H is isometrically isomorphic with $H_{\mathbb{R}}$ or $H_{\mathbb{C}}$. If H is real, $H_{\mathbb{C}}$ may be regarded as the complexification of H. Let H be real and $T \in B(H)$. Then $T_{\mathbb{R}} \in B(H_{\mathbb{R}})$ and $T_{\mathbb{C}} \in B(H_{\mathbb{C}})$ are defined by $[T_{\mathbb{R}}]_{\mathcal{A}} = [T_{\mathbb{C}}]_{\mathcal{A}} = [T]_{\mathcal{A}}$. Now, if $u, v \in H_{\mathbb{R}}$, then $||u + iv||^2 = ||u||^2 + ||v||^2$ and $\langle T_{\mathbb{C}}(u + iv), u + iv\rangle = \langle T_{\mathbb{R}}u, u\rangle + \langle T_{\mathbb{R}}v, v\rangle$. It follows that $||T_{\mathbb{C}}|| = ||T_{\mathbb{R}}||$ and that $T_{\mathbb{C}}$ is a bounded operator extending $T_{\mathbb{R}}$ to $H_{\mathbb{C}}$. Moreover, if $\lambda \in \mathbb{R}$, then $T_{\mathbb{R}} - \lambda I$ is injective (resp. bijective) if and only if $T_{\mathbb{C}} - \lambda I$ is so. For this reason, the spectrum of T is defined only for its complexification and is denoted by $\sigma(T)$ which is equal to $\sigma(T_{\mathbb{C}})$ as defined for operators on complex Banach spaces.

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FUNCTIONAL CALCULUS. PART II: SELFADJOINT OPERATORS

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ABSTRACT. The functional calculus is extended to bounded or unbounded symmetric operators.

Bounded selfadjoint operators. Throughout this section, we assume H is a real or complex Hilbert space and $A \in B(H)$ is a bounded selfadjoint operator; that is, $A = A^*$ and $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in H$. Moreover, we assume all direct sums are orthogonal. The results of the remainder of the paper are mostly from unproven claims done in the introduction of [2]. I am indebted and grateful to the beneficial comments by the *referee* as well as the *editor* of [2] which pressed me to improve my impression of the subject.

It follows from [1] that

$$||p(A_{\mathbb{F}})|| = ||p||_{\sigma(A)} \ \forall\ p \in \mathbb{F}[x], \tag{0.1}$$

where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. By uniform density of polynomials in $C_{\mathbb{F}}(\sigma(A))$, we obtain an isometric *-algebra homomorphism $F : C_{\mathbb{F}}(\sigma(A)) \to B(H_{\mathbb{F}})$, called continuous functional calculus on A, such that $F(p) = p(A)$ for all $p \in \mathbb{F}[x]$. Denoting $F(f)$ by $f(A)$, we have

$$||f(A_{\mathbb{F}})|| = ||f||_{\sigma(A)} \ \forall f \in \mathbb{C}_{\mathbb{F}}(\sigma(A)). \tag{0.2}$$

It follows that $\mathbf{id}(A_{\mathbb{F}}) = A_{\mathbb{F}}$ and $\mathbf{1}(A_{\mathbb{F}}) = I_{H_{\mathbb{F}}}$.

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It is well-known that the invariant subspaces of a bounded symmetric or selfadjoint operator are reducing; that is, if $\mathcal{M} \subset H_{\mathbb{F}}$ is an invariant subspace of $A_{\mathbb{F}}$, then $A_{\mathbb{F}}\mathcal{M}^{\perp} \subset \mathcal{M}^{\perp}$. In particular, there exists a (not necessarily countable) collection $\{\phi_j : j \in \mathbb{J}\}$ of unit vectors in (the *real* Hilbert space) $H_{\mathbb{R}}$ such that

$$H_{\mathbb{F}} = \oplus_{j\in\mathbb{J}}\bar{Z}(\phi_j, A_{\mathbb{F}}), \tag{0.3}$$

where $\bar{Z}(\phi_j, A_{\mathbb{F}}) \subset H_{\mathbb{F}}$ is the closure of $Z(\phi_j, A_{\mathbb{F}})$ in $H_{\mathbb{F}}$.

Consider the bilinear forms

$$\langle f, g\rangle_j := \langle f(A_{\mathbb{F}})\phi_j, g(A_{\mathbb{F}})\phi_j\rangle, \tag{0.4}$$

which define $\mathbb{F}$-valued semi-inner products on $C_{\mathbb{F}}(\sigma(A))$ $(j \in \mathbb{J})$. Moreover, if $f \in C_{\mathbb{F}}(\sigma(A))$ and $\langle f, f\rangle_j = 0$ for all $j \in \mathbb{J}$, then $f \equiv 0$.

Each semi-inner product in (0.4) yields a functional $\langle f, \mathbf{1}\rangle_j$ acting as a positive linear functional on $C_{\mathbb{F}}(\sigma(A))$ and, hence, bringing about a regular Borel probability measure μ_j supported on $\sigma(A)$ such that

$$||f(A_{\mathbb{F}})\phi_j||^2 = \int |f(t)|^2 d\mu_j(t)\ \forall f \in C_{\mathbb{F}}(\sigma(A))\ \forall j \in \mathbb{J}. \tag{0.5}$$

In particular, $\{f_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^2_{\mathbb{F}}(\mu_j)$ if and only if $\{f_k(A_{\mathbb{F}})\phi_j\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\bar{Z}(\phi_j, A_{\mathbb{F}})$. Thus, we may and shall assume without loss of generality that

$$\bar{Z}(\phi_j, A_{\mathbb{F}}) = L^2_{\mathbb{F}}(\mu_j)\ \forall j \in \mathbb{J} \text{ and, hence, } H_{\mathbb{F}} = \oplus_j L^2_{\mathbb{F}}(\mu_j). \tag{0.6}$$

The next step is to unleash the functional calculus (0.2) to go beyond continuous functions. Let $B_{\mathbb{F}}(\sigma(A))$ be the algebra of all bounded $\mathbb{F}$-valued Borel functions defined on $\sigma(T)$. For $f \in B_{\mathbb{F}}(\sigma(A))$, define

$$f(A_{\mathbb{F}}) \in B(H_{\mathbb{F}}) \text{ by } f(A_{\mathbb{F}})g = fg(= \oplus_j fg_j) \tag{0.7}$$

for all $g(= \oplus_j g_j) \in H_{\mathbb{F}}(= \oplus_j L^2_{\mathbb{F}}(\mu_j))$. The mapping

$$f \mapsto f(A_{\mathbb{F}}) : B_{\mathbb{F}}(\sigma(A)) \to B(H_{\mathbb{F}})$$

is an isometric *-algebra homomorphism such that $\mathbf{1}(A_{\mathbb{F}}) = I$, $\mathbf{id}(A_{\mathbb{F}}) = A_{\mathbb{F}}$ and

$$\begin{aligned} f_1(A_{\mathbb{F}})f_2(A_{\mathbb{F}}) &= f_2(A_{\mathbb{F}})f_1(A_{\mathbb{F}}) \\ &= (f_1f_2)(A_{\mathbb{F}})\ \forall f_1, f_2 \in B_{\mathbb{F}}(\sigma(A)). \end{aligned} \tag{0.8}$$

Moreover, for $f \in B_{\mathbb{F}}(\sigma(A))$, the spectrum of the operator $f(A_{\mathbb{F}})|_{L^2_{\mathbb{F}}(\mu_j)}$ is equal to the μ_j-essential range of f; moreover, $f(A_{\mathbb{F}})$ is symmetric (resp. positive) if and only if $f(t) \in \mathbb{R}$ (resp. $f(t) \geq 0$) for all $t \in \sigma(A)$.

Theorem 0.1. (Known) *With the notation established in the previous paragraphs, define the set-function E sending Borel sets $\Delta \subset \mathbb{R}$ to symmetric idempotents $\chi_{\Delta}(A_{\mathbb{F}}) \in B(H_{\mathbb{F}})$. It is a weakly σ-additive spectral measure associated to $A_{\mathbb{F}}$ in the following sense.*

1. *$E(\emptyset) = 0$, $E(\sigma(A)) = I$ and $A_{\mathbb{F}}E(\Delta) = E(\Delta)A_{\mathbb{F}}$, for all Borel sets $\Delta \subset \mathbb{R}$.*
2. *For all $g, h \in H_{\mathbb{F}}$ and for all families $\{\Delta_n \subset \mathbb{R} : n \in \mathbb{N}\}$ of mutually disjoint Borel sets,*
$$\langle E(\cup_{n\in\mathbb{N}}\Delta_n)g, h\rangle = \Sigma_{i\in\mathbb{N}}\langle E(\Delta_n)g, h\rangle \tag{0.9}$$
3. *For each pair $g, h \in H_{\mathbb{F}} = \oplus_j L^2_{\mathbb{F}}(\mu_j)$, the set function*
$$\omega_{g,h}(\Delta) := \langle E(\Delta)g, h\rangle = \int_{\Delta} \Sigma_j g_j h_j^* d\mu_j \tag{0.10}$$
defines a Borel $\mathbb{F}$-valued measure such that
$$\langle f(A_{\mathbb{F}})g, h\rangle = \int_{\sigma(A)} f(t)d\omega_{g,h}(t)\ \forall f \in B_{\mathbb{F}}(\sigma(A)).$$

(*Warning: To avoid confusion with the closure sign $\bar{h}$, we have used the notation h^* instead of the complex conjugate $\bar{h}$; in case $g = h$, the set function $\omega_{g,g}$ is a positive Borel measure.*)

In the light of the spectral measure E, we can define the algebra $L^\infty_{\mathbb{F}}(E)$ to be the class of all E-essentially bounded $\mathbb{F}$-valued functions defined a.e.[E]. Thus, by Theorem 0.1, if $f \in L^\infty_{\mathbb{F}}(E)$,

$$\begin{aligned}(\inf_{\lambda\in\Delta}|f(\lambda)|^2)||E(\Delta)g||^2 &\leq ||f(A_{\mathbb{F}})E(\Delta)g||^2 \\ &\leq (\sup_{\lambda\in\Delta}|f(\lambda)|^2)||E(\Delta)g||^2,\end{aligned} \tag{0.11}$$

which is valid for every Borel set $\Delta \subset \mathbb{R}$ and every $g \in H_{\mathbb{F}}$. In particular, $E(\{r\})H_{\mathbb{F}}$ consists of all eigenvectors (including 0) of $A_{\mathbb{F}}$ corresponding to $r \in \mathbb{R}$.

Another useful consequence of Theorem 0.1 is the production of a rich family

$$\{E(\Delta)H_{\mathbb{F}} \equiv \oplus_j L^2_{\mathbb{F}}(\mu_j|_{\Delta}) : \Delta \subset \mathbb{R} \text{ Borel}\} \tag{0.13}$$

of invariant subspaces for $A_{\mathbb{F}}$, which provide decompositions such as

$$H_{\mathbb{F}} = \mathcal{M}_1 \oplus \mathcal{M}_2 \text{ and } f(A_{\mathbb{F}}) = f(S_1) \oplus f(S_2) \tag{0.14}$$

whenever $\sigma(A) = \Delta_1\dot{\cup}\Delta_2$ and $f \in L^\infty_{\mathbb{F}}(\mu)$, where $S_k = A_{\mathbb{F}}|_{\mathcal{M}_k}$ and $\mathcal{M}_k \equiv \oplus_j L^2_{\mathbb{F}}(\mu_j|_{\Delta_k})$ $(k = 1, 2)$.

The functional calculus for bounded positive operators also helps us in finding the positive square root of a positive operator. Let

$s : [0, \infty) \to [0, \infty)$ be the positive square root function. Then $A_{\mathbb{F}}^{1/2} = s(A_{\mathbb{F}})$ is called the positive square root of $A_{\mathbb{F}}$ which is the unique bounded symmetric operator satisfying

$$(A_{\mathbb{F}}^{1/2})^2 = A_{\mathbb{F}} \text{ and } A_{\mathbb{F}}^{1/2} \geq 0. \tag{0.15}$$

Unbounded positive operators. The pseudo-inverse of a positive operator is included in the class of all unbounded positive operators. Fortunately, the latter class of operators are pseudo inverses of bounded selfadjoint operators. In fact, if T is a (densely defined) unbounded positive selfadjoint operator on a Hilbert space H, then $A = (I + T)^{-1}$ is a bounded selfadjoint operator with $\sigma(A) \subset [0, 1]$ and one may write $T = \phi(A)$ if an unbounded function like $\phi(x) = x^{-1} - 1$ is admitted to our functional calculus. To legitimize such a functional calculus, we define $F_{\mathbb{F}}(E)$ to be the class of all $\mathbb{F}$-valued Borel functions defined on $\sigma(A)$ a.e.[E], in which, as usual, f_1 and f_2 are taken to be equal if $f_1 = f_2$ a.e.$[E]$. For every $f \in F_{\mathbb{F}}(E)$, define

$$\mathcal{D}(f(A_{\mathbb{F}})) = \{g \in H_{\mathbb{F}} : fg \in H_{\mathbb{F}}\} \text{ and } f(A_{\mathbb{F}})g = fg \tag{0.16}$$

for all $g \in \mathcal{D}(f(A_{\mathbb{F}}))$. The linear transformation $f(A_{\mathbb{F}})$ is a densely defined closed operator; that is $\overline{\mathcal{D}(f(A_{\mathbb{F}}))} = H_{\mathbb{F}}$ and $f(A_{\mathbb{F}})g_n \to f(A_{\mathbb{F}})g$ whenever $g_n \to g$ and $f(A_{\mathbb{F}})g_n$ is a Cauchy sequence. Moreover, $(f(A_{\mathbb{F}}))^* = f^*(A_{\mathbb{F}})$ and, if f is real-valued, then it is symmetric; that is, $\langle f(A_{\mathbb{F}})g, h\rangle = \langle g, f(A_{\mathbb{F}})h\rangle$ for all $g, h \in \mathcal{D}(f(A_{\mathbb{F}}))$. It is not difficult to see that for $f_1 \in L^\infty_{\mathbb{F}}(\mu)$ and $f_2 \in F_{\mathbb{F}}(\mu)$,

$$\overline{f_1(A_{\mathbb{F}})f_2(A_{\mathbb{F}})} = (f_1 f_2)(A_{\mathbb{F}}) = f_2(A_{\mathbb{F}})f_1(A_{\mathbb{F}}), \tag{0.17}$$

and $f_2^2(A_{\mathbb{F}}) = (f_2(A_{\mathbb{F}}))^2$.

Thus, not only the pseudo-inverse of a bounded selfadjoint operator is defined, the functional calculus can be extended for an arbitrary unbounded selfadjoint operator T via $f(T) = fo\phi(A)$. The details will appear elsewhere.

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LEFT-LOOKING VERSION OF RIF PRECONDITIONER WITH COMPLETE PIVOTING STRATEGY

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Abstract. In this paper, we use a complete pivoting strategy for the left-looking version of RIF preconditioner and study effect of this pivoting.

1. Introduction

Consider the linear system of equations of the form $Ax = b$ where the coefficient matrix $A \in \mathbb{R}^{n\times n}$ is nonsingular, large, sparse and nonsymmetric and also $x, b \in \mathbb{R}^n$. Krylov subspace methods can be used to solve this system [3]. An implicit preconditioner M for this system is an approximation of matrix A, *i.e.*, $M \approx A$. This preconditioner can be used as the right preconditioner for this system. In this case, instead of solving the original system $Ax = b$, it is better to solve the right preconditioned system $AM^{-1}u = b$, where $x = M^{-1}u$, by the Krylov subspace methods. ILU preconditioners are examples of implicit preconditioners. These type of preconditioners are in the form of $M = LDU$ where L and U^T are unit lower triangular matrices and

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D is a diagonal matrix. In this paper, we propose pivoting for the left-looking version of RIF which is an implicit preconditioner.

2. Complete pivoting strategy for the left-looking version of RIF preconditioner

Suppose that no dropping is applied in Algorithm 1. Also suppose that Π_k and Σ_k, for $1 \leq k \leq i-1$, are the computed row and column permutation matrices at steps 1 to $i-1$ of this algorithm. At the beginning of step i of this algorithm, at first, lines 2 and 3 are set. Then, the internal *while* loop is run and in line 6 the parameter *iter* is set equal to *iter* $+1$. After that, vector $z_i^{(i-1)}$ is computed in lines 7-11. The pivot element $q_i^{(i-1)}$ is computed in line 12. The essential relation $q_j^{(i-1)} = e_j^T(\Pi A \Sigma) z_i^{(i-1)}$, for $j \geq i+1$, gives the chance to compute the vector $(q_{i+1}^{(i-1)}, \cdots, q_n^{(i-1)})$ in lines 13-15. This vector is used to apply the row pivoting strategy in lines 16-22 and matrix Π is updated. Since the balance of the column pivoting has been ruined, then $satisfied\_p$ is set to false in line 18. After the row pivoting, $satisfied\_q$ is set to true in line 23. Next, vector $w_i^{(i-1)}$ is computed in lines 26-30. In line 31, the pivot element $p_i^{(i-1)}$ is set equal to $q_i^{(i-1)}$. The key relation $p_j^{(i-1)} = (w_i^{(i-1)})^T(\Pi A \Sigma) e_j$, for $j \geq i+1$, is used to compute the vector $(p_{i+1}^{(i-1)}, \cdots, p_n^{(i-1)})$ in lines 32-34. After that, the column pivoting strategy is applied in lines 35-41 and matrix Σ is updated. Since the balance of the row pivoting has been ruined, then $satisfied\_q$ is set to false in line 37. After the column pivoting, $satisfied\_p$ should be set equal to true in line 42 and the algorithm will alternate between the row and the column pivoting until the desired pivot element will be computed. After the internal while loop of Algorithm 1, the i-th row and the i-th column of matrices L and U are computed and dropped in lines 45-48. After that, element $q_i^{(i-1)}$ is defined as the (i,i) entry of matrix D, i.e., d_{ii}. At the end of step n of Algorithm 1, the factorization $\Pi A \Sigma \approx M = LDU$ will be computed where $\Pi = \Pi_{n-1} \cdots \Pi_1$ and $\Sigma = \Sigma_1 \cdots \Sigma_{n-1}$. Matrix M is termed the left-looking version of RIF preconditioner with complete pivoting.

3. Numerical Results

In this section, we report the results of $GMRES(30)$ method to solve the original and the right preconditioned linear systems. Preconditioners are left-looking RIF with and without pivoting. In Table 1, notation $LLRIF$ is used for the left-looking version of RIF and $LLRIFP(1.0)$ indicates the left-looking version of RIF with pivoting

Algorithm 1

Input: Let $A \in \mathbb{R}^{n\times n}$, $U = L = I_n$, $\Pi = \Sigma = I_n$, $\tau_w, \tau_z, \tau_l, \tau_u \in (0,1)$ be drop tolerances and prescribe a tolerance $\alpha \in (0,1]$.

Output: $\Pi A \Sigma \approx LDU$.

for $i = 1$ to n **do**
  $m_i = n_i = 0$, $iter = 0$
  $satisfied\_p = satisfied\_q = false$
  while not $satisfied\_q$ **do**
    $z_i^{(0)} = e_i$
    $iter = iter + 1$
    for $j = 1$ to $i-1$ **do**
      $p_i^{(j-1)} = e_j^T(\Pi A \Sigma) z_i^{(j-1)}$
      $z_i^{(j)} = z_i^{(j-1)} - (\frac{p_i^{(j-1)}}{p_j^{(j-1)}}) z_j^{(j-1)}$
      for all $l \leq j$, if $|z_{li}^{(j)}| < \tau_z$, then set $z_{li}^{(j)} = 0$
    end for
    If $iter = 1$, then set $q_i^{(i-1)} = e_i^T(\Pi A \Sigma) z_i^{(i-1)}$. Otherwise set $q_i^{(i-1)} = p_i^{(i-1)}$
    for $j = i+1$ to n **do**
      $q_j^{(i-1)} = e_j^T(\Pi A \Sigma) z_i^{(i-1)}$
    end for
    if $|q_i^{(i-1)}| < \alpha\ max_{m \geq i+1} |q_m^{(i-1)}|$ **then**
      $m_i = m_i + 1$, $\pi_{m_i}^{(i-1)} = I_n$.
      $satisfied\_p = false$
      choose k such that $|q_k^{(i-1)}| = max_{m \geq i+1} |q_m^{(i-1)}|$.
      Interchange rows i and k of $\pi_{m_i}^{(i-1)}$ and elements $q_i^{(i-1)}$ and $q_k^{(i-1)}$
      $\Pi = \pi_{m_i}^{(i-1)} \Pi$
    end if
    $satisfied\_q = true$
    if not $satisfied\_p$ **then**
      $w_i^{(0)} = e_i$
      for $j = 1$ to $i-1$ **do**
        $q_i^{(j-1)} = (w_i^{(j-1)})^T (\Pi A \Sigma) e_j$
        $w_i^{(j)} = w_i^{(j-1)} - (\frac{q_i^{(j-1)}}{q_j^{(j-1)}}) w_j^{(j-1)}$
        for all $l \leq j$, if $|w_{li}^{(j)}| < \tau_w$, then set $w_{li}^{(j)} = 0$
      end for
      $p_i^{(i-1)} = q_i^{(i-1)}$
      for $j = i+1$ to n **do**
        $p_j^{(i-1)} = (w_i^{(i-1)})^T (\Pi A \Sigma) e_j$
      end for
      if $|p_i^{(i-1)}| < \alpha\ max_{m \geq i+1} |p_m^{(i-1)}|$ **then**
        $n_i = n_i + 1$, $\sigma_{n_i}^{(i-1)} = I_n$
        $satisfied\_q = false$
        choose l such that $|p_l^{(i-1)}| = \max_{m \geq i+1} |p_m^{(i-1)}|$
        Interchange columns i and l of $\sigma_{n_i}^{(i-1)}$ and elements $p_i^{(i-1)}$ and $p_l^{(i-1)}$
        $\Sigma = \Sigma \sigma_{n_i}^{(i-1)}$
      end if
      $satisfied\_p = true$
    end if
  end while
  for $j = 1$ to $i-1$ **do**
    $L_{ij} = \frac{q_i^{(j-1)}}{q_j^{(j-1)}}$, $U_{ji} = \frac{p_i^{(j-1)}}{p_j^{(j-1)}}$
    If $|L_{ij}| < \tau_l$, then set $L_{ij} = 0$. Also if $|U_{ji}| < \tau_u$, then set $U_{ji} = 0$.
  end for
  $d_{ii} = q_i^{(i-1)}$
end for
Return $L = (L_{ij})_{1 \leq i,j \leq n}$, $U = (U_{ij})_{1 \leq i,j \leq n}$, $D = diag(d_{ii})_{1 \leq i \leq n}$, Π and Σ.

TABLE 1.

	properties		*No preconditioning*		$LLRIFP(1.0)$			$LLRIF$		
	n	nnz	it	$Itime$	$density$	it	$Ttime$	$density$	it	$Ttime$
$fpga\_dcop\_12$	1220	5892	+	+	1.842	45	0.078	0.9628	79	0.062
$raefsky6$	3402	137845	1353	5.72	0.995	5	0.484	0.274	7	0.406
$sherman4$	1104	3786	558	0.36	1.802	26	0.031	1.239	46	0.031
$fpga\_dcop\_14$	1220	5892	+	+	1.813	77	0.093	0.954	1144	1.296
$epb3$	84617	463625	+	+	1.560	207	20.43	1.005	316	24.312
$poisson3Da$	13514	352762	261	3.50	2.863	60	5.390	0.337	130	3.765

that uses $\alpha = 1.0$ for the row and the column pivoting. We have considered 6 linear systems with the coefficient matrices from reference [2]. The right hand side vector of these systems is Ae where $e = [1, \cdots, 1]^T$. The code of left-looking version of RIF with pivoting is written in $Fortran$ 77 and the codes of $GMRES$ and left-looking version of RIF are downloaded from the *Sparskit* [4] and *Sparslab* [1] packages. In all the tests, parameters τ_w, τ_z, τ_l and τ_u are set equal to 0.1. In Table 1, n is the dimension and nnz is the number of nonzero entries of the matrix. In columns 4 and 5 of this table, it and $Itime$ are the number of iterations and the iteration time of $GMRES(30)$ method with no preconditioning. In columns 7 and 10 and in columns 8 and 11 of this table, it is the number of iterations and $Ttime$ is the total time of $GMRES(30)$ method that solves the right preconditioned linear systems. For all the tests of this table, the convergence criterion is satisfied when the relative residual is less than 10^{-8}. A + sign indicates that this criterion has not been satisfied in 5000 number of iterations. The parameter *density* in this table is defined as $density = \frac{nnz(L)+nnz(U)}{nnz(A)}$ where $nnz(L)$ and $nnz(U)$ are the number of nonzero entries of L and U factors. Numerical experiments indicate that for all matrices, $LLRIFP(1.0)$ is denser than $LLRIF$. The results also indicate that $LLRIFP(1.0)$ makes the $GMRES(30)$ method converge in less number of iterations than $LLRIF$.

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ON MODULES OF LINEAR TRANSFORMATIONS

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Abstract. Let D be a division ring, $\mathcal{V}, \mathcal{W}$ right or left vector spaces over D, and $\mathcal{L}(\mathcal{V}, \mathcal{W})$ the $\mathcal{L}(\mathcal{W})$-$\mathcal{L}(\mathcal{V})$ bimodule of all right (resp. left) linear transformations from $\mathcal{V}$ into $\mathcal{W}$. We prove some basic results about certain submodules of $\mathcal{L}(\mathcal{V}, \mathcal{W})$. For instance, we show, among other results, that a right submodule (resp. left submodule) of $\mathcal{L}(\mathcal{V}, \mathcal{W})$ is finitely generated whenever its image (resp. coimage) is finite-dimensional.

1. Introduction

Throughout this note, unless otherwise stated, D denotes a division ring, $\mathcal{V}$ and $\mathcal{W}$ right (resp. left) vector spaces over D, and $\mathcal{L}(\mathcal{V}, \mathcal{W})$ the set of all right (resp. left) linear transformations $A : \mathcal{V} \longrightarrow \mathcal{W}$ such that $A(x + y) = Ax + Ay$ and $A(x\lambda) = (Ax)\lambda$ (resp. $A(\lambda x) = \lambda(Ax)$) for all $x, y \in \mathcal{V}$ and $\lambda \in D$. It is well-known that $\mathcal{L}(\mathcal{V}, \mathcal{W})$ forms an abelian group under the adition of linear transformations. When $\mathcal{V} = \mathcal{W}$, we use the symbol $\mathcal{L}(\mathcal{V})$ to denote $\mathcal{L}(\mathcal{V}, \mathcal{W})$. It is easy to see that the set $\mathcal{L}(\mathcal{V})$ forms a ring under the addition and multiplication of linear transformations which are, respectively, defined

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by $(A+B)(x) := Ax + Bx$ and $(AB)(x) := A(Bx)$. It is also easily verified that $\mathcal{L}(\mathcal{V}, \mathcal{W})$ is a right $\mathcal{L}(\mathcal{V})$-module (resp. left $\mathcal{L}(\mathcal{W})$-module) via the multiplication of linear transformations. Throughout, by saying $\mathcal{I}$ is a right (resp. left) submodule of $\mathcal{L}(\mathcal{V}, \mathcal{W})$, we mean $\mathcal{I}$ is a right $\mathcal{L}(\mathcal{V})$-submodule (resp. left $\mathcal{L}(\mathcal{W})$-submodule) of $\mathcal{L}(\mathcal{V}, \mathcal{W})$. By the *image* and the *kernel* of the family $\mathcal{F} \subseteq \mathcal{L}(\mathcal{V}, \mathcal{W})$, denoted by $\operatorname{im}(\mathcal{F})$ and $\ker(\mathcal{F})$, respectively, we mean $\langle \{Ax : A \in \mathcal{F}, x \in \mathcal{V}\} \rangle$ and $\bigcap_{A \in \mathcal{F}} \ker A$. The *coimage* and *cokernel* of the family $\mathcal{F}$, denoted by $\operatorname{coim}(\mathcal{F})$ and $\operatorname{coker}(\mathcal{F})$, respectively, are defined as $\mathcal{V}/\ker \mathcal{F}$ and $\mathcal{W}/\operatorname{im}(\mathcal{F})$. As is usual, we use the symbol $\mathcal{V}'$ for $\mathcal{L}(\mathcal{V}, D)$. The members of $\mathcal{V}'$ are called linear functionals on $\mathcal{V}$. Also, when $\mathcal{V}$ is a right (resp. left) vector space, $\mathcal{V}'$ is a left (resp. right) vector space over D endowed with the addition and scalar multiplication defined by $(f+g)(x) := f(x) + g(x)$ and $(\lambda f)(x) := \lambda f(x)$ (resp. $(f\lambda)(x) := f(x)\lambda$) for all $x \in \mathcal{V}$ and $\lambda \in D$. *The second dual* of $\mathcal{V}$, denoted by $\mathcal{V}''$, is the dual of $\mathcal{V}'$. The space $\mathcal{V}''$ has the same chirality as that of $\mathcal{V}$ over D. It is easily seen that $\mathcal{V}$ naturally imbeds into $\mathcal{V}''$ via the natural mapping $\widehat{\ }: \mathcal{V} \to \mathcal{V}''$ ($x \mapsto \widehat{x}$) defined by $\widehat{x}(f) = f(x)$ for all $f \in \mathcal{V}'$, and that the natural mapping is an isomorphism of the vector spaces $\mathcal{V}$ and $\mathcal{V}''$ if and only if the space $\mathcal{V}$ is finite-dimensional. For a collection $\mathcal{C}$ of vectors in a right (resp. left) vector space $\mathcal{V}$ over D, $\langle \mathcal{C} \rangle$ is used to denote *the right (resp. left) linear subspace spanned by* $\mathcal{C}$. For a subset S of $\mathcal{V}$, we define $S^{\perp} := \{f \in \mathcal{V}' : f(S) = 0\}$. It is plain that $S^{\perp}$ is a subspace of $\mathcal{V}'$. For $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, $T' \in \mathcal{L}(\mathcal{W}', \mathcal{V}')$ denotes the *adjoint* of T which is defined by $(T'f)(v) := f(Tv)$ where $f \in \mathcal{W}', v \in \mathcal{V}$. For a subset $\mathcal{S}$ of $\mathcal{L}(\mathcal{V}, \mathcal{W})$, it is not difficult to see that the map $\phi : (\ker \mathcal{S})^{\perp} \longrightarrow \left(\frac{\mathcal{V}}{\ker \mathcal{S}}\right)'$ defined by $\phi f(x + \ker \mathcal{S}) = f(x)$, where $f \in (\ker \mathcal{S})^{\perp}$ and $x \in \mathcal{V}$, is an isomorphism of vector spaces. Therefore,

$$(\ker \mathcal{S})^{\perp} \cong \left(\frac{\mathcal{V}}{\ker \mathcal{S}}\right)'.$$

By a weak right (resp. left) submodule of $\mathcal{L}(\mathcal{W}', \mathcal{V}')$, we mean $\mathcal{I}' := \{T' \in \mathcal{L}(\mathcal{W}', \mathcal{V}') : T \in \mathcal{I}\}$, where $\mathcal{I}$ is a left (resp. right) submodule of $\mathcal{L}(\mathcal{V}, \mathcal{W})$. By definition $\mathcal{L}'(\mathcal{V}, \mathcal{W}) := \{T' \in \mathcal{L}(\mathcal{W}', \mathcal{V}') : T \in \mathcal{L}(\mathcal{V}, \mathcal{W})\}$. An important subset of $\mathcal{L}(\mathcal{V}, \mathcal{W})$ is the class of rank-one linear transformations. It can be shown that every rank-one linear transformation in $\mathcal{L}(\mathcal{V}, \mathcal{W})$ is of the form $x \otimes f$ for some $x \in \mathcal{W}$ and $f \in \mathcal{V}'$, where $(x \otimes f)(y) := xf(y)$ or $(x \otimes f)(y) := f(y)x$ depending on whether the space $\mathcal{W}$ is a right or a left vector space over D. It is readily checked that $(x \otimes f)' = f \otimes \widehat{x}$. Also, every finite-rank linear transformation is a finite sum of rank-one linear transformations. We use the symbol $\mathcal{F}(\mathcal{V}, \mathcal{W})$ to denote the set, in fact the bi-module, of all finite-rank linear transformations from $\mathcal{V}$ into $\mathcal{W}$. As is usual, $|A|$ is used to denote the cardinal number of the set A. The following is a standard observation. If $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, then there are subsets B_1 and B_2 of $\mathcal{V}$ such that B_1 and $B_1 \cup B_2$ are bases for $\ker A$ and $\mathcal{V}$, respectively. Moreover, $T(B_2)$ is basis for $\mathrm{im}(T)$ so that $\mathrm{im}(T) = \langle T(B_2)\rangle$. In fact, $\mathcal{V} = \langle B_1\rangle \oplus \langle B_2\rangle$. We refer the reader to [2] and [5] for general references on rings, modules, and linear algebra over division rings.

2. Main results

Lemma 2.1. (i) *Let $S \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ and $T \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$. Then,* $\ker S = \ker T$ *iff there exists an injective linear transformation $P \in \mathcal{L}(\mathcal{W}, \mathcal{Z})$ or $Q \in \mathcal{L}(\mathcal{Z}, \mathcal{W})$ such that $T = PS$ or $S = QT$ depending on whether* $\dim \mathrm{coker}(S) \leq \dim \mathrm{coker}(T)$ *or* $\dim \mathrm{coker}(T) \leq \dim \mathrm{coker}(S)$*, respectively.*

(ii) *Let $S \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ and $T \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$. Then,* $\ker S \subseteq \ker T$ *iff there exists a $P \in \mathcal{L}(\mathcal{W}, \mathcal{Z})$ such that $T = PS$.*

(iii) *Let $S_i \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ $(1 \leq i \leq n)$ and $T \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$. Then,* $\bigcap_{i=1}^n \ker S_i \subseteq \ker T$ *iff there exist $P_i \in \mathcal{L}(\mathcal{W}, \mathcal{Z})$ such that $T = P_1S_1 + \cdots + P_nS_n$.*

Lemma 2.2. (i) *Let $S \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{W})$. Then,* $\mathrm{im}(S) = \mathrm{im}(T)$ *iff there exists a surjective linear transformation $P \in \mathcal{L}(\mathcal{Z}, \mathcal{V})$ or $Q \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$ such that $T = SP$ or $S = TQ$ depending on whether* $\dim \ker S \leq \dim \ker T$ *or* $\dim \ker T \leq \dim \ker S$*, respectively.*

(ii) *Let $S \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{W})$. Then,* $\mathrm{im}(S) \subseteq \mathrm{im}(T)$ *iff there exists a $P \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$ such that $S = TP$.*

(iii) *Let $S \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ and $T_i \in \mathcal{L}(\mathcal{Z}, \mathcal{W})$ $(1 \leq i \leq n)$. Then,* $\mathrm{im}(S) \subseteq \mathrm{im}(\{T_i\}_{i=1}^n)$ *iff there exist $P_i \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$ such that $S = T_1P_1 + \cdots + T_nP_n$.*

Lemma 2.3. *Let $\mathcal{V}$ and $\mathcal{W}$ be two vector spaces over D and $\mathcal{C} \subseteq \mathcal{L}(\mathcal{V}, \mathcal{W})$. Then the following holds.*

(i) $(\mathrm{im}\ \mathcal{C})^\perp = \ker(\mathcal{C}')$*, where $\mathcal{C}' = \{T' : T \in \mathcal{C}\}$. In other words, $\langle \bigcup_{T \in \mathcal{C}} T\mathcal{V}\rangle^\perp = \bigcap_{T \in \mathcal{C}} \ker T'$.*

(ii) *If $\mathcal{C} = \{T_i\}_{i=1}^n$, where $n \in \mathbb{N}$, then* $(\ker \mathcal{C})^\perp = \mathrm{im}(\mathcal{C}')$*. In other words, $(\bigcap_{i=1}^n \ker T_i)^\perp = \langle \bigcup_{i=1}^n T_i'\mathcal{W}'\rangle$.*

Lemma 2.4. (i) *Let $S_i \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ $(1 \leq i \leq n)$ and $T \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$. Then, $\bigcap_{i=1}^n \ker S_i \subseteq \ker T$ iff $\mathrm{im}(T') \subseteq \mathrm{im}(\{S_i'\}_{i=1}^n)$ iff there exist $P_i \in \mathcal{L}(\mathcal{W}, \mathcal{Z})$ such that $T = P_1S_1 + \cdots + P_nS_n$.*

Theorem 2.5. *Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over a division ring D and $\mathcal{I}$ be a right submodule (resp. left submodule) of $\mathcal{L}(\mathcal{V}, \mathcal{W})$. If $\mathcal{I}$ finitely generated or $\mathcal{W}$ (resp. $\mathcal{V}$) is finite-dimensional, then $\mathcal{I} = \{T \in$*

$\mathcal{L}(\mathcal{V},\mathcal{W}) : T\mathcal{V} \subseteq \mathrm{im}(\mathcal{I})\}$ (resp. $\mathcal{I} = \{T \in \mathcal{L}(\mathcal{V},\mathcal{W}) : T\ker(\mathcal{I}) = \{0\}\}$). Moreover, if $\dim\mathcal{V} \geq \dim\mathcal{W}$ (resp. $\dim\mathcal{V} \leq \dim\mathcal{W}$), then every such right (res. left) submodule is principal. In particular, if $\mathcal{V} = \mathcal{W}$, then every finitely generated one-sided ideal of $\mathcal{L}(\mathcal{V})$ is principal.

Theorem 2.6. *Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over a division ring D and $\mathcal{I}$ be a right submodule (resp. left submodule) of $\mathcal{L}(\mathcal{V},\mathcal{W})$. If the image (resp. coimage) of $\mathcal{I}$ is finite-dimensional, then $\mathcal{I}$ is finitely generated, and hence $\mathcal{I} = \{T \in \mathcal{L}(\mathcal{V},\mathcal{W}) : T\mathcal{V} \subseteq \mathrm{im}(\mathcal{I})\}$ (resp. $\mathcal{I} = \{T \in \mathcal{L}(\mathcal{V},\mathcal{W}) : T\ker\mathcal{I} = \{0\}\}$). Moreover, if $\dim\mathcal{V} \geq \dim\mathcal{W}$ (resp. $\dim\mathcal{V} \leq \dim\mathcal{W}$), then $\mathcal{I}$ is principal.*

Theorem 2.7. *Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over a division ring D and $\mathcal{I}$ be a right submodule of $\mathcal{L}(\mathcal{V},\mathcal{W})$. Then, $\mathcal{I} \cap \mathcal{F}(\mathcal{V},\mathcal{W}) = \{T \in \mathcal{F}(\mathcal{V},\mathcal{W}) : \mathrm{im}(T) \subseteq \mathrm{im}(\mathcal{I})\}$.*

Remark. We conjecture that the counterpart of the theorem holds for left submodules of $\mathcal{L}(\mathcal{V},\mathcal{W})$. That is, *if $\mathcal{I}$ is a left submodule of $\mathcal{L}(\mathcal{V},\mathcal{W})$, then $\mathcal{I} \cap \mathcal{F}(\mathcal{V},\mathcal{W}) = \{T \in \mathcal{F}(\mathcal{V},\mathcal{W}) : \ker\mathcal{I} \subseteq \ker T\}$.*

Lemma 2.8. *Let $\mathcal{V},\mathcal{W},\mathcal{X},\mathcal{Y}$ be vector spaces over a division ring D and $S \in \mathcal{L}(\mathcal{V},\mathcal{W})$ and $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$. Then, $\mathrm{rank}(S) \leq \mathrm{rank}(T)$ iff there exist $P \in \mathcal{L}(\mathcal{Y},\mathcal{W})$ and $Q \in \mathcal{L}(\mathcal{V},\mathcal{X})$ such that $S = PTQ$. Moreover, if $S \in \mathcal{L}(\mathcal{V},\mathcal{W})$ and $T_i \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ ($1 \leq i \leq n$), then $\mathrm{rank}(S) \leq \sum_{i=1}^{n} \mathrm{rank}(T_i)$ iff there exist $P_i \in \mathcal{L}(\mathcal{Y},\mathcal{W})$ and $Q_i \in \mathcal{L}(\mathcal{V},\mathcal{X})$ such that $S = \sum_{i=1}^{n} P_i T_i Q_i$.*

Theorem 2.9. *Let $\mathcal{V}$ and $\mathcal{W}$ be infinite-dimensional vector spaces over a division ring D. Then the nontrivial bi-submodules of $\mathcal{L}(\mathcal{V},\mathcal{W})$ are of the form*

$$\{T \in \mathcal{L}(\mathcal{V},\mathcal{W}) : \mathrm{rank}(T) < e\},$$

for some unique infinite cardinal number $e \leq \min(\dim\mathcal{V}, \dim\mathcal{W})$.

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COMMUTING GRAPH OF OPERATORS SPACE

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Abstract. In this paper, we introduce the commuting graph of a subset of operators on a Hilbert space. Then, we study the connectivity and the diameter of the commuting graphs of the set of all bounded operators, the set of all projections, and the set of all nilpotent operators on a separable Hilbert space.

1. Introduction

For a ring $\mathfrak{R}$, we denote the *center* of $\mathfrak{R}$ by $Z(\mathfrak{R})$. If X is either an element or a subset of $\mathfrak{R}$, then $C_{\mathfrak{R}}(X)$ denotes the *centralizer* of X in $\mathfrak{R}$. For each non-commutative ring $\mathfrak{R}$, we associate a graph, with the vertex set $\mathfrak{R} \setminus Z(\mathfrak{R})$ and join two vertices x and y if and only if $x \neq y$ and $xy = yx$. This graph has been introduced in [1], is called the *commuting graph* of $\mathfrak{R}$, and is denoted by $\Gamma(\mathfrak{R})$. If $\mathfrak{X}$ is a subset of $\mathfrak{R}$, then $\Gamma(\mathfrak{X})$ denotes the induced subgraph of $\Gamma(\mathfrak{R})$ on $\mathfrak{X} \setminus Z(\mathfrak{R})$; that is the subgraph of $\Gamma(\mathfrak{R})$ with vertex set $\mathfrak{X} \setminus Z(\mathfrak{R})$. If D is a division

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ring and m, n are natural numbers, then we denote the ring of all $n \times n$ matrices over D by $M_n(D)$.

Let G be a graph, a *path* P is a sequence $v_0e_1v_1e_2 \dots e_kv_k$ whose terms are alternately distinct vertices and distinct edges in G, such that for any i, $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i. We say u is *connected to* v in G if there exists a path between u and v. The graph G is called *connected* if there exists a path between any two distinct vertices of G.

2. Main results

First, we recall some theorems in commuting graphs of some subsets of matrix algebra.

Theorem 2.1. [2, Theorem 5] *Let D be a division ring and $n \geqslant 3$. If $\mathcal{A}_n$ is the set of all non-invertible matrices in $M_n(D)$, then* $\operatorname{diam} \mathit{\Gamma}(\mathcal{A}_n) = 4$.

Theorem 2.2. . [2, Theorem 6] *Let F be a field and $n \geqslant 3$ If $\mathcal{T}_n$ is the set of all triangularizable matrices in $M_n(F)$, then* $\operatorname{diam} \mathit{\Gamma}(\mathcal{T}_n) = 4$.

Corollary 2.3. [2, Corollary 7] *Let F be an algebraically closed field and $n \geqslant 3$. Then* $\operatorname{diam} \mathit{\Gamma}(M_n(F)) = 4$.

The $C-$commuting graph is a very good generalization of commuting graph. It was defined as follows.

Definition 2.4. [5, Definition 2.1] For a division ring D, $n \in \mathbb{N}$, and $\mathcal{C} \subseteq M_n(D)$, a pair of matrices A and B in $M_n(D)$ is called $\mathcal{C}-$*Commuting* if $AB - BA = C$, for some $C \in \mathcal{C}$. Especially, a pair of matrices A and B are commuting matrices, if they are $\{0\}-$commuting.

Definition 2.5. [5, Definition 2.2] For a division ring D with center F, $n \in \mathbb{N}$, and $\mathcal{C} \subseteq M_n(D)$, the $\mathcal{C}-$*Commuting graph* of $M_n(D)$, denoted by $\Gamma_{\mathcal{C}}(M_n(D))$, is a graph with vertex set $M_n(D) \backslash FI$ such that distinct vertices A and B are adjacent if and only if they are $\mathcal{C}-$Commuting, where $FI = \{ \alpha I \mid \alpha \in F \}$. Especially, the $\{0\}-$Commuting graph of $M_n(D)$ is the commuting graph of $M_n(D)$ that was defined in [**?**].

Theorem 2.6. [5, Theorem 2.3]*Let D be a division ring with center F and $n \geqslant 3$ a natural number. Then the following hold:*

(i) *If D is non-commutative and $\mathcal{C}_1$ is the set of all matrices in $M_n(D)$ such that their ranks are at most 1, then $\Gamma_{\mathcal{C}_1}(M_n(D))$ is a connected graph.*

(ii) *If D is commutative and $\mathcal{C}_2$ is the set of all matrices in $M_n(D)$ such that their ranks are at most 2, then $\Gamma_{\mathcal{C}_2}(M_n(D))$ is a connected graph.*

Theorem 2.7. [5, Theorem 2.9] *Let F be a field with* $\operatorname{char} F = 0$ *and* $n \geqslant 3$ *a natural number. If $\mathcal{D}_n$ is the set of all diagonalizable matrices in $M_n(F)$, then $\Gamma_{\mathcal{D}_n}(M_n(F))$ is a connected graph.*

Now, we are going to generalize the commuting graph to a subset of operators on a Hilbert space.

Definition 2.8. Let H be a Hilbert space and S be a set of operators on H. The *commuting graph* of S, denoted by $\Gamma(S)$, is the graph with vertex set $S\backslash Z(S)$ such that distinct vertices a and b are adjacent if and only if $ab = ba$, where $Z(S)$ is the center of S.

Theorem 2.9. *Let $\mathcal{A}$ be the set of all non-zero bounded operators with zero as their eigenvalues on a separable Hilbert space $\mathcal{H}$ such that* $\dim(\mathcal{H}) \geqslant 3$. *Then $\Gamma(\mathcal{A})$ is a connected graph.*

Theorem 2.10. *Let $\mathcal{P}$ be the set of all non-zero, non-scalar projections on a separable Hilbert space $\mathcal{H}$ that* $\dim(\mathcal{H}) \geqslant 3$. *Then $\Gamma(\mathcal{P})$ is a connected graph.*

Theorem 2.11. *Let $\mathcal{N}$ be the set of all non-zero, nilpotent operators on a separable Hilbert space $\mathcal{H}$ that* $\dim(\mathcal{H}) \geqslant 3$. *Then $\Gamma(\mathcal{N})$ is a connected graph.*

Theorem 2.12. *Let $\mathcal{F}$ be the set of all non-zero finite rank operators on a separable Hilbert space $\mathcal{H}$ that* $\dim(\mathcal{H}) \geqslant 3$. *Then $\Gamma(\mathcal{F})$ is a connected graph.*

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THREE $2-(v,4,1)$ BLOCK DESIGNS HAVING A PRESCRIBED NUMBER OF BLOCKS IN COMMON

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Abstract. In this paper we investigate the existence of three $S(2,4,v)$ designs with prescribed number of blocks in common. Let $b_v = \frac{v(v-1)}{12}$ and $I_3[v] = \{0,1,\ldots,b_v\} \setminus \{b_v-7, b_v-6, b_v-5, b_v-4, b_v-3, b_v-2, b_v-1\}$. Let $J_3[v] = \{k|$ there exist three $S(2,4,v)$ designs with k same common blocks$\}$. We show that $J_3[v] \subseteq I_3[v]$ for any positive integer $v \equiv 1,4 \pmod{12}$.

1. Introduction

A *Stiener system* $S(2,4,v)$ is a pair $(\mathcal{V},\mathcal{B})$ where $\mathcal{V}$ is a v-element set and $\mathcal{B}$ is a family of 4-element subsets of $\mathcal{V}$ called *blocks*, such that each 2-element subsets of $\mathcal{V}$ is contained in exactly one block.

Two Steiner systems $S(2,4,v)$, $(\mathcal{V},\mathcal{B})$ and $(\mathcal{V},\mathcal{B}_1)$ are said to *intersect* in s blocks if $|\mathcal{B}\cap\mathcal{B}_1| = s$. The intersection problem for $S(2,4,v)$ designs can be extended in this way: determine the sets $\overline{J_\mu[v]}(J_\mu[v])$ of all integers s such that there exists a collection of μ (≥ 2) $S(2,4,v)$ designs mutually intersecting in s blocks (in the same set of s blocks). This generalization is called μ-way intersection problem. Clearly $\overline{J_2[v]} = J_2[v] = J[v]$ and $J_\mu[v] \subseteq \overline{J_\mu[v]} \subseteq J[v]$.

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* Speaker.

The intersection problem for $\mu = 2$ was considered by Colbourn, Hoffman, and Lindner in [2]. They determined the set $J[v](J_2[v])$ completely for all values $v \equiv 1, 4 \pmod{12}$, with some possible exceptions for $v = 25$, 28 and 37. Let $[a, b] = \{a, a+1, ..., b-1, b\}$, $b_v = \frac{v(v-1)}{12}$, and $I[v] = [0, b_v] \setminus ([b_v - 5, b_v - 1] \cup \{b_v - 7\})$. It is shown in [2]; that:

(1) $J[v] \subseteq I[v]$ for all $v \equiv 1, 4 \pmod{12}$.

(2) $J[v] = I[v]$ for all admissible $v \geq 40$.

(3) $J[13] = I[13]$ and $J[16] = I[16] \setminus \{7, 9, 10, 11, 14\}$.

(4) $I[25] \setminus \{31, 33, 34, 37, 39, 40, 41, 42, 44\} \subseteq J[25]$ and $\{42, 44\} \not\subseteq J[25]$.

(5) $I[28] \setminus \{44, 46, 49, 50, 52, 53, 54, 57\} \subseteq J[28]$.

(6) $I[37] \setminus (\{64, 66, 76, 82, 84, 85, 88\} \cup [90, 94] \cup [96, 101]) \subseteq J[37]$.

In this paper we investigate the three way intersection problem for $S(2, 4, v)$ designs. Also we apply some recursive constructions. But some small orders ($v \in \{25, 28, 37, 40\}$) are remained to characterize.

2. Necessary conditions

A (v, k, t) *trade* of volume s consists of two disjoint collections T_1 and T_2, each of s blocks, such that for every t-subset of blocks, the number of blocks containing these elements (t-subset) are the same in both T_1 and T_2. A (v, k, t) trade of volume s is *Steiner* when for every t-subset of blocks, the number of blocks containing these elements are at most one. A μ-way (v, k, t) trade $T = \{T_1, T_2, \dots, T_\mu\}$, $\mu \geq 2$ is a set of pairwise disjoint (v, k, t) trade.

In every collection the union of blocks must cover the same set of elements. This set of elements is called the *foundation* of the trade. Its notation is found (T) and $r_x =$ no. of blocks in a collection which contain the element x.

By definition of the trade, if $b_v - s$ is in $J_3[v]$, then it is clear that there exists a 3-way Steiner $(v, 4, 2)$ trade of volume s. Consider three $S(2, 4, v)$ designs (systems) intersecting in $b_v - s$ same blocks (of size four). The remaining set of blocks (of size four) form disjoint partial quadruple systems, containing precisely the same pairs, and each has s blocks. Rashidi and Soltankhah in [5] established that there do not exist a 3-way Steiner $(v, 4, 2)$ trade of volume s, for $s \in \{1, 2, 3, 4, 5, 6, 7\}$. So we have the following lemma:

Lemma 2.1. $J_3[v] \subseteq I_3[v]$.

3. Recursive constructions

Let K be a set of positive integers . A *group divisible design* K-GDD (as GDD for short) is a triple $(\mathcal{X}, \mathcal{G}, \mathcal{A})$ satisfying the following properties: (1) $\mathcal{G}$ is a partition of a finite set $\mathcal{X}$ into subsets (called groups); (2) $\mathcal{A}$ is a set of subsets of $\mathcal{X}$ (called blocks), each of cardinality from K, such that a group and a block contain at most one common element; (3) every pair of elements from distinct groups occurs in exactly one block. If $\mathcal{G}$ contains u_i groups of size g_i, for $1 \le i \le s$, then we denote by $g_1^{u_1} g_2^{u_2} \dots g_s^{u_s}$ the *group type* (or type) of the GDD. If $K = \{k\}$, we write $\{k\}$-GDD as k-GDD.

Theorem 3.1. *(Weighting construction). Let $(\mathcal{X}, \mathcal{G}, \mathcal{A})$ be a GDD with groups $G_1,\ G_2, \dots,\ G_s$. Suppose that there exists a function $w\ :\ X \to\ Z^+ \cup \{0\}$ (a weight function) so that for each block $A = \{x_1, \dots, x_k\} \in \mathcal{A}$ there exist three K-GDDs of type $[w(x_1),\ \dots\ w(x_k)]$ with b_A common blocks. Then there exist three K-GDDs of type $[\sum_{x \in G_1} w(x), \dots, \sum_{x \in G_s} w(x)]$ which intersect in $\sum_{A \in \mathcal{A}} b_A$ blocks.*

Theorem 3.2. *(Filling construction (i)). Suppose that there exist three 4-GDDs of type $g_1 g_2 \dots g_s$ which intersect in b blocks. If there exist three $S(2,4,g_i+1)$ designs with b_i common blocks for $1 \le i \le s$, then there exist three $S(2,4,\sum_{i=1}^s g_i + 1)$ designs with $b + \sum_{i=1}^s b_i$ common blocks.*

Theorem 3.3. *(Filling construction (ii)). Suppose that there exist three 4-GDDs of type $g_1 g_2 \dots g_s$ which intersect in b blocks. If there exist three $S(2,4,g_i+4)$ designs containing b_i common blocks for $1 \le i \le s$. Also all designs containing a block y. Then there exist three $S(2,4,\sum_{i=1}^s g_i + 4)$ designs with $b + \sum_{i=1}^s b_i - (s-1)$ common blocks.*

Let there be three $S(2,4,v)$ designs with a common parallel class, then $J_{p3}[v]$ for $v \equiv 4 \pmod{12}$ denotes the number of blocks shared by these $S(2,4,v)$ designs , in addition to those shared in the parallel class.

