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The 15th Seminar on Differential Equations and Dynamical Systems

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Keynote Talks

The 15th Seminar on
Differential Equations and Dynamical Systems

6-8 March 2021, University of Guilan, Rasht, Iran

SOME NEW TRENDS IN FRACTIONAL CALCULUS

DUMITRU BALEANU

Abstract. After 325 years the fractional calculus is a very important branch of mathematics with several top level applications in various fields of science and engineering. In this talk I will review some open problems within this emerging field and I will present some illustrative examples from mathematical biology.

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ATTRACTORS AND ATTRACTING INVARIANT GRAPHS

F. H. GHANE

Abstract. Many problems in dynamical systems deal with the study of the limit behavior of orbits, leading to the study of the so-called attractors. Understanding the structure of attractors is a major goal in the area. We consider random dynamical systems forced by a deterministic external factor. In general, systems of this kind are modeled, in discrete time, as skew products. Skew products are used as a tool for the construction of robust or generic examples of complicated behaviors. Here, we study invariant graphs which are the natural substitutes of attracting fixed points of autonomous systems and they play a central role in the theory. The main goal is to obtain skew products having attracting invariant graphs with a complicated dynamics exhibiting nice ergodic properties such as the existence of SRB measures whose supports lie on invariant graphs. The other goal here is to describe the structure of invariant graphs and study the properties of the pinching set, the set of points where the values of all of the invariant graphs coincide.

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Keywords: Milnor attractor, invariant graph, ergodic measure, SRB measure.
AMS Mathematical Subject Classification [2010]: 37C70, 37H99, 37D25 .

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DIFFERENT ASPECTS OF ASYMPTOTIC BEHAVIOR OF ZEROS OF MITTAG-LEFFLER FUNCTIONS

A.JODAYREE AKBARFAM

Abstract. In this paper, asymptotic behavior of zeros of Mittag-Leffler functions by using integral representation of Mittag-Leffler function has been studied. By using the results, the distribution of eigenvalues of FSLPS was derived.

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TRAJECTORY METRICS AND TRAJECTORY MANIFOLDS

M. MOULAEE

Abstract. In this talk we have an overview on our recent research works on the geometry of a class of the nature flows. This study leads us to define trajectory metrics and trajectory manifolds. In this direction we present an extension of the Levi-Civita connections. In this talk we answer to the following questions:
1. Suppose we have a system of ordinary differential equations on a smooth manifold M, can we find a linear connection ∇ on M such that each orbit of this system be a geodesic in (M, ∇)?
2. If there is such connection on M, is there any pseudo-Riemannian metric corresponding to ∇?

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LAPLACE AND P-LAPLACE PROBLEMS

A. RAZANI

Abstract. In this talk, we review Laplace and p-Laplace problems. Next we study the Lane-Emden problem and present some recent results.

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Talks

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THE ABLOWITZ-KAUP-NEWELL-SEGUR HIERARCHY AND RECURSION OPERATOR

FATEMEH AHMADI ZEIDABADI AND SAYED MOHAMMAD HOSEINI

Abstract. Integrable equations are an important class of nonlinear wave equations. Integrable equations are ubiquitous in mathematically-oriented scientific fields, such as physics and engineering. By using the general form Lax pair and the zero curvature equation, we can obtain two recursive operator to construct these equations. That, we will see that by giving values to free parameters in these two relation, integrable equations with different properties can be constructed.

1. Introduction

Integrable equations are an important class of nonlinear wave equations. Notable examples include the Korteweg–de Vries (KdV) equation, the Nonlinear Schrödinger (NLS) equation, the sine-Gordon equation, the Kadomtsev– Petviashvili (KP) equation, and many others. These equations have the remarkable property that their solitary waves collide elastically; i.e., they pass through each other without change of shape or speed. In addition, these equations can be integrated exactly by the inverse scattering transform method. Lax realized that if a nonlinear wave equation is the compatibility condition between two linear operators (now called the Lax pair) [3], then this equation would be integrable by the inverse scattering transform method. Based on this idea, Zakharov and Shabat showed that the NLS equation is also integrable [4]. Subsequently, Ablowitz et al. developed a method to derive a large class of integrable equations such as an integrable hierarchy [2]. From that time on,

Keywords: Integrable equation, Soliton wave, NLS equation, KdV equation.
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numerous advances have been made on integrable equations, and this field is still very active today.

2. AKNS hierarchy

To make the recursion operator, we consider write the Zakharov-Shabat system [1] in the following generalized form:

$$Y_x = My, \qquad M = \begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix}, \tag{2.1}$$

where (q, r) are functions of (x, t). Suppose the temporal equation for Y is

$$Y_t = Ny, \qquad N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.2}$$

then by cross-differentiation, we get the zero curvature equation,

$$M_t - N_x + [M, N] = 0, \tag{2.3}$$

$$-A_x - rB + qC = 0 \tag{2.4}$$

$$q_t - B_x - 2i\zeta B - q(A - D) = 0, \tag{2.5}$$

$$r_t - C_x + 2i\zeta C + r(A - D) = 0, \tag{2.6}$$

$$-D_x + rB - qC = 0. \tag{2.7}$$

$$A = \partial^{-1}(qC - rB) + A_0, \tag{2.8}$$

$$D = \partial^{-1}(rB - qC) + D_0, \tag{2.9}$$

$$\begin{pmatrix} q \\ -r \end{pmatrix}_t = i(2\zeta - L_R)\begin{pmatrix} B \\ C \end{pmatrix} + (A_0 - D_0)\begin{pmatrix} q \\ r \end{pmatrix}, \tag{2.10}$$

where

$$L_R = i\begin{pmatrix} \partial - 2q\partial^{-1}r & 2q\partial^{-1}q \\ -2r\partial^{-1}r & -\partial + 2r\partial^{-1}q \end{pmatrix}, \tag{2.11}$$

$$\begin{pmatrix} B \\ C \end{pmatrix} = \sum_{j=1}^{n}\begin{pmatrix} b_j \\ c_j \end{pmatrix}(2\zeta)^{n-j}, \tag{2.12}$$

and take

$$D_0 = -A_0 = i\zeta(2\zeta)^{n-1}. \tag{2.13}$$

$$\begin{pmatrix} q \\ -r \end{pmatrix}_t = -iL_R\begin{pmatrix} b_n \\ c_n \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} q \\ r \end{pmatrix}, \quad \begin{pmatrix} b_j \\ c_j \end{pmatrix} = L_R\begin{pmatrix} b_{j-1} \\ c_{j-1} \end{pmatrix}. \tag{2.14}$$

From these equations, we obtain the AKNS hierarchy

$$i\begin{pmatrix} q \\ -r \end{pmatrix}_t = L_R^n \begin{pmatrix} q \\ r \end{pmatrix}, \qquad n = 1, 2, ... \tag{2.15}$$

2.1. **The famous non-linear differential equation.** The hierarchy (2.15) contains many more equations when one takes different integers n or imposes different relations between q and r. For instance, taking $n = 2$ and $r = -q^*$, then (2.15) becomes the NLS equation

$$iq_t + q_{xx} + 2|q|^2 q = 0, \tag{2.16}$$

and when $n = 3$ and $r = 1$, (2.15) gives the KdV equation

$$q_t + q_{xxx} + 6qq_x = 0, \tag{2.17}$$

when $n = 3$ and $r = q$, (2.15) gives the modified KdV equation

$$q_t + q_{xxx} + 6q^2 q_x = 0, \tag{2.18}$$

and when $n = 3$ and $r = q$, (2.15) gives the complex modified KdV equation

$$q_t + q_{xxx} + 6|q|^2 q_x = 0, \tag{2.19}$$

2.2. **The KdV hierarchy.** By taking $r = 1$ and odd integers of n, i.e., $n = 2m + 1$, one can easily show by induction that the AKNS hierarchy (2.15) reduces to

$$\begin{pmatrix} q \\ 1 \end{pmatrix}_t = -\begin{pmatrix} K^m q_x \\ 0 \end{pmatrix}, \qquad m = 1, 2, ..., \tag{2.20}$$

i.e.,

$$q_t + K^m q_x = 0, \qquad m = 1, 2, ..., \tag{2.21}$$

which is the KdV hierarchy. Here the recursion operator K is

$$K = \partial^2 + 4q + 2q_x \partial^{-1}, \tag{2.22}$$

In other words, one can also leave q and r independent. In that case, coupled integrable equations for (q, r) would be obtained. For instance, if we take$n = 2$, then the AKNS hierarchy (2.15) becomes the following coupled Schrödinger equations:

$$\begin{aligned} iq_t + q_{xx} - 2q^2 r = 0, \\ ir_t - r_{xx} + 2r^2 q = 0. \end{aligned} \tag{2.23}$$

$$i\begin{pmatrix} q \\ -r \end{pmatrix}_t = \Omega(L_R) \begin{pmatrix} q \\ r \end{pmatrix}, \tag{2.24}$$

3. Conclusions

Using a Lax pair for nonlinear equations a recursive relation for Soliton equations which have a wide range of applications from mathematics to physics has been found.

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CONSERVATION LAWS FOR THE MATRIX NONLINEAR SCHRÖDINGER EQUATION

FATEMEH AHMADI ZEIDABADI AND SAYED MOHAMMAD HOSEINI

Abstract. Matrix nonlinear Schrödinger equation is a famous integrable equation, thus it is solvable with Inverse Scattring Transform (IST) method. Here, by using the *Lax pair*, the recursive relation for the conservation laws is found. Some of the first conserved quantities are explicitely determined. The procedure has been succesfully apllied to construct the conserved quantities for the normal nonlinear Schrödinger equation, but the results for matrix versions have not been reported in literature.

1. Introduction

Nonlinear PDEs that admit conservation laws arise in many disciplines of the applied sciences including physical chemistry, fluid mechanics, particle and quantum physics, plasma physics, elasticity, gas dynamics, electromagnetism, magnetohydrodynamics, nonlinear optics, and the biosciences. Conservation laws are fundamental laws of physics that maintain that a certain quantity will not change in time during physical processes. Familiar conservation laws include conservation of momentum, mass (matter), electric charge, or energy.
There are many reasons to compute conserved densities and fluxes of PDEs explicitly. Invariants often lead to new discoveries as was the case in soliton theory. For example, an infinite sequence of conserved densities is a predictor of the existence of

Keywords: Integrable equation, NLS equation, IST theory , Conservation Laws.
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solitons and complete integrability which means that the PDE can be solved with the Inverse Scattering Transform (IST) method [2, 4, 5].
We consider the Matrix nonlinear Schrödinger (MNLS) equation of the form

$$i\partial_t \mathcal{Q} + \partial_x^2 \mathcal{Q} + 2\mathcal{Q}\mathcal{Q}^\dagger \mathcal{Q} = 0, \qquad \mathcal{Q} = \begin{pmatrix} \phi_+ & \phi_0 \\ \phi_0 & \phi_- \end{pmatrix}. \tag{1.1}$$

This equation is integrable, *i.e.*, is the compatibility condition of two linear equations.

2. Lax pair for MNLS equation

In this subsection, we derive Lax-pair by AKNS hierarchy. The scattering operator associated with this hierarchy is the following higher-order degenerate system:

$$Y_x = WY, \qquad W = \begin{pmatrix} -i\zeta I_k & q \\ r^T & i\zeta I_m \end{pmatrix}, \qquad Y_t = HY, \qquad H = \begin{pmatrix} A & B \\ C^T & D \end{pmatrix} \tag{2.1}$$

where q and r are $k \times m$ matrix functions of (x, t), and I_k, I_m are kth- and mth-order identity matrices. And the matrix W has the same block structure as the matrix H. Then using the zero curvature equation

$$W_t - H_x + [W, H] = 0 \tag{2.2}$$

, we get

$$\partial_x Y = ik[\lambda, Y] + \hat{\mathcal{Q}}Y, \qquad \partial_t Y = 2ik^2[\lambda, Y] + VY, \tag{2.3}$$

where $\lambda = diag(-1, -1, 1, 1)$ and $\hat{\mathcal{Q}} = \begin{pmatrix} 0 & \mathcal{Q} \\ -\mathcal{Q}^\dagger & 0 \end{pmatrix}$, $\quad V = 2k\hat{\mathcal{Q}} + i\begin{pmatrix} \mathcal{Q}\mathcal{Q}^\dagger & \mathcal{Q}_x \\ \mathcal{Q}_x^\dagger & -\mathcal{Q}^\dagger\mathcal{Q} \end{pmatrix}$,
and k is a spectral parameter.

3. Rank-one soliton solution

To obtain the rank-one soliton solution of the matrix NLS equation (1.1), we consider the single pair k_1 and k_1^* of zeros and the eigenvector $|1\rangle$. The eigenvector takes the form

$$|1\rangle = (e^{-ik_1x - 2ik_1^2t} n_1, e^{-ik_1x - 2ik_1^2t} n_2, e^{ik_1x + 2ik_1^2t} n_3, e^{ik_1x + 2ik_1^2t} n_4)^T, \tag{3.1}$$

where $n_a, a = 1, .., 4$ are complex numbers. We set $k_1 = \mu + i\nu$ and find the solution of the RH problem.
Indeed, the rank-one soliton (1.1) can be represented as

$$\mathcal{Q} = 2\nu \begin{pmatrix} e^{-i\chi}\cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & e^{i\chi}\sin^2\theta \end{pmatrix} e^{i\varphi} \; sech \; z, \tag{3.2}$$

where

$$\cos\theta = \frac{|n_1|}{\sqrt{(|n_1|^2 + |n_2|^2)}} = \frac{|n_3|}{\sqrt{(|n_3|^2 + |n_4|^2)}}, \quad \chi = arg(n_3) - arg(n_4), \tag{3.3}$$

$$\varphi = -2\mu x - 4(\mu^2 - \nu^2)t + \varphi_0, \qquad \varphi = arg(n_1) - arg(n_4) = arg(n_2) - arg(n_3), \tag{3.4}$$

$$z = 2\nu(x + 4\mu t) + \rho, \quad e^{2\rho} = \frac{|n_1|^2 + |n_2|^2}{|n_3|^2 + |n_4|^2}, \tag{3.5}$$

The soliton amplitude is determined by the parameter ν, and its velocity is equal to 4μ . The parameters ρ and φ_0 give the initial position of the soliton center and its initial phase, respectively. The angle θ determines the normalized population of atoms in different spin states, while the phase factor $e^{i\chi}$ is responsible for the relative phases between the components $\phi_\pm$ and ϕ_0.

4. Infinite number conservation laws

Let us consider a solution $Y = (y_1, y_2, y_3, y_4)^T$ to the Lax pair (2.3). Defining

$$\mu^{(1)} = \frac{y_1}{y_4}, \qquad \mu^{(2)} = \frac{y_2}{y_4}, \qquad \mu^{(3)} = \frac{y_3}{y_4}, \tag{4.1}$$

then it is easy to find from this Lax pair that

$$(Lny_4)_x = i\zeta - \phi_0^* \mu^{(1)} - \phi_-^* \mu^{(2)}. \tag{4.2}$$

Using the temporal equation (2.3) of this hierarchy, an analogous equation for $(lny_4)_t$ can also be derived. Cross-differentiating these two equations with respect to t and x, respectively, we see that if we expand $\mu^{(1)}$, $\mu^{(2)}$ and $\mu^{(3)}$ into the power series,

$$\mu^{(j)}(x, t, \zeta) = -\sum_{n=1}^{\infty} \frac{\mu_n^{(j)}(x, t)}{(-2i\zeta)^n}, \quad j = 1, 2, 3. \tag{4.3}$$

then $\phi_0^* \mu_n^{(1)} + \phi_-^* \mu_n^{(2)}$ would be the density of a local conservation law, and an infinite number of conserved quantities are

$$I_n = \int_{-\infty}^{\infty} (\phi_0^* \mu_n^{(1)} + \phi_-^* \mu_n^{(2)}) \mathrm{d}x. \tag{4.4}$$

To determine the expansion coefficients $\mu_n^{(1,2)}$, we use the scattering equation (2.3). Simple calculations show that $\mu^{(1,2,3)}$ satisfy the following coupled *Riccati* system :

$$\mu_x^{(1)} = -(2i\zeta - \phi_-^* \mu^{(2)}) \mu^{(1)} + \phi_0^* (\mu^{(1)})^2 + \phi_0 + \phi_+ \mu^{(3)}, \tag{4.5}$$

$$\mu_x^{(2)} = -(2i\zeta - \phi_0^* \mu^{(1)}) \mu^{(2)} + \phi_-^* (\mu^{(2)})^2 + \phi_- + \phi_0 \mu^{(3)}, \tag{4.6}$$

$$\mu_x^{(3)} = (\phi_0^* \mu^{(1)} + \phi_-^* \mu^{(2)}) \mu^{(3)} - (\phi_+^* \mu^{(1)} + \phi_0^* \mu^{(2)}). \tag{4.7}$$

Then by inserting $\mu^{(1,2,3)}$' s expansions (4.3) into this Riccati system and equating terms of the same power in $(-2i\zeta)^n$, we find that $\mu_n^{(1,2)}$ are given as

(4.8)

$$\mu_1^{(1)} = \phi_0, \quad \mu_2^{(1)} = \mu_{1x}^{(1)} - \phi_+ \mu_1^{(3)}, \quad \mu_1^{(2)} = \phi_- \quad \mu_2^{(2)} = \mu_{1x}^{(2)} - \phi_0 \mu_1^{(3)}, \quad \mu_1^{(3)} = -\int_0^x (\phi_+^* \phi_0 + \phi_0^* \phi_-) \mathrm{d}x,$$

and

$$\mu_{k_x}^{(1),(2)} = \mu_{k+1}^{(1),(2)} + \phi_{+,0}\mu_k^{(3)} - \phi_{-,0}^*\sum_{\ell=1}^{k-1}\mu_\ell^{(2)}\mu_{k-\ell}^{(1)} - \phi_{0,-}^*\sum_{\ell=1}^{k-1}\mu_\ell^{(1),(2)}\mu_{k-\ell}^{(1),(2)}, \tag{4.9}$$

$$\mu_{k_x}^{(3)} = -\phi_+^*\mu_k^{(1)} - \phi_0^*\mu_k^{(2)} - \phi_0^*\sum_{\ell=1}^{k-1}\mu_\ell^{(3)}\mu_{k-\ell}^{(1)} - \phi_-^*\sum_{\ell=1}^{k-1}\mu_\ell^{(3)}\mu_{k-\ell}^{(2)}. \tag{4.10}$$

Then the infinite number of conserved quantities (4.4) are obtained. The first two conserved quantities are

$$I_1 = \int_{-\infty}^{\infty}(\phi_0^*\phi_0 + \phi_-^*\phi_-)\mathrm{d}x, \quad I_2 = \int_{-\infty}^{\infty}(\phi_0^*\mu_2^{(1)} + \phi_-^*\mu_2^{(2)})\mathrm{d}x, \tag{4.11}$$

which are the mass (or power) of MNLS equation (1.1). Higher conserved quantities can be similarly calculated. If one also wishes to obtain the flux functions of local conservation laws, then one can first use the temporal equation of the Lax pair to derive the $(lny_4)_t$ equation, then insert the expansions (4.3). The coefficients of $(lny_4)_t$ at various orders of ζ^{-n} would then be the fluxes of local conservation laws.

5. Conclusion

An important property of the MNLS equation is that it possesses an infinite number of conservation laws. In the above derivation, the conserved quantities (4.4) were obtained only from the first equation of the Lax pair.

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ERGODICITY OF GEODESIC FLOWS ON HECKE SURFACE

D. AHMADI AND S. LAMEI

Abstract. Let $\mathbb{D}$ be the Poincare disc and G_α be the Hecke group. The geodesic flow on Hecke surface $\mathbb{D}/G_\alpha$ has rich dynamics. We projected this dynamics to the boundary of $\mathbb{D}$ and introduced an invariant ergodic measure both for the geodesic flows and the induced map on the boundary.

1. Introduction

Let $\mathbb{D} = \{z = x+iy : |z| < 1\}$ be the Poincare disc with hyperbolic metric $ds = \frac{|dz|}{1-|z|^2}$. With this metric, the geodesics are the diameters and the arcs perpendicular to the unit circle. Consider the Hecke group G_α with generators $T_\alpha(z) = z+\alpha$ and $S(z) = \frac{-1}{z}$. Here, $\alpha = 2\cos(\frac{\pi}{q})$ where $q \in \mathbb{N} \setminus \{1, 2\}$. The Hecke surface M_α is defined as $\mathbb{D}/G_\alpha$ which is topologically a sphere with two singularities and a cusp. Let SM_α be the unit tangent bundle of M_α. For any vector in SM_α there exists a unique geodesic tangent to v. Any geodesic γ has a repelling endpoint w and an attracting one u. For any disjoint $w, u \in \partial\mathbb{D}$ there exists a unique oriented geodesic $\gamma = (w, u)$ connecting w to u. Geodesic flow moves the set of vectors in SM_α tangent to a geodesic in the direction of the geodesic with speed one.

2. A suitable cross section to code the geodesic flows on M_α

Let F_α be the Dirichlet fundamental region associated to the Hecke surface M_α. The region F_α is a hyperbolic triangle with sides on geodesics. For any side s_i, there exists an element (generator) g_i of G_α such that $g_i(s_i)$ lies on another side s_j. To verify the dynamics of the geodesic flows on M_α we use the Morse method.

From here on fix an α. Let $u_i(w)$ be the repelling endpoint of a geodesic with attracting endpoint w passing through the vertex v_i. For each side s_i, $1 \le i \le 3$ set

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1. $C_{i1} = \{(w,\, u)\, :\, a_i < w < b_{i-1},\, u_{i+1}(w) < u < u_i(w)\}$, consisting of the geodesics cutting the inside of side s_i and exiting F.
2. $C_{i2} = \{(w,\, u)\, :\, a_i < w < b_i,\, u = u_i(w)\}$, consisting of the geodesics (w, u) passing through v_i with $a_i < w < b_{i-1}$ and $b_i < u < a_{i-1}$.
3. $C_{i3} = \{(w,\, u)\, :\, b_{i-1} < w < b_i,\, u = u_i(w)\}$, consisting of the geodesics (w, u) passing through v_i with $b_{i-1} < w < b_i$ and $a_{i-1} < u < a_i$.

Set $C_i = \cup_{j=1}^{3} C_{ij}$ and $C = \cup_{i=1}^{3} C_i$. See Figure 1. To each oriented geodesic $\gamma = (w, u)$ in $\mathbb{D}$ we correspond a point $(w, u) \in \mathbb{R}^2$. Therefore, all the points in $\mathbb{D}$ correspond to the set $I^2 \setminus \{(w,\, u)\, : w = u\}$ where I is the unit interval. Suppose the sides s_i of the region F_α lie on geodesics $(a_i,\, b_i)$, $1 \leq i \leq 3$. The maps T_α and S induce a map T_C on C defined as follows:

$$T_C(w,\, u) = \begin{cases} S(w,\, u) = (\frac{-1}{w},\, \frac{-1}{u}) & \text{on } C_1 \\ T_\alpha^{-1}(w,\, u) = (w - \alpha,\, u - \alpha) & \text{on } C_2 \\ T_\alpha(w,\, u) = (w + \alpha,\, u + \alpha) & \text{on } C_3. \end{cases}$$

By the successive actions of T_α and S on F_α, the copies of F_α tiles the entire disc $\mathbb{D}$. There are 6 copies of F_α around each vertex v_1 and v_2 and there ate infinitely many copies of F_α around the vertex v_3. According to this tiling the following lemma arises.

Lemma 2.1. *Let* $c_1 = T_\alpha(a_1)$, $c_2 = T_\alpha(a_2) c_3 = T_\alpha(a_3)$, $c_4 = T_\alpha^{-1}(a_1)$, $c_5 = T_\alpha^{-1}(a_3)$. *Then*

$$\begin{aligned} T_\alpha(a_1,\, b_1) &= (c_1,\, c_3), & T_\alpha(a_2,\, b_2) &= (c_2,\, b_2), \\ T_\alpha(a_3,\, b_3) &= (b_2,\, a_2), & T_\alpha(c_4,\, c_1) &= (a_1,\, b_1), \\ T_\alpha(a_3,\, c_5) &= (a_3,\, b_3), & S(a_1,\, b_1) &= (b_1,\, a_1), \\ S(a_2,\, b_2) &= (c_4,\, c_1), & S(a_3,\, b_3) &= (c_1,\, c_3). \end{aligned}$$

Let $C'_i = T_C(C_i)$ and $C' = \cup_{i=1}^{3} C'_i$. Since the sides of C are curvilinear, we construct a rectangular region R with a map $T_R : R \to R$ such that T_C and T_R are conjugate. Let R be the rectangular part as in Figure 1. let R_i, $0 \geq i \leq 2$ be the smaller rectangles from left to right in Figure 1. Define T_R as follows

$$T_R(w,\, u) = \begin{cases} S(w,\, u) = (\frac{-1}{w},\, \frac{-1}{u}) & \text{on } R_0 \cup R_1 \\ T_\alpha^{-1}(w,\, u) = (w - \alpha,\, u - \alpha) & \text{on } R_2 \\ T_\alpha(w,\, u) = (w + \alpha,\, u + \alpha) & \text{on } R_3. \end{cases}$$

Consider the regions

$$\begin{aligned} U_0 =& \{(w,\, u) \in C \setminus R\, :\, w \in [b_3,\, c_1)\} \subset C_1, \\ U_1 =& \{(w,\, u) \in C \setminus R\, :\, w \in [c_1,\, a_2)\} \subset C_1, \\ U_2 =& \{(w,\, u) \in C \setminus R\, :\, w \in [c_3,\, b_2)\} \subset C_2, \\ U_3 =& \{(w,\, u) \in C \setminus R\, :\, w \in [a_3,\, c_4)\} \subset C_3. \end{aligned}$$

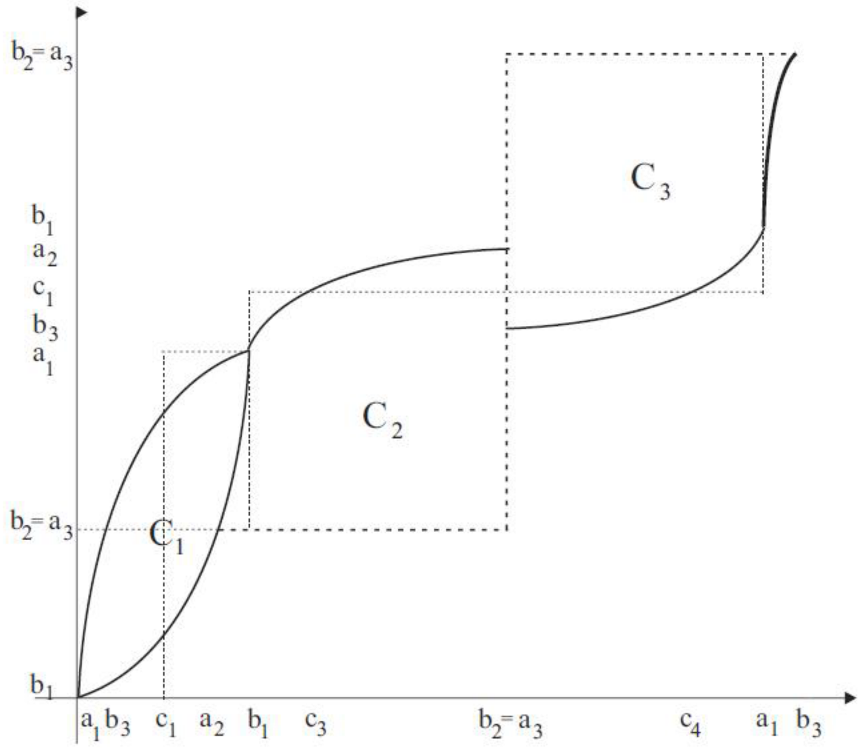

FIGURE 1. The border of C and R are shown by thick and dashed lines respectively. The common border is shown in thick dashed lines.

Theorem 2.2. *The map $\Phi : C \to R$ defined as*

$$\Phi = \begin{cases} ST & \text{on } U_0 \\ ST^{-1} & \text{on } U_1 \\ TSTS & \text{on } U_2 \\ T^{-1}ST^{-1}S & \text{on } U_3 \\ Id & \text{on } C \cap R. \end{cases}$$

forms a conjugacy between T_C and T_R.

3. PROJECTING THE DYNAMICS ON COORDINATES

Theorem 3.1. *The dynamical systems (Σ_R, T_R) is 1-step Markov space.*

The rectangles $\mathcal{P}_R = \{R_0, R_1, R_2, R_3\}$ form a partition for R. Let $I_0 = [a_1, c_1)$, $I_1 = [c_1, b_1)$, $I_2 = [b_1, b_2)$, $I_3 = [b_2, b_3)$ be the intervals on w-axis and $J_0 = [b_1, b_2)$, $J_1 = [b_2, a_1)$, $J_2 = [a_1, c_1)$, $J_3 = [c_1, b_2)$ the intervals on u-axis. Both $\mathcal{P}_I$ and $\mathcal{P}_J$ are intervals on $\partial\mathbb{D}$. The maps T_R and T_R^{-1} induce factor maps f and g on $\cup_{i=0}^{3} I_i$ and $\cup_{j=0}^{3} J_j$ respectively. Here, $T_R(w, u) = (f(w), \cdot)$ and $T_R^{-1}(w, u) = (\cdot, g(u))$.

Theorem 3.2. *The map T_R preserves the measure $dm = \frac{|dw|\,|du|}{|w-u|^2}$. For a Borel subset A of $\partial\mathbb{D}$, let $\mu(A) = \int_A h(w)|dw|$ and $\nu(A) = \int_A k(u)|du|$ where*

$$h(w) = \begin{cases} \int_{b_1}^{b_2} \frac{|du|}{|w-u|^2} & \text{on } w \in I_0 \\ \int_{b_2}^{a_1} \frac{|du|}{|w-u|^2} & \text{on } w \in I_1 \\ \int_{b_2}^{c_1} \frac{|du|}{|w-u|^2} & \text{on } w \in I_2 \\ \int_{c_1}^{b_2} \frac{|du|}{|w-u|^2} & \text{on } w \in I_3, \end{cases}$$

and

$$h(w) = \begin{cases} \int_{a_1}^{c_1} \frac{|dw|}{|w-u|^2} & \text{on } u \in J_0 \\ \int_{c_1}^{b_2} \frac{|dw|}{|w-u|^2} & \text{on } u \in J_1 \\ \int_{b_1}^{b_2} \frac{|dw|}{|w-u|^2} & \text{on } u \in J_2 \\ \int_{b_2}^{b_3} \frac{|dw|}{|w-u|^2} & \text{on } u \in J_3. \end{cases}$$

The maps f and g preserve the measures μ and ν.

Theorem 3.3. *The maps T_C, T_R, f and g are ergodic with respect to Lebesgue measure.*

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ALGEBRAIC STABILITY OF THE NUMERICAL METHODS FOR ODES

RANA AKBARI AND GHOLAMREZA HOJJATI

Abstract. For general linear methods (GLMs) as to include linear multistep and Runge–Kutta methods as special cases, a concept known as algebraic stability is defined. The algebraic stability property is based on a non-linear test problem which extends present results on Runge–Kutta methods and linear multistep methods. In this paper, algebraic stability properties of a number of particular methods in GLMs are studied.

1. Introduction

In considering the applicability of a numerical method for the solution of stiff ordinary differential equations, it is appropriate to investigate stability of general linear methods with respect to the standard test equation. A general linear method used for the numerical solution of a non-autonomous initial value problem (IVP)

$$\begin{aligned} &y'(x) = f(t, y(t)), \qquad f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m, \\ &y(0) = 0, \end{aligned} \tag{1.1}$$

has both multistage and multistep structure. We consider a multistep method in which at the start of the nth integration step, r points in $\mathbb{R}^m$, say $y^{[n]} = [y_i^{[n-1]}]_{i=1}^r$ contain all the information about the solution that is to be used in carrying out the step. At the end of a step for calculating the corresponding information, we assume that s internal stages are needed and we denote the values computed in these stages as $Y^{[n]} = [Y_i^{[n]}]_{i=1}^s$ and the derivatives evaluated by $f(Y^{[n]}) = [f(t_{n-1} + c_i h, Y_i^{[n]})]_{i=1}^s$.

Keywords: General linear methods, Algebraic stability, A-stability, Linear multistep methods, Runge–Kutta methods.

The vector $y^{[n]} = [y_i^{[n]}]_{i=1}^r$ denotes the output vector at the step number n.
GLMs with s internal stages and r external stages are defined by

$$\begin{aligned} Y^{[n]} &= h(A \otimes I_m) f(Y^{[n]}) + (U \otimes I_m) y^{[n-1]}, \\ y^{[n]} &= h(B \otimes I_m) f(Y^{[n]}) + (V \otimes I_m) y^{[n-1]}. \end{aligned} \tag{1.2}$$

This formulation of GLMs was introduced by Burrage and Butcher [8] in 1980. This paper is concerned with the stability analysis of GLMs. Several types of stability are considered using more general test problems. The study is split into two sections including: the linear stability with respect to the standard linear test equation and the non-linear stability (algebraic stability) with respect to the non-autonomous test equation. It should be noted that, in this paper we have not introduced a new method, but a property of the concepts of stability for the GLMs was studied. Based on this criterion, GLMs with this proprety could be constructed. However, due to the page limit of this seminar, we have not appotunity to present examples of the constructed methods with this property.

2. Linear stability of GLMs

To analyze the linear stability properties of GLMs, it is applied to the linear test problem of Dahlquist [2], $y' = \xi y, \xi \in \mathbb{C}$.
Then (1.2) follows that $Y^{[n]} = (I - zA)^{-1} U y^{[n-1]}$, therefore $y^{[n]} = M(z) y^{[n-1]}$, where the matrix $M(z)$ is defined by

$$M(z) = V + zB(I - zA)^{-1}U,$$

and is called the stability matrix of GLM. For certain methods applied to stiff problems, the stepsize necessary for stability may be excessively small. This means stability rather than accuracy is restricting the stepsize. To ensure there is no restriction on the stepsize, A-stability condition is desired. A GLM is A-stable if for all $z \in \mathbb{C}^-$, $I - zA$ is non-singular and $M(z)$ is a stable matrix.

3. Algebraic stability of GLMs

A-stability deals with the behavior of the method when applied to a linear, autonomous differential equation. This concept is valuable, but it has a disadvantage. It says little about the behavior of the method applied to problems that are either non-autonomous or non-linear or both. Recently, other stability definitions have been suggested for overcoming this weakness. In the case of linear multistep methods, the stability of non-linear problems has been studied through the idea of G-stability [3] while in the case of Runge–Kutta methods, the same type of model has been used under the name B-stability [5]. The concepts of AN-, BN-, B-, and algebraic stability were investigated by Butcher [1, 6, 7]. The concept of AN-stability is a generalization of A-stability and is related to the scalar, linear, and non-autonomous test equation

$$\begin{cases} y' = \xi(t)y, & t \geq 0, \\ y(0) = y_0, \end{cases}$$

$\mathrm{Re}(\xi(t)) \leq 0$, where $\xi(t)$ has an arbitrarily complex-valued. Consider (1.1) for $t \geq 0$, where $(f(t,y_1) - f(t,y_2))^T(y_1 - y_2) \leq 0$, for all $t \geq 0$ and $y_1, y_2 \in \mathbb{R}^m$. Applying GLM to test equation, we obtain

$$y^{[n+1]} = \mathbf{S}(\xi)y^{[n]}, \qquad n = 0, 1, \ldots.$$

where $\xi = \mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_s) = \mathrm{diag}(h\xi(t_n + c_1 h), \ldots, h\xi(t_n + c_s h))$, and

$$\mathbf{S}(\xi) = \mathbf{V} + \mathbf{B}\xi(\mathbf{I} - \mathbf{A}\xi)^{-1}\mathbf{U}.$$

To define AN-, G- and algebaric stability, let $G = [g_{ij}]_{i,j=1}^{r}$ be a real, symmetric and positive definite matrix, and for a vector $y \in \mathbb{R}^{mr}$, consider the inner product norm

$$\|y\|_G^2 = \sum_{i=1}^{r}\sum_{j=1}^{r} g_{ij} y_i^T y_j.$$

Definition 3.1. GLM is said to be AN-stable if for all $\xi = \mathrm{diag}(\xi_1, \ldots, \xi_s)$, there exists a real, symmetric, and positive definite matrix G such that

$$\|\mathbf{S}(\xi)y\|_G \leq \|y\|_G,$$

so that $\xi_i = \xi_j$ whenever $c_i = c_j$ and such that $\mathrm{Re}(\xi_i) \leq 0$ for $i = 1, 2, \ldots, s$.

Definition 3.2. GLM is said to be G-stable if for two numerical solutions, $\{y^{[n]}\}_{n=0}^{N}$ and $\{\widetilde{y}^{[n]}\}_{n=0}^{N}$, there exists a real, symmetric, and positive definite matrix $G \in \mathbb{R}^{r\times r}$ such that

$$\|y^{[n+1]} - \widetilde{y}^{[n+1]}\|_G \leq \|y^{[n]} - \widetilde{y}^{[n]}\|_G,$$

where $\|.\|_G$ is the norm defined for all $h > 0$. For given $G \in \mathbb{R}^{r\times r}$ and $D \in \mathbb{R}^{s\times s}$, define the matrix M by the formula

$$M := \left[\begin{array}{c|c} DA + A^T D - B^T G B & DU - B^T G V \\ \hline U^T D - V^T G B & G - V^T G V \end{array}\right].$$

Definition 3.3. GLM is said to be algebraically stable if there exist a real, symmetric, and positive definite matrix G and a real, diagonal, and positive definite matrix D such that the matrix M is nonnegative definite [8].

There are some relations between the types of stability defined above. Algebraic stability is equivalent to AN-stability and both of them imply A-stability [4].
For example, we study the property of algebraic stability on the implicit trapezoidal method given by

$$Y_1 = y_n + \frac{1}{2}hf(t_{n-1} + \frac{h}{2}, Y_1),$$
$$y_{n+1} = y_n + hf(t_{n-1} + \frac{h}{2}, Y_1).$$

Coefficients partitioned matrix for this method is given by

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{c|c} \frac{1}{2} & 1 \\ \hline 1 & 1 \end{array}\right].$$

We find

$$M = \left[\begin{array}{c|c} \frac{D}{2} + \frac{D}{2} - G & D - G \\ \hline D - G & 0 \end{array}\right], \tag{3.1}$$

which is a nonnegative definite matrix with $G = D = 1$. Therefore, the implicit trapezoidal method is algebraically stable.

As the another example, for the implicit Euler method

$$\begin{aligned} Y_1 &= y_n + hf(t_{n-1} + h, Y_1), \\ y_{n+1} &= y_n + hf(t_{n-1} + h, Y_1), \end{aligned}$$

where can be represented by a partitioned matrix

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & 1 \end{array}\right],$$

this yields

$$M = \left[\begin{array}{c|c} 2D - G & D - G \\ \hline D - G & 0 \end{array}\right].$$

It can be shown that the matrix M is nonnegative definite for $G = D = 1$. Therefore, the implicit Euler method is algebraically stable.

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GLOBAL EXISTENCE, ASYMPTOTIC BEHAVIOR AND BLOW-UP OF SOLUTION FOR NONLINEAR KIRCHHOFF TYPE EQUATION WITH BOUNDARY FEEDBACK CONTROL

HAJAR ANSARI AND MAHMOUD HESAARAKI

Abstract. The goal of this paper is to study the initial-boundary value problem for a nonlinear Kirchhoff type equation with boundary feedback control. For this problem, we show the local and global existence and uniqueness of solution via Faedo-Galerkin method and potential well theory. We also consider the asymptotic behavior of solutions. Making use of integral inequalities and multiplier technique, we establish exponential decay of solutions. By taking advantage of a method used in Vitillaro, we also show that the energy function grows-up as exponential function when $t \to +\infty$. Moreover, the blow-up of solutions is established for arbitrary initial energy by using the modified concavity method.

1. Introduction

Let $\Omega \subset \mathbf{R}^n$, $n \geq 1$, be a bounded domain with a boundary $\partial\Omega$ of class C^2. Assume that (Γ_0, Γ_1) is a partition of $\partial\Omega$, with Γ_0 and Γ_1 to have positive Lebesgue measures and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. Let ν be the unit outward normal on Γ_1. In this paper, we consider

Keywords: Kirchhoff type wave equation, blow-up of solution, boundary feedback control, exponential decay of solution, grow-up of solution .

AMS Mathematical Subject Classification [2010]: 35L10 , 35L20, 35L70 .

the following nonlinear Kirchhoff type wave equation with boundary feedback control,

$$
\begin{cases}
u_{tt} - M(\|\nabla u\|^2)\Delta u - \lambda\Delta u_t + \delta u_t + \eta \displaystyle\sum_{i=1}^{n} u_{tx_i} = \mu |u|^{p-1}u, & (x,t) \in \Omega_T, \\
u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), & x \in \Omega, \\
u(x,t) = 0, & (x,t) \in \Sigma_0, \\
M(\|\nabla u\|^2)\dfrac{\partial u}{\partial \nu} + \lambda \dfrac{\partial u_t}{\partial \nu} + g(u_t) = 0, & (x,t) \in \Sigma_1.
\end{cases} \tag{1.1}
$$

This problem describes transverse vibration of an axially moving string. Here T is a positive constant or $T = \infty$, $\Omega_T = \Omega \times (0,T)$, $\Sigma_0 = \Gamma_0 \times (0,T)$, $\Sigma_1 = \Gamma_1 \times (0,T)$ and $M(\|\nabla u\|^2) = m_0 + \alpha \|\nabla u(t)\|^{2\gamma}$ with $m_0 > 0$, $\alpha \geq 0$ is the tension in the string. While $\gamma, \lambda, \delta, \eta > 0$ and $p > 1$ are constants. Moreover, $u(x,t)$ is the transversal displacement of the strip at spatial coordinate x and time t, $-\lambda\Delta u_t$ and δu_t is called a strong damping term and a weak one, respectively. Furthermore, u_0 and u_1 are the initial displacement and velocity of the string respectively. Finally, $\eta \sum_{i=1}^{n} u_{tx_i}$ is the Coriolis force term. If the transport speed of the string is positive, $\eta > 0$, the control problem (1.1) is called downstream boundary control and if the transport speed of the string is negative, $\eta < 0$, the control problem (1.1) is called upstream boundary control. The boundary velocity feedback control, g, is assumed to be a nondecreasing C^1 function and $g(0) = 0$. Furthermore, there exist positive constants a and b such that

$$
\begin{cases}
(g(s) - g(r))(s - r) \geq a|s - r|^2, & \forall s \in \mathbf{R}, \\
|g(s)| \leq b|s|, & \forall s \in \mathbf{R}.
\end{cases} \tag{1.2}
$$

Recently, Gazzola and Squassina [2] considered the following equation

$$
u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = |u|^{p-2}u, \tag{1.3}
$$

with the Dirichlet boundary condition on a bounded domain Ω. They proved global existence of solutions with suitable initial data. They also showed blow-up of solutions with high-energy initial data. Bilgin and Kalantarov [1] considered the following initial boundary value problem

$$
u_{tt} - \nabla\big((a_0 + a|\nabla u|^{m-2})\nabla u\big) - b\Delta u_t = g(x,t,u,\nabla u) + |u|^{p-2}u,
$$

with the Dirichlet boundary condition on a bounded domain Ω. They also gave some sufficient conditions on initial data for which blow up occurs in a finite time. Kim et al. [3] considered

$$
u_{tt} - M(x,t,\|\nabla u\|^2)\Delta u + \rho(x,t,u_t,\nabla u,\nabla u_t) = 0,
$$

with boundary feedback control on a bounded domain, $\Omega \subset \mathbf{R}^n$, with smooth boundary. They have proven existence and uniqueness of strong solution by using some techniques of functional analysis.

2. Main results

In this section, we state our main results. First, we study the local existence of solutions of problem (1.1). We show that this problem has a unique weak local solution by using Galerkin method.

Theorem 2.1. *(local existence) Assume that $p > 1$ for $n = 1, 2$ and $1 < p \leq \frac{n}{n-2}$ for $n \geq 3$ and (1.2) holds. Furthermore,*

$$\begin{cases} u_0 \in H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega) \ \text{ and } \ u_1 \in H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega), \\ (m_0 + \alpha\|\nabla u_0\|^{2\gamma})\dfrac{\partial u_0}{\partial \nu} + \lambda \dfrac{\partial u_1}{\partial \nu} + g(u_1) = 0. \end{cases} \tag{2.1}$$

Then problem (1.1) has a unique local strong solution, $u(t)$ such that

$$\begin{cases} u \in C\big([0, T_0], H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)\big) \cap C^1\big([0, T_0], H^1_{\Gamma_0}(\Omega)\big), \\ u_t \in C\big([0, T_0], H^1_{\Gamma_0}(\Omega)\big) \cap L^2\big(\Omega \times (0, T_0)\big). \end{cases}$$

Moreover, at least one of the following statements holds

(i) $T_0 = +\infty$.

(ii) $\lim_{t \to T_0^-} \|u_t(t)\|^2 + \|\nabla u(t)\|^2 = +\infty$.

Now, we study global existence of solutions for problem (1.1) and show that this problem has a unique weak global solution.

Theorem 2.2. *Suppose the assumptions in Theorem 2.1 hold and $p > 1$ for $n = 1, 2$ and $1 < p \leq \frac{n}{n-2}$ for $n \geq 3$ and (1.2). Let u be the solution to problem (1.1). Then problem (1.1) has a unique global strong solution.*

In the following, we will state that blow-up occurs for the unique solution of problem (1.1) for suitable initial conditions when $g(u_t)$ is linear i.e. $g(u_t) = bu_t$. We will use the concavity argument developed by Levine [4].

Theorem 2.3. *Let $u(t)$ be the solution of problem (1.1), under the assumptions $p \geq 2\gamma + 1$, $\delta, \lambda > \max\big\{\frac{\sqrt{n}B^2_{\frac{1}{2}}\eta}{2}, \frac{(p+5+\sigma)\sqrt{n}B^2_{\frac{1}{2}}\eta + (p+1)bB^2_{\frac{1}{2}}}{2(p-1)-\sigma}, \lambda^*\big\}$, $b > \max\big\{\frac{(p+1)\sqrt{n}\eta\epsilon}{2\sqrt{n}\eta\epsilon-(p+5)}, \frac{\sqrt{n}\eta}{2}\big\}$, $0 < \eta < \sqrt{\frac{m_0(p-1)(p+3)}{2n}}$ and $u_0, u_1 \in H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)$ where σ, ϵ and λ^* will be introduced in the proof. Assume that either one of the following statements holds*
(i) $I_2(0) \leq 0$ *and* $0 < E(0) < d$;
(ii) $E(0) \leq 0$;
Then the solution $u(x, t)$ blows up in finite time t_1, provided that the initial conditions satisfy $\int_\Omega u_0 u_1 dx > 0$. Moreover,

$$t_1 \leq \frac{2}{\sqrt{(p-1-\sqrt{n}\epsilon\eta)\vartheta}} \ln \frac{\gamma_1 \Phi(0) + \frac{p-1-\sqrt{n}\epsilon\eta}{4}\Phi'(0)}{\gamma_2 \Phi(0) + \frac{p-1-\sqrt{n}\epsilon\eta}{4}\Phi'(0)},$$

where $\Phi(0) = \|u_0\|^2 + H_0 + 1$, $2H_0 = \left(\delta - \frac{1}{2}\sqrt{n}B_{\frac{1}{2}}^2\eta\right)\|u_0\|^2 + \left(\lambda - \frac{1}{2}\sqrt{n}B_{\frac{1}{2}}^2\eta\right)\|\nabla u_0\|^2 + b\|u_0\|^2 dx$,$\Phi'(0) = 2\int_\Omega u_0u_1dx + \delta\|u_0\|^2 + \lambda\|\nabla u_0\|^2 + b\|u_0\|^2 + \eta\int_{\Gamma_1}(u_0^2, ..., u_0^2).\nu ds$ *and* γ_1, γ_2 *and* ϑ *will be introduced in the proof.*

Now, we show that the energy function grows-up exponentially as time goes to infinity, provided that the initial data are large enough. The key ingredient in the proof is the method which is used by Vitillaro in [5].

Theorem 2.4. *(exponential decay) Suppose the assumptions in Theorem 2.2 hold. Then the solution u to problem* (1.1) *satisfies the following energy decay estimates*

$$E(t) \leq (1 - \kappa\epsilon)^{-1}(1 + \kappa\epsilon)^{-1}L(0)\exp\big(-\sigma(1 + \kappa\epsilon)^{-1}t\big), \quad t \geq 0,$$

where $L(0) = E(0) + \epsilon\int_\Omega u_0u_1dx + \frac{\epsilon\lambda}{2}\|\nabla u_0\|^2$ *and* σ, κ, ϵ *are positive constants, which will be determined in the proof.*

In the following, we established an exponential growth result for certain solutions with positive initial energy to problem (1.1).

Theorem 2.5. *Suppose that the assumption in Theorem 2.2 hold and* $p > 2\gamma + 1$ *. Let u be a solution to problem* (1.1) *with initial data satisfying* $\int_\Omega u_0u_1dx > 0$, $< E(0) < d$ *and* $\|\nabla u_0\| > \beta_0$. *Then u grows up as an exponential function, where* β_0 *will be introduced in the proof.*

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ADVANCES IN COMBINATION THERAPIES OF CANCER

MEHRDAD ANVARI, HAMIDREZA MARASI, AND HOSSEIN KHEIRI

Abstract. Targeting tumor microenvironment is the best way to extinguish cancer cells that leads to novel therapies of cancer. The goal of immunotherapy is to direct the patient's own immune system to fight cancer. In recent years, the focus on further development has shifted to combination therapies. In this paper, we review some successfull approaches that tackle and address this challenge through mechanistic modeling and simulation. In these models, a systems biology network is built and then estimated antitumor effects to reproduce several observed, in vivo, experimental results using a system of ordinary differential equations, in silico.

1. Introduction

Cancer is the second cause of death in the world and the third in Iran (after cardiovascular disease and accidents). In 2020, the number of new cases in Iran was 131191 person and the number of death was 79136 person. Top 5 most frequent cancers was breast (12.9%) , stomach (11.2%), colorectum (9.1%), lung (8%) and prostate (6.8%) [8]. This shows importance of cancer in Iran and it's rising trend in recent years.

1.1. **Cancer-Immunity Cycle.** Cancer is a disease of the DNA that leads to uncontrolled growth of cells from mutations in the some specific genes such as oncogenes, tumor suppressor genes, and repair genes [9]. However, immune system can recognizes and controls tumor via a fundamental process namely Cancer-Immunity Cycle

Keywords: Mathematical Oncology, Tumor Heterogeneity, Cancer Immunotherapy, Systems Biology, Cancer Stem Cells.

AMS Mathematical Subject Classification [2010]: 92C42, 92B05.

(CIC) [2]. Dendritic cells (DCs) capture the tumor specific antigens (TSAs) that released by dying cancer cells. Then, DCs present this proteins on MHC class I and II molecules to T-cells that result in the activation of of effector T-cells. This activated T-cells traffic to and infiltrate the tumor site and then after recognition of cancer cells, kill them resulting in release of additional TSAs or TAAs[1] and therfore repeating the cycle[2]. The process that immune system keeps a tumor in check is called cancer immunosurveillance. cancer can evolve another process to escape from immune response namely immunoediting [5].

Cancer is complex and traditional therapies i.e. surgery, chemotherapy and radiotherapy, have failed in treatment of cancer cells, specifically at eliminating cancer stem cells [7, 2]. Manipulation in several points of CIC, which results in targeting cancer cells specifically, gives different novel and effective methods that called immunotherapy of cancer. Thsese therapies are classified with respect to treatment strategies: non-specific Immune stimulation, adoptive cell transfer, immune-checkpoint blockade and vaccination strtegies such as DC vaccines, CAR T-cell Therapy, Checkpoint therapy via targeting PD1/PD-L1 and CTLA4 checkpoint inhibitors, and etc [2, 6, 3].

1.2. **Cancer Stem Cells.** It is proven in several last years that most tumors are composed of different types of cells and are heterogeneous [3]. The cancer stem cell theory states that this heterogeneity is, at least in part, due to cancer stem cells (CSCs). CSCs are a sub-population of tumor which have significant potential for self-renewal and capacity to differentiate into all other types of cancer cells that found in several types of cancer [7, 3]. Therefore, A tumor comprise of a small number of CSCs and other non CSCs (nCSCs). In fact, cancer cells within a tumor form a hierarchy of cancer cell types that CSCs are in apex and fully differentiated cancer cells (DCCs) located at the end of pyramid. Progenitor cancer cells (PCCs), that have some potential for self-renewal and can produce some of the cancer cells (not as strong as CSCs), lie between CSCs and DCCs [7]. For further reading and an in-depth discussion of biological context, the reader is referred to [2, 6, 3].

1.3. **Mathematical modeling of cancer-immune interaction.** Despite initial success, immunotherapies are not effective in all patients. Thus, we need mechanistic understanding of the reasons for patient variability and development of diagnostic methods for pateint selection. From these reasons, the focus is on combination therapies. However, we faced a vast number of possible combination targets and regimens that is a important challenge.

Konstorum et al [5] reviewed several modeling approaches under the umbrella of the major challenges, which consists of tumor classification, optimal treatment scheduling, and combination therapy design. Chelliah et al [1] classified several work

[1]Tumor-Associated Antigens

using quantitative systems pharmacology (QSP) to tackle the challenge of combination therapy selection. The majority of these dynamic mathemathical models (QSP models) are developed in an ordinary differential equation (ODE) framework. The authours claimed that there are 119, 17 and 5, repectively, ODE, PDE and other spatial QSP models that researchers used in articles.

Kirschner and Panetta [4] constructed a three dimensional ODE system that describes the dynamics of effector (E) and tumor (T) cells, and the cytokine IL-2 (I_L):

$$\begin{aligned}
\frac{dE}{dt} &= cT - \mu_2 E + \frac{p_1 E I_L}{g_1 + I_L} + s_1, \\
\frac{dT}{dt} &= r_2(T)T - \frac{aET}{g_2 + T}, \\
\frac{dI_L}{dt} &= \frac{p_2 ET}{g_3 + T} - \mu_3 I_L + s_2
\end{aligned} \tag{1.1}$$

Where c is the antigenicity of the tumor. The term $r_2(T)T$ models the growth of the tumor and can be exponential, logistic or gompertzian growth.

Sigal et al [7] construct a 7 dimentional ODE system that models the interaction of population of CSCs (S), nCSCs (P), CSC-specific (T_S) and nCSC-specific (T_P) activated T-cells, CSC-specific (D_S) and nCSC-specific (D_P) mature DCs and the concentration of chemotherapeutic agent (C):

$$\begin{aligned}
\frac{dS}{dt} &= \alpha_S S + \rho_{PS} P - \rho_{SP} S - \beta_S S T_S - \delta_S S - \Gamma_S S C \\
\frac{dP}{dt} &= \alpha_P P + \alpha_{SP} S + 2\rho_{SP} S - \rho_{PS} P - \beta_P P T_P - \delta_P P - \Gamma_P P C \\
\frac{dT_S}{dt} &= \kappa_{T_S} T_S^n \frac{D_S}{s_{T_S} + D_S} - \delta_{T_S} T_S \\
\frac{dT_P}{dt} &= \kappa_{T_P} T_P^n \frac{D_P}{s_{T_P} + D_P} - \delta_{T_P} T_P \\
\frac{dD_S}{dt} &= \gamma_{D_S} DS - \beta_{D_S} D_S T_S - \delta_{D_S} D_S \\
\frac{dD_P}{dt} &= \gamma_{D_P} DP - \beta_{D_P} D_P T_P - \delta_{D_P} D_P \\
\frac{dC}{dt} &= -e_c C
\end{aligned} \tag{1.2}$$

2. Main results

In (1.1), if we consider therapy with s_1, there is cirtical point s_1^{crit} that for $s > s_1^{crit}$ the tumor-free equilibrium is stable. The bifurcation diagram of s_1^{crit} vs. c (drawn in **XPPAUTO**) represent regions where depending on c, the tumor will either die or survive. The model shown IL-2 doesn't affect seriously on dynamics of system.

In (1.2), simulations (using **ODE15s** toolbox in **MATLAB**), with respect to Host-specific, Non-host-specific and Drug-specific parameters, reveals that:

- CSCs leads to significant tumor growth, whereas nCSCs leads to negligible tumor growth
- Vaccination with CSC-specific DCs is more effective than nCSC-specific and mixed Vaccine

- CSC-specific T-cell therapy is more effective than nCSC-specific and mixed Vaccine
- Combination of chemotherapy with CSC-specific immunotherapy (both DC vaccine and T-cell therapy) is more successfull, compared with nCSC-specific
- T-cell treatment befor chemotherapy is less effective over short time-scale and is more effective over long time-scale
- DC treatment after chemotherapy is more effective over both short time-scale and long time-scale
- Chemotherapy enhances tumorigenicity whereas CSC-specific immunotherapy decreases it

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The 15^{th} Seminar on
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THE RADIUS OF COMPARISON IN C*-DYNAMICAL SYSTEMS

MOHAMMAD B. ASADI

Abstract. In this talk, we will review the notion of radius of comparison, as an algebraic approach for study of C*-dynamical systems.

1. Introduction

A C*-dynamical system is a triple (A, G, α) consisting of a C*- algebra A, a locally compact group G, and a continuous homomorphism $\alpha\colon G \to \mathrm{Aut}(A)$. In our setting, C*-dynamical system is usually shortened to just "dynamical system". It is known that a locally compact transformation group (G, X) gives us a dynamical system with the commutative C*-algebra $A = C(X)$. Also, all dynamical systems with a commutative C*-algebra A arise from locally compact transformation groups.

The radius of comparison, based on the Cuntz semigroup, is a numerical invariant which was introduced by Andrew Toms in 2006 to study a specific class of C*-algebras [10]. The Cuntz semigroup and the radius of comparison both play an important role in the Elliott program for the classification of C*-algebras. Roughly speaking, among a special class of C*-algebras, the classifiable ones are those whose Cuntz semigroups are easily accessible or whose radii of comparison are zero. With the near completion of the Elliott program, attention is turning to nonclassifiable C*-algebras [4, 7, 8, 9, 12].

Keywords: C*-algebras; Crossed products; Weak tracial Rokhlin property; Radius of comparison.

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The main goal of this talk is to give a quick overview of the relation between the radius of the comparison of the simple unital C*-algebra A and the radius of the comparison of the crossed product associated with the dynamical system (A, G, α) when G is a finite group and α is sufficiently outer.

2. Main results

We begin with the following definitions for the convenience of the reader.

Definition 2.1. A C*-algebras is a Banach $*$-algebras over $\mathbb{C}$ with an involution satisfying $\|aa^*\| = \|a\|^2$.

Definition 2.2. An *approximately homogeneous (AH) algebra* is the inductive limits of C*-algebras of the form

$$p_1 M_{n_1}(C(X_1))p_1 \oplus p_2 M_{n_2}(C(X_2))p_2 \oplus \ldots \oplus p_l M_{n_l}(C(X_l))p_l$$

where $l, n_1 \ldots n_l$ are in $\mathbb{Z}_{>0}$, $X_1, \ldots, X_l$ are compact Hausdorff spaces, and where p_j is a (non-zero) projection in $M_{n_j}(C(X_j))$ for each j.

Definition 2.3. Let G be a group, taken with the discrete topology, and let A be a C*-algebra. An *action* of G on A is a group homomorphism $\alpha\colon G \to \mathrm{Aut}(A)$, written $g \mapsto \alpha_g$.

Definition 2.4. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A.

(1) The *crossed product*, denoted $A \rtimes_\alpha G$, is the universal unital C*-algebra generated by a copy of A and a unitary representation $g \mapsto u_g$ of G which implements the action, in the sense that $u_g a u_g^* = \alpha_g(a)$ for all $g \in G$ and $a \in A$.
(2) The *fixed point algebra*, denoted A^α, is

$$A^\alpha = \big\{a \in A\colon \alpha_g(a) = a \text{ for all } g \in G\big\}.$$

Definition 2.5. Let A be a unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. The action α has the *Rokhlin property* if, for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $p_g \in A$ for $g \in G$ such that

(1) $\|\alpha_g(p_h) - p_{gh}\| < \varepsilon$ for all $g, h \in G$.
(2) $\|p_g a - a p_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
(3) $\sum_{g \in G} p_g = 1$.

We call $(p_g)_{g \in G}$ a *family of Rokhlin projections* for α, F and ε.

A generalization of the Rokhlin property under the name "weak tracial Rokhlin property", which uses positive elements instead of projections, is fairly common

For a C*-algebra A, we denote the set of positive elements of A by A_+.

Definition 2.6. Let A be a C*-algebra. Let $m, n \in \mathbb{Z}_{>0}$, let $a \in M_n(A)_+$, and let $b \in M_m(A)_+$. We say that a is *Cuntz subequivalent* to b in A, written $a \precsim_A b$, if there exists a sequence $(x_k)_{k=1}^{\infty}$ in $M_{n,m}(A)$ such that $\lim_{k\to\infty} x_k b x_k^* = a$.

We let $\mathrm{T}(A)$ denote the set of normalized traces on a unital C*-algebra. For a given $\tau \in \mathrm{T}(A)$, one can extend it to $M_n(A)$ by $\tau_n((a_{jk})) = \sum_{j=1}^{n} \tau(a_{jj})$.

The following is Definition 6.1 of [10], except that we allow $r = 0$ in (2). This change makes no difference.

Definition 2.7. Let A be a stably finite exact unital C*-algebra.

(1) For every $\tau \in \mathrm{T}(A)$ and every $a \in \bigcup_{k=1}^{\infty} M_k(A)_+$, define

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}}).$$

(2) For $r \in [0, \infty)$, A has r-*comparison* if whenever $a, b \in \bigcup_{k=1}^{\infty} M_k(A)_+$ satisfy $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in \mathrm{T}(A)$, then $a \precsim_A b$.

(3) The *radius of comparison* of A, denoted $\mathrm{rc}(A)$, is

$$\mathrm{rc}(A) = \inf\big(\big\{r \in [0,\infty) \colon A \text{ has } r\text{-comparison}\big\}\big).$$

(We take $\mathrm{rc}(A) = \infty$ if there is no r such that A has r-comparison.)

The following conjecture is proposed by N. Christopher Phillips (see [1]).

Conjecture 2.8. Let A be a stably finite unital C*-algebra and let X be a compact metric space. Then

$$\mathrm{rc}(A) \leq \mathrm{rc}\left(C(X) \otimes A\right) \leq \frac{1}{2}\dim(X) + \mathrm{rc}(A) + 1, \tag{2.1}$$

where $\dim(X)$ is the covering dimension of X.

In [1], we prove that the left-hand side of (2.1) is true for any stably finite unital C*-algebra A and any compact metric space X. We further give a class of AH algebras A with $\mathrm{rc}\left(C(X) \otimes A)\right) = \mathrm{rc}(A)$. So, (2.1) is also true if A is chosen from this class of AH algebras [1].

Theorem 2.9. *(Theorem 4.5 of [3]) Let A be an infinite-dimensional stably finite simple unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite G on A which has the weak tracial Rokhlin property. Then*

$$\mathrm{rc}(A^\alpha) \leq \mathrm{rc}(A) \quad \textit{and} \quad \mathrm{rc}\left(A \rtimes_\alpha G\right) \leq \frac{1}{\mathrm{card}(G)} \cdot \mathrm{rc}(A).$$

In Section 6 of [3], the authors give a example of a dynamical system (A, G, α) in which $G = \mathbb{Z}/2\mathbb{Z}$, A is a simple unital AH algebra, α has the Rokhlin property, $\mathrm{rc}(A) > 0$, $\mathrm{rc}(A^\alpha) = \mathrm{rc}(A)$, and $\mathrm{rc}\big(A \rtimes_\alpha G\big) = \frac{1}{2}\mathrm{rc}(A)$.

The most obvious open problem is whether equality always holds in Theorem 2.9.

Question 2.10. Let G be a finite group, let A be an infinite-dimensional stably finite simple unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A which has the weak tracial Rokhlin property. Does it follow that

$$\mathrm{rc}(A^{\alpha}) = \mathrm{rc}(A) \quad \text{and} \quad \mathrm{rc}\left(A \rtimes_{\alpha} G\right) = \frac{1}{\mathrm{card}(G)} \cdot \mathrm{rc}(A)?$$

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EQUICONTINUITY AND ITS WEAKER FORMS IN FORT TRANSFORMATION SEMIGROUPS

FATEMAH AYATOLLAH ZADEH SHIRAZI AND SEYED MOHAMMADREZA MORTAZAVI

Abstract. In the following text we prove that all fort transformation groups are weakly almost periodic, also we compare two notions of equicontiuity and weakly almost periodicity in categories of infinite Fort transformation groups and infinite Fort transformation semigroups via a diagram and several counterexamples.

1. Introduction

By a transformation semigroup (resp. group) (S, X) we mean a compact Hausdorff topological space X, and a subsemigroup (resp. subgroup) S of continuous self–maps on X containing identity map $id_X : \underset{x \mapsto x}{X \to X}$. In transformation semigroup (S, X) we denote the closure of S in X^X with pointwise convergence topology (product topology) by $E(S, X)$ and call it the enveloping semigroup or Ellis semigroup of (S, X). It is well–known that $E(S, X)$ is a semigroup under the operation of composition of maps. We say (S, X) is weakly almost periodic if $E(S, X)$ consists of continuous self–maps on X [3]. For more details on transformation semigroups and their enveloping semigroup we refer the interested reader to [5].
Suppose $b \in F$ and equipp it with topology $\{U \subseteq F : b \notin U \vee (F \setminus U \text{ is finite})\}$, we call F a Fort topological space with particular point b [7]. In brief words a Fort topological

Keywords: enveloping semigroup, Ellis semigroup, Fort space, transformation group, weakly almost periodic.

AMS Mathematical Subject Classification [2010]: 54H15, 54H20.

space, is just one–point–compactification of a discrete space. Fort transformation semigroups and dynamical systems have been studied in several papers like [2].

2. Main results

In the following text suppose:

- F is an infinite Fort space with particular point b,
- $\mathcal{G} := Homeo(F, F)$ is the collection of all homeomorphisms $h : F \to F$,
- $\mathcal{S} := C(F, F)$ is the collection of all continuous self–maps $g : F \to F$.

Remark 2.1. $f : F \to F$ is continuous if and only if one of the following conditions holds [1]:

- $F \setminus f^{-1}(f(b))$ is finite,
- $f(b) = b$ and for all $x \in F \setminus \{b\}$ the set $f^{-1}(x)$ is finite.

Also b is unique limit point of F, thus $f : F \to F$ is a homeomorphism if and only if it is bijective and $f(b) = b$.

In transformation group (G, F) and transformation semigroup (S, F) we have $G \subseteq \mathcal{G}$ and $S \subseteq \mathcal{S}$, hence $E(G, F) \subseteq E(\mathcal{G}, F)$ and $E(S, F) \subseteq E(\mathcal{S}, F)$.
In the following lemma we prove that $E(\mathcal{G}, F)$ is a collection of continuous self–maps on X, therefore enveloping semigroup of a Fort transformation group is a collection of continuous self–maps on F and it is weakly almost periodic.

Lemma 2.2. *$E(\mathcal{G}, F) \subseteq \{p \in F^F : \exists D \subseteq F \setminus \{b\}\ (p|_D : D \to F \setminus \{b\}$ is one–to–one and $p|_{F \setminus D} = b)\}$.*

Proof. Suppose $p \in E(\mathcal{G}, F)$. There exists a net $\{g_\alpha\}_{\alpha \in \Gamma}$ in $\mathcal{G}$ with $\lim\limits_{\alpha \in \Gamma} g_\alpha = p$ (in F^F). Thus $pb = \lim\limits_{\alpha \in \Gamma} g_\alpha b = \lim\limits_{\alpha \in \Gamma} b = b$, hence $pb = b$. Let $D := p^{-1}(F \setminus \{b\})$, by $pb = b$ we have $b \notin D$. For $z, w \in D$ with $b \neq y := pz = pw$ we have $\lim\limits_{\alpha \in \Gamma} g_\alpha z = pz = y$ and $\lim\limits_{\alpha \in \Gamma} g_\alpha w = pw = y$ and $\{y\}$ is an open subset of F, thus there exists $\beta \in \Gamma$ such that

$$\forall \alpha \geq \beta \ \ (g_\alpha z \in \{y\} \wedge g_\alpha w \in \{y\})$$

in particular $g_\beta z = y = g_\beta w$, since $g_\beta : F \to F$ is a homeomorphism (and bijective) we have $z = w$. Hence $p|_D : D \to F \setminus \{b\}$ is one–to–one. □

Theorem 2.3. *Fort transformation group (G, F) is weakly almost periodic.*

Proof. Using Remark 2.1 and Lemma 2.2 we have

$$E(G, F) \subseteq E(\mathcal{G}, F) \subseteq C(F, F)(= \mathcal{S}) \ .$$

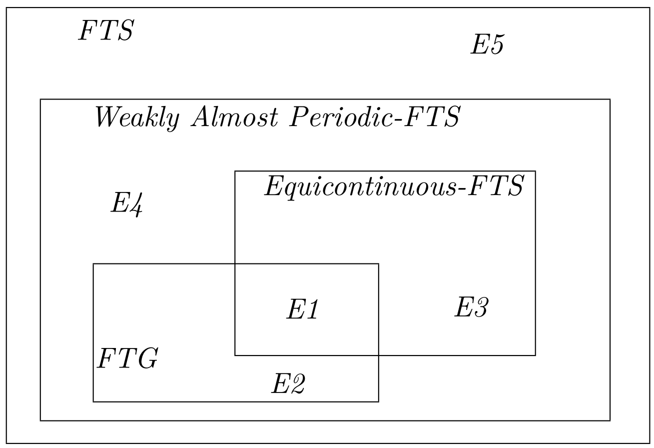
□

Remark 2.4. We say the transformation semigroup (S,X) with uniform structure $\mathcal{H}$ is equicontinuous if for every $\alpha \in \mathcal{H}$ there exists $\beta \in \mathcal{H}$ such that $\{(sx,sy) : s \in S, (x,y) \in \beta\} =: S\beta \subseteq \alpha$, in equicontinuous transformation semigroup (S,X) all elements of $E(S,X)$ are continuous (for more details on equicontinuous transformation semigroups see [5], also for details on uniform spaces we refer the interested reader to [4, 6]. In Fort space F for $D \subseteq F \setminus \{b\}$ let

$$\alpha_D := ((F \setminus D) \times (F \setminus D)) \cup \{(a,a) : a \in D\} \, ,$$

then $\mathcal{H}_F := \{\alpha \subseteq F \times F :$ there exists finite subset D of $F \setminus \{b\}$ with $\alpha_D \subseteq \alpha\}$ is unique compatible uniform structure on F.

Corollary 2.5. *We have the following diagram:*



where:

- *"FTS" denotes the class of Fort transformation semigroups*
- *"Weakly Almost Periodic-FTS" denotes the class of weakly almost periodic Fort transformation semigroups*
- *"Equicontinuous-FTS" denotes the class of equicontinuous Fort transformation semigroups*
- *"FTG" denotes the class of Fort transformation groups*

and Ei s are the following examples:

E1. *For $G = \{id_F\}$, thransformation group (G,F) is clearly equicontinuous.*

E2. *By Theorem all Fort transformation groups and in particular $(\mathcal{G},F)$ is weakly almost periodic. However it is not equicontinuous, since in equicontinuous transformation groups, enveloping semigroup is a group of continuous maps and consists of one–to–one map [5] and one can prove easily that constant map b belonges to $E(\mathcal{G},F)$.*

E3. *Suppose $C_b : F \to F$ is constant map b, then for $S = \{id_F, C_b\}$ transformation semigroup (F,S) is an equicontinuous and $E(S,F) = S(= \{id_F, C_b\})$ (note that (S,F) is not a transformation group).*

E4. *Let* $S = E(\mathcal{G}, F)(\subseteq \mathcal{S})$*, then* (S, F) *is not a transformation group and* $E(S, F) = S$*, by (E2),* (S, F) *is not equicontinuous too (and it is weakly almost periodic).*

E5. *Choose one–to–one sequence* $\{a_n\}_{n\geq 1}$ *in* $F \setminus \{b\}$*, choose* $c \in F \setminus \{b\}$*, for all* $n \geq 1$ *define* $f_n : F \to F$ *with*

$$f_n(x) := \begin{cases} c & x = a_2, a_4, \ldots, a_{2n} \ , \\ x & \text{otherwise} \ , \end{cases}$$

then $\{f_n : n \geq 1\} \subseteq \mathcal{S}$ *(use remark 2.1)also* $f : F \to F$ *with*

$$f(x) = \begin{cases} c & x = a_2, a_4, a_6, \ldots \ , \\ x & \text{otherwise} \ , \end{cases}$$

is not continuous and $f = \lim\limits_{n\to\infty} f_n \in E(\mathcal{S}, F)$*, hence* $(\mathcal{S}, F)$ *is not weakly almost periodic.*

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The authors wish to dedicate this text to our Lady Fatima-Zahra (AS) daughter of Holy Prophet Muhammad (PBUH).

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ON ORBIT SPACE OF FORT TRANSFORMATION GROUPS

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Abstract. In the following text we prove that in Fort transformation group (G, F) the following statements are equivalent:

- $\frac{F}{G}$ is a Fort space,
- $\frac{F}{G}$ is T_1,
- for all $x \in F$, Gx is finite,

Moreover if F is an infinite space, then two spaces F and $\frac{F}{G}$ are homeomorphic.

1. Introduction

A transformation group (G, X), X is a compact Hausdorff topological space and a subgroup G of homeomorphisms on X containing identity map $id_X : \underset{x \mapsto x}{X \to X}$. In transformation group (G, X), for $x \in X$,we have $Gx := \{gx : g \in G\}$. Consider the following two equivalence relations on X:

- for $x, y \in X$ let $x \Re y$ iff $Gx = Gy$ (or equivalently $Gx \cap Gy \neq \emptyset$),
- for $x, y \in X$ let $x \widetilde{\Re} y$ iff $\overline{Gx} = \overline{Gy}$.

We denote qoutient space $\frac{X}{\Re}$ by $\frac{X}{G}$ (hence $\frac{X}{G} = \{Ga : a \in X\}$) and call it orbit space (of (G, X)), in addition we denote qoutient space $\frac{X}{\widetilde{\Re}}$ by $\frac{X}{\widetilde{G}}$ and call it quasi–orbit space (of (G, X)) [3]. For more details on transformation groups and orbit spaces see [2, 4]. Suppose $b \in F$ and equipp it with topology $\{U \subseteq F : b \notin U \vee (F \setminus U \text{ is finite})\}$, it is called F a Fort topological space with particular point b [5].

Keywords: Fort space, orbit space, quasi–orbit space, transformation group.
AMS Mathematical Subject Classification [2010]: 54H15, 54H20.

In the following text we pay attention to orbit space and quasi–orbit space approach in Fort transformation groups.

2. Main results

In the following text we compare orbit and quasi–orbit spaces arised from Fort transformation groups, moreover we charachterize all T_1 orbit spaces arised from Fort transformation groups.

Theorem 2.1. *We prove $\frac{F}{G} = \frac{F}{\widetilde{G}}$ when (G, F) is a Fort transformation group.*

Proof. Suppose F is a Fort space with particular point b. For $x \in F$ we have:

$$\overline{Gx} = \begin{cases} Gx & Gx \text{ is finite },\\ Gx \cup \{b\} & Gx \text{ is infinite }, \end{cases}$$

moreover if F is infinite, then $Gb = \{b\}$, thus for all $x, y \in F$ we have:

$$Gx = Gy \Leftrightarrow \overline{Gx} = \overline{Gy}$$

which complets the proof. □

Lemma 2.2. *In Fort transformation group (G, F) with particular point b of F:*

1. *for all $x \in F \setminus \{b\}$, $\{xG\}$ is an open subset of $\frac{F}{G}$,*
2. *$\frac{F}{G}$ is a T_0 space with at most one limit point.*

Proof. If F is finite, then it and all of its quotient spaces are discrete, so the lemma is valid in this case. Suppose $\pi : \underset{a \mapsto Ga}{F \to \frac{F}{G}}$ is natural quotient map and b is particular point of infinite Fort space F, thus $Gb = \{b\}$ and for all $x \in F \setminus \{b\}$, we have $Gx \subseteq F \setminus \{b\}$ hence $Gx = \pi^{-1}(\{xG\})$ is an open subset of F, which completes the proof of (1). Since all points of $\frac{F}{G} \setminus \{Gb\}$ are isolated poits of $\frac{F}{G}$ (by (1)), so (2) is valid too. □

Theorem 2.3. *In Fort transformation group (G, F) with particular point b for F, the following statements are equivalent:*

1. *$\frac{F}{G}$ is a Fort space with particular point bG,*
2. *$\frac{F}{G}$ is Hausdorff,*
3. *$\frac{F}{G}$ is T_1,*
4. *for all $x \in F$, Gx is finite,*
5. *for all $x \in F \setminus \{b\}$, Gx is finite,*
6. *for all $x \in F \setminus \{b\}$, $\{Gx\}$ is a closed subset of $\frac{F}{G}$.*

Proof. It is clear, all of the above statements in 2.3 are valid for finite space F.
In the rest of this proof **suppose F is infinite** and $\pi : \underset{a \mapsto Ga}{F \to \frac{F}{G}}$ is natural quotient map. Moreover since F is infinite, we have $Gb = \{b\}$, thus "(4) $\Leftrightarrow$ (5)". Since

every Fort space is a Hausdorff space, clearly we have "(1) $\Rightarrow$ (2)" and "(2) $\Rightarrow$ (3)". Using definition of T_1 space (points are closed) and quotient topology, clearly we have "(6) $\Leftrightarrow$ (3)".
"(3) $\Rightarrow$ (5)": Choose $x \in F \setminus \{b\}$ with infinite Gx, $\pi^{-1}(\{Gx\}) = Gx$ is not a closed subset of F, (since $\overline{Gx} \setminus Gx = \{b\} \neq \emptyset$), thus $\{Gx\}$ is not a closed subset of $\frac{F}{G}$. Therefore $\frac{F}{G}$ is not T_1.
"(4) $\Rightarrow$ (1)": Since F is infinite, $Gb = \{b\}$ and for $x \in F$, so $b \in Gx$ iff $x = b$. For $U \subseteq \frac{F}{G}$ we have (note that for all $w \in \frac{F}{G}$, $\pi^{-1}(w)$ is finite):

$$\begin{array}{rcl} U \text{ is an open subset of } \dfrac{F}{G} & \Leftrightarrow & \pi^{-1}(U) \text{ is an open subset of } F \\ & \Leftrightarrow & b \notin \pi^{-1}(U) \vee (F \setminus \pi^{-1}(U) \text{ is finite }) \\ & \Leftrightarrow & Gb = \{b\} \not\subseteq \pi^{-1}(U) \vee (\pi^{-1}(\dfrac{F}{G} \setminus U) \text{ is finite }) \\ & \Leftrightarrow & Gb \notin U \vee (\bigcup\{\pi^{-1}(w) : w \in \dfrac{F}{G} \setminus U\} \text{ is finite }) \\ & \Leftrightarrow & Gb \notin U \vee (\dfrac{F}{G} \setminus U \text{ is finite }) \end{array}$$

therefore $\{U \subseteq \frac{F}{G} : Gb \notin U \vee (\frac{F}{G} \setminus U \text{ is finite})\}$ is qoutient topology on $\frac{F}{G}$, which completes the proof. $\square$

Remark 2.4. Two Fort spaces are homeomorphic iff they are equipotent [1].

Theorem 2.5. *Suppose F, K are two Fort spaces. the following statements are equivalent:*

1. *K is (homeomorphic with) an orbit space of F,*
2. *K is (homeomorphic with) a quasi–orbit space of F,*
3. *$card(K) = card(F)$ or $card(K) \leq card(F) < \aleph_0$.*

Proof. (1) and (2) are equivalent by Theorem 2.1.
"(1) $\Rightarrow$ (3)": Suppose K is an orbit space of F, thus there exists transformation group (G, F) and homeomorphism $K \to \frac{F}{G}$, therefore $card(K) = card(\frac{F}{G}) \leq card(F)$. Since $\frac{F}{G}$ is a Fort space, by theorem 2.3 for all $x \in F$ the set Gx is finite, and $\frac{F}{G}$ is a partition of F to its finite subsets, hence F is finite iff $\frac{F}{G}$ (and K) is finite. Suppose $\frac{F}{G}$ and F are infinite, then $card(\frac{F}{G}) = card(\frac{F}{G})\aleph_0$ and we have:

$$\begin{array}{rcl} card(F) & = & card(\bigcup \dfrac{F}{G}) \leq card(\dfrac{F}{G}) \sup\{card(w) : w \in \dfrac{F}{G}\} \\ & \leq & card(\dfrac{F}{G})\aleph_0 = card(\dfrac{F}{G}) \leq card(F) \end{array}$$

hence $card(F) = card(\frac{F}{G})(= card(K))$.
"(3) $\Rightarrow$ (1)": If $card(F) = card(K)$, then by Remark 2.4, F and K are homeomorph, on the other hand clearly for $G = \{id_F\}$ to spaces F and $\frac{F}{G}$ are homeomorph, hence

K and $\frac{F}{G}$ are homeomorph.
Now suppose $m = card(K) \leq card(F) = n \in \mathbb{N}$. Let $K = \{c_1, \ldots, c_m\}$ and $F = \{a_1, \ldots, a_n\}$. Let $G = \{f : F \overset{bijective}{\to} F : \forall i < n\, f(a_i) = a_i\}$ therefore $\frac{F}{G}$ is discrete finite space $\{\{a_i\} : i < n\} \cup \{\{a_i : i \geq n\}\}$ with m elements and homeomorph with K. ☐

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The authors wish to dedicate this text to our Lady Fatima-Zahra (AS) daughter of Holy prophet Muhammad (Holy Prophet).

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BIRKHOFF RECURRENT POINT AND SENSITIVE SEMIFLOWS

ALI BARZANOUNI

Abstract. We introduce the notion of sensitivity for a semiflow on uniform space $(X,\mathcal{U})$. We show that if X is regular uniform space and semiflow (T,X) is non-minimal, topologically transitive and the set of Birkhoff recurrent points is dense in X, then (T,X) is sensitive.

1. Introduction

A *dynamical system or semiflow* is a triple (X,T,π) where X is a topological space, T is a topological semigroup which acts on X, such that the action π is continuous. In this talk, we denote it by (T,X).

In applications of the topological dynamical systems theory, in fact we are often concerned with only semiflows, not flows. For example, for a flow $\pi:\mathbb{R}\times M^n\to M^n$ $(t,x)\mapsto tx$ on a manifold M^n induced by a vector field, we are usually interested to the dynamics like recurrence and almost periodicity of trajectories as $t\to+\infty$ or $t\to-\infty$, not $|t|\to\infty$. In this case, we need to consider the invertible semiflow $\pi_+:\mathbb{R}_+\times M^n\to M^n$(see [4])

Although dynamics on a metric phase space is an important case, yet in many interesting cases we have to face with non-metric phase spaces. For example, the universal dynamics are usually defined on compact T2 non-metrizable phase spaces (see [2]). The Stone-Cech compactification βT of the phase group or semigroup T is also an important phase space which is compact T_2 non-metrizable in general.

KeyWords : Semiflow, Birkhoff Recurrent point, Sensitive.
AMS Mathematical Subject Classification [2010]: :. Primary: 37 B05, 37B99

In this paper,we recall the notions of almost periodicity, uniformly recurrence and Birkhoff recurrence for a semiflow and study relation between them. Also we introduce the notion of sensitivity for a semiflow on uniform space $(X, \mathcal{U})$. We show that if X is regular uniform space and semiflow (T, X) is non-minimal, topologically transitive and the set of Birkhoff recurrent points is dense in X, then (T, X) is sensitive.

Given (T, X), $x \in X$, subsets V, U of X and subsets K, A of T, we write

(1) $Tx = \{tx : t \in T\}$,
(2) $K^{-1}A = \{t \in T : \text{ there is } k \in K \text{ with } kt \in A\}$,

Definition 1.1. Let (T, X) be a semiflow.

(1) A point $x \in X$ is said to be almost periodic point of von Neumann for (T, X) if for every open set U of x, the set of return times $N(x, U)$ is GH-syndetic in T, this means that there is compact set $K \subseteq T$ with $T = KN(, U)$.
(2) A point $x \in X$ is said to be uniformly recurrence for (T, X) if for every open set U of x, the set of return times $N(x, U)$ is syndetic in the sense of (F) in T, this means that there is compact set $K' \subseteq T$ with $T = K^{-1}N(, U)$.

Although the notions of Definition 1.1 agree to each other in the flow and cyclic systems case, the following, we give an example to show that there is no relation between them in the case of semiflow.

Example 1.2. (1) (Almost periodic point of Neumann for (T, X) does not imply uniformly recurrent.) Note that T is the compact subset of all continuous mappings on $[0, 1]$. Let $f_\alpha : [0, 1] \to [0, 1]$ be defined by $f_\alpha(x) = \alpha x$ and take $r \in (0, 1)$. Take $T = \{f_\alpha : \alpha \in [0, r] \cup \{1\}\}$. It is easy to see that $T = T\{f_1\}$.This implies that $x = r$ is Almost periodic point of Neumann for (T, X) while it is not uniformly recurrent. Because for every open set U of $x = r$, $f_1 \in N(r, U)$, while $N(r, V) = \{f_1\}$ where $V = (r - r^2, 1)$.

(2) (Uniformly recurrent does not imply Almost periodic point of Neumann) Let L_n be the line segment that connect $(0, n)$ to $(1, 0)$. Let $T = \bigcup_{n=0}^{\infty} L_n$. We use the following notion for the element of T: for $t \in [0, 1)$, denote the intersection L_n with the line $x = t$ by t^n and the point $(1, 0)$ will be denoted by 1. We introduce a commutative operation on T by:

$$s^m + t^n = \max\{s, t\}^{\max\{m,n\}}, s^m + 1 = 1.$$

We equip T with the metric induced from the plane. Then T is a topological semigroup. It is easy to see that $A \subseteq T$ is syndetic in the sense of F if and only if $1 \in A$. Also one can check that $A = \{1\}$ is not GH-syndetic. Let (T, T) be the semiflow in which T acts on itself by the operation. It is clear that $x = 1$ is uniformly recurrent while it is not almost periodic point of von Neumann for (T, T).

Note that these notions agree to each other in the case of$T = \mathbb{Z}_+$ or $\mathbb{R}_+$, see [1, Proposition 2.8.]. Also if $T = \mathbb{Z}_+^n$ or $\mathbb{R}_+^n$, then almost periodic of von Neumann is uniformly recurrent.

- Let $\Lambda \subseteq X$. We say that Λ is an invariant subset of (T, X) if $Tx \subseteq \Lambda$ for all $x \in \Lambda$
- Λ is minimal subset of (T, X) if and only if $cls_X Tx = \Lambda$ for $x \in X$,
- A point $x \in X$ is called a minimal point if $cls_X Tx$ is minimal set of (T, X)
- If X is minimal set, then we say that (T, X) is a minimal semiflow.

Note that if (T, X) is a semiflow on a locally compact space X which is either Hausdorff or regular, then $cls_X Tx_0$ is compact for every almost periodic point of von Neumann $x_0 \in X$ of (T, X), but in the following we show that it may be $cls_X Tx_0$ is not minimal.

Example 1.3. Let $(T, [0, 1])$ be as semiflow given in Example 1.2. Then $x = \{r\}$ is almost periodic point of von Neumann, hence $\overline{Tr} = \{\alpha r : \alpha \in [0, r] \cup \{1\}\}$ is compact set while it is not minimal, because $\{0\}$ is a minimal subset of $\overline{Tr}$.

Definition 1.4. Let $(X,\mathcal{U})$ be a uniform space, see [3] for definition. The point $x \in X$ is Birkhoff recurrent for semiflow (T, X) if for every entourage $U \in \mathcal{U}$ there is compact set $K \subseteq T$ such that

$$Tx \subseteq U[K(tx)], \forall t \in T$$

This means that for a Birkhoff recurrent point x, any orbit arc $K(tx)$ approximates the entire orbit Tx with a precision to within U. Theorem 1.3 in [1] shows that if X is a compact metric space, then a point $x \in X$ is Birkhoff recurrent whenever x is uniformly recurrent for (T, X).

2. Main Results

Definition 2.1. Let $(X,\mathcal{U})$ be a uniform space and (T, X) be a semiflow. (T, X) is sensitive if there is a symmetric entourage $U \in \mathcal{U}$ such that for every $x \in X$ and every open set $\dot{A}$ of x in X, there exist $y \in \dot{A}$ and $t \in T$ with $(tx, ty) \notin U$.

If the phase space X has an isolated point, then every semiflow (T, X) is clearly nonsensitive. Hence, we assume that X has no isolated point. It is easy to see that definition of sensitivity for semiflow (T, X) is equivalent with the following definition.

- (T, X) is sensitive if there is a symmetric entourage $U \in \mathcal{U}$ such that for every open set $\dot{A}$ of x in X, there exist $y, z \in \dot{A}$ and $t \in T$ with $(tz, ty) \notin U$.

We say that semiflow (T, X) is sensitive on $Y \subseteq X$ if there is a symmetric entourage $U \in \mathcal{U}$ such that for every $x \in Y$ and every open set $\dot{A}$ of x in X, there exist $y \in \dot{A} \cap Y$ and $t \in T$ with $(tx, ty) \notin U$.

Example 2.2. Let $(T, [0, 1])$ be a sensitive semiflow such that $t0 = 0$ for all $t \in T$. Take $X = [-1, 1]$ and define $t \in T$ on $[-1, 0]$ as identity map. Then (T, X) is semiflow on $[0, 1]$ while it is nonsensitive.

In the following, we show that sensitivity (T, X) on Y can be extend to $\overline{Y}$.

Lemma 2.3. *If Y is dense in X and (T, X) is sensitive on Y, then (T, X) is sensitive.*

The system (T, X) is topologically transitive, if for every two open sets U, V of X, there is $t \in T$ with $tU \cap V \neq \emptyset$. Note that it may be there is a point-transitive for semiflow (T, X) while (T, X) is not topologically transitive. For example let $T = (\mathbb{Q}_+, +)$ be the nonnegative rational numbers semigroup and $X = \mathbb{R}_+ \cup \{+\infty\}$ with the usual one-compactification topology. Define $T \times X \to X$ by $(t, x) \mapsto t + x$. Then $\overline{T0} = \mathbb{R}_+$ while for open sets $U = (10, 12)$ and $V = (5, 6)$ we have $tU \cap V = \emptyset$ for all $t \in T$. This implies that $(T, \mathbb{R}_+)$ has a transitive point while it is not topologically transitive. However, if X is polish space and the set of transitive point be a dense G_δ set of X, then we can say that (T, X) is topologically transitive.

There is no relation between notions of topologically transitive and sensitivity for a semiflow (T, X). Let (T, T) be the semiflow as given Example. It is easy to see that (T, T) is sensitive semiflow while it is not topologically transitive. Also let semigroup T generated by irrational rotation map R_α and constant map f, then $(T, \mathbb{S}^1)$ is topologically transitive while it is nonsensitive.

Proposition 2.4. *Let $(X, \mathcal{U})$ be a uniform space which is regular and (T, X) be a semiflow which satisfies the following conditions:*

1. *(T, X) is non-minimal,*
2. *(T, X) is topologically transitive,*
3. *The set of Birkhoff recurrent point for (T, X) is dense in X.*

Then (T, X) is sensitive.

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MOUNTAIN PASS SOLUTION FOR A FRACTIONAL (p, q)-LAPLACIAN EQUATION

FERESHTEH BEHBOUDI AND ABDOLRAHMAN RAZANI

Abstract. The main purpose of this paper is to establish a Mountain Pass type solution to a boundary value problem involving the fractional (p, q)-Laplacian operator in a smooth bounded domain in $\mathbb{R}^N$.

1. Introduction

It is worth noticing that the Mountain Pass Theorem of A. Ambrosetti and P. H. Rabinowitz [1, Theorem 2.1] and its generalizations are usefule to find critical points. Note that if $p \neq q$ the differential operator $\Delta_p + \Delta_q$ is not homogeneous, some technical difficulties arise when applying the methods to elliptic problems. The authers in [2] study the existence of two weak solutions for a singular (p, q)-Laplacian problem.

In the nonlocal framework, the fractional Laplacian is the simplest example. Qiu and Xiang [7] obtain the solutions for fractional p-Laplacian problem as follows

$$\begin{cases} (-\Delta)_p^s u = f(x, u) & x \in \Omega \\ u = 0 & in \quad \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $(-\Delta)_p^s$ is the fractional p-Laplacian problem, Ω is open bounded domain in $\mathbb{R}^N$ with the Lipschitz boundary, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

Keywords: Fractional (p, q)-Laplacian operator, Mountain Pass Theorem.
AMS Mathematical Subject Classification [2010]: 35R11, 35J35, 35J92.

In this paper, we are interested to study a fractional (p, q)-Laplacian problem as

$$\begin{cases} (-\Delta)_p^r u + (-\Delta)_q^s u = f(x, u) & x \in \Omega \\ u = 0 & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain with $N \geq 2$, $0 < s < r < 1$, $1 < q < p < \frac{N}{r}$ and $(\Delta)_m^j$ is a nonlocal operator known as the fractional m-Laplacian, may be defined for u smooth enough, for every $x \in \mathbb{R}^N$ as follows

$$(-\Delta)_m^j u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{m-2}(u(x) - u(y))}{|x - y|^{N+jm}} dy,$$

where $B_\epsilon(x)$ is the ball in $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ and with radius $\epsilon > 0$. We refer to [4] for the details and the introduction of this operator. Moreover, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carath*é*odory function with regularity assumptions on Ω. For this problem, homogeneous Dirichlet boundary conditions, that is $u = 0$ in $\mathbb{R}^N \setminus \Omega$. Therefore the Dirichlet boundary conditions is given in $\mathbb{R}^N \setminus \Omega$ and not simply on $\partial\Omega$.
By [5], for any $p \in (1, \infty)$ and $r \in (0, 1)$ we have the fractional Sobolev space $W^{r,p}(\mathbb{R}^N)$ with the norm defined by

$$\|u\|_{W^{r,p}(\mathbb{R}^N)} := \Big(\int_{\mathbb{R}^N} |u(x)|^p dx + \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+rp}} dxdy \Big)^{\frac{1}{p}}.$$

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded Lipschitz domain. So, we consider the closed space

$$W_0^{r,p}(\Omega) = \{u \in W^{r,p}(\mathbb{R}^N) : u(x) = 0 \ \ a.e. \ \ in \ \ \mathbb{R}^N \setminus \Omega\},$$

endowed with the Gagliardo norm as follow

$$\|u\|_{W_0^{r,p}(\Omega)} := \Big(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+rp}} dxdy \Big)^{\frac{1}{p}},$$

for all measurable functions $u : \mathbb{R}^N \longrightarrow \mathbb{R}$. The norm in the Lebesgue space $L^p(\Omega)$ is

$$\|u\|_{L^p(\Omega)} = \Big(\int_\Omega |u(x)|^p dx \Big)^{\frac{1}{p}}.$$

For notational simplicity, we set $X := W_0^{r,p}(\Omega)$, $\|.\|_p := \|.\|_{L^p(\Omega)}$, $\|.\|_{r,p} := \|.\|_{W_0^{r,p}(\Omega)}$ and $X^* := W_0^{-r,p'}(\Omega)$ denotes the dual space of X, where $\frac{1}{p} + \frac{1}{p'} = 1$.

The natural space to study fractional (p, q)-Laplacian problems is fractional Sobolev space $X(= W_0^{r,p}(\Omega))$. It is easy to check that $(X, \|.\|_{r,p})$ is a uniformly convex reflexive Banach space and that in [6] by Ho et al., the embedding $X \hookrightarrow L^k(\Omega)$ is continuous for any $k \in [1, p_r^*]$ if $N > rp$, for any $k \in [1, \infty[$ if $N = rp$ and into $L^\infty(\Omega)$ if $N < rp$. The embedding is compact for any $k \in [1, p_r^*[$ if $N \geq rp$ and into $L^\infty(\Omega)$ if $N < rp$, where $p_r^* := \frac{Np}{N-rp}$ if $rp < N$ and $p_r^* := \infty$ if $rp \geq N$.

Proposition 1.1. [3] *Let $1 < q \leq p \leq \infty$ and $0 < s < r < 1$. Let Ω be a smooth bounded domain in $\mathbb{R}^N$ where $N > rp$ and $u : \Omega \to \mathbb{R}^N$ be a measurable function. Then*

$$\|u\|_{s,q} \leq C\|u\|_{r,p}, \qquad \forall u \in W_0^{r,p}(\Omega),$$

for some suitable positive constant $C = C(N, r, s, p, q) \geq 0$. In particular,

$$W_0^{r,p}(\Omega) \subseteq W_0^{s,q}(\Omega).$$

We suppose that the Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions

(f_1) $|f(x,t)| \leq a_1 + a_2|t|^{b-1}$, *a.e.* $x \in \Omega$, $t \in \mathbb{R}$,
where $a_1, a_2 > 0$ and $b \in (p, p_r^*)$.

(f_2) $\lim_{|t|\to 0} \frac{f(x,t)}{|t|^{p-1}} = 0$ uniformly in $x \in \Omega$.

(f_3) $0 < \mu F(x,t) \leq tf(x,t)$, *a.e.* $x \in \Omega$, $t \in \mathbb{R}$, $|t| \geq \rho$,
where $\mu > p$, $\rho > 0$ and $F(x,\xi) := \int_0^\xi f(x,t)dt$ for every $(x,\xi) \in \Omega \times \mathbb{R}$.

The Mountain Pass Theorem given by Ambrosetti and Rabinowitz is a main tool to prove the existence of solution for our problem.

Theorem 1.2. *Let $r \in (0,1)$, $N > rp$ and Ω be an open bounded set of $\mathbb{R}^N$ with Lipschitz boundary and let f be a Carathéodory function satisfying $(f_1) - (f_3)$. Then the problem* (1.1) *admits a Mountain Pass type solution $u \in X$ which is not identically zero, and such that $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$.*

2. Mountain Pass type solution

We know the problem (1.1) has a variational structure, and the Euler-Lagrange equation of the functional $I : X \to \mathbb{R}$ defined as follows

$$I(u) := \frac{1}{p}\|u\|_{r,p}^p + \frac{1}{q}\|u\|_{s,q}^q - \int_\Omega F(x, u(x))dx, \quad \forall u \in X.$$

The functional I is Fréchet differentiable in $u \in X$ and every $v \in X$

$$\begin{aligned} I'(u)(v) = &\int_{\mathbb{R}^{2N}} \frac{\big(|u(x)-u(y)|^{p-2}(u(x)-u(y))\big)\big(v(x)-v(y)\big)}{|x-y|^{N+rp}}dxdy \\ &+ \int_{\mathbb{R}^{2N}} \frac{\big(|u(x)-u(y)|^{q-2}(u(x)-u(y))\big)\big(v(x)-v(y)\big)}{|x-y|^{N+sq}}dxdy \\ &- \int_\Omega f(x,u(x))v(x)dx, \quad \forall u, v \in X. \end{aligned}$$

Hence, the weak solution of (1.1) is a critical point of the functional I.

Here, we can show that the functional I has the Mountain Pass geometriy and satisfies the compactness condition of Palais-Smale. Firstly, we start by proving the geometric features of the functional I.

Proposition 2.1. *Assume that f is a Carathéodory function satisfying conditions f_1 and f_2. Then there exist $\rho > 0$ and $\alpha > 0$ such that for any $u \in X$ with $\|u\|_{r,p} = \rho$ it results that $I(u) \geq \alpha$.*

Proposition 2.2. *Suppose f is a Carathéodory function satisfying conditions f_1-f_3. Then there exists $e \in X$ such that $e \geq 0$ a.e. in $\mathbb{R}^N$ with $\|e\|_{r,p} > \rho$ and $I(e) < \alpha$, where ρ and α are given in Proposition 2.1.*

From Propositions 2.1 and 2.2 we can conclude that the geometry of the Mountain Pass Theorem is fulfilled with I. In order to apply Mountain Pass Theorem, by using following propositions we prove that I satisfies the Palais-Smale condition.

Proposition 2.3. *Suppose that f is a Carathéodory function satisfying conditions f_1-f_3. Let $c \in \mathbb{R}$ and $\{u_n\}$ be a sequence in X such that*

$$I(u_n) \to c, \quad sup\{| \prec I'(u_n), v \succ | : v \in X, \|v\|_{r,p} = 1\} \to 0,$$

as $n \to +\infty$. Then $\{u_n\}$ is bounded in X.

Proposition 2.4. *Assume that f is a Carathéodory function satisfying conditions f_1 and f_2. Then there exist $\rho > 0$ and $\alpha > 0$ such that for any $u \in X$ with $\|u\|_{r,p} = \rho$ it implies that $I_\lambda(u) \geq \alpha$.*

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FRACTIONAL ORDER MATHEMATICAL MODELING OF COVID 19 EPIDEMIC

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Abstract. Two transmission pathway exists for the spread of COVID-19 virus including direct and indirect pathways. The direct pathway is the person to person transmission between susceptibles and infectious individuals. Furthermore infected individuals shed virus on the objects, and new infections arise through touching contaminated objects in the environment, which is called the indirect transmission pathway. We propose an $SADO_IR$ fractional ode model with both direct and indirect transmission pathways which includes a compartment for the contaminated objects and also includes, compartments for susceptibles, asymptomatic infectious, detected infectious, and recovered individuals.

1. Introduction

Coronavirus disease 2019 (COVID-19) has now become an unprecedented challenge around the world, a disease that began on December 23, 2019, in the wuhan province of china and quickly spread to all parts of the world. The world health organization declared it a global pandemic on March 11, 2020. In addition to the high number of cases and deaths in the world, this pandemic has had dramatic and unprecedented effects on many aspects of human life, including economic and industrial activities, education, health, religion, recreation, and entertainment. Since the first days of the disease, many efforts have been made to understand and analyze the dynamics of the spread of the disease and its influential factors, see [2, 4].

Keywords: Fractional ODE Model, COVID-19 virus, Direct and Indirect Transmission.
AMS Mathematical Subject Classification [2010]: 92D30, 92C60.

Coronavirus transmits through two transmission pathways. Direct pathway in which infection occurs by droplet inhalation, i.e., aerosol pathway. Furthermore infected individuals contaminate clean objects in the environment by touching and droplet shedding on objects through coughing, sneezing, or exhaling. And then pathogen transferred from contaminated objects to hands and then to mouth, nose, and eyes of individuals which causes new infectious cases, this route is called the indirect pathway. Some studies have been highlighted the effect of the indirect transmission pathway in influenza and other infectious diseases, see [1, 3].

2. Model Formulation

Definition 2.1. Let E and F denote ideal spaces over (T, μ) and (S, ν) respectively. An operator $I : E \to F$ is called integral if there exists a measurable function $K(s,t), (s \in S, t \in T)$ such that for every $x \in E$ the value $y = Ix$ is the function

$$y(s) = \int_T K(s,t)x(t)d\mu(t)$$

The function $K(s,t)$ is referred to as the kernel of the integral operator I.

Definition 2.2. Suppose that $\alpha > 0, t > a, \alpha, a, t \in \mathbb{R}$, Caputo fractional derivative $f(t)$ on interval $[a,t]$ of order α is defined as follows [5]

$${}_a^C D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha < n \in \mathbb{N}, \\ \frac{d^n}{dx^n} f(t), & \alpha = n \in \mathbb{N}, \end{cases}$$

where D^α refers to Caputo fractional differential operator of order α.

Representation property for Caputo operator holds

$${}_a^C D^\alpha f(t) = I_a^{n-\alpha} D^{(n)} f(t),$$

then

$${}_a^C D^\alpha f(t) = I_a^{n-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau,$$

which $I_a^{n-\alpha} f^{(n)}(t)$ is Riemann-Liouville fractional integral operator Our model has the following compartments. The class S consists of susceptible individuals. The class A consists of two types of individuals in the community, asymptomatic infectious individuals, i.e., infected individuals who have no symptoms of the disease and individuals with symptoms of the disease which are not detected by the healthcare system. The class D consists of infectious individuals who are detected by the healthcare system i.e. confirmed cases, and the recovered individuals form the class R. We assume that all detected cases are hospitalized until recovery or death. We also consider two compartments for the objects, clean objects O_C and infectious i.e., contaminated objects O_I.

TABLE 1. Description of variables and parameters of the model

Symbol	Description
S	Susceptible individuals
A	Infectious individuals without symptoms + individuals with symptoms which are not detected by the healthcare system
D	Infectious individuals who are detected by the healthcare system i.e., confirmed cases
R	Recovered individuals
O_I	Contaminated objects
Λ	Recruitment rate into the susceptible population
μ	Natural death rate
m	Death rate due to COVID-19
β_1	Direct transmission rate
β_3	Indirect transmission rate
δ	Rate of objects contamination
δ_1	Rate of exacerbation of symptoms of the disease
γ	Rate loss of immunity
γ_1	Rate of recovery of confirmed cases
ν	Identification rate of infectious individuals
ξ	Rate of recovery of asymptomatic infectious individuals
$\frac{1}{\xi_1}$	Mean lifetime of virus on the objects

Individuals in the class A spread the disease between the community through two routes: (I) direct route with the flow $\beta_1 SA$ in which β_1 is the probability of virus transmission through direct contact. And (II) indirect route, i.e., the transmission of infection by touching contaminated objects.

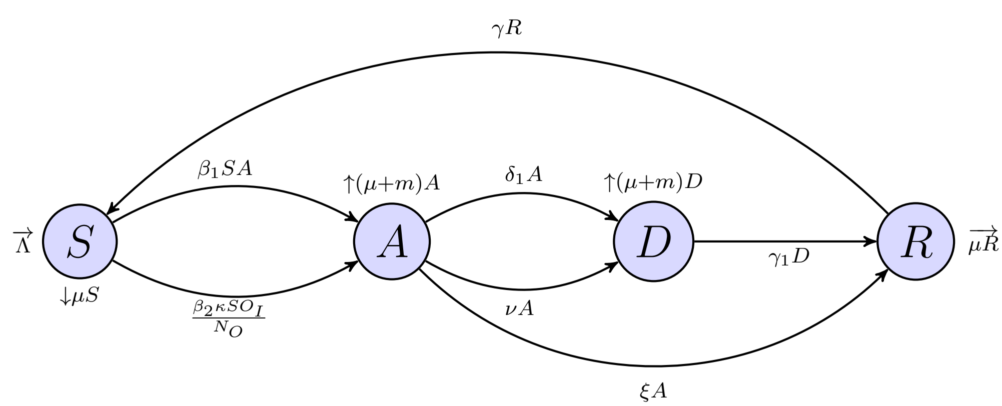

FIGURE 1. The flowchart of the model

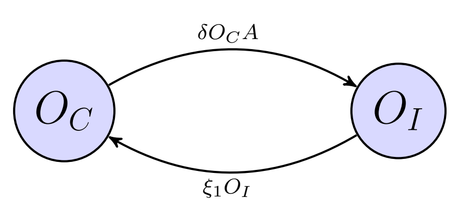

FIGURE 2. The flowchart of the objects

Based on the flow diagram of the model depicted in the above figures, we have the following system of fractional Model of Corona virus as follow

$$
\begin{cases}
{}_0^C D_t^\alpha S(t) = \Lambda + \gamma R - \beta_1 SA - \frac{\beta_2 \kappa S O_I}{N_O} - \mu S \\
{}_0^C D_t^\alpha A(t) = \beta_1 SA + \frac{\beta_2 \kappa S O_I}{N_O} - (\mu + \delta_1 + \nu + \xi + m)A \\
{}_0^C D_t^\alpha D(t) = \delta_1 A + \nu A - (\mu + m + \gamma_1)D \\
{}_0^C D_t^\alpha O_I(t) = \delta(N_O - O_I)A + \xi_1 O_I \\
{}_0^C D_t^\alpha R(t) = \xi A + \gamma_1 D - (\mu + \gamma)R
\end{cases} \tag{2.1}
$$

where D^α refer to the Caputo fractional derivative of order $0 < \alpha \leq 1$, and with following initial conditions

$$S(0) = S_0,\ A(0) = A_0,\ D(0) = D_0,\ O_I(0) = 0,\ R(0) = R_0. \tag{2.2}$$

We also use the notation $\beta_3 = \frac{\beta_2 \kappa}{N_O}$. The main goal of this work is to investigate a theoretical and numerical study for the proposed model. In the next sub-section some theoretical issue of the proposed model have been considered.
In the first result, we prove that the solutions are non-negative.

2.1. Non-Negative Solution.

Let

$$\mathbb{R}_+^5 = \{X \in \mathbb{R}^5,\ X \geq 0\} \text{ and } X(t) = \big(S(t), A(t), D(t), O_I(t), R(t)\big)^T.$$

Let $f \in C[a,b]$ and $D^\alpha f \in C[a,b]$ for $0 < \alpha \leq 1$, we use the generalized mean value theorem (GMVT) to investigate the positively of the solution of (2.1)-(2.2).

Lemma 2.3. *The proposed model* (2.1)-(2.2) *has a unique solution* $X(t)$ *and the solution remains on* $\mathbb{R}_+^5$.

Proof. For the positively of solution, we show that all components are bounded in the positive guadrent due to the fact we are dealing with population model. We need to show that the domain $\mathbb{R}_+^5$ is positively invariant. Since

$$
\begin{aligned}
&{}_0^C D_t^\alpha S(t)|_{S=0} = \Lambda + \delta R \geq 0, \\
&{}_0^C D_t^\alpha A(t)|_{A=0} = \frac{\beta_2 \kappa S O_I}{N_O} \geq 0, \\
&{}_0^C D_t^\alpha D(t)|_{D=0} = \delta_1 A + \nu A \geq 0, \\
&{}_0^C D_t^\alpha O_I(t)|_{O_I} = 0 \geq 0, \\
&{}_0^C D_t^\alpha R(t)|_{R=0} = \xi A + \gamma_1 D \geq 0.
\end{aligned}
$$

In view of GMVT, in each hyperplane bounding the non-negative octant, the vector field point into $\mathbb{R}^5_+$.

3. Existence and Uniqueness of Solution for the Caputo Fractional Model

In this section, we will investigate the existence of unique solution for fractional model (2.1) with help of fixed point theory.
By applying the fractional integral operator on both sides of Eq. (2.1), we obtain

$$\begin{aligned}
S(t) - S(0) &= {}_0I_t^\alpha\{\Lambda + \gamma R - \beta_1 SA - \frac{\beta_2\kappa S O_I}{N_O} - \mu S\},\\
A(t) - A(0) &= {}_0I_t^\alpha\{\beta_1 SA + \frac{\beta_2\kappa S O_I}{N_O} - (\mu + \delta_1 + \nu + \xi + m)A\},\\
D(t) - D(0) &= {}_0I_t^\alpha\{\delta_1 A + \nu A - (\mu + m + \gamma_1)D\},\\
O_I(t) - O_I(0) &= {}_0I_t^\alpha\{\delta(N_O - O_I)A + \xi_1 O_I\},\\
R(t) - R(0) &= {}_0I_t^\alpha\{\xi A + \gamma_1 D - (\mu + \gamma)R\}.
\end{aligned}$$

Now, by using the definition of the fractional integral operator ${}_0I_t^\alpha = {}_0D_t^{-\alpha}$, we have

$$\begin{aligned}
S(t) - S(0) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} A_1(\tau, S)d\tau,\\
A(t) - A(0) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} A_2(\tau, A)d\tau,\\
D(t) - D(0) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} A_3(\tau, D)d\tau,\\
O_I(t) - O_I(0) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} A_4(\tau, O_I)d\tau,\\
R(t) - R(0) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} A_5(\tau, R)d\tau, \qquad (3.1)
\end{aligned}$$

where the kernels A_1, A_2, A_3, A_4 and A_5 are defined as follows:

$$\begin{aligned}
A_1(t, S(t)) &= \Lambda + \gamma R - \beta_1 SA - \frac{\beta_2\kappa S O_I}{N_O} - \mu S,\\
A_2(t, A(t)) &= \beta_1 SA + \frac{\beta_2\kappa S O_I}{N_O} - (\mu + \delta_1 + \nu + \xi + m)A,\\
A_3(t, D(t)) &= \delta_1 A + \nu A - (\mu + m + \gamma_1)D,\\
A_4(t, O_I(t)) &= \delta(N_O - O_I)A + \xi_1 O_I\\
A_5(t, R(t)) &= \xi A + \gamma_1 D - (\mu + \gamma)R, \qquad (3.2)
\end{aligned}$$

Theorem 3.1. *All the kernels A_1, A_2, A_3, A_4 and A_5 in Eq. (3.2) will be satisfied for Lipschitz and contracts conditions if the following conditions hold:* $0 <$

$Max\{m_1, m_2, m_3, m_4, m_5\} < 1$ *where* m_1, m_2, m_3, m_4 *and* m_5 *are Lipschitz constant in accordance with* A_1, A_2, A_3, A_4 *and* A_5.

Theorem 3.2. *Assume that*

$$\frac{b^\alpha}{\Gamma(\alpha)} m_i < 1, \ i = 1, 2, \cdots, 5,$$

then, the proposed fractional order mathematical model has a unique solution for $t \in [0, b]$ *for any positive fixed* b.

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INVARIANT ANALYSIS OF A SPECIAL CASE OF FOKKER-PLANK EQUATION OF THE HESTON MODEL

HAMID ERFANIAN ORAEI DEHROKHI AND ELHAM DASTRANJ

Abstract. The Heston model, is a model where the stock model price dynamics is governed by a geometrical (multiplicative) Brownian motion with stochastic variance. In this paper a corresponding special case of Fokker-Plank equation is solved via the Lie group theory of differential equations.The symmetry operators of the equation are computed. Then, the method of group-invariant solutions is applied in order to find some exact solutions of the equation.

1. Introduction

Geometric Brownian motion (GBM) is a very important concept in financial mathematics. This stochastic differential equation is known as Black-Scholes model financial markets. This market is free of arbitrage possibilities and be specified by the following stochastic differential equation

$$dS_t = \alpha S_t dt + \sigma S_t dW_t,$$

where the drift parameter α and the volatility σ are assumed to be constants. The above SDE is called geometric Brownian motion. Various mathematical models with stochastic volatility have been discussed in literature. Lie group theory plays a very important role in geometric analysis of differential equations and there are lots of papers and books have been presented about this subject [1, 2]. Also Lie symmetries method have many efficient applications in physics and mathematics. As an important

Keywords: Heston model, symmetry group, invariant transformation, reduction.
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application of symmetry operators the reduction procedure could be mentioned. This is possible from a similarity variable obtaining from the symmetries.

The Fokker-Plank equations which will be considered in the sequel is a linear second order PDE written by

$$\frac{\partial P}{\partial t}=\frac{1}{2}\left(\frac{\partial P}{\partial x}+\frac{\partial^2 P}{\partial x^2}\right), \tag{1.1}$$

where P is a smooth function of (x,t). The paper is organized as follows: the second paper is devoted for computing the symmetry operators of (1.1). In the third eqution the method of group-invariant solutions is applied in order to construct exact solutions of the equation by using the invariant transformations extracted from symmetries.

2. Symmetries of Eq. (1.1)

A PDE with $p-$independent and $q-$dependent variables has a Lie point transformations $\tilde{x}_i=x_i+\xi_i(x,P)+\mathcal{O}(\epsilon^2),\tilde{P}_\alpha=P_\alpha+\eta_\alpha(x,P)+\mathcal{O}(\epsilon^2)$, where $\xi_i=\frac{\partial\tilde{x}_i}{\partial\epsilon}|_{\epsilon=0}$ for $i=1,...,p$ and $\eta_\alpha=\frac{\partial\tilde{P}_\alpha}{\partial\epsilon}|_{\epsilon=0}$ for $\alpha=1,...,q$. The action of the Lie group can be considered by its associated infinitesimal generator

$$X=\sum_{i=1}^{p}\xi_i(x,u)\frac{\partial}{\partial x_i}+\sum_{\alpha=1}^{q}\eta_\alpha(x,u)\frac{\partial}{\partial u_\alpha}, \tag{2.1}$$

on the total space of PDE (the space containing independent and dependent variables). Furthermore, the characteristic of the vector field (2.1) is given by $Q^\alpha(x,u^{(1)})=\eta_\alpha(x,P)-\sum_{i=1}^{p}\xi_i(x,P)\frac{\partial P^\alpha}{\partial x_i}$ and its $n-$th prolongation is determined by

$$X^{(n)}=\sum_{i=1}^{p}\xi_i(x,P)\frac{\partial}{\partial x_i}+\sum_{\alpha=1}^{q}\sum_{|J|=j=0}^{n}\eta_\alpha^J(x,u^{(j)})\frac{\partial}{\partial P_J^\alpha}, \tag{2.2}$$

where $\eta_\alpha^J=D_JQ^\alpha+\sum_{i=1}^{p}\xi_iP_{J,i}^\alpha$ are the prolong coefficients.

The symmetry operators of the Eq. (1.1) is spanned by the general operator $X=\xi(x,t,P)\frac{\partial}{\partial x}+\tau(x,t,P)\frac{\partial}{\partial t}+\eta(x,t,P)\frac{\partial}{\partial P}$ with the second prolong coefficients

$$X^{(2)}=X+\eta^x\frac{\partial}{\partial P_x}+\eta^t\frac{\partial}{\partial P_t}+\eta^{xx}\frac{\partial}{\partial P_{xx}}+\eta^{xt}\frac{\partial}{\partial P_{xt}}+\eta^{tt}\frac{\partial}{\partial P_{tt}}, \tag{2.3}$$

computed by (2.2). Using the invariance condition, applying (2.3) on (1.1) yields the following symmetry operators: $X_1=\frac{\partial}{\partial x},X_2=\frac{\partial}{\partial t},X_3=\frac{\partial}{\partial P},X_4=(x-t)\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t},X_5=2t\frac{\partial}{\partial x}+(2x+t)P\frac{\partial}{\partial P},X_6=8xt\frac{\partial}{\partial x}+8t^2\frac{\partial}{\partial t}-(4x^2+t^2+4(x+1)t)P\frac{\partial}{\partial P}$.

3. Reductions and Exact solutions

One of the most important applications of the group theory of differential equations is to construct the exact solutions of differential equations called group-invariant solutions. This is based finding some new variables which are invariants under the operaors $X_i's$. These variables reduce the Eq. (1.1) to an ODE. Then, the solution of obtained ODEs give the group-invariant solutions of the Eq. (1.1).

4. Exact solution via X_1

The operator X_1 gives the invariants $y = x, u(y) = P$. These variables reduce the equation to $u'' + u' = 0$. Thus the exact solution in this case is $P = a + be^{-x}$.

5. Exact solution via X_2

Consider the operator X_2. This symmetry has the invariants $y = t, u(y) = P$. So, the reduced is $u' = 0$ and the solution is just a constant.

6. Exact solution via X_3

In this case we do not have any implicit solution. Because the operator shows that any translation on the solutions is a new solution. This is abvviously based on the lineariry of the Eq. (1.1).

7. Exact solution via X_4

The symmetry X_4 has two invariants

$$y = \frac{2x+t}{2\sqrt{t}}, \qquad u(y) = P. \tag{7.1}$$

Inserting variables (7.1) to (1.1) give the following reduced equation:

$$yu' + u'' = 0.$$

So, the exact solution is

$$P = a + \operatorname{erf}\left(\frac{2x+t}{2\sqrt{2t}}\right) + b,$$

where erf is the error function.

8. Exact solution via X_5

Two new invariants

$$y = t, \qquad u(y) = P \exp\left\{\frac{x(x+t)}{2t}\right\}, \tag{8.1}$$

are obtained from operator X_5. Inserting variables (8.1) to (1.1) give the following reduced equation:

$$8yu' + (y+4)u = 0.$$

Consequently, the following exact solution is derived:

$$P = \frac{a}{\sqrt{t}} \exp\left(-\frac{1}{8}t\right).$$

9. Exact solution via X_6

Finally, this operator has two invariants

$$y = \frac{t}{x}, \qquad u(y) = \sqrt{x} P \exp\left\{ \frac{(2x+t)^2}{8t} \right\}. \tag{9.1}$$

Inserting variables (9.1) to (1.1) give the following reduced equation:

$$u^2 u'' + 3uu' + \frac{3}{4}u = 0.$$

So, the following solution is concluded:

$$P = \sqrt{\frac{x}{t}} \left(\frac{ax}{t} + b \right).$$

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ERGODICITY OF EXPANDING SEQUENTIAL DYNAMICAL SYSTEMS

A. EHSANI AND J. NAZARIAN SARKOOH

Abstract. In this paper, sequential dynamical systems given by sequences of the mappings on a compact connected Riemannian manifold are studied. We prove that each uniformly Hölder continuous uniformly expanding sequential dynamical system built from a sequence of conformal $C^{1+\alpha}$ local diffeomorphisms on a compact connected Riemannian manifold possesses an infinite countable Markov partition and by using this fact we show that this system is Lebesgue ergodic in mean.

1. Introduction

The time-dependent systems, so-called non-autonomous or sequential, yield very flexible models than autonomous cases for the study and description of real world processes. In fact, many physical, biological, and economical problems are non-autonomous. There have been major efforts in establishing a general theory of such systems in recent years (see [1, 2, 4, 5, 6, 7, 8]), but a global theory is still out of reach.

In this paper, first we consider a sequential dynamical system (φ_n) on a Borel probability space $(M, \mathcal{A}, \lambda)$, where M is a compact connected Riemannian manifold, λ and $\mathcal{A}$ are the normalized Lebesgue measure and Borel sigma-algebra on M, respectively, and (φ_n) is a sequence of conformal $C^{1+\alpha}$ local diffeomorphisms $\varphi_n : M \to M$ that are *non-singular* on $(M, \mathcal{A}, \lambda)$. The time evolution of a sequential dynamical system

Keywords: Sequential dynamical systems; Markov partition; Expanding map; Ergodic in mean..

AMS Mathematical Subject Classification [2010]: 37A25.

(φ_n) on M is defined by composing the maps φ_n in the obvious way. In general,

$$\theta_{k,n} := \varphi_{k+n-1} \circ \cdots \circ \varphi_{k+1} \circ \varphi_k \text{ for } k, n \in \mathbb{N} \text{ and } \theta_{k,0} := \mathrm{id}_M. \tag{1.1}$$

In this work, we provide a special infinite countable Markov partition for uniformly Hölder continuous uniformly expanding sequential dynamical systems which is achieved primarily through a finite cover.

We recall the definition of Lebesgue-ergodicity in mean property for sequential dynamical systems used by J. P. Conze and A. Raugi in [2].

Definition 1.1. A sequential dynamical system (φ_n) given by a sequence of non-singular transformations φ_n on a Borel probability space $(M, \mathcal{A}, \lambda)$ is *ergodic in mean* if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \left[\lambda(\theta_k^{-1}(A) \cap B) - \lambda(\theta_k^{-1}(A))\lambda(B)\right] = 0$$

for any measurable sets $A, B \in \mathcal{A}$.

J. P. Conze and A. Raugi in [2] considered (A) Transformations $\varphi : x \to \beta x + \alpha \mod 1$, with $\beta > 2$ and (B) Transformations $\varphi : x \to \beta x \mod 1$, with $\beta \geq 1 + a$, for some $a > 0$, on interval $I = [0, 1]$ and show the ergodicity in mean property for sequential dynamical systems built from such transformations.

Theorem 1.2. [2] *For both families of transformations (A) and (B), any sequence* (φ_n) *is ergodic in mean.*

Our aim is to extend this theorem to higher dimensions. Indeed, by using the Markov Partition, we prove the Lebesgue-ergodicity in mean for uniformly Hölder continuous uniformly expanding sequential dynamical systems built from a sequence of non-singular conformal $C^{1+\alpha}$ local diffeomorphisms on a Borel probability space $(M, \mathcal{A}, \lambda)$.

1.1. Preliminaries.

Definition 1.3. A C^1 local diffeomorphism $\varphi : M \to M$ is *expanding* if there exist $\sigma > 1$ and some Riemannian metric on M such that $\|D\varphi(x)v\| \geq \sigma\|v\|$, for every $x \in M$ and every vector v tangent to M at the point x.

For an expanding C^1 local diffeomorphism $\varphi : M \to M$, there exist constants $\sigma > 1$ and $\rho > 0$ such that for every $p \in M$ the image of the ball $B(p, \rho)$ contains a neighborhood of the closure of $B(f(p), \rho)$ and $d(\varphi(x), \varphi(y)) \geq \sigma d(x, y)$, for every $x, y \in B(p, \rho)$. Also, for any pre-image x of any point $y \in M$, there exists a C^1 map $h : B(y, \rho) \to M$ such that $\varphi \circ h = \mathrm{id}$, $h(y) = x$ and

$$d(h(y_1), h(y_2)) \leq \sigma^{-1} d(y_1, y_2) \text{ for every } y_1, y_2 \in B(y, \rho). \tag{1.2}$$

The factors σ and ρ will be called the *expansion factor* and *injectivity constant* of the expanding C^1-local diffeomorphism φ, respectively (see [9]).

For any $\sigma > 1$ and $\rho > 0$, we denote by $\mathcal{E}(\sigma, \rho)$ the set of all expanding onto C^1 local diffeomorphisms on M with expanding factor σ and injectivity constant ρ.

Definition 1.4. We say that a sequential dynamical system (φ_n) on M is *uniformly expanding* if there exist $\sigma > 1$ and $\rho > 0$ such that $\varphi_n \in \mathcal{E}(\sigma, \rho)$ for each $n \in \mathbb{N}$.

Definition 1.5. We say that a sequential dynamical system (φ_n) given by a sequence of $C^{1+\alpha}$ local diffeomorphisms φ_n on M is *uniformly Hölder continuous* if there exist constants $C > 0$ and $\alpha > 0$ such that for every $x, y \in M$ and each $n \in \mathbb{N}$ one has that

$$d(D\varphi_n(x), D\varphi_n(y)) < Cd(x, y)^\alpha, \tag{1.3}$$

where the constants C and α are independent of n.

Definition 1.6. Let (φ_n) be a sequential dynamical system on M. A countable collection $\mathcal{C} = \{C_n : n \in I\}$ composed of closed subsets of M together with a collection $\Theta = \{\theta_{k_n} : n \in I\}$ is called a *countable Markov partition* for (φ_n) if:

- $\lambda(M \setminus \bigcup_{n\in I} C_n) = 0$ and $C_n = \mathrm{Cl}(\mathrm{int}C_n)$;
- $\mathrm{int}C_n \bigcap \mathrm{int}C_j = \emptyset$ whenever $n \neq j$;
- if $\theta_{k_n}(\mathrm{int}C_n) \bigcap \mathrm{int}C_j \neq \emptyset$, then $\theta_{k_n}(C_n) \supset C_j$.

Moreover, we say that a countable Markov partition $(\mathcal{C}, \Theta) = (C_n, \theta_{k_n})_{n\in I}$ has the *finite images property* (FIP) if $\mathcal{B} = \{\theta_{k_n}(\mathrm{int}(C_n)) : n \in I\}$ consists of finitely many open subsets $\{B_1, \dots, B_N\}$ of M. We recall that a C^1 local diffeomorphism f on M is *conformal* if there exists a function $\alpha : M \to \mathbb{R}$ such that for all $x \in M$ we have $Df(x) = \alpha(x)\mathrm{Isom}(x)$, where $\mathrm{Isom}(x)$ denotes an isometry of T_xM. Clearly, $\alpha(x) = \|Df(x)\| = m(Df(x))$, for all $x \in M$, where $m(A) = \|A^{-1}\|^{-1}$ denote the co-norm of a linear transformation A.

2. Main results

Theorem 2.1. *Any uniformly Hölder continuous uniformly expanding sequential dynamical system (φ_n) given by a sequence of conformal $C^{1+\alpha}$ local diffeomorphisms φ_n on M, admits a countable Markov partition with FIP.*

Corollary 2.2. *For every sequential dynamical system (φ_n) enjoying the assumptions of Theorem A, there exists a sequence of countable Markov partitions with FIP so that their diameters tends to zero.*

Notice that, in general, when the transformations φ_n depend on n, there is no joint invariant measure, but it is convenient to make for the sequential dynamical systems (φ_n) the following property: for every $\epsilon > 0$, there exists $\eta(\epsilon) > 0$ such that

$$\forall B \in \mathcal{A}, \lambda(B) < \eta(\epsilon) \Longrightarrow \lambda(\theta_n^{-1}(B)) < \epsilon \text{ for all } n \geq 1. \tag{2.1}$$

As an application of Theorem 2.1, we provide the following result.

Theorem 2.3. *Every uniformly Hölder continuous uniformly expanding sequential dynamical system* (φ_n) *given by a sequence of non-singular conformal* $C^{1+\alpha}$ *local diffeomorphisms* φ_n *on a Borel probability space* $(M, \mathcal{A}, \lambda)$ *enjoying the condition (2.1) is ergodic in mean.*

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PARABOLIC CONTOUR TO NUMERICAL APPROXIMATION OF FRACTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATION OF FIRST KIND

MAHSHID FOROUGHIPOUR AND MOHAMMAD SHAFI DAHAGHIN

Abstract. In this paper, we consider the fractional Volterra integro-differential equations of first kind and represent the solution of this type equations in the form of contour integral in the complex plane. Then we select the parabolic contour as an optimal contour to approximate this integral. Further, an example to show absolute errors for various fractional order by using our numerical scheme on parabolic contour is given.

1. Introduction and Preliminaries

The fractional Volterra integro-differential equations have many applications in in physics, engineering, economics, diffusion problems. Finding the exact solutions of fractional Volterra integro-differential equations is sometimes difficult. So, numerical methods are proposed to obtain the solution of such types of equations [1, 3]. In this paper, by applying Laplace transform we get the solution fractional Volterra integro-differential equation of first kind in the form of contour integral in the complex plane. Then we select the parabolic contour and use trapezoidal rule with equal step size to approximate this integral. Finally, an example to application of the method is given.