Lemma 3.4. *Let G be a GDD on $v = 3s + 6t$ elements with b blocks of size 4 and group type $3^s 6^t$, $s \ge 1$. For $1 \le i \le b$, let $a_i \in J_{p3}[16]$. For $1 \le i \le s-1$, let $c_i + 1 \in J_3[16]$ and let $c_s \in J_3[16]$. For $1 \le i \le t$, let $d_i + 1 \in J_3[28]$. Then there exist three $S(2,4,4v+4)$ designs with precisely $\sum_{i=1}^b a_i + \sum_{i=1}^s c_i + \sum_{i=1}^t d_i$ blocks in common.*

The *flower* of an element is the set of blocks containing that element. Let $J_{f3}[v]$ denote the number of blocks shared by three $S(2,4,v)$ designs, in addition to those in a required common flower.

Lemma 3.5. *Let G, B be a GDD of order v with b_4 blocks of size 4, b_5 blocks of size 5 and group type $4^s 5^t$. For $1 \le i \le b_4$, let $a_i \in J_{f3}[13]$. For $1 \le i \le b_5$, let $c_i \in J_{f3}[16]$. For $1 \le i \le s$, let $d_i \in J_3[13]$ and for $1 \le i \le t$, let $e_i \in J_3[16]$. Then there exist three $S(2,4,3v+1)$ designs intersecting in precisely $\sum_{i=1}^{b_4} a_i + \sum_{i=1}^{b_5} c_i + \sum_{i=1}^s d_i + \sum_{i=1}^t e_i$ blocks.*

Lemma 3.6. [3]. *The necessary and sufficient conditions for the existence of a 4-GDD of type g^n are: (1) $n \ge 4$, (2) $(n-1)g \equiv 0 \pmod 3$, (3) $n(n-1)g^2 \equiv 0 \pmod{12}$, with the exception of $(g,n) \in \{(2,4),(6,4)\}$, in which case no such GDD exists.*

Lemma 3.7. [?]. *There exists a $(v, \{4, 7^*\}, 1)$-PBD with exactly one block of size 7 for any positive integer $v \equiv 7, 10 \pmod{12}$ and $v \ne 10, 19$.*

Lemma 3.8. [3]. *A 4-GDD of type $12^u m^1$ exists if and only if either $u = 3$ and $m = 12$, or $u \ge 4$ and $m \equiv 0 \pmod 3$ with $0 \le m \le 6(u-1)$.*

4. Ingredients

Lemma 4.1. $J_3[13] = I_3[13]$.
Lemma 4.2. $J_3[16] = I_3[16] \setminus \{7, 9, 10, 11, 12\}$.
Lemma 4.3. $\{0, 23, 29, 50\} \subseteq J_3[25]$.
Lemma 4.4. $\{1, 63\} \subseteq J_3[28]$.
Lemma 4.5. *There exist three 4-GDDs of type* 4^4 *with* i *common blocks,* $i \in \{0, 1, 2, 4, 16\}$.
Corollary 4.6. $\{0, 1, 2, 4, 16\} \subseteq J_{p3}[16]$.
Lemma 4.7. *There exist three 4-GDDs of type* 3^5 *with* i *common blocks,* $i \in \{0, 1, 3, 15\}$.
Lemma 4.8. *There exist three 4-GDDs of type* 3^4 *with* i *common blocks,* $i \in \{0, 1, 9\}$.
Corollary 4.9. $\{0, 1, 9\} \subseteq J_{f3}[13]$.

5. main result

We obtain the following Theorem by applying recursive constructions and ingredients.

Theorem 5.1.
(1) $J_3[v] \subseteq I_3[v]$ *for all* $v \equiv 1, 4 \pmod{12}$.
(2) $J_3[v] = I_3[v]$ *for all admissible* $v \geq 49$.

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CONTINUOUS-TIME GARCH(1,1) PROCESS DRIVEN BY A HEAVY TAIL LÉVY PROCESS

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Abstract. Gaussian distribution is not a proper distribution to model many of the real-life time series. Heavy tail distributions such as the Pareto distribution have proved instrumental in providing a model for considering observation in finance, insurance, telecommunications, metrology and hydrology. Continuous-time GARCH(1,1) which is an extension of discreet one can be obtain using an extended version of Black-Scholes model given by a stochastic differential equation. In this paper, we consider the tail behavior of COGARCH(1,1) process driven by a heavy tail Lévy process.

1. Introduction

The classical pricing model is the Black-Scholes model given by the stochastic differential equation (SDE)

$$dS_t = rS_t dt + \sigma S_t dB_t\,, \qquad S_0 = x \in \mathbb{R}\,, \tag{1.1}$$

where $r \in \mathbb{R}$ is the stock-appreciation rate, $\sigma > 0$ is the volatility and B is a standard Brownian motion. The Black-Scholes model is based on the assumption that the relative price changes of the asset form a Gaussian process with stationary and independent increments. There exists a lot of extension for this model in the literature. We refer the

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reader to [1] and the collection of papers in [2] for specific examples and references. The theory of regular variation provides a proper and unified background for considering multivariate extremes when heavy tails are present; see [5] for the basic theory and [2] and [3] for some recent developments. One of interesting model that can be construct using an extension to the process (1.1) is continuous-time GARCH(1,1).

Klüppelberg, et al [4] considered discrete-time GARCH(1,1) model and replaced the noise variables by a Lévy process L with jumps $\Delta L_t = L_t - L_{t-}\,,\ t \geq 0$. This yields a stochastic volatility model of the type

$$\begin{aligned} dS_t &= \sqrt{V}_t dL_t^{(1)}\,, \\ dV_{t+} &= \beta dt + V_t e^{X_{t-}} d(e^{-X_t})\,, \end{aligned} \tag{1.2}$$

where $\beta > 0$ and V is left-continuous. This continuous-time GARCH (1,1) model is called a COGARCH(1,1) model. Now if $L^{(i)}$, $i = 1, 2$ are two independent Lévy process and an auxiliary càdlàg process X is defined by

$$X_t = \eta t - \sum_{0<s\leq t} \log(1 + \lambda e^{\eta} (\Delta L_s^{(2)})^2)\,, \tag{1.3}$$

for $\eta > 0$ and $\lambda \geq 0$, Then, the stationary volatility process has representation

$$V_t = \left(\beta \int_0^t e^{X_s} ds + V_0\right) e^{-X_{t-}}\,, \quad t \geq 0\,, \tag{1.4}$$

with $\beta > 0$ and $V_0 =^d \beta \int_0^\infty e^{-X_t} dt$, independent of L. The auxiliary process $(X_t)_{t\geq 0}$ itself is a spectrally negative Lévy process of bounded variation with drift $\gamma_X = \eta$, no Gaussian component, and Lévy measure ν_X given by

$$\nu_X[0,\infty) = 0\,, \nu_X(-\infty, -x] = \nu\Big(\{y \in \mathbb{R} : |y| \in \sqrt{(e^x - 1)/(\lambda e^{\eta})}\}\Big)\,. \tag{1.5}$$

Gaussian distribution fails to model many of the real-life time series which display clusters of outlying observations. Heavy tail distributions such as the Pareto distribution have proved instrumental in providing a model for a wide spectrum of bursty phenomena which are observed in finance, insurance, telecommunications, metrology and hydrology. A formulation of regular variation for multivariate stochastic processes on the unit interval on $\mathbb{D}$ and the tail behavior for filtered regularly varying lévy processes was considered in [3]. The aim of this paper is to consider the tail behavior of the COGARCH(1,1) model driven by heavy tail Lévy process.

2. Main results

Let $\mathbb{R}, \mathbb{Q}$ and $\mathbb{N}$ be the sets of real, rational and natural numbers, respectively. Regular variation on $\mathbb{D} = \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ is defined as the space of cadalag functions. This space is equipped with the J_1-metric that makes it complete and separable. We denote by $S_{\mathbb{D}}$ the subspace $\{\mathbf{x} \in \mathbb{D} : |\mathbf{x}|_\infty = 1\}$ (where $|\mathbf{x}|_\infty = \sup_{\mathbf{t}\in\mathbb{R}_+^d} |\mathbf{x}(\mathbf{t})|$) equipped with the subspace topology. Define $\overline{\mathbb{D}}_0 = (0, \infty] \times S_{\mathbb{D}}$ where $(0, \infty]$ is equipped with the metric $\rho(x, y) = |1/x - 1/y|$ making it complete and separable. Then, $\overline{\mathbb{D}}_0$ equipped with the metric $max\{\rho(x^*, y^*)\ ,\ d^\circ_\infty(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})\}$ is a complete separable metric space (CSMS). Hult and Lindskog [3] have shown that regular variation on $\mathbb{D}[0, 1]$ is naturally expressed in terms of so-called $\widehat{w}$-convergence of boundedly finite measures on $(0, \infty] \times S_{\mathbb{D}[0,1]}$. A boundedly finite measure assigns a finite measure to bounded sets. A sequence of boundedly finite measures $\{m_n : n \in N\}$ on a complete separable metric space E converges to m in the $\widehat{w}$- topology, $m_n \xrightarrow{\hat{w}} m$, if $m_n(B) \to m(B)$ for every bounded Borel set B with $m(\partial B) = 0$. A stochastic process $\mathbf{X} = \{\mathbf{X}_t : t \in [0, 1]\}$ with sample paths in $\mathbb{D}$ is said to be regularly varying if there exists a sequence $\{a_n\}, 0 < a_n \uparrow \infty$ and a nonzero boundedly finite measure m on $\mathcal{B}(\overline{\mathbb{D}}_0)$ with $m(\overline{\mathbb{D}}_0 \setminus \mathbb{D}) = 0$ such that, as $n \to \infty$

$$n\mathbb{P}(a_n^{-1}\mathbf{X} \in \cdot) \xrightarrow{\hat{w}} m(\cdot) \quad on \quad \mathcal{B}(\overline{\mathbb{D}}_0) \tag{2.1}$$

denoted by $\mathbf{X} \in RV(\{a_n\}, m, \overline{\mathbb{D}}_0)$.

Now, we show that the COGARCH(1,1) model driven by regularly varying Lévy process is also a regularly varying random process.

Theorem 2.1. *Let $L^{(1)}$ be a Lévy process with bounded jump and S_t be a COGARCH(1,1) random process (1.2) such that its auxiliary sequence satisfying $X \in RV_\alpha((a_n), \mu_1, \mathbb{R}_0)$ and $S_0 \in RV_\alpha((a_n), \mu_2, \overline{\mathbb{D}}_0)$. Then, $S_t \in RV_\alpha((a_n), \mu^*, \overline{\mathbb{D}}_0)$.*

Proof. Let $h : \mathbb{D} \to \mathbb{R}$ be defined by

$$h(\mathbf{x}) = \left(\beta \int_0^t \left(e^{x_s} + e^{-x_s}\right) ds + \beta \int_t^\infty e^{-x_s} ds\right) e^{-x_{t-}}\,. \tag{2.2}$$

The mapping h satisfies the conditions of Theorem 8 in [3] and V_t distributed as $h(X_t)$. Therefore, $V_t \in RV_\alpha((a_n), \mu_3, \mathbb{R}_0^d)$ with $\mu_3(\cdot) = \mu\ o\ h^{-1}(\cdot)$. But, by Theorem 5.2.1 the SDE (1.2) has unique solution $S_t = \int_0^t \sqrt{V_t} dL_t^{(1)}$. With a similar arguments as in the proof of Theorem 3.4 in [3], we can show that $\int_0^t \sqrt{V_t} dL_t^{(1)} \in RV_\alpha((a_n), \mu_4, \overline{\mathbb{D}}_0)$ with

$$\mu^*(\cdot) = E(\mu\{x \in \bar{\mathbb{R}}_0 : (\sqrt{V} - \sqrt{V}_U - \Delta\sqrt{V}_U) x 1_{[U,1]} \in B\})\,,$$

where U is uniformly distributed on $[0,1)$ and independent of V , and this completes the proof of Theorem. $\square$

The following result is also consider certain conditions under which the COGARCH(1,1) process has heavy tail.

Theorem 2.2. *Let $L^{(1)} \in RV_\alpha((a_n), \mu_1, \mathbb{R}_0)$ be a regularly varying Lévy process and S_t be a COGARCH(1,1) random process (1.2) such that its auxiliary sequence $EV^{(\alpha+\epsilon)/2} < \infty$ for some $\epsilon > 0$. Then, $S_t \in RV_\alpha((a_n), \mu_1^*, \overline{\mathbb{D}}_0)$.*

Proof. Using Theorem 3.4 in [3], we can obviously obtain that $\int_0^t \sqrt{V_t}\, dL_t^{(1)} \in RV_\alpha((a_n), \mu_1^*, \overline{\mathbb{D}}_0)$ with

$$\mu_1^*(\cdot) = E(\mu\{x \in \bar{\mathbb{R}}_0 : (\Delta\sqrt{V}_U)x1_{[U,1]} \in B\}) ,$$

where U is uniformly distributed on $[0,1)$ and independent of V , and this completes the proof of Theorem. $\square$

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OPERATOR EQUATION IN RANDOM BANACH ALGEBRAS

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Abstract. In this paper, we apply a fixed point theorem to solve the operator equation $AxBx = x$ in random Banach algebras under a nonlinear contraction.

1. Introduction

The purpose of this paper is to give a comprehensive text to the study of random Banach algebras such as the study of fixed point theory and operator equations in random Banach algebras.

In the other hand, the first important hybrid fixed point theorem due to Krasnoselskii [3] which combines the metric fixed point theorem of Banach with the topological fixed point theorem of Schauder in a Banach space has several applications to nonlinear integral equations that arise in the inversion of the perturbed differential equations. Many attempts have been made to improve and weaken the hypotheses of Krasnoselskii's fixed point theorem (see [1]). The study of the nonlinear integral equations in Banach algebras was initiated by Dhage [2] via fixed point theorems. In this paper, we apply a fixed point theorem to solve the operator equation $AxBx = x$ in the random Banach algebras under a nonlinear contraction.

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2. Random Normed Algebras

In the section, we adopt the usual terminology, notations and conventions of the theory of random normed spaces, then we consider random normed algebras. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0,1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all t in $\mathbb{R}$. For example an element for Δ^+ is the distribution function ε_a given by

$$\varepsilon_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a. \end{cases}$$

The maximal element for Δ^+ in this order is the distribution function ε_0.

Definition 2.1. ([5]) A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:
(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,1) = a$ for all $a \in [0,1]$;
(d) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1,0)$ (the Lukasiewicz t-norm).

Definition 2.2. ([5]) A *random normed space* (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm and μ is a mapping from X into D^+ such that the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \neq 0$;
(RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

Every normed space $(X, \|.\|)$ defines a random normed space (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 2.3. A *random normed algebra* (X, μ, T, T') is a random normed space (X, μ, T) with algebraic structure such that

(RN-4) $\mu_{xy}(ts) \geq T'(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s > 0$. in which T' is a continuous t-norm.

Every normed algebra $(X, \|\cdot\|)$ defines a random normed algebra (X, μ, T_M, T_P), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$ if and only if

$$\|xy\| \leq \|x\|\|y\| + s\|y\| + t\|x\| \ (x, y \in X;\ t, s > 0).$$

This space is called the induced random normed algebra.

Definition 2.4. Let (X, μ, T) be an RN-space.
(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.
(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_m - x_n}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.
(3) An RN-space (X, μ, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

3. MAIN RESULT

A mapping $T : X \longrightarrow X$ is called random $\mathcal{D}$-Lipschitzian if there exists a continuous and nondecreasing function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$\mu_{T(x) - T(y)}(t) \geq \mu_{x-y}(\phi(t)), \tag{1}$$

for $x, y \in X$ and $t > 0$, where $\phi(0) = 0$.

We call the function ϕ random $\mathcal{D}$-function of T on X. If ϕ is not necessarily nondecreasing and satisfies $\phi(r) < r$, for $r > 0$, the mapping T is called a nonlinear contraction with a contraction function ϕ. If $\phi(t) = \alpha t$ in the (1), the mapping $T : X \longrightarrow X$ (1) is called random Lipschitzian with Lipschitz constant α.

Let (X, μ, T) be a random Banach space and let $T : X \longrightarrow X$. Then T is called a compact operator if $\overline{T(X)}$ is a compact subset of X. Again T is called totally bounded if for any bounded subset S of X, $T(S)$ is a totally bounded set of X. Further, T is called completely continuous if it is continuous and totally bounded. Note that every compact operator is totally bounded, but the converse may not be true [4].

Theorem 3.1. *Let S be a closed, convex and bounded subset of a random Banach algebra (X, μ, T) and let $A : X \longrightarrow X$, $B : S \longrightarrow X$ be two operators such that*

(a) *A is random $\mathcal{D}$-Lipschitzian with a random $\mathcal{D}$-function ϕ,*

(b) *B is completely continuous, and*

(c) *$x = AxBy \Longrightarrow x \in S$, for all $y \in S$.*

Then the operator equation

$$AxBx = x \tag{2}$$

has a solution.

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CONVERGENCE RESULTS FOR GENERALIZED CONJUGATE GRADIENT METHOD FOR THE MATRIX EQUATION $AXB = C$

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Abstract. In this paper, we introduce an iterative method for solving the matrix equation $AXB = C$ in which A and B are symmetric positive definite. This method is based on global Conjugate Gradient method which is equivalent to an orthogonal projection. We study some convergence analysis of this method, such as expression and upper bound for the error and minimization property.

1. Introduction

In this paper, we consider the following matrix equations

$$AXB = C, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{s\times s}$ and $C \in \mathbb{R}^{n\times s}$. Furthermore, we assume that the matrices A and B are symmetric positive definite (SPD).

Some convergence properties of the special case $B = I_s$, whether or not $s = 1$, have been studied by many authors. For more details, we refer to [2, 4, 5] and the references therein. For example, the Conjugate Gradient (CG) method is a well-known method that was originally developed to solve the equation $Gx = g$ where G is a SPD matrix of

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suitable dimension. Let x_0 be an initial guess, $r_0 = g - Gx_0$ be its corresponding residual and x_* be the solution of $Gx = g$. The CG constructs a sequence of approximate solutions $\{x_m\}$ satisfying

$$\|x_* - x_m\|_G = \min_{x \in x_0 + \mathcal{K}_m(G, r_0)} \|x_* - x\|_G,$$

where $\mathcal{K}_m(G, r_0) = span\{r_0, Gr_0, \ldots, G^{m-1}r_0\}$ is the Krylov subspace and $\|\cdot\|_G$ denotes the well-known $G-$norm.

The aim of this paper is to study the benefits that the CG method can bring for solving the matrix equation (1.1). First, we described generalized Conjugate Gradient (GCG) method which is based on the Conjugate Gradient method. Then, we present that the GCG is equivalent to an orthogonal projection. Finally, we gave some theoretical results for the GCG, such as expression, minimization property and some upper bounds for the $(A, B)-$norm of the GCG error.

Let $X, Y \in \mathbb{R}^{n\times s}$ and $Z \in \mathbb{R}^{p\times q}$. The Kronecker product $X \otimes Z$ is defined by $X \otimes Z = [x_{ij}Z]$. The vector $\mathrm{vec}(X)$ denotes the vector of $\mathbb{R}^{ns}$ defined by $\mathrm{vec}(X) = [x_1^T, \ldots, x_s^T]^T$ where x_i is the ith column of X. The Frobenius inner product is defined by $\langle X, Y\rangle_F = \mathrm{trace}(Y^T X)$.

Proposition 1.1. *The map* $\langle\cdot,\cdot\rangle_{(A,B)} : \mathbb{R}^{n\times s} \times \mathbb{R}^{n\times s} \to \mathbb{R}$ *defined as follows is an inner product denoted by* (A, B)*-inner product.*

$$\langle X, Y\rangle_{(A,B)} = \mathrm{trace}(Y^T AXB).$$

In addition, the associated norm of the (A, B)*-inner product is*

$$\|X\|_{(A,B)} = \|\mathrm{vec}(X)\|_{B\otimes A}, \qquad \forall X \in \mathbb{R}^{n\times s}.$$

Let $E = [E_1, E_2, \ldots, E_p] \in \mathbb{R}^{n\times ps}$ and $F = [F_1, F_2, \ldots, F_l] \in \mathbb{R}^{n\times ls}$, where E_i and F_j are $n\times s$ matrices. The matrices $E^T \diamond F$ and $E^T \diamond_{(A,B)} F$ are defined by $(E^T \diamond F)_{ij} = \langle E_i, F_j\rangle_F$ and $(E^T \diamond_{(A,B)} F)_{ij} = \langle E_i, F_j\rangle_{(A,B)}$, respectively. We should add that the $\diamond$ product is introduced in [1]. Finally, the following definition of [3, p. 411] is a basic tool for us.

Definition 1.2. Let $p_k(x, y) = \sum_{i,j=0}^{k} c_{ij}x^i y^j$ be a polynomial in two variables with real coefficients c_{ij}. If $C \in \mathbb{R}^{m\times m}$ and $D \in \mathbb{R}^{n\times n}$, then $p_k(C : D)$ is defined a matrix of the form $p_k(C : D) = \sum_{i,j=0}^{k} c_{ij}(C^i \otimes D^j)$.

2. Main results

Suppose that $G = B \otimes A$, $x_0 \in \mathbb{R}^{ns}$ and $r_0 = g - Gx_0$. It can be shown that the mth iterate x_m of the CG minimizes $\|x_* - x\|_{B\otimes A}$ over $x_0 + \mathcal{K}_m(B \otimes A, r_0)$ if and only if

$$x_m \in x_0 + \mathcal{K}_m(B \otimes A, r_0) \quad \text{such that} \quad g - (B \otimes A)x_m \perp \mathcal{K}_m(B \otimes A, r_0).$$

Suppose that $\mathcal{GK}_i(A, V, B) = \text{span}\{V, AVB, \ldots, A^{i-1}VB^{i-1}\}$. Hence, the map $T : \mathcal{GK}_i(A, V, B)) \to \mathcal{K}_i(G, v)$ given by $X \mapsto T(X) = \text{vec}(X)$ is an isomorphism. On the other hand, the matrix equation (1.1) can equivalently be written in the form of $(B \otimes A)\text{vec}(X) = \text{vec}(C)$. Now, in view of these facts, we introduce generalized Conjugate Gradient (CGC) method as follows.

Algorithm 1 The generalized Conjugate Gradient (CGC) method.

1: Choose X_0 and a tolerance $\varepsilon > 0$.
2: Compute $R_0 = C - AX_0B$ and set $P_0 = R_0$.
3: **for** $i = 0, 1, 2 \ldots$ **do**
4: $\quad \alpha_i = \frac{\langle R_i, R_i \rangle_F}{\langle P_i, P_i \rangle_{(A,B)}}$,
5: $\quad X_{i+1} = X_m + \alpha_i P_i$,
6: $\quad R_{i+1} = R_i - \alpha_i A P_i B$,
7: $\quad \beta_i = \frac{\langle R_{i+1}, R_{i+1} \rangle_F}{\langle R_i, R_i \rangle_F}$,
8: $\quad P_{i+1} = R_{i+1} + \beta_i P_i$,
9: $\quad$ If $\|R_i\|_F \le \varepsilon$ then stop and X_{i+1} is a desirable approximation of the solution, else goto 3.
10: **end for**

From the description of the GCG method, the following result is obtained.

Theorem 2.1. *Let $X_0 \in \mathbb{R}^{n \times s}$ and $R_0 = C - AX_0B$. The GCG method constructs the approximate solutions X_m satisfying*

$$X_m \in X_0 + \mathcal{GK}_m(A, R_0, B), \tag{2.1}$$

such that

$$C - AX_mB \perp_F \mathcal{GK}_m(A, R_0, B). \tag{2.2}$$

We observe that the CGC method is equivalent to an orthogonal projection. In the following, by using this fact, we will derive some convergence properties of the GCG method. Let $\mathcal{P}_m = [P_0, P_1, \ldots, P_{m-1}]$ and $\mathcal{V}_{m+1} = [V_0, V_1, \ldots, V_m]$ in which the matrices P_i introduced in Algorithm 1 and $V_i = R_i/\|R_i\|_F$. If we define $\mathcal{T}_m = \mathcal{V}_m^T \diamond_{(A,B)} \mathcal{P}_m$ and $\tilde{\mathcal{T}}_m = \mathcal{V}_{m+1}^T \diamond_{(A,B)} \mathcal{P}_m$, then the following results are established.

Theorem 2.2. *Assume that $m+1$ steps of the CGC method have been executed. Then we have*

(I) *The matrix $\tilde{\mathcal{T}}_m$ is a upper Hessenberg matrix.*

(II) $A\mathcal{P}_m(I_m \otimes B) = \mathcal{V}_m(\mathcal{T}_m \otimes I_s) + t_{m+1,m}V_m(e_m^T \otimes I_s)$, *where $t_{m+1,m}$ is the $(m+1, m)$ entry of $\tilde{\mathcal{T}}_m$.*

(III) $X_m = X_0 + \mathcal{P}_m(y_m \otimes I_s)$ *such that* $\mathcal{T}_m y_m = \mathcal{V}_m^T \diamond R_0$.

(IV) $R_m = -t_{m+1,m}(e_m^T y_m)V_m$ *and* $\|R_m\|_F = |t_{m+1,m}(e_m^T y_m)|$.

We present that the (A, B)-norm of the GCG error has the minimization property.

Theorem 2.3. *X_m is the mth approximation of the GCG method if and only if*

$$\|X_* - X_m^{or}\|_{(A,B)} = \min_{X \in X_0 + \mathcal{GK}_m(A,R_0,B)} \|X_* - X\|_{(A,B)},$$

where X_* is the solution of the matrix equation (1.1).

By considering the minimization property of the $(A, B)-$norm of the GCG error, we will try to establish more properties of the error.

Theorem 2.4. *The error $X_* - X_m$ satisfies the following relations*

$$\|X_* - X_m\|_{(A,B)} = \|R_m\|_F \sqrt{V_{m+1}^T \diamond_{(A^{-1},B^{-1})} V_{m+1}}, \tag{2.3}$$

$$\|X_* - X_m\|_{(A,B)} \leq \|R_m\|_F \sqrt{\frac{\kappa(A)\kappa(B)}{\|A\|_2 \|B\|_2}}, \tag{2.4}$$

$$\|X_* - X_m\|_{(A,B)} \leq \min_{\substack{p \in \mathbb{P}_m \\ p(0,0)=1}} \max_{\substack{\lambda \in \sigma(B) \\ \mu \in \sigma(A)}} |p(\lambda, \mu)| \|X_* - X_0\|_{(A,B)}, \tag{2.5}$$

$$\|X_* - X_m\|_{(A,B)} \leq \frac{\|R_m\|_F}{\|R_0\|_F} \sqrt{\kappa(A)\kappa(B)} \|X_* - X_0\|_{(A,B)}, \tag{2.6}$$

where $\mathbb{P}_m$ denotes the set of polynomials in two variables x, y of the form $p_m(x, y) = \sum_{i=0}^m c_i x^i y^i$, and $\kappa(Z) = \|Z\|_2 \|Z^{-1}\|_2$.

Remark 2.5. It is important to notice that the presented bounds in the above theorem depend on the initial guess and the spectral informations of A and B.

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AN ORTHOGONALITY IN NORMED LINEAR SPACES BASED ON ANGULAR DISTANCE INEQUALITY

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Abstract. In this paper, we present an orthogonality in a normed linear space which is based on an angular distance inequality. Main properties of this orthogonality is disscussed. We also find another version of the Singer orthogonality in terms of an angular distance inequality and compare these orthogonalities with each other.

1. Introduction

The notion of orthogonality goes a long way back in time and various extensions have been introduced over the last decades. In particular, proposing the notion of orthogonality in normed linear spaces has been the object of extensive efforts of many mathematicians. The most natural notion of orthogonality arises in the case where the norm $\|.\|$ derives from an inner product. In this case $x \perp y$ if and only if $\langle x, y\rangle = 0$. The notion of orthogonality in an inner product space has interesting properties. Some of them are listed as follows:

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(1) $\lambda x \perp \mu x$ if and only if $\|\lambda \mu x\| = 0$ for all $\lambda, \mu \in \mathbb{R}$ (Non-degeneracy).
(2) For every $x, y \in X, x \neq 0$, there exists a real number a such that $x \perp ax + y$ $(ax + y \perp x)$ (*Right* (*left*) *existence*).
(3) $x \perp y$ implies $\mu x \perp \lambda y$ for all $x, y \in X$ and $\lambda, \mu \in \mathbb{R}$ (Homogeneity).

In 1957, Singer [4] introduced the following orthogonality in a normed linear space:

$$x \perp_S y \quad \text{either} \quad \|x\|\|y\| = 0 \quad \text{or} \quad \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|.$$

Singer orthogonality is based on the concept of angular distance between nonzero vectors x and y in a normed linear space $(X, \|.\|)$ which was introduced as $\alpha[x, y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$ by Clarkson [5] in 1936. Some other known orthogonalities in normed linear spaces can be found in [1, 2] and references therein. If the norm $\|.\|$ derives from an inner product $\langle ., . \rangle$, then the generalized orthogonality relations reduce to the classical condition $\langle x, y \rangle = 0$. One way to obtain characterizations of inner product spaces and other geometric properties of the space by means of properties of the norms is to force the generalized orthogonalities to fulfill some of the above properties of orthogonality. [3]

Our new definition is also based on the concept of angular distance, in fact it is formulated as an angular distance inequality. In this paper $(X, \|.\|)$ always denotes a real normed linear space and S_X is the corresponding unit sphere.

2. Orthogonality and angular distance

In this section, we present an orthogonality in a normed linear space $(X, \|.\|)$, which is based on an angular distance inequality.

Definition 2.1. Let $(X, \|.\|)$ be a normed linear space and $x, y \in X$. We say that x is orthogonal to y and we will denote it by the notation $x \perp_+ y$, if $\|x\|\|y\| = 0$ or the following two statements hold:

(i) $\{x, y\}$ is linearly independent.

(ii) $$\alpha[x + ty, y] + \alpha[x + ty, -y] \leq \alpha[x, y] + \alpha[x, -y] \qquad \textit{for all } t \in \mathbb{R}. \tag{2.1}$$

We note that if $\{x, y\}$ is an independent set, then $x + ty \neq 0$, for all $t \in \mathbb{R}$ and so inequality (2.1) is well defined.

Lemma 2.2. *Let X be an inner product space and let $x, y \in X$ be two independent vectors and $t \in \mathbb{R}$ be arbitrary. Then the following two*

inequalities are equivalent.
(i) $\alpha[x+ty,y]+\alpha[x+ty,-y]\le\alpha[x,y]+\alpha[x,-y]$,
(ii) $\alpha[x+ty,y]\alpha[x+ty,-y]\le\alpha[x,y]\alpha[x,-y]$.

In the following theorem we will show that in an inner product space Definition 2.1 is equivalent with the standard definition of orthogonality.

Theorem 2.3. *Let X be an inner product space and $x,y\in X$ be two independent vectors. Then $x\perp_+ y$ if and only if $\langle x,y\rangle=0$.*

It is obvious that the orthogonality satisfies non-degeneracy, continuity and simplification. Now we will discuss homogeneity and left existence properties of the orthogonality.

Theorem 2.4. *The orthogonality is left existent and homogenous.*

The following example shows that the orthogonality is not right existent in general.

Example 2.5. Let X be a Minkowski plane with the l_∞ norm and let $x=(0,1)$ and $y=(1,0)$. Then $x\not\perp_+ ax+y$ for all $a\in\mathbb{R}$.

3. Another version of Singer orthogonality

In this section we express another orthogonality relation in terms of an angular distance inequality. We show that this notion of orthogonality is equivalent with Singer orthogonality. In fact this notion of orthogonality can be considered as another version of Singer orthogonality in terms of an angular distance inequality.

Definition 3.1. Let $(X,\|.\|)$ be a normed linear space and $x,y\in X$. We say that x is orthogonal to y and we will denote it by the notation $x\perp_- y$ if $\|x\|\|y\|=0$ or the following two statements hold:

(i) $\{x,y\}$ is linearly independent.

(ii) $$|\alpha[x,-y]-\alpha[x,y]|\le|\alpha[x+ty,-y]-\alpha[x+ty,y]|\quad for\ all\ t\in\mathbb{R}. \tag{3.1}$$

We note that if $\{x,y\}$ is an independent set, then $x+ty\neq 0$ for all $t\in\mathbb{R}$ and so inequality (3.1) is well defined.

Theorem 3.2. *Let $(X,\|.\|)$ be a normed linear space. Then the orthogonality $\perp_-$ is equivalent with the Singer orthogonality.*

Lemma 3.3. *Let $x,y\in X$ be two linearly independent vectors and let $h(t)=\alpha[x+ty,-y]-\alpha[x+ty,y]$. Then there exists a unique $t_0\in\mathbb{R}$ such that*

$$|h(t_0)|\le|h(t)|,\quad for\ \ all\ \ t\in\mathbb{R}.$$

Now by using this version of Singer orthogonality we define acute and obtuse angles as follows:

Definition 3.4. Let $(X, \|.\|)$ be a normed linear space and $x, y \in X$ be two independent vectors. The angle between x and y is called an acute angle if there exists a unique number $t_0 \in (-\infty, 0)$ such that

$$|\alpha[x+t_0y, -y]-\alpha[x+t_0y, y]| \leq |\alpha[x+t, -y]-\alpha[x+t, y]| \text{ for all } t \in \mathbb{R}. \tag{3.2}$$

The angle between x and y is called an obtuse angle if there exists a unique number $t_0 \in (0, \infty)$ such that

$$|\alpha[x+t_0y, -y]-\alpha[x+t_0y, y]| \leq |\alpha[x+t, -y]-\alpha[x+ty, y]| \text{ for all } t \in \mathbb{R}. \tag{3.3}$$

Remark 3.5. Let X be an inner product space and $x, y \in X$ be two independent vectors. Then the angle between x, y is acute (obtuse) in terms of Definition 3.4 if and only if the angle is acute (obtuse) with the standard definitions of acute and obtuse angles in X.

Proposition 3.6. *With respect to Definition 3.4, the angle between two linearly independent vectors x and y is acute (obtuse) if and only if the angle between y and x is acute (obtuse).*

Although in an inner product space the orthogonality relations $\perp_+$ and $\perp_-$ (Singer orthogonality) coincide with each other, the following remark shows that these orthogonalities are different in a general normed linear space.

Remark 3.7. Let X be a Minkowski plane with the l_∞ norm and let $x = (1, 0)$ and $y = (0, 1)$. Then $x \perp_S y$, but $x \not\perp_+ y$. On the other hand if we consider the vectors $x = (1, 1)$ and $y = (0, 1)$ we will see that $x \perp_+ y$ but $x \not\perp_S y$.

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APPROXIMATION OF THE MATRIX GEOMETRIC MEAN

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Abstract. The computation of the geometric mean of two positive definite matrices is motivated by the need to solve some nonlinear matrix equations. However, the integral representation is an applicable tool for evaluating geometric mean of two matrices. In this article, an efficient algorithm is proposed to the computation of the matrix geometric mean is considered by exploiting some approximation approaches.

1. Introduction

Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian positive definite matrices. The geometric mean of two matrices A and B is defined by [1]

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}. \tag{1.1}$$

However, there is an agreement about the definition of the matrix geometric mean, G, of two positive definite matrices A and B, viz $G = A(A^{-1}B)^{1/2}$ [3]. Other definitions of G and also some numerical scheme for the estimation of $A\#B$ have been given in [1, 2]. In this paper, we discuss the computation of geometric mean of two positive definite matrices. A new integral representation will be proposed and it will be approximated by using composite trapezoid rule and Gauss

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quadrature method. An algorithm will be recommended for computing matrix geometric mean with feasible accuracy.

2. Integral representation

Consider the function $\varphi : \mathbb{C} \to \mathbb{C}^{n\times n}$ that is defined as $\varphi(z) = P+Qz+Pz^2$, where $P, Q \in \mathbb{C}^{n\times n}$, and P is not the null matrix [3]. The function $\varphi(z)$ is a quadratic matrix polynomial which is called a "quadratic palindromic matrix polynomial". Iannazzo and Meini [3] have stated that the invertibility domain of the palindromic Laurent matrix polynomial $\mathcal{L}(z) = Pz^{-1} + Q + Pz$, is obtained by multiplying $\varphi(z)$ by z^{-1}. They show that the Laurent matrix polynomial is invertible in an annulus containing the unit circle, and its inverse $\mathcal{H}(z) = \mathcal{L}^{-1}(z)$ has a power series expansion $\mathcal{H}(z) = H_0 + \sum_{i=1}^{\infty} H_i(z^i + z^{-i})$. Since the function $\mathcal{H}(z) = \mathcal{L}^{-1}(z)$ is convergent in an annulus containing the unit circle, it is expressed that

$$A\#B = H_0 = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}(e^{i\theta})d\theta \tag{2.1}$$

where $\mathcal{H}(e^{i\nu}) = (Q + 2P\cos\theta)^{-1}$. By using equation (2.1), an integral form for $A\#B$ can be obtained. If $P = \frac{1}{4}(A^{-1} - B^{-1})$ and $Q = \frac{1}{2}(A^{-1} + B^{-1})$, therefore $\mathcal{H}(e^{i\nu}) = 2B\left((1+\cos\theta)B + (1-\cos\theta)A\right)^{-1} A$. Hence, using equation (2.1), we give a general integral representation as follows

$$A\#B = \frac{1}{\pi}B\left(\int_0^{2\pi} \left((1+\cos\theta)B + (1-\cos\theta)A\right)^{-1} d\theta\right)A. \tag{2.2}$$

Note that the proposed formula for $A\#B$ is completely different to that given in [3].

2.1. Composite trapezoid rule.

In order to calculate the integral (2.2) more accurately, one can split the interval of integration $[a, b]$ into N smaller uniform subintervals, and then apply the trapezoidal rule for each one. The composite trapezoidal rule is expressed as follows [4]:

$$\int_a^b f(\xi)d\xi \approx \frac{h}{2}\left(f(\xi_0) + 2\sum_{j=1}^{N-1} f(\xi_j) + f(\xi_n)\right) \tag{2.3}$$

where $h = \frac{b-a}{N}$ and $\xi_j = jh, (j = 1, \ldots, N-1)$. Since $f(\xi_0) = f(\xi_n)$ the equation (2.3) can be reformulated as

$$\int_0^{2\pi} f(\xi)d\xi \approx \frac{2\pi}{N}\sum_{j=0}^{N-1} f(\xi_j). \tag{2.4}$$

Thus, we can propose the following identity

$$\int_0^{2\pi} ((1+\cos\theta)B + (1-\cos\theta)A)^{-1}\, d\theta \approx \frac{2\pi}{N}\sum_{j=0}^{N-1} \Psi(\theta_j) \tag{2.5}$$

where $\Psi(\theta) = ((1+\cos\theta)B + (1-\cos\theta)A)^{-1}$. Consequently, by simple arrangement, the applicable formula can be obtained as

$$A\#B \approx \frac{2}{N} B \left(\sum_{j=0}^{N-1} \left((1+\cos(\frac{2\pi}{N}j))B + (1-\cos(\frac{2\pi}{N}j))A \right)^{-1} \right) A. \tag{2.6}$$

It is pointed out that the elements of $(1+\cos(\frac{2\pi}{N}j))B+(1-\cos(\frac{2\pi}{N}j))A$ can be evaluated by multiplying the values of $(1 + \cos(\frac{2\pi}{N}j)), (1 - \cos(\frac{2\pi}{N}j))$ to the elements of the matrices A, B.

2.2. **Quadratures rule.** An n-point Gaussian quadrature rule yields an exact result for polynomials of degree $2n-1$ or less by an appropriate choice of the points t_j and weights w_j for $j = 1, \ldots, n$. The domain of integration is taken as $[-1, 1]$, so the rule is stated as [4]

$$\int_{-1}^{1} f(\zeta)d\zeta \approx \sum_{j=1}^{n} w_j f(t_j) \tag{2.7}$$

This, an integral over $[a, b]$ has to be changed into an integral over $[-1, 1]$ before the Gaussian quadrature rule can be applied. This change of interval can be done by considering the following variable change $\zeta = \frac{b-a}{2}t + \frac{b+a}{2}$. Then we can write precisely as

$$\begin{aligned} \textstyle\int_a^b f(\zeta)d\zeta &= \tfrac{b-a}{2} \textstyle\int_{-1}^{1} f(\tfrac{b-a}{2}t + \tfrac{b+a}{2})dt \\ &\approx \tfrac{b-a}{2} \textstyle\sum_{j=1}^{n} w_j f(\tfrac{b-a}{2}t_j + \tfrac{b+a}{2}). \end{aligned} \tag{2.8}$$

By using $\zeta = \pi t + \pi$, we have

$$\int_0^{2\pi} f(\zeta)d\zeta \approx \pi\left((1+\cos(\pi t+\pi))B + (1-\cos(\pi t+\pi))A\right)^{-1} dt. \tag{2.9}$$

Finally, we obtain

$$A\#B \approx B \left(\sum_{i=1}^{n} w_i \left((1-\cos(\pi t_i))B + (1+\cos(\pi t_i))A\right)^{-1} \right) A. \tag{2.10}$$

Notice that in our implementations, we have used 64-points Gaussian quadratures rule to ocquire better accuracy.

2.3. **New algorithm.** Assume $\widetilde{A} = \frac{A}{\|A\|}$ and $\widetilde{B} = \frac{B}{\|B\|}$ from which it is clear that $\rho(\widetilde{A}) < 1$ and $\rho(\widetilde{B}) < 1$. Thus, according to properties of the matrix geometric mean of two matrices it can be written

$$\widetilde{A}\#\widetilde{B} = \frac{A}{\|A\|}\#\frac{B}{\|B\|} = \frac{1}{\sqrt{\|A\|.\|B\|}}(A\#B)$$

Therefore, the proposed procedure for the evaluation of the matrix geometric mean is given in following algorithm.

Algorithm: `(Computing the geometric mean of two matrices)`

(1) Set $\widetilde{A} = \frac{A}{\|A\|}$ and $\widetilde{B} = \frac{B}{\|B\|}$;

(2) Using the relation (2.6) or (2.10), compute $\widetilde{A}\#\widetilde{B}$;

(3) Evaluate $A\#B = \sqrt{\|A\|.\|B\|}(\widetilde{A}\#\widetilde{B})$;

(4) End.

It should be mentioned that in the implementation, any matrix norm can be considered. Clearly, the computation expenditure is the inversion of a positive matrix, that is n^3 operations, for each node of the quadrature and two matrix multiplication at the end.

3. Conclusions

According to our implementations, it is concluded that accuracy will be considerably improved by increasing the points in composite trapezoid rule. Furthermore, by increasing the dimension naturally condition number becomes large and consequently accuracy will decline. Finally, the CPU time in seconds was measured and it was seen that CPU time also significantly rose by increasing dimension particularly in ill-conditioned matrices.

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A PRESERVER PROBLEM IN STATISTICAL QUANTUM

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Abstract. Let $\mathscr{H}$ be a finite dimensional complex Hilbert space, $\mathbb{B}(\mathscr{H})_+$ be the set of all positive semi-definite operators (matrices) on $\mathscr{H}$ and φ is a (not necessarily linear) map of $\mathbb{B}(\mathscr{H})_+$ preserving the generalized Helmholtz free energy. In this paper, under suitable conditions we prove that there exists either a unitary or an anti-unitary operator U on $\mathscr{H}$ such that $\varphi(A) = UAU^*$ for any $A \in \mathbb{B}(\mathscr{H})_+$.

1. Introduction

Let $\mathscr{H}$ be a finite dimensional complex Hilbert space. Let $\mathbb{B}(\mathscr{H})$ denote the algebra of all bounded linear operators (matrices) on a finite dimensional Hilbert space $\mathscr{H}$ equipped with Hilbert-Schmidt inner product. As usual, an operator $A \in \mathbb{B}(\mathscr{H})$ is called positive semi-definite if $\langle Ax, x\rangle \geq 0$. The set of all positive semi-definite operators on $\mathscr{H}$ is denoted by $\mathbb{B}(\mathscr{H})_+$. Denote $\mathbb{S}(\mathscr{H})$ the space of all statistical operators on $\mathscr{H}$, i.e., positive semi-definite operators with unit trace. The spectrum of an operator A is denoted by $\mathrm{sp}(A)$. If $|x\rangle$ and $\langle y|$ are in $\mathscr{H}$, then $|x\rangle\langle y|$ stands for the rank one operator defined by

$$(|x\rangle\langle y|)\ |z\rangle \quad = \quad \langle z|y\rangle\ |x\rangle\,.$$

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Note that

$$\mathrm{sp_p}(|x\rangle\langle y|) \quad = \quad \{0, \langle z|y\rangle\},$$

where $\mathrm{sp_p}$ means the point spectrum. Recall that the point spectrum $\mathrm{sp_p}(A)$ of an operator A is $\{\lambda \in \mathbb{C} : Ax = \lambda x \text{ for some } x \neq 0\}$.
The support of an operator A is standing for the orthogonal complement of the kernel of A and denoted by supp A. For $-I \leq A \leq I$ and $\lambda \in (-1,0) \cup (0,1)$, we denote the generalized exponential function by $\exp_\lambda(A) = (I + \lambda A)^{\frac{1}{\lambda}}$. As the inverse function of $\exp_\lambda(.)$, for $A \geq 0$ and $\lambda \in (-1,0) \cup (0,1)$, we denote the generalized logarithmic function by $\ln_\lambda(A) = \frac{A^\lambda - I}{\lambda}$. The Tsallis relative entropy and the Tsallis entropy for non-negative matrices A and B are defined by $\mathrm{S}_\lambda(A\|B) = \mathrm{tr}[A^{1-\lambda}(\ln_\lambda A - \ln_\lambda B)]$ and $\mathrm{S}_\lambda(A) = -\mathrm{S}_\lambda(A, I)$ respectively. Also, Umegaki relative entropy is defined by $\lim_{\lambda \longrightarrow 0} \mathrm{S}_\lambda(A) = \mathrm{S}(A) = -\mathrm{tr}(A \log A)$ and $\lim_{\lambda \longrightarrow 0} S_\lambda(A\|B) = \mathrm{S}(A\|B) = \mathrm{tr}(A(\log A - \log B)$. Here log stands for the logarithm with base 2.
Recall that the quantum observables are modeled by selfadjoint operators. In what follows suppose that the energy operator H is a selfadjoint operator. The statistical mean value of the state described by the statistical operator A is $\mathrm{E} = \mathrm{tr}(AH)$ and the free energy of the state is $\mathrm{tr}(AH) - \theta \mathrm{S}(A)$, where $\theta = kT$, k is the Boltzmann constant and T is the absolute temperature.
In thermodynamics, the Helmholtz free energy is a thermodynamic potential which measures the useful work obtainable from a closed thermodynamic system at a constant temperature and volume. Hiai and Petz [2, 4] extended the thermodynamic inequality by proving
$\log \mathrm{tr}(e^{H + \log B}) = \max\{\mathrm{tr}(AH) - S(A\|B) : A \in \mathbb{B}(\mathscr{H})_+\}$, where $B > 0$ and H is a selfadjoint matrix. There are some related works in the literature: cf. [1, 4] and references therein. In 2008, Furuichi [1] studied on some matrix trace inequalities related to the Tsallis relative entropy. He defined a generalized Helmholtz free energy by $F_\lambda(A, H) = \mathrm{tr}(A^{1-\lambda}H) - \|H\|\mathrm{S}_\lambda(A)$. Now, we promote it by
$F_\lambda(A, B, H) = \mathrm{tr}(A^{1-\lambda}H) - \|H\|\mathrm{S}_\lambda(A\|B)$, where $\lambda \in (-1,0) \cup (0,1)$.

2. Main results

Lemma 2.1. *Let $\varphi : \mathbb{B}(\mathscr{H})_+ \longrightarrow \mathbb{B}(\mathscr{H})_+$ be a map such that*

$$F_\lambda(A, B, H) \quad = \quad F_\lambda(\varphi(A), \varphi(B), \varphi(H)).$$

Then $trAH = tr\ \varphi(A)\varphi(H)$, where A is a rank one projection of $\mathbb{B}(\mathscr{H})_+$.

Proposition 2.2. *Let $A, B \in \mathbb{B}(\mathscr{H})_+$ be two positive semi-definite operators. Then* $S_\lambda(A\|B) = \sum_{a\in sp(A)} \sum_{b\in sp(B)} a^{1-\lambda}(\ln_\lambda a - \ln_\lambda b)\ tr\ P_a Q_b$.

Remark 2.3. If A is a positive non-invertible operator in $\mathbb{B}(\mathscr{H})_+$, then we can define A^{-1} is the generalized inverse defined on the range of A, i.e., $AA^{-1} = A^{-1}A = \text{supp } A$. So that we can adopt the convention that $0^{-1} = 0$, and also $\dfrac{0}{0} = 0$ and $0.\infty = 0$. We know if $a \neq 0$ is a positive real number then:

$$\left[a^{1-\lambda}(\ln_\lambda a - \ln_\lambda b)\right] = -a \ln_\lambda ba^{-1}. \tag{2.1}$$

Therefore, by considering that the above convention the equality (2.1) is true even if $a = 0$. Now we claim that $F_\lambda(A, B, H) < \infty$ if and only if supp $A \subseteq$ supp B.

Lemma 2.4. *Let A and B be two positive operators in $\mathbb{B}(\mathscr{H})_+$. Then $F_\lambda(A, B, H)$ is finite if and only if supp $A \subseteq$ supp B.*

Lemma 2.5. *Let $\varphi : \mathbb{B}(\mathscr{H})_+ \longrightarrow \mathbb{B}(\mathscr{H})_+$ be a map such that $F_\lambda(A, B, H) = F_\lambda(\varphi(A), \varphi(B), \varphi(H))$. Then φ preserves the rank of the elements of $\mathbb{B}(\mathscr{H})_+$.*

Lemma 2.6. *Let $\varphi : \mathbb{B}(\mathscr{H})_+ \longrightarrow \mathbb{B}(\mathscr{H})_+$ be a unital map such that $F_\lambda(A, B, H) = F_\lambda(\varphi(A), \varphi(B), \varphi(H))$ and $\|H\| = \|\varphi(H)\|$. Then*

$$tr \ln_\lambda B = tr \ln_\lambda \varphi(B), \tag{2.2}$$

for any full rank operator $B \in \mathbb{B}(\mathscr{H})_+$.

Theorem 2.7. *Let $\mathscr{H}$ be a two dimensional complex Hilbert space and $\varphi : \mathbb{B}(\mathscr{H})_+ \longrightarrow \mathbb{B}(\mathscr{H})_+$ is a unital trace preserving map such that $F_\lambda(A, B, H) = F_\lambda(\varphi(A), \varphi(B), \varphi(H))$ and $\|H\| = \|\varphi(H)\|$. Then:*
(1) φ preserves spectrum of full rank operators.
(2) The quantity $F_\lambda(A, B, H)$ can only enlarge the convex hull of the spectrum of operator energy H and numbers depends to sp(B).
(3) φ preserves orthogonality between rank one projections.
(4) If $B = \alpha P + \beta Q$, where P, Q are mutually orthogonal rank-one projections and α, β are arbitrary positive real numbers with $\alpha + \beta = 1$ and $\alpha > \beta$ then $\varphi(B) = \varphi(\alpha P + \beta Q) = \alpha\varphi(P) + \beta\varphi(Q)$.
(5) φ preserves the nonzero transition probability between rank-one projections.

Remark 2.8. Before we proved the next lemma, we notice a remark. Let $\mathscr{H}$ and $\mathscr{K}$ be two complex Hilbert space. If $U : \mathscr{K} \to \mathscr{H}$ is a unitary operator and $\varphi : \mathbb{B}(\mathscr{H})_+ \longrightarrow \mathbb{B}(\mathscr{K})_+$ is a trace preserving map, then $U\varphi(.)U^* : \mathbb{B}(\mathscr{H})_+ \to \mathbb{B}(\mathscr{H})_+$ is well defined. Indeed, let $A \in \mathbb{B}(\mathscr{H})_+$, since φ preserves positivity then $U\varphi(A)U^*$ is positive semi-definite. Furthermore, $\mathrm{tr}(U\varphi(A)U^*) = \mathrm{tr}(\varphi(A)) = \mathrm{tr}(A)$, where $A \in \mathbb{B}(\mathscr{H})_+$. More important, the map $U\varphi(.)U^* : \mathbb{B}(\mathscr{H})_+ \to \mathbb{B}(\mathscr{H})_+$ preserves the generalized Helmholtz free energy and norm of operator energy.

Lemma 2.9. *If $\varphi : \mathbb{B}(\mathscr{H})_+ \longrightarrow \mathbb{B}(\mathscr{H})_+$ is a unital map such that preserves the generalized Helmholtz free energy, norm of operator energy and trace of operators, then φ preserves the nonzero transition probability between rank-one projections.*

Theorem 2.10. *If $\varphi : \mathbb{B}(\mathscr{H})_+ \longrightarrow \mathbb{B}(\mathscr{H})_+$ is a unital trace preserving map such that $\|H\| = \|\varphi(H)\|$ and $F_\lambda(A, B, H) = F_\lambda(\varphi(A), \varphi(B), \varphi(H))$, where $\lambda \in (-1, 0)$. Then there exists either a unitary or an anti-unitary operator U on $\mathscr{H}$ such that $\varphi(A) = UAU^*$ for any $A \in \mathbb{B}(\mathscr{H})_+$.*

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MINIMAL ENERGY SOLUTIONS FOR A CLASS OF SEMILINEAR ELLIPTIC PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. In this paper, we deal with the existence and multiplicity of positive minimal energy solutions for a class of semilinear elliptic problems with nonlinear boundary conditions. By extracting the Palais-Smale sequences in the Nehari manifold, it is proved that there exists λ^* such that for $\lambda \in (0, \lambda^*)$, the given boundary value problem has at least two positive solutions.