Keywords: Fractional Volterra integro-differential equations, Numerical approximation, Parabolic contour.

AMS Mathematical Subject Classification [2010]: 26A33, 34C20.

Definition 1.1. The Caputo fractional derivative of order $\alpha > 0$ $(m-1 < \alpha < m,\ m \in \mathbb{N})$ is defined as [2]

$$ {}^{C}D_{t_0}^{\alpha} f(t) = I_{t_0}^{m-\alpha} \frac{d^m}{dt^m} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} (t-\tau)^{m-1-\alpha} \frac{d^m}{dt^m} f(\tau) d\tau. \tag{1.1} $$

where $I_{t_0}^{m-\alpha}$ is the Riemann-Liouville fractional integral defined as

$$ I_{t_0}^{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} (t-\tau)^{m-\alpha-1} f(\tau) d\tau. \tag{1.2} $$

2. Numerical approximation of fractional Volterra integro-differential equation of first kind

The Volterra integro-differential equation of first kind is given by [5]

$$ \int_0^t k_1(t,\tau) f(\tau) d\tau + \int_0^t k(t,\tau) f^n(\tau) d\tau = g(t). \tag{2.1} $$

Here, we consider the case in which $k_1(t,\tau) = 0$ and generalize the above equation to Volterra integro-differential equation. So, the equation (2.1) becomes as follows

$$ \int_0^t k(t,\tau) {}^{C}D_{t_0}^{\alpha} f(\tau) d\tau = g(t). \tag{2.2} $$

We consider the convolution type of equation, so the kernel will be of the form $k(t,\tau) = k(t-\tau)$, and the equation (2.2) becomes

$$ \int_0^t k(t-\tau) {}^{C}D_{t_0}^{\alpha} f(\tau) d\tau = g(t). \tag{2.3} $$

Taking the Laplace transform on the both side of (2.3) with respect to t, we have

$$ \mathcal{L}\{\int_0^t k(t-\tau) {}^{C}D_{t_0}^{\alpha} f(\tau) d\tau; s\} = \mathcal{L}\{g(t); s\}. \tag{2.4} $$

By using the convolution theorem and the following formula (the Laplace transform of Caputo derivative) [2]

$$ \mathcal{L}\{{}^{C}D_{t_0}^{\alpha} f(t); s\} = s^{\alpha} \mathcal{L}\{f(t); s\} - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(t_0), \quad m-1 < \alpha < m,\ m \in \mathbb{N}, \tag{2.5} $$

we get

$$ \begin{aligned} & \mathcal{L}\{k(t); s\} \mathcal{L}\{{}^{C}D_{t_0}^{\alpha} f(t); s\} = \mathcal{L}\{g(t); s\} \\ & \Rightarrow K(s)\Big(s^{\alpha} \mathcal{L}\{f(t); s\} - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(t_0)\Big) = G(s) \\ & \Rightarrow F(s) = \frac{G(s)}{k(s)s^{\alpha}} + \frac{\sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(t_0)}{k(s)s^{\alpha}}, \end{aligned} $$

where $F(s) = \mathcal{L}\{f(t); s\}$, $G(s) = \mathcal{L}\{g(t); s\}$ and $K(s) = \mathcal{L}\{k(t); s\}$. Taking inverse Laplace, the problem reduces to compute the following integral in the complex plane

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds. \tag{2.6}$$

We need to select the contour of integration to approximate the path from $c - i\infty$ to $c + i\infty$. To this end we consider parabolic contour [4, 6]. The parametric equation of parabolic contour is given by

$$s = \beta\Big((1-c)^2 - \zeta^2 + 2i\beta\zeta(1-c)\Big), \quad -\infty < \zeta < +\infty, \tag{2.7}$$

where β and c are parameters and need to be optimized for better accuracy. The numerical solution can be represented in the following form

$$f(t) = \frac{1}{2\pi i}\int_{\Gamma} F(s(\zeta))e^{s(\zeta)t}s'(\zeta)ds. \tag{2.8}$$

If we use equal weight quadrature rule, i.e trapezoidal rule with step size h, then the equation (2.8) can be approximated as

$$f_N(t) = \frac{h}{2\pi i}\sum_{j=-N}^{N} F(s(\zeta_j))e^{s(\zeta_j)t}s'(\zeta_j), \quad 1 < \alpha < 2, \quad \zeta_j = jh.$$

3. Example

Consider the following fractional Volterra integro-differential equation of first kind

$$\begin{cases} \int_0^t \cos(t-\tau)\,{}^C D_{t_0}^{\alpha} f(\tau)d\tau = 2\sin(t), \\ t_0 = 0, \quad f(0) = 0, \quad f'(0) = 0, \end{cases} \tag{3.1}$$

where $\alpha \in (1, 2)$. The exact and analytical solution to the fractional Volterra integro-differential (3.1) is

$$f(t) = \frac{2t^{\alpha}}{\Gamma(\alpha+1)}, \quad 1 < \alpha < 2.$$

To give the approximate solution of (3.1), we use the presented numerical scheme. Applying the Laplace transform to equation (3.1) with respect to t, we have

$$\begin{aligned}
&\mathcal{L}\{\int_0^t \cos(t-\tau)\,{}^C D_{t_0}^{\alpha} f(\tau)d\tau; s\} = 2\mathcal{L}\{\sin(t); s\} \\
&\Rightarrow \mathcal{L}\{\cos(t) * {}^C D_{t_0}^{\alpha} f(t); s\} = 2\mathcal{L}\{\sin(t); s\} \\
&\Rightarrow \mathcal{L}\{\cos(t); s\}\mathcal{L}\{{}^C D_{t_0}^{\alpha} f(t); s\} = \frac{2}{1+s^2} \\
&\Rightarrow \frac{s}{s^2+1}\Big(s^{\alpha}F(s) - s^{\alpha-1}f(0) - s^{\alpha-2}f'(0)\Big) = \frac{2}{1+s^2}.
\end{aligned}$$

Therefore, we get

$$F(s) = \frac{2}{s^{\alpha+1}}, \quad 1 < \alpha < 2. \tag{3.2}$$

From the above equation and equation (2), we get the approximate solution as

$$f_N(t) = \frac{h}{\pi i} \sum_{j=-N}^{N} \frac{1}{(s(\zeta_j))^{\alpha+1}} e^{s(\zeta_j)t} s'(\zeta_j), \quad 1 < \alpha < 2, \quad \zeta_j = jh.$$

For our numerical experiments, we use the parameters $c = 0.7$, $h = \frac{3}{N}$, $t = 0.001$, $\beta = \frac{\pi N}{12t}$. Table (1) shows the absolute value error for approximate solution.

TABLE 1. Absolute error of fractional Volterra integro-differential equation of first kind

N	α	β	c	h	t	Absolute error
100	1.99	$\frac{\pi N}{12t}$	0.7	$\frac{3}{N}$	0.001	$1.9058e-21$
100	1.98	$\frac{\pi N}{12t}$	0.7	$\frac{3}{N}$	0.001	$8.4704e-22$
100	1.97	$\frac{\pi N}{12t}$	0.7	$\frac{3}{N}$	0.001	$2.1176e-21$
50	1.9	$\frac{\pi N}{12t}$	0.7	$\frac{3}{N}$	0.001	$3.6844e-15$
50	1.8	$\frac{\pi N}{12t}$	0.7	$\frac{3}{N}$	0.001	$4.9916e-15$
50	1.5	$\frac{\pi N}{12t}$	0.7	$\frac{3}{N}$	0.001	$1.1693e-14$

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INVESTIGATING THE TEMPERATURE OF FRACTIONAL HEAT EQUATION BY USING DOUBLE LAPLACE TRANSFORMS

MAHSHID FOROUGHIPOUR AND MOHAMMAD SHAFI DAHAGHIN

Abstract. In this paper, we consider the fractional heat conduction equation in a rod and obtain the temperature of this equation by using double Laplace transforms. Then we get the important result that the temperature depends only on time not the length of rod. Therefore, with the help of this feature we will be able to design the proper and small temperature sensors.

1. Introduction

Fractional calculus is a generalization of classical differentiation and integration to arbitrary order. In recent years, fractional calculus has been a fruitful field of research in science and engineering. Fractional-order models are found to be more adequate than integer-order models in some real world problems. In this paper, we consider the fractional heat conduction equation in a rod. Then we apply double Laplace transforms to obtain the solution of this equation. Further, we show that the temperature is free from length and does not depend distance from one end.

Keywords: Fractional heat conduction equation, Laplace transform.
AMS Mathematical Subject Classification [2010]: 26A33.

2. Preliminaries

Definition 2.1. The Caputo fractional derivative of order $\alpha > 0$ $(m-1 < \alpha < m,\ m \in \mathbb{N})$ is defined as [2]

$$ {}^{C}D_{t_0}^{\alpha} f(t) = I_{t_0}^{m-\alpha} \frac{d^m}{dt^m} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} (t-\tau)^{m-1-\alpha} \frac{d^m}{dt^m} f(\tau) d\tau, \tag{2.1}$$

where $I_{t_0}^{m-\alpha}$ is the Riemann-Liouville fractional integral defined as

$$ I_{t_0}^{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} (t-\tau)^{m-\alpha-1} f(\tau) d\tau. \tag{2.2}$$

The double Laplace transform of a function $f(x, y)$ of two variables x and y defined in the first quadrant of the xy plane is defined by the double integral as follows

$$\begin{aligned} \mathcal{L}_2\{f(x,y); p_1, p_2\} &= \mathcal{L}\Big\{\mathcal{L}\{f(x,y); x \to p_1\}; y \to p_2\Big\} \\ &= \int_0^{\infty}\int_0^{\infty} e^{-p_1 x} e^{-p_2 y} f(x,y) dx dy = F(p_1, p_2), \quad \Re(p_1)\Re(p_2) > 0. \end{aligned}$$

Further, the inverse double Laplace Transform is defined by the complex double integral

$$\mathcal{L}_2^{-1}\{F(p_1,p_2); x, y\} = \frac{-1}{4\pi^2} \int_{c_1 - i\infty}^{c_1 + i\infty} e^{p_1 x} dp_1 \int_{c_2 - i\infty}^{c_2 + i\infty} e^{p_2 x} F(p_1, p_2) dp_2,$$

where $F(p_1, p_2)$ is an analytic function for all p_1 and p_1 in the region defined by the inequalities $\Re(p_1) > c_1$ and $\Re(p_2) > c_2$ where c_1 and c_2 are arbitrary suitable constant.

3. Fractional Heat Conduction Equation

Consider the following fractional heat conduction equation

$$\begin{cases} \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \kappa \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \\ u(0,t) = 1, \quad u(x,0) = 1, \quad u_x(0,t) = 0, \end{cases} \tag{3.1}$$

where the operator $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ indicates the Caputo fractional derivative of order $\alpha \in (0,1)$, $u(x,t)$ denotes temperature, t and x denote time and a spatial coordinate, respectively, f is the input heat, κ is the thermal conductivity. Here, we take the thermal conductivity as unity ($\kappa = 1$), that is on unit thermal conductivity. Since when we input heat in a rod then temperature is directly proportional to input heat. So, $u(x,t) = \lambda f(x,t)$, where λ is heat resistive coefficient.

Theorem 3.1. *The solution (the temperature) to the fractional heat conduction equation* (3.1) *on unit thermal conductivity is given by*

$$ u(x,t) = E_{\alpha}(\frac{1}{\lambda} t^{\alpha}), \tag{3.2}$$

where $E_\alpha(\frac{1}{\lambda}t^\alpha) = \sum_{n=0}^{\infty} \frac{(\frac{1}{\lambda})^n t^{\alpha n}}{\Gamma(\alpha n+1)}$ *is the classical Mittag-Leffler function* [1].

Proof. We denote with $\tilde{u}(x, p_2) = \mathcal{L}\{u(x,t); p_2\}$ the Laplace transform with respect to the time variable t and $\tilde{u}(p_2, t) = \mathcal{L}\{u(x,t); p_1\}$ the Laplace transform with respect to the space variable x. Taking double Laplace transform of (3.1), we have

$$\mathcal{L}_2\{\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}; p_1, p_2\} = \kappa \mathcal{L}_2\{\frac{\partial^2 u(x,t)}{\partial x^2}; p_1, p_2\} + \frac{1}{\lambda}\mathcal{L}_2\{u(x,t); p_1, p_2\}, \tag{3.3}$$

by using the following formula (the Laplace transform of Caputo derivative) [2]

$$\mathcal{L}\{{}^C D_{t_0}^\alpha f(t); p\} = p^\alpha \mathcal{L}\{f(t); p\} - \sum_{k=0}^{m-1} p^{\alpha-k-1} f^{(k)}(t_0), \quad m-1 < \alpha < m,\ m \in \mathbb{N}, \tag{3.4}$$

we get

$$\mathcal{L}\{p_2^\alpha \tilde{u}(x, p_2) - p_2^{\alpha-1} u(x,0); p_1\} = \kappa \mathcal{L}\{\frac{\partial^2 \tilde{u}(x,p_2)}{\partial x^2}; p_1\} + \frac{1}{\lambda}\mathcal{L}\{\tilde{u}(x,p_2); p_1\}.$$

According to the initial condition $u(0,t) = 1$, $u(x,0) = 1$ and $u_x(0,t) = 0$, we get $\tilde{u}(0,p_2) = \frac{1}{p_2}$, $\hat{u}(p_1, 0) = \frac{1}{p_1}$ and $\tilde{u}_x(0, p_2) = 0$. So,

$$\begin{aligned} &p_2^\alpha \hat{\tilde{u}}(p_1,p_2) - p_2^{\alpha-1}\frac{1}{p_1} = \kappa\Big(p_1^2 \hat{\tilde{u}}(p_1,p_2) - p_1 \tilde{u}(0,p_2) - \tilde{u}_x(0,p_2)\Big) + \frac{1}{\lambda}\hat{\tilde{u}}(p_1,p_2) \\ &\Rightarrow \hat{\tilde{u}}(p_1,p_2) = \frac{p_2^{\alpha-1}}{p_1\big(p_2^\alpha - \kappa p_1^2 - \frac{1}{\lambda}\big)} - \frac{\kappa p_1}{p_2\big(p_2^\alpha - \kappa p_1^2 - \frac{1}{\lambda}\big)}. \end{aligned}$$

For $\kappa = 1$ (unit thermal conductivity), the above equation can be written as

$$\hat{\tilde{u}}(p_1,p_2) = \frac{p_2^{\alpha-1}}{p_1\big(p_2^\alpha - p_1^2 - \frac{1}{\lambda}\big)} - \frac{p_1}{p_2\big(p_2^\alpha - p_1^2 - \frac{1}{\lambda}\big)}. \tag{3.5}$$

Inverting the Laplace transform along p_2 and using the following relation [2]

$$\mathcal{L}^{-1}\Big\{\frac{p^{\alpha-\beta}}{p^\alpha - \eta}; p \to z\Big\} = z^{\beta-1}E_{\alpha,\beta}(\eta z^\alpha), \quad |\eta p^{-\alpha}| < 1, \tag{3.6}$$

where $E_{\alpha,\beta}(\eta z^\alpha) = \sum_{n=0}^{\infty} \frac{(\eta z^\alpha)^n}{\Gamma(\alpha n+\beta)}$ is the Mittag-Leffler function with two parameters [1], we have

$$\begin{aligned}
\hat{u}(p_1,t) &= \mathcal{L}^{-1}\{\hat{\hat{u}}(p_1,p_2); p_2 \to t\} \\
&= \frac{1}{p_1}\mathcal{L}^{-1}\{\frac{p_2^{\alpha-1}}{p_2^\alpha - \left(p_1^2+\frac{1}{\lambda}\right)}; p_2 \to t\} - p_1\mathcal{L}^{-1}\{\frac{p_2^{-1}}{p_2^\alpha - \left(p_1^2+\frac{1}{\lambda}\right)}; p_2 \to t\} \\
&= \frac{1}{p_1}E_\alpha\Big((p_1^2+\frac{1}{\lambda})t^\alpha\Big) - p_1 t^\alpha E_{\alpha,\alpha+1}\Big((p_1^2+\frac{1}{\lambda})t^\alpha\Big) \\
&= \frac{1}{p_1}\sum_{n=0}^{\infty}\frac{(p_1^2+\frac{1}{\lambda})^n t^{\alpha n}}{\Gamma(\alpha n+1)} - p_1 t^\alpha \sum_{n=0}^{\infty}\frac{(p_1^2+\frac{1}{\lambda})^n t^{\alpha n}}{\Gamma(\alpha n+\alpha+1)} \\
&= \Big(\frac{1}{p_1} + \frac{(p_1^2+\frac{1}{\lambda})t^\alpha}{p_1\Gamma(\alpha+1)} + \frac{(p_1^2+\frac{1}{\lambda})^2 t^{2\alpha}}{p_1\Gamma(2\alpha+1)} + \dots\Big) \\
&- \Big(p_1\frac{t^\alpha}{\Gamma(\alpha)} + \frac{p_1(p_1^2+\frac{1}{\lambda})t^{2\alpha}}{\Gamma(2\alpha)} + \frac{p_1(p_1^2+\frac{1}{\lambda})^2 t^{3\alpha}}{\Gamma(2\alpha)} + \dots\Big).
\end{aligned}$$

Finally, taking inverse Laplace with respect to p_1, we obtain

$$u(x,t) = 1 + \frac{\frac{1}{\lambda}t^\alpha}{\Gamma(\alpha+1)} + \frac{\frac{1}{\lambda^2}t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\frac{1}{\lambda^3}t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{\lambda}t^\alpha\right)^n}{\Gamma(\alpha n+1)} = E_\alpha(\frac{1}{\lambda}t^\alpha).$$

$\square$

4. Conclusion

In the present paper, we introduce the fractional heat equation and get the temperature of this equation by using double Laplace transforms. Then we show that the temperature of rod by heat conduction is depend only on time because $E_\alpha(\frac{1}{\lambda}t^\alpha)$ is a function independent of length. For the future research, we will extend our results for fractional heat conduction equation in a rod and design temperature sensors and heaters into car. We intend to develop a sensor which is length independent and has a small size to fit easily in the car. Also, we will discuss on temperature conduction rate and the advantages of fractional model with respect to integer-order one.

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NUMERICAL SOLUTION FOR FRACTIONAL BRATU'S INITIAL VALUE PROBLEM

FATEME GHOMANJANI

ABSTRACT. In this paper, a method for solving fractional Bratu's initial value problem (FBIVP) is presented. The main idea behind this work is the use of the Bezier curve method (BCM). To show the efficiency of the developed method, numerical results are presented. The study shows that this method is effective and is a simple technique to solve FBIVP.

1. Introduction

Fractional differential equations (FDEs) are modelled in different fields of science and engineering such as control engineering, electromagnetism, image processing, fluid flow, statistical mechanics, In general, most of FDEs do not have exact solutions, therefore some investigators studied various methods for finding approximate solutions. These techniques include, Adomian decomposition method (ADM) [1], homotopy perturbation method (HPM) [2], Bezier curve method (BCM) [3], finite difference method, etc. For example, He [1] proposed HPM when this method is an approach which searches for an analytical approximate solution of linear and nonlinear problems, also, the differential transform method (DTM) is applied to use for solving FBIVP [4]. BCM is used for solving dynamical systems, (see [5]). Also BCM is used for solving delay differential equations and switched systems (see [6], [7]). Also, to solve the quadratic Riccati differential equation and the Riccati differential-difference equation, BCM is utilized (see [6]). In this study, BCM is extended for

Keywords: Bezier curve, Fractional Bratu-type equation..
AMS Mathematical Subject Classification [2010]: 46E22, 34A08..

solving FBIVP as follows:

$$
\begin{aligned}
&D^{\alpha}_{0_+}x(t) + \lambda e^{x(t)} = 0,\ 1 < \alpha \leq 2,\ 0 < t < 1, \\
&x(0) = x_0,\ x'(0) = x'_0,
\end{aligned} \tag{1.1}
$$

where $D^{\alpha}_{0_+}$ is derivative operator of fractional order α, $x(t)$ is unknown function on $[0,1]$, $x'(0)$ is the derivative of x with respect to t at $t = 0$, and x_0, x'_0 and λ are given constant. Also, ADM was investigated for solving FBIVP in [8]. Babolian et al. [9] presented reproducing kernel method (RKM) for solving FBIVP. The outline of this sequel is as follows: In Section 2, background materials are stated. Section 3 is devoted to a numerical example for the precision of the proposed technique. Finally, the conclusion is presented in Section 4.

2. Background materials

2.1. **Problem statement.** One may consider the following FBIVP:

$$
\begin{aligned}
&D^{\alpha}_{0_+}x(t) + \lambda e^{x(t)} = 0,\ 1 < \alpha \leq 2,\ 0 < t < 1, \\
&x(0) = x_0,\ x'(0) = x'_0,
\end{aligned} \tag{2.1}
$$

By BCM, one may have

$$
x_n(t) \simeq \sum_{i=0}^{n} c_i B_{i,n}(t),\ 0 \leq t \leq 1, \tag{2.2}
$$

by Eqs. (2.1) and (2.2), one may have

$$
D^{\alpha}_{0_+}\left(\sum_{i=0}^{n} c_i B_{i,n}(t)\right) \quad = \quad -\lambda e^{\left(\sum_{i=0}^{n} c_i B_{i,n}(t)\right)} \tag{2.3}
$$

Hence

$$
R(x, c_0, c_1, \ldots, n) \quad = \quad \sum_{i=0}^{n} c_i D^{\alpha}_{0_+} B_{i,n}(x) + \lambda e^{\left(\sum_{i=0}^{n} c_i B_{i,n}(t)\right)}
$$

Suppose

$$
\begin{aligned}
S(x, c_0, c_1, \ldots, c_n) \quad &= \quad \int_0^1 R(x, c_0, c_1, \ldots, n)^2 w_1(x)dx,\ \text{where } w_1(x) = 1, \\
S(x, c_0, c_1, \ldots, c_n) \quad &= \quad \int_0^1 \left(\sum_{i=0}^{n} c_i D^{\alpha} B_{i,n}(x) + \lambda e^{\left(\sum_{i=0}^{n} c_i B_{i,n}(t)\right)}\right)^2 dx,
\end{aligned}
$$

then, one may have

$$\frac{\partial S}{\partial c_i} = 0, \ 0 \leq i \leq n,$$

therefore

$$\int_0^1 \left(\sum_{i=0}^{n} c_i D^{\alpha} B_{i,n}(x) + \lambda e^{\left(\sum_{i=0}^{n} c_i B_{i,n}(t) \right)} \right) \times \left(D^{\alpha} B_{i,n}(x) \right.$$

$$+ \ \lambda e^{\left(\sum_{i=0}^{n} c_i B_{i,n}(t) \right)} \left(B_{i,n}(x) \right) \Bigg) dx = 0, \tag{2.4}$$

by Eq. (2.4), one can obtain a system of $n+1$ linear equations with $n+1$ unknown coefficients c_i. Also by utilizing many subroutine algorithm for solving this linear equations, one can find the unknown coefficients c_i, $i = 0, 1, \ldots, n$.

3. Numerical applications

In this section, some numerical examples are presented to illustrate the proposed method.

Example 3.1. The following FBIVP is considered (see [9]):

$$D_{0_+}^{\alpha} x(t) - 2e^{x(t)} = 0, \ 1 < \alpha \leq 2, \ 0 < t < 1,$$
$$x(0) = 0, \ x'(0) = 0,$$
$$x_{exact}(t) = -2ln(cos(t)), \ \text{for } \alpha = 2.$$

The value of approximate solution with the stated technique is more accurate than that with the stated technique in [9] (see Table 1 and Table 2).

Table 1. The comparison of approximation solution of the this method and RKM in [9] with $\alpha = 1.9$ and $n = 5$ for Example 3.1

t	approximate solution of proposed method	approximate solution of RKM
0.1	0.01031561203	1.3082×10^{-2}
0.2	0.04026954618	5.1424×10^{-2}
0.3	0.09121393595	1.1325×10^{-1}
0.4	0.1643473163	1.9923×10^{-1}
0.5	0.2611684808	3.1127×10^{-1}
0.6	0.3839303388	4.5103×10^{-1}
0.7	0.5360937724	6.2324×10^{-1}
0.8	0.7227814938	8.3787×10^{-1}
0.9	0.9512319022	11.051×10^{-1}
1.0	1.2312529410	14.299×10^{-1}

Example 3.2. The following FBIVP is considered (see [9]):

$$D_{0_+}^{\alpha} x(t) - e^{2x(t)} = 0, \ 1 < \alpha \leq 2, \ 0 < t < 1,$$
$$x(0) = 0, \ x'(0) = 0,$$
$$x_{exact}(t) = ln(sec(t)), \ \text{for } \alpha = 2.$$

Using the described technique, we have Table 3.

Table 2. The comparison of between the error solution of the this method and RKM in [9] with $\alpha = 2$ for Example 3.1

t	error of proposed method	error of RKM
0.1	0.00029890083	1.6674×10^{-5}
0.2	0.00000000000	3.100×10^{-7}
0.3	0.00016937595	1.131×10^{-6}
0.4	0.00011072190	212.0×10^{-6}
0.5	0.00000000000	2.900×10^{-6}
0.6	0.00000000000	4.100×10^{-6}
0.7	0.00007774260	6.500×10^{-6}
0.8	0.00000000000	7.500×10^{-6}
0.9	0.00034701520	3.350×10^{-5}
1.0	0.00000000000	4.370×10^{-5}

Table 3. The comparison of between the approximate solution of the this method and RKM in [9] with $\alpha = 1.9$ and $n = 5$ for Example 3.2

t	approximate solution of proposed method	approximate solution of RKM
0.1	0.006541208907	6.54110×10^{-3}
0.2	0.023350951330	2.57120×10^{-2}
0.3	0.048440451460	5.66250×10^{-2}
0.4	0.082229018710	9.96150×10^{-2}
0.5	0.126834069800	1.55630×10^{-1}
0.6	0.185361150900	2.25520×10^{-1}
0.7	0.261193959900	3.11620×10^{-1}
0.8	0.357284368200	4.18893×10^{-1}
0.9	0.475442443200	5.52550×10^{-1}
1.0	0.615626470300	7.14940×10^{-1}

4. Conclusions

In this study, BCM is used to solve a class of FBIVP. The achieved results by the BCM are in good agreement with the given exact solutions. The study shows that the method is effective and is a simple technique to solve FBIVP.

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SOLVING A QUADRATIC RICCATI DIFFERENTIAL EQUATION

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Abstract. An effective algorithm for solving quadratic Riccati differential equation (QRDE) is presented. This technique is based on Genocchi polynomials (GPs). The properties of Genocchi polynomials are stated, and operational matrices of derivative are constructed. Collocation method based on this operational matrix is used. The findings show that the technique is accurate and simple to use.

1. Introduction

Riccati differential equations (RDEs) play significant role in many fields of applied science [1]. Recently, various iterative methods are employed for the numerical and analytical solution of functional equations such as Adomian's decomposition method (ADM) (see [2]), and differential transform method (DTM) [3]. The GPs are non-orthogonal polynomials, which were first applied to solve fractional calculus problem (FCP) involving differential equation [4], this GPs were successfully applied to solve different kinds of problems in numerical analysis, system of Volterra integro-differential equation [5] and fractional Klein-Gordon equation [6], differential topology (differential structures on spheres), theory of modular forms (Eisenstein series). The outline of this sequel is as follows: In Section 2, Some basic preliminaries are stated. Explanation of the problem is explained in Section 3. Some numerical results are provided in Section 4. Finally, Section 5 will give a conclusion briefly.

Keywords: Genocchi polynomials; Operational matrix of derivatives.
AMS Mathematical Subject Classification [2010]: 0096, 3003, 49K15..

2. Some basic preliminaries

Genocchi numbers (G_n) and Genocchi polynomials ($G_n(x)$) have been extensively studied in various papers, (see [7]). The classical Genocchi polynomials $G_n(x)$ are usually defined by the following form

$$\frac{2te^{xt}}{e^t+1}=\sum_{i=0}^{\infty}G_n(x)\frac{t^n}{n!},\ \ (|t|<\pi), \tag{2.1}$$

where

$$\begin{aligned}
&G_n(x)=\sum_{k=0}^{n}\binom{n}{k}G_kx^{n-k},\\
&G_1=1, G_2=0, G_3=0, G_4=1, G_5=0, G_6=-3, G_7=0,\\
&G_8=17, G_9=0, G_{10}=-155, G_{11}=0, G_{12}=2073, G_{2n+1}=0, n\in N,\\
&G_1(x)=1, G_3(x)=3x^2-3x, G_3(x)=3x^2-3x,\\
&G_n(x+1)+G_n(x)=2nx^{n-1}, n\ge 0,
\end{aligned} \tag{2.2}$$

3. Explanation of the problem

Firstly, Riccati differential equation (RDE) is considered

$$\begin{aligned}
&y'(x)=p(x)-q(x)y(x)+r(x)y^2(x),\ \ x_0\le x\le x_f,\\
&y(x_0)=\alpha,
\end{aligned} \tag{3.1}$$

where $p(x)$, $q(x)$ and $r(x)$ are continuous, x_0, x_f and α are arbitrary constants, and $y(x)$ is unknown function.

Now, the collocation method based on Genocchi operational matrix of derivatives to solve numerically RDEs is presented. Our strategy is utilizing GPs to approximate the solution $y(x)$ by $y_N(x)$ is as given below.

$$\begin{aligned}
&y(x)\approx y_N(x)=\sum_{n=1}^{N}c_nG_n(x)=G(x)C,\\
&C^T=[c_1,c_2,\ldots,c_n],\\
&G(x)=[G_1(x),G_2(x),\ldots,G_N(x)],\\
&M=\begin{bmatrix}0&0&0&\ldots&0&0&0,\\2&0&0&\ldots&0&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&\ldots&N-1&0&0\\0&0&0&\ldots&0&N&0\end{bmatrix}
\end{aligned} \tag{3.2}$$

$$G'(x)^T = MG^T(x), \Rightarrow G'(x) = G(x)M^T,$$

$$\vdots$$

$$G^{(k)}(x) = G(x)(M^T)^k, \tag{3.3}$$

then, the k-th derivative of $y_N(x)$ can be stated as

$$y_N^{(k)}(x) = G^{(k)}(x)C = G(x)(M^T)^k C, \tag{3.4}$$

by Eqs. (3.1) and (3.4), we have

$$G(x)M^T C = p(x) - q(x)G(x)C + r(x)(G(x)C)^2, \tag{3.5}$$

to obtain $y_N(x)$, one may use the collocation points $x_j = \frac{j-1}{N}$, $j = 1, 2, \ldots, N-1$. These equations can be solved by Maple 15 software.

Lemma 3.1. *If $y(x) \in C^{n+1}[0,1]$ and $U = Span\{G_1(x), G_2(x), \ldots, G_N(x)\}$, then $G(x)C$ is the best approximation of $y(x)$ out of U when*

$$\|y(x) - G(x)C\| \leq \frac{h^{\frac{2n+3}{2}} R}{(n+1)!\sqrt{2n+3}}, \; x \in [x_i, x_{i+1}] \subset [0,1],$$

where $R = \max_{x\in[x_i,x_{i+1}]} |y^{(n+1)}(x)|$ and $h = x_{i+1} - x_i$.

Proof. See [7]. □

4. Numerical applications

Example 4.1. First, the following RDE is considered (see [8])

$$y'(x) = 1 + 2y(x) - y^2(x), \; 0 \leq x \leq 1,$$
$$y(0) = 1,$$
$$y_{exact}(x) = 1 + \sqrt{2}\tanh\left(\sqrt{2}x + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1})\right), \; y_{exact}(0) = 2 \times 10^{-10} \simeq 0.$$

One may achieve $y_{approx}(x) = 0.4836486196 + 1.959259361x + .1873135074x^2 - 0.5349351716x^3$ with this technique by $n = 4$. Table 1 demonstrates the absolute error of the this technique.

5. Conclusions

In this paper, GPs stated for solving the RDEs. The stated technique is computationally attractive. A numerical example is included to explain the validity of this technique.

Table 1. The absolute error of the this method for Example 4.1

x	error of y
0.1	0.01576255020
0.2	0.01957152970
0.3	0.01520160000
0.4	0.007259874300
0.5	$7.000000000 \times 10^{-10}$
0.6	0.003753795400
0.7	0.003263092300
0.8	$3.000000000 \times 10^{-10}$
0.9	0.002664043000
1.0	0.0

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PROVIDE A METHOD FOR SOLVING LINEAR INTEGRAL EQUATIONS

MARYAM HAGHSHENAS

Abstract. In this article, we presents a method numerical for solution linear integral equations of the volterra with the taylor polynomial method. this method starts by differentiation from the sides of the integral equation and using taylor polynomials and finally reaches the approximate answer of the linear integral equation by transforming the original form into a matrix relation.

1. Introduction

Integral equations appear in many problems of engineering, physics, chemistry, biology, etc [2],[5],[9]. These equations can be used as another representation for problems with the initial or boundary value of the solution of differential, which are divided into two types of fredholm and volterra integral equations [3].

2. Introduction of the integral equation

In general, integral equations are defined as follows:

$$\phi(x)y(x) = f(x) + \lambda \int_a^b k(x,t,y(t))dt$$

Keywords: Linear integral equations, Linear integral equations of type fredholm, Linear integral equations of type volterra, taylor polynomials and series.

AMS Mathematical Subject Classification [2010]: 13F55, 05E40, 05C65 (at least 1 and at most 3).

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and

$$\phi(x)y(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t,y(t))dt$$

Which order represent the equations of fredholm and volterra with the parametr known λ. Now, if the kernel k relative to y is linear, then it will be a linear integral equation, otherwise it will be a nonlinear integral equation. Therefore, in a linear integral equation of volterra, $k(x,t)$ and $f(x)$ are known functions and $y(x)$ is an unknown function. So far, several methods have been proposed to solve linear integral equations [1],[4],[6],[7]. Which In the next section, we present a solution method for the linear integral equation of the volterra type using the taylor polynomial method.

3. Present a numerical method for solving Volterra linear integral equations

This method involves taking n differentiation from both sides of the integral equation and using taylor polynomials to localize the unknown function $y(x)$. the linear integral equation of volterra is assumed as follows:

$$y(x) = f(x) + \lambda \int_a^x k(x,t)y(t)dt \tag{3.1}$$

Assuming that the answer is a taylor polynomial of degree N around $x = c$ as follows:

$$y(x) = \sum_{k=0}^{N} \frac{y^k(c)(x-c)^k}{k!} \qquad a \le c \le x \tag{3.2}$$

Find the unknown coefficients $y^k(c)$ for $k = 0, 1, 2, ..., N$. Suppose:

$$I^n(x) = \frac{d^n}{dx^n} \int_a^x k(x,t)y(t)dt \tag{3.3}$$

And for, n = 0

$$I^0(x) = I(x) = \int_a^x k(x,t)y(t)dt$$

By differentiation n times (relative to x) from the sides of the equation 3.1 we have:

$$y^n(x) = f^n(x) + \lambda I^n(x) \tag{3.4}$$

According to Leibniz's rule, $I^n(x)$ is obtained as follows[10]:

$$I^n(x) = \sum_{i=0}^{n-1} [h_i(x)y(x)]^{(n-i-1)} + \int_a^x \frac{\partial^n k(x,t)}{\partial x^n} y(t)dt \qquad n \ge 1 \tag{3.5}$$

That:

$$h_i(x) = \frac{\partial^i k(x,t)}{\partial x^i}|_{t=x} \tag{3.6}$$

By calculating $[h_i(x)y(x)]^{(n-i-1)}$ from the Leibniz rule and replacing it in the 3.5 formula we will have:

$$I^n(x) = \sum_{m=0}^{n-1} \sum_{i=0}^{n-m-1} \binom{n-i-1}{m} h_i^{(n-m-i-1)}(x) y^m(x) + \int_a^x \frac{\partial^n k(x,t)}{\partial x^n} y(t) dt \tag{3.7}$$

Now, we write the expansion of taylor $y(t)$ around $t = c$

$$y(t) = \sum_{m=0}^{\alpha} \frac{y^m(c)}{m!} (t-c)^m \tag{3.8}$$

By replacing $x = c$ in the equation 3.4 and using relations 3.7 , 3.8 we reach the following relation:

$$\begin{aligned} y^n(c) = f^n(c) + \lambda \sum_{m=0}^{n-1} \sum_{i=0}^{n-m-1} \binom{n-i-1}{m} h_i^{(n-m-i-1)}(c) y^m(c) \\ + \lambda \int_a^c \frac{\partial^n k(x,t)}{\partial x^n}|_{t=c} [\sum_{m=0}^{\alpha} \frac{y^m(c)(t-c)^m}{m!}] dt \end{aligned}$$

We now define H_{nm} and T_{nm} for $n = 1, 2, ...$ and $m = 0, 1, 2, n-1$ as follows:

$$H_{nm} = \begin{cases} \sum_{i=0}^{n-m-1} \binom{n-i-1}{m} h_i^{(n-m-i-1)}(c) & n > m \\ 0 & n \leq m \end{cases} \tag{3.9}$$

And for the values of $n, m = 0, 1, ...$ we define:

$$T_{nm} = \frac{1}{m!} \int_a^c \frac{\partial^n k(x,t)}{\partial x^n}|_{x=c} (t-c)^m dt \tag{3.10}$$

Then:

$$y^n(c) = f^n(c) + \lambda \{ \sum_{m=0}^{n-1} (H_{nm} + T_{nm}) y^m(c) + \sum_{m=n}^{\alpha} T_{nm} y^n(c) \} \tag{3.11}$$

The relation 3.11 is a system consisting of infinite linear equations. Assuming that $y(x)$ is approximated by taylor polynomials of degree N, then $m, n = 0, 1, 2, ...N$. Thus the above relation to a system is reduced by $N+1$ linear equation and $N+1$ unknown coefficient $y^0(c), y^1(c), ..., y^N(c)$ witch by converting it into a matrix and using numerical methods, a unique solution is obtained as follows for the linear integral equation 3.1

$$y(x) = \sum_{n=0}^{N} \frac{1}{n!} y^n(c)(x-c)^n \tag{3.12}$$

Example. The following integral equation is assumed

$$y(x) = 1 + x + \lambda \int_0^x (x-t) y(t) dt$$

Suppose the answer is a taylor polynomial of degree 5. we have:

$$f(x) = 1 + x, \qquad k(x,t) = x - t, \qquad a = 0, \qquad N = 5$$

By replacing, $c = a = 0$ and according to the relations 3.6, 3.9:

$$\begin{array}{lllll} H_{10} = 0, & H_{20} = 1, & H_{21} = 0, & H_{30} = 0, & H_{31} = 0, \\ H_{32} = 0, & H_{40} = 0, & H_{41} = 0, & H_{42} = 1, & H_{43} = 0, \\ H_{50} = 0, & H_{51} = 0, & H_{52} = 0, & H_{53} = 1, & H_{54} = 0. \end{array}$$

And using the relation 3.10:

$$T_{nm} = 0 \qquad n, m = 0, 1, ..., 5$$

and $f^n(x)$ for $n = 0, 1, ..., 5$ around $x = c = 0$:
$f^0(0) = 1, \qquad f^1(0) = 1, \qquad f^2(0) = f^3(0) = f^4(0) = f^5(0) = 0$
By forming a system of equations and solving it:
$y^0(0) = 1 = y^1(0) = 1, \qquad y^2(0) = 1 = y^3(0) = \lambda, \qquad y^4(0) = 1 = y^5(0) = \lambda^2$ So the answer to integral equation 3.12 is as follows

$$y(x) = 1 + x + \lambda[\frac{x^2}{2!} + \frac{x^3}{3!}] + \lambda^2[\frac{x^4}{4!} + \frac{x^5}{5!}]$$

If $\lambda = 1$ and N is large enough, the exact answer $y(x) = exp(x)$ is obtained.

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CHEBYSHEV POLYNOMIALS AND ITS APPLICATION IN SOLVING INTEGRAL EQUATIONS

MARYAM HAGHSHENAS

Abstract. Due to the maximum use of integral equations in various sciences, so far several solution methods have been proposed for them. Which we examine in this article a solution method for integral equations using the chibyshev iteration method. That its use is based on the bases of chibyshev and thus reaches a approximate solution for solving integral equations.

1. Introduction

In this article, we refers first shortly describe the functions of chibyshev and its bases. Then while introducing fredholm integral equations, we paying explanation a numerical method for solving fredholm linear integral equation of the second.

2. Chibyshev polynomials

An important class of orthogonal functions are the chibyshev polynomials $T_n(x)$, which are defined on interval [-1,1] and are obtained from the following return relations:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x), \qquad n \geq 2 \end{aligned}$$

Keywords: chibyshev polynomials, fredholm linear integral equation, chibyshev iteration method .

APPLICATION CHEBYSHEV POLYNOMIALS.

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It is worth noting that chibyshev polynomials have various properties and many theorems have been proposed for them[1]. Also chebyshev polynomials are important in approximation theory and provides an approximation that is close to the best polynomial approximation to a continuous function under the maximum norm, is called the minimax criterion.

In practice, chibyshev polynomials of order n building an orthogonal basis for the space of polynomials up to degree n. That the base of chibyshev functions $\{T_0, T_1, ..., T_n\}$ can be written in terms of base $\{1, x, ..., x^n\}$ and contrariwise.

3. Introduction of Fredholm Linear Integral Equation

A fredholm integral equation is defined as follows:

$$\phi(x)y(x) = f(x) + \lambda \int_a^b k(x, t, y(t))dt$$

In the above equation if the kernel k is linear with respect to y, then a fredholm linear integral equation is obtained. That according to $\phi(x)$ are divided into two categories.

(1) If $\phi(x) = 0$, then the fredholm linear integral equation of the first kind is called.
(2) If $\phi(x) \neq 0$, then, dividing the sides by $\phi(x)$ and placing $F(x) = \frac{f(x)}{k(x)}$ and $K(x,t) = \frac{k(x,t)}{\phi(x)}$ will give:

$$y(x) = F(x) + \lambda \int_a^b K(x, t)y(t)dt$$

This equation is called a fredholm linear integral equation of the second kind.

Many of the above equations do not have analytical solutions and inevitably we have to use numerical methods to solve them. There are several numerical methods to solve these equations [4],[3],[7],[8],[9],... . In the following, we present a solution method for fredholm linear integral equation of the second kind[6].

4. Chibyshev iteration method for solving the Fredholm linear integral equation of the second kind

Consider the equation as follows:

$$y(x) = f(x) + \lambda \int_{-1}^1 k(x, t)y(t)dt \tag{4.1}$$

Assuming that f on the interval [-1,1] is piecewise smooth and $k(x,t)$ is of the function class which are piecewise smooth for $k = 0, 1, ...$ over the interval [-1,1]. We will explain chibyshev iteration method.

Consider the initial approximation $y_0(x)$ for $y(x)$ as the chibyshev series and use the following iteration:

$$y_i(x) = f(x) + \lambda \int_{-1}^{1} k(x,t) y_{i-1}(t) dt \qquad i = 1, 2, ... \tag{4.2}$$

where in:

$$y_{i-1}(x) = \sum_{j=0}^{N} a_j T_j(x) \tag{4.3}$$

Obtained from previous iterations. Suppose the i-th iteration is below chibyshev expansion:

$$y_i(x) = \sum_{j=0}^{N} A_j T_j(x) \tag{4.4}$$

We also write $f(x)$ as an extension of chibyshev:

$$f(x) = \sum_{j=0}^{N} f_j T_j(x) \tag{4.5}$$

And we put:

$$\int_{-1}^{1} k(x,t) T_k(t) dt = \sum_{j=0}^{N} \beta_{kj} T_j(x) \tag{4.6}$$

In relation 4.6 we first approximate the left integral with a numerical integration formula and and then write the resulting function according to chibyshev bases and obtain the coefficients β_{kj}. By replacing 4.3, 4.4, 4.5, 4.6 in 4.2 and using the linearly independent we will have :

$$A_j = f_j + \lambda \sum_{k=0}^{N} \beta_{kj} a_k \qquad j = 0, 1, 2, ..., N \tag{4.7}$$

With the following matrix form:

$$A = f + \lambda B_a \tag{4.8}$$

Where in a,f,A are vectors of coefficients a_j,f_j,A_j and the matrix $B = (B_{ij})$ is:

$$B_{ij} = \begin{cases} \frac{1}{2}\beta_{0i} & j = 0 \\ \beta_{ji} & j = 1, ..., N \end{cases} \tag{4.9}$$

Where in $i = 0, 1, ..., N$. We can now find the unknown coefficients A_j and obtain the answer of the integral equation by replacing it in the relation 4.4.

5. Conclusions

In this paper to solve a fredholm linear integral equation of the second, we use chebyshev polynomials and its itration and by applying the original form of the equation in the form of matrix relations, we arrive at an approximate answer.

The use of chibyshev polynomials because of the easy calculation of derivatives and their integrals, has been the attention of many researchers in the field of solving integral equations this way.

Also, the advantage of chibyshev method over other methods is that by a simple modification is also used for other equations of the second type.

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AN ITERATIVE COMPACT FINITE DIFFERENCE METHOD FOR THE NUMERICAL SOLUTION OF NON-ISOTHERMAL REACTION-DIFFUSION MODEL IN A SPHERICAL CATALYST

AZAM SADAT HASHEMI, GHASEM BARID LOGHMANI, AND MOHAMMAD HEYDARI

ABSTRACT. In this investigation, an iterative scheme to find the approximate solution of the non-isothermal reaction-diffusion model in a spherical catalyst is proposed. For this purpose, the governing differential equation is converted to a sequence of linear differential equations via quasilinearization method (QM). Then, applying compact finite difference method at each iteration, the obtained linearized differential equations are solved numerically to compute the approximate solution.

1. INTRODUCTION

One of the basic and most important equations in astronomy is the Lane-Emden equation which describes the equilibrium density distribution in self-gravitating sphere of polytropic isothermal gas. Moreover, recently observed that the density profile of dark matter can be modeled by the isothermal Lane-Emden equation. Similar to other linear and nonlinear singular problems, this equation is numerically challenged because of its singularity at the origin. Recently, Rach et al. [5] and Wazwaz et al. [6] developed efficient methods to solve this equation.

In this study, we consider the non-isothermal reaction-diffusion problem which can be characterized by a Lane-Emden boundary value problem with strongly nonlinear term as [5]:

$$y''(x) + \frac{2}{x}y'(x) - \phi^2 y(x) \exp\left(\frac{\gamma\beta(1-y(x))}{1+\beta(1-y(x))}\right) = 0, \ x \in [0,1] \tag{1.1}$$

$$y'(0) = 0, \ y(1) = 1, \tag{1.2}$$

which γ, β and ϕ denote the dimensionless activation energy, heat of reaction and the Thiele modulus as evaluated at the surface of the spherical catalyst pellet, respectively.

Keywords: Non-isothermal reaction-diffusion model, Compact finite difference method, Quasilinearization method.

AMS Mathematical Subject Classification [2010]: 13F55, 05E40, 05C65.

2. Linearization method

As we know, the QM stated by Bellman and Kalaba [1] is a useful technique to compute the solutions of the nonlinear differential equations as a sequence of linear differential equations. Consider nonlinear boundary value problem as follows:

$$y'' = f(x, y, y'), \quad x \in [a, b], \tag{2.1}$$

$$c_1 y(a) + c_2 y'(a) = c_3,\ c_4 y(b) + c_5 y'(b) = c_6, \tag{2.2}$$

where $c_i,\ i = 1 \dots 6$ are real constants. Applying the QM for one variable nonlinear BVP (2.1) and (2.2), one can obtain the following sequence of linear differential equations

$$\begin{gathered} y''_{n+1} = f(x, y_n, y'_n) + (y_{n+1} - y_n) f_y(x, y_n, y'_n) + (y'_{n+1} - y'_n) f_{y'}(x, y_n, y'_n), \\ c_1 y_{n+1}(a) + c_2 y'_{n+1}(a) = c_3,\ c_4 y_{n+1}(b) + c_5 y'_{n+1}(b) = c_6. \end{gathered} \tag{2.3}$$

Theorem 2.1. [4] *Suppose that $\delta y_{n+1} = y_{n+1} - y_n$ denotes the difference between two subsequence iterations in (2.3). Then the QM is convergent quadratically, i.e.*

$$\|\delta y_{n+1}\| \le k \|\delta y_n\|^2.$$

3. Compact finite difference method

Compact finite difference method (CFDM) is a high accurate way to approximate the solution of differential equations numerically. As stated by Lele [3], the finite difference approximation of first and second derivatives are provided as a linear combination of the function values on given nodes.
Suppose that $y_i = y(x_i),\ i = 0, 1, \dots, N$ are the function values at the nodes $x_i = a + ih,\ i = 0, 1, \dots, N$ where $N \in \mathbb{N}$ and $h = (b - a)/N$. According to [3], fourth order triangular scheme for first and second derivatives are stated as follows:

$$\hat{\alpha} y'_{i-1} + y'_i + \hat{\alpha} y'_{i+1} = \hat{b}_1 \frac{y_{i+1} - y_{i-1}}{2h} + \hat{b}_2 \frac{y_{i+2} - y_{i-2}}{4h}, \tag{3.1}$$

$$\tilde{\alpha} y''_{i-1} + y''_i + \tilde{\alpha} y''_{i+1} = \tilde{b}_1 \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \tilde{b}_2 \frac{y_{i+2} - 2y_i + y_{i-2}}{4h^2}, \tag{3.2}$$

where

$$\hat{b}_1 = \frac{2}{3}(\hat{\alpha} + 2), \quad \hat{b}_2 = \frac{1}{3}(4\hat{\alpha} - 1), \quad \tilde{b}_1 = \frac{4}{3}(-\tilde{\alpha} + 1), \quad \tilde{b}_2 = \frac{1}{3}(10\tilde{\alpha} - 1).$$

In this way, $\hat{\alpha} = \frac{1}{4}$ and $\tilde{\alpha} = \frac{1}{10}$ yield Padé standard scheme for first and second derivatives, respectively.

4. Description of the method

The aim of this section is to solve the singular nonlinear BVP (1.1)-(1.2), numerically. The Numerical solution of the equation (1.1) will be computed in two steps. At first, the QM (2.3) is applied to obtain the linear structure of (1.1). So we have

$$y''_{n+1}(x) + \frac{2}{x} y'_{n+1}(x) - g_y(x, y_n(x)) y_{n+1}(x) = g(x, y_n(x)) - g_y(x, y_n(x)) y_n(x), \tag{4.1}$$

$$y'_{n+1}(0) = 0,\ y_{n+1}(1) = 1, \tag{4.2}$$

where $g(x,y)=\phi^2 y\exp\left(\frac{\gamma\beta(1-y)}{1+\beta(1-y)}\right)$. In the second step, using the CFDM, we present a discrete system for solving linear differential equation (4.1) as follows:

$$\left\{\begin{array}{l} M_1Y'=\frac{1}{h}A_1Y+\frac{1}{h}B_1,\\ M_2Y''=\frac{1}{h^2}A_2Y+\frac{1}{h^2}B_2,\\ Y''+B_3Y'+C_3Y=C,\end{array}\right. \tag{4.3}$$

where $Y=[y_0,y_1,\ldots,y_{N-1}]^T$, B_3 and C_3 are diagonal matrices and M_1 and M_2 are tridiagonal invertible matrices if $\hat{\alpha}$ and $\tilde{\alpha}$ don't equal to 1. Note that A_1 and A_2 are tridiagonal matrices but their first and last rows may be recalculated due to boundary conditions (2.2) to meet the order of the accuracy. In this way, the first and last rows of Matrix A_1 and A_2 have the following relations

$$\left\{\begin{array}{l} y_1'+\hat{\alpha}y_2'=\frac{1}{h}\left(\hat{\alpha}_0y_0+\hat{\alpha}_1y_1+\hat{\alpha}_2y_2+\hat{\alpha}_3y_3+\hat{\alpha}_4y_4\right),\\ \hat{\alpha}y_{N-2}'+y_{N-1}'=\frac{1}{h}\left(\hat{\beta}_0y_{N-4}+\hat{\beta}_1y_{N-3}+\hat{\beta}_2y_{N-2}+\hat{\beta}_3y_{N-1}+\hat{\beta}_4y_N\right),\end{array}\right. \tag{4.4}$$

$$\left\{\begin{array}{l} y_1''+\tilde{\alpha}y_2''=\frac{1}{h^2}\left(\tilde{\alpha}_0y_0+\tilde{\alpha}_1y_1+\tilde{\alpha}_2y_2+\tilde{\alpha}_3y_3+\tilde{\alpha}_4y_4+\tilde{\alpha}_5y_5\right),\\ \tilde{\alpha}y_{N-2}''+y_{N-1}''=\frac{1}{h^2}\left(\tilde{\alpha}_5y_{N-5}+\tilde{\alpha}_4y_{N-4}+\tilde{\alpha}_3y_{N-3}+\tilde{\alpha}_2y_{N-2}+\tilde{\alpha}_1y_{N-1}+\tilde{\alpha}_0y_N\right),\end{array}\right. \tag{4.5}$$

where

$$\left\{\begin{array}{l} \hat{\alpha}_0=\frac{1}{12}\hat{\alpha}-\frac{1}{4},\hat{\alpha}_1=-\frac{2}{3}\hat{\alpha}-\frac{5}{6},\hat{\alpha}_2=\frac{3}{2},\hat{\alpha}_3=\frac{2}{3}\hat{\alpha}-\frac{1}{2},\hat{\alpha}_4=-\frac{1}{12}\hat{\alpha}+\frac{1}{12},\\ \hat{\beta}_0=\frac{1}{12}\hat{\alpha}-\frac{1}{12},\hat{\beta}_1=-\frac{2}{3}\hat{\alpha}+\frac{1}{2},\hat{\beta}_2=-\frac{3}{2},\hat{\beta}_3=\frac{2}{3}\hat{\alpha}+\frac{5}{6},\hat{\beta}_4=-\frac{1}{12}\hat{\alpha}+\frac{1}{4},\\ \tilde{\alpha}_0=\frac{-1}{12}\tilde{\alpha}+\frac{5}{6},\tilde{\alpha}_1=\frac{4}{3}\tilde{\alpha}-\frac{5}{4},\tilde{\alpha}_2=\frac{-5}{2}\tilde{\alpha}-\frac{1}{3},\tilde{\alpha}_3=\frac{4}{3}\tilde{\alpha}+\frac{7}{6},\tilde{\alpha}_4=\frac{-1}{12}\tilde{\alpha}-\frac{1}{12},\tilde{\alpha}_5=\frac{1}{12}.\end{array}\right. \tag{4.6}$$

Other rows of A_1 and A_2 are calculated by relations (3.1) and (3.2), respectively. Hence, the vector Y is obtained by solving the linear system $AY=B$ where $A=\frac{1}{h^2}M_2^{-1}A_2+\frac{1}{h}B_3M_1^{-1}A_1+C_3$ and $B=C-\frac{1}{h^2}M_2^{-1}B_2-\frac{1}{h}B_3M_1^{-1}B_1$.
According to the above discussion, we have

$$\left(\frac{1}{h^2}M_2^{-1}A_2+\frac{1}{h}B_3M_1^{-1}A_1+C_3\right)Y=C-\frac{1}{h^2}M_2^{-1}B_2-\frac{1}{h}B_3M_1^{-1}B_1. \tag{4.7}$$

Multiplying both sides of equation (4.7) by h^2 implies that if $h\to 0$ then the coefficient matrix A tends to the matrix $M_2^{-1}A_2$ and we have $A_2Y=B_2$. Using elementary row operations, A_2 will be converted to a tridiagonal matrix. So, assuming that $\tilde{\alpha}\neq 1,10$, the matrix A_2 is invertible (see Theorem 6.31 in [2]).

5. Numerical results

To show the proposed method efficiency and accuracy, we set $\gamma=1.4$, $\beta=1.2$, $\phi=0.5$, 1, $1.5, 2$ and apply the method with five iterations ($n=5$) and $N=100$ nodes. Here, we consider $\hat{a}=\frac{1}{4}$ and $\tilde{a}=\frac{1}{10}$ to achieve fourth order accuracy. Figure 1 shows the curve of the approximate solution and the numerical solutions obtained by the `bvp4c` command in Matlab. As we see, the solutions obtained by `bvp4c` confirm our computations accuracy. Also, Table 1 shows the absolute error for $\phi=1$.

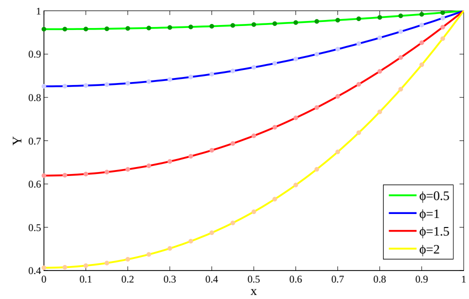

FIGURE 1. The curve of the solutions for $\gamma = 1.4$, $\beta = 1.2$ and $\phi = 0.5$, 1, 1.5 and 2. The approximate solutions computed by the proposed method (line), The numerical solutions computed by bvp4c command (dots).

TABLE 1. The numerical solutions obtained by the proposed method and bvp4c command.

x_i	Proposed mathod	`bvp4c`	Absolute error
0	0.8255878400407376	0.825587521202478	3.19E-7
0.2	0.8326016348176813	0.832601671882272	3.71E-8
0.4	0.8536485109187895	0.853649232046992	7.21E-7
0.6	0.8887110638014090	0.888713706859066	2.64E-6
0.8	0.9376449750278584	0.937654365467653	9.39E-6
1	1	1	0

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INFINITELY MANY SOLUTIONS FOR $(p_1(x), \cdots, p_n(x))$-LAPLACIAN SYSTEMS

S. HEIDARI

Abstract. The paper deals with the study of the existence of infinitely many weak solutions for $(p_1(x), \cdots, p_n(x))$-Laplacian -like system, originated from a capillary phenomenon. Our technical approach is based on variational methods and Ricceri's critical points principle modified by Bonanno.

1. Introduction

The purpose of this article is to establish the existence of infinitely many weak solutions for $(p_1(x), \cdots, p_n(x))$-Laplacian-like system originated from capillary phenomenon of the following form:

(1.1)
$$\begin{cases} -div\big((1+\dfrac{|\nabla u_i|^{p_i(x)}}{\sqrt{1+|\nabla u_i|^{2p_i(x)}}})|\nabla u_i|^{p_i(x)-2}\nabla u_i\big) = \lambda F_{u_i}(x, u_1, u_2, \cdots, u_n) \quad \text{in} \quad \Omega, \\ u_i = 0 \qquad \text{on} \quad \partial\Omega, \end{cases}$$

for $1 \leq i \leq n$ where Ω is an open bounded domain in $\mathbb{R}^N (N \geq 2)$, with smooth boundary, λ is positive parameter and $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a function such that the mapping $(t_1, t_2, \cdots, t_n) \to F(x, t_1, t_2, \cdots, t_n)$ is in C^1, in $\mathbb{R}^n$ for all $x \in \Omega$, F_{t_i} denotes the partial derivative of F, with respect to t_i and F_{t_i} is continuous in $\Omega \times \mathbb{R}^n$, for

Keywords: :$(p_1(x), \cdots, p_n(x))$-Laplacian systems, variational methods, critical points.
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$i = 1, 2, \cdots, n$. $p_i(x) \in C(\bar{\Omega})$ $(i = 1, 2, \cdots, n)$ with

$$N < p_i^- := \inf_{x \in \bar{\Omega}} p_i(x) \leq p_i^+ := \sup_{x \in \bar{\Omega}} p_i(x) < +\infty.$$

The study of capillary phenomena has gained some attention recently. This increasing interest is motivated not only by fascination in naturally-occurring phenomena such as motion of drops, bubbles, and waves but also its importance in applied fields ranging from industrial and biomedical and pharmaceutical to microfluidic systems. In [3] the authors investigate existence of infinitely many solutions to the fourth-order boundary-value problem by means of a variational theorems of Ricceri and Bonanno. We recall some background facts concerning the variable exponent Lebesgue and Sobolev spaces (see in [1, 2]) and introduce some notation. Set $C_+(\Omega) := \{h \in C(\bar{\Omega}) : h(x) > 1, \forall x \in \bar{\Omega}\}$. For every $p(\cdot) \in C_+(\Omega)$, we define the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty\}.$$

which is a Banach space under the following Luxemburg norm,

$$|u|_{p(\cdot)} = \inf\{\mu > 0 : \int_\Omega |\frac{u(x)}{\mu}|^{p(x)} dx \leq 1\}.$$

We define the variable exponent Sobolev space by

$$W^{1,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

equipped with the norm: $\|u\|_{1,p(\cdot)} := |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}$, becomes a separable, reflexive uniformly convex Banach space. Now, we introduce $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ which can be renormed by the equivalent norm: $\|u\|_{p(\cdot)} := |\nabla u|_{p(\cdot)}$

This space is separable and reflexive Banach space.

Proposition 1.1. *For $u \in W_0^{1,p(\cdot)}(\Omega)$, the following inequalities hold:*

$$\|u\|_{p(\cdot)}^{p^-} \leq \int_\Omega |\nabla u|^{p(x)} dx \leq \|u\|_{p(\cdot)}^{p^+} \qquad \textit{if} \quad \|u\|_{p(\cdot)} \geq 1, \tag{1.2}$$

$$\|u\|_{p(\cdot)}^{p^+} \leq \int_\Omega |\nabla u|^{p(x)} dx \leq \|u\|_{p(\cdot)}^{p^-} \qquad \textit{if} \quad \|u\|_{p(\cdot)} \leq 1, \tag{1.3}$$

Proposition 1.2. *(see [4]) If $\Omega \subset \mathbb{R}^N$ is a bounded domain, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ is compact whenever $N < p^-$.*

Here and in the sequel, we let X be the Cartesian product of n Sobolev spaces $W_0^{1,p_i(\cdot)}(\Omega)$ for $i = 1, 2, \cdots, n$. i.e., $X = \prod_{i=1}^n W_0^{1,p_i(\cdot)}(\Omega)$, endowed with the norm

$$\|(u_1, u_2, \cdots, u_n)\| = \sum_{i=1}^n \|u\|_{p_i(\cdot)},$$

where $\|u_i\|_{p_i(\cdot)} = \inf\{\mu > 0 : \int_\Omega |\frac{\nabla u_i}{\mu}|^{p_i(x)} dx \leq 1\}$.

On the space $C^0(\bar{\Omega})$ we consider the norm $\|u\|_\infty := \sup_{x\in\bar{\Omega}} |u(x)|$. Put

$$C := \max\{ \sup_{u_i \in W_0^{1,p_i(\cdot)} \setminus \{0\}} \frac{\max_{x\in\bar{\Omega}} |u_i(x)|_\infty}{\|u_i\|_{p_i(\cdot)}} : \text{for } 1 \leq i \leq n\}.$$

Since $N < p_i^-$ for $1 \leq i \leq n$, from Proposition 1.2, the embedding $X \hookrightarrow C^0(\bar{\Omega}) \times \cdots \times C^0(\bar{\Omega})$ is compact, so that $C < +\infty$. Define the functionals $\Phi, \Psi : X \to \mathbb{R}$, by

$$\Phi(u) = \sum_{i=1}^{n} \int_\Omega \Big(\frac{1}{p_i(x)} |\nabla u_i|^{p_i(x)} + \frac{\sqrt{1 + |\nabla u_i|^{2p_i(x)}}}{p_i(x)} \Big) dx,$$

and $\Psi(u) = \int_\Omega F(x, u_1, \cdots, u_n) dx$, and set $I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$, for all $u = (u_1, \cdots, u_n) \in X$. For all $\sigma > 0$, we define $Q(\sigma) := \{(t_1, t_2, \cdots, t_n) \in \mathbb{R}^n, \sum_{i=1}^n |t_i| \leq \sigma\}$ and $p_* = \min\{p_i^-; i = 1, 2, \cdots, n\}$.

2. Main results

In this section, first we recall multiple critical points theorem of Bonanno, which is our main tool.

Theorem 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strong continuous, sequentially weakly lower semi-continuous, and coercive, and Ψ is sequentially weakly upper-semi-continuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)) - \Psi(u)}{r - \Phi(u)}$$

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then:

(a) *If $\gamma < +\infty$ then, for each $\lambda \in (0, \frac{1}{\gamma})$, the following alternative holds: either*

(a1) *$I_\lambda := \Phi - \lambda\Psi$ possesses a global minimum, or*

(a2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to\infty} \Phi(u_n) = +\infty$.*

(b) *If $\delta < +\infty$ then, for each $\lambda \in (0, \frac{1}{\delta})$, the following alternative holds: either*

(b1) *there is a global minimum of Φ that is a local minimum of I_λ, or*

(b2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ that weakly converges to a global minimum of Φ.*

The following result is obtained by applying Theorem 2.1, we introduce the suitable hypothesis and establish an open interval of positive parameters such that problem (1.1) admits infinitely many weak solutions.