1. Introduction

In this paper the existence and multiplicity of positive solutions for the following semilinear elliptic problem is discussed

$$\begin{cases} -\Delta u + m(x)u = f(x,u) & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \lambda g(x,u) - h(x)u^p & x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where $\lambda > 0$, $0 \leq p \leq 1$, $\Omega \subset \mathbb{R}^N (N > 2)$ is a bounded domain with the smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial n}$ is the outer normal derivative. Also $m(x) \in C(\Omega)$ and $h(x) \in C(\partial\Omega)$ are nonnegative bounded functions and the basic assumptions for the $f(x, u)$ and $g(x, u)$ are the following:

(f1) $f(x, u) \in C^1(\Omega \times \mathbb{R})$ such that $f(x, 0) \geq 0$, $f(x, 0) \not\equiv 0$ and there exists $C_1 > 0$ such that, $f(x, |u|) \leq C_1(1 + |u|^q)$, where $0 < q < 1$.

(f2) $f_u(x, u) \in L^\infty(\Omega \times \mathbb{R})$ and for $u \in L^2(\Omega)$, $\int_\Omega \frac{\partial}{\partial u} f(x, t|u|)u^2 dx$ has the same sign for every $t \in (0, \infty)$.

(g1) $g(x, u) \in C^1(\partial\Omega \times \mathbb{R})$ and for $u \in L^2(\partial\Omega)$, $\int_{\partial\Omega} \frac{\partial}{\partial u} g(x, t|u|)u^2 dx$ has the same sign for every $t \in (0, \infty)$.

(g2) There exists constant $C_2 > 0$ such that for all $(x, u) \in (\partial\Omega \times \mathbb{R}^+)$, $G(x, u) \leq \frac{1}{r} g(x, u)u \leq \frac{1}{r(r-1)} g_u(x, u)u^2 \leq C_2 u^r$ where $2 < r < \frac{2N}{N-2}$ and $G(x, u) = \int_0^u g(x, s)ds$.

(g3) $g(x, 0) \geq 0$, $\lim_{t\to\infty} \frac{g(x,t|u|)|u|}{t^{r-1}} = \eta(x, u)$ uniformly respect to (x, u), where $\eta(x, u) \in C(\partial\Omega \times \mathbb{R}^+)$ and $|\eta(x, u)| > \theta > 0$, a.e. for all $(x, u) \in (\partial\Omega \times \mathbb{R}^+)$.

2. Main results

Define the Sobolev space $W = W^{1,2}(\Omega)$ endowed the norm $\|u\|_W = (\int_\Omega (|\nabla u|^2 + m(x)|u|^2)dx)^{\frac{1}{2}}$ and we use the standard $L^p(\Omega)$ spaces whose norms are denoted by $\|u\|_p$. We denote S_r the best sobolev constant for the embedding of W into $L^r(\Omega)$, so for $1 \leq p < 2^*$ ($2^* = \frac{2N}{N-2}$ if $N > 2$, $2^* = \infty$ if $N \leq 2$), we have

$$\frac{(\|u\|_W^2)^{p+1}}{(\int_\Omega |\, u \,|^{p+1}\, dx)^2} \geq \frac{1}{S_{p+1}^{2(p+1)}}. \tag{2.1}$$

The Euler functional associated with problem (1.1) is $I_\lambda : W \to \mathbb{R}$ such that

$$I_\lambda(u) = \frac{1}{2}\|u\|_W^2 - \int_\Omega F(x, |u|)dx - \lambda \int_{\partial\Omega} G(x, |u|)dx + H(u), \tag{2.2}$$

where

$$F(x, u) = \int_0^u f(x, s)ds \ , \quad H(u) = \int_{\partial\Omega} h(x)|u|^{p+1}dx, \tag{2.3}$$

and $G(x, u)$ is introduced in (g2). The critical points of the functional I_λ are in fact weak solutions of problem (1.1).

Definition 2.1. $u \in W(\Omega)$ is said to be a weak solution of problem (1.1), whenever for any $\varphi \in W(\Omega)$

$$-\int_\Omega \big(\nabla u.\nabla\varphi + m(x)u\varphi\big)dx = \int_\Omega f(x,u)\varphi dx + \lambda\int_{\partial\Omega} g(x,u)\varphi dx - \int_{\partial\Omega} h(x)|u|^{p-1}u\varphi dx.$$

If I_λ is bounded below and has a minimizer on W, then this minimizer is a critical point of I_λ, so it is a solution of the corresponding elliptic problem. However, the energy functional I_λ is not bounded below on the whole space W, but is bounded on an appropriate subset of W and a minimizer on this set gives rise to a solution of problem (1.1). In order to obtain the existence results, we introduce the Nehari manifold

$$\mathcal{N}_\lambda(\Omega) = \{u \in W \setminus \{0\} : \ \langle I'_\lambda(u), u\rangle = 0\},$$

where $\langle,\rangle$ denotes the usual duality between W and $W^{-1}(\Omega)$, here W^{-1} is the dual space of the Sobolev space W. Note that $N_\lambda(\Omega)$ contains every nonzero solution of problem (1.1). so we have the following theorem.

Theorem 2.2. *I_λ is coercive and bounded below on $\mathcal{N}_\lambda(\Omega)$.*

It can be proved that the points in $\mathcal{N}_\lambda(\Omega)$ correspond to the stationary points of the fibering map $\varphi_u(t) : [0,\infty) \to \mathbb{R}$ defined by $\varphi_u(t) = I_\lambda(tu)$, which were introduced by Drabek and Pohozaev in [4]. Then by (2.1), (2.2) and (2.3) we have

$$\varphi_u(t) = \frac{t^2}{2}\|u\|_W^2 - \int_\Omega F(x,t|u|)dx - \lambda\int_{\partial\Omega} G(x,t|u|)dx - \frac{t^{p+1}}{p+1}G(u).$$

It is easy to see that $\varphi'_u(t) = 0$ if and only if $tu \in \mathcal{N}_\lambda(\Omega)$. In particular, $u \in \mathcal{N}_\lambda(\Omega)$ if and only if $\varphi'_u(1) = 0$. Thus, it is natural to split $\mathcal{N}_\lambda$ into three parts $\mathcal{N}_\lambda^+, \mathcal{N}_\lambda^-$ *and* $\mathcal{N}_\lambda^0$ corresponding to local minima, local maxima and points of inflection of fibering map at $t = 1$.

The following lemma shows that minimizers for $I_\lambda(u)$ on $N_\lambda(\Omega)$ are usually critical points for I_λ, as proved by Aghajani et al. in [2].

Lemma 2.3. *Let u_0 be a local minimizer for $I_\lambda(u)$ on $\mathcal{N}_\lambda(\Omega)$ such that $u_0 \notin \mathcal{N}_\lambda^0(\Omega)$, then u_0 is a critical point of I_λ.*

Lemma 2.4. *There exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, we have $\mathcal{N}_\lambda^0 = \emptyset$.*

Definition 2.5. A sequence $u_n \subset W$ is called a Palais-Smale sequence if $I_\lambda(u_n)$ is bounded and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. If $I_\lambda(u_n) \to c$ and

$I'_\lambda(u_n) \to 0$, then u_n is a $(PS)_c - sequence$. It is said that the functional I_λ satisfies the Palais-Smale condition (*or* $(PS)_c - condition$), if each Palais-Smale sequence ($(PS)_c - sequence$) has a convergent subsequence.

The following lemma shows that a $(PS)_c - sequence$ is bounded in W, as proved by Aghajani et al. in [1] and [2].

Lemma 2.6. *If* $\{u_n\}$*is a* $(PS)_c-$*sequence for* I_λ, *then* $\{u_n\}$ *is bounded in* W.

Now we are going to describe the nature of the derivative of the fibering map for all possible signs of $\int_{\partial\Omega} \frac{\partial}{\partial u} g(x, t|u|)u^2 dx$.

Lemma 2.7. *There exists* $\lambda_1 > 0$ *such that when* $\lambda < \lambda_1$, *then* $\varphi_u(t)$ *and* $\varphi'_u(t)$ *take on positive values for all non-zero* $u \in W$.

Corollary 2.8. *If* $\int_{\partial\Omega} \frac{\partial}{\partial u} g(x, t|u|)u^2 dx \le 0$ *for* $u \in W \setminus \{0\}$, *then there exists* t_1 *such that* $t_1 u \in N^+_\lambda$ *and* $I_\lambda(t_1 u) = \varphi_u(t_1) < 0$.

Corollary 2.9. *If* $\int_{\partial\Omega} \frac{\partial}{\partial u} g(x, t|u|)u^2 dx > 0$ *for* $u \in W \setminus \{0\}$, *and* $\lambda < \lambda_1$, *then there exist* $t_1 < t_2$ *such that* $t_1 u \in N^+_\lambda$, $t_2 u \in N^-_\lambda$ *and* $I_\lambda(t_1 u) < 0$.

According to the definition of weakly continuous and weak to strong continuous operators introduced in [3], we have the following theorem.

Theorem 2.10. *For* $0 < \lambda < \lambda^* := min\{\lambda_0, \lambda_1\}$;
(i) there exists a minimizer of I_λ *on* $\mathcal{N}^+_\lambda(\Omega)$,
(ii) there exists a minimizer of I_λ *on* $\mathcal{N}^-_\lambda(\Omega)$.

Using Rellich theorem [3], Lemma 2.3, Lemma 2.4 and Theorem 2.10 we have the following corollary.

Corollary 2.11. *Problem (1.1) has at least two positive solutions for* $0 < \lambda < \lambda^*$.

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METRIZABILITY OF ALGEBRAIC CONE METRIC SPACES

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Abstract. In this paper, we prove that the topology induced by algebraic cone metric coincides with the topology induced by the metric obtained via a nonlinear scalarization function, i.e. any algebraic cone metric space is metrizable.

1. Introduction

The concept of metric and any concept related to metric play a very important role not only in pure mathematics but also in other branches of science involving mathematics especially in computer science, information science, and biological science.
Ordered normed spaces and cones have applications in applied mathematics and optimization theory [2]. A useful approach for analyzing a vector optimization problem is to reduce it to a scalar optimization problem. Nonlinear scalarization functions play an important role in this reduction in the context of non-convex vector optimization problems. Recently this has been applied by Du [3] to investigate the

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equivalence of vectorial versions of fixed point theorems of contractive mappings in TVS-cone metric spaces and scalar versions of fixed point theorems in general metric spaces in usual sense. Our aim is to generalize the notion of nonlinear scalarization function to prove that any algebraic cone metric space with its inherited topology is metrizable, i.e. the topology induced by the algebraic cone metric coincides with the topology induced by an appropriate metric. Also normability of the algebraic cone normed spaces with their induced topologies is discussed. Note that in [4], we introduced the notion of algebraic cone metric and it has been shown that every algebraic cone metric space has a Hausdorff topology.

2. Main results

Definition 2.1. [5] Let E be a real vector space and P be a convex subset of E. A point $x \in P$ is said to be an algebraic interior point of P if for each $v \in E$ there exists $\epsilon > 0$ such that $x + tv \in P$, for all $t \in [0, \epsilon]$.

Suppose that E is a real vector space with its zero vector θ and $P \subset E$ is a non-empty set such that $P + P \subset P$, $\lambda P \subset P$ $(\lambda \geq 0)$, $P \cap (-P) = \{\theta\}$. In this case we will say that P is an algebraic cone in E. For a given algebraic cone P in E, a partial ordering $\preceq_a$ on E with respect to P is defined by $x \preceq_a y$ if and only if $y - x \in P$. Furthermore, we write $x \ll_a y$ whenever $y - x \in aintP$ and we say that (E, P) is an algebraic cone space. If for each $x \in E$ and $y \in P \setminus \{\theta\}$ there exists $n \in \mathbb{N}$ such that $x \preceq_a ny$, we say that (E, P) has the Archimedean property.

Lemma 2.2. *Let (E, P) be an algebraic cone space and $aintP \neq \emptyset$. Then*
(i) $P + aintP \subset aintP$.
(ii) $\alpha aintP \subset aintP$, for each scalar $\alpha > 0$.
(iii) For any $x, y, z \in X$, $x \preceq_a y$ and $y \ll_a z$ imply that $x \ll_a z$.

Definition 2.3. Let (E, P) be an algebraic cone space, $aintP \neq \emptyset$ and $d_a : X \times X \to E$ be a vector-valued function that satisfies:
(ACM1) For all $x, y \in X$, $\theta \preceq_a d_a(x, y)$ and $d_a(x, y) = \theta$ if and only if $x = y$,
(ACM2) $d_a(x, y) = d_a(y, x)$ for all $x, y \in X$,
(ACM3) $d_a(x, y) \preceq_a d_a(x, z) + d_a(z, y)$ for all $x, y, z \in X$.
Then d_a is called an algebraic cone metric on X and (X, d_a) is said to be an algebraic cone metric space.

The nonlinear scalarization function $\xi_e : E \to \mathbb{R}$ is defined as follows:

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - P\}$$

for all $y \in E$, where E is a topological vector space and P is a closed convex cone in E such that $P \cap (-P) = \{\theta\}$ (see [1]).
In this approach real vector spaces are used as the domain of the nonlinear scalarization function, instead of topological vector spaces.
For any $y \in E$, put

$$M_{e,y} = \{r \in \mathbb{R} : y \in re - P\}.$$

Definition 2.4. Let E be a real vector space and P be an algebraic cone in E. For a given $e \in aintP$, the nonlinear scalarization function is defined by:

$$\xi_e(y) = \inf M_{e,y}.$$

Lemma 2.5. *For any $e \in aintP$, the function ξ_e has the following properties:*
1) $\xi_e(\theta) = 0$.
2) $\xi_e(e) = 1$.
3) $y \in P$ implies $\xi_e(y) \geq 0$.
4) $\xi_e(y) < r$ if and only if $y \in re - aintP$.
5) if $y_1 \preceq_a y_2$, then $\xi_e(y_1) \leq \xi_e(y_2)$, for each $y_1, y_2 \in E$.
6) ξ_e is subadditive on E.
7) ξ_e is positively homogeneous on E (i.e. $\xi_e(\lambda y) = \lambda\xi_e(y)$, for each $y \in E$).
8) $\xi_e(y) > 0$ for each $y \in aintP$.

By using the idea of Wie-shie Du [3], we can assert the following theorem.

Theorem 2.6. *Let (X, d_a) be an algebraic cone metric space and $e \in aintP$. Then $d_e : X \times X \to [0, \infty)$ defined by $d_e = \xi_e \circ d_a$ is a metric.*

Lemma 2.7. *Let (X, d_a) be an algebraic cone metric space, $e \in aintP$, $d_e = \xi_e \circ d_a$, and $x \in X$. Then $B_{d_e}(x, r) = B_a(x, re)$, where $B_{d_e}(x, r) = \{y \in X : d_e(x, y) < r\}$ and $B_a(x, re) = \{y \in X : d_a(x, y) \ll_a re\}$.*

In the sequel we assume that (E, P) has the Archimedean property. Let (X, d_a) be an algebraic cone metric space. Then, for each $c_1, c_2 \in aintP$, there exists $c \in aintP$ such that $c \ll_a c_1$ and $c \ll_a c_2$.

Note that this implies that the collection $\{B_a(x, c) : c \in aintP,\ x \in X\}$ forms a basis for τ_a the topology of X which is induced by d_a. Consequently, if $x \in X$ and $\{x_n\}$ is a sequence in X, then $\{x_n\}$ converges to x with respect to τ_a if and only if for every $\theta \ll_a c$ there exists a natural number N such that for all $n > N$, $d_a(x_n, x) \ll_a c$.

Definition 2.8. Let (X, d_a) be an algebraic cone metric space and $\{x_n\}$ be a sequence in X. Then
(i) $\{x_n\}$ is a Cauchy sequence whenever for every $\theta \ll_a c$ there exists a natural number N such that for all $m, n > N$, $d_a(x_n, x_m) \ll_a c$.
(ii) (X, d_a) is said to be a complete algebraic cone metric space if every Cauchy sequence is convergent.

Let (X, d_a) be an algebraic cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Let d_e be the same as in Theorem 2.6, then it is easy to see that
(i) $\{x_n\}$ converges to x if and only if $d_e(x_n, x) \to 0$ as $n \to \infty$;
(ii) $\{x_n\}$ is a Cauchy sequence in (X, d_a) if and only if $\{x_n\}$ is a Cauchy sequence (in usual sense) in (X, d_e);
(iii) (X, d_a) is complete algebraic cone metric space if and only if (X, d_e) is a complete metric space.

Theorem 2.9. *Let (X, d_a) be an algebraic cone metric space. Then there exists a metric on X which induces the same topology on X as the topology induced by d_a.*

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A NOTE ON NORM COMPARISON FOR HEINZ MEANS AND THE A-L-G INTERPOLATION MEANS

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Abstract. In this paper we compare the Heinz means

$$\frac{1}{2}\left(H^{\frac{1}{2}+\beta}XK^{\frac{1}{2}-\beta}+H^{\frac{1}{2}-\beta}XK^{\frac{1}{2}+\beta}\right)$$

and the A-L-G interpolation means $M_\alpha(H,K)X$.

1. Introduction

We assume that H, K are positive operators on a separable Hilbert space $\mathcal{H}$ throughout, and let $N(s,t)$ be a continuous non negative function on $[0,\infty)\times[0,\infty)$. Firstly, let us assume $\dim\mathcal{H}=n<\infty$ for simplicity, and

$$H=U\mathrm{diag}(s_1,s_2,\cdots,s_n)U^* \quad and \quad K=V\mathrm{diag}(t_1,t_2,\cdots,t_n)V^*$$

be diagonalization of positive matrices H, K. In [2] Kosaki introduced $N(H,K)X\in\mathbb{M}_n(\mathbb{C})$ the $n\times n$ matrices, by

$$N(H,K)X=U([N(s_i,t_j)]_{ij}\circ(U^*XV))V^*,$$

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* Speaker.

for each $X = [X_{ij}] \in \mathbb{M}_n(\mathbb{C})$, where $\circ$ means the Hadamard product. In general situation, we assume that

$$H = \int_0^{\|H\|} s dE_s \quad \textit{and} \quad K = \int_0^{\|K\|} t dF_t$$

are the respective spectral decomposition. Then, the notion of double integral transformations

$$N(H,K)X = \int_0^{\|H\|} \int_0^{\|K\|} N(s,t) dE_s X dF_t \tag{1.1}$$

2. Main results

Theorem 2.1. *We assume that continuous non-negative homogeneous functions $M(s,t)$ and $N(s,t)$ satisfy*

$$\frac{M(e^x,1)}{N(e^x,1)} = \hat{\nu}(x) \left(\int_{-\infty}^{\infty} e^{ixy} d\nu(y) \right) \tag{2.1}$$

with a signed measure ν on $\mathbf{R}$.
(i) When $H, K \geqslant 0$ are nonsingular, we have

$$M(H,K)X = \int_{-\infty}^{\infty} H^{ix}(N(H,K)X)K^{-ix} d\nu(x).$$

(ii) When $M(1,0) = M(0,1) = 0$, we have

$$M(H,K)X = \int_{-\infty}^{\infty} (Hs_H)^{ix}(N(H,K)X)(Ks_K)^{-ix} d\nu(x).$$

(iii) When either (a) $M(s,t) = M(t,s)$ and $N(s,t) = N(t,s)$ or (b) $M(s,t) = -M(t,s)$ and $N(s,t) = -N(t,s)$, we have

$$M(H,K)X = \int_{x\neq 0} (Hs_H)^{ix}(N(H,K)X)(Ks_K)^{-ix} d\nu(x) + \nu(0)N(H,K)X.$$

Here, s_H for instance means the support projection of H, and $(Hs_H)^{ix}$ is understoood as a unitary operator on $s_H\mathcal{H}$.

Theorem 2.2. *We assume one of the three conditions in Theorem2.1 is satisfied (with the validity of (1.1) of course). If $N(s,t)$ is a Schur multiplier relative to (H,K), then so is $M(s,t)$ and moreover we have*

$$|||M(H,K)X||| \leqslant |\nu|(\mathbf{R}) \times |||N(H,K)X|||$$

for each unitarily invariant norm $|||.|||$ and $X \in B(\mathcal{H})$. In particular, when $\hat{\nu}$ is positive definite (i.e., ν is a positive measure by Bochner's theorem), we have

$$|||M(H,K)X||| \leqslant \hat{\nu}(0) \times |||N(H,K)X|||\,.$$

In [3, 4] Hiai and Kosaki considered the one parameter family $\{M_\alpha\}_{-\infty\leqslant\alpha\leqslant\infty}$ of means corresponding to the following scalar means (A-L-G interpolation means):

$$M_\alpha(s,t) = \frac{\alpha-1}{\alpha} \times \frac{s^\alpha - t^\alpha}{s^{\alpha-1} - t^{\alpha-1}}.$$

and by (1.1) we have the following for operators

$$M_{1/2}(H,K)X = H^{1/2}XK^{1/2} \qquad (\textit{the geometric mean}),$$

$$M_1(H,K)X = \int_0^1 H^x X K^{1-x} dx \qquad (\textit{the logarithmic mean}),$$

$$M_2(H,K)X = \frac{1}{2}(HX + XK) \qquad (\textit{the arithmetic mean}).$$

Among other things the monotonicity

$$|||M_{\alpha'}(H,K)X||| \leqslant |||M_\alpha(H,K)X||| \ \textit{ as long as } \alpha' \leqslant \alpha \tag{2.2}$$

were proved in [2, 3] also recall the celebrated Heinz inequality (see [1])

$$\left|\left|\left| H^{\frac{1}{2}+\beta'} X K^{\frac{1}{2}-\beta'} + H^{\frac{1}{2}-\beta'} X K^{\frac{1}{2}+\beta'} \right|\right|\right| \leqslant \left|\left|\left| H^{\frac{1}{2}+\beta} X K^{\frac{1}{2}-\beta} + H^{\frac{1}{2}-\beta} X K^{\frac{1}{2}+\beta} \right|\right|\right|$$

where $\beta' \leqslant \beta$.

Lemma 2.3. *Let $\alpha, \beta \geqslant 0$ and $\alpha + \beta < 1$. Then the function* $\dfrac{\sinh(\alpha t)\cosh(\beta t)}{\sinh(t)} (\alpha \neq 0)$ *is positive definite if and only if $0 \leqslant \beta \leqslant \frac{1}{2}$.*

Lemma 2.4. *(i) For $\alpha \in [0,1)$ we have*

$$\frac{s\cosh(\alpha s)}{\sinh(s)} = \frac{\pi}{2}\int_{-\infty}^{\infty} \frac{1 + \cosh(\pi t)\cosh(\pi\alpha)}{(\cosh(\pi t) + \cosh(\pi\alpha))^2} e^{-ist} dt,$$

$$\frac{s\sinh(\alpha s)}{\sinh(s)} = \frac{i\pi}{2}\int_{-\infty}^{\infty} \frac{\sinh(\pi t)\sinh(\pi\alpha)}{(\cosh(\pi t) + \cosh(\pi\alpha))^2} e^{-ist} dt.$$

(ii) The function $\dfrac{s\cosh(\alpha s)}{\sinh(s)}$ *is positive definite if and only if $0 \leqslant \alpha \leqslant \frac{1}{2}$ and*

$$\int_{-\infty}^{\infty} \left| \frac{i\pi}{2} \times \frac{\sinh(\pi s)\sinh(\pi\alpha)}{(\cosh(\pi s) + \cosh(\pi\alpha))^2} \right| ds = \tan(\pi\alpha/2).$$

In the following theorem we compute Fourier transforms of function $f(x) = M(e^x,1)/N(e^x,1)$ between scalar Heinz means and A-L-G means. Then based on Bochner's theorem we determine if these ratio are positive definite. Note that this information give us many useful norm inequalities (see Theorem 2.1 and Theorem 2.2).

Theorem 2.5. *Let $\alpha \geqslant \frac{1}{2}$ and $0 \leqslant \beta \leqslant \frac{1}{2}$.*
(i) When $\alpha > \frac{2}{3}$, the inequality

$$\frac{1}{2}\left|\left|\left|H^{\frac{1}{2}+\beta}XK^{\frac{1}{2}-\beta}+H^{\frac{1}{2}-\beta}XK^{\frac{1}{2}+\beta}\right|\right|\right| \leqslant |||M_\alpha(H,K)X|||$$

is valid if and only if $\beta \leqslant \frac{\alpha}{4}$. In particular, the inequality is always valid for $\alpha \geqslant 2$.
(ii) When $\alpha \leqslant \frac{2}{3}$, the norm inequality is valid if and only if $\beta \leqslant \alpha - \frac{1}{2}$. I n particular, the estimate

$$\frac{1}{2}\left|\left|\left|H^{\frac{1}{2}+\beta}XK^{\frac{1}{2}-\beta}+H^{\frac{1}{2}-\beta}XK^{\frac{1}{2}+\beta}\right|\right|\right| \leqslant |||M_1(H,K)X|||\,,$$

is valid if and only if $\beta \leqslant \frac{1}{4}$.

Proposition 2.6. *Let $0 \leqslant \beta < \frac{1}{2}$. Then*

$$\frac{1}{2}\left|\left|\left|H^{\frac{1}{2}+\beta}XK^{\frac{1}{2}-\beta}+H^{\frac{1}{2}-\beta}XK^{\frac{1}{2}+\beta}\right|\right|\right| \leqslant \tan(\pi\beta)\,|||M_1(H,K)X|||\,.$$

Corollary 2.7. *Let A, B be self adjoint operators. Then*

$$|||AX - XB||| \leqslant \pi \left|\left|\left|e^{-A/2}\left(\int_0^1 e^{xA}Xe^{(1-x)B}dt\right)e^{-B/2}\right|\right|\right|$$
$$\left(= \left|\left|\left|\int_{\frac{-1}{2}}^{\frac{1}{2}} e^{xA}Xe^{(1-x)B}dt\right|\right|\right| \right)$$

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THE GENERALIZATION OF THE PERRON-FROBENIUS THEOREM FOR NONNEGATIVE TENSORS TO THE CLASS OF REAL TENSORS

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Abstract. In this paper a new quantity for real tensors, the sign-real spectral radius, is defined and investigated. A various characterizations, bounds and some properties are derived. In certain aspects our quantity shows similar behavior to the spectral radius of a nonnegative tensor. In fact, we generalize the Perron-Frobenius Theorem for nonnegative tensors to the class of real tensors.

1. Introduction

An mth order tensor is an m-way array whose entries accessed via m indices. Higher order tensors are generalizations of matrices, a matrix is a second order tensor. Many important ideas, notions, and results have been successfully extended from matrices to higher order tensors. A real mth-order n-dimensional tensor $\mathbb{A}$ consists of n^m real entries:

$$a_{i_1 i_2 \dots i_m} \in R,$$

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* Speaker.

where $i_j \in \{1, ..., n\}$ for $j \in \{1, ..., m\}$.
We denote the set of all real mth-order n-dimensional tensors by Γ.
For a vector $x \in R^n$, we use x_i to denote its components, and $x^{[m]}$ to denote a vector in R^n such that

$$x_i^{[m]} = x_i^m,$$

for all i.
$\mathbb{A}x^{m-1}$ for a vector $x \in R^n$ denotes a vector in R^n, whose ith component is

$$\left(\mathbb{A}x^{m-1}\right)_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} x_{i_2} \ldots x_{i_m}.$$

For a tensor $\mathbb{A}$ of order $m \geq 2$ and dimension $n \geq 2$, if there exist $\lambda \in C$ and $x \in C^n \backslash \{0\}$ such that

$$\mathbb{A}x^{m-1} = \lambda x^{[m-1]} \tag{1.1}$$

holds, then λ is called an eigenvalue of $\mathbb{A}$, x is called a corresponding eigenvector of $\mathbb{A}$ with respect to λ, and (λ, x) is called an eigenpair of $\mathbb{A}$.
If x is real, then λ is also real. In this case, λ and x are called an H-eigenvalue of $\mathbb{A}$ and an H-eigenvector of $\mathbb{A}$ associated with the H-eigenvalue λ, respectively. In the case $m = 2$, (1.1) reduces to the definition of eigenvalues and corresponding eigenvectors of a square matrix. This definition was introduced by Qi [1].

The spectral radius of tensor $\mathbb{A}$ is defined by Yang and Yang in [3] as follows:

Definition 1.1. The spectral radius of tensor $\mathbb{A}$ is defined as

$$\rho(\mathbb{A}) = \max\{\,|\lambda| \;:\; \lambda \; is\, an\, eigenvalue\, of\, \mathbb{A}\}\,.$$

He proved that the spectral radius of a nonnegative tensor, is an eigenvalue of it.
The purpose of this paper is to extend Perron Frobenius Theorem for nonegative tensors to general real tensors.

Definition 1.2. Let $\mathbb{A} \in \Gamma$, the real spectral radius is defined by

$$\rho_0(\mathbb{A}) := \max\,\{\,|\lambda| \,:\, \lambda \in \sigma(\mathbb{A}) \cap R\}\,,$$

where $\rho_0(\mathbb{A}) = 0$ if $\mathbb{A}$ has no real eigenvalues.

It easily follows that

$$\rho_0^s(\mathbb{A}) = \max_{S \in \varphi} \rho_0(S\mathbb{A})$$

where φ is the set of real diagonal orthogonal matrices.
This quantity is called the sign-real spectral radius for real tensors.

In this paper, first we investigate the properties of this quantity which is similar to the spectral radius for nonnegative ones. Also we will show that $\rho_0^s(\mathbb{A})$ is a real eigenvalue $S\mathbb{A}$ for some $S \in \varphi$ and the associated eigenvector is nonnegative (see Theorem 2.6).
Also it will be shown $\rho_0^s(\mathbb{A}) = \rho(\mathbb{A})$ when $\mathbb{A}$ be a nonnegative tensor.

2. Main results

Definition 2.1. [2] Let $\mathbb{A}$ (and $\mathbb{B}$) be an order $m \geq 2$ (and order $k \geq 1$), dimension n tensor, respectively. The product $\mathbb{A}\mathbb{B}$ is defined to be the following tensor $\mathbb{C}$ of order $(m-1)(k-1)+1$ and dimension n:

$$c_{i\alpha_1\ldots\alpha_{m-1}} = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} b_{i_2\alpha_1}\ldots b_{i_m\alpha_{m-1}} \ (i \in [n]\,,\ \alpha_1,\ldots,\alpha_{m-1} \in [n]^{k-1}).$$

It is easy to check from the definition that $I_n\mathbb{A} = \mathbb{A} = \mathbb{A}I_n$, where I_n is the identity matrix of order n.

Theorem 2.2. *Let $\mathbb{A} \in \Gamma$, $S_1, S_2 \in \varphi$, D is diagonal nonsingular matrix then*
(**i**) *For all $S_1, S_2 \in \varphi$ we have $\rho_0^s(\mathbb{A}) = \rho_0^s(\mathbb{A}S_2) = \rho_0^s(S_1\mathbb{A}) = \rho_0^s(S_1\mathbb{A}S_2)$.*
(**ii**) *For all α where α is a real number , $\rho_0^s(\alpha\mathbb{A}) = |\alpha|\ \rho_0^s(\mathbb{A})$.*
(**iii**) $\rho_0^s(D\mathbb{A}) = \rho_0^s(\mathbb{A}D)$.
(**iv**) $\rho_0^s(\mathbb{A}) = \rho_0^s(D^{-1}\mathbb{A}D)$.

Note. If $\mathbb{A} \in \Gamma$ and $\mathbb{A} \geq 0$ then $\rho_0^s(\mathbb{A}) = \rho(\mathbb{A})$.

Theorem 2.3. *For upper or lower triangular tensor $\rho_0^s(\mathbb{A}) = \max\limits_i |a_{i\ldots i}|$.*

Note. Let $\mathbb{A}, \mathbb{B} \in \Gamma$. In general $\rho_0^s(\mathbb{A}\mathbb{B}) \neq \rho_0^s(\mathbb{B}\mathbb{A})$ and $\rho_0^s(\mathbb{A}) \neq \rho_0^s\left(Q^T\mathbb{A}Q\right)$ for any orthogonal matrix Q.
A signature tensor $\mathbb{S}$ of order m and dimension n, is a diagonal tensor with diagonal entries $+1$ or -1. we denote the set of all signture tensors of order m and dimension n, by ϕ_n.

Theorem 2.4. *Let $\mathbb{A}, \mathbb{B} \in \Gamma$ and $\mathbb{S}, \mathbb{T} \in \phi_n$. For direct product of tensors we have*
(**i**). $\rho_0^s(\mathbb{A} \otimes \mathbb{S}) = \rho_0^s(\mathbb{S} \otimes \mathbb{A}) = \rho_0^s(\mathbb{S} \otimes \mathbb{A} \otimes \mathbb{T}) = \rho_0^s(\mathbb{A})$.

(ii). $\rho_0^s(\mathbb{A} \otimes \mathbb{B}) = \rho_0^s(\mathbb{B} \otimes \mathbb{A})$.
(iii). $\rho_0^s(\mathbb{A})\,\rho_0^s(\mathbb{B}) \leq \rho_0^s(\mathbb{A} \otimes \mathbb{B})$.

Theorem 2.5. *Let $\mathbb{A} \in \Gamma$, then for every $S \in \varphi$ there is $T \in \varphi$ such that $|S\mathbb{A}x^{m-1}| = (TS\mathbb{A})\,x^{m-1}$. In particular $|\mathbb{A}x^{m-1}| = (T\mathbb{A})\,x^{m-1}$.*

Theorem 2.6. *Let $\mathbb{A} \in \Gamma$, then for every $T \in \varphi$ there exists some $S \in \varphi$ such that $S\mathbb{A}$ has an eigenvector in that orthant corresponding to a real nonnegative eigenvalue, i.e.,*

$$\forall T \in \varphi \; \exists \; S \in \varphi \; \exists \; 0 \neq x \in R^n :$$
$$x \geq 0 \text{ and } S\mathbb{A}(Tx)^{m-1} = \lambda(Tx)^{[m-1]} \, for\, some \;\; 0 \leq \lambda \in R.$$

Lemma 2.7. *Let $\mathbb{A}, \mathbb{B} \in \Gamma$ be diagonal, also suppose that D be a $n \times n$ diagonal matrix, then we have*
(i). $\det(\mathbb{A}\mathbb{B}) = \det(\mathbb{A})\det(\mathbb{B})$.
(ii). $\det(D\mathbb{A}) = \det(D\mathbb{I})\det(\mathbb{A})$, *where $\mathbb{I}$ is a unit tensor.*

Theorem 2.8. *Let $\mathbb{A}$ be a real mth-order n-dimensional tensor then there are $S, T \in \varphi$ and $0 \neq x \in R^n$ with $x \geq 0$ and*

$$(S\mathbb{A}T)\,x^{m-1} = \rho_0^s(\mathbb{A})\,x^{[m-1]}.$$

Corollary 2.9. *If $\rho_0^s(\mathbb{A}) = 0$ then* $\det(\mathbb{A}) = 0$.

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$\Delta-$FRAMES AND THE BELOW BOUNDEDNESS OF MATRIX OPERATORS ON THE SEQUENCE SPACE bv_p

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Abstract. Denote by $L_{bv_p,bv_p}(\mathrm{A})$, the supremum of those l, satisfying the inequality $\|\mathrm{A}x\|_{bv_p} \geq l\,\|x\|_{bv_p}$, where $A = (a_{n,k})_{n,k\geq 1}$ is a non-negative matrix, $x \geq 0$ and $x \in bv_p$. In this paper, we focus on the evaluation of $L_{bv_p,bv_p}(\mathrm{A}^t)$ for a lower triangular matrix A with increasing rows, where $0 < p < 1$. A general lower estimate is obtained. As a consequence, we apply our result to the weighted mean matrices and the Nörlund matrices. Also, in this paper the concepts of $\Delta-$frame, dual $\Delta-$frame, $\Delta-$Beseel sequence, $\Delta-$Riesz and dual $\Delta-$Riesz basis which are all related to the space bv_2, are introduced. We investigate some properties of these classes of frames and also characterize all of them. In addition, we present some sequences that are frame and are not $\Delta-$frame and vice versa. Also, we give some sequences which are both frame and $\Delta-$ frame.

1. Introduction

For $0 < p < \infty$, the sequence space bv_p consisting of all sequences $\{x_k\}_{k=1}^{\infty}$, such that $\{x_k - x_{k-1}\}_{k=1}^{\infty}$ belongs to the space ℓ^p, where $x_0 = 0$. More precisely, the bv_p is the space of all real or complex sequences

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* Speaker.

whose $\mathbf{\Delta}-$transforms are in the space ℓ^p, where $\mathbf{\Delta}$ denotes the matrix $\mathbf{\Delta} = (\mathbf{\Delta}_{n,k})_{n,k\geq 1}$ defined by

$$\mathbf{\Delta}_{n,k} = \begin{cases} (-1)^{n-k} & n-1 \leq k \leq n, \\ 0 & 1 \leq k < n-1 \, or \, k > n. \end{cases}$$

This space was introduced and studied by Altay and Başar in [1]. They proved that the sequence space bv_p is linearly isomorphic to the space ℓ^p and that the space bv_2 is a Hilbert space.

In this paper, we first consider the lower bound problem for some matrix mapping on the space bv_p. The lower bound involved here, is the number $L_{bv_p,bv_p}(\mathrm{A})$, which is defined as the supremum of those l, obeying the following inequality

$$\|\mathrm{A}x\|_{bv_p} \geq l\|x\|_{bv_p},$$

where $x \geq 0$, $x \in bv_p$ and $\mathrm{A} = (a_{n,k})_{n,k\geq 1}$ is a non-negative lower triangular matrix operator selfmap of the space bv_p, $0 < p < 1$. Our result gives a lower estimate for $L_{bv_p,bv_p}(A^t)$ in term of the constant M, defined by

$$(a_{n,k} - a_{n,k-1}) \leq M(a_{n,j} - a_{n,j-1}), \quad 1 \leq k \leq j \leq n. \tag{1.1}$$

We use the convention that any term with zero subscript is equal to naught. Here $M \geq 1$ and we shall assume that M is the smallest value appeared in (1.1). If (1.1) is fails, we set $M = \infty$.

Another purpose of this paper is to introduce the concepts of $\Delta-$frame, dual $\Delta-$frame, $\Delta-$Beseel sequence, $\Delta-$Riesz basis and dual $\Delta-$Riesz basis which are all related to the space bv_2. We investigate some properties of these classes of frames and also characterize all of them. Moreover, we present some sequences that are both frame and $\Delta-$frame, that are frame and no $\Delta-$frame and that are $\Delta-$frame and no frame.

2. Main results

Theorem 2.1. *Let $0 < p < 1$ and $A = (a_{n,k})_{n,k\geq 1}$ be a lower triangular matrix with non-negative entries. If $a_{n,k} \leq a_{n,k+1}(1 \leq k < n)$, then*

$$L_{bv_p,bv_p}\left(A^t\right) \ \geq \ \frac{1}{2}pM^{p-1}\left(\inf_{j\geq 1} a_{j,j}\right). \tag{2.1}$$

Here M is defined by (1.1).

In the following corollaries we apply Theorem 2.1 to some famous classes of non-negative lower triangular matrices such as weighted mean matrices and Nörlund matrices, where the weighted mean matrices, $(A_W^{WM}) = (a_{n,k})_{n,k\geq 1}$ and the Nörlund matrices, $(A_W^{NM}) = (b_{n,k})_{n,k\geq 1}$, are defined by

$$a_{n,k} = \begin{cases} \frac{w'_n}{W'_n} & 1 \leq k \leq n, \\ 0 & \text{otherwise}, \end{cases} \quad \& \quad b_{n,k} = \begin{cases} \frac{w'_{n-k+1}}{W'_n} & 1 \leq k \leq n, \\ 0 & \text{otherwise}. \end{cases}$$

Here $W'_n = \sum_{k=1}^n w'_k$ and $w' = (w'_n)$ is a non negative sequence with $w'_1 > 0$.

Corollary 2.2. *Let $0 < p < 1$ and $w' = (w'_n)$ be an increasing non-negative sequence of real numbers. Then*

$$L_{bv_p,bv_p}\left(\left(A_W^{WM}\right)^t\right) \geq \frac{1}{2} p M^{p-1} \left(\inf_{n\geq 1} \frac{w'_n}{w'_1 + \dots + w'_n}\right),$$

where M is defined by (1.1).

Corollary 2.3. *Let $0 < p < 1$ and $w' = (w'_n)$ be a decreasing non-negative sequence of real numbers with $w'_1 > 0$. Then*

$$L_{bv_p,bv_p}\left(\left(A_W^{NM}\right)^t\right) \geq \frac{1}{2} p M^{p-1} \left(\inf_{n\geq 1} \frac{w'_1}{w'_1 + \dots + w'_n}\right),$$

where M is defined by (1.1).

In the rest of this paper we suppose the $\mathcal{H}$ is separeble Hilbert space.

Definition 2.4. A $\mathbf{\Delta}-$frame for $\mathcal{H}$ is a sequence $\{f_k\}_{k=1}^\infty$ for which there exist positive real numbers A and B such that

$$A\left\|f\right\|^2 \leq \|\{\langle f, f_k\rangle\}_{k=1}^\infty\|_{bv_2} \leq B\left\|f\right\|^2, \quad \forall f \in H \tag{2.2}$$

If it satisfies the upper condition in (2.2), it is called a $\mathbf{\Delta}-$Bessel sequence in $\mathcal{H}$.

Proposition 2.5. *Let $\{f_k\}_{k=1}^\infty$ be a $\mathbf{\Delta}-$frame for $\mathcal{H}$ with the lower $\mathbf{\Delta}-$frame bound A. Then it satisfies the lower frame condition with the lower frame bound $A/4$*

Theorem 2.6. *In a Hilbert space $\mathcal{H}$,*

(i) *There exists a frame which is not a $\mathbf{\Delta}-$frame.*
(ii) *There exists a $\mathbf{\Delta}-$frame which is not a frame.*
(iii) *There exist sequences which are both frame and $\mathbf{\Delta}-$frame.*

Proof. (i) Consider the sequence $\{f_k\}_{k=1}^\infty = \{e_1, e_1, -e_1, -e_2, e_3, e_3, \cdots\}$, where $\{e_k\}_{k=1}^\infty$ is an orthonormal basis for $\mathcal{H}$. Then $\{f_k\}_{k=1}^\infty$ is a tight frame with frame bound 2. On the other hand, for any $f \in \mathcal{H}$, we have

$$\|\{\langle f, f_k\rangle\}_{k=1}^\infty\|_{bv_2}^2 = |\langle f, e_1\rangle|^2 + \sum_{k=1}^\infty |\langle f, e_k + e_{k+1}\rangle|^2.$$

Now by the same reason as in ([2], Example 5.1.10), we conclude that $\{f_k\}_{k=1}^{\infty}$ does not satisfy the lower $\mathbf{\Delta}-$frame condition.
(ii) The sequence

$$\{f_k\}_{k=1}^{\infty} = \{e_1,\ e_1 + e_2,\ e_1 + e_2 + e_3,\ \cdots\},$$

is a $\mathbf{\Delta}-$frame which is not a frame.
(iii) Let $\{f_k\}_{k=1}^{\infty}$ be an arbitrary frame in $\mathcal{H}$ with the frame bound A and B. Then the sequence $\{g_k\}_{k=1}^{\infty} = \{f_1, 0, f_2, 0, f_3, 0, ...\}$ is both a frame for $\mathcal{H}$ with the same frame bounds as $\{f_k\}_{k=1}^{\infty}$ and a $\mathbf{\Delta}-$frame for $\mathcal{H}$ with the $\mathbf{\Delta}-$frame bounds A and $2B$. □

Definition 2.7. Consider the sequence $\{g_k\}_{k=1}^{\infty}$ of vectors in $\mathcal{H}$. (i) The sequence $\{g_k\}_{k=1}^{\infty}$ is a $\Delta-$complete if $\overline{span}\{g_k - g_{k-1}\}_{k=1}^{\infty} = \mathcal{H}$. (ii) The sequence $\{g_k\}_{k=1}^{\infty}$ is a $\Delta-$basis($\Delta-$ Schauder basis) for $\mathcal{H}$ if for each $f \in \mathcal{H}$ there exist unique scalar coefficients $\{c_k(f)\}_{k=1}^{\infty}$ such that $f = \sum\limits_{k=0}^{\infty} c_k(f)\,(g_k - g_{k-1})$. (iii) A $\Delta-$basis $\{g_k\}_{k=1}^{\infty}$ is an $\Delta-$orthonormal basis if $\{g_k\}_{k=0}^{\infty}$ is an $\Delta-$orthonormal system, i.e., if

$$\langle g_i - g_{i-1}, g_j - g_{j-1}\rangle = \delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Theorem 2.8. *Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Then (i) The $\mathbf{\Delta}-$orthonormal bases are the sets $\{U\,(e_1 + e_2 + \cdots + e_k)\}_{k=1}^{\infty}$, where U is an unitary operator on $\mathcal{H}$. (ii) A $\mathbf{\Delta}-$Riesz basis for $\mathcal{H}$ is a family of the form $\{U(e_1 + e_2 + ... + e_k)\}_{k=1}^{\infty}$, where U is a bounded bijective operator on $\mathcal{H}$. (iii) The $\mathbf{\Delta}-$frames for $\mathcal{H}$ are precisely the families $\{\mathrm{U}\,(e_1 + e_2 + \cdots + e_k)\}_{k=1}^{\infty}$, where U is bounded and surjective operator on $\mathcal{H}$.*

Theorem 2.9. *Let $\{f_k\}_{k=1}^{\infty}$ be a $\mathbf{\Delta}-$frame for $\mathcal{H}$. The dual $\mathbf{\Delta}-$frames of $\{f_k\}_{k=1}^{\infty}$ are precisely the families*

$$\{g_k\}_{k=1}^{\infty} = \left\{ \mathrm{S}^{-1} f_k + h_k - \sum_{j=1}^{\infty} \left\langle \mathrm{S}^{-1} f_k - \mathrm{S}^{-1} f_{k-1}, f_j - f_{j-1} \right\rangle h_j \right\}_{k=1}^{\infty},$$

where $\{h_k\}_{k=1}^{\infty}$ is a $\mathbf{\Delta}-$Bessel sequence in $\mathcal{H}$.

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RECOGNITION OF THE NON-COMMUTING GRAPH BY THE NON-CENTRALIZER GRAPH

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Abstract. We define the non-centralizer graph associated to a finite group G, as the graph whose vertices are the elements of G, and whose edges are obtained by joining two distinct vertices if their centralizers are not equal. We denote this graph by Υ_G. The non-centralizer graph is used to study the properties of the non-commuting graph of an AC-group.

1. Introduction

Graphs play an important role in the mathematics, providing visual means that help us to better understand other mathematical objects that they are connected with. Associating a graph to a group and using information on one of the two objects to solve a problem for the other is an interesting research topic. On the other hand, many recent problems in group theory are related to the notion of commutativity, like for instance the problem to determine the probability that two elements of a group commute, or to find how many centralizers can a group have. For results related to the problem of counting the centralizers of a group, we refer the reader to the work of S. M. Belcastro and G. J. Sherman [5], A. R. Ashrafi [2], A. R. Ashrafi and B. Taeri [3], A. Abdollahi, S. M. Jafarian Amiri, and A. Mohammadi Hassanabadi [1].

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The non-commuting graph of a group G was first considered by Paul Erdös in 1975. We denote this graph by Γ_G, and recall that the vertices of Γ_G are the elements of G, and that two distinct vertices are joined by an edge whenever they do not commute. Of course, there are some other ways to construct a graph associated to a given group or semigroup. In this paper we will define the non-centralizer graph Υ_G of the group G to be the graph whose vertices are the elements of G and whose edges are obtained by joining two distinct vertices if their centralizers are not equal. The non-centralizer graph and the non-commuting graph of a group G are closely related. By studying the non-centralizer graph, one may describe the structure of the non-commuting graph associated to large classes of groups.

We discuss general properties of the graph. Moreover, we observe that the non-centralizer and the non-commuting graph associated to an AC-group are isomorphic. Since the non-centralizer graph is a complete k-partite graph, the non-commuting graph of an AC-group is completely determined. Finally, we prove that if $\Upsilon_G \cong \Upsilon_S$, then $G \cong S$, where S is a simple group not isomorphic to $B_n(q)$ or $C_n(q)$.

Throughout the paper, graphs are simple and all the notations and terminologies about the graphs are standard.

2. MAIN RESULTS

For a group G, $C_G(x) = \{y \in G : xy = yx\}$ is the centralizer of the element $x \in G$. Let us start with the following definition.

Definition 2.1. Let G be a group. We construct a graph whose vertices are the elements of G and whose edges are obtained by joining any two vertices x and y whenever $C_G(x) \neq C_G(y)$. We call this graph *the non-centralizer graph* of G, and we denote it by Υ_G.

If G is an abelian group, then Υ_G is an empty graph. Therefore throughout the paper all the groups are finite non-abelian unless otherwise mentioned. It is clear that for a non-central element x we have $\deg(x) \geq |G| - |C_G(x)|$ and $\deg(\mathrm{z}) = |G| - |Z(G)|$ for a central element z. If we consider the induced subgraph of Υ_G associated to the non-abelian group G with vertex set $G \setminus Z(G)$, then we have non-central vertices with degree $\deg(x) \geq |G| - |C_G(x)|$. Let us denote this subgraph by $\Upsilon_{G \setminus Z(G)}$. We conclude $\mathrm{diam}(\Upsilon_G) = \mathrm{diam}(\Upsilon_{G \setminus Z(G)}) = 2$ and $\mathrm{girth}(\Upsilon_G) = \mathrm{girth}(\Upsilon_{G \setminus Z(G)}) = 3$. We deduce that Υ_G and $\Upsilon_{G \setminus Z(G)}$ are connected.

Suppose $\mathrm{Cent}(G) = \{C_G(g) | g \in G\}$. A group G is called n-centralizer if $|\mathrm{Cent}(G)| = n$. It is clear that Υ_G is a complete $|\mathrm{Cent}(G)|$-partite graph.

Theorem 2.2. *Let G be an AC-group. Then $\Upsilon_{G \setminus Z(G)} \cong \Gamma_{G \setminus Z(G)}$.*

By the above result and the non-centralizer induced subgraph definition, we deduce that non-commuting graph $\Gamma_{G \setminus Z(G)}$ associated to a non-abelian AC-group is a complete $|\mathrm{Cent}(G)|$-partite graph.

Proposition 2.3. *Assume that Υ_G is the non-centralizer graph associated to the group G. Then we have*

(i) *If the central factor of G is of order p^2, then Υ_G is a complete $(p+2)$-partite graph.*

(ii) *Let p be the smallest prime dividing $|G|$. If $|G : Z(G)| = p^3$, then Υ_G is complete (p^2+p+2)-partite or (p^2+2)-partite graph.*

Theorem 2.4. *Let G be a group which satisfies one of the hypothesis (i) or (ii) of Proposition 2.3. Then Γ_G is either a $(p+2)$-partite graph or a (p^2+p+2)-partite, or a (p^2+2)-partite graph, respectively.*

Proof. By [4, Lemma 2.1] we see that a group whose central factor is of order pqr, where p, q, r are primes not necessarily distinct, is an AC-group. The proof follows now by Theorem 2.2 and Proposition 2.3. □

If $\Upsilon_G \cong \Upsilon_H$, then $N(G) = N(H)$, where $N(X)$ is the set $\{n \in \mathbb{N} |\ X$ has a conjugacy class C, such that $|C| = n\}$ and X is a group. Therefore we deduce the following result.

Theorem 2.5. *Let G be a group.*

(i) *If $\Upsilon_G \cong \Upsilon_{S_3}$, then $G \cong S_3$.*

(ii) *If $\Upsilon_G \cong \Upsilon_{A_n}$, G is a simple group and $n \geq 5$, then $G \cong A_n$.*

(iii) *If $\Upsilon_G \cong \Upsilon_H$, $G \neq B_n(q)$ or $C_n(q)$ is a simple group, then $G \cong H$.*

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MAXIMAL POSITIVE DEFINITE SOLUTIONS OF A MATRIX EQUATION

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Abstract. We study the nonlinear matrix equation $X+A^\star X^{-1}A+B^\star X^{-1}B=I$, where A and B are square matrices. some iterations for finding maximal positive definite solutions of these equations are given. Also some convergence results for these iteratins are given.

1. Introduction

In this paper, we consider the matrix equation

$$X + A^\star X^{-1} A + B^\star X^{-1} B = I, \tag{1.1}$$

where A *and* B are square matrices, and I is the identity matrix. Trying to solve special linear systems [1] leads to solving nonlinear matrix equations of the above types as follows:

For a linear system $Mx = f$ with $M = \begin{pmatrix} I & 0 & A \\ 0 & I & B \\ A^\star & B^\star & I \end{pmatrix}$ positive

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definite, we rewrite $M = \tilde{M} + K$, where

$$\tilde{M} = \begin{pmatrix} X & 0 & A \\ 0 & X & B \\ A^\star & B^\star & I \end{pmatrix}, K = \begin{pmatrix} I-X & 0 & 0 \\ 0 & I-X & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover, we decompose $\tilde{M}$ to the LU decomposition

$$\tilde{M} = \begin{pmatrix} X & 0 & A \\ 0 & X & B \\ A^\star & B^\star & I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ A^\star X^{-1} & B^\star X^{-1} & I \end{pmatrix} \begin{pmatrix} X & 0 & A \\ 0 & X & B \\ 0 & 0 & X \end{pmatrix}.$$

Such a decomposition of $\tilde{M}$ exists if and only if X is a positive definite solution of the matrix equations (1.1). Solving the linear system $\tilde{M}y = f$ is equivalent to solving two linear systems with a lower and upper block triangular system matrix. To compute the solution of $Mx = f$ from y, the Woodbury formula [2] can be applied.
The matrix Eq. (1.1) was studied in [3], and based on some conditions they proved that the matrix Eq. (1.1) has positive definite solutions. They also, proposed two iterative methods to find the Hermitian positive definite solutions of the Eq. (1.1). They did not analyze the convergence rate of proposed algorithms. In this paper we propose two algorithms. We will show that Algorithm (2.2) is more accurate than Algorithm pointed out in [3]. Also, Algorithm (2.4) needs less operation in comparison with these two algorithms.
The following notations are used throughout the rest of the paper. The notation $A \geq 0 (A > 0)$ means that A is Hermitian positive semidefinite (positive definite). For Hermitian matrices A and B, we write $A \geq B (A > B)$ if $A - B \geq 0 (> 0)$. Similarly, by $\lambda_1(A)$ and λ_n we denote respectively, the maximal and the minimal eigenvalues of A. The norm used in this paper is the spectral norm of the matrix A, i.e., $\|A\| = (\lambda_1(A^\star A))^{\frac{1}{2}}$.