Theorem 2.2. *Assume that*

(I) $F(x, t_1, \cdots, t_n) \geq 0$ *for every* $(x, t_1, \cdots, t_n) \in \Omega \times \mathbb{R}_+^n$

(II) *there exist a point* $x_0 \in \Omega$ *and* $R_2 > R_1 > 0$, *and*

$$A < LB,$$

where $L := \min\{L_{p_i^+}, i = 1, 2, \cdots, n\}$, *and*

$$L_{p_i^+} := \frac{2\Gamma(1+\frac{N}{2})}{\left(C\sum_{i=1}^{n}(p_i^+)^{\frac{1}{p_i^-}}\right)^{p_*}\pi^{\frac{N}{2}}(R_2^N - R_1^N)}\left(1 - \frac{2}{(R_2-R_1)^{p_i^+}+2}\right)$$

$$A := \liminf_{\sigma\to+\infty} \frac{\int_\Omega \sup_{(t_1,\cdots,t_n)\in Q(\sigma)} F(x, t_1, \cdots, t_n)dx}{\sigma^{p_*}}$$

$$B := \limsup_{(t_1,\cdots,t_n)\to(+\infty,\cdots,+\infty)} \frac{\int_{B(x^0,R_1)} F(x, t_1, \cdots, t_n)dx}{\sum_{i=1}^{n}\frac{t_i^{p_i^+}}{p_i^-}}.$$

Then for each

$$\lambda \in \Lambda := \frac{2}{\left(C\sum_{i=1}^{n}(p_i^+)^{\frac{1}{p_i^-}}\right)^{p_*}}\left]\frac{1}{LB}, \frac{1}{A}\right[$$

system (1.1) *has an unbounded sequence of weak solutions in* X.

Acknowledgement

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REDUCED FORMS AND GROUP-INVARIANT SOLUTIONS OF CLARKE'S PDE

S. REZA HEJAZI

Abstract. Symmetry operators of Clarke equation are computed by using the Lie algorithm method. The characteristic equations of symmetries give the invariant transformations in order to find the reductions. This is based on inserting the obtained transformations to Clarke equation for finding the reduced equations.

1. Introduction

The method of point transformations is a powerful tool in order to find exact solutions for nonlinear partial differential equations. It happens that many PDEs of physical importance are nonlinear and Lie classical symmetries admitted by nonlinear PDEs are useful for finding invariant solutions. One of the most important application of symmetry's method is the reducing systems of differential equations, i.e., finding equivalent systems of differential equations of simpler form, that is called reduction. This method provides a systematic computational algorithm for determining a large classes of special solutions. The solutions of the obtained equivalent system will correspond to solutions of the original system. There is a lot of papers in the literature for this process and one can find the classical reduction method in [3, 4].

In combustion, Clarke's equation is a third-order nonlinear partial differential equation, first derived by John Frederick Clarke in 1978. The equation describes the thermal explosion process, including both effects of constant-volume and constant-pressure processes, as well as the effects of adiabatic and isothermal sound speeds.

Keywords: symmetry group, invariant transformation, reduction.

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The equation reads as

$$D_t^2(D_t\theta - \gamma e^\theta) = D_x^2(D_t\theta - \gamma e^\theta), \tag{1.1}$$

where θ is the non-dimensional temperature perturbation and γ the specific heat ratio. The term $\theta_t - e^\theta$ describes the explosion at constant pressure and the term $\theta_t - \gamma e^\theta$ describes the explosion at constant volume. Similarly, the term $D_t^2 - D_x^2$ describes the wave propagation at adiabatic sound speed and the term $\gamma D_t^2 - D_x^2$ describes the wave propagation at isothermal sound speed. Molecular transports are neglected in the derivation. The organization of the present paper is outlined as follows: the symmetry operators of the Eq. (1.1) are computed in the second chapter. The method of reduction process is coming in chapter 3. Also this is stated for the Clarke's equation to construct the reduced forms of the equation.

2. Symmetry operators of the Eq. (1.1)

Symmetry plays a very important role in various fields of nature. As is known to all, Lie method is an effective method and a large number of equations are solved with the aid of this method. There are still many authors who use this method to find the exact solutions of nonlinear differential equations. Since this method has powerful tools to find exact solutions of nonlinear problems [1, 2]

The Lie algebra of infinitesimal symmetries is the set of vector fields in the form of

$$X = \xi(x,t,\theta)\frac{\partial}{\partial x} + \tau(x,t,\theta)\frac{\partial}{\partial t} + \phi(x,t,\theta)\frac{\partial}{\partial \theta}, \tag{2.1}$$

with the second prolongation

$$\begin{aligned} X^{(3)} = X &+ \phi^x\frac{\partial}{\partial\theta_x} + \phi^t\frac{\partial}{\partial\theta_t} + \phi^{xx}\frac{\partial}{\partial\theta_{xx}} + \phi^{xt}\frac{\partial}{\partial\theta_{xt}} + \phi^{tt}\frac{\partial}{\partial\theta_{tt}} \\ &+\phi^{xxx}\frac{\partial}{\partial\theta_{xxx}} + \phi^{xxt}\frac{\partial}{\partial\theta_{xxt}} + \phi^{xtt}\frac{\partial}{\partial\theta_{xtt}} + \phi^{ttt}\frac{\partial}{\partial\theta_{ttt}}, \end{aligned} \tag{2.2}$$

where $\phi^x, \phi^t, \phi^{xx}, \phi^{xt}, \phi^{tt}, \phi^{xxx}, \phi^{xxt}, \phi^{xtt}, \phi^{ttt}$ are prolongation coefficients computed by $\theta^J = D_J(\theta - \xi\theta_x - \tau\theta_t) + \xi\theta_{xJ} + \tau\theta_{tJ}$. Using the invariance condition, i.e. applying the second prolongation (2.2) on Eq. (1.1)

$$X^{(3)}(D_t^2(\theta_t - \gamma e^\theta) - D_x^2(\theta_t - \gamma e^\theta)) \equiv 0, \qquad \text{mod } (1.1),$$

gives the following three symmetry operators for the Clarke's equation:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta}.$$

3. Reduced forms of the Clarke's equation

The first advantage of symmetry operator method is to construct new solutions from known solutions. To do this, the operators are considered and their corresponding invariants are determined. The Clarke's equation expressed in the coordinates (x, t, θ) so to reduce this equation is to search for its form in specific coordinates. Those coordinates will be constructed by searching for independent invariants (y, u) corresponding to any operators. So using the chain rule, the expression of the equation in the new coordinate allows us to the reduced equation. Here we will obtain some invariant solutions with respect to symmetries. First we obtain the similarity variables for each term of symmetries, then we use this method to reduced the equation.

3.1. **Space translation invariance (X_1).** Integrating the characteristic system $\frac{dx}{ds} = 1$, for the parameter s gives the invariant transformation

$$y = t, \qquad \theta = u(t).$$

Inserting these new variables to the Eq. (1.1) reduces the equation to the ODE

$$u_{yyy} - \gamma u_{yy} e^u - \gamma u_y^2 e^u = 0.$$

This ODE has not any implicit solution. But, we can find a series for the solution around zero. Thus, the following group-invariant solution would be written for Eq. (1.1):

$$\begin{aligned}
\theta(t) &= \theta(0) + \mathrm{D}(\theta)(0)\,t + \frac{\left(D^{(2)}\right)(\theta)(0)}{2} t^2 + \left(\frac{\gamma\,(\mathrm{D}(\theta)(0))^2 \mathrm{e}^{\theta(0)}}{6} \right.\\
&\left. + \frac{\gamma\,\left(D^{(2)}\right)(\theta)(0)\,\mathrm{e}^{\theta(0)}}{6} \right) t^3 + \left(\frac{\gamma^2\,(\mathrm{D}(\theta)(0))^2 \left(\mathrm{e}^{\theta(0)}\right)^2}{24} + \frac{\gamma^2 \left(\mathrm{e}^{\theta(0)}\right)^2 \left(D^{(2)}\right)(\theta)(0)}{24} \right.\\
&\left. + \frac{\gamma\,(\mathrm{D}(\theta)(0))^3 \mathrm{e}^{\theta(0)}}{24} + \frac{\gamma\,\mathrm{D}(\theta)(0)\left(D^{(2)}\right)(\theta)(0)\,\mathrm{e}^{\theta(0)}}{8} \right) t^4 \\
&+ \left(\frac{\gamma\,(\mathrm{D}(\theta)(0))^2 \left(D^{(2)}\right)(\theta)(0)\,\mathrm{e}^{\theta(0)}}{20} + \frac{\gamma\,\left(\left(D^{(2)}\right)(\theta)(0)\right)^2 \mathrm{e}^{\theta(0)}}{40} \right.\\
&+ \frac{\gamma^2\,(\mathrm{D}(\theta)(0))^3 \left(\mathrm{e}^{\theta(0)}\right)^2}{24} + \frac{7\,\gamma^2 \mathrm{D}(\theta)(0)\left(\mathrm{e}^{\theta(0)}\right)^2 \left(D^{(2)}\right)(\theta)(0)}{120} \\
&\left. + \frac{\gamma^3 \left(\mathrm{e}^{\theta(0)}\right)^3 (\mathrm{D}(\theta)(0))^2}{120} + \frac{\gamma^3 \left(\mathrm{e}^{\theta(0)}\right)^3 \left(D^{(2)}\right)(\theta)(0)}{120} + \frac{\gamma\,(\mathrm{D}(\theta)(0))^4 \mathrm{e}^{\theta(0)}}{120} \right) t^5 + O\left(t^6\right)).
\end{aligned}$$

3.2. **Time translation invariance (X_2).** The solution of the characteristic system $\frac{dt}{ds} = 1$, gives the new invariants

$$y = x, \qquad \theta = u(x).$$

Replacing these inavriants by the old variables in Eq. (1.1) yields the equation:

$$u_{yy} + u_y^2 = 0.$$

Consequently, the following group-invariant solution is concluded:

$$\theta = \ln(\alpha + \beta x).$$

3.3. **Scaling invariance on space and time (X_3).** The corresponding invariants for this operator is computed by the integrating of the system

$$\frac{dx}{ds} = x, \qquad \frac{dt}{ds} = t, \qquad \frac{d\theta}{ds} = -1.$$

The solution of the system gives the following invariants:

$$y = \frac{t}{x}, \qquad u(y) = \ln x + \theta.$$

By inserting these new variables to the Eq. (1.1), the following ODe is constructed:

$$(y^2 - 1)u_{yyy} + (-y^2 + 4y + \gamma)u_{yy} + (-y^2 + \gamma)u_y^2 + 2(1 - 2y)u_y = 2.$$

Thus, the group-invariant solution in this case satisfies in the following integral equation:

$$xe^{\theta}\left(\frac{t}{x} + 1\right)^{\alpha}\left(\frac{t}{x} - 1\right)^{\beta} - \int_0^{\frac{t}{x}} (\eta + 1)^{\alpha - 1}(\eta - 1)^{\beta - 1}(\eta^2 + \gamma)d\eta = \delta.$$

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EXACT SOLUTIONS OF BOUSSINESQ TYPE EQUATION VIA INVARIANCE PROPERTIES

S. REZA HEJAZI AND HAMID ERFANIAN ORAEI DEHROKHI

Abstract. Lie symmetry analysis is applied in order to find exact solutions of Boussinesq type equation. For this aim we use the infinitesimal invariance condition for constructing the symmetry operators for the considered PDE. Then, some exact solutions are computed by use of the group-inavriant transformations of the symmetries.

1. Introduction

Lie group theory plays a very important role in geometric analysis of differential equations and there are lots of papers and books have been presented about this subject, [1, 4, 5, 6]. Also Lie symmetries method have many efficient applications in physics and mathematics. As an important application of symmetry operators is the reduction procedure. This is possible from a similarity variable obtaining from the symmetries. In fluid dynamics, the Boussinesq approximation for water waves is an approximation valid for weakly non-linear and fairly long waves. The approximation is named after Joseph Boussinesq, who first derived them in response to the observation by John Scott Russell of the wave of translation (also known as solitary wave or soliton). The 1872 paper of Boussinesq introduces the equations now known as the Boussinesq equations. This is a fourth order non-linear PDE written

Keywords: fluid mechanics, symmetry group, invariant transformation, reduction.
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by

$$u_{tt} - u_{xx} - 2\alpha(uu_x)_x - \beta u_{xxtt} = 0. \tag{1.1}$$

In this paper the Eq. (1.1) is cosidered for $\alpha = \beta = 1$. In the second chapter the symmetry operators of the equation is derived by using the invariance condition in the group method. The third chapter is devoted for construction the group-invariant transformations in order to reduce the Eq. (1.1). Finally, some exact solutions called group-invariant solutions are extracted from the reduced forms.

2. Lie Symmetries of the Boussinesq type equation

In this article we focus on the Eq. (1.1) for finding Lie symmetries of the equation. Let us consider a one-parameter Lie group of infnitesimal transformations in (x, t, u) given by

$$\begin{aligned} x^* &= x + s\xi(x,t,u) + \mathcal{O}(s^2), \\ t^* &= t + s\tau(x,t,u) + \mathcal{O}(s^2), \\ u^* &= u + s\phi(x,t,u) + \mathcal{O}(s^2), \end{aligned} \tag{2.1}$$

where s is the group parameter. The transformations (2.1) leave invariant the set of solutions of Eq. (1.1). This yields to the overdetermined linear system of equations for the coefficients of infinitesimals. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u},$$

with the fourth prolongation

$$X^{(4)} = X + \phi^x\frac{\partial}{\partial u_x} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{tt}\frac{\partial}{\partial u_{tt}} + \phi^{xxtt}\frac{\partial}{\partial u_{xxtt}}, \tag{2.3}$$

the prolongation formula are illustrated in [2, 6] exactly. Having determined the in

nitesimals, the symmetry coefficients are found by solving the invariant surface condition

$$X^{(4)}(u_{tt} - u_{xx} - 2\alpha(uu_x)_x - \beta u_{xxtt}) \equiv 0. \tag{2.4}$$

The solutions of the system (2.4) give coeffcients function for infinitesimal generators of the one-parameter Lie group of the point symmetries for the Eq. (1.1) as follows:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = t\frac{\partial}{\partial t} - (2u+1)\frac{\partial}{\partial u}.$$

3. Group-invariant solutions of the Boussinesq type equation

When we confronted with a complicated system of partial differential equations in some physically important problem, the discovery of any explicit solutions whatsoever is of great interest. Explicit solutiona can be used as models for physical experiments, as benchmarks for testing numerical methods, etc., and often reflect the asymptotic or dominant bahaviour of more general types of solutions. The method used to find group-invariant solutions, generalizing the well-known techniques for finding similarity solutions, provide a systematic computational method for determining large classes of special solutions. These group-invariant solutions are characterized by their invariance under some symmetry group of the system of partial differential equations; the more symmetrical the solution, the easier it is to construct, see [3, 6] for more detsails.

3.1. **Group-invariant solution through X_1.** The solution of the characteristic system $\frac{dx}{ds} = 1$ gives the invariant transformations

$$z = t, \qquad f(z) = u. \tag{3.1}$$

Inserting the invariants (3.1) to Eq. (1.1), the equation

$$f'' = 0.$$

Thus, the group-invariant soltion

$$u = at + b,$$

is concluded.

3.2. **Group-invariant solution through X_2.** The symmetry X_2 has the characteristic system $\frac{dt}{ds} = 1$. This system has the new invariants

$$z = x, \qquad f(z) = u. \tag{3.2}$$

So, the invariants (3.2) reduces the Eq. (1.1) to

$$(2f + 1)f'' + 2f'^2 = 0.$$

This equation gives the following group-invariant solution:

$$u = -\frac{1}{2}\left(1 \pm \sqrt{4ax + 4b + 1}\right).$$

3.3. **Group-invariant solution through X_3.** The invariants of the thirs symmetry is derived by solving the characterstic system

$$\frac{dt}{ds} = t, \qquad \frac{du}{ds} = 2u + 1.$$

Consequently, the new invariants are:

$$z = x, \qquad f(z) = \frac{t^2}{2}(2u + 1). \tag{3.3}$$

Inserting the invariants (3.3) to Eq. (1.1) yields the following reduced equation:

$$(f+3)f'' + f'^2 - 3f = 0.$$

This equation gives the following group-invariant solution:

$$\pm \int^{u(x)} \frac{\eta + 3}{\sqrt{2\eta^3 + 9\eta^2 + a}} d\eta - x - b = 0.$$

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EFFICIENT EXPLICIT METHODS FOR NONSTIFF ORDINARY DIFFERENTIAL EQUATIONS

ARASH JALILIAN AND GHOLAMREZA HOJJATI

Abstract. In this paper, we use Albrecht technique to construct a class of variable stepsize general linear methods which have large region of absolute stability. Such methods are considered as an alternative to the Nordsieck technique.

1. Introduction

Consider the initial-value problem (IVP) for a system of autonomous ordinary differential equations (ODEs)

$$\begin{cases} y'(x) = & f(y(x)), \quad x \in [x_0, \overline{x}], \\ y(x_0) = & y_0, \end{cases} \tag{1.1}$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$, $y : \mathbb{R} \to \mathbb{R}^m$, and m is the dimensionality of the system and f is sufficiently smooth function. In 1966, Butcher [1] provided general linear methods (GLMs) as a middle ground between two traditional methods; Runge–Kutta (RK) methods and linear multistep methods (LMMs). To achieve the optimized calculations and effective implementation, using the variable stepsize technique is necessary. In this direction, Jackiewicz [2] studied a class of variable stepsize diagonally implicit multistage integration methods with Runge–Kutta stability (RKS) property for the numerical solution of ODEs by using Albrecht's technique [3]. The goal of this paper

Keywords: Initial value problem, General linear methods, Zero-stability, Variable stepsize.

AMS Mathematical Subject Classification [2010]: 65L05.

is to construct and implement a class of variable stepsize GLMs (VS-GLMs) without RKS property which have large region of stability. Consider a nonuniform mesh

$$x_{-\rho} < \cdots < x_{-1} < x_0 < x_1 < \cdots < x_N, \quad x_N > \bar{x}, \tag{1.2}$$

and assume $h_n = x_{n+1} - x_n$, $n = -\rho, \ldots, 0, \ldots, N+1$, $\sigma_{n,i} = \frac{h_{n-i}}{h_n}$, $i = 1, 2, \ldots, \rho$. Here, the points $x_{-\rho}, \ldots, x_{-1}$ are introduced in order to simplify the formulas of order conditions. It should be noted that we only consider the grids $x_0, x_1, \ldots, x_N$ and start the integration process at x_ρ for some integer ρ. In this paper, we are going to find methods of the form

$$\begin{cases} Y^{[n+1]} = h_n(A(\sigma_n) \otimes I_m) f(Y^{[n+1]}) + (U(\sigma_n) \otimes I_m) y^{[n]}, \\ y^{[n+1]} = h_n(B(\sigma_n) \otimes I_m) f(Y^{[n+1]}) + (V(\sigma_n) \otimes I_m) y^{[n]}, \end{cases} \tag{1.3}$$

$n = 0, 1, \ldots, N-1$ where $Y_i^{[n+1]} \simeq y(x_n + c_i(\sigma_n)h_n)$, $i = 1, 2, \ldots, s$, $c(\sigma_n) = [c_1(\sigma_n), \ldots, c_s(\sigma_n)]^T$ and the starting values $y_i^{[0]}$, $i = 1, 2, \ldots, r$ are approximations to linear combinations of $y(x_{-\rho})$, $y(x_{-\rho+1})$, $\ldots$, $y(x_0)$. The coefficients matrices $A(\sigma_n) \in \mathbb{R}^{s\times s}$, $U(\sigma_n) \in \mathbb{R}^{s\times r}$, $B(\sigma_n) \in \mathbb{R}^{r\times s}$, $V(\sigma_n) \in \mathbb{R}^{r\times r}$ and the vector $c(\sigma_n) \in \mathbb{R}^s$ depend on the ratios of the current stepsize and the past stepsizes.

Definition 1.1. The method (1.3) is zero-stable if the product $\prod_{j=0}^{n} V(\sigma_j)$ is bounded uniformly with respect to n, i.e. $\| \prod_{j=0}^{n} V(\sigma_j)\| \le L$, where L is a scaler.

2. Local discretization errores Of VS-GLMs

In this section, we assume that the stage vector $Y^{[n]}$ is an approximation of at least one to the vector $z_1(x_n) := y(x_n + bh_{n-1})$ where y is the solution to system equation (1.1) and $b = c(\sigma_{n-1}) - e$, with $c(\sigma_{n-1}) = [c_1(\sigma_{n-1}), \ldots, c_s(\sigma_{n-1})]^T$, and $e = [1, \ldots, 1]^T$. To obtain the order conditions for VS-GLMs (1.3), we assume that $y^{[n]} = \sum_{l=0}^{\rho} \beta_l y(x_{n-l}) + \mathcal{O}(h^{p+1})$ where

$$h = \max_{0\le n\le N-1} \quad |h_n|$$

and for some vectors $\beta_l = [\beta_{i,l}]_{i=1}^{r}$, and require that $y^{[n+1]} = \sum_{l=0}^{\rho} \beta_l y(x_{n-l+1}) + \mathcal{O}(h^{p+1})$ for the same vectors β_l. It means that the correct function is defined by $z_2(x_n) := \sum_{l=0}^{\rho} \beta_l y(x_{n-l})$. This leads to a method of order p and provide a starting procedure to compute the initial vector $y^{[0]}$ such that

$$y^{[0]} = \sum_{l=0}^{\rho} \beta_l y(x_{-l}) + \mathcal{O}(h^{p+1}).$$

Define $h_n d^{[n+1]}$ and $h_n \widehat{d}^{[n+1]}$ as the local discretization errors by

$$\begin{cases} z_1(x_{n+1}) = h_n A(\sigma_n) f(z_1(x_{n+1})) + U(\sigma_n) z_2(x_n) + h_n d^{[n+1]}, \\ z_2(x_{n+1}) = h_n B(\sigma_n) f(z_1(x_{n+1})) + V(\sigma_n) z_2(x_n) + h_n \widehat{d}^{[n+1]}. \end{cases} \tag{2.1}$$

Substituting the equivalent values $z_1(x_{n+1})$, $z_2(x_n)$ and $z_2(x_{n+1})$ into (2.1) and then expanding both sides of equations (2.1) about the point x_n, we obtain

$$\begin{cases} h_n d^{[n+1]} = C_0(\sigma_n)y(x_n) + \sum_{\mu=1}^{p} C_\mu(\sigma_n)h_n^\mu y^{(\mu)}(x_n) + \mathcal{O}(h^{p+1}), \\ h_n \widehat{d}^{[n+1]} = \widehat{C}_0(\sigma_n)y(x_n) + \sum_{\mu=1}^{p} \widehat{C}_\mu(\sigma_n)h_n^\mu y^{(\mu)}(x_n) + \mathcal{O}(h^{p+1}), \end{cases}$$

where coefficients $C_\mu(\sigma_n)$ and $\widehat{C}_\mu(\sigma_n)$, $\mu = 0, 1, \ldots, p$ are given by

$$\begin{cases} C_0(\sigma_n) = \displaystyle\sum_{l=0}^{\rho} \beta_l - U(\sigma_n) \sum_{l=0}^{\rho} \beta_l, \\ C_\mu(\sigma_n) = \dfrac{C(\sigma_n)}{\mu!} - A(\sigma_n)\dfrac{C(\sigma_n)^{\mu-1}}{(\mu-1)!} - \dfrac{(-1)^\mu}{\mu!}U(\sigma_n)\displaystyle\sum_{l=0}^{\rho}(\sum_{\nu=1}^{l}\sigma_{n,\nu})^\mu \beta_l, \end{cases}$$

and

$$\begin{cases} \widehat{C}_0(\sigma_n) = (I_{r\times r} - V(\sigma_n))\displaystyle\sum_{l=0}^{\rho} \beta_l, \\ \widehat{C}_\mu(\sigma_n) = \dfrac{\beta_0}{\mu!} - \dfrac{(-1)^\mu}{\mu!}\displaystyle\sum_{l=2}^{\rho}\beta_l(\sum_{\nu=1}^{l-1}\sigma_{n,\nu})^\mu - B(\sigma_n)\dfrac{C(\sigma_n)^{\mu-1}}{(\mu-1)!} \\ \qquad\qquad - \dfrac{(-1)^\mu}{\mu!}V(\sigma_n)\displaystyle\sum_{l=1}^{\rho}\beta_l(\sum_{\nu=1}^{l}\sigma_{n,\nu})^\mu, \end{cases}$$

where $\mu = 1, 2, \ldots, p$.

3. Construction of VS-GLM with $p = q = s = r = 2$

In this section, we are going to construct an explicit variable stepsize method with $p = q = s = r = 2$ which their coefficients matrices take the form

$$\left[\begin{array}{c|c} A(\sigma_n) & U(\sigma_n) \\ \hline B(\sigma_n) & V(\sigma_n) \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 0 & u_{11}(\sigma_n) & 1-u_{11}(\sigma_n) \\ a_{21}(\sigma_n) & 0 & u_{21}(\sigma_n) & 1-u_{21}(\sigma_n) \\ \hline b_{11}(\sigma_n) & b_{12}(\sigma_n) & v_1(\sigma_n) & 1-v_1(\sigma_n) \\ b_{21}(\sigma_n) & b_{22}(\sigma_n) & v_1(\sigma_n) & 1-v_1(\sigma_n) \end{array}\right],$$

where

$$\begin{cases} a_{21}(\sigma_n) = \frac{(\sigma_{n,1}+1)}{\sigma_{n,1}}, \quad u_{11}(\sigma_n) = 1, \quad u_{21}(\sigma_n) = \frac{(\sigma_{n,1}^2-1)}{\sigma_{n,1}^2}, \\ b_{11}(\sigma_n) = \frac{1}{2}\sigma_{n,1}^2(1-v_1(\sigma_n)) - \sigma_{n,1}(v_1(\sigma_n)-1) + \frac{1}{2}, \\ b_{12}(\sigma_n) = \frac{1}{2}\sigma_{n,1}^2(v_1(\sigma_n)-1) + \frac{1}{2}, \\ b_{21}(\sigma_n) = (\frac{1}{2}\sigma_{n,1}^2 + \sigma_{n,1})(1-v_1(\sigma_n)), \\ b_{22}(\sigma_n) = \frac{1}{2}\sigma_{n,1}^2(v_1(\sigma_n)-1). \end{cases} \tag{3.1}$$

Using MATLAB's subroutine `fminsearch`, we obtain methods with a large region of stability by choosing an appropriate value $v_1(\sigma_n) = \frac{1043}{3221}$.

4. Numerical experiment

In order to show the efficiency of constructed method, we solve the non-stiff problem

$$\begin{cases} y'(x) = -y(x) - 5\sin(5x)\exp(-x), & x \in [0,1], \\ y(0) = 1, \end{cases} \tag{4.1}$$

with the exact solution is $y(x) = \exp(-x)\cos(5x)$, on the generated meshes according to the pattern $h_{n+1} = R^{q_n} h_n$, $n = 0, 1, \ldots$, where $q_n = 1$, if $n = 0, 1, 4, 5, 8, 9, \ldots$ and otherwise $q_n = -1$. Numerical results for explicit VS-GLM without RKS property $(Error_1, N_1)$ and with RKS property $(Error_2, N_2)$ are listed in the Table 1 where $Error_1$ and $Error_2$ are global errors at the end point, and N_1 and N_2 are the number of stepsizes for both methods.

Table 1. Numerical results of explicit VS-GLMs of order 2 with and without RKS property for $R = 1.5$

h_0	$Error_1$	N_1	$Error_2$	N_2
$\frac{1}{20}$	3.38E-3	13	3.06E-2	13
$\frac{1}{40}$	3.67E-4	26	5.27E-4	26
$\frac{1}{80}$	1.05E-4	51	1.26E-4	51

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EFFICIENT METHODS FOR SOLVING SECOND KIND VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we introduce a new approach for the numerical solution of Volterra integro-differential equations. This approach contains a combination of general linear methods and Runge–Kutta quadrature rule. Some numerical experiments are provided to show the robustness of the proposed methods.

1. Introduction

The purpose of this work is to design a new approach for the numerical solution of second-kind system of Volterra integro-differential equations (VIDEs)

$$\begin{cases} y'(t) = f(t, y(t), z(t)), \quad t \in [t_0, T], \\ y(t_0) = y_0, \end{cases} \tag{1.1}$$

with $z(t) := \int_{t_0}^{t} K(t, s, y(s))\mathrm{d}s$, in which $y : \mathbb{R} \to \mathbb{R}^m$ is the unknown function, $K : S \times \mathbb{R}^m \to \mathbb{R}^{\overline{m}}$ with $S = \{(t, s) : t_0 \le s \le t \le T\}, m, \overline{m} \in \mathbb{N}$, and $f : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{\overline{m}} \to \mathbb{R}^m$ are the given functions. To guarantee the existence of a unique solution to these equations, it is assumed that f satisfies the Lipschitz conditions with respect to y and z whereas K satisfies the Lipschitz condition with respect to y [4].

Keywords: Ordinary differential equations, Volterra integro-differential equations, General linear methods, Natural Runge–Kutta methods, Linear stability.

AMS Mathematical Subject Classification [2010]: 65L05, 65L20.

The proposed methods for (1.1) take the following form

$$\left\{\begin{array}{l} Y^{[n]} = h(A\otimes I)f(t_{n-1}+ch, Y^{[n]}, Z^{[n]}) + (U\otimes I)y^{[n-1]}, \\ y^{[n]} = h(B\otimes I)f(t_{n-1}+ch, Y^{[n]}, Z^{[n]}) + (V\otimes I)y^{[n-1]}, \end{array}\right. \tag{1.2}$$

in which $Z^{[n]} = [Z_i^{[n]}]_{i=1}^s \in \mathbb{R}^{ms}$ is an approximation to $z(t_{n-1}+ch) = [z(t_{n-1}+c_ih)]_{i=1}^s \in \mathbb{R}^{ms}$, as

$$Z_i^{[n]} = \widetilde{F}_n(t_{n-1}+c_ih) + h\sum_{j=1}^{s}\alpha_{ij}K(t_{n-1}+d_{ij}h, t_{n-1}+e_{ij}h, \sum_{l=1}^{s}\beta_{ijl}Y_l^{[n]}), \tag{1.3}$$

for $i = 1,2,\ldots,s$, with

$$\widetilde{F}_n(t_{n-1}+c_ih) = h\sum_{\kappa=1}^{n-1}\sum_{j=1}^{s} v_jK(t_{n-1}, t_{\kappa-1}+\xi_jh, u(t_{\kappa-1}+\xi_jh)),$$

as an approximation to the tail

$$\int_{t_0}^{t_{n-1}} K(t_{n-1}+c_ih, s, y(s))\mathrm{d}s,$$

and u is a natural continuous extension of the numerical solution by the interpolation formula of degree $d \le p$

$$u(t_{n-1}+\theta h) = \sum_{j=1}^{s} w_j(\theta)Y_j^{[n]},\ n = 1,\ \ldots, N,\ \theta \in [0,1],$$

where $w_j(\theta)$ are polynomials of degree d with $[p/2] \le d \le \min\{s-1, p\}$. Here, the values of $\xi_j, v_j, \alpha_{ij}, d_{ij}, e_{ij}$, and β_{ijl} are the same as appearing in the structure of the natural Volterra Runge–Kutta (NVRK) methods for VIEs in [2].

2. Stability analysis of NVDGLMs

The stability properties of the NVDGLMs (1.2)-(1.3) are analyzed with respect to the basic test problem [1]

$$\left\{\begin{array}{l} y'(t) = \gamma y(t) + \lambda\displaystyle\int_0^t y(s)ds, \quad t\ge 0, \\ y(0) = 1, \end{array}\right. \tag{2.1}$$

with γ and λ as real parameters. By applying the methods (1.2)-(1.3) to (2.1), we get

$$\left\{\begin{array}{ll} Y_i^{[n]} = h\displaystyle\sum_{j=1}^{s}a_{ij}(\gamma Y_j^{[n]} + \lambda Z_j^{[n]}) + \sum_{j=1}^{r}u_{ij}y_j^{[n-1]}, & i = 1,2,\ \ldots, s, \\ y_i^{[n]} = h\displaystyle\sum_{j=1}^{s}b_{ij}(\gamma Y_j^{[n]} + \lambda Z_j^{[n]}) + \sum_{j=1}^{r}v_{ij}y_j^{[n-1]}, & i = 1,2,\ \ldots,\ r, \end{array}\right. \tag{2.2}$$

$$Z_i^{[n]} = h\sum_{\kappa=1}^{n-1}\sum_{j=1}^{s} v_j u(t_{\kappa-1}+\xi_j h) + h\sum_{j=1}^{s}\alpha_{ij}\sum_{l=1}^{s}\beta_{ijl}Y_l^{[n]}$$
$$= h\sum_{\kappa=1}^{n-1}\sum_{\ell=1}^{s} v_\ell^{(w)}Y_\ell^{[\kappa]} + h\sum_{\ell=1}^{s}\alpha_{i\ell}^{(\beta)}Y_\ell^{[n]},$$

where $i = 1,2,\dots,s$, with

$$\alpha_{i\ell}^{(\beta)} = \sum_{j=1}^{s}\alpha_{ij}\beta_{ijl}, \quad v_\ell^{(w)} = \sum_{j=1}^{s} v_j w_\ell(\xi_j).$$

By replacing the vector

$$Z^{[n]} = h\alpha^{(\beta)}Y^{[n]} + h\Big(\sum_{\kappa=1}^{n-1}\sum_{\ell=1}^{s} v_\ell^{(w)}Y_\ell^{[\kappa]}\Big)e,$$

with $e = [1\ 1\ \cdots\ 1]^T \in \mathbb{R}^s$ in (2.2), these relations can be written more compactly as

$$\begin{cases} Y^{[n]} = \zeta AY^{[n]} + \eta A\alpha^{(\beta)}Y^{[n]} + \eta\overline{Y}^{[n-1]}Ae + Uy^{[n-1]}, \\ y^{[n]} = \zeta BY^{[n]} + \eta B\alpha^{(\beta)}Y^{[n]} + \eta\overline{Y}^{[n-1]}Be + Vy^{[n-1]}, \end{cases}$$

in which $\zeta := h\gamma$, $\eta := h^2\lambda$, and

$$\overline{Y}^{[n-1]} := \sum_{\kappa=1}^{n-1} v^{(w)^T}Y^{[\kappa]}.$$

The last relations can be written in matrix form as

$$\begin{bmatrix} y^{[n]} \\ \overline{Y}^{[n]} \end{bmatrix} = M(\zeta,\eta)\begin{bmatrix} y^{[n-1]} \\ \overline{Y}^{[n-1]} \end{bmatrix}, \quad n \geq 1,$$

where

(2.3)
$$M(\zeta,\eta) = \begin{bmatrix} (\zeta B + \eta B\alpha^{(\beta)})R(\zeta,\eta)U + V & \eta(\zeta B + \eta B\alpha^{(\beta)})R(\zeta,\eta)Ae + \eta Be \\ v^{(w)^T}R(\zeta,\eta)U & \eta v^{(w)^T}R(\zeta,\eta)Ae + 1 \end{bmatrix},$$

for which $R(\zeta,\eta) := (I_s - \zeta A - \eta A\alpha^{(\beta)})^{-1}$ with I_s as the identity matrix of order s. Also, the stability function $\Phi(w,\zeta,\eta)$ of the methods is defined by

$$\Phi(w,\zeta,\eta) = \det(wI_{s+1} - M(\zeta,\eta)). \tag{2.4}$$

Definition 2.1. The set of S in (ζ,η)-plane is said to be the absolute stability region of the NVDGLMs (1.2)-(1.3) if for every pair $(\zeta,\eta) \in S$, all the roots $w_i = w_i(\zeta,\eta), i = 1,2,\dots,s+1$, of the stability function $\Phi(w,\zeta,\eta)$, defined by (2.4), lie inside the unit circle with only simple roots on the boundary.

Definition 2.2. The NVDGLM (1.2)-(1.3) is said to be A_0-stable if the absolute stability region S of the method includes $\mathbb{R}^- \times \mathbb{R}^-$.

Definition 2.3. The NVDGLM (1.2)-(1.3) is said to be $A_0(\alpha)$-stable, $\alpha \in [0, \pi/2)$, if its stability region S contains the set $\{(\zeta, \eta) \in \mathbb{R}^- \times \mathbb{R}^- : 0 \leq \arctan(\eta/\zeta) < \pi/2\}$.

3. Numerical experiments

Denote the global error by $e_N(h) := \|y(T) - y_N\|_\infty$ and the numerical estimation to the order of convergence p by $O_N := \log_2(e_N(h)/e_N(h/2))$ to show the accuracy and efficiency of the methods.

Consider the linear VIDE [3, 5],

$$\begin{cases} y'(t) = 1 + 2t - y(t) + \displaystyle\int_0^t t(1+2t)e^{s(t-s)}y(s)\mathrm{d}s, \\ y(0) = 1, \end{cases} \tag{3.1}$$

with $t \in [0, 1]$ and exact solution $y(t) = e^{t^2}$. The results illustrate that the errors decrease with the orders $p = 3$, and 4 confirming the theoretical expectations.

Table 1. Numerical results of NVDGLM of orders $p = 3$, and 4 for the problem (3.1).

h		2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
Method with $p=3$	e_N	2.41×10^{-2}	1.87×10^{-3}	1.39×10^{-4}	1.12×10^{-5}	1.00×10^{-6}	1.04×10^{-7}
	O_N		3.69	3.75	3.63	3.49	3.27
Method with $p=4$	e_N	2.90×10^{-2}	1.07×10^{-3}	4.22×10^{-5}	1.67×10^{-6}	6.34×10^{-8}	3.66×10^{-9}
	O_N		4.76	4.66	4.66	4.72	4.11

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A HERMITE INTERPOLATION METHOD TO SOLVE A NON-ISOTHERMAL REACTION-DIFFUSION MODEL

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Abstract. A numerical method is proposed to approximate the solution of the problem of non-isothermal reaction-diffusion in a spherical catalyst which is a basic equation in many industrial structures and can be modeled by a nonlinear singular boundary value problem. The method is based on the two-point Taylor formula as a special case of the Hermite interpolation method and utilizes L'Hospital's rule to overcome the singular behavior of the problem. The obtained results confirm the efficiency of the proposed computational procedure.

1. Introduction

The Lane- Emden equation was first considered in astrophysics for investigating the thermal behavior of a spherical cloud of a gas under the gravitation of its molecules [2]. This equation can effectively model several systems in science and engineering and appears frequently in various fields, such as the theory of stellar structure and the isothermal gas spheres [4]. Also, the Lane-Emden equation has attracted the attention of many researchers due to its singular behavior at $x = 0$, which is the main difficulty to gain its solution. Since the exact solution of such problems is often unavailable, many efforts have been devoted to approximate the solution of the Lane-Emden equation [6, 7].

In this work, we estimate the dimensionless concentration of chemical species inside a spherical catalyst, which can be modeled by the following Lane-Emden BVP [4]

$$\begin{cases} u''(x) + \dfrac{2}{x}u'(x) - \phi^2 u(x) e^{\frac{\gamma\beta(1-u(x))}{1+\beta(1-u(x))}} = 0, \\ u'(0) = 0, \quad u(1) = 1, \end{cases} \tag{1.1}$$

where γ, β and ϕ represent the dimensionless activation energy, dimensionless heat of reaction, and the Thiele modulus, respectively. Eq. (1.1) is an important problem in

Keywords: Hermite interpolation, Boundary value problems, Lane-Emden equation.
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the design of catalytic reactors and arises in the modeling of chemical and biochemical phenomena [5, 1].

The proposed method in this paper is based upon a special type of Hermite interpolation, namely the two-point Taylor formula (TTF). The TTF produces an approximation to a function by utilizing the values of the function and its derivatives up to an adequate order at the endpoints of the domain. In order to overcome the singular behavior of Eq. (1.1) at $x = 0$, L'Hospital's rule is applied for computing each order of derivatives at this point. The use of L'Hospital's rule provides a recurrence relation to calculate different orders of derivatives at $x = 0$. The needed values of derivatives at $x = 1$ are computed in a different way. The main feature of the proposed method is that the final system of algebraic equations contains only two unknowns, regardless of the number of the used basis functions.

2. Description of the method

Consider the Hermite interpolation problem in which a unique polynomial $\mathrm{P} \in \mathbb{P}_{2n-1}$ is constructed to interpolate a function f such that

$$f^{(r)}(x) = \mathrm{P}^{(r)}(x), \quad 0 \le r \le n-1, \quad x \in \{0, 1\}.$$

An explicit form of this Hermite approximation was obtained by Phillips [3].

Theorem 2.1. [3] Suppose that $f \in C^{2n}[0,1]$, then f can be approximated using the polynomial $\mathrm{P}_{2n-1}(f;x) \in \mathbb{P}_{2n-1}$ as

$$\mathrm{P}_{2n-1}(f;x) = \sum_{j=0}^{n-1} \left[C_{n,j}(x) f^{(j)}(0) + (-1)^j C_{n,j}(1-x) f^{(j)}(1) \right], \tag{2.1}$$

where

$$C_{n,j}(x) = \frac{x^j}{j!}(1-x)^n \sum_{k=0}^{n-j-1} \binom{n+k-1}{k} x^k.$$

Furthermore, the error term of this approximation is in the form

$$\mathrm{R}_n(f;x) = \frac{f^{(2n)}(\xi_x)}{(2n)!} x^n (1-x)^n, \quad \xi_x \in (0,1).$$

In order to approximate the solution of problem (1.1) using the polynomial interpolant (2.1), the values of $u^{(j)}(0)$ and $u^{(j)}(1), j = 0, 1, \ldots, n-1$ are needed. $u'(0)$ and $u(1)$ are available as boundary conditions. If we denote $u(0)$ and $u'(1)$ by δ_1 and δ_2, respectively, and calculate $u^{(j)}(0)$ and $u^{(j)}(1), j = 2, 3, \ldots, n-1$ in the unknowns δ_1 and δ_2, then all the derivatives can be computed by solving the system of equations

$$\begin{cases} \mathrm{Res}(x_1) = 0, \\ \mathrm{Res}(x_2) = 0, \end{cases} \tag{2.2}$$

where $x_1, x_2 \in (0,1)$ are two arbitrary points, and

$$\mathrm{Res}(x) = \mathrm{P}''_{2n-1}(u;x) + \frac{2}{x}\mathrm{P}'_{2n-1}(u;x) - \phi^2 \mathrm{P}_{2n-1}(u;x) e^{\frac{\gamma\beta(1-\mathrm{P}_{2n-1}(u;x))}{1+\beta(1-\mathrm{P}_{2n-1}(u;x)))}}. \tag{2.3}$$

3. Computation of derivatives

3.1. **The derivatives at $\mathbf{x = 1}$.** We know that $u(1) = 1$ and $u'(1) = \delta_2$, thus $u''(1)$ can be computed from Eq. (1.1) as $u''(1) = -2\delta_2 + \phi^2$. Subsequently the values of $u^{(j)}(1), j = 3, 4, \ldots, n-1$, can be obtained in the unknown δ_2 by consecutive differentiating (1.1).

3.2. **The derivatives at $\mathbf{x = 0}$.** Due to the singularity of problem (1.1) at $x = 0$, the derivatives of u at this point can not be computed in the same way as $u^{(j)}(1), j = 2, 3, \ldots, n-1$. We know that $u(0) = \delta_1$ and $u'(0) = 0$, therefore, by utilizing L'Hospital's rule on the singular term of (1.1), one can obtain

$$u''(0) = \frac{1}{3}\phi^2 \delta_1 e^{\frac{\gamma\beta(1-\delta_1)}{1+\beta(1-\delta_1)}}.$$

Consecutive differentiating Eq. (1.1) and the use of the general Leibniz rule lead to

$$\begin{aligned} u^{(k+2)}(x) &= \left(\frac{-2}{x}u'(x)\right)^{(k)} + \left(\phi^2 u(x) e^{\frac{\gamma\beta(1-u(x))}{1+\beta(1-u(x))}}\right)^{(k)} \\ &= -2\sum_{i=0}^{k}\binom{k}{i}(-1)^i i! x^{-(i+1)} u^{(k-i+1)}(x) + \left(\phi^2 u(x) e^{\frac{\gamma\beta(1-u(x))}{1+\beta(1-u(x))}}\right)^{(k)}. \end{aligned} \tag{3.1}$$

Applying L'Hospital's rule for $k = 1, 2, \ldots, n-3$, implies that

$$-2\sum_{i=0}^{k}\binom{k}{i}(-1)^i i! x^{-(i+1)} u^{(k-i+1)}(x)|_{x=0} = \frac{-2}{k+1}u^{(k+2)}(0),$$

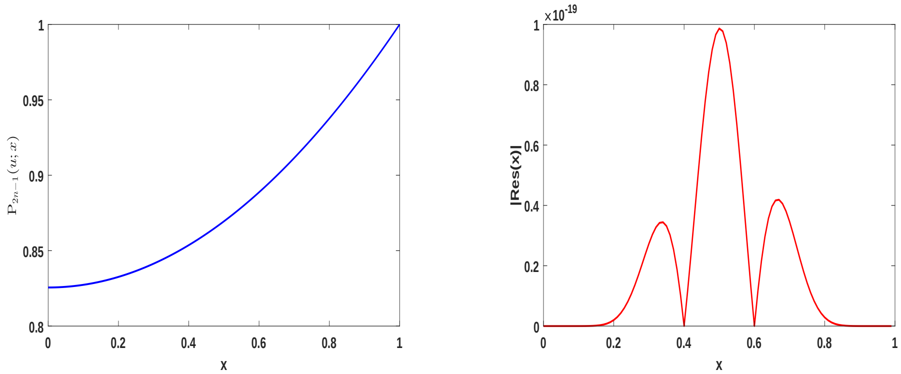
which together with (3.1) gives

$$\frac{k+3}{k+1}u^{(k+2)}(0) = \left(\phi^2 u(x) e^{\frac{\gamma\beta(1-u(x))}{1+\beta(1-u(x))}}\right)^{(k)}_{x=0}.$$

After computing the values of $u^{(j)}(0)$ and $u^{(j)}(1), j = 0, 1, \ldots, n-1$, in the unknowns δ_1, δ_2, and solving the system of equations (2.2) to calculate these unknowns, polynomial (2.1) produces an approximation to the solution of problem (1.1).

4. Numerical results

In this section, the results of the TTF for solving Eq. (1.1) by taking $\beta = 1.2, \gamma = 1.4$ and $\phi = 1$ are reported. The TTF approximation and the residual function (2.3) by taking $n = 15$ are displayed in Figure 1. Also the TTF approximation for $n = 10$, and the computed residual error for some selected points are reported in Table 1. The results indicate the capability of the TTF for solving problem (1.1).



(a) (b)

FIGURE 1. (a) Approximate solution, and (b) Residual error of the TTF using $n = 15$ for problem (1.1).

TABLE 1. The results of the TTF using $n = 10$ for solving Eq. (1.1).

x	$P_{2n-1}(u;x)$	Res(x)
0	0.82559	0.00000
0.2	0.83260	-5.41e-14
0.4	0.85365	1.87e-27
0.6	0.88871	-1.20e-25
0.8	0.93764	-6.86e-14
1	1.00000	-8.26e-24

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THE OCCURRENCE OF RIDDLED BASIN AND BLOWOUT BIFURCATION IN A TWO-SPECIES COMPETITION MODEL

SOHRAB KARIMI AND FATEME HELEN GHANE

ABSTRACT. A competition model of two competing species in population biology is studied. Our model a parametric family has a chaotic attractor A for which the basin of attraction is riddled. We verify the occurrence of riddling basin and blowout bifurcation by varying the normal parameter. The uncertainty exponent and the stability index are applied to quantify the degree of riddling basin.

1. INTRODUCTION

Here, we study a two-species competition model in which the density dependent growth functions are both exponential and rational. This competition model is a difference equation system of the form

$$F_{c_1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\exp\left(2.8 - 0.1(x+y)\right) \\ \dfrac{c_1 y}{(x+y+1.2),} \end{pmatrix}. \tag{1.1}$$

where c_1 is a positive parameter. The species with the rational growth function possesses a single non-trivial equilibrium density which is always stable but species with the exponential growth function produces chaos. [Alexander *et al.*(1992)] observed that the competition model given by (1.1), for some choice of parameter, has an attractor A with a riddled basin of attraction.

Keywords: competition model, riddled basin of attraction, normal Lyapunov exponent..

AMS Mathematical Subject Classification [2010]: 13F55, 05E40, 05C65 (at least 1 and at most 3).

When a riddled basin occurs, for every initial condition in the basin of one attractor there are arbitrary near conditions which tend to the basin of any of the other attractors.

To prove the basin $\mathcal{B}(A)$ is riddled, the following two facts must be verified:

(1) there exists a set of points of positive measure that are attracted to A;
(2) sufficiently many points are repelled from A.

Lyapunov exponents allow us to verify these conditions.

Here, we analyse the population model (1.1) and estimate the range of values of the parameter c_1 such that the attractor A is asymptotically stable, has a riddled basin, or becomes a chaotic saddle. Moreover, we show that by varying the normal parameter c_1, the system undergos a sequence of bifurcations including a "blowout bifurcation", and a "bifurcation to normal repulsion".

2. The riddled basin and blowout bifurcation

Definition 2.1. An attractor A has a *riddled basin* if for all $x \in \mathcal{B}(A)$ and $\varepsilon > 0$, one has

$$m(B_\varepsilon(x) \cap \mathcal{B}(A))m(B_\varepsilon(x) \cap \mathcal{B}(A)^c) > 0. \tag{2.1}$$

An compact invariant set A is called an *asymptotically stable attractor* if it is Lyapunov stable and the basin of attraction $\mathcal{B}(A)$ contains a neighbourhood of A. An invariant transitive set A is a *chaotic saddle* if there exists a neighbourhood U of A such that $\mathcal{B}(A) \cap U \neq \emptyset$ but $m(\mathcal{B}(A)) = 0$.

The next theorem is our main result.

Theorem 2.2. *(*[Karimi & Ghane(2020), Theorem 3]*) Consider the smooth parametric family F_{c_1} given by (1.1) and the invariant set A which is an asymptotically stable attractor for $F_{c_1}|_N$. Then, under $F_{c_1} : \mathbb{R}^2 \to \mathbb{R}^2$, the following possibilities might occur:*

(a) *The invariant set A is an asymptotically stable attractor, whenever* $0 < c_1 < 1$.
(b) *A is a Milnor (essential) attractor whenever* $c_1 < 20.28$. *In addition, A is a Milnor attractor with riddling basin whenever* $20.24 < c_1 < 20.30$.
(c) *A is a chaotic saddle whenever* $20.35 < c_1 < 29.2$.

Definition 2.3. For the smooth parametric family F_{c_1} given by (1.1) with the normal parameter c_1, we say that (see Definition 3.8 of [Ashwin *et al.*(1996)])

(1) $\widetilde{c}_1$ is a *"point of loss of asymptotic stability"* if $\Lambda_{max} < 0$ for $c_1 < \widetilde{c}_1$ and $\Lambda_{max} > 0$ for $c_1 > \widetilde{c}_1$.
(2) $\widetilde{c}_1$ is a *"blowout bifurcation point"* if $\Lambda_{SBR} < 0$ for $c_1 < \widetilde{c}_1$ and $\Lambda_{SBR} > 0$ for $c_1 > \widetilde{c}_1$.

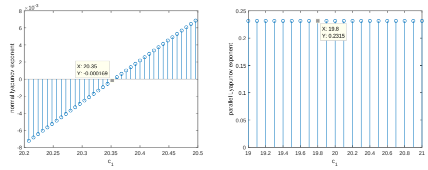

Figure 1. Bifurcation diagrams for Lyapunov exponents

(3) $\widetilde{c}_1$ is a *"point of bifurcation to normal repulsion"* if $\lambda_{min} < 0$ for $c_1 < \widetilde{c}_1$ and $\lambda_{min} > 0$ for $c_1 > \widetilde{c}_1$.

Corollary 2.4. *(*[Karimi & Ghane(2020), Corollary 3.1]*) For the parametric family F_{c_1} given by (1.1) and the invariant set A as above, there exist parameter values $1 < c_a < c_b < c_c < 29.2$ such that F_{c_1} possesses the following bifurcations:*

(1) *there exists a loss of asymptotic stability at c_a;*
(2) *there exists a blowout bifurcation at c_b;*
(3) *there exists a bifurcation to normal repulsion at c_c.*

3. The existence of a riddled basin

Systems with multiple attractors and riddling basin of attractions have sensitive dependence on initial conditions and small variations in initial conditions induce a switch between the different asymptotic attractors.

Here, we examine the behavior of the path F_{c_1} to show that it can verify the emergence conditions of riddled basins for some values of c_1. The dynamic of such system is described by two Lyapunov exponents: the first is the parallel Lyapunov exponent which describe the evolution on the x-axis (as an invariant set), which must be positive, and the other is the normal Lyapunov exponent which characterizes [Alexander *et al.*(1992), Ashwin *et al.*(1996)] the evolution transverse to the x-axis. If the normal Lyapunov exponent is negative [Alexander *et al.*(1992)], then A is a Milnor attractor and its basin has positive Lebesgue measure. This causes the attractor A is transversely stable in the phase space $\mathbb{R}^2$.

Fig. 1 shows the evolution of the Lyapunov exponents of the model by varying the parameter c_1. For some parameter region, these figures display the occurrence of riddled basins. Indeed, Fig. 1 illustrates that if $20.24 < c_1 \leq 20.30$, then the typical orbits on A, for infinitesimal perturbations, having the negative normal Lyapunov exponents.

For occurrence the riddled basin, we must show that the normal Lyapunov exponent, although negative for almost any orbit on A, experiences finite time fluctuations that are positive. Verifying of this condition can be done by computing the finite-time Lyapunov exponents.

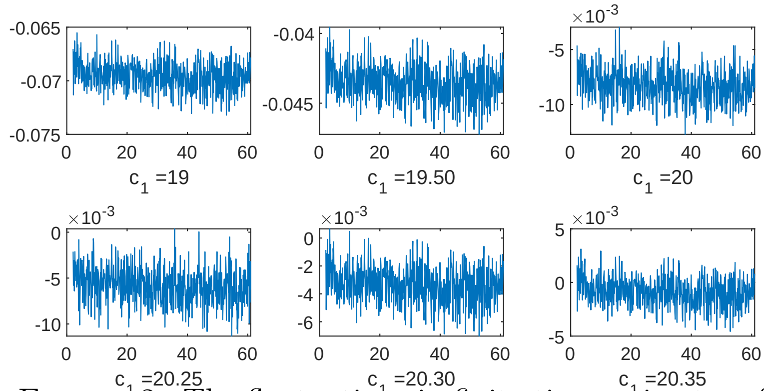

FIGURE 2. The fluctuations in finite time estimates of $\lambda_{\perp}$.

Fig. 2 shows that if $20.24 < c_1 \leq 20.30$, the finite time fluctuations of the normal Lyapunov exponent are positive. We refer to the previous section that we have shown the existence of a period sink p of period two for $19 < c_1 < 21$. By these observations, for parameter values $20.24 < c_1 \leq 20.30$, the riddled basins occur.

Fig. 2 displays the fluctuations in finite time estimates of $\lambda_{\perp}$.

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NONLOCAL DIFFUSION PROBLEM AND NUMERICAL SOLUTION FOR TWO DIMENSIONAL WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS

ROGHAYEH KATANI

Abstract. In this paper, we consider a class of two-dimensional singular Volterra integral equations, which arise in practice by considering a diffusion problem with an output flux which is nonlocal in time. This class of integral equations often have non-smooth solutions, particularly at $t = 0$ that decries the order of convergence. To overcome this difficulty, we propose a simple smoothing change of variable and then we use a Nystrom method for solving obtained integral equation.

1. Introduction

Consider the nonlocal diffusion problem

$$
\begin{aligned}
&\frac{\partial c}{\partial t}(x,t) = \frac{\partial^2 c}{\partial x^2}(x,t), \qquad 0 < x < 1, \quad t > 0, \\
&\frac{\partial c}{\partial t}(0,t) = 0, \qquad t > 0, \\
&\frac{\partial c}{\partial x}(1,t) = -\int_0^t \frac{1}{\sqrt{\pi(t-\tau)}} c(1,\tau)\,d\tau, \qquad t > 0,
\end{aligned}
\tag{1.1}
$$

with initial condition $c(x,0) = c_0$.
By using Laplace Transforms and some calculations, this problem transform to second

Keywords: diffusion problem, two-dimensional Volterra integral equations, weakly singular, smoothing, Gauss-Radau quadrature, Jacobi weight..

AMS Mathematical Subject Classification [2010]: 65R20, 45G10..

Volterra integral equation

$$y(t) = f(t) + \int_0^t \int_0^\tau \frac{k(t,\tau,\sigma)y(\sigma)}{(t-\tau)^\alpha(\tau-\sigma)^\beta} d\sigma d\tau, \qquad 0 \le \alpha, \beta < 1, \tag{1.2}$$

where $(t,\tau,\sigma) \in \Omega = \{0 \le \sigma \le \tau \le t \le T\}$ and $y(0) = f(0)$ [3].
This has provided the motivation for a more general study of this class of integral equations. In [3], for spacial case $k(t,\tau,\sigma) \equiv 1$, $f(t) \equiv 1$, an analytic solution to this integral equation in terms of a variant of the Mittag-Leffler function has obtained. In this paper we propose a numerical method for this class of integral equations. Since Volterra integral equations with weakly singular kernels typically have solutions which are non-smooth near the initial point of the interval of integration [2], most numerical methods are unable to obtain an optimal rate of convergence. Therefore for overcome this difficulty we use smoothing technique [1] and then using Gauss-Radau quadrature for Jacobi weight functions for solving the resulted equation.

2. The smoothing transformation

By interchanging the integrals and using the change variable $s = 2\frac{\tau-\sigma}{t-\sigma} - 1$, (1.2) leads to

$$y(t) = f(t) + 2^{\alpha+\beta-1} \int_0^t \int_{-1}^1 \frac{k(t, 1/2(t-\sigma)(s+1)+\sigma, \sigma)y(\sigma)}{(1-s)^\alpha(1+s)^\beta(t-\sigma)^{\alpha+\beta-1}} ds d\sigma. \tag{2.1}$$

Now, we employ the ideas of Baratella and Orsi [1] to eliminate the singularity of the solution at the origin. Thus consider the change variables $\gamma(t) = t^q$, $\gamma(s) = s^q$ in the above equation, with suitable positive constant q, then we have

$$\begin{aligned} y(\gamma(t)) = f(\gamma(t)) + 2^{\alpha+\beta-1} &\int_0^t (\gamma(t)-\gamma(\sigma))^{1-\alpha-\beta} \\ &\int_{-1}^1 \frac{k(\gamma(t), 1/2(\gamma(t)-\gamma(\sigma))(s+1)+\gamma(\sigma), \gamma(\sigma))y(\gamma(\sigma))}{(1-s)^\alpha(1+s)^\beta} ds \gamma'(\sigma) d\sigma. \end{aligned} \tag{2.2}$$

The equation (2.2) can be simplified as

$$Y(t) = F(t) + 2^{\alpha+\beta-1} \int_0^t (t-\sigma)^{1-\alpha-\beta} \delta_{\alpha,\beta}(t,\sigma)\gamma'(\sigma)Y(\sigma) \int_{-1}^1 \frac{K(t,s,\sigma)}{(1-s)^\alpha(1+s)^\beta} ds d\sigma, \tag{2.3}$$

where $F(t) := f(\gamma(t))$, $Y(t) := y(\gamma(t))$,$K(t,s,\sigma) := k(\gamma(t), 1/2(\gamma(t)-\gamma(\sigma))(s+1) + \gamma(\sigma), \gamma(\sigma))$ and

$$\delta_{\alpha,\beta}(t,\sigma) := \begin{cases} (\frac{\gamma(t)-\gamma(\sigma)}{t-\sigma})^{1-\alpha-\beta}, & t \ne \sigma, \\ (\gamma'(t))^{1-\alpha-\beta}, & t = \sigma. \end{cases}$$

Eq. (2.3) has a kernel which is still weakly singular but has a smooth solution and the smoothness of this solution can be increased by an appropriate choice of q.

3. Numerical method

We approximate the second integral in Eq. (2.3) by using Gauss-Radau quadrature rule for Jacobi wight function [4], therefore

$$Y(t) = F(t) + 2^{\alpha+\beta-1}\int_0^t (t-\sigma)^{1-\alpha-\beta}\delta_{\alpha,\beta}(t,\sigma)\gamma'(\sigma)Y(\sigma)[e_0K(t,1,\sigma) + \sum_{j=1}^{m} d_jK(t,s_j,\sigma)]d\sigma, \tag{3.1}$$

where s_j, $j = 1, ..., m$ are the zeroes of the Jacobi polynomial $P_m^{1-\alpha,\beta}$,

$$e_0 = 2^{1-\alpha-\beta}\frac{\Gamma(-\alpha+1)\Gamma(-\beta+1+m)\Gamma(1+m)\Gamma(2-\alpha)}{\Gamma(m+2-\alpha)\Gamma(2+m-\alpha-\beta)\Gamma(1)},$$

$$d_j = 2^{-\alpha-\beta}\frac{\Gamma(m-\alpha+2)\Gamma(m-\beta+1)(2m-\alpha-\beta+3)^2(1+s_j)}{\Gamma(m+1)\Gamma(m-\alpha-\beta+2)(m+1)^2(m-\alpha-\beta+2)^2[P_{m+1}^{1-\alpha,-\beta}(s_j)]^2}.$$

Now, let $t_i = ih$, $i = 0, 1, ..., N$ be mesh points with step length $h = \frac{1}{N}$, then by the change variable $\sigma = \frac{t_i}{2}x + \frac{t_i}{2}$, we can write

$$\begin{aligned} Y(t_i) &= F(t_i) + 2^{2(\alpha+\beta)-3}t_i^{2-\alpha-\beta}e_0 \\ &\quad \int_{-1}^{1}(1-x)^{1-\alpha-\beta}\delta_{\alpha,\beta}(t_i,t_i2x+t_i2)\gamma'(t_i2x+t_i2)K(t_i,1,t_i2x+t_i2)Y(\frac{t_i}{2}x+\frac{t_i}{2})dx \\ &\quad + 2^{2(\alpha+\beta)-3}t_i^{2-\alpha-\beta}\sum_{j=1}^{m}d_j \\ &\quad \int_{-1}^{1}(1-x)^{1-\alpha-\beta}\delta_{\alpha,\beta}(t_i,t_i2x+t_i2)\gamma'(t_i2x+t_i2)K(t_i,s_j,t_i2x+t_i2)Y(\frac{t_i}{2}x+\frac{t_i}{2})dx, \end{aligned} \tag{3.2}$$

again the Gauss-Radau quadrature rule leads to

$$\begin{aligned} Y(t_i) &= F(t_i) + 2^{2(\alpha+\beta)-3}t_i^{2-\alpha-\beta}e_0[c_0\delta_{\alpha,\beta}(t_i,t_i)\gamma'(t_i)K(t_i,1,t_i)Y(t_i) \\ &\quad + \sum_{l=1}^{n}b_l\delta_{\alpha,\beta}(t_i,t_i2x_l+t_i2)\gamma'(t_i2x_l+t_i2)K(t_i,1,t_i2x_l+t_i2)Y(\frac{t_i}{2}x_l+\frac{t_i}{2})] \\ &\quad + 2^{2(\alpha+\beta)-3}t_i^{2-\alpha-\beta}\sum_{j=1}^{m}d_j[c_0\delta_{\alpha,\beta}(t_i,t_i)\gamma'(t_i)K(t_i,s_j,t_i)Y(t_i) \\ &\quad + \sum_{l=1}^{n}b_l\delta_{\alpha,\beta}(t_i,t_i2x_l+t_i2)\gamma'(t_i2x_l+t_i2)K(t_i,s_j,t_i2x_l+t_i2)Y(\frac{t_i}{2}x_l+\frac{t_i}{2})],\ i = 1, 2, ..., N, \end{aligned} \tag{3.3}$$

where x_l, $l = 1, ..., n$ are the zeroes of the Jacobi polynomial $P_n^{2-\alpha-\beta,0}$,

$$c_0 = 2^{2-\alpha-\beta}\frac{\Gamma(3-\alpha-\beta)\Gamma(2-\alpha-\beta)\Gamma(1+n)^2}{\Gamma(3-\alpha-\beta+n)\Gamma(3-\alpha-\beta+n)\Gamma(1)},$$

$$b_l = 2^{1-\alpha-\beta}\frac{\Gamma(n-\alpha-\beta+3)\Gamma(n+1)(2n+4-\alpha-\beta)^2(1+x_l)}{\Gamma(n+1)\Gamma(n-\alpha-\beta+3)(n+1)^2(n-\alpha-\beta+3)^2[P_{n+1}^{2-\alpha-\beta,0}(x_l)]^2}.$$

In system (3.3) the values $Y(\frac{t_i}{2}x_l + \frac{t_i}{2})$, $i = 1, 2, ..., N$, are unknown, but they can be approximated by Lagrange interpolation at the points $t_0, t_1, ..., t_N$, that is, $Y(\frac{t_i}{2}x_l +$

$\frac{t_i}{2}) \approx \sum_{i'=0}^{n} L_{i'}(\frac{t_i}{2}x_l + \frac{t_i}{2})Y(t_{i'})$. Thus we have

$$Y(t_i) = F(t_i) + 2^{2(\alpha+\beta)-3} t_i^{2-\alpha-\beta} c_0 \delta_{\alpha,\beta}(t_i,t_i)\gamma'(t_i)[e_0 K(t_i,1,t_i) + \sum_{j=1}^{m} d_j K(t_i,s_j,t_i)]Y(t_i)$$
$$+ 2^{2(\alpha+\beta)-3} t_i^{2-\alpha-\beta} \sum_{l=1}^{n} \Big[b_l \delta_{\alpha,\beta}(t_i, \frac{t_i}{2}x_l + \frac{t_i}{2})\gamma'(\frac{t_i}{2}x_l + \frac{t_i}{2})$$
$$[e_0 K(t_i,1,\frac{t_i}{2}x_l + \frac{t_i}{2}) + \sum_{j=1}^{m} d_j K(t_i,s_j,\frac{t_i}{2}x_l + \frac{t_i}{2})] \, [\sum_{i'=0}^{N} l_{i'}(\frac{t_i}{2}x_l + \frac{t_i}{2})Y(t_{i'})] \Big], \tag{3.4}$$

which can be simplified as

$$Y(t_i) = F(t_i) + A_i Y(t_i) + B_{i,0}Y(t_0) + B_{i,1}Y(t_1) + B_{i,2}Y(t_2) + ... + B_{i,N}Y(t_N), \quad i = 1,2,...,N, \tag{3.5}$$

where

$$A_i := 2^{2(\alpha+\beta)-3} t_i^{2-\alpha-\beta} c_0 \delta_{\alpha,\beta}(t_i,t_i)\gamma'(t_i)[e_0 K(t_i,1,t_i) + \sum_{j=1}^{m} d_j K(t_i,s_j,t_i)]$$
$$B_{i,i'} := 2^{2(\alpha+\beta)-3} t_i^{2-\alpha-\beta} \sum_{l=1}^{n} \Big[b_l \delta_{\alpha,\beta}(t_i, \frac{t_i}{2}x_l + \frac{t_i}{2})\gamma'(\frac{t_i}{2}x_l + \frac{t_i}{2})$$
$$[e_0 K(t_i,1,\frac{t_i}{2}x_l + \frac{t_i}{2}) + \sum_{j=1}^{m} d_j K(t_i,s_j,\frac{t_i}{2}x_l + \frac{t_i}{2})] \, l_{i'}(\frac{t_i}{2}x_l + \frac{t_i}{2}) \Big].$$

Therefore relation (3.5) gives rise to a system of N linear equations which can be rewritten in matrix form

$$\begin{bmatrix} A_1 + B_{1,1} - 1 & B_{1,2} & \dots & B_{1,N} \\ B_{2,1} & A_2 + B_{2,2} - 1 & \dots & B_{2,N} \\ \vdots & & & \\ B_{N,1} & B_{N,2} & \dots & A_N + B_{N,N} - 1 \end{bmatrix} \begin{bmatrix} Y(t_1) \\ Y(t_2) \\ \vdots \\ Y(t_N) \end{bmatrix} = \begin{bmatrix} -F(t_1) - B_{1,0}Y(t_0) \\ -F(t_2) - B_{2,0}Y(t_0) \\ \vdots \\ -F(t_N) - B_{N,0}Y(t_0) \end{bmatrix}.$$

Finally, by solving this system (by the Neumann Lemma, this system has a unique solution for sufficiently small h), the values of the unknown function are obtained at the mesh points.

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APPLICATION OF THE DARBOUX THEORY OF INTEGRABILITY IN INVERSE PROBLEMS

NAJMEH KHAJOEI AND MAHAMMAD REZA MOLAEI

Abstract. The main purpose of this paper is to study qualitative properties of 3-dimensional differential equations via Darboux theory of integrability. We take advantage of Darboux theorem to make a normal form of polynomial differential systems which have at least three invariant algebraic surfaces and to prove that if there are the odd number of polynomial differential systems whose their inverse Jacobian multipliers are determined, we can make a new polynomial differential system of the first systems. The inverse Jacobian multiplier of new system can be obtained by the inverse Jacobian multipliers of the first systems.

1. Introduction and preliminaries

There are two main problems in the theory of ordinary differential equations, direct and inverse problems. In this paper, the main focuse is on the second one which is related to finding a class of differential systems that satisfies a given set of properties. By using Darboux theory of integrability we prove two theorems to determine the qualitative behavior of polynomial differential systems in $\mathbb{R}^3$.
Let $\mathbb{K}[x,y,z]$ be the ring of polynomials in the variables x, y and z with coefficients in $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. In this paper we study some properties of the polynomial differential systems in $\mathbb{R}^3$ which are defined by

$$\dot{x} = P(x,y,z),\ \dot{y} = Q(x,y,z),\ \dot{z} = R(x,y,z) \tag{1.1}$$

Keywords: Darboux theorem, Invariant surface, Normal form.
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where P, Q and R are relatively prime polynomials in $\mathbb{R}[x,y,z]$. The degree of this system is defined by $m = max\{deg(P), deg(Q), deg(R)\}$. We can naturally associate to system (1.1) the vector field

$$X(x,y,z) = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y} + R(x,y,z)\frac{\partial}{\partial z}.$$

A polynomial $f(x,y,z) \in \mathbb{R}[x,y,z]$ is called a Darboux polynomial for system (1.1) if

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z} = Kf,$$

for some polynomial $K(x,y,z) \in \mathbb{R}[x,y,z]$. $K(x,y,z)$ is called a cofactor of f . It is simple to prove that the degree of K is less than or equal to $m-1$. If $f(x,y,z)$ is a Darboux polynomial of system (1.1), then the algebraic surface $f = 0$ in $\mathbb{R}^3$ is an invariant algebraic surface, i.e. if a solution curve has an initial point in it, then it remains on it for all time.

A nonconstant real function $H(x,y,z,t) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is an invariant of the system (1.1) if it is constant on all solution curves $(x(t), y(t), z(t))$ of the system (1.1), i.e., $H(x(t), y(t), z(t), t)$ is a constant for all the values of t for which the solution $(x(t), y(t), z(t))$ is defined. It is clear if H is differentiable on $\mathbb{R}^3 \times \mathbb{R}$ then H is an invariant of the system (1.1) if and only if along every solution of the system (1.1) we have

$$P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} + R\frac{\partial H}{\partial z} + \frac{\partial H}{\partial t} \equiv 0.$$

If the invariant H is independent of the time, then it is called a first integral. A first integral H is called a polynomial (rational) first integral if it is a polynomial (a rational function).

Let us define the concept of inverse Jacobian multiplier and its vanishing set. Consider the C^1 differential system (1.1) defined on an open set $U \subseteq \mathbb{R}^3$. Suppose U_0 is an open subset of U. A function $V \in C^1(U_0)$ is an inverse Jacobian multiplier of system (1.1) if $X(V) = V div(X)$, where $div(X) = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$.

2. Main results

The main results of this papers are Theorems 2.2 and 2.3.

Proposition 2.1. [1] *Darboux theorem: Suppose that the polynomial differential system (1.1) of degree m admits p irreducible invariant algebraic surfaces $f_i = 0$ with cofactors K_i, for $i = 1, ..., p$, and q exponential factors $F_j = exp(g_j/h_j)$ with cofactors L_j, for $j = 1, ..., q$.*

I) There exist $\alpha_i, \beta_j \in \mathbb{C}$ not zero such that $\Sigma_{i=1}^{p}\alpha_i K_i + \Sigma_{j=1}^{q}\beta_j L_j = o$, if and only if the real (multivalued) function $V = f_1^{\alpha_1}...f_p^{\alpha_p}F_1^{\beta_1}...F_q^{\beta_q}$ is a first integral of system 1.1.

II) There exist $\alpha_i, \beta_j \in \mathbb{C}$ not zero such that $\Sigma_{i=1}^{p}\alpha_i K_i + \Sigma_{j=1}^{q}\beta_j L_j = div(X)$, if and only if the real (multivalued) function $V = f_1^{\alpha_1}...f_p^{\alpha_p}F_1^{\beta_1}...F_q^{\beta_q}$ is an inverse Jacobi multiplier of system 1.1.

Theorem 2.2. [2] *Let $X_i = P_i(x,y,z)\partial_x + Q_i(x,y,z)\partial_y + R_i(x,y,z)\partial_z$, with $i = 1,2,...,n$ that n is an odd real number, be C^1 vector fields defined on an open subset $U \subseteq \mathbb{R}^3$, which have C^2 inverse Jacobi multiplier $V_i(x,y,z)$, respectively. Then, the vector field, $X = P(x,y,z)\partial_x + Q(x,y,z)\partial_y + R(x,y,z)\partial_z$ with*

$$P = \sum_{i=1}^{n} \lambda_i P_i(\Pi_{j=1,i\neq j}^{n} V_j) + \lambda_0 \frac{\partial(\Pi_{i=1}^{n} V_i)}{\partial y} + \lambda_0 \frac{\partial(\Pi_{i=1}^{n} V_i)}{\partial z},$$
$$Q = \sum_{i=1}^{n} \lambda_i Q_i(\Pi_{j=1,i\neq j}^{n} V_j) - \lambda_0 \frac{\partial(\Pi_{i=1}^{n} V_i)}{\partial x} - \lambda_0 \frac{\partial(\Pi_{i=1}^{n} V_i)}{\partial z},$$
$$R = \sum_{i=1}^{n} \lambda_i R_i(\Pi_{j=1,i\neq j}^{n} V_j) - \lambda_0 \frac{\partial(\Pi_{i=1}^{n} V_i)}{\partial x} + \lambda_0 \frac{\partial(\Pi_{i=1}^{n} V_i)}{\partial y},$$

where λ_i is an arbitrary real number for each $i \in \{0,1,2,...,n\}$ has the inverse Jacobi multiplier $V(x,y,z)$ given by $V(x,y,z) = \Pi_{i=1}^{n} V_i(x,y,z)$.

In the next theorem We make normal form of polynomial differential systems in $\mathbb{R}^3$ which have at least three invariant algebraic surfaces also we obtain the most general normal form that has the finite number of invariant algebraic surfaces.

Theorem 2.3. [3] *Let $f_j(x,y,z)$ for $j = 1,...,r$ with $r \geq 3$ be polynomials that*

$$\bar{J} = det \begin{pmatrix} f_{1x} & f_{1y} & f_{1z} \\ f_{2x} & f_{2y} & f_{2z} \\ f_{3x} & f_{3y} & f_{3z} \end{pmatrix}$$

be non-zero , and let the determinant of the matrix

$$(f_1 f_2 f_3)^{-2} \begin{pmatrix} \{x, f_1, f_2\} & \{x, f_1, f_3\} & \{x, f_2, f_3\} \\ \{y, f_1, f_2\} & \{y, f_1, f_3\} & \{y, f_2, f_3\} \\ \{z, f_1, f_2\} & \{z, f_1, f_3\} & \{z, f_2, f_3\} \end{pmatrix} \tag{2.1}$$

be non-zero. Then the polynomial differential system

$$\begin{cases} \dot{x} = \displaystyle\sum_{k=1<l=2}^{r+3} \omega_{k,l}\{x, f_k, f_l\} \prod_{m=1,m\neq k,l}^{r} f_m \\ \dot{y} = \displaystyle\sum_{k=1<l=2}^{r+3} \omega_{k,l}\{y, f_k, f_l\} \prod_{m=1,m\neq k,l}^{r} f_m \\ \dot{z} = \displaystyle\sum_{k=1<l=2}^{r+3} \omega_{k,l}\{z, f_k, f_l\} \prod_{m=1,m\neq k,l}^{r} f_m \end{cases} \tag{2.2}$$

has $f_j = 0$ as invariant algebraic surfaces for $j = 1, ..., r$ where $f_{r+1} = z, f_{r+2} = y$, $f_{r+3} = x$, and each $\omega_{k,l}$ is an arbitrary polynomial.

In the former theorem if we replace $f_{r+1} = z, f_{r+2} = y$ and $f_{r+3} = x$ then the system (2.2) can be written by the following form

$$\begin{cases} \dot{x} = \sum_{k=1<l=2}^{r} \omega_{k,l}\{x, f_k, f_l\} \prod_{m=1, m\neq k,l}^{r} f_m + \omega_{r+1,r+2} \prod_{m=1}^{r} f_m \\ \dot{y} = \sum_{k=1<l=2}^{r} \omega_{k,l}\{y, f_k, f_l\} \prod_{m=1, m\neq k,l}^{r} f_m + \omega_{r+1,r+3} \prod_{m=1}^{r} f_m \\ \dot{z} = \sum_{k=1<l=2}^{r} \omega_{k,l}\{z, f_k, f_l\} \prod_{m=1, m\neq k,l}^{r} f_m + \omega_{r+2,r+3} \prod_{m=1}^{r} f_m \end{cases} \tag{2.3}$$

Example 2.4. As an example consider the following system

$$\begin{cases} \dot{x} = -2x^3y^2z - 4x^3y^2z^2 + 2xy^4z + 4xy^4z^2 + 2xy^2z^2 + 4xy^2z^3 \\ \dot{y} = 2x^4yz - 2x^2y^3z - 2x^2yz^2 \\ \dot{z} = 4zx^4y^2 - 4x^2y^4z - 4y^2z^2x^2. \end{cases}$$

This system has $f_1 = x^2 + y^2 + z^2 - 1 = 0$, $\mathcal{P} = f_2 = y^2 - z = 0$ (parabolic cylinder) and $\mathcal{H} = f_3 = x^2 - y^2 - z = 0$ (hyperbolic parabolic) as invariant algebraic surfaces.

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GENERALIZED SYNCHRONIZATION OF A NON-AUTONOMOUS UNIFIED CHAOTIC SYSTEM

BEHNAZ KOOCHECK SHOOSHTARI AND MOHAMMADREZA MOLAEI

Abstract. In this article the generalized synchronization of the unidirectionally coupled mechanism between two different systems of a non-autonomous unified chaotic system with continuous periodic switch is considered. Its stability is analyzed by estimating the entire Lyapunov characteristic exponent spectrum and the auxiliary system approach.Then an excellent example from this type of coupling is introduced that when the parameter becomes greater, the generalized synchronization tends to identical synchronization. Numerical and graphical works are done with Mathematica.

1. Introduction

Chaos synchronization is a fundamental important topic in the nonlinear science. This phenomenon has been widely studied in coupled autonomous chaotic dynamical systems in various fields as secure communication, biological systems and information processing since it has both the theoritical and applied significance. There are additional complication to synchronization of chaotic systems when they are non-autonomous. Synchronization of chaotic systems [20, 22], in a generalized concept, leads to very rich dynamical behavior to compare with identical synchronization [6, 9, 22, 23, 27] in coupled chaotic dynamical systems. This type of synchronization often implies a more complicated connection between the synchronized trajectories of coupled chaotic dynamical systems in the phase space. In this case dynamical

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variables from one chaotic system equates with a function of the variables of another chaotic system in coupled scheme. This kind of synchronization was called mutual synchronization by Afraimovich et. al [3] or generalized synchronization (GS) by Rulkov et.al [25]. In physical world, the application of generalized synchronization may wider or more practical than those of identical synchronization.
A great class of synchronized systems has unidirectionally coupled mechanism,that possess drive-response (or master-slave) configuration:

$$\frac{dX(t)}{dt} = F(X(t)), \tag{1.1}$$

$$\frac{dY(t)}{dt} = G(Y(t), X(t), C), \tag{1.2}$$

where, $X(t) \in \mathbb{R}^n, Y(t) \in R^m$ are dynamical variables.The systems in (1.1) and (1.2) are referred to as drive system and response system, respectively. Since the chaotic dynamics of the drive system does not depend on the response system, therefore the coupling between the systems is unidirectional. The coupling of the response system to the drive system is determined by the coupling matrix C. For $C = 0$, the response variable $Y(t)$ is independent of the drive variable $X(t)$ and both systems evolve on distinct chaotic attractors. For $C \neq 0$, two systems have the feature of GS [25], if there exists a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, a manifold $R = \{(X, Y) : Y = T(X)\}$, synchronization manifold, and a subset $B = B_X \times B_Y \subseteq R^n \times R^m$ witch $R \subseteq B$ such that all trajectories of the coupled systems with initial conditions in the basin B approach R as $t \to \infty$ [1, 13, 25]. When the response system is different from the drive system, identical synchronization is usually impossible and we may observe generalized synchronization in this case, which includes identical synchronization as a particular case when $T(X)$ is the identity function [6, 9, 13, 23].

According to Kocarev and Parlitz[13], the following theorem gives us the conditions for occurring GS in the unidirectional coupling.

Theorem 1.1. *The generalized synchronization (GS) occurs between two systems (1.1)and (1.2), if and only if for all $(X_0, Y_0) \in B$, the driven system $\frac{dY(t)}{dt} = G(Y(t), X(t), C)$ is asymptotically stable, i.e.,*

$$\forall Y_{01}, Y_{02} \in B_Y : \lim_{t\to\infty} ||Y(t, X_0, Y_{01}) - Y(t, X_0, Y_{02})|| = 0.$$

The consequence of this theorem is that the response is predictable, that is we can predict the behavior of $Y(t)$, based on our knowledge of $X(t)$ and $T(.)$. Once transients die out, the existence [21] of a transformation $T(.)$, guarantees the capability to predict the $Y(t)$ from $X(t)$ alone. This relation between $X(t)$ and $Y(t)$ need not maintain everywhere in the phase space, but require maintain only on the attractor. Except in[25], one rarely will be able to produce an explicit formula for exhibiting the mapping $T(.)$.

Recently, the binary generalized synchronization has reported, when the certain values of the coupling strength two unidirectionally coupled dynamical systems generateing the aperiodic binary sequences are in the generalized synchronization regime[14]. The inverse generalized synchronization problem for different dimensional chaotic dynamical systems is studied[11, 18]. Applicable, a chaotic communication system based on employment of the regime of generalized synchronization between the transmitter and receiver is developed[24].

In [8], a non-autonomous unified chaotic system with continuous periodic switch between Chen [7, 16] and Lorenz [15, 16] systems was introduced, that is,

$$\begin{cases} \dot{x} = (25cos^2(\omega t) + 10)(y - x); \\ \dot{y} = (28 - 35cos^2(\omega t))x - xz + (29cos^2(\omega t) - 1)y; \\ \dot{z} = xy - \frac{cos^2(\omega t)+8}{3} z. \end{cases} \tag{1.3}$$

The dynamical behaviors of this system and the identical (complete) synchronization of the bidirectionally coupled between two identical systems of this type were investigated[8].

In this article, the main attention is focused on detection of GS in drive-response configuration of non-autonomous unified chaotic systems of type (1.3). Three examples will be presented for detecting generalized synchronization. The first and second examples are unidirectional coupled mechanisms between two unified chaotic dynamical systems of type (1.3), with very different parameters. In the first example, the stability of GS will be analyzed by estimating the entire LCE spectrum. In the second one, with the same drive and response systems of the first example, the auxiliary system approach will be used for detecting of GS. The third one, is unidirectional coupled mechanism between system (1.3), as drive system and the system introduced in [12] as response system. In this example we will observe when the parameter becomes greater, the generalized synchronization tends to identical synchronization. In each of the examples, numerical simulations are carried out to observe the status of drive and resp–+onse systems before and after synchronization in various beautiful figures of two and three dimensional.

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SUB-SUPER SOLUTIONS FOR A NON-LOCAL PROBLEM

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ABSTRACT. We consider the problem

$$\begin{cases} (-\Delta)^s u = u|u|^{p-2} + \mu u|u|^{q-2} & \text{in } \Omega, \\ u = 0, & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where Ω is a bounded domain with continuous boundary in $\mathbb{R}^n$, $0 < s < 1$ and $1 < q < 2 < p$ with $2s < n$. Our plan is to make use of a new variational principle and prove the existence of at least one solution for this problem in the presence of a weak sub and a weak super solution.

1. INTRODUCTION

Over the last few decades from the basic method of sub- and supersolutions (see [4]), it is well-known that if Ω is a bounded domain in $\mathbb{R}^n$, g is smooth and if there exist smooth sub- and supersolutions u_1 and u_2 of

$$\begin{cases} -\Delta u = g(x,u) & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $u_1 \leq u_2$, then problem (1.1) admits a classical solution u_0 such that $u_1 \leq u_0 \leq u_2$. In this paper we are interested in a non-local version of the above problem. Indeed, we study a concave-convex problem involving the fractional Laplacian operator

$$\begin{cases} (-\Delta)^s u = u|u|^{p-2} + \mu u|u|^{q-2}, & \text{in } \Omega, \\ u = 0, & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.2}$$

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AMS Mathematical Subject Classification [2010]: 35J60, 35R11.

where Ω is a bounded domain with continuous boundary in $\mathbb{R}^n$, $1<q<2<p$ and $0<s<1$ with $2s<n$. Here $(-\Delta)^s$ denotes the fractional Laplace operator, which is defined as the following singular integral

$$(-\Delta)^s u(x) = C(n,s)P.V.\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}dy \quad (x\in\mathbb{R}^n),$$

where $P.V.$ is the principal value of the integral and $C(n,s)=\left(\int_{\mathbb{R}^n}\frac{1-\cos(\xi_1)}{|\xi|^{n+2s}}d\xi\right)^{-1}$ with $\xi=(\xi_1,...,\xi_n)\in\mathbb{R}^n$. Now, let us recall some notations and results about the theory of fractional Sobolev spaces. Let $s\in(0,1)$ be the fractional exponent, Ω be a bounded domain in $\mathbb{R}^n$, $n>2s$, and denote by Q the set $Q:=(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)$, where $\Omega^c:=\mathbb{R}^n\setminus\Omega$. The space $\mathbb{H}^s(\Omega)$ is the linear space of Lebesgue measurable functions $u:\mathbb{R}^n\to\mathbb{R}$ such that $u|_\Omega$ belongs to $L^2(\Omega)$, and the map $(x,y)\mapsto\dfrac{u(x)-u(y)}{|x-y|^{\frac{n}{2}+s}}$ is in $L^2(Q,dxdy)$. We also define

$$\mathbb{H}^s_0(\Omega)=\{u\in\mathbb{H}^s(\Omega): u=0\ a.e.\ in\ \mathbb{R}^n\setminus\Omega\}.$$

Using Lemma 1.28 in [2], we can consider the norm on $\mathbb{H}^s_0(\Omega)$ defined as

$$\|u\|_{\mathbb{H}^s_0(\Omega)}=\Big(\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy\Big)^{\frac{1}{2}}. \tag{1.3}$$

Lemma 1.1. [2, *Lemma* 1.29] $\big(\mathbb{H}^s_0(\Omega),\|.\|_{\mathbb{H}^s_0(\Omega)}\big)$ *is a Hilbert space with scalar product*

$$\langle u,v\rangle_{\mathbb{H}^s_0(\Omega)}:=\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}dxdy.$$

Proposition 1.2. [2, *Lemma* 4.6]*(non-local maximum principle) Let $u\in\mathbb{H}^s(\Omega)$ satisfy $(-\Delta)^s u\geq 0$ in the weak sense, with $u\geq 0$ in $\mathbb{R}^n\setminus\Omega$. Then $u\geq 0$ in $\mathbb{R}^n$.*

Lemma 1.3. [1, *Lemma* 2.5] *Assume that $f\in L^p(\Omega)$ for some $p>\dfrac{2n}{n+2s}$. Then the problem*

$$\begin{cases}(-\Delta)^s u=f, & in\ \Omega,\\ u=0, & on\ \mathbb{R}^n\setminus\Omega,\end{cases}$$

has a unique weak solution $u\in\mathbb{H}^s_0(\Omega)$.

Here is our result.

Theorem 1.4. *Let Ω be a domain in $\mathbb{R}^n$, $1<q<2<p$ and $0<s<1$ with $2s<n$. Assume that $u_1\in\mathbb{H}^s_0(\Omega)\cap L^p(\Omega)$ is a weak sub-solution and $u_2\in\mathbb{H}^s_0(\Omega)\cap L^p(\Omega)$ is a weak super-solution of (1.2) with $u_1\leq u_2$. Then there exists a weak solution u_0 of (1.2) such that $u_1\leq u_0\leq u_2$.*

We shall be proving Theorem 1.4 by the following result established in [3].

Theorem 1.5. *Let V be a reflexive Banach space and K be a convex and weakly closed subset of V. Let $\Psi : V \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semi-continuous function which is Gáteaux differentiable on K and let $\Phi \in C^1(V, \mathbb{R})$. Assume that the following two assertions hold:*

(*i*) *The functional I_K defined by $I_K = \Psi_K - \Phi$, where $\Psi_K(u) = \begin{cases} \Psi(u) & u \in K, \\ +\infty & u \notin K, \end{cases}$ has a critical point $u_0 \in V$, i.e.,*

$$\Psi_K(v) - \Psi_K(u_0) \geq \langle D\Phi(u_0), v - u_0 \rangle \quad (v \in V).$$

(*ii*) *there exists $v_0 \in K$ such that $D\Psi(v_0) = D\Phi(u_0)$,*

Then $u_0 \in K$ is a solution of the equation $D\Psi(u) = D\Phi(u)$.

2. Main results

In this section, to apply the new variational principle, we shall consider the Banach space $V = \mathbb{H}_0^s(\Omega) \cap L^p(\Omega)$ equipped with the norm $\|u\|_V = \|u\|_{\mathbb{H}_0^s(\Omega)} + \|u\|_{L^p(\Omega)}$. Set $\Phi(u) = \dfrac{1}{p} \int_\Omega |u|^p dx + \dfrac{\mu}{q} \int_\Omega |u|^q dx$ and $\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} \dfrac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy$. Moreover, let $K := \big\{ u \in V : u_1 \leq u \leq u_2 \ a.e. \ in \ \Omega \big\}$ and define

$$I_K(u) = \begin{cases} \Psi(u) - \Phi(u) & u \in K, \\ +\infty & u \notin K. \end{cases} \tag{2.1}$$

Proof of Theorem 1.4. We shall apply Theorem 1.5 for functions Φ and Ψ and the convex set K. First, we show that there exists $u_0 \in K$ such that $I_K(u_0) = \inf_{u \in V} I_K(u)$. For this, set $\beta = \inf_{u \in V} I_K(u)$. In view of the definition of I_K, it is clear that $\beta = \inf_{u \in K} I_K(u)$. For every $u \in K$ we have

$$\Phi(u) = \frac{1}{p} \int_\Omega |u|^p dx + \frac{\mu}{q} \int_\Omega |u|^q dx \leq \frac{1}{p} \int_\Omega \big| |u_1| + |u_2| \big|^p dx + \frac{\mu}{q} \int_\Omega \big| |u_1| + |u_2| \big|^q dx.$$

Since $|u_1| + |u_2| \in L^p(\Omega)$ and Ψ is nonnegative, one can conclude that $\beta > -\infty$. Now suppose that $\{u_n\}$ is a sequence in K such that $I_K(u_n) \to \beta$. So the sequence $\{\Psi(u_n)\}$ is bounded, because of the boundedness of Φ. Thus, by definition of the norm on $\mathbb{H}_0^s(\Omega)$, $\{u_n\}$ is bounded in $\mathbb{H}_0^s(\Omega)$ and after passing to a subsequence if it is necessary, by [2, *Lemma* 1.31] there exists some $u_0 \in \mathbb{H}_0^s(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u_0 && \text{weakly in } \mathbb{H}_0^s(\Omega), \\ &u_n(x) - u_0(x) \to 0 && a.e. \ x \in \Omega, \qquad \|u_n - u_0\|_{L^2(\Omega)} \to 0, \end{aligned} \tag{2.2}$$

as $n \to +\infty$. Since $\{u_n\} \subseteq K$, $u_1(x) \leq u_n(x) \leq u_2(x)$ for all $n \in \mathbb{N}$ and a.e. $x \in \Omega$. Then by (2.2) we can get $u_1(x) \leq u_0(x) \leq u_2(x)$ and $|u_0(x)| \leq |u_1(x)| + |u_2(x)|$ a.e.

$x \in \Omega$. Therefore,

$$\Big(\int_{\Omega} |u_0(x)|^p dx\Big)^{\frac{1}{p}} \le \Big(\int_{\Omega} \big(|u_1(x)| + |u_2(x)|\big)^p dx\Big)^{\frac{1}{p}} = \Big\| |u_1| + |u_2| \Big\|_{L^p(\Omega)}, \tag{2.3}$$

which implies that $u_0 \in L^p(\Omega)$ and then $u_0 \in K$.

It also follows from $\|u_n - u_0\|_{L^q(\Omega)} \to 0$ together with the dominated convergence theorem that $\Phi_{n\to+\infty}(u_n) = \Phi(u_0)$.

On the other hand, $\Psi(u_0) \le \liminf_{n\to+\infty} \Psi(u_n)$, then $I_K(u_0) \le \liminf_{n\to+\infty} I_K(u_n)$. So $u_0 \in K$ is a critical point of I_K and the proof of (i) in Theorem 1.5 is complete.

To verify condition (ii) in Theorem 1.5, we shall show that there exists $v_0 \in K$ such that $(-\Delta)^s v_0 = D\Phi(u_0) = u_0|u_0|^{p-2} + \mu u_0|u_0|^{q-2}$ in the weak sense, that is

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(v_0(x)-v_0(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}} dxdy = \int_{\Omega} D\Phi(u_0)\eta\, dx \qquad (\eta \in V).$$

By Lemma 1.3 we obtain the existence of $v_0 \in \mathbb{H}_0^s(\Omega)$ such that $(-\Delta)^s v_0 = D\Phi(u_0)$.

Since u_1 is a subsolution to problem (1.2) and $u_0 \in K$, we can get

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u_1(x)-u_1(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}} dxdy \le \int_{\Omega} D\Phi(u_1)\eta(x)dx \le \int_{\Omega} D\Phi(u_0)\eta(x)dx,$$

for all non-negative function $\eta \in H_0^s(\Omega)$. In the same way,

$$\begin{aligned}\int_{\Omega} D\Phi(u_0)(x)\eta(x)dx &\le \int_{\Omega} D\Phi(u_2)(x)\eta(x)dx \\ &\le \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u_2(x)-u_2(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}} dxdy,\end{aligned}$$

for all non-negative function $\eta \in H_0^s(\Omega)$. Therefore, for all non-negative function $\eta \in H_0^s(\Omega)$ we can write

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u_1(x)-u_1(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}} dxdy &\le \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(v_0(x)-v_0(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}} dxdy \\ &\le \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u_2(x)-u_2(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}} dxdy.\end{aligned}$$

On the other word, $(-\Delta)^s u_1 \le (-\Delta)^s v_0 \le (-\Delta)^s u_2$ in the weak sense for which together with the maximum principle (Proposition 1.2), we can conclude that $u_1 \le v_0 \le u_2$. As in (2.3), we can show that $v_0 \in L^p(\Omega)$ and then $v_0 \in K$.

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AN N-TO-1 SMALE HORSEHSOE

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Abstract. We give the construction of a 2-to-1 Smale horseshoe which is topologically conjugated with a $(2,4)-$zip-shift map. This example represents the first two-dimensional hyperbolic endomorphism, topologically conjugated with a local homeomorphism.

1. Introduction

In 1970, Stephen Smale constructed a brilliant example that has become fundamental in the study of dynamical systems [6]. The so called *Smale Horseshoe*, which characterizes a class of hyperbolic chaotic diffeomorphisms, is known as the hallmark of deterministic chaos. In 1969, M. Shub performed a comprehensive study on endomorphisms of compact differentable manifolds and presented, mainly the structural stability of expanding maps [5].

In this paper, we present the construction of an n-to-1 Smale horseshoe, as an extended version of the prior Smale horseshoe, without any use of inverse limit techniques. The presence of a strong transversality regardless of the choice of the orbits, together with the density of periodic hyperbolic points seems pleasant. It is well known that these two conditions are equivalent to the structural stability for diffeomorphisms in the C^1- topology [4].

The relevance of this construction is based on a natural and intrinsic topological conjugacy with some extended shift map (a local homeomorphism). The topological conjugation, which in the context of hyeprbolic endomorphisms of dimension greater than one, is happening for the first time in the past 40 years, sheds some new light on distinguishing the real dynamics of hyperbolic endomorphisms.

Keywords: Smale horseshoe, Bernoulli Transformations, Zip shift map.
AMS Mathematical Subject Classification [2010]: 28D05, 37B10, 37D45.

Let $Q = [0,1] \times [0,1] \subset \mathbb{R}^2$ be the unit square and $f : Q \to Q$ an n-to-1 local diffeomorphism. Assume that there exist n rectangles $Q_i = [0,1] \times [i/n, (i+1)/n], i = 0 \cdots, n-1$ that $Q = \sqcup_{i=1}^{n} Q_i$ and $f_{|_{Q_i}} : Q_i \to f(Q_i)$ is a diffeomorphism (see Figure 1 for $n = 2$). For the sake of simplicity, we consider the standard *two-legged* horseshoe (with two vertical strips), but the reader can feel free to construct a horseshoe set starting with any number of strips, where $k \in \mathbb{N}, k > 2$. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map,

$$A = \begin{bmatrix} \beta & 0 \\ 0 & \alpha \end{bmatrix}$$

where $0 < \beta < 1/n$, $\alpha > 1$ and $\alpha\,\beta = n$. Let $m = \alpha/n$ and map g, fold $A(Q)$ *mod-m*

FIGURE 1. First and second images of Q over a 2-to-1 map f

into a horseshoe shape and bring it back to the square. Denote the composition of these three transformations by f and let $\Lambda_1 := f(Q) \cap Q$. Since f is an n-to-1 map, $f^{-1}(\Lambda_1)$ is a set of n distinct vertical horseshoes.(see Figure 2) we denote the 2 horizontal strips obtained from the first forward iteration in $f(Q) \cap Q$ by $H_j, j = a, b$. By a second forward iterate one obtains 4 new strips $H_{i_1\,i_2}$, that $H_{i_1\,i_2} \subset H_i$ and $i_1, i_2 \in \{a, b\}$ (with n^n hidden copies).

Let $V^1, V^{1'}, \cdots, V^n, V^{n'}$, indicate the $2n$ backward vertical strips obtained from $f^{-1}(Q) \cap Q$ and V^{i_1, i_2} with $i_1, i_2 \in \{1, \cdots, n, 1', \cdots, n'\}$, the $(2n)^2$ vertical strips from $f^{-2}(Q) \cap Q$. One can continue this process. The limit set is called a "*2-to-1 horseshoe set*". This horseshoe set is a Cantor set that has all geometric properties of the original Smale horseshoe (with $n = 1$). Obviously, comparing the 1-to-1 Smale horseshoe with the n-to-1 horseshoe, there are some differences as well as interesting topological and dynamical similarities. For example, they are different in topological entropy or in the number of periodic points of period k, it can be seen that the topological entropy of an n-to-1 horseshoe map is equal to $\log 2n$ and the number of periodic points with period k is $(2n)^k$. on the other hand, both are topologically transitive and mixing. Both contain an infinite number of periodic orbits of arbitrary periods and an infinite number of non-periodic orbits.

FIGURE 2. First pre-image of Q over a 2-to-1 map f

2. Zip shift Space

In this section we describe some symbolic dynamics which is an extension of the well-known two sided and one-sided shift homeomorphisms. We define this extended shift space on two sets of symbols (alphabets). The sets P and S can contain any number of elements such that $\#P \leq \#S$. One can consider the set P as a re-symbolized subset of S. In order to have a well defined map we use a serjective factor map $\tau : S \to P$. Let $P = \{a_1, a_2, \cdots, a_p\}$ and $S = \{b_1, b_2, \cdots, b_s\}$, be two collections of symbols that $s \geq p$. Define, $\Sigma_S := \prod_{i=-\infty}^{+\infty} S_i$, where $S_i = S$. Consider the two-sided Bernoulli shift $\sigma : \Sigma_S \to \Sigma_S$. Then to any point $\bar{s} = (s_i) \in \Sigma_S$ correspond a point $\bar{x} = (x_i)_{i\in\mathbb{P}}$, such that

$$x_i = \begin{cases} y_i \in S & \forall i \geq 0 \\ \tau(y_i) \in P & \forall i < 0. \end{cases} \tag{2.1}$$

then we define the *zip shift space* as the set

$$\Sigma = \Sigma_{P,S} := \{x = (x_i)_{i\in\mathbb{Z}} \ : x_i \text{ satisfies (2.1)}\}.$$

For $\bar{s} = (s_i)$, $\bar{t} = (t_i) \in \Sigma$, and $N(\bar{s}, \bar{t}) = \min_i\{|i|, s_i \neq t_i, i \in \mathbb{Z}\}$, the $d(\bar{s}\,, \bar{t}) = \frac{1}{2^{N(\bar{s}\,,\bar{t})}}$ is a metric on Σ and $\mathcal{N}_r(\bar{s}) = \{\bar{t} \in \Sigma : d(\bar{t}, \bar{s}) < r\}$ represents a neighborhood of $\bar{s}$ with diameter $2r$. Obviously, if $N \in \mathbb{N} \cup \{0\}$ is the smallest integer with $\frac{1}{2^N} < r$, then

$$\mathcal{N}_r(\bar{s}) = \{\bar{t} = (t_n)_{n\in\mathbb{Z}} \in \Sigma \ : \ s_i = t_i, \ |i| \leq N\}.$$

For $i, n \in \mathbb{Z}$ and $\ell \in \mathbb{N} \cup \{0\}$, one can define a cylinder set C_i^l as following.

$$C_i^l = [s_i, \cdots, s_{i+\ell}] = \{\bar{k} = (k_n) \in \Sigma : k_{i-\ell} = s_{i-\ell}, \cdots, k_i = s_i, \cdots, k_{i+\ell} = s_{i+\ell}\}$$

The set of all cylinder sets, form a basis for the product topological space (Σ, d). Note that in the definition of the cylinder set, $s_{i+j}, i \leq j \leq i+l$, once $i+j < 0$, the $s_{i+j} \in P$ and once $i+j \geq 0$ the $s_{i+j} \in S$. It is not difficult to verify that the metric space (Σ, d) is compact, totally disconnected and perfect, indeed it is a Cantor set. The following known Lemma is easy to verify [7].

Lemma 2.1. *For $\bar{s}\,, \bar{t} \in \Sigma$,*

- *suppose that $d(\bar{s}\,, \bar{t}) < 1/(2^{M+1})$. Then $s_i = t_i$ for all $|i| < M$.*
- *suppose that $s_i = t_i$ for $|i| \leq M$. Then $d(\bar{s}\,, \bar{t}) \leq 1/(2^M)$.*

One defines the extended shift map as follows.

Definition 2.2 (Zip shift map). Let (Σ, d) and τ be as above. Then

$$\begin{aligned} \sigma_\tau :&\Sigma \longrightarrow \Sigma, \\ &(s_n) \longmapsto (s_{n+1}) = (\cdots s_{-k} \cdots s_{-1}\, \tau(s_0)\,.\, s_1 \cdots s_k \cdots). \end{aligned} \tag{2.2}$$

It is obvious that $\sigma_\tau(\Sigma) = \Sigma$. Using Lemma 2.1 and the fact that the set of all cylinder sets forms a basis for the space Σ, it can be seen that σ_τ is a local homeomorphism. We call (Σ, σ_τ) *zip shift space* with $(p, s)-$symbols. Here p and l are respectively the cardinality of P and S.

Example 2.3. Consider the 2-to-1 map represented in Figure 2. Then $S = \{1, 1', 2, 2'\}$ and $P = \{a, b\}$. The map τ is defined as, $\tau(1) = \tau(2) = a$ and $\tau(1^{'}) = \tau(2^{'}) = b$. Let $\bar{t} \in \Sigma$. For instance take $\bar{t} = (\dots a\, b\, a\, b\, b\, .\, 1\, 0\, 1^{'}\, 1\, 1\, \dots)$. Then

$$\sigma_\tau((\dots\, a\, b\, a\, b\, b\, .\, 1\, 2\, 1^{'}\, 1\, 2^{'}\, \dots)) = (\dots\, a\, b\, a\, b\, b\, a\, .\, 2\, 1^{'}\, 1\, 2^{'}\, \dots).$$

In [2] we adapted the Conely-Moser conditions [7, 3], for an n-to-1 local diffeomorphism, on which the dynamics is topologically conjugate to a (full) zip shift map. The main theorem of this work expresses the following result.

Theorem 2.4. *Suppose that f is an n-to-1 local diffeomorphism, which satisfies the Assumptions 1 and 2. Then, f has an invariant Cantor set Λ, that $f_{|_\Lambda}$ is topologically conjugate to a zip shift map with $(2, 2n)-$symbols, i.e. there exists some topological conjugacy map ϕ such that $\phi \circ f = \sigma_\tau \circ \phi$.*

Proof. The proof of this theorem is proceeding on two steps.
Step 1: Construction of the conjugacy map ϕ between the horseshoe map $f : Q \to Q$ and the zip shift map $\sigma_\tau : \Sigma \to \Sigma$ and showing that ϕ is a homeomorphism.
Step 2: To show that $\phi \circ f = \sigma_\tau \circ \phi$. □

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ON ONE DIMENSIONAL PIECEWISE SMOOTH MAPS: MIXING AND DENSITY OF ORBITS

ROYA MAKROONI, MEHDI POURBARAT, AND NEDA ABBASI

Abstract. In this paper we consider a family of one dimensional discontinuous maps known as Baker like maps with $N > 2$ branches and investigate the problem of existence of chaos in the sense of Devaney. In fact, we prove that under some conditions a semi-baker map has the property of topologically mixing and its periodic points are dense.

1. Introduction

The essential feature of a piecewise smooth system, both continuous and discontinuous, which may greatly influence the dynamics, is the presence of a so-called *switching manifold* at which the system's function changes its definition ([1]). Nowadays many results related to dynamics of piecewise smooth systems (PWS for short) and in particular about chaotic dynamics in such systems are already published, especially those related to piecewise linear maps. For example in [6] and [3] some relevant properties of the chaotic sets related to the piecewise linear maps with constant slope, known as $\beta-$ Transformation, are determined.

This paper is devoted to an important class of piecewise smooth expanding maps of an interval into itself, constituted by $N \geq 2$ branches called Baker like maps. Regarding Baker like maps with $N \geq 2$ branches, in [2] the authors give the necessary and sufficient conditions for a discontinuous expanding map to be chaotic in the whole

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interval in terms of homoclinic bifurcations. Also in [7] for Baker like map with infinitely many branches the existence of full measure unbounded chaotic attractors which are persistent under parameter perturbation (also called robust) has been proved.

Here we consider $N > 2$ and give analytical sufficient conditions under which the system is chaotic in the sense of Devaney. This subject is not new in the literature. The basic tools are related to a Baker like map on the interval, with two branches that corresponds to Lorenz maps, which has been deeply studied since many years and is nowadays of common knowledge ([4], [5]). In the present paper a rigorous proof to the existence of chaos in the sense of Devaney in that system is presented in terms of the derivatives of the branches. Here we improve the results of chaotic dynamics of Baker like maps with two branches in [8], which already published by same authors.

Definition 1.1. Let $I = [0,\ 1]$, and $r \in \mathbb{N} \cup \{\infty\}$. The map $f : I \longrightarrow I$ is called a piecewise C^r-smooth one dimensional dynamical system if there exist the points $0 = \xi_0 < \xi_1 < ... < \xi_{N-1} < \xi_N = 1$ such that $f|_{(\xi_i,\ \xi_{i+1})}$ is C^r-smooth.

The PWS map f is topologically transitive in I, if for any pair of non-empty open sets U and V of I there exists a natural number n such that $f^n(U) \cap V \neq$. It is topologically mixing in I if for any pair of non-empty open sets U and V in X there exists a natural number n such that for any $m > n$, $f^m(U) \cap V \neq$.

The PWS map $f : I \longrightarrow I$ is chaotic in the sense of Devaney if $Per(f)$ is dense in I, it is topologically transitive and has sensitive dependence on initial conditions in I.

The 1D piecewise C^1-smooth map which we are interested in defined as follows

Definition 1.2. Let N be a natural number and λ a real number both bigger than one and real numbers $\xi_0 = 0 < \xi_1 ... < \xi_{N-1} < \xi_N = 1$ are given. For each $1 \leq i \leq N$ suppose that $I_i = [\xi_{i-1},\ \xi_i)$ and $f_i :\ I_i \longrightarrow [0,\ 1)$ is a differetiable map satisfying ${f_i}' > \lambda$. Also, suppose that f_i is surjective for all $1 < i < N$ and $f_{1-}(\xi_1) = 1$ and $f_{N+}(\xi_{N-1}) = 0$. Then map $f :\ I \longrightarrow I$ given by

$$f(x) := f_i(x) \quad x \in I_i, \quad f(1) := f_{N-}(1) \tag{1.1}$$

is called a Baker-like map with N branches and expanding rate λ.

2. Main results

This section corresponds to Baker-like maps with more than two branches. According to the following theorem, these maps are topologically mixing and $Per(f)$ is dense in I, and so full chaos occur in this class.

Theorem 2.1. *Suppose that f is a Baker-like map with rate $\lambda > 2$ and $N \geq 3$. Then for each interval $(a,\ b) \subset [0.\ 1]$, there are one interval $J \subset (a,\ b)$ and one natural number k such that f^k is continuous on J and $f^k(J) = (0,\ 1)$.*

Proof. Without losing generality, we may assume that $(a,\ b) \cap \{\xi_i\}_{i=0}^N$ is an empty set. Let $U_0 := (a,\ b)$ and $k_1 \in \mathbb{N}$ be the smallest number such that

$$f^{k_1}(U_0) \cap \{\xi_i\}_{i=0}^N \neq \emptyset.$$

Notice that f is an expanding map with $\lambda > 2$, and by using mean value theorem there is alway such k_1. Moreover, f is a continuous and non decreasing map on the intervals $U_0, f(U_0), \cdots f^{k_1-1}(U_0)$, this also gives $\xi_0, \xi_N \notin f^{k_1}(U_0)$. Let $0 < l_1 < N$ be the smallest number such that $\xi_{l_1} \in f^{k_1}(U_0)$. Assume

$$U_1 := \begin{cases} \big(f^{k_1}(a),\ \xi_{l_1}\big), \ 2\big|\big(f^{k_1}(a),\ \xi_{l_1}\big)\big| > \big|f^{k_1}(U_0)\big| \\ \big(\xi_{l_1},\ f^{k_1}(b)\big), \ 2\big|\big(f^{k_1}(a),\ \xi_{l_1}\big)\big| \leq \big|f^{k_1}(U_0)\big| \\ \quad (\xi_{l_1},\ \xi_{l_1+1}), \ (\xi_{l_1},\ \xi_{l_1+1}) \subset f^{k_1}(U_0) \end{cases}.$$

By using mean value theorem, we have $|U_1| \geq \frac{\left|f^{k_1}(U_0)\right|}{2} > \frac{\lambda}{2}|U_0|$. By induction method, we construct finite sequences of the numbers $\{l_i\}_{i=1}^n$, $\{k_i\}_{i=1}^n$ and the intervals $\{U_i\}_{i=0}^n$ as follows. Suppose $i \geq 1$, and the numbers l_i, k_i and the interval U_i are characterized. If $U_i \neq (\xi_{l_i},\ \xi_{l_i+1})$, then let $k_{i+1} \in \mathbb{N}$ be the smallest number such that

$$f^{k_{i+1}}(U_i) \cap \{\xi_i\}_{i=0}^N \neq \emptyset.$$

Let $0 < l_{i+1} < N$ be the smallest number that $\xi_{l_{i+1}} \in f^{k_{i+1}}(U_i)$ and

$$U_{i+1} := \begin{cases} \big(f^{k_{i+1}}(a_*),\ \xi_{l_{i+1}}\big), \ 2\big|\big(f^{k_{i+1}}(a_*),\ \xi_{l_{i+1}}\big)\big| > \big|f^{k_{i+1}}(U_i)\big| \\ \big(\xi_{l_{i+1}},\ f^{k_{i+1}}(b_*)\big), \ 2\big|\big(f^{k_{i+1}}(a_*),\ \xi_{l_{i+1}}\big)\big| \leq \big|f^{k_{i+1}}(U_i)\big| \\ \quad (\xi_{l_{i+1}},\ \xi_{l_{i+1}+1}), \ (\xi_{l_{i+1}},\ \xi_{l_{i+1}+1}) \subset f^{k_{i+1}}(U_i) \end{cases},$$

where $a_* := \inf U_i$ and $b_* := \sup U_i$. With this notation, we have $|U_{i+1}| > \frac{\lambda}{2}|U_i|$ and this gives $|U_{i+1}| > (\frac{\lambda}{2})^{i+1}|U_0|$. Hence with $\lambda > 2$, there is a natural number n such that $U_n = (\xi_{l_n},\ \xi_{l_n+1})$. On the other hand, f^{k_n} is a continuous and non decreasing map on the interval U_n, so $V_{n-1} := f^{-k_n}(U_n)$ is a non empty interval of U_{n-1}. For given $1 < j \leq n$, let $V_{n-j} := f^{-k_{n-j}}(V_{n-j+1})$ which is a non empty interval of U_{n-j}. Now take $k := \sum_{j=0}^n k_j + 1$ and $J := V_0$ that is a non empty interval of $(a,\ b)$. According to the above discussion, we have immediately the following result. The map f^k is continuous on J since $V_j \subset U_j$, and moreover $f^k(J) = (0,\ 1)$ which completes the proof of the lemma. □

In the following example, map f has derivative smaller or equal to $\sqrt{2}$ and it loses topological transitivity on $[0, 1]$.

Example 2.2. Let $0 < \epsilon < \frac{1}{4}$, and consider Baker-like map f with 3 branches as follows:

$$f(x) = \begin{cases} f_1^\epsilon(x) = \frac{x}{1-2\epsilon} + \frac{1-3\epsilon}{1-2\epsilon}, & 0 \leq x < \epsilon \\ f_2^\epsilon(x) = \frac{x}{1-2\epsilon} - \frac{\epsilon}{1-2\epsilon}, & \epsilon \leq x < 1-\epsilon \\ f_3^\epsilon(x) = \frac{x}{1-2\epsilon} - \frac{1-\epsilon}{1-2\epsilon}, & 1-\epsilon \leq x \leq 1 \end{cases}.$$

For all $x \in I$, we have $1 < f'(x) = \frac{1}{1-2\epsilon} < 2$. Since $\alpha = \frac{1-3\epsilon}{1-2\epsilon}$ and $\beta = \frac{\epsilon}{1-2\epsilon}$, it is easy to check that $\beta < \xi_1 < \frac{1}{2} < \alpha < \xi_2$ and $x = \frac{1}{2}$ is a repelling fixed point of f. $I_0 =?$. Thus $f(I_0) = [\alpha,\ \xi_2) \cup [\xi_2,\ 1]$. This gives $f^2(I_0) \subset f(I_0) \cup I_0$, since $\alpha < f(\alpha)$. Hence $f^n(I_0) \cap (\xi_1,\ \frac{1}{2}) = \emptyset$, for all $n \in \mathbb{N}$ and so f is not transitive in I.

In order to construct a PWS map f with more than three branches, let $\epsilon < \epsilon_1 < \frac{1}{4}$ satisfying $\frac{\epsilon_1}{1-2\epsilon_1} < \epsilon$. Then let f be a Baker-like map with N branches such that f_1^{ϵ}, $f_2^{\epsilon_1}$ and $f_3^{\epsilon_1}$ are three of them. The same argument can be posed to show that f is not transitive in I.

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FLIP BIFURCATION IN PIECEWISE SMOOTH MAPS

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ABSTRACT. In this paper a family of one-dimensional discontinuous invertible maps is considered and it is shown that the presence of the vertical and horizontal asymptotes in the function causes several nonstandard bifurcation structures. Also it is proved that depending on the exponent of the hyperbolic branch, the boundaries of a periodicity region are related either to subcritical, or supercritical, or degenerate flip bifurcations of the related cycle.

1. Introduction

In the last decade, *piecewise smooth* dynamical systems have received much attention from researchers of various theoretical and applied fields. In fact, a large number of applied models characterized by sharp switching between several states are ultimately described by nonsmooth systems, continuous or discontinuous. Known examples are switching electronic circuits, such as DC–DC converters, mechanical systems with impacts or stick–slip motion, relay control systems.

The essential feature of a piecewise smooth (PWS for short) system, both continuous and discontinuous, which may greatly influence the dynamics, is the presence of a so-called *switching manifold* at which the system's function changes its definition. Varying some parameter, an invariant set of the system can collide with such a manifold leading to a particular kind of bifurcations called *border collision bifurcation* (BCB henceforth). This term introduced by Nusse and Yorke ([7], [8]) is generally used to denote a bifurcation in discrete time systems, that is, in maps, caused by

Keywords: Piecewise smooth map, Border collision bifurcation, Flip bifurcation.
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a fixed point or periodic point colliding with a switching manifold (The term C-bifurcation is also used, see [1]). It is well known that a BCB can lead, for example, to a sharp transition from an n-cycle to a cycle of any other period or directly to chaos. Beside this, an important effect of a BCB in PWS maps is related to the fact that a cycle can appear/disappear not only due to a standard 'smooth' bifurcation, which are typical for smooth systems, but also due to a nonsmooth bifurcation, that is, due to a BCB ([2]).

Simplifying assumptions in applications often lead to piecewise linear systems which can possess quite complicated dynamics and have an obvious advantage for the investigation (see e.g. [1]). However, many applied models include specific nonlinearities which are essential for the description of their dynamics. In particular, quite a lot of attention has been paid to a model of impact oscillators defined by a map with the square-root nonlinearity, first described by Nordmark ([5], [6]). In the present paper we consider that one-dimensional (1D for short) piecewise smooth map defined by the two functions, $f_L(x)$ and $f_R(x)$, as follows:

$$x \longmapsto f(x) = \begin{cases} f_L(x) = ax + \mu & \text{if} \quad x \leq 0 \\ f_R(x) = bx^{-\gamma} + \mu & \text{if} \quad x > 0 \end{cases} \tag{1.1}$$

where a, b, γ and μ are real parameters. For $\gamma < 0$ then map f is *continuous* at the border point $x = 0$. In particular, the case $\gamma = -1/2$ related to the square-root nonlinearity is associated with Nordmark' systems and used to describe grazing bifurcations.

The map (1.1) is investigated also in [9] where, besides the cases with $\gamma < 0$, the authors considered also the discontinuous case with $\gamma > 0$. This leads to particular maps in which the function $f_R(x)$ defined on the right branch has a vertical asymptote at the discontinuity point $x = 0$. The particular case with $\gamma = 1/2$ is considered in [10], but without improving the results already given in [9].

It is well known that an important property of the map is its invertibility/noninvertibility. It is clear that if a 1D map, smooth or nonsmooth, is *invertible and continuous*, then its dynamics are rather trivial (the only possible asymptotic behavior of an orbit is either divergence or convergence to a fixed point or 2-cycle). In contrast, 1D *invertible discontinuous* maps can have more interesting dynamics. In fact, this subject up to now is not well studied, especially when the map has vertical and/or horizontal asymptotes, as the map (1.1) for $\gamma > 0$. Obviously due to invertibility the map cannot be chaotic. However, interesting bifurcation structures can be observed, which cannot occur in invertible continuous maps (see, e.g., [1], [2]).

2. Main results

The object of the present work is to investigate the dynamics of the one-dimensional (1D) discontinuous invertible piecewise smooth map coming from an application in engeneering defined as follows

$$x \longmapsto f(x) = \begin{cases} f_L(x) = ax + 1 & \text{if} \quad x \leq 0 \\ f_R(x) = bx^{-\gamma} + 1 & \text{if} \quad x > 0 \end{cases} \tag{2.1}$$

for $\gamma > 0$, $b < 0$, that is, in the discontinuous case. Here we consider the range of parameters in which the map is invertible, which is related to $a < 0$ ([4]). For the noninvertible case, when $a > 0$ (see [3]), the preliminary study performed in [9] has been developed.

To distinguish between different cycles of the map f, it is convenient to use their symbolic representations obtained associating the symbols L and R to the definition domains of the branches $f_L(x)$ and $f_R(x)$, respectively, that is, to the intervals $I_L = (-\infty, 0]$, $I_R = (0, +\infty)$. Then any orbit of f can be represented by its itinerary by using the symbol L when a point belongs to I_L and R when a point belongs to I_R. Before describing the basic cycles of the map f let us first discuss its fixed points.

Theorem 2.1. *Let $-\frac{1}{\gamma}\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1} < b < 0$. Then map f has two fixed points belonging to I_R, namely, $x = x_R^u$ which is unstable and $x = x_R^s$ which is stable, satisfying $x_R^u < x_R^* < x_R^s < 1$, where $x = x_R^* = \frac{\gamma}{\gamma+1} < 1$ is the fixed point of f at the parameter values satisfying*

$$\Phi_R: \quad b = b_R, \quad b_R = -\frac{1}{\gamma}\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1} \tag{2.2}$$

related to the fold bifurcation in $f_R(x)$.

Note that for any $\gamma > 0$ the value of b related to the fold bifurcation is larger than -1.

The rank-1 preimage of the discontinuity point $x = 0$ (which is unique due to invertibility of f), that is, the point $x = O^{-1}$ defined as $O^{-1} = (-b)^{\frac{1}{\gamma}}$ plays an important role for the dynamics of f :

- if $O^{-1} \geq 1$, that holds for $b \leq -1$, an orbit can have at most two consecutive points in I_R necessarily followed by just one point in I_L;
- if $O^{-1} < 1$, that holds for $-1 < b < 0$, an orbit can have several consecutive points in I_R followed by just one point in I_L.

These two different cases leading to different dynamics of map f are distinguished as *Range I* (for $b \leq -1$) and *Range II* (for $-1 < b < 0$).

We summarize the results related to the dynamics of map f in Range I in the following

Theorem 2.2. ***(Bifurcations of the*** $\mathbf{2}$***-cycle).*** *Let* $a < 0, \gamma > 0$ *and* $b < -1$*. Then map* f *given in (2.1) has a unique* 2*-cycle* $\{x_{0,2}, x_{1,2}\}$ *with* $x_{0,2} < 0$, $1 < x_{1,2} < (-b)^{\frac{1}{\gamma}}$ *which undergoes a flip bifurcation for parameter values satisfying*

$$\begin{aligned} b &= \frac{1}{a\gamma}\left(\gamma\frac{a+1}{\gamma-1}\right)^{\gamma+1} \quad && for \quad \gamma \neq 1 \\ a &= -1 && for \quad \gamma = 1 \end{aligned}$$

Moreover, this flip bifurcation is subcritical for $0 < \gamma < 1$*, degenerate for* $\gamma = 1$*, and supercritical for* $\gamma > 1$*. Increasing* b*, the* 2*-cycle* $\{x_{0,2}, x_{1,2}\}$ *disappears due to a regular BCB occurring at* $b = -1$*.*

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SOLVING LINEAR OPTIMAL CONTROL PROBLEMS WITH QUARTIC B-SPLINES

M. MATINFAR AND M. DOSTI

Abstract. In this article, we apply an impressive method for solving linear optimal control problem based on approximate solution. Hamilton-Jacobi equation are applied to linear optimal control problem convert to systems of first-order equations. The main idea of our scheme is approximation derivative. The results of scheme are made in pleasant agreement with analytic solutions. The accuracy of the proposed method is demonstrated by absolute error.

1. Introduction

Consider Hamiltonian for Equation (1.1) as

$$
\begin{aligned}
&\dot{x}(t) = Ax(t) + Bu(t),\ x(t_0) = x_0,\\
&J = \frac{1}{2}x^T(t_f)Sx(t_f) + \frac{1}{2}\int_{t_0}^{t_f}(x^TQx + u^TRu)dt,
\end{aligned}
\tag{1.1}
$$

where $x \in \mathbf{R}^n, y \in \mathbf{R}^m, A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times m}$. The control $u(t)$ is an admissible control if it is piecewise continuous in t for each t. Its values belong to a given closed subset U of $\mathbf{R}^+$. The input $u(t)$ is derived by minimizing the quadratic performance index J, where S and Q are symmetric positive semi-definite and $\mathbf{R}$ is symmetric positive definite. As we will point out in Section 2, we can achieve the optimal control law, $u^*(t) = -k(t)x(t)$, for system (1) by using Pontryagin's maximum principle [1].

Keywords: B-spline, Optimal Control.
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In this article, we try to find an approximate value for $k(t)$ by means of B-spline method.

We try to develope numerical method that is accurate and efficient than other method. In this work, we obtain an approximate value for k(t) by Quartic B-spline function. B-spline function is very useful in the study of approximation theory and its applications. The B-spline basis has been used to build up the approximation solutions for some differential equations.

2. Main results

Consider Hamiltonian for Equation (1.1) as

$$H(x,u,\lambda,t) = \frac{1}{2}(x^T Qx + u^T Ru) + \lambda^T(Ax + Bu), \tag{2.1}$$

where $\lambda \in \mathbf{R}^n$. is known as the costate variable. The optimal control by Pontryagin's maximum principle is obtained when $u(t)$ is not subjected to any constraint by solving

$$\frac{\partial H}{\partial u} = Ru + B^T\lambda = 0, \tag{2.2}$$

where λ is a solution of the adjoint equation

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -Qx - A^T\lambda, \tag{2.3}$$

with the terminal condition

$$\lambda(t_f) = Sx(t_f). \tag{2.4}$$

Thus, it follows from Equation (2.2) that the optimal control law is

$$u^*(t) = -R^{-1}B^T\lambda(t), \tag{2.5}$$

where $\lambda(t)$ is the solution of the Hamiltonian system[1]. The optimal control can be implemented as a closed loop if the solution to the adjoint Equation (2.3) is assumed like Equation (2.4) as a linear function of the states in the form

$$\lambda(t) = p(t)x(t),\ p(t_f) = S. \tag{2.6}$$

By using Equations above, since the above equation must hold for all nonzero x(t), p(t) must satisfy the matrix Riccati equation,

$$-\dot{p}(t) = p(t)A + A^Tp(t) + Q - p(t)BR^{-1}B^Tp(t),\ p(t_f) = S. \tag{2.7}$$

By considering Equations (2.5) and (2.6), we can see that the optimal control law is given by [1] $u^*(t) = -R^{-1}B^Tp(t)x(t)$, and as shown in [2], p(t) can be computed using the following relation

$$p(t) = W(t)V^{-1}(t), \tag{2.8}$$

TABLE 1. The values of $Q_i, Q_i^{'}, Q_i^{''}$.

t	t_{i-2}	t_{i-1}	t_i	t_{i+1}	t_{i+2}	t_{i+3}
Q_i	0	1	11	11	1	0
$Q_i^{'}$	0	$\frac{4}{h}$	$\frac{12}{h}$	$\frac{-12}{h}$	$\frac{-4}{h}$	0

where

$$\begin{pmatrix} \dot{V}(t) \\ \dot{W}(t) \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} V(t) \\ W(t) \end{pmatrix}, \tag{2.9}$$

with conditions, $V(t_f) = I$ and $W(t_f) = S$. This system of equations involving the first-order derivatives can be computed by employing the Quartic B-spline method in the following section. We consider a mesh $a = t_0 < t_1 < t_2 < ... < t_f = b$ as a uniform partition of the solution domain $a \leq t \leq b$ by the knots t_m and $h = t_m - t_{m-1}$, $m = 1, 2, ..., N$ throughout paper. The Q_m and its first derivative vanish outside the interval $[x_{m-1}, x_{m+2}]$. An interval $[x_m, x_{m+1}]$ is covered by three successive Quadratic B-splines. Now the solution of the problems are considered as follows:

$$V(t) = \sum_{m=-2}^{f+1} C_m Q_m(t), W(t) = \sum_{m=-2}^{f+1} D_m Q_m(t), \tag{2.10}$$

where C_m and $D_m, m = -2, ..., f+1$ are unknown Constant coefficients. Substituting Equation 2.10 into Equation 2.9 then we continue with the discretization of the system of Equation and the value of spline functions at the knots $t_{i=0}^{f}$ are determined using Table 1. we obtain

$$\begin{cases} (-\frac{4}{h} - A)C_{m-2} + (-\frac{12}{h} - 11A)C_{m-1} + (\frac{12}{h} - 11A)C_m + (\frac{4}{h} - A)C_{m+1} \\ +BR^{-1}B^T D_{m-2} + 11BR^{-1}B^T D_{m-1} + 11BR^{-1}B^T D_m + BR^{-1}B^T D_{m+1} = 0, \\ QC_{m-2} + 11QC_{m-1} + 11QC_m + QC_{m+1} + (-\frac{4}{h} + A^T)D_{m-2} + (-\frac{12}{h} + 11A^T) \\ D_{m-1} + (\frac{12}{h} + 11A^T)D_m + (\frac{4}{h} + A^T)D_{m+1} = 0, \end{cases} \tag{2.11}$$

with conditions

$$C_{f-2} + 11C_{f-1} + 11C_f + C_{f+1} = I, \tag{2.12}$$

$$D_{f-2} + 11D_{f-1} + 11D_f + D_{f+1} = S. \tag{2.13}$$

The matrix equation are obtained from left and right hand sides of Equation 2.11-2.13, respectively as follows

$$AZ = N, \tag{2.14}$$

where $Z = [C_{-2}, ..., C_f, C_{f+1}, D_{-2}, ..., D_f, D_{f+1}]^T$, $N = [0,0,0,0,...,0,0,I,S]^T$ and

$$A = \left[\begin{array}{ccccccccccccccc}
r_1 & r_2 & r_3 & r_4 & 0 & ... & 0 & r_5 & 11r_5 & 11r_5 & r_5 & 0 & 0 & ... & 0 \\
Q & 11Q & 11Q & Q & 0 & ... & 0 & r_6 & r_7 & r_8 & r_9 & 0 & 0 & ... & 0 \\
0 & r_1 & r_2 & r_3 & r_4 & 0 & ... & 0 & r_5 & 11r_5 & 11r_5 & r_5 & 0 & ... & 0 \\
0 & Q & 11Q & 11Q & Q & 0 & ... & 0 & r_6 & r_7 & r_8 & r_9 & 0 & ... & 0 \\
\vdots & & & & & & & & & & & & & & \\
0 & 0 & ... & r_1 & r_2 & r_3 & r_4 & 0 & ... & 0 & 0 & r_5 & 11r_5 & 11r_5 & r_5 \\
0 & 0 & ... & Q & 11Q & 11Q & Q & 0 & ... & 0 & 0 & r_6 & r_7 & r_8 & r_9 \\
0 & 0 & ... & 1 & 11 & 11 & 1 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\
0 & 0 & ... & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 1 & 11 & 11 & 1
\end{array}\right].$$

where

$r_1 = -\frac{4}{h} - A,\ r_2 = -\frac{12}{h} - 11A,\ r_3 = \frac{12}{h} - 11A,\ r_4 = \frac{4}{h} - A,\ r_5 = BR^{-1}B^T,\ r_6 = -\frac{4}{h} + A^T,\ r_7 = -\frac{12}{h} + 11A^T,\ r_8 = \frac{12}{h} + 11A^T,\ r_9 = \frac{4}{h} + A^T.$

The B-spline solution of given system is obtained by solving the above matrix equation.