2. Main results

Lemma 2.1. [5] *If C and P are Hermitian matrices of the same order with $P > 0$, then $CPC + P^{-1} \geq 2C$.*

Algorithm 2.2. *Let*

$$\begin{cases} X_0 = Y_0 = I, \\ Y_{n+1} = 2Y_n - Y_n X_n Y_n, \\ X_{n+1} = I - A^\star Y_{n+1} A - B^\star Y_{n+1} B, \quad n = 0, 1, 2, \ldots. \end{cases} \tag{2.1}$$

Theorem 2.3. *Assume that the equation (1.1) has a positive definite solution, then Algorithm (2.1) defines a monotically decreasing matrix sequence $\{X_n\}$ converging to X_∞ which is the maximal Hermitian positive definite solution of the equation (1.1). Also, sequence $\{Y_n\}$ defined in Algorithm (2.1) defines a monotically increasing sequence converges to X_∞^{-1}.*

Algorithm 2.4. *Take*

$$\begin{cases} X_0 = I, Y_0 = I \\ Y_{n+1} = (I - X_n)Y_n + I, \\ X_{n+1} = I - A^* Y_{n+1} A - B^* Y_{n+1} B, \quad n = 0, 1, 2, \dots. \end{cases} \tag{2.2}$$

Theorem 2.5. *If Eq. (1.1) has a positive definite solution and the two sequences $\{X_n\}$ and $\{Y_n\}$ are determined by Algorithm (2.4), then $\{X_n\}$ is monotone decreasing and converges to the maximal Hermitian positive definite solution X_∞. Also, sequence $\{Y_n\}$ defined in Algorithm (2.4) is a monotically increasing sequence converges to X_∞^{-1}.*

Theorem 2.6. *If the Eq. (1.1) has a positive definite solution and after n iterative steps of the Algorithm (2.4), the inequality $\|I - X_n Y_n\| < \epsilon$ implies*

$$\|X_n + A^\star X_n^{-1} A + B^\star X_n^{-1} B - I\| \le \epsilon(\|A\|^2 + \|B\|^2)\|X_\infty^{-1}\|.$$

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MIXED BELL AND STIRLING NUMBER

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Abstract. Stirling numbers of the second kind and Bell numbers are intimately linked through the roles they play in enumerating partitions of n-sets. As an extension of this problem, we consider $b_1+b_2+\ldots+b_n$ balls with b_1 balls labelled 1, b_2 balls labelled 2, …, b_n balls labelled n and $c_1+c_2+\ldots+c_k$ cells with c_1 cells labelled 1, c_2 cells labelled 2, …, c_k cells labelled k, in this paper we count the number of ways to partition the set of these balls into of these cells. As an application, for a positive integer m we evaluate the number of ways to write m as the form $m_1 \cdot m_2 \cdot \ldots \cdot m_k$, where $k \geqslant 1$.

1. Introduction

Stirling numbers of the second kind, denoted by $\left\{ n \atop k \right\}$, are the number of partitions of a set with n distinct elements into k disjoint non-empty

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sets. The recurrence relation

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$

with the initial value $\left\{ {n \atop 1} \right\} = 1$ determines these numbers. As an alternative definition, we can say that $\left\{ {n \atop k} \right\}$'s are the unique numbers satisfying

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x(x-1)(x-2)...(x-k+1).$$

An introduction on Stirling numbers can be found in [5]. Bell numbers, denoted by B_n, are the number of all partitions of a set with n distinct elements into disjoint non-empty sets. Thus

$$B_n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\}.$$

These numbers also satisfy the recurrence relation

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

See [1].

Values of B_n are given in Sloane on-line Encyclopaedia of Integer Sequences [2] as the sequence $A000110$. The sequence $A008277$ also gives the triangle of Stirling numbers of the second kind. There are a number of well-known results associated with them in [1, 3, 5] and [4].

In this paper we consider the following new problem.

- ⋆ Consider $b_1 + b_2 + \ldots + b_n$ balls with b_1 balls labelled 1, b_2 balls labelled 2, …, b_n balls labelled n and $c_1 + c_2 + \ldots + c_k$ cells with c_1 cells labelled 1, c_2 cells labelled 2, …, c_k cells labelled k. Evaluate the number of ways to partition the set of these balls into cells of these types.

In the present paper, we just consider the following two special cases of the mentioned problem: $b_1 = \ldots = b_n = 1, c_1, \ldots, c_k \in \mathbb{N}$ and $b_1, \ldots, b_n \in \mathbb{N}, c_1 = \ldots = c_k = 1$.

Definition 1.1. A multiset is a pair (A, m) where A is a set and $m : A \to \mathbb{N}$ is a function. The set A is called the set of underlying elements of (A, m). For each $a \in A$, $m(a)$ is called the multiplicity of a.

A formal definition for a multiset can be found in [4]. Let $A = \{1, 2, \ldots, n\}$ and $m(i) = b_i$ for $i = 1, 2, \ldots, n$. We denote the multiset

(A,m) by $\mathcal{A}(b_1,\ldots,b_n)$. Under this notation, the problem of partitioning $b_1+b_2+\ldots+b_n$ balls with b_1 balls labelled 1, b_2 balls labelled 2, $\ldots$, b_n balls labelled n into $c_1+c_2+\ldots+c_k$ cells with c_1 cells labelled 1, c_2 cells labelled 2, $\ldots$, c_k cells labelled k can be formulated as follows.

Definition 1.2. Let $\mathcal{B}=\mathcal{A}(b_1,\ldots,b_n)$ and $\mathcal{C}=\mathcal{A}(c_1,\ldots,c_k)$. Then the number of ways to partition $\mathcal{B}$ balls into non-empty $\mathcal{C}$ cells is denoted by $\left\{{\mathcal{B} \atop \mathcal{C}}\right\}$. These numbers are called the mixed Stirling numbers of the second kind. If cells are allowed to be empty, then we denote the number of ways to partition these balls into these cells by $\left\{{\mathcal{B} \atop \mathcal{C}}\right\}_0$.

2. Main results

The Case $b_1=\ldots=b_n=1$:
Note that if $b_1=b_2=\ldots=b_n=1$ and $c_1=k, c_2=\ldots=c_k=0$ then $\left\{{\mathcal{B} \atop \mathcal{C}}\right\}=\left\{{n \atop k}\right\}$ and $\left\{{\mathcal{B} \atop \mathcal{C}}\right\}_0=\sum_{i=1}^{k}\left\{{n \atop i}\right\}$. We denote $\sum_{i=1}^{k}\left\{{n \atop i}\right\}$ by $\left\{{n \atop k}\right\}_0$. Moreover, if $b_1=b_2=\ldots=b_n=1$ and $c_1=n, c_2=\ldots=c_k=0$ then $\left\{{\mathcal{B} \atop \mathcal{C}}\right\}_0=B_n$.

Definition 2.1. Let n,k and r be positive integers, $b_1=b_2=\ldots=b_n=1$ and $c_1=r, c_2=\ldots=c_k=1$. Then we denote $\left\{{\mathcal{B} \atop \mathcal{C}}\right\}$ by $B(n,k,r)$. These numbers are called the mixed Bell numbers. In this case $\left\{{\mathcal{B} \atop \mathcal{C}}\right\}_0$ is also denoted by $B_0(n,k,r)$.

Proposition 2.2. *Let n,k and r be positive integers. Then*

$$B_0(n,k,r)=\sum_{\ell=0}^{n}\binom{n}{\ell}\left\{{\ell \atop r}\right\}_0(k-1)^{n-\ell}.$$

Proposition 2.3. *Let n,k and r be positive integers. Then*

$$B(n,k,r)=\sum_{\ell=r}^{n-k+1}\binom{n}{\ell}\left\{{\ell \atop r}\right\}\left\{{n-\ell \atop k-1}\right\}(k-1)!.$$

Proposition 2.4. *Let n,k and r be positive integers. Then*

$$B(n,k,r)=\sum_{0\leqslant s\leqslant r, 0\leqslant t\leqslant k-1}(-1)^{t+\varepsilon_s}\binom{k-1}{t}B_0(n,k-t,r-s),$$

where

$$\varepsilon_s=\begin{cases}0 & \text{if } s=0\\ 1 & \text{otherwise}\end{cases}$$

Proposition 2.5. *Let n,k and r be positive integers. Then*

$$B(n,k,r)=B(n-1,k,r-1)+(k-1)B(n-1,k-1,r)+(k-1+r)B(n-1,k,r).$$

These results can be easily extended to the following general facts.

Theorem 2.6. *Let* $b_1 = \ldots = b_n = 1, c_1, \ldots, c_k \in \mathbb{N}, \mathcal{B} = \mathcal{A}(b_1, \ldots, b_n)$ *and* $\mathcal{C} = \mathcal{A}(c_1, \ldots, c_k)$. *Then*

$$\left\{\begin{matrix}\mathcal{B}\\ \mathcal{C}\end{matrix}\right\} = \sum_{\ell_1+\ldots+\ell_k=n} \frac{n!}{\ell_1!\ldots\ell_k!}\left\{\begin{matrix}\ell_1\\ c_1\end{matrix}\right\}\left\{\begin{matrix}\ell_2\\ c_2\end{matrix}\right\}\ldots\left\{\begin{matrix}\ell_k\\ c_k\end{matrix}\right\}.$$

Theorem 2.7. *Let* $b_1 = \ldots = b_n = 1, c_1, \ldots, c_k \in \mathbb{N}, \mathcal{B} = \mathcal{A}(b_1, \ldots, b_n)$ *and* $\mathcal{C} = \mathcal{A}(c_1, \ldots, c_k)$. *Then*

$$\left\{\begin{matrix}\mathcal{B}\\ \mathcal{C}\end{matrix}\right\} = \sum_{1\leqslant i\leqslant k, 0\leqslant j_i\leqslant c_i} (-1)^{\sharp(j_1,\ldots,j_k)}\left\{\begin{matrix}\mathcal{B}\\ \mathcal{C}_{j_1,\ldots,j_k}\end{matrix}\right\}_0,$$

where $\sharp(j_1, \ldots, j_k)$ *is the number of* i*'s such that* $j_i \neq 0$ *and* $\mathcal{C}_{j_1,\ldots,j_k} = \mathcal{A}(c_1 - j_1, \ldots, c_k - j_k)$.

Theorem 2.8. *Let* $b_1 = \ldots = b_n = 1, c_1, \ldots, c_k \in \mathbb{N}, \mathcal{B} = \mathcal{A}(b_1, \ldots, b_n)$, $\mathcal{C} = \mathcal{A}(c_1, \ldots, c_k), \mathcal{B}' = \mathcal{A}(b_2, \ldots, b_n)$ *and* $\mathcal{C}_j = \mathcal{A}(c_1, \ldots, c_{j-1}, c_j - 1, c_{j+1}, \ldots, c_k)$. *Then*

$$\left\{\begin{matrix}\mathcal{B}\\ \mathcal{C}\end{matrix}\right\} = (c_1 + \ldots + c_k)\left\{\begin{matrix}\mathcal{B}'\\ \mathcal{C}\end{matrix}\right\} + \sum_{j=1}^{k}\left\{\begin{matrix}\mathcal{B}'\\ \mathcal{C}_j\end{matrix}\right\}$$

3. The Case $c_1 = \ldots = c_k = 1$

Theorem 3.1. *Let* $b_1, \ldots, b_n \in \mathbb{N}, c_1 = \ldots = c_k = 1, \mathcal{B} = \mathcal{A}(b_1, \ldots, b_n)$ *and* $\mathcal{C} = \mathcal{A}(c_1, \ldots, c_k)$. *Then*

$$\left\{\begin{matrix}\mathcal{B}\\ \mathcal{C}\end{matrix}\right\}_0 = \prod_{j=1}^{n}\binom{b_j + k - 1}{k - 1}.$$

Theorem 3.2. *Let* $b_1, \ldots, b_n \in \mathbb{N}, c_1 = \ldots = c_k = 1, \mathcal{B} = \mathcal{A}(b_1, \ldots, b_n)$ *and* $\mathcal{C} = \mathcal{A}(c_1, \ldots, c_k)$. *Then*

$$\left\{\begin{matrix}\mathcal{B}\\ \mathcal{C}\end{matrix}\right\} = \sum_{i=1}^{n}(-1)^i\binom{k}{i}\prod_{j=1}^{n}\binom{b_j + k - i - 1}{k - i - 1}.$$

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ON GEOMETRIC MEANS OF n-MATRICES AND RELATED RESULTS

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Abstract. For $n \geq 3$, three kinds of geometric mean of n-positive definite matrices and their properties are introduced. Especially, we shall consider some matrix inequalities for one of the geometric means of n-matrices. It includes famous Ando-Hiai anf Furuta inequalities. Moreover as recent topics, we introduce converses of Loewner-Heinz inequality via geometric mean of n-matrices.

1. Introduction

Let $\mathbb{M}_m$ be the set of all m–by$m$ matrices over $\mathbb{C}$, and let $\mathbb{H}_m, \mathbb{P}_m \subset \mathbb{M}_m$ be the set of all Hermitian and positive definite matrices, respectively. For $A, B \in \mathbb{P}_m$, weighted geometric mean $A\sharp_\alpha B$ is well known as

$$A\sharp_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}} \quad \text{for } \alpha \in [0,1]. \tag{1.1}$$

Especially, in the case of $\alpha = \frac{1}{2}$, we call $A\sharp_{\frac{1}{2}}B$ geometric mean, and denote it by $A\sharp B$, simply. If A and B are non-invertible positive matrices, then geometric mean is defined by

$$A\sharp_\alpha B = \lim_{\varepsilon \to +0}(A+\varepsilon I)\sharp_\alpha(B+\varepsilon I) \quad \text{for } \alpha \in [0,1].$$

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The problem of extending two-variable geometric mean to multi-variable was long standing. In this talk, we shall introduce recent topics on matrices means of more than three matrices. Especially, we focus on matrix inequalities, and we will introduce extensions of some important matrix inequalities.

This talk is organized as follows:

(1) Three types of geometric means of more than three matrices ans their basic properties.
(2) Ando-Hiai and Furuta inequalities.
(3) Power mean and its properties.
(4) Recent results: converses of Loewner-Heinz inequality.

2. Three types of Geometric means and their basic properties

To extend the definition of geometric mean of two matrices into more than three matrices has not been succeeded for a long time. In fact, some definitions have given, but they do not have some important properties (for example, monotonicity or permutation invariance). Recently, a nice definition of geometric mean of n-matrices was given in [1]. Here we shall introduce its definition in the case of 3-matrices as follows:

Theorem 2.1 ([1]). *Let $A, B, C \in \mathbb{P}_m$. Define three sequences $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$ and $\{C_n\}_{n=0}^{\infty}$ as follows: $A_0 = A$, $B_0 = B$, $C_0 = C$ and*

$$A_{n+1} = B_n \sharp C_n, \quad B_{n+1} = C_n \sharp A_n, \quad C_{n+1} = A_n \sharp B_n.$$

Then $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$ and $\{C_n\}_{n=0}^{\infty}$ converge to the same limit

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = G.$$

Here we denote the limit G by $\mathfrak{G}_{alm}(A, B, C)$, and we call it ALM mean.

ALM mean has the following nice properties: Let $A_1, ..., A_n \in \mathbb{P}_m$.

(P1) If $A_1, \ldots, A_n$ commute with each other, then

$$\mathfrak{G}_{alm}(A_1, \ldots, A_n) = A_1^{\frac{1}{n}} \cdots A_n^{\frac{1}{n}}.$$

(P2) Joint homogeneity.

$$\mathfrak{G}_{alm}(a_1 A_1, \ldots, a_n A_n) = \prod_{i=1}^{n} a_i^{\frac{1}{n}} \mathfrak{G}_{alm}(A_1, \ldots, A_n)$$

for positive numbers $a_i > 0$ $(i = 1, \ldots, n)$.

(P3) Permutation invariance. For any permutation σ on $\{1, 2, \ldots, n\}$,

$$\mathfrak{G}_{alm}(A_1, \ldots, A_n) = \mathfrak{G}_{alm}(A_{\pi(1)}, \ldots, A_{\pi(n)}).$$

(P4) Monotonicity. For each $i = 1, 2, \ldots, n$, if $B_i \le A_i$, then

$$\mathfrak{G}_{alm}(B_1, \ldots, B_n) \le \mathfrak{G}_{alm}(A_1, \ldots, A_n).$$

(P5) Continuity. For each $i = 1, 2, \ldots, n$, let $\{A_i^{(k)}\}_{k=1}^{\infty} \subset \mathbb{P}_m$ be sequences such that $A_i^{(k)} \to A_i$ as $k \to \infty$. Then

$$\mathfrak{G}_{alm}(A_1^{(k)}, \ldots, A_n^{(k)}) \to \mathfrak{G}_{alm}(A_1, \ldots, A_n) \quad \text{as } k \to \infty.$$

(P6) Congruence invariance. For any invertible matrix S,

$$\mathfrak{G}_{alm}(S^*A_1S, \ldots, S^*A_nS) = S^*\mathfrak{G}_{alm}(\omega; A_1, \ldots, A_n)S.$$

(P7) Joint concavity. For $0 \le \lambda \le 1$,

$$\begin{aligned}\mathfrak{G}_{alm}(\lambda A_1 + (1-\lambda)A_1', \ldots, \lambda A_n + (1-\lambda)A_n') \\ \ge \lambda\mathfrak{G}_{alm}(A_1, \ldots, A_n) + (1-\lambda)\mathfrak{G}_{alm}(A_1', \ldots, A_n').\end{aligned}$$

(P8) Self-duality.

$$\mathfrak{G}_{alm}(A_1^{-1}, \ldots, A_n^{-1})^{-1} = \mathfrak{G}_{alm}(A_1, \ldots, A_n).$$

(P9) Determinantial identity.

$$\det \mathfrak{G}_{alm}(A_1, \ldots, A_n) = \prod_{i=1}^{n} (\det A_i)^{\frac{1}{n}}.$$

(P10) Arithmetic-geometric-harmonic means inequalities.

$$\left(\frac{1}{n}\sum_{i=1}^{n} A_i^{-1}\right)^{-1} \le \mathfrak{G}_{alm}(A_1, \ldots, A_n) \le \frac{1}{n}\sum_{i=1}^{n} A_i.$$

Although ALM mean has many nice properties, it is too difficult to compute. To improve about this problem, Bini-Meini-Poloni and Izumino-Nakamura gave another definition of geometric mean which satisfies the same ten properties of ALM mean as above. Here we shall introduce the definition of the geometric mean only three matrices case as follows:

Theorem 2.2 ([3, 4]). *Let $A, B, C \in \mathbb{P}_m$. Define three sequences $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$ and $\{C_n\}_{n=0}^{\infty}$ by $A_0 = A$, $B_0 = B$, $C_0 = C$ and*

$$A_{n+1} = (B_n \sharp C_n)\sharp_{\frac{1}{3}} A_n, \quad B_{n+1} = (C_n \sharp A_n)\sharp_{\frac{1}{3}} B_n, \quad C_{n+1} = (A_n \sharp B_n)\sharp_{\frac{1}{3}} C_n.$$

Then $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$ and $\{C_n\}_{n=0}^{\infty}$ converge the same limit.

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = G$$

Here we denote the limit G by $\mathfrak{G}_{bmp}(A, B, C)$, and we call it BMP mean.

ALM and BMP means are defined by the analytic way, however, the following last geometric mean is defined by the geometric way. To introduce the definition of the third geometric mean, we will need some preparation. For $A, B \in \mathbb{M}_m$, define an inner product $\langle A, B\rangle$ by $\langle A, B\rangle = \mathrm{tr}A^*B$, where $\mathrm{tr}X$ means the trace of a matrix X. Then $\mathbb{M}_m$ is an inner product space equipped with the norm $\|A\|_2 = (\mathrm{tr}A^*A)^{\frac{1}{2}}$, moreover $\mathbb{P}_m$ is a differential manifold, and we can consider the geodesic $[A, B] \subset \mathbb{P}_m$ which includes $A, B \in \mathbb{P}_m$. It can be parameterized as follows:

Theorem 2.3 ([2]). *Let $A, B \in \mathbb{P}_m$. Then there exists a unique geodesic $[A, B]$ joining A and B. It has a parametrization*

$$\gamma(t) = A\natural_t B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^t A^{\frac{1}{2}}, \quad t \in [0, 1].$$

Furthermore, we have a distance $\delta_2(A, B)$ between A and B along the geodesic $[A, B]$ as

$$\delta_2(A, B) = \|\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\|_2.$$

We call the metric $\delta_2(A, B)$ between A and B the Riemannian metric. A vector $\omega = (w_1, w_2, \ldots, w_n)$ is called a probability vector if and only if its components satisfy $\sum_{i=1}^n w_i = 1$ and $w_i > 0$ for $i = 1, 2, \ldots, n$, and let $\Delta_n \subset (0, 1)^n$ be the set of all probability vectors. Then the weighted Riemannian mean is defined as follows:

Definition 2.4 ([2]). Let $A_1, \ldots, A_n \in \mathbb{P}_m$, and $\omega = (w_1, \ldots, w_n) \in \Delta_n$. Then the weighted Riemannian mean $\Lambda(\omega; A_1, \ldots, A_n)$ is defined by

$$\Lambda(\omega; A_1, \ldots, A_n) = \mathrm{arg}zw \min_{zwX\in\mathbb{P}_m} \sum_{i=1}^n w_i\delta_2^2(A_i, X),$$

where $\mathrm{argmin}f(X)$ means the point X_0 which attains minimum value of the function $f(X)$.

Riemannian mean also satisfies the above ten properties, too. Moreover Riemannian mean has two types of characterization

Theorem 2.5 (Moakher, 2005, Lawson-Lim, 2010). *Let $A_1, \ldots, A_n \in \mathbb{P}_m$, and $\omega = (w_1, \ldots, w_n) \in \Delta_n$. Then $X = \Lambda(\omega; A_1, \ldots, A_n)$ is the unique positive solution of the following matrix equation:*

$$w_1 \log X^{\frac{-1}{2}}A_1X^{\frac{-1}{2}} + \cdots + w_n \log X^{\frac{-1}{2}}A_nX^{\frac{-1}{2}} = 0.$$

The matrix equation in Theorem 2.5 is called the Karcher equation. So we sometimes call Reimannian mean the Karcher mean. Let $A_1, \cdots, A_n \in \mathbb{P}_m$, and $\omega = (w_1, \cdots, w_n) \in \Delta_n$. We choose a natural number i from $\{1, \cdots, n\}$ according to the probability w_i. If i_k is chosen in the k-th trial, we set $X_k = A_{i_k}$, then we define the random walk $\{S_k\}$ as

$$S_1 = X_1, S_2 = S_1 \sharp X_2, S_3 = S_2 \sharp_{\frac{1}{3}} X_3, \cdots, S_k = S_{k-1} \sharp_{\frac{1}{k}} X_k.$$

Theorem 2.6 (Strum, 2003, Lawson-Lim, 2010, Holbrook, 2012)**.** *As we run through all possible outcome of the above procedure, almost always the sequence $\{S_k\}_{k\in\mathbb{N}}$ will converge to $\Lambda(\omega; A_1, \cdots, A_n)$, the weighted Reimannian mean.*

3. Ando-Hiai and Furuta inequalities.

There are many interesting results on the geometric mean of two-matrices. Especially, the following result is well known as the Ando-Hiai inequality:

Theorem 3.1 (Ando and Hiai, 1994)**.** *Let A and B be positive matrices. For any $\alpha \in [0,1]$, $A\sharp_\alpha B \leq I$ implies $A^p \sharp_\alpha B^p \leq I$ for all $p \geq 1$.*

We remark that the order is defined by positive semi-definiteness. (In the whole paper, we use this order.) For $A, B \in \mathbb{P}_m$, the order $\log A \geq \log B$ is called chaotic order. The following relation is very famous since $\log t$ is an operator monotone function:

$$A \geq B \quad \Longrightarrow \quad \log A \geq \log B.$$

It is well known that the following characterization of chaotic order:

Theorem 3.2 (Ando, 1987, Fujii-Furuta-Kamei, 1993, M. Uchiyama, 1999)**.** *Let $A, B \in \mathbb{P}_m$. Then they are mutually equivalent:*

(1) $\log A \geq \log B$,
(2) $A^p \geq (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}}$ *for all* $p \geq 0$,
(3) $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for all* $p, r \geq 0$.

Ando-Hiai inequality and the above characterization of chaotic order play important roles in the theory of matrix (operator) inequalities. We will introduce extensions of the above results to more than three matrices. Before, we shall introduce an important result:

Theorem 3.3 (Y. 2012)**.** *Let $A_1, \ldots, A_n \in \mathbb{P}_m$, and $\omega = (w_1, \ldots, w_n) \in \Delta_n$. Then*

$$w_1 \log A_1 + \cdots + w_n \log A_n \leq 0 \quad \Longrightarrow \quad \Lambda(\omega; A_1, \ldots, A_n) \leq I.$$

Using Theorem 3.3, we shall obtain extensions of Ando-Hiai inequality and characterization of chaotic order as follows:

Theorem 3.4 (Y. 2012)**.** *Let $A_1, \ldots, A_n \in \mathbb{P}_m$. For any $\omega \in \Delta_n$,*

$$\Lambda(\omega; A_1, \ldots, A_n) \leq I \quad \Longrightarrow \quad \Lambda(\omega; A_1^p, \ldots, A_n^p) \leq I$$

for all $p \geq 1$.

Theorem 3.5 (Y. 2012, Lim-Y. 2013)**.** *Let $A_1, ..., A_n \in \mathbb{P}_m$, and let $\omega = (w_1, ..., w_n) \in \Delta_n$. Then the following assertions are mutually equivalent;*

(1) $\sum_{i=1}^n w_i \log A_i \leq 0$;
(2) $\Lambda(\omega; A_1^p, ..., A_n^p) \leq I$ *for all* $p > 0$;
(3) $\Lambda(\omega_{\mathbf{p}}; A_1^{p_1}, ..., A_n^{p_n}) \leq I$ *for all* $\mathbf{p} = (p_1, ..., p_n) \in (0, \infty)^n$,

where $\omega_{\mathbf{p}} = (c\frac{w_1}{p_1}, ..., c\frac{w_n}{p_n})$ *and* $\frac{1}{c} = \sum_{i=1}^n \frac{w_i}{p_i}$.

One might expect that the above results can be satisfied by other geometric means (ALM and BMP means) But we have a negative answer for the problem.

Theorem 3.6 (Y. 2012, Lim-Pálfia 2012)**.** *Let $A_1, \ldots, A_n \in \mathbb{P}_m$ and $\omega \in \Delta_n$. If $\mathfrak{G}(\omega; A_1, \ldots, A_n)$ is a weighted geometric mean satisfying all properties of (P1)–(P10). If the weighted geometric mean satisfies Theorem 3.3, then the weighted geometric mean $\mathfrak{G}$ coincides with the weighted Riemannian mean.*

Corollary 3.7. *Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$, $\omega \in \Delta_n$ and $\mathfrak{G}(\omega; A_1, \ldots, A_n)$ be a weighted geometric mean satisfying all properties of (P1)–(P10). If the weighted geometric mean satisfies Theorem 3.4, then the weighted geometric mean $\mathfrak{G}$ coincides with the weighted Riemannian mean.*

Corollary 3.8. *ALM and BMP means do not satisfy Theorems 3.3 and 3.4.*

Moreover Theorem 3.2 is well known result as the most essential part of the famous Furuta inequality. Hence, we obtain an extension of Furuta inequality for several variables by using Theorem 3.3.

Theorem 3.9 (Ito 2012)**.** *Let $\mathbb{A} = (A_1, ..., A_n) \in \mathbb{P}_m^n$ and $q > 0$. Then $0 < A_n^q \leq A_i^q$ $(i = 1, ..., n-1)$ implies*

$$\Lambda(\omega; A_1^{-p_1}, ..., A_{n-1}^{-p_{n-1}}, A_n^{p_n}) \leq A_n^q \leq A_i^q$$

for all $p_i \geq 0$, $i = 1, ..., n-1$ and $p_n > q$, where $\hat{\omega} = \left(\frac{1}{p_1+q}, ..., \frac{1}{p_{n-1}+q}, \frac{n-1}{p_n-q}\right)$ and $\omega = \frac{\hat{\omega}}{\|\hat{\omega}\|_1}$.

In fact, Furuta inequality can be obtained as a two-matrices case of Theorem 3.9, that is, if $B \leq A$, then

$$A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}A^{\frac{-r}{2}} = \Lambda(\left(\frac{p-1}{p+r}, \frac{1+r}{p+r}\right); A^{-r}, B^p) \leq B \leq A$$

holds for $p \geq 1$, $r \geq 0$.

4. Power mean and its properties

It is well known that for $t \in [-1,1] \setminus \{0\}$, $f_t(x) = \left(\frac{1+x^t}{2}\right)^{\frac{1}{t}}$ is an operator monotone function on $[0,+\infty)$. Moreover $f_t(x)$ is increasing on t and $\lim_{t\to 0} f_t(x) = x^{\frac{1}{2}}$. From this fact, we can obtain power mean of two matrices: Let $A, B \in \mathbb{P}_m$ and $w \in (0,1)$. Then for $t \in [-1,1] \setminus \{0\}$ the power mean $P_t((1-w,w); A, B)$ is defined by

$$P_t((1-w,w); A, B) = A^{\frac{1}{2}} \left[(1-w)I + w(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^t\right]^{\frac{1}{t}} A^{\frac{1}{2}}. \quad (4.1)$$

It is known that $P_t((1-w,w); A, B)$ is increasing for $t \in [-1,1] \setminus \{0\}$ and $\lim_{t\to 0} P_t((1-w,w); A, B) = A\sharp B$. So the power mean interpolates among Arithmetic-Geometric-Harmonic means. Lim and Pálfia have extended the power mean into more than three matrices as follows:

Theorem 4.1 (Power mean [5])**.** *Let $A_1, ..., A_n \in \mathbb{P}_m$ and $\omega = (w_1, \ldots, w_n) \in \Delta_n$. Then for $t \in (0,1]$, the following matrix equation has a unique positive definite solution:*

$$X = \sum_{i=1}^{n} w_i (X \sharp_t A_i).$$

We write this solution $P_t(\omega; A_1, ..., A_n)$, and call it the ω-weighted power mean of order t. If $t \in [-1,0)$, we define the power mean by

$$P_t(\omega; A_1, ..., A_n) = P_{-t}(\omega; A_1^{-1}, ..., A_n^{-n})^{-1}$$

.

Let $A_1, ..., A_n \in \mathbb{P}_m$ and $\omega = (w_1, ..., w_n) \in \Delta_n$. It is easy to see that

$$P_1(\omega; A_1, ..., A_n) = \sum_{i=1}^{n} w_i A_i \quad \text{and} \quad P_{-1}(\omega; A_1, ..., A_n) = \left(\sum_{i=1}^{n} w_i A_i^{-1}\right)^{-1}.$$

We write $P_1(\omega; A_1, ..., A_n)$ and $P_{-1}(\omega; A_1, ..., A_n)$ by $\mathcal{A}(\omega; A_1, ..., A_n)$ and $\mathcal{H}(\omega; A_1, ..., A_n)$, respectively.

In the case of two matrices, the power mean coincides with (4.1).

Power mean has similar properties as geometric means in Section 2. Here we shall introduce some properties of the power mean in the view point of matrix inequalities.

Theorem 4.2 (Lim-Y. 2013)**.** *Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}_m^n$ and let $\omega = (w_1, \ldots, w_n) \in \Delta_n$. Then*

(1) *for* $t \in (0,1]$, $\sum_{i=1}^{n} w_i A_i^t \leq I$ *implies* $P_t(\omega;\mathbb{A}) \leq I$;
(2) *for* $t \in [-1,0)$, $\sum_{i=1}^{n} w_i A_i^t \leq I$ *implies* $P_t(\omega;\mathbb{A}) \geq I$.

We obtain a variant of Ando-Hiai inequality on power means.

Corollary 4.3 (Lim-Y. 2013)**.** *Let* $\mathbb{A} = (A_1, ..., A_n) \in \mathbb{P}_m^n$ *and* $\omega \in \Delta_n$. *Then for* $t \in (0,1]$,

(1) $P_t(\omega;\mathbb{A}) \leq I$ *implies* $P_{\frac{t}{p}}(\omega; A_1^p, ..., A_n^p) \leq I$ *for all* $p \geq 1$,
(2) $P_{-t}(\omega;\mathbb{A}) \geq I$ *implies* $P_{-\frac{t}{p}}(\omega; A_1^p, ..., A_n^p) \geq I$ *for all* $p \geq 1$.

For $G, H : \Delta_n \times \mathbb{P}_m^n \to \mathbb{P}_m$, we define $G \leq H$ if $G(\omega;\mathbb{A}) \leq H(\omega;\mathbb{A})$ for all $\omega \in \Delta_n$ and $\mathbb{A} \in \mathbb{P}_m^n$. We note that $\mathcal{H} \leq \mathcal{A}$, the arithmetic-harmonic means inequality.

Corollary 4.4 (Lim-Pálfia 2012, Lim-Y. 2013)**.** *For* $0 < s \leq t \leq 1$,

$$\mathcal{H} = P_{-1} \leq P_{-t} \leq P_{-s} \leq \cdots \leq P_s \leq P_t \leq P_1 = \mathcal{A}$$

Furthermore, for $\mathbb{A} \in \mathbb{P}_m^n$ *and* $\omega \in \Delta_n$, *the limit of power means* $\lim_{t\to 0} P_t(\omega;\mathbb{A})$ *exists and coincides with the Karcher mean* $\Lambda(\omega;\mathbb{A})$.

5. Recent results: converses of Loewner-Heinz inequality

A real continuous function $f(t)$ defined on a real interval I is said to be *operator monotone*, provided $A \leq B$ implies $f(A) \leq f(B)$ for any two bounded self-adjoint operators A and B whose spectra are in I. The Loewner- Heinz inequality means the power function t^a is operator monotone on $[0,\infty)$ for $0 < a < 1$. $\log t$ is operator monotone on $(0,\infty)$ too. A continuous function f defined on I is called an *operator convex function* on I if $f(sA + (1-s)B) \leq sf(A) + (1-s)f(B)$ for every $0 < s < 1$ and for every pair of bounded self-adjoint operators A and B whose spectra are both in I. An *operator concave function* is likewise defined. If $I = (0,\infty)$, then $f(t)$ is operator monotone on I if and only if $f(t)$ is operator concave and $f(\infty) > -\infty$. This implies that every operator monotone function on $(0,\infty)$ is operator concave. σ is said to be symmetric if $A\sigma B = B\sigma A$ for every A, B. σ is symmetric if and only if $f(t) = tf(1/t)$.

Theorem 5.1 (M. Uchiyama-Y. to appear in JMAA)**.** *Let* $f(t)$ *be an operator monotone function on* $(0,\infty)$ *with* $f(1) = 1$, *and let* A *and* B *be bounded self-adjoint operators. Let* σ *be an operator mean satisfying* $! \leq \sigma \leq \nabla$. *Then* $A \leq B$ *if and only if* $f(\lambda A + I)\sigma f(-\lambda B + I) \leq I$ *for all sufficiently small* $\lambda \geq 0$.

We remark that a symmetric operator mean σ, that is $A\sigma B = B\sigma A$ for every A and B, satisfies $! \leq \sigma \leq \nabla$. The following corollary may make the above statement clearer.

Corollary 5.2 (M. Uchiyama-Y. to appear in JMAA). *Let $f(t)$ and $h(t)$ be operator monotone functions on $(0, \infty)$ with $f(1) = h(1) = 1$, and let A and B be bounded self-adjoint operators. Let σ and δ be operator means satisfying $! \leq \sigma \leq \nabla$ and $! \leq \delta \leq \nabla$. Then $h(\lambda A + I)\delta h(\lambda B + I) \leq I$ for all sufficiently small $\lambda \geq 0$ if and only if $f(\lambda A + I)\sigma f(\lambda B + I) \leq I$ for all sufficiently small $\lambda \geq 0$.*

Theorem 5.3 (M. Uchiyama-Y. to appear in JMAA). *Let $f(t)$ be a non-constant operator monotone function on $(0, \infty)$ with $f(1) = 1$, and let A and B be bounded self-adjoint operators. Then the following are equivalent:*

(1) $A \leq B$,
(2) $\|x\|^2 \leq \|f(\lambda A + I)^{\frac{-1}{2}} x\| \|f(-\lambda B + I)^{\frac{-1}{2}} x\|$ *for all $x \in \mathcal{H}$ and all sufficiently small $\lambda \geq 0$,*
(3) $\|x\|^2 \leq \|e^{-pA} x\| \|e^{pB} x\|$ *for all $x \in \mathcal{H}$ and all $p \geq 0$.*

We use the symbol $P_0(\omega; A_1, ..., A_n)$ instead of $\Lambda(\omega; A_1, ..., A_n)$.

Theorem 5.4 (M. Uchiyama-Y. to appear in JMAA). *Let $A_1, ..., A_n$ be Hermitian matrices, and $\omega = (w_1, ..., w_n) \in \Delta_n$. Let $f(t)$ be a non-constant operator monotone function on $(0, \infty)$ with $f(1) = 1$. Then the following are equivalent:*

(1) $\displaystyle\sum_{i=1}^{n} w_i A_i \leq 0$,
(2) $P_1(\omega; f(\lambda A_1 + I), ..., f(\lambda A_n + I)) = \displaystyle\sum_{i=1}^{n} w_i f(\lambda A_i + I) \leq I$ *for all sufficiently small $\lambda \geq 0$,*
(3) *for each $t \in [-1, 1]$, $P_t(\omega; f(\lambda A_1 + I), ..., f(\lambda A_n + I)) \leq I$ for all sufficiently small $\lambda \geq 0$.*

Corollary 5.5 (M. Uchiyama-Y. to appear in JMAA). *Let $A_1, ..., A_n$ be Hermitian matrices, and $\omega = (w_1, ..., w_n) \in \Delta_n$. Let $f(t)$ be a non-constant operator monotone function on $(0, \infty)$ with $f(1) = 1$. Then the following are equivalent:*

(1) $\displaystyle\sum_{i=1}^{n} w_i A_i \leq 0$,
(2) $\displaystyle\sum_{i=1}^{n} w_i f(\lambda A_i + I) \leq I$ *for all sufficiently small $\lambda \geq 0$,*

(3) $\Lambda(\omega; f(\lambda A_1 + I), ..., f(\lambda A_n + I)) \leq I$ *for all sufficiently small* $\lambda \geq 0$.

We especially consider the probability vector $\omega = (\frac{1}{n}, ..., \frac{1}{n})$ to obtain a multi-variable case of Theorem 5.3.

Theorem 5.6 (M. Uchiyama-Y. to appear in JMAA)**.** *Let* $A_1, ..., A_n$ *be Hermitian matrices, and let* f *be a non-constant operator monotone function on* $(0, \infty)$ *with* $f(1) = 1$*. Then the following are equivalent:*

(1) $\displaystyle\sum_{i=1}^{n} A_i \leq 0$,

(2) $\|x\|^n \leq \displaystyle\prod_{i=1}^{n} \|f(\lambda A_i + I)^{\frac{-1}{2}} x\|$ *for all sufficiently small* $\lambda \geq 0$ *and all* $x \in \mathcal{H}$,

(3) $\|x\|^n \leq \displaystyle\prod_{i=1}^{n} \|e^{-pA_i} x\|$ *for all* $x \in \mathcal{H}$ *and all* $p \geq 0$.

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SOME RESULTS ON THE GENERALIZED NUMERICAL RANGES OF NON SQUARE MATRICES

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Abstract. In this paper, we introduce the notion of $k-$rank numerical range of a non square matrix $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \in M_{n\times m}$ $(n > m)$ with respect to the matrix $I_{n,m} = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$. Some algebraic and geometrical properties are investigated.

1. Introduction

Let $M_{n\times m}$ be the vector space of all $n \times m$ complex matrices. For the case $n = m$, $M_{n\times n}$ is denoted by M_n, i.e. the algebra of all $n \times n$ complex matrices. By [3, Proposition 1.1], for a positive integer $1 \leq k \leq n$, the k-rank numerical range of $A \in M_n$ is defined and denoted by

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : X^*AX = \lambda I_k \ for \ some \ X \in \chi_{n,k}\},$$

where $\chi_{n,k} = \{X \in M_{n\times k} : X^*X = I_k\}$, i.e., the set of all $n\times k$ isometry matrices. The sets $\Lambda_k(A)$, where $k \in \{1, \ldots, n\}$, are generally called higher rank numerical range of A. Apparently, for k=1, $\Lambda_k(A)$ reduces to the classical numerical range of A, namely,

$$\Lambda_1(A) = W(A) := \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\},$$

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which has been studied extensively for many decades. It is useful in studying and understanding matrices and operators, and has many applications in numerical analysis, differential equations, systems theory, etc. It is readily verified

$$W(A) = \Lambda_1(A) \supseteq \Lambda_2(A) \supseteq \cdots \supseteq \Lambda_n(A).$$

The numerical range $W(A) = \Lambda_1(A)$ is a nonempty, compact and convex subset of $\mathbb{C}$. The higher-rank numerical ranges can, of course, be empty. But the compactness and convexity still hold in general. Moreover, see [1],

$$\Lambda_k(A) = \bigcap_{X \in \chi_{n,n-k+1}} W(X^*AX). \tag{1.1}$$

Stampfli and Williams [5, Theorem 4], and later Bonsall and Duncan [2, Lemma 6.22.1], observed that the numerical range of $A \in M_n$, $W(A)$, can be written

$$W(A) = \{\mu \in \mathbb{C} : \|A - \lambda I_n\|_2 \geq |\mu - \lambda| \quad \forall \lambda \in \mathbb{C}\},$$

where $\|.\|_2$ denotes the spectral matrix norm (i.e., the matrix norm subordinate to the Euclidean vector norm), and I_n is the $n \times n$ identity matrix. By this idea, Chorianopoulos, Karanasios and Psarrakos in [4] recently observed that the numerical range of $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \in M_{n\times m}$ $(n > m)$ with respect to matrix $I_{n,m} = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$, $W_{\|.\|}(A; I_{n,m})$, can be written

$$W_{\|.\|_2}(A; I_{n,m}) = \{\mu \in \mathbb{C} : \| \begin{pmatrix} A_1 & 0 \\ A_2 & 0 \end{pmatrix} - \lambda \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} \|_2 \geq |\mu - \lambda|,$$

$\forall \lambda \in \mathbb{C}\}$. Where the matrices

$$\begin{pmatrix} A_1 & 0 \\ A_2 & 0 \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$$

are $n \times n$. It is clear that $W_{\|.\|_2}(A; I_n) = W(A) = \Lambda_1(A)$, where $A \in M_n$. In this paper, we are going to define the definition and some general properties of $\Lambda_{k.\|.\|_2}(A; I_{n,m})$.

2. Main results

It is natural to use a formula analogous to (1.1) to propose a definition of the $k-$ rank numerical range of non square matrix $A \in M_{n\times m}$ $(n > m)$ with respect to $I_{n,m}$.

Definition 2.1. Let $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \in M_{m\times m}(n > m)$ with $A_1 \in M_{m\times m}$, $A_2 \in M_{(n-m)\times m}$ and $I_{n,m} = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$ and $1 \leq k \leq m$ be a positive integer. The k-rank numerical range of A with respect to $I_{n,m}$ is defined and denoted by

$$\Lambda_{k,\|\cdot\|_2}(A; I_{n,m}) = \{\mu \in \mathbb{C} : \| \begin{pmatrix} Y^*(A_1 - \lambda I_m)Y \\ A_2 Y \end{pmatrix} \|_2 \geq |\mu - \lambda| \ \forall \lambda \in \mathbb{C},$$

$\forall\ Y \in \chi_{m,m-k+1}\}$.

In the following Propositions, without loss of generality, we assume that $n > m, A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ with $A_1 \in M_{m\times m}$, $A_2 \in M_{(n-m)\times m}$ and $I_{n,m} = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$.

In the following Proposition, we state the relationship between $\Lambda_{k,\|\cdot\|_2}(A; I_{n,m})$ and $\bigcap_{S\in M_{n\times(n-m)}} \Lambda_{k,\|\cdot\|_2}(A; I_{n,m})$.

Proposition 2.2. *Let $A \in M_{n\times m}$ and $1 \leq k \leq m$ be a positive integer. Then*

$$\Lambda_{k,\|\cdot\|_2}(A; I_{n,m}) \subseteq \bigcap_{S\in M_{n\times(n-m)}} \Lambda_{k,\|\cdot\|_2}(A; I_{n,m}).$$

Proof. Let $\mu \in \Lambda_{k,\|\cdot\|_2}(A; I_{n,m})$. Using Definition (2.1), the result holds. $\square$

In the following Proposition, we show that the intersection coincides with the higher rank numerical range A_1, $\Lambda_k(A_1)$.

Proposition 2.3. *Let $A \in M_{n\times m}$ and $1 \leq k \leq m$ be a positive integer. Then*

$$\Lambda_k(A_1) = \bigcap_{S\in M_{n\times(n-m)}} \Lambda_k(\begin{pmatrix} A & S \end{pmatrix}).$$

By the above Proposition, the next corollary follows readily.

Corollary 2.4. *Let $A \in M_{n\times m}$ and $1 \leq k \leq m$ be a positive integer. If $\mu \in \Lambda_k(A_1)$, Then*

$$\Lambda_k(A_1) = \bigcap_{\theta\in[0,2\pi]} \Lambda_k(\begin{pmatrix} A_1 & e^{i\theta}A_2^* \\ A_2 & \mu I_{n-m} \end{pmatrix}).$$

In the following Theorem, we show that the set $\Lambda_{k,\|.\|_2}(A; I_{n,m})$ coincides with the higher rank numerical range of the submatrix A_1, $\Lambda_k(A_1)$.

Theorem 2.5. *Let $A \in M_{n\times m}$ and $1 \leq k \leq m$ be a positive integer. Then*

$$\Lambda_{k,\|\cdot\|_2}(A; I_{n,m}) = \Lambda_k(A_1).$$

If we replace the matrix $I_{n,m}$ by $bI_{n,m}$ for some nonzero scaler $b \in \mathbb{C}$, then we have the following result.

Proposition 2.6. *Let $A \in M_{n\times m}$, $1 \leq k \leq m$ be a positive integer and $b \in \mathbb{C}$. Then the followig holds:*
(i) If $|b| = 1$, then $\Lambda_{k,\|\cdot\|_2}(A; bI_{n,m}) = b^{-1}\Lambda_{k,\|\cdot\|_2}(A; I_{n,m})$;
(ii) If $|b| < 1$, then $\Lambda_{k,\|\cdot\|_2}(A; bI_{n,m}) \subseteq b^{-1}\Lambda_{k,\|\cdot\|_2}(A; I_{n,m})$;
(iii) If $|b| > 1$, then $\Lambda_{k,\|\cdot\|_2}(A; bI_{n,m}) \supseteq b^{-1}\Lambda_{k,\|\cdot\|_2}(A; I_{n,m})$.

Proposition 2.7. *Let $A \in M_{n\times m}$, $1 \leq k \leq m$ be a positive integer and a, $b \in \mathbb{C}$. Then $\Lambda_{k,\|\cdot\|_2}(aA + bI_{n,m}; I_{n,m}) = a\Lambda_{k,\|\cdot\|_2}(A; I_{n,m}) + b$.*

In the following Proposition, we state the relation between the $\Lambda_{k,\|\cdot\|_2}(A; I_{n,m})$ and $\Lambda_{k,\|\cdot\|_2}(I_{n,m}; A)$.

Proposition 2.8. *Let $A \in M_{n\times m}$, $1 \leq k \leq m$ be a positive integer.*

Then

$$\{\mu^{-1} \in \mathbb{C} : \mu \in \Lambda_{k,\|\cdot\|_2}(A; I_{n,m}), |\mu| \geq 1\} \subseteq \Lambda_{k,\|\cdot\|_2}(I_{n,m}; A).$$

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APPROXIMATE ROBERTS ORTHOGONALITY

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Abstract. In a real normed space we introduce two notions of approximate Roberts orthogonality as follows:

$$x \perp_R^{\varepsilon} y \text{ iff } \Big| \|x+ty\|^2 - \|x-ty\|^2 \Big| \leq 4\varepsilon \|x\| \|ty\| \text{ for all } t \in \mathbb{R}\,;$$

and

$$x\, {}^{\varepsilon}\!\perp_R y \text{ iff } \Big| \|x+ty\| - \|x-ty\| \Big| \leq \varepsilon(\|x+ty\| + \|x-ty\|) \text{ for all } t \in \mathbb{R}\,.$$

We study class of linear mappings preserving the approximately Roberts orthogonality of type ${}^{\varepsilon}\!\perp_R$.

1. Introduction

One of the important ideas playing a fundamental role in geometry of normed spaces is the concept of orthogonality. Many mathematicians have introduced different types of orthogonality for normed linear spaces. Suppose $(\mathcal{X}, \|.\|)$ is a real normed linear space whose dimension is at least two. In 1934, Roberts introduced the first orthogonality type: $x \in \mathcal{X}$ is said to be orthogonal in the sense of Roberts to $y \in \mathcal{X}$ $(x \perp_R y)$ if $\|x + ty\| = \|x - ty\|$ for all $t \in \mathbb{R}$. Later, in 1935, Birkhoff introduced one of the most important orthogonality types: x is said to be Birkhoff orthogonal to y $(x \perp_B y)$ if $\|x + ty\| \geq \|x\|$ for all $t \in \mathbb{R}$. James in 1945 introduced isosceles orthogonality: x is

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said to be isosceles orthogonal to y ($x \perp_I y$) if $\|x+y\| = \|x-y\|$. Trivially, the implications $\perp_R \Longrightarrow \perp_I$ and $\perp_R \Longrightarrow \perp_B$ hold while each of the reciprocals of these implications holds only in real-valued inner product spaces. Also $\perp_B$ and $\perp_I$ are independent and they coincide only in real-valued inner product spaces. There are many other types of orthogonality in normed linear spaces. For example the following mapping was introduced by Miličič:

$$\langle .|. \rangle_g \ : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$$

$$\langle y|x \rangle_g = \frac{1}{2}(\rho'_+(x,y) + \rho'_-(x,y)),$$

where mappings $\rho'_+, \rho'_- : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$ are defined by

$$\rho'_\pm(x,y) = \lim_{t \to 0^\pm} \frac{\|x+ty\|^2 - \|x\|^2}{2t} \qquad (x, y \in \mathcal{X}).$$

In addition, the ρ-orthogonality $x \perp_\rho y$ means $\langle y|x \rangle_g = 0$. Note that if $(\mathcal{X}, \langle .|. \rangle)$ is a real-valued inner product space, then all above orthogonalities coincide with the usual orthogonality $\perp$ derived from $\langle .|. \rangle$ [1].

2. Main results

Let $\varepsilon \in [0,1)$ and x, y be elements of a real normed space $\mathcal{X}$. In a real-valued inner product space it is natural to consider the approximate orthogonality (ε-orthogonality) $x \perp^\varepsilon y$ defined by

$$|\langle x|y \rangle| \leq \varepsilon \|x\| \|y\|.$$

This fact motivated Dragomir [2] to give the following definition of the approximate Birkhoff orthogonality $x \, {}^\varepsilon\!\!\perp_B y$ as follows:

$$\|x+ty\| \geq (1-\varepsilon)\|x\| \qquad (t \in \mathbb{R}).$$

Chmieliński [3] introduced another approximate Birkhoff orthogonality $x \perp_B^\varepsilon y$ by

$$\|x+ty\|^2 \geq \|x\|^2 - 2\varepsilon \|x\| \|ty\| \qquad (t \in \mathbb{R}).$$

Now, we propose two definitions of approximate Roberts orthogonality (ε-R-orthogonality) [5]. Let us say $x \perp_R^\varepsilon y$ if

$$\Big| \|x+ty\|^2 - \|x-ty\|^2 \Big| \leq 4\varepsilon \|x\| \|ty\| \qquad (t \in \mathbb{R}).$$

We also define $x \, {}^\varepsilon\!\!\perp_R y$ if

$$\Big| \|x+ty\| - \|x-ty\| \Big| \leq \varepsilon(\|x+ty\| + \|x-ty\|) \qquad (t \in \mathbb{R}).$$

The following results state some basic properties of ε-R-orthogonality.

Theorem 2.1. *Let $\mathcal{X}$ be an arbitrary real normed space and $x, y \in \mathcal{X}$. Then*

(a) $x \perp_R^{\varepsilon} y \Longrightarrow x \,{}^{\varepsilon}\!\!\perp_R y$;
(b) $x \,{}^{\varepsilon}\!\!\perp_R y \Longrightarrow x \,{}^{\varepsilon}\!\!\perp_B y$;
(c) $x \perp_R^{\varepsilon} y \Longrightarrow x \,{}^{\varepsilon}\!\!\perp_B y$;
(d) $x \perp_R^{\varepsilon} y \Longrightarrow x \perp_B^{\varepsilon} y$.

Theorem 2.1(a) shows that in an arbitrary normed space the second ε-R-orthogonality is weaker than the first ε-R-orthogonality, but the converse is not true in general (see the example below).

Example 2.2. Consider the space $(\mathbb{R}^2, |||.|||)$ equipped with the maximum norm $|||(x,y)||| := \max\{|x|, |y|\}$. Let $x = (1,0), y = (\frac{1}{2}, 1), \varepsilon = \frac{11}{29}$. One can show that $x \,{}^{\varepsilon}\!\!\perp_R y$, but not $x \perp_R^{\varepsilon} y$ (take, for example, $t = \frac{2}{3}$). Thus $x \,{}^{\varepsilon}\!\!\perp_R y \nRightarrow x \perp_R^{\varepsilon} y$.

Proposition 2.3. *If $\mathcal{X}$ is a real-valued inner product space and $x, y \in \mathcal{X}$, then*

$$x \perp^{\varepsilon} y \Longleftrightarrow x \perp_R^{\varepsilon} y \Longleftrightarrow x \,{}^{\zeta}\!\!\perp_R y,$$

where $\zeta = \frac{1-\sqrt{1-\varepsilon^2}}{\varepsilon}$.

Here is another property of the Roberts orthogonality.

Proposition 2.4. *Suppose that there are two equivalent norms on a real normed space $\mathcal{X}$, i.e., $m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$ for all $x \in \mathcal{X}$ and some $0 < m \leq M$. Then*

$$x \perp_{R,1} y \Longrightarrow x \,{}^{\xi}\!\!\perp_{R,2} y,$$

for all $x, y \in \mathcal{X}$, where $\xi = \frac{M-m}{M+m}$.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are real normed spaces and let $\delta, \varepsilon \in [0,1)$. We say that a linear mapping $T : \mathcal{X} \to \mathcal{Y}$ preserves the approximate Roberts orthogonality (δ-ε-R-orthogonality) if for each $x, y \in \mathcal{X}$, $x \,{}^{\delta}\!\!\perp_R y$ implies that $Tx \,{}^{\varepsilon}\!\!\perp_R Ty$. Approximately orthogonality preserving mappings in the framework of normed spaces have been recently studied. In the case where $\delta = 0$, Mojškerc and Turnšek [4] verified the properties of mappings that preserve approximate Birkhoff orthogonality. Also Chmieliński and Wójcik [3] studied some properties of mappings that preserve approximate isosceles-orthogonality and ρ-orthogonality in the case when $\delta = 0$. Recently Zamani and Moslehian [5] studied approximate Roberts orthogonality preserving mappings.
We state some prerequisites for the next theorem. For a bounded linear mapping $T : \mathcal{X} \to \mathcal{Y}$, let $\|T\| = \sup\{\|Tx\|;\ \|x\| = 1\}$ denote the operator norm and $[T] := \inf\{\|Tx\|;\ \|x\| = 1\}$.

Theorem 2.5. *If a linear mapping $T : \mathcal{X} \to \mathcal{Y}$ satisfies*

$$\frac{1-\varepsilon}{1+\varepsilon}\gamma\|x\| \leq \|Tx\| \leq \frac{1+\varepsilon}{1-\varepsilon}\gamma\|x\|$$

for all $x \in \mathcal{X}$ and $\gamma \in [[T], \|T\|]$, then

$$x \,{}^{\delta}\!\perp_R y \Longrightarrow Tx \,{}^{\theta}\!\perp_R Ty \qquad (x, y \in \mathcal{X}),$$

where $\theta = \frac{\delta+\varepsilon}{1+\delta\varepsilon}$.

Corollary 2.6. *If a nonzero bounded linear mapping $T : \mathcal{X} \to \mathcal{Y}$ satisfies $\|T\| \leq \frac{1+\varepsilon}{1-\varepsilon}[T]$, then*

$$x \,{}^{\delta}\!\perp_R y \Longrightarrow Tx \,{}^{\theta}\!\perp_R Ty \qquad (x, y \in \mathcal{X}),$$

where $\theta = \frac{\delta+\varepsilon}{1+\delta\varepsilon}$.

A linear mapping $U : \mathcal{X} \to \mathcal{Y}$ is said to be an approximate similarity if it is a non-zero-scalar multiple of an approximate linear isometry, or equivalently it satisfies

$$|\lambda|(1-\varphi_1(\varepsilon))\|w\| \leq \|Uw\| \leq |\lambda|(1+\varphi_2(\varepsilon))\|w\|$$

for some unitary U, some $\lambda \in \mathbb{R} \setminus \{0\}$ and for all $w \in \mathcal{X}$, where $\varphi_1(\varepsilon) \to 0$ and $\varphi_2(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Theorem 2.7. *Let $U : \mathcal{X} \to \mathcal{Y}$ be an approximate similarity. If a bounded linear mapping $T : \mathcal{X} \to \mathcal{Y}$ satisfies $\|T - U\| \leq \varepsilon\|U\|$, then*

$$x \,{}^{\delta}\!\perp_R y \Longrightarrow Tx \,{}^{\theta}\!\perp_R Ty \qquad (x, y \in \mathcal{X})$$

for any $\delta \in [0, 1)$, where $\theta = \frac{2\delta+2\varepsilon+(1-\delta)\varphi_1(\varepsilon)+(1+\delta+2\varepsilon)\varphi_2(\varepsilon)}{2+2\delta\varepsilon-(1-\delta)\varphi_1(\varepsilon)+(1+\delta+2\delta\varepsilon)\varphi_2(\varepsilon)}$.

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THE SOLUTION OF INTERVAL SYSTEM OF MATRIX EQUATIONS

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Abstract. In this paper, we consider an interval system of matrix equations as

$$\begin{cases} A_{11}X + YA_{12} = C_1, \\ A_{21}X + YA_{22} = C_2. \end{cases}$$

We define a solution set for this system and then we present a direct method and an iterative method for solving this interval system.

1. Introduction

Interval matrix equations and interval systems of matrix equations have many applications in sciences and enginearing, such as electromagnetic scattering, structural mechanics and computation of the frequency response matrix in control theory.
A sampel of these systems of matrix equations is as the following

$$\begin{cases} \mathbf{A}_{11}X + Y\mathbf{A_{12}} = \mathbf{C_1}, \\ \mathbf{A}_{21}X + Y\mathbf{A_{22}} = \mathbf{C_2}, \end{cases} \tag{1.1}$$

where $\mathbf{A_{ij}}$ and $\mathbf{C_i}$, for $i, j = 1, 2$, are interval matrices.
Note that bold-face letters are used to show intervals. Some samples of interval matrix equations such as $\mathbf{A}X = \mathbf{B}$ and the interval sylvester equation $\mathbf{A}X + X\mathbf{B} = \mathbf{C}$, have been considered previously. See for

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example [1] and [5].
In this paper, we denote the set of all $m\times n$ interval matrices by $\mathbb{IR}^{m\times n}$. For the interval matrix $\mathbf{A}=[\underline{A},\overline{A}]$, the center matrix denoted by $\check{\mathbf{A}}$ and the radius matrix denoted by $rad\mathbf{A}$ are respectively defined as

$$\check{\mathbf{A}}=\frac{1}{2}(\underline{A}+\overline{A})\ ,\ rad\mathbf{A}=\frac{1}{2}(\overline{A}-\underline{A}).$$

We assume that the reader is familiar with a basic interval arithmetic and interval operators on the interval matrices, otherwise see [2]. An $n\times n$ interval matrix $\mathbf{A}=[\underline{A},\overline{A}]$ is said to be regular if each $A\in\mathbf{A}$ is regular. For two interval matrices $\mathbf{A}\in\mathbb{IR}^{m\times n}$ and $\mathbf{B}\in\mathbb{IR}^{k\times t}$, the Kronecker product denoted by $\otimes$ is defined by the $mk\times nt$ block interval matrix

$$\mathbf{A}\otimes\mathbf{B}=\begin{bmatrix}\mathbf{a}_{11}\mathbf{B} & \cdots & \mathbf{a}_{1n}\mathbf{B}\\ \vdots & \ddots & \vdots\\ \mathbf{a}_{m1}\mathbf{B} & \cdots & \mathbf{a}_{mn}\mathbf{B}\end{bmatrix},$$

and $vec(\mathbf{A})$ is defined as an mn-interval vector and obtained by stacking the columns of $\mathbf{A}$ i.e.

$$vec(\mathbf{A})=(\mathbf{A}_{.1},\mathbf{A}_{.2},\cdots,\mathbf{A}_{.n})^T,$$

where $\mathbf{A}_{.j}$ is the j^{th} column of $\mathbf{A}$.

2. Main results

Consider the interval system of matrix equations (1.1). We define the solution set for the system (1.1) by

$$\Sigma(X,Y)=\{(X,Y):\ X,Y\in\mathbb{R}^{m\times n},A_{i1}X+YA_{i2}=C_i \text{ for some } A_{i1}\in\mathbf{A_{i1}},A_{i2}\in\mathbf{A_{i2}},C_i\in\mathbf{C_i};i=1,2\}.\quad(2.1)$$

Similar to solving of interval linear systems [4, 3], the solution set of an interval system is generally of a complicated structure. But if $\Sigma(X,Y)$ is bounded , we look for an enclosure of this set i.e. for a pair of interval matrices $(\mathbf{X},\mathbf{Y})$ satisfying

$$\Sigma(X,Y)\subseteq(\mathbf{X},\mathbf{Y}).$$

In sequel, we will present a direct method to obtain an enclosure of $\Sigma(X,Y)$ and conditions under which $\Sigma(X,Y)$ is bounded. Also we will present an iterative algorithm to solve this important problem.

We consider interval linear system of equations

$$\mathbf{G}z=\mathbf{d},\qquad(2.2)$$

where $\mathbf{G} \in \mathbb{IR}^{2mn \times 2mn}$ and $\mathbf{d} \in \mathbb{IR}^{2mn}$ are as the following

$$\mathbf{G} = \begin{bmatrix} \mathbf{I_n} \otimes \mathbf{A_{11}} & \mathbf{A_{12}}^T \otimes \mathbf{I_m} \\ \mathbf{I_n} \otimes \mathbf{A_{21}} & \mathbf{A_{22}}^T \otimes \mathbf{I_m} \end{bmatrix}, \ \mathbf{d} = \begin{bmatrix} vec(\mathbf{C_1}) \\ vec(\mathbf{C_2}) \end{bmatrix}.$$

We assume that Γ is the solution set of (2.2). We define

$$\Theta = \{ \begin{bmatrix} vec(X) \\ vec(Y) \end{bmatrix} : \ (X, Y) \in \Sigma(X, Y) \}, \tag{2.3}$$

It is clear that

$$\Theta \subseteq \Gamma.$$

Therefore, by solving interval linear system (2.2) and finding interval vector $\mathbf{z}$ as an enclosure of it's solution set, we can specify the columns of the interval matrices $\mathbf{X}$ and $\mathbf{Y}$ which $(\mathbf{X}, \mathbf{Y})$ is an enclosure of $\Sigma(X, Y)$. Rohn's direct or iterative methods [3, 4] are recommended to solve the interval linear system (2.2).

The next theorem give us a necessary and sufficient condition for boundedness of $\Sigma(X, Y)$.

Theorem 2.1. *Suppose that $\mathbf{A_{11}}$ in the interval system of matrix equations is regular. Then for all interval matrices $\mathbf{C_1}$ and $\mathbf{C_2}$, the solution set of (1.1) is bounded if and only if $(\mathbf{A_{22}^T} \otimes \mathbf{A_{11}}) - (\mathbf{A_{12}^T} \otimes \mathbf{A_{21}})$ is regular.*

Example 2.2. Consider the interval system of matrix equations (1.1) in which

$$\mathbf{A_{11}} = \begin{bmatrix} [1,2] & [2,2.5] \\ [-2,-1] & [5,6] \end{bmatrix}, \ \mathbf{A_{12}} = \begin{bmatrix} [0.1,0.3] & [2,2.5] \\ [-0.5,0.2] & [0.1,0.3] \end{bmatrix}, \ \mathbf{C_1} = \begin{bmatrix} [3,4] & [-1,0] \\ [1,1] & [6,8] \end{bmatrix},$$

$$\mathbf{A_{21}} = \begin{bmatrix} [0.1,0.3] & [0,0.5] \\ [-0.4,-0.3] & [0.2,0.2] \end{bmatrix}, \ \mathbf{A_{22}} = \begin{bmatrix} [3,4] & [-4,-3] \\ [1,2] & [6,7] \end{bmatrix}, \ \mathbf{C_2} = \begin{bmatrix} [6,7] & [1,3] \\ [8,9] & [6,8] \end{bmatrix}.$$

Here, $\big((\mathbf{A_{22}^T} \otimes \mathbf{A_{11}}) - (\mathbf{A_{12}^T} \otimes \mathbf{A_{21}})\big)$ is regular. By using the above direct method, we find that the enclosure of the solution set is a pair of interval matrices of $(\mathbf{X}_{Di}, \mathbf{Y}_{Di})$ where

$$\mathbf{X}_{Di} = \begin{bmatrix} [0.1211, 4.1360] & [-7.9006, 0.9944] \\ [0.0360, 1.4322] & [-2.3282, 2.1786] \end{bmatrix}, \ \mathbf{Y}_{Di} = \begin{bmatrix} [0.4394, 2.3682] & [0.2358, 2.2193] \\ [0.8141, 3.1215] & [0.9311, 3.2972] \end{bmatrix}.$$

In the rest of this section an iterative method for solving the interval system of matrix equations (1.1) is presented. Let us rewrite (1.1) as the following:

$$\begin{cases} \mathbf{A_{11}} X = \mathbf{C_1} - Y \mathbf{A_{12}}, \\ Y \mathbf{A_{22}} = \mathbf{C_2} - \mathbf{A_{21}} X. \end{cases}$$

Now we define the following iteration equations:

$$\begin{cases} \mathbf{A_{11}} X_{k+1} = \mathbf{C_1} - Y_k \mathbf{A_{12}}, \\ Y_{k+1} \mathbf{A_{22}} = \mathbf{C_2} - \mathbf{A_{21}} X_{k+1}. \end{cases} \quad k \geq 0 \tag{2.4}$$

Assume that an initial guess $\mathbf{Y}_0$ is given.

Algorithm:

step 1. Choose Y_0. set $k = 0$.

step 2. Solve the interval matrix equations

$$\mathbf{A_{11}} X_{k+1} = \mathbf{C_1} - Y_k \mathbf{A_{12}} \ , \ \mathbf{A}_{\mathbf{22}}^T Y_{k+1}^T = \mathbf{C_2}^T - X_{k+1}^T \mathbf{A}_{\mathbf{21}}^T$$

and find the interval matrices $\mathbf{X}_{k+1}$ and $\mathbf{Y}_{k+1}$.

step 3. If

$$\max\{\| \underline{X}_{k+1} - \underline{X}_k \|, \| \overline{X}_{k+1} - \overline{X}_k \|, \| \underline{Y}_{k+1} - \underline{Y}_k \|, \| \overline{Y}_{k+1} - \overline{Y}_k \|\} < \epsilon,$$

stop.

Otherwise, set

$$\mathbf{X}_k = \mathbf{X}_{k+1} \ , \ \mathbf{Y}_k = \mathbf{Y}_{k+1} \text{ if } k = 0, \tag{2.5}$$

$$\mathbf{X}_k = \mathbf{X}_{k+1} \cap \mathbf{X}_k \ , \ \mathbf{Y}_k = \mathbf{Y}_{k+1} \cap \mathbf{Y}_k \text{ if } k \geq 1. \tag{2.6}$$

Set $k = k + 1$, and go to the step 2.

Example 2.3. Consider the interval system of matrix equations in example (2.2). With $\mathbf{Y}_0 = 0$, by performing 3 iterations of the above algorithm we have:

$$\mathbf{X}_{It} = \begin{bmatrix} [0.8333, 2.7501] & [-3.3334, -0.7499] \\ [0.3448, 1.0001] & [0.3636, 0.8616] \end{bmatrix}, \ \mathbf{Y}_{It} = \begin{bmatrix} [0.6265, 1.9906] & [0.4882, 1.8031] \\ [1.0279, 2.7385] & [1.2390, 2.8742] \end{bmatrix}.$$

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AN ITERATIVE METHOD FOR THE LEAST SQUARES SYMMETRIC SOLUTION OF THE LINEAR MATRIX $AXB = C$

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Abstract. An iterative method is presented to solve the minimum Frobenius norm residual problem: $min\|AXB - C\|$ with unknown symmetric matrix X. By this iterative method, for any initial symmetric matrix X_0, a solution X^* can be obtained within finite iteration steps in the absence of roundoff errors, and the solution X^* with least norm can be obtained by choosing a special kind of initial symmetric matrix. Given numerical examples are show that the iterative method is quite efficient.

1. Introduction

Denoted by $R^{m\times n}$ and $SR^{n\times n}$ be the set of $m \times n$ real matrices and the set of $n \times n$ real symmetric matrices, respectively. Denoted by the

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* Speaker.

superscripts T and + be the transpose and MoorePenrose generalized inverse of matrices, respectively. In space $R^{m\times n}$, we define inner product as: $\langle A, B\rangle = trace(B^TA)$ for all $A, B \in R^{m\times n}$. Then the norm of a matrix A generated by this inner product is, obviously, Frobenius norm and denoted by $\|A\|$. We consider the solution of the minimum residual problem

$$\underbrace{min}_{X\in SR^{n\times n}} \quad \|AXB - C\| \tag{1.1}$$

with $A \in R^{m\times n}$,$B \in R^{n\times p}$,$C \in R^{m\times p}$. We also consider the solution of the matrix nearness problem

$$\underbrace{min}_{X\in S_X} \|X - \overline{X}\| \tag{1.2}$$

where $\overline{X} \in R^{m\times n}$ is given matrix and S_X is the solution set of the minimum residual problem (1.1). The approach taken in both papers is use the generalized singular value decomposition (GSVD) of matrices. The necessary and sufficient conditions for the existence of and the expressions for the solution of the matrix equation were established. Presented an iterative method for finding the symmetric solution of the matrix equation $AXB = C$. They have been proved that the iteration method can be terminated within finite iteration steps for any initial matrix, and that the solution with least Frobenius norm can be obtained by choosing a special kind of initial iteration matrix. Because ofA, B and C occurring in practice are usually obtained from experiments, it is difficult for them to satisfy the solvability conditions of the above matrix equation. Therefore, consider the minimum Frobenius norm residual problem (1.1) is necessary. Given numerical examples are show that the iterative method is quite efficient.

Algorithm 1.1. We give a pseudocode of the algorithm an following :
1.Input matrices $A \in R^{m\times n}$,$B \in R^{n\times p}$,$C \in R^{m\times p}$and $X_0 \in SR^{n\times n}$;
2.Calculate

$$R_0 = A^TCB^T + BC^TA - A^TAX_0BB^T - BB^TX_0A^TA; \tag{1.3}$$

$$P_0 = A^TAR_0BB^T + BB^TR_0A^TA; \tag{1.4}$$

$$k := 0 \tag{1.5}$$

3.If $R_k = 0$,then stop;else,$k = k + 1$;

$$X_k = X_{k-1} + \frac{\|R_{k-1}\|^2}{\|P_{k-1}\|^2}P_{k-1} \tag{1.6}$$

$$R_k = A^TCB^T + BC^TA - A^TAX_0BB^T - BB^TX_0A^TA; \tag{1.7}$$

$$= R_{k-1} + \frac{\|R_{k-1}\|^2}{\|P_{k-1}\|^2}(A^TAP_{k-1}BB^T - BB^TP_{k-1}A^TA); \tag{1.8}$$

$$P_k = A^AR_kBB^T + BB^TR_kA^TA + \frac{\|R_k\|^2}{\|R_{k-1}\|^2}P_{k-1}; \tag{1.9}$$

4.Go to step 3.
By using Algorithm, let initial matrix $\overline{X}_0 = A^TAHBB^T + BB^THA^TA$, where H is arbitrary symmetric matrix, or more especially, let $\overline{X}_0 = 0 \in R_{n\times n}$, we can obtain the unique least Frobenius norm solution $\widetilde{X}^*$ of the linear matrix equation. Once the above matrix $\widetilde{X}^*$ is obtained, the unique solution$\widehat{X}$ of the matrix nearness problem can be obtained. In this case,$\widehat{X}$ can be expressed $\widehat{X} = \widetilde{X}^* + \dfrac{\overline{X} + \overline{X^T}}{2}$.

Example 1.2. Given matrix A, B and C respectively as follows:

$$\begin{bmatrix} 4 & 3 & -1 & 3 & 1 & -3 & 2 \\ 3 & -2 & 3 & -4 & 3 & 2 & 1 \\ 3 & -1 & 3 & -1 & 3 & 2 & 1 \\ 4 & 3 & -1 & 3 & 1 & -3 & 2 \\ 3 & -1 & 3 & -1 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 4 & -3 & -3 & 4 & 4 \\ 5 & -3 & 5 & 5 & -3 & -3 \\ -6 & 2 & -6 & -6 & 2 & 2 \\ -8 & 4 & -8 & -8 & 4 & 4 \\ 4 & -5 & 4 & 3 & -2 & -7 \\ -3 & 2 & -3 & -3 & 2 & 2 \\ -1 & -2 & -1 & -1 & -2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 43 & -54 & 73 & -54 & 51 & -54 \\ -31 & 37 & -61 & 37 & -53 & 37 \\ 43 & -54 & 73 & -54 & 51 & -54 \\ -31 & 37 & -61 & 37 & -53 & 37 \\ 47 & -54 & 73 & -54 & 21 & -54 \\ -31 & 27 & -61 & 27 & -53 & 27 \end{bmatrix}$$

It is easy shown that the above matrix equation (1.1) is consistent . By 58 steps , we obtain the unique minimum norm solution pair of Eq.(1.1) as follows: We have X_{58} respectively as follow:

$$\begin{bmatrix} 1.0650 & 0.2510 & -0.9062 & 0.6469 & 0.6130 & -1.8154 & 0.5729 \\ 0.2510 & -0.6516 & -0.0189 & 0.4239 & 1.8937 & 0.8660 & -1.3207 \\ -0.9062 & -0.0189 & 1.9641 & 0.3755 & -2.2609 & 0.4210 & 2.2353 \\ 0.6469 & 0.4239 & 0.3755 & -0.3307 & -0.2146 & -0.4136 & 1.0401 \\ 0.6130 & 1.8937 & -2.2609 & -0.2146 & -2.6651 & -4.3017 & 2.3216 \\ -1.8154 & 0.8660 & 0.4210 & -0.4136 & -4.3017 & -1.0648 & 2.4271 \\ 0.5729 & -1.3207 & 2.2353 & 1.0401 & 2.3216 & 2.4271 & -0.4410 \end{bmatrix}$$

With $\|R_{58}\| = \|A^TCB^T + BC^TA - A^TAX_{58}BB^T - B^TBX_{58}AA^T\|^2 = 6.2572 \times 10^{-11}$

And the minimum residual

$$\underbrace{min}_{X} \|AXB - C\| = \|AX_{58}B - C\| = 179.0445. \tag{1.10}$$

2. Main results

In this article,we first introduce an iterative method, that is, Algorithm for solving the minimum residual problem (1.1) with unknown symmetric matrix X. We then show that, for any initial symmetric matrix X_0, the matrix sequence X_k generated by Algorithm converges to its a solution within at most n^2 iteration steps in the absence of roundoff errors. We also show that if let the initial matrix $\widetilde{X}_0 = A^TAHBB^T + BB^THA^TA$, where H is arbitrary symmetric matrix, then the solution X^* obtained by the iterative method is the least Frobenius norm solution. We also consider using Algorithm for solving the matrix nearness problem (1.2). Given two examples and many other examples we have tested by $MATLAB$ confirm our theoretical results in this paper. This is an important problem which we should study in future.

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NOTE TO THE GRADIENT-BASED ITERATIVE ALGORITHM FOR SOLVING SYLVESTER TENSOR EQUATIONS

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Abstract. More recently, Chen and Lu [Math. Probl. Eng., DOI: 10.1155/2013/819479] have proposed an iterative algorithm to solve the Sylveter tensor equation. More precisely, the gradient-based iterative algorithm has been developed for finding the unique solution of the mentioned Sylveter tensor equation. In this paper we demonstrate that how an oblique projection technique can be applied to propound a modified algorithm which surpasses the proposed algorithm in the earlier refereed work without setting the restriction of the existence of the unique solution.

1. Introduction and preliminaries

It is known that a tensor is a multidimensional array. In [3], the authors have elaborated an overview of higher-order tensors and their decomposition. The order of a tensor is the number of dimensions which is called by modes or ways. During this paper, matrices (tensors of order two) are signified by capital letters. Higher-order tensors (here order three) are indicated by Euler script letters, e.g., $\mathcal{X}$. A key operation for a tensor is the tensor-matrix multiplication. The 1-mode

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tensor product of tensor $\mathcal{X} \in \mathbb{R}^{I\times J\times K}$ by a matrix $A \in \mathbb{R}^{P\times I}$ is denoted by $\mathcal{X} \times_1 A$ which is a $P \times J \times K$ tensor and its entries are given by $(\mathcal{X} \times_1 A)(p, j, k) = \sum_{i=1}^{I} x_{ijk}a_{pi}$. Similarly, the elements of the 2-mode multiplication of $\mathcal{X}$ by a matrix $B \in \mathbb{R}^{Q\times J}$ are expounded by $(\mathcal{X} \times_2 B)(i, q, k) = \sum_{j=1}^{J} x_{ijk}b_{qj}$. In an analogous manner the 3-mode multiplication can be determined.

Consider the following Sylvester tensor equation

$$\mathcal{A}(\mathcal{X}) := \mathcal{X} \times_1 A + \mathcal{X} \times_2 B + \mathcal{X} \times_3 C = \mathcal{D}, \tag{1.1}$$

where the matrices $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{l\times l}$ and the tensor $\mathcal{D} \in \mathbb{R}^{m\times n\times l}$ are given.

Definition 1.1. The inner product of two tensors $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{I\times J\times K}$ is defined by $\langle \mathcal{X}, \mathcal{Y}\rangle = \sum\limits_{i=1}^{I}\sum\limits_{j=1}^{J}\sum\limits_{k=1}^{K} x_{ijk}y_{ijk}$ and the corresponding induced norm is given by $\|\mathcal{X}\|^2 = \langle \mathcal{X}, \mathcal{X}\rangle$.

As known, n-mode multiplication commutes with respect to the inner product, i.e., $\langle \mathcal{X}, \mathcal{Y} \times_n A\rangle = \left\langle \mathcal{X} \times_n A^T, \mathcal{Y}\right\rangle$; for further details see [3].

It is not difficult to verify that (1.1) is equivalent to the following linear system of equations

$$\mathcal{W}x = b,$$

with $x = \text{vec}(\mathcal{X})$, $b = \text{vec}(\mathcal{D})$ and

$$\mathcal{W} = I_l \otimes I_n \otimes A + I_l \otimes B \otimes I_m + C \otimes I_n \otimes I_m,$$

where $\otimes$ denotes the Kronecker product, I_n stands for the identity matrix of order n and the "vec" operator stacks the column of a matrix (or a tensor) to form a vector. It is well-known that the linear system $\mathcal{W}x = b$ is consistent if and only if (1.1) is consistent. Evidently, the size of the linear system $\mathcal{W}x = b$ would be huge even for moderate values of l, m and n. Consequently, it is more desirable to apply an iterative method based on tensor format for solving (1.1).

2. Main results

In [1], the authors have investigated the convergence of the presented method for solving (1.1) under the hypophysis that it has a unique solution $\mathcal{X}^*$. Afterwards, a gradient-based approach [2] has been evolved for solving (1.1). In fact the propounded gradient-based algorithm produces the sequence of approximate solutions $\{\mathcal{X}^k\}_{k=1}^{\infty}$ to (1.1) by the following recursive formulas

$$\mathcal{X}^k = \mathcal{X}^{k-1} + \frac{\mu}{3}\left(\mathcal{R}^{k-1} \times_1 A^T + \mathcal{R}^{k-1} \times_2 B^T + \mathcal{R}^{k-1} \times_3 C^T\right),$$

where $\mathcal{R}^{k-1} = \mathcal{D} - \mathcal{X}^{k-1} \times_1 A - \mathcal{X}^{k-1} \times_2 B - \mathcal{X}^{k-1} \times_3 C$ and the initial value $\mathcal{X}^0$ is given. A sufficient condition is established for parameter μ, i.e.,

$$0 < \mu < 2/(\|A\|_2^2 + \|B\|_2^2 + \|C\|_2^2),$$

under which the sequence $\{\mathcal{X}^k\}_{k=1}^{\infty}$ is convergent to the unique solution $\mathcal{X}^*$ of (1.1) for arbitrary initial value $\mathcal{X}^0$.

2.1. **Main contribution.** In [1], Chen and Lu have not discussed the necessary condition for the convergence of the proposed algorithm to solve (1.1). Meanwhile the proposed algorithm relies on a fixed parameter denoted by μ whose optimum value has not been derived. In the current paper, we apply an oblique projection technique to ameliorate the speed of convergence of the gradient-based algorithm to solve the Sylvester tensor equation (1.1) without setting the restriction that (1.1) has a unique solution. Our proposed approach do not depend on a fixed parameter and the produced approximated solution, at each iterate, satisfies an optimality property.

2.2. **Application of an oblique projection technique.** Instead of using a fixed parameter μ, we select the parameter in a progressive manner. As a matter of fact, at each iterate say kth iteration, we exploit a projection technique to derive the parameter such that the norm of the residual tensor $\mathcal{R}^{k+1} = \mathcal{D} - \mathcal{X}^{k+1} \times_1 A - \mathcal{X}^{k+1} \times_2 B - \mathcal{X}^{k+1} \times_3 C$ corresponding to the new approximation $\mathcal{X}^{k+1}$ is minimized over $S_k = \{\overline{\mathcal{X}} \mid \overline{\mathcal{X}} = \mathcal{X}^k + \alpha \mathcal{P}^k \text{ for } \alpha > 0\}$ where

$$\mathcal{P}^k = \mathcal{R}^k \times_1 A^T + \mathcal{R}^k \times_2 B^T + \mathcal{R}^k \times_3 C^T.$$

For the fundamental concepts of the projection techniques for solving the linear system of equations, we refer the reader to Chapter 5 of [4]. In fact, we aim to find α^* such that the residual tensor $\mathcal{R}^{k+1}$ associated with $\mathcal{X}^{k+1} = \mathcal{X}^k + \alpha^* \mathcal{P}^k$ satisfies

$$\|\mathcal{R}^{k+1}\| = \min_{\overline{\mathcal{X}} \in S_k} \|\mathcal{D} - \overline{\mathcal{X}} \times_1 A - \overline{\mathcal{X}} \times_2 B - \overline{\mathcal{X}} \times_3 C\|.$$

To this end, we may prove that it is sufficient to determine α^* such that $\langle \mathcal{R}^{k+1}, \mathcal{A}(\mathcal{P}^k) \rangle = 0$ where $\mathcal{A}(\mathcal{P}^k) = \mathcal{P}^k \times_1 A + \mathcal{P}^k \times_2 B + \mathcal{P}^k \times_3 C$. Or equivalently, we may set

$$\alpha^* = \frac{\langle \mathcal{R}^k, \mathcal{A}(\mathcal{P}^k) \rangle}{\langle \mathcal{A}(\mathcal{P}^k), \mathcal{A}(\mathcal{P}^k) \rangle}, \qquad \langle \mathcal{A}(\mathcal{P}^k), \mathcal{A}(\mathcal{P}^k) \rangle \neq 0.$$

Evidently S_k includes the $(k+1)$th approximate solution obtained by the gradient-based iterative algorithm for $\alpha = \frac{\mu}{3}$.

Now we present the following useful theorem which turns out that $\mathcal{P}^k = 0$ implies that $\mathcal{R}^k = 0$, i.e., $\mathcal{X}^k$ satisfies (1.1).

Theorem 2.1. *Suppose that (1.1) is consistent. Presume that $\mathcal{X}_*$ is an arbitrary solution of (1.1). Then,* $\left\langle \mathcal{P}^k, \mathcal{X}_* - \mathcal{X}^k \right\rangle = \left\langle \mathcal{R}^k, \mathcal{R}^k \right\rangle$.

Straightforward computations demonstrate that

$$\left\langle \mathcal{R}^{k+1}, \mathcal{R}^{k+1} \right\rangle = \left\langle \mathcal{R}^k, \mathcal{R}^k \right\rangle \left[1 - \frac{\left\langle \mathcal{R}^k, \mathcal{A}(\mathcal{P}^k)) \right\rangle^2}{\langle \mathcal{A}(\mathcal{P}^k), \mathcal{A}(\mathcal{P}^k)) \rangle \langle \mathcal{R}^k, \mathcal{R}^k \rangle} \right].$$

From the Cauchy-Schwarz inequality and in view of the above relation, we deduce that $\|\mathcal{R}^{k+1}\| \leq \|\mathcal{R}^k\|$. It is not difficult to verify that in the above relation the inequality holds strictly if $\mathcal{P}^k \neq 0$. This fact follows from the subsequent relation,

$$\left\langle \mathcal{R}^k, \mathcal{A}(\mathcal{P}^k) \right\rangle = \left\langle \mathcal{P}^k, \mathcal{P}^k \right\rangle.$$

Note that the above equality reveals that without loss of generality we may assume that $\left\langle \mathcal{R}^k, \mathcal{A}(\mathcal{P}^k) \right\rangle \neq 0$, because otherwise $\mathcal{P}^k = 0$ which indicates that $\mathcal{X}^k$ is an exact solution of (1.1) and hence there is no need to compute the new approximate solution.

Our proposed approach for solving (1.1) is given by

$$\mathcal{X}^k = \mathcal{X}^{k-1} + \frac{\left\langle \mathcal{R}^k, \mathcal{A}(\mathcal{P}^k) \right\rangle}{\langle \mathcal{A}(\mathcal{P}^k), \mathcal{A}(\mathcal{P}^k) \rangle} \left(\mathcal{R}^{k-1} \times_1 A^T + \mathcal{R}^{k-1} \times_2 B^T + \mathcal{R}^{k-1} \times_3 C^T \right), \tag{2.1}$$

for $k = 1, 2, \ldots,$ where the initial value $\mathcal{X}^0$ is given. Note that in the above, as pointed out, we may assume that $\left\langle \mathcal{A}(\mathcal{P}^k), \mathcal{A}(\mathcal{P}^k) \right\rangle \neq 0$. Now we present the next theorem which shows that the sequence of approximate solutions produced by the recursive formula (2.1) converges to a solution of (1.1).

Theorem 2.2. *Presume that the Sylvester tensor equation (1.1) is consistent. Then for $k = 0, 1, 2, \ldots,$ the following statements hold*

- *If $\left\langle \mathcal{A}(\mathcal{P}^k), \mathcal{A}(\mathcal{P}^k) \right\rangle \neq 0$ then $\|\mathcal{R}^k\| < \|\mathcal{R}^{k-1}\|$.*
- *If $\left\langle \mathcal{A}(\mathcal{P}^k), \mathcal{A}(\mathcal{P}^k) \right\rangle = 0$ then $\mathcal{X}^k$ is a solution of (1.1).*

In practice, we continue computing the approximate solutions by the recursive formula (2.1) while $\|\mathcal{R}^k\| \geq \epsilon$ where ϵ is a give tolerance.

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MATRIX IDR(S)-PROTO

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Abstract. The IDR(s)-proto based on the induced dimension reduction theorem, is a new class of efficient algorithms for large nonsymmetric linear systems. IDR(s) generates residuals that are forced to be in a sequence of nested subspaces. In this paper, a variant of the IDR theorem is given, then the matrix IDR(s),based on the variant IDR(s) theorem, is proposed.

1. Introduction

We consider the linear system with multiple right hand side

$$AX = B$$

where $A \in \mathbb{C}^{n\times n}$ a large sparse and the cofficient matrix A is nonsingular of order n, $X = [x_1, x_2, \ldots, x_m]$ and $B = [b_1, b_2, \ldots, b_m] \in \mathbb{C}^{n\times m} m \ll n$. In 1980, Sonneveld proposed the iterative method IDR [2] for solving such systems. The prototype IDR(s) algorithm that is described in Sonneveld and Van Gijzen [3] is only one of many possible

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* Speaker.

variants.One of the possibilities to make alternative matrix IDR methods is a different computation of the intermediate residuals. These intermediate residuals are not uniquely defined and their computation leaves freedom to derive algorithmic variants. In this paper we describe a general framework for deriving matrix IDR(s)-proto method. Next section starts with reviewing the matrix IDR theorem, then it explains how the theorem can be used to compute the first residual in $\mathcal{G}_{j+1}$ and corresponding approximation for the solution, given sufficient matrices in $\mathcal{G}_j$. Throughout this paper the following notation is used. $\|.\|$ is the Euclidean norm. Tr(.) denotes trace of square matrix. $\mathcal{N}(P)$ indicats the nullspace of matrix P.

2. Matrix IDR(s)-Proto

In this section we consider the nonsymmetric linear systems with multiple right-hand sides. In order to propose the matrix version of IDR(s) theorem , which is an extension of original IDR(s) theorem.

2.1. **Matrix krylov subspace.**

At the first, we define the matrix krylov subspace and then we say IDR(s) theorem.

Definition 2.1. Subspace $\mathbb{K}_k(A, R_0)$ generated by A and R_0

$$\mathbb{K}_k(A, R_0) := \{\sum_{i=0}^{k-1} \alpha_i A^i R_0 \ : \ \alpha_i \in \mathbb{C}\}$$

is called the matrix krylov subspace.

Definition 2.2. Let A and B are $n \times n$ matrices, it is called that A and B are F-orthogonal if $\langle A, B\rangle_F = 0$.

Theorem 2.3. *([1]). Let A be any matrix in $\mathbb{C}^{n\times n}$ and let $\mathcal{G}_0 = \mathbb{K}(A, R_0)$. Let S be any (proper) subspace of $\mathbb{C}^n$ such that S and $\mathcal{G}_0$ do not share a nontrivial invariant subspace of A and define the sequance $\mathcal{G}_j$, $j = 1, 2, \ldots$ as*

$$\mathcal{G}_j = (I - \omega_j A)(\mathcal{G}_{j-1} \cap S)$$

where the $\omega_j' s$ are nonzero scalars, then

(1) $\mathcal{G}_j \subset \mathcal{G}_{j-1}$, $\forall j > 0$
(2) $\mathcal{G}_j = \{0\}$ *for some* $j \leq n$

under the assumptions of theorem 2.3 , the system will be solved after at most n dimensions reduction steps. First, let the matrices $R_1, \ldots, R_s$ are computed by any matrix krylov subspace method. Now we want to compute R_{n+1} in $\mathcal{G}_{j+1}$ if

$$R_{n+1} = (I - \omega_{j+1} A)V_n \quad with \quad V_n \in \mathcal{G}_j \cap S$$

Now, if we choose

$$V_n = R_n - \sum_{i=1}^{s} \gamma_i G_{n-i} \ , \tag{2.1}$$

where $G_{n-i} = R_{n-i} - R_{n-i+1}$ and $\gamma_i \in \mathbb{C}$. Without loss of generality, we assume the space S to be the left null space of some (full rank) $n \times sm$ matrix P :

$$P = [P_1 \quad P_2 \quad \dots \quad P_s] \ , \quad S = \mathcal{N}(P^H)$$

Now suppose that R_n , $G_{n-i} \in \mathcal{G}_j$, $i = 1, \dots, s$. This implies that $V_n \in \mathcal{G}_j$. If we choose γ_i , $i = 1, \dots s$ such that $V_n \in S$, then by theorem 2.3 , we have $R_{n+1} \in \mathcal{G}_{j+1}$. Now we show that V_n is in S , in order to that

$$P^H V_n = 0 \tag{2.2}$$

Combining (2.1) and (2.2) yields a matrix equation with unknown matrix $(\gamma_s I_m, \dots, \gamma_1 I_m)^T$, i.e ,

$$P^H R_n = P^H [G_{n-s} \quad G_{n-s+1} \quad \dots \quad G_{n-1}] \begin{pmatrix} \gamma_s I_m \\ \vdots \\ \gamma_1 I_m \end{pmatrix}.$$

Suppose that after n iterations we have exactly s matrices $G_i \in \mathcal{G}_j$, $i = n-s, n-s+1, \dots, n-1$ and s corresponding matrices U_i with $G_i = AU_i$ at our disposal. Then we define setting $c = (\gamma_s, \dots, \gamma_1)^T$ and $dG_n = [G_{n-s}, \dots, G_{n-1}]$, we get

$$P^H dG_n (c \otimes I_m) = P^H R_n.$$

Corresponding to dG_n , we have

$$dU_n = [U_{n-s} \quad U_{n-s+1} \quad \dots \quad U_{n-1}].$$

Then the computation of residual R_{n+1} in $\mathcal{G}_{j+1}$ can be executed by following algorithm :

Calculate :$c \in \mathbb{C}^s$ from $(P^H dG_n)(c \otimes I_m) = P^H R_n$,

$$V_n = R_n - dG_n (c \otimes I_m) \ ,$$

$$R_{n+1} = V_n - \omega_{j+1} A V_n \ .$$

Therefore, the new residual satisfies

$$R_{n+1} = R_n - \omega_{j+1} A V_n - dG_n (c \otimes I_m). \tag{2.3}$$

Then, from (2.3), approximate solution X_{n+1} can be updated as

$$X_{n+1} = X_n + \omega_{j+1} V_n + dU_n (c \otimes I_m) \ .$$

In the computation of R_{n+1} in $\mathcal{G}_{j+1}$, we may choose ω_{j+1} freely , but the same value must be used in computations of the subsequent residuals in $\mathcal{G}_{j+1}$. The parameter ω_{j+1} is the value that minimizes the norm of R_{n+1}. This minimizing yields

$$\omega_{j+1} = \frac{\langle V_n, AV_n\rangle_F}{\langle AV_n, AV_n\rangle_F} \quad .$$

We can write the matrix IDR(s)-proto as follows:

Algorithm 1 (*Matrix IDR(s)-proto algorithm*)

Enter the matrix $A \in \mathbb{C}^{n\times n}$, the matrices $X_0, B \in \mathbb{C}^{n\times m}$ and $P \in \mathbb{C}^{n\times sm}$ and s, TOL .
Compute $R_0 = B - AX_0$.
for $i = 1, \dots, s$ do:
  $V = AR_{i-1}, \omega = \frac{\langle V, R_{i-1}\rangle_F}{\langle V, V\rangle_F},\ U(:,(i-1)m+1:im) = \omega R_{i-1}$.
  $G(:,(i-1)m+1:im) = -\omega V$.
  $X_i = X_{i-1} + U(:,(i-1)m+1:im), R_i = R_{i-1} + G(:,(i-1)m+1:im)$.
end for.
$j = 1, i = s,\ M = P^H G,\ h = P^H R_i$;
while $max_{j=1:m} \frac{\|R_i(:,j)\|}{\|B(:,j)\|} > TOL$ do :
  For $k = 0, \dots, s$ do:
    Solve C from $M(C\otimes I_m) = h,\ Q = -G(C\otimes I_m), V = R_i + Q$;
    If $k == 0$ then :
      $T = AV, \omega = \frac{\langle T, V\rangle_F}{\langle V, V\rangle_F}, G(:,(j-1)m+1:jm) = Q - \omega T$.
      $U(:,(j-1)m+1:jm) = -U(C\otimes I_m) + \omega V$.
    else
      $U(:,(j-1)m+1:jm) = -U(C\otimes I_m) + \omega V$.
      $G(:,(j-1)m+1:jm) = -AU(:,(j-1)m+1:jm)$.
    end
    $R_{i+1} = R_i + G(:,(j-1)m+1:jm), X_{i+1} = X_i + U(:,(j-1)m+1:jm)$.
    $\Delta m = P^H G(:,(j-1)m+1:jm), M(:,(j-1)m+1:jm) = \Delta m$.
    $h = h + \Delta m, i = i+1, j = j+1$.
    $j = j-1 \quad ./.\ s+1$; ($./.$ *is the modulo operation*).
  end for
end while

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SOME INEQUALITIES INVOLVING CLOSED RANGE OPERATORS

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Abstract. Suppose $S, X \in \mathfrak{B}(\mathcal{H})$ such that S is closed range operator. We establish several unitarily invariant norm inequalities involving closed range operators. In particular, we show

$$|||S^*XS^\dagger - S^\dagger XS^*||| \leq \frac{1}{2}\|S\|^2\ |||S^*XS^\dagger + S^\dagger XS^*|||\,.$$

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and let $\mathfrak{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on $\mathcal{H}$ with the operator norm $\|\cdot\|$ and the identity $I_{\mathcal{H}}$. The cone of positive operators is denoted by $\mathfrak{B}^+(\mathcal{H})$. For $A \in \mathfrak{B}(\mathcal{H})$ we denote by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, the null-space and the range of A. A unitarily invariant norm $|||\,.\,|||$ is defined on a norm ideal $\mathfrak{L}_{|||\,.\,|||}$ of $\mathbb{B}(\mathcal{H})$ associated with it and has the unitary invariance property $|||UXV||| = |||X|||$, where U,V are arbitrary unitaries and $X \in \mathfrak{L}_{|||\,.\,|||}$, and the submultiplicativity property $|||AXB||| \leq \|A\|\,|||X|||\,\|B\|$, where $A, B \in \mathfrak{B}(\mathcal{H})$ and $X \in \mathfrak{L}_{|||\,.\,|||}$.

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* Speaker.

Definition 1.1. Let $\mathfrak{A}$ be an algebra with an involution $*$ and $a \in \mathfrak{A}$. If there exist an element $x \in \mathfrak{A}$ such that

$$axa = a \quad xax = x$$

$$(ax)^* = ax \quad (xa)^* = xa,$$

then x is called Moore-Penrose inverse of a and denoted by $a^\dagger$.

It is easy to show that the Moore-Penrose inverse of an element a is unique.
If $a \in \mathfrak{A}$ is Moore-Penrose invertible, then
(1) If a is invetible, then $a^\dagger = a^{-1}$,
(2) a^* is Moore-Penrose invertible and $a^{*\dagger} = a^{\dagger *}$,
(3) $(aa^*)^\dagger = a^{*\dagger}a^\dagger$,
(4) $a^{\dagger\dagger} = a$.
Harte and Mbekhta [2] proved that if $\mathcal{H}$ be a Hilbert space and $S \in \mathfrak{B}(\mathcal{H})$, then following statements are equivalent

(i) S has a Moore-Penrose inverse.
(ii) $\mathcal{R}(S)$ is closed.

Davies [1] showed that

$$\|AX - XA\|_p \leq \lambda_p \|AX + XA\|_p \ \ (1 < p < \infty), \tag{1.1}$$

where $A \in \mathfrak{B}^+(\mathcal{H})$, $X \in \mathfrak{B}(\mathcal{H})$ and λ_p is a constant depended only on p, and it is equal to one when $p = 2$.
As consequence of this inequality we have [3]

$$\|R^*XT^{-1} - R^{-1}XT^*\|_p \leq \lambda_p \|R^*XT^{-1} + R^{-1}XT^*\|_p \ \ (1 < p < \infty), \tag{1.2}$$

where $X, R, T \in \mathfrak{B}(\mathcal{H})$ such that R, T are positive invertible and λ_p is a constant depended only on p, and it is equal to one when $p = 2$.
For all 2×2 matrices and unitarily invariant norms $|||\,.\,|||$ we have

$$\left|\left|\left|RXR^{-1} - R^{-1}XR\right|\right|\right| \leq \left|\left|\left|RXR^{-1} + R^{-1}XR\right|\right|\right| \tag{1.3}$$

where R is positive definite, see [3].
Our analysis in this paper is related to that utilized in [4, 5], and depends the properties of closed range operators and the submultiplicativity property of unitarily invariant norms.

Lemma 1.2. *Let $S \in \mathfrak{B}(\mathcal{H})$ has closed range. Then S has the matrix decomposition with respect to the orthogonal decompositions of spaces $\mathcal{H} = \mathcal{R}(S^*) \oplus \mathcal{N}(S)$ and $\mathcal{H} = \mathcal{R}(S) \oplus \mathcal{N}(S^*)$:*

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(S^*) \\ \mathcal{N}(S) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(S) \\ \mathcal{N}(S^*) \end{bmatrix},$$

where S_1 is invertible.

We have following lemma for closed range selfadjoint operator.

Lemma 1.3. *Let $S, Y \in \mathfrak{B}(\mathcal{H})$ such that S is closed range selfadjoint. Then*

$$\|SXS^\dagger - S^\dagger XS\|_p \leq \lambda_p \|SXS^\dagger + S^\dagger XS\|_p \ \ (1 < p < \infty), \tag{1.4}$$

where λ_p is a constant depended only on p.

Lemma 1.4. [4, Lemma 2.2] *Let $S \in \mathfrak{B}(\mathcal{H})$ be a closed range operator and $S = U|S|$ be the polar composition of S. Then $S^\dagger = |S|^\dagger U^*$ and $|S|^\dagger = S^\dagger U$.*

2. Main results

Theorem 2.1. *Let $X, R, T \in \mathfrak{B}(\mathcal{H})$ such that R, T are closed range operators. Then*

$$\|R^* XT^\dagger - R^\dagger XT^*\|_p \leq \lambda_p \|R^* XT^\dagger + R^\dagger XT^*\|_p \qquad (1 < p < \infty).$$

Lemma 2.2. *Let $A, B \in \mathfrak{B}^+(\mathcal{H})$ and $X \in \mathfrak{B}(\mathcal{H})$. Then*

$$|||AX - XB||| \leq \max\{\|A\|, \|B\|\}\, |||X|||,$$

for all unitarily invariant norms $|||\,.\,|||$.

McIntosh inequality for unitarily invariant norm states that

$$|||A^*AX + XB^*B||| \geq 2|||AXB^*|||, \tag{2.1}$$

where $A, B, X \in \mathfrak{B}(\mathcal{H})$. The author in [4] showed that

$$\left\|S^\dagger XS + SXS^\dagger\right\| \geq \|PXP\|,$$

where $S, X \in \mathfrak{B}(\mathcal{H})$ such that S is closed range selfadjoint operator and $P = SS^\dagger$. In the following lemma, we have an another version of this inequality for unitarily invariant norms.

Lemma 2.3. *Let $S, R, X \in \mathfrak{B}(\mathcal{H})$ such that S, R are closed range operators. Then*

$$|||S^* XR^\dagger + S^\dagger XR^*||| \geq 2|||S^\dagger XR^\dagger|||.$$

for all unitarily invariant norms $|||\,.\,|||$.

Theorem 2.4. *Let $S, R, X \in \mathfrak{B}(\mathcal{H})$ such that S, R are closed range and $|||\,.\,|||$ is unitarily invariant norm. Then*

$$\left|\left|\left|S^* XR^\dagger - S^\dagger XR^*\right|\right|\right| \leq \frac{1}{2} \max\left\{\|S\|^2, \|R\|^2\right\} \left|\left|\left|S^* XR^\dagger + S^\dagger XR^*\right|\right|\right|. \tag{2.2}$$

In particular

$$|||S^*XS^\dagger - S^\dagger XS^*||| \leq \frac{1}{2}\|S\|^2 \, |||S^*XS^\dagger + S^\dagger XS^*||| . \tag{2.3}$$

Theorem 2.5. *Let $S, X \in \mathfrak{B}(\mathcal{H})$ such that S is closed range operator and $|||\,.\,|||$ is unitarily invariant norm. Then*

$$|||S^\dagger XS^* - S^*XS^\dagger||| \leq \frac{1}{2}\left(\|S^*S - S^\dagger S\| + \|SS^* - SS^\dagger\|\right) |||S^\dagger XS^* + S^*XS^\dagger||| .$$

In particular, if $S = S^*$, then

$$|||S^\dagger XS - SXS^\dagger||| \leq \|S^2 - SS^\dagger\| \, |||S^\dagger XS + SXS^\dagger||| .$$

Remark 2.6. If $S \in \mathfrak{B}(\mathcal{H})$ be an invertible selfadjoint operator, then we have

$$|||S^{-1}XS - SXS^{-1}||| \leq \|S^2 - I\| \, |||S^{-1}XS + SXS^{-1}|||, \tag{2.4}$$

where $X \in \mathfrak{B}(\mathcal{H})$.

In [5], showed that if $S \in \mathfrak{B}(\mathcal{H})$ be a invertible selfadjoint operator, then

$$|||S^{-1}XS - SXS^{-1}||| \leq \left(\|S\| \, \|S^{-1}\| - 1\right) |||S^{-1}XS + SXS^{-1}||| , \tag{2.5}$$

where $X \in \mathfrak{B}(\mathcal{H})$.

If we put $S = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 3 \end{pmatrix}$, then $\|S^2 - I\| = 8 < \|S\| \, \|S^{-1}\| - 1 = 14$. If $S = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, then $\|S\| \, \|S^{-1}\| - 1 = \frac{1}{2} < \|S^2 - I\| = 8$. Thus inequality (2.4) is different from inequality (2.5).

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NUMERICAL SOLUTION OF FREDHOLM INTEGRAL EQUATION

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Abstract. In this paper, quadratic rules for obtaining approximate solution of definite integrals using spline quasi-interpolants will be illustrated. The method is applied for solving the linear Fredholm integral equation of the second kind. Then we give a few test examples to illustrate the accuracy and the implementation of the method.

1. Introduction

Integration of a function on bounded interval or on a certain region is an important operation for many physical problems. Most procedures for approximating the value of definite integral use polynomials that approximate the function.

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* Speaker.