Example 2.1. According to system (1.1), we have $A = -2, B = 1, S = 1, Q = 1, R = 1$ and $t_f = 1$. The analytical solution of this example is

$$k(t) = \frac{\sqrt{5}\cosh\sqrt{5}(1-t) - \sinh\sqrt{5}(1-t)}{\sqrt{5}\cosh\sqrt{5}(1-t) + 3\sinh\sqrt{5}(1-t)},$$

Accuracy of algorithms is shown by calculating the absolute error. The results are reported in table (2).

TABLE 2. Results at $h = 0.0005$ in Example 4.1.

	Exact Solution	Estimat Solution	Absolute Error
$t = 0$	0.2435	0.2435	2.8178×10^{-6}
$t = 0.4$	0.2811	0.2811	8.5718×10^{-6}
$t = 0.8$	0.5197	0.5197	5.6842×10^{-5}

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ENCODING A 2-TO-1 BAKER MAP

POUYA MEHDIPOUR AND NEEMIAS MARTINS

Abstract. We encode a 2-to-1 Baker map using the so called zip shift map, defined on an extended two-sided symbolic space. Moreover we define an $(m, l)-$Bernoulli transformation and show that the 2-to-1 Baker map is a $(2, 4)-$Bernoulli transformation. Indeed it is ergodic and strongly mixing. This encoding can be extended to an n-to-1 Baker map.

1. Introduction

Analysis of the mathematical models is one of the main tools to study chaotic dynamics. The transformation, known as the *Baker´s map*, provides a paradigm for deterministic chaos [1]. from a physical point of view, many invertible systems are known to be equivalent to the Baker's transformation. This process can be produced by flipping the coin, which is known as the Bernoulli Process. In ergodic theory, such transformations are known as Bernoulli transformations [3].

In [2] the $(m, l)-$Bernoulli transformations (finite-to-1) are introduced, which are the prototype version of the Bernoulli transformations, that are measure theoretically conjugated with a zip shift map instead of a shift homeomorphism. In this work, we encode a 2-to-1 Baker map and show that it is a $(2, 4)-$Bernoulli transformation. Indeed it is strongly mixing and an ergodic transformation.

2. An Extended Two-sided Shift map.

The zip shift space and the zip shift maps are defined in [1].The zip shift map is a local homeomorphism which is an extension of the known shift homeomorphism.

Let $Z = \{a_1, a_2, \cdots, a_m\}$ and $S = \{0, 1, \dots, l-1\}$, be two collections of symbols that $l \geq m$ and $\kappa : S \to Z$ a surjective factor map. Define, $\Sigma_S := \prod_{i=-\infty}^{+\infty} S_i$, where $S_i = S$.

Keywords: Deterministic Chaos, Bernoulli Transformations, Zip shift map.

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Consider the two-sided Bernoulli shift $\sigma : \Sigma_S \to \Sigma_S$. Then to any point $\bar{t} = (t_i) \in \Sigma_S$ correspond a point $\bar{x} = (x_i)_{i\in\mathbb{Z}}$, such that

$$x_i = \begin{cases} t_i \in S & \forall i \geq 0 \\ \kappa(t_i) \in Z & \forall i < 0. \end{cases} \tag{2.1}$$

Define $\Sigma = \Sigma_{Z,S} := \{x = (x_i)_{i\in\mathbb{Z}} \ : x_i \text{ satisfies}\,(2.2)\}$ and $\sigma_\kappa : \Sigma \to \Sigma$ such that,

$$x_i = \begin{cases} t_{i+1} & \text{if } i \neq -1, \\ \kappa(t_0) & \text{if } i = -1, \end{cases} \tag{2.2}$$

is the shift map defined on Σ. We call it the "Zip shift map". The pair (Σ, σ_κ) is called the *Zip shift space* on (m, l) symbols. We may abuse this notation to recall a $(m, l)-$zip shift map. By that, we mean the zip shift map defined on a zip shift space with $(m, l)-$symbols.

Example 2.1. Let $S = \{0, 1, 2, 3\}$ and $Z = \{a, b\}$ and a corresponded factor map $\kappa : S \to Z$ that $\kappa(0) = \kappa(2) = a$ and $\kappa(1) = \kappa(3) = b$. Let $(t_n) = (\cdots a\, b\, a\, b\, b\,\mathbf{.}\,1\,0\,1^{'}\,1\,1\,0'\,\cdots)$. One can verify that

$$\sigma_\kappa((\cdots a\, b\, a\, b\, b\,\mathbf{.}\,1\,0\,1^{'}\,1\,1\,0'\,\cdots)) = (\cdots a\, b\, a\, b\, b\, b\,\mathbf{.}\,0\,1^{'}\,1\,1\,0'\,\cdots).$$

We will show that this zip shift map is isomorphic (mod-0) with a 2-to-1 Baker map represented in Figure 1.

Let $\overline{d} : \Sigma \times \Sigma \to [0, 1]$ be given as, $\overline{d}(x, y) = \frac{1}{2^{M(x,y)}}$ for $x \neq y$ and $\overline{d}(x, y) = 0$ for $x = y$. with $M(x, y) = \min\{|i|;\, x_i \neq y_i\}$. Then $(\Sigma, \overline{d})$ is a metric space and induces a topology, on Σ. We equip the Σ with a $\sigma-$algebra and turn it into a probability measure space. Let define the basic cylinder sets as follows.

$$C_i^{s_i} = \{(t_n) \in \Sigma \,|\, t_i = s_i, i \in \mathbb{Z},\ s_i \in Z, \text{ if}, i < 0, \text{ and}\, s_i \in S, \text{ if}, i \geq 0\}. \tag{2.3}$$

The $C_i^{s_i}$ presents the set of all sequences, that have s_i in the $i-$th entry. For $i, n \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$, one can define a cylinder set as follows.

$$C_i^{s_i,\cdots,s_{i+k}} = \{(t_n) \in \Sigma : t_i = s_i, \cdots, t_{i+k} = s_{i+k}\} = C_i^{s_i} \cap C_{i+1}^{s_{i+1}} \cdots \cap C_{i+k}^{s_{i+k}},$$

The set of all such cylinder sets, generate a basis for the topology induced by $(\Sigma, \overline{d})$. Let $\mathcal{C}$ denotes the smallest $\sigma-$algebra (Borel $\sigma-$algebra) generated by the family of all cylinder sets. In order to change the $(\Sigma, \mathcal{C})$ into a probability measure space, one considers $P = (p_0, \ldots, p_{l-1})$ as a probability distribution on S and defines the measure of the basic cylinder sets as following.

$$\mu(C_i^{s_i}) = p_i, \ \ i \geq 0 \text{ and}, \ \sum_{i=0}^{l-1} p_i = 1.$$

Observe that, considering the factor map $\kappa : S \to Z$, for $i < 0$, $\mu(C_i^{s_i}) = kp_i$, with, $k = \#(\kappa^{-1}(s_i))$ and, $\sum_{i=1}^{m} p_i = 1$. Then,

$$\begin{aligned} \mu(C_i^{s_i,\cdots,s_{i+k}}) &= \mu(\{(t_n) \in \Sigma : t_i = s_i, \cdots, t_{i+k} = s_{i+k}\}) \\ &= \mu(C_i^{s_i} \cap C_{i+1}^{s_{i+1}} \cdots \cap C_{i+k}^{s_{i+k}}) \\ &= p_i. \ \ldots \ .p_{i+k} \end{aligned}$$

and $(\Sigma, \mathcal{C}, \mu)$ is a probability measure space.

Proposition 2.2. *Let $P = (p_0, \dots, p_{l-1})$ be a probability distribution for the elements of the set $S = \{0, 1, \dots, l-1\}$ and $(\Sigma, \mathcal{C}, \mu)$ a measure space defined on $\Sigma = \Sigma_{Z,S}$. Then $\sigma_\kappa : \Sigma \to \Sigma$ preserves the probability measure μ.*

Proposition 2.3. *Let $\sigma_\kappa : \Sigma \to \Sigma$ be the zip shift map. Then, $\sigma_\kappa(\Sigma) = \Sigma$ and σ_κ is a local homeomorphism.*

Proposition 2.4. *The zip shift map $\sigma_\kappa : \Sigma \to \Sigma$ is strongly mixing and ergodic.*

Definition 2.5 (**$(m, l)-$Bernoulli Transformations**)**.** The locally invertible measure preserving transformation $T : X \to X$ defined on a Lebesgue space $(X, \mathcal{B}, \mu)$ have $(m, l)-$*Bernoulli property* or is a $(m, l)-$*Bernoulli transformation* if it is isomorphic (mod 0) with an $(m, l)-$zip shift map.

Corollary 2.6. *As measure theoretical conjugacy preserves the strong mixing property, the $(m, l)-$Bernoulli transformations are strongly mixing and ergodic.*

FIGURE 1. A 2-to-1 Baker map

Example 2.7. Let $T : Q \to Q$, represents a 2-to-1 Baker map with $Q = [0, 1] \times [0, 1]$.

$$T(x, y) = \begin{cases} (4x, \frac{1}{2}y) & 0 \le x < \frac{1}{4},\ 0 \le y \le 1 \\ (4x-1, \frac{1}{2}y + \frac{1}{2}) & \frac{1}{4} \le x < \frac{1}{2},\ 0 \le y \le 1 \\ (4x-2, \frac{1}{2}y) & \frac{1}{2} \le x < \frac{3}{4},\ 0 \le y \le 1 \\ (4x-3, \frac{1}{2}y + \frac{1}{2}) & \frac{3}{4} \le x \le 1,\ 0 \le y \le 1. \end{cases} \tag{2.4}$$

Theorem 2.8. *The 2-to-1 Baker map is a $(2, 4)-$Bernoulli Transformation.*

Proof. The 2-to-1 Baker map is a transformation isomorphic (mod 0) with a $(2, 4)-$zip shift map defined on $\Sigma_{Z,S}$. Where, $Z = \{a, b\}$, $S = \{0, 1, 2, 3\}$. The factor code map $\kappa : S \to Z$ is defined as in Example 2.1. We give some numeric representation to a, b in which we can see $a = a_0$ and $b = b_1$. Let define the measurable transformation $\rho : \Sigma_{Z,S} \to Q$ as follows.

$$\rho((s_i)) = \Big(\sum_{i=1}^{\infty} \frac{s_{i-1}}{4^i} \mod 1, \sum_{i=1}^{\infty} \frac{s_{-i}}{2^i} \mod 1 \Big). \tag{2.5}$$

The map ρ corresponds any (s_i) almost uniquely to a point $s = (x, y) \in Q$. It turns out that the forward and backward orbits of some points have more than one representation, we represent them by ∂. Let $Q_\partial = \bigcup_{-\infty}^{\infty} T^n(\partial)$ and $\Sigma_\partial = \rho^{-1}(Q_\partial)$. It is not difficult to verify that $\sigma_\kappa(\Sigma_\partial) = \Sigma_\partial$ and $m(Q_\partial) = 0$. Let $\Sigma_0 = (\Sigma \setminus \Sigma_\partial)$ and $Q_0 = Q \setminus Q_\partial$. Consider the Lebesgue space $(Q, \overline{\mathcal{B}}, m)$ with the Borel $\sigma-$algebra generated by subsets of the form $[0, x] \times [0, y]$. Then the 2-to-1 Baker transformation preserves the Lebesgue measure. Moreover $\mu(\Sigma_0) = m(Q_0) = 1$ and $\rho : \Sigma_0 \to Q_0$ is a measure- preserving isomorphism. Observe that the push forward (use ρ^{-1}) of the Lebesgue measure defined on X_0 induces a probability measure μ on Σ_0. By Proposition 2.4, σ_κ preserves the probability measure μ. Furthermore, $\rho \circ \sigma_\kappa = \bar{T} \circ \rho$. To show that, one rewrites the 2-to-1 Baker map in the form of,

$$T(x,y) = \big(4x \mod 1, \frac{1}{2}(y + \kappa(s_0))\big).$$

Note that $k(s_0)$ is considered by numeric representations of $a, b \in Z$. Then,

$$\begin{aligned} T(\rho((s_i))) &= T\big(\sum_{i=1}^{\infty} \frac{s_{i-1}}{4^i} \mod 1, \sum_{i=1}^{\infty} \frac{s_{-i}}{2^i} \mod 1\big) \\ &= \big(4\sum_{i=1}^{\infty} \frac{s_{i-1}}{4^i} \mod 1, \frac{\kappa(s_0)}{2} + \sum_{i=1}^{\infty} \frac{s_{-i}}{2^{i+1}} \mod 1\big) \\ &= \big(\sum_{i=1}^{\infty} \frac{s_i}{4^i} \mod 1, \sum_{i=1}^{\infty} \frac{s_{-i+1}}{2^i} \mod 1\big) = \rho(\sigma_\kappa(s_i)), \end{aligned}$$

where, by statement 2.5,

$$\rho(\sigma_\kappa((s_i))) = \rho((s_{i+1})) = \big(\sum_{i=1}^{\infty} \frac{s_i}{4^i} \mod 1, \sum_{i=1}^{\infty} \frac{s_{-i+1}}{2^i} \mod 1\big).$$

Thus the 2-to-1 Baker map is a $(2,4)-$Bernoulli transformation. Indeed by Corollary 2.6 is strongly mixing and ergodic. □

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RECURRENT WEIGHTED SHIFTS AND SUBSPACE-HYPERCYCLICITY

MANSOOREH MOOSAPOOR

Abstract. In this paper, we consider the subspace-hypercyclicity of recurrent weighted shifts. By using weighted shifts, we show that there are subspace-hypercyclic operators that are not recurrent. Also, we show that if T is a recurrent bilateral weighted shift, then T^n and λT are subspace-hypercyclic for any natural number n and any scalar λ with $|\lambda| = 1$.

1. Introduction

Let X be a Banach space. Let T be a bounded linear operator or briefly an operator on X. We say that T is hypercyclic if there exists $x \in X$ such that $orb(T, x)$ is dense in X, where

$$orb(T, x) = \{x, Tx, T^2x, ...\}.$$

On a complete and separable metric space X, this is equivalent to saying that there exists $n \geq 0$ such that $T^{-n}U \cap V \neq \phi$ for any open and nonempty sets U and V. Hypercyclicity is a notable concept in the dynamical systems and related to the invariant subspace problem. Another important concept in dynamical systems is recurrence.

Keywords: recurrent operators, weighted shifts, hypercyclic operators, subspace-hypercyclic operators.

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Definition 1.1. ([3]) We say that an operator T is recurrent if for any nonempty open set U of X, there exists $n \geq 0$ such that $T^{-n}U \cap U \neq \phi$.

By Definition1.1, it is clear that hypercyclic operators are recurrent. So, the set of hypercyclic operators is a subset of the set of recurrence operators. A new concept in dynamical systems is subspace-hypercyclicity. We say an operator T is subspace-hypercyclic with respect to a closed and nontrivial subspace M, if there exists $x \in X$ such that

$$\overline{orb(T,x) \cap M} = M.$$

This notion is introduced by Madore and Martinez-Avendano in [5]. They show that subspace-hypercyclic operators exist only on infinite dimensional spaces. Bamerni, Kadets and Kilicman showed in [1] that hypercyclic operators are subspace-hypercyclic and by this, they answered to a question posed in [5].

One can see also [6] and [7] for more information about subspace-hypercyclic operators.

Theorem 1.2. *([1]) Every hypercyclic operator T on X is subspace-hypercyclic for a subspace M of X.*

Let $\{w_n\}_{n\in\mathbb{N}}$ be a bounded sequence of positive numbers. We say $T : l^p \to l^p$, $1 \leq p \leq \infty$, is a unilateral backward weighted shift with weight sequence $\{w_n\}$, if for any $n \geq 1$, we have

$$T(e_n) = w_n e_{n-1} \quad \text{and} \quad T(e_1) = 0,$$

where $\{e_n\}_{n\in\mathbb{N}}$ is the canonical basis. Similarly, we say T, is a bilateral backward weighted shift with weight sequence $\{w_n\}_{n\in\mathbb{Z}}$, if for any $n \in \mathbb{Z}$, we have $T(e_n) = w_n e_{n-1}$, where $\{e_n\}_{n\in\mathbb{Z}}$ is the canonical basis and $\{w_n\}_{n\in\mathbb{Z}}$ be a bounded sequence of positive numbers.

For any $1 \leq p < \infty$, recurrent weighted shifts exist on every l^p. But there is no recurrent weighted shift on $l^\infty(\mathbb{N})$ or $l^\infty(\mathbb{Z})$([3]).

Now, we interested in knowing relations between recurrent weighted shifts and subspace-hypercyclicity. In the next section, we state some theorems and corollaries about these relations.

2. Main results

We start this section with a theorem about unilateral weighted backward shifts.

Theorem 2.1. *([8]) Let T be a unilateral weighted backward shift and I be the identity operator. Then $I + T$ is hypercyclic.*

In the next theorem, it is showd that we make non recurrent operators by weighted backward shifts.

Theorem 2.2. *([3]) Let $T : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})$ be a unilateral weighted backward shift and I be the identity operator. Then $I + T$ is not recurrent.*

Now we state our first corollary.

Corollary 2.3. *There are subspace-hypercyclic operators that are not recurrent.*

Proof. Let T be a unilateral weighted backward shift on $l^\infty(\mathbb{N})$. By Theorem 2.1, $I+T$ is hypercyclic. So by Theorem 1.2, there is a closed and nontrivial subspace M of $l^\infty(\mathbb{N})$ such that $I+T$ is subspace-hypercyclic with respect to it. But by Theorem 2.2, $I+T$ is not recurrent and this completes the proof. □

Costakis and Parissis proved in [2] that for a bilateral weighted backward shift on $l^2(\mathbb{Z})$, hypercyclicity is equivalent to recurrency.

Theorem 2.4. *([2]) Let $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ be a bilateral weighted backward shift. Then T is hypercyclic if and only if T is recurrent.*

So, we can make subspace-hypercyclic operators by using recurrent operators as it is proved in the next corollary.

Corollary 2.5. *Let T be a recurrent bilateral weighted backward shift on $l^2(\mathbb{Z})$. Then T is subspace-hypercyclic.*

Proof. By Theorem 2.4, hypercyclicity and recurrency of T are equivalent. So, T is a hypercyclic operator. Now, By Theorem 1.2, we can conclude that T is subspace-hypercyclic. □

It is an open problem in subspace-hypercyclicity that can we conclude subspace-hypercyclicity of λT from subspace-hypercyclicity of T, where λ is an scalar with $|\lambda| = 1$? Or can we conclude subspace-hypercyclicity of T^n from subspace-hypercyclicity of T, where n is a natural number greater than 1?([5]) In the next theorem we partially answer to this questions.

Theorem 2.6. *Let $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ be a recurrent bilateral weighted backward shift. Then:*

(i) *λT is subspace-hypercyclic for any scalar λ with $|\lambda| = 1$.*
(ii) *T^n is subspace-hypercyclic for any $n \in \mathbb{N}$.*

Proof. Let T be a recurrent bilateral weighted shift. By Theorem 2.4, T is hypercyclic. Hence, λT and T^n are hypercyclic for any $n \in \mathbb{N}$ and any scalar λ with $|\lambda| = 1$ ([4]). Now, by Theorem 1.2, these operators are subspace-hypercyclic. □

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TERNARY DYNAMICAL SYSTEMS ON HILBERT C^*-MODULES

MAYSAM MOSADEQ

ABSTRACT. Let M and N be Hilbert A-modules, $\varphi : A \to B$ be an isomorphism of C^*-algebras and $T : M \to N$ be a bijective linear operator. We define the Hilbert B-module structure implemented by (φ, T) on N. Denoting by $N_{(\varphi,T)}$ the Hilbert B-module N implemented by (φ, T), we investigate some relations between M and the implemented Hilbert B-module $N_{(\varphi,T)}$. We also, introduce the concepts of ternary derivation and ternary dynamical system and correspond to each ternary dynamical system $\{\alpha_t\}_{t\in\mathbb{R}}$ on M, a dynamical system $\{\beta_t\}_{t\in\mathbb{R}}$ on $N_{(\varphi,T)}$ and a unique ternary derivation δ_2 of $N_{(\varphi,T)}$ such that δ_2 is the generator of $\{\beta_t\}_{t\in\mathbb{R}}$. Furthermore, we show that if M is full, then there are a unique C^*-dynamics $\{\psi_t\}_{t\in\mathbb{R}}$ on B and a unique derivation d_2 of B such that d_2 is the generator of $\{\psi_t\}_{t\in\mathbb{R}}$ and consequently, δ_2 is a d_2-derivation.

1. Introduction

A Hilbert C^*-module over the C^*-algebra A is an algebraic left A-module M equipped with an A-valued inner product $\langle .,. \rangle$, which is A-linear in the first and conjugate linear in the second variable for which M is a Banach space with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. The Hilbert module M is called full if the closed linear span $\langle M, M\rangle$ of all elements of the form $\langle x, y\rangle$ $(x, y \in M)$ is equal to A. For example, each C^*-algebra$A$ is a left Hilbert A-module by $< a, b >= ab^*$. Additionally, if $a \in A$

Keywords: C^*-Dynamical systems, generalized derivation, Hilbert C^*-module, ternary derivation, one parameter group.

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and $\{e_\lambda\}$ is an approximate identity for A, then $\lim_\lambda < a, e_\lambda >= \lim_\lambda ae_\lambda^* = a$. Hence, $< M, M >= A$ and therefore, M is a full Hilbert A-module.

Throughout the paper, let M be a (left) Hilbert A-module. We know that a map $T : M \to M$ is adjointable if there exists a map $T^* : M \to M$, called the adjoint map of T, that fulfills $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in M$. This definition implies that each adjointable map is a bounded A-linear operator (see [4, p. 8]). The referred to [4] for more details on Hilbert C^*-modules.

The classical C^*-dynamical systems are expressed by means of strongly continuous one parameter groups of $*$-automorphisms on C^*-algebras. On the other hand, the infinitesimal generator d of a C^*-dynamical system is a derivation. Recently, some generalized notions of derivations have been investigated in the the setting of Hilbert C^*-modules in which it can be pointed to "ternary derivations" and "generalized derivations" (see [1, 2, 7] and references therein).

A *ternary derivation* is a densely defined linear map δ of M into M such that $\delta(\langle x, y\rangle z) = \langle x, y\rangle\delta(z) + (\langle\delta(x), y\rangle + \langle x, \delta(y)\rangle)z$ for each $x, y, z \in D(\delta)$ where $D(\delta)$ is a ternary subalgebra of M in the sense that $D(\delta)$ is invariant under the ternary product $(x, y, z) \to \langle x, y\rangle z$ (i.e. $\langle x, y\rangle z \in D(\delta)$ for every $x, y, z \in D(\delta)$). As an example, let δ be an adjointable operator with the adjoint $-\delta$. Then, δ is a ternary derivation.

As another extension of derivations, it can be pointed to the notion of "generalized derivations" as follows.

A linear mapping δ from a dense subspace $D(\delta)$ of a full Hilbert A-module M into M is called a *generalized derivation* if there exists a mapping d from a dense subalgebra $D(d)$ of A into A for which $D(\delta)$ is an algebraic left $D(d)$-module, $\delta(ax) = a\delta(x) + d(a)x$ for each $x \in D(\delta)$ and $a \in D(d)$.

This definition implies that d is a derivation, [1]. It is notable that every ternary derivation δ of full Hilbert A-module M is also a generalized derivation (see [2, Theorem 3.5]).

In each case of generalization of derivation, a noted point drawing the attention of analysts is trying to represent a suitable dynamical system whose generator is exactly the desired extended derivation as well as being an extension of a C^*-dynamical system. Some approaches to preparing new dynamical systems and their applications have been explained in [1, 5, 6, 7, 9] and references therein.

In the remainder of this section, we introduce some important classes of operators between Hilbert modules.

Let $\varphi : A \to B$ be an injective morphism of C^*-algebras. A surjective map $T : M \to N$ is called *φ-unitary* if T is a *φ-morphism* in the sense that $\langle T(x), T(y)\rangle = \varphi(\langle x, y\rangle)$ for all $x, y \in M$.

It is notable that each φ-unitary map T is necessarily a linear operator, a φ-module map and an isometry. Moreover, if N is a full Hilbert B-module, then φ is surjective and so it is an isomorphism (see [3, Remark 2.9]).

Following [2], we call a bijective linear map $T : M \to N$ a *ternary isomorphism* if $T(\langle x, y\rangle z) = \langle T(x), T(y)\rangle T(z)$ for all $x, y, z \in M$. As an example of a ternary isomorphism, let $T : M \to N$ be a φ-unitary operator. Hence, T is a bijection and $T(\langle x, y\rangle z) = \varphi(\langle x, y\rangle)T(z) = \langle T(x), T(y)\langle T(z)$ for all $x, y, z \in M$. We denote by $TU(M)$ the group of all bounded ternary automorphisms of M onto M.

2. The implemented Hilbert modules and their associated ternary dynamical systems

Theorem 2.1. *Let M and N be Hilbert A-modules, $\varphi : A \to B$ be a linear isomorphism of C^*-algebras and $T : M \to N$ be a bijective linear operator. We consider the module action $b.y := T(\varphi^{-1}(b).T^{-1}(y))$ on N. Then, N via the inner product $<< y, w >>:= \varphi(< T^{-1}(y), T^{-1}(w) >)$ is a Hilbert B-module called, the Hilbert B-module implemented by (φ, T) or briefly the implemented Hilbert B-module. We denote the so-called implemented Hilbert B-module by $N_{(\varphi,T)}$.*

Lemma 2.2. *Let $N_{(\varphi,T)}$ be the implemented Hilbert B-module. Then, T is a φ-morphism. More precisely, T is a φ-unitary operator and consequently, is a ternary isomorphism.*

Lemma 2.3. *M is a full Hilbert A-module if and only if the implemented Hilbert B-module $N_{(\varphi,T)}$ is full.*

Corollary 2.4. *Let M be a full Hilbert A-module and $b \in B$. If $by = 0$ for every $y \in N_{(\varphi,T)}$, then $b = 0$.*

Theorem 2.5. *If D be a ternary subalgebra of a Hilbert A-module M. Then, $T(D)$ is a ternary subalgebra of the implemented Hilbert B-module $N_{(\varphi,T)}$. Moreover, if δ_1 is a ternary derivation of full Hilbert A-module M, then the mapping $\delta_2 : T\left(D(\delta_1)\right) \subseteq N_{(\varphi,T)} \to N_{(\varphi,T)}$ defined by $\delta_2\left(T(x)\right) := T\left(\delta_1(x)\right)$ is a ternary derivation and consequently, is also a generalized derivation.*

Definition 2.6. A *one parameter group* $\{\alpha_t\}_{t\in\mathbb{R}}$ of bounded ternary automorphisms on M is a group homomorphism $t \mapsto \alpha_t$ from the additive group $\mathbb{R}$ into the group $TU(M)$ of bounded ternary automorphisms on M. The one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ is called strongly continuous if $\lim\limits_{t\to 0} \alpha_t(x) = x$, for all $x \in M$. In this case, $\{\alpha_t\}_{t\in\mathbb{R}}$ is called a *ternary dynamical system* on M. We define the generator δ_1 of the ternary dynamical system $\{\alpha_t\}_{t\in\mathbb{R}}$ as a mapping $\delta_1 : D(\delta_1) \subseteq M \to M$ such that $\delta_1(x) = \lim\limits_{t\to 0} \dfrac{\alpha_t(x) - x}{t}$, where $D(\delta) = \{x \in M : \lim\limits_{t\to 0} \dfrac{\alpha_t(x) - x}{t} exists\}$.

Before we state the main result of this paper, we need the following useful theorem.

Theorem 2.7. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a ternary dynamical system on a Hilbert A-module M and δ_1 be its generator. Then, $D(\delta_1)$ is a dense ternary subalgebra of M and δ_1 is a ternary derivation. In particular, if M is full, then there exists a unique derivation d_1 on the dense subalgebra $< D(\delta_1), D(\delta_1) >$ of A such that d_1 turns δ_1 into a d_1 derivation.*

Now, we are ready to present the following main result.

Theorem 2.8. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a ternary dynamical system on M and $\delta_1 : D(\delta_1) \subseteq M \to M$ be its generator. Then, there one can associate with $\{\alpha_t\}_{t\in\mathbb{R}}$ a ternary dynamical system $\{\beta_t\}_{t\in\mathbb{R}}$ on $N_{(\varphi,T)}$ and a unique ternary derivation δ_2 of $N_{(\varphi,T)}$ such that δ_2 is the generator of $\{\beta_t\}_{t\in\mathbb{R}}$. Furthermore, if M is full, then there are a unique C^*-dynamics $\{\psi_t\}_{t\in\mathbb{R}}$ on B and a unique derivation d_2 of B such that d_2 is the generator of $\{\psi_t\}_{t\in\mathbb{R}}$ and consequently, δ_2 is a d_2-derivation.*

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CONFLATING TWO KINDS OF DERIVATTIONS TO CONSTRUCT THE GENERATOR OF A NEW DYNAMICAL SYSTEM

MAYSAM MOSADEQ

Abstract. In this paper, we follow to convolve the notions of generalized derivations and σ-derivations to construct a new form of derivations entitled "hyper generalized derivation". Inspiring the structures of generalized dynamics and σ-dynamics, we also demonstrate a suitable dynamical system whose infinitesimal generator is exactly the so-called hyper generalized derivation as well as being an extension of generalized dynamics and σ-dynamics.

1. Introduction

Let A be a Banach space. A one parameter family $\{\varphi_t\}_{t\in\mathbb{R}}$ of bounded linear operators on A is called a one parameter group if $\varphi_0 = I_A$ and $\varphi_{t+s} = \varphi_t\varphi_s$ for every $t, s \in \mathbb{R}$. The one parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ is called uniformly (resp. strongly) continuous if $\lim\limits_{t\to 0} \| \varphi_t - I \| = 0$ (resp. $\lim\limits_{t\to 0} \varphi_t(a) = I(a)$, for each $a \in A$). The infinitesimal generator d of the one parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ is a mapping $d : D(d) \subseteq A \to A$ such that $d(a) = \lim\limits_{t\to 0} \dfrac{\varphi_t(a) - a}{t}$ where $D(d) = \{a \in A : \lim\limits_{t\to 0} \dfrac{\varphi_t(a) - a}{t} exists\}$.

The classical C^*-dynamical systems are expressed by means of strongly continuous one parameter groups of $*$-automorphisms on C^*-algebras. Also, the infinitesimal generator d of a C^*-dynamical system is a $*$-derivation. Recently, various generalized notions of derivations have been investigated in the context of Banach algebras in

Keywords: C^*-Dynamical systems, generalized derivation, generalize module map, σ-derivation, one parameter group.

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which it can be pointed to "generalized derivations" and " σ-derivations". From now to the end of this paper, let A be a Banach algebra and M be an A-bimodule.

A linear mapping $\delta : M \to M$ is said to be a *bi-generalized derivation* if there are derivations $d_1, d_2 : A \to A$ such that $\delta(ax) = a\delta(x) + d_1(a)x$ and $\delta(xa) = \delta(x)a + xd_2(a) \quad (x \in M,\, a \in A)$. We call δ a (d_1, d_2)-derivation. As an example, let $c, b \in A$ and define $\delta_{c,b}(x) = cx - xb$. Then, $\delta_{c,b}$ is a (d_c, d_b)-derivation, called *inner generalized derivation*, where $d_c(a) = ca - ac$, $d_b(a) = ba - ab$ (see [1, 9] and references therein).

As another extension of derivations, it can be pointed to the notion of σ-derivation as follows.

Let $\sigma : A \to A$ be a linear operator. A linear map $d : A \to A$ is called a *σ-derivation* if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in A$. For instance, let σ be an endomorphism and c be an arbitrary element of A. Then, the mapping $d_c : A \to A$ defined by $d_c(a) = [c, \sigma(a)]$ is a σ-derivation which is called *inner*.

The reader is referred to [2, 4, 8, 10] for more information on σ-derivations.

In each case of generalization of derivation, a noted point drawing the attention of analysts is trying to represent a suitable dynamical system whose generator is exactly the desired extended derivation as well as being an extension of a C^*-dynamic. Some approaches to preparing new dynamical systems have been explained in [6, 7, 8, 9].

In order to construct an extension of a C^*-dynamical system on modules associated with generalized derivations it is enough to manifest an extension of automorphisms, called generalized module maps.

A *generalized module map*, is a linear mapping $\alpha : M \to M$ that fulfills $\alpha(ax) = \varphi(a)\alpha(x)$ and $\alpha(xa) = \alpha(x)\psi(a)$ $(a \in A,\, x \in M)$ for some homomorphisms $\varphi, \psi : A \to A$. This map is called a *(φ, ψ)-module map*. In the case that φ and ψ are automorphisms and α is a bijection, then α is said to be a *generalized isomorphism* or more precisely, a *(φ, ψ)-isomorphism* [9].

Let $\{\varphi_t\}_{t\in\mathbb{R}}$ and $\{\psi_t\}_{t\in\mathbb{R}}$ be C^*-dynamical systems on a C^*-algebra A. Suppose that $\{\alpha_t\}_{t\in\mathbb{R}}$ is a uniformly continuous one parameter group of bounded linear operators on M. If α_t is a (φ_t, ψ_t)-isomorphism for each $t \in R$, then it is called a *(φ_t, ψ_t)dynamics* or a *generalized dynamics* on M.

It has been proved in [9] that generalized derivations are appeared as the generators of generalized dynamics on modules.

Before we describe the structure of dynamical systems whose generator are σ-derivations, we need the following useful concept.

Let A be a Banach space and σ be a bounded linear operator on A. A one parameter family $\{\varphi_t\}_{t\in\mathbb{R}}$ of bounded linear operators on A is called a *σ-one parameter group* if $\varphi_0 = \sigma$ and $\varphi_{s+t} = \varphi_s\varphi_t$ $(s, t \in \mathbb{R})$. The σ-one parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ is called uniformly continuous if $\lim_{t\to 0} \| \varphi_t - \sigma \| = 0$. The generator d of the σ-one parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ is a mapping $d : D(d) \subseteq A \to A$ such that $d(a) = \lim_{t\to 0} \frac{\varphi_t(a) - \sigma(a)}{t}$, where $D(d) = \{a \in A : \lim_{t\to 0} \frac{\varphi_t(a) - \sigma(a)}{t} exists\}$.

σ-one parameter semigroups were introduced by Janfada in 2008. The reader is referred to [3] for more details.

If $\{\varphi_t\}_{t\in\mathbb{R}}$ is a σ-one parameter group, then one can easily check that φ_t is σ-bijective in the sense that $\varphi_t(A) = \sigma(A)$ and $ker(\alpha_t) = ker(\sigma)$ for each $t \in \mathbb{R}$.

Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be a uniformly continuous σ-one parameter group of linear $*$-endomorphisms on a C^*-algebra A. An immediate consequence of the σ-bijective feature of $\{\varphi_t\}_{t\in\mathbb{R}}$ is that by substituting $\sigma = I_A$, we obtain a C^*-dynamical system. In 2013, the author introduced the aforementioned extension of C^*-dynamical systems and called it a σ-C^*-dynamics. It has been proved in [5] that the generator d of the σ-C^*-dynamics $\{\varphi_t\}_{t\in\mathbb{R}}$ is a σ-derivation.

2. HYPER GENERALIZED DYNAMICAL SYSTEMS ON MODULES AND THEIR GENERATORS

Throughout this section, let $\sigma_j : A \to A$ $(j = 1, 2)$ be a linear endomorphism.

Definition 2.1. A linear mapping $\delta : M \to M$ is said to be a *hyper generalized derivation* if there are a σ_j-derivation $d_j : A \to A$ $(j = 1, 2)$ and a linear (σ_1, σ_2)-module map $\gamma : M \to M$ such that $\delta(ax) = \sigma_1(a)\delta(x) + d_1(a)\gamma(x)$ and $\delta(xa) = \delta(x)\sigma_2(a) + d_2(a)\gamma(x)$ for each $a \in A$, $x \in M$.

In the case that $\sigma_j = I_A$ $(j = 1, 2)$ and $\gamma = I_M$, the hyper generalized derivation δ is nothing more than a bi-generalized derivation. Also, since the Banach algebra A is an A-module, considering $M := A$, $\gamma = \sigma_j$ and $\delta := d_j$ $(j = 1, 2)$, it follows that δ is indeed a σ-derivation. Therefore, our notion covers the both concepts of generalized derivation and σ-derivation.

Example 2.2. Let b, c be two arbitrary elements of A and $\gamma : M \to M$ be a linear (σ_1, σ_2)-module map Then, the mapping δ defined by $\delta(x) = c\gamma(x) - \gamma(x)b$ is a hyper generalized derivation which is called *inner*.

Definition 2.3. Let $\{\varphi_t\}_{t\in\mathbb{R}}$ (resp. $\{\psi_t\}_{t\in\mathbb{R}}$) be a σ_1-C^*-dynamics (resp. σ_2-C^*-dynamics) on a C^*-algebra A and γ be a bounded linear (σ_1, σ_2)-module map on M. A *hyper generalized dynamical system* on M is a uniformly continuous γ-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ on M such that α_t is a (φ_t, ψ_t)-module map for each $t \in R$.

Theorem 2.4. *The generator δ of a hyper generalized dynamical system is a hyper generalized derivation.*

Definition 2.5. A generalized module map $\alpha : M \to M$ is said to be a *hyper quasi inner* if there exist unitary elements u, v in a C^*-algebra A and a (σ_1, σ_2)-module map $\gamma : M \to M$ such that $\alpha(x) = u\gamma(x)v^*$ for each $x \in M$.

Theorem 2.6. *Let $\{u_t\}_{t\in\mathbb{R}}$ and $\{v_t\}_{t\in\mathbb{R}}$ be uniformly continuous one parameter groups of unitary elements in a C^*-algebra A and $\alpha_t : M \to M$ be the γ-one parameter group $\alpha_t(x) = u_t\gamma(x)v_t^*$ with the generator δ. Then, δ is an inner hyper generalized derivation.*

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ON THE SPARSE MULTI-SCALE SOLUTION OF TIME-VARYING DELAY SYSTEMS

BEHZAD NEMATI SARAY AND MEHRDAD LAKESTANI

ABSTRACT. An efficient algorithm based on wavelet Galerkin method is proposed for solving the time-varying delay systems. According to the useful properties of Alpert's multiwavelets, a time-varying system is reduced to the sparse linear system of algebraic equations. So the computational cost is reduced using thresholding. The results illustrate, by selecting the appropriate threshold while the number of nonzero coefficients reduces and the speed of process increases, the error will not be less than a certain amount. The analysis of convergence have been investigated and the efficiency and accuracy of the proposed method are illustrated by several examples.

1. INTRODUCTION

During this work we will describe an algorithm for the numerical solution of Delay system based on Alpert's multiwavelets and their sparseness property. This method aims to solve any equation similar to the form

$$\begin{aligned} &\mathbf{x}'(t) = e(t)\mathbf{x}(t) + f(t)\mathbf{x}(t-\eta) + g(t)\mathbf{u}(t), \qquad 0 \le t \le 1,\\ &\mathbf{x}(0) = \mathbf{x}_0,\\ &\mathbf{x}(t) = \chi(t), \qquad -\eta \le t < 0 \end{aligned} \tag{1.1}$$

where $\mathbf{u}(t) \in \mathbb{R}^n$, $e(t)$, $f(t)$ and $g(t)$ are known matrices of appropriate dimensions [2] and $\mathbf{x}(t) \in \mathbb{R}^m$ is unknown. Also $\chi(t)$ is a known function and $\mathbf{x}_0$ is a constant specified vector.

Keywords: Wavelet Galerkin method, Delay systems, Multiwavelets.

AMS Mathematical Subject Classification [2010]: 65T60, 34K28, 65L60.

2. Wavelet Galerkin Method (WGM)

At first we integrate both sides of jth equation of (1.1) for $j = 1, \cdots, m$ in the interval $[0, t]$ as

(2.1)
$$\left[\mathbf{x}(t) - \mathbf{x}(0) - \int_0^t \left(e(s)\mathbf{x}(s) + f(s)\mathbf{x}(s-\eta) + g(s)\mathbf{u}(s)\right) ds\right]_j = 0, \qquad 0 \le t \le 1,$$

here we use the $[.]_j$ symbol to show the jth equation. We will be able to expand $\upsilon_{1,j}(t) = [e(t)\mathbf{x}(t)]_j$, $\upsilon_{2,j}(t) = [f(t)\mathbf{x}(t-\nu)]_j$ and $\upsilon_{3,j}(t) = [g(t)\mathbf{u}(t)]_j$ by multiwavelets as follows

$$\mathcal{P}_J^r(\upsilon_{i,j})(t) \approx E_{ij}^T \Phi_J^r(t) = E_{ij}^T T_J^{-1} \Psi_J^r(t), \qquad i = 1, 2, 3, \quad j = 1, \cdots, m. \tag{2.2}$$

Applying the operational matrix of integration I_ϕ for multi-scaling functions $\Phi_J^r(t)$ and I_ψ for multiwavelets $\Psi_J^r(t)$ introduced in [3], we can write

(2.3)
$$\mathcal{P}_J^r\left(\int_0^x \upsilon_{i,j}\right)(t) \approx E_{ij}^T I_\phi \Phi_J^r(t) = E_{ij}^T I_\psi T_J^{-1} \Psi_J^r(t), \qquad i = 1, 2, 3, \quad j = 1, \cdots, m,$$

where E_{ij} for $i = 1, 2, 3$, and $j = 1, \cdots m$ are $n \times 1$ with $n = r2^J$ vectors. Also we have assumed that

$$\mathcal{P}_J^r(\mathbf{x}_j)(t) = X_j^T \Psi_J^r(t), \quad j = 1, \cdots, m, \tag{2.4}$$

where X_j is $n \times 1$ vector. Equation (2.1) is linear and so we can write

$$E_{ij}^T I_\psi T_J^{-1} = X_j^T \Upsilon_{ij}, \qquad i = 1, 2, \tag{2.5}$$

where Υ_{ij} for $i = 1, 2$ and $j = 1, \cdots, m$ are $n \times n$ matrices. For each equation of (2.1), we have

$$X_j^T \left(I_n - \Upsilon_{1j} - \Upsilon_{2j}\right) \Psi_J^r(t) = \left(X_{0_j}^T + E_{3j}^T I_\psi T_J^{-1}\right) \Psi_J^r(t), \quad j = 1, \cdots, m, \tag{2.6}$$

where $[\mathbf{x}(0)]_j = X_{0_j}^T \Psi_J^r(t)$ and I_n is a identity matrix. To use Galerkin method, we multiply (2.6) by $\Psi_J^r(t)$ and integrate over $[0, 1]$. So with the orthonormality property of our bases, we obtain

$$X_j^T \left(I_n - \Upsilon_{1j} - \Upsilon_{2j}\right) = X_{0_j}^T + E_{3j}^T I_\psi T_J^{-1}, \quad j = 1, \cdots, m. \tag{2.7}$$

Let the $mn \times 1$ matrix X be partitioned into the $1 \times n$ blocks X_j for $j = 1, \cdots, m$. Writing $\mathcal{M}_i = [\Upsilon_{i1}, \Upsilon_{i2}, \cdots, \Upsilon_{im}]$ for $i = 1, 2$ and using (2.1)-(2.5) and Galerkin method, one can write after rearranging

$$X^T \underbrace{\left(I_{mn} - \mathcal{M}_1 - \mathcal{M}_2\right)}_{\mathcal{M}} = \underbrace{{X_0}^T + E \otimes I_\psi}_{\mathcal{N}}, \tag{2.8}$$

TABLE 1. The percentage of sparsity and L_2 error.

		$\varepsilon = 0$		$\varepsilon = 10^{-6}$		$\varepsilon = 10^{-4}$	
r	J	S_ε	L_2	S_ε	L_2	S_ε	L_2
	3	0	$8.01e-9$	65.75%	$8.01e-9$	68.63%	$2.84e-6$
5	4	0	$2.51e-10$	83.19%	$8.28e-8$	85.91%	$2.94e-6$
	5	0	$7.85e-12$	91.20%	$8.57e-8$	93.70%	$2.94e-6$
	3	0	$4.28e-13$	69.36%	$1.69e-7$	73.73%	$3.28e-5$
7	4	0	$3.35e-15$	86.54%	$1.69e-7$	89.41%	$3.28e-5$
	5	0	$2.62e-17$	93.51%	$1.69e-7$	95.41%	$3.28e-5$

where I_{mn} is a identity matrix, $E = [E_{31}^T, E_{32}^T, \cdots E_{3m}^T]$ and X_0 is obtained from initial condition (1.1) as

$$\mathbf{x}_0 = {X_0}^T \otimes \Psi_J^r(t),$$

where $\otimes$ denotes Kronecker product.

Therefore, we have

$$X^T \mathcal{M} = \mathcal{N}, \tag{2.9}$$

where $\mathcal{M}$ and $\mathcal{N}$ are the $mn \times mn$ matrices. The solution to this system of linear equations gives rise to obtain the approximate solution of the equation (2.1).

Theorem 2.1. *Assume* $\mathbf{x}$ *and* $\mathbf{x}_J$ *are the exact and approximate solutions of linear system (2.1), respectively and* $\mathbf{e}_J = \mathbf{x} - \mathbf{x}_J$. *Also presume that* $e(t)$, $f(t)$, $g(t)$ *and* $\mathbf{u}(t)$ *are known matrices functions which are integrable on* $[0, 1]$. *Then one has*

$$\|\mathbf{e}(t)\|_2 \leq \xi 2^{J(1/2-r)} \frac{2\sqrt{r}}{4^r r!} \sup_{t\in[0,1]} |\mathbf{x}^{(r)}(t)|, \quad 0 \leq t \leq 1, \tag{2.10}$$

where η *and* ξ *are positive constant. Consequently,* $\mathbf{e}(t) \to 0$ *when* $J \to \infty$.

3. Numerical examples

Consider the time-varying delay equation

$$x'(t) = -x(t) - 2x(t - \frac{1}{4}) + 2u(t - \frac{1}{4}), \tag{3.1}$$

$$x(t) = u(t) = 0, \quad -\frac{1}{4} \leq t \leq 0, \quad u(t) = 1, \qquad t > 0. \tag{3.2}$$

The exact solution for this example is reported in [1]. In Table 3, we demonstrate the numerical results with different values of r, J, and ε. Table 3, shows a comparison of the errors in some points computed using the proposed method ($\|e_J\|$ defined in Theorem 3.1) with some other methods. Fig. 1, shows the sparse matrix together with its histogram (which shows the distribution of entries of sparse matrix) for $r = 6$ and $J = 5$ by thresholding parameter $\varepsilon = 10^{-6}$.

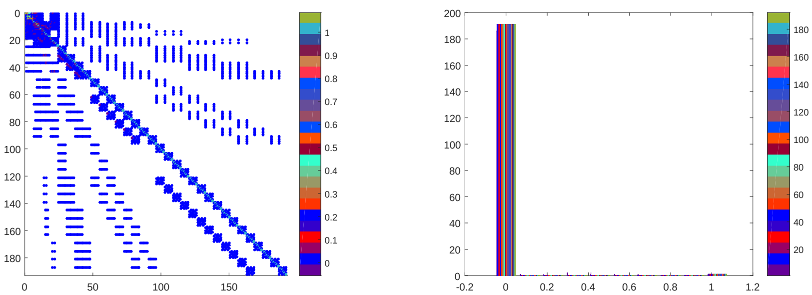

FIGURE 1. The histogram (right) and sparse matrix (left) after thresholding with $\varepsilon = 10^{-4}$, $r = 6$ and $J = 5$.

TABLE 2. Comparison of the global error $\mathbf{e}_J$.

	$t = 0.6$	$t = 0.9$	$t = 1$
[2](first scheme)	$1.09e-7$	$1.87e-6$	$6.00e-7$
[2](second scheme)	$3.07e-8$	$9.67e-7$	$1.00e-7$
[?]	$6.67e-10$	$3.25e-4$	$1.25e-3$
$WGM(r = 7, J = 4)$	$1.15e-39$	$1.58e-38$	$6.73e-38$

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TOPOLOGICAL SHADOWING FOR INDUCED MAPS OF UNIFORM HYPERSPACES

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Abstract. In this note we investigate the relation between dynamical properties of a map on a topological space and the induced map on topological hyperspace.

1. Intrduction

The shadowing problem is related to the following question: Under which conditions, for any pseudo-orbit of f a closed real orbit can be found? A dynamical system on a non-compact metric space may have the shadowing property with respect to one metric but not with respect to another one that induces the same topology. Hence one prefers to have a theory that is independent of any choice of the metric. Recently, many authors have extended various notions of dynamical properties on metric spaces for homeomorphisms on uniform spaces [4, 1, 2]. In this note we investigate shadowing properties of induced dynamical systems.

2. Preliminaries

Let X be a non-empty set. A *uniformity* $\mathscr{U}$ on the set X is a subset of the product $X \times X$ that satisfies ([3], Chapetr 8):

U1) for any $E_1, E_2 \in \mathscr{U}$, the intersection $E_1 \cap E_2$ is also contained in $\mathscr{U}$, and if $E_1 \subset E_2$ and $E_1 \in \mathscr{U}$, then $E_2 \in \mathscr{U}$;

Keywords: topological shadowing, topological h-shadowing, hyperspace.
AMS Mathematical Subject Classification [2010]: 37B65.

U2) every set $E \in \mathscr{U}$ contains the diagonal $\Delta_X = \{(x,x) : x \in X\}$;
U3) if $E \in \mathscr{U}$, then $E^T = \{(y,x) : (x,y) \in E\} \in \mathscr{U}$;
U4) for any $E \in \mathscr{U}$ there exists $\hat{E} \in \mathscr{U}$ such that $\hat{E} \circ \hat{E} \subset E$, where

$$\hat{E} \circ \hat{E} = \{(x,y) : \exists z \in X \text{ with } (x,z) \in \hat{E}, (z,y) \in \hat{E}\}.$$

The set X with a uniformity $\mathscr{U}$ on it is called *uniform space* and denoted by $(X, \mathscr{U})$. Let $(X, \mathscr{U})$ be a uniform space. Then each element of $\mathscr{U}$ is called *entourage* of X. An entourage E is called *symmetric* if $E = E^T$. We can, of cours, take the whole of $\mathscr{U}$ as a base, but more usefully we can take the symmetric entourage of $\mathscr{U}$ as a base. so we have $E \circ \hat{E} = \hat{E} \circ E$ for all entourages $E, \hat{E} \in \mathscr{U}$. If $x \in X$ and $A \subset X$ and $E \in \mathscr{U}$, then the set $E[x] = \{y \in X : (x,y) \in E\}$ is said to be the *cross-section* of E at a point x and

$$E[A] = \{y \in X : \exists a \in A, \ \textit{such that } y \in E[a]\}.$$

Moreover, for every $n \in \mathbb{N}$ denote

$$E^n := E \circ E \circ \cdots \circ E \ (n \text{ times})$$

If $\tau = \{A \subset X : \forall a \in A, \exists E \in \mathscr{U} \ \textit{such that } E[a] \subset A\}$, then τ is a topology on X that is called the *induced topology* by uniformity $\mathscr{U}$. From now on, when we say that the uniform space $(X, \mathscr{U})$ satisfies a certain topological property, we mean that the topological space (X, τ) has the same property.

In a uniform space $(X, \mathscr{U})$, we define hyperspaces of X as follows:

1) $\mathscr{K}(X) = \{A \subset X : A \textit{ is closed and non-empty}\}$,
2) $C(X) = \{A \in \mathscr{K}(X) : A \textit{ is connected}\}$,
3) $C_n(X) = \{A \in \mathscr{K}(X) : \ A \textit{ has at most } n \textit{ components}\}, n \in \mathbb{N}$;
4) $F_n(X) = \{A \in \mathscr{K}(X) : A \textit{ has at most } n \textit{ points}\}, n \in \mathbb{N}$
5) $F(X) = \bigcup_{n=1}^{\infty} F_n(X) - \textit{the collection of all finite subsets of } X.$

Let $(X, \mathscr{U})$ be a uniform space and $E \in \mathscr{U}$. If

$$2^E = \{(A, A') \in \mathscr{K}(X) \times \mathscr{K}(X) : A \subset E[A'], \ A' \subset E[A]\},$$

then it is easy to prove that the set $\mathfrak{B} = \{2^E : E \in \mathscr{U}\}$ is a base for a uniformity on $\mathscr{K}(X)$, that denoted by

$$2^{\mathscr{U}} = \{\mathcal{U} \subset \mathscr{K}(X) \times \mathscr{K}(X) : \text{there exists } E \in \mathscr{U} \text{ such that } 2^E \subset \mathcal{U}\}.$$

It is known that the topology induced by $2^{\mathscr{U}}$ coincides with the *Vietoris topology.* We recall that a Vietoris topology on a set X is a topology with a base of all the sets of the form

$$\mathcal{V}(U_1, U_2, \ldots, U_k) = \{B \in \mathscr{K}(X) : B \subset \bigcup_{i=1}^{k} U_i \ \textit{and } B \cap U_i \neq \emptyset \ \textit{for } i = 1, 2, \ldots, k\},$$

where $U_1, U_2, \ldots, U_k$ are open in $(X, \tau_{\mathscr{U}})$ [3]. If $(X, \mathscr{U})$ is compact and Hausdorff, then $\mathscr{K}(X)$ is also compact and Hausdorff.
Let $f : X \longrightarrow X$ be a map. We define $\tilde{f} : \mathscr{K}(X) \to \mathscr{K}(X)$ by $\tilde{f}(A) = f(A)$ for all $A \in \mathscr{K}(X)$. We also define the maps $C_n(f), C(f), f_n$ *and* $f^{<\omega}$ from $\mathscr{K}(X)$ into $\mathscr{K}(X)$ as follows:

$$C_n(f) = \tilde{f}|_{C_n(X)}, \quad C(f) = \tilde{f}|_{C(X)}, \quad f_n = \tilde{f}|_{F_n(X)}, \quad f^{<\omega} = \tilde{f}|_{F(X)}.$$

These functions are also called *induced maps* by f. We denot the product space $\underbrace{X \times X \times \cdots \times X}_{n-times}$ by $X^{(n)}$ and define $f^{(n)} : X^{(n)} \to X^{(n)}$ by $f^{(n)}(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n))$.

3. Topological shadowing

Definition 3.1. Let $(X, \mathscr{U})$ be a uniform space, and $f : X \to X$ a continuous map and $D \in \mathscr{U}$. Then a *topological D-pseudo orbit* is a (finite or infinite) sequence of points $\{x_0, x_1, x_2, \ldots\}$ such that $(f(x_i), x_{i+1}) \in D$, for $i \geq 0$.
For $E \in \mathscr{U}$, a point $z \in X$ *topological E-shadows* the (finite or infinite) sequence $\{x_0, x_1, x_2, \ldots\}$ of points of X if $(f^j(z), x_j) \in E$, for $j \geq 0$.
f has *topological shadowing* if for every entourage $E \in \mathscr{U}$ there is an entourage $D \in \mathscr{U}$ such that every topological D-pseudo orbit is topological E-shadowed by some point in X. In the case that only finite pseudo orbits are shadowed, we say that f has topological finite shadowing.

Proposition 3.2. *Let $(X, \mathscr{U})$ be compact uniform space, and $f : X \longrightarrow X$ be a continuous function. If Y is a dense and invariant subset of X, then f has topological shadowing iff $f|Y$ has topological shadowing .*

Proposition 3.3. *Let $(X, \mathscr{U})$ be a uniform space and let $f : X \longrightarrow X$ be a continuous function. Let $n \geq 1$. If one of the maps in the collection $\{C(f), f_n, f^{<\omega}, \tilde{f} \mid n \in \mathbb{N}\}$ has topological shadowing, then f has topological shadowing.*

Proposition 3.4. *Let $(X, \mathscr{U})$ be a uniform space and let $f : X \longrightarrow X$ be a continuous function. If f has topological shadowing, then $f^{<\omega}$ has topological shadowing.*

Theorem 3.5. *Let $(X, \mathscr{U})$ be a uniform space and let $f : X \longrightarrow X$ be a continuous function. Then f has topological shadowing, iff 2^f has topological shadowing.*

4. Topological h-shadowing

Definition 4.1. Let $(X, \mathscr{U})$ be compact uniform space, and $f : X \longrightarrow X$ be a continuous function. f has *topological h-shadowing* if for any entourage $E \in \mathscr{U}$ there is an entourage $D \in \mathscr{U}$ such that for every topological finite D-pseudo orbit

$\{x_0, x_1, x_2, \ldots, x_r\}$ there is a point $z \in X$ such that $(f^i(z), x_i) \in E$, for $i < r$ and $f^r(z) = x_r$.

Proposition 4.2. *Let $(X, \mathscr{U})$ be a uniform space and let $f : X \longrightarrow X$ be a continuous function.Let $n \geq 1$. If one of the maps in the collection $\{f_n, f^{<\omega}, \tilde{f}$
$\mid n \in \mathbb{N}\}$, has topological h-shadowing, then f has topological h-shadowing.*

Proposition 4.3. *Let $(X, \mathscr{U})$ be a uniform space and let $f : X \longrightarrow X$ be a continuous function. If f has topological h-shadowing, then $f^{<\omega}$ has topological h-shadowing.*

Theorem 4.4. *Let $(X, \mathscr{U})$ be a uniform space and let $f : X \longrightarrow X$ be a continuous function. Then $2^f : 2^X \to 2^X$, has topological h-shadowing, iff $f^{<\omega} : F(X) \to F(X)$ has topological h-shadowing.*

Theorem 4.5. *Let $(X, \mathscr{U})$ be a uniform space and let $f : X \longrightarrow X$ be a continuous function. Then f has topological h-shadowing, iff 2^f has topological h-shadowing.*

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PERMANENCY IN PREDATOR-PREY MODELS OF LESLIE TYPE WITH RATIO-DEPENDENT SIMPLIFIED HOLLING TYPE-IV FUNCTIONAL RESPONSE

HASSAN QOLIZADEH AMIRABAD AND OMID RABIEIMOTLAGH

Abstract. We consider a predator-prey model of Leslie type with ratio-dependent simplified Holling type-IV functional response. First, We show that permanency of the system holds automatically for some values of parameters. Second, we establish the sufficient conditions for the global stability of interior equilibrium by constructing Lyapunov function. As a result, the permanency of the system can be solved.

1. Introduction

we study the following predator-prey system of Leslie type with ratio-dependent simplified Holling type IV functional response

$$
\begin{aligned}
&\dot{x} = rx(1-\tfrac{x}{K}) - \underbrace{\left(\frac{m\frac{x}{y}}{a+\frac{x^2}{y^2}}\right)}_{p(x/y)} y \;=\; rx(1-\tfrac{x}{K}) - \tfrac{m\,xy^2}{ay^2+x^2},\\
&\dot{y} = sy(1-\tfrac{y}{hx}),\\
&x(0)>0, y(0)>0.
\end{aligned}
\tag{1.1}
$$

Keywords: Predator-prey of Leslie type system, Global asymptotic stability, Permanency .

AMS Mathematical Subject Classification [2010]: 03C99 , 37G10.

where x and y represent respectively prey and predator densities which grow in a logistic manner with intrinsic growth rate r and s respectively. The function $p(x/y)$ is known as ratio-dependent functional response.

2. PERMANENCY

For simplicity,we write the system in nondimensional form . Let $\tau = rt$, $\bar{x} = x/K$ and $\bar{y} = my/rK$. we obtain the following system (after eliminate the bar)

$$\begin{array}{l} \dot{x} = x(1-x) - \frac{nxy^2}{by^2+x^2}, \\ \dot{y} = \delta y(\beta - \frac{y}{x}), \\ x(0) > 0, y(0) > 0, \end{array} \tag{2.1}$$

where $n = r/m$,$b = an$, $\delta = \frac{s}{r}$ and $\beta = mh/r$.
System (2.1) has two fixed points, boundary equilibrium $E_0 = (1, 0)$ and interior equilibrium $E_1 = (x^*, \beta x^*)$ with $x^* = (1 + (b-n)\beta^2)/1 + b\beta^2$. Interior equilibrium E_1 is biologically admissible (we simply say E_1 exists) iff $x^* > 0$ or equivalently $\beta^2(n-b) < 1$. This implies that if $b \geq n$ then E_1 exists and if $0 < b < n$, then E_1 exists if $\beta\sqrt{(n-b)} < 1$.
In this section we discuss the permanency of system (1.1) which is defined as follows.

Definition 2.1. The system (2.1) is said to be permanent if there exist positive constants r and s with $0 < r \leq s$ such that for any solution $(x(t), y(t))$ with $x(0) > 0$ and $y(0) > 0$;

$$\begin{array}{l} min\{\liminf_{t\to\infty} x(t), \liminf_{t\to\infty} y(t)\} \geq r, \\ max\{\limsup_{t\to\infty} x(t), \limsup_{t\to\infty} y(t)\} \leq s. \end{array}$$

In order to prove the permanency for system (2.1), we first need the following lemma.

Lemma 2.2. *(See([1]).) Solutions of system (2.1) are positive and bounded, furthermore there exists $T > 0$ such that $0 < x(t) < M$, $0 < y(t) < N$, for all $t > T$, Where $M > 1, N > M\beta$.*

We have the following result.

Lemma 2.3. *If $b > n$ then system (2.1) is permanent.*

Proof. Suppose that $b > n$, From the first equation of system (2.1) $\dot{x} > x(1 - x - \frac{n}{b})$. Hence, by the comparison theorem $\liminf_{t\to\infty} x(t) \geq 1 - \frac{n}{b} \equiv x_0$. Also, it is easy to see that $\dot{y} > \delta(\beta - \frac{2}{x_0}y)$.Thus, by the comparison theorem $\liminf_{t\to\infty} y(t) \geq \frac{\beta x_0}{2} \equiv y_0$. Let $s = max\{M, N\}$ and $r = min\{x_0, y_0\}$, from Definition 2.1 system (2.1) is permanent. □

3. Local and Global Stability of E_1

we use the following transformation which obtain locally equivalent system to system (2.1) at $E_1 = (x^*, y^*)$. Let $x = x$,$u = \frac{y}{x}$ and $p(u) = nu^2/(bu^2+1)$ Then we have

$$\begin{aligned}\dot{x} &= x(1-x) - p(u)x,\\ \dot{u} &= u(x - 1 + p(u) + \delta\beta - \delta u),\\ x(0) &> 0, u(0) > 0.\end{aligned} \tag{3.1}$$

In the rest of this section, we consider system (3.1) instead of system (2.1). Obviously, interior equilibrium $E_1 = (x^*, y^*)$ of system (2.1) is turned to $E_1 = (x^*, u^*)$ for system (3.1), where $x^* = 1 - p(\beta)$, $u^* = \beta$.
The Jacobi matrix of system (3.1) at E_1 takes the form

$$J(E_1) = \begin{pmatrix} -1+p(\beta) & (-1+p(\beta))p^{'}(\beta) \\ \beta & \beta(-\delta + p^{'}(\beta)) \end{pmatrix} = \begin{pmatrix} -x^* & \frac{-2x^*(1-x^*)^2}{\beta^3} \\ \beta & \frac{-n\delta\beta^3 + 2(1-x^*)^2}{n\beta^2} \end{pmatrix}.$$

The determinant and the Trace of $J(E_1)$ are computed as:
$Det(J(E_1)) = \beta\delta(1 - p(\beta)) = \delta\beta x^* > 0$,
$Tr(J(E_1)) = p(\beta) + \beta p^{'}(\beta) - 1 - \beta\delta = \frac{2x^{*2} - (4+n\beta^2)x^* + 2 - n\beta^3\delta}{n\beta^2}$.