2. Quadrature rules based on a quadratic spline quasi-interpolants

Let $X_n := \{x_k,\ 1 \leq k \leq n+2\}$ be the uniform partition of the interval $I = [a, b]$ in to $n+1$ equal subintervals, i. e. $x_1 = a$, $x_{n+2} = b$, $x_k = a+(k-3/2)h, 2 \leq k \leq n+1$ with $h = \frac{b-a}{n}$. We consider the space $S_2 = S_2(I, X_n)$ of quadratic splines of class C^1 on this partition. Its canonical basis is formed by the $n+2$ normalized B-splines, $\{B_k,\ k \in J\}$ with $J := \{1, 2, \cdots, n+2\}$. We consider the following quadratic spline quasi-interpolant (abbr.dQI) defined in [1] by

$$Q_2 f = \sum_{k\in J} \mu_k(f) B_k, \tag{2.1}$$

where $\mu_k(f)$ are given by $\mu_1(f) = f_1,\ \mu_{n+2}(f) = f_{n+2}$,

$$\mu_2(f) = -1/3f_1 + 3/2f_2 - 1/6f_3 = \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3,$$

$$\mu_{n+1}(f) = -1/6f_n + 3/2f_{n+1} - 1/3f_{n+2} = \beta_3 f_n + \beta_2 f_{n+1} + \beta_1 f_{n+2}, \tag{2.2}$$

$$\mu_j(f) = -1/8f_{j-1} + 5/4f_j - 1/8f_{j+1} = \gamma_1 f_{j-1} + \gamma_2 f_j + \gamma_3 f_{j+1},\ 3 \leq j \leq n \tag{2.3}$$

with $f_i = f(t_i)$, $t_1 = a$, $t_{n+2} = b$, $t_i = a + (i - 3/2)h,\ 2 \leq i \leq n+1$. The author in [2] constructs a new and simple quadrature rule of convergence order $O(h^4)$. Here, we will construct quadrature rules based on this type of dQI and of convergence order $O(h^l)$, with $l \geq 4$ (see [2]). For $j = 2, \ldots, n+1$, we retain the same values of $\mu_j(f)$, given by (2.2), (2.3), and for $j = 1$ and $j = n+2$, we set $\mu_1(f) = \sum_{i=1}^m \alpha_i f_i,\ \mu_{n+2}(f) = \sum_{i=1}^m \alpha_i f_{n+3-i}$, where m is an odd integer such that $3 \leq m \leq n+2$, and $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ are real parameters. We consider the quadrature rule defined by $\mathcal{I}_{Q_2}^m(f) := \int_I Q_2 f(x)dx$. This quadrature formula with m correction points and based on intergrating quasi-interpolant Q_2 can be written as

$$\mathcal{I}_{Q_2}^m(f) = h\sum_{i=1}^{m} v_i^{(m,2)}(f_i + f_{n+3-i}) + h\sum_{i=m+1}^{n+2-m} f_i. \tag{2.4}$$

In Table 2, we list correction weights $\{v_i^{(m,2)}, i = 1, \ldots, m.\}$

3. Application to fredholm integral equations

In this section, we describe an application of the quadrature rules to numerical solution of fredholm integral equation of the second kind. The equation in which we are interested is given by

$$u(x) - \int_a^b k(x,s)u(s)ds = f(x),$$

where $k(.,.) \in C([a,b] \times [a,b])$ and $f \in C([a,b])$ are known functions and u is the function to be determind. The method associated with the quadrature formula $\int_a^b f(s)ds = \sum_{l=1}^{n+2} w_l f(x_l)$, consists in looking for a solution u satisfying

$$u(x) - \sum_{j=1}^{n+2} w_j k(x,t_j)u(t_j) = f(x),$$

by replace x_i, we have

$$u(x_i) - \sum_{j=1}^{n+2} w_j k(x_i,t_j)u(t_j) = f(x_i).$$

Denoting by u the vector with components $u(x_i)$, by f the vector with components $f(x_i)$, and by A the matrix with entries $A(i,j) := K(x_i,t_j)$, and by B the matrix with entries

$$B(i,j) = \begin{cases} w[i], & i = j, \\ 0, & i \neq j, \end{cases} \tag{3.1}$$

where

$$w_i = \begin{cases} hv_i^{(m,2)} & i = 1,\ldots,m, \\ h & i = m+1,\ldots,n-m+2, \\ hv_{n+3-i}^{(m,2)} & i = n-m+3,\ldots,n+2. \end{cases}$$

then the above equations are equivalent to the linear system $(I - AB)u = f$. Moreover, if we define the function

$$u_n^{(m,2)}(x) - \sum_{j=1}^{n+2} w_j k(x,t_j)u_j = f(x),$$

then it is clear that u satisfies $u(x_i) = u_i$, so u is an interpolant of the computed values at data points.

Example 3.1. Let us consider the following fredholm integral equation of the second kind

$$u(x) - \int_0^{\Pi/2} \sin(30x)\cos(31s)u(s)ds = \sin(x) + (1/30)\sin(30x)$$

where the exact solution is $u(x) = \sin(x)$. In Table 1, numerical results are presented for rules $\mathcal{I}_{Q_2}^m(f)$, $m = 5, 7, 9, 13$.

Table 1: The error $\| u - u_n^{(m,2)} \|$

$m = 13$	$m = 9$	$m = 7$	$m = 5$	n
$1.9611E-05$	$2.39239E-05$	$1.09203E-04$	$1.12294E-04$	32
$7.4029E-10$	$9.55238E-08$	$1.38778E-07$	$1.5237E-06$	64
$6.6724E-14$	$3.58377E-11$	$1.55629E-09$	$4.69372E-08$	128

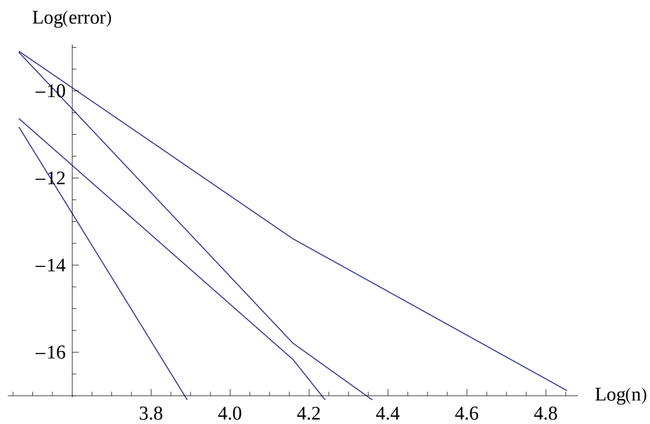

Fig. 1. error

Table 2: Quadrature weights $v_i^{(m,2)}$.

$m=5$	$m=7$	$m=9$	$m=13$	$m=17$	$m=21$
0.13079	0.137415	0.14009	0.141489	0.14114	0.1402697
0.835937	0.81902	0.810952	0.80607	0.80761	0.811803
1.04496	1.06982	1.08705	1.101459	1.09415	1.06952
0.986146	0.96034	0.932189	0.89773	0.92695	1.050334

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SPLINE QUASI-INTERPOLANTS METHOD FOR SOLVING TRIPLE INTEGRALS

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Abstract. In this paper, quadratic rules for obtaining approximate solution of definite integrals as well as single and triple integrals using spline quasi-interpolants will be illustrated. The method is applied to a few test examples.

1. Introduction

Triple integration of a function on bounded interval is an important operation for many physical problems.

2. Quadrature rules based on a quadratic spline quasi-interpolants for calculating triple integrals

Let $X_n := \{x_k,\ 0 \le k \le n\}$ be the uniform partition of the interval $I = [a, b]$ into n equal subintervals, i.e. $x_k := a + kh$, with $h = \frac{b-a}{n}$. We consider the space $S_2 = S_2(I, X_n)$ of quadratic splines of class C^1. Its canonical basis is formed by the $n + 2$ normalized B-splines, $\{B_k,\ k \in J\}$ with $J := \{1, 2, \cdots, n + 2\}$. Consider the following quadratic spline quasi-interpolant defined in [1] by $Q_2 f = \sum_{k\in J} \mu_k(f) B_k$,

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* Speaker.

where $\mu_1(f) = f_1,\ \mu_{n+2}(f) = f_{n+2}\ \mu_2(f) = \frac{-1}{3}f_1 + \frac{3}{2}f_2 - \frac{1}{6}f_3$, $\mu_{n+1}(f) = \frac{-1}{6}f_n + \frac{3}{2}f_{n+1} - \frac{1}{3}f_{n+2}$, $\mu_j(f) = \frac{-1}{8}f_{j-1} + \frac{5}{4}f_j - \frac{1}{8}f_{j+1},\ 3 \leq j \leq n$ with $f_i = f(t_i),\ t_i = a + (i-3/2)h,\ 2 \leq i \leq n+1,\ t_{n+2} = b,\ t_1 = a$. Here, we will construct quadrature rules based on this type of dQI and of convergence order $O(h^l)$, $l \geq 4$ (see [2]). For $j = 2, \dots, n+1$, we retain the same values of $\mu_j(f)$ and for $j = 1$ and $j = n+2$, we set $\mu_1(f) = \sum_{i=1}^{m} \alpha_i f_i,\ \ \mu_{n+2}(f) = \sum_{i=1}^{m} \alpha_i f_{n+3-i}$ where m is an odd integer and $3 \leq m \leq n+2$. Consider the quadrature rule defined by $\mathcal{I}_{Q_2}^{m}(f) := \int_l Q_2 f(x)dx$. This quadrature formula with m correction points can be written as $\mathcal{I}_{Q_2}^{m}(f) = h\sum_{i=1}^{m} v_i^{m,2}(f_i + f_{n+3-i}) + h\sum_{i=m+1}^{n+2-m} f_i$. Obtain

$$
\begin{aligned}
\mathcal{I}_{Q_2}^{m,m',m''}(f) &= h_x h_y h_z \Big(\sum_{j=m'+1}^{n'+2-m'} \sum_{i=1}^{m} \sum_{k=m''+1}^{n''+2-m''} v_i^{(m)} (f_{i,j,k} + f_{n+3-i,j,k}) + \\
& \sum_{j=m'+1}^{n'+2-m'} \sum_{i=1}^{m} \sum_{k=1}^{m''} v_i^{(m)} v_k^{(m'')} ((f_{i,j,k} + f_{i,j,n''+3-k}) \\
& + (f_{n+3-i,j,k} + f_{n+3-i,j,n''+3-k})) \\
& + \sum_{j=1}^{m'} \sum_{i=m+1}^{n+2-m} \sum_{k=m''+1}^{n''+2-m''} v_j^{(m')} v_i^{(m)} f_{i,n'+3-j,k} + \sum_{j=1}^{m'} \sum_{i=m+1}^{n+2-m} \sum_{k=m''+1}^{n''+2-m''} v_j^{(m')} f_{i,j,k} \\
& + \sum_{j=m'+1}^{n'+2-m'} \sum_{i=m+1}^{n+2-m} \sum_{k=1}^{m''} v_k^{(m'')} (f_{i,j,k} + f_{i,j,n''+3-k}) \\
& + \sum_{j=1}^{m'} \sum_{i=1}^{m} \sum_{k=m''+1}^{n''+2-m''} v_j^{(m')} v_i^{(m)} (f_{i,j,k} + f_{n+3-i,j,k}) \\
& + \sum_{j=1}^{m'} \sum_{i=1}^{m} \sum_{k=1}^{m''} v_j^{(m')} v_i^{(m)} v_k^{(m'')} ((f_{n+3-i,j,k} + f_{n+3-i,j,n''+3-k}) + \\
& (f_{n+3-i,n'+3-j,k} + f_{n+3-i,n'+3-j,n''+3-k})) + \sum_{j=m'+1}^{n'+2-m'} \sum_{i=m+1}^{n+2-m} \sum_{k=m''+1}^{n''+2-m''} f_{i,j,k} \\
& + \sum_{j=1}^{m'} \sum_{i=1}^{m} \sum_{k=m''+1}^{n''+2-m''} v_j^{(m')} v_i^{(m)} (f_{i,n'+3-j,k} + f_{n+3-i,n'+3-j,k}) \\
& + \sum_{j=1}^{m'} \sum_{i=1}^{m} \sum_{k=1}^{m''} v_j^{(m')} v_i^{(m)} v_k^{(m'')} ((f_{i,n'+3-j,k} + f_{i,n'+3-j,n''+3-k})
\end{aligned}
$$

$$+\left(f_{i,j,k}+f_{i,j,n''+3-k}\right)))+\sum_{j=1}^{m'}\sum_{i=m+1}^{n+2-m}\sum_{k=1}^{m''}v_j^{(m')}v_k^{(m'')}$$
$$((f_{i,n'+3-j,k}+f_{i,n'+3-j,n''+3-k})+(f_{i,j,k}+f_{i,j,n''+3-k}))$$

Gregory rules:
$GR_{k,n}(f)=h\sum_{i=0}^{n}f_{n,i}+h\sum_{i=0}^{k}\left[(-1)^{i+1}\sum_{j=i}^{k}\binom{j}{i}\mathcal{L}_{j+1}\right](f_{n,i}+f_{n,n-i})$
with $f_{n,i}=f(x_{n,i}), h=\frac{b-a}{n},\ x_{n,i}=a+ih,\ i=0,1,\ldots,n$. The Laplace coefficients $\mathcal{L}_j$, can be computed from $\sum_{v=1}^{\mu}\frac{\mathcal{L}_v}{\mu-v+1}=\frac{1}{\mu+1},\ \mu=1,2,\ldots$

$$GR_{m-1,m'-1,m'',n,n',n''}=h_xh_yh_z(\sum_{j=0}^{n'}\sum_{i=0}^{n}\sum_{p=0}^{n''}f_{n,i,n',j,n'',p}+$$
$$\sum_{j=0}^{n'}\sum_{i=0}^{n}\sum_{p=0}^{m''-1}\sum_{k=p}^{m''-1}(-1)^{p+1}\binom{k}{p}\mathcal{L}_{k+1}(f_{n,i,n',j,n'',p}+f_{n,i,n',j,n'',n''-p})+$$
$$\sum_{j=0}^{n'}\sum_{i=0}^{m-1}\sum_{k'=i}^{m-1}\sum_{p=0}^{m''-1}\sum_{k=p}^{m''-1}(-1)^{p+i+2}\mathcal{L}_{k+1}\binom{k}{p}\mathcal{L}_{k'+1}\binom{k'}{i}$$
$$((f_{n,i,n',j,n'',p}+f_{n,i,n',j,n'',n''-p})+(f_{n,n-i,n',j,n'',p}+f_{n,n-i,n',j,n'',n''-p}))$$
$$+\sum_{j=0}^{m'-1}\sum_{k=j}^{m'-1}\sum_{i=0}^{n}\sum_{p=0}^{m''-1}\sum_{k'=p}^{m''-1}(-1)^{p+j+2}\mathcal{L}_{k'+1}\binom{k'}{p}\mathcal{L}_{k+1}\binom{k}{j}$$
$$((f_{n,i,n',j,n'',p}+f_{n,i,n',j,n'',n''-p})+(f_{n,i,n',n'-j,n'',p}+f_{n,i,n',n'-j,n'',n''-p}))$$
$$+\sum_{j=0}^{m'-1}\sum_{k=j}^{m'-1}\sum_{i=0}^{n}\sum_{p=0}^{n''}(-1)^{j+1}\binom{k}{j}\mathcal{L}_{k+1}(f_{n,i,n',j,n'',p}+f_{n,i,n',n'-j,n'',p})$$
$$+\sum_{j=0}^{n'}\sum_{i=0}^{m-1}\sum_{k'=i}^{m-1}\sum_{p=0}^{n''}(-1)^{i+1}\binom{k'}{i}\mathcal{L}_{k'+1}f_{n,i,n',j,n'',p}+$$
$$\sum_{j=0}^{m'-1}\sum_{k=j}^{m'-1}\sum_{i=0}^{m-1}\sum_{k'=i}^{m-1}\sum_{p=0}^{n''}(-1)^{i+j+2}\mathcal{L}_{k'+1}\binom{k'}{i}\mathcal{L}_{k+1}\binom{k}{j}f_{n,n-i,n',n'-j,n'',p}+$$
$$\sum_{j=0}^{m'-1}\sum_{k=j}^{m'-1}\sum_{i=0}^{m-1}\sum_{k'=i}^{m-1}\sum_{p=0}^{m''-1}\sum_{l=p}^{m''-1}(-1)^{i+j+p+3}\mathcal{L}_{k'+1}\binom{k'}{i}\mathcal{L}_{k+1}\binom{k}{j}\mathcal{L}_{l+1}\binom{l}{p}$$
$$((f_{n,i,n',j,n'',p}+f_{n,i,n',j,n'',n''-p})+(f_{n,n-i,n',j,n'',p}+f_{n,n-i,n',j,n'',n''-p}))$$

$$+\sum_{j=0}^{m'-1}\sum_{k=j}^{m'-1}\sum_{i=0}^{m-1}\sum_{k'=i}^{m-1}\sum_{p=0}^{n''}(-1)^{i+j+2}\mathcal{L}_{k'+1}\binom{k'}{i}\mathcal{L}_{k+1}\binom{k}{j}$$

$$(f_{n,n-i,n',j,n'',p}+f_{n,i,n',n'-j,n'',p})+$$

$$\sum_{j=0}^{m'-1}\sum_{k=j}^{m'-1}\sum_{i=0}^{m-1}\sum_{k'=i}^{m-1}\sum_{p=0}^{m''-1}\sum_{l=p}^{m''-1}(-1)^{i+j+p+3}\mathcal{L}_{k'+1}\binom{k'}{i}\mathcal{L}_{k+1}\binom{k}{j}\mathcal{L}_{l+1}\binom{l}{j}$$

$$(f_{n,i,n',n'-j,n'',p}+f_{n,i,n',n'-j,n'',n''-p}+f_{n,n-i,n',n'-j,n'',p}+f_{n,n-i,n',n'-j,n'',n''-p})$$

$$+\sum_{j=0}^{m'-1}\sum_{k=j}^{m'-1}\sum_{i=0}^{m-1}\sum_{k'=i}^{m-1}\sum_{p=0}^{n''}(-1)^{i+j+2}\mathcal{L}_{k'+1}\binom{k'}{i}\mathcal{L}_{k+1}\binom{k}{j}f_{n,i,n',j,n'',p}+$$

$$+\sum_{j=0}^{n'}\sum_{i=0}^{m-1}\sum_{k'=i}^{m-1}\sum_{p=0}^{n''}(-1)^{i+1}\binom{k'}{i}\mathcal{L}_{k'+1}f_{n,n-i,n',j,n'',p}).$$

Example 2.1. Consider $f(x) = 12xy^2z^3$. In Tables 1 and 2, numerical results are presented for the rules $\mathcal{I}_{Q_2}^{m,m',m''}$ and $GR_{m-1,m'-1,m''-1,n,n',n''}$.

Table 1: $\left|\int_0^2\int_{-1}^2\int_0^3 f(x,y,z)dydxdz - \mathcal{I}_{Q_2}^{m,m',m''}(f(x,y))\right|$

$(m=m'=m'')m=9$	$m=7$	$m=5$	$n=n'=n''$
$1.13687E-13$	$2.27374E-13$	$2.27374E-13$	64
$2.27374E-13$	$2.27374E-13$	$2.27374E-13$	128
$2.27374E-13$	$3.41061E-13$	$2.27374E-13$	256

Table 2: $\left|\int_0^2\int_{-1}^2\int_0^3 f(x,y,z)dydxdz - GR_{m-1,m'-1,m''-1,n,n',n''}(f(x,y))\right|$

$(m=m'=m'')m=9$	$m=7$	$m=5$	$n=n'=n''$
$2.27374E-13$	$2.27374E-13$	$2.27374E-13$	64
$2.27374E-13$	$2.27374E-13$	$2.27374E-13$	128
$2.27374E-13$	$2.27374E-13$	$2.27374E-13$	256

Table 3: Quadrature weights $v_i^{(m,2)}$.

$m=5$	$m=7$	$m=9$	$m=13$	$m=17$
0.13079	0.137415	0.14009	0.141489	0.14114
0.835937	0.81902	0.810952	0.80607	0.80761
1.04496	1.06982	1.08705	1.101459	1.09415

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TWO NEW FAMILIES OF ITERATIVE METHODS OF OPTIMAL EIGHT-ORDER CONVERGENCE FOR SOLVING NONLINEAR EQUATIONS

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Abstract. In this paper, two new families of iterative methods of optimal eight-order for solving nonlinear equations by using weight function methods is presented. Per iteration the new methods require three evaluations of the function and one evaluation of its first derivative. Therefore, these families of methods Have the efficiency index which equals 1.682. Kung and Traub conjectured that a multipoint iteration without memory based on n evaluations could achieve optimal convergence order 2^{n-1}. Thus, we provide two new class which agrees with the conjecture of Kung-Traub for $n = 4$.

1. Introduction

In this paper, we consider iterative methods to find a simple root α of a nonlinear equation $f(x) = 0$, where $f : D \subset \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval D. This problem is a prototype for many nonlinear numerical problems. Newton's method is the most widely used algorithm for dealing with such problems, and it is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

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* Speaker.

which converges quadratically in some neighborhood of α (see [1, 2]).

We use the symbols $\rightarrow$, O, and $\sim$ according to the following conventions [1]: If $\lim_{x_n\rightarrow\infty} g(x_n) = C$, we write $g(x_n) \rightarrow C$ or $g \rightarrow C$. If $\lim_{x\rightarrow a} g(x) = C$, we write $g(x) \rightarrow C$ or $g \rightarrow C$. If $\frac{f}{g} \rightarrow C$, where C is a nonzero constant, we write $f = O(g)$ or $f \sim Cg$. Let $f(x)$ be a function defined on an interval I, where I is the smallest interval containing $k+1$ distinct nodes $x_1, x_2, \ldots, x_k$. The divided difference $f[x_0, x_1, \ldots, x_k]$ with kth-order is defined as follows:
$f[x_0] = f(x_0)$
$f[x_0, x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0}, \cdots, f[x_0, x_1, \ldots, x_k] = \frac{f[x_1,x_2,\ldots,x_k]-f[x_0,x_1,\ldots x_{k-1}]}{x_k-x_0}$.
Moreover, we recall the definition of efficiency index (EI) as $E = p^{1/n}$, where p is the order of convergence and n is the total number of function evaluations per iteration.

2. The methods and analysis of convergence

In order to construct new methods, we consider an iteration scheme of the form,

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(y_n)}{f'(y_n)}, \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f'(z_n)}. \end{cases}$$

This scheme has convergent order eighth and it requires six function evaluations . it has an efficiency index of $8^{\frac{1}{6}} = 1.414$, which is the same as Newton's method. To improve the efficiency index, we approximate $f'(z_n)$ by the divided difference [3]

$$f'(z_n) \approx \frac{f[x_n, z_n]f[y_n, z_n]}{f[x_n, y_n]}.$$

Now, we present two new families of iterative methods of optimal eight-order by using the method of weight functions as follows:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(y_n)}{L(t_1)f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{f(z_n)f[x_n, y_n]}{\{H(t_2) \times P(t_3)\}\, f[x_n, z_n]f[y_n, z_n]}, \end{cases} \tag{2.1}$$

where $L(t_1), H(t_2)$, and $P(t_3)$ are three real-valued weight functions when

$$t_1 = \frac{f(y)}{f(x)}, \qquad t_2 = \frac{f(z)}{f(x)}, \qquad t_3 = \frac{f(z)}{f(y)},$$

and

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - K(u_1)\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \{G(u_2) \times Q(u_3)\} \dfrac{f(z_n) f[x_n, y_n]}{f[x_n, z_n] f[y_n, z_n]}, \end{cases} \tag{2.2}$$

where $K(u_1), G(u_2)$, and $Q(u_3)$ are three real-valued weight functions when

$$u_1 = \frac{f(y)}{f(x)}, \qquad u_2 = \frac{f(z)}{f(x)}, \qquad u_3 = \frac{f(z)}{f(y)},$$

without the index n, should be chosen such that the order of convergence arrives at the optimal level eight.

Theorem 2.1. *Assume that $f \in C^5(D)$. Suppose $\alpha \in D$, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. If the initial point x_0 is sufficiently close to α, then the sequence x_n generated by any method of the family (2.1) has eight-order of convergence to α if L, H and P are any functions with*

$$\begin{array}{llll} L(0) = 1, & L'(0) = -2, & L''(0) = 0, & \left|L^{(3)}(0)\right| < \infty, \\ H(0) = 1, & H'(0) = -1, & & \\ P(0) = 1, & P'(0) = 0, & |\mathrm{P}''(0)| < \infty, & \end{array}$$

where its error equation reads

$$\begin{aligned} e_{n+1} = -\frac{1}{6} \big(& c_2(c_2^2 - c_3)((L^{(3)}(0) - 3(6 + P''(0)))c_2^4 \\ & + 6(4 + P''(0))c_2^2 c_3 - 3P''(0)c_3^2 - 6c_2c_4) \big) e_n^8 + O(e_n^9) \end{aligned}$$

In what follows, we give some weight function forms of scheme (2.1):

Method 1–1.

$$L(t) = 1 - 2\sin(t), \quad H(t) = 1 + \sin(t)\cos(t) - 2te^{-t}, \quad P(t) = e^{t^2}$$

Method 1–2.

$$L(t) = -t^2 + e^{-2t-t^2}, \quad H(t) = \cos(t)e^{-t}, \quad P(t) = 1 - t + \sin(t)$$

Method 1–3.

$$L(t) = -2\sin(t) + e^{t^3}, \quad H(t) = e^{-t}, \quad P(t) = \cos(t)$$

Theorem 2.2. *Assume that $f \in C^5(D)$. Suppose $\alpha \in D$, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. If the initial point x_0 is sufficiently close to α, then the sequence x_n generated by any method of the family (2.2) has eight-order of convergence to α if K, G and Q are any functions with*

$$\begin{aligned}
&K(0) = 1, \quad K'(0) = 2, \quad K''(0) = 8, \quad \left|K^{(3)}(0)\right| < \infty,\\
&G(0) = G'(0) = 1,\\
&Q(0) = 1, \quad Q'(0) = 0, \quad |Q''(0)| < \infty,
\end{aligned}$$

where its error equation reads

$$\begin{aligned}
e_{n+1} = \frac{1}{6}c_2(c_2^2 - c_3)&\left((K^{(3)}(0) - 3(10 + Q''(0)))c_2^4\right.\\
&\left.+6(-4 + Q''(0))c_2^2c_3 - 3Q''(0)c_3^2 + 6c_2c_4\right) e_n^8 + O(e_n^9)
\end{aligned}$$

In what follows, we give some weight function forms of scheme (2.2):

Method 2–1.

$$K(t) = 1 + t + 4t^2 + \sin(t), \quad G(t) = 1 + t, \quad Q(t) = e^{t^2}$$

Method 2–2.

$$K(t) = 2\sin(t) + e^{4t^2+t^3}, \quad G(t) = \cos(t)e^t, \quad Q(t) = 1 - t + \sin(t)$$

Method 2–3.

$$K(t) = t + 2t^2 + \sin(t) + e^{2t^2}, \quad G(t) = \cos(t) + \sin(t), \quad Q(t) = \cos(t)$$

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TWO-POINT METHODS WITH MEMORY FOR SOLVING NONLINEAR EQUATIONS

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Abstract. In this paper, based on iterative methods without memory proposed by Hafiz and Bahgat, two iterative methods with memory is presented. We show that the order of convergence is increased without any additional function evaluations. Numerical comparisons are made to show the performance of the presented methods.

1. Introduction

Let f be a sufficiently smooth function of single variable in some neighborhood D of α, where α satisfies $f(\alpha) = 0$. Traub [2] considered the iterative function of order two

$$x_{n+1} = x_n - \frac{\beta f(x_n)^2}{f(x_n + \beta f(x_n)) - f(x_n)} \tag{1.1}$$

where $\beta \neq 0$ is a real constant. Hafiz and Bahgat developed some iterative methods [1], in the following form:

$$\begin{cases} y_n = x_n - \dfrac{\beta f(x_n)^2}{f(x_n + \beta f(x_n)) - f(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)^2 + f(y_n)^2}{P_0(x_n)(f(x_n) - f(y_n))}, \end{cases} \tag{1.2}$$

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$$\begin{cases} y_n = x_n - \dfrac{\beta f(x_n)^2}{f(x_n + \beta f(x_n)) - f(x_n)}, \\ x_{n+1} = y_n - \dfrac{2f(y_n)P_1(x_n, y_n)}{2P_1^2(x_n, y_n) - f(y_n)P_2(x_n, y_n)}, \end{cases} \tag{1.3}$$

where $\beta \neq 0$ is a real constant and

$$\begin{aligned} P_0(x_n) &= \tfrac{f(x_n+\beta f(x_n))-f(x_n)}{\beta f(x_n)}, \\ P_1(x_n, y_n) &= 2\left(\tfrac{f(y_n)-f(x_n)}{y_n-x_n}\right) - \tfrac{f(x_n+\beta f(x_n))-f(x_n)}{\beta f(x_n)}, \\ P_2(x_n, y_n) &= \tfrac{2}{y_n-x_n}\left(\tfrac{f(y_n)-f(x_n)}{y_n-x_n} - \tfrac{f(x_n+\beta f(x_n))-f(x_n)}{\beta f(x_n)}\right). \end{aligned}$$

In this paper we use iterative methods without memory proposed by Hafiz and Bahgat as the base for constructing considerably faster methods employing information from the current and previous iteration without any additional evaluations of the function.

2. Two-point methods with memory

In [?], Petković et al. have presented some methods through the following forms of β_n

$$\beta_n = -\frac{1}{\overline{f}'(\alpha)} = -\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \tag{2.1}$$

$$\beta_n = -\frac{1}{\overline{f}'(\alpha)} = -\frac{x_n - y_{n-1}}{f(x_n) - f(y_{n-1})}, \tag{2.2}$$

$$\beta_n = -\frac{1}{\overline{f}'(\alpha)} = -\frac{1}{N_2'(x_n)}, \tag{2.3}$$

where $\overline{f}'(\alpha)$ denotes an approximation to $f'(\alpha)$, and Newton's interpolatory polynomial of second degree is $N_2(t) = N_2(t; x_n, y_{n-1}, x_{n-1})$. The derivative $N'_2(t)$ at $t = x_n$ is calculated by using the following formula:

$$\begin{aligned} N'_2(x_n) &= \left[\tfrac{d}{dt}N_2(t)\right]_{t=x_n} = \left[\tfrac{d}{dt}(f(x_n) + f[x_n, y_{n-1}](t - x_n)\right. \\ &\quad \left. + f[x_n, y_{n-1}, x_{n-1}](t - x_n)(t - y_{n-1})\right]_{t=x_n} \\ &= f[x_n, y_{n-1}] + f[x_n, y_{n-1}, x_{n-1}](x_n - y_{n-1}), \end{aligned}$$

Now we replace constant parameters β in the iterative formulas (1.2) and (1.3) by the varying β_n defined by (2.1), (2.2) and (2.3). Then multipoint methods with memory becomes

$$\begin{cases} y_n = x_n - \dfrac{\beta_n f(x_n)^2}{f(x_n + \beta_n f(x_n)) - f(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)^2 + f(y_n)^2}{P_0(x_n)(f(x_n) - f(y_n))}, \end{cases} \tag{2.4}$$

$$\begin{cases} y_n = x_n - \dfrac{\beta_n f(x_n)^2}{f(x_n + \beta_n f(x_n)) - f(x_n)}, \\ x_{n+1} = y_n - \dfrac{2f(y_n)P_1(x_n, y_n)}{2P_1^2(x_n, y_n) - f(y_n)P_2(x_n, y_n)}, \end{cases} \tag{2.5}$$

where

3. Convergence theorems

Theorem 3.1. *Let the function $f(x)$ be sufficiently differentiable in a neighborhood of its simple zero α. If an initial approximation x_0 is sufficiently close to α and the parameter β_n in (2.4) is recursively calculated by the forms given in (2.1)-(2.3). Then, the R-order of convergence of the Steffensen-like method with memory (2.4) with the corresponding expressions (2.1)-(2.3) of β_n is at least 3.30278, 3.73205 and 4, respectively.*

Theorem 3.2. *Let the function $f(x)$ be sufficiently differentiable in a neighborhood of its simple zero α. If an initial approximation x_0 is sufficiently close to α and the parameter β_n in (2.5) is recursively calculated by the forms given in (2.1)-(2.3). Then, the R-order of convergence of the Steffensen-like method with memory (2.5) with the corresponding expressions (2.1)-(2.3) of β_n is at least 3.30278, 3.73205 and 4, respectively.*

4. Numerical results

In this section we demonstrate the convergence behavior of the methods with memory (2.4) and (2.5), where β_n is calculated by one of the formulas (2.1)-(2.3). Numerical computations reported here have been carried out in a *Mathematica* 8.0 environment. We calculate the computational order of convergence r_c using the formula [4]

$$r_c \approx \frac{\log |f(x_n)/f(x_{n-1})|}{\log |f(x_{n-1})/f(x_{n-2})|},$$

taking into consideration the last three approximations in the iterative process. we use the following examples (selected from [3]):
$f_1(x) = e^{x^2+x\cos x-1}\sin \pi x + x\log(x\sin x+1), \alpha = 0, x_0 = 0.6, \gamma_0 = -0.1$
$f_2(x) = \log(x^2-2x+2)+e^{x^2-5x+4}\sin(x-1), \alpha = 1, x_0 = 1.35, \gamma_0 = -0.1.$
It is obvious from these tables that recursive calculation by the Newton interpolation (2.3) gives the best results.

TABLE 1. Some two-point methods with memory for example $f_1(x)$.

Methods	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	r_c
(2.4), (2.1)	0.109e − 1	0.278e − 6	0.237e − 21	3.2826
(2.4), (2.2)	0.109e − 1	0.440e − 7	0.332e − 27	3.7333
(2.4), (2.3)	0.109e − 1	0.209e − 7	0.520e − 31	4.1303
(2.5), (2.1)	0.109e − 1	0.273e − 6	0.224e − 21	3.2834
(2.5), (2.2)	0.109e − 1	0.431e − 7	0.307e − 27	3.7335
(2.5), (2.3)	0.109e − 1	0.203e − 7	0.471e − 31	4.1304

TABLE 2. Some two-point methods with memory for example $f_2(x)$.

Methods	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	r_c
(2.4), (2.1)	0.622e − 2	0.516e − 6	0.689e − 20	3.4037
(2.4), (2.2)	0.622e − 2	0.391e − 7	0.157e − 26	3.7321
(2.4), (2.3)	0.622e − 2	0.191e − 7	0.394e − 30	4.1183
(2.5), (2.1)	0.678e − 2	0.667e − 6	0.163e − 19	3.4026
(2.5), (2.2)	0.678e − 2	0.504e − 7	0.399e − 26	3.7284
(2.5), (2.3)	0.678e − 2	0.241e − 7	0.100e − 29	4.1114

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CHARACTERIZING JORDAN DERIVATIONS OF SOME ALGEBRAS THROUGH ZERO PRODUCTS

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Abstract. Let $\mathcal{A}$ be an algebra, $\mathcal{M}$ be an $\mathcal{A}$-bimodule and let $D : \mathcal{A} \to \mathcal{M}$ be a linear map satisfying $D(a)b + aD(b) + D(b)a + bD(a) = 0$ whenever $a, b \in \mathcal{A}$ are such that $ab = ba = 0$. If $\mathcal{A}$ is a C^*-algebra with D continuous or $\mathcal{A}$ is a matrix algebra, then there exists a derivation $\delta : \mathcal{A} \to \mathcal{M}$ such that D is equal to δ plus elementary operators.

1. Introduction

Throughout this article all algebras and vector spaces will be over the complex field $\mathbb{C}$. Let $\mathcal{A}$ be an algebra and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Recall that a linear map $D : \mathcal{A} \to \mathcal{M}$ is said to be a *Jordan derivation* if $D(ab + ba) = D(a)b + aD(b) + D(b)a + bD(a)$ for all $a, b \in \mathcal{A}$. It is called a *derivation* if $D(ab) = D(a)b + aD(b)$ for all $a, b \in \mathcal{A}$. Each map $I_m : \mathcal{A} \to \mathcal{M}$ given by $I_m(a) = am - ma$, where $m \in \mathcal{M}$ is a derivation which will be called an *inner derivation*. Clearly, each derivation is a Jordan derivation. The converse is, in general, not true.

The question under what conditions that a map becomes a (Jordan) derivation attracted much attention of mathematicians. Every Jordan derivation from a 2-torsion free semiprime ring into itself is a derivation. By a classical result of Jacobson and Rickart every Jordan derivation on a full matrix ring over a 2-torsion free unital ring is a derivation.

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Also, there have been a number of papers concerning the study of conditions under which (generalized or Jordan) derivations of rings can be completely determined by the action on some sets of points. For more studies concerning these problems we refer the reader to [1, 2] and the references therein.

In this paper, following [1], we consider the subsequent condition on an linear map D from an algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$:

$$a, b \in \mathcal{A}, \quad ab = ba = 0 \Rightarrow D(a)b + aD(b) + D(b)a + bD(a) = 0. \qquad (*)$$

Our purpose is to investigate whether the condition $(*)$ characterizes Jordan derivations. Particularly, in this note we give some results from [2] and [1] which characterize linear maps satisfying $(*)$ on full matrix algebras and C^*-algebras, respectively.

Remark 1.1. Each of the following conditions on a linear map $D : \mathcal{A} \to \mathcal{M}$ implies $(*)$, which have been considered by a number of authors (see, for instance, [3]):

$$a, b \in \mathcal{A}, \quad ab + ba = 0 \Rightarrow D(a)b + aD(b) + D(b)a + bD(a) = 0.$$

$$a, b \in \mathcal{A}, \quad ab = ba = 0 \Rightarrow D(a)b + aD(b) + D(b)a + bD(a) = 0.$$

Therefore, main results still holds with each of the above conditions replaced by $(*)$.

2. Main results

Our main results characterize linear maps satisfying $(*)$ on two classes of algebra: Matrix algebras and C^*-algebras. For the case of matrix algebras we describe linear maps satisfying $(*)$ without any continuity assumption on the maps, but for the case of C^*-algebras we assume that linear maps be continuous.

2.1. Matrix algebras.

From this point up to the last section $M_n(\mathcal{A})$, for $n \geq 2$, is the algebra of all $n \times n$ matrices over a unital algebra $\mathcal{A}$ and $\mathcal{M}$ is a unital $M_n(\mathcal{A})$-bimodule. In this section, we discuss the linear maps from $M_n(\mathcal{A})$ into $\mathcal{M}$ satisfying $(*)$. The results are from [2].

We shall denote the elements of $M_n(\mathcal{A})$ by bold letters and the identity matrix by $\mathbf{1}$. Also, $\mathbf{e}_{ij}$ for $1 \leq i, j \leq n$ is the matrix unit, $a\mathbf{e}_{ij}$ is the matrix whose (ij)th entry is a and zero elsewhere, where $a \in R$ and $1 \leq i, j \leq n$, and $a_{i,j}$ is the (ij)th entry of $\mathbf{a} \in M_n(\mathcal{A})$.

The following is our main result in this section.

Theorem 2.1. *Let $D : M_n(\mathcal{A}) \to \mathcal{M}$ be a linear map satisfying*

$$\mathbf{a}, \mathbf{b} \in M_n(\mathcal{A}), \quad \mathbf{ab} = \mathbf{ba} = 0 \Rightarrow D(\mathbf{a})\mathbf{b} + \mathbf{a}D(\mathbf{b}) + D(\mathbf{b})\mathbf{a} + \mathbf{b}D(\mathbf{a}) = 0.$$

Then there exist a derivation $\delta : M_n(\mathcal{A}) \to \mathcal{M}$ such that $D(\mathbf{a}) = \delta(\mathbf{a}) + \mathbf{a}D(\mathit{1})$ and $\mathbf{a}D(\mathit{1}) = D(\mathit{1})\mathbf{a}$ for each $\mathbf{a} \in M_n(\mathcal{A})$.

Let $\mathcal{A}$ be an algebra and $\mathcal{N}$ be a unital $\mathcal{A}$-bimodule. Let $M_n(\mathcal{N})$ be the set of all $n \times n$ matrices over $\mathcal{N}$, then $M_n(\mathcal{N})$ has a natural structure as unital $M_n(\mathcal{A})$-bimodule. Any derivation $d : \mathcal{A} \to \mathcal{N}$, induces a derivation $\bar{d} : M_n(\mathcal{A}) \to M_n(\mathcal{N})$ given by $\bar{d}(\mathbf{a}) = \mathbf{n}$, where $n_{i,j} = d(a_{i,j})$. We can show that if $\delta : M_n(\mathcal{A}) \to M_n(\mathcal{N})$ is a derivation, then there is an inner derivation $I_{\mathbf{g}} : M_n(\mathcal{A}) \to M_n(\mathcal{N})$ and a derivation $d : \mathcal{A} \to \mathcal{N}$ such that $\delta = \bar{d} + I_{\mathbf{g}}$. So by Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let $\mathcal{A}$ be an algebra and $\mathcal{N}$ be a unital $\mathcal{A}$-bimodule. Let $D : M_n(\mathcal{A}) \to M_n(\mathcal{N})$ be a linear mapping. If D satisfies $(*)$, then there is an inner derivation $I_{\mathbf{g}} : M_n(\mathcal{A}) \to M_n(\mathcal{N})$ and a derivation $d : \mathcal{A} \to \mathcal{N}$ such that $D(\mathbf{a}) = \bar{d}(\mathbf{a}) + I_{\mathbf{g}}(\mathbf{a}) + \mathbf{a}D(\mathit{1})$ and $\mathbf{a}D(\mathit{1}) = D(\mathit{1})\mathbf{a}$ for each $\mathbf{a} \in M_n(\mathcal{A})$.*

2.2. **C^*-algebras.** In this section we consider continuous linear maps on C^*-algebras satisfying $(*)$. The results are from [1].

The main idea of the proof of main result consists in considering continuous bilinear maps $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$ satisfying

$$a, b \in \mathcal{A}, \quad ab = ba = 0 \Rightarrow \phi(a, b) = 0. \qquad (**).$$

Theorem 2.3. *Let $\mathcal{A}$ be a C^*-algebra, let $\mathcal{X}$ be a Banach space and let $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$ be a continuous bilinear map satisfying $(**)$. Then*

$$\phi(ab, cd) - \phi(a, bcd) + \phi(da, bc) + \phi(dab, c) = 0, \quad a, b, c \in \mathcal{A},$$

and there exist continuous linear maps $\Phi, \Psi : \mathcal{A} \to \mathcal{X}$ such that

$$\phi(ab, c) - \phi(b, ca) + \phi(bc, a) = \Phi(abc), \quad a, b, c \in \mathcal{A},$$

and

$$\phi(a, b) + \phi(b, a) = \Psi(ab + ba), \quad a, b \in \mathcal{A}.$$

Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{M}$ be an essential Banach A-bimodule (essential i.e. $\overline{\mathcal{A}\mathcal{M}} = \mathcal{M}$) and $D : \mathcal{A} \to \mathcal{M}$ be a continuous linear map satisfying $(*)$. Define a continuous bilinear map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ by $\phi(a, b) = D(a)b + aD(b) + D(b)a + bD(a)$. Then ϕ satisfies $(**)$ and hence by applying Theorem 2.3 and a result from [4] which showed that every continuous Jordan derivation from a C^*-algebra into any Banach bimodule is a derivation, the main result of this section will be proved.

Theorem 2.4. *Let $\mathcal{A}$ be a C^*-algebra, let $\mathcal{M}$ be an essential Banach A-bimodule and let $D : \mathcal{A} \to \mathcal{M}$ be a continuous linear map satisfying*

$$a, b \in \mathcal{A}, \quad ab = ba = 0 \Rightarrow D(a)b + aD(b) + D(b)a + bD(a) = 0.$$

Then there exist a derivation $\delta : \mathcal{A} \to \mathcal{M}$ and a bimodule homomorphism $\varphi : \mathcal{A} \to \mathcal{M}$ such that $D = \delta + \varphi$.

Note that a bimodule homomorphism $\varphi : \mathcal{A} \to \mathcal{M}$ is a linear map such that $\varphi(ab) = a\varphi(b) = \varphi(a)b$ for all $a, b \in \mathcal{A}$.

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NUMERICAL SOLUTION OF NONLINEAR B-EQUATION VIA RADIAL BASIS FUNCTIONS

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Abstract. Radial basis functions (RBFs), have important role in solving partial differential equations as a mesh free method. Interpolation with RBFs, is very suitable because they depend only on distance of distributed points. In this paper we solve nonlinear modified Camassa-Holm and Degasperis-Procesi equations numerically using radial basis functions.

1. Introduction

Interpolation of a given set of points is an important problem specially in higher dimensional domains. Radial basis functions are very efficient instruments for interpolating a scattered set of points which have been used in last 20 years. Meshless methods are a class of numerical methods for solving partial differential equations. In these methods mesh generation on the spatial domain of the problem is not needed. For more information see [1]. In this paper, we solve nonlinear b-equations [2], using radial basis functions (RBFs) method, numerically. Some well-known radial basis functions (RBFs) are listed in Table 1. Which r is the Euclidean distant between $x^* \in \mathbb{R}^d$ and any $x \in \mathbb{R}^d$ i.e. $\|x - x^*\|_2$.

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TABLE 1. Some well-known functions that generate RBFs.

Gaussian (GA)	$\phi(r) = \exp(-r^2/c)$
Hardy Multiquadric (MQ)	$\phi(r) = \sqrt{1+(rc)^2}$

2. Main Results

In this section we use radial basis functions method for numerical solution of b-equation via collocation technique. Consider nonlinear b-equation as

$$u_t - u_{xxt} + (b+1)u^2u_x = bu_xu_{xx} + uu_{xxx},\ x \in [a_1, b_1], t \in (0, T], \quad (2.1)$$

with the following initial and boundary conditions:

$$u(x, 0) = f(x),\ x \in [a_1, b_1], \quad (2.2)$$

$$u(a_1, t) = g_1(t),\ u(b_1, t) = g_2(t),\ t \in (0, T]. \quad (2.3)$$

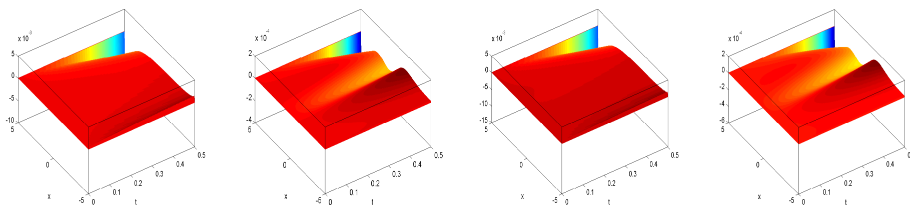
where $[a_1, b_1] \subset \mathbb{R}$. We solve the problem (2.1)-(2.3) in $[a_1, b_1]$ using the collocation technique. Let $\widetilde{u}^n(x) = \sum_{i=1}^{N} \lambda_i^n \phi_i(x),\ n = 1, 2, ...,$ be interpolant of solution of (2.1) in time step n, witch $\lambda_i, i = 1, ..., N$ are unknown. Derivatives of the interpolant $\widetilde{u}^n(x)$ in term of x, may be calculated in a straightforward manner. After discretization of (2.1)-(2.3) in term of variable t and applying collocation technique, we have matrix form of system as follows:

$$Az^{n+1} = B, \quad (2.4)$$

witch z is unknown, A is a $N \times N$ matrix and B is known acording to the initial and boundary conditions.

Example 2.1. Considering $b = 2$, in (2.1), we have mCH equation. Let $a_1 = -5, b_1 = 5, T = 0.5$ in (2.1) and $f(x) = -2sech^2(x/2), g_1(t) = -2sech^2(-5/2-t), g_2(t) = -2sech^2(5/2-t)$ in (2.2)-(2.3). In Figure 1, $(a.1)$ and $(a.2)$, which is drawn with matlab, the error function $u - \tilde{u}$ is plotted using GA-RBF with $c = 0.03$ and the set of collocation points $x_i = -5 + hi, i = 1, ..., N$ and $\Delta t = 0.0001$ for $0 < t \leq 0.5$ for $N = 101$ and $N = 201$. In Table 2 and 3, some value of the shape parameter c, the condition number of matrix A, the RMS error and the max error are listed for GA-RBF and MQ-RBF and equidistant collocation points where N is the number of collocation points which are placed in $-5 \leq x \leq 5$ and $0 \leq t \leq 0.5$ for $\Delta x = 0.1$ and $\Delta t = 0.0001$. Form the contents of the Table 2 and 2, it is clear that the choice of the shape parameter has an important role in approximation of solution precision of equation. In Table 4, some approximation of the solution

FIGURE 1. Plot of error function $u-\tilde{u}$ for set of collocation points $N = 101$ for $(a.1),(b.1)$ and $N = 201$ for $(a.2),(b.2)$ using GA-RBF with $c = 0.03$ in Example 1 , 2, (plotting with matlab program).



$(a.1)$ $(a.2)$ $(b.1)$ $(b.2)$

of mCH equation are computed at $t = 0.5$ with $x_0 = -4.95$, $\Delta x = 0.1$, $\Delta t = 0.0001$, $-5 \leq x \leq 5$ and $N = 101$ using the MQ-RBFs and GA-RBFs with $c = 0.9$ and $c = 0.07$, respectively.

Example 2.2. Considering $b = 3$, in (2.1), we have mDP equation. Let $a_1 = -5, b_1 = 5, T = 0.5$ in (2.1) and $f(x) = -(15/8)sech^2(x/2), g_1(t) = -(15/8)sech^2(-5/2 - (5/4)t), g_2(t) = -(15/8)sech^2(5/2 - (5/4)t)$ in (2.2)-(2.3). In Figure 1, $(b.1)$ and $(b.2)$, the error function $u - \tilde{u}$ is plotted using GA-RBF with $c = 0.03$ and the set of collocation points $x_i = -5 + hi, i = 1, ..., N$ and $\Delta t = 0.0001$ for $0 < t \leq 0.5$ for $N = 101$ and $N = 201$. In Table 5 and 6, some value of the shape parameter c, the condition number of matrix A, the RMS error and the max error are listed for GA-RBF and MQ-RBF and equidistant collocation points where N is the number of collocation points which are placed in $-5 \leq x \leq 5$ and $0 \leq t \leq 0.5$ for $\Delta x = 0.1$ and $\Delta t = 0.0001$. In Table 7, some approximation of the solution of mDP equation are computed at $t = 0.5$ with $x_0 = -4.95$, $\Delta x = 0.1$, $\Delta t = 0.0001$, $-5 \leq x \leq 5$ and $N = 101$, using the MQ-RBFs and GA-RBFs with $c = 0.9$ and $c = 0.07$, respectively.

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TABLE 2. Some values of shape parameter c, E^2, E^∞.

shape parameter c	E^2	E^∞	condition number
0.08	1.70e-03	7.49e-04	3.27e+06
0.09	1.10e-03	4.60e-04	3.34e+07
0.1	7.49e-04	2.84e-04	3.44e+08

TABLE 3. Some values of shape parameter c, E^2, E^∞.

shape parameter c	E^2	E^∞	condition number
0.9	3.99e-04	9.09e-05	1.64e+15
1	3.99e-04	9.09e-05	5.28e+13
2	4.16e-04	9.07e-05	1.10e+07

TABLE 4. tsome approximation mCH equation solution in $-5 \le x \le 5$.

	GA-RBFs	MQ-RBFs
x	$\lvert u-\widetilde{u}\rvert$	$\lvert u-\widetilde{u}\rvert$
-4.95	1.00e-04	3.80e-07
-4.85	1.00e-04	1.07e-06
-3.65	4.46e-05	1.18e-05
-0.15	1.26e-05	1.64e-05
$+2.55$	1.00e-04	6.82e-06
$+4.95$	1.10e-03	6.10e-07

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TABLE 5. Some values of shape parameter c, E^2, E^∞.

shape parameter c	E^2	E^∞	condition number
0.07	3.60e-03	1.60e-03	3.23e+05
0.1	1.10e-03	3.76e-04	3.44e+08

TABLE 6. Some values of shape parameter c, E^2, E^∞.

shape parameter c	E^2	E^∞	condition number
1	6.13e-04	1.32e-04	5.27e+13
2	6.26e-04	1.31e-05	1.11e+07

TABLE 7. some approximation mDP equation solution in $-5 \le x \le 5$.

	GA-RBFs	MQ-RBFs
x	$\lvert u-\widetilde{u}\rvert$	$\lvert u-\widetilde{u}\rvert$
-4.95	1.00e-04	5.00e-07
-4.85	1.00e-04	1.40e-06
-0.05	1.00e-05	2.50e-06
$+0.05$	1.00e-05	1.22e-05
$+0.15$	1.00e-05	2.72e-05
$+3.75$	5.00e-04	9.80e-06
$+4.95$	1.40e-03	7.00e-07

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LINEAR FUNCTIONS PRESERVING SR-MAJORIZATION (RSR-MAJORIZATION) ON $\mathbb{R}^2$ ($\mathbb{R}_2$)

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ABSTRACT. For $x, y \in \mathbb{R}^n$ ($x, y \in \mathbb{R}_n$), we say that x sr-majorized (rsr-majorized) by y (write as $x \prec_{sr} y$ ($x \prec_{rsr} y$)) if for some symmetric row stochastic matrix R with all its main diagonal entries equal, $x = Ry$ ($x = yR$). In this paper, the structure of all linear functions $T : \mathbb{R}^2 \to \mathbb{R}^2$ ($T : \mathbb{R}_2 \to \mathbb{R}_2$), preserving (or strongly preserving) $\prec_{sr}$ ($\prec_{rsr}$) are characterized.

1. INTRODUCTION

The theory of majorization is a powerful mathematical tool which naturally arises in various areas of mathematics, statistics, the quantum theory, and etc. In recent years, this concept has been attended specially and many papers in this topic have been published. For example, one can see [1, 2, 3].

The following notation will be fixed throughout the paper.
The set of all $m \times n$ real matrices is denoted by $\mathbf{M}_{m,n}$. The set of all $n \times 1$ real column vectors is denoted by $\mathbb{R}^n$. The set of all $1 \times n$ real row vectors is denoted by $\mathbb{R}_n$. The collection of all $n \times n$ symmetric row

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stochastic matrices with all its main diagonal entries equal is denoted by $\mathcal{R}_n^{sr}$. The standard basis of $\mathbb{R}^n$ is denoted by $\{e_1, \ldots, e_n\}$. The standard basis of $\mathbb{R}_n$ is denoted by $\{\varepsilon_1, \ldots, \varepsilon_n\}$. The summation of all components of a vector x is denoted by $tr(x)$. The set $\{1, \ldots, k\} \subset \mathbb{N}$ is denoted by $\mathbb{N}_k$. The transpose of a given matrix A is denoted by A^t. The matrix representation of a linear function $T : \mathbb{R}^n \to \mathbb{R}^n$ ($T : \mathbb{R}_n \to \mathbb{R}_n$) with respect to the standard basis is denoted by $[T]$. The set $\{\sum_{i=1}^m \lambda_i a_i \mid m \in \mathbb{N},\ \lambda_i \geq 0,\ \sum_{i=1}^m \lambda_i = 1,\ a_i \in A,\ \ i \in \mathbb{N}_m\}$, where $A \subseteq \mathbb{R}^n$ ($\mathbb{R}_n$), is denoted by Co(A).
Let $\sim$ be a relation on $\mathbf{M}_{m,n}$. A linear function $T : \mathbf{M}_{m,n} \longrightarrow \mathbf{M}_{m,n}$ is said to be a linear preserver (or strong linear preserver) of $\sim$ if $x \sim y$ implies that $T(x) \sim T(y)$ (or $x \sim y$ if and only if $T(x) \sim T(y)$).

1.1. **Row stochastic and sr-majorization (rsr-majorization).** We introduce the relation $\prec_{sr}$ ($\prec_{rsr}$) on $\mathbb{R}^n$ ($\mathbb{R}_n$) and we study some properties of sr-majorization (rsr-majorization) on $\mathbb{R}^2$ ($\mathbb{R}_2$).

Definition 1.1. A matrix R with nonnegative entries is called row stochastic if the sum of entries of each row of R is equal to one.

Now we pay attention to the symmertic row stochastic matrices and introduce a new type of majorization on $\mathbb{R}^n$ ($\mathbb{R}_n$).

Definition 1.2. For $x, y \in \mathbb{R}^n$ ($x, y \in \mathbb{R}_n$), it is said that x is sr-majorized (rsr-majorized) by y, and write as $x \prec_{sr} y$ ($x \prec_{rsr} y$), if there exists $R \in \mathcal{R}_n^{sr}$ such that $x = Ry$ ($x = yR$).

The following proposition gives an equivalent condition for $\prec_{sr}$ ($\prec_{rsr}$) on $\mathbb{R}^2$ ($\mathbb{R}_2$).

Proposition 1.3. *Let $x = (x_1, x_2)^t$, $y = (y_1, y_2)^t \in \mathbb{R}^2$ ($x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}_2$). Then $x \prec_{sr} y$ ($x \prec_{rsr} y$) if and only if $x_i \in Co\{y_1, y_2\}$ ($i \in \mathbb{N}_2$) and $tr(x) = tr(y)$.*

Proof. First, suppose that $x \prec_{sr} y$. Then there exists $R \in \mathcal{R}_2^{sr}$ such that $x = Ry$. We see that $R = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$, for some a, $b \geq 0$ such that $a + b = 1$. It is seen that $x_i \in Co\{y_1, y_2\}$ ($i \in \mathbb{N}_2$) and $tr(x) = tr(y)$.

Next, assume that $x_i \in Co\{y_1, y_2\}$ ($i \in \mathbb{N}_2$) and $tr(x) = tr(y)$. So $x_1 = \alpha y_1 + (1-\alpha)y_2$ and $x_2 = \beta y_1 + (1-\beta)y_2$, for some $0 \leq \alpha, \beta \leq 1$. Since $tr(x) = tr(y)$, we deduce $(1-\alpha-\beta)(y_1 - y_2) = 0$. If $y_1 \neq y_2$; Then $\alpha + \beta = 1$, and put $R = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$. It is clear that $R \in \mathcal{R}_2^{sr}$ and $x = Ry$. Therefore, $x \prec_{sr} y$. If $y_1 = y_2$; Put $R = I_2$ and see $x = Ry$. Hence $x \prec_{sr} y$.
In a similar fashion as in the proof of $\prec_{sr}$, prove the statement for $\prec_{rsr}$. $\square$

Some properties of $\prec_{sr}$ on $\mathbb{R}^2$ are stated in the following proposition. These statements hold about $\prec_{rsr}$ too.

Proposition 1.4. *Let $x = (x_1, x_2)^t$, $y = (y_1, y_2)^t \in \mathbb{R}^2$. Then*

(a) $x \prec_{sr} y \nRightarrow y \prec_{sr} x$.
(b) $x \prec_{sr} y$ *and* $y \prec_{sr} x \nRightarrow x = y$.
(c) $x \prec_{sr} y$ *and* $y \prec_{sr} z \Rightarrow x \prec_{sr} z$.

Proof. Proof, which is easy, is omitted for the sake of brevity. □

2. Main results

In this section, we will characterize all linear functions that preserve (or strongly preserve) sr-majorization (rsr-majorization) on $\mathbb{R}^2$ ($\mathbb{R}_2$). The following theorem characterizes all the linear preservers of $\prec_{sr}$ on $\mathbb{R}^2$.

Theorem 2.1. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function, and $[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then T preserves $\prec_{sr}$ if and only if*

(a) *if $(a-c)(b-d) \geq 0$, then $a = b$ and $c = d$. That is, $[T] = \begin{pmatrix} a & a \\ c & c \end{pmatrix}$.*
(b) *if $(a-c)(b-d) \leq 0$, then $a = d$ and $b = c$. That is, $[T] = \begin{pmatrix} a & c \\ c & a \end{pmatrix}$.*

Proof. First, assume that T preserves $\prec_{sr}$. As $e_1 \prec_{sr} e_2$ and $e_2 \prec_{sr} e_1$, so $Te_1 \prec_{sr} Te_2$ and $Te_2 \prec_{sr} Te_1$, and hence $a, c \in Co\{b, d\}$ and $b, d \in Co\{a, c\}$. First, consider $(a-c)(b-d) \geq 0$. This means ($a \geq c$ and $b \geq d$) or ($a \leq c$ and $b \leq d$). Without loss of generality assume that $a \geq c$ and $b \geq d$. Now, since $a, c \in Co\{b, d\}$, we conclude that $d \leq c \leq a \leq b$, and as $b, d \in Co\{a, c\}$, we deduce $c \leq d \leq b \leq a$. Hence $a = b$ and $c = d$. Now, if $(a-c)(b-d) \leq 0$, by a similar process, we observe that $a = d$ and $c = b$.

Next, suppose that (a) or (b) holds. Let $x = (x_1, x_2)^t$, $y = (y_1, y_2)^t \in \mathbb{R}^2$, and let $x \prec_{sr} y$. If (a) holds, then $Tx = tr(x)(a, c)^t$ and $Ty = tr(y)(a, c)^t$. As $tr(x) = tr(y)$, we see $Tx \prec_{sr} Ty$. If (b) holds, then $Tx = (ax_1 + cx_2, cx_1 + ax_2)^t$ and $Ty = (ay_1 + cy_2, cy_1 + ay_2)^t$. Since $x_i \in Co\{y_1, y_2\}$ $(i \in \mathbb{N}_2)$ and $tr(x) = tr(y)$, so $x_1 = \alpha y_1 + (1-\alpha)y_2$ and $x_2 = \beta y_1 + (1-\beta)y_2$, for some $0 \leq \alpha, \beta \leq 1$. Since $tr(x) = tr(y)$, we have $(1 - \alpha - \beta)(y_1 - y_2) = 0$. If $y_1 \neq y_2$, then $\alpha + \beta = 1$. If $y_1 = y_2$, then $x_i = y_i$ $(i \in \mathbb{N}_2)$, and so $\alpha = 1$ and $\beta = 0$. Thus $(Tx)_1 = \alpha(Ty)_1 + (1-\alpha)(Ty)_2$, and hence $(Tx)_1 \in Co\{(Ty)_1, (Ty)_2\}$. Similarly, $(Tx)_2 \in Co\{(Ty)_1, (Ty)_2\}$. Therefore, $Tx \prec_{sr} Ty$. □

The following theorem characterizes all the linear preservers of $\prec_{rsr}$ on $\mathbb{R}_2$.

Theorem 2.2. *Let $T : \mathbb{R}_2 \to \mathbb{R}_2$ be a linear function, and $[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then T preserves $\prec_{rsr}$ if and only if*

(a) *if $(a-b)(c-d) \geq 0$, then $a = c$ and $b = d$. That is, $[T] = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$.*

(b) *if $(a-b)(c-d) \leq 0$, then $a = d$ and $b = c$. That is, $[T] = \begin{pmatrix} a & c \\ c & a \end{pmatrix}$.*

Proof. In a similar fashion as in the proof of Theorem 2.1, we can prove it. □

We need the following lemma to prove the last result of this section.

Lemma 2.3. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ ($T : \mathbb{R}_2 \to \mathbb{R}_2$) be a linear function. If T strongly preserves $\prec_{sr}$ ($\prec_{rsr}$), then T is invertible.*

Proof. Let $x \in \mathbb{R}^2$ ($\mathbb{R}_2$) and let $Tx = 0$. Since $Tx = T0$ and T strongly preserves $\prec_{sr}$ ($\prec_{rsr}$), it shows that $x \prec_{sr} 0$ ($x \prec_{rsr} 0$). So $x = 0$ and hence T is invertible. □

The following theorem characterizes all the linear functions $T : \mathbb{R}^2 \to \mathbb{R}^2$ ($T : \mathbb{R}_2 \to \mathbb{R}_2$) which strongly preserves $\prec_{sr}$ ($\prec_{rsr}$).

Theorem 2.4. *A linear function $T : \mathbb{R}^2 \to \mathbb{R}^2$ ($T : \mathbb{R}_2 \to \mathbb{R}_2$) strongly preserves $\prec_{sr}$ ($\prec_{rsr}$) if and only if $[T] = \begin{pmatrix} a & c \\ c & a \end{pmatrix}$, where $a^2 - c^2 \neq 0$.*

Proof. We prove this statement for $\prec_{sr}$. One can prove it for $\prec_{rsr}$ in a similar fashion. First, suppose that T strongly preserves $\prec_{sr}$. So T is invertible and T preserves $\prec_{sr}$, and hence by Theorem 2.1, $[T] = \begin{pmatrix} a & c \\ c & a \end{pmatrix}$, where $a^2 - c^2 \neq 0$.

For the converse, it is enough to prove that $x \prec_{sr} y$ whenever $T(x) \prec_{sr} T(y)$. Let $x,y \in \mathbb{R}^2$ such that $T(x) \prec_{sr} T(y)$. Thus there exists $R \in \mathcal{R}_2^{sr}$ such that $T(x) = RT(y)$. Hence $x = (T^{-1}RT)y$. Since $T^{-1}RT = R$, we have $x = Ry$. Therefore, $x \prec_{sr} y$. □

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CONTINUOUS REPRESENTABILITY OF INTERVAL ORDERS

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ABSTRACT. In this article, we study the continuous representability of interval orders. For this mean, we introduce the interval order and consider its representation.

1. Introduction

If we introduce a total preorder on a topological space $(X, \mathcal{T})$, we will deal with the concepts such CRP (or *useful topology*) and SRP (or *completely useful topology*)[1]. Now we introduce the notion of an interval order, *completely useful topology in the sense of an interval order* (*i.o. completely useful topology* or $SRP-I.O$) and *useful topology in the sense of an interval order (i.o. useful topology* or $CRP - I.O$). Gianni Bosi and Magali E. Zuanon in [2] studied upper semicontinuous representability of interval order and showed that every *i.o. completely useful topology* $\mathcal{T}$ on a set X is *completely useful* also they proved that a second countable topology $\mathcal{T}$ on a set X is *i.o. completely useful* then they conclude that for a metrizable topology, *i.o. completely usefulness* and separability are equivalent. In this paper, we study continuous representability of interval order and show these theorems for *i.o. useful topology*.

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* Speaker.

Definition 1.1. An asymmetric binary relation $\prec$ on a set X is called an **interval order** if the following condition obtains:

$$IO \quad (y \prec x \quad and \quad y' \prec x') \Rightarrow (y' \prec x \quad or \quad y \prec x').$$

Definition 1.2. If $\prec$ is an asymmetric binary relation on X and there exist mappings $u, v : X \to \mathbb{R}$ such that $y \prec x$ if and only if $v(y) < u(x)$, then we say that $\prec$ is represented by the function pair (u, v), and that (u, v) is a representation of $\prec$; if X is a topological space and the functions u, v are continuous, we call (u, v) a continuous representation of $\prec$. If $\prec$ is represented by the function pair (u, v), then $\prec$ is an interval order and $u \leq v$.

Definition 1.3. An interval order $\prec$ on a topological space $(X, \mathcal{T})$ is said to be continuous (respectively, upper semicontinuous) if the sets $G_{\prec}(x) = \{a \in X : x \prec a\}$ and $L_{\prec}(x) = \{a \in X : a \prec x\}$ (respectively, $L_{\prec}(x) = \{a \in X : a \prec x\}$) are open subset in X for every $x \in X$.

In similar way, the continuity of a total preorder is defined.

Definition 1.4. A topology $\mathcal{T}$ on X satisfies the *continuous representability property for interval orders* $(CRP - I.O)$ if for every continuous interval order $\prec$ defined on X, there exists a pair of continuous functions $u, v : X \to \mathbb{R}$ (where X is endowed with the topology $\mathcal{T}$ and $\mathbb{R}$ with the usual topology), such that $x \prec y \Leftrightarrow v(x) < u(y)\, (x, y \in X)$. Also, $\mathcal{T}$ satisfies the *semicontinuous representability property for interval orders* $(SRP - I.O)$ if for every semicontinuous interval order $\prec$ defined on X such that all the sets $L(x) = \{a \in X : a \prec x\}\, (x \in X)$ are $\mathcal{T}$-open, there exists a pair functions $u, v : X \to \mathbb{R}$ such that $x \prec y \Leftrightarrow v(x) < u(y)\, (x, y \in X)$ and v is upper semicontinuous.

Definition 1.5. If $(X, \precsim)$ is a preordered set then a real-valued function $u : X \to \mathbb{R}$ is said to be order-monomorphism if for every $x, y \in X$, $[x \precsim y \Leftrightarrow u(x) \leq u(y)]$.

Definition 1.6. The topology $\mathcal{T}$ on X is said to have the continuous representability property (CRP) (respectively, semicontinuous representability property (SRP)) if every continuous total preorder (respectively, semicontinuous total preorder) $\precsim$ defined on X admits a representation by means of a continuous (respectively, semicontinuous) order-monomorphism.

Definition 1.7. Let $\prec$ be an interval order on a set X, and define corresponding relations $\prec^*$ and $\prec^{**}$ by

$$y \prec^* x \text{ if and only if } \exists z \in X\, (y \prec z \precsim x),$$
$$y \prec^{**} x \text{ if and only if } \exists z \in X\, (y \precsim z \prec x).$$

We also define $\precsim^*$, $\precsim^{**}$ to be the negations of $\prec^*$, $\prec^{**}$ respectively; so
$$y \precsim^* x \text{ if and only if } \forall z \in X\,(z \precsim y \Rightarrow z \precsim x),$$
$$y \precsim^{**} x \text{ if and only if } \forall z \in X\,(x \precsim z \Rightarrow y \precsim z).$$

2. Main Results

Definition 2.1. An interval order $\prec$ on a set X is said to be **i.o. separable** if there exists a countable subset $D \subseteq X$ such that for all $x, y \in X$ with $x \prec y$ there exists $d \in D$ such that $x \prec d \precsim^{**} y$.

Proposition 2.2. *Let $\prec$ be an asymetric binary relation defined on a nonempty set X. Then the following statements are equivalent:*[5]

a) *$\prec$ is an interval order;*
b) *$\precsim^*$ is a total preorder;*
c) *$\precsim^{**}$ is a total preorder.*

In addition, $\precsim$ is transitive if and only if $\precsim$, $\precsim^*$ and $\precsim^{**}$ coincide.

Theorem 2.3. *Let X be a nonempty set endowed with an interval order $\prec$. Then the following statements are equivalent:*[5]

a) *$\prec$ is i.o. separable;*
b) *$\prec$ is representable;*
c) *$\prec$ is representable by a pair of real-valued functions (u, v) such that u is a representation of the total preorder $\precsim^{**}$ and v is a representation of the total preorder $\precsim^*$.*

We say that the preorder $\precsim$ on X is order separable in the sense of Debreu if there exists a countable subset $Z \subseteq X$ such that if $x \prec y$, then there exists $z \in Z$ with $x \precsim z \precsim y$. In [3] has been proved that CRP is equivalence with the order separability in the sense of Debreu.

Lemma 2.4. *Every i.o. useful topology $\mathcal{T}$ on a set X is useful.*

Proof. Let $\mathcal{T}$ be an i.o. useful topology on a set X and consider an continuous total preorder $\precsim$ on $(X, \mathcal{T})$. Then by 2.2, $\precsim$, $\precsim^*$ and $\precsim^{**}$ coincide hence $\prec$ is an continuous interval order which is representable by means of a pair (u, v) of real-valued functions on X. In this case $\prec$ is representable and according to 2.3 is i.o. separable and therefore there exists a countable subset $D \subseteq X$ such that for all $x, y \in X$ with $x \prec y$ there exists $d \in D$ such that $x \prec d \precsim y$. Hence, $\precsim$ is order separable in the sense of Debreu and there exists an continuous utility function on the totally preordered topological space $(X, \mathcal{T})$. $\square$

A gap of a subset S of the real line $\mathbb{R}$ is a maximal nondegenerate interval which is disjoint from S and has a lower bound and an upper bound in S.

Lemma 2.5. *If S is a subset of the extended real line, then there exists a strictly increasing function $g : \bar{\mathbb{R}} \to \mathbb{R}$ such that all the Debreu gaps of $g(S)$ are open.*[3]

Lemma 2.6. *A second countable topology $\mathcal{T}$ on a set X is i.o. useful.*

Proof. Let $\mathcal{T}$ be a second countable topology on a set X and consider an continuous interval order $\prec$ on X. Then $\prec$ according to 2.3 is representable by a pair of real-valued functions $(u^{'}, v^{'})$ on X where $u^{'}$ is a representation for the total preorder $\precsim^{**}$ and $v^{'}$ is a representation for the total preorder $\precsim^{*}$ according to 2.5 there exists a strictly increasing function $h : \mathbb{R} \to \mathbb{R}$ such that every gap of $(hov^{'})(X)$ is open. Observe that the total preorder $\precsim^{*}$, $\precsim^{**}$ is continuous since the interval order $\prec$ is continuous and for every $x \in X$ we have that $L_{\prec^{*}}(x) = \bigcup_{\{\xi \in X, \xi \precsim x\}} L_{\prec}(\xi)$, $G_{\prec^{*}}(x) = \bigcup_{\{\xi \in X, x \prec \xi\}} G_{\prec}(\xi)$. Hence, if we let $U = hou^{'}$ and $V = hov^{'}$ it is immediate that the pair (U, V) of real-valued functions on X represents the interval order $\prec$ because $x \prec y \Leftrightarrow v^{'}(x) < u^{'}(y) \Leftrightarrow hov^{'}(x) < hou^{'}(y) \Leftrightarrow V(x) < U(y)$. □

Proposition 2.7. *Let $\mathcal{T}$ be a metrizable topology on a set X. Then the following conditions are equivalent:*
(a) $\mathcal{T}$ is i.o. useful; *(b) $\mathcal{T}$ is separable.*

Proof. (a)⇒(b) If $\mathcal{T}$ is an i.o. useful and metrizable topology on a set X then from 2.4 we have that $\mathcal{T}$ is completely useful and therefore it is separable [4]. (b)⇒(a) If $\mathcal{T}$ is a metrizable and separable topology on a set X then $\mathcal{T}$ is second countable since separability and second countability in metric spaces are equivalent, therefore it is i.o. completely useful by 2.6. This consideration completes the proof. □

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BANACH SPACES AND CONTINUOUS REPRESENTABILITY PROPERTY

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Abstract. In this article we define the Banach space endowed with weak topology as a topological space that satisfies continuous representability property (CRP).

1. Introduction

Let $(X, \mathcal{T})$ be a topological space. We define a total preorder $\precsim$ (i. e., a reflexive, transitive and complete binary relation) on X. We denote $x \prec y$ instead of $-(y \precsim x)$. In this case, we deal with the Continuous Representability Property (CRP) concept. The considering of this concept and recognition of the topological spaces that satisfy CRP, is important. In here, we define the Banach space endowed with weak topology as a space that satisfies continuous representability property (CRP).

Definition 1.1. If $(X, \precsim)$ is a preordered set and $\mathcal{T}$ is a topology on X, then the preorder $\precsim$ is said to be $\mathcal{T}$-continuous on X if for each $x \in X$ the sets $\{a \in X : x \precsim a\}$ and $\{b \in X : b \precsim x\}$ are $\mathcal{T}$-closed in X.

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* Speaker.

Definition 1.2. If $(X, \precsim)$ is a preordered set then a real-valued function $u : X \to \mathbb{R}$ is said to be order-monomorphism if for every $x, y \in X$, $[x \precsim y \Leftrightarrow u(x) \leq u(y)]$.

Definition 1.3. If $(X, \mathcal{T})$ be a topological space, then the total preorder $\precsim$ on X is said to be continuously representable if there exists an order-monomorphism that is continuous with respect to the topology $\mathcal{T}$ on X and the usual topology on the real line $\mathbb{R}$.

Definition 1.4. Given a nonempty set X endowed with a topology $\mathcal{T}$. The topology $\mathcal{T}$ on X is said to have the continuous representability property (CRP) if every continuous total preorder $\precsim$ defined on X admits a representation by means of a continuous order-monomorphism.

Definition 1.5. A topological space $(X, \mathcal{T})$ is said to be **separably connected** if for every $a, b \in X$ there exists a connected and separable subset $C(a, b) \subseteq X$ such that $a, b \in C(a, b)$.

Remark 1.6.

a) A separably connected set X is, in particular, connected because once fixed an element $x_0 \in X$ we have that $X = \bigcup_{x \in X} C_{\{x, x_0\}}$.
b) There are topological spaces that are connected but not separably connected: An example is the lexicographic square $[0, 1] \times [0, 1]$, endowed with the order topology that comes from the lexicographic ordering on $\mathbb{R}^2$.
c) It is obvious that a path-connected topological space is, in particular, separably connected, because every path is clearly a separable subset. The converse, however, does not hold: Consider the following example: $X = \{0\} \times [-1, 1] \cup \{(x, \sin(\dfrac{1}{x})) : x \in [0, 1]\}$, endowed with the Euclidean topology as a subset of the plane $\mathbb{R}^2$. It is well known that this set X is connected, but not path-connected. Moreover it is separable because $\mathbb{R}^2$ is metric and separable, and the separability is hereditary on metric spaces. Therefore, X is separably connected.

Definition 1.7. If X be a set of points and $\mathcal{F}$ is a collection of real-valued functions on X, there is always a weakest topology on X such that every function in $\mathcal{F}$ is continuous. This topology is called the weak topology generated (or induced) by $\mathcal{F}$.

2. Main results

Definition 2.1. Let X be a nonempty set endowed with a topology $\mathcal{T}$. The topological space $(X, \mathcal{T})$ is said to satisfy the **coauntble chain**

condition (*ccc*) if every family of pairwise disjoint $\mathcal{T}$-open subsets is countable.

The next Lemma prove that if a topological space satisfies (*ccc*) then every subsets of its also will satisfy (*ccc*).

Lemma 2.2. *Let A be a dense subset of a topological space B. If $\{U_\gamma : \gamma \in \Gamma\}$ is a disjoint collection of open sets in A, then there is a disjoint collection of open sets $\{V_\gamma : \gamma \in \Gamma\}$ in B such that $U_\gamma \subset V_\gamma$ for each $\gamma \in \Gamma$.*[3]

Theorem 2.3. *In the Cartesian product separable spaces any family of pairwise disjoint nonempty open sets is countable.*[2]

The following Lemma is an consequence of 2.2 and 2.3.

Lemma 2.4. *Let X be a Banach space endowed with its weak topology $\mathcal{W}$. Then $(X, \mathcal{W})$ satisfies the countable chain condition (ccc).*[1]

Lemma 2.5. *Let $(X, \mathcal{W})$ be a separably connected topological space that satisfies the countable chain condition ccc. Then it also satisfies the continuous representability property CRP.*[1]

Theorem 2.6. *Let X be a Banach space endowed with its weak topology $\mathcal{W}$. Then $(X, \mathcal{W})$ satisfies the continuous representability property CRP.*[1]

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ON THE COLLECTION WPFN OF MATRICES

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Abstract. WPFn is the set of all real $n \times n$ matrices for which the spectral radius is an eigenvalue and the corresponding eigenvector is nonnegative. Here we characterize WPFn for which the spectral radius is simple and strictly dominant. Also, by this characterization, we prove that $A \in WPFn$ if and only if A is eventually nonnegative, when spectral radius is simple and strictly dominant.

1. Introduction

Consider a matrix $A \in \mathbb{R}^{n \times n}$ that satisfies the property $Av = \rho(A)v$ where $v \neq 0$ is a (entrywise) nonnegative vector and $\rho(A)$ is the spectral radius of A, i.e. A has a nonnegative right eigenvector whose corresponding eigenvalue is the spectral radius of A. This property is known as Perron-Frobenius property and the corresponding eigenvector v is called the Perron-Frobenius eigenvector of A. If in addition to this property, the eigenvector v is (entrywise) positive and $\rho(A)$ is simple and strictly dominant, then we say that A has the strong Perron-Frobenius property. For those two properties we refer the reader to [2].

We denote by WPFn, the collection of all matrices $A \in \mathbb{R}^{n \times n}$ such that $\rho(A)$ is an eigenvalue of A and A has a left and a right nonnegative eigenvector. Also, PFn is fixed for the collection of all matrices $A \in$

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* Speaker.

$\mathbb{R}^{n\times n}$ that $\rho(A)$ is positive, simple and strictly dominant eigenvalue of A, in addition, A has a left and right positive eigenvector with respect to $\rho(A)$.

Theorem 1.1. *The following assertions are true*

(i) $A \in WPFn$ *if and only if* A *and* A^T *posses the Perron-Frobenius property.*

(ii) $A \in PFn$ *if and only if* A *and* A^T *posses the strong Perron-Frobenius property.*

2. Main results

The following definitions are from [3]. The first one was for positivity which we define it similarly for nonnegativity in order to characterize WPFn.

Definition 2.1. If S is an invertible matrix such that the first column of S is positive (resp. nonnegative) and the first row of S^{-1} is positive (resp. nonnegative), S is called *initially positive (resp. initially nonnegative).*

Definition 2.2. The $n \times n$ matrix K of the form

$$K = \begin{bmatrix} \lambda & 0 \\ 0 & K' \end{bmatrix}$$

in which $\lambda > \rho(K')$. We call K *dominant direct sum* and show it by DDS.

A characterization of PFn is as follows,

Theorem 2.3. [3] $A \in PFn$ *if and only if* $A = SKS^{-1}$ *in which* S *is initially positive and* K *is DDS.*

Now, we state the following characterization for WPFn.

Theorem 2.4. $A \in WPFn$ *and* $\rho(A)$ *is positive, simple and strictly dominant if and only if* $A = SKS^{-1}$ *in which* S *is initially nonnegative and* K *is DDS.*

Proof. Let $A = SKS^{-1}$ such that $K = \begin{bmatrix} \rho(A) & 0 \\ 0 & K' \end{bmatrix}$ is DDS and S is initially nonnegative. Since $A = SKS^{-1}$, the first column of S is a right eigenvector of A with respect to $\rho(A)$ which is nonnegative. Similarly, since $A^T = (S^{-1})^T K^T S^T$, the first column of $(S^{-1})^T$ is a right eigenvector of A^T with respect to $\rho(A)$ which is also nonnegative. Therefore A and A^T acsses the Perron-Frobenius property. By Theorem1.1, $A \in WPFn$.

Conversely, let $A \in WPFn$ and let $A = PJP^{-1}$ that $P = (P_1 \dots P_n)$ and $J = \begin{bmatrix} \rho(A) & 0 \\ 0 & A' \end{bmatrix}$. Hence $AP = PJ$ and hence $AP_1 = \rho(A)P_1$. So $P_1 = \alpha x$ where x is nonnegative Perron-Frobenius eigenvector of A for some $\alpha \in \mathbb{R}$. Hence we can assume $P_1 \geq 0$. $A^T = (P^{-1})^T J^T P^T$ where $(P^T)^{-1} = (P'_1, \dots, P'_n)$ such that $(P'_1)^T$ is the first row of P^{-1}. Thus $A^T(P^T)^{-1} = (P^T)^{-1}J^T$ and hence $A^T P'_1 = \rho(A)P'_1$. Consider $P'_1 = \beta x$, where $\beta \in \mathbb{R}$. Since $P^{-1}P = I$, $\delta_1(P_1^{-1})T + \dots + \delta_n(P_n^{-1})T = (1, 0, \dots, 0)^T$ and hence $(P'_1)^T P_1 = 1$, $P_1 \geq 0$, $P'_1 = \beta x$. We know that $P'_1 \geq 0$ or $P'_1 \leq 0$. Therefore $P'_1 \geq 0$. Let $S = P$, which is initially nonnegative. $\square$

Definition 2.5. A square matrix A is called eventually nonnegative if there exists integer k_0 such that A^k is (entrywise) nonnegative, for all $k \geq k_0$.

Theorem 2.6. [1]*If the matrix $A \in \mathbb{R}^{n\times n}$ is nonnilpotent eventually nonnegative then $A \in WPFn$.*

The following example show that converse of Theorem2.6 is not true.

Example 2.7. Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

whose eigenvalues are $\lambda_1 = 2, \lambda_2 = 0, \lambda_3 = -2$ and the Perron-Frobenius eigenvector is $v = [1, 1, 2]^T$. $A \in WPFn$ but A is not eventually nonnegative since $traceA^k < 0$, for odd integer numbers k.

Theorem 2.8. *Let $A \in \mathbb{R}^{n\times n}$ and $\rho(A)$ be a positive, simple, strictly dominant eigenvalue. Then the following assertions are equivalent*

(i) $A \in WPFn$

(ii) $A = SKS^{-1}$, *where K is DDS and S is initially nonnegative.*

(iii) *A is eventually nonnegative.*

Proof. By Theorems 2.4 and 2.6, its enough to prove (ii) $\Rightarrow$ (iii). $A = SKS^{-1}$ and

$$K = \begin{bmatrix} \lambda & 0 \\ 0 & K' \end{bmatrix}.$$

By Shur's Theorem without loss of generality we can assume K' to be upper triangular and $\rho(K') < \rho(A)$. $A = S(\rho(A) \oplus K')S^{-1}$, hence

$$\frac{1}{\rho(A)^l}A^l = S(1 \oplus \frac{1}{\rho(A)^l}K')S^{-1}.$$

Therefore $\frac{1}{\rho(A)^l}A^l \to S(1 \oplus 0)S^{-1} \geq 0$, as $l \to \infty$, since S is initially nonnegative. $\square$

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FUZZY LINEAR INDEPENDENCE AND EFFECT OF LINEAR TRANSFORMATIONS ON THE FUZZY LINEAR INDEPENDENCE

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Abstract. In this paper, we investigate Fuzzy linear independence and examine whether basic linear transformations in $\mathbb{R}^2$, such as rotations, scales, reflections, shears preserve fuzzy linear independence. We provide some examples to the contrary.

1. Introduction

we find an attempt at the definition of fuzzy linear independence in [3]. It is a well-known fact in linear algebra, that invertible linear transformations preserve linear independence of vectors (see [2]). One would like to examine whether this is also true for fuzzy linearly independent vectors. As the examples in the next section show, this is actually not the case. But first, some preliminary definitions: Lubczonok in [1] defines a fuzzy vector space to be a pair (V, μ), where V is a vector space and μ is a membership function $\mu : V \to [0, 1]$ that satisfies:

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$$\mu(ax+by) \geq \mu(x) \wedge \mu(y)$$

where $x, y \in V, a, b \in R$.

2. Main results

In [1],We let (V, μ) be a Fuzzy vector space. We say that a finite set of vectors $\{x\}_{i=1}^{N}$ is fuzzy linearly independent in V if and only if $\{x\}_{i=1}^{N}$ is linearly independent in V and for all $\{a_i\}_{i=1}^{N} \subset \mathbb{R}$,

$$\mu(\sum_{i=1}^{N} a_i x_i) = \bigwedge_{i=1}^{N} \mu(a_i x_i)$$

. A set of vectors is fuzzy linearly independent in V if all finite subsets of it are fuzzy linearly independent in$V$. In this section, we present example wish show that a set of vectors is fuzzy linear independent if the above conditions is true, then using one of this examples we show that basic invertible linear transformations (which preserve classical linear independence) do not in general preserve fuzzy linear independence.

Example 2.1. Let $(\mathbb{R}^2, \mu)$, where $\mu : V \to [0,1]$ is defined as follows: $\mu(x,y) = 1$ where $x = 0, y = 0$, $\mu(x,y) = 1$ where $x = 0, y \neq 0$, $\mu(x,y) = 0$ where $x \neq 0, y = 0$, $\mu(x,y) = 0$ where $x \neq 0, y \neq 0$. for all$(x,y) \in \mathbb{R}^2$. Consider the vectors $x = (1,0)$ and $y = (0,1)$ which are clearly linearly independent, as well as fuzzy linearly independent as one can easily check.
$\mu(x+y) = \mu((1,0)+(0,1)) = \mu(1,1) = 0 = 0 \wedge 1 = \mu(x) \wedge \mu(y)$

Example 2.2. Consider $(\mathbb{R}^2, \mu)$ where $\mu[(0,0)] = 1, \mu[(0, \mathbb{R} \backslash \{0\})] = \frac{1}{2}$ and $\mu[\mathbb{R}^2 \backslash (0, \mathbb{R})] = \frac{1}{4}$. It is easily checked that vectors $x = (1,0)$ and $y = (-1,1)$ are linearly independent but are not fuzzy linearly independent in $(\mathbb{R}^2, \mu)$. This example also illustrates a situation where $\mu(x) = \mu(y)$ and $\mu(x+y) > \mu(x)$.

Theorem 2.3. *Let (V, μ) be Fuzzy vector spaces. Then any set of vectors $\{x\}_{i=1}^{N} \subset V \backslash \{0\}$ which has distinct $\mu-$values is linearly and fuzzy linearly independent.*

Proof. We prove the proposition by induction on$\mathbb{N}$. In case $\mathbb{N} = 1$ we have only one vector and clearly the statement is true.

Now suppose that the proposition is true for$\mathbb{N}$. Let $\{x\}_{i=1}^{N+1}$ be a set of vectors in $V \backslash \{0\}$ with distinct μ-values. By inductive hypothesis we have: $\{x\}_{i=1}^{N}$ is fuzzy linearly independent. Suppose that

$\{x\}_{i=1}^{N+1}$ is not linearly independent and thus $x_{N+1} = \sum_{i\in S} a_i x_i$ where $S \subset \{1, ..., N\}, S \neq \varnothing$ and for all $i \in S, a_i \neq 0$. Then

$$\mu(x_N + 1) = \bigwedge_{i\in S} \mu(a_i x_i) = \bigwedge_{i\in S} \mu(x_i)$$

and hence $\mu(x_{N+1}) \in \{\mu(x_i)\}_{i=1}^{N}$. this contradicts the fact that $\{x\}_{i=1}^{N+1}$ has distinct μ-values. Therefore $\{x\}_{i=1}^{N+1}$ is linearly independent.Finally in [2, Propositions 2.3(ii), 2.4 and 2.5] clearly show that $\{x\}_{i=1}^{N+1}$ is fuzzy linearly independent. □

we present examples in [1] which show that basic invertible linear transformations (which preserve classical linear independence) do not in general preserve fuzzy linear independence. This shows that extra conditions are probably required, in addition to linearity, invertibility, etc, in order to preserve fuzzy linear independence. We examine four types of linear transformations: rotations, reflections, shears and scales:

Example 2.4. Let $(\mathbb{R}^2, \mu)$, where $\mu : V \to [0, 1]$ and the vectors x and y are all defined as in Example 2-1. Again, these vectors are clearly linearly independent, as well as fuzzy linearly independent as one can easily check. Rotating these vectors by 45° counterclockwise, under $R_{45^\circ} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$, we get $x' = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$ and $y' = \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$. these vectors are obviously still linearly independent, but not fuzzy linearly independent. Indeed, for $a = b \neq 0$, we have:
$\mu(ax' + by') = \mu(a \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} + a \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}) = \mu \begin{pmatrix} 0 \\ a\sqrt{2} \end{pmatrix} = 1 \neq 0 = 0 \bigwedge 0 = \mu(ax') \bigwedge \mu(by')$

Which shows that these vectors are not fuzzy linearly independent.

Example 2.5. Let $(\mathbb{R}^2, \mu)$, where $\mu : V \to [0, 1]$ and the vectors x and y are all defined as in Example 2.1. Again, these vectors are clearly linearly independent, as well as fuzzy linearly independent, but reflecting these vectors about the y-axis under $L_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, we get $x' = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $y' = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.These vectors are obviously still linearly independent, but not fuzzy linearly independent, because for $a = 0, b \neq 0$, we have:
$\mu(ax' + by') = \mu(0 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \mu \begin{pmatrix} 0 \\ b \end{pmatrix} = 1 \neq 0 = 0 \bigwedge 1 = \mu(ax') \bigwedge \mu(by')$

Which shows that these vectors are not fuzzy linearly independent.

Example 2.6. Let $(\mathbb{R}^2, \mu)$, where $\mu : V \to [0,1]$ and the vectors x and y are all defined as in Example 2.1. Again, these vectors are clearly linearly independent, as well as fuzzy linearly independent. Applying a shear $H_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k \neq 0$,to these vectors, we get $x' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $y' = \begin{pmatrix} k \\ 1 \end{pmatrix}$.These vectors are obviously still linearly independent, but not fuzzy linearly independent, because for $a \neq 0, b = -\frac{a}{k}$, we have:
$\mu(ax' + by') = \mu(a \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{a}{k} \begin{pmatrix} k \\ 1 \end{pmatrix} = \mu \begin{pmatrix} 0 \\ -\frac{a}{k} \end{pmatrix} = 1 \neq 0 = 0 \bigwedge 0 = \mu(ax') \bigwedge \mu(by')$

Which shows that these vectors are not fuzzy linearly independent.

Example 2.7. Let $(\mathbb{R}^2, \mu)$, where $\mu : V \to [0,1]$ and the vectors x and y are all defined as in Example 2.1. Again, these vectors are clearly linearly independent, as well as fuzzy linearly independent. Applying a scale $S_{c,d} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$, $c, d > 0$,to these vectors, we get $x' = \begin{pmatrix} c \\ 0 \end{pmatrix}$ and $y' = \begin{pmatrix} 0 \\ d \end{pmatrix}$. These vectors are obviously still linearly independent, but not fuzzy linearly independent, because for $a = 0, b \neq 0$, we have:
$\mu(ax' + by') = \mu(0 \begin{pmatrix} c \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ d \end{pmatrix}) = \mu \begin{pmatrix} 0 \\ bd \end{pmatrix} = 1 \neq 0 = 0 \bigwedge 1 = \mu(ax') \bigwedge \mu(by')$

Which shows that these vectors are not fuzzy linearly independent.

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SOME RESULT OF ORTHONORMAL BASES IN HILBERT C^*-MODULES

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Abstract. The paper describes some basic properties of adjointable operator and φ- morphisms on orthonormal basis for Hilbert C^*-modules. Moreover by minimal projection property of orthonormal basis for Hilbert C^*-modules, we obtain some results.

1. Introduction.

A (left) *pre-Hilbert C^*-module* over a C*-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module $\mathcal{X}$ equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$, $(x, y) \mapsto \langle x, y \rangle$, which is $\mathcal{A}$-linear in the first variable x (and conjugate-linear in y) and has the properties for all x, y in $\mathcal{X}$ and a in $\mathcal{A}$:

$$\langle x, y \rangle = \langle y, x \rangle^*, \ \langle ax, y \rangle = a\langle x, y \rangle,$$

$$\langle x, x \rangle \geq 0 \ \text{ with equality only when } x = 0.$$

A pre-Hilbert $\mathcal{A}$-module $\mathcal{X}$ is called a *Hilbert $\mathcal{A}$-module* if $\mathcal{X}$ is a Banach space with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. As well as its scalar-valued norm $\| \ . \ \|$, an inner-product $\mathcal{A}$-module $\mathcal{X}$ has an $\mathcal{A}$-valued norm $| \ . \ |$, given by $|x| = \langle x, x \rangle^{\frac{1}{2}}$. Notice that the norm on $\mathcal{X}$ makes $\mathcal{X}$ into a normed $\mathcal{A}$-module.

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* Speaker.

Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $\mathcal{Y}$ is a closed submodule of $\mathcal{X}$. We say that Y is an orthogonality complemented if $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Y}^{\perp}$, where $\mathcal{Y}^{\perp} := \{y \in \mathcal{X} : \langle x, y\rangle = 0 \text{ for all } x \in \mathcal{Y}\}$ denotes the orthogonal complement of $\mathcal{Y}$ in $\mathcal{X}$. The reader is referred to [3, 4] and the references cited therein for more details.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. Then $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the set of all maps $T : \mathcal{X} \to \mathcal{Y}$ for which there is a map $T^* : \mathcal{Y} \to \mathcal{X}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for each $x \in \mathcal{X}$, $y \in \mathcal{Y}$. It is known that any element T in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(xa) = (Tx)a$ for $x \in \mathcal{X}$ and $a \in \mathcal{A}$. we use the notations $\mathcal{L}(\mathcal{X})$ in place of $\mathcal{L}(\mathcal{X}, \mathcal{X})$.

If $\mathcal{A}$ is a C^*-algebra, then $\mathcal{A}$ itself is a Hilbert $\mathcal{A}$-module if we define $\langle a, b\rangle = ab^*$ for each $a, b \in \mathcal{A}$. Given elements $x, y \in \mathcal{X}$ we define $\Theta_{x,y} : \mathcal{X} \to \mathcal{X}$ by $\Theta_{x,y}(z) = \langle z, y\rangle x$ for each $z \in \mathcal{X}$, then $\Theta_{x,y} \in B(\mathcal{X})$, with $(\Theta_{x,y})^* = \Theta_{y,x}$. The closure of the span of $\{\Theta_{x,y} : \ x, y \in \mathcal{X}\}$ in $B(\mathcal{X})$ is denoted by $\mathcal{K}(\mathcal{X})$, and elements from this set will be called $\mathcal{A}$-compact operators.

We denote by $\langle \mathcal{X}, \mathcal{X}\rangle$ the closed linear span of all elements in the underlying C^*-algebra $\mathcal{A}$ of the form $\langle x, y\rangle$, $x, y \in \mathcal{X}$. Obviously, $\langle \mathcal{X}, \mathcal{X}\rangle$ is an ideal in $\mathcal{A}$ and $\mathcal{X}$ is said to be a full $\mathcal{A}$-module if $\langle \mathcal{X}, \mathcal{X}\rangle = \mathcal{A}$.

2. Orthonormal bases in Hilbert C*-modules.

In this section we like to admit behavior adjointable operator and φ-morphism on orthonormal basis for Hilbert C^*-modules.

The concept of an orthonormal basis of a Hilbert C^*-module is introduced by D. Bakic, B. Guljas in [1]. The system of vectors $\{\varepsilon_i\}_{i\in I}$ in a Hilbert C^*-module $\mathcal{X}$ over a C^*-algebra $\mathcal{A}$ is said to be an *orthonormal basis* for $\mathcal{X}$ if it satisfies the following conditions:
(1) $p_i = \langle \varepsilon_i, \varepsilon_i\rangle$ is a minimal projection such that $p_i \mathcal{A} p_i = \mathbb{C} p_i$ for every $i \in I$,
(2) $\langle \varepsilon_i, \varepsilon_j\rangle = 0$ for all $i, j \in I$, $i \neq j$,
(3) $\{\varepsilon_i\}_{i\in I}$ generates a dense submodule of $\mathcal{X}$.
If $\{\varepsilon_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{X}$ then immediately, the previous definition implies that if $x \in \mathcal{X}$ satisfies $p_i = \langle x, x\rangle$ for some projection (not necessarily minimal) $p_i \in \mathcal{A}$, then $\langle p_i x - x, p_i x - x\rangle = 0$ so $p_i x = x$. In particular, the same is true for all basic vectors in $\mathcal{X}$.

Theorem 2.1. (see [1, Theorem 1.]) Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and let $\{\varepsilon_i\}_{i\in I}$ be an orthonormal system in $\mathcal{X}$. The following statements are mutually equivalent:

(i) $\{\varepsilon_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{X}$.
(ii) $x = \sum_{i\in I} \langle x, \varepsilon_i\rangle \varepsilon_i$.

(iii) $\langle x, x\rangle = \sum_{i\in I}\langle x, \varepsilon_i\rangle\langle \varepsilon_i, x\rangle$.
(iv) $\langle x, y\rangle = \sum_{i\in I}\langle x, \varepsilon_i\rangle\langle \varepsilon_i, y\rangle$.
for each $x, y \in \mathcal{X}$.

Proposition 2.2. *Suppose $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathcal{A}$-module and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be surjective. If $\{\varepsilon_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{X}$ and $\{\nu_j\}_{j\in J}$ is an orthonormal basis for $\mathcal{Y}$ then for each $x \in \mathcal{X}$,*

$$T(x) = \sum_{j\in J}\sum_{i\in I}\langle x, \varepsilon_i\rangle\langle \varepsilon_i, T^*(\nu_j)\rangle\nu_j = \sum_{i\in I}\sum_{j\in J}\langle x, \varepsilon_i\rangle\langle \varepsilon_i, T^*(\nu_j)\rangle\nu_j$$

Theorem 2.3. *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has a closed range and T^* is surjective. If $\mathcal{X}$ be full Hilbert $\mathcal{A}$-module then $\mathcal{Y}$ is too.*

Lemma 2.4. *If $\mathcal{X}$ be Hilbert $\mathcal{A}$-module $\{\varepsilon_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{X}$. Then $\sum_{i\in I}\Theta_{\varepsilon_i,\varepsilon_i}$ is identity operator in $\mathcal{L}(\mathcal{X})$.*

Theorem 2.5. *Let $\mathcal{A}$ be a unital commutative C^*-algebra and be Hilbert C^*-module over itself. If $\{\varepsilon_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{A}$. Then $\sum_{i\in I} p_i = 1$.*

Definition 2.6. Let $\mathcal{X}$ be a Hilbert C^*-module over $\mathcal{A}$, and $\mathcal{I}$ an ideal in $\mathcal{A}$ The associated ideal submodule $\mathcal{X}_{\mathcal{I}}$ is defined by

$$\mathcal{X}_{\mathcal{I}} = \overline{[\mathcal{I}\mathcal{X}]} = \overline{[\{bx : \ x \in \mathcal{X}, b \in \mathcal{I}\}]}.$$

(the closed linear span of the action of $\mathcal{I}$ on $\mathcal{X}$).

Clearly, $\mathcal{X}_{\mathcal{I}}$ is a closed submodule of $\mathcal{X}$. It can be also regarded as a Hilbert C^*-module over $\mathcal{I}$.

Lemma 2.7. (see [2, Proposition 1.3])Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, $\mathcal{I}$ an ideal in $\mathcal{A}$, and $\mathcal{X}_{\mathcal{I}}$ the associated ideal submodule. Then

$$\mathcal{X}_{\mathcal{I}} = \{x \in \mathcal{X} \ : \ \langle x, x\rangle \in \mathcal{I}\} = \{x \in \mathcal{X} \ : \ \langle x, v\rangle \in \mathcal{I}, \forall v \in \mathcal{X}\}.$$

If $\mathcal{X}$ is full, then $\mathcal{X}_{\mathcal{I}}$ is full as a Hilbert $\mathcal{I}$-module.

Theorem 2.8. *Suppose $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module, $\{\varepsilon_i\}_{i\in I}$ be an orthonormal base for $\mathcal{X}$, and $\mathcal{I}$ be an ideal of $\mathcal{A}$. If $\langle \varepsilon_i, \varepsilon_i\rangle \in \mathcal{I}$ for all $i \in I$, then $\mathcal{X} = \mathcal{X}_{\mathcal{I}}$.*
Moreover if $\mathcal{X}$ is full, then $\mathcal{I}$ contain approximate unite of $\mathcal{A}$.

Lemma 2.9. (see [2, Proposition 1.2]) Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, and $\mathcal{I}$ an ideal in $\mathcal{A}$. Then

$$\mathcal{X}_{\mathcal{I}} = \mathcal{I}\mathcal{X} = \{bx \ : \ \ x \in \mathcal{X}, b \in \mathcal{I}\}.$$

Theorem 2.10. *Suppose $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module, $\{\varepsilon_i\}_{i\in I}$ be an orthonormal base for $\mathcal{X}$, and $\mathcal{I}$ be an ideal of $\mathcal{A}$. If $\langle \varepsilon_i, \varepsilon_i\rangle \in \mathcal{I}$ for all $i \in I$, then $\mathcal{I}E$ is dense in $\mathcal{X}$.*

3. Morphism on Orthonormal bases.

In this section we like admit result in φ- morphisms in this case that $\mathcal{X}$ is a Hilbert $\mathcal{A}$- module with orthonormal bases.

Definition 3.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert C^*-modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a morphism of C^*-algebras. A map $\Phi : \mathcal{X} \to \mathcal{Y}$ is said to be a φ-morphism of Hilbert C^*-modules if $\langle \Phi(x), \Phi(y)\rangle = \varphi(\langle x, y\rangle)$ for all $x, y \in \mathcal{X}$.

It is also easy to show that each φ-morphism is necessarily a linear operator and a module map in the sense $\Phi(ax) = \varphi(a)\Phi(x)$, for all $a \in \mathcal{A}$, $x \in \mathcal{X}$.

Theorem 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, and $\mathcal{X}$ be Hilbert $\mathcal{A}$-module with orthonormal basis $\{\varepsilon_i\}_{i\in I}$ and $\mathcal{Y}$ be Hilbert $\mathcal{B}$-module and $\Phi : \mathcal{X} \to \mathcal{Y}$ be a surjective φ-morphism. Then $\{\Phi(\varepsilon_i)\}_{i\in I}$ is an orthonormal basis for $\mathcal{Y}$.*

Theorem 3.3. *Let $\mathcal{A}$, $\mathcal{B}$ are C^*-algebras, $\varphi : \mathcal{A} \to \mathcal{B}$ be a $*$-morphism, $\mathcal{X}$ be left Hilbert $\mathcal{A}$ and $\mathcal{B}$ modules with orthonormal basis $\{\varepsilon_i\}_{i\in I}$, and $\Phi : \mathcal{X} \to \mathcal{X}$. If $\langle \Phi(x), \varepsilon_i\rangle = \varphi(\langle x, \varepsilon_i\rangle)$ for all $x \in \mathcal{X}$. Then Φ is φ-morphism.*

Corollary 3.4. *Let $\mathcal{A}$, $\mathcal{B}$ are C^*-algebras, $\varphi : \mathcal{A} \to \mathcal{B}$ be a $*$-morphism, $\mathcal{X}$ be left Hilbert $\mathcal{A}$ and $\mathcal{B}$ modules with orthonormal basis $\{\varepsilon_i\}_{i\in I}$, and $\Phi : \mathcal{X} \to \mathcal{X}$. If $\langle \Phi(x), \varepsilon_i\rangle = \varphi(\langle x, \varepsilon_i\rangle)$ for all $x \in \mathcal{X}$. Then $\Phi(x) = \sum_{i\in I}\langle \Phi(x), \varepsilon_i\rangle \Phi(\varepsilon_i)$.*

Theorem 3.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, and $\mathcal{X}$ be full Hilbert $\mathcal{A}$-module and $\mathcal{Y}$ be Hilbert $\mathcal{B}$- module and $\Phi : \mathcal{X} \to \mathcal{Y}$ be $\varphi-$morphism, then there exists sequence $\{x_i\}$ in $\mathcal{X}$ such that $\sum_{i=1}^{\infty} |\Phi(ax_i)|^2 =| \varphi(a) |^2$ for some $a \in \mathcal{A}$.*

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A METHOD FOR THE DECOMPOSITION OF VECTOR SPACES

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Abstract. In this note we establish a general method for the decomposition of vector spaces by considering relatively prime polynomials instead of irreducible ones.

1. Introduction

Let H be a Hilbert space. In [3] we introduce a kind of closed subsets $D \subseteq H \oplus H$ for any Hilbert space H, which has also a kind of maximality and for which every $h \in H$ has a unique representation $h = x - y$ for some $(x, y) \in D$. Then as a consequence we prove the standard decomposition of the Hilbert spaces. In this study a general method for the decomposition of vector spaces is given which is based on the decomposition of polynomials. Let F be a field and T be a linear operator defined on the finite dimensional F vector space V. Let χ be a polynomial in $F[x]$ such that $\chi(T) = 0$ and $\chi = p_1^{r_1} \ldots p_k^{r_k}$ where $p_i (i = 1, \ldots, k)$ are relatively prime polynomials of $F[x]$ and r_i

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are positive integers. Let $W_i = \ker p_i(T)^{r_i}$ for $i = 1, ..., k$, then it is shown that $V = W_1 \oplus ... \oplus W_k$.

Theorem 1.1. [1] *Let V be a F vector space, then there exist $n \in N$ and some linear subspaces $V_i(1 \leq i \leq n)$ such that $V = \oplus_{i=1}^{n} V_i$ if and only if there exist linear operators $p_i : V \to V(1 \leq i \leq n)$ such that $p_i \circ p_j = 0$ for $i \neq j$ and $\sum_{i=1}^{n} p_i = id_V$. Moreover $V_i = Imp_i(1 \leq i \leq n)$.*

2. Main results

Theorem 2.1. *Let V be a nonzero finite dimensional vector space over the field F. Let $T : V \to V$ be a linear operator, and $\chi \in F[x]$ be a polynomial with coefficient in F, such that $\chi(T) = 0$. Let*

$$\chi = p_1^{l_1} ... p_k^{l_k} \qquad l_1, ..., l_k \in N$$

for relatively prime polynomials $p_1, ..., p_k \in F[x]$ and

$$W_i = \ker p_i(T)^{l_i}(1 \leq i \leq k)$$

then $V = \oplus_{i=1}^{k} W_i$.