Lemma 3.1. *Let the condition $\beta^2(n-b) < 1$ hold:*
(i) If $n\delta\beta^3 < 2$ and $\beta\delta > p(\beta) + \beta p^{'}(\beta) - 1$ or $n\delta\beta^3 \geq 2$ then the interior equilibrium E_1 exists and is locally asymtotical stable.
(ii) If $n\delta\beta^3 < 2$ and $\beta\delta < p(\beta) + \beta p^{'}(\beta) - 1$ then the interior equilibrium E_1 exists and is an unstable focus or node.

Now, we introduce a Lyapunov function for the system (3.1) and use it to prove its global stability.

Theorem 3.2. *Assume that $\beta^2(n-b) < 1$, $n\delta\beta^3 > 2$ and $b > \frac{\beta^2}{16-8n\beta^2} > 0$, then all solutions of system (3.1) with initial condition $x_0 > 0, y_0 > 0$ satisfy*
$\liminf_{t\to\infty} x(t) = x^*, \liminf_{t\to\infty} y(t) = y^*$.

Proof. Note that, the condition of (i,ii) ensure local stability of E_1. We construct the following Lyapunov function

$$V(t) = \int_{x^*}^{x} \frac{\xi - x^*}{\xi} d\xi + \int_{\beta}^{u} \frac{p(\eta) - p(\beta)}{\eta} d\eta$$

The time derivative of V along the solutions of system (3.1) is

$$\dot{V} = -(x - x^*)^2 - (p(u) - p(\beta))(u - \beta)\frac{\delta(b\beta^2+1)(bu^2+1) - n(u+\beta)}{(b\beta^2+1)(bu^2+1)},$$

by using the conditions (ii) and (iii) we obtain

$$\delta(b\beta^2+1)(bu^2+1) - n(u+\beta) > \frac{2bu^2 - n\beta u - n\beta^2 + 2}{\beta} > 0.$$

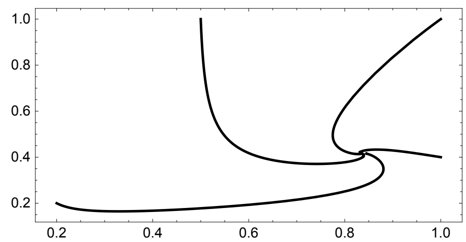

FIGURE 1. parametric plot with $(b, \beta, \delta) = (2, 0.5, 1)$ and initial conditions $(x_0, y_0) = (1,1), (0.2, 0.2), (0.5, 1), (1, 0.4)$.All solutions converge to $E_1 = (5/6, 5/12)$ and stay away from the axis.

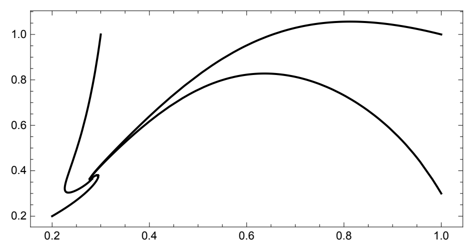

FIGURE 2. parametric plot with $(b, \beta, \delta) = (0.8, 1.3, 1)$ and initial conditions $(x_0, y_0) = (1,1), (0.2, 0.2), (0.3, 1), (1, 0.3)$.All solutions converge to $E_1 \simeq (0.281, 0.366)$ in agreement with the theoretical results.

We know that derivation function $p^{'}(u) > 0$ for $u > 0$. Hence $\dot{V}(t) < 0$ along the solutions of system (3.1) in the first quadrant except E_1. □

we present numerical simulations of system (2.1). We choose parameters $b = 2, \beta = 1/2, \delta = 1 = n$ such that $\beta^2(n - b) < 1$ and $b > n$. By Lamma 2.3 the permanency holds. The biological interpretation is that, the prey and the predator populations will survive over the long term. The simulations are shown in Fig.1. Now we choose $b = 0.8, \beta = 1.3, \delta = 1 = n$ such that $\beta^2(n - b) < 1, b < n$. The conditions of lamma 3.1 and theorem 3.2 are held. Thus, local stability implies global stability. As a result, we still have permanency. For example see Fig.2.

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RIDDLED BASINS IN A PARAMETRIC NONLINEAR SYSTEM

M. RABIEE AND F. H. GHANE

Abstract. In this paper, a parameter family of maps of the plane living two different lines invariant is stuedid. Our model is a generalization of piecewise linear model defined by Ott et al. [4]. to a nonlinear system. We verify the local dynamic stability of the chaotic attractors using normal Lyapunov exponents. By varying the parameter, we show the occurrence of riddled basin and hysteretic blowout bifurcation. It is shown that the system presents a complex fractal boundary between the initial conditions leading to each of the two attractors. To verify the occurrence of riddled basin, we conjugate our system to a random walk model and using properties of this model we describe riddled basin and blowout in detail. Numerical simulations are presented graphically to confirm the validity of our results.

1. Introduction

There has been a lot of recent interest in the global dynamics of systems with multiple attractors, with the recognition that the structure of basins of attraction may be very complicated. It is very common for dynamical systems to have more than one attractor. Among of them, we focus on systems having multiple attractors may present basins of attraction densely blended, a phenomenon called *riddling*. Ott et al. [3] introduced non-linear dynamical systems with a simple symmetry contain riddled basins. Also, the conditions of occurrence riddled basins are defined in Alexander et al. [1] and then generalized by Ashwin et al. [2].

Keywords: riddled basin of attraction, blowout bifurcation, skew product, normal Lyapunov exponent..

AMS Mathematical Subject Classification [2010]: 37C05,37C40, 37C70,.

In this article, we examine the behavior of a parametric system of the plane and demonstrate the emergence conditions of riddled basins. Our model is a one parameter family of piecewise C^2 skew product maps of the form

$$F_\beta : \mathbb{I} \times \mathbb{I} \to \mathbb{I} \times \mathbb{I}, \ F_\beta(x, y) := (f(x), g_\beta(x, y)), \tag{1.1}$$

where $\mathbb{I}$ is the unit interval $[0, 1]$, f is an expanding Markov map given by

$$f(x) = \begin{cases} 2x & \text{for} \quad 0 \leq x \leq 1/2, \\ 2x - 1 & \text{for} \quad 1/2 < x \leq 1 \end{cases} \tag{1.2}$$

and

$$g_\beta(x, y) := \begin{cases} g_1(y) = y + y(1 - y)(y - \beta) & \text{for} \quad 0 \leq x \leq 1/2, \\ g_2(y) = y - y(1 - y)(y - 1/2) & \text{for} \quad 1/2 < x \leq 1. \end{cases} \tag{1.3}$$

Here, $\beta \in (0, 1)$ is a fixed point of g_1. The following subspaces

$$N_0 := \{(x, y) : \quad 0 \leq x \leq 1, \quad y = 0\}, \tag{1.4}$$

$$N_1 := \{(x, y) : \quad 0 \leq x \leq 1, \quad y = 1\}, \tag{1.5}$$

are invariant by F_β, for each β. The parameter β varies the transverse dynamics without changing the dynamics on the invariant subspaces N_0 and N_1. We will show that, for some values of β, the parametric family F_β exhibit two attractors lying in invariant subspaces N_i with a qualitative dynamics which depends on the initial conditions. Attractors in our model exhibit a complex attracting basin structure that is riddled by holes.

Additionally, we will show that by varying the parameter β, it is possible one of the chaotic sets in the invariant subspaces looses the stability when the parameter passes through a critical value β_c. This happens if the largest normal Lyapunov exponent cross zero at β_c. If the normal Lyapunov exponent is negative, there is a set of positive measure which is forward asymptotic to the attractor. However, there can still be an infinite set of trajectories in the neighborhood of the attractor that are repelled from it. In particular, if the normal Lyapunov exponent is small and negative, the blowout bifurcation occurs and the riddling basin can be observed near the bifurcation point β_c. The aim of this study is to analyze the multiple attractors with riddled basins of attraction in our model. It can be done by conjugating the system to a random walk model, then using properties of this model we investigate the system at riddled basin and blowout in detail.

2. Main results

Let M be a compact connected smooth Riemannian manifold and m denotes the normalized Lebesgue measure. Let $F : M \to M$ be a continuous map and $A \subset M$ is a compact F-invariant set (i.e. $F(A) = A$). *The basin of attraction* of A, we denote

it by $\mathcal{B}(A)$, is the set of points whose ω-limit set iscontained in A.
The compact invariant set A is called an *asymptotically stable attractor* if it is Lyapunov stable and the basin of attraction $\mathcal{B}(A)$ contains a neighbourhood of A.

We say that A is a *Milnor attractor* if $\mathcal{B}(A)$ has non-zero Lebesgue measure and there is no compact proper subset A' of A whose basin coincides with $\mathcal{B}(A)$ up to a set of zero measure. Here, we deal with chaotic attractors. A compact F-invariant set A is a *chaotic attractor* if A is a transitive Milnor attractor and supports an ergodic measure μ but is not uniquely ergodic. In particular, at least one of the Lyapunov exponents (with respect to μ) is positive.

Some of dynamical systems having chaotic attractors with densely intertwined basins of attraction, which we call it *riddled basin*. An attractor A has a *riddled basin* if for all $x \in \mathcal{B}(A)$ and $\varepsilon > 0$, one has

$$m(B_\varepsilon(x) \cap \mathcal{B}(A))m(B_\varepsilon(x) \cap \mathcal{B}(A)^c) > 0, \tag{2.1}$$

where $B_\varepsilon(x)$ is the ε-neighborhood of x and m denotes the Lebesgue measure.
Assume F is a smooth map defined on a smooth manifold M and let $N \subset M$ be an n-dimensional embedded submanifold and forward invariant by F, with $n < m$. We consider the restriction of F to N, denoted by $F_{|N}$. Moreover, we assume that A is a chaotic attractor for F. For a vector $v \neq 0$ with base point x, *the Lyapunov exponent* $\lambda(x, v)$ at the point x in the direction of v is defined to be

$$\lambda(x, v) = \lim_{n\to\infty} \frac{1}{n} \log \|d_x F^n(v)\|_{T_{F^n(x)}M} \tag{2.2}$$

whenever the limit exists. Since N is an embedded submanifold, we can take a smooth splitting of the tangent bundle TM in a neighbourhood of N of the form $T_xM = T_xN \oplus (T_xN)^\perp$, when $x \in N$. To simplify the notation, we write $TM_n := T_{F^n(x)}M$. Given $x \in A$; $v \in T_xM = T_xN \oplus T_xN^\perp$, the *parallel Lyapunov exponent* at x in the direction of v to be

$$\lambda_\|(x, v) = \lim_{n\to\infty} \frac{1}{n} \ln \| \pi_{(TN_n)} \circ d_xF^{(n)} \circ \pi_{TN_0}(v) \|_{TM}, \tag{2.3}$$

where π_V is the orthogonal projection onto a subspace V. Similarly, the *normal Lyapunov exponent* at x in the direction of v to be

$$\lambda_\perp(x, v) = \lim_{n\to\infty} \frac{1}{n} \ln \| \pi_{(TN_n)^\perp} \circ d_xF^{(n)} \circ \pi_{(TN_0)^\perp}(v) \|_{TM_n} . \tag{2.4}$$

Let A be an asymptotically stable attractor under $F_{|N}$. An invariant ergodic probability measure μ is called an SRB measure for A if its support is A and has absolutely continuous conditional measures on unstable manifolds (with respect to the Riemannian measure). An attractor A is an *SRB-attractor* if it supports an SRB measure. Let μ be an F-invariant ergodic probability measure supported in A, with normal

Liapunov exponents $\lambda^1_\perp(\mu) < \cdots < \lambda^s_\perp(\mu)$. The *normal stability index* Λ_μ of μ is

$$\Lambda_\mu := \lambda^s_\perp(\mu). \tag{2.5}$$

Let A be a chaotic attractor that supports an SRB measure μ_{SRB}. For the occurrence of a riddled basin, we require that at least one of the parallel Lyapunov exponents $\lambda_\|$ is positive but the maximal normal Lyapunov exponent Λ_{SRB} is slightly negative. The following conditions [1] verify the occurrence of riddling basin:

(H1) there is another attractor not belonging to N;
(H2) the attractor A is transversally stable in M, i.e. for typical orbits on the attractor the Lyapunov exponents for infinitesimal perturbations along the directions transversal to the invariant submanifold N are all negative;
(H3) a set of unstable periodic orbits imbedded in A is transversely unstable.

Theorem 2.1. *Let the one parameter family F_β and the chaotic attractors A_i, $i = 0, 1$, be as above. Then we get the following:*

(a) *if $\beta \in (\frac{1}{3}, \frac{1}{2})$, then $\mathcal{B}(A_0)$ is riddled with $\mathcal{B}(A_1)$;*
(b) *if $\beta \in (\frac{1}{2}, \frac{2}{3})$, then $\mathcal{B}(A_1)$ is riddled with $\mathcal{B}(A_0)$;*

where $\mathcal{B}(A_i)$ is the basin of attraction of A_i, for $i = 0, 1$.

Corollary 2.2. *$F_{1/2}$ exhibit three Milnor attractors A_i, $i = 0, 1/2, 1$. Moreover, $\mathcal{B}(A_{1/2})$ is riddled with both basins $\mathcal{B}(A_0)$ and $\mathcal{B}(A_1)$.*

Corollary 2.3. *F_β exhibits a (subcritical) hysteretic blowout bifurcation on passing through $\beta = 1/3$ and $\beta = 2/3$.*

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ON BIMODAL POLYNOMIALS WITH A NON-HYPERBOLIC FIXED POINT

M. RABII AND M. AKBARI

ABSTRACT. We consider the real polynomials of degree $d+1$ with a fixed point of multiplicity $d \geq 2$. Such polynomials are conjugate to $f_{a,d}(x) = ax^d(x-1)+x,\ a \in \mathbb{R} \setminus \{0\}$. In this family, the point 0 is always a non-hyperbolic fixed point. We prove that for given d, d', and a, where d and d' are positive even numbers and a belongs to a special subset of $\mathbb{R}^-$, there is $a' < 0$ such that $f_{a,d}$ is topologically conjugate to $f_{a',d'}$. Then we extend the properties that we have studied in case $d = 2$ to this family for every even $d > 2$.

1. INTRODUCTION

Among the C^1 multi-modal maps, polynomials are typical. It has been shown that each C^1 l-modal map is semi-conjugate to a polynomial l-modal map (see [2, Chapter II, Theorem 6.4]). Therefore it is useful to investigate the dynamical behavior of polynomials. In [4], the dynamics of the family of complex polynomials $f(z) = z^3 + az^2 + z = z^2(z+a) + z$, with $f'(0) = 1$ is studied. In [1], a family of real polynomials $f_a(x) = ax^2(x-1) + x$ is studied. Note that in this case we also have $f_a'(0) = 1$. In this paper we consider the family of polynomials $f_{a,d}(x) = ax^d(x-1)+x$ where $a < 0$ is a real number and $d \geq 2$ is an even integer. Each map of this family has a fixed point of multiplicity d. The main features of $f_{a,d}$ are similar to $f_{a',2}$. The question is whether these similarities make $f_{a,d}$ and $f_{a',d'}$ conjugate. The main tool

Keywords: l-modal map, non-hyperbolic fixed point, order preserving bijection, topological conjugacy.

AMS Mathematical Subject Classification [2010]: 37E05, 37E15.

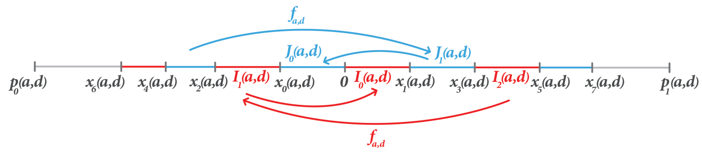


FIGURE 1. The images of $I_i(a,d)$'s and $J_i(a,d)$'s under $f_{a,d}$

in this paper is Corollary 3.1 of Chapter II of [2], which states the conditions under which two l-modal maps are conjugate on a compact interval. For this purpose, the parameter line $a<0$ is partitioned into some subintervals such that the behavior of the critical points are different in these subintervals.

2. THE COMMON PROPERTIES OF $f_{a,d}$

The common properties of the family $f_{a,d}(x) = ax^d(x-1)+x$ are stated in the following proposition. We use the notation $I \sqsubseteq J$ for two intervals I and J when $x<y$ for all $x \in I$ and all $y \in J$.

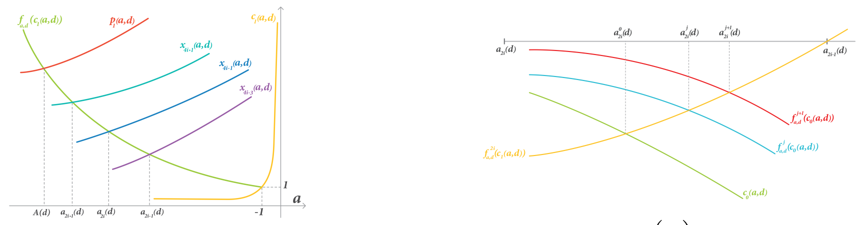
Proposition 2.1. (1) *$f_{a,d}(x)=0$ has only two non-zero solutions $x_0(a,d)$ and $x_1(a,d)$. Moreover, $x_0(a,d)<0<1<x_1(a,d)$.*

(2) *$f'_{a,d}(x)=0$ has only two solutions $c_0(a,d)$ and $c_1(a,d)$ where $c_0(a,d)$ is a local minimum point and $c_1(a,d)$ is a local maximum point of $f_{a,d}$. Moreover, $x_0(a,d)<c_0(a,d)<0<\frac{d}{d+1}<c_1(a,d)<x_1(a,d)$.*

(3) *There are an increasing bounded sequence $\big(x_{2n+1}(a,d)\big)_{n\geq 0}$ in $\big[x_1(a,d),\infty\big)$ and a decreasing bounded sequence $\big(x_{2n}(a,d)\big)_{n\geq 0}$ in $\big(-\infty, x_0(a,d)\big]$ such that if $p_0(a,d)=\lim_{n\to\infty} x_{2n}(a,d)$ and $p_1(a,d)=\lim_{n\to\infty} x_{2n+1}(a,d)$, then(see Figure 1)*

$$f_{a,d}\big(p_0(a,d)\big)=p_1(a,d), \qquad f_{a,d}\big(p_1(a,d)\big)=p_0(a,d).$$

Set $J_0(a,d)=\big[x_0(a,d),0\big]$, $I_0(a,d)=\big(0,x_1(a,d)\big)$, and for $n\geq 1$ $J_{2n-1}(a,d)=\big[x_{2n-3}(a,d),x_{2n-1}(a,d)\big]$, $J_{2n}(a,d)=\big[x_{4n}(a,d),x_{4n-2}(a,d)\big]$, $I_{2n-1}(a,d)=\big(x_{2n-2}(a,d),x_{2n-4}(a,d)\big)$, $I_{2n}(a,d)=\big(x_{4n-1}(a,d),x_{4n+1}(a,d)\big)$. Then $\big(p_0(a,d),p_1(a,d)\big)=\big(\cup_{n\geq 0} I_n\big)\cup\big(\cup_{n\geq 0} J_n\big)$, $c_0(a,d)\in J_0(a,d)$, $c_1(a,d)\in I_0(a,d)$, $f_{a,d}\big(J_n(a,d)\big)=J_{n-1}(a,d)$, and $f_{a,d}\big(I_n(a,d)\big)=I_{n-1}(a,d)$. Moreover, $\cdots I_{2n+1}(a,d)\sqsubseteq J_{2n}(a,d)\sqsubseteq\cdots\sqsubseteq J_0(a,d)\sqsubseteq I_0(a,d)\sqsubseteq\cdots\sqsubseteq I_{2n}(a,d)\sqsubseteq J_{2n+1}(a,d)\cdots$,
and for every n the orbit of any point of the interval $J_n(a,d)$ converges to 0, and $\lim_{n\to\infty}|f^n_{a,d}(x)|=\infty$ for each $x\notin\big[p_0(a,d),p_1(a,d)\big]$.



(A) (B)

FIGURE 2. A schematic diagram of (A) the locations of the sequence $(a_i(d))_{i\geq 1}$ and $A(d)$ and (B) the locations of the sequence $\left(a^j_{2i}(d)\right)_{j\geq 0}$ in the interval $(a_{2i}(d), a_{2i-1}(d))$.

3. Partitioning the parameter space

The critical point $c_0(a,d)$ and its iterations always belong to $J_0(a,d)$. The critical point $c_1(a,d)$ always belongs to $I_0(a,d)$ but the parameter a determines whether the orbit of the critical point $c_1(a,d)$ meets $I_n(a,d)$'s, $J_n(a,d)$'s, or neither. To determine the position of the orbit of the critical point $c_1(a,d)$, for given even $d \geq 2$, we prove there is a unique parameter $A(d) < -1$ such that $p_1(A(d),d) = f_{A(d),d}\big(c_1(A(d),d)\big)$. Therefore, for each $a \in (A(d),0)$ we have $f_{a,d}\,(c_1(a,d)) \in \cup_{n\geq 0}\,((I_n(a,d) \cup (J_n(a,d))$. We partition the parameter interval $(A(d),0)$ to determine for which parameters $f_{a,d}\,(c_1(a,d))$ belongs to $I_n(a,d)$ and for which parameters $f_{a,d}\,(c_1(a,d))$ belongs to $J_n(a,d)$. Accordingly, we present a decreasing sequence of parameters $\big(a_i(d)\big)_{i\geq 1} \subseteq (A(d),-1)$ (see Figure 2(A)) such that if $a \in [a_{2i}(d), a_{2i-1}(d)]$, then $f_{a,d}(c_1(a,d))$ belongs to the interval $J_{2i-1}(a,d)$ and if $a \in (a_{2i+1}(d), a_{2i}(d))$, then $f_{a,d}(c_1(a,d))$ belongs to the interval $I_{2i}(a,d)$. If $a \in [a_{2i}(d), a_{2i-1}(d)]$ for some $i \geq 1$, i.e., $f_{a,d}(c_1(a,d))$ belongs to the interval $J_{2i-1}(a,d)$, then i is the smallest index that $f^{2i}_{a,d}\big(c_1(a,d)\big) \in J_0(a,d)$ that is the immediate basin of the non-hyperbolic fixed point 0. Now, we divide the interval $(a_{2i}(d), a_{2i-1}(d))$ into an infinite number of subsets to distinguish the position of the orbit of $c_1(a,d)$ with respect to the orbit of $c_0(a,d)$. In fact, we prove that there is a sequence $\big(A^j_{2i}(d)\big)_{j\in\mathbb{Z}}$ of sets that partitions the interval $(a_{2i}(d), a_{2i-1}(d))$ such that if $a \in A^j_{2i}(d)$ for some $j \geq 0$, then $f^j_{a,d}\big(c_0(a,d)\big) \leq f^{2i}_{a,d}\big(c_1(a,d)\big) < f^{j+1}_{a,d}\big(c_0(a,d)\big)$ and if $a \in A^{-j}_{2i}(d)$ for some $j \geq 1$, then $f^j_{a,d}\big(c_0(a,d)\big) \leq f^{2i+1}_{a,d}\big(c_1(a,d)\big) < f^{j+1}_{a,d}\big(c_0(a,d)\big)$ (see Figure 2(B)). Our information about the position of the critical orbits respect to each other enables us to define an order preserving bijection that is applied in Corollary 3.1 of Chapter II of [2].

The case $a \in (a_{2i+1}(d), a_{2i}(d))$, i.e., $f_{a,d}(c_1(a,d))$ belongs to the interval $I_{2i}(a,d)$ is more complicated since the orbit of this critical point may meet the intervals $I_n(a,d)$'s, repeatedly. In this case, for a special parameter, we study the topological conjugacy in the family $f_{a,d}$. Also, topological conjugacy is investigated when $f_{a,d}(c_1(a,d)) = p_1(a,d)$. Based on the above, we prove:

Main Theorem: *Suppose that even integers $d, d' \geq 2$ are given and for $a \in \mathbb{R}^-$ one of the following conditions is satisfied.*

(1) $f_{a,d}(c_1(a,d)) \in J_{2i-1}(a,d)$ *for some* $i \geq 1$.
(2) $f_{a,d}(c_1(a,d)) \in I_{2i}(a,d)$ *and* $f_{a,d}^{2i+1}\big(c_1(a,d)\big) = c_1(a,d)$ *for some* $i \geq 0$.
(3) $f_{a,d}(c_1(a,d)) = p_1(a,d)$.

Then there is an $a' \in \mathbb{R}^-$ *such that* $f_{a,d}$ *and* $f_{a',d'}$ *are topologically conjugate.*

We employ the Main Theorem and Corollaries 3.3, 3.4, and Proposition 1 of [3] and conclude the following corollary.

Corollary 3.1. *Let $d \geq 2$ be an even integer and a be a negative real number.*

(1) *If* $f_{a,d}(c_1(a,d)) \in J_1(a,d)$, *then the entropy of* $f_{a,d}$ *is* $\log 2$.
(2) *For each* $n \geq 1$, *the entropy of* $f_{a,d}$ *is constant when* $f_{a,d}(c_1(a,d)) \in J_{2n-1}(a,d)$.
(3) *If* $f_{a,d}(c_1(a,d)) = p_1(a,d)$, *then the entropy of* $f_{a,d}$ *is* $\log(1+\sqrt{2})$.

Table 1 represented in [3] that shows an estimation of the topological entropy of $f_{a,2}$, in different cases for $1 \leq n \leq 5$, can be used for $f_{a,d}$ when $d \geq 2$ is an even number.

TABLE 1. The estimation of the topological entropy of $f_{a,d}$, for $1 \leq n \leq 5$.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$f_{a,d}\big(c_1(a,d) \in J_{2n-1}(a,d)$	$\log 2$	$\log 2.360$	$\log 2.406$	$\log 2.413$	$\log 2.415$
$f_{a,d}\big(c_1(a,d)\big) \in I_{2n}(a,d)$ $f_{a,d}^{2n+1}\big(c_1(a,d)\big) = c_1(a,d)$	$\log 2.207$	$\log 2.384$	$\log 2.410$	$\log 2.414$	$\log 2.415$

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A STUDY ON THE ECOLOGICAL INITIAL VALUE PROBLEMS: THE EXPONENTIAL AND LOGISTIC GROWTH RATES OF SINGLE SPECIES MODELS HAVING HARVESTED FACTOR

MOHAMMAD HOSSEIN RAHMANI DOUST AND ATENA GHASEMABADI

Abstract. Because of no existing interspecific interaction in the single species, one is able to see that this is the simplest model. Meanwhile, by adding some assumptions, we see that it has so many practical applications in nature and any branch of sciences. In this article, some dynamic models of single species are studied. First, Picard's iteration method for the exponential growth rate is analyzed. In continuation, some logistic models for both cases without harvesting and having harvested factor which is constant or variable are studied. Indeed, the solution and stability of equilibria for the said models are analyzed.

1. Introduction

The study of population change started by Leonardo of Pisa Fibonacci. Accounting for the local spatial processes indeed brings the theory of single species population growth a step closer to the growth of real spatially structured populations. The population growth of infection of virus and worm in computer networks is analyzed by help of logistic modeling [1]. Moreover, It is shown that the susceptible population does not vanish when it is only under the effect of infection meanwhile in the polluted environment, it can extinct [3]. An application of exponential and logistic growths for single-cell models that are incapable or misleading for inferring population dynamics

Keywords: Harvesting Factor, Logistic Equation, Picard's Iteration Method, Single Species, Stability..

AMS Mathematical Subject Classification [2010]: 34D20, 65L05, 92D40..

as there is no any interactions between cells via metabolites or physical contact, nor competition for limited resources such as nutrients or space is studied in [2]. Some more text on applications of single species with exponential or logistic growth rate having harvesting factor can be seen in [4], [5], [6] and [8]. Exponential growth is associated with the name of Thomas Robert Malthus (1766-1834) who first realized that any species can potentially increase in numbers according to a geometric series [7]. Indeed, there is no a science branch that no need to help of exponential growth. Some of applications are microbiology (growth of bacteria), conservation biology (restoration of disturbed populations), insect rearing (prediction of yield), plant or insect quarantine (population growth of introduced species), fishery (prediction of fish dynamics), biochemical(radioactivity).

2. Main results

In this section, we construct and analyze some ecological I.V.Ps. of single species models in some cases which are based on P. Verhulst idea and harvesting factor. Because of limitation of page numbers in the extended abstract, the proof of theorems are removed.

2.1. **Exponential Model Having Constant Harvesting Factor.** By considering constant harvesting factor h and initial population $x(0) = x_0$, we get:

$$\frac{dx}{dt} = rx - h \ , \ x(0) = x_0. \tag{2.1}$$

In the following theorem, the solution of I.V.Ps. (2.1) by Picard's iteration method is analyzed:

Theorem 2.1. *The series solution of I.V.P.* (2.1) *is*

$$x(t) = \frac{h}{r} + (x_0 - \frac{h}{r}) \sum_{k=0}^{k=\infty} \frac{(rt)^k}{k!}$$

2.2. **Logistic Model without Harvesting Factor.** Consider parameters M and r as the carrying capacity of the environment and the rate of growth for small population numbers, respectively. The expression $M-x$ is equal to the number of individuals that may be added to a population at a given time, and we saw that $M-x$ is divided by M is the fraction of the carrying capacity available for further growth.

$$\frac{dx}{dt} = rx(1 - \frac{x}{M}) \ , \ x(0) = x_0. \tag{2.2}$$

The above I.V.P has the following properties:

(i) For case of $0 < x < M$, it increases, and for case of $x > M$ it decreases;
(ii) If $x_0 < M$, then population grows and approaches to M asymptotically;
(iii) If $x_0 > M$, then population decreases, again approaching M asymptotically;

(iv) If $x_0 = M$, then the population remains in time at $x = M$;
(v) Equilibrium point$x = M$ is globally stable.

2.3. **Logistic Modeling with Constant Harvesting Factor.** Having assumed the positive constant number h of population removed per each duration, we obtain:

$$\frac{dx}{dt} = rx(1 - \frac{x}{M}) - h \ , \ x(0) = x_0. \tag{2.3}$$

Theorem 2.2. *Ecological I.V.P.* (2.3) *has two equilibria* x_1 *and* x_2 *as follows:*

$$x_1 = \frac{M}{2}(1 - \sqrt{1 - \frac{4h}{rM}}) \ , \ x_2 = \frac{M}{2}(1 + \sqrt{1 - \frac{4h}{rM}}).$$

Moreover, the above equilibrium points are unstable and asymptotic stable, respectively.

2.4. **Logistic Modeling with Fractional Harvesting Factor.** By making assumption harvesting factor $f(x) = h\frac{x}{1+x}$, we model the following I.V.P.:

$$\frac{dx}{dt} = rx(1 - \frac{x}{M}) - h\frac{x}{x+1} \ , \ x(0) = x_0. \tag{2.4}$$

Theorem 2.3. *The following statements for logistic modeling I.V.P.* (2.4) *are true:*
(i) It has three equilibria x_1*,* x_2 *and* x_3 *as follows:*

$$x_1 = 0, \ \ x_{2,3} = \frac{M-1}{2} \pm \frac{M}{2}\sqrt{(\frac{1}{M} - 1)^2 + \frac{4}{M}(1 - \frac{h}{r})};$$

(ii) The equilibria x_2 *and* x_3 *are real number provided*

$$\frac{h}{r} \leq 1 + \frac{M}{4}(1 - \frac{1}{M})^2;$$

(iii) It is impossible that both equilibria x_2 *and* x_3 *are positive;*
(iv) The solution of this I.V.P. is as follows:

$$x^{B_1}(x - x_2)^{B_2}(x - x_3)^{B_3} = {x_0}^{B_1}(x_0 - x_2)^{B_2}(x_0 - x_3)^{B_3} exp(\frac{-rt}{M})$$

(v) In case of $M = r = h = 1$*, I.V.P. has equilibria* $x = 0$ *which is stable point.*

2.5. **Logistic Modeling with Cubic Harvesting Factor.** By considering harvesting function $f(x) = x^3$, we get:

$$\frac{dx}{dt} = rx(1 - \frac{x}{M}) - hx^3 \ , \ x(0) = x_0. \tag{2.5}$$

Theorem 2.4. *The following statements for logistic modeling I.V.P.* (2.5) *are true:*
(i) It has three equilibria x_1*,* x_2 *and* x_3 *as follows:*

$$x_1 = 0, \ \ x_{2,3} = \frac{r \pm \Delta}{2hM} \ \ where \ \ \Delta = \sqrt{r^2 + 4hrM^2};$$

(ii) The solution of this I.V.P. is the following implicit function:

$$x^{\frac{h}{r}}(x-x_2)^{\frac{r-\Delta}{-2\Delta}}(x-x_3)^{\frac{r+\Delta}{-2\Delta}} = x_0^{\frac{h}{r}}(x_0-x_2)^{\frac{r-\Delta}{-2\Delta}}(x_0-x_3)^{\frac{r+\Delta}{-2\Delta}};$$

(iii) In case of $M = r = h = 1$, the solution of this I.V.P. is decreasing function. Moreover, the equilibria $x_1 = 0$ is asymptotically stable.

3. Conclusion

The discussed models in this work have so many practical application. In deed, by help of these models, one may predict, check and defence to spread of viruses, microbe and bacteria such as Black Death, Spanish Flu, HIV/AIDS, Swine Flu, Ebola virus, Zika virus, Corona viruses such as: SARS-Cov, MERS-Cov, COVID-19 in a community. We worked out the series solution for exponential modeling for both cases of having constant harvesting factor or simple model without harvesting factor. Moreover, making some conditions for single species of logistic modeling, we found out they have asymptotic stable solutions. The important result is: "carrying capacity, growth rate, harvesting factor are important subjects to stability the equilibria".

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SECOND DERIVATIVE MULTISTAGE METHODS

SOHEILA RASI AND GHOLAMREZA HOJJATI

Abstract. In this paper, we describe construction of a class of explicit second derivative Runge–Kutta methods which have extensive region of absolute stability. Examples of such methods with $p = q = s = 2$ are given in which p and q stand for order and stage order, and s is the number of stages. These methods, because of their extensive stability region, can compete with the traditional explicit Runge–Kutta methods of the same order in solving initial value problems.

1. Introduction

Two-derivative Runge–Kutta (TDRK) methods for solving intial value problems (IVPs) of ordinary differential equations (ODEs)

$$\begin{cases} y'(x) = f(y(x)), \quad x \in [x_0, \overline{X}], \\ y(x_0) = y_0, \end{cases} \tag{1.1}$$

are defined by

$$\begin{cases} Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j) + \sum_{j=1}^{s} \widehat{a}_{ij} g(Y_j), \quad i = 1, 2, \ldots, s, \\ y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f\big(Y_i\big) + \sum_{i=1}^{s} \widehat{b}_i g(Y_i), \end{cases} \tag{1.2}$$

Keywords: Ordinary differential equations, Two-derivative Runge–Kutta methods, Order conditions, Stability.

where $y'' = g(y) := f'(y)f(y)$ with $f, g : \mathbb{R}^m \to \mathbb{R}^m$ and the internal stage value Y_i approximate $y(x_n + c_i h)$ and y_{n+1} is the update value which approximates $y(x_n + h)$. The coefficients of these methods can be represented by Butcher tableau

$$\begin{array}{c|c||c} c & A & \widehat{A} \\ \hline & b^T & \widehat{b}^T \end{array}$$

where $A = (a_{ij})_{s\times s}$, $\widehat{A} = (\widehat{a}_{ij})_{s\times s}$ and the vector forms $b = (b_i)_{s\times 1}$, $\widehat{b} = (\widehat{b}_i)_{s\times 1}$ are the vectors of weights and $c = (c_i)_{s\times 1}$ is the abscissa vector. The internal stage vector is $Y = [Y_1, \ldots, Y_s]^T$, he vectors of the first and second derivatives evaluated at the internal stage points are $F(Y) = [f(Y_1), \ldots, f(Y_s)]^T$ and $G(Y) = [g(Y_1), \ldots, g(Y_s)]^T$ respectively; the TDRK method (1.2) can be written in a more compact vector form

$$\begin{aligned} Y &= e \otimes y_n + h(A \otimes I_m)F(Y) + h^2(\widehat{A} \otimes I_m)G(Y), \\ y_{n+1} &= y_n + h(b^T \otimes I_m)F(Y) + h^2(\widehat{b}^T \otimes I_m)G(Y). \end{aligned} \tag{1.3}$$

2. Order conditions and stability properties of TDRK methods

Chan and Tsai [2] derived the order conditions of TDRK methods based on the tree theory, including mappings and composition of trees, developed in [1, 3]. The order conditions of TDRK methods up to order four are given in Figure 1.

Tree	**Order condition**
	$b^T e = 1$
	$b^T c + \widehat{b}^T e = \frac{1}{2}$
	$b^T c^2 + 2\widehat{b}^T c = \frac{1}{3}$
	$b^T Ac + b^T \widehat{c} + \widehat{b}^T c = \frac{1}{6}$
	$b^T c^3 + 3\widehat{b}^T c^2 = \frac{1}{4}$
	$b^T cAc + b^T c\widehat{c} + \widehat{b}^T c^2 + \widehat{b}^T Ac + \widehat{b}^T \widehat{c} = \frac{1}{8}$
	$b^T Ac^2 + 2b^T \widehat{A}c + \widehat{b}^T c^2 = \frac{1}{12}$
	$b^T A^2 c + b^T A\widehat{c} + b^T \widehat{A}c + \widehat{b}^T Ac + \widehat{b}^T \widehat{c} = \frac{1}{24}$

Figure 1. The order conditions of TDRK methods up to order four.

Also, the stage order conditions are given by

$$Ac^{k-1} + (k-1)\widehat{A}c^{k-2} = \frac{c^k}{k}, \quad k = 1, \ldots, q.$$

The stability properties of (1.2) are studied by applying (1.1) to the linear test problem

$$y' = \xi y,$$

where $\xi \in \mathbb{C}$. This leads to the matrix recurrence relation

$$y_{n+1} = R(z) y_n,$$

in which

$$R(z) = 1 + (zb^T + z^2 \widehat{b}^T)(I - zA - z^2 \widehat{A})^{-1} e,$$

with $z = h\xi \in \mathbb{C}$, is the stability function. In our proposed methods, this function contains some free parameters which are used to construct methods with a large region of absolute stability.

3. Construction of the methods

After satisfying the appropriate order and stage order conditions, we find the free parameters such that the resulting method has a large area of absolute stability. Here, we illustrate the construction of methods with $s = p = q = 2$. Such methods with the abscissa vector $c = [0 \ \ c_2]^T$ are given by the Butcher tableau

$$\begin{array}{c|cc||cc} 0 & & & & \\ c_2 & a_{21} & & \widehat{a}_{21} & \\ \hline & b_1 & b_2 & \widehat{b}_1 & \widehat{b}_2 \end{array}$$

with seven parameters which must satisfy the order and stage order conditions

$$b_1 + b_2 = 1, \qquad \frac{1}{2} b_2 + \widehat{b}_1 + \widehat{b}_2 = \frac{1}{2},$$
$$a_{21} = \frac{1}{2} \qquad \widehat{a}_{21} = \frac{1}{8}$$

The explicit two-stage method of order 2 with tableau

$$\begin{array}{c|cc||cc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{8} & 0 \\ \hline & 2(\widehat{b}_1 + \widehat{b}_2) & 1 - 2(\widehat{b}_1 + \widehat{b}_2) & \widehat{b}_1 & \widehat{b}_2 \end{array}$$

is defined by the equations. The stability function of the resulting two-parameter methods is

$$R(z) = 1 + z + \frac{1}{2} z^2 + \left(\frac{1}{8} - \frac{1}{4}\widehat{b}_1 + \frac{1}{4}\widehat{b}_2\right) z^3 + \frac{1}{8} \widehat{b}_2 z^4.$$

The method with a large region of absolute stability can be found with the values $\widehat{b}_1 = 0.1844$ and $\widehat{b}_2 = 0.0412$. The area of the stability region of resulted method is approximately 37.8559. This region is plotted in Figure 2 and compared with that for the explicit Runge–Kutta method with $p = s = 2$.

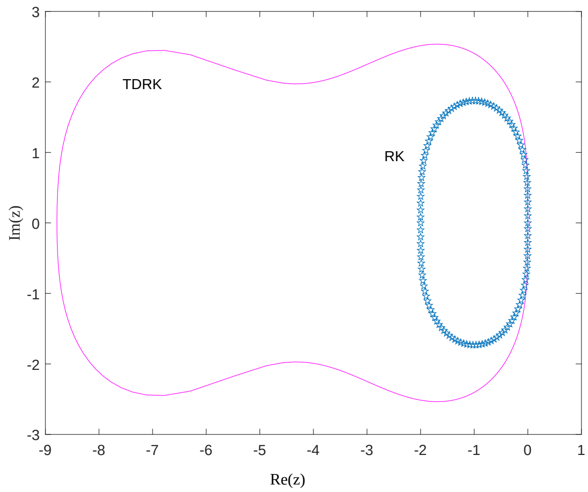

Figure 2. Stability regions TDRK and RK methods of the order two.

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دانشگاه گیلان

A DIRICHLET p-LAPLACIAN PROBLEM

F. SAFARI AND A. RAZANI

Abstract. Here using a new variational principle, we demonstrate the existence of at least one positive radial solution of a generalized p-Laplacian problem in the unit ball.

1. Introduction

An important task of hydrodynamics engineering throughout the 18^{th} century was to build reliable water supplies for fast growing urban centers. Numerous mathematical problems in this area derived and formulated. Henry Darcy, a French hydraulic engineer interested in purifying water supplies using sand filters, conducted experiments to determine the flow rate of water through the filters. He coducted the column experiments that established Darcy's law for flow in sands. Beginning in the 1870s, many engineers discovered that, if the fluid flow is in turbulent regime, the linear Darcy law does not provide the correct relationship between the pressure slope (force), $\mathbf{F}$, and the velocity, $\mathbf{v}$. Oscare Smreker suggested the power law $\mathbf{F} = \text{const.}|\mathbf{v}|^{p-2}\mathbf{v},\ p > 2$. It is now evident, that the operator $-\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is created by nonlinear power law. The quasilinear elliptic partial differential equations, where the nonlinear dependence involves not only the unknown function, but also all of its derivatives except those of the highest order are so-called p-Laplace equations of the form

$$-\Delta_p u = f(x, u, \nabla u).$$

Keywords: Variational principle, p-Laplace problem, Mountain Pass Geometry, Palais-Smale compactness condition.

AMS Mathematical Subject Classification [2010]: 35J15, 58E30.

Here the existence of at least one positive radial solution of the following Dirichlet problem

$$
\begin{cases} -\Delta_p u + \lambda |u|^{p-2}u = a(|x|)|u|^{q-2}u & x \in B, \\ u > 0 & x \in B, \\ u = 0 & x \in \partial B, \end{cases} \tag{1.1}
$$

is studied, where B is the unit ball centered at the origin in $\mathbb{R}^N$, $N \geq 3$, λ is a positive constant, $p < q < p^*$, and $a \in L^\infty(0,1)$ is an increasing positive, non constant radial function.

2. Preliminieries

In this section, we recall some important definitions and results from convex analysis. Fix $1 \leq p \leq \infty$. The Sobolev Space $W^{1,p}(B)$ consist of all locally summable functions $u : B \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq 1$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(B)$. If $u \in W^{1,p}\ (B)$, then its norm is defined by

$$
\|u\|_{W^{1,p}\ (B)} := \begin{cases} \Big(\sum\limits_{|\alpha|\leq 1} \int\limits_B |D^\alpha u|^p dx \Big)^{\frac{1}{p}} & 1 \leq p < \infty, \\ \sum\limits_{|\alpha|\leq 1} \operatorname*{ess\,sup}\limits_{x\in B} |D^\alpha u| & p = \infty. \end{cases}
$$

$W_0^{1,p}(B)$ is the closure of $C_0^\infty(B)$ in $W^{1,p}(B)$. As a subspace of $W_o^{1,p}(B)$ where we focus on it in this paper, let us define

$$
D^{1,p}(B) = \{u : B \to \mathbb{R}; u \in L^{p^*}(B), |\nabla u| \in L^p(B)\},
$$

where p^* is Sobolev conjugated of p and the vector $\nabla u = (\dfrac{\partial u}{\partial x_1}(x), ..., \dfrac{\partial u}{\partial x_n}(x))$ is the gradiant of u at $x = (x_1, ..., x_n)$. This space is equiped with the norm

$$
\|u\| = (\int_B |\nabla u|^p + |u|^p dx)^{\frac{1}{p}},
$$

and $D_0^{1,p}(B)$ be the cluster of $C_0^\infty(B)$ in $D^{1,p}(B)$, where according to the Poincaré inequality equiped with the norm $\|u\| = (\int_B |\nabla u|^p dx)^{\frac{1}{p}}$.

Definition 2.1. Let V be a real Banach space , $\varphi \in C^1(V, \mathbb{R})$ and $\psi : V \to (-\infty, +\infty]$ be a proper (i.e. Dom $\psi \neq \emptyset$), convex and lower semi continuous function. Define the function $\psi_K : V \to (-\infty, +\infty]$ by

$$
\psi_K(u) := \begin{cases} \psi(u) & u \in K, \\ \\ +\infty & u \notin K. \end{cases} \tag{2.1}
$$

A point $u \in V$ is called a critical point of $I := \psi_K - \varphi$, if $u \in Dom(\psi)$ and it satisfies the following inequality

$$\langle D\varphi(u), u - v\rangle + \psi_K(v) - \psi_K(u) \geq 0, \quad \text{for all } v \in V. \tag{2.2}$$

Definition 2.2. We say that I satisfies the Palais-Smale Compactness condition (PS) if every sequence $\{u_n\}$ such that

- $I[u_n] \to c \in \mathbb{R}$,
- $\langle D\varphi(u_n), u_n - v\rangle + \psi(v) - \psi(u_n) \geq -\varepsilon_n\|v - u_n\|$, for all $v \in V$,

where $\varepsilon_n \to 0$, then $\{u_n\}$ possesses a convergent subsequence (see [2])

The following theorem is Mountain Pass Geometry (MPG) Theorem.

Theorem 2.3. *Suppose that $I : V \to (-\infty, +\infty]$ is of the form $I := \psi_K - \varphi$, and satisfies PS condition and the following conditions:*

(i) $I(0) = 0$,

(ii) *there exists $e \in V$ such that $I(e) \leq 0$,*

(iii) *there exists positive constant ρ such that $I(u) > 0$, if $\|u\| = \rho$.*

Then I has a critical value $c \leq \rho$ which is characterized by $c = \inf_{g\in\Gamma} \sup_{t\in[0,1]} I[g(t)]$, where $\Gamma = \{g \in C([0, 1], V); g(0) = 0\ ,\ g(1) = e\}$.

3. Main results

Following lemma is one of our results to prove the main.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $p > 1$. Let $p' = \frac{p}{p-1}$ be the conjugate exponent of p. Then for every $h \in L^{p'}(\Omega)$, the problem*

$$\begin{cases} -\Delta_p u + \lambda|u|^{p-2}u = h(x) & u \in K, \\ \\ u = 0 & \text{on } \partial K \end{cases} \tag{3.1}$$

has a (weak) solution.

Definition 3.2. The triple (ψ, φ, K) satisfies the point -wise invariance condition at a point $u \in V$ if there exists a convex Gâteaux differentiable function $G : V \to \mathbb{R}$ and a point $v \in K$ such that

$$D\psi(v) + DG(v) = D\varphi(u) + DG(u).$$

Next theorem is a new variational principle that we utilize to verify the main result.

Theorem 3.3. *Let V be a reflexive Banach space and K be a convex and weakly closed convex subset of V. Let $\psi : V \to \mathbb{R}\cup\{+\infty\}$ be a convex, lower semi continuous function which is Gâteaux differentiable on K, and let $\varphi \in C^1(V, \mathbb{R})$. Assume that the following two assertions hold:*

(i) *The functional* $I_K : V \to \mathbb{R} \cup \{+\infty\}$ *defined by* $I_K(w) = \psi_K(w) - \varphi(w)$ *has a critical point* $u \in V$ *as in Definition 3.2,*

(ii) *the triple* (ψ_K, φ, K) *satisfies in the point-wise invariance condition at the point* u.

Then $u \in K$ *is a solution of the equation*

$$D\psi_K(u) = D\varphi(u).$$

Here is our approach.

Theorem 3.4. *Assume* B *is the unit ball centered at the origin in* $\mathbb{R}^N, N \geq 3,\ p < q < p^*$ *and* $a \in L^\infty(0,1)$ *is an increasing positive nonconstant radial function. Then the problem (1.1) admits at least one radial increasing solution*

Proof. We consider $V = D^{1,2}_{0,r}(B) \cap L^p_a(B)$ and equip it with the suitable norm to become a reflexive Banach space. By using MPG theorem we show that I_K which is Euler-Lagrang functional restricted to closed convex set K has a critical point, where K includes increasing radial functions. Then we find v corresponding to the u, as $D\psi_K(v) = D\varphi(u)$. Finally, Using Theorem 3.3 we show that u is a solution of problem (1.1).

□

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EXISTENCE OF SOLUTIONS FOR A SYSTEM OF MULTI-POINS BOUNDARY VALUE PROBLEMS BY FIXED POINT THEORY

R. SAHANDI TOROGH

Abstract. In this paper we investigated the existence of positive solutions for a class of multi points boundary value problems. We define a cone and an operator. Then, we establish some conditions under which this problem can be solvable. By some fixed point theorems in cones, we prove the existence of solution to this system.

1. Introduction

This paper is concerned with a multi-point boundary value problem system. We study the existence of positive solutions for a class of boundary problems:

$$\begin{cases} \triangle_{p_1} u + f_1(u) + g_1(v) = 0 \quad (1.1) \\ \triangle_{p_2} v + f_2(u) + g_2(v) = 0 \end{cases}$$

$$\begin{cases} u(0) = \sum_{i=1}^{n} a_i u(\xi_i) \quad , \ u(1) = \sum_{i=1}^{n} a_i u(\eta_i) \ (1.2) \\ v(0) = \sum_{i=1}^{n} b_i v(\xi_i) \quad , \ v(1) = \sum_{i=1}^{n} b_i v(\eta_i) \end{cases}$$

where

$$\triangle_{p_i} s = \phi_{p_i}(s'), \phi_{p_i}(s) = |s|^{p_i - 2} s, p_i > 1, \phi_{q_i} = (\phi_{p_i})^{-1}, \frac{1}{p_i} + \frac{1}{q_i} = 1, a_i \geq 0, b_i \geq 0, 0 \leq \sum_{i=1}^{n} a_i < 1,$$

$$0 \leq \sum_{i=1}^{n} b_i < 1, 0 < \xi_1 < \xi_2 < ... < \xi_n < \frac{1}{2}, \xi_i + \eta_i = 1, i = 1, 2, ..., n.$$

Keywords: boundary value problem, positive solution, fixed point index, Jensen's inequality).

AMS Mathematical Subject Classification [2010]: 34B20; 34B15; 47H11; 47H07.

and $f_i, g_i \in C([0,+\infty),[0,+\infty))$.
Recently, the existence and multiplicity of positive solutions to boundary value problems have been studied by many authors [1-9, 11].
In [1] author, studied the existence of positive solution for the following system:
$$\begin{cases} \varphi_{p_1}(u_1') + h_1(t)f_1(u_1,u_2) = 0 \\ \varphi_{p_2}(u_2') + h_2(t)f_2(u_1,u_2) = 0 \\ u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0 \end{cases}$$
A completely continuous operator is defined and the existence of multiple solutions is proved.

2. The preliminary lemmas

Definitions.[2]. Let E be a real Banach space. A nonempty convex closed set $K \subset E$ is said to be a cone provided that
$i)au$ for all $u \in K$ and $a \geq 0$ and
$ii)u, -u \in K$ implies that $u = 0$.
Let $E := C([0,1], \mathbf{R})$ and

$$K := \{u \in E : u(t)0, t \in [0,1]\}, \|u\| := max\{|u(t)| : t \in [0,1]\}$$

and $\|(u,v)\| := max\{\|u\|, \|v\|\}, (u,v) \in E \times E$
$B_r = \{(u,v) \in E^2 : \|(u,v)\| < r\}$ for $r > 0$. Then $(E, \| \ . \ \|)$ is a real Banach space and K, K^2 are cones.
We define the following operators:

$$L(u,v) = (L_1(u,v)(t), L_2(u,v)(t))$$

such that

$$L_1(u,v)(t) := \frac{1}{1-\sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i \int_0^t \varphi_{p_1}^{-1}(\int_s^1 (f_1(u(r)) + g_1(v(r)))dr)ds$$

$$L_2(u,v)(t) := \frac{1}{1-\sum_{i=1}^{n} b_i} \sum_{i=1}^{n} b_i \int_0^t \varphi_{p_2}^{-1}(\int_s^1 (f_2(u(r)) + g_2(v(r)))dr)ds$$

Hence $L : K^2 \to K^2$ is an completely continuous operator.
The main tool of this paper is the following theorem.

Theorem 2.1. *([10]). Let E be a real Banach space and $K \subset E$ a cone. Suppose that $\Omega \subset E$ is a bounded open set and $T : \overline{\Omega} \to K$ is a completely continuous operator. Let $x_0 \in K \setminus \{0\}$*
(I) If $x - Tx \neq \eta x_0$ for $\eta \geq 0, x \in \delta\Omega \cap K$, then $i(T, \Omega \bigcap k, k) = 0$, where i indicates the fixed point index on K.
(II) Let E be a real Banach space and K a cone in E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and $T : \overline{\Omega} \to K$ is a completely continuous operator. If

$x - \eta Tx \neq \eta x_0$ *for* $\eta \in [0,1], x \in \delta\Omega \cap K,$ *then* $i(T, \Omega \bigcap k, k) = 1.$

Remark 2.2. Suppose that $x \in K$ is concave on [0,1] , $\parallel x \parallel = x(1)$. then $\parallel x \parallel \leq \frac{\pi^2}{4}\int_0^1 x(t) sin\frac{\pi}{2}tdt.$

Lemma 2.3. *(Jensen's Integral Inequality for Nonnegative Concave Functions) Suppose that* $u([a,b], \boldsymbol{R}), \phi(R^+, R^+)$. *If* ϕ *is cocave, then*

$$\phi(\frac{1}{b-a}\int_a^b u(t)dt) \geq \frac{1}{b-a}\int_a^b \phi(u(t))dt.$$

In particular, if $b - a \leq 1$, *then we have* $(\int_a^b u(t)dt)^\alpha \geq (b-a)^{\alpha-1}\int_a^b u^\alpha(t)dt \geq \int_a^b u^\alpha(t)dt$ *for* $0 < \alpha \leq 1$. *Let* C *is a cone in a Banach space* $(K, \parallel . \parallel)$, *the Bonsall cone spectral radius of* T *is defined by* $R_C(T) := lim_{m\to\infty} \parallel T^m \parallel^{\frac{1}{m}} = inf_{m\geq 1} \parallel T^m \parallel^{\frac{1}{m}}$

Lemma 2.4. *([4]), Let* C *is a cone in a Banach space* $(K, \parallel . \parallel)$, *and* $T : C \to C$ *is a countiuous homogeneous. If* $R_C(T) < 1, u, u_0 \in C$ *satisfy* $u \leq Tu + u_0$, *then* $u \leq (I-T)_C^{-1}u_0$, *where the Bonsall cone spectral radius of* T *is* $R_C(T) := lim_{m\to\infty} \parallel T^m \parallel^{\frac{1}{m}} = inf_{m1} \parallel T^m \parallel^{\frac{1}{m}}$ *and* $(I-T)_C^{-1}$ *is the inverse operator of* $I - T$ *on* C.

Let $\Theta > 1$, we define : $\Theta' := min\{2, \Theta\}, \Theta'' := max\{2, \Theta\}, \Theta_{.} = \frac{\Theta'-1}{\Theta-1}, \Theta^{.} = \frac{\Theta''-1}{\Theta-1}$

3. Main results

Suppose that f, g satisfy:
$A_1)p_1, p_2 > 1, f_i, g_i \in C([0,+\infty), [0,+\infty))$
A_2) There are two constants $\alpha > \frac{\pi^4}{16}, d > 0$ and two nonnegative functions $m_1, n_1 \in C([0,+\infty), [0,+\infty))$ such that
$i)m_1^{p_{1.}}$ is concave on $[0,+\infty)$
$ii)f_1(u) + g_1(v) \geq m_1(v^{p_2'-1}) - c, f_2(u) + g_2(v) \geq n_1(u^{p_1'-1}) - c$ for all $u, v \in [0,+\infty)$,
$iii)m_1^{p_{1.}}(n_1^{p_{2.}}(w)) \geq \iota w - d$ for all $w \in [0,+\infty)$
A_3)There are nonnegative constants $\alpha_1, \beta_1, \gamma_1, \delta_1, R$ such that $R_{K^2}(T_1) < 1$ and

$$f_1(u) + g_1(v) \leq \alpha_1 u^{p_1-1} + \beta_1 v^{p_1-1}, f_2(u) + g_2(v) \leq \gamma_1 u^{p_2-1} + \delta_1 v^{p_2-1}$$

for $u, v \in [0, R, t \in [0,1]$ we define $T_1 : K^2 \to K^2$ by $T_1(u,v)(t) =$

$$(\frac{1}{1-\sum_{i=1}^n a_i}\sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1}(\int_s^1 (\alpha_1 u^{p_1-1}(r) + \beta_1 v^{p_1-1}(r)dr)ds,$$

$$\frac{1}{1-\sum_{i=1}^n b_i}\sum_{i=1}^n b_i \int_0^t \varphi_{p_2}^{-1}(\int_s^1 (\gamma_1 u^{p_2-1}(r) + \delta_1 v^{p_2-1}(r)dr)ds)$$

Theorem 3.1. *Let assumptions A_1, A_2, A_3 be satisfied. Then the problem (1.1) has at least one positive solution.*

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A REFINED UPPER BOUND FOR ENTROPY OF STOCHASTIC PROCESS

YAMIN SAYYARI

Abstract. Estimation of Shannon's entropies of stochastic process from numerical simulation of long orbits is difficult. Our aim within this paper is to present a strong upper bound for the Shannon's entropy of information sources.

1. Introduction

In [2, 3, 5], the authors presented a strong upper bound for the classical Shannon entropy. Our purpose within this work is to present a strong upper bound for the Shannon entropy of information sources, refining recent results from the literature.

Let $X \neq \emptyset$ be a set. Then $(X, \mathcal{F}, \mu)$ is called measure probability space if $\mathcal{F}$ is an σ-algebra of subsets of X, μ is a measure on X, and $\mu(X) = 1$.

A stochastic process $\mathbf{S}$ is a sequence $(S_n)_{n=1}^{\infty}$ of the random variables $S_n : X \longrightarrow A$, where $n \in \mathbb{N}$. For given $L \geq 1$ we define a mapping $p : A^L \to [0,1]$ by $p(s_1^L) = \mu\{\omega \in X : S_1(\omega) = s_1, ..., S_L(\omega) = s_L\}$. The Shannon entropy of order L and the Shannon entropy of source $\mathbf{S}$ are respectively defined by

$$H_\mu(S_1^L) = -\frac{1}{L} \sum_{s_1^L \in A^L} p(s_1, ..., s_L) \log p(s_1, ..., s_L), \text{ and } h_\mu(\mathbf{S}) = \lim_{L \to \infty} H_\mu(S_1^L).$$

In this paper we use the symbol s_1^L instead of notation $(s_1, ..., s_L)$ and Let $p(s_1^L) \neq 0$ for every $L \in \mathbb{N}$.

Keywords: entropy, Shannon's entropy, information source, stochastic process, Random variable..

AMS Mathematical Subject Classification [2010]: 37B40, 94A17, 26A51.

2. Main results

Theorem 2.1. *Let $I=[a,b]$ be an interval, $H : A^L \longrightarrow I$ be a function, and $f : I \longrightarrow \mathbb{R}$ be a convex function, then*

$$\begin{aligned}&\sum_{s_1^L\in A^L} p(s_1^L)f(H(s_1^L)) - f(\sum_{s_1^L\in A^L} p(s_1^L)H(s_1^L))\\&\geq \max\{p(r_1^L)f(H(r_1^L)) + p(t_1^L)f(H(t_1^L)) + p(u_1^L)f(H(u_1^L))\\&-(p(r_1^L)+p(t_1^L)+p(u_1^L))f(\frac{p(r_1^L)H(r_1^L)+p(t_1^L)H(t_1^L)+p(u_1^L)H(u_1^L)}{p(r_1^L)+p(t_1^L)+p(u_1^L)})\},\end{aligned}$$

where the maximum is taken over all distinct $r_1^L, t_1^L, u_1^L \in A^L$.

Proof. Choose arbitrary $t_1^L, r_1^L, u_1^L \in A^L$. So,

$$\begin{aligned}&f(\sum_{s_1^L\in A^L} p(s_1^L)H(s_1^L)) = f(\sum_{s_1^L\neq r_1^L,t_1^L,u_1^L\in A^L} p(s_1^L)H(s_1^L))\\&+(p(r_1^L)+p(t_1^L)+p(u_1^L))(\frac{p(r_1^L)H(r_1^L)+p(t_1^L)H(t_1^L)+p(u_1^L)H(u_1^L)}{p(r_1^L)+p(t_1^L)+p(u_1^L)})\\&\leq \sum_{s_1^L\neq r_1^L,t_1^L.u_1^L\in A^L} p(s_1^L)f(H(s_1^L))\\&+(p(r_1^L)+p(t_1^L)+p(u_1^L))f(\frac{p(r_1^L)H(r_1^L)+p(t_1^L)H(t_1^L)+p(u_1^L)H(u_1^L)}{p(r_1^L)+p(t_1^L)+p(u_1^L)}).\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{s_1^L\in A^L} p(s_1^L)f(H(s_1^L)) - f(\sum_{s_1^L\in A^L} p(s_1^L)H(s_1^L))\\&\geq p(r_1^L)f(H(r_1^L)) + p(t_1^L)f(H(t_1^L)) + p(u_1^L)f(H(u_1^L))\\&-(p(r_1^L)+p(t_1^L)+p(u_1^L))f(\frac{p(r_1^L)H(r_1^L)+p(t_1^L)H(t_1^L)+p(u_1^L)H(u_1^L)}{p(r_1^L)+p(t_1^L)+p(u_1^L)}).\end{aligned}$$

Since $s_1^L, t_1^L \in A^L, u_L^1$ are arbitrary,

$$\begin{aligned}&\sum_{s_1^L\in A^L} p(s_1^L)f(H(s_1^L)) - f(\sum_{s_1^L\in A^L} p(s_1^L)H(s_1^L))\\&\geq \max\{p(r_1^L)f(H(r_1^L)) + p(t_1^L)f(H(t_1^L)) + p(u_1^L)f(H(u_1^L))\}\\&-(p(r_1^L)+p(t_1^L)+p(u_1^L))f(\frac{p(r_1^L)H(r_1^L)+p(t_1^L)H(t_1^L)+p(u_1^L)H(u_1^L)}{p(r_1^L)+p(t_1^L)+p(u_1^L)})\}.\end{aligned}$$

On the other hand,

$$
\begin{aligned}
& f(\frac{p(r_1^L)H(r_1^L)+p(t_1^L)H(t_1^L)+p(u_1^L)H(u_1^L)}{p(r_1^L)+p(t_1^L)+p(u_1^L)}) \\
&= f(\frac{p(r_1^L)+p(t_1^L)}{p(r_1^L)+p(t_1^L)+p(u_1^L)}\frac{p(r_1^L)H(r_1^L)+p(t_1^L)H(t_1^L)}{p(r_1^L)+p(t_1^L)}+\frac{p(u_1^L)H(u_1^L)}{p(r_1^L)+p(t_1^L)+p(u_1^L)}) \\
&\leq \frac{p(r_1^L)+p(t_1^L)}{p(r_1^L)+p(t_1^L)+p(u_1^L)}f(\frac{p(r_1^L)H(r_1^L)+p(t_1^L)H(t_1^L)}{p(r_1^L)+p(t_1^L)}) \\
&+ \frac{p(u_1^L)}{p(r_1^L)+p(t_1^L)+p(u_1^L)}f(H(u_1^L)).
\end{aligned}
$$

So,

$$
\begin{aligned}
& (p(r_1^L)+p(t_1^L)+p(u_1^L))f(\frac{p(r_1^L)H(r_1^L)+p(t_1^L)H(t_1^L)+p(u_1^L)H(u_1^L)}{p(r_1^L)+p(t_1^L)+p(u_1^L)}) \\
&\leq (p(r_1^L)+p(t_1^L))f(\frac{p(r_1^L)H(r_1^L)+p(t_1^L)H(t_1^L)}{p(r_1^L)+p(t_1^L)})+(p(u_1^L))f(H(u_1^L)).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& p(r_1^L)f(H(r_1^L))+p(t_1^L)f(H(t_1^L))+p(u_1^L)f(H(u_1^L)) \\
&- (p(r_1^L)+p(t_1^L)+p(u_1^L))f(\frac{p(r_1^L)H(r_1^L)+p(t_1^L)H(t_1^L)+p(u_1^L)H(u_1^L)}{p(r_1^L)+p(t_1^L)+p(u_1^L)}) \\
&\geq p(r_1^L)f(H(r_1^L))+p(t_1^L)f(H(t_1^L))\} \\
&- (p(r_1^L)+p(t_1^L))f(\frac{p(r_1^L)H(r_1^L)+p(t_1^L)H(t_1^L)}{p(r_1^L)+p(t_1^L)}),
\end{aligned}
$$

which completes the proof. □

In order to present the generalization we define some notation, as follows:

$$T_k := \max\{\sum_{i=1}^{k} p({r_i}_1^L)f(H({r_i}_1^L)) - (\sum_{i=1}^{k} p({r_i}_1^L))f(\frac{\sum_{i=1}^{k} p({r_i}_1^L)H({r_i}_1^L)}{\sum_{i=1}^{k} p({r_i}_1^L)})\}$$

where $2 \leq k \leq N^L - 1$, the maximum is taken over all distinct ${r_i}_1^L \in A^L$.

Theorem 2.2. *Let $I = [a,b]$ be an interval, $H : A^L \longrightarrow I$ be a function, $|A| = N$ and $f : I \longrightarrow \mathbb{R}$ be a convex function, then*

$$0 \leq T_2 \leq T_3 \leq ... \leq T_{N^L-1} \leq \sum_{s_1^L \in A^L} p(s_1^L)f(H(s_1^L)) - f(\sum_{s_1^L \in A^L} p(s_1^L)H(s_1^L)).$$

Proof. The proof is similar to the proof of Theorem 2.1. □

Theorem 2.3. $h_\mu(\mathbf{S}) \leq \log N - \max_k\{\lim_{L\to\infty} \frac{1}{L} log[\{\frac{k}{\sum_{i=1}^{k} p({r_i}_1^L)}\}^{\sum_{i=1}^{k} p({r_i}_1^L)}]$
$\times [\prod_{i=1}^{k}\{p({r_i}_1^L)\}^{p({r_i}_1^L)}]\}$

Proof. Since

$$-LH_\mu(S_1^L)+\log(N^L)\geq \max_k\{-\sum_{i=1}^k p(r_{i1}^L)\log(\frac{1}{p(r_{i1}^L)})+(\sum_{i=1}^k p(r_{i1}^L))$$
$$\times\log(\frac{k}{\sum_{i=1}^k p(r_{i1}^L)})\}=\max_k\{\log(\prod_{i=1}^k\{p(r_{i1}^L)\}^{p(r_{i1}^L)})$$
$$+\log[\{\frac{k}{\sum_{i=1}^k p(r_{i1}^L)}\}^{\sum_{i=1}^k p(r_{i1}^L)}]\},$$

$$H_\mu(S_1^L)\leq \log N-\max_k\{\frac{1}{L}log[\{\frac{k}{\sum_{i=1}^k p(r_{i1}^L)}\}^{\sum_{i=1}^k p(r_{i1}^L)}][\prod_{i=1}^k\{p(r_{i1}^L)\}^{p(r_{i1}^L)}]\}.$$

Therefore,

$$h_\mu(\mathbf{S})$$
$$\leq \log N-\lim_{L\to\infty}\max_k\{\frac{1}{L}log[\{\frac{k}{\sum_{i=1}^k p(r_{i1}^L)}\}^{\sum_{i=1}^k p(r_{i1}^L)}][\prod_{i=1}^k\{p(r_{i1}^L)\}^{p(r_{i1}^L)}]\}$$
$$\leq \log N-\max_{2\leq k\leq N^L-1}\{\lim_{L\to\infty}\frac{1}{L}log[\{\frac{k}{\sum_{i=1}^k p(r_{i1}^L)}\}^{\sum_{i=1}^k p(r_{i1}^L)}][\prod_{i=1}^k\{p(r_{i1}^L)\}^{p(r_{i1}^L)}]\}.$$

□

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ON $\underline{d}$-SHADOWING PROPERTY OF SEMIGROUP ACTIONS

ZAHRA SHABANI

ABSTRACT. In this talk, we introduce the notion of $\underline{d}$-shadowing property of semigroup actions on the compact metric spaces and investigate its relation with average shadowing and ergodic shadowing properties.

1. Introduction

The concept of shadowing was originated from the Anosov closing lemma and because of its rich consequences, shadowing plays an important role in the general qualitative theory of dynamical systems. It is considerably developed in recent years and many authors have studied several kinds of shadowing including ergodic shadowing [3], $\underline{d}$-shadowing [1], and average shadowing [4], which have the common motivation of studying the behavior of a dynamical system by using the closeness of approximate orbits and true orbit.

The continuous actions associated with finitely generated semigroups on compact metric spaces are also called iterated function systems(IFSs). Bahabadi [2] introduced the notions of shadowing and average shadowing properties for free semigroup actions(IFSs). He obtained that a semigroup with average shadowing property is chain transitive. Wu, Wang, and Liang [6] improved this result and showed that the average shadowing property for a semigroup actions implies chain mixing.

Ergodic shadowing and pseudo orbital specification properties for finitely generated semigroup actions were introduced in [5], and it was proved that these properties

Keywords: semigroup actions, ergodic shadowing, $\underline{d}$-shadowing.
AMS Mathematical Subject Classification [2010]: 37B05, 37C50.

are equivalent to the semigroup being topologically mixing and having the ordinary shadowing property. Here, we obtain the following result.

Theorem 1.1. *Let G be a semigroup generated by the family $\{id, g_1, \dots, g_m\}$ of continuous maps on the compact metric space X where g_i is surjective for some $i \in \{1, \dots, m\}$. Then the following properties on G are equivalent:*

(1) *ergodic shadowing,*
(2) *$\underline{d}$-shadowing and ordinary shadowing,*
(3) *chain mixing and ordinary shadowing,*
(4) *topologically mixing and ordinary shadowing,*
(5) *average shadowing property and ordinary shadowing,*

2. Main results

In this section, we first describe some notations and definitions, and then we state our main results. Let $G_1 = \{id, g_1, \dots, g_m\}$ be a finite collection of continuous maps on the compact metric space X. Th symbolic dynamic is a way to display the elements of semigroup G associated with this family. Let Σ^m be the space of infinite sequences of m symbols $\{1, \dots, m\}$, that is, $\Sigma^m = \{\omega = \omega_0\omega_1\omega_2 \dots : \omega_i \in \{1, \dots, m\}\}$. For any sequence $\omega = \omega_0\omega_1\omega_2 \dots \in \Sigma^m$, take $g^0_\omega := id$ and for any $n > 0$, $g^n_\omega(x) := g_{\omega_{n-1}} \circ \dots \circ g_{\omega_0}(x)$.

For a sequence $\xi = \{x_i\}_{i \geq 0}$, $\delta > 0$, and $\omega = \omega_0\omega_1\omega_2 \dots \in \Sigma^m$, put

$$Np(\xi, \omega, \delta) = \{i \in \mathbb{Z}^+ : d(g_{\omega_i}(x_i), x_{i+1}) \geq \delta\}, \quad Np_n(\xi, \omega, \delta) = Np(\xi, \omega, \delta) \cap [0, n).$$

Given a sequence $\xi = \{x_i\}_{i \geq 0}$ and a point $z \in X$, consider

$$Ns(\xi, \omega, z, \delta) = \{i \in \mathbb{Z}^+ : d(g^i_\omega(z), x_i) \geq \epsilon\}, \quad Ns_n(\xi, \omega, z, \delta) = Ns(\xi, \omega, z, \delta) \cap [0, n).$$

Definition 2.1. Let $\delta > 0$ and let $\xi = \{x_i\}_{i \geq 0} \subset X$. We say that ξ is a (δ, ω)-*ergodic pseudo orbit* of G for some $\omega = \omega_0\omega_1 \dots \in \Sigma^m$ provided that the set $Np(\xi, \omega, \delta)$ has zero density (see [5]).

Definition 2.2. A semigroup G has the $\underline{d}$-*shadowing property* if for each $\epsilon > 0$ there exists $\delta > 0$ such that every (δ, ω)-ergodic pseudo orbit ξ of G can be $\underline{d}$-ϵ-shadowed by some point z in X, that is, there exists $\varphi \in \Sigma^m$ with $\varphi_i = \omega_i$ for $i \in Np^c(\xi, \omega, \delta)$ such that $\underline{d}(Ns(\xi, \varphi, z, \epsilon)) > 0$.

Here, we state our main results.

Theorem 2.3. *Suppose that $g_1, \dots, g_m$ are continuous maps on a compact metric space X such that one of them is surjective. If the semigroup G generated by these maps has the $\underline{d}$-shadowing property, then it is chain transitive.*

Proof. Let $x, y \in X$ and $\epsilon > 0$ be given. Let $\delta > 0$ be as in the definition of the $\underline{d}$-shadowing property. Set $m_0 = -1$ and consider $M = \{m_i\}_{i\in\mathbb{N}}$ an increasing subsequence of natural numbers with $d(M) = 0$ such that $\mathbb{N}$ partitions into two subsets M_1 and M_2 satisfying

$$M_1 = \{0, 1, 2, \ldots, m_1\} \cup \{m_2+1, m_2+2, \ldots, m_3\} \cup \{m_4+1, m_4+2, \ldots, m_5\} \cup \ldots$$

and $M_2 = \mathbb{N} \setminus M_1$ with $\overline{d}(M_1) = \overline{d}(M_2) = 1$. Fix a sequence $\omega = \omega_0\omega_1 \ldots \in \Sigma^m$. Suppose that $g_\ell, \ell \in \{1, \ldots, m\}$ is surjective. For each $j \geq 0$, consider the sequence

$$y_i := \begin{cases} g_\omega^{i-m_{2j}-1}(x), & m_{2j}+1 \leq i \leq m_{2j+1}, \\ g_\ell^{i-m_{2j+2}}(y), & m_{2j+1}+1 \leq i \leq m_{2j+2}. \end{cases}$$

Then $\{y_i\}_{i\geq 0}$ is a (δ, η)-ergodic pseudo orbit of G for some $\eta \in \Sigma^m$. The $\underline{d}$-shadowing property of G yields that there are a sequence $\gamma \in \Sigma^m$ and a point $z \in X$ such that $Ns(\{y_i\}_{i\geq 0}, \gamma, z, \epsilon)$ has positive lower density, which implies that $Ns(\{y_i\}_{i\geq 0}, \gamma, z, \epsilon) \cap M_i \neq \emptyset$, for $i = 1, 2$. Thus we can find positive integers i, j, r, s with $r < s$ such that

$$d(g_\gamma^r(z), g_\omega^i(x)) < \epsilon, \quad d(g_\gamma^s(z), g_\ell^{-j}(y)) < \epsilon.$$

One can see that the

$$x, g_{\omega_0}(x), \ldots, g_\omega^{i-1}(x), g_\gamma^r(z), g_\gamma^{r+1}(z), \ldots, g_\gamma^{s-1}(z), g_\ell^{-j}(y), g_\ell^{-j+1}(y), y$$

is an (ϵ, ζ)-chain from x to y. □

Theorem 2.4. *If a semigroup G has the $\underline{d}$-shadowing property, then the semigroup G^k has the $\underline{d}$-shadowing property for any $k \in \mathbb{N}$.*

Proof. Given fixed $\epsilon > 0$ and $k \in \mathbb{N}$. Take $\eta < \epsilon/k$ such that $d(x, y) < \eta$ implies that $d(g_w^i(x), g_w^i(y)) < \epsilon/k$, for any $i = 1, \ldots, m$ and $w \in \mathcal{A}_k^m$. Thus, for any finite (η, w)-chain $\{r_0, \ldots, r_k\}$, we have $d(g_w^i(r_0), r_i) < \epsilon/2, i = 0, 1, \ldots, k$. Let $\delta < \eta$ be an η-modulus of ergodic shadowing property for the semigroup G. Suppose that $\xi = \{x_0, x_1, \ldots, x_{m_0}; x_{m_0+1}, \ldots, x_{m_1}; x_{m_1+1}, \ldots\}$ is a δ-ergodic pseudo orbit for the semigroup G^k. It can be verified that there is a sequence $\omega = \omega_0\omega_1 \ldots \in \Sigma^m$ such that

$$\{i \in \mathbb{Z}^+; d(g_{\omega_{(i+1)k-1}} \circ \cdots \circ g_{\omega_{ik}}(x_i), x_{i+1}) \geq \delta\} \subseteq \{m_i\}_{i\in\mathbb{Z}^+}.$$

Take a sequence $\{y_i\}_{i\geq 0}$ with

$$y_i := \begin{cases} x_{i/k}, & i = nk \quad \text{for some} \quad n \in \mathbb{Z}^+ \\ g^j_{\sigma^{nk}\omega}(x_n), & i = nk + j \quad \text{for some} \quad n \in \mathbb{Z}^+, 0 < j < k. \end{cases}$$

Then $\{y_i\}_{i\geq 0}$ is a δ-ergodic pseudo orbit for the semigroup G. The $\underline{d}$-shadowing property of G implies that there are a point $z \in X$ and $\gamma \in \Sigma^m$ such that $\{y_i\}_{i\geq 0}$

is $\underline{d}$-η-shadowed by $\{g_\gamma^i(z)\}_{i\geq 0}$. Since $M=\{m_i\}_{i\in\mathbb{Z}^+}$ has zero density, we can assume that $Ns(\{y_i\}_{i\geq 0},\gamma,z,\eta)\cap M=\emptyset$. If ever, $m\in Ns(\{y_i\}_{i\geq 0},\gamma,z,\eta)\cap M$, then $d(g_\gamma^m(z),y_m)<\eta$. Let $m=nk+j$ for some $n\in\mathbb{Z}^+$ and $0<j<k$. Then

$$\{g_\gamma^{m-1}(z),g^j_{\sigma^{nk}\omega}(x_n),g^{j+1}_{\sigma^{nk}\omega}(x_n),\ldots,g^{k-1}_{\sigma^{nk}\omega}(x_n),x_{n+1}\}$$

is an (η,w)-chain with $w=\gamma_{m-1}\omega_{nk+1}\ldots\omega_{(n+1)k-1}\omega_{(n+1)k}\in\mathcal{A}_k^m$. Therefore, we have

$$d(g_\eta^{m+k-j}(z),x_{n+1})=d(g_w^{k-j}(g_\gamma^{m-1}(z)),x_{n+1})<\epsilon.$$

Set $B_1:=Ns(\{y_i\}_{i\geq 0},\gamma,z,\eta)$. By an induction argument, it is obtained that the sequence ξ is (η,ϵ)-traced by the orbit of z under the semigroup G^k along a set $B\subset\mathbb{Z}^+$ with $k\underline{d}(B)\geq\underline{d}(B_1)$. This terminates the proof. $\square$

Using Theorem 2.3 and Theorem 2.4, the following result is obtained.

Corollary 2.5. *Let G be a semigroup generated by a finite family of continuous self-maps on a compact metric space X such that one of the generators is a surjective map. If G has $\underline{d}$-shadowing property, then it is chain mixing.*

Theorem 2.6. *Let G be a semigroup generated by the family $\{id,g_1,\ldots,g_m\}$ on a compact metric space X, satisfies that g_i is surjective for some $i\in\{1,\ldots,m\}$ and let G have the shadowing property. Then G is topologically mixing if and only if it has the average shadowing property.*

Proof of Theorem 1.1. In [5, Theorem 1.1], it was proved that (1) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4). The remaining implications obtain from Theorem 2.6 and Corollary 2.5. $\square$

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CONSTRUCTION OF IMPLICIT-EXPLICIT SGLMS

M. SHARIFI

Abstract. In this paper, we construct a class of numerical methods for solving initial value problems of differential equations which have both non-stiff and stiff parts. For such systems the non-stiff part and stiff part can be solved by explicit general linear methods (GLMs) and implicit second derivative general linear methods (SGLMs), respectively. Assuming that the implicit part of methods is $L-$stable, we construct methods up to order four with large absolute stability regions and show their efficiency by applying to some well-known problems.

1. Introduction

For many systems of ordinary differential equations (ODEs) there are natural splitting of the right hand side of differential systems into two parts. Such systems can be written in the form

$$\begin{cases} y'(x) = f_1\big(y(x)\big) + f_2\big(y(x)\big), & x \in [x_0, \bar{x}], \\ y(x_0) = y_0, \end{cases} \tag{1.1}$$

where $f_1(y)$ represents the non-stiff part and $f_2(y)$ represents the stiff part of (1.1). The non-stiff part is solved by the explicit GLMs were introduced by Butcher [5]

Keywords: IMEX methods, General linear methods, Second derivative of GLMs, Order conditions, Stability analysis.

which take the form

$$
\begin{aligned}
Y_i^{[n]} &= h\sum_{j=1}^{s} a_{ij}F\big(Y_j^{[n]}\big) + \sum_{j=1}^{r} u_{ij}y_j^{[n-1]},\ i=1,2,\dots,s,\\
y_i^{[n]} &= h\sum_{j=1}^{s} b_{ij}F\big(Y_j^{[n]}\big) + \sum_{j=1}^{r} v_{ij}y_j^{[n-1]},\ i=1,2,\dots,r,
\end{aligned}
\tag{1.2}
$$

for $n = 1,2,\dots,N$ and $a_{ii} = 0$ for $j \geq i$. Here $\big[Y_i^{[n]}\big]_{i=1}^{s}$ are approximations of stage order q to $y(x_{n-1}+c_ih)$, i.e. $Y_i^{[n]} = y(x_{n-1}+c_ih)+\mathcal{O}(h^{q+1})$, and $\big[y_i^{[n]}\big]_{i=1}^{r}$ are approximations of order p to the linear combinations of the derivative of the solution y at the point x_n,

$$
y_i^{[n]} = \sum_{k=0}^{p} q_{ik}h^k y^{(k)}(x_n),
$$

for some real vectors $q_k = [q_{ik}]_{i=1}^{r}$, $k = 0,1,\dots,p$. The stiff part is solved by the implicit SGLMs were introduced by Butcher and Hojjati [6] and further studies on these methods were done by Abdi and Hojjati, see [1, 2]. SGLMs take the form

$$
\begin{aligned}
Y_i^{[n]} &= h\sum_{j=1}^{s} \bar{a}_{ij}F\big(Y_j^{[n]}\big) + h^2\sum_{j=1}^{s} \hat{a}_{ij}G\big(Y_j^{[n]}\big) + \sum_{j=1}^{r} u_{ij}y_j^{[n-1]},\ i=1,2,\dots,s,\\
y_i^{[n]} &= h\sum_{j=1}^{s} \bar{b}_{ij}F\big(Y_j^{[n]}\big) + h^2\sum_{j=1}^{s} \hat{b}_{ij}G\big(Y_j^{[n]}\big) + \sum_{j=1}^{r} v_{ij}y_j^{[n-1]},\ i=1,2,\dots,r,
\end{aligned}
\tag{1.3}
$$

where $\bar{a}_{ii} = \lambda > 0$, $\hat{a}_{ii} = \mu < 0$ and $G(\cdot) = F'(\cdot)F(\cdot)$.

2. A class of IMEX-SDIMSIMs

First, we consider the transformation $y = x + z$, where

$$
x' = f_1(x+z), \qquad z' = f_2(x+z). \tag{2.1}
$$

For the system (2.1), the non-stiff part and stiff part will be solved by the explicit DIMSIMs and implicit SDIMSIMs respectively, i.e.,

$$
\begin{aligned}
X^{[n]} &= h(A\otimes I_m)f_1\big(X^{[n]}+Z^{[n]}\big) + (U\otimes I_m)x^{[n-1]},\\
x^{[n]} &= h(B\otimes I_m)f_1\big(X^{[n]}+Z^{[n]}\big) + (V\otimes I_m)x^{[n-1]},
\end{aligned}
\tag{2.2}
$$

and

$$
\begin{aligned}
Z^{[n]} &= h(\bar{A}\otimes I_m)f_2\big(X^{[n]}+Z^{[n]}\big) + h^2(\hat{A}\otimes I_m)g\big(X^{[n]}+Z^{[n]}\big) + (U\otimes I_m)z^{[n-1]},\\
z^{[n]} &= h(\bar{B}\otimes I_m)f_2\big(X^{[n]}+Z^{[n]}\big) + h^2(\hat{B}\otimes I_m)g\big(X^{[n]}+Z^{[n]}\big) + (V\otimes I_m)z^{[n-1]}.
\end{aligned}
\tag{2.3}
$$

Combining (2.2) and (2.3) leads to the class of so-called IMEX-SDIMSIMs defined by

(2.4)
$$\begin{aligned} Y^{[n]} &= h(A\otimes I_m)f_1\big(Y^{[n]}\big) + h(\bar{A}\otimes I_m)f_2\big(Y^{[n]}\big) + h^2(\hat{A}\otimes I_m)g\big(Y^{[n]}\big) + (U\otimes I_m)y^{[n-1]},\\ y^{[n]} &= h(B\otimes I_m)f_1\big(Y^{[n]}\big) + h(\bar{B}\otimes I_m)f_2\big(Y^{[n]}\big) + h^2(\hat{B}\otimes I_m)g\big(Y^{[n]}\big) + (V\otimes I_m)y^{[n-1]}, \end{aligned}$$

for $n = 1, 2, \ldots, N$ with $g(\cdot) = f_2'(\cdot)\big(f_1(\cdot) + f_2(\cdot)\big)$.

It was proved in [4] that if explicit and implicit method has order p and stage order $p = q$, then the overall method (2.4) has also order p and stage order $p = q$.

3. Stability analysis of IMEX-SDIMSIMs

To analyze the stability properties of IMEX-SDIMSIMs, we will imply (2.4) to the test equation

$$y'(x) = \lambda_0 y(x) + \lambda_1 y(x), \quad x \geq 0,$$

where λ_0 and λ_1 are complex parameters corresponding to the non-stiff part and stiff part of (1.1), we obtain $y^{[n]} = M(z_0, z_1)y^{[n]}$, $n = 0, 1, \ldots$, where $z_0 = \lambda_0 h,\ z_1 = \lambda_1 h$. Here $M(z_0, z_1)$ is the stability matrix defined by

$$M(z_0, z_1) = V + (z_0 B + z_1 \bar{B})(I - z_0 A - z_1 \bar{A} - z_1^2 \hat{A})^{-1} U, \tag{3.1}$$

and also we define the stability function as the stability polynomial of $M(z_0, z_1)$, by

$$p(w, z_0, z_1) = \det\big(wI - M(z_0, z_1)\big),$$

where $w \in \mathbb{C}$. We say that the IMEX-SDIMSIMs (2.4) is stable for given $z_0, z_1 \in \mathbb{C}$ if all roots $w_i(z_0, z_1)$ of stability polynomial $p(w, z_0, z_1)$ are inside of the unit circle. We will be mainly interested in IMEX-SDIMSIMs which are $A-$, $A(\alpha)-$, or $L-$stable with respect to the implicit part $z_1 \in \mathbb{C}$.

4. Numerical experiment

Our test problem is the famous van der Pol system

$$\begin{aligned} y_1' &= y_2,\\ y_2' &= \left((1 - y_1^2)y_2 - y_1)\right)/\epsilon, \end{aligned} \tag{4.1}$$

$x \in [0, 055139]$, where the first component is non-stiff and the second component is stiff for small values of the parameter ϵ. The initial values are

$$y_1(0) = 2, \quad y_2 = -\frac{2}{3} + \frac{10}{81}\epsilon - \frac{292}{2187}\epsilon^2 - \frac{1814}{19683}\epsilon^3 + O(\epsilon^4).$$

In Figure 1 we have presented the results of numerical experiments for $\epsilon = 10^{-6}$ with the methods IMEX-SDIMSIMs of orders $p = 2,\ 3,\ 4$ with $\alpha = \frac{\pi}{2}$, and also $p = 4$ with $\alpha = \frac{\pi}{4}$.

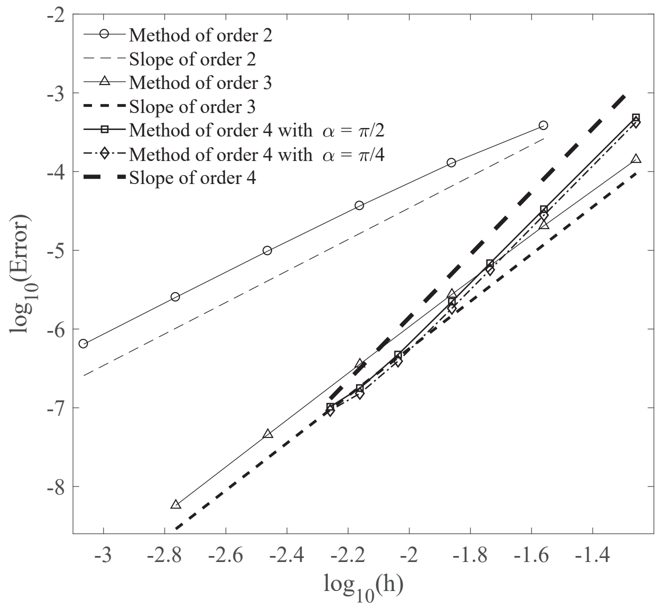

FIGURE 1. Numerical results of the IMEX-SDIMSIMs of orders $p = 2, 3$, 4 with $\alpha = \pi/2$, and $p = 4$ with $\alpha = \pi/4$ for the problem (4.1) with $\epsilon = 10^{-6}$.

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STABILITY ANALYSIS OF A NONSTANDARD FINITE DIFFERENCE SCHEME FOR FRACTIONAL-ORDER CANCER MODEL

M. J. MOHAMMAD TAGHIZADEH, M. AMINIAN, M. NAMJOO, AND M. KARAMI

Abstract. In this paper, fractional-order form of cancer model is introduced. At first, the positivity and boundedness of the model are discussed. Afterwards, the nonstandard finite difference (NSFD) scheme is implemented to study the dynamic behaviours in the fractional-order cancer model. In continuation, the stability analysis of the equilibrium points model are discussed in detail. Numerical results show that the NSFD scheme ie effective when applied to fractional-order cancer model.

1. Introduction

In the last years, fractional calculus has been widely considered in many fields of science, engineering and medical [5]. Fractional calculus provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Also, the fractional ordinary differential equations (FODEs) is widely used in many medical models and engineering research. For this reason, from many years ago researchers have been interested in solving FODEs. Mathematical modeling provides a tool to better understand the transmission dynamics of cancer disease. However, the exact solution of the fractional-order cancer model cannot be easily derived and consequently a numerical method must be utilized. Whenever continuous dynamic system have been converted into discrete system, the properties of continuous system is not transferred fully to the discrete system in the case of large stepsize in the discrete system. However, if we use the NSFD scheme, the properties of the continuous system can be preserved into its discrete system. This paper is devoted to the construct a NSFD scheme for solving the fractional-order cancer model. A sensible model for cancer model at time t is a system of form

$$(1)\qquad \begin{cases} N'(t) = rN - NI, \\ I'(t) = \sigma - \dfrac{1}{r}I - dN + NI, \\ N(0) = N_0, \quad I(0) = I_0 \,. \end{cases}$$

In this model, $N(t)$ is the number of cells population at time t and $I(t)$ stands the number of lymphocyte population at time t. Here r is rate the propagation of cancer cells, σ is a lymphocyte

Keywords: Boundedness, Grunwald-Letnikov derivative, Nonstandard finite difference scheme, Positivity, Stability.

AMS Mathematical Subject Classification [2010]: 37M05, 34A03, 34H10.

current to the interplay location between the populations and d is death rate of the lymphocyte population.

2. Grunwald-Letnikov fractional derivative and NSFD scheme

Derivatives of fractional-order have introduced in several ways. In this paper, we consider Grunwald-Letnikov (GL) approach. The GL derivative for one dimensional fractional-order differential equation takes the following form

$$D^{\alpha}x(t) = f(t, x(t)),\ x(0) = x_0,\ t \in [0, t_f] \tag{2}$$

$$D^{\alpha}x(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{[\frac{t}{h}]} (-1)^j \binom{\alpha}{j} x(t - jh),$$

where $0 < \alpha \leq 1$, D^{α} denotes the GL fractional derivative and h is stepsize. Therefore, Eq. (2) is discretized in the following form

$$\sum_{j=0}^{n+1} c_j^{\alpha} x_{n+1-j} = f(t_{n+1}, x_{n+1}),\ n = 0, 1, 2, \dots$$

where $t_{n+1} = (n+1)h$ and c_j^{α} are the GL coefficients defined as

$$c_j^{\alpha} = (1 - \frac{1+\alpha}{j})c_{j-1}^{\alpha},\ c_0^{\alpha} = h^{-\alpha},\ j = 1, 2, 3, \dots$$

The nonstandard discretization technique is a general scheme where derivative term $\frac{dy}{dt}$ is replaced by $\frac{y(t+h)-y(t)}{\Phi(h)}$ where the function $\Phi(h)$ satisfies $\Phi(h) = h + O(h^2)$. Moreover, linear and nonlinear terms should be represented by nonlocal discrete representation on the discrete computational lattice [1, 2, 3, 4]. By applying the NSFD technique and using the GL discretization method, we obtain

$$x_{n+1} = c_0^{-\alpha} \left(-\sum_{j=0}^{n+1} c_j^{\alpha} x_{n+1-j} + f(t_{n+1}, x_{n+1}) \right),\ n = 0, 1, 2, \dots$$

where $c_0^{\alpha} = (\Phi(h))^{-\alpha}$.

3. Positivity and boundedness

In this section, we prove positivity and boundedness of the solution model of (1).

Theorem 3.1. *Assume that $\sigma > d/r$. If $N(0) > 0$ and $I(0) > d$, then for all $t \geq 0$, $N(t) \geq 0$ and $I(t) \geq d$.*

Proof. Obviously, set $A = \{(0, i) | i \in \mathbb{R}\}$, is invariant. So, for all $t \geq 0$, $N(t) \geq 0$. Let $B = \{t \geq 0 | I(t) < d\}$, we will show that $B = \emptyset$. Suppose that $B \neq \emptyset$ and $b = \inf(B)$. Since $I(0) > d$, and $I(t)$ is continuous, it follows that $b > 0$, $I(b) = d$. Hence, we have $I'(b) = \sigma - d/r > 0$. Therefore, there exists an $\varepsilon > 0$ such that $I'(t) > 0$, for all $t \in (b - \varepsilon, b + \varepsilon) \subseteq (0, +\infty)$. Consequently, $I(t) > I(b) = d$, for all $t \in (b, b + \varepsilon)$. This leads to contradiction with the $b = \inf(B)$. □

Theorem 3.2. *Suppose that $\sigma > d/r > 1$. Then for all $t \geq 0$, $N(t) + I(t) \leq N(0) + I(0) + \sigma/m$, where m is defined by* $\min\{d - r, 1/r\}$.

Proof. Define $K(t) = N(t) + I(t)$. It is evident that $K'(t) + (d - r)N + \frac{1}{r}I = \sigma$. Therefore $K'(t) + mK(t) \leq \sigma$. By Gronwall inequality, we have $K(t) \leq K(0) + \sigma/m$, which completes the proof. □

4. A NSFD scheme for fractional-order cancer model

In this section, we apply an NSFD scheme for solving fractional-order cancer model as the form

$$\begin{cases} D^{\alpha}N(t) = rN - NI, \\ D^{\alpha}I(t) = \sigma - \dfrac{1}{r}I - dN + NI, \\ N(0) = N_0, \quad I(0) = I_0\,. \end{cases} \tag{3}$$

where the state variables and parameters are defined as the integer order cancer model. Applying nonstandard finite difference scheme by replacing the stepsize h by a function $\Phi(h)$ and using the Grunwald-Letnikov approximation, yields the following equations

$$\begin{cases} N_{k+1} = \dfrac{-\sum_{j=0}^{k+1} c_j^{\alpha_1} N_{k+1-j} + rN_k - N_k I_k}{c_0^{\alpha_1}}, \\ I_{k+1} = \dfrac{-\sum_{j=0}^{k+1} c_j^{\alpha_2} I_{k+1-j} + \sigma - dN_{k+1} + N_{k+1}I_{k+1}}{c_0^{\alpha_2} + \frac{1}{r}}, \end{cases}$$

where $c_0^{\alpha_1} = (\Phi_1(h))^{-\alpha_1}$, $c_0^{\alpha_2} = (\Phi_2(h))^{-\alpha_2}$, with $\Phi_1(h) = \dfrac{e^{rh}-1}{r}$, $\Phi_2(h) = r(e^{rh}-1)$. Equilibrium points of system are $E_1 = (0, r\sigma)$ and $E_2 = (\frac{1-\sigma}{r-d}, r)$.

Theorem 4.1. *System* (3) *is*

(i) *locally asymptotically stable around* E_1 *if and only if* $\sigma > 1$.
(ii) *locally asymptotically stable around* E_2 *if and only if* $\sigma < 1$ *and* $\frac{d-r\sigma}{r-d} < 0$.

5. Numerical results and conclusion

Analytical studies always remain incomplete without numerical verification of the results. In this section, numerical results from the implementation of NSFD scheme for the fractional-order cancer model are presented. In Figure 1 is depicted phase trajectory of the fractional-order cancer system for different values of orders and the parameters $r = \sigma = 2$, $d = 0.2$ with initial conditions $N_0 = 2.1$, $I_0 = 2.7$ for simulating time 1000s and stepsize $h = 0.2$. Figure 1 illustrates that the equilibrium point $E_1 = (0, 4)$ is locally stable. In Figure 2 is depicted phase trajectory of the fractional-order cancer model for different values of orders and the parameters $r = 2$, $\sigma = 0.8$, $d = 1$ with initial condition $N_0 = 2.1$ and $I_0 = 2.7$ for simulating time 1000s and stepsize $h = 0.5$. The Figure 2 confirms that (N_k, I_k) will approach the equilibrium point solution $E_2 = (0.2, 2)$ as $k \to +\infty$. The results show that the numerical solutions of the NSFD scheme meet the properties that the physically relevant solutions should have.

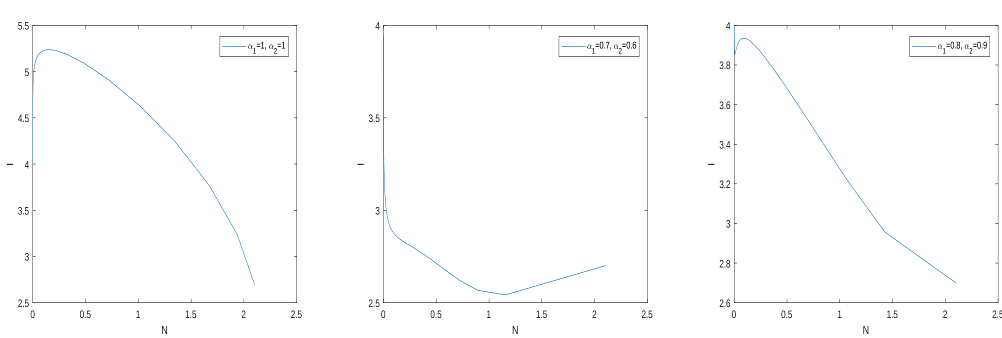

FIGURE 1. Plot of the populations N and I for different values $0 < \alpha_i \leq 1$, $i = 1, 2$ with stepsize $h = 0.2$.

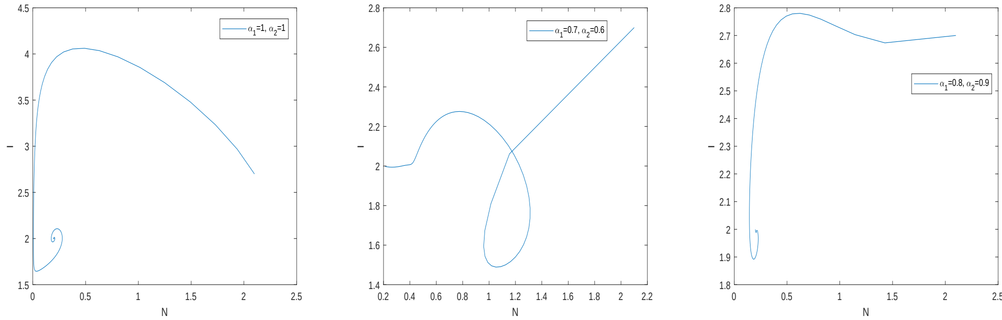

FIGURE 2. Plot of the populations N and I for different values $0 < \alpha_i \leq 1$, $i = 1, 2$ with stepsize $h = 0.5$.

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TIME-REVERSIBLE NUMERICAL METHODS FOR SOLVING HAMILTONIAN SYSTEMS

BEHNAZ TALEBI AND ALI ABDI

Abstract. Symmetric general linear methods (GLMs) are constructed to approximately preserve invariants for Hamiltonian systems. This paper considers 6th-order composite symmetric GLMs, which are applicable to time-reversible differential equations. For this purpose, a nonlinear transformation which is used to convert the GLM to a canonical form in which its starting and finishing methods are trivial. Efficiency and capability of the method are shown by some numerical experiments.

1. Introduction

Symmetry is an essential property of numerical methods with regards to order of accuracy and geometric properties of the solution. A lot of attention has been devoted in recent to Hamiltonian systems. These systems form the most important class of ordinary differential equations (ODEs) in numerical geometric integrals.
Nowadays, numerical solvers for Hamiltonian problems prefer methods of high order, which can preserve quadratic and Hamiltonian invariants over long-time without suffering from the parasitic solutions. In recent, symmetric GLMs, are mostly limited to 4th-order [1]. Moreover, the construction of higher order methods is difficult. The main aim of this paper is to construct and test high-order composite symmetric GLMs with zero parasitic solutions, base on the theory of composition for one-step methods,

Keywords: Ordinary differential equations, Symmetric general liner methods, Hamiltonian problems.

AMS Mathematical Subject Classification [2010]: 65L05, 65L06.

such as Runge-Kutta methods [2].

In the Hamiltonian formulation of mechanics, we describe the system with variables (p, q), where q is a vector describing the position of the system and p a vector describing its momentum.

The Hamiltonian $H(p, q)$ is a function which generates the equations of motion via

$$\dot{p} = -H_q(p, q), \quad \dot{q} = H_p(p, q). \tag{1.1}$$

1.1. **General Linear Methods.** GLMs are numerical methods for solving the initial value problem

$$\begin{aligned} &y'(t) = f(y(t)), \quad t \in [t_0, T], \\ &y(t_0) = y_0, \end{aligned} \tag{1.2}$$

where $f : [t_0, T] \times \mathbb{R}^m \to \mathbb{R}^m$. A GLM is formed of s-many stage equations and r-many update equations, with $r, s \in \mathbb{N}$. The method with time-step h acts upon a set of inputs $y^{[n-1]} = [y_i^{[n-1]}]_{i=1}^r \in X$ and generates outputs $y^{[n]} = [y_i^{[n]}]_{i=1}^r \in X$ via the following equations

$$\begin{aligned} Y_i^{[n]} &= h\sum_{j=1}^{s} a_{ij} f(Y_j^{[n]}) + \sum_{j=1}^{r} u_{ij} y_j^{[n-1]}, \quad i = 1, 2, \ldots, s, \\ y_i^{[n]} &= h\sum_{j=1}^{s} b_{ij} f(Y_j^{[n]}) + \sum_{j=1}^{r} v_{ij} y_j^{[n-1]}, \quad i = 1, 2, \ldots, r. \end{aligned} \tag{1.3}$$

1.2. **Symmetry.** A numerical integration scheme(one-step method) $\mathbf{R}_h$ is symmetric if it is equal to its adjoint $\mathbf{R}_h^\star$, defined such that $\mathbf{R}_h^\star := \mathbf{R}_{-h}^{-1}$.

For GLMs, the action of computing the adjoint often rearrange the inputs of the method.

Definition 1.1. Consider the GLM $\mathcal{M}_h$ and its adjoint method $\mathcal{M}_h^\star : X^m \longrightarrow X^m$ defined such that $\mathcal{M}_h^\star := \mathcal{M}_{-h}^{-1}$. Then $\mathcal{M}_h$ is symmetric if there exists an involution matrix $L \in \mathbb{C}$ such that

$$\mathcal{M}_h(y) = L\mathcal{M}_h^\star(Ly), \quad y \in X^m.$$

2. Canonical Form For GLMs

Canonical methods have the important property that their inputs are independent of h. Thus, we can compose multiple canonical methods of different time-steps.

A GLM is said to be canonical if its starting and finishing methods are given by the preconsistency vectors u and ω^H.

Theorem 2.1. *Every GLM $\mathcal{M}_h$, with starting and finishing methods, $\mathcal{S}_h$ and $\mathcal{F}_h$, is equivalent to a canonical GLM defined by the composition*

$$\mathbf{C}_h = T_h^{-1} \circ \mathcal{M}_h \circ T_h,$$

where $T_h, T_h^{-1} : X^m \longrightarrow X^m$ are respectively determined by the GLM tableaux

$$\left[\begin{array}{c|c} A_S & U_F \\ \hline B_S & I \end{array}\right], \quad and \quad \left[\begin{array}{c|c} A_S - U_F B_S & U_F \\ \hline -B_S & I \end{array}\right], \tag{2.1}$$

where A_S, U_F, B_S are the coefficients matrices of the starting and finishing methods, and I is the $m \times m$ identity matrix.

Remark 2.2. Suppose that $\mathcal{M}_h$ is a symmetric GLM. If $\mathcal{S}_h$ and $\mathcal{F}_h$ are symmetric, then $\mathbf{C}_h$ is symmetric.

2.1. **Composition of canonical methods.** Composition is a technique applied to numerical methods to construct higher-order methods.
Let Φ_h denote a one-step method. Then, a composition method with step sizes $\gamma_1 h$, $\gamma_2 h$,...,$\gamma_s h$ is given by

$$\Psi_h = \Phi_{\gamma_s h} \circ ... \circ \Phi_{\gamma_1 h}, \tag{2.2}$$

where γ_1 ,..., γ_s are real numbers.
Consider a canonical GLM $\mathbf{C}_h$ with an invertible matrix V. As explained before, the inputs to a canonical method are h-independent. Thus, the generalisation of the composition (2.2) to GLMs is in the form

$$\mathcal{C}_h := \mathbf{C}_{\gamma_s h} \circ ... \circ V^{-1}\mathbf{C}_{\gamma_2 h} \circ V^{-1}\mathbf{C}_{\gamma_1 h}.$$

3. Numerical experiments

In this section we present the results of numerical experiments to show efficiency of the proposed method of order 6. Computational experiments are done by applying method on the following Hénon-Heiles problem and the famous Lorenz equations.
P1. The Hénon-Heiles problem which is used as a simple model of the motion of a star inside a galaxy with the Hamiltonian given by

$$H(p,q) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2) + q_1^2 q_2 - \frac{1}{3}q_2^3.$$

P2. The famous Lorenz equations provide a simple example of a chaotic system. They are given by

$$\begin{cases} y_1' = \sigma(y_2 - y_1), \\ y_2' = r y_1 - y_2 - y_1 y_3, \\ y_3' = y_1 y_2 - b y_3, \end{cases}$$

where σ, r and b are positive parameters. Following Lorenz, we set $\sigma = 10$, $b = 8/3$, $r = 28$, $y(0) = [0, 1, 0]^T$ and $t \in [0, 100]$. The results of numerical experiments for these problems are presented in Figures 1 and 2. It is known that Hénon-Heiles problem is a Hamiltonian problem and by Fig 1 we can see that the method preserves the structure for the tolerance equals to $tol = 10^{-14}$.

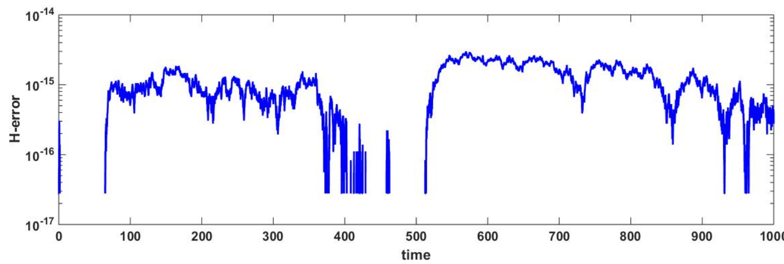

FIGURE 1. Errors in the Hamiltonian H for Hénon-Heiles problem with $tol = 10^{-14}$, and $T = 1000$.

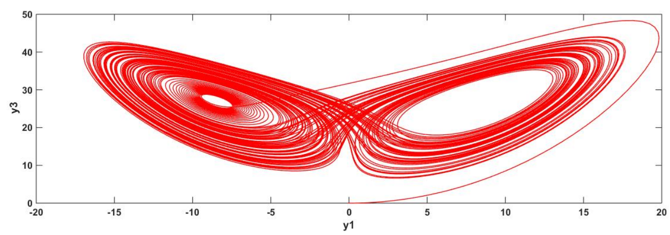

FIGURE 2. Lorenz in the $y_1 \times y_3$ plane with $tol = 10^{-14}$, and $T = 100$.

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STOCHASTIC ANALYSIS OF A BLOOD GLUCOSE REGULATORY MODEL

T. WAEZIZADEH AND F. MOHABATI

ABSTRACT. In this paper, we consider a stochastic model of Glucose-Insulin regulatory system. This model is derived from its deterministic model by virtue of introducing random noises. Ultimately, a set of numerical simulations are provided in order to corroborate and calibrate our results.

1. Introduction

According to the World Health Organization, Diabetes Mellisent is a prevalent illness owing to the enormous number of diabetic individuals [3]. Two prime causes of this disease are detected as the inability of the pancreas to release adequate insulin and the inability of cells to respond the insulin. The mechanism via which these major and critical factors prepare the ground for Diabetes is raising the glucose concentration level in the blood to levels much higher compared with the normal range. In order to deepen and improve the understanding of Diabetes, insulin sensitivity and glucose-insulin regulatory system have been studied by many scholars. Such models are often based on negative feedback loop which naturally exists in biological systems [3], and they have been established by applying the deterministic approach so far. A detectable issue is the investigation of the glucose-insulin regulatory system from a stochastic perspective in which a couple of external leading factors such as stress and diet that have profound and detrimental effects on the operation of pancreas in releasing sufficient insulin are considered. In this paper, we assume that

Keywords: Stochastic Model, Diabete, Milstein method..
AMS Mathematical Subject Classification [2010]: 34K50, 65C20, 92B05..

stochastic perturbations are of white noise type that are directly proportional to the concentration of insulin and glucose.

2. Formulation of Mathematical Model

There are varied ways to include random factors in a model and obtain a stochastic model from a deterministic one. In this article, the approach presented by Beddington and May [1] is selected. In this respect, white noise stochastic perturbation is employed. Noticeably, this type of noise is directly proportional to $x(t)$ and $y(t)$, and has influences on $\frac{dx(t)}{dt}$ and $\frac{dy(t)}{dt}$, where $x(t)$ and $y(t)$ are the concentration of glucose and respectively insulin in the blood.

Making use of this approach, we obtain the SDE model 2.2 that is similar to its deterministic version 2.1. We introduce stochastic perturbation terms into the equations of the model to include the impacts of haphazardly fluctuating environments in the model. The corresponding glucose-insulin regulatory model has the following form

$$\left\{\begin{array}{ll} \dfrac{dx}{dt} & = p - \alpha(x - x^*) - \beta(y - y^*) \\ \dfrac{dy}{dt} & = q + \gamma(x - x^*) - \delta(y - y^*) \end{array}\right. . \tag{2.1}$$

Let $x^* = \frac{p\delta - \beta q}{\beta\gamma + \alpha\delta}$ and $y^* = \frac{p\gamma + \alpha q}{\beta\gamma + \alpha\delta}$. x^* and y^* indicate the base level of glucose and insulin concentration, respectively and p, q, α, β, γ and δ are positive constants. p and q denote the rate at which the glucose and respectively the insulin intravenous are injected into the extracellular fluid volume. α and β evaluate the effects of sensitivity of liver glycogen storage and the influences of utilization of tissue glucose on glucose and insulin concentration, respectively. Moreover, γ is the rate at which the sensitivity of pancreatic insulin output affects the glucose concentration. In addition, δ is the effective rate of the sensitivity of insulinase activity on the insulin concentration.

We set $p^* = p + \alpha x^* + \beta y^*$ and $q^* = q - \gamma x^* + \delta y^*$, therefore, the model 2.1 changes into the form

$$\left\{\begin{array}{ll} \dfrac{dx}{dt} & = p^* - \alpha x - \beta y \\ \dfrac{dy}{dt} & = q^* + \gamma x - \delta y \end{array}\right. .$$

Consequently, the stochastic form of the model is obtained as

$$\left\{\begin{array}{ll} dx & = (p^* - \alpha x - \beta y)dt + \sigma_1 x dB_1(t) \\ dy & = (q^* + \gamma x - \delta y)dt + \sigma_2 y dB_2(t) \end{array}\right. , \tag{2.2}$$

where σ_1 and σ_2 are real constants which indicate the intensification of environmental fluctuations, and $B_1(t)$ and $B_2(t)$ are one-dimensional independent standard Brownian motions.

The state space of the model is

$$\Gamma = \{(x,y) \in \mathbb{R}_+^2 : 0 < x, y \leq K\}, \tag{2.3}$$

where K is the maximum value of insulin and glucose in the blood. Throughout this paper, $\mathbb{E}(.)$ denotes the expectation of the corresponding random variable.

3. Dynamics of the Model

In this section, we show that the unique positive global solution of the SDE model 2.2 exists.

Theorem 3.1. *If $(x(0), y(0)) \in \Gamma$ is an arbitrary initial condition for Model 2.2, then there is a unique solution $(x(t), y(t))$ in which for all $t > 0$, it remains in Γ with probability one.*

Proof. At first, we show that the coefficients of Model 2.2 satisfy the local Lipschitz condition. Assume that $\sigma : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ and $b : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ are mappings for which $\sigma(t,x,y) = (\sigma_1 x, \sigma_2 y)$ and $b(t,x,y) = (p^* - \alpha x - \beta y, q^* + \gamma x - \delta y)$. The claim is that there is a constant $D > 0$ such that the Inequality 3.1 holds.

$$|b(t,x,y) - b(t,x',y')| + |\sigma(t,x,y) - \sigma(t,x',y')| \leq D|(x,y) - (x',y')|, \tag{3.1}$$

where $x, y \in \mathbb{R}_+$ and $t \in [0,T]$.
It is sufficient to set:

$$D^2 = 1/4 \max\{\alpha^2 + \gamma^2 + \theta^2, \beta^2 + \delta^2 + 1, \sigma_1^2, \sigma_2^2\}.$$

Therefore, the local Lipschitz condition for the coefficients is satisfied. That is to say that there exists a unique local solution for Model 2.2 on $[0, \tau_e)$, where τ_e is the explosion time, and it is guaranteed by Itô's formula that this solution is positive. Now, we intend to show that this local solution is in fact global, that means $\tau_e = \infty$ a.s.
Suppose that $n_0 > 0$ is sufficiently large so that $x(0)$ and $y(0)$ lies in the interval $[\frac{1}{n_0}, n_0]$ for any integer $n > n_0$. Define the stopping time as

$$\tau_n = \inf\Big\{t \in [0, \tau_e] : \min\{x(t), y(t)\} \leq \frac{1}{n} \text{ or } \max\{x(t), y(t)\} \geq n\Big\}. \tag{3.2}$$

Let $\inf \emptyset = \infty$, where $\emptyset$ represents the empty set. The sequence $\{\tau_n\}_{n \geq 1}$ is increasing. If we set $\tau_\infty := \lim_{n\to\infty} \tau_n$, we get $\tau_\infty \leq \tau_e$ a.s.

Subsequently, we show that $\tau_\infty = \infty$ a.s. In case this statement is not true, there exists a constant $T > 0$ such that for any $\varepsilon \in (0,1)$, $\mathcal{P}\{\tau_\infty \leq T\} > \varepsilon$. Therefore, there is an integer $n_1 \geq n_0$ such that

$$\mathcal{P}\{\tau_n \leq T\} \geq \varepsilon, \quad n \geq n_1. \tag{3.3}$$

Define a C^2–function $V : \mathbb{R}_+^2 \to \mathbb{R}_+$ by $V(x, y) = x + y - 1 - \ln y$ which is a non-negative function.

By the Itô's formula, we have

$$LV = (p^* - \alpha x - \beta y) + (1 - \frac{1}{y})(q^* + \gamma x - \delta y) + \frac{\sigma_2^2}{2}. \tag{3.4}$$

In case $q^* > 0$, we get

$$LV < p^* + q^* + \gamma x + \delta + \frac{\sigma_2^2}{2}.$$

For $q^* \leq 0$, it is evident that

$$LV < p^* + \gamma x - \frac{q^*}{y} + \delta + \frac{\sigma_2^2}{2}.$$

Thus, there is a positive constant M such that $LV \leq M$. Hence,

$$\mathrm{d}V(x, y) \leq M\mathrm{d}t + \sigma_1 x \mathrm{d}B_1(t) + \sigma_2(y - 1)\mathrm{d}B_2(t), \tag{3.5}$$

which implies that

$$\begin{aligned} \int_0^{\tau_n \wedge T} \mathrm{d}V(x(r), y(r)) \leq & \int_0^{\tau_n \wedge T} M\mathrm{d}t + \int_0^{\tau_n \wedge T} \sigma_1 x(r)\mathrm{d}B_1(r) \\ & + \int_0^{\tau_n \wedge T} \sigma_2(y(r) - 1)\mathrm{d}B_2(r), \end{aligned} \tag{3.6}$$

where $\tau_n \wedge T = \min\{\tau_n, T\}$. Applying the expectation operator to Inequality 3.6, we get

$$\mathbb{E}V(x(\tau_n \wedge T), y(\tau_n \wedge T)) \leq V(x(0), y(0)) + MT. \tag{3.7}$$

Let $\Omega_n := \{\tau_n \leq T\}$ for $n \geq n_1$. From 3.3, we have $\mathcal{P}(\Omega_n) \geq \varepsilon$. We also get

$$V(x(\tau_n, \omega), y(\tau_n, \omega)) \geq (n - 1 - \ln n) \wedge (\frac{1}{n} - 1 - \ln \frac{1}{n}).$$

It then results from 3.7 that

$$\begin{aligned} V(x(0), y(0)) + MT &\geq \mathbb{E}[I_{\Omega_n} V(x(\tau_n), y(\tau_n))] \\ &\geq \varepsilon\Big((n - 1 - \ln n) \wedge (\frac{1}{n} - 1 - \ln \frac{1}{n})\Big), \end{aligned}$$

where I_{Ω_n} is the characteristic function of Ω_n. Allowing $n \to \infty$ leads to

$$\infty > V(x(0), y(0)) + MT = \infty \quad a.s., \tag{3.8}$$

that is a contradiction. Ergo, $\tau_\infty = \infty$. This means that the solution of Model 2.2 will not expand at a finite time with probability one. It completes the proof. $\square$

4. Numerical Simulations and Discussions

In this section, by utilizing the Milstein method mentioned in [2], we present a set of numerical examples to illustrate the dynamical behavior of Model 2.2. In this method, we discretize the equations as

$$x_{k+1} = x_k + (p^* - \alpha x_k - \beta y_k)\Delta t + \sigma_1 x_k \sqrt{\Delta t}\xi_k + \frac{\sigma_1^2}{2} x_k^2(\xi_k^2 - 1)\Delta t$$

and

$$y_{k+1} = y_k + (q^* + \gamma x_k - \delta y_k)\Delta t + \sigma_1 y_k \sqrt{\Delta t}\eta_k + \frac{\sigma_1^2}{2} x_k^2)\eta_k^2 - 1)\Delta t,$$

where ξ_k and η_k are the Gaussian random variables $N(0,1)$ for $k = 1, 2, \ldots, n$. As follows, we provide an example via which the solution is investigated for two diverse cases.

Example 4.1. Assume that $\alpha = 0.0106$, $\beta = 0.7261$, $\gamma = 0.0014$, $\delta = 0.06$, $p = 1.5$ and $q = 0.1$. Again, two cases are perceived.
In the first case, consider $\sigma_1 = 0.0001$ and $\sigma_2 = 0.01$. As Figure 1 illustrates, the level of glucose is on the rise on account of the glucose consumption during the laboratory test, nonetheless, the concentration of insulin produced simultaneously by pancreas is not sufficient to keep the amount of glucose in the blood under control. This leads to maintain the high concentration of the glucose in the blood in the long run. For this reason, this case can be considered as diabetic. Considering the parameters such as $\sigma_1 = 0.0007$ and $\sigma_2 = 0.1$. Therefore, as Figures 1 and 2 indicate, the solutions are unstable and the level of the glucose and insulin concentrations in the blood is exceedingly high.

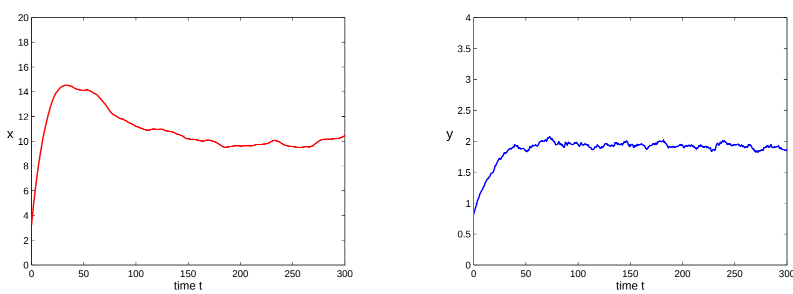

Figure 1. Stochastic trajectories of the model with parameter values $\alpha = 0.0106$, $\beta = 0.7261$, $\gamma = 0.0014$, $\delta = 0.06$, $p = 1.5$, $q = 0.1$, $\sigma_1 = 0.0001$ and $\sigma_2 = 0.01$.

The presented example demonstrate that the more the system affected by the noise, the more the solutions experience fluctuations. Ergo, it takes a longer period of time for the considered individual to recover his/her normal status. That is why, some factors such as the stress can be considered as a chief cause of the dramatic

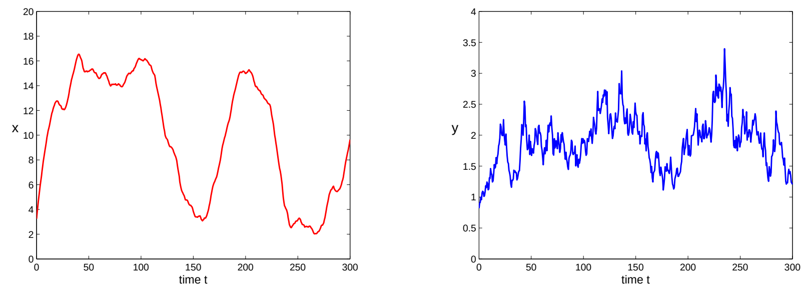

FIGURE 2. Stochastic trajectories of the model with parameter values $\alpha = 0.0106$, $\beta = 0.7261$, $\gamma = 0.0014$, $\delta = 0.06$, $p = 1.5$, $q = 0.1$, $\sigma_1 = 0.0007$ and $\sigma_2 = 0.1$.

fluctuations in the system that creates serious and tragic circumstances in which the patient's general state of health tends to critical and precarious conditions.

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GEOMETRICAL STRUCTURE OF INVARIANT GRAPHS

M.ZAJ

Abstract. In this paper, we address the existence and ergodicity of non-uniformly hyperbolic attracting sets for a certain class of smooth endomorphisms on the solid torus. Such systems have formulation as a skew product system defined by planar diffeomorphisms, with average contraction condition, forced by any expanding circle map. These attractors are invariant graphs of upper semicontinuous maps that support exactly one physical measure. Under some conditions including negative fiber Lyapunov exponents, we prove the existence of unique non-uniformly hyperbolic attracting invariant graphs for these systems which attract positive orbits of almost all initial points.

1. Introduction

Many problems in dynamical systems deal with the study of the limit behavior of orbits, leading to the study of the so-called attractors. Understanding the structure of attractors is a major goal in the area. Here we study attractors of random dynamical systems from the topological and ergodic angles. Let us observe that there are several different definitions for the concept of an attractor depending of the point on view, see [5], our precise setting will be stated later.

The situation that we are interested in is random dynamical systems forced by a deterministic external factor. In general, systems of this kind are modeled, in discrete

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time, as skew products

$$F(b,x) = (h(b), f_b(x)), \tag{1.1}$$

where h is the base transformation (also driving system or forcing process) and f_b is the fiber transformation (the forced system). The theory of such systems has been developed as a way to model systems that are forced by random perturbations, see [1, 2, 3]. In this work, we introduce a certain class of skew products that exhibit an attracting invariant graph with a complicated dynamic and a nice ergodic property. Indeed, the aim of this paper is to discuss a class of smooth endomorphisms on the solid torus admitting robust non-uniformly hyperbolic attracting sets which are invariant graphs and supporting a unique physical measure. To establish this result, first we prove it for a special class of skew products over expanding circle maps with weak contractive planar fiber maps. In our approach, we also prove the occurrence of a master-slave synchronization. A *master-slave synchronization* means the convergence of orbits starting at different initial points when iterated by the same sequence of diffeomorphisms which is explained by a single attracting invariant graph for the skew product system [6, 7]. We also show that the attractors are the support of unique invariant ergodic physical measures.

2. Main results

Throughout this paper we assume that X is a compact ball of $\mathbb{R}^2$ and S^1 is the unit circle. Take the solid torus $\mathbb{T} = S^1 \times X$. Denote by $\mathcal{C}(\mathbb{T})$ the space of all skew products over φ with the fiber X, where φ is a linear expanding map on the circle, i.e. the maps of the form

$$F : (t,x) \mapsto (\varphi(t), f_t(x)),\ t \in S^1,\ x \in X, \tag{2.1}$$

where $\varphi(t) = kt \pmod 1$ for some suitable positive integer k. Here, f_t is a diffeomorphism onto its image, depending C^0-continuously on the base parameter t. The metric on $\mathcal{C}(\mathbb{T})$ is defined by

$$\operatorname{dist}(F, \widetilde{F}) := \sup_t \operatorname{dist}_{C^1(X)}(f_t^{\pm 1}, \widetilde{f}_t^{\pm 1}). \tag{2.2}$$

In this article, we generalize the concept of a bony graph [4] to our setting. We say that a closed invariant set of a skew product F is a *bony graph* if it intersects almost every fiber at a single point and any other fiber at a compact connected set which is called a bone. A bony graph can be represented as a disjoint union of two sets, K and Γ, where K denotes the union of the bones. The projection of K by the natural projection map to the base has zero measure, while Γ is the graph of some measurable function from base to the fiber. Let Δ be a maximal attractor of F. We say that Δ is a *semicontinuous bony graph* (SCBG) if Δ is a bony graph and the graph function is upper semicontinuous.

Theorem 2.1. [8] *There exists a nonempty open set $\mathcal{U}$ in $\mathcal{C}(\mathbb{T})$ such that any skew product F belonging to $\mathcal{U}$, admits a non-uniformly hyperbolic attractor Δ_F such that*

(1) Δ_F *is either a continuous invariant graph or a semicontinuous bony graph,*
(2) Δ_F *has a negative maximal Lyapunov exponent,*
(3) Δ_F *supports a unique physical measure.*

In particular, it is Bernoulli and so mixing. Also, the open set $\mathcal{U}$ can be chosen as a neighborhood of a particular weakly (and not uniformly) hyperbolic skew product in the space $\mathcal{C}(\mathbb{T})$.

As we mentioned before, when the maps are uniformly contracting in the fiber direction it is extremely easy to prove the existence of a continuous invariant graph supporting an ergodic physical measure. In most cases it is difficult to get sharp uniform bounds for the contraction rate of fiber maps, instead, this is usually expressed in terms of the most negative Lyapunov exponent, which is an averaged quantity. It is important to know the regularity of the graph function in the case in which the contraction on the fiber is given in terms of Lyapunov exponents. Here, we consider the non-uniform case and construct physical measures supported on non-uniformly hyperbolic attractors of smooth endomorphisms and show that these attractors have the form of an invariant graph. In particular, the graph function is upper semicontinuous.

For positive integer $k > 2$, $\mathcal{E}(\mathbb{T})_k$ denote the space of all C^1-smooth k-to-1 coverings (endomorphisms) of $\mathbb{T}$ by itself equipped with C^1-topology.

Theorem 2.2. [8] *There exists a nonempty open set $\mathcal{W} \subset \mathcal{E}(\mathbb{T})_k$ such that any $\mathcal{F}$ belonging to $\mathcal{W}$ admits a non-uniformly hyperbolic attractor which is either an invariant continuous graph or an invariant semicontinuous bony graph with negative maximal fiber Lyapunov exponent. Moreover, there exists a unique physical measure supported on the closure of the graph. In particular, it is Bernoulli and so mixing. Furthermore, the open set $\mathcal{W}$ can be chosen as a neighborhood of a particular weakly (and not uniformly) hyperbolic system in the space $\mathcal{E}(\mathbb{T})_k$.*

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