Proof. The theorem is clear for $k = 1$. Let $k \geq 2$ and $q_i = \frac{\chi}{p_i^{l_i}} = \prod_{j \neq i} p_j^{l_j}$ for $i = 1, ..., k$. There are $a_1, ..., a_k \in F[x]$ such that $a_1 q_1 + ... + a_k q_k = 1$. Let $r_i = a_i q_i (1 \leq i \leq k)$ then $r_1 + ... + r_k = 1$ and consequently $r_1(T) + ... + r_k(T) = id_V$. Since

$$r_i r_j = a_i a_j q_i q_j = a_i a_j p_i^{l_i} p_j^{l_j} \prod_{m \neq i,j} p_m^{2l_m}$$

we have $r_i(T) r_j(T) = 0$ for $i \neq j$. Therefore according to the Theorem 1.1 we have $V = \oplus_{i=1}^{k} \mathrm{Im} r_i(T)$.
Now we show that $\mathrm{Im} r_i(T) = \ker p_i(T)^{l_i}$ for $1 \leq i \leq k$. Since $p_i^{l_i} q_i = \chi$ and $r_i = a_i q_i$ we have $p_i(T)^{l_i} r_i(T) = a_i(T)\chi(T) = 0$ and $\mathrm{Im} r_i(T) \subseteq \ker p_i(T)^{l_i}$. Moreover it can be easily seen that $r_j(T) = a_j(T) q_j(T) = a_j(T) \prod_{i \neq j} p_i(T)^{l_i}$, therefore

$$\begin{aligned} \ker p_i(T)^{l_i} &\subseteq \bigcap_{j \neq i} \ker r_j(T) \subseteq \ker \sum_{j \neq i} r_j(T) \\ &= \ker(id_V - r_i(T)) = \mathrm{Im} r_i(T) \end{aligned}$$

□

with the above notations,

Definition 2.2. The subspaces $W_i(1 \leq i \leq k)$, are called the T primary components of V under χ.

Theorem 2.3. *If $\chi = p_1^{l_1}...p_k^{l_k}, \kappa = p_1^{m_1 l_1}...p_k^{m_k l_k}, l_i, m_i \in N(1 \leq i \leq k)$ for relatively prime polynomials p_i and $T : V \to V$ be a linear operator such that $\chi(T) = 0$, then the T primary components of V under χ and κ are same.*

Proof. let $W_i = \ker p_i(T)^{l_i}$ and $U_i = \ker p_i(T)^{m_i l_i}$, then $W_i \subseteq U_i$ for $(1 \leq i \leq k)$ and $\oplus_{i=1}^k W_i = \oplus_{i=1}^k U_i$ by Theorem 2.1, therefore $W_i = U_i$ for $(1 \leq i \leq k)$. □

Theorem 2.4. *Let V be a nonzero finite dimensional F vector space and $T : V \to V$ be a linear operator. Define*

$$I = \{\chi \mid \chi \in F[x], \chi(T) = 0\}$$

Then I is finitely generated ideal of $F[x]$.

Proof. Let m_T be the minimal monic polynomial of T. Obviously m_T is unique and $m_T \mid \chi$ for any $\chi \in I$, so $I =< m_T >$. □

The polynomial χ as in the Theorem 2.1 exists, since it can be the characteristic or minimal polynomial of T over F. If $T, S \in L(V, V)$ are two linear operator, then $T \circ S$ and $S \circ T$ have exactly the same characteristic polynomial [2]. Note that the annihilation of $T \circ S$ by the polynomial χ does not implies the annihilation of $S \circ T$ by the same polynomial. In fact

Example 2.5. let $S(x, y, z) = (z, z, 0), T(x, y, z) = (z, z, z)$ then $T \circ S = 0$ and $S \circ T = S$ therefore $\chi(T \circ S) = 0$ for $\chi(x) = 0$ but $\chi(S \circ T) \neq 0$.

Theorem 2.6. *Let V be a nonzero F finite dimensional vector space, then the linear operator $T : V \to V$ is diagonalizable iff there exists a natural number k and some distinct values $\lambda_1, ..., \lambda_k \in F$ such that $\chi(x) = (x - \lambda_1)...(x - \lambda_k)$ annihilates T.*

Proof. If there exists distinct values $\lambda_1, ..., \lambda_k$ such that $\chi(x) = (x - \lambda_1)...(x - \lambda_k)$ annihilates T, then V is a direct sum of the subspaces $W_i = \ker(T - \lambda_i id_V)$ by Theorem 2.1. But if $0 \neq x \in W_i$ then $T(x) = \lambda_i x$ and x is an eigenvector corresponding to the eigenvalue λ_i, now the union of the bases of $W_1, ..., W_k$ is a basis β for V, and T is diagonalizable. Conversely let V has a basis of eigenvectors of T and $\lambda_1, ..., \lambda_k \in F$ are its distinct eigenvalues and $\chi(x) = (x - \lambda_1)...(x - \lambda_k)$. Obviously $\chi(T)$ annihilates the elements of β and therefore $\chi(T)(x) = 0$ for all $x \in V$, thus $\chi(T) = 0$. □

Theorem 2.7. *Let V be a nonzero finite dimensional F vector space, $f, g : V \to V$ are diagonalizable linear operators, then there exists a basis for V consists of eigenvalues of both f and g iff $f \circ g = g \circ f$.*

Proof. Let $\beta_V = \{v_1, ..., v_n\}$ be a basis of V such that each v_i is an eigenvector of both f and g. Therefore there exists $\lambda_i, \mu_i \in F$ such that $f(v_i) = \lambda_i v_i$ for $(1 \leq i \leq n)$, moreover

$$\begin{aligned} f \circ g(v_i) &= f(\mu_i v_i) = \mu_i f(v_i) = \mu_i \lambda_i v_i = \lambda_i \mu_i v_i \\ &= \lambda_i g(v_i) = g(\lambda_i v_i) = g \circ f(v_i) \end{aligned}$$

Conversely let $f \circ g = g \circ f$, since f is diagonalizable there exists some distinct values $\lambda_1, ..., \lambda_k$ and f annihilator polynomial $\chi(x) = (x - \lambda_1)...(x - \lambda_k)$ such that $V = \oplus_{i=1}^{n} W_i$ for $W_i = \ker(f - \lambda_i id_V)$. If $v_i \in W_i$ then

$$f \circ g(v_i) = g \circ f(v_i) = g(\lambda_i v_i) = \lambda_i g(v_i)$$

Thus W_i is a g-invariant subspace of V. Let $g_i = g\mid_{W_i}$ be the restriction of g to W_i, since g is diagonalizable so is g_i by Theorem 2.6, and there exists a basis B_i of eigenvectors of g_i in W_i. But any eigenvector of g_i is an eigenvector of g and any element of W_i is an eigenvector of f, thus $B = \cup_{i=1}^{k} B_i$ is a basis for V consists of eigenvectors of f and g. $\square$

Corollary 2.8. *Let A, B are two diagonalizable $n \times n$ matrices. Then they are simultaneously diagonizable if and only if $AB = BA$.*

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A NOTE ON CONSTRUCTING UMVUE FOR GENERAL VARIANCE IN SIMPLE QUADRATIC AND CUBIC NEF'S ON $\Re^d$

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Abstract. This paper concerns some modifications and generalizations of previous results, related to constructing uniformly minimum variance unbiased (UMVU) estimators for general variance function of some Natural Exponential Families (NEF's). Some illustrating examples are provided for this result.

1. Introduction

Let L be a linear vector space with finite dimension, and L_* be its dual and let $L_* \times L \longrightarrow \Re : (\theta, \mathbf{x}) \longmapsto < \theta, \mathbf{x} >$ be the duality bracket. If μ is a positive Radon measure on L, we denote the Laplace transform of μ by

$$L_\mu(\theta) = \int_{\Re^n} \exp < \theta, \mu > \mu(d\mathbf{x}) \leq +\infty$$

The interior of convex set $D(\mu) = \{\theta \in L_*, L_\mu(\theta) < \infty\}$ is denoted by $\theta(\mu)$. We denote by M_L, the set of all measures μ such that $\theta(\mu)$ is not empty and it is not concentrated on affine hyperplane of the space. $k_\mu(\theta) := logL_\mu(\theta)$ for all $\theta \in \theta(\mu)$ is called cumulate function of μ. The set $F = F(\mu) = \{P(\theta, \mu)(d\mathbf{x}) := exp[< \theta, \mu > -k_\mu(\theta)]\mu(d\mathbf{x})\}$

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is called natural exponential family generated by μ. Since μ is in M_L, k_μ is strictly convex and real analytic on $\theta(\mu)$ .If we define $\mathbf{x} \bigotimes \mathbf{y}$ as the product matrix and $\Re^d$ as a space with finite dimension, we have $k_\mu(\theta) = \int_{\Re^d} XP(\theta, \mu)\mu(d\mathbf{x})$. Thus $k'_\mu(\theta) : \theta(\mu) \longmapsto M_F$ is a diffeomorphism called the mean domain of F. Let $\psi_\mu : M_F \longrightarrow \theta(\mu)$ be its inverse function , and for $\mathbf{M}$ in $M_F, P(\mathbf{M}, F) = P(\psi_{\mu(\mathbf{M})}, \mu)$. Then $V_F(\mathbf{M})$, the covariance operator of $P(\mathbf{M}, F)$, is:

$$V_F(\mathbf{M}) = k''_\mu(\psi_\mu(\mathbf{M})) = (\psi'_\mu(\mathbf{M}))^{-1}$$

If we wish to consider a natural exponential family (NEF), first we need to estimate its variance covariance matrix. One solution is estimating the general variance (i.e determinant of variance covariance matrix). This has been considered in several papers such as [5]. Let $F = \{P(\mathbf{M}, F); \mathbf{M} \in M_F\}$ be a natural exponential family on $\Re^d$ and $V_F(\mathbf{M})$ denote the covariance operator of the probability distribution $P(\mathbf{M}, F)$. Function $\mathbf{M} \longrightarrow V_F(\mathbf{M})$ is called the variance function and $detV_F(\mathbf{M})$ is general variance. Unbiased estimators with uniformly minimum variance of general variance have been described in [2, 3]. In section 2, we shall extend a result in [4] to multivariate case. This finding determines a relation between variance covariance matrix of two NEF's. Using this result and a result in [1], finally we find two new simple Cubic NEF's and their UMVUE of general variances on $\Re^d$.

2. Cubic NEF's

If μ in $\mathbf{M}_F$ is concentrated on $\aleph^d$, the NEF generated by μ is instigated as Multivariate NEF concentrated on $\aleph^d$. We could exemplify this type of NEF by

$$f_\mu(\mathbf{z}) = \sum_{k \in \aleph^d} \mu(k)\mathbf{z}^k \tag{2.1}$$

where $\mu(d\mathbf{x}) = \sum_{k \in \aleph^d} \mu(k)\delta_k(d\mathbf{x})$ Clearly μ is in $\mathbf{M}_F$ if $R(\mu) > 0$, where $R(\mu) > 0$ is a vector that is the radius of convergence of the entire (2.1). Perceptibly, if μ is in $\mathbf{M}_F$ then $\theta(\mu) = (-\infty, [R(\mu)]_1) \times ... \times (-\infty, [R(\mu)]_d)$ where $(-\infty, [R(\mu)]_1)$ is the ith component of $R(\mu)$ we have:

$$\log f_\mu(e^\theta) \forall\theta \in \theta(\mu)$$

$$p(\log \mathbf{z}, \mu)(d\mathbf{x}) = \sum_{k \in \aleph^d} \frac{\mu_k \mathbf{z}^k}{f_\mu(\mathbf{z})} \delta_k(d\mathbf{x})$$

if $0 < z_i < [R(\mu)]_i$. We need to mention some more notations and definitions here. Suppose that U^* is an open set on $\Re^d$ that contain 0. We say that $f : U^* \mapsto \Re$ is an analytic function if and only if

$f(\mathbf{z}) = \sum_{n\in\aleph^d} f_n \mathbf{z}^n$, where f_n is $[\mathbf{z}^n]f(\mathbf{z})$ i.e. the coefficient of $\mathbf{z}^n$ in $f(\mathbf{z})$ when expanded.

Now, we generalize the result in [4] to the multivariate case. This may well be used in constructing new multivariate NEF's .

Theorem 2.1. *Let $p > 0$, $g_\mu(\mathbf{z}) = \sum_{n\in\aleph^d} g_n \mathbf{z}^n$ with radius of convergence $R(f) > 0$ on $\Re^d$. Suppose*
(i) $g_n \geq 0$ for all $i \in \aleph^d$, $g_0 > 0$, $g_1 > 0$ and $|J(\mathbf{z})| > 0$.
(ii) $\mu_n = [\mathbf{z}^n][(g(\mathbf{z})^{n+p}|J(\mathbf{z})|)] \geq 0$ where $[\mathbf{z}^n][g]$ is the nth coefficient of $\mathbf{x}^n$ when we expand g.
Let F and F_1 be the NEF generated by $\mu = \sum_{n\in\aleph^d} \mu_n\delta_n$ And $\nu = \sum_{n\in\aleph^d} g_n\delta_n$ respectively. Then, if $M_{F_1} = D_2$ and $M_F = D_3$ where $D_2 = (0, b'_1) \times ... \times (0, b'_d)$ is an open set in $\Re^d$, and $D_3 = (0, b_1) \times ... \times (0, b_d)$ where $b_i = \infty$ if $b'_i \geq 1$ and $b_i = pb'_i/(1-b'_i)$ if $0 < b'_i < 1$. Furthermore,

$$V_F(\mathbf{M}) = diag[\frac{(\mathbf{M}+p)^3}{p^2}]V_{F_1}(\frac{\mathbf{M}}{\mathbf{M}+p}).$$

Proof. Since $\mu_0 = (g_\mu)^p|J(.)| > 0$, $\mu_1 = [\mathbf{z}][g^p + 1(.)|J(.)|] = (p+1)(g_1^p(.)|J(.)| + \frac{\partial|J(.)|}{\partial\mathbf{z}} g(.)) \geq (p+1)g_1^p|J(.)| > 0$. Thus, since $k_\mu(\theta) = \log g_\mu(e^\theta)$, $\theta(\mu) = D_{\mathbf{x}} = (-\infty, R(\mu))^d$. Thus, $k'_\mu(\theta) = [e_i^\theta \frac{\partial g_\mu}{\partial\theta_i}(e^\theta)]/g_\mu(e^\theta)$ and therefore $\mathbf{M}_F = (0, b_1)\times...\times(0, b_d)$ where $b_i = \lim_{Z_i\to\infty} \frac{\partial g_\mu(e^\theta)}{\partial Z_i}\frac{1}{g_\mu(e^Z)} \times diag[e^Z]$ where $Z = (Z_1, ..., Z_d)$.
Let ψ_1 and ψ be the primitives of $\mathbf{M} \mapsto diag[\mathbf{M}][V_F(\mathbf{M})]^{-1}$ and $\mathbf{M} \mapsto [V_F(\mathbf{M})]^{-1}$ on D_3 respectively. Obviously we have

$$exp(\psi_1)(\mathbf{M}) = \sum_{n\ \epsilon\ \aleph^d} \mu_n \exp(n\psi)$$

for $\mathbf{M}\ \epsilon\ D_3$ Now put $w = exp(\psi(\mathbf{M}))$ and define h as in (**??**), then by corollary 3.2 we have

$$goh(w) = [w]^{-1}h(w) = \exp(\frac{\psi_1(\mathbf{M})}{p})$$

$$h(w) = [w].exp(\frac{\psi_1(\mathbf{M})}{p})) = [\exp\psi(\mathbf{M})]\exp(\frac{\psi_1(\mathbf{M})}{p}) \tag{2.2}$$

where $[\mathbf{w}]^{-1} = [w_1^{-1}, ..., w_d^{-1}]$ and $[\mathbf{w}]\mathbf{w}' = [w_1w'_1, ..., w_dw'_d]$. Thus, we obtain

$$g(\exp[\psi(\mathbf{M}) + \frac{\psi_1(\mathbf{M})}{p}]) = \exp(\frac{\psi_1(\mathbf{M})}{p})$$

Differentiating with respect to m_i we obtain

$$\frac{\partial}{m_i} g(\exp[\psi(\mathbf{M}) + \frac{\psi_1(\mathbf{M})}{p}]) = \exp(-\psi(\mathbf{M}))(\frac{m_i}{m_+ p})$$

Since $\frac{\partial}{m_i} g(.) \geq 0$, g is increasing in m_i on $(0, R(g))$ and its inverse g^{-1} is defined on

$$(g(0), g(R(g)))$$

Therefore, we have

$$[\frac{\mathbf{M}}{\mathbf{M}+p}] \exp(\psi(\mathbf{M})) = (g' o g^{-1})(\exp \frac{\psi_1(\mathbf{M})}{p})$$

Let us define $\varphi : D^* \to \Re^n$ where

$$\varphi(z) = g' o g^{-1}(z).[g^{-1}].[z]^{-1} \qquad \forall z \in D^*$$

Since φ is the combination of two one to one function, $w \to g'(w).[w]$ $[g(w)]^{-1}$ and $z \to g^{-1}(z)$ therefore φ to be invertible. Now suppose $G : D \times D \to \Re^n$ to be the reciprocal of φ. So we could write

$$G([\frac{m_i}{m_i + p}]) = \exp \frac{\psi_1(\mathbf{M})}{p}$$

Since $k_{\mu_g}(\theta) = \log g_\mu(e^\theta)$ and $\frac{\partial k_{\mu_g}}{\partial \theta_i} = m_i$ we have $\theta = \log((g^{-1} o G)(m))$ and therefore we have

$$[G(m)]^{-1} G'(m) = diag[\mathbf{M}][V_{F_1}(\mathbf{M})]^{-1}$$

The remaining part is proved as in [4].

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SOLVING AN SPECIAL TYPE OF INFINITE BOUNDARY INTEGRO-DIFFERENTIAL EQUATIONS BY USING LAGUERRE POLYNOMIALS

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Abstract. In this paper, we present a matrix method to solve infinite boundary integro-differential equations (IBI-DE) of the second kind with degenerate kernel in terms of Laguerre polynomials. Properties of these polynomials and operational matrix of integration are first presented.

1. Introduction

The integral equation is called infinite boundary integro-differential equation if one or both of it's limits are improper. Many problems of theoretical physics, electromagnetics, scattering problems, boundary integral equations [1, 2] leads to (IBI-DE) of the second kind of the

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* Speaker.

form

$$\begin{cases} u'(x) - (\vartheta\ u)(x) = f(x), \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where

$$(\vartheta\ u)(x) := \lambda \int_0^\infty e^{-t} k(x,t) u(t) dt,\ 0 \le x, t < \infty.$$

In Eq. (1.1), λ is parameter and $f(x)$ is continuous function and the kernel $k(x,t)$ might has singularity in the region $D = \{(x,t) : 0 \le x, t < \infty\}$, and $u(x)$ is the unknown function which to be determined. Many researchers have developed the approximate method to solve infinite boundary integral equation using Galerkin and collocation methods with Laguerre and Hermite polynomials as a bases function [3, 4]. But for Eq. (1.1), work have been not done.

Definition 1.1. The Laguerre polynomials of nth-degree are defined on the interval $[0,\infty)$ as [4]:

$$L_n(x) = \frac{1}{n!} e^x \partial_x^n (x^n e^{-x}),\ n = 0, 1, ...$$

1.1. **Function Approximation.** A function $f(x) \in L^2[0,\infty)$ defined over $[0,\infty)$ may be represented by the Laguerre polynomials series as follows:

$$f(x) = \sum_{i=0}^{\infty} f_i L_i(x). \tag{1.2}$$

If the infinite series in (1.2) is truncated up to term n, then (1.2) can be written as

$$f(x) \simeq \sum_{i=0}^{n} f_i L_i(x) = F^T L_x, \tag{1.3}$$

where $f_0, f_1, \cdots, f_n$ are arbitrary coefficients, F and L_x are $(n+1)\times 1$ vectors given by $F = [f_0,\ f_1, \dots,\ f_n]^T$ and $L_x = [L_0(x),\ L_1(x), \dots,\ L_n(x)]^T$. But F^T can be obtained by

$$F^T < L_x, L_x >=< f, L_x >,$$

where

$$< f, L_x >= \int_0^\infty w(x) f(x) L_x^T dx = [< f, L_0 >, \cdots < f, L_n >],$$

where $w(x)$ is the weight function e^{-x}, and $< L_x, L_x >$ is a $(n+1)\times(n+1)$ matrix which is said the dual matrix of L_x denoted by Q and will be introduced in the following. Therefore

$$Q =< L_x, L_x >= \int_0^\infty e^{-x} L_x L_x^T dx, \tag{1.4}$$

then

$$F^T = \{\int_0^\infty w(x) f(x) L_x^T dx\} Q^{-1}.$$

Similarly, a function of two variables, $k(x,t) \in L^2([0,\infty) \times [0,\infty))$ may be approximated as follows:

$$k(x,t) \cong \sum_{i=0}^{n} \sum_{j=0}^{n} k_{i,j} L_i(x) L_j(t) = L_x^T K L_t,$$

where K is a $(n+1) \times (n+1)$ matrix and we can see

$$K = Q^{-1} < L_x, < k(x,t), L_t >> Q^{-1}.$$

1.2. **Operational matrix of integration.** The main objective of this subsection is to find the integration of the Laguerre vector L_x defined in Eq. (1.3).

Theorem 1.1. *Let L_x be the Laguerre vector then*

$$\int_0^x L_t dt \simeq P L_x,$$

where P is the $(n+1) \times (n+1)$ operational matrix for integration as follows:

$$P = \begin{bmatrix} \Omega(0,0) & \Omega(0,1) & \Omega(0,2) & \cdots & \Omega(0,n) \\ \Omega(1,0) & \Omega(1,1) & \Omega(1,2) & \cdots & \Omega(1,n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega(n,0) & \Omega(n,1) & \Omega(n,2) & \cdots & \Omega(n,n) \end{bmatrix},$$

where

$$\Omega(i,j) = \sum_{k=0}^{i} \sum_{r=0}^{j} \frac{(-1)^{k+r} i! j! \Gamma(k+r+2)}{(i-k)!(j-r)!(k+1)!k!(r!)^2}.$$

2. Main results

In this section, we consider (IBI-DE) of the second kind in (1.1) and approximate to solution by means of finite Laguerre series defined in (2.1). The aim is to find Laguerre coefficients, we approximate functions $f(x)$, $k(x,t)$ and $u'(x)$ with respect to Laguerre polynomials (basis) by the way mentioned in before section as follows:

$$f(x) \cong F^T L_x, \;\; u'(x) \cong C'^T L_x, \;\; u(0) = C_0^T L_x, \;\; k(x,t) \cong L_x^T K L_t.$$

Then we have the following linear system of algebraic equations:

$$(I - \lambda K P^T)C' = F + \lambda K C_0.$$

We can find the vector C', so

$$C^T = C'^T P + C_0^T \Longrightarrow u(x) \simeq C^T L_x. \tag{2.1}$$

Example 2.1. consider the following infinite boundary integro-differential equation of second kind:

$$\begin{cases} u'(x) = 3x^2 + sin(x) - \frac{1}{2}(4 + cos(x)) + \int_0^\infty e^{-t} sin(x-t)u(t)dt, \\ u(0) = 1, \end{cases} \tag{2.2}$$

and the exact solution is $u(x) = x^3 - 2x + 1$.
Next, we simplify to obtain

$$\begin{bmatrix} c'_0 \\ c'_1 \\ c'_2 \\ c'_3 \end{bmatrix} = \begin{bmatrix} 309/389 & -6/3389 & 34/389 & 40/389 \\ -68/389 & 345/3389 & -10/389 & 34/389 \\ -28/389 & -41/3389 & 362/389 & 14/389 \\ 6/389 & -19/3389 & -22/389 & 386/389 \end{bmatrix} \begin{bmatrix} 17/4 \\ -25/2 \\ 43/8 \\ -3/8 \end{bmatrix}.$$

The solution of this equation is

$$c'_0 = 4,\ c'_1 = -12,\ c'_2 = 6,\ c'_3 = 0.$$

By substituting the obtained coefficients in (2.1) the solution of (2.2) becomes

$$u(x) \simeq 5L_0(x) - 16L_1 + 18L_2(x) - 6L_3(x) = x^3 - 2x + 1 \tag{2.3}$$

which is the exact solution. Also, if we choose $n \geq 4$, we get the same approximate solution as obtained in equation (2.3).
Corollary: If the exact solution to equation (1.1) be a polynomial, then the proposed method will obtain in the real solution.

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NUMERICAL SOLUTION OF CONVECTION-DIFFUSION EQUATION USING QUINTIC B-SPLINES COLLOCATION METHOD VIA ALGEBRAIC SOLUTION WITH NEUMANN'S BOUNDARY CONDITIONS

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Abstract. In this work we will discuss the solution of convection-diffusion partial differential equations with Neumann's boundary conditions by using the collocation method with quintic B-splines.we discretize the time derivative with Crank Nicolson scheme and handle spatial derivatives with quantic B-splines. The implementation of algorithm for this method is very easy and economical. The accuracy of the numerical solutions indicates that the presented method is well suited for convection-diffusion equations with Neumann's boundary conditions.

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* Speaker.

1. Introduction

In this paper, we consider the numerical solution of the following one dimensional convection-diffusion Equation [2]

$$\frac{\partial u}{\partial t} + \varepsilon \frac{\partial u}{\partial x} = \gamma \frac{\partial^2 u}{\partial x^2} , \quad 0 \le x \le L , \;\; 0 \le t \le T \tag{1.1}$$

with initial condition

$$u(x,0) = \varphi(x)$$

and Neumann's boundary conditions are as follows :

$$\left(\frac{\partial u}{\partial x}\right)_{(0,t)} = g_0(t) , \quad \left(\frac{\partial u}{\partial x}\right)_{(L,t)} = g_1(t)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{(0,t)} = p_0(t) , \quad \left(\frac{\partial^2 u}{\partial x^2}\right)_{(L,t)} = p_1(t) , \quad t \in [0,T]$$

where the parameter γ is the viscosity coefficient and ε is the phase speed and both are assumed to be positive. φ, g_0 and g_1 are known functions of sufficient smoothness. The convection-diffusion problems arise in many important applications in science and engineering such as fluid motion, heat transfer, astrophysics, oceanography, meteorology, semiconductors, hydraulics, pollutant and sediment transport, and chemical engineering.

In quintic B-splines collocation method the approximate solution can be written as a linear combination of quintic B-splines basis functions for the approximation space under consideration.

We consider a mesh

$$0 = x_0 < x_1, \cdots , x_{N-1} < x_N = L$$

as a uniform partition of the solution domain $0 \le x \le L$ by the knots x_j with $h = x_{j+1} - x_j = \dfrac{L}{N}$, $j = 0, 1, \cdots , N-1$ Our numerical treatment for solving equation (1.1) using the collocation method with quintic B-splines is to find an approximate solution $U^N(x,t)$ to the exact solution $u(x,t)$ in the form:

$$U^N(x,t) = \sum_{j=-2}^{N+2} \alpha_j(t) \cdot B_j(x)$$

Where $\alpha_j(t)$ are unknown time dependent quantities to be determined from the boundary conditions and Collocation from the differential equation. The quintic B-splines $B_j(x)$ at the knots is given by [3]

$$B_j(x) = \frac{1}{h^5}\begin{cases} (x-x_{i-3})^5 \\ x\in[x_{i-3},x_{i-2}) \\ \\ (x-x_{i-3})^5 - 6(x-x_{i-2})^5 \\ x\in[x_{i-2},x_{i-1}) \\ \\ (x-x_{i-3})^5 - 6(x-x_{i-2})^5 + 15(x-x_{i-1})^5 \\ x\in[x_{i-1},x_{i}) \\ \\ (x_{i+3}-x)^5 - 6(x_{i+2}-x)^5 + 15(x_{i+1}-x)^5 \\ x\in[x_{i},x_{i+1}) \\ \\ (x_{i+3}-x)^5 - 6(x_{i+2}-x)^5 \\ x\in[x_{i+1},x_{i+2}) \\ \\ (x_{i+3}-x)^5 \\ x\in[x_{i+2},x_{i+3}) \\ \\ 0 & \text{O.W} \end{cases}$$

2. Main results

To gain insight into the performance of the presented method, a numerical example is given in this section with L_2 and L_∞ errors.

Example 2.1. We consider the following equation [1]

$$\frac{\partial u}{\partial t} + \varepsilon\frac{\partial u}{\partial x} = \gamma\frac{\partial^2 u}{\partial x^2}\ , \quad 0\le x\le 1\ , \quad 0\le t\le 5$$

With the initial condition $\varphi(x) = \exp(\alpha x)$ and boundary conditions are

$$\left(\frac{\partial u}{\partial x}\right)_{(0,t)} = \alpha\exp(\beta t)\ , \quad \left(\frac{\partial u}{\partial x}\right)_{(1,t)} = \alpha\exp(\alpha+\beta t)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{(0,t)} = \alpha^2\exp(\beta t)\ , \quad \left(\frac{\partial^2 u}{\partial x^2}\right)_{(L,t)} = \alpha^2\exp(\alpha+\beta t)\ , \quad t\in[0,5]$$

In our computation,we take $h = 0.1$, $T = 5$, $k = 0.01$ .The results are computed for different time levels.

TABLE 1. $\varepsilon = 0.1$, $\gamma = 0.02$, $\alpha = 1.17712434446770$, $\beta = -0.09$

$Times$	L_2	L_∞
0.2	$1.5071e-09$	$4.4740e-09$
0.4	$2.9762e-09$	$8.8216e-09$
0.6	$1.3712e-08$	$2.0855e-08$
0.8	$2.7140e-07$	$3.3157e-07$
1.0	$2.0370e-06$	$2.0370e-06$
2.0	$1.2257e-04$	$1.8490e-04$
3.0	$4.6561e-04$	$7.5580e-04$
4.0	$8.6074e-04$	$1.4453e-03$
5.0	$1.1842e-03$	$2.0235e-03$

TABLE 2. $\varepsilon = 3.5$, $\gamma = 0.022$, $\alpha = 0.02854797991928$, $\beta = -0.0999$

$Times$	L_2	L_∞
0.2	$1.6489e-09$	$5.4572e-09$
0.4	$3.2615e-09$	$1.0806e-08$
0.6	$4.8422e-09$	$1.6048e-08$
0.8	$6.3916e-09$	$2.1187e-08$
1.0	$7.9103e-09$	$2.6225e-08$
2.0	$1.5065e-08$	$4.9956e-08$
3.0	$2.1540e-08$	$2.1540e-08$
4.0	$2.7399e-08$	$9.0863e-08$
5.0	$3.2701e-08$	$1.0845e-07$

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PROJECTIONS ON A LEFT QUATERNIONIC HILBERT SPACE

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Abstract. In this paper, we study projections on a quaternionic Hilbert space and prove similar properties of projections on complex Hilbert spaces in quaternionic vesion.

1. Introduction

Throughout this paper, $\mathbb{H}$ stands for the field of quaternions, i.e. all elements of the form $\mathbf{q} = x_0 + x_1 i + x_2 j + x_3 k$, where x_0, x_1, x_2 and x_3 are real numbers, x_0 is called real part of $\mathbf{q}$ and is denoted by $\mathrm{Re}(\mathbf{q})$ and i, j, k are the so-called imaginary units with the following multiplication rules:

$$i^2 = j^2 = k^2 = -1, \;\; ij = -ji = k, \;\; jk = -kj = i, \;\; \text{and} \;\; ki = -ik = j.$$

As we see, multiplication is not commutative in $\mathbb{H}$. The quaternionic conjugate of $\mathbf{q}$ is defined by $\overline{\mathbf{q}} = x_0 - x_1 i - x_2 j - x_3 k$, and $|\mathbf{q}| = \sqrt{\mathbf{q}\overline{\mathbf{q}}} = \sqrt{{x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2}$ is the absolute value of $\mathbf{q}$. For any two quaternions $\mathbf{q}_1$ and $\mathbf{q}_2$, $\overline{\mathbf{q}_1\mathbf{q}_2} = \overline{\mathbf{q}_2}\;\overline{\mathbf{q}_1}$. See [2] for more properties of the quaternions.

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* Speaker.

Let H be a linear vector space over the field of quaternions under left scalar multiplication. In order to emphasis more on the left scalar multiplication, a subspace of H is addressed as a left $\mathbb{H}$-linear subspace. A function $\langle .,. \rangle : \mathsf{H} \times \mathsf{H} \longrightarrow \mathbb{H}$, that satisfies the following properties:

(i) $\overline{\langle f, g \rangle} = \langle g, f \rangle$,
(ii) $\langle f, f \rangle > 0$ unless $f = 0$,
(iii) $\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle$,
(iv) $\langle \mathbf{q}f, g \rangle = \mathbf{q}\langle f, g \rangle$,

for all $f, g, h \in \mathsf{H}$ and $\mathbf{q} \in \mathbb{H}$, is called an inner product. The quaternionic norm of $f \in \mathsf{H}$ is defined by $\|f\| = \sqrt{\langle f, f \rangle}$ and satisfies all properties of a norm including Cauchy-Schwartz inequality (see [2], Proposition 2.2) and the parallelogram law:

$$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2), \quad f, g \in \mathsf{H}. \tag{1.1}$$

If $(\mathsf{H}, \|.\|)$ is a Hilbert space, then it is called a "left quaternionic Hilbert space". Most of the properties of complex Hilbert spaces are valid in left quaternionic Hilbert spaces, such as the Riesz representation theorem (see [5]). In fact, if $h : \mathsf{H} \longrightarrow \mathbb{H}$ is a left linear functional on a left quaternionic Hilbert space H, then there is a unique vector $x' \in \mathsf{H}$ such that $\| h \| = \| x' \|$ and $h(x) = \langle x, x' \rangle$. By left linearity, we mean that, for $\mathbf{p} \in \mathbb{H}$ and $x, y \in \mathsf{H}$, $h(\mathbf{p}x + y) = \mathbf{p}h(x) + h(y)$, and boundedness is eaquivalent to finiteness of

$$\|h\| := \sup\{|h(x)|, \|x\| = 1, x \in \mathsf{H}\}.$$

In a similar manner, one can define a left linear operator $T : \mathsf{H} \longrightarrow \mathsf{H}$, a bounded left linear operator and its norm. The set of all bounded left linear operators on H, that will be denoted by $\mathfrak{B}(\mathsf{H})$ throughout this paper, is a complete normed space (see [2, Proposition 2.11] for properties of $\mathfrak{B}(\mathsf{H})$). The adjoint of $T \in \mathfrak{B}(\mathsf{H})$, is the unique operator $T^* \in \mathfrak{B}(\mathsf{H})$, such that, for all $f, g \in \mathbb{H}$, $\langle Tf, g \rangle = \langle f, T^*g \rangle$. We refer the interested reader to [2, Theorem 2.15 and Remark 2.16] for all properties of the adjunction including $\|T\| = \|T^*\|$. It is worthwhile to note that, the adjoint operation is not an involution on $\mathfrak{B}(\mathsf{H})$, as $(\mathbf{q}T)^* \neq \overline{\mathbf{q}}T^*$ for a non-real $\mathbf{q} \in \mathbb{H}$. For the information of curious readers, Ghiloni et al. [2] considered a quaternionic construction for involution and Banach C^*-algebra and proved that $\mathfrak{B}(\mathsf{H})$ with the adjoint operation is a quaternionic two-sided Banach C^*-algebra with unity ([2], Theorem 3.4).

In the next section, we will study projections on a left quaternionic Hilbet space and seek properties to them, similar to those on complex Hilbert spaces.

2. Orthogonal Projections and Projections

In this section, we aim to investigate properties of projections on a left quaternionic Hilbert space. We confess that we pursue sections 2 and 4 of chapter I and section 3 of chapter II of [1] to extend results in complex case to the quaternionic one. We remind that the familiar concepts of convexity and orthogonality are the same as in any vector space and inner product space, respectively.

The quaternionic version of [3, Proposition 2.2.1] is proved similarly by the parallelogram law (1.1).

Theorem 2.1. *If K is a non-empty closed convex subset of a left quaternionic Hilbert space H and $h \in \mathsf{H}$, then there is a unique $h_0 \in K$ such that*

(i) $\|h - h_0\| = \inf\{\|h - k\| : k \in K\}$,
(ii) $Re\langle h_0, h - h_0\rangle \geq Re\langle f, h - h_0\rangle, f \in K$.

The next theorem is also proved as its complex version (see for instance the proof of [4, Lemma 3.3-2]).

Theorem 2.2. *Let M be a closed left $\mathbb{H}$-linear subspace of a left quaternionic Hilbert space H, $h \in \mathsf{H}$ and h_0 be the unique element of M such that $\|h - h_0\| = \inf\{\|h - k\| : k \in K\}$. Then $h - h_0 \perp M$. Conversely, if $h_0 \in M$ and $h - h_0 \perp M$, then $\|h - h_0\| = \inf\{\|h - k\| : k \in K\}$.*

In the following theorem, we summarize properties of P_M, the orthogonal projection of H onto a left $\mathbb{H}$-linear subspace M of H.

Theorem 2.3. *For a closed left $\mathbb{H}$-linear subspace M of H and $h \in \mathsf{H}$, let $P_M h$ be the unique point in M such that $h - P_M h \perp M$. Then*

(i) *P_M is a left linear operator on H,*
(ii) *$\|P_M h\| \leq \|h\|$ for every $h \in \mathsf{H}$,*
(iii) *$P_M^2 = P_M$,*
(iv) *$\ker P_M = M^\perp$ and $\mathrm{Ran P_M = M}$.*

For a subset A of H, let $\langle A\rangle$ denote the set of all finite left $\mathbb{H}$-linear combinations of elements of A. The closure of $\langle A\rangle$ is denoted by $\vee A$ and is called the closed left $\mathbb{H}$-linear span of A. It is easy to see that both $\langle A\rangle$ and $\vee A$ are left $\mathbb{H}$-linear subspaces of H. Also $\vee A$ is the smallest closed left $\mathbb{H}$-linear subspaces of H that contains A.

Proposition 2.4. *For an orthonormal set $\{e_1, ..., e_n\}$ in H, let $M = \vee\{e_1, ..., e_n\}$ and P_M be the orthogonal projection of H onto M, then for all $h \in \mathsf{H}$,*

$$P_M h = \sum_{k=1}^{n} \langle h, e_k\rangle e_k.$$

Corollary 2.5. ***Bessel's Inequality***. *If $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal set in H and $h \in \mathsf{H}$, then*

$$\sum_{n=1}^{+\infty} |\langle h, e_n\rangle| \leqslant \|h\|^2.$$

According to [2, Propositions 2.5 and 2.6], a left quaternionic Hilbert space admits an orthonormal basis and all such bases have the same cardinality.

A left linear operator $E \in \mathfrak{B}(\mathsf{H})$ that satisfies $E^2 = E$ is called an idempotent. An idempotent P for which $\ker P = (\mathrm{RanP})^{\perp}$ is said to be a projection.

Proposition 2.6. *E is an idempotent if and only if $I - E$ is an idempotent, in this case,* $\mathrm{RanE} = \ker(\mathrm{I} - \mathrm{E})$ *and* $\ker E = \mathrm{Ran}(\mathrm{I} - \mathrm{E})$ *and both* RanE *and* $\ker E$*, are $\mathbb{H}$-left linear subspaces of H and so* $\mathsf{H} = \mathrm{RanE} \oplus \ker \mathrm{E}$.

Following the proof of [1, Proposition 3.3], with slight modification, we obtain the same results for the quaternionic case.

Proposition 2.7. *For a non-zero idempotent E on H, the following are equivalent:*

(i) *E is a projection.*
(ii) *E is the orthogonal projection of H onto* RanE.
(iii) $\|E\| = 1$.
(iv) *E is self-adjoint, i.e. $E^* = E$,*
(v) *E is normal, i.e. $E^*E = EE^*$,*
(vi) *$\langle Eh, h\rangle \geq 0$ for all $h \in \mathsf{H}$.*

During this paper, we have studied properties of few kinds of left linear operators on a left quaternionic Hilbert sapce. The properties were similar to their corresponding ones in complex setting. Although, continuing the study to the spectral theory of such operators, drastic changes will happen, which needs more challenges [2, Section 4].

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LINEAR ALGEBRAIC METHODS APPLIED FOR POLYNOMIAL DIVISION

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Abstract. The current paper studies the division of a polynomial of degree n by a polynomial of degree m in the Lagrange basis and finding the quotient and remainder directly without changing the basis. The algorithms proceed by setting up systems of linear equations for the coefficients of the quotient and remainder. Then solutions are computed by using LU factorization method.

1. Introducing the System

Basic operations for polynomials represented in bases other than the usual power basis are being intensely studied. Division of univariate polynomials with respect to different bases has been widely studied. The idea of alternative bases is motivated by the desire to avoid computational cost and numeric errors incurred by converting between different polynomial bases [1]. Some well known polynomial division algorithms can be implemented using triangular and Toeplitz matrix

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inversion, which mostly proceed by setting up systems of linear equations for the coefficients of the quotient and remainder. Then solutions are computed by using SVD and LU factorization, respectively. (see [2, 3, 4])

The purpose of this work is the division of a polynomial of degree n by a polynomial of degree m in the Lagrange basis and finding the quotient and remainder directly without changing the basis. Starting from two polynomials $F(x)$ a polynomial of degree n given on the nodes $X = \{x_0, \cdots, x_n\}$, (the dividend) and nonzero polynomial $G(x)$ of degree m, given on the same nodes (the divisor), produces the quotient $Q(x)$ and remainder $R(x)$ of degrees $n-m$ and $m-1$, respectively, such that

$$F(x) = G(x)Q(x) + R(x). \tag{1.1}$$

Finding the coefficient of $Q(x)$ and $R(x)$ leads to the system of linear equations:

$$\left[\begin{array}{ccc|ccc} g_0 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & g_n & & & 1 \\ \hline W_{0,1} & \dots & W_{0,n+1} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ W_{m-1,1} & \dots & W_{m-1,n+1} & 0 & \dots & 0 \\ 0 & \dots & 0 & W_{0,1} & \dots & W_{0,n+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & W_{n-m,1} & \dots & W_{n-m,n+1} \end{array}\right] \left[\begin{array}{c} q_0 \\ \vdots \\ q_n \\ \hline r_0 \\ \\ \vdots \\ \\ r_n \end{array}\right] = \left[\begin{array}{c} f_0 \\ \vdots \\ f_n \\ \hline 0 \\ \\ \vdots \\ \\ 0 \end{array}\right], \tag{1.2}$$

in terms of $2n+2$ unknowns $q_0, q_q, \cdots, q_n,\ r_0, r_1, \cdots, r_n$, where

$$W_{k,j} = (-1)^k w_{j-1} S_{j-1}^k,\ k = 0, \cdots, max\{n-m, m-1\},\ j = 1, \cdots, n+1, \tag{1.3}$$

and S_{j-1}^k is the i**th order elementary symmetric function** associated with $X = \{x_0, \cdots, x_n\} - \{x_{j-1}\}$, and $w_j, j = 0, \cdots, n$ are the Barycentric weights of the Lagrange basis corresponding to x_j.

2. Solving the main system

In the matrix form, the system (1.2) can be presented in the matrix form:

$$\underbrace{\begin{pmatrix} G & I \\ M & N \end{pmatrix}}_{C} \begin{pmatrix} Q \\ R \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}, \tag{2.1}$$

where C is a $(2n+2)\times(2n+2)$ block matrix, consists of $(n+1)\times(n+1)$ matrices G, I, M, N, presented in (1.2). We are now in a position to solve the linear system (2.1) for the quotient and remainder $Q = [q_i]_{i=0}^n$ and $R = [r_i]_{i=0}^n$, and the right-hand vector is of size $2n+2$ in which the first $(n+1)$ elements are $F = [f_i]_{i=0}^n$. There exist several linear algebraic approaches for solving the system (1.2) without converting from Lagrange basis to the monomial basis, and they can have widely varying computational complexity. Since LU factors of the block coefficient matrix of the system (2.1) are structured, easy and cheap to find (which is shown in [3]) we first discuss the solvability of the system by LU method, then we plan to find the best possible symbolic-numeric method.

We plan to use the LU decomposition of the matrix C in (2.1) for solving the system. According to Maple results, the LU factorization of the coefficient matrix can be presented in the block form, as follows:

$$L = \begin{pmatrix} I & 0 \\ T & V \end{pmatrix}, \qquad U = \begin{pmatrix} G & I \\ 0 & V' \end{pmatrix}, \tag{2.2}$$

where G is a $(n+1)\times(n+1)$ diagonal matrix of diagonal elements g_i, I is the identity matrix of $(n+1)\times(n+1)$, and in what follows, we present the $(n+1)\times(n+1)$ matrices T, V and V' in terms of $W_{k,j}$ introduced in (1.3).

In the matrix $T_{(n+1)\times(n+1)}$, the first m rows are the nonzero rows, and can be presented as $T = \sum_{i=1}^{m}\sum_{j=1}^{n+1} \frac{W_{i-1,j}}{g_{j-1}}\, \mathcal{E}_{i,j}$.

To proceed in depth, we almost only concentrate on the special case $n = 4$ and $m = 1$ for presenting V and V'. The matrix V is a lower triangular matrix which can be shown as follows:

$$V_{i,j} = \begin{cases} 1, & i = j = 1 \\ -g_0\frac{W_{i-2,1}}{W_{0,1}} & i = 2,\cdots,5,\ j = 1 \\ \\ \frac{\chi_{i-2,j,j-1}^{j-1}}{\chi_{j-2,j,j-1}^{j-1}}, & i = 2,\cdots,5,\ j = 2,\cdots,5 \end{cases} \tag{2.3}$$

where the values of χ are recursively defined as:

$$\begin{aligned}
\chi^{(1)}_{i,j,k} &= W_{i,j} - \frac{g_0 W_{0,j} W_{i,j-k}}{g_{j-1} W_{0,j-k}}, \\
\chi^{(2)}_{i,j,k} &= \chi^{(1)}_{i,j,k} - \frac{\chi^{(1)}_{0,j,1}\chi^{(1)}_{i,2,1}}{\chi^{(1)}_{0,2,1}}, \\
\chi^{(3)}_{i,j,k} &= \chi^{(2)}_{i,j,k} - \frac{\chi^{(2)}_{1,j,3}\chi^{(2)}_{i,3,2}}{\chi^{(2)}_{1,3,2}},
\end{aligned} \tag{2.4}$$

and the matrix V' is an upper triangular matrix:

$$V'_{i,j} = \begin{cases}
\chi^{(1)}_{i,j,k} = \frac{-W_{i-1,j}}{g_{j-1}}, & i = 1, j = 1, \cdots, 5 \\
\chi^{(2)}_{i,j,k} = W_{i-2+k,j} - \frac{g_0 W_{i-2+k,1}}{g_{j-1}} & i = 2, j = 2, \cdots, 5, k = 0 \\
\chi^{(3)}_{i,j,k} = \chi^{*(2)}_{i,j,k} - \frac{\chi^{*(2)}_{i,i-1,k}\chi^{2}_{i,j,0}}{\chi^{2}_{i,i-1,0}} & i = 3, j = 3, \cdots, 5, k = 1 \\
\chi^{(n)}_{i,j,k} = \chi^{(n-1)}_{i,j,k} - \frac{\chi^{(n-1)}_{i,i-1,k}\chi^{(n-1)}_{i,j,0}}{\chi^{(n-1)}_{i,i-1,0}}, & i = 4, j = 4, 5, k = 1, n = 4, 5
\end{cases} \tag{2.5}$$

where $\chi^{*(2)}_{i,j,k} = W_{1,j} - \frac{g_0 W_{1,1} W_{i-2+k,j}}{W_{0,1} g_{j-1}}$. The matrix for other m and n can be calculated in the same way.

Finally, applying the LU decomposition method leads to the solution of the system (1.2). These results strengthen the motivation for examining algorithms for polynomial division. The similar discussion can be done directly in different bases, including the monomial basis, the Lagrange basis, the orthogonal bases, the Newton basis, the Pochhammer basis and the Bernstein basis.

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THE SOLUTION OF POSITIVE DEFINITE OR DIAGONLLY DOMINANT SYSTEMS

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Abstract. We propose a new direct method to solve linear systems. This method is based on the Sherman-Morrison formula and uses a finite iterative formula. If the coefficient matrix of linear system be a diagonally dominant or positive definite matrix, then this method can be actually carried out.

1. Introduction

Numerical linear algebra is a fundamental tool in several approximation techniques of many different fields. We present a novel method for linear system solution, based on Sherman-Morrison formula. The Sherman-Morrison formula and the corresponding generalization given by the Sherman-Morrison-Woodburry formula have been used in several applications, such as, the solution of special linear systems, the solution of linear systems arising in mathematical, ... [2].
Let N be a positive integer. Let $A \in \mu_R(N,N)$ and $b \in R^N$ that $\mu_R(N,N)$ is the space of real matrices having N rows and N columns and R^N is the N-dimensional real Euclidean space. We always suppose that A is a nonsingular matrix, that is $det(A) \neq 0$, and we consider the linear system

$$Ax = b \tag{1.1}$$

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Let $x = x^*$be the solution of (1.1).
The Sherman-Morrison formula, that gives the inverse of a rank-one perturbation of a matrix from the knowledge of unperturbed inverse matrix is given as folowing:
given$B \in \mu_R(N,N)$, $u, v \in R^N$, such that $det(B) \neq 0$, $det(B+uv^t) \neq 0$, we have

$$(B+uv^t)^{-1} = B^{-1} - \frac{B^{-1}uv^tB^{-1}}{1+v^tB^{-1}u}$$

see [3, p .25] for detailes.

1.1. **The numeical method.** An arbitrary matrix A can be always expressed as sum of a given A_0 and M rank-one matrices, that is

$$A = A_0 + u_1v_1^t + u_2v_2^t + ... + u_Mv_M^t \tag{1.2}$$

where u_j, v_j, $j = 1, 2, ..., M$. Then nember M depends on A, A_0, and on u_j, v_j, $j = 1, 2, ..., M$, however, we can always choose $M \leq N$ [4].

Remark 1.1. Let $A \in \mu_R(N,N)$ be a given matrix having diagonal entries different from zero. Then (1.2) can be always given by: $A_0 = diag(A)$, that is, the diagonal matrix with diagonal entries of A, $M = N$, and, for j=1,2,...,N, u_j equal to jth column of matrix $A - A_0$, and $v_j = e_j$, that is the jth element of the canonical basis of R^N, i.e. $e_j(j) = 1$, $j \neq k$, and $e_j(k) = 0$, if $j \in 1, ..., N$ [4].

Let us consider the coefficient matrix A in (1.1) expressed as in (1.2). For $l = 0, 1, ..., M$, let $A_l = A_0 + u_1v_1^t + ... + u_lv_l^t \in \mu_R(N,N)$, $x = x_l \in R^N$ be the solution of $A_lx = b$, and for $k = l+1, ..., M$, $y = y_{l,k} \in R^N$ be the solution of $A_ly = u_k$. When $l = M$ vector x_l is the solution of (1.1), more over for each $l > 0$, we have, $A_l = A_{l-1} + u_lv_l^t$, so from the Sherman-Morrison formula, we obtain following algorithm.

Algorithm1.1. Given nonsingular matrices $A, A_0 \in \mu_R(N,N)$, and given vector $u_j, v_j \in R^N$, $j = 1, 2, ..., M$, such that (1.2) holds, compute the approximate solution $x = x_M \in R^N$ of linear system (1.1) as the result following steps:
(i) compute the solution $x = x_0 \in R^N$ of $A_0x = b$;
(ii) if $M > 0$, then for $k = 1, 2, ..., M$, compute the solution $y = y_{0,k} \in R^N$ of $A_0y = u_k$, go to step (iii), otherwise go to step (v);
(iii) for $l = 1, 2, ..., M-1$ compute:

$x_l = x_{l-1} - \dfrac{v_l^t x_{l-1}}{1+v_l^t y_{l-1,l}} y_{l-1,l}$

$y_{l,k} = y_{l-1,k} - \dfrac{v_l^t y_{l-1,k}}{1+v_l^t y_{l-1,l}} y_{l-1,l},\ k = l+1, l+2, ..., M$

(iv) compute:

$x_M = x_{M-1} - \dfrac{v_M^t x_{M-1}}{1+v_M^t y_{M-1,M}} y_{M-1,M}$

(v) stop.

2. Main results

We have a breakdown of the algorithm when $1 + v_l^t y_{l-1,l} = 0$ for some $l = 1, 2, ..., M$, So that condition $1 + v_l^t y_{l-1,l} \neq 0$,$l = 1, 2, ..., M$ is necessary to assure that proposed algorithm can be actually carried out [4]. Following this thought we have:

Theorem 2.1. *Let $A_l \in \mu_R(N, N)$, $l = 0, 1, ..., M-1$ be nonsingular matrices computed by proposed algorithm can be actually carried out, and for vector $x_M \in R^N$ computed by this algorithm, we have $x_M = x^*$.*

Proof. See [4]. □

From the Sherman-Morrison-Woodbury formula, by denoting with $U, V \in \mu_R(N, N)$, the matrices having columns $u_l, v_l \in R^N$, $l = 1, 2, ..., M$, respectively, we have $A = A_0 + UV^t$ and for the solution of (1.1), we have

$$x^* = A_0^{-1}b + A_0^{-1}U(I_M + V^t A_0^{-1}U)^{-1}V^t A_0^{-1}b \tag{2.1}$$

that I_M is the M-dimensional unique matix [4]. Then, by attention to (2.1), we have following remark.

Remark 2.1. Algorithm 1.1 can be see as LU decomposition [1, p. 174] of matrix $I_M + V^t A_0^{-1}U$ appearing in (2.1) [4].

From Theorem 1.1, and Remark 2.1, we obtain following corollary.

Corollary. Let $A \in \mu_R(N, N)$ be a diagonally dominant matrix or a positive definite matrix. Let $A_0 \in \mu_R(N, N)$, and $u_l, v_l \in R^N$, $l = 1, 2, ..., M$ be chosen according to Remark1.1. Then proposed algorithm can be actually carried out, and we have $x_M = x^*$.

Proof. See [4]. □

2.1. **Numerical result.** We show some results obtained from an example with the method proposed in this paper. Consider following example

Example 2.2. Let

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$$

Matrix A is a diagonally dominant matrix, but A is symetric, therefore, A is a positive definite matrix, see [1, p .32]. Then, by proposed algorithm, the relative error is almost zero.

We believe that the proposed method can be a very interesting tool, however further investigations need to be considered.

